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SEMESTER - II (CBCS)

MATHEMATICAL FOUNDATION FOR COMPUTER SCIENCE 2

SUBJECT CODE : MCA21

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M.C.A.	Semester – II
Course Name: Mathematical Foundation for Computer Science 2	**Course Code: MCA21**

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UNIT I

1

LINEAR PROGRAMMING PROBLEM

Unit structure

1.0 OBJECTIVES

After going through this chapter, students will able to:

- Learn how to develop linear programming models for simple problems.
- Formulate a given simplified description of a suitable real world problem as a linear programming model in general, standard and canonical forms.
- Understand the importance of extreme points in obtaining the optimal solution.
- Solve a two-dimensional linear programming problem graphically.

1.1 INTRODUCTION

Linear programming is one of the most popular and widely used quantitative techniques. Linear programming deals with the optimization {maximization or minimization) of a function of variables. It consists of linear functions which are subject to constraints in the form of linear equations or in the form inequalities. Linear programming is considered an important technique to find optimum resource utilization. Linear programming has been used to solve problems involving assignment of jobs to machines, product mix, advertising media selection, least cost diet, transportation, investment opportunity selection and many others.

Linear programming is widely used in Mathematics and some other fields such as economics, business, telecommunication and manufacturing fields. In this chapter, we will learn meaning and formulation of linear programming, its components and various methods to solve linear programming problems i.e. LPP.

Following are important terms used in LPP:

1. **Optimization:** It means 'minimization' or 'maximization' of some mathematical function of any number of variables.

2. **Objective Function:** Any mathematical function to be optimized is called an Objective function

3. **Decision variables:** Variables appearing in an objective function are called decision variables.

4. **Optimum Solution:** The set of values of the decision variables giving the 'maximum' or the 'minimum' value of an objective function is called the optimum solution.

5. **Constrained and unconstrained Optimization:** If the decision variable are subject to some limitations (constraints) on the values which they can take the optimization is called constrained optimization otherwise the optimization called unconstrained optimization.

6. **Mathematical Programing:** Method of finding solution of a constrained optimization problem is called mathematical programming.

Example 1: Find the minimum of $z = 2x_1 + 5x_2 + 7x_3$

Example 2: Find the maximum of $z = 2x_1^2 + 5x_2^3 + 3x_3$

Subject to $x_1 + x_2 + 3x_3 \leq 10$

$3x_1 + 5x_2 + 8x_3 \leq 20$

$2x_1 + x_2 + 4x_3 \leq 30$

And $x_1, x_2, x_3 \geq 0$

In Example 1, the objective function is linear in x_1, x_2 and x_3. In Example 2 the objective function is non-linear but the constraints are linear function of the decision variables.

7. **Linear Programming:** If the objective function as well as all the constraints are linear function of the decision variables, an optimization problem is called Linear Programming Problem (LPP).

Example 3: Maximize $z = x_1 + 5x_2 + 7x_3$ Objective Function

Subject to $x_1 + 3x_2 + 3x_3 \leq 15$

$4x_1 + 5x_2 + 8x_3 \leq 25$ Constraints

$3x_1 + x_2 + 4x_3 \leq 30$

And $x_1, x_2, x_3 \geq 0$ Non negativity condition

8. **Coefficient of Objective Function** represent 'profit' per unit of the decision variables for the maximization problem and they represent 'cost' for a minimization problem.

9. **Coefficient of the variables in the constraints** are called technological coefficient / requirements.

10. **Constants on the R. H. S. of the constraints** represent the resource availabilities in different departments.

11. **General Statement of an LPP:**

Optimize (Minimize / maximize)

$Z = C_1x_1 + C_2x_2 + \dots\dots\dots\dots + C_nx_n$

$= \sum_{j=1}^{n} C_j x_j$

Subject to

$a_{11}x_1 + a_{11}x_1 + \dots\dots\dots\dots\dots\dots + a_{1n}x_n \geq, =, \leq b_1$

$a_{21}x_1 + a_{22}x_1 + \dots\dots\dots\dots\dots\dots + a_{2n}x_n \geq, =, \leq b_2$

.

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$A_{m1}x_1 + a_{m2}x_1 + \text{-----------------} + a_{mn}x_n \geq, =, \leq b_m$

And $x_1, x_2, \cdots\cdots\cdots\cdots\cdots.\ xn \geq 0$

In the matrix notation the same problem can be stated as follows:-

Optimize z = C'x

Sunject to Ax $\geq, =, \leq$ B

Where

C = $[C_1\ C_2 \ldots C_n\ ]$, x = $[x_1\ x_2 \ldots x_n\ ]$, A = $[a_{11}\ a_{12} \ldots a_{1n}\ a_{21}\ a_{22} \ldots a_{2n} \ldots\ldots\ldots\ldots a_{m1}\ a_{m2}\ .\ a_{mn}\ ]$ and B = $[b_1\ b_2 \ldots b_m\ ]$

And C' = Transpose of C

Some more Definitions:

1. **Feasible Area:** Area bounded by all the constraints along with non-negativity conditions is called the feasible area.

Max or min $Z = \sum_{j=1}^{n} C_j x_j$

Subject to

$$\sum_{j=1}^{n} a_{ij} x_j \geq, =, \leq b_i;\ i = 1, 2 \ldots m$$

And $x_j \geq 0;\ j = 1, 2 \cdots n$

2. **Feasible Solution:** Any set of values of decision variables satisfying all the constraints is called a feasible solution.

   Number of feasible solution can be infinite. If there is no such solution, the problem is said to have an infeasible solution.

3. **Alternate Optimum Solution:** If there are more than one set of values of decision variables giving the optimum values of the objective function, then the problem is said to have alternate optimum solution.

4. **Unbounded Solution:** If for some problem the value of the objective function can be increased or decreased infinitely, then such a solution is called an unbounded solution.

5. **Corner Point: In LPP, intersection of two** (any) Constraints is called a corner point of the feasible area.

6. **Degenerate Solution:** In an LPP an optimum solution always lies at some corner point of the feasible solution. If more than two constraints are passing through the corner point, it means a redundant (excess/extra/not needed for the solution) constraint. In such situation degeneracy occur.

1.2 FORMULATION OF LINEAR PROGRAMMING PROBLEM

Example 4: A company manufactures two products A and B, Each using Machine I and Machine II. Processing time per unit of A are 5 and 6 hours respectively and that of B are 4 and 5 respectively. Maximum number of hours available are 20 for machine I and 22 for machine II. Per unit, profit of A and B are Rs. 6 and Rs. 5 respectively. Formulate LPP to determine production of A and B for maximum profit.

Solution: Given A and B are products and I and II are products.

Processing time per unit of A are 5 and 6 hours respectively and that of B are 4 and 5 respectively.

Available time 20 hrs for machine I and 22 hrs for machine II

Profit of product A and product B is Rs. 6 and Rs. 5 respectively.

Machine	Machine hrs/unit		Total Available Time
	Product A	Product B	
I	5	4	20
II	6	5	22
Profit/unit(Rs)	6	5	-

Let x_1 and x_2 be the number of units of product A and product B to be produced respectively. The problem can be stated as follows:

Maximize z = 6 $x_1 + 5\ x_2$

Subject to $5x_1 + 4x_2 \leq 20$

$6x_1 + 5x_2 \leq 30$

And $x_1, x_2 \geq 0$

Example 5: ABC Company engaged in diet food. Two food A and B contains three kinds of vitamins Vit A, Vit B and Vit C. One unit of A costs Rs. 30 and B costs Rs. 60. One unit of A contains 36 unit of vit A, 3 units of vit B and 20 units of vit C. One unit of B contains 6 unit of vit A, 12 units of vit B and 10 units of vit C. Minimum requirement of vit A and vit B ia 108 and 36 units and of vit C should not exceed 100 units. Formulate LPP.

Solution: Given A and B are two types of food.

Cost per unit of A is Rs. 30 and that of B is Rs. 60.

Machine	Machine hrs / unit		Requirement
	A	B	
Vit A	36	6	108
Vit B	3	12	36
Vit C	20	10	100
Cost/unit(Rs)	30	60	

Let x_1 and x_2 be the number of units of food A and food B. The problem can be stated as follows:

Min $z = 30\ x_1 + 60\ x_2$

Subject to $36\ x_1 + 6\ x_2 \geq 108$

$3\ x_1 + 12\ x_2 \geq 36$

$20\ x_1 + 10\ x_2 \leq 100$

And $x_1, x_2 \geq 0$

Example 6: CCD at Pune runs 24 hours coffee shop. Each waiter has to work in continuity 8 hrs. But to have continuity in work, waiter are added at every 4 hrs of interval to work along those who have already completed 4 hrs. The requirement for different 4 hrs period are estimated as follows. Formulate an LPP.

Duration (Hrs)	Minimum number of waiter required
00 – 04	4
04 – 08	5
08 – 12	7
12 – 16	6
16 – 20	7
20 – 24	3

Solution: Let xj = number of waiters reporting at the beginning of the jth time interval, j = 1, 2, …., 6

Min $z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$

Subject to $x_6 + x_1 \geq 4$

$x_1 + x_2 \geq 5$

$x_2 + x_3 \geq 7$

$x_3 + x_4 \geq 6$

$x_4 + x_5 \geq 7$

$x_5 + x_6 \geq 3$

And $x_1, x_2 \geq 0$

1.3 GRAPHICAL METHOD

Once a problem is formulated as mathematical model, the next step is to solve the problem to get the optimal solution. If the number of decision variables is only two, a linear programming problem can be solved graphically.

Steps for Graphical Method:

1. Formulate the mathematical model for the given problem.
2. Draw the x_1 and x_2 axes. The non-negativity conditions $x_1 \geq 0$ and $x_2 \geq 0$ implies that solution area lies only in the 1st quadrant of x_1x_2.
3. Plot each of the constraint on the graph. For plotting any line (for each constraint), choose two points:

   One on x_1 axis putting $x_2 = 0$ and other on x_2 axis putting $x_1 = 0$
4. Identify the feasible region that satisfies all the constraints. For the constraints with ≤ sign,

solution area lies below the line (towards the origin) and that of $\geq$ sign, solution area lies above the line (that side of line not facing the origin). The area common to all the constraints is called feasible region. This area may be bounded or unbounded. Show the feasible region by shading.

5. Find the optimum solution.

Example 7: Solve the following LPP by graphical method.

Maximize z = $4\,x_1 + 7\,x_2$

Subject to $6x_1 + 12x_2 \leq 60$

$5x_1 + 5x_2 \leq 30$

And $x_1, x_2 \geq 0$

Solution: The non-negativity conditions $x_1 \geq 0$ and $x2 \geq 0$

∴ Solution area lies in first quadrant.

Constraints are of $\leq$ type, Solution area lies below the line i. e. towards the origin.

For plotting the line,

Constraint 1: $6x_1 + 12x_2 \leq 60$

When $x_1 = 0$, $x_2 = 5$; $x_1 = 10$, $x_2 = 0$

Constraint 2: $5x_1 + 5x_2 \leq 30$

When $x_1 = 0$, $x_2 = 6$; $x_1 = 6$, $x_2 = 0$

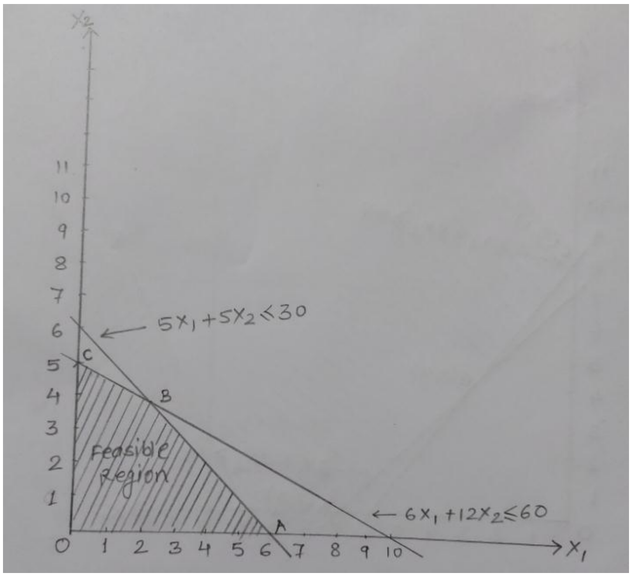

Fig 1.1

Find the intersection of constraints.

Here the point of intersection is B (2, 4).

The feasible area is shown shaded and given by the polygon OABC.

Here the feasible area is bounded/closed.

Optimum solution: Optimum solution always lies at a corner point of the feasible region/area.

∴ Find the values of the objective function z at the corner points O, A, B and C.

Maximize z = $4x_1 + 7x_2$

Corner Point	Value of z
O ($x_1 = 0, x_2 = 0$)	4 x 0 + 7 x 0 = 0
A ($x_1 = 6, x_2 = 0$)	4 x 6 + 7 x 0 = 24
B ($x_1 = 2, x_2 = 4$)	4 x 2 + 7 x 4 = 36
C ($x_1 = 0, x_2 = 5$)	4 x 0 + 7 x 5 = 35

Since z = 36 is maximum at ($x_1 = 2, x_2 = 4$), this is the optimum solution of the given LPP.

Example 8: Solve the following LPP by graphical method.

Minimize z = $2x_1 + 3x_2$

Subject to $2x_1 + 2x_2 \geq 12$

$6x_1 + 3x_2 \geq 18$

And $x_1, x_2 \geq 0$

Solution: The non-negativity conditions $x_1 \geq 0$ and $x_2 \geq 0$

∴ Solution area lies in first quadrant.

Constraints are of ≥ type, Solution area lies above the line.

For plotting the line,

Constraint 1: $2x_1 + 2x_2 \geq 14$

When $x_1 = 0, x_2 = 7$; $x_1 = 7, x_2 = 0$

Constraint 2: $6x_1 + 8x_2 \geq 48$

When $x_1 = 0, x_2 = 6$; $x_1 = 8, x_2 = 0$

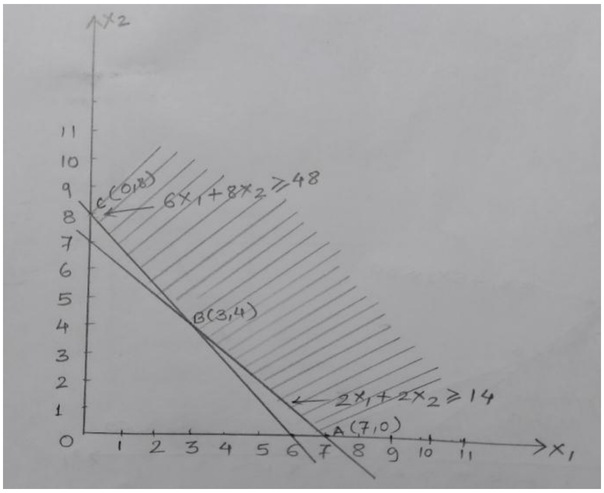

Fig 1.2

Find the intersection of constraints.

Here the point of intersection is B (3, 4).

Shade the feasible area. Here the feasible area is unbounded.

Find the values of the objective function z at the corner points A(7, 0), B(3, 4). and C(0, 8).

Minimum z = $2 x_1 + 3 x_2$

Corner Point Value of z

A ($x_1 = 7, x_2 = 0$) 2 x 7 + 3 x 0 = 14

B ($x_1 = 3, x_2 = 4$) 2 x 3 + 3 x 4 = 18

C ($x_1 = 0, x_2 = 8$) 2 x 0 + 3 x 8 = 24

Since z = 14 is minimum at ($x_1 = 3, x_2 = 4$), this is the optimum solution of the given LPP.

Example 9: A company produces pen A and pen B. Profit is Rs. 5 and Rs. per pen respectively. Raw material requirement of pen A is twice of pen B. Supply of raw material is sufficient for 1000 pens of type A and B. Pen needs special clips, 400 such clips are available for A and for B 700 clips are available. Formulate problem of LPP and solve it for maximizing profit by graphical method.

	Types of Pen		Availability
	A	B	
Raw Material	2	1	1000
Profit (Rs.)	5	4	-

Solution: First, we formulate the LPP as follows.

Objective Function,

$$\text{Max } z = 5x_1 + 4x_2$$

Subject to $2x_1 + x_2 \leq 1000$

$x_1 \leq 400$

$x_2 \leq 700$

And $x_1, x_2 \geq 0$

The non-negativity conditions $x_1 \geq 0$ and $x_2 \geq 0$

∴ Solution area lies in first quadrant.

Constraints are of ≤ type, Solution area lies below the line i. e. towards the origin.

For plotting the line,

Constraint 1: $2x_1 + x_2 \leq 1000$

When $x_1 = 0$, $x_2 = 1000$; $x_1 = 500$, $x_2 = 0$

Constraint 2: $x_1 \leq 400$

∴ $x_1 = 400$

Constraint 3: $x_2 \leq 700$

∴ $x_2 = 700$

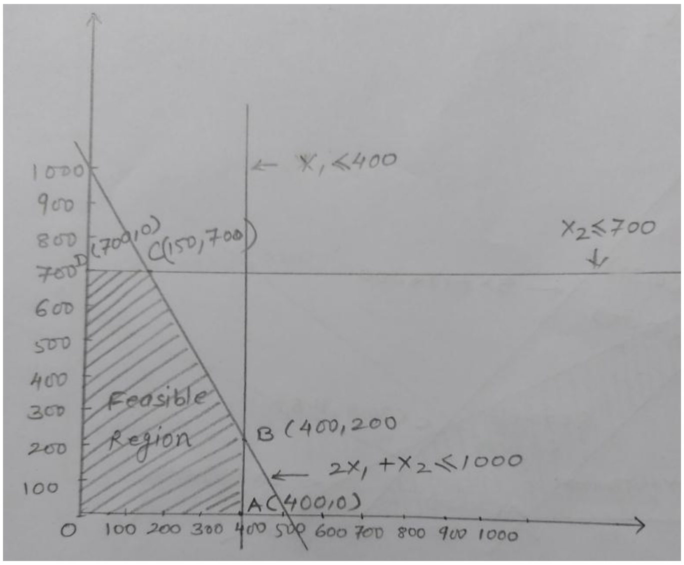

Fig 1.3

Find the intersection of constraints.

Here the points of intersection are B (400, 200) and C (150, 700)

The feasible area is shown shaded and given by the polygon OABCD.

Here the feasible area is bounded/closed.

Optimum solution: Optimum solution always lies at a corner point of the feasible region/area.

∴ Find the values of the objective function z at the corner points O, A, B, C and D.

Maximize $z = 5x_1 + 4x_2$

Corner Point	Value of z
O ($x_1 = 0, x_2 = 0$)	5 x 0 + 4 x 0 = 0
A ($x_1 = 400, x_2 = 0$)	5 x 400 + 4 x 0 = 2000
B ($x_1 = 400, x_2 = 200$)	5 x 400 + 4 x 200 =2800
C ($x_1 = 150, x_2 = 700$)	5 x 150 + 4 x 700 = 3550
D ($x_1 = 700, x_2 = 0$)	5 x 700 + 4 x 0 = 3500

Since z = 3550 is maximum at ($x_1 = 150$, $x_2 = 700$), this is the optimum solution of the given LPP.

Example 10: Solve the following LPP by graphical method.

Max $z = 200x + 300y$

Subject to $5x + 4y \le 200$

$3x + 5y \le 150$

$5x + 4y \ge 100$

And $x, y \ge 0$

Solution: The non-negativity conditions $x \ge 0$ and $y \ge 0$

∴ Solution area lies in first quadrant.

Constraints are of ≤ type as well as ≥ type

For plotting the line,

Constraint 1: $5x + 4y \le 200$

When x = 0, y = 50; x = 40, y = 0

Constraint 2: $3x + 5y \le 150$

When x = 0, y = 30; x = 50, y = 0

Constraint 3: $5x + 4y \ge 100$

When x = 0, y = 25; x = 20, y = 0

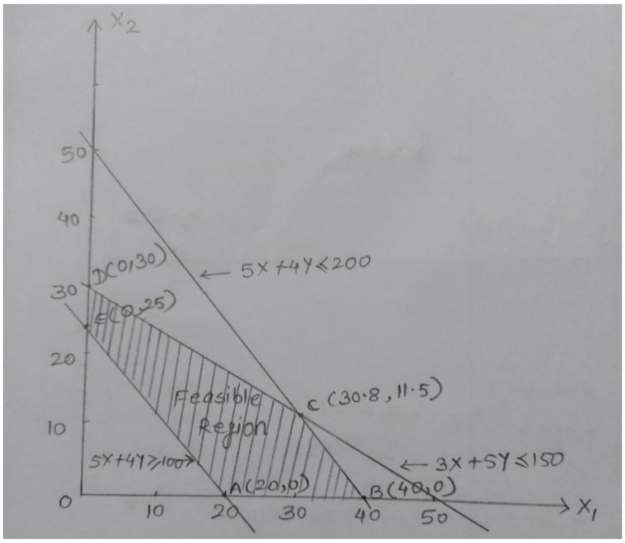

Fig 1.4

Find the intersection of constraints.

Here the point C is the intersection of constraint 1 and constraint 2.

To find the coordinate of point C, we solve constraint 1 and constraint 2 simultaneously.

i.e. Constraint 1 x 3 ⇒ 15x + 12y = 600

and Constraint 2 x 5 ⇒ 15x + 25y =750

∴ x = 30.8 and y = 11.5

The feasible area is shown shaded and given by the polygon ABCDE.

Optimum solution: Optimum solution always lies at a corner point of the feasible region/area.

∴ Find the values of the objective function z at the corner points A, B, C, D and E.

Max z = 200 x + 300 y

Corner Point	Value of z
A (x = 20, y = 0)	200 x 20 + 300 x 0 = 4000
B (x = 40, y = 0)	200 x 40 + 300 x 0 = 8000
C (x = 30.8, y = 11.5)	200 x 30.8 + 300 x 11.5 = 9610
D (x = 0, y = 30)	200 x 0 + 300 x 30 = 9000
A (x = 0, y = 25)	200 x 0 + 300 x 25 = 7500

Since z = 9610 is maximum at (x = 30.8, y = 11.5), this is the optimum solution of the given LPP.

1.4 SPECIAL CASES IN LPP

1. Redundant Constraint: A constraint in a given programming problem is said to be redundant if the feasible region of the problem is unchanged by deflecting that constraint.

Example 11: Solve the following LPP by graphical method.

Max z = $25x_1 + 20x_2$

Subject to $6x_1 + 4x_2 \leq 240$

$2x_1 + 4x_2 \leq 200$

$3x_1 + 4x_2 \leq 360$

And $x_1, x_2 \geq 0$

Solution: The non-negativity conditions $x_1 \geq 0$ and $x_2 \geq 0$

∴ Solution area lies in first quadrant.

For plotting the line,

Constraint 1: $6x_1 + 4x_2 \leq 240$

When $x_1 = 0, x_2 = 60$; $x_1 = 40, x_2 = 0$

Constraint 2: $2x_1 + 4x_2 \leq 200$

When $x_1 = 0, x_2 = 50$; $x_1 = 100, x_2 = 0$

Constraint 3: $3x_1 + 4x_2 \leq 360$

When $x_1 = 0, x_2 = 90$; $x_1 = 120, x_2 = 0$

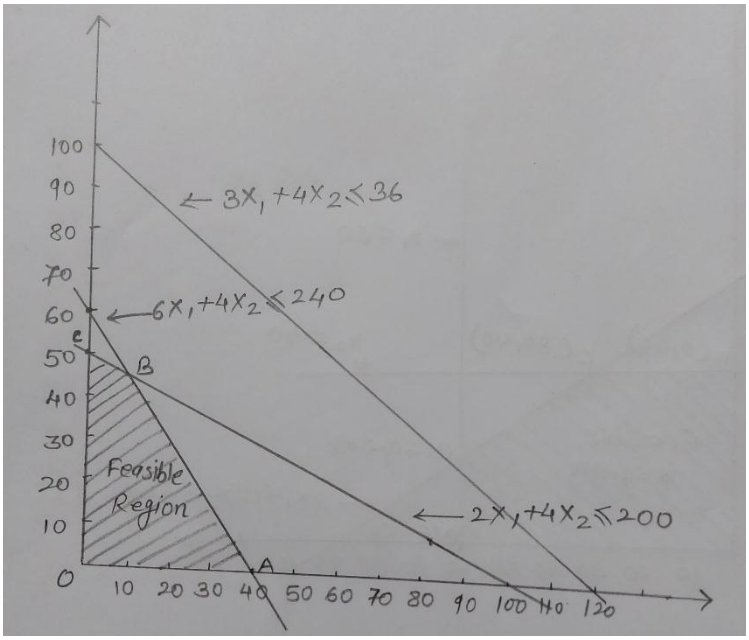

Fig 1.5

In LPP, every constraint forms boundary of the feasible region. However if any constraint does not define boundary of feasible region is called redundant constraint. In above example third constraint does not define the boundary of feasible region. Therefore, it is redundant constraint. It is not necessary for the solution and can be eliminate from the problem.

2. Multiple Solution: So far we have seen that the optimal solution of any LPP lies at corner point of the feasible region and solution is unique. However, in certain cases a given LPP may have more than one optimal solution

Example 12: Solve the following LPP by graphical method.

` Max z = $100x_1 + 150\, x_2$

Subject to $8x_1 + 12x_2 \leq 720$

$x_1 \leq 60$

$x_2 \leq 40$

And $x_1, x_2 \geq 0$

Solution: The non-negativity conditions $x_1 \geq 0$ and $x_2 \geq 0$

∴ Solution area lies in first quadrant.

For plotting the line,

Constraint 1: $8x_1 + 12x_2 \leq 720$

When $x_1 = 0, x_2 = 60$; $x_1 = 90, x_2 = 0$

Constraint 2: $x_1 \leq 60$

$x_1 = 6$

Constraint 3: $x_2 \leq 40$

$x_2 = 40$

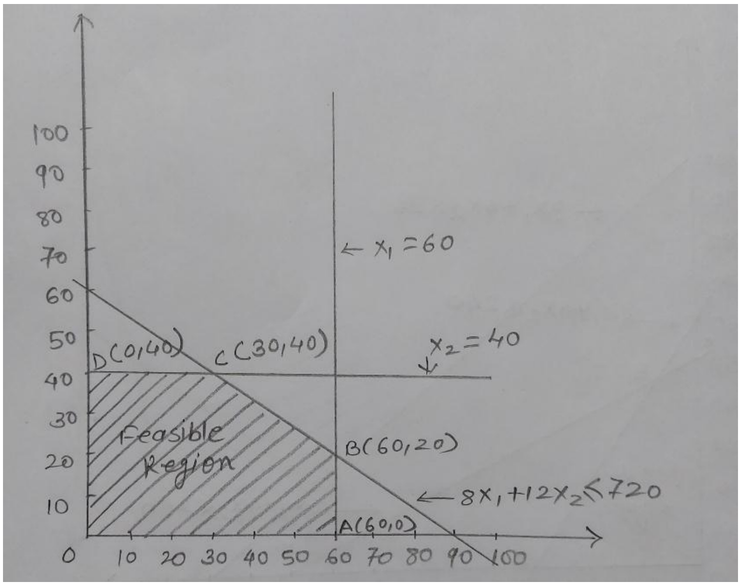

Fig 1.6

The feasible area is shown shaded and given by the polygon OABCD.

Optimum solution always lies at a corner point of the feasible region/area.

∴ Find the values of the objective function z at the corner points O, A, B, C and D.

Max z = $100x_1 + 150x_2$

Corner Point	Value of z
O ($x_1 = 0, x_2 = 0$)	100 x 0 + 150 x 0 = 0
A ($x_1 = 60, x_2 = 0$)	100 x 60 + 150 x 0 =6000
B ($x_1 = 60, x_2 = 20$)	100 x 60 + 150 x 20 = 9000
C ($x_1 = 30, x_2 = 40$)	100 x 30 + 150 x 40 = 9000
D ($x_1 = 0, x_2 = 40$)	100 x 0 + 150 x 40 = 6000

Thus, the maximum value of Z is 9000 and it occurs at two corner points, B and C.

The problem has multiple solution.

3. Unbounded Solution: LPP may have unbounded solution which means it has no limit on constraints. The common feasible region is not bounded at any respect.

Example 13: Solve the following LPP by graphical method.

Max z = $30x_1 + 15x_2$

Subject to $4x_1 + 6x_2 \geq 12$

$4x_1 + x_2 \geq 4$

And $x_1, x_2 \geq 0$

Solution: The non-negativity conditions $x_1 \geq 0$ and $x_2 \geq 0$

∴ Solution area lies in first quadrant.

For plotting the line,

Constraint 1: $4x_1 + 6x_2 \geq 12$

When $x_1 = 0, x_2 = 2$; $x_1 = 3, x_2 = 0$

Constraint 2: $4x_1 + x_2 \geq 4$

When $x_1 = 0, x_2 = 4$; $x_1 = 1, x_2 = 0$

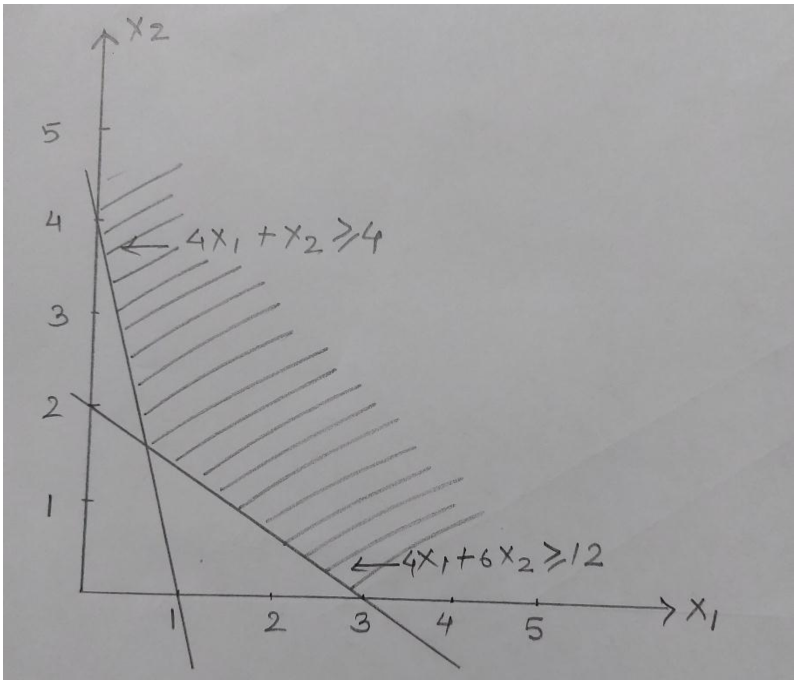

Fig 1.7

In the given graph, feasible region has no upper boundary. Hence solution is infinitely large (as we want to find Max). This is called unbounded solution.

4. Infeasible Solution: If it is not possible to find a feasible solution that satisfies all the constraints, then LPP is said to have an infeasible solution or inconsistence.

Example 14: Solve the following LPP by graphical method.

Max z = $25x_1 + 35x_2$

Subject to $3x_1 + 5x_2 \geq 15$

$2x_1 + 3x_2 \leq 6$

And $x_1, x_2 \geq 0$

Solution: The non-negativity conditions $x_1 \geq 0$ and $x_2 \geq 0$

∴ Solution area lies in first quadrant.

For plotting the line,

Constraint 1: $3x_1 + 5x_2 \geq 15$

When $x_1 = 0$, $x_2 = 3$; $x_1 = 5$, $x_2 = 0$

Constraint 2: $2x_1 + 3x_2 \leq 6$

When $x_1 = 0$, $x_2 = 2$; $x_1 = 3$, $x_2 = 0$

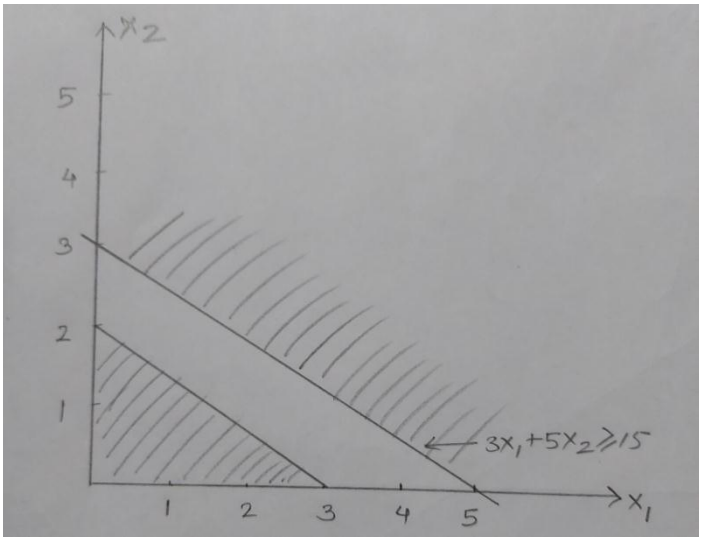

Fig 1.8

As there is no common region. There is no point that satisfies both the quadrant. Thus, the given LPP has no feasible solution.

Limitation of Graphical Method: It is possible to find a graphic solution of LPP, even if number of decision variables are THREE, but it will be cumbersome. For more than THREE variable case, a graphic solution will be impossible.

1.5 SUMMARY

Linear programming is widely used technique in various problems of management. The problem should have well defined objective function with decision variables. The objective function is of maximization when we deal with profit and it is of type minimization when we deal with the cost. The decision variable interact with each other through set of constraints.

A linear programming problem with two-decision variable can be solved by graphical method. Any non-negative solution which satisfies all the constraints is the feasible solution of the problem. The collection of all feasible solution is called feasible region. The optimum solution of LPP always lies at a corner point of the feasible region/area. In some problems, there may be more than one solution. It is also possible that LPP has no finite solution, sometimes problem may have infeasible solution. . If more than three decision variables are there in LPP then graphical method will be impossible.

1.6 REFERENCES

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1.7 EXERCISE

Exercise 1: A nutrition program for babies have proposed two types of food (Food I and Food II) available in the standard packets of 50 grams. The cost per packet are Rs. 2 and 3 respectively. Vitamin availability in each of food per packet and minimum requirement for each type of vitamin are given in the following table.

Vitamins	**Vitamins availability per packet**		**Minimum Requirement**
	I	II	
A	2	1	6
B	6	2	14
Cost/packet	2	3	-

Formulate LPP to determine the optimal combination of food types with minimum cost such that the minimum requirement of vitamins in each type is satisfied.

Exercise 2: In a multispecialty hospital, nurses report for duty at the end of hour period, as shown in the table. Each nurse will work for 8 hours after reporting for the duty.

Interval No.	**Time Period**		**Minimum number of nurses required**
	From	To	
1	12 mid night	4 am	18
2	4 am	8 am	23
3	8 am	12 noon	35

4	12 noon	4 pm	30
5	4 pm	8 pm	22
6	8 pm	12 mid night	15

Formulate an LPP such that total number of nurses reporting for duty is minimized.

Exercise 3: Five items are to be loaded on a ship. The weight, volume and return per unit of each item are given in the table below. The ship cannot carry a weight more than 120 units and volume more than 115 units. Formulate an LPP.

Item No.	Weight	Volume	Return (Rs)
1	6	2	5
2	7	8	9
3	3	5	7
4	3	6	6
5	7	4	3

Exercise 4: Solve the following LPP by graphical method.

Max z = $8x_1 + 9x_2$

Subject to $11x_1 + 9x_2 \leq 9900$

$7x_1 + 12x_2 \leq 8400$

$6x_1 + 16x_2 \leq 9600$

And $x_1, x_2 \geq 0$

Exercise 5: Solve the following LPP by graphical method.

Max z = $4x_1 + 8x_2$

Subject to $2x_1 + 4x_2 \leq 48$

$x_1 + 3x_2 \leq 42$

$x_1 + x_2 \leq 21$

And $x_1, x_2 \geq 0$

Exercise 6: Solve the following LPP by graphical method.

Minimize z = $10x_1 + 20x_2$

Subject to $x_1 + 3x_2 \geq 3$

$x_1 + x_2 \geq 2$

And $x_1, x_2 \geq 0$

Exercise 7: Solve the following LPP by graphical method.

Max z = $60x_1 + 100\,x_2$

Subject to $x_1 + x_2 \leq 9$

$x_1 \geq 2$

$x_2 \geq 3$

$30\,x_1 + 60\,x_2 \leq 360$

And $x_1, x_2 \geq 0$

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2

SIMPLEX METHOD, BIG M METHOD AND TWO PHASE METHOD

Unit structure

Self-Learning Topics: Special Cases of LPP

2.0 OBJECTIVES

After going through this chapter, students will able to:

- Identify and write a linear problem in standard form.
- Convert inequality constraints to equations.
- Know the use and interpretation of slack, surplus and artificial variables.
- Check for optimality (to determine entering and leaving variable) by performing pivoting operations.
- Identify the optimal solution.

2.1 INTRODUCTION

Linear programming is a very well-known method in the field of optimization theory. In last chapter we used graphical method to solve the linear programming problem. But when numbers of decision variables are more than two, it is very difficult to solve LPP by graphical method. To deal with this difficulty, a highly efficient method for solving LPP is used. This method can applied for any number i. e. more than two decision variables and is called Simplex Method.

2.2 SIMPLEX METHOD

Simplex method is one of the most popular and used algorithms in optimization theory. The Simplex method was developed during Second World War by Dr. George Dantzig. His linear programming models helped the Allied forces with transportation and scheduling problems.

This method uses the basic concepts of matrix algorithm and theory of equation.

To solve a linear programming problem using the simplex method:

1. First we write the problem in standard canonical form by converting inequalities into equations by introducing slack variables.
2. Write objective function and constraints in the matrix form. The table contains n decision variables and m (equal to number of the constraints) slack variables. The variables with non-zero values are called the basic variables, their constraint coefficient make an identity matrix.
3. Construct the initial simplex table.
4. Check for optimality (Determine entering variable and select leaving variable) and identify pivot element.
5. Create a new tableau (next iteration)
6. Identify optimal values.

Some definitions:

1. **Basic variable:** A variable whose coefficient is "unity: in only one of the constraints and "zero" in the remaining constraints is called a Basic variable.
2. **Basis:** The set of all basic variables (= m, the number of constraints) is called the Basis.
3. **Canonical Form:** If in the standard form of an LPP, each constraint has a Basic variable, then such form of the LPP is called Canonical Form.
4. **Basic Feasible Solution:** Any solution of an LPP where exactly m (equal to number of the constraints) variables have non zero values satisfying all the constraints, is called a Basic Feasible Solution.
5. **Basic variables:** The variables with non-zero values are called the vasic variables, their constraint coefficients make Identity Matrix.

Example 1: Max $Z = 6x_1 + 8x_2$

Sub to $5x_1 + 10x_2 \leq 60$

$4x_1 + 4x_2 \leq 40$

And $x_1, x_2 \geq 0$

Solution:

Step 1: write the problem in standard canonical form by converting inequalities into equations by introducing slack variables Si's for each inequality. The coefficient of each slack variable is unity. All these slack variables will also be appear in the objective function with zero coefficient.

Max Z = $6x_1 + 8x_2 + 0S_1 + 0S_2$

Sub to $5x_1 + 10x_2 + S_1 = 60$

$4x_1 + 4x_2 + S_2 = 40$

And $x_1, x_2, S_1, S_2 \geq 0$

Step 2: Write objective function and constraints in the matrix form.

Table 1

Variables	x_1	x_2	S_1	S_2	b_i
Constraints	5	10	1	0	60
	4	4	0	1	40
C_j	6	8	0	0	z

From above Table 1, it is clear that the problem now amounts to solve m equations in m + n variables (m slack variables and n decision variables) such that z is maximum.

Here m = 2 and n = 2

Put $x_1 = 0$ and $x_2 = 0$ in above equations,

We get, $S_1 = 60$ and $S_2 = 40$ ---------------------(1)

This is initial basic feasible solution.

Step 3: Construct the initial simplex table as follows:

Table 2: Initial Simplex Table

CB_i	**Basic Variables**	**Variables**				**Solution (bi)**	**Ratio**
		x_1	**x_2**	**S_1**	**S_2**		
0	S_1	5	**10**	1	0	60	$\frac{60}{10} = 6$ **
0	S_2	4	4	0	1	40	$\frac{40}{4} = 10$
	C_j	6	8	0	0	-	
	Z_j	0	0	0	0	0	
	$C_j - Z_j$	6	8*	0	0	-	

First column in Table 2 is Objective Column (old value) which contain coefficient of objective function of Basic Variables (Current solution (CB_i))

Second column is Basic Variable column. In the initial table slack variables are Basic Variables.

Here, S_1 and S_2 are basic variables.

To the right of basic variables column, columns of decision variables x_1 x_2 , …and slack variables S_1, S_2, … will be there.

First row contain coefficient of decision and slack variables of the first constraint.

Second row contain coefficient of decision and slack variables of the second constraint.

Next column is solution column which contain value of basic variable.

To find the scope of improvement, calculate row title Z_j and $C_j - Z_j$

Z_j is the present contribution of the jth variable to the objective function z where

$$Z_j = \sum_{i=1}^{m} (CB_i)(a_{ij}),\ j = 1, 2, \ldots, m + n$$

$$= \sum_{i=1}^{2} (CB_i)(a_{ij}),\ j = 1, 2, \ldots, 4$$

i.e Z_1= (0.5) + (0.4) = 0; Z_2= (0.10) + (0.4) = 0;

Z_3= (0.1) + (0.0) = 0; Z_4= (0.0) + (0.1) = 0

Where a_{ij} is the technological coefficient of the j^{th} variable in the i^{th} row (constraint).

$C_j - Z_j$ is the change in the value of the objective function per unit of the jth variable.

In the above Table 2, all $Z_j = 0$

Step 4: Check for optimality:

To determine entering variable:

a. **Maximization Problem:** If all ($C_j - Z_j$) values are ≤ 0, then present solution is optimal. Otherwise, select the variable with max ($C_j - Z_j$) as the entering variable. Column corresponding to the selected variable is called the Key Column and is marked by *.

For the present problem, all ($C_j - Z_j$) ≥ 0, therefore the current solution is not optimal.

Max ($C_j - Z_j$) = 8 and corresponds to x_2. Hence x_2 enters the solution and the 2^{nd} column is the key column.

b. **Minimization Problem:** If all $C_j - Z_j$ values are ≥ 0, then present solution is optimal. Otherwise the variable with highesr negative value will enter the solution. Column corresponding to the selected variable is called the Key Column and is marked by *.

Selecting the leaving variable: Feasibility Conditions:

In a basic feasible solution, number of non-zero variable must be equal to the number of constraints. Thus, one of the basic variables has to leave the solution and take the position of entering variable. i.e. should become equal to 0.

The following steps to be taken:

i. For each row, find the ratio of the elements of the solution column to the elements of the key column.

i.e. Ratio = $\frac{\textit{Elements of the solution column}}{\textit{Elements of the key column}}$ i.e. 60/ 10 = 6; 40/4 = 10

ii. The row corresponding to the smallest positive ratio will be the Key Row and the corresponding basic variable will leave the solution.

For the above problem, the smallest +ve ratio is 6 corresponds to the 1^{st} row.

Hence, row 1 is the Key Row

The corresponding basic variable S_1 will be the leaving variable.

Note: If there is tie in selecting Key-Column and Key-Row, the element can be selected arbitrarily.

The intersection of Key Column and Key Row is called **Key Element or Pivot Element.**

For the above example, Key Element is **10**.

Step 5: Next Iteration:

i. Replace the outgoing variable by the entering variable in the basic variable column and calculate , Key Row = $\frac{\textit{Key Row}}{\textit{Key Element}}$

R_1 (Key Row) = R_1 / (Key Element) i.e. R_1 (Key Row) = R_1 / 10

ii. Bring the column corresponding to the entering variable in the form of the outgoing variable as follows:

New values of the next iteration table are given by (in each row)

i.e. New i^{th} Row = Old i^{th} Row - (New element of R_i) x (Key Row)

For above example,

R_2(New) = R_2(old) – 4 R_1 (New)

$a_{21} = 4 - 4\ (1/2) = 2$; $a_{22} = 4 - 4\ (1) = 0$;

$a_{23} = 0 - 4\ (1/10) = -2/5$; $a_{24} = 1 - 4\ (0) = 2$;

$b_2 = 40 - 4(6) = 16$

Therefore, the Iteration 1 will be as follows:

Table 3: Iteration 1

CB_i	Basic Variables	x_1	x_2	S_1	S_2	Solution (bi)	Ratio
8	x_2	1/2	1	1/10	0	6	$\frac{6}{1/2}$ =12
0	S_2	**2**	0	-2/5	1	16	$\frac{16}{2}$ =8 **
	C_j	6	8	0	0	-	
	Z_j	4	8	4/5	0	48	
	C_j - Z_j	2*	0	-4/5	0	-	

After computing New values of aij, bi and completing the entries of CBi and Basic variable columns,

i. Compute the values of Z_j

$$Z_j = \sum_{i=1}^{m} (CB_i)(a_{ij}), j = 1, 2, \ldots, m + n$$

$$= \sum_{i=1}^{2} (CB_i)(a_{ij}), j = 1, 2, \ldots, 4$$

i.e $Z_1 = 8.(1/2) + (2 . 0) = 4$; $Z_2 = 8.(1) + (2 . 0) = 8$;

$Z_3 = 8.(1/10) + (-2/5).0 = 4/5$; $Z_4 = 8.0 + 1.0 = 0$

ii. Find C_j - Z_j

iii. Test for optimality (Decide the entering variable if required)

iv. Test for feasibility (Decide the leaving variable if required)

[Note that these above steps (i to iv) form the part of each iteration]

From the above Table 3, all values of $(C_j - Z_j)$ are $\nleq 0$.

The only +ve value is 2, corresponds to the 1st column.

Thus, 1st column is the key column and variable x_1 enters the solution.

Minimum +ve ratio of solution column to key column is 8 and corresponding variable S_2 leaves the solution.

The key element (element at the intersection of key column and key row) is 2.

Now go for second iteration.

R_2 (Key Row) = R_2 / (Key Element)

R_1(New) = R_1(old) – ½ R_2 (New)

$a_{11} = 1/2 – (1/2)1 = 2$; $a_{12} = 1 – (1/2)0 = 0$;

$a_{13} = 1/10 + (1/2)1/5 = 1/5$; $a_{14} = 0 – (1/2)1/2 = ¼$;

$b_1 = 6 - (1/2)8 = 2$

Compute the values of Z_j and Find $C_j - Z_j$

Table 4 below gives the values of second iteration using the method for first iteration.

Table 4: Iteration 2

CB i	**Basic Variables**	**Variables**				**Solution (bi)**	**Ratio**
		x_1	**x_2**	**S_1**	**S_2**		
8	x_2	0	1	1/5	-1/4	2	No need to calculate
6	x_1	1	0	-1/5	1/2	8	
	C_j	6	8	0	0	-	
	Z_j	6	8	2/5	1	64	
	$C_j - Z_j$	0	0	-2/5	-1	-	

Now all $(C_j - Z_j) \leq 0$

∴ Present solution is optimal.

No need for feasibility test. No need to find ratio column.

Here $x_1 = 2$ and $x_2 = 8$

Max $Z = 6x_1 + 8x_2 = 6 \times 2 + 8 \times 8 = 64$

Example 2: Max $Z = 4x_1 + 3x_2 + 6x_3$

Sub to $2x_1 + 3x_2 + 2x_3 \leq 440$

$4x_1 + 3x_3 \leq 470$

$2x_1 + 5x_3 \leq 430$

and $x_1, x_2, x_3 \geq 0$

Solution:

Step 1: write the problem in standard canonical form by converting inequalities into equations by introducing slack variables Si's for each inequality. The coefficient of each slack variable is unity. All these slack variables will also be appear in the objective function with zero coefficient.

Max $Z = 4x_1 + 3x_2 + 6x_3 + 0S_1 + 0S_2 + 0S_3$

Sub to $2x_1 + 3x_2 + 2x_3 + S_1 = 440$

$4x_1 + 3x_3 + S_2 = 470$

$2x_1 + 5x_3 + S_3 = 430$

and $x_1, x_2, x_3 \geq 0$

Here Slack variables, m = 3 and decision variables, n = 3

Put $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$ in above equations,

We get, $S_1 = 440$, $S_2 = 470$ and , $S_3 = 430$ ---------------------(1)

This is initial basic feasible solution.

Step 3: Construct the initial simplex table as follows:

Table 1: Initial Simplex Table

CB_i	Basic Variables	Variables						Solution (bi)	Ratio
		x_1	x_2	x_3	S_1	S_2	S_3		
0	S_1	2	3	2	1	0	0	440	440/2 220
0	S_2	4	0	**3**	0	1	0	470	**156.66****
0	S_3	2	5	0	0	0	1	430	∞
	C_j	4	3	6	0	0	0	-	
	Z_j	0	0	0	0	0	0	0	
	$C_j - Z_j$	4	3	**6***	0	0	0	-	

Compute $Z_j = \sum_{i=1}^{m} (CB_i)(a_{ij})$, j = 1, 2, …..

For above example,

$Z_1 = 0.2 + 0.4 + 0.2 = 0$; $Z_2 = 0.3 + 0.0 + 0.5 = 0$;

$Z_3 = 0.2 + 0.3 + 0.0 = 0$; $Z_4 = 0.1 + 0.0 + 0.0 = 0$;

$Z_5 = 0.0 + 0.1 + 0.0 = 0$; $Z_6 = 0.0 + 0.0 + 0.1 = 0$

Compute $C_j - Z_j$

Since it is a maximization problem, optima will be if $(C_j - Z_j) \leq 0$

Now all $(C_j - Z_j) \nleq 0$

Largest +ve value 6 corresponds to third column.

Thus column 3 is the key column and x_3 is the entering variable.

Now find entries of the ratio.

i.e. Ratio = $\frac{Elements\ of\ the\ solution\ column}{Elements\ of\ the\ key\ column}$

i. e. 440/2 = 220; 470/ 3 =156.66; 430/0 = ∞

Row corresponding to the smallest +ve ratio will be the key row.

Here it is 156.66 and corresponds to R_2.

Thus, R_2 is the Key Row and therefore S_2 is the outgoing variable.

The key element is 3. [Element at the intersection of Key row and Key Column]

Next Iteration: Iteration 1:

Replace the outgoing variable by the entering variable in the basic variable column

i.e. Replace S_2 by x_3 and put the value of C_3 at CB_2.

and calculate Key Row = $\frac{Key\ Row}{Key\ Element}$

R_2 (Key Row) = R_2 / (Key Element) i.e.$R_2 = R_2/3$

Bring the column corresponding to the entering variable in the form of the outgoing variable as follows:

New values of the next iteration table are given by (in each row)

i.e. New i^{th} Row = Old i^{th} Row - (New element of R_i) x (Key Row)

For above example,

$R_1(New) = R_1(old) - 2 R_2 (New)$

$a_{11} = 2 - 2(4/3) = -2/3$; $a_{12} = 3 - 2(0) = 3$;

$a_{13} = 2 - 2(1) = 0$; $a_{14} = 1 - 2(0) = 1$;

$a_{15} = 0 - 2(1/3) = -2/3$; $a_{16} = 0 - 2(0) = 0$;

$b_1 = 440 - 2(156.66) = 126.68$

For above example, in row R3, element below key element is already 0.

Therefore, no need to do calculations for R3.

Find Z_j

$Z_j = \sum_{i=1}^{m} (CB_i)(a_{ij})$, j = 1, 2, …..

$Z_1 = 0.(-2/3) +6.(4/3) + 0.\ 2 = 8$; $Z_2 = 0.3 +6.0 + 0.\ 5 = 0$;

$Z_3 = 0.0 +6.1 + 0.\ 0 = 6$; $Z_4 = 0.1 +6.0 + 0.\ 0 = 0$;

$Z_5 = 0.(-2/3) +6.(1/3) + 0.\ 0 = 2$; $Z_6 = 0.0 +6.0 + 0.\ 1 = 0$

and $C_j - Z_j$

Therefore, the Iteration 1 will be as follows:

Table 2: Iteration 1

CB_i	Basic Variables	Variables						Solution (bi)	Ratio
		x_1	x_2	x_3	S_1	S_2	S_3		
0	S_1	-2/3	**3**	0	1	-2/3	0	126.68	42.22**
6	x_3	4/3	0	1	0	1/3	0	156.66	∞
0	S_3	2	5	0	0	0	1	430	86
	C_j	4	3	6	0	0	0	-	
	Z_j	8	0	6	0	2	0	936.96	
	$C_j - Z_j$	-4	3*	0	0	-2	0	-	

Since it is a maximization problem, optima will be if $(C_j - Z_j) \leq 0$

Now all $(C_j - Z_j) \nleq 0$

Largest +ve value 3 corresponds to second column.

Thus column 2 is the key column and x_2 is the entering variable.

Now find entries of the ratio.

i.e. Ratio = $\frac{\textit{Elements of the solution column}}{\textit{Elements of the key column}}$

i. e. 126.68/3 = 42.22;   156.66/ 0 = ∞; 430/5 = 86

Row corresponding to the smallest +ve ratio will be the key row.

Here it is 42.22 and corresponds to R_1.

Thus, R_1 is the Key Row and therefore S_1 is the outgoing variable.

The key element is 3. [Element at the intersection of Key row and Key Column]

Next Iteration: Iteration 2

Replace the outgoing variable by the entering variable in the basic variable column

i.e. Replace S_1 by x_2 and put the value of C_2 at CB_2.

and calculate

Key Row = $\frac{\textit{Key Row}}{\textit{Key Element}}$

R_1 (Key Row) = R_1 / (Key Element)

i.e.$R_1 = R_1/3$

Bring the column corresponding to the entering variable in the form of the outgoing variable as follows:

New values of the next iteration table are given by (in each row)

i.e. New i^{th} Row = Old i^{th} Row - (New element of R_i) x (Key Row)

For above example,

R_3(New) = R_3(old) – 5 R_1 (New)

$a_{31} = 2 – 5 (-2/9) = 28/9$; $a_{32} = 5 – 5 (1) = 0$;

$a_{33} = 0 – 5 (0) = 0$; $a_{34} = 0 – 5 (1/3) = -5/3$;

$a_{35} = 0 – 5 (-2/9) = 10/9$; $a_{36} = 1 – 5 (0) = 1$;

$b_3 = 430 – 5 (42.22) = 208.9$

For above example, in row R2, element below key element is already 0.

Therefore, no need to do calculations for R2.

Find Z_j

$Z_j = \sum_{i=1}^{m} (CB_i)(a_{ij})$, j = 1, 2, …..

$Z_1 = 3.(-2/9) +6.(4/3) + 0.(28/9) =-10/3$; $Z_2 = 3.1 +6.0 + 0. 0 = 3$;

$Z_3 = 3.0 +6.1 + 0. 0 = 6$; $Z_4 = 3.(1/3) +6.0 + 0.(-5/3) =1$;

$Z_5 = 3.(-2/9) +6.(1/3) + 0.(10/9) = 4/3$; $Z_6 = 3.0 +6.0 + 0. 1 = 0$

and $C_j - Z_j$

Therefore, the Iteration 2 will be as follows:

Table 3: Iteration 2

CB_i	**Basic Variables**	**Variables**						**Solution (bi)**	**Ratio**
		x_1	**x_2**	**x_3**	**S_1**	**S_2**	**S_3**		
3	x_2	-2/3	1	0	1/3	-2/9	0	42.22	
6	x_3	4/3	0	1	0	1/3	0	156.66	
0	S_3	28/9	0	0	-5/3	10/9	1	218.9	
	C_j	4	3	6	0	0	0	-	
	Z_j	22/3	3	6	1	4/3	0	1066.62	
	$C_j - Z_j$	-10/3	0	0	-1	-4/3	0	-	

All $(C_j - Z_j) \leq 0$

∴ The present solution is optimal.

Here $x_1 = 0$, $x_2 = 42.22$ and $x_3 = 156.66$

Max $Z = 4x_1 + 3x_2 +6x_3 = (4 \times 0) + (3 \times 42.22) + (6 \times 156.66) = 1066.62$

2.3 BIG M METHOD

If some of the constraints are of '=' or '≥' type, then in the standard form such constraint will not have basic variables (and we will not have a canonical form). To overcome this difficulty the Big M Method is used, where M is a large positive quantity. For the constraints with '=' or '≥' sign we introduce new variables called the 'Artificial Variables'. There are two methods to deal with the artificial variables.

1. Big M Method
2. Two Phase Method

These artificial variables are fictitious and cannot have any physical meaning.

Following points to be consider while using artificial variable:

- Only one artificial variable will appear in any constraint.
- Coefficient of an artificial variable is unity in that constraint.
- Number of artificial variable is equal to the number of constraints of the type '=' or '≥'
- Cost coefficient of an artificial variable will be (+M) for a minimization problem and (-M) for a maximization problem.
- Once an artificial variable goes out of the solution, it is not allowed to re-enter. For this purpose, the corresponding column is deleted from the Simplex Table for further calculations usually.

Example 3: Min $Z = 2x_1 + 3x_2$

Sub to $x_1 + x_2 \geq 6$

$7x_1 + x_2 \geq 14$

and $x_1, x_2 \geq 0$

Solution: Write the problem in the standard form.

Min $Z = 2x_1 + 3x_2 + 0S_1 + 0S_2$

Sub to $x_1 + x_2 - S_1 = 6$

$7x_1 + x_2 - S_2 = 14$

and $x_1, x_2, S_1, S_2 \geq 0$, Si's are called surplus variables.

As the value of $S_1 = S_2 = -1$ so these are not basic variable and therefore this is not a canonical form.

To convert it into canonical form, introduce artificial variables A_1 and A_2 with cost +M, where M is a large +ve number.

Now the problem can be expressed as,

Min Z = $2x_1 + 3x_2 + 0S_1 + 0S_2 + MA_1 + MA_2$

Sub to $x_1 + x_2 - S1 + A_1 = 6$

$7x_1 + x_2 - S2 + A_2 = 14$

and $x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$

Now A_1 and A_2 are Basic Variables and the solution is $A_1 = 6$ and $A_2 = 14$

We make the initial simplex table.

Table – 1: Initial Simplex Table

CBi	**Basic Variables**	**Variables**						**Solution (bi)**	**Ratio**
		x_1	x_2	S_1	S_2	A_1	A_2		
M	A_1	1	**1**	-1	0	1	0	6	$\frac{6}{1}$ =6
M	A_2	**7**	1	0	-1	0	1	14	$\frac{14}{7}$ =2**
	C_j	2	3	0	0	M	M	-	
	Z_j	8M	2M	-M	-M	M	M	20M	
	$C_j - Z_j$	2- 8M*	3-2M	M	M	0	0	-	

Compute $Z_j = \sum_{i=1}^{m} (CB_i)(a_{ij})$, j = 1, 2, …..

For above

$Z_1 = M.1 + M.7 = 8M$; $Z_2 = M.1 + M.1 = 2M$;

$Z_3 = M.(-1) + M.0 = -M$; $Z_4 = M.0 + M.(-1) = -M$;

$Z_5 = M.1 + M.0 = M$; $Z_6 = M.0 + M.1 = M$

Compute $C_j - Z_j$

Since it is a minimization problem optima will be if $(C_j - Z_j) \geq 0$

[If $C_j - Z_j$ include M, ignore the numeric term for big M method]

Now all $(C_j - Z_j) \ngeq 0$

Largest –ve value -8M corresponds to first column.

Thus column 1 is the key column and x_1 is the entering variable.

Now find entries of the ratio.

i.e. Ratio = $\frac{\textit{Elements of the solution column}}{\textit{Elements of the key column}}$

Row corresponding to the smallest +ve ratio will be the key row.

Here it is 2 and corresponds to R_2.

Thus, R_2 is the Key Row and therefore A_2 is the outgoing variable.

The key element is 7. [Element at the intersection of Key row and Key Column]

Next Iteration: Iteration 1

Replace the outgoing variable by the entering variable in the basic variable column

i.e. Replace A_2 by $x_1$and put the value of C_1 at CB_2.

and calculate Key Row = $\frac{\textit{Key Row}}{\textit{Key Element}}$

R_2 (Key Row) = R_2 / (Key Element) i.e. $R_2 = R_2/7$

Bring the column corresponding to the entering variable in the form of the outgoing variable as follows:

New values of the next iteration table are given by (in each row)

i.e. New i^{th} Row = Old i^{th} Row - (New element of R_i) x (Key Row)

For above example,

R_1(New) = R_1(old) – R_2 (New)

$a_{11} = 1 – 1 = 0$; $a_{12} = 1 – 1/7 = 6/7$;

$a_{13} = -1 – 0 = - 1$; $a_{14} = 0 – (-1/7) = 1/7$;

$a_{15} = 1 – 0 = 1$; $a_{16} = 0 – (1/7) = - 1/7$;

$b_1 = 6 - 2 = 4$

Find Z_j and $C_j - Z_j$

Therefore, the Iteration 1 will be as follows:

Table 2: Iteration 1

C_{B_i}	Basic Variables	Variables x_1	x_2	S_1	S_2	A_1	A_2	Solution (bi)	Ratio
M	A_1	0	**6/7**	-1	1/7	1	-1/7	4	$\frac{4}{\frac{6}{7}}=14/3$ **
2	x_1	**1**	1/7	0	-1/7	0	1/7	2	$\frac{2}{\frac{1}{7}}=14$
	C_j	2	3	0	0	M	M	-	
	Z_j	2	$\frac{2}{7}+\frac{6M}{7}$	-M	$\frac{-2}{7}+\frac{M}{7}$	M	$\frac{-2}{7}-\frac{M}{7}$	4M +4	
	$C_j - Z_j$	0	$\frac{19}{7}-\frac{6M}{7}$ *	M	$\frac{2}{7}-\frac{M}{7}$	0	$\frac{-2}{7}+\frac{8M}{7}$	-	

Note: We could have deleted the column 5 corresponding to the outgoing artificial variable. Removing the outgoing artificial variable for calculations of the next iteration does not affect future process of finding the optima, but reduces the computations.

All $(C_j - Z_j) \ngeq 0$

Largest –ve value corresponds to second column.

Thus column 2 is the key column and x_2 is the entering variable.

Now find entries of the ratio.

i.e. Ratio = $\frac{\textit{Elements of the solution column}}{\textit{Elements of the key column}}$

Row corresponding to the smallest +ve ratio will be the key row.

Here it is 14/3 and corresponds to R_1.

Thus outgoing variable is A_1 and key element is 6/7.

Next Iteration: Iteration 2

In the Simplex Table [Table 3], replace the outgoing variable by the entering variable in the basic variable column

i.e. Replace A_1 by x_2 and put the value of C_2 at CB_1.

and calculate Key Row = $\frac{\textit{Key Row}}{\textit{Key Element}}$

R_1(Key Row) = R_1 / (Key Element) i.e. $R_1 = R_2/(6/7)$

New values of the next iteration table are given by (in each row)

i.e. New i^{th} Row = Old i^{th} Row - (New element of R_i) x (Key Row)

For above example,

R_2(New) = R_2(old) – (1/7)R_1 (New)

a_{21} = 1 – (1/7).0 = 1; a_{22} = 1/7 – (1/7).1 = 0;

a_{23} = 0 –(1/7).(-7/6) 0 = 1/6; a_{24} = (-1/7) – (1/7).(1/6) = -1/6;

b_2 = 2 – (1/7)(14/3) = 4/3

Find Z_j and C_j - Z_j

Therefore, the Iteration 2 will be as follows:

[Note: Here we remove the columns A_1 and A_2]

Table 3: Iteration 2

CB_i	**Basic Variables**	**Variables**				**Solution (bi)**	**Ratio**
		x_1	**x_2**	**S_1**	**S_2**		
3	x_2	0	1	-7/6	**1/6**	14/3	(14/3)/(1/6) =28 **
2	x_1	**1**	0	1/6	-1/6	4/3	(-ve)
	C_j	2	3	0	0		
	Z_j	2	3	-19/6	1/6	50/3	
	C_j - Z_j	0	0	19/6	-1/6 *	-	

Now there is still one C_j - Z_j value corresponds to column 4 which is < 0.

Thus column 4 is the key Column and S_2 is the entering variable.

The smallest +ve ratio [in this case only +ve ratio] is 28 corresponds to R_1.

Thus R_1 is the Key Row, x_2 is the outgoing variable and 1/6 is the Key Element.

Next Iteration: Iteration 3

In the Simplex Table [Table 4], replace the outgoing variable by the entering variable in the basic variable column

i.e. Replace x_2 by S_2 and put 0 in the place of 3 (CB_1).

and calculate

Key Row = $\frac{Key\ Row}{Key\ Element}$

R_1(Key Row) = R_1 / (Key Element)

i.e. Divide Key Row R_1 by the Key Element [R_1 = R_2/(1/6)]

New values of the next iteration table are given by (in each row)

i.e. New i^{th} Row = Old i^{th} Row - (New element of R_i) x (Key Row)

For above example,

R_2(New) = R_2(old) – (-1/6)R_1 (New)

$a_{21} = 1 - (-1/6).0 = 1$; $a_{22} = 0 - (-1/6).6 = 1$;

$a_{23} = (1/6) - (-1/6).(-7) = -1$; $a_{24} = (-1/6) - (-1/6).1 = 0$;

$b_2 = 4/3 - (-1/6)(28) = 4/3$

Find Z_j and $C_j - Z_j$

Therefore, the Iteration 3 will be as follows:

Table 4: Iteration 3

CB_i	Basic Variables	x_1	x_2	S_1	S_2	Solution (bi)
0	S_2	0	6	-7	1	28
2	x_1	1	1	-1	0	6
	C_j	2	3	0	0	
	Z_j	2	3	-2	0	12
	$C_j - Z_j$	0	0	2	0	-

All $(C_j - Z_j) \geq 0$

∴ The present solution is optimal.

Here $x_1 = 6$, $S_2 = 28$ and $x_2 = 0$

Min $Z = 2x_1 + 3x_2 = 2 \times 6 + 3 \times 0 = 12$

Example 4: Max $Z = 10x_1 + 15x_2$

Sub to $2x_1 + 2x_2 \leq 160$

$x_1 + 2x_2 \leq 120$

$4x_1 + 2x_2 \leq 280$

$x_2 \geq 45$

and $x_1, x_2 \geq 0$

Solution: Write the problem in the standard form.

Max $Z = 10x_1 + 15x_2 + 0S_1 + 0S_2 + 0S_3 + 0S_4$

Sub to $2x_1 + 2x_2 + S_1 = 160$

$x_1 + 2x_2 + S_2 = 120$

$4x_1 + 2x_2 + S3 = 280$

$x_2 - S_4 = 45$

and $x_1, x_2, S_1, S_2, S_3, S_4 \geq 0$

Since m = 4 there must be 4 basic variables and there are only 3.

The last $x_2 - S_4 = 45$ does not have a basic variable.

Introducing artificial variable A_1, with cost coefficient as – M.

Max $Z = 10x_1 + 15x_2 + 0S_1 + 0S_2 + 0S_3 + 0S_4 - M A_1$

Sub to $2x_1 + 2x_2 + S_1 = 160$

$x_1 + 2x_2 + S_2 = 120$

$4x_1 + 2x_2 + S3 = 280$

$x_2 - S_4 + A_1 = 45$

and $x_1, x_2, S_1, S_2, S_3, S_4, A_1 \geq 0$

Now A_1 is Basic Variables and the solution is $A_1 = 45$

We make the initial simplex table.

Table – 1: Initial Simplex Table:

C B_i	Basic Variables	Variables							Solution (bi)	Ratio
		x_1	x_2	S_1	S_2	S_3	S_4	A_1		
0	S_1	2	2	1	0	0	0	0	160	$\frac{160}{2}$ =80
0	S_2	1	2	0	1	0	0	0	120	$\frac{120}{2}$ = 60
0	S_3	4	2	0	0	1	0	0	280	$\frac{280}{2}$ = 140
- M	A_1	0	**1**	0	0	0	-1	1	45	$\frac{45}{1}$ =45**
	C_j	10	15	0	0	0	0	-M		
	Z_j	0	-M	0	0	0	M	-M	-45M	
	$C_j - Z_j$	10	15+M*	0	0	0	-M	0	-	

Compute $Z_j = \sum_{i=1}^{m} (CB_i)(a_{ij})$, j = 1, 2, …..

Compute $C_j - Z_j$

Since it is a maximization problem optima will be if $(C_j - Z_j) \leq 0$

[If $C_j - Z_j$ include M, ignore the numeric term for big M method]

Now all $(C_j - Z_j) \nleq 0$

Largest +ve value (15 + M) corresponds to second column.

Thus column 2 is the key column and x_2 is the entering variable.

Now find entries of the ratio.

i.e. Ratio = $\frac{Elements\ of\ the\ solution\ column}{Elements\ of\ the\ key\ column}$

Row corresponding to the smallest +ve ratio will be the key row.

Here it is 45 and corresponds to R_4.

Thus, R_4 is the Key Row and therefore A_1 is the outgoing variable.

The key element is 1. [Element at the intersection of Key row and Key Column]

Next Iteration: Iteration 1

Replace the outgoing variable by the entering variable in the basic variable column

i.e. Replace A_1 by x_2 and calculate

Key Row = $\frac{Key\ Row}{Key\ Element}$

R_4 (Key Row) = R_4/ (Key Element)

i.e. $R_4 = R_4/1$]

Bring the column corresponding to the entering variable in the form of the outgoing variable as follows:

New values of the next iteration table are given by (in each row)

i.e. New i^{th} Row = Old i^{th} Row - (New element of R_i) x (Key Row)

For above example,

R_1(New) = R_1(old) – $2R_4$ (New)

R_2(New) = R_2(old) – $2R_4$ (New)

R_3(New) = R_3(old) – $2R_4$ (New)

Find Z_j and C_j - Z_j

Therefore, the Iteration 1 will be as follows:

Table 2: Iteration 1

C_{B_i}	Basic Variables	x_1	x_2	S_1	S_2	S_3	S_4	Solution (bi)	Ratio
0	S_1	2	0	1	0	0	2	70	$\frac{70}{2} = 35$
0	S_2	1	0	0	1	0	**2**	30	$\frac{30}{2} =$ 15**
0	S_3	4	0	0	0	1	2	190	$\frac{190}{2} = 95$
15	X_2	0	1	0	0	0	-1	45	-ve value
	C_j	10	15	0	0	0	0		
	Z_j	0	15	0	0	0	-15	675	
	$C_j - Z_j$	10	0	0	0	0	15*	-	

Note: We have deleted the column 7 corresponding to the outgoing artificial variable (A_1).

All $(C_j - Z_j) \nleq 0$

Largest +ve value corresponds to sixth column.

Thus column 6 is the key column and S_4 is the entering variable.

Now find entries of the ratio.

i.e. Ratio = $\frac{\textit{Elements of the solution column}}{\textit{Elements of the key column}}$

Row corresponding to the smallest +ve ratio will be the key row.

Here it is 15 and corresponds to R_2.

Thus outgoing variable is S_2 and key element is 2.

Next Iteration: Iteration 2

In the Simplex Table [Table 3], replace the outgoing variable by the entering variable in the basic variable column

i.e. Replace S_2 by S_4 and and calculate

Key Row = $\frac{\textit{Key Row}}{\textit{Key Element}}$ i.e $R_2 = R_2/2$

New values of the next iteration table are given by (in each row)

i.e. New i^{th} Row = Old i^{th} Row - (New element of R_i) x (Key Row)

For above example,

R_1(New) = R_1(old) – $2R_2$ (New)

$R_3(New) = R_3(old) - 2R_2\,(New)$

$R_4(New) = R_4(old) + R_2\,(New)$

Find Z_j and $C_j - Z_j$

Therefore, the Iteration 2 will be as follows:

Table 3: Iteration 2

C B_i	**Basic Variables**	**Variables**						**Solution (bi)**	**Ratio**
		x_1	**x_2**	**S_1**	**S_2**	**S_3**	**S_4**		
0	S_1	1	0	1	-1	0	0	40	$\frac{40}{1} = 40$
0	S_4	**1/2**	0	0	1/2	0	1	15	$\frac{15}{1/2} =$ 30**
0	S_3	3	0	0	-1	1	2	160	$\frac{160}{3} =$ 53.33
15	X_2	1/2	1	0	1/2	0	0	60	$\frac{60}{1/2} =$ 120
	C_j	10	15	0	0	0	0		
	Z_j	15/2	15	0	0	0	0	900	
	$C_j - Z_j$	5/2*	0	0	-15/2	0	0	-	

All $(C_j - Z_j) \nleq 0$

Largest +ve value corresponds to first column.

Thus column 1 is the key column and S_1 is the entering variable.

Now find entries of the ratio.

i.e. Ratio = $\frac{\textit{Elements of the solution column}}{\textit{Elements of the key column}}$

Row corresponding to the smallest +ve ratio will be the key row.

Here it is 30 and corresponds to R_2.

Thus outgoing variable is S_4 and key element is 1/2.

Next Iteration: Iteration 3:

In the Simplex Table [Table 3], replace the outgoing variable by the entering variable in the basic variable column

i.e. Replace S_4 by x_1 and calculate Key Row = $\frac{\textit{Key Row}}{\textit{Key Element}}$ i.e. $R_1 = R_1/(1/2)$

New values of the next iteration table are given by (in each row)

i.e. New i^{th} Row = Old i^{th} Row - (New element of R_i) x (Key Row)

For above example,

$R_1(New) = R_1(old) – R_2 (New)$

$R_3(New) = R_3(old) – 3R_2 (New)$

$R_4(New) = R_4(old) – 1/2 R_2 (New)$

Find Z_j and $C_j - Z_j$

Therefore, the Iteration 3 will be as follows:

Table 4: Iteration 3

C B_i	**Basic Variables**	**Variables**						**Solution (bi)**	**Ratio**
		x_1	**x_2**	**S_1**	**S_2**	**S_3**	**S_4**		
0	S_1	0	0	1	-2	0	-2	10	
10	x_1	1	0	0	1	0	2	30	
0	S_3	0	0	0	-4	1	-6	70	
15	x_2	0	1	0	0	0	-1	45	
	C_j	10	15	0	0	0	0		
	Z_j	10	15	0	0	0	0	975	
	$C_j - Z_j$	0	0	0	0	0	0	-	

Now, all $(C_j - Z_j) \leq 0$

∴ Solution is optimal.

$x_1 = 30$ and $x_2 = 45$

Max $Z = 10x_1 + 15x_2$

$= 10 (30) + 15 (45) = 975$

2.4 TWO PHASE METHOD

This is an alternative method of Big M method for computerized solution.

Following are the steps of Two phase Method:

Step 0 [Initial step]: Obtain canonical form [by introducing slack/surplus and artificial variables.]

Phase 1:

Step 1: Form the modified problem

Min z = sum of artificial variables i.e. min $z = A_1 + A_2 + …. + A_n$

Subject to the same set of constraints and non-negativity conditions.

Note: Whatever the original problem (max /min) the modified phase 1 problem will always be a minimization problem.

Step 2: Apply simplex method till the optimality is reached.

Step 3: Check

a) If the value of objective function (sum of artificial variables (i.e. z) is 0 in the optimal table.

b) If the modified $z \neq 0$ then the problem has no feasible solution.

c) If the modified $z = 0$, go to Phase 2.

Phase 2:

Step 1: From the optimal table of Phase 1, drop all the columns of artificial variables which are non-basic.

Step 2: If some artificial variables are in the basic solution with value 0 then in the original objective function put their cost coefficient = 0

Step 3: Find the optimal solution of the original problem using the optimal solution of the Phase 1 as the initial solution. All cost coefficients will be originals.

Example 5: Min $Z = 12x_1 + 18x_2 + 15x_3$

Sub to $4x_1 + 8x_2 + 6x_3 \geq 64$

$3x_1 + 6x_2 + 12x_3 \geq 96$

and $x_1, x_2 \geq 0$

Solution:

Step 0: Min z = Min $Z = 12x_1 + 18x_2 + 15x_3 + 0\,S_1 + 0\,S_2 + M\,A_1 + MA_2$

Sub to $4x_1 + 8x_2 + 6x_3 - S_1 + A_1 = 64$

$3x_1 + 6x_2 + 12x_3 - S_2 + A_2 = 96$

and $x_1, x_2, x_3, S_1, S_2, A_1, A_2 \geq 0$

This is required canonical form.

Phase 1:

Step 1: Form the modified problem

Min Z_m = sum of artificial variables

Min $Z_m = A_1 + A_2$ (cost coe. Of all other variables = 0)

Sub to $4x_1 + 8x_2 + 6x_3 - S_1 + A_1 = 64$

$$3x_1 + 6x_2 + 12x_3 - S_2 + A_2 = 96$$

and $x_1, x_2, x_3, S_1, S_2, A_1, A_2 \geq 0$

Step 2: Find the optimal solution of this modified problem

Table – 1: Initial Simplex Table

C_{B_i}	Basic Variables	x_1	x_2	x_3	S_1	S_2	A_1	A_2	Solution (bi)	Ratio
1	A_1	4	8	6	-1	0	1	0	64	$\frac{64}{6}$ = 32/3
1	A_2	3	6	**12**	0	-1	0	1	96	$\frac{96}{12}$ = 8**
	C_j	0	0	0	0	0	1	1		
	Z_{mj}	7	14	18	-1	-1	1	1	160	
	C_j - Zm_j	-7	-14	-18*	1	1	0	0	-	

All (C_j - Zm_j) $\ngeq$ 0.

x_3 is entering variable , A_2 is outgoing variable and 12 is Key Element

Table – 2: Iteration 1

C_{B_i}	Basic Variables	x_1	x_2	x_3	S_1	S_2	A_1	Solution (bi)	Ratio
1	A_1	5/2	**5**	0	-1	1 /2	1	16	$\frac{16}{5}$ **
0	x_3	1/4	1/2	1	0	- 1/12	0	8	$\frac{8}{1/2}$ = 16
	C_j	0	0	0	0	0	1		
	Z_{mj}	5/2	5	0	-1	1 /2	1	16	
	C_j - Z_{mj}	-5/2	-5*	0	0	-1/ 2	0	-	

All (C_j - Zm_j) $\ngeq$ 0.

X_2 is entering variable , A_1 is outgoing variable and 5 is Key Element

Table – 3: Iteration 2

C_{B_i}	Basic Variables	Variables					Solution (bi)	Ratio
		x_1	x_2	x_3	S_1	S_2		
0	x_2	1 /2	1	0	-1/5	1/10	16/5	
0	x_3	0	0	1	1/10	-2/15	32/5	
	C_j	0	0	0	0	0		
	Z_{mj}	0	0	0	0	0	0	
	$C_j – Z_{mj}$	0	0	0	0	0	-	

Step 3: Since all $(C_j - Zm_j) \geq 0$.

This solution is optimal.

The value of $Z_{mj} = 0$ therefore the problem is feasible and we can go to Phase II

Phase II:

Steps 1 and 2: [All cost coefficients will be originals]

Table – 4: Initial Table Phase II

CB_i	Basic Variables	Variables					Solution (bi)	Ratio
		x_1	x_2	x_3	S_1	S_2		
18	x_2	1 /2	1	0	-1/5	1/10	16/5	
15	x_3	0	0	1	1/10	-2/15	32/5	
	C_j	12	18	15	0	0		
	Z_j	9	18	15	-21/10	-1/5	768/5	
	$C_j – Z_j$	3	0	0	21/10	1/5	-	

Since, all $(C_j - Z_j) \geq 0$.

The current solution (the one given by Phase I itself) is optimal.

Optimal Solution: $x_1 = 0$, $x_2 = 16/5$ and $x_3 = 32/5$

Min $Z = 12x_1 + 18x_2 + 15x_3$

$= 12(0) + 18(16/5) + 15(32/5) = 768/5$

[**Note:** The optimal solution of Phase 1 need not be optimal for the original problem and phase II will usually need more than one iterations.]

2.5 SUMMARY

The Simplex method is an approach to solving linear programming models by hand. After studying this chapter, students are able to introduce slack variables, surplus variables and artificial variable to convert the inequalities into equalities and able to get canonical form. Also able to find optimal solution using tableau and pivot variables (Key Element).

Students are able to used Big M Method and Two Phase Method to find a best feasible solution by adding artificial variable. In Two Phase method, the whole procedure of solving a linear programming problem involving artificial variables is divided into two phases. In phase I, form a new objective function by assigning zero to every original variable. Then we try to eliminate artificial variables. The solution at the end of phase I serves as a basic feasible solution for Phase II. In Phase II the original objective function is introduced and by using usual method we find an optimal solution.

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2.7 EXERCISE

Q.1 Solve by Simplex Method

a) Max $Z = 10x_1 + 4x_2$

Sub to $20x_1 + 10x_2 \leq 1200$

$40x_1 + 10x_2 \leq 1600$

and $x_1, x_2, x_3 \geq 0$ [Ans: $x_1 = 20, x_2 = 80$, Max Z = 520]

b) Max Z = 25 x + 20 y

Sub to $6x + 4y \leq 3600$

$2x + 4y \leq 2000$

and $x, y \geq 0$ [Ans: $x = 400$, $y = 300$, Max $Z = 16000$]

c) Max $Z = 3x_1 + 2x_2 + 5x_3$

Sub to $x_1 + 2x_2 + x_3 \leq 430$

$3x_1 + 2x_3 \leq 460$

$x_1 + 4x_3 \leq 420$

and $x_1, x_2, x_3 \geq 0$ [Ans: $x_1 = 250$, $x_2 = 550/8$, $x_2 = 170/4$, Max $Z =$ 1100]

d) Max $Z = 12x_1 + 15x_2 + 14x_3$

Sub to $6x_1 + 5x_2 + 10x_3 \leq 76$

$2x_1 + x_2 - x_3 \leq 20$

$3x_1 - 3x_2 + 6x_3 \leq 50$

and $x_1, x_2, x_3 \geq 0$

e) Max $Z = 10x_1 + 15x_2 + 20x_3$

Sub to $2x_1 + 4x_2 + 6x_3 \leq 24$

$3x_1 + 9x_2 + 6x_3 \leq 30$

and $x_1, x_2, x_3 \geq 0$ [Ans: $x_1 = 6$, $x_2 = 0$, $x_3 = 2$, Max $Z = 100$]

Q.2 Solve by Big - M Method

a) Min $Z = 2x_1 + 3x_2$

Sub to $3x_1 + 5x_2 \geq 30$

$5x_1 + 3x_2 \geq 60$

and $x_1, x_2 \geq 0$ [Ans: $x_1 = 12$, $x_2 = 0$, $S_1 = 6$, Min $Z = 24$]

b) Min $Z = 25x_1 + 30x_2$

Sub to $4x_1 + 3x_2 \geq 60$

$2x_1 + 3x_2 \geq 36$

and $x_1, x_2 \geq 0$ [Ans: $x_1 = 12$, $x_2 = 4$, Min $Z = 420$]

c) Min $Z = 12x_1 + 20x_2$

Sub to $6x_1 + 8x_2 \geq 100$

$7x_1 + 12x_2 \geq 120$

and $x_1, x_2 \geq 0$ [Ans: $x_1 = 15, x_2 = 5/4$, Min Z = 205]

d) Min $Z = 10x_1 + 15x_2 + 20x_3$

Sub to $2x_1 + 4x_2 + 6x_3 \geq 24$

$3x_1 + 9x_2 + 6x_3 \geq 30$

and $x_1, x_2 \geq 0$

Q.2 Solve by Two Phase Method.

a) Max $Z = 5x_1 - 4x_2 + 3x_3$

Sub to $2x_1 + x_2 - 6x_3 = 20$

$6x_1 + 5x_2 + 10x_3 \leq 76$

$8x_1 - 3x_2 + 6x_3 \leq 50$

and $x_1, x_2, x_3 \geq 0$ [Ans: $x_1 = 55/7$, $x_2 = 30/7$, Max Z = 155/7]

b) Max $Z = x_1 + 2x_2 + x_3$

Sub to $x_1 + x_2 + x3 = 7$

$2x_1 - 5x_2 + x_3 \geq 10$

and $x_1, x_2, x_3 \geq 0$ [Ans: $x_1 = 45/7$, $x_2 = 4/7$, $x_3 = 0$, Max Z = 53/7]

Self-Learning Topics: Special Cases of LPP

1. **Tie between key column:** If $(C_j - Z_j)$ is most positive for two variables indicating two entering variables are possible then select the column with largest positive $(C_j - Z_j)$ from left to right of matrix.

	x_1	x_2	S_1	S_2
$C_j - Z_j$	-10	8 select	8	3

Example: Max $Z = 10x_1 + 4x_2 + 6x_3$

Sub to $3x_1 + 2x_2 + 6x_3 \leq 240$

$2x_1 + 3x_2 + 3x_3 \leq 270$

$x_1 \leq 60$

and $x_1, x_2, x_3 \geq 0$

2. **Unbounded Solution:** If all **(**$C_j - Z_j$) is positive indicating that solution is not optimal but all the ratios are negative (negative or 0) indicating that no variable is ready to come out. This is the case of unbounded solution.

Example: Max Z = $4x_1 + x_2 + 3x_3 + 5x_4$

Sub to $4x_1 - 6x_2 - 5x_3 - 4x_4 \geq 20$

$-3x_1 - 2x_2 + 4x_3 + 4x_4 \leq 10$

$-8x_1 - 3x_2 + 3x_3 + 2x_4 \leq 20$

and $x_1, x_2, x_3, x_4 \geq 0$

3. **Infeasible Solution:** If artificial variable is in basic variable with non zero value i.e. artificial variable is the part of the solution. Hence it is called infeasible solution.

Example: Min Z = $x_1 + 2x_2 + x_3$

Sub to $x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 \leq 1$

$\frac{3}{2}x_1 + 2x_2 + x_3 \geq 8$

and $x_1, x_2, x_3 \geq 0$

4. **Alternate Solution:** If **(**$C_j - Z_j$) = 0 for non basic variable and all other **(**$C_j - Z_j$) $\leq$ 0 or negative then it indicates case of alternate solution.

To get alternate solution, do one more iteration by taking key column as aentering column with **(**$C_j - Z_j$) = 0 for non-basic variable.

Example: Max Z = $30x_1 + 20x_2$

Sub to $x_1 + 2x_2 \leq 80$

$3x_1 + 2x_2 \leq 120$

and $x_1, x_2 \geq 0$

*****

UNIT II

3

TRANSPORTATION PROBLEM

Unit Structure

3.1 OBJECTIVE

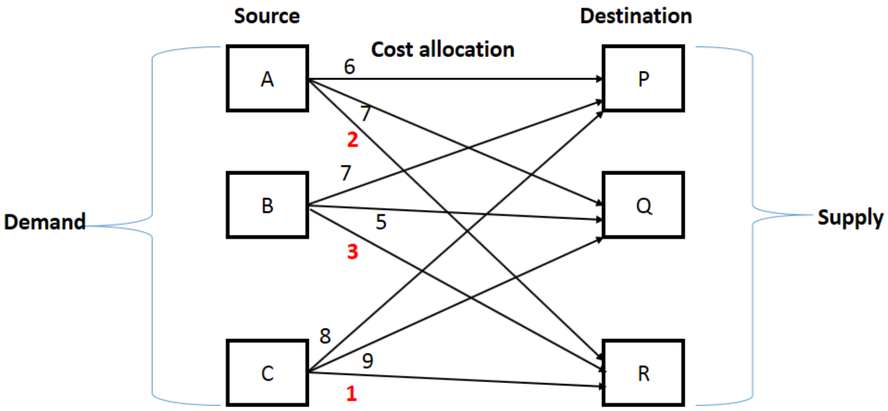
This chapter would make you understand the following concepts:

- Discuss the concept of transportation problem
- Definition of transportation problem
- Three cases occur in transportation problem
- Method to find initial basic solution with example
- Method to find optimal solution with example

3.2 INTRODUCTION

In mathematical science "operation research" acts important role. When we are perform any operation means perform some action to solve problem. In this unit we are discuss transportation problem technique. In transportation problem transport goods from setup sources /origin to set of destination in such a way that total transportation cost is minimised.

Discuss transportation concept with the help of example. Suppose a manufacturer of soap company has three plants situated at place A,B and C. Suppose his buyer are located in different region P,Q and R where he has to supply then the soap(goods).Also assume the demand in the three region and the production in different plants per unit time period are known and the cost out transporting one box to each center is given. The manufacturer's problem is to determine as to how he should rate this his product from his plant to the market place so that the total cost involved in the transportation is minimised. In other word how many soap boxes are deliver from A to P, Q and R; how many from B to P, Q and R; how many from C to P,Q and R to achieve 8 and list transportation cost is minimum.



The place where the soap goods originate are called source or origin and place where they are supplied are called destination. In the above example the manufacturer is to decide as how many units to be transported from different origin to different destinations so the total cost is minimum and this is the concept of transportation problem.

3.3 DEFINITION OF TRANSPORTATION PROBLEM (HITCH COCK PROBLEM)

Transportation problem is linear programming problem. Consider a transportation problem with m origin n destination with their respective available capacity a1, a2, a3….am and respect to demand as b1, b2…. bn.

Mathematically transportation problem may be stated as follows:

Minimise (Total cost) Z= $\sum_{i=1}^{m} \; . \; \sum_{j=1}^{n}$ .CijXij

Subject to $\sum_{j=1}^{n} \quad Cij = a1$

$\sum_{j=1}^{n} \quad C2j = a2$

Supply/Capacity

$\sum_{j=1}^{n} \quad C3j = a3$……………..up to $\sum_{j=1}^{n} \quad Cmj = am$

Constraint

Similarly $\sum_{i=1}^{m} \quad Ci1 = b1$

$\sum_{i=1}^{m} \quad Ci2 = b2$
Demand / Requirement

$\sum_{i=1}^{m} \quad Ci3 = b3$ ....up to $\sum_{i=1}^{m} \quad Cin = bn$ Constraint

The problem is to transport the goods at minimum cost from source to destination. it is called transportation problem.

The transportation problem can be stated as in the following tabular form:

	1	2	3	……… …	N	Supply
1	C11 X11	C12 X12	C13 X13	……… ….	C1n X1n	a1
2	C21 X21	C22 X22	C23 X23	……… …	C2n X2n	a2
3	C31 X31	C32 X32	C33 X33	……… …	C3n X3n	a3
Demand	b1	b2	b3	……… …	bn	

According to the tabular form,

ai = Quantity of product available at origin i

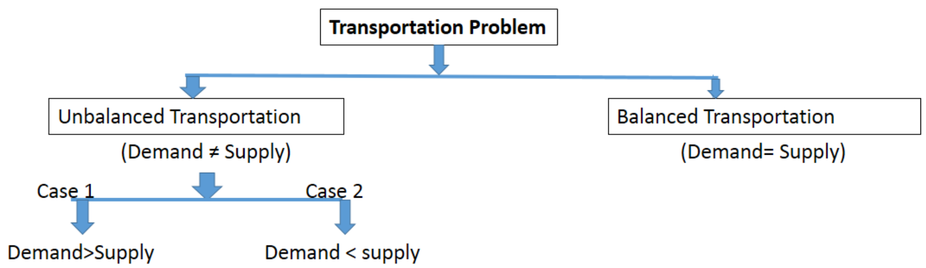
bj = Quantity of product available at origin j

Cij = Cost of transportation one unit of the product from I th origin to j th destination

Xij = Quantity transported from i th origin to j th destination.

Then the problem is to determine the transportation operation so is to minimise the total transportation cost satisfying supply and demand condition.

Transportation problem can be divided into two categories.



3.3.1 Balanced Transportation Problem:

In balance transportation problem check sum of availability of constrain (demand) equal to the sum of requirements (supply) means check

$\sum_{i=1}^{m} bi = \sum_{j=1}^{n} ai$

Demand = Supply

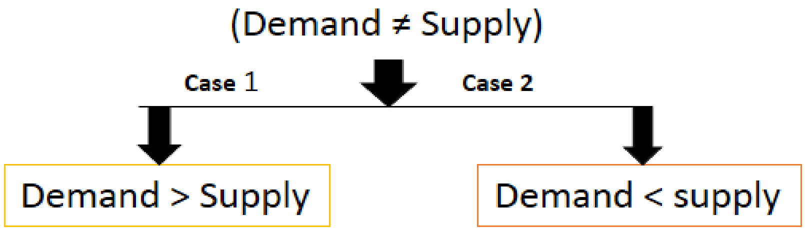
Consider a given example with three origin and three destination.

	P	**Q**	**R**	**Supply**
A	3	6	7	90
B	3	5	2	10
C	5	2	5	20
Demand	60	30	30	60+30+30=**<u>120</u>** Equal to 90+10+20=**<u>120</u>**

$\therefore$ *Demand = Supply*
$\therefore$ *This type of probem is called balanced transportation problem.*

3.3.2 Unbalanced Transportation Problem:

Unbalanced transportation problem arises due to shortage of raw material, improper planning, shortage of transport and time scheduling. In unbalanced transportation problem check sum of availability of constraints is equal or not equal to sum of requirement.it it is not equal then either it can be greater than or less than. In unbalanced transportation have two cases:



3.3.2. a. Demand is grater is greater than supply:

For balancing the unbalanced the problem can be solved by introducing the dummy source (row) added. The unit transportation cost to dummy row are assigned to zero value and perform demand - supply and place the result to the corresponding dummy destination. Discuss the concept with the help of example:

	P	**Q**	**R**	**Supply**
A	3	6	7	90
B	3	5	2	10
C	5	2	5	20
Demand	50	50	40	50+50+40=**<u>140</u>**> 90+10+20=**<u>120</u>**

The problem is unbalanced transportation problem.

$\therefore$ *Demand > Supply*

$$\sum_{i=1}^{m} bi > \sum_{j=1}^{n} ai$$

140 > 120

To convert it into balanced transportation problem include a dummy source(row).

	P	Q	R	Supply
A	3	6	7	90
B	3	5	2	10
C	5	2	5	20
D1	0	0	0	20
Demand	50	50	40	

Demand-supply (40-20=20)

After check demand = Supply.

	P	Q	R	Supply
A	3	6	7	90
B	3	5	2	10
C	5	2	5	20
D1	0	0	0	20
Demand	50	50	40	140=140

Yes, Problem converted to balanced transportation problem.

3.3.2. b. Demand is less than supply:

In demand variation may be occur in real life because of customer of preference, New product comes in market, product cost etc. Due to this demand variation our problem is converted into unbalanced problem.

To solve this unbalanced the problem (demand is less than supply) and destination (column) with supply. The unit transportation cost for dummy column are assigned to zero value and for subtract supply from demand and place the result to corresponding dummy supply.

	P	Q	R	S	Supply
A	5	4	2	6	10
B	8	3	5	7	50
C	10	11	3	1	30
D	9	12	6	3	20
Demand	30	40	20	10	**100 < 110**

The given problem is unbalanced transportation problem.

Demand < Supply

$$\sum_{i=1}^{m} \quad bi < \sum_{j=1}^{n} \quad ai$$

To convert it into balanced transportation problem include a destination (column).

	P	**Q**	**R**	**S**	**S1**	**Supply**
A	5	4	2	6	0	10
B	8	3	5	7	0	50
C	10	11	3	1	0	30
D	9	12	6	3	0	20
Demand	30	40	20	10	$\boxed{10}$	

Yes the problem converted to balanced transportation problem.

	P	**Q**	**R**	**S**	**S1**	**Supply**
A	5	4	2	6	0	10
B	8	3	5	7	0	50
C	10	11	3	1	0	30
D	9	12	6	3	0	20
Demand	30	40	20	10	10	**110=110**

Remember:

Demand> Supply

Add a dummy row with cost zero and supply equal to (demand- supply)

Demand<Supply

Add column with cost zero and demand equal to (supply - demand)

3.4 SOLUTION OF TRANSPORTATION PROBLEM

The solution of transportation problem can be obtained in two stages: A) initial solution B) optimal solution.To update initial basic feasible solution using any one of the three.

A-1) North West Corner Rule

A-2) least cost method/Matrix minima method

A-3) Vogel's approximation method

To obtain optimal solution using any one of the following:

B-1) MODI method

B-2) Stepping Stone Method

3.4.1 Initial Feasible Solution:

In previous chapter we have learnt about solve transportation problem with the help of simplex simplex method is complex to solve problem. Hindi lesson we will look for an alternate solution to solve transportation problem. To solve the transportation problem first check total capacity equal to total requirement that is balanced problem.

The existence of initial feasible solution necessary and sufficient condition-

1) Rim condition- $\sum_{i=1}^{m} \quad bi = \sum_{j=1}^{n} \quad ai$

2) Number of allocation=m+n-1

(Where m is number of rows and n number of column)

If the above condition is satisfied the solution obtained is a non-degenerate. When number of positive allocation in initial basic solution less than m+n-1 the solution is said to be degenerate solution.

Number of allocation< m+n-1……….. degenerate solution

Number of allocation=m+n-1…………non- degenerate solution

The initial feasible solution can be obtained by following three method

1. North West Corner Method

2. Least cost method/ Matrix minima method

3. Vogel's approximation method/ penalty method

3.4.1. a. NCWR method:

Tabular form of transportation problem:

	1	**2**	**3**	**……………..**	**n**	**Supply**
1	C11 X(1,1)	C12 X(1,2)	C13 X(1,3)	…………….	C1n X(1,n)	a1
2	C21 X(2,1)	C22 X(2,2)	C23 X(2,3)	…………….	C2n X(2,n)	a2
3	C31 X(3,1)	C32 X(3,2)	C33 X(3,3)	……………	C3n X(3,n)	a3
Demand	b1	b2	b3	……………..	bn	

Consider a transportation method Xij denote the unit transportation cost from i[th] origin to j[th] destination a1, a2, a3,………,an are respective supply and b1,b2,b3……bn are respect demand.

Steps for the methods are:

1. Begin with the upper left hand corner X(1,1) cell of transportation table and allocate as many unit as possible equal to minimum between available capacity/ supply and demand/ requirement.

2. Balance the supply and demand units in the respective rows and column allocation.

3.

a) If the supply for the first row is pending then move down (↓) to the first sale in second row and first column 21 and locate minimum (bi-ai, a2) then go to step to for balance the supply and demand unit.

b) If the demand for the first column is satisfied then moved horizontally (→) to the next sale in the second column and first row and supplied the quantity as per step 1 then go to step to for adjust the supply and demand.

4. If for any cell supply equal demand then rim condition will occur in this case move to next allocation can be made in sale other in the next row or column.

5. Continue the procedure until the total available quantity is fully allocated to the same as required hence we have basic feasible solution with m+n-1 positive allocation.

Question-Obtained initial basic feasible solution for the following problem using the NCWR method.

	w1	**w2**	**w3**	**w4**	**supply**
A	6	7	9	3	70
B	11	5	2	8	55
C	10	12	4	7	90
demand	85	35	50	45	

Solution: Above problem is balanced transportation problem. Start with the top cell aw1 and allocate min(85,70)=70. First allocation **70** done here. Check b1>a1 we move to cell(B,w1) here we are located min(55,85 -70)=15Second allocation satisfies **15**. Hence we move horizontally to (B,w2) we allocate min(55-15,35)=35Third allocation satisfies **35**. According to step 3 b capacity of B still remains thus we move cell (B,w3) allocate min(55-15-35,50)=5

Forth allocation satisfies **5**.Requirement of B is over. Move vertically down to cell(c,w3) allocate min(90,50-5)=45.Fifth allocation satisfy **45**.Finally allocate **45** units.

	1	2	3	n	Supply	
1	6 **70**	7	9	3	~~70~~ -	0
2	11 **15**	**35** 5	2 **5**	8	~~55~~ -	~~40~~ ~~5~~ 0 - -
3	10	12 ~~35~~	4 **45**	7 **45**	~~90~~ -	~~45~~ 0 -
Demand	~~85~~ - ~~15~~ 0	0 -	~~50~~ - ~~45~~ 0	~~45~~ - 0		

Check positive location m=3,n=4 we have m+n-1=3+4-1=6 positive allocation.

Calculate total cost-70*6+15*11+35*5+5*2+45*4+45*7=1265.

3.4.1. b. Least cost method:

This method consider lowest cost this method give optimum solution then NCWR.

Step 1: select the sale with lowest transportation cost among all rows and column of table.

Step 2: allocate as many unit as possible to the sale determined in step 1 and eliminate that rom on column either supply is exhausted or demand satisfied.

Step 3: search the next minimum cost in the table and repeat from step 2- repeat above step till all supply and demand are exhausted.

Question: Obtain initial basic solution for the following problem

	w1	**w2**	**w3**	**w4**	supply
A	6	7	9	3	70
B	11	5	2	8	55
C	10	12	4	7	90
demand	85	35	50	45	

Solution:

In given problem minimum cost is to which is unique. We select cell (B,W3). Allocate min(50,55)=50. Requirement of w3 is satisfied and reduced the capacity 55 by 50 and eliminate the column w3.

	w1	**w2**	**w3**	**w4**	supply	
A	6	7	9	3	70	
B	11	5	2 **50**	8	~~55~~	5
C	10	12	4	7	90	
demand	85	35	~~50~~	45		
			0			

Search for next minimum in the remaining table. Next minimum cost is 3. We select (A,W4) can allocate min(45,70)=45 and reduce the capacity 75 by 45. Eliminate column w4.

	w1	**w2**	**w3**	**w4**	supply	
A	6	7	9	3 ***45***	~~70~~	25
B	11	5	2 ***50***	8	~~55~~	5
C	10	12	4	7	90	
demand	85	35	~~50~~	~~45~~		
			0	0		

Next minimum cost element is 5. Select cell (B,w2) and allocate min(35,5)=5. Satisfies the requirement of w2 and eliminate the row B.

	w1	**w2**	**w3**	**w4**			
A	6	7	9	3 ***45***	~~70~~	25	
B	11	5 ***5***	2 ***50***	8	~~55~~	~~5~~	0
C	10	12	4	7	90		

	85	~~35~~	~~50~~	~~45~~	
		30	0	0	

Search next minimum cost element is 6. Select cell (A,w1) allocate min(85,25)=25 and reduce the capacity 85 by 25. Eliminate row A.

	w1	**w2**	**w3**	**w4**		
A	6 ***25***	7	9	3 ***45***	~~70~~	~~25~~ 0
B	11	5 ***5***	2 ***50***	8	~~55~~	~~5~~ 0
C	10	12	4	7	90	
	~~85~~	~~35~~	~~50~~	~~45~~		
	60	30	0	0		

Next minimum cost is 10 select cell (C,w1) and allocate min(60,90)=60 han reduce the capacity and eliminate column w1.

	w1	**w2**	**w3**	**w4**		
A	6 ***25***	7	9	3 ***45***	~~70~~	~~25~~ 0
B	11	5 ***5***	2 ***50***	8	~~55~~	~~5~~ 0
C	10 ***60***	12	4	7	~~90~~	30
	~~85~~	~~35~~	~~50~~	~~45~~		
	~~60~~ 0	30	0	0		

Last minimum cost is 12 in the remaining table we select (C, w2) cell. Allocate cell and satisfy the requirement of w3 warehouse.

	w1	**w2**	**w3**	**w4**		
A	6 ***25***	7	9	3 ***45***	~~70~~	~~25~~ 0
B	11 	5 ***5***	2 ***50***	8	~~55~~	~~5~~ 0
C	10 ***60***	12 ***30***	4	7	~~90~~	~~30~~ 0

	~~85~~	~~35~~	~~50~~	~~45~~	
	~~60~~	~~30~~	0	0	
	0	0			

There is exactly 3 + 4 - 1= 6 allocation.

Total cost is: 25*6+45*3+5*5+50*2+60*10+30*12=1370

The cost is less than the cost determine in the solution by NCWR method. Hence this method very useful and practical oriented.

3.4.1.C. Vogel's approximation method:

This method is also called Penalty method. This method is a pray for the other two because the initial solution is a much closed to optimal solution.

Steps:

1. Find the difference between smallest cost and next smallest cost in particular row or column or penalty. Calculate this penalty for each row and column in the transportation table.

2. Select largest penalty of all transportation table. Hindi rope or column choose the sale that how the smallest cost and locate the maximum possible quantity to the cell.

3. Adjust the row capacity and column requirement by amount a locate to the cell in step 2.

4. If only one row or one column is left, we directly allocate the demand.

Question-solve the following transportation problem for initial basic solution using VAM method:

	w1	**w2**	**w3**	**w4**	**supply**
A	6	7	9	3	70
B	11	5	2	8	55
C	10	12	4	7	90
demand	85	35	50	45	

Solution:

Calculate the difference between smallest cost and next smallest cost i.e calculate penalty.

Penaulty

	w1	**w2**	**w3**	**w4**	**supply**					
A	6 ***70***	7	9	3	~~**70**~~	0	3			
B	11	5 ***35***	2 ***20***	8	~~**55**~~	~~20~~ 0	3	3	**6** ←	
C	10 ***15***	12	4 ***30***	7 ***45***	~~**90**~~	~~60~~ ~~15~~ 0	3	3	3	3
demand	~~**85**~~	~~**35**~~	~~**50**~~	~~**45**~~						
	~~**15**~~ **0**	**0**	~~**30**~~ **0**	**0**						
	4↑	2	2	4						
	1	**7** ↑	2	1						
	1		2	1						
	1									

Select largest penalty of all in above penalty 4 is the largest penalty but it is not unique. Select any one between them within this column w1 select smallest cost cell (A,w1) and allocate maximum possible quantity to this cell.

Remove particular row/ column in which supply/demand is exhausted.

Similar way to calculate penalty for remaining table and allocate cost up to one row and one cost left directly or locate the demand.

Number of allocation=m+n-1

Solution is a non- degenerate.

Total cost is:

70*6+35*5+20*2+30*4+45*7+10*15=1220

Initial solution is nearest to optimal solution.

3.5 OPTIMALITY TEST: (APPROACH TO OPTIMAL SOLUTION)

Once, we get the basic feasible solution for a transportation problem, the next duty is to test whether the solution got is an optimal one or not?

Optimal solution is achieved when there is no other alternative solution give lower cost.

An optimality test can be applied to the feasible solution only if it satisfies the following conditions:

(i) It contains exactly m+n-1 allocations where m and n represent the number of rows and columns, respectively, of the transportation table.

(ii) These allocations are independent.

This can be done by two methods:

(a) By Modified Distribution Method, or MODI method.

(b) By Stepping Stone Method

3.5.1 Modified Distribution Method (MODI):

In MODI, we can get the opportunity costs of empty cells without writing the loop. After getting the opportunity cost of all the cells, we have to select the cell with highest positive opportunity Cost for including it in the modified solution.

For modified solution the method follows following steps:

1. First find an initial feasible solution using a suitable method.

2. Check optimality conditions for the current basic feasible solution if it has exactly

   (m+n-1) independent allocations, write the cost matrix for only the allocated cells.

3. Row 1, row 2,…, row i of the cost matrix are assigned with variables U1, U2, …, Ui and the

   Column 1, column 2,…, column j are assigned with variables V1, V2, …,Vj respectively.

4. Using this cost matrix, determine the set of m+n numbers ui(i =1,2,…,m) and vj (j=1,2,..,n) such that for each occupied cell (i, j), $Cij=ui+vj$, taking one of ui or vj as zero.

5. Fill the vacant cells using $ui+vj$.

6. Compute all unit cost differences

   $\Delta ij=Cij-(ui+vj)$

   by subtracting the values so obtained in Step 5 from the corresponding values of the original cost matrix.

7. Examine the sign of each Δij

- If all $\Delta ij>0$, for all i, j; then the current basic feasible solution is an optimal solution.

- (b) If all $\Delta ij \geq 0$, for all i, j and at least one Δij is zero then the current basic feasible solution is an optimal solution and multiple basic initial feasible solution exists.
- (c) If some of $\Delta ij < 0$ for some (i, j), the current solution is not optimal. Then select the cell having the most negative Δij and tick it.

Question: In initial basic feasible solution is obtained by Matrix Minimum Method and is shown in table:

Distribution Centre						
		D1	**D2**	**D3**	**D4**	**Supply**
Plant	**P1**	19	30	50	12 7	7
	P2	70 3	30	40 7	60	10
	P3	40 2	10 8	60	20 8	18
Requirement		5	8	7	15	

Initial basic feasible solution

12 X 7 + 70 X 3 + 40 X 7 + 40 X 2 + 10 X 8 + 20 X 8 = 894.

Calculating u_i and v_j using $u_i + v_j = c_{ij}$

Substituting $u_1 = 0$, we get

$u_1 + v_4 = c_{14} \Rightarrow 0 + v_4 = 12$ or $v_4 = 12$
$u_3 + v_4 = c_{34} \Rightarrow u_3 + 12 = 20$ or $u_3 = 8$
$u_3 + v_2 = c_{32} \Rightarrow 8 + v_2 = 10$ or $v_2 = 2$
$u_3 + v_1 = c_{31} \Rightarrow 8 + v_1 = 40$ or $v_1 = 32$
$u_2 + v_1 = c_{21} \Rightarrow u_2 + 32 = 70$ or $u_2 = 38$
$u_2 + v_3 = c_{23} \Rightarrow 38 + v_3 = 40$ or $v_3 = 2$

Distribution centre							
		D1	**D2**	**D3**	**D4**	**Supply**	**u_i**
Plant	**P1**	19	30	50	12 7	7	0
	P2	70 3	30	40 7	60	10	38
	P3	40 2	10 8	60	20 8	18	8
Requirement		5	8	7	15		
v_j		32	2	2	12		

Calculating **opportunity cost** using $c_{ij} – (u_i + v_j)$

Unoccupied cells	**Opportunity cost**
(P_1, D_1)	$c_{11} – (u_1 + v_1) = 19 – (0 + 32) = –13$
(P_1, D_2)	$c_{12} – (u_1 + v_2) = 30 – (0 + 2) = 28$
(P_1, D_3)	$c_{13} – (u_1 + v_3) = 50 – (0 + 2) = 48$
(P_2, D_2)	$c_{22} – (u_2 + v_2) = 30 – (38 + 2) = –10$
(P_2, D_4)	$c_{14} – (u_2 + v_4) = 60 – (38 + 12) = 10$
(P_3, D_3)	$c_{33} – (u_3 + v_3) = 60 – (8 + 2) = 50$

Distribution centre							
		D1	**D2**	**D3**	**D4**	**Supply**	u_i
Plant	**P1**	-13 19	28 30	48 50	7 12	7	0
	P2	3 70	-10 30	7 40	10 60	10	38
	P3	2 40	8 10	50 60	8 20	18	8
Requirement		5	8	7	15		
v_j		32	2	2	12		

Now choose the smallest (most) negative value from opportunity cost (i.e., –13) and draw a closed path from P1D1. The following table shows the closed path.

Distribution centre							
		D1	**D2**	**D3**	**D4**	**Supply**	u_i
	P1	-13 19	28 30	48 50	7 12	7	0
Plant	**P2**	3 70	-10 30	7 40	10 60	10	38
	P3	2 40	8 10	50 60	8 20	18	8
Requirement		5	8	7	15		
v_j		32	2	2	12		

Choose the smallest value with a negative position on the closed path(i.e., 2), it indicates the number of units that can be shipped to the entering cell. Now add this quantity to all the cells on the corner points of the closed path marked with plus signs and subtract it from those cells marked with minus signs. In this way, an unoccupied cell becomes an occupied cell.

Now again calculate the values for u_i & v_j and opportunity cost. The resulting matrix is shown below

Distribution centre							
		D1	**D2**	**D3**	**D4**	**Supply**	**u_i**
Plant	**P1**	2 19	28 30	61 50	5 12	7	0
	P2	3 70	-23 30	7 40	-3 60	10	51
	P3	13 40	8 10	63 60	10 20	18	8
Requirement		5	8	7	15		
v_j		19	2	–11	12		

Choose the smallest (most) negative value from opportunity cost (i.e., –23). Now draw a closed path from P2,D2 .

Distribution centre							
		D1	**D2**	**D3**	**D4**	**Supply**	**u_i**
Plant	**P1**	2 19	28 30	61 50	5 12	7	0
	P2	3 70	-23 30	7 40	-3 60	10	51
	P3	13 40	8 10	63 60	10 20	18	8
Requirement		5	8	7	15		
v_j		19	2	–11	12		

Now again calculate the values for u_i & v_j and opportunity cost

Distribution center							
		D1	**D2**	**D3**	**D4**	**Supply**	**u_i**
Plant	**P1**	5 19	28 30	38 50	2 12	7	0
	P2	23 70	3 30	7 40	20 60	10	28

	P3	13 40	10 5	40 60	20 13	18	8
Requirement		5	8	7	15		
v_j		19	2	12	12		

Total cost -19*5+30*3+10*5+40*7+12*2+20*13=799

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UNIT III

4

ASSIGNMENT MODEL

Unit Structure

4.0 OBJECTIVES

After studying the unit you should be able to:

- Formulate mathematically an assignment problem
- Optimal solution can be obtained by using Hungarian Method
- Special cases of assignment problem viz maximization case, , unbalanced assignment problem, restricted assignment problem, alternate optimal solution and travelling salesman problem can be solved.

4.1 ASSIGNMENT MODEL

An assignment problem is a type of linear programming problem or you can say that it is a special case of transportation problem. The objective of assignment model is to allocate number of facilities to number of jobs on one to one basis i.e. one facility can be assigned to only one job, so as to minimize the costs and maximize the profits of the jobs. One can notice situation of this kind in many areas like assigning salesmen in different areas for sales, number of taxis to number of customers, assigning teachers to classes etc.

4.2 MATHEMATICAL REPRESENTATION OF THE ASSIGNMENT PROBLEM

The assignment problem can be represented by $n \times n$ matrix, where n facilities are assigned to n jobs. The matrix is called as cost matrix (cij) where cij is the cost of assigning i^{th} facility to j^{th} job.

$$\begin{bmatrix} C11 & C12 & C13\ldots\ldots\ldots C1n \\ C21 & C22 & C23\ldots\ldots\ldots C2n \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ Cn1 & C2n & C3n\ldots\ldots\ldots Cnn \end{bmatrix}$$

Mathematically, the assignment model can be represented as:

Xij = 1: if i^{th} acility is assigned to j^{th} job

= 0: if i^{th} facility is not assigned to j^{th} job

Minimize Z = $\sum_{i=1}^{n} \sum_{j=1}^{n} CijXij \quad \{i = 1,2, \ldots \ldots n, j = 1,2, \ldots \ldots . n$

Subject to restrictions (constraints)

$\sum_{i=1}^{n} Xij = 1 \;\; if\; j = 1,2 \ldots \ldots . . n$
$\{One\ facility\ is\ assigned\ to\ jth\ job$

$\sum_{j=1}^{n} Xij = 1 \;\; if\; i = 1,2 \ldots \ldots . . n$
$\{One\ job\ is\ assigned\ to\ ith\ facility$

4.2 HUNGARIAN METHOD FOR SOLVING ASSIGNMENT PROBLEM

The Hungarian method is an efficient method of solving assignment problem. The method involves the following steps:

Step 1:

Prepare a cost matrix of the given problem. If the number of rows and number of columns are equal then matrix will be called as balanced matrix. If number of rows and number of columns are not equal then matrix is unbalanced matrix. To convert an unbalanced matrix to balance matrix, a dummy row/column is introduced so that matrix becomes balanced (a square matrix).The cost entries of dummy are always zero.

Step 2:

Row Reduction: Subtract the smallest element of each row from all the elements of the row. This will result in at least one zero in each row.

Column Reduction: Subtract the smallest element of each column from all the elements of the column. This will result in at least one zero in each column. Now each row and column will have at least one zero.

Step3:

Investigate the rows and columns sequentially, select a row or column with a single zero element and encircle it (or square it). Mark a X (cross) in the cells having zero element, lying in the same column or row so that no further assignments can be done in that particular column or row.

Again select the next row and encircle (or square) the zero element and mark a X (cross) in the cells having zero element, lying in the same column or row so that no further assignments can be done in that particular column or row.

Repeat the process for the remaining rows till all the zero's are assigned in each row and column. Each row or column must have one encircled (or square) zero element in it.

Step 4:

If the number of assignments (zero's circled or squared) are equal to number of row/column, that means we have reached an optimal solution. The total cost of assignment is obtained by adding original cost values from the original given cost matrix.

If the number of assignments (zero's circled or squared) are not equal to number of row/column that means no optimal solution is obtained. Go to step 5.

Step5:

Draw the minimum number of horizontal or vertical straight lines to cover all the zeroes from the reduced cost matrix (matrix obtained using step 2). If the number of straight lines are equal to size of matrix (i.e number of rows/columns of matrix), the present solution is optimal solution else go to step 6.

Step 6:

Investigate the uncovered elements of the matrix (elements not covered by any lines), find the smallest element from uncovered elements. Hence subtract all the uncovered elements from this smallest uncovered element and further add the smallest uncovered element to the elements at the intersection point of two lines. Covered elements will remain unaffected, no change in their values.

Step 7:

Perform the step 3 and repeat the entire process until optimal solution is obtained.

Problem 1:

Solve the following assignment problem using Hungarian Method. The given cost matrix represents the combination of 4 different jobs on 4 different machines.

Table A.2 Jobs

	A	B	C	D
Machine 1	20	24	38	22
2	10	20	14	16
3	24	28	26	22
4	16	30	22	18

Solution:

Since the number of rows and columns are equal we can say that problem is balanced type.

Step 1:

Choose the minimum element in each row and subtract it from every element of that row. In this case, 20, 10, 22 and 16 are minimum element from 1st, 2nd, 3rd and 4th row respectively. In this way we get row reduced matrix.

Table A.3 Jobs
Row Reduced Matrix

	A	B	C	D
Machine 1	0	4	18	2
2	0	10	4	6
3	2	6	4	0
4	0	14	6	2

Step 2:

Choose the minimum element in each column and subtract it from every element of that column. In this case, 0, 4, 4 and 0 are minimum element from 1st, 2nd, 3rd and 4th column respectively. In this way we get column reduced matrix.

Table A.4 Jobs
Column Reduced Matrix

	A	B	C	D
Machine 1	0	0	14	2
2	0	6	0	6
3	2	2	0	0
4	0	10	2	2

Step 3:

Table A.5 Jobs
Assignment Matrix

	A	B	C	D
Machine 1	$\cancel{0}$	0	14	2
2	$\cancel{0}$	6	0	6
3	2	2	$\cancel{0}$	0
4	0	10	2	2

Starting with the first row (Machine 1) in table A.5, we examine each rows until we found a row with single zero (Machine 4). We make an assignment by encircling that cell. Then we cross other zeroes in that particular column and row to eliminate the possibility of making further assignments in that column and row. We repeat the process for other rows till all the rows have exactly one encircled zero (and columns also). If all the rows (and columns also) contains exactly one encircled zero or in other words total number of assignments are equal to size of matrix, we can say that we have reached an optimal solution.

From table A.5 it is evident that total number of assignments (encircled zeroes) are 4 and size of matrix is also 4, hence we have obtained an optimal solution.

The pattern of assignments for the combination of machine and jobs are as follows:

Table A.6

Assignment Schedule

	Jobs	Cost
Machine 1	B	24
2	C	14
3	D	22
4	A	16
	Total	76

The optimal cost is Rs 76.

Problem 2: Solve the following assignment problem using the Hungarian method. The cost matrix in the table B.1 represents the combination of employees and their job status.

Table B.1 Jobs

		A	B	C	D	E
	1	14	10	18	16	22
	2	18	24	14	22	20
Employees	3	16	10	8	12	18
	4	14	6	12	18	10
	5	8	12	14	10	22

Solution: The matrix consist of 5 rows (1,2,3,4,5) and 5 columns (A,B,C,D,E), since the number of rows and columns are equal, we can say that matrix is balanced and of minimization (as cost is given) type.

Step 1:

Choose the minimum element in each row and subtract it from every element of that row. In this case, 10, 14, 8, 6 and 8 are minimum element from 1st, 2nd, 3^{rd} 4^{th} and 5^{th} row respectively. In this way we get row reduced matrix.

Table B.2 Jobs

Row Reduced Matrix

		A	B	C	D	E
	1	4	0	8	6	12
	2	4	10	0	8	6
Employees	3	8	2	0	4	10
	4	8	0	6	12	4
	5	0	4	6	2	14

Step 2:

Choose the minimum element in each column and subtract it from every element of that column. In this case, 0, 0, 0,2 and 4 are minimum element from 1st, 2nd, 3rd 4th and 5th column respectively. In this way we get column reduced matrix.

Table B.3 Jobs

Column Reduced Matrix

		A	B	C	D	E
	1	4	0	8	4	6
	2	4	10	0	6	2
Employees	3	8	2	0	2	6
	4	8	0	6	10	0
	5	0	4	6	0	10

Step 3:

Starting with the first row (Employees 1) in table B.3, we examine each rows until we found a row with single zero (Employees 1). We make an assignment by encircling that cell. Then we cross other zeroes in that particular column and row to eliminate the possibility of making further assignments in that column and row. We repeat the process for other rows till all the rows have exactly one encircled zero (and columns also). If all the rows (and columns also) contains exactly one encircled zero or in other words total number of assignments are equal to size of matrix, we can say that we have reached an optimal solution.

From table B.4 it is evident that total number of assignments (encircled zeroes) are 4 and size of matrix is 5. The 3rd row (Employees 3) does not have a single assignment (encircled zero), hence we have not obtained an optimal solution.

The pattern of assignments for the combination of employees and jobs are as follows:

Table B.4 Jobs

Assignment Matrix

		A	B	C	D	E
	1	4	0	8	4	6
	2	4	10	0	6	2
Employees	3	8	2	~~0~~	2	6
	4	8	~~0~~	6	10	0
	5	0	4	6	~~0~~	10

Step 4:

Consider Table B.4 and draw minimum number of lines to cover all the zeroes. We can draw vertical or horizontal lines to cover all the zeroes. To get minimum number of lines we start with row/column which has maximum zeroes. Hence we draw 1st horizontal line in the 4th or 5th row as both the rows has two zeroes. So first horizontal line is drawn on 4th row and then on 5th row. Now column C consist of two zeroes, to cover these zeroes we draw a vertical line to cover the zeroes. Then column B consist of two zeroes out of which one zero (employees 4 and job B) has already been covered, so only remaining zero is 1st row (employees 1 and job B), so we can draw either vertical or horizontal line. Drawing either of the lines (vertical or horizontal) will not affect the final outcome. Here we are drawing a vertical line (employees 1 and job B).After drawing the lines we will get the following matrix i.e. Table B.5

Table B.5 Jobs

		A	B	C	D	E
	1	4	0	8	4	6
	2	4	10	0	6	2
Employees	3	8	2	0	2	6
	4	8	0	6	10	0
	5	0	4	6	0	10

Identify the smallest value which is not covered by any line (minimum uncovered value).From table B.5,it is evident that minimum uncovered value is 2, this value will be subtracted from all the uncovered values and will be added to the values at the point of intersection (where the two lines are intersecting) and the remaining values (covered values) will remain the same . The resultant matrix is shown in the table B.6.

Table B.6 Jobs

		A	B	C	D	E
	1	2	0	8	2	4
	2	2	10	0	4	0
Employees	3	6	2	0	0	4
	4	8	2	8	10	0
	5	0	6	8	0	10

Step 6:

Examine all the rows one by one and find the row which have a single zero. The 1st row and 4th row have a single zero so we will encircle it and cross other zeroes in that particular column to eliminate the possibility of making further assignments in that column .Now 2nd row has a single zero to be encircled (row-Employee-2, column-Job-C) as another zero (lying on row-Employee-2, column-Job-E) cannot be considered because it has been crossed. We will examine the 3rd row and 5th row in a same way and encircle the zeroes. Make sure each row consist of only one encircled zeroes which indicates that all the columns will have single encircled zeroes.

Table B.7 Jobs

Assignment Matrix

		A	B	C	D	E
Employees	1	2	0	8	2	4
	2	2	10	0	4	~~0~~
	3	6	2	~~0~~	0	4
	4	8	2	8	10	0
	5	0	6	8	~~0~~	10

From Table B.7, number of assignments are 5 and size of matrix is also 5, hence we can say we have reached an optimal solution. The assignment schedule is given in Table B.8

Table B.8

Assignment Schedule

	Jobs	Cost
Machine1	B	10
2	C	14
3	D	12
4	E	10
5	A	14
	Total	60

The optimal cost is Rs 60.

Problem 3: Solve the following assignment problem using the Hungarian method. Three jobs A, B and C are to be assigned to the machines. The processing cost of each job – machine combination matrix is given in the table C.1

Table C.1 Machines

		I	II	III	IV
Job	A	6	9	11	15
	B	1	3	5	6
	C	2	4	6	8

Solution: The matrix consist of 3 rows (1, 2, 3) and 4 columns (I, II, III, IV), since the number of rows and columns are not equal, we can say that matrix is not balanced and of minimization (as cost is given) type. To apply Hungarian Method the matrix should be of balanced type, so we will introduce dummy row (as one row is less than the number of columns) under the name Dummy where all the entries will be zeroes. Refer Table C.2 now the number of rows including dummy are 4 and number of columns are also 4. Hence the matrix is balanced and of minimization type (cost is given). We can now apply Hungarian Method.

Table C.2 Machines

Job	A	6	9	11	15
	B	1	3	5	6
	C	2	4	6	8
	Dummy	0	0	0	0

Step 1:

Choose the minimum element in each row and subtract it from every element of that row. In this case, 6, 1, 2 and 0 are minimum element from 1st, 2nd, 3rd and 4th row respectively. In this way we get row reduced matrix.

Table C.3 Machines

Row Reduced Matrix

		I	II	III	IV
Job	A	0	3	5	9
	B	0	2	4	5
	C	0	2	4	6
	Dummy	0	0	0	0

Step 2:

Choose the minimum element in each column and subtract it from every element of that column. In this case, 0, 0, 0 and 0 are minimum element from 1st, 2nd, 3rd and 4th column respectively. In this way we get column reduced matrix.

Table C.4 Machines

Column Reduced Matrix

		I	II	III	IV
Job	A	0	3	5	9
	B	0	2	4	5
	C	0	2	4	6
	Dummy	0	0	0	0

Step 3:

Starting with the first row (Job A) in table C.4, we examine each rows until we found a row with single zero (Job A). In this case 1st has single zero and we encircle it and marking cross on all other zeroes lying in the first column. Now 2nd and 3rd row does not have zero to get encircled but 4th row has choice of three zeroes. We can select any zero out of the three zeroes and encircled it. We make an assignment by encircling that cell. Then we cross other zeroes in that particular column and row to eliminate the possibility of making further assignments in that column and row.

From table C.5 it is evident that total number of assignments (encircled zeroes) are 2 and size of matrix is 4. The 2nd and 3rd row (Job 2 and Job 3) does not have a single assignment (encircled zero), hence we have not obtained an optimal solution.

The pattern of assignments for the combination of jobs and machines are as follows:

Table C.5 Machines

Assignment Matrix

		I	II	III	IV
	A	0	3	5	9
Job	B	~~0~~	2	4	5
	C	~~0~~	2	4	6
	Dummy	~~0~~	0	~~0~~	~~0~~

Consider Table C.6 and draw minimum number of lines to cover all the zeroes. We can draw vertical or horizontal lines to cover all the zeroes. To get minimum number of lines we start with row/column which has maximum zeroes. Hence we draw 1st horizontal line in the Dummy row as it has four zeroes. So first horizontal line is drawn on Dummy row and then 1st column consist of four zeroes, to cover these zeroes we draw a vertical line to cover the zeroes. After drawing the lines we will get the following matrix i.e. Table C.6.

Table C.6 Machines

		I	II	III	IV
	A	0	3	5	9
Job	B	0	2	4	5
	C	0	2	4	6
	Dummy	0	0	0	0

Identify the smallest value which is not covered by any line (minimum uncovered value).From table C.6,it is evident that minimum uncovered value is 2, this value will be subtracted from all the uncovered values and will be added to the values at the point of intersection (where the two lines are intersecting) and the remaining values (covered values) will remain the same. The resultant matrix is shown in the table C.7

Table C.7 Machines

		I	II	III	IV
Job	A	0	1	3	7
	B	0	0	2	3
	C	0	0	2	4
	Dummy	2	0	0	0

.

Step 4:

Now we can do the assignments. Starting with the first row (Job A) in table C.8, we examine each rows until we found a row with single zero (Job A). In this case 1st row has single zero and we encircle it and marking cross on all other zeroes lying in the first column. Now 2nd row has single zero to be encircled and 3rd row does not have zero to get encircled but 4th row has choice of two zeroes. We can select any zero out of the two zeroes and encircled it. We make an assignment by encircling that cell. Then we cross other zeroes in that particular column and row to eliminate the possibility of making further assignments in that column and row.

Table C.8 Machines

		I	II	III	IV
Job	A	0	1	3	7
	B	~~0~~	0	2	3
	C	~~0~~	~~0~~	2	4
	Dummy	2	~~0~~	0	~~0~~

The number of assignments are 3 and size of matrix is 4, hence optimal solution is not obtained.

Step 5:

Consider Table C.8 and draw minimum number of lines to cover all the zeroes. We can draw vertical or horizontal lines to cover all the zeroes. We draw 1st horizontal line in the Dummy row as it has three zeroes. So first horizontal line is drawn on Dummy row and then 1st column consist of three zeroes and 2nd column consist of two zeroes, and to cover these zeroes we draw a vertical lines After drawing the lines we will get the following matrix i.e. Table C.9

Table C.9 Machines

		I	II	III	IV
Job	A	0	1	3	7
	B	0	0	2	3
	C	0	0	2	4
	Dummy	2	0	0	0

Step 6:

Identify the smallest value which is not covered by any line (minimum uncovered value).From table C.9,it is evident that minimum uncovered value is 2, this value will be subtracted from all the uncovered values and will be added to the values at the point of intersection (where the two lines are intersecting) and the remaining values (covered values) will remain the same. The resultant matrix is shown in the table C.10

Table C.10 Machines

		I	II	III	IV
Job	A	0	1	1	4
	B	0	0	0	1
	C	0	0	0	2
	Dummy	4	2	0	0

Step 7:

Now we can do the assignments. Starting with the first row (Job A) in table C.11, we examine each rows until we found a row with single zero (Job A). In this case 1^{st} row has single zero and we encircle it and marking cross on all other zeroes lying in the first column. Now 2^{nd} row has two's zero out of which one is to be encircled, so we choose zero lying on 2^{nd} row and 2^{nd} column (Job-B and Machine –II) and marking cross on the zeroes lying in the 2^{nd} column. 3^{rd} row have single zero lying on 3^{rd} row and 3^{rd} column (Job-C and Machine –III) to get encircled and 4^{th} row has also single zero lying on 4^{th} row and 4^{th} column (Job-Dummy and Machine –IV) to get encircled. After, we obtained the resultant matrix i.e. Table C.11

Table C.11 Machines

Assignment Matrix

		I	II	III	IV
Job	A	0	1	1	4
	B	~~0~~	0	0	3
	C	~~0~~	~~0~~	0	4
	Dummy	4	2	0	0

Number of assignments are 4 and the size of matrix is also 4, we can say that optimal solution is obtained.

Table C.12

Assignment Schedule

	Jobs	Cost
Machine A	I	6
B	II	3
C	III	6
Dummy	IV	0
	Total	15

The optimal cost is Rs 15

Problem 4: A sales manager has to assign 4 salesman to four territories. The following table shows the annual profit (in Rs lakh) that has to be generated by each salesman in each territory. Find optimum assignment of salesman and territory to maximize the profit.

Table D.1 Territory

		T1	T2	T3	T4
Salesman	A	70	54	56	74
	B	56	68	58	80
	C	70	48	64	66
	D	48	64	50	56

Solution: The matrix consist of 4 rows (A, B, C.D) and 4 columns (T1, T2, T3, T4), since the number of rows and columns are equal, we can say that matrix is balanced and of maximization type (as profit is given) type. To apply Hungarian Method the matrix should be of minimization type, so we will create a regret matrix by selecting the maximum value from the entire matrix i.e. 80 and subtracting each value by 80. The matrix so obtained will be consider as Regret Matrix. We can now apply Hungarian Method to the matrix given in Table D.2

Table D.2 Territory

Regret Matrix

		T1	T2	T3	T4
Salesman	A	10	26	24	6
	B	24	12	22	0
	C	10	32	16	14
	D	32	16	30	24

Step 1:

Prepare a Row Reduced matrix by selecting a minimum element from each row of regret matrix and subtracting it from each element of that row. The minimum elements from each row are 6,0,10 and 16. The following matrix will be obtained (Table D.3).

Table D.3 Territory

Row Reduced Matrix

		T1	T2	T3	T4
Salesman	A	4	20	18	0
	B	24	12	22	0
	C	0	22	6	4
	D	16	0	14	8

Step 2:

Prepare a Column Reduced matrix by selecting a minimum element from each column of Row Reduced matrix (Table D.3) and subtracting it from each element of that column. The minimum elements from each column are 0, 0, 6 and 0. The following matrix will be obtained (Table D.4).

Table D.4 Territory

Column Reduced Matrix

		T1	T2	T3	T4
	A	4	20	18	0
Salesman	B	24	12	22	0
	C	0	22	6	4
	D	16	0	14	8

Step 3:

Now we can do the assignments. Starting with the first row (Salesman A) in table D.4, In this case 1st row has single zero and we encircle it and marking cross on all other zeroes lying in the fourth column. 3rd row have single zero lying on 3rd row and 1st column (Salesman -C and Territory-T1) to get encircled and 4th row has also single zero lying on 4th row and 2nd column (Salesman –D and Territory-T2) to get encircled. After, we obtained the resultant matrix i.e. Table D.5

Table D.5 Territory

Assignment Matrix

		T1	T2	T3	T4
	A	4	20	12	0
Salesman	B	24	12	16	~~0~~
	C	0	22	~~0~~	4
	D	16	0	8	8

The number of assignments are 3 and size of matrix is 4, hence optimal solution is not obtained.

Step 5:

Draw minimum number of lines to cover all the zeroes.

Table D.6 Territory

		T1	T2	T3	T4
	A	4	20	12	0
Salesman	B	24	12	16	0
	C	0	22	0	4
	D	16	0	8	8

Step 6:

Select the minimum uncovered element from Table D.6 and subtract from other uncovered elements and minimum uncovered element to the elements where the two lines are intersecting. In this case minimum uncovered element is 4. The following matrix (Table D.7) will be obtained:

Table D.7 Territory					
Asignment Matrix					
		T1	T2	T3	T4
Salesman	A	0	16	8	0
	B	20	8	12	0
	C	0	22	0	8
	D	16	0	8	12

Step 7:

Now we can do the assignments. Starting with the first row (Salesman A) in table D.8, In this case 1st row has single zero and we encircle it and marking cross on all other zeroes lying in the first column. 2nd row has single zero to be encircled, 3rd row and 4th row have single zeroes to be encircled.. After, we obtained the resultant matrix i.e. Table D.8

Table D.8 Territory
Assignment Matrix

		T1	T2	T3	T4
	A	0	16	8	0
Salesman	B	20	8	12	0
	C	~~0~~	22	0	8
	D	16	0	8	12

The number of assignments are 4 and size of matrix is 4, hence optimal solution is obtained.

Step 8:

The assignment schedule is given in Table D.

Table D.9
Assignment Schedule

	Territory	Cost
Salesman 1	T1	70
2	T4	80
3	T3	64
4	T2	64
	Total	278

The optimal profit is Rs 278.

Problem 5: A manager wants to assign four jobs to four operators and from his experience he knows that two operators are inefficient to do two specific jobs. This is indicated by '- 'in the cost matrix. Determine the optimal assignment from the following matrix which gives cost (in hundred rupees) for employing operators on different jobs.

Table E.1 Jobs

		A	B	C	D
	I	8	14	10	12
Operator	II	_	16	14	12
	III	6	_	10	6
	IV	16	12	8	4

Solution: The matrix consist of 4 rows (I, II, III, IV) and 4 columns (A, B, C, D), since the number of rows and columns are equal, we can say that matrix is balanced and of minimization type (as cost is given).Since operator II cannot perform Job A and operator III cannot perform Job B, hence it is a prohibited or restricted problem. Cost of prohibited or restricted problem is taken as 'M' which indicates infinity. The value 'M' will remain as it is throughout the solution. No assignments or no change in values can happen to value 'M'. The matrix so obtained given in Table E.2

Table E.2 Jobs

		A	B	C	D
	I	8	14	10	12
Operator	II	M	16	14	12
	III	6	M	10	6
	IV	16	12	8	4

Step 1:

Prepare a Row Reduced matrix by selecting a minimum element from each row of given matrix and subtracting it from each element of that row. The minimum elements from each row are 8,12,6 and 4. The following matrix will be obtained (Table E.3).

Table E.3 Jobs

Row Reduced Matrix

		A	B	C	D
	I	0	6	2	4
Operator	II	M	4	2	0
	III	0	M	4	0
	IV	12	8	4	0

Step 2:

Prepare a Column Reduced matrix by selecting a minimum element from each column of Row Reduced matrix (Table E.3) and subtracting it from each element of that column. The minimum elements from each column are 0, 6, 2 and 0. The following matrix will be obtained (Table E.4).

Table E.4 Jobs

Column Reduced Matrix

		A	B	C	D
	I	0	2	0	4
Operator	II	M	0	0	0
	III	0	M	2	0
	IV	12	4	2	0

Step 3:

Now we can do the assignments. Starting with the forth row as it has only single zero and we encircle it and marking cross on all other zeroes lying in the 4th column , 3rd row has single zero to be encircled, 1st row and 2nd rows have zeroes to be encircled.. After, we obtained the resultant matrix i.e. Table E.5

Table E.5 Jobs

Assignment Matrix

		A	B	C	D
	I	~~0~~	2	0	4
Operator	II	M	0	~~0~~	~~0~~
	III	0	M	2	~~0~~
	IV	12	4	2	0

Step 4:

The assignment schedule is given in the following Table E.6.

Table E.6

Assignment Schedule

	Jobs	Cost
Operator 1	C	10
2	B	16
3	A	6
4	D	4
	Total	36

The optimal profit is Rs 36.

Problem 6: A sales manager has to assign salesman to three territories. He has four salesman of varying candidates. The manager assesses the possible profit for each salesman in each territory as given below:

Table F.1 Territory

		T1	T2	T3
	S1	120	134	180
Salesman	S2	160	166	190
	S3	140	144	164
	S4	170	190	220

Solution: The matrix consist of 4 rows (S1, S2, S3, S4) and 3 columns (T1, T2, T3), since the number of rows and columns are not equal, we can say that matrix is unbalanced and of maximization type (as profit is given) type.

Step 1:

To apply Hungarian Method the matrix should be balanced and of minimization type, so we have to add dummy column with all entries as zeroes (Table F.2) and further have to create a regret matrix by selecting the maximum value from the matrix (Table F.3) i.e. 220 and subtracting each value by 220. The matrix so obtained will be consider as Regret Matrix. We can now apply Hungarian Method to the matrix

Table F.2 Territory

		T1	T2	T3	Dummy
Salesman	S1	120	134	180	0
	S2	160	166	190	0
	S3	140	144	164	0
	S4	170	190	220	0

Table F.3 Territory
Regret Matrix

		T1	T2	T3	Dummy
Salesman	S1	100	86	40	220
	S2	60	54	30	220
	S3	80	76	56	220
	S4	50	30	0	220

Step 2:

Prepare a Row Reduced matrix by selecting a minimum element from each row of given matrix and subtracting it from each element of that row. The minimum elements from each row are 40, 30, 56, and 0. The following matrix will be obtained (Table F.4).

Table F.4 Territory
Row Reduced Matrix

		T1	T2	T3	Dummy
Salesman	S1	60	46	0	180
	S2	30	24	0	190
	S3	24	20	0	164
	S4	50	30	0	220

Step 3:

Prepare a Column Reduced matrix by selecting a minimum element from each column of Row Reduced matrix (Table F.4) and subtracting it from each element of that column. The minimum elements from each column are 24, 20, 0, and 164. The following matrix will be obtained (Table F.5).

Table F.5 Territory					
Column Reduced Matrix					
		T1	T2	T3	Dummy
Salesman	S1	36	26	0	16
	S2	6	4	0	26
	S3	0	0	0	0
	S4	26	10	0	56

Step 4:

Now we can do the assignments. Starting with the first row (Salesman S1) in table F.6, In this case 1st row has single zero and we encircle it and marking cross on all other zeroes lying in the 3rd column. 3rd row has single zero to be encircled, 2nd row and 4th row have no zeroes to be encircled.. After, we obtained the resultant matrix i.e. Table F.6

Table F.6 Territory					
Assignment Matrix					
		T1	T2	T3	Dummy
Salesman	S1	36	26	0	16
	S2	6	4	~~0~~	26
	S3	~~0~~	0	~~0~~	~~0~~
	S4	26	10	~~0~~	56

The number of assignments are 2 and size of matrix is 4, hence no optimum solution is obtained.

Step 5:

Draw minimum number of lines to cover all the zeroes.

Table F.7 Territory					
		T1	T2	T3	Dummy
Salesman	S1	36	26	0	16
	S2	6	4	0	26
	S3	0	0	0	0
	S4	26	10	0	56

Step 6:

Select the minimum uncovered element from Table F.7 and subtract from other uncovered elements and minimum uncovered element to the elements where the two lines are intersecting. In this case minimum uncovered element is 4. The following matrix (Table F.8) will be obtained:

Table F.8 Territory

		T1	T2	T3	Dummy
Salesman	S1	32	22	0	12
	S2	2	0	0	22
	S3	0	0	4	0
	S4	22	6	0	52

Step 7:

Now we can do the assignments. Starting with the first row (Salesman S1) in table F.9, In this case 1st row has single zero and we encircle it and marking cross on all other zeroes lying in the 3rd column. 2nd row and 3rd row has single zero to be encircled, and 4th row have no zeroes to be encircled. After, we obtained the resultant matrix i.e. Table F.9

Table F.9 Territory

		T1	T2	T3	Dummy
Salesman	S1	32	22	0	12
	S2	2	0	~~0~~	22
	S3	~~0~~	~~0~~	4	0
	S4	22	6	~~0~~	52

Number of assignments are 3 and size of matrix is 4 hence optimal solution is not obtained

Step 8:

Draw minimum number of lines to cover all the zeroes.

Table F.10 Territory

		T1	T2	T3	Dummy
Salesman	S1	32	22	0	12
	S2	2	0	0	22
	S3	0	0	4	0
	S4	22	6	0	52

Select the minimum uncovered element from Table F.10 and subtract from other uncovered elements and add minimum uncovered element to the elements where the two lines are intersecting. In this case minimum uncovered element is 6. The following matrix (Table F.11) will be obtained:

Table F.11 Territory

		T1	T2	T3	Dummy
Salesman	S1	24	16	0	6
	S2	0	0	6	22
	S3	0	0	10	0
	S4	16	0	0	46

Step 9:

Now we can do assignments on zeroes of each row. Every row must contain only one encircled zero.

Table F.12 Territory

		T1	T2	T3	Dummy
Salesman	S1	24	16	0	6
	S2	0	~~0~~	6	22
	S3	~~0~~	~~0~~	10	0
	S4	16	0	~~0~~	46

Number of assignments are 4 and size of matrix is 4 hence we reached an optimal solution.

Step 10:

The assignment schedule is given in the table F.13

Table F.13
Assignment Schedule

	Territory	Cost
Salesman S1	T3	180
S2	T1	160
S3	T4	164
S4	T2	190
	Total	694

The optimal profit is Rs 694.

Problem 7: The following matrix gives information about the cost of performing jobs on different machines. Find the optimum assignment.

Table G.1 Machines

		A	B	C	D	E
Jobs	1	12	19	16	20	19
	2	20	23	20	21	18
	3	22	25	24	_	25
	4	20	22	21	20	13

The matrix consist of 4 rows (1, 2, 3, 4) and 5 columns (A, B, C, D, E) since the number of rows and columns are not equal, we can say that matrix is not balanced and of minimization (as cost is given) type. To apply Hungarian Method the matrix should be of balanced type, so we will introduce dummy row (as one row is less than the number of columns) under the name Dummy where all the entries will be zeroes. Refer Table G.2 now the number of rows including dummy are 5 and number of columns are also 5. Hence the matrix is balanced and of minimization type (cost is given). We can now apply Hungarian Method.

Table G.2 Machines

		A	B	C	D	E
Jobs	1	12	19	16	20	19
	2	20	23	20	21	18
	3	22	25	24	M	25
	4	20	22	21	20	13
	Dummy	0	0	0	0	0

Step 1:

Choose the minimum element in each row and subtract it from every element of that row. In this case, 12, 18, 22, 13 and 0 are minimum element from 1st, 2nd, 3rd, 4th and 5th row respectively. In this way we get row reduced matrix.

Table G.3 Machines

Row Reduced Matrix

		A	B	C	D	E
Jobs	1	0	7	4	8	7
	2	2	5	2	3	0
	3	0	3	2	M	3
	4	8	9	8	7	0
	Dummy	0	0	0	0	0

Step 2:

Choose the minimum element in each column and subtract it from every element of that column. In this case, 0, 0, 0 and 0 are minimum element from 1st, 2nd, 3rd and 4th column respectively. In this way we get column reduced matrix.

Table G.4 Machines

Column Reduced Matrix

		A	B	C	D	E
Jobs	1	0	7	4	8	7
	2	2	5	2	3	0
	3	0	3	2	M	3
	4	8	9	8	7	0
	Dummy	0	0	0	0	0

From table G.5 it is evident that total number of assignments (encircled zeroes) are 3 and size of matrix is 5. The 3rd and 4th row (Job 3 and Job 4) does not have a single assignment (encircled zero), hence we have not obtained an optimal solution.

The pattern of assignments for the combination of jobs and machines are as follows:

Table G.5 Machines

Assignment Matrix

		A	B	C	D	E
Jobs	1	0	7	4	8	7
	2	2	5	2	3	0
	3	~~0~~	3	2	M	3
	4	8	9	8	7	~~0~~
	Dummy	~~0~~	0	~~0~~	~~0~~	~~0~~

Step 3:

Consider Table G.6 and draw minimum number of lines to cover all the zeroes. We can draw vertical or horizontal lines to cover all the zeroes.

Table G.6 Machines

		A	B	C	D	E
Jobs	1	0	7	4	8	7
	2	2	5	2	3	0
	3	0	3	2	M	3
	4	8	9	8	7	0
	Dummy	0	0	0	0	0

Identify the smallest value which is not covered by any line (minimum uncovered value).From table G.6,it is evident that minimum uncovered value is 2, this value will be subtracted from all the uncovered values and will be added to the values at the point of intersection (where the two lines are intersecting) and the remaining values (covered values) will remain the same. The resultant matrix is shown in the table G.7

Table G.7 Machines

		A	B	C	D	E
Jobs	1	0	5	2	6	5
	2	2	3	0	1	0
	3	0	1	0	M	3
	4	8	7	6	5	0
	Dummy	2	0	0	0	2

Step 4:

Now we can do the assignments. Starting with the first row (Job 1) in table G.8, we examine each rows until we found a row with single zero (Job 1). In this case 1st row has single zero and we encircle it and marking cross on all other zeroes lying in the first column. Now 2nd row and 3rd row has single zero to be encircled but 4th row does not have zero to get encircled.

Table G.8 Machines

Assignment Matrix

		A	B	C	D	E
Jobs	1	0	5	2	6	5
	2	2	3	~~0~~	1	0
	3	~~0~~	1	0	M	3
	4	8	7	6	5	~~0~~
	Dummy	2	~~0~~	~~0~~	0	2

The number of assignments are 4 and size of matrix is 5, hence solution is not optimal.

Step 8:

Draw minimum number of lines to cover all the zeroes.

Table G.9 Machines

		A	B	C	D	E
Jobs	1	0	5	2	6	5
	2	2	3	0	1	0
	3	0	1	0	M	3
	4	8	7	6	5	0
	Dummy	2	0	0	0	2

Identify the smallest value which is not covered by any line (minimum uncovered value).From table G.9, it is evident that minimum uncovered value is 1, this value will be subtracted from all the uncovered values and will be added to the values at the point of intersection (where the two lines are intersecting) and the remaining values (covered values) will remain the same. The resultant matrix is shown in the table G.10

Table G.10 Machines

		A	B	C	D	E
Jobs	1	0	4	2	5	5
	2	2	2	0	0	0
	3	0	0	0	M	3
	4	8	6	6	4	0
	Dummy	3	0	1	0	3

Step 9:

Now we can do assignments on zeroes of each row. Every row must contain only one encircled zero.

Table G.11 Machines						
Assignment Matrix						
		A	B	C	D	E
Jobs	1	0	4	2	5	5
	2	2	2	0	~~0~~	~~0~~
	3	~~0~~	0	~~0~~	M	3
	4	8	6	6	4	0
	Dummy	3	~~0~~	1	0	3

Number of assignments are 5 and size of matrix is 5 hence we reached an optimal solution.

Step 10:

The assignment schedule is given in the table G.12

Table G.12		
Assignment Schedule		
	Territory	Cost
Jobs 1	A	12
2	C	21
3	B	25
4	E	13
Dummy	D	0
	Total	71

The optimal cost is Rs 71.

Problem 8: Solve the following assignment problem using the Hungarian method. The cost matrix in the table F.1 represents the combination of worker and the time taken to finish their respective jobs.

Table F.1 Time (in min)

		T1	T2	T3	T4
Worker	W1	60	40	20	30
	W2	20	30	50	10
	W3	50	80	60	40
	W4	30	30	20	30

The matrix consist of 4 rows (W1, W2, W3, W4) and 4 columns (T1, T2, T3, T4), since the number of rows and columns are equal, we can say that matrix is balanced and of minimization (as cost is given) type.

Step 1:

Choose the minimum element in each row and subtract it from every element of that row. In this case20, 10, 40 and 20 are minimum element from 1st, 2nd, 3rd and 4th row respectively. In this way we get row reduced matrix.

Table F.2 Time (in min)

Row Reduced Matrix

		T1	T2	T3	T4
Worker	W1	40	20	0	10
	W2	10	20	40	0
	W3	10	40	20	0
	W4	10	10	0	10

Step 2:

Prepare a Column Reduced matrix by selecting a minimum element from each column of Row Reduced matrix (Table F.4) and subtracting it from each element of that column. The minimum elements from each column are 24, 20, 0, and 164. The following matrix will be obtained (Table F.3)

Table F.3 Time (in min)

Column Reduced Matrix

		T1	T2	T3	T4
Worker	W1	30	10	0	10
	W2	0	10	40	0
	W3	0	30	20	0
	W4	0	0	0	10

Step 3:

Now we can do the assignments. Starting with the first row (Job 1) in table F.4, we examine each rows until we found a row with single zero (W1). In this case 1st row has single zero and we encircle it and marking cross on all other zeroes lying in the 3rd column. Now 2nd row has choice of two zeroes, we are selecting zero from 1st column (T1), 3rd row and 4th row has single zero to be encircled. The assignment matrix is given in Table F.4 (A)

Table F.4(A) Time (in min)

Assignment Matrix

		T1	T2	T3	T4
Worker	W1	30	10	0	10
	W2	0	10	40	0
	W3	~~0~~	30	20	0
	W4	~~0~~	0	~~0~~	10

OR

1st row has single zero and we encircle it and marking cross on all other zeroes lying in the 3rd column. Now 2nd row has choice of two zeroes, we are selecting zero from 4th column (T4), 3rd row and 4th row has single zero to be encircled. The assignment matrix is given in Table F.4 (B)

Table F.4 (B) Time (in min)
Alternate Assignment Matrix

Worker		T1	T2	T3	T4
	W1	30	10	0	10
	W2	~~0~~	10	40	0
	W3	0	30	20	~~0~~
	W4	~~0~~	0	~~0~~	10

Step 4:

The assignment schedule for Table F.4 (A) is:

Assignment Schedule of Table F.4(A)

Worker	Job	Time (in min)
W1	T3	20
W2	T1	20
W3	T4	40
W4	T2	30
	Total	110

The optimal cost is Rs 110.

The assignment schedule for Table F.4 (B) is:

OR

Assignment Schedule of Table F.4(B)

Worker	Job	Time (in min)
W1	T3	20
W2	T4	10
W3	T1	50
W4	T2	30
	Total	110

The optimal cost is Rs 110.

4.3 TRAVELLING SALESMAN PROBLEM

Travelling Salesman Problem is a special case of assignment problem with certain restrictions .Consider a salesman who is assigned a job of visiting n different cities. He knows the distance between all pairs of cities. He is allowed to visit each of the cities only once. The travel should be continuous and he should be come back to the city from where he started using the shortest route. These restrictions imply that no assignment should be made along the diagonal and no city should be travelled more than once.

Table H.1 Cities

To ➡ From ⬇	1	2	3	4	5
1	_	70	60	80	40
2	70	_	80	50	60
3	60	80	_	90	70
4	80	50	90	_	80
5	40	60	70	80	_

Solution: **The** matrix consist of 5 rows (1, 2, 3, 4, 5) and 5 columns (1, 2, 3, 4, 5), since the number of rows and columns are equal, we can say that matrix is balanced and of minimization type (as cost is given).Since salesman 1 cannot travel to city 1 and salesman 2 cannot travel to city 2, salesman 3cannot travel to city 3, salesman 4 cannot travel to city 4 and salesman 5 cannot travel to city 5, so we assigning value 'M' which. No assignments or no change in values can happen to value 'M'. The matrix so obtained given in Table H.2

Table H.2 Cities

To ➡ From ⬇	1	2	3	4	5
1	M	70	60	80	40
2	70	M	80	50	60
3	60	80	M	90	70
4	80	50	90	M	80
5	40	60	70	80	M

Step 1:

Prepare a Row Reduced matrix by selecting a minimum element from each row of given matrix and subtracting it from each element of that row. The following matrix will be obtained (Table H.3).

Table H.3 Cities Row Reduced Matrix

To ➡ From ⬇	1	2	3	4	5
1	M	30	20	40	0
2	20	M	30	0	10
3	0	20	M	30	10
4	30	0	40	M	30
5	0	20	30	40	M

Step 2:

Prepare a Column Reduced matrix by selecting a minimum element from each column of Row Reduced matrix (Table H.3) and subtracting it from each element of that column.

Table H.4 Column Reduced Costs Matrix

To → From ↓	1	2	3	4	5
1	M	30	0	40	0
2	20	M	10	0	10
3	0	20	M	30	10
4	30	0	20	M	30
5	0	20	10	40	M

Step 3:

Now we can do the assignments. Starting with the first row we obtained the resultant matrix i.e. Table H.5

Table H.5 Assignment Matrix

To → From ↓	1	2	3	4	5
1	M	30	0	40	0
2	20	M	10	0	10
3	0	20	M	30	10
4	30	0	20	M	30
5	0	20	10	40	M

The number of assignments are 3 and size of matrix is 4, hence optimal solution is not obtained.

Step 4:

Draw minimum number of lines to cover all the zeroes.

Table H.6

To → From ↓	1	2	3	4	5
1	M	30	0	40	0
2	20	M	10	0	10
3	0	20	M	30	10
4	30	0	20	M	30
5	0	20	10	40	M

Step 5:

Select the minimum uncovered element from Table H.6 and subtract from other uncovered elements and minimum uncovered element to the elements where the two lines are intersecting. In this case minimum uncovered element is 10. The following matrix (Table H.7) will be obtained:

Table H.7

To ➡ / From ⬇	1	2	3	4	5
1	M	40	0	50	0
2	20	M	0	0	0
3	0	20	M	30	0
4	30	0	10	M	20
5	0	20	0	40	M

The assignment is 1→3, 2 → 4, 3→ 5, 4→ 2, 5→ 1. The route is not continuous it has two sub routes 2→4→2 and 1→3→5→1. The total distance travelled is 50+50= 100 and 60+70 +40 = 170 i.e. is 100+170 = 270 Kms.

We resolve the problem by considering the first sub route 2→4→2 by making 2→4 not then 4→2 not possible by assigning large value M to these routes. The new matrix is obtained

Table H.8

To ➡ / From ⬇	1	2	3	4	5
1	M	40	0	50	0
2	20	M	0	M	0
3	0	20	M	30	0
4	30	0	10	M	20
5	0	20	0	40	M

The number of assignments are 3 and size of matrix is 4, hence optimal solution is not obtained.

Step 6:

Draw minimum number of lines to cover all the zeroes.

Table H.9

To ➡ / From ⬇	1	2	3	4	5
1	M	40	0	50	0
2	20	M	0	M	0
3	0	20	M	30	0
4	30	0	10	M	20
5	0	20	0	40	M

Select the minimum uncovered element from Table H.9 and subtract from other uncovered elements and minimum uncovered element to the elements where the two lines are intersecting. In this case minimum uncovered element is 20. The following matrix (Table H.10) will be obtained

Step 5:

Table H.10

To →					
From ↓	1	2	3	4	5
1	M	40	0	50	0
2	20	M	0	M	0
3	0	20	M	0	0
4	30	0	10	M	20
5	0	20	0	40	M

The optimum assignment is

1 →3, 2→5, 3→4, 4→2, 5→1 i.e. 1→3→4→2→5→1

The total distance is 60+90+50+40 = 300Kms

Step 6 :

Consider 4→2 is not possible by putting a large value M at (4, 2)

Table H.11

To →					
From ↓	1	2	3	4	5
1	M	40	0	50	0
2	20	M	0	0	0
3	0	20	M	30	0
4	30	M	10	M	20
5	0	20	0	40	M

Prepare row reduced matrix.

Table H.12 Row Reduced Matrix

To →					
From ↓	1	2	3	4	5
1	M	40	0	50	0
2	20	M	0	0	0
3	0	20	M	30	0
4	20	M	0	M	10
5	0	20	0	40	M

Prepare column reduced matrix.

Table H.13 Column Reduced Matrix

To →					
From ↓	1	2	3	4	5
1	M	20	0	50	0
2	20	M	0	0	0
3	0	0	M	30	0
4	20	M	0	M	10
5	0	0	0	40	M

Table H.14 Assignment Matrix

To → / From ↓	1	2	3	4	5
1	M	20	0	50	0
2	20	M	0	0	0
3	0	0	M	30	0
4	20	M	0	M	10
5	0	0	0	40	M

The optimum assignment is

1 →5, 2→4, 3→1, 4→3, 5→2 i.e. 1→5→2→4→3→1

The total distance is 40 + 60 + 50 + 90 + 60 = 300Kms.

1→3→4→2→5→1 or 1→5→2→4→3→1 are feasible routes which gives the same distance travelled (300Kms). The salesman can choose either of the two routes.

4.5 SUMMARY

Assignment problem is a special case of transportation problem where resources are allocated to equal number of activities to minimize the cost or maximize the profit. A special method called Hungarian Method is used to solve assignment problem. Hungarian Method is applied on a balanced matrix i.e. number of resources and number of activities should be same. If matrix is unbalanced, by introducing dummy row/column, a matrix can be made balanced. Another type of assignment problem is Maximization Problem (Profit/Revenue etc.) has to convert into minimization case by creating regret matrix. In Prohibited case, no reduction or assignment can be done in restricted cells.

4.5 REFERENCES

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4.6 EXERCISE

Problem 1: Solve the following assignment problem to obtain optimal solution.

Table B.1 Jobs (Costs in Rs)

		A	B	C	D
Employees	1	20	56	38	26
	2	30	60	12	56
	3	80	42	40	34
	4	42	56	52	24

Problem 2: Solve the following assignment problem to obtain optimal solution.

Table B.1 Jobs (Costs in Rs)

		A	B	C	D
Employees	1	30	32	33	42
	2	33	39	34	45
	3	40	29	37	38
	4	29	37	30	33

Problem 3: Solve the following assignment problem to obtain optimal solution.

Table B.1 Time (in hours)

		A	B	C	D	E
Tasks	1	20	19	17	21	18
	2	28	26	24	27	25
	3	26	23	25	26	28
	4	16	13	15	17	14

Problem 4: Solve the following assignment problem to obtain optimal solution.

Table B.1 Jobs (Costs in Rs)

		A	B	C	D
Employees	1	62	52	42	47
	2	42	47	57	37
	3	57	72	62	52
	4	47	47	42	47

Problem 5: Solve the following assignment problem to obtain optimal solution.

Table B.1 Jobs (Costs in Rs)

		A	B	C	D
Employees	1	28	36	30	33
	2	35	43	-	38
	3	32	28	34	30
	4	30	29	33	25

4.7 SELF-LEARNING TOPIC

Applications of Assignment Problem in Daily Life:

It involves assignment of people to projects, jobs to machines, workers to jobs and teachers to classes etc, while minimizing the total assignment costs. One of the important characteristics of assignment problem is that only one job (or worker) is assigned to one machine (or project). An assignment problem is a special type of linear programming problem where the objective is to minimize the costs or time of completing a number of jobs by a number of persons.

Though assignment problems finds applicability in various diverse business situations, like assigning machines to factory orders, in assigning sales/marketing people to sales territories. In assigning contracts to bidders to systematic bid evaluation. In assigning teachers to classes, accountants to accounts of the clients.

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UNIT IV

5

GAME THEORY

Unit structure

5.0 OBJECTIVE

After going through this chapter, students will able to:

- Understand the situation of conflict.
- Estimate the probabilities of earning profits by reducing the probabilities of losses.
- Examine and solve the game theory problems using Maximin-Minimax and dominance principle.
- Understand the key elements and underlying mathematical concepts of game theory.
- Understand strengths and weaknesses of game theory.

5.1 INTRODUCTION

John Von Newman and Oscar Morgenstern developed the concept of Game Theory in 1928. A game is a conflicting situation where two or more than two players are involved. Game theory is a branch of Applied Mathematics which provides devices to examine the situations where players make decisions that are independent. The approach of game theory to analyze the opponent's best counter strategy to the one's own best strategy and to formulate the appropriate defensive measures. Game theory deals with competitive situations of decision making under uncertainty. A solution to game describes the optimal decisions of the players, who may have similar, mixed or contrasts interest, and the outcomes that may result from these decisions. The games can be classified as two person and n person game, zero sum and non- zero sum game, constant sum game, co-operative and non-cooperative games, pure strategy and mixed strategy games etc. Any game in which the gains of the winners are equal the losses of losers is called a zero sum game. A game in which there is a difference between the gains and losses is called a non-zero sum game.

5.2 ESSENTIAL FEATURES OF GAME THEORY

1) There are finite number of competitors or players say ‘n’ with conflicting interest.

2) Every player has a finite number of alternative choices or strategies available to them.

3) Before the game begins, each player knows the strategies available to himself/herself and the ones available to his/her opponents.

4) To each play there corresponds particular outcome called pay off which determines a set of gains to each player. Loss is considered as negative gain.

5) Each player attempts to maximize gains or minimizes losses.

5.3 OPERATING CHARACTERISTICS OF GAME THEORY

1) Players: A set of competitors or decision makers or opponent are termed as players. Any game involves two or more than two players in a game. A game having two players contesting or playing any game against each other is termed as” two person sum game” and if in a game more than two contestants are playing that game is termed as n – person sum game.

2) Strategies: Strategy in game theory refers to a situation where a player has predetermined set of rules or finite number of alternatives or course of action available to him. There are two types of strategies :

- **Pure Strategy:** The same strategy or rules or course of action adopted by player every time is termed as pure strategy. The objective of player for adopting pure strategy is to maximize gains and minimizes the losses. Therefore the pure strategy is considered as best strategy.

- **Mixed Strategy:** In a mixed strategy, a player adopts various combination of strategies to maximize the gains and make keep guessing the opponents about his next alternative or course of action in a particular situation or event. For e.g. in a game of cricket a bowler cannot throw a ball of same technique every time as batsman will become alert make more runs. Thus the mixed strategies is a selection among pure strategies with some fixed probabilities. The objective of player is to maximize the gains and to minimize the losses by adopting strategies and each strategy having a fixed probability.

3) **Payoff:** Payoff is the outcome of a game. Payoff can be either gain, draw or loss. The payoff of a game depends upon the strategies or course of action adopted by player. The tabular representation of payoff of all players for all strategies gives payoff matrix.

If X and Y are two players playing a game and ‘r’ indicates the strategies available to X and ‘n’ indicates the strategies available to Y, then the size of payoff matrix will be r x n.

r = number of rows

n = number of columns

Table A.1			
Player Y			
Player X	I	II	III
I	4	7	6
II	-3	5	-4

Interpretation:

- There are two players X and Y playing a game.
- X has two alternatives or strategies and Y has three alternatives or strategies.
- Elements such as 4, 7, 6 and -3, 5,-4 represents the payoff. When player X opts for strategy I and player Y also opts for strategy I, then payoff to X is 4. In short, the gain for player X will be of 4 units and loss of 4 units to player B. If X opts for strategy II and player Y opts for strategy I, then payoff to X is -4. So in this case , the gain for player Y will be of 4 units and loss of 4 units to player X (loss of X and gain of Y).In other words we can say that positive values indicates gain of player X and loss to player Y and vice versa.

4) **Row Minima:** The minimum payoff value from each row is Row Minima. From Table A.1:

Row Minima for player X:

Row 1: 4

Row 2:-4

5) **Column Maxima:** The maximum payoff value from each column will be Column Minima. From Table A.1:

Column Maxima for player Y:

Column 1	Column 2	Column 3
4	7	6

6) **Maximin:** Selecting the maximum value out of Row Minima. For player X (Row Minima are 4 and -4) maximin is 4.

7) **Minimax:** Selecting the minimum value out of Column Maxima. For player Y (Column Maxima are 4, 7, 6) minimax is 7.

8) **Saddle Point:** The saddle points occurs when minimax of rows and maximin of columns are equal in a payoff matrix.

Minimax of Row = Maximin of Column

9) **Optimal Strategy:** A strategy or course of action that puts a player in a preferred position irrespective of the strategies followed by opponents or competitors is called optimum strategy.

10) **Value of the game:** The expected outcome per play when players follow their optimal strategies is called value of the game. The game is fair if the value of game is zero and if the value of game is not zero then game is considered as unfair.

11) **Two-Person Zero-Sum Game:** In a game, two persons or competitors are involved where the gain of one player is equal to the loss of another player. The payoff matrix is rectangular in form, so two-person-sum-game is also called as react angular game or matrix. There are two types of two-person-sum-game, if most preferred position obtained by adopting single strategy by players then game is called as pure strategy game and if most preferred position obtained by adopting mixed strategies by players then game is called as mixed strategy game. If n-person are playing and the sum of the game is zero then such game is called as n-person –zero sum game. The features of two-person-sum-game are:

- Only two players are involved.
- Each player has a finite number of strategies to use.
- Each strategy results in a payoff.
- Each payoff results in a payoff matrix.
- Total payoff at the end of game of two-person-sum-game is always zero.

5.4 CLASSIFICATION OF GAME THEORY

Pure Strategy Games (Saddle Points):

Minimax and Maximin Principle: When a single strategy followed by player A and similarly a single strategy followed by another player B every time when they play a game against each other, such a strategy is pure strategy.

The objective of player A is to maximize his minimum gains and this strategy is called as maximin strategy and his corresponding gains are called as maximin or lower value of the game. On the other hand the strategy of player will be minimize the maximum losses incurred to him during game. This strategy adopted by player B is called as minimax strategy and his corresponding loss are called as minimax or upper value of the game.

When minimax value = maximin value, we get a saddle point or equilibrium point and the value of the game is given by this saddle point. If maximin value = minimax value =0, then game is called as fair game and if maximin value = minimax value ≠ 0 then game is said to be strictly determinable.

The steps involved in finding the saddle point is:

- Select a minimum element from each row and write down under the heading of Row Minima heading prepare at the right side of matrix) and ring the largest of them..
- Select a maximum element from each column and write down under the heading of Column Maxima heading prepare at the down side of matrix) and ring the smallest of them.
- If these two ringed elements are same, the cell corresponds to intersection of row and column is a saddle point. The element in the cell will be the value of the game.
- If there are more than one saddle points then there will be more than one solution.
- If these two ringed elements are not same, there will be no saddle point.

Problem 1: Find the optimal strategies for player A and player B in the following game. Also find the value of the game.

	Strategy of Player B		
	B1	B2	B3
Strategy of Player A	160	130	135
	-20	10	150
	230	110	20
	230	130	150

Solution:

From each row, select minimum payoff value (Row Minima).

From each column, select maximum payoff value (Column Maxima).

	Strategy of Player B			
	B1	B2	B3	Row Minima
Strategy of Player A	160	130	135	130
	-20	10	150	-20
	230	110	20	20
Column Maxima	230	130	150	

Ring the maximum payoff value out of minimum in Row Minima.

Maximin Value = 130

Ring the minimum payoff value out of maximum in Column Maxima.

Minimax Value = 130.

Thus, Maximin Value = Minimax Value = 130.

The matrix has a saddle point in a cell (A1, B2).

Value of the game = 130.

Thus the optimal strategy for A = A1 (1, 0, 0)

Thus the optimal strategy for B = B2 (0, 1, 0)

Problem 2: Find the optimal strategies for player A and player B in the following game. Also find the value of the game.

		Firm B				
		B1	B2	B3	B4	B5
Player A's Strategy	A1	6	-3	12	18	21
	A2	-3	24	6	12	36
	A3	48	24	18	42	36
	A4	3	33	-12	6	3

Solution:

From each row, select minimum payoff value (Row Minima).

From each column, select maximum payoff value (Column Maxima).

		Firm B					
		B1	B2	B3	B4	B5	Row Minima
Player A's Strategy	A1	6	-3	12	18	21	-3
	A2	-3	24	6	12	36	-3
	A3	48	24	18	42	36	18
	A4	3	33	-12	6	3	-12
Column Maxima		48	33	18	42	36	

Ring the maximum payoff value out of minimum in Row Minima.

Maximin Value = 18

Ring the minimum payoff value out of maximum in Column Maxima.

Minimax Value = 18.

Thus, Maximin Value = Minimax Value = 18.

The matrix has a saddle point in a cell (A3, B3).

Value of the game = 18.

Thus the optimal strategy for A = A3 (0, 0, 1, 0, 0)

Thus the optimal strategy for B = B3 (0, 0, 0, 1, 0)

Problem 3: Find the optimal strategies for player A and player B in the following game. Also find the value of the game.

		Player B's Strategy		
		B1	B2	B3
Player A's Strategy	A1	18	5	6
	A2	9	8	10
	A3	-4	7	3

Solution:

From each row, select minimum payoff value (Row Minima).

From each column, select maximum payoff value. (Column Maxima)

		Player B's Strategy			
		B1	B2	B3	Row Minima
Player A's Strategy	A1	18	5	6	5
	A2	9	8	10	8
	A3	-4	7	3	-4
Column Maxima		18	8	10	

Ring the maximum payoff value out of minimum in Row Minima.

Maximin Value = 8

Ring the minimum payoff value out of maximum in Column Maxima.

Minimax Value = 8.

Thus, Maximin Value = Minimax Value = 8.

The matrix has a saddle point in a cell (A2, B2).

Value of the game = 8.

Thus the optimal strategy for A = A2 (0, 1, 0)

Thus the optimal strategy for B = B2 (0, 1, 0)

Problem 5: Find the optimal strategies for player A and player B in the following game. Also find the value of the game.

		Player B's Strategy				
		B1	B2	B3	B4	B5
Player A's Strategy	A1	12	6	4	11	3
	A2	9	8	7	9	10
	A3	5	7	6	6	11
	A4	8	9	5	5	4

Solution:

From each row, select minimum payoff value (Row Minima).

From each column, select maximum payoff value (Column Maxima).

		Player B's Strategy					
		B1	B2	B3	B4	B5	Row Minima
Player A's Strategy	A1	12	6	4	11	3	3
	A2	9	8	7	9	10	7
	A3	5	7	6	6	11	5
	A4	8	9	5	5	4	4
Column Maxima		12	9	7	11	11	

Ring the maximum payoff value out of minimum in Row Minima.

Maximin Value = 7

Ring the minimum payoff value out of maximum in Column Maxima.

Minimax Value = 7.

Thus, Maximin Value = Minimax Value = 7.

The matrix has a saddle point in a cell (A2, B3).

Value of the game = 7.

Thus the optimal strategy for A = A2 (0, 1, 0, 0, 0)

Thus the optimal strategy for B = B3 (0, 0, 1, 0, 0)

Principle of Dominance:

If pure strategies does not exist, the next step is reduce the size of matrix by eliminating a course of action (rows/columns) by the rule of dominance.

Rule1: If all the elements of a row say p^{th} are less than or equal to corresponding elements of another row say q^{th} row, it indicates that p^{th} row is dominated by q^{th} row. Delete the dominated row.

Rule2: If all the elements of a column say p^{th} are less than or equal to corresponding elements of another column say q^{th} column, it indicates that p^{th} column is dominated by q^{th} column. Delete the dominated column.

Problem 5: Find the optimal strategies for player A and player B in the following game. Also find the value of the game.

Strategy of Player A \ Strategy of Player B	B1	B2	B3	B4
A1	16	20	18	28
A2	20	22	16	24
A3	26	24	28	26

Solution:

Strategy of Player A \ Strategy of Player B	B1	B2	B3	B4
A1	16	20	18	28
A2	20	22	16	24
A3	26	24	28	26

In the game, column B4 is dominated by column B2, eliminating B4, the reduced matrix is given below:

Strategy of Player A \ Strategy of Player B	B1	B2	B3
A1	16	20	18
A2	20	22	16
A3	26	24	28

In the game, row A1 is dominated by row A3, eliminating row A1, the reduced matrix is given below:

Strategy of Player A \ Strategy of Player B	B1	B2	B3
A2	20	22	16
A3	26	24	28

In the game, row A2 is dominated by row A3, eliminating row A2, the reduced matrix is given below:

Strategy of Player A \ Strategy of Player B	B1	B2	B3
A3	26	24	28

In the game, column B3 is dominated by column B1, eliminating B3, the reduced matrix is given below:

Strategy of Player A \ Strategy of Player B	B1	B2
A3	26	24

In the game, column B1 is dominated by column B2, eliminating B1, the reduced matrix is given below:

Strategy of Player A \ Strategy of Player B	B2
A3	24

(A3, B2) is the saddle point. Hence, the value of the game is 24 as it represents the best pay-off for both the players.

Problem 6: Find the optimal strategies for player A and player B in the following game. Also find the value of the game.

		Player B's Strategy				
		B1	B2	B3	B4	B5
Player A's Strategy	A1	4	8	6	16	2
	A2	18	16	14	18	16
	A3	10	14	12	12	14
	A4	16	18	10	10	4

Solution:

		Player B's Strategy				
		B1	B2	B3	B4	B5
Player A's Strategy	A1	4	8	6	16	2
	A2	18	16	14	18	16
	A3	10	14	12	12	14
	A4	16	18	10	10	4

In the game, row A1 is dominated by column A2, eliminating A1, the reduced matrix is given below:

		Player B's Strategy				
		B1	B2	B3	B4	B5
Player A's Strategy	A2	18	16	14	18	16
	A3	10	14	12	12	14
	A4	16	18	10	10	4

In the game, row A3 is dominated by column A4, eliminating A3, the reduced matrix is given below:

		Player B's Strategy				
		B1	B2	B3	B4	B5
Player A's Strategy	A2	18	16	14	18	16
	A4	16	18	10	10	4

In the game, column B1 is dominated by column B3, eliminating B1, the reduced matrix is given below:

		Player B's Strategy			
		B2	B3	B4	B5
Player A's Strategy	A2	16	14	18	16
	A4	18	10	10	4

In the game, column B2 is dominated by column B3, eliminating B2, the reduced matrix is given below:

		Player B's Strategy		
		B3	B4	B5
Player A's Strategy	A2	14	18	16
	A4	10	10	4

In the game, column B4 is dominated by column B5, eliminating B4, the reduced matrix is given below:

	Player B's Strategy		
		B3	B5
Player A's Strategy	A2	14	16
	A4	10	4

In the game, row A4 is dominated by column A2, eliminating A4, the reduced matrix is given below:

	Player B's Strategy		
		B3	B5
Player A's Strategy	A2	14	16

In the game, column B5 is dominated by column B3, eliminating B5, the reduced matrix is given below:

	Player B's Strategy	
		B3
Player A's Strategy	A2	14

(A2, B3) is the saddle point. Hence, the value of the game is 14 as it represents the best pay-off for both the players.

Problem 7: Find the optimal strategies for player A and player B in the following game. Also find the value of the game.

	Strategy of Player B				
		I	II	III	IV
Strategy of Player A	A	12	14	13	18
	B	14	15	12	16
	C	17	16	18	17

Solution:

In the game, column IV is dominated by column II, eliminating IV, the reduced matrix is given below:

	Strategy of Player B			
		I	II	III
Strategy of Player A	A	12	14	13
	B	14	15	12
	C	17	16	18

In the game, row A is dominated by row C, eliminating row A, the reduced matrix is given below:

	Strategy of Player B			
		I	II	III
Strategy of Player A	B	14	15	12
	C	17	16	18

In the game, row B is dominated by row C, eliminating row B, the reduced matrix is given below:

	Strategy of Player B			
Strategy of Player A		I	II	III
	C	17	16	18

Column III is dominated by column I, eliminating column III, and the reduced matrix is given below:

	Strategy of Player B		
Strategy of Player A		I	II
	C	17	16

Column I is dominated by column II, eliminating column I, the reduced matrix is given below:

(C, II) is the saddle point. Hence, the value of the game is 16 as it represents the best pay-off for both the players.

5.5 SUMMARY

The game theory is a branch of Applied Mathematics devised to make decisions in complex circumstances. Game theory has been widely used in Social Sciences, Biology, and Economics and in Engineering, Political Science etc. Game theory is the mathematical study of strategy and conflict, where a player success depend in making choices depends on the choice of other. The game theory module explains the concept of competitive situations, strategy, course of action, payoff etc. The module also explains the maximin-minimax principle, saddle point and principle of dominance etc.

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5.7 EXERCISE

Problem 1: Solve the following 2 × 2 game

		Player B's Strategy	
		B1	B2
Player A's Strategy	A1	-10	5
	A2	5	-10

Problem 2: Solve the following 3 × 3game

		Player B's Strategy		
		B1	B2	B3
Player A's Strategy	A1	2	-2	-2
	A2	-2	-2	6
	A3	-2	4	-2

Problem 3: Solve the following 4 × 4 game by dominance principle.

		Player B's Strategy			
		B1	B2	B3	B4
Player A's Strategy	A1	40	70	30	10
	A2	35	25	20	5
	A3	45	55	5	15
	A4	60	65	15	20

Problem 4: Solve the following game by dominance principle.

		Player B's Strategy				
		B1	B2	B3	B4	B5
Player A's Strategy	A1	7	9	8	13	10
	A2	9	10	7	11	12
	A3	12	11	13	12	11
	A4	8	6	12	9	7

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6

DECISION THEORY

Unit structure

Self Learning Topics: Decision tree for decision-making problem.

6.0 OBJECTIVES

After going through this chapter, students will able to:

- Understand the decision making process
- Describe a decision problem in terms of decision alternatives, state of nature and payoffs
- Construct a payoff table and an opportunity loss table.
- Use payoff tables to analyze decision problems
- Understand the types of decision making environment
- Describe the decision environments of certainty and uncertainty

6.1 INTRODUCTION

Decision theory is used to determine optimal strategies among the several decision alternatives. While performing various management functionalities like planning, organizing, directing, coordinating and controlling, manager has to take several decisions. Manager has to take the decision under uncertain, risky conditions. We may define Decision theory as a process, which results in the selection from set of alternative courses of action, that course of action, which is considered to meet the objectives of the decision problem.

The decision taken by the manager decide the future of business, right decisions will have beneficial effect while the wrong ones may prove to be unsuccessful. Therefore, it is important to choose the appropriate decision. Decision theory provides a rational approach to the managers in dealing with the problems challenged with partial, imperfect or uncertain future conditions.

Steps in Decision Theory:

Systematic approach of decision-making consists of following steps:

1. Clearly define the problem
2. Define the objectives
3. List out all the possible alternatives
4. Identify all possible outcomes for each alternative
5. Identify the payoff for each alternative and outcome combination
6. Use a decision model to choose best alternative

6.2 ELEMENTS OF DECISION MAKING PROBLEM

1. **Decision Maker:** Decision maker is an individual or group of individuals who make decisions.

2. **Courses of action or strategies:** The alternative course of action or strategies, are the acts that are available to decision maker.

Eg**.** To order Product A, number of units to be order for that product depending upon demand. .

3. **State of Nature or outcomes:** The events identify the occurrences which are outside of the decision maker's control and which determine the level of success for a given act. These events are called State of Nature or outcomes.

Eg. State of nature is the level of market demand for a particular item during stipulated time period.

4. **Payoff:** Each combination of a course of action and a state of nature is associated with a payoff. They are also known as conditional profit values.

5. **Payoff Table:** Payoff Table consists of the states of nature (outcome or events) which are mutually exclusive and collectively exhaustive and a set of given courses of action (strategies). Payoff is calculated for each combination of state of nature and course of action.

Following table shows the general form of Payoff Table:

Table 1: General form of Payoff Table

States of Nature (Events)	**Conditional Pay-off Courses of Action (Strategies)**			
	A_1	A_2	---------	A_n
S_1	P_{11}	P_{12}	--------	P_{1n}
S_2	P_{21}	P_{22}	----------	P_{2n}
.	.	.		.
.	.	.		.
S_m	P_{m1}	P_{m2}	---------	p_{mn}

6. **Regret or Opportunity Loss Table:** The opportunity loss is defined as the difference between the highest possible profit for the state of nature and the actual profit obtained for the particular action taken.

Consider a fixed state of nature Si. The pay offs corresponding to the n strategies are given by pi1, pi2, …, pin. Suppose Mi is the maximum of these quantities. Then A_1 is used by the decision maker then there is a loss of opportunity of $M_1 – p_{i1}$ and so on.

Following table showing opportunity loss ca be calculated as:

Table 2: General form of Regret Table

States of Nature (Events)	**Conditional Opportunity Loss Courses of Action (Strategies)**			
	A_1	A_2	---------	A_n
S_1	M_1 - P_{11}	M_1 - P_{12}	--------	M_1 - P_{1n}
S_2	M_2 - P_{21}	M_2 - P_{22}	----------	M_2 - P_{2n}
.	.	.		.
.	.	.		.
S_m	M_m - P_{m1}	M_m -P_{m2}	---------	M_m - p_{mn}

6.3 TYPES OF DECISION MAKING ENVIRONMENTS

The main aim of decision theory is to help the decision maker in selecting best course of action from the available courses of action. Decisions are made under four types of environments: certainty, risk, uncertainty and conflict.

1. **Decision making under certainty:** In this environment, there is complete certainty about future. It is easy to analyze the situation the situation and make good decision. Since the decision maker has the knowledge about the future outcome, he simply chooses the alternative having optimum payoff.

For example, suppose a person has Rs. 100000 to invest for one year period. One alternative is to open a savings account in bank with rate of interest of 3 percent and another one is to invest in fixed deposit with 5.5 percent rate of interest. If both alternatives are secure and guaranteed, then there is a certainty that fixed deposit is better alternative.

The various techniques used for solving problems under certainty are:

i. System of equations

ii. Linear programming

iii. Integer programming

iv. Dynamic programming

v. Queuing models

vi. Inventory models

2. **Decision making under risk:** In this environment, the decision maker is aware of all the possible states of nature but does not know their occurrences with certainty. Decision maker knows or can estimate the probabilities of their occurrences. This is based on his experience or on the theoretical probability distributions of the states.

3. **Decision making under uncertainty:** In this environment, more than one states of nature exist but the decision maker lacks sufficient knowledge to allow him assign probabilities to the various states of nature.

4. **Decision making under conflict:** In this environment, two or more opponents with conflicting objectives try to make decisions with each trying to gain at the cost of the others.

6.4 DECISION MAKING UNDER RISK

Here, decision maker is aware of all the possible states of nature but does not know their occurrences with certainty. Thus, the decision maker knows or just estimate the probabilities of their occurrences. This is based on his/her past experience or on the theoretical probability distributions of the states. Under conditions of risk, a number of decision criteria are available.

A. Expected Monetary Value (EMV):

This is most generally used decision criterion under the condition of risk. The main purpose of this criterion is to optimize the expected payoff i.e. either maximization of expected profit or minimization of expected loss. Here in a payoff table, conditional payoffs and their associated probabilities are given for all state of nature.

Following are the steps for calculating Expected Monetary Value (EMV):

1. Construct the payoff table using all state of nature and all possible courses of action.

2. List the conditional payoff values and corresponding probabilities of the occurrences of each state of nature.

3. Calculate the Expected Monetary Value as,

$$\text{EMV } (S_i) = \sum_{j=1}^{n} P_j x_{ij}$$

4. Select the course of action corresponds to optimal EMV.

Example 1: Consider the following payoff table along with the probabilities of each state:

	States		
	A_1	A_2	A_3
Strategies	**Probabilities of States**		
	0.4	0.5	0.1
S_1	15	13	-12
S_2	20	10	5
S_3	35	-15	7

Solution: Calculate Expected Monetary Value (EMV) for each strategy as

EMV $(S_i) = \sum_{j=1}^{n} P_j x_{ij}$

For strategy S_1,

EMV (S_1) = (0.4) (15) + (0.5) (13) + (0.1) (-12)

= 6.0 + 4.5 -1.2= 9.3

EMV (S_2) = (0.4) (20) + (0.5) (10) + (0.1) (5)

= 8.0 + 5.0 +1.5= **14.5**

EMV (S_2) = (0.4) (35) + (0.5) (-15) + (0.1) (7)

= 14.0 – 7.5 +0.7 7.2

The maximum EMV is 14.5 corresponding to S_2.

Hence S_2 is optimal strategy.

B. Expected Value with Perfect Information (EVPI):

Before taking any decision, if decision maker has a perfect information about the states of nature and associated probabilities then he/ she can calculate the Expected Value with Perfect Information. To calculate Expected Value with Perfect Information, select the best alternative for each state of nature and multiply its payoff with the given probability.

Example 2:

	States		
	A_1	A_2	A_3
Strategies	**Probabilities of States**		
	0.4	0.5	0.1
S_1	15	13	-12
S_2	20	10	5
S_3	35	-15	7

Solution: Best strategy for state A_1 is S_3 with payoff 35 and probability 0.4.

Best strategy for state A_2 is S_1 with payoff 13 and probability 0.5

Best strategy for state A_3 is S_3 with payoff 7 and probability 0.1

Expected Value with Perfect Information = (35) (0.4) + (13) (0.5) + (7) (0.1) = 21.2

The Expected Value with Perfect Information, EVPI, is the difference between expected value with perfect information and the expected value without perfect information (maximum EMV).

∴ EVPI = Expected Value with Perfect Information – Max (EMV)

For above example (From example 1 and example2),

EVPI = 21.2 - 14.5 = 6.7

6.5 DECISION MAKING UNDER UNCERTAINTY

In this environment, the probabilities associated with occurrences of different states of nature are not known. Decision maker has no way of calculating the expected payoff for his courses of action or strategies.

For example, when a new product is introduced in market or new plant is set up; in this situation, no previous data is available.

The course of action is depend upon the personality of decision maker and policies of organization. Various criteria are available under Decision making Under Uncertainty as follows:

A. Maximax Criterion or Criteria of Optimism:

In this method first find the maximum payoff associated with each course of action or alternative strategy and then selects the alternative with maximum number. Since it deals with maximum possible profit or gain, it is also called as Optimistic decision criterion.

Example 3: Select the best strategy to maximize the payoff for following.

States of Nature	Courses of Action			
	A_1	A_2	A_3	A_4
S_1	25	40	60	80
S_2	10	70	150	130
S_3	20	40	60	100
S_4	15	20	30	40

Solution: First we identify the maximum payoff associated with each course of action.

Then select the maximum of the maximum values in order to select the best (optimal) course of action.

States of Nature	Courses of Action				Maximum of each Row
	A_1	A_2	A_3	A_4	
S_1	25	40	60	80	80
S_2	10	70	150	130	**150**
S_3	20	40	60	100	100
S_4	15	20	30	40	40

Hence the decision maker selects the Strategy 2 i. e. S_2 to maximize the payoff.

Example 4: Mr. Sharma has 4 alternatives each of which can followed by any of the four events. The payoff for each combination is given in the table

Alternatives	Conditional Events			
	A_1	A_2	A_3	A_4
P	16	0	-20	12
Q	-8	24	25	-4
R	28	12	0	16
S	20	-18	25	8

Determine which alternative Mr. Sharma should select to get maximum payoff

Solution: Identify the maximum payoff associated with each alternative.

Then select the maximum of the maximum values in order to select the best (optimal) course of action.

Alternatives	Conditional Events				Maximum of each Row
	A_1	A_2	A_3	A_4	
P	16	0	-20	12	16

Q	-8	24	25	-4	25
R	28	12	0	16	**28**
S	20	-18	25	8	25

Hence Mr. Sharma selects the alternative R to maximize the payoff.

B. Maximin Criterion or Criteria of Pessimism:

In this method, decision maker first find the minimum payoff associated with each course of action or alternative strategy and then selects the alternative with maximum number. Since in this criterion decision maker find the alternative strategy that has minimum possible loss, it is also called as Pessimistic decision criterion.

Example 5: Select the optimal strategy to maximin the payoff for following.

States of Nature	**Courses of Action**			
	A_1	**A_2**	**A_3**	**A_4**
S_1	25	40	60	80
S_2	10	70	150	130
S_3	20	40	60	100
S_4	15	20	30	40

Solution: First determine the minimum payoff associated with each course of action.

Then select the minimum of the maximum values in order to select the best (optimal) course of action.

States of Nature	**Courses of Action**				**Minimum of each Row**
	A_1	**A_2**	**A_3**	**A_4**	
S_1	25	40	60	80	**25**
S_2	10	70	150	130	10
S_3	20	40	60	100	20
S_4	15	20	30	40	15

Hence decision maker selects the alternative S_1 under maximin criterion.

C. Minimax Regret Criterion (Sevage Criteria):

Leonard Sevage developed this criterion. The basic steps involved in this criterion are,

I. Consider the largest payoff for each state of nature

II. Find out the regret for each strategy as:

Regret = Largest payoff - Payoff for the strategy

III. Identify the maximum regret fir each strategy.

IV. Find out the minimum regret from these maximum regrets

V. The strategy corresponds to minimax regret is the optimal strategy

Example 6: Select the best strategy using Minimax Regret Criterion.

States of Nature	Courses of Action			
	A_1	A_2	A_3	A_4
S_1	25	40	60	80
S_2	10	70	150	130
S_3	20	40	60	100
S_4	15	20	30	40

Solution: First find out the regrets for each state of nature as:

Regret = Largest payoff - Payoff for the strategy

For A_1, Largest payoff is 25.

∴ Regret for $S_1 = 25 – 25 = 0$; Regret for $S_2 = 25 - 10 = 15$

Regret for $S_3 = 25 – 20 = 5$; Regret for $S_4 = 25 – 15 = 10$

Similarly, For A_2, Largest payoff is 70

∴ Regret for $S_1 = 70 – 40 = 30$; Regret for $S_2 = 70 – 70 = 0$

Regret for $S_3 = 70 – 40 = 30$; Regret for $S_4 = 70 – 20 = 50$

For A_3, Largest payoff is 150.

∴ Regret for $S_1 = 150 – 60 = 90$; Regret for $S_2 = 150 – 150 = 0$

Regret for $S_3 = 150 – 60 = 90$; Regret for $S_4 = 150 – 30 = 12\ 0$

And for A_4, Largest payoff is 130

∴ Regret for $S_1 = 130 – 80 = 50$; Regret for $S_2 = 130 – 130 = 0$

Regret for $S_3 = 130 – 100 = 30$; Regret for $S_4 = 130 – 40 = 90$

We get the regret table as follows:

States of Nature	A_1	A_2	A_3	A_4	Maximum Regret
S_1	0	30	90	50	90
S_2	15	0	0	0	**15**
S_3	5	30	90	30	90
S_4	10	50	120	90	120

Minimum regret among all the maximum regret is 15 corresponding to strategy S_2. Hence strategy S_2 is optimal strategy.

D. Hurwicz Criterion (Criterion of Realism):

In this method, decision maker take into consideration both the maximum and minimum for each alternative and assign them weight according to degree of of optimism or pessimism. The alternative which maximizes the sum of these weighted payoffs is then selected. It uses a coefficient of optimism α (alpha), $0 \leq \alpha \leq 1$, resentencing the decision maker's degree of optimism. Value of α is estimated on the basis of past experience. α = 0 implies pessimism and α = 1 implies optimism.

It consists of following steps:

1. Estimate or identify an appropriate value of α.
2. Determine the maximum payoff and minimum payoff of each alternative.
3. Obtain Expected profit, P = α maximum + (1- α) minimum.
4. Select the alternative which has maximum value of P.
5. Strategy corresponding to this maximum value of P is an optimal Strategy.

Example 7: Consider the previous example.

States of Nature	**Courses of Action**			
	A_1	**A_2**	**A_3**	**A_4**
S_1	25	40	60	80
S_2	10	70	150	130
S_3	20	40	60	100
S_4	15	20	30	40

Solution: Assume the coefficient of optimism, $\alpha = 0.6$

$\therefore (1 - \alpha) = 1 - 0.6$

$= 0.4$

States of Nature	**Courses of Action**				**Maximum of Row**	**Minimum of Row**	**P = α maximum + (1- α) minimum**
	A_1	**A_2**	**A_3**	**A_4**			
S_1	25	40	60	80	80	25	58
S_2	10	70	150	130	150	10	**94**
S_3	20	40	60	100	100	20	68

S_4	15	20	30	40	40	15	30

∴ The strategy corresponding to the maximum expected profit of 94 corresponds to S_2. So S_2 is optimal strategy.

E. Laplace Criterion (Criterion of Rationality (Bayes' Criterion):

Since the probabilities associated with the occurrences of the states of nature are not known, assign equal probabilities to all the events of each alternative and select identify the alternative associated with the maximum expected payoff.

It consists of following steps:

1. Find expected value for each strategy as,

$$S_i = \frac{1}{n}[P_1 + P_2 + \ldots + P_n],$$

Where P_i denotes the payoffs and n denotes number of events.

2. Find out maximum value among these.
3. Strategy corresponding to this maximum value is an optimal Strategy.

Example 8: Consider the same example.

States of Nature	**Courses of Action**			
	A_1	A_2	A_3	A_4
S_1	25	40	60	80
S_2	10	70	150	130
S_3	20	40	60	100
S_4	15	20	30	40

Solution: For each strategy calculate expected value as follows:

States of Nature	**Courses of Action**				**Average Expected Profit**
	A_1	A_2	A_3	A_4	
S_1	25	40	60	80	$\frac{25+40+60+80}{4} = \frac{205}{4} = 51.25$
S_2	10	70	150	130	$\frac{10+70+150+130}{4} = \frac{360}{4} = \mathbf{90}$
S_3	20	40	60	100	$\frac{20+40+60+100}{4} = \frac{220}{4} = 55$
S_4	15	20	30	40	$\frac{15+20+30+40}{4} = \frac{205}{4} = 26.25$

The maximum average expected profit is 90 corresponding to strategy S_2.

Hence, S_2 is optimal strategy.

6.6 SUMMARY

Decision theory is used to determine optimal strategies among the several decision alternatives. The decision maker analyzes the possible outcomes resulting from his available alternatives in two dimensions: value (by means of utility theory) and probability of occurrence. He/she always selects the alternative that he expects to have the highest value. Decision Theory suggests the decision maker to trust more strongly than ever on his own preferences and judgments. Decision theory provides a rational approach to the managers in dealing with the problems challenged with partial, imperfect or uncertain future conditions.

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6.8 EXERCISE

Exercise 1: Calculate Expected Monetary Value and Expected Value with Perfect Information for following.

a)

	States				
	N_1	**N_2**	**N_3**	**N_4**	**N_5**
Strategies	**Probabilities of States**				
	0.10	**0.15**	**0.20**	**0.25**	**0.30**
S_1	150	120	110	60	30
S_2	150	170	140	110	80
S_3	150	170	190	160	130
S_4	150	170	190	210	180
S_5	150	170	190	210	230

	States			
	A_1	**A_2**	**A_3**	**A_4**
Strategies	**Probabilities of States**			
	0.10	**0.20**	**0.40**	**0.30**
S_1	60	45	40	35
S_2	60	50	35	40
S_3	60	50	45	50
S_4	60	50	45	55

b)

Exercise 2: Consider the following pay off matrix. Determine best alternative using maximax criterion, maximin criterion and **Minimax Regret Criterion**.

a)

Alternatives	**Conditional Events**			
	A_1	**A_2**	**A_3**	**A_4**
X_1	14	24	40	54
X_2	20	18	20	50
X_3	56	40	28	56
X_4	32	48	42	34

b)

Alternatives	**Conditional Events**			
	A_1	**A_2**	**A_3**	**A_4**
P	10	20	36	50
Q	16	14	-16	-23
R	20	-18	24	25
S	30	-22	38	30

c)

Alternatives	**Conditional Events**			
	A_1	**A_2**	**A_3**	**A_4**
X_1	-20	-40	-60	-80
X_2	3	-17	-35	-65
X_3	2	5	-15	-36
X_4	3	5	-15	10

d)

Alternatives	**Conditional Events**		
	A_1	**A_2**	**A_3**
X_1	-75	125	195
X_2	-175	150	185
X_3	-225	135	225

Exercise 3: Consider the following payoff table. Find which strategy is optimal using

1. Maximin Criterion

2. Maximax Criterion
3. Minimax Regret Criterion
4. Laplace Criterion

a)

States of Nature	Payoffs in Rs. Courses of Action			
	A_1	**A_2**	**A_3**	**A_4**
S_1	40	20	20	18
S_2	50	-5	15	17
S_3	60	30	4	10

b)

States of Nature	Payoffs in Rs. Courses of Action		
	A_1	**A_2**	**A_3**
S_1	50000	10000	13000
S_2	30000	25000	0
S_3	10000	10000	10000

Self-Learning Topics: Decision tree for decision-making problem

Some times we have take decisions under the situation where there are multiple stages. They are considered by a sequence of decisions with each decision inducing the next. Such problems called sequential decision problems. These problems are analyzed and solved with the help of Decision Tree.

Decision Tree is the device for presenting a diagrammatic presentation of sequential and multi-dimensional aspects of a particular decision problem for systematic analysis and evaluation It requires the decision maker to examine all possible outcomes, whether desirable or undesirable. It displays the logical relationship between the parts of a complex decision. But it becomes highly complicated when interdependent alternatives and dependent variables are present in the problem.

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UNIT V

7

SIMULATION

Unit Structure

7.1 OBJECTIVE

This chapter would make you understand the following concepts:

- What is simulation and discuss various model of simulation.
- With the help of simulation step how to solve problem**.**
- Discuss Advantages and disadvantages and application of simulation.
- Definition and step of Monte Carlo Method

7.2 INTRODUCTION TO SIMULATION

In mathematical science we are discuss various types of problem with solution. In the chapter we are discuss a new term “simulation”. Simulation is more popular and powerful and it deal with a real system problem. In fact simulation can be extremely general tomb since the idea applies across many field, industries and application deals with model of system. A system is a facility or process either actual or planned. Real World system are too complex to evaluate analytically. In simulation we use a computer to evaluate model numerically and data are gathered in order to estimate the desired true characteristics of the model.

Example of Simulation: Human Patient Simulator (HPS) use by trainee to practice of surgery before doing actual. On HPS perform demonstration of new drugs and new equipment by pharmaceutical and medical equipment companies. It is beneficial in medical college to teach basic skill in medical science. All trainee fill practical standard in a controlled, safe and even enjoyable environment.

7.3 BASIC TERMS

System:

Collection of entities that are joined together for accomplishment of some purpose.

Example-Bank system and objects are customer, passbook, cashier, computer...

- Entity-It is an object of an interest in a system.
- Attribute-It is property of an entity.
- Activity-A process that causes changes in the system.

There are two types of acitivity-1) Deterministic Activity and 2) Stochastic Activity

Deterministic Activity-The outcome of activity can be described completely in term of its input. Randomness does not affect the behavior of the system.

Stochastic Activity-The effect of the activity vary randomly over various possible outcome.

- State-collection of variable necessary to describe the system at any time relative to the objective of the study.

Example-Bank System:

State- Number of customer waiting in the queue for opening account, first customer done process and next customer arrived time start.

- Event-Instantaneous occurrence that may change the state of the system.

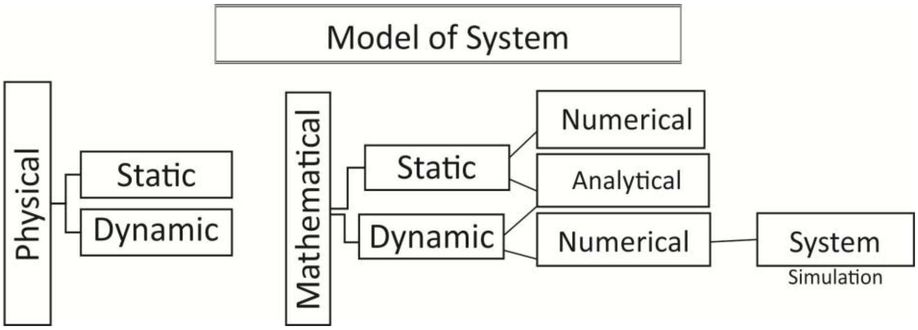
There are mainly five types:

1) Endogenous system 2) Exogenous system 3) Open system 4) Closed system 5) Discrete system 6) Continuous system

1) Endogenous system-In this system activity and events occurring within a system.
2) Exogenous system- In this system activity and events outside the system boundary i.e those variables in the environment, which affect the system.
3) Open system-It is a system, which has an exogenous activity.
4) Closed system- It is a system, which does not have any exogenous activity.
5) Discrete system-State of variable change only at a discrete set of point.
6) Continuous system-State of variable change continuously over time.

7.4 MODEL OF A SYSTEM

- A model is an abstraction of real system that represent information about a system. Model are help to studying of a system.
- Model are classified as follows:



7.4 A) Physical Model:

- This model used for mechanical & electrical and hydraulic system. Physical model represent, representation of an object of interest which shows how the object looks and how well it performs in the real world.

- A physical model is a scaled-up model of the object under study which may be small or large and it’s an exact replica of the original design but smaller and its physical characteristics resemble the physical characteristics of the original object or system.

- The system attribute are reflected in the physical laws that derive the model.
- E.g. Scale model, Prototype plants.

7.4. B) Mathematical Model:

- In this model use Symbolic notation and mathematical equation to represent a system.
- Mathematical models are collections of variables, equations, and starting values that form a cohesive representation of a process or behavior.
- Mathematics has the potential to prove general results, these results depend critically on the form of equations used.
- Small changes in the structure of equations may require enormous changes in the mathematical methods.
- Using computers to handle the model equations may never lead to elegant results, but it is much more robust against alterations.

7.5 STEPS IN SIMULATION

Steps of simulation help to study simulation study for find solution of complex problem. When we are solve the problem with the help of steps its more clear to find solution.

Steps are:

1. Problem formulation:

The initial step involves defining the goals of the study and determine what needs to be solved. The problem is further defined through objective observations of the process to be studied. Care should be taken to determine if simulation is the appropriate tool for the problem under investigation.

2. Setting of objectives and overall project plan:

Project plan include how many number of people involved, the cost of the study, and time frame to complete the work of each stage with expected results at the end of each of each stage. This step define" How we should approach the problem." The tasks for completing the project are broken down into work packages with a responsible party assigned to each package.

3. Model conceptualization:

How the actual system behaves and determining the basic requirements of the model are necessary in developing the right model. Creating a flow chart of how the system operates facilitates the understanding of what variables are involved and how these variables interact.

4. Data collection:

In this step collect the data necessary to run the simulation. This data is collected according to system layout and operating procedures.

5. Model translation:

Translate a model to programming language or simulation software to create simulation.

6. Verification:

In this step the semantics and syntax of the model.

7. Validation:

Validation ensures that no significant difference exists between the model and the real system and that the model reflects reality. Repeat this process until model accuracy is reached to the acceptable level. Validation can be achieved through statistical analysis.

8. Experimental design:

Experimentation involves developing the alternative model(s), executing the simulation runs, and statistically comparing the alternative(s) system performance with that of the real system. For each scenario that is to be simulated, decisions need to be made concerning the length of the simulation run, the number of runs (also called replications), and the manner of initialization, as required.

9. Production runs and analysis:

Production runs, and their subsequent analysis, are used to estimate measures of performance for the scenarios that are being simulated and comparing alternative system configuration.

10. Repetition:

On the basis of analysis of runs completed in previous steps, the analyst determine if additional runs are required or not. If additional runs are required then what design experiments it should follows.

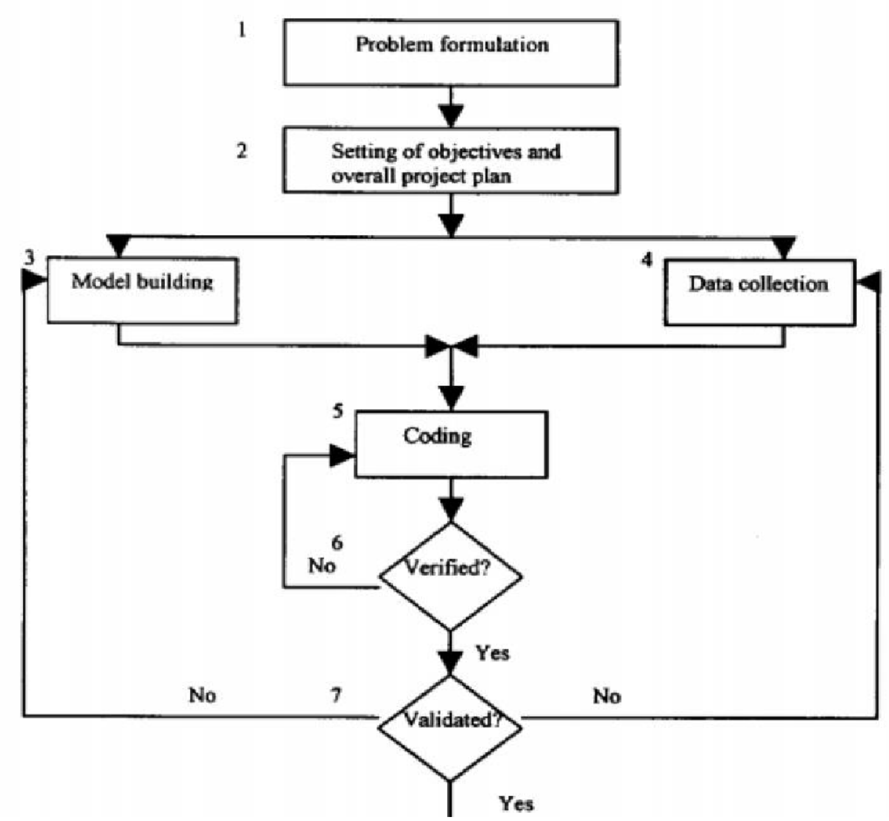
11. Document and report:

There are two types of documentation: program and progress.

Program document is necessary to understand how program operate. Documentation consists of the written report and/or presentation. Documentation is necessary for numerous reasons.

12. Implementation:

The success of implementation phase completely depends upon how well the previous all steps are performed. If the client has been involved throughout the study period, and the simulation analyst has followed all of the steps rigorously, then the likelihood of a successful implementation is increased.



7.6 ADVANTAGES OF SIMULATION

1. Simulation model designed for training purpose allow to learning new system without the cost.

2. Hypothesis of new phenomenon check for feasibility.

3. Without using resources and capital check new hardware design, physical layout, and transportation system.

4. With help of simulation concept observe the internal states of the system.

5. Simulation can help in better understanding system operation.

6. ‘‘What-if’’ question can be answered which is useful in the design of new system and modification or deployment of a model.

7. Time factor can be compressed or expanded allowing for a speedup or slowdown of the system under investigation.

8. Simulation help to learn about real system, without having the system at all.

9. In many system actual environment is not possible that time simulation help to create environment to perform task first virtually. E.g. Nuclear explosion simulation system.
10. Decision making problem are too complex to be solved by mathematical programming.
11. Simulation relative free from mathematics can easily be understood by the operating personal and non-technical situation.
12. Simulation model are comparatively flexible and modified environment according to real situation.

7.7 DISADVANTAGES OF SIMULATION

1. Non-technical personal not handle simulation system.
2. Simulation result may be difficult to interpret.
3. Simulation modeling and analysis can be time consuming and expensive.
4. Time may be needed to make sense of the results.
5. People's reaction to the model or simulation might not be realistic
6. Mistake may be made in the programming or rules of the simulation or model.

7.8 APPLICATION OF SIMULATION

1. Manufacturing and material handling system
2. Computer system
3. Human system
4. Communication system
5. Business application
6. Production and inventory system.
7. Military application
8. Logistic transportation and distribution application
9. Health system
10. Construction engineering

7.9 MONTE CARLO METHOD

- Monte Carlo simulation is a powerful method for studying the behavior of a system, as expressed in a mathematical model on a computer.
- Monte Carlo simulations usually based on computer generation of pseudo random numbers
- Monte Carlo Method random sampling of values for uncertain variables that are "plugged into" the simulation model and used to calculate outcomes of interest.
- Monte Carlo simulation is especially helpful when there are several different sources of uncertainty that interact to produce an outcome.
- The three primary techniques for effective Multiple Probability Simulation are – predictive approach, probability distribution, and repeated simulations.
- For example, if we're dealing with uncertain market demand, competitors' pricing, and variable production and raw materials costs at the same time, it can be very difficult to estimate the impacts of these factors — in combination — on Net Profit.

7.9.1. Steps of Monte Carlo Method:

1. IDENTIFY THE TRANSFER EQUATION
2. DEFINE THE INPUT PARAMETERS
3. SET UP SIMULATION
4. ANALYZE PROCESS OUTPUT

7.9.2 Example of Monte Carlo Method:

Jon's is a dentist who schedule all her patients for 30 minutes appoinments. Some of the patients take more or less than 30 minutes depending on the type of dental work to be done .The following summary shows the various categories of work their probabilities and the time actually needed to complete the work:

Category	Time Required	No. of Patients
Filling	45 min	40
Crown	60 min	15
Cleaning	15 min	15
Extracting	45 min	10
Checkup	15 min	20

Simulation the dentist clinic for four hours and find out average waiting time the patients as well as the idleness of the doctors. Assume that all the patients show up at the clinic at exactly their schedule arrival.

Use the following random number for handling the above problem

40, 82,11,34,25,66,17,79.

Solution:

To solve the problem using Monte Carlo break down problem into five steps:

1. Establishing probability distribution
2. Cumulative probability distribution
3. Setting random number interval
4. Generating random number
5. To find the answer of question asked using the above four steps.

To solve above problem follows the steps

Step 1-Eshtablising probability:

Category	Time Required	No. of Patients	Probability
Filling	45 min	40	0.40
Crown	60 min	15	0.15
Cleaning	15 min	15	0.15
Extracting	45 min	10	0.10
Checkup	15 min	20	0.20

1 patient=30 min i.e. in four hours check 8 patients.

Step 2- Cumulative Probability distribution:

Category	Probability	Cumulative Prob.
Filling	0.40	0.40
Crown	0.15	0.55
Cleaning	0.15	0.70
Extracting	0.10	0.80
Checkup	0.20	1.00

According to cumulative probability we are set interval.

Step 3-Setting Random Number interval:

Category	Probability	Cumulative Prob.	Random No. Interval
Filling	0.40	0.40	00-39
Crown	0.15	0.55	40-54
Cleaning	0.15	0.70	55-69
Extracting	0.10	0.80	70-79
Checkup	0.20	1.00	80-99

Step 4-Generating Random number and prepared table:

Patient	Scheduled Arrival	Random Number	Category	Service time needed
1	8:00	40	Crown	60 min
2	8:30	82	Checkup	15 min
3	9:00	11	Filling	45 min
4	9:30	34	Filling	45 min
5	10:00	25	Filling	45 min
6	10:30	66	Cleaning	15 min
7	11:00	17	Filling	45 min

Step 5-Find Average waiting time and idleness of doctor:

Patient	Scheduled Arrival	Service Duration	Service End	Random Number	Waiting(in minute)	Idle time
1	8:00	60 min		40	0	0
2	8:30	15 min		82	30	0
3	9:00	45 min		11	15	0
4	9:30	45 min		34	30	0
5	10:00	45 min		25	45	0
6	10:30	15 min		66	60	0
7	11:00	45 min		17	45	0
8	11:30	45 min		79	60	0
				Total=	285	

Average waiting time =285/8 =35.625 min

Idleness of doctor= 0 minutes.

7.10 QUEUING

A queuing system consists of one or more servers that provide service of some sort to arriving customers. Customers who arrive to find all servers busy generally join one or more queues (lines) in front of the servers, hence the name queuing systems such as bank-teller service, computer systems, manufacturing systems, maintenance systems, communications systems and so on Common to all of these cases are the arrivals of objects requiring service and the attendant delays when the service mechanism is busy.

Basic Terminology:

Queuing Model

It is a suitable model used to represent a service-oriented problem, where customers arrive randomly to receive some service, the service time being also a random variable.

- Arrival

The statistical pattern of the arrival can be indicated through the probability distribution of the number of the arrivals in an interval.

- Service Time

The time taken by a server to complete service is known as service time.

- Queue Discipline

It is the order in which the members of the queue are offered service. i.e, It is the rule accordingly to which customers are selected for service when queue has been formed. The most common disciplines are:

1. First come First Services (FCFS)
2. First in First Out (FIFO)
3. Last in First Out (LIFO)
4. Selection for service in Random Order (SIRO)

Notation for Queues:

Since all queues are characterised by arrival, service and queue and its discipline, the queue system is usually described in shorten form by using these characteristics. The general notation is:

[A/B/s]:{d/e/f}

Where,

A = Probability distribution of the arrivals B = Probability distribution of the departures s = Number of servers (channels) d = The capacity of the queue(s) e = The size of the calling population f = Queue ranking rule (Ordering of the queue)

- **Different types of Queuing Model**

1. (M/M/1): (“/FIFO) system
2. (M/M/C): (“/FIFO) system
3. (M/M/1): (“/FIFO) system
4. (M/M/1): (“/FIFO) system

7.10.1 [M/M/1]:{//FCFS} Queue System(Single Channel Queuing System):

This is a queuing model in which the arrival is Marcovian and departure distribution is also Marcovian, number of server is one and size of the queue is also Marcovian, no. of server is one and size of the queue is infinite and service discipline is 1st come 1st serve (FCFS) and the calling source is also finite .

Applying the single-server model:

1. Analyze service times.

2. Analyze arrival rates
3. Determine queue capacity
4. Determine size of source population
5. Choose model from SINGLEQ worksheet.

Single-server equations:

Arrival rate = ë

Service rate = ì

Mean number in queue = ë2/(ì(ì-ë))

Mean number in system = ë /(ì-ë)

Mean time in queue = ë /(ì(ì-ë))

Mean time in system = 1/(ì-ë)

Utilization ratio = ë /ì

Single-server queuing identities:

A. Number units in system = arrival rate * mean time in system

B. Number units in queue = arrival rate * mean time in queue

C. Mean time in system = mean time in queue + mean service time

Note: Mean service time = 1/ mean service rate

If we can determine only one of the following, all other values can be found by substitution:

- Number units in system or queue
- Mean time in system or queue

7.10.2 Example of single server queue model:

A help desk in the computer lab serves students on a first-come, first served basis. On average, 15 students need help every hour. The help desk can serve an average of 20 students per hour.

Solution: Based on this description, we know:

μ = 20 students/hour (average service time is 3 minutes)

λ = 15 students/hour (average time between student arrivals is 4 minutes)

Average Utilization is $p = \frac{\lambda}{\mu} = \frac{15}{20} = 0.75 \; or \; 75\%$

Average Number of Student in the System, and in Line is

$$L = \frac{\lambda}{\mu - \lambda} = \frac{15}{20 - 15} = 3\ students$$

$$L_q = pL = 0.75(3) = 2.25\ students$$

Average Time in the System & in Line is.

$$W = \frac{1}{\mu - \lambda} = \frac{1}{20 - 15} = 0.2 \text{ hours} \quad or\ 12 \text{ minutes}$$

$$W_q = pW = 0.75(0.2) = 0.15 \text{ hours} \quad or\ 9 \text{ minutes}$$

Probability of n Students in the Line

$$P_0 = (1 - p)p^0 = (1 - 0.75)1 = 0.25$$

$$P_1 = (1 - p)p^1 = (1 - 0.75)0.75 = 0.188$$

$$P_2 = (1 - p)p^2 = (1 - 0.75)0.75^2 = 0.141$$

$$P_3 = (1 - p)p^3 = (1 - 0.75)0.75^3 = 0.105$$

$$P_4 = (1 - p)p^4 = (1 - 0.75)0.75^4 = 0.079$$

Single Server: Spreadsheet Approach

Key Formulas:

B9: =1/B5

B10: =1/B6

B13: =B5/B6

B14: =1-B13

B15: =B5/(B6-B5)

B16: =B13*B15

B17: =1/(B6-B5)

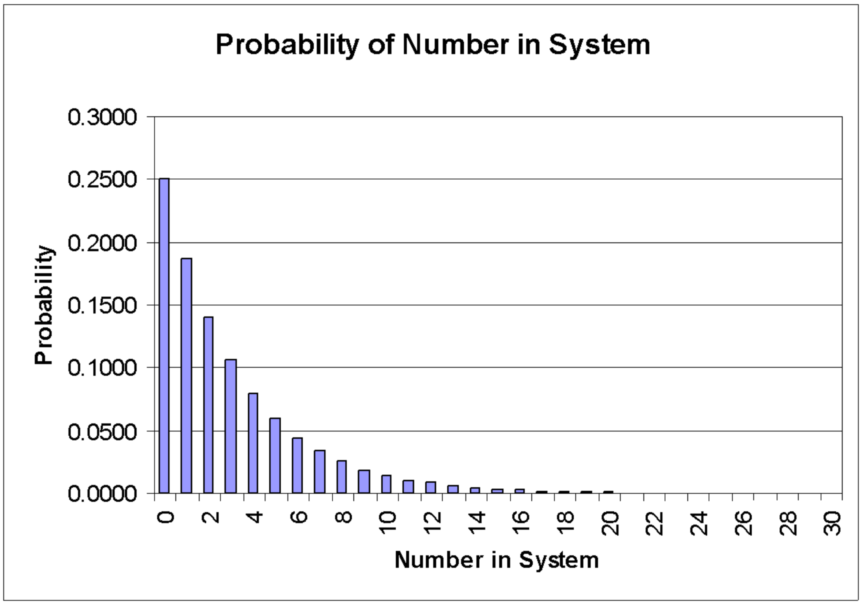
B18: =B13*B17

B22: =(1-B$13)*(B13^B21)

	A	B	C
1	**Queuing Analysis: Single Server**		
2			
3	**Inputs**		
4	Time unit	hour	
5	Arrival Rate (lambda)	15	customers/hour
6	Service Rate (mu)	20	customers/hour
7			
8	**Intermediate Calculations**		
9	Average time between arrivals	0.066667	hour
10	Average service time	0.05	hour
11			
12	**Performance Measures**		
13	Rho (average server utilization)	0.75	
14	P0 (probability the system is empty)	0.25	
15	L (average numberin the system)	3	customers
16	Lq (average number waiting in the queue)	2.25	customers
17	W (average time in the system)	0.2	hour
18	Wq (average time in the queue)	0.15	hour
19			
20	**Probability of a specific number of customers in the system**		
21	Number	2	
22	Probability	0.140625	

Use Data Table (tracking B22) to easily compute the probability of n customers in the system.

Single Server: Probability of *n* Students in the System:



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8

QUEUING MODELS

Unit structure

Self-Learning Topics: Understanding Kendle‟s notation in queuing theory.

8.0 OBJECTIVES

After going through this chapter, students will able to:

- Know the examples of various queuing system structures.
- Understand the key elements and underlying mathematical concepts queuing models.
- Discuss various operating characteristics of queuing system
- Examine and solve the single-server queuing model and multiple-server queuing model problems.
- Understand economic trade-offs associated with designing and managing queuing systems.
- Understand strengths and weaknesses of Queuing Models.

8.1 INTRODUCTION

Queuing Theory is a branch of mathematics that studies how lines form, how they function and why they malfunction. It is essential to the study of how people behave when they have to wait in line to make a purchase or receive a service. Also what sort of queue structure is required to move people most efficiently. Queue form because resources are limited. For example how many supermarkets required in a particular area to avoid queuing, how many buses or trains needed if queues were to be avoided? A Queuing model is created in order to anticipate queue length and waiting time. Queuing model was first developed by a Danish engineer A.

K. Erlang. He considered the problem of determining how many telephone circuits were necessary to provide for a service that would prevent customers from waiting too long for an available circuit. In designing queuing systems we need to aim a balance between service to customers and economic considerations (not too many servers).

In short we can say that whenever there is service to be provided and there are individuals/items/unit needing that service, we usually find a line formed of units waiting for the service. Such a waiting line is called a Queue.

Following are few examples of queue:

1. Railway Ticket Booking
2. Bank counter
3. Toll Gate
4. Library
5. Airport
6. Maintenance shop

8.2 ESSENTIAL FEATURES OF QUEUING SYSTEMS

A. Elements of Queuing System: A schematic representation of a Queuing System is given below.

Waiting line of customers (Queue)

Thus, basically there are only two elements of a Queuing System.

1. **Customers:** Arriving randomly (in the Queue if either customer is waiting otherwise straightway go to the service counter).

2. **Services:** Individual / unit giving the desired service (at any service point there can be more than one service)

Note 1: When there are more than one service, there may be separate queue for each server (eg. railway ticket booking) or there may be only one queue and customers approach a free server in the order of their arrival.

Note 2: Sometimes service is provided sequentially in stages. For example, in a bank, a customer presents his/her cheque to the counter clerk. After checking the balance, the counter clerk gives a taken to the customer and passes the cheque / withdrawal form to the supervisor for signature verification. Then the supervisor sends the cheque to the cashier for payment. Such Queues are called Queue in Jordan.

B. Queue and Service Discipline:

Queue Discipline: It is determined by customers.

Bulk Arrival: Usually customers arrive at the service point singly at random but in certain case they may arrive in groups. Such arrival are called Bulk Arrivals.

Balking: In on arrival, a customer things that the queue is too long he or she may decide not to joint the cube this is called Balking.

Reneging: The customer joins the queue. After sometime leaves if he or she thinks the waiting time could be longer than he or she was prepared for.

Jockeying: This situation occurs when there are multiple servers each having a separate queue. The customer joins a particular queue but after some time he or she joins another queue due to the smaller size or fastest service by the server

Service Discipline: This is determined by service provider. It is the manner in which the customer will be provided the required service.

First Come First Served (FCFS)/ First In First Out (FIFO): This is the most common manner of service and is followed in general

Last Come First Served (LCFS)/Last In First Out (LIFO): This normally happens when the customer are doing sheets or papers or files piled up.

Priorities: customer arriving at the service point is taken at the head of the queue by passing all waiting customers. eg profusely bleeding patient/ an emergency case of a hospital or VIP. The priorities can be of two type:

1. **Pre-emptive:** On arrival, the service to the customer starts immediately by putting the customer already getting the service in abeyance.

2. **Non- Pre-emptive:** On arrival the customer is taken at the head of the queue but his service will start only after the customer already getting service leaves.

8.3 OPERATING CHARACTERISTICS OF QUEUING SYSTEM

A. Terminology:

Arrival Rate: The rate (per unit of time) at which customers arrive at the service point. The mean rate for arrivals is denoted by λ.

Service Rate: The rate (customers per unit of time) at which one service channel can perform the service. The mean service rate is denoted by μ (number of customers per unit of time). We can use the term departure rate also.

Probability distribution of arrival and departure service straight through one service channel: The number of arrivals per unit of time and number of units (customers) served by the service channel follows some probability distribution. It is assumed that arrival pattern and service pattern are independent.

Note 1: The most common probability distribution to describe arrival and departure (service) pattern is Poison (also called Markorians denoted by M).

Note 2: If number of occurrences (number of customers arriving per unit of time or number of customer serviced per unit of time) follow a poison distribution then inter arrival time (time between two successive arrivals) and the service time follow exponential distribution.

B. Some more Definition and Notations:

1. λ = mean number of arrivals per time period
2. μ = mean number of customers (or units) serviced per unit of time
3. $1/\lambda$ = mean inter arrival time (mean time between arrivals)
4. $1/\mu$ = mean time per customer served
5. ρ = average utilization of the service facility (Service utilization Factor) defined as $\rho = \lambda / \mu$
6. L_q = Mean (Expected) number of customers (waiting) in the Queue
7. L_s =Mean (Expected) number of customers in the system (waiting + receiving service)
8. W_q = Mean (Expected) Waiting Time (Time spent in the queue before the start of the service)
9. W_s = Mean time spent in the system (waiting time + service)
10. n = Maximum number of customers permitted in the system
11. P_n = Probability that there are n customers in the system
12. P_0 = Probability of zero (No) customers in the system
13. $1- P_0$ = Probability of the system being busy
14. P_n (t) = Probability that there are n customers in the system at time t
15. GD: General Discipline (of service)

C. Notation of a Queuing System: (P/Q/R: (X/Y/Z))

Where,

P = Arrival pattern probability distribution.

Q = Service pattern exponential distribution.

R = Number of servers / number of service channels

X = Source discipline (FCFS, LCFS etc)

Y = Maximum number of customers permitted in the system

Z = size of source (population of customers)

For example (M/M/K): (FCFS/∞/∞) means

P = Poisson,

Q = Poisson,

R = k services

X = First come first serve (FCFS)

Y = System capacity is infinite (any number of customers can arrive)

Z = size of the source population is infinite i. e. the possible number of customers are infinite.

8.4 CLASSIFICATION OF QUEUING MODELS

1. (M/M/1): (GD/∞/∞)
2. (M/M/k): (GD/∞/∞)
3. (M/M/1): (GD/N/∞)
4. (M/M/k): (GD/N/∞)
5. (M/M/1): (GD/N/N)
6. (M/M/k): (GD/N/N)

8.4.1 Model 1: Single Server Poisson Arrivals with Exponential Service Infinite Population Model (M/M/1): (FCFS/∞/∞):

The following are the following are the assumptions to be made in this type of model:

1. The queue discipline is FCFS i.e. first come first served.
2. Arrival rate follows Poisson distribution and service time follows exponential distribution.

3. Arrival rate and service rate are both independent of the number of customers in the waiting line.

4. Mean arrival rate λ is less than the mean service rate μ.

The basic characteristics of this model are as follows:

S.no	Parameters	System	Queue
1	Average waiting time per customer	$Ws = \frac{1}{\mu-\lambda}$	$Wq = \frac{1}{\mu-\lambda} \times \frac{\lambda}{\mu}$
2	Average (expected) length (Number of customers)	$Ls = \frac{\lambda}{\mu-\lambda}$	$Lq = \frac{\lambda}{\mu-\lambda} \times \frac{\lambda}{\mu}$
3	Non empty queue length	$\frac{1}{\mu-\lambda} + \frac{\lambda}{\mu}$	$\frac{\mu}{\mu-\lambda}$
4	Probability that there will be n customers in the system	$(1-\frac{\lambda}{\mu})\left(\frac{\lambda}{\mu}\right)n$	$\left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)n+1$
5	Probability that there will be more than n customers in the system		$\left(\frac{\lambda}{\mu}\right)n+1$
6	Probability that waiting time is more than t.	$e^{-(}\mu-\lambda)t$	$\left(\frac{\lambda}{\mu}\right)e^{-(}\mu-\lambda)t$

Remarks:

1. The average waiting time indicates waiting time in queue not in system unless specified.

2. Probability that a service channel is busy or utilization factor for the system, ρ = λ / μ

Problem 1: A self-serviced cafeteria manned by single cashier. During peak hours, customers arrive at a rate of 24 customer per hour. The average number of customer that can be serviced by cashier is 28 per hour. Calculate:

i. The probability that cashier is idle.

ii. The average number of customers in the queuing system.

iii. The average time a customer spends in the system.

iv. The average number of customers in the queue.

v. The average time a customer spends in the queue waiting for service.

Solution:

According to the given information

Mean arrival rate, λ = 24 customer per hour

Mean service rate, μ = 28 customer per hour

i. The probability that cashier is idle.

$$Po = 1 - \frac{\lambda}{\mu} = 1 - \frac{24}{28} = 0.143$$

ii. The average number of customers in the queuing system.

$$Ls = \frac{\lambda}{\mu - \lambda}$$

$$= \frac{24}{28-24} = 6$$

ii. The average time a customer spends in the system.

$$Ws = \frac{1}{\mu - \lambda}$$

$$= \frac{1}{28-24}$$

$$= \frac{1}{4} \text{ hour or 15 minutes}$$

iii. The average number of customers in the queue.

$$Lq = \frac{\lambda}{\mu - \lambda} \times \frac{\lambda}{\mu}$$

$$= \frac{24}{28-24} \times \frac{24}{28}$$

$$= 5.142$$

iv. The average time a customer spends in the queue waiting for service.

$$Wq = \frac{1}{\mu - \lambda} \times \frac{\lambda}{\mu}$$

$$= \frac{1}{28-24} \times \frac{24}{28}$$

$$= \frac{24}{112} \text{ hour} = 12.85 \text{ minutes}$$

Problem 2: Arrivals at a certain telephone booth follows Poisson distribution with an average time of 10 minutes between one arrival and the next. The length of phone call follows Exponential distribution with a mean of 4 minutes. Calculate:

i. Expected queue length.

ii. The average number of customers in the queuing system.

iii. The average time a customer has to wait in the queue.

iv. The average time a customer has to spend in the system.

Solution:

According to the given information:

Mean arrival rate, λ = 6 customer per hour (one customer in 10 minutes than 6 customers in 60 minutes

Mean service rate, μ = 15 customer per hour

i. Expected queue length.

$$Lq = \frac{\lambda}{\mu-\lambda} \times \frac{\lambda}{\mu.}$$

$$= \frac{6}{15-6} \times \frac{6}{15}$$

$$= \frac{4}{15} \text{ customer}$$

ii. The average number of customers in the queuing system.

$$Ls = \frac{\lambda}{\mu-\lambda}$$

$$= \frac{6}{15-6}$$

$$= \frac{2}{3} \text{ customer}$$

iii. The average time a customer has to wait in the queue.

$$Wq = \frac{1}{\mu-\lambda} \times \frac{\lambda}{\mu.}$$

$$= \frac{1}{15-6} \times \frac{6}{15}$$

$$= \frac{1}{45} \text{hour}$$

iv. The average time a customer has to spend in the system.

$$Ws = \frac{1}{\mu-\lambda}$$

$$= \frac{1}{15-6}$$

$$= \frac{1}{9} \text{ hour}$$

Problem 3: Customers arrives at a certain airline reservation booking counter manned by a single clerk at a rate of 6 per hour, assuming arrival of customer follows Poisson distribution. The clerk can serve a customer on an average of 4 minutes, with an exponentially distributed service time. Calculate:

i. What is the probability that the system is busy?

ii. What is the average time a customer spends in the system?

iii. What is the average length of the queue?

iv. What is the number of customer in the system?

Solution:

According to the given information:

Mean arrival rate, λ = 6 customer per hour

Mean service rate, μ = 15 customer per hour (one customer in 4 minutes, so in one hour mean service time will be 15 customer per hour)

i. What is the probability that the system is busy?

ρ = Probability of the system being busy $= \frac{\lambda}{\mu}$.

$$= \frac{6}{15}$$

$=$ 0.6 i.e. 60% of the time system is busy.

ii. What is the average time a customer spends in the system?

$$Ws = \frac{1}{\mu-\lambda}$$

$$= \frac{1}{15-6}$$

$$= \frac{1}{9} \text{ hour}$$

iii. What is the average length of the queue?

$$Lq = \frac{\lambda}{\mu-\lambda} \times \frac{\lambda}{\mu}.$$

$$= \frac{6}{15-6} \times \frac{6}{15}$$

$$= \frac{4}{15} \text{ customer}$$

iv. What is the number of customer in the system?

$$Ls = \frac{\lambda}{\mu - \lambda}$$

$$= \frac{6}{15-6}$$

$$= \frac{2}{3} \text{ customer}$$

Problem 5: Arrival of customers at a petrol pump follows Poisson distribution with an average time of 4 minutes between arrivals. The service at petrol pump follows Exponential distribution, and the mean time taken to service a unit is 1 minute. Find the following:

i. Expected queue length.

ii. The average number of customers in the system.

iii. The average time a customer has to wait in queue.

iv. The average time a customers has to spend in the system.

Solution:

According to the given information:

Mean arrival rate, λ = 15 customer per hour

Mean service rate, µ = 60 customer per hour (one customer in 1 minutes, so in one hour mean service time will be 60 customer per hour)

i. Expected queue length.

$$Lq = \frac{\lambda}{\mu - \lambda} \times \frac{\lambda}{\mu}$$

$$= \frac{15}{60-15} \times \frac{15}{60}$$

$$= \frac{1}{12} \text{ customer}$$

ii. The average number of customers in the system.

$$Ls = \frac{\lambda}{\mu - \lambda}$$

$$= \frac{15}{60-15}$$

$$= \frac{15}{45}$$

$$= \frac{1}{3} \text{ customer}$$

iii. The average time a customer has to wait in queue.

$$Wq = \frac{1}{\mu-\lambda} \times \frac{\lambda}{\mu}$$

$$= \frac{1}{60-15} \times \frac{15}{60}$$

$$= \frac{1}{180}\text{hour}$$

iv. The average time a customers has to spend in the system.

$$\text{Ws} = \frac{1}{\mu-\lambda}$$

$$= \frac{1}{60-15}$$

$$= \frac{1}{45}\text{ hour}$$

Problem 6: Arrival of customers at a barber shop manned by single person follows Poisson distribution with an average time of 15 minutes between arrivals. The customer spend on an average of 10 minutes in the barber's shop follows Exponential distribution. Find the following:

i. What is the probability that the barber is busy all the time?

ii. The expected number of customer in the barber's shop?

iii. How much time can a customer expect to wait for his turn?

iv. How much time can a customer expect to spend in the shop?

Solution:

According to the given information:

Mean arrival rate, λ = 4 customer per hour

Mean service rate, μ = 6 customer per hour

i. The probability that the barber is busy all the time?

ρ = Probability of the barber being busy $= \frac{\lambda}{\mu}$

$$= \frac{4}{6}$$

$$= \frac{2}{3}$$

ii. The expected number of customer in the barber's shop?

$$Ls = \frac{\lambda}{\mu - \lambda}$$

$$= \frac{4}{6-4}$$

$$= \frac{4}{2}$$

$= 2$ customer per hour

iii. The time customer expects to wait for his turn

$$Wq = \frac{1}{\mu-\lambda} \times \frac{\lambda}{\mu}$$

$$= \frac{1}{6-4} \times \frac{4}{6}$$

$= \frac{1}{3}$ hour

iv. The time customer expect to spend in the shop

$$Ws = \frac{1}{\mu-\lambda}$$

$$= \frac{1}{6-4}$$

$= \frac{1}{2}$ hour

8.4.1 Model 3: Finite Queue Length Model (M/M/1): (FCFS/N/∞):

This model is different from model 1, this model is applicable for when the queue can accommodate only a limited number of customers or the maximum number of customers in the system is limited to N. New arrivals can join the queue only if space is available, otherwise leaves the queue without joining the queue. For the queuing system, the arrival rate of customer follows Poisson distribution and are served on the basis of first come first served, service rate follows exponential distribution. Arrivals are from infinite population and forms finite queue length.

Example for this model are queue formation at cinema ticket counter, clinics, petrol filling pumps etc.

1. Probability of zero customer in the queue system:

$$Po = \frac{1-\lambda/\mu}{1-\left(\frac{\lambda}{\mu}\right)N+1}.$$

2. Probability of n customer in the queue system:

$$Pn = (\lambda / \mu)^n.Po \qquad 0 < n \leq N.$$

3. Average number of customers in the queue:

$$Lq = \sum_{n=1}^{n=N} (n-1)Pn$$

4. Average number of customers in the queue system:

$$Ls = \sum_{n=1}^{n=N} n.Pn$$

5. Average time a customer spends in the system:

$$Ws = \frac{Ls}{\lambda\prime} \text{ where } \lambda' = \lambda\,(1\text{-}Pn).$$

6. Average time a customer spends in the queue:

7. $Wq = \dfrac{Lq}{\lambda\prime}$

Problem 4: At a railway station only one train can stand at a time. The railway yard can accommodate only two trains to wait and other trains if there were given a signal to leave. Arrivals of trains follows Poisson distribution at an average rate of 6 per hour. The railway yard can accommodate on an average of 12 trains per hour. Assuming exponential service distribution, find the steady state probabilities for the various number of trains in the system.

Solution:

According to the given information:

Mean arrival rate, λ = 6 customer per hour

Mean service rate, μ = 12 customer per hour)

ρ = Probability of the system being busy $= \dfrac{\lambda}{\mu.}$

$$= \frac{6}{12}$$

$=$ 0.5 i.e. 50% of the time

The maximum queue length is 2 i.e. the maximum number of trains in the system is 3 = N (one train will stand and two can remain in queue)

Probability of no train in the queue system:

$$Po = .\frac{1-\frac{\lambda}{\mu.}}{1-\left(\frac{\lambda}{\mu.}\right)N+1} = \frac{1-0.5}{1-(0.5)3+1} = 0.53$$

Probability of 3 trains in the queue system:

Pn = $(\lambda / \mu)^n.Po$

$P_1 = (0.5)(0.53) = 0.27$; $P_2 = (0.5)^2(0.53) = 0.1325$; $P_3 = (0.5)^3(0.53) = 0.06625$

Average number of trains in the queue system:

$$Ls = \sum_{n=1}^{n=N} n.Pn$$

Ls = 1(0.27) +2(0.1325) + 3(0.06625) = 0.73

Problem 5: At a railway station, trains arrives at the yard every 12 minutes and the service time is 30 minutes. If the line capacity of yard is only 4 trains assuming arrival and service follows Poisson and Exponential distribution respectively. Find

i. What is the probability that the yard is empty?

ii. The average number of trains in the system

Solution:

According to the given information:

Mean arrival rate, λ = 5 trains per hour

Mean service rate, μ = 2 trains per hour

N = 4

ρ = Probability of the yard being busy $= \frac{\lambda}{\mu} = \frac{5}{2} = 2.5$

i. The probability that the yard is empty

Probability of no train in the queue system:

$$Po = \frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)N+1} = \frac{\frac{\lambda}{\mu}-1}{\left(\frac{\lambda}{\mu}\right)N+1-1}$$

$$= \frac{2.5-1}{(2.5)4+1-1} = \frac{1.5}{(2.5)5-1}$$

$$= \frac{1.5}{96.65625} = 0.01551$$

ii. The average time a customer spends in the system

Probability of 4 trains in the queue system:

Pn = $(\lambda / \mu)^n.Po$

$P1 = (2.5)(0.01551) = 0.038775$; $P2 = (2.5)^2 (0.01551) = 0.0969375$;

$P3 = (2.5)^3(0.01551) = 0.24234375$; $P4 = (2.5)^4(0.01551) = 0.605859375$

Average number of trains in the queue system:

$Ls = \sum_{n=1}^{n=N} n.Pn$

$Ls = 1(0.038775) + 2(0.0969375) + 3(0.24234375) + 4(0.605859375)$

$= 3.383$

8.4 SUMMARY

A queuing theory is a branch of mathematics that answers how queues are formed? How it works? and discusses about the mathematical analysis of queuing process . Queuing theory examines every aspect of queue process, including arrival process, service process, and number of servers, number of customers and number of system.

A real life applications of queuing theory finds helpful in wide range of business. It helps in maintaining traffic flow, faster and improved customer services, scheduling and managing inventory, streamline staffing needs etc.

8.5 REFERENCES

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8.6 EXERCISE

Problem 1: The arrival rate of customer in a bank on an average is 15/hr under Poisson law. The teller of a bank can serve one customer in 3 minutes under exponential law. Find:

i. The probability that teller is busy.

ii. The average number of customers in the queuing system.

iii. The average time a customer spends in the system.

iv. The average number of customers in the queue.

Problem 2: The arrival rate of customer in a bank on an average is 15/hr under Poisson law. The teller of a bank can serve one customer in 3 minutes under exponential law. Find:

i. The probability that teller is busy.

ii. The average number of customers in the queuing system.

iii. The average time a customer spends in the system.

iv. The average number of customers in the queue

Problem 3: A supermarket has a single cashier. Arrival rate of customers follows Poisson distribution with an average rate of 28 customer per hour. The average number of customers that can be serviced by cashier is 32 per hour assuming exponential service distribution. Calculate:

i. The probability that cahier is idle.

ii. The average number of customers in the queuing system.

iii. The average time a customer spends in the system.

iv. The average number of customers in the queue

v. The average time a customer spends in the queue waiting for service.

Problem 4: Goods train comes to yard at the rate of 34 trains per day and the inter arrival time follows Poisson distribution. The service train for each train assumed to be exponential with an average of 40 minutes. If the yard can admit 5 trains at a time. Calculate:

i. What is the probability that the yard is empty?

ii. The average number of trains in the system

Self-Learning Topics: Understanding Kendle‟s notation in queuing theory

D.G Kendall and later A. Lee (1966) introduced useful notations for queuing models. The complete notations can be expressed as :

(a/b/c): (d/e/f)

Where,

a = arrival distribution

b = departure distribution

c = number of parallel service channels in the system

d = service discipline

e = maximum number of customer allowed in the system

f = calling source or population

M = Markovian arrival or departure distribution

Ek = Erlangian service time distribution with parameter k

GI = general independent arrival distribution

G = general departure distribution

D = deterministic interarrival or service times

FCFS = first come first served

LCFS = last come first served

SIRO = service in random order

GD = general service discipline

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