5/8/21

Program: Master of Computer Applications

Curriculum Scheme: MCA 2 year Course

Examination: MCA 1 SEMESTER II

Course Code: MCA21 and Course Name: Mathematical Foundation for Computer Science 2

Time: 2 HRS Max. Marks: 80

Section I - MCQS (40 Marks) – 40 Minutes Section II – Subjective (40 Marks) – 80 Minutes

The timings If the Examination Time is 10:00 am to 12:00 noon

Section I – 11:00 am – 11:40 am Section II – 11:40 am – 1:00 pm

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SECTION II

Q2.Solve any two out of three (10 Marks each)

a. Use a graphical method to solve the following LP problem.

Maximize $Z = 3X_1 + 2X_2$

Subject to constraints

$X_1 - X_2 >= 1$

$X_1 + X_2 >= 3$

$X_1, X_2 >= 0$

b. Use VAM to solve the following transportation problem

Distribution Centre

		D1	D2	D3	D4
Plant	P1	2	3	11	7
	P2	1	0	6	1
	P3	5	8	15	9

c. In a supermarket four salesmen A, B, C, D are available for four counters W, X, Y, Z. Each salesman can handle one counter at a time. Service time in (hrs) by the salesmen for each counter is given below. Assign the salesman to the counters so that the service time is minimized.

	Salesman				
		A	B	C	D
Counters	W	5	3	2	8
	X	7	9	2	6
	Y	6	4	5	7
	Z	5	7	7	8

1 | Page

Q3. Solve any two out of three (10 Marks each)

a. Use Simplex method to solve

Maximize $Z = 16 X_1 + 17 X_2 + 10 X_3$

Subject to constraints

$X_1 + X_2 + 4X_3 <= 200$

$2X_1 + X_2 + X_3 <= 360$

$X_1 + 2X_2 + 2X_3 <= 240$

$X_1, X_2, X_3 >= 0$

b. A garage mechanic finds the time spent on his jobs has exponential distribution with mean 45 minutes. If he repairs cars in the order in which they come in, which follows poisson distribution with a mean of 5 per 8 hour day. What is the mechanic's idle time each day. How many jobs are ahead of the average number of cars which came in?

c. Find the optimum strategy for Player A & B and value of the game where payoff matrix is given as follows

Player B

		B1	B2	B3
Player A	A1	7	3	1
	A2	1	7	3
	A3	0	1	7

Program: Master of Computer Applications

Curriculum Scheme: MCA 2 YEAR COURSE

Examination: MCA SECOND YEAR SEMESTER-2 JANUARY-2022 (ATKT)

Course Code: MCA21 and Course Name: Mathematical Foundation for Computer Science 2

Time: 2:00 pm to 4:00 pm (2 Hrs) Max. Marks: 80

Section I - MCQS (40 Marks) – 40 Minutes (2:00 pm to 2:40 am)

Section II – Subjective (40 Marks) – 80 Minutes (2:40 am to 4:00 pm)

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SECTION II

Q.2 Solve any **TWO** questions out of three questions. **[20 Marks]**

1. Use simplex method to solve the following LPP

Maximize Z = 30x1+40x2

subject to
$$60x_1 + 120x_2 \leq 12000$$
$$8x_1 + 5x_2 \leq 600$$
$$3x_1 + 4x_2 \leq 500$$
and $x_1, x_2 \geq 0$

2. Reduce the game by dominance property and solve using algebraic method

	Player B			
Player A	1	7	2	4
	0	3	7	8
	5	2	6	10

3. A company has factories F1, F2 and F3 which supply warehouses W1, W2, W3. Weekly factory capacities are 200, 160 and 90 units respectively. Weekly warehouse requirements are 180, 120 and 150 units respectively.

Find the initital basic feasible solution (IBFS) using Vogel's Approximation Method. Unit shipping costs are

	W1	W2	W3
F1	16	20	12
F2	14	8	18
F3	26	24	16

Q.3 Solve any **TWO** questions out of three questions. **[20 Marks]**

1. Using two-phase method solve the following LPP

Maximize $Z = 2x1 + 3x2 - 5x3$

subject to $x1 + x2 + x3 = 7$

$2x1 - 5x2 + x3 \geq 10$

and $x1, x2, x3 \geq 0$

2. An automobile dealer wishes to put four repairmen to four jobs. The repairmen have somewhat different kinds of skills and they exhibit different levels of efficiency from one job to another. The dealer has estimated the number of man hours that would be required for each man job combination. This is given in the matrix form in the following table.

		Jobs			
		A	B	C	D
Men	1	5	3	2	8
	2	7	9	2	6
	3	6	4	5	7
	4	5	7	7	8

Find the optimum assignment that will result in minimum man hours required.

3. At a railway reservation booking window, customers arrive randomly at the average rate of 16 per hour approximated to Poisson's distribution. If service time is exponentially distributed with a mean of 20 per hour, determine:

a) Percentage utilization capacity

b) Probability that there are at least 3 customers in the queue

c) Average time spent in the system

d) Average number of customers waiting in the line

e) Probability that there are 5 customers in the system

Mathematical Foundation for Computer Science 2

<table>
<tr><th rowspan="1">Course Code</th><th>Course Name</th><th colspan="4">Teaching Scheme</th><th colspan="3" rowspan="2">Credits Assigned</th></tr>
<tr><td rowspan="7">MCA21</td><td rowspan="7">Mathematical Foundation for Computer Science 2</td><td colspan="4">Contact Hours</td></tr>
<tr><td colspan="3">Theory</td><td>Tutorial</td><td>Theory</td><td>Tutorial</td><td>Total</td></tr>
<tr><td colspan="3">3</td><td>1</td><td>3</td><td>1</td><td>4</td></tr>
<tr><td colspan="7">Examination Scheme</td></tr>
<tr><td colspan="3">Theory</td><td rowspan="2">Term Work</td><td rowspan="2">End Sem Exam</td><td rowspan="2">Total</td></tr>
<tr><td>CA</td><td>Test</td><td>AVG</td></tr>
<tr><td>20</td><td>20</td><td>20</td><td>25</td><td>80</td><td>125</td></tr>
</table>

Pre-requisite: Basic knowledge of Mathematics and Statistics

Course Objectives: The course aim to

Sr.No	Course Objective
1	Study the formulation of Linear programming problems and obtain the optimum solution using various methods.
2	Solve the transportation, assignment problems and obtain their optimal solution
3	Use competitive strategy for analysis and learn to take decisions in various business environments
4	Understand queuing and simulation models and analyze their performance in real world systems

Course Outcomes: On successful completion of course learner/student will be able to

Sr.No.	**Outcome**	**Bloom Level**
CO1	Formulate mathematical model for a broad range of problems in business and industry.	Creating
CO 2	Apply mathematics and mathematical modeling to forecast implications of various choices in real world problems	Applying
CO 3	Think strategically and decide the optimum alternative from various available options	Evaluating
CO 4	Evaluate performance parameters of a real system using various methods	Evaluating

Reference Books:

Reference No	Reference Name
1	Hamdy A. Taha, University of Arkansas, “Operations Research: An Introduction”, Pearson, 9th Edition, ©2011, ISBN-13: 9780132555937
2	Sharma, S.D. and Sharma, H. , “Operations Research: Theory, methods and Applications”,KedarNath Ram Nath, 2010, 15, reprint
3	J. K. Sharma, “Operations Research : Theory And Applications” , Macmillan India Limited, 2006 (3 Edition),ISBN 1403931518, 9781403931511
4	S. C. Gupta, “Fundamentals of Statistics” – Himalaya Publishing House, 2017, 7th edition, ISBN 9350515040, 9789350515044
5	Prem Kumar Gupta & D S Hira, S. Chand publications , “Operations Research”, 7/e, ISBN-13: 978-8121902816, ISBN-10: 9788121902816
6	A. Ravindran, Don T. Phillips, James J. Solberg, “Operations Research: Principles and Practice”, 2nd Edition, January 1987, ISBN: 978-0-471-08608-6
7.	Frederick S. Hillier, Gerald J. Lieberman, Introduction to Operations Research , McGraw-Hill, 2001,Edition7, illustrated,ISBN 0071181636, 9780071181631

Linear Programming Problem

Module-1

Syllabus

Introduction, Formulation of linear programming problem and basic feasible solution: graphical method, Simplex method, artificial variables, Big M method, Two Phase method.

Self Learning Topics:special cases of LPP

Introduction

- **Operations research** (**OR**) is a discipline that deals with the application of advanced analytical methods to help make better decisions.
- Employing techniques from other mathematical sciences, such as mathematical modeling, statistical analysis, and mathematical optimization, operations research arrives at optimal or near-optimal solutions to complex decision-making problems.
- Operation Research is a relatively new discipline. The contents and the boundaries of the OR are not yet fixed. Therefore, to give a formal definition of the term Operations Research is a difficult task.
- The OR starts when mathematical and quantitative techniques are used to substantiate the decision being taken.
- In our daily life we make the decisions even without noticing them. The decisions are taken simply by common sense, judgment and expertise without using any mathematical or any other model in simple situations.

Introduction (contd…)

- But the decision we are concerned here with are complex and heavily responsible. Examples are public transportation network planning in a city having its own layout of factories, residential blocks or finding the appropriate product mix when there exists a large number of products with different profit contributions and production requirement etc.
- In the decades after the two world wars, the tools of operations research were more widely applied to problems in business, industry and society. Since that time, operational research has expanded into a field widely used in industries ranging from petrochemicals to airlines, finance, logistics, and government, moving to a focus on the development of mathematical models that can be used to analyse and optimize complex systems, and has become an area of active academic and industrial research.
- The modern field of operational research arose during World War II. In the World War II era, operational research was defined as "a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control". Other names for it included operational analysis (UK Ministry of Defence from 1962) and quantitative management.
- During the Second World War close to 1,000 men and women in Britain were engaged in operational research. About 200 operational research scientists worked for the British Army.

History of OR

- With expanded techniques and growing awareness of the field at the close of the war, operational research was no longer limited to only operational, but was extended to encompass equipment procurement, training, logistics and infrastructure. Operations Research also grew in many areas other than the military once scientists learned to apply its principles to the civilian sector.
- With the development of the simplex algorithm for linear programming in 1947 and the development of computers over the next three decades, Operations Research can now "solve problems with hundreds of thousands of variables and constraints. Moreover, the large volumes of data required for such problems can be stored and manipulated very efficiently."
- Much of operations research (modernly known as 'analytics') relies upon stochastic variables and a therefore access to truly random numbers. Fortunately the cybernetics field also required the same level of randomness. The development of increasingly better random number generators has been a boon to both disciplines. Modern applications of operations research include city planning, football strategies, emergency planning, optimizing all facets of industry and economy, and undoubtedly with the likelihood of the inclusion of terrorist attack planning and definitely counter-terrorist attack planning.

Linear Programming

This is a constrained optimization technique, which optimize some criterion within some constraints. In Linear programming the objective function (profit, loss or return on investment) and constraints are linear. There are different methods available to solve linear programming.

Linear Programming is a special and versatile technique which can be applied to a variety of management problems viz. Advertising, Distribution, Investment, Production, Refinery Operations, and Transportation analysis.

The linear programming method is applicable in problems characterized by the presence of decision variables.

The objective function and the constraints can be expressed as **linear functions** of the **decision variables**.

The decision variables represent quantities that are, in some sense, controllable inputs to the system being modeled.

An objective function represents some principal objective criterion or goal that measures the effectiveness of the system such as maximizing profits or productivity, or minimizing cost or consumption.

Linear Programming

- Solving a linear programming problem means determining actual values of the decision variables that optimize the objective function subject to the limitation imposed by the constraints.
- The main important feature of linear programming model is the presence of linearity in the problem.
- The use of linear programming model arises in a wide variety of applications.

Linear Programming

Let us look at the steps of defining a Linear Programming problem generically:

1. Identify the decision variables
2. Write the objective function
3. Mention the constraints
4. Explicitly state the non-negativity restriction

For a problem to be a linear programming problem, the decision variables, objective function and constraints all have to be linear functions.

If all the three conditions are satisfied, it is called a **Linear Programming Problem**.

Example

Suppose an industry is producing two types of products P1 and P2. The profit per kg of Rs. 30 and Rs. 40 for the two products respectively. These two products require processing in three types of machines. Following table shows the available machine hours per day and time required on each machine to produce 1 kg of P1 and P2. Formulate the problem in LP model

Profit/Kg	P1 Rs.30	P2 Rs.40	Total available Machine hours/day
Machine 1	3	2	600
Machine 2	3	5	800
Machine 3	5	6	1100

Example

A company owns two flour mills viz. A and B, which have different production capacities for high, medium and low quality flour. The company has entered a contract to supply flour to a firm every month with at least 8, 12 and 24 quintals of high, medium and low quality respectively. It costs the company Rs.2000 and Rs.1500 per day to run mill A and B respectively. On a day, Mill A produces 6, 2 and 4 quintals of high, medium and low quality flour, Mill B produces 2, 4 and 12 quintals of high, medium and low quality flour respectively. How many days per month should each mill be operated in order to meet the contract order most economically.

Problem

Consider a chocolate manufacturing company that produces only two types of chocolate – A and B. Both the chocolates require Milk and Choco only. To manufacture each unit of A and B, the following quantities are required:

- Each unit of A requires 1 unit of Milk and 3 units of Choco
- Each unit of B requires 1 unit of Milk and 2 units of Choco

The company kitchen has a total of 5 units of Milk and 12 units of Choco. On each sale, the company makes a profit of

- Rs 6 per unit A sold
- Rs 5 per unit B sold.

Now, the company wishes to maximize its profit. Formulate LPP?

Profit: Max $Z = 6X+5Y$

$X+Y \le 5$

$3X+2Y \le 12$

$X \ge 0$ & $Y \ge 0$

Problem

A farmer has recently acquired a 110 hectares piece of land. He has decided to grow Wheat and barley on that land. Due to the quality of the sun and the region's excellent climate, the entire production of Wheat and Barley can be sold. He wants to know how to plant each variety in the 110 hectares, given the costs, net profits and labor requirements according to the data shown below:

Variety	Cost (Price/Hec)	Net Profit (Price/Hec)	Man-days/Hec
Wheat	100	50	10
Barley	200	120	30

The farmer has a budget of Rs. 10,000 and availability of 1,200 man-days during the planning horizon. Formulate the LPP.

Max $Z = 50X + 120Y$
$100X + 200Y \leq 10{,}000$
$10X + 30Y \leq 1200$
$X + Y \leq 110$
$X \geq 0, Y \geq 0$

Graphical Method

Graphical solution is limited to linear programming models containing only two decision variables.

<u>Procedure</u>

Step I: Convert each inequality as equation

Step II: Plot each equation on the graph

Step III: Shade the ‘Feasible Region’. Highlight the common Feasible region.

- ❑ *Feasible Region: Set of all possible solutions.*

Step IV: Compute the coordinates of the corner points (of the feasible region). These corner points will represent the ‘Feasible Solution’.

- ❑ *Feasible Solution: If it satisfies all the constraints and non negativity restrictions.*

Graphical Method

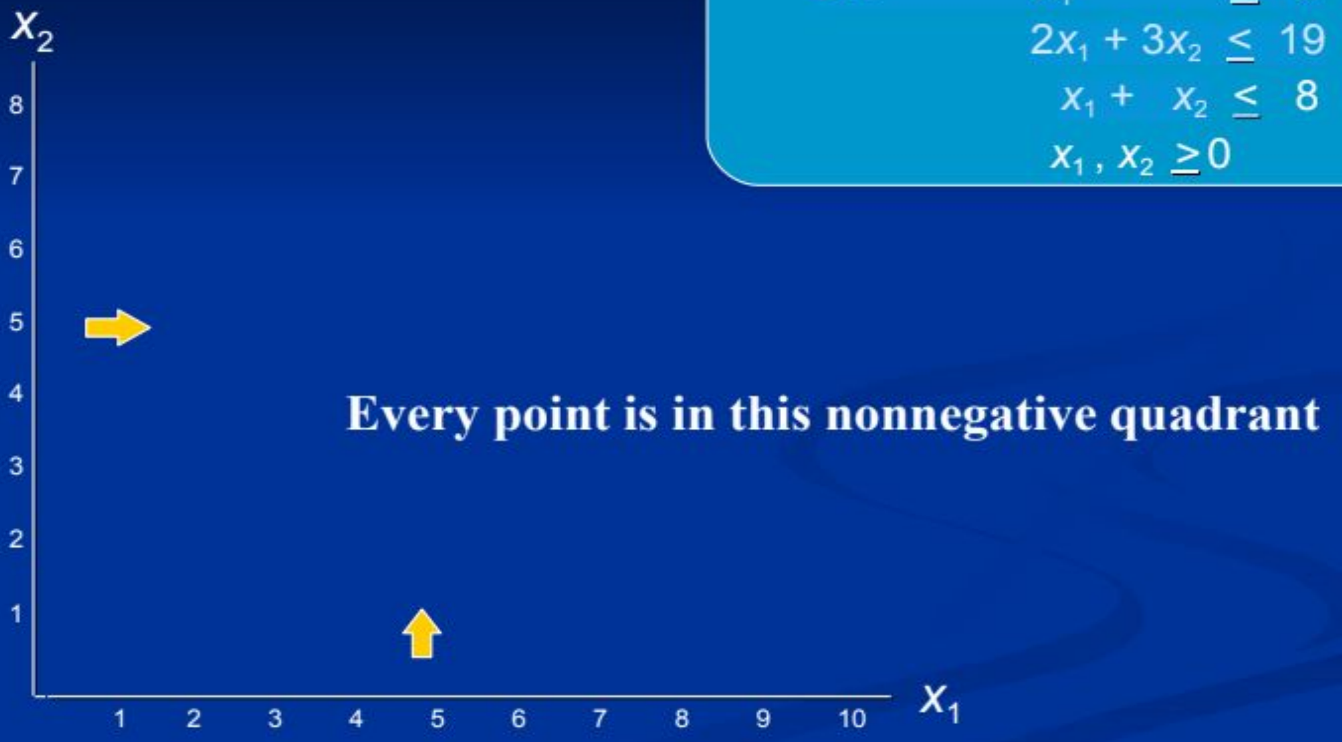
Procedure (Cont...)

Step V: Substitute the coordinates of the corner points into the objective function to see which gives the Optimal Value. That will be the 'Optimal Solution'.

- ❑ Optimal Solution: If it optimizes (maximizes or minimizes) the objective function.
- ❑ Unbounded Solution: If the value of the objective function can be increased or decreased indefinitely, Such solutions are called Unbounded solution.
- ❑Inconsistent Solution: It means the solution of problem does not exist. This is possible when there is no common feasible region.

Example

$$\begin{aligned} \text{Max} \quad & z = 5x_1 + 7x_2 \\ \text{s.t.} \quad & x_1 \le 6 \\ & 2x_1 + 3x_2 \le 19 \\ & x_1 + x_2 \le 8 \\ & x_1, x_2 \ge 0 \end{aligned}$$

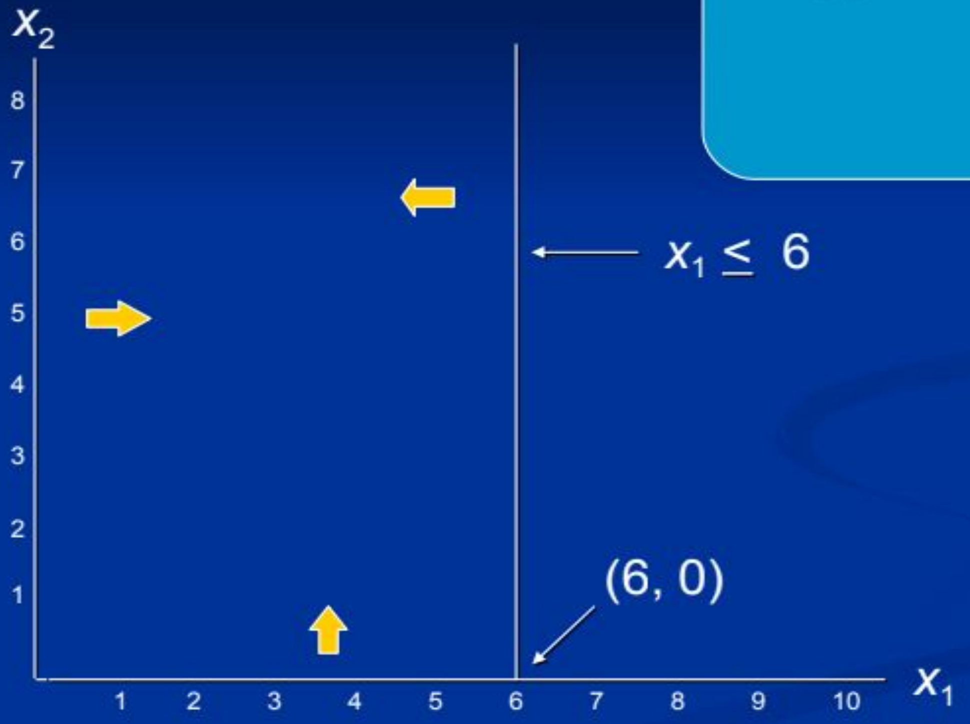


Example (Cont…)

$$\text{Max} \quad z = 5x_1 + 7x_2$$

s.t.

$$\begin{aligned} x_1 &\le 6 \\ 2x_1 + 3x_2 &\le 19 \\ x_1 + x_2 &\le 8 \\ x_1, x_2 &\ge 0 \end{aligned}$$



Example (Cont…)

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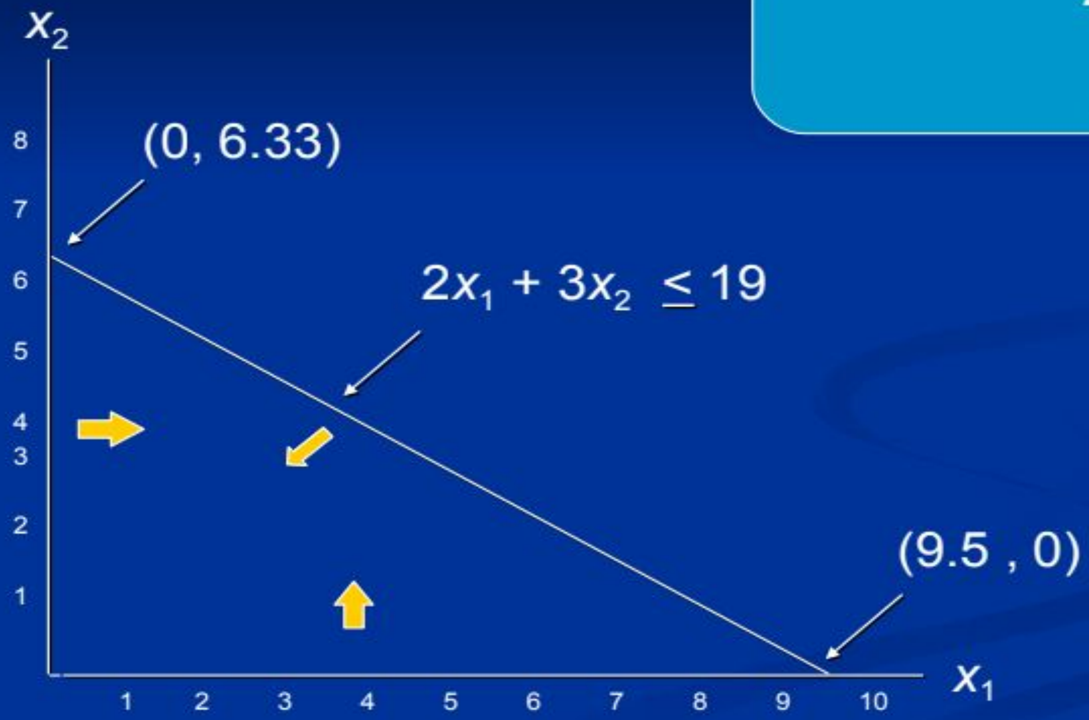
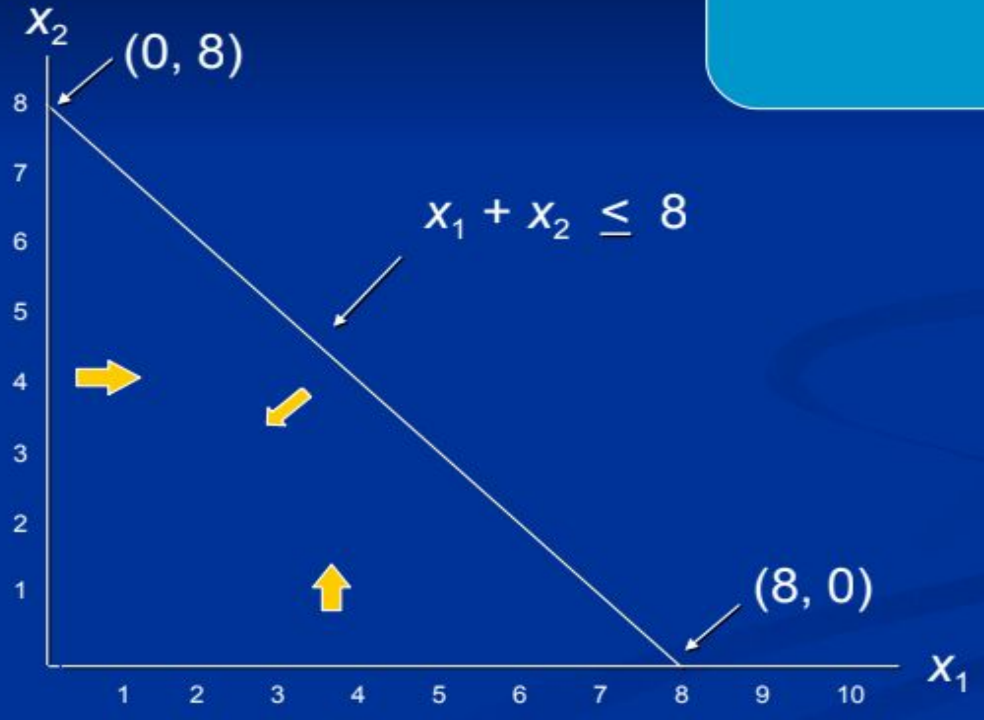


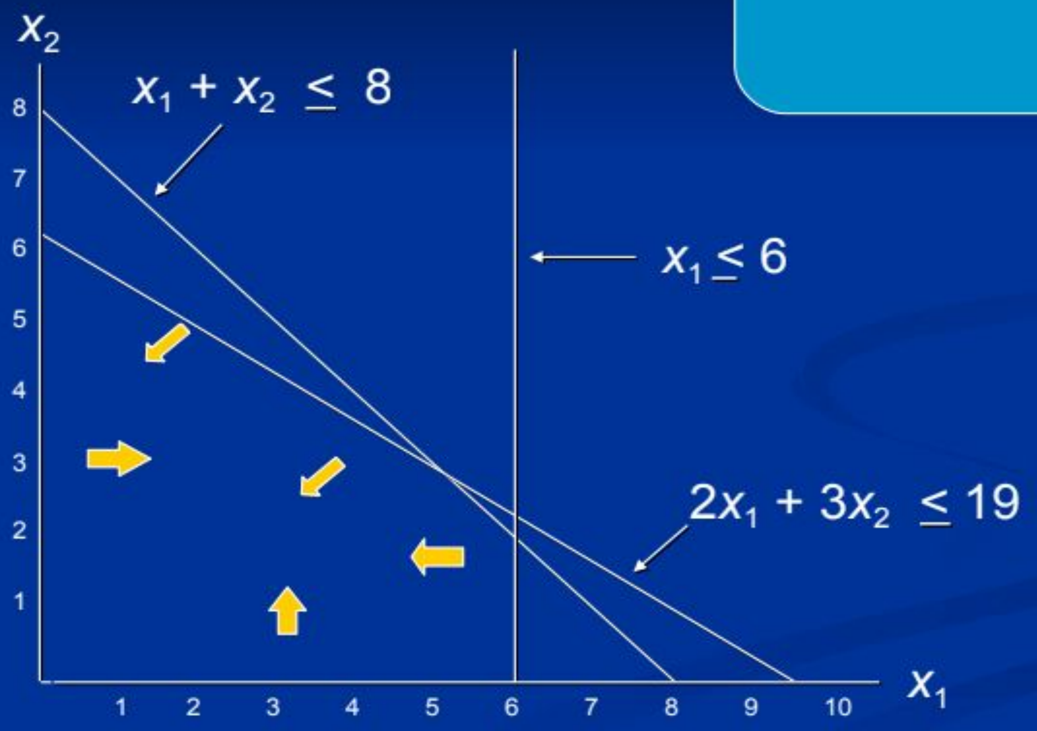
Example (Cont…)

$$\begin{array}{lrcl} \text{Max} & z = 5x_1 + 7x_2 & & \\ \text{s.t.} & x_1 & \leq & 6 \\ & 2x_1 + 3x_2 & \leq & 19 \\ & x_1 + x_2 & \leq & 8 \\ & x_1, x_2 & \geq & 0 \end{array}$$

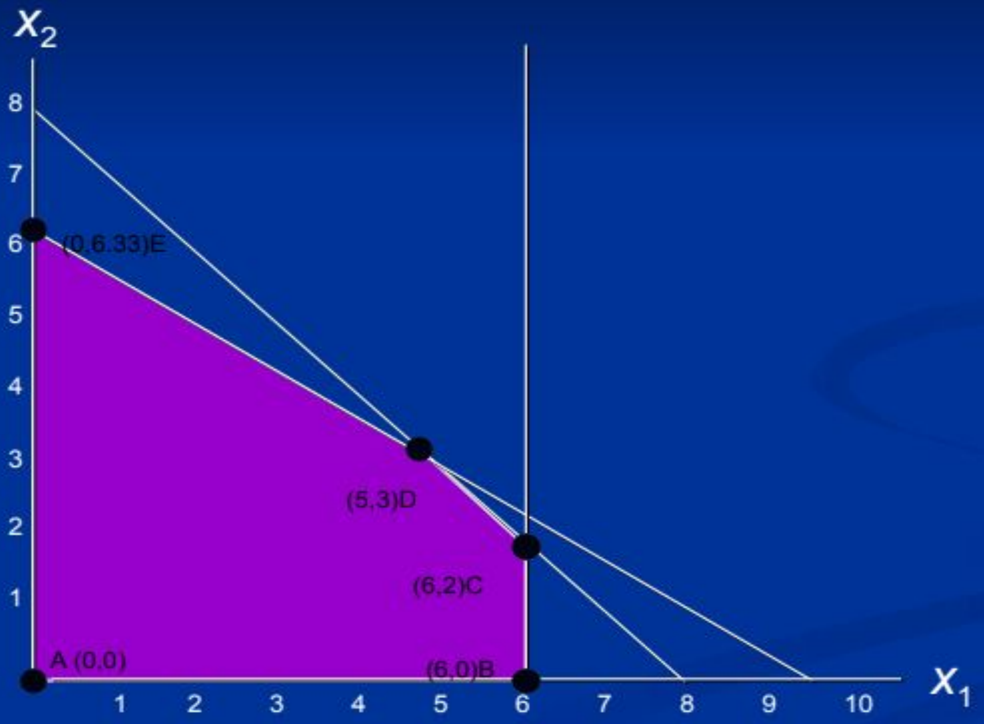


Example (Cont…)

$$\begin{aligned} \text{Max} \quad & z = 5x_1 + 7x_2 \\ \text{s.t.} \quad & x_1 \le 6 \\ & 2x_1 + 3x_2 \le 19 \\ & x_1 + x_2 \le 8 \\ & x_1, x_2 \ge 0 \end{aligned}$$



Example (Cont…)



Objective Function : Max $Z = 5x_1 + 7x_2$

Corner Points	Value of Z
A – (0,0)	0
B – (6,0)	30
C – (6,2)	44
D – (5,3)	46
E – (0,6.33)	44.33

Optimal Point : (5,3)
Optimal Value : 46

Example

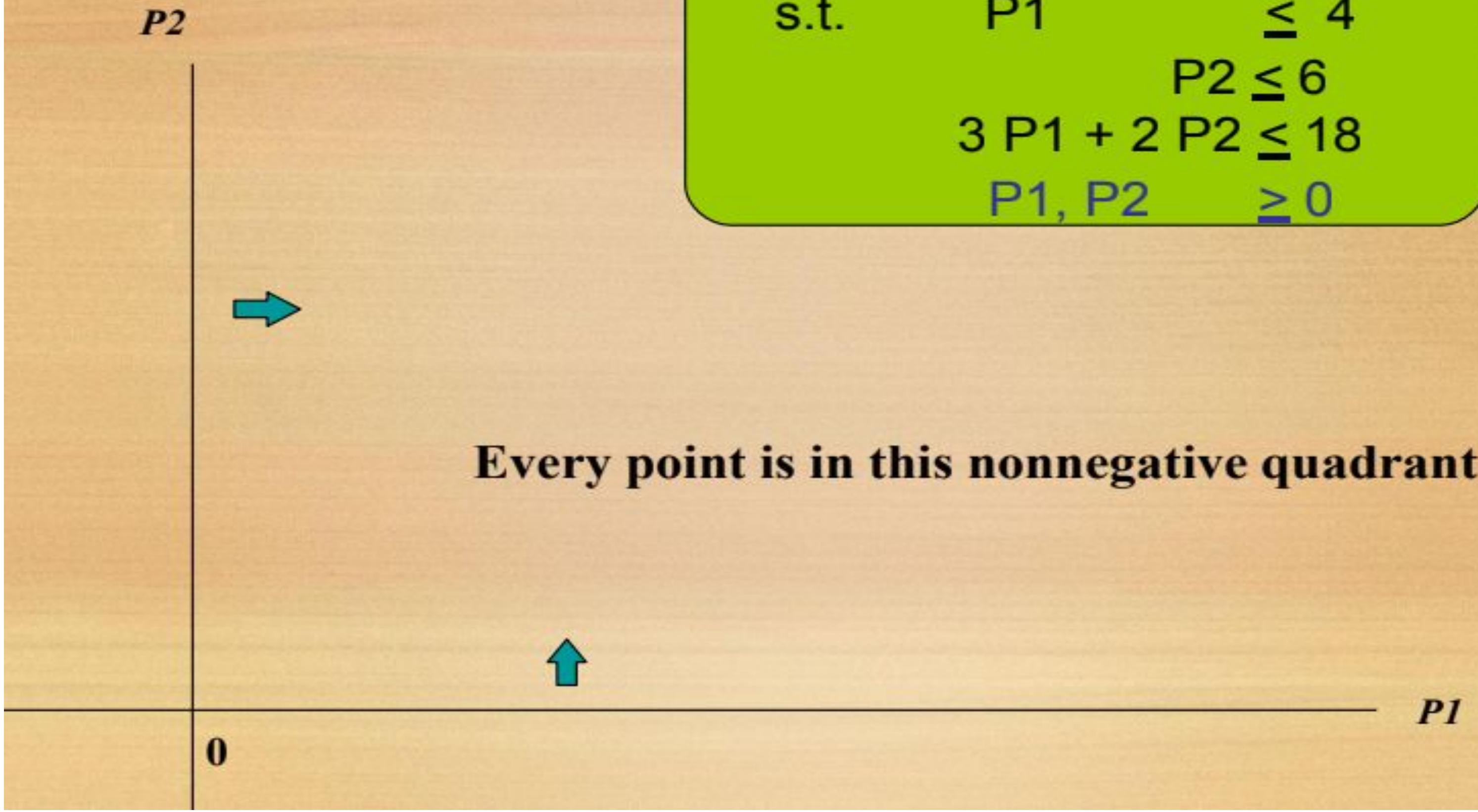
$$
\begin{aligned}
\text{Max} \quad & Z = 3P1 + 5P2 \\
\text{s.t.} \quad & P1 \le 4 \\
& P2 \le 6 \\
& 3P1 + 2P2 \le 18 \\
& P1, P2 \ge 0
\end{aligned}
$$

Example

Max $Z = 3P1 + 5P2$

s.t.

$$P1 \le 4$$
$$P2 \le 6$$
$$3P1 + 2P2 \le 18$$
$$P1, P2 \ge 0$$



Example (Cont…)

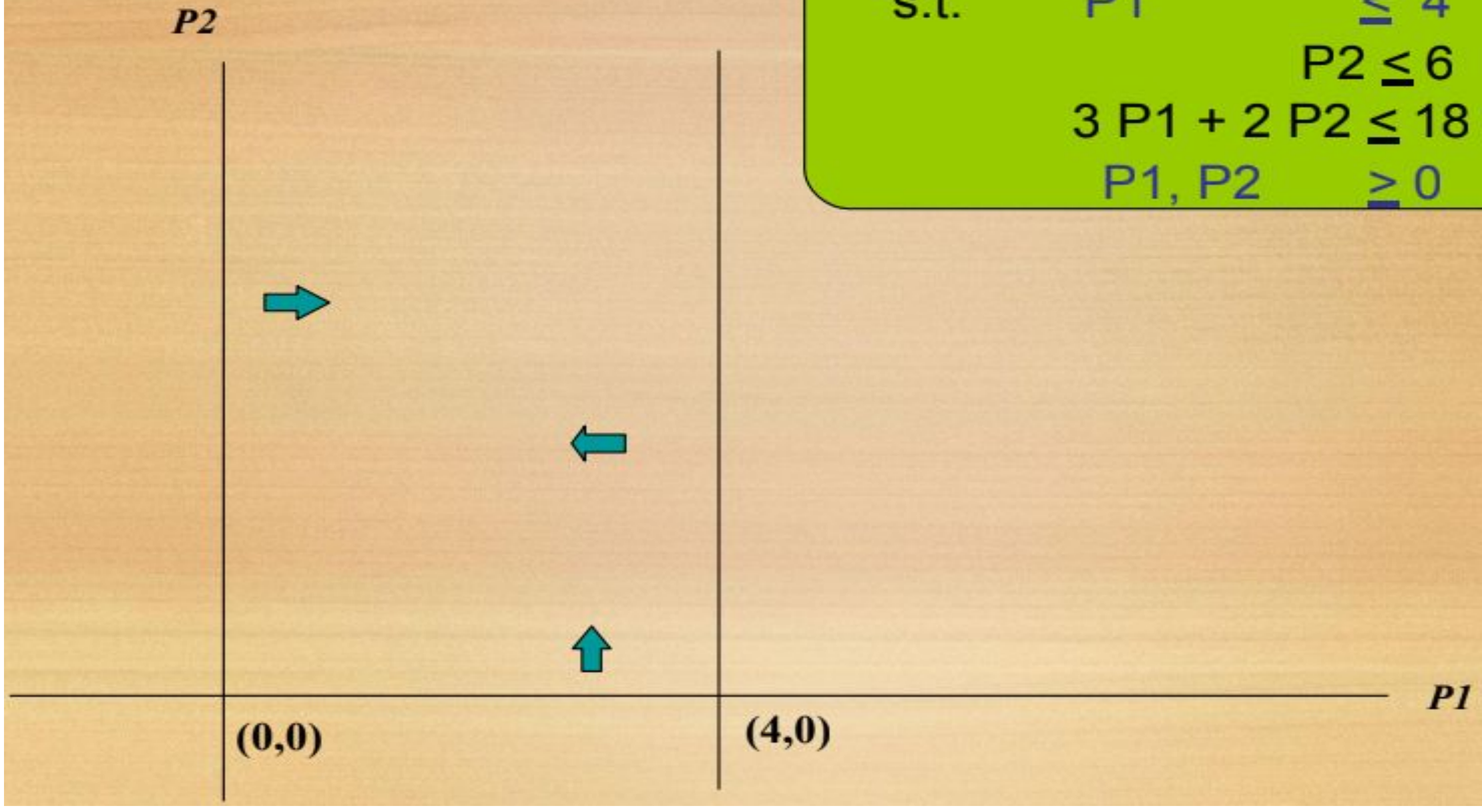
Max $Z = 3P1 + 5P2$

s.t. $P1 \leq 4$

$P2 \leq 6$

$3P1 + 2P2 \leq 18$

$P1, P2 \geq 0$



Example (Cont…)

Max $Z = 3\,P1 + 5\,P2$

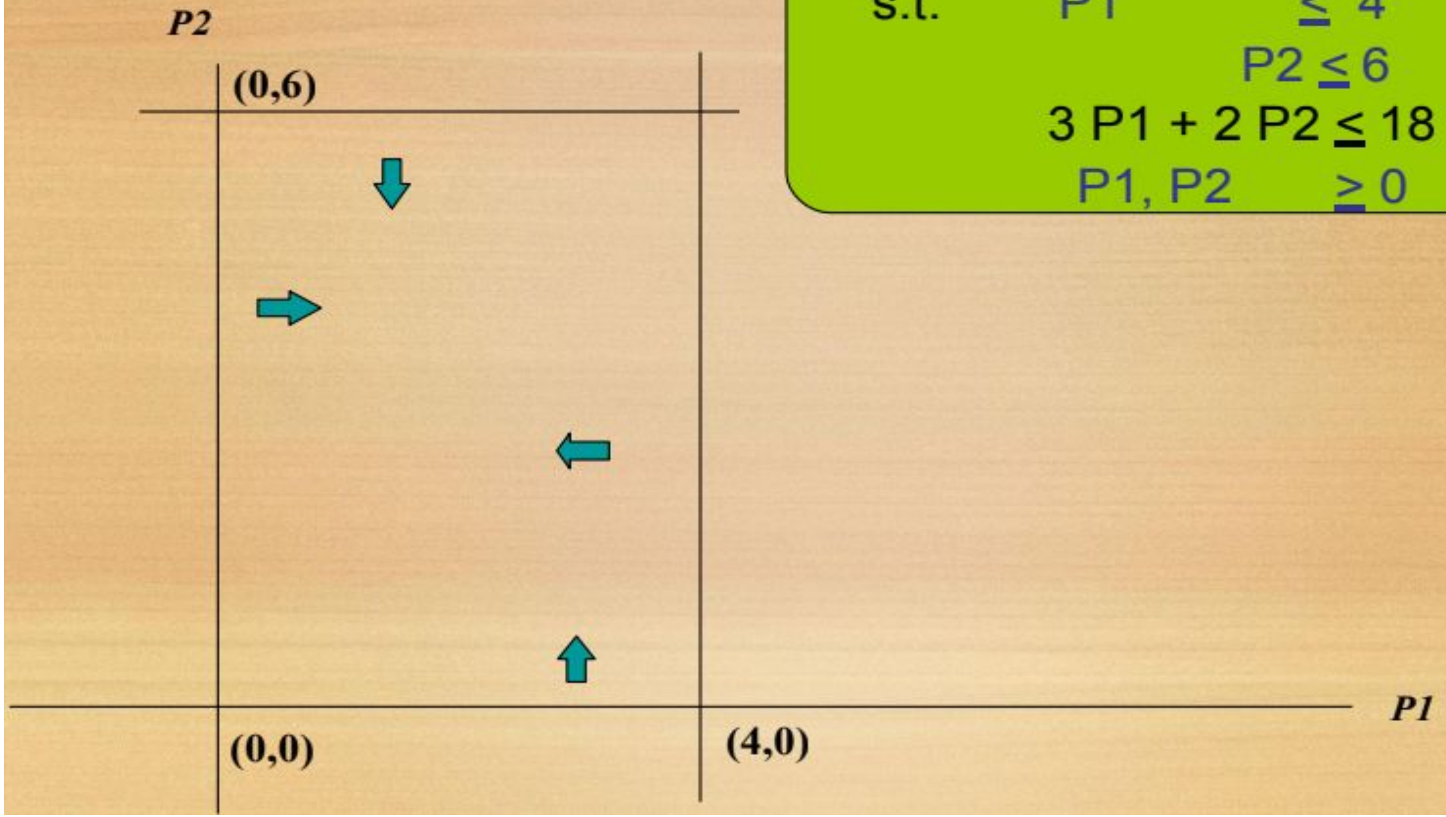
s.t.

$P1 \leq 4$

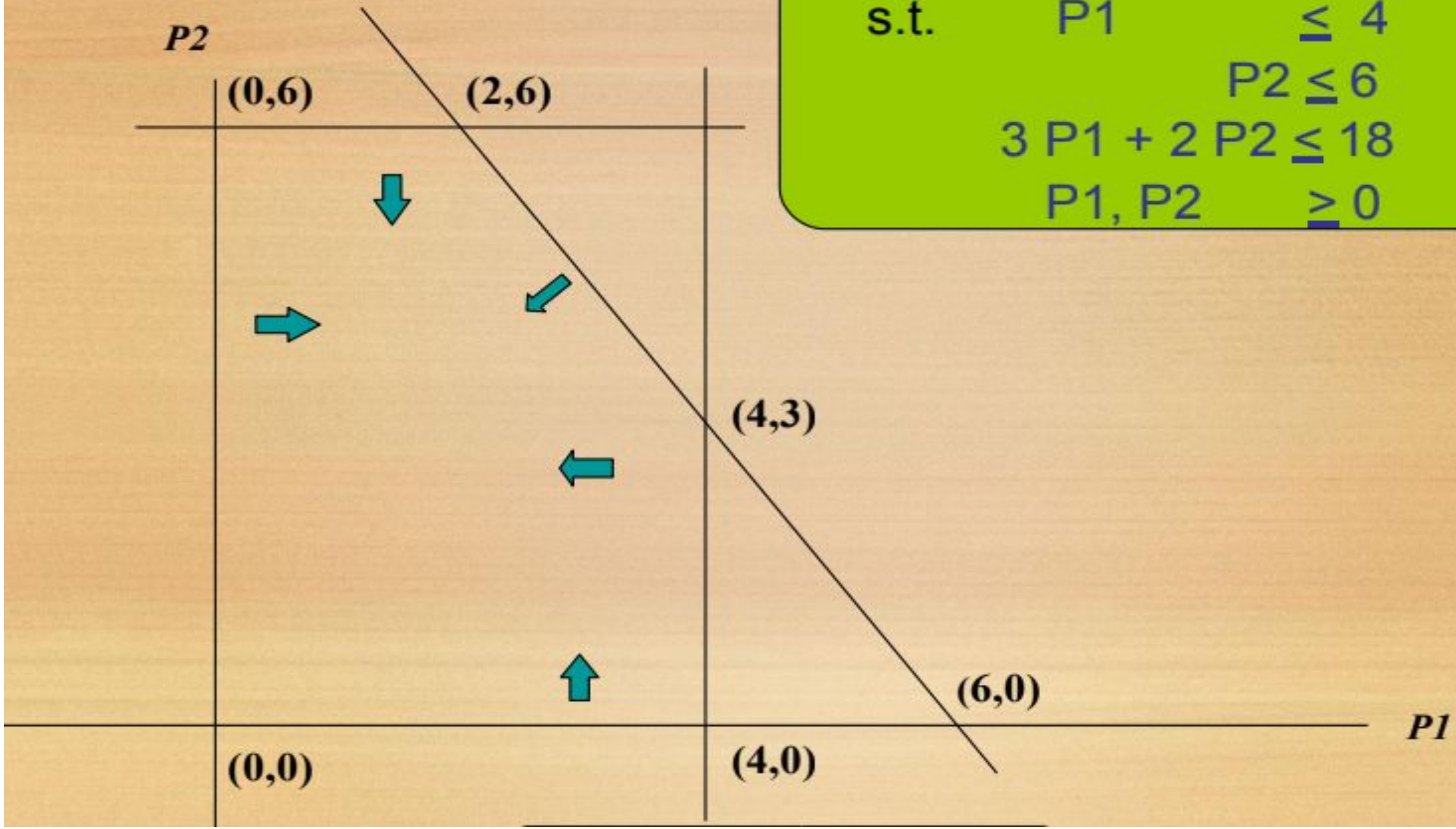
$P2 \leq 6$

$3\,P1 + 2\,P2 \leq 18$

$P1, P2 \geq 0$



Example (Cont…)



Max $Z = 3P1 + 5P2$

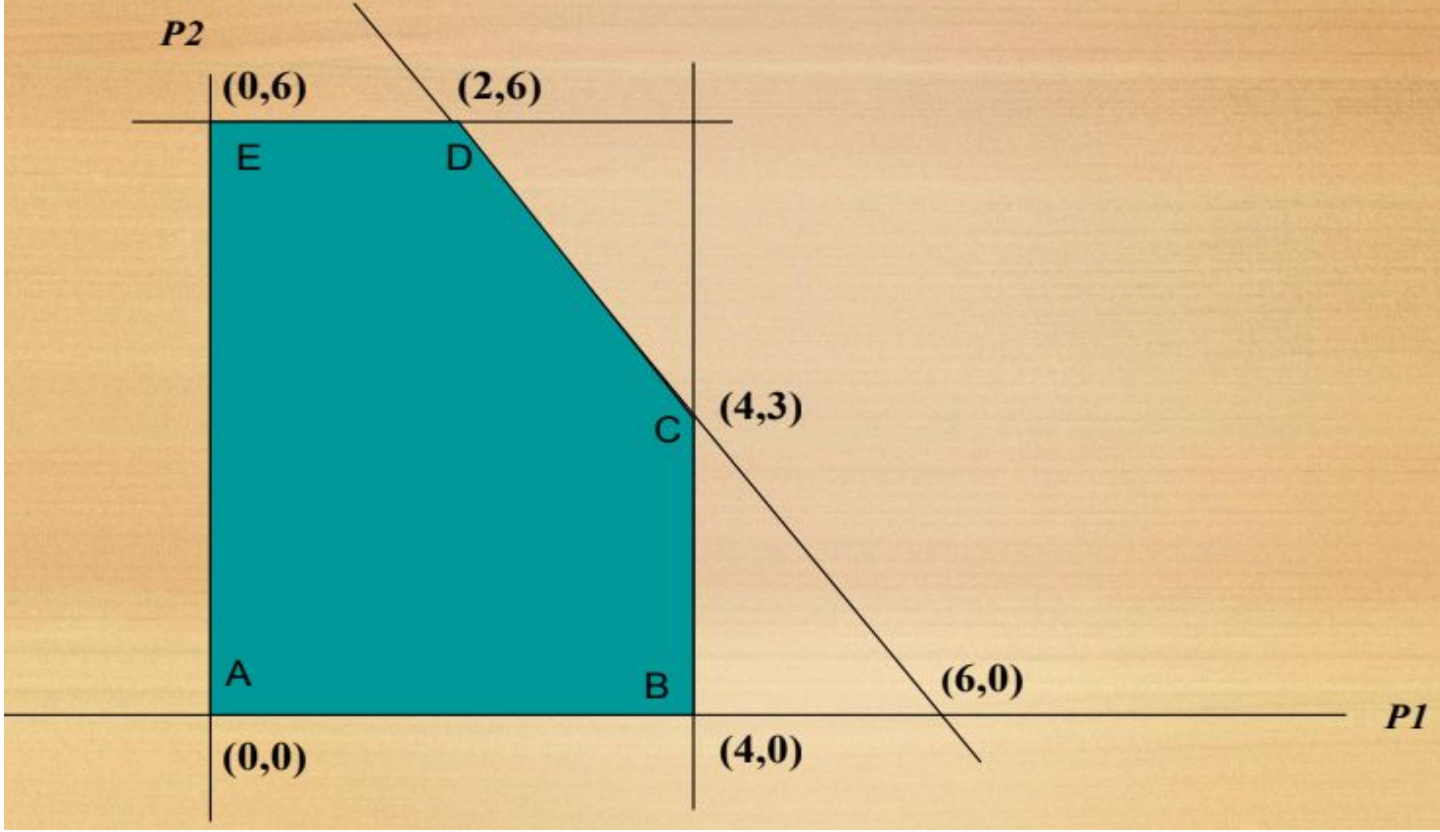
s.t. $P1 \leq 4$

$P2 \leq 6$

$3P1 + 2P2 \leq 18$

$P1, P2 \geq 0$

Example (Cont…)



Example (Cont…)

Objective Function : Max $Z = 3\ P1 + 5\ P2$

Corner Points	**Value of Z**
A – (0,0)	0
B – (4,0)	12
C – (4,3)	27
D – (2,6)	36
E – (0,6)	30

Optimal Point : (2,6)
Optimal Value : 36

Example

$$\begin{aligned} \textbf{Min} \quad z &= 5x_1 + 2x_2 \\ \textbf{s.t.} \quad 2x_1 + 5x_2 &\geq 10 \\ 4x_1 - x_2 &\geq 12 \\ x_1 + x_2 &\geq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Example

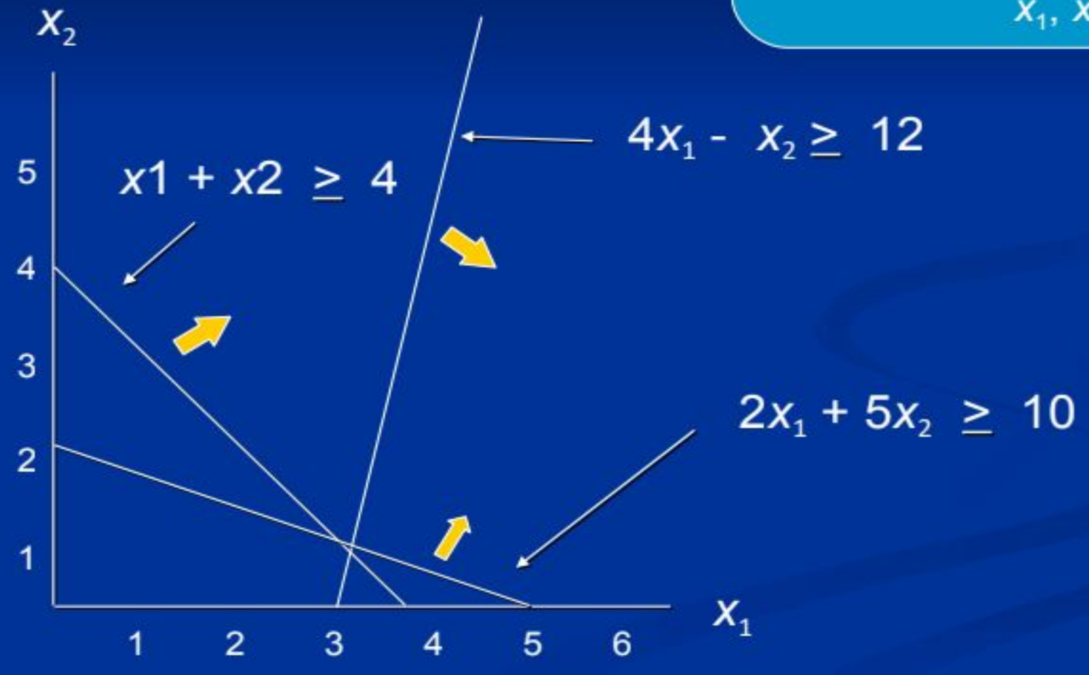
$$\text{Min} \quad z = 5x_1 + 2x_2$$

$$\text{s.t.} \quad 2x_1 + 5x_2 \geq 10$$

$$4x_1 - x_2 \geq 12$$

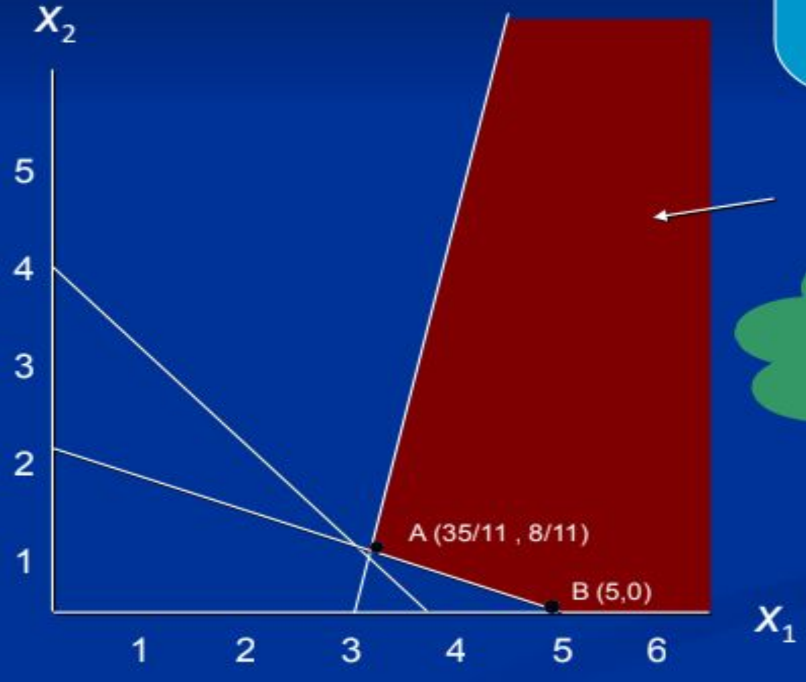
$$x_1 + x_2 \geq 4$$

$$x_1, x_2 \geq 0$$



Example (Cont…)

$$\begin{aligned} \text{Min} \quad z &= 5x_1 + 2x_2 \\ \text{s.t.} \quad 2x_1 + 5x_2 &\geq 10 \\ 4x_1 - x_2 &\geq 12 \\ x_1 + x_2 &\geq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$



Feasible Region

This is the case of 'Unbounded Feasible Region'

Example (Cont…)

Objective Function : **Min** $Z = 5x_1 + 2x_2$

Corner Points	**Value of Z**
A – (35/11, 8/11)	191/11 (17.364)
B – (5,0)	25

Optimal Point : (35/11,8/11)
Optimal Value : 191/11=17.364

Example

$$
\begin{aligned}
\textbf{Max} \quad z &= 3x_1 + 4x_2 \\
\textbf{s.t.} \quad x_1 + x_2 &\geq 5 \\
3x_1 + x_2 &\geq 8 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

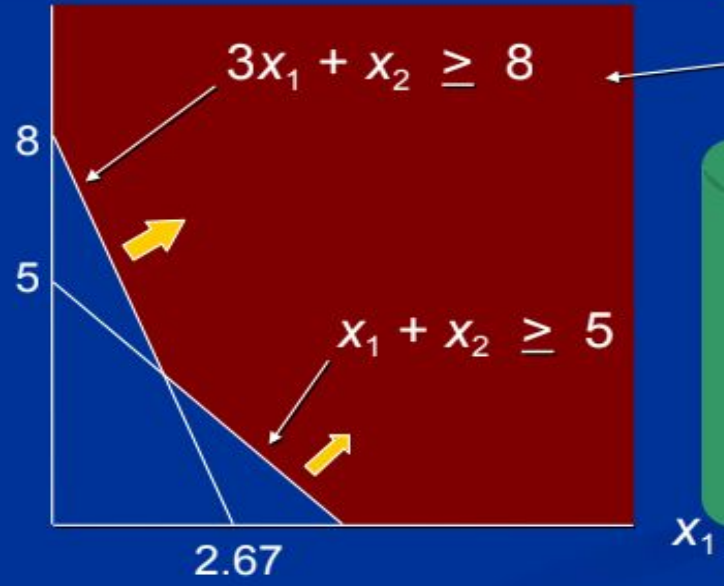
Example

$$\text{Max} \quad z = 3x_1 + 4x_2$$

$$\text{s.t.} \quad x_1 + x_2 \geq 5$$

$$3x_1 + x_2 \geq 8$$

$$x_1, x_2 \geq 0$$



Feasible Region

This feasible region is unbounded, hence z can be increased infinitely.

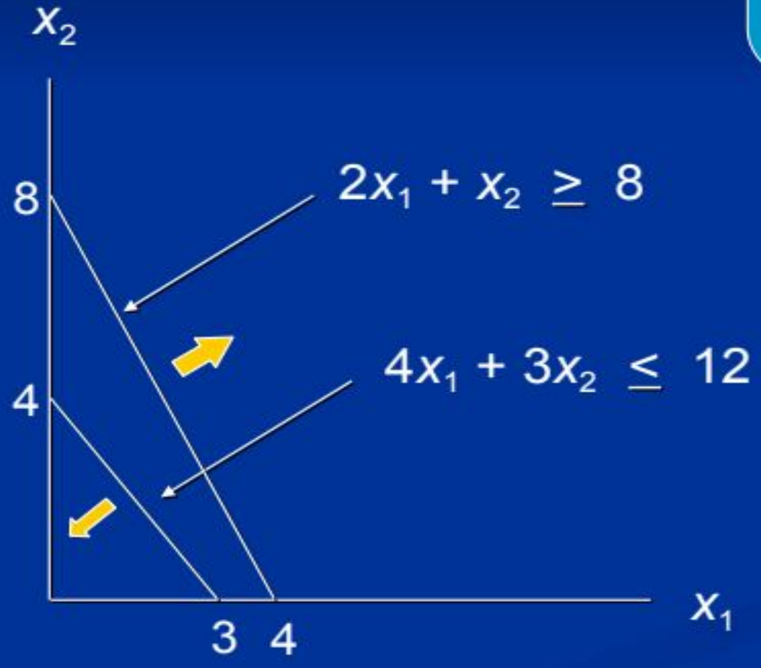
So this problem is having a Unbounded Solution.

Example

$$\begin{aligned} \text{Max} \quad z &= 2x_1 + 6x_2 \\ \text{s.t.} \quad 4x_1 + 3x_2 &\leq 12 \\ 2x_1 + x_2 &\geq 8 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Example

$$\begin{aligned} \text{Max} \quad & z = 2x_1 + 6x_2 \\ \text{s.t.} \quad & 4x_1 + 3x_2 \le 12 \\ & 2x_1 + x_2 \ge 8 \\ & x_1, x_2 \ge 0 \end{aligned}$$



In this example, common feasible region does not exist and hence the problem is not having a optimal solution.

This is the case of infeasible solution.

Simplex Method

- When decision variables are ***more than 2***, we always use Simplex Method
- **Slack Variable**: Variable added to a ≤ constraint to convert it to an equation (=).
  - A slack variable represents unused resources
  - A slack variable contributes nothing to the objective function value.
- **Surplus Variable:** Variable subtracted from a ≥ constraint to convert it to an equation (=).
  - A surplus variable represents an excess above a constraint requirement level.
  - Surplus variables contribute nothing to the calculated value of the objective function.

LINEAR PROGRAMMING PROBLEM (LPP)

Example 1: Solve following LPP using Simplex method

Maximize $Z = 7X_1 + 6X_2$ ← Objective Function

$$\left.\begin{aligned} \text{Subject to, } X_1 + X_2 &\le 4; \\ 2X_1 + X_2 &\le 6; \end{aligned}\right\} \text{Constraints}$$

Where; $X_1, X_2 \ge 0$ ← Non negativity constraints

Convert into Standard form

Maximize $Z = 7X_1 + 6X_2$

Subject to, $X_1 + X_2 \le 4;$

$2X_1 + X_2 \le 6;$

Where; $X_1, X_2 \ge 0$

Standard form:

Maximize $Z = 7X_1 + 6X_2 + 0S_1 + 0S_2$

Subject to, $X_1 + X_2 + S_1 = 4;$

$2X_1 + X_2 + S_2 = 6,$

Where; $X_1, X_2, S_1, S_2 \ge 0.$

Simplex Table:

C_j		7	6	0	0	b
C_B	B.V.	X_1	X_2	S_1	S_2	
0	S_1	1	1	1	0	4
0	S_2	2	1	0	1	6
Z_j		0	0	0	0	
$C_j - Z_j$						

$Z_j = \sum C_B a_{ij}$

$Z_1 = 0\times1 + 0\times2$

$Z_1 = 0$

$S_1 = 4$
$S_2 = 6$
Zmax = 0

Initial Basic Feasible solution

Simplex Table:

$$Z_j = \sum C_B a_{ij}$$

C_j		7	6	0	0	b	Min. Ratio = b/pivot col.
C_B	B.V.	X_1	X_2	S_1	S_2		
0	S_1	1	1	1	0	4	4/1 = 4
0	S_2	2	1	0	1	6	6/2 = 3
Z_j		0	0	0	0		
$C_j - Z_j$		7	6	0	0		

Pivot Row

Pivot column

Key element

Incoming variable

Outgoing variable

C_j		7	6	0	0	b	Min. Ratio
CB	B.V.	X_1	X_2	S_1	S_2		
0	S_1	1	1	1	0	4	4/1 = 4
0	S_2	2	1	0	1	6	6/2 = 3
Z_j		0	0	0	0		
$C_j - Z_j$		7	6	0	0		

Pivot row

Pivot Column

Key element

	C_j	7	6	0	0	b	Min. Ratio
	CB	B.V.	X_1	X_2	S_1	S_2		
R1	0	S_1	1	1	1	0	4	4/1 = 4
R2	0	S_2	2	1	0	1	6	6/2 = 3
	Z_j		0	0	0	0		
	$C_j - Z_j$		7	6	0	0		

$$R2^{new} = R2 / 2$$

$$R1^{new} = R1 - R2^{new}$$

C_j		7	6	0	0	b	Min. Ratio
C_B	B.V.	X_1	X_2	S_1	S_2		
0	S_1	0	0.5	1	-0.5	1	1/0.5 = 2
7	X_1	1	0.5	0	0.5	3	3/0.5 = 6
Z_j		7	3.5	0	3.5		
$C_j - Z_j$		0	2.5	0	-3.5		

$$Z_j = \sum C_B a_{ij}$$

$\leftarrow R2^{new}$

Key element

C_j		7	6	0	0	b	Min. Ratio
C_B	B.V.	X_1	X_2	S_1	S_2		
0	S_1	0	0.5	1	-0.5	1	1/0.5 = 2
7	X_1	1	0.5	0	0.5	3	3/0.5 = 6
Z_j		7	3.5	0	3.5		
$C_j - Z_j$		0	2.5	0	-3.5		

$Z_j = \sum C_B a_{ij}$

$R1^{new} = 2 \times R1$

$R2^{new} = R2 - 0.5 \times R1^{new}$

C_j		7	6	0	0	b
C_B	B.V.	X_1	X_2	S_1	S_2	
6	X_2	0	1	2	-1	2
7	X_1	1	0	-1	1	2
Z_j		7	6	5	1	
$C_j - Z_j$		0	0	-5	-1	

$R1^{new}$

$Z_j = \sum C_B a_{ij}$

Final Table

C_j		7	6	0	0	b
C_B	B.V.	X_1	X_2	S_1	S_2	
6	X_2	0	1	2	-1	2
7	X_1	1	0	-1	1	2
Z_j		7	6	5	1	
$C_j - Z_j$		0	0	-5	-1	

$$Z_j = \sum C_B a_{ij}$$

$X_1 = 2$
$X_2 = 2$

$Z = 7X_1 + 6X_2 + 0S_1 + 0S_2$
$Z_{max} = 7\times2 + 6\times2 + 0 = 26$

Optimal Solution:

$X_1 = 2$
$X_2 = 2$
$Z = 26$

Problem

$$\text{Max. } Z = 13x_1 + 11x_2$$

Subject to constraints:

$$4x_1 + 5x_2 \leq 1500$$

$$5x_1 + 3x_2 \leq 1575$$

$$x_1 + 2x_2 \leq 420$$

$$x_1, x_2 \geq 0$$

Optimal Solution is : $\mathbf{x_1 = 270, x_2 = 75, Z = 4335}$

Problem

$$\text{Max. } Z = 3x_1+5x_2+4x_3$$

Subject to constraints:

$$2x_1+3x_2 \leq 8$$

$$2x_2+5x_3 \leq 10$$

$$3x_1+2x_2+4x_3 \leq 15$$

$$x_1, x_2, x_3 \geq 0$$

Example

$$\text{Min.. } Z = x_1 - 3x_2 + 2x_3$$

Subject to constraints:

$$3x_1 - x_2 + 3x_3 \le 7$$

$$-2x_1 + 4x_2 \le 12$$

$$-4x_1 + 3x_2 + 8x_3 \le 10$$

$$x_1, x_2, x_3 \ge 0$$

Convert the problem into maximization problem

$$\text{Max.. } Z' = -x_1 + 3x_2 - 2x_3 \quad \text{where } Z' = -Z$$

Subject to constraints:

$$3x_1 - x_2 + 3x_3 \leq 7$$

$$-2x_1 + 4x_2 \leq 12$$

$$-4x_1 + 3x_2 + 8x_3 \leq 10$$

$$x_1, x_2, x_3 \geq 0$$

Let S_1, S_2 and S_3 be three slack variables.

Modified form is:

$$Z' = -x_1 + 3x_2 - 2x_3 + 0S_1 + 0S_2 + 0S_3$$

$$3x_1 - x_2 + 3x_3 + S_1 = 7$$

$$-2x_1 + 4x_2 + S_2 = 12$$

$$-4x_1 + 3x_2 + 8x_3 + S_3 = 10$$

$$x_1, x_2, x_3 \geq 0$$

Initial BFS is : $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $S_1 = 7$, $S_2 = 12$, $S_3 = 10$

and $Z = 0$.

Cj		-1	3	-2	0	0	0		Min. Ratio
CB	B.V	X1	X2	X3	S1	S2	S3	b	
0	S1	3	-1	3	1	0	0	7	-
0	S2	-2	$\boxed{4}$	0	0	1	0	12	12/4=3
0	S3	-4	3	8	0	0	1	10	10/3=3.33
	Zj	0	0	0	0	0	0		
Cj - Zj		-1	3	-2	0	0	0		

Therefore, x_2 is the entering variable and S_2 is the departing variable.

Cj		-1	3	-2	0	0	0		Min. Ratio
CB	B.V	X1	X2	X3	S1	S2	S3	b	
0	S1	3	-1	3	1	0	0	7	-
0	S2	-2	4	0	0	1	0	12	12/4=3
0	S3	-4	3	8	0	0	1	10	10/3=3.33
	Zj	0	0	0	0	0	0		
Cj - Zj		-1	3	-2	0	0	0		

R1
R2
R3

NR2= R2/4
NR1= R1 + NR2
NR3= R3- 3 NR2

Cj		-1	3	-2	0	0	0		Min. Ratio
CB	B.V	X1	X2	X3	S1	S2	S3	b	
0	S1	5/2	0	3	1	1/4	0	10	4
3	X2	-1/2	1	0	0	1/4	0	3	-
0	S3	-5/2	0	8	0	-3/4	1	1	-
	Zj	-3/2	3	0	0	3/4	0		
Cj - Zj		1/2	0	-2	0	-3/4	0		

Cj		-1	3	-2	0	0	0		Min. Ratio
CB	B.V	X1	X2	X3	S1	S2	S3	b	
0	S1	5/2	0	3	1	1/4	0	10	4
3	X2	-1/2	1	0	0	1/4	0	3	-
0	S3	-5/2	0	8	0	-3/4	1	1	-
	Zj	-3/2	3	0	0	3/4	0		
Cj - Zj		1/2	0	-2	0	-3/4	0		

R1

R2

R3

Therefore, x_1 is the entering variable and S_1 is the departing variable.

$NR_1=2R_1/5$

$NR_2= R_2 + NR_1/2$

$NR_3= R_3+ 5\ NR_2/2$

Cj		-1	3	-2	0	0	0	
CB	B.V	X1	X2	X3	S1	S2	S3	b
-1	X1	1	0	6/5	2/5	1/10	0	4
3	X2	0	1	3/5	1/5	3/10	0	5
0	S3	0	0	11	1	-1/2	1	11
	Zj	-1	3	3/5	1/5	4/5	0	
Cj - Zj		0	0	-13/5	-1/5	-4/5	0	

All Δj are -ve or 0 so stop.

Solution

$X1 = 4$, $X2 = 5$, $X3 = 0$

$Z' = 11$

$Z = -11$

It is the case of degenerate solution as one of the decision variable is equal to 0.

Example

$$\text{Max.. } Z = 3x_1 + 4x_2$$

Subject to constraints:

$$x_1 - x_2 \leq 1$$

$$-x_1 + x_2 \leq 2$$

$$x_1, x_2 \geq 0$$

Let S_1 and S_2 be two slack variables

.

Modified form is:

$$Z = 3x_1 + 4x_2 + 0 \cdot S_1 + 0 \cdot S_2$$

$$x_1 - x_2 + S_1 = 1$$

$$-x_1 + x_2 + S_2 = 2$$

$$x_1, x_2, S_1, S_2 \geq 0$$

Initial BFS is : $x_1 = 0$, $x_2 = 0$, $S_1 = 1$, $S_2 = 2$

Basic Variable	CB	3 x_1	4 x_2	0 S_1	0 S_2	Sol.	Ratio
S_1	0	1	-1	1	0	1	-
S_2	0	-1	1	0	1	2	2
-	Z	0	0	0	0		
	Δ	3	4	0	0		

Therefore, x_2 is the entering variable and S_2 is the departing variable.

Basic Variable		3	4	0	0	Sol.	Ratio
	CB	x_1	x_2	S_1	S_2		
S_1	0	0	0	1	1	3	-
x_2	4	-1	1	0	1	2	-
	Z	-4	4	0	4		
	Δ	7	0	0	-4		

x_1 is the entering variable, but as in x_1 column every no. is less than equal to zero, ratio cannot be calculated. Therefore given problem is having a **unbounded solution**.

Artificial Variable

- Artificial variable is added to the LHS of an equation of a $\geq$ and = constraint in order to convert it to an equation (=)
- Artificial variable can not take negative value just like other variables and are assigned non-negativity restrictions, i.e. $A_i \geq 0$
- An artificial variable is fictitious and do not have any physical meaning
- Assign –**M** to an artificial variable in the objective function of a maximization problem and assign +**M** to an artificial variable in the objective function of a minimization problem, where M is a big penalty or large coefficient

In minimization problem, the artificial variable has largest coefficient with positive value, therefore the objective function will be minimum when its value is 0

Test for optimality

- **Maximization problem**: If all the values in the Cj-Zj row are ≤ 0, the solution is optimal
- **Minimization problem**: If all the values in the Cj-Zj row are ≥ 0, the solution is optimal

If the solution is not optimal, then it is improved till the optimal solution is obtained.

Note: *If in the final optimal solution, any of the non-basic variable has a 0 Cj-Zj value, then the solution is multiple optimal otherwise a unique solution*

Example Big M Method

Minimize $Z = 20x_1 + 30x_2$

Subject to constraints:

$x_1 + x_2 \geq 50$

$x_1 + 2x_2 \geq 60$

$x_1, x_2 \geq 0$

Introducing the surplus and artificial variable to the constraints to augment the original LPP

Minimize $Z = 20x_1 + 30x_2 + MA_1 + MA_2 - 0S_1 - 0S_2$

Subject to:

$x_1 + x_2 - S_1 + A_1 = 50$

$x_1 + 2x_2 - S_2 + A_2 = 60$

$x_1, x_2, S_1, S_2, A_1, A_1 \geq 0$

Locate the key element

		C_j	20	30	0	0	M	M	
C_j	Basic Variable	Solution bi	x_1	x_2	S_1	S_2	A_1	A_2	Ratio $= \frac{b_i}{a_{ij}}$
M	A_1	50	1	1	-1	0	1	0	50
M	A_2	60	1	2	0	-1	0	1	30 →
	Z_j	110M	2M	3M	-M	-M	M	M	
	C_j-Z_j		20-2M	30-3M	M	M	0	0	

Pivot row= row/key element

$$\textbf{New value} = \text{Old value} - \frac{\text{Corresponding key row value} * \text{Corresponding key column value}}{\text{Key element}}$$

Table 2

		C_j	20	30	0	0	M	M	
C_j	Basic Variable	Solution bi	x_1	x_2	S_1	S_2	A_1	A_2	Ratio $= \frac{b_i}{a_{ij}}$
M	A_1	20	1/2	0	-1	1/2	1	-1/2	40 →
30	x_2	30	1/2	1	0	-1/2	0	1/2	60
	Z_j	20M+900	M/2+15	30	-M	M/2-15	M	-M/2+15	
	C_j-Z_j		5-M/2	0	M	15-M/2	0	3M/2-15	

Table 3 (Final table)

		C_j	20	30	0	0	M	M	
C_j	Basic Variable	Solution bi	x_1	x_2	S_1	S_2	A_1	A_2	Ratio $= \frac{b_i}{a_{ij}}$
20	x_1	40	1	0	-2	1	2	-1	
30	x_2	10	0	1	1	-1	-1	1	
	Z_j	1100	20	30	-10	-10	10	10	
	C_j-Z_j		0	0	10	10	M-10	M-10	

All the values in the C_j-Z_j row are ≥ 0, therefore the solution is minimum and cant not be improved further. The optimal solution is x_1=40, x_2=10 and minimum value of objective is 1100

Example

Maximize $Z = 10x_1+12x_2$

Subject to:

$x_1+x_2 = 5$

$x_1 \geq 2$

$x_2 \leq 4$

$x_1, x_2 \geq 0$

Introducing the slack, surplus and artificial variable to augment the original LPP

Maximize $Z = 10x_1+ 12x_2 - MA_1 - MA_2 - 0S_1+ 0S_2$

Subject to:

$x_1+x_2 + A_1= 5$

$x_1-S_1+ A_2= 2$

$x_2+S_2= 4$

Table -1

		C_j	10	12	0	0	-M	-M	
C_j	Basic Variable	Solution bi	x_1	x_2	S_1	S_2	A_1	A_2	Ratio $= \frac{b_i}{a_{ij}}$
-M	A_1	5	1	1	0	0	1	0	5
-M	A_2	2	1	0	-1	0	0	1	2 →
0	S_2	4	0	1	0	1	0	0	∞
	Z_j	-7M	-2M	-M	M	0	-M	-M	
	C_j-Z_j		10+2M	12+M	-M	0	0	0	

↑

Pivot row= row/key element

$$\textbf{New value} = \text{Old value} - \frac{\text{Corresponding key row value} * \text{Corresponding key column value}}{\text{Key element}}$$

Table -2

		C_j	10	12	0	0	-M	-M	
C_j	Basic Variable	Solution bi	x_1	x_2	S_1	S_2	A_1	A_2	Ratio $= \frac{b_i}{a_{ij}}$
-M	A_1	3	0	1	1	0	1	-1	3 →
10	x_1	2	1	0	-1	0	0	1	∞
0	S_2	4	0	1	0	1	0	0	4
	Z_j	20-3M	10	-M	-M-10	0	-M	M+10	
	C_j-Z_j		0	12+M	M+10	0	0	-2M-10	

Table-3

		C_j	10	12	0	0	-M	-M	
C_j	Basic Variable	Solution bi	x_1	x_2	S_1	S_2	A_1	A_2	Ratio $= \frac{b_i}{a_{ij}}$
12	x_2	3	0	1	1	0	1	-1	
10	x_1	2	1	0	-1	0	0	1	
0	S_2	1	0	0	-1	1	-1	1	
	Z_j	56	10	12	2	0	12	-2	
	C_j-Z_j		0	0	-2	0	-M-12	-M+2	

All the values in the C_j-Z_j row are non-positive, therefore the solution is maximum and cant not be improved further. The optimal solution is $x_1=2$, $x_2=3$ and maximum value of objective is 56

MIXED CONSTRAINTS

Example

Max. $Z = 2X_1 + 3X_2 + 4X_3$

Subject to

$$3X_1 + X_2 + 4X_3 <= 600$$
$$2X_1 + 4X_2 + 2X_3 >= 480$$
$$2X_1 + 3X_2 + 3X_3 = 540$$
$$X_1 , X_2 , X_3 >= 0$$

Slack , Surplus and Artificial variables are introduced with 0 coefficient to Slack and Surplus variables & -M to Artificial variables in the objective function

Max. $Z = 2X_1 + 3X_2 + 4X_3 + 0S_1 + 0S_2 - MA_1 - MA_2$

Subject to

$$3X_1 + X_2 + 4X_3 + S_1 + 0S_2 = 600$$
$$2X_1 + 4X_2 + 2X_3 + 0S_1 - S_2 + A_1 + 0A_2 = 480$$
$$2X_1 + 3X_2 + 3X_3 + 0S_1 + 0S_2 + 0A_1 + 1A_2 = 540$$

Simplex Table I

Cj			2	3	4	0	0	-M	-M	
	Basic variable	Solution Value	X_1	X_2	X_3	S_1	S_2	A_1	A_2	Min. Ratio
0	S_1	600	3	1	4	1	0	0	0	600
-M	A_1	480	2	4	2	0	-1	1	0	120
-M	A_2	540	2	3	3	0	0	0	1	180
	Zj	-1020M	-4M	-7M	-5M	0	M	-M	-M	
		Cj - Zj	2 + 4M	3 + 7M	4 +5M	0	-M	0	0	

NR2=R2/4
NR1=R1 - NR2
NR3= R3 - 3xNR2

Simplex Table II

Cj			2	3	4	0	0	-M	-M	
	Basic variable	**solution Value**	**X1**	**X2**	**X3**	**S1**	**S2**	**A1**	**A2**	**Min Ratio**
0	S1	480	5/2	0	7/2	1	1/4	-1/4	0	137.1
3	X2	120	1/2	1	1/2	0	-1/4	1/4	0	240
-M	A2	180	1/2	0	3/2	0	3/4	-3/4	1	120
	Zj	360- 180M	- M/2 +3/2	3	-3M/ 2 + 3/2	0	-3M/4 -3/4	3M/4 + 3/4	-M	
		Cj-Zj	M/2 +1/2	0	3M/2 + 5/2	0	3M/4+ 3/4	-7M/4 -3/4	0	

Simplex Table II

Cj			2	3	4	0	0	-M	-M	
	Basic variable	solution Value	X1	X2	X3	S1	S2	A1	A2	Min Ratio
0	S1	480	5/2	0	7/2	1	1/4	-1/4	0	137.1
3	X2	120	1/2	1	1/2	0	-1/4	1/4	0	240
-M	A2	180	1/2	0	3/2	0	3/4	-3/4	1	120
	Zj	360-180M	- M/2 +3/2	3	-3M/2 + 3/2	0	-3M/4 -3/4	3M/4 + 3/4	-M	
		Cj-Zj	M/2 +1/2	0	3M/2 + 5/2	0	3M/4+ 3/4	-7M/4 -3/4	0	

Simplex Table III

NR3=2xR3/3
NR1=R1 - 7xNR3/2
NR2= R2 - NR3/2

Cj			2	3	4	0	0	-M	-M
	Basic variable	solution Value	X1	X2	X3	S1	S2	A1	A2
0	S1	20	4/3	0	0	1	-3/2	3/2	-7/3
3	X2	60	1/3	1	0	0	-1/2	1/2	-1/3
4	X3	120	1/3	0	1	0	1/2	-1/2	2/3
	Zj	660	7/3	3	4	0	1/2	-1/2	5/3
		Cj-Zj	-1/3	0	0	0	-1/2	-M+ 1/2	-M -5/3

Solution:

X1=0
X2=60
X3= 120

Z= 660

Two Phase Method

Steps:

1. *Convert the original problem into standard form*
2. *Setup phase-I problem by:*
    - a. *Add artificial variables where it is needed*
    - b. *Create phase-I objective function as min* $Z' = \Sigma A_i$
    - c. *Convert new objective function into standard form*
    - d. *Don't forget non-negativity constraints*

Two Phase Method contd….

3. Setup the tableau and legitimize

4. Find optimal solution of phase-I (Remember: minimization problem)

5. Original problem is infeasible if optimal solution of phase-I has any of the artificial variables as nonzero. Otherwise, end phase-I, begin phase-II

6. Replace the original objective function, legitimize, and find optimal solution by regular simplex iterations

Example

Use two phase simplex method to solve the following LPP.

$$Minimize\ Z = x_1 + x_2$$

Subject to,

$$2x_1 + x_2 \geq 4$$

$$x_1 + 7x_2 \geq 7$$

$$and\ x_1, x_2 \geq 0$$

Standard form Min Z = x1 + x2 +0.s1 +0.s2 + A1 + A2

2x1 + x2 - s1 + A1 = 4

x1 + 7x2 - s2 + A2 =7

x1,x2,s1,s2,A1,A2 >=0

Phase 1

Min Z = A1 + A2

Table 1

Cj			0	0	0	0	1	1	
	Basic variable	**solution Value**	**X1**	**X2**	**S1**	**S2**	**A1**	**A2**	**Min. Ratio**
1	A1	4	2	1	-1	0	1	0	4
1	A2	7	1	7	0	-1	0	1	1
	Zj	11	3	8	-1	-1	1	1	
		Cj-Zj	-3	-8	1	1	0	0	

Table 1

C_j			0	0	0	0	1	1	
	Basic variable	**solution Value**	**X1**	**X2**	**S1**	**S2**	**A1**	**A2**	**Min. Ratio**
1	A1	4	2	1	-1	0	1	0	4
1	A2	7	1	7	0	-1	0	1	1
	Zj	11	3	8	-1	-1	1	1	
		Cj-Zj	-3	-8	1	1	0	0	

Table 2

C_j			0	0	0	0	1	1	
	Basic variable	**solution Value**	**X1**	**X2**	**S1**	**S2**	**A1**	**A2**	**Min. Ratio**
1	A1	3	13/7	0	-1	1/7	1	-1/7	21/13
0	X2	1	1/7	1	0	-1/7	0	1/7	7
	Zj	3	13/7	0	-1	1/7	1	-1/7	
		Cj-Zj	-13/7	0	1	-1/7	0	6/7	

Table 2

Cj			0	0	0	0	1	1	
	Basic variable	**solution Value**	**X1**	**X2**	**S1**	**S2**	**A1**	**A2**	**Min. Ratio**
1	A1	3	13/7	0	-1	1/7	1	-1/7	21/13
0	X2	1	1/7	1	0	-1/7	0	1/7	7
	Zj	3	13/7	0	-1	1/7	1	-1/7	
		Cj-Zj	-13/7	0	1	-1/7	0	6/7	

Table 3

Cj			0	0	0	0	1	1
	Basic variable	**solution Value**	**X1**	**X2**	**S1**	**S2**	**A1**	**A2**
0	X1	21/13	1	0	-7/13	1/13	7/13	-1/13
0	X2	10/13	0	1	1/13	-2/13	-1/13	2/13
	Zj	0	0	0	0	0	0	0
		Cj-Zj	0	0	0	0	1	1

Table 3

Cj			0	0	0	0	1	1
	Basic variable	solution Value	X1	X2	S1	S2	A1	A2
0	X1	21/13	1	0	-7/13	1/13	7/13	-1/13
0	X2	10/13	0	1	1/13	-2/13	-1/13	2/13
	Zj	0	0	0	0	0	0	0
		Cj-Zj	0	0	0	0	1	1

Phase 2

Standard form Min Z = x1 + x2 +0.s1 +0.s2

2x1 + x2 - s1 = 4

x1 + 7x2 - s2 =7

x1,x2,s1,s2,A1,A2 >=0

Table 1

Cj			1	1	0	0
	Basic variable	solution Value	X1	X2	S1	S2
1	X1	21/13	1	0	-7/13	1/13
1	X2	10/13	0	1	1/13	-2/13
	Zj	31/13	1	1	-6/13	-1/13
		Cj-Zj	0	0	6/13	1/13

Solution

X1 = 21/13

X2 = 10/13

Z= 31/13

Example

- Solve given LPP by Two-Phase Method

$$Max\,Z = 5x_1 - 4x_2 + 3x_3$$

$$\begin{aligned} \text{Subject to}\quad 2x_1 + x_2 - 6x_3 &= 20 \\ 6x_1 + 5x_2 + 10x_3 &\le 76 \\ 8x_1 - 3x_2 + 6x_3 &\le 50 \end{aligned}$$

Phase I

$$Min\ Z' = 0x_1 + 0x_2 + 0x_3 + A_1$$

$$\text{Subject to} \quad 2x_1 + x_2 - 6x_3 + A_1 = 20$$

$$6x_1 + 5x_2 + 10x_3 + S_1 = 76$$

$$8x_1 - 3x_2 + 6x_3 + S_2 = 50$$

		C_J	0	0	0	0	0	1
C_B	Basis Variable	B	X_1	X_2	X_3	S_1	S_2	A_1
1	A_1	20	2	1	-6	0	0	1
0	S_1	76	6	5	10	1	0	0
0	S_2	50	8	-3	6	0	1	0

Table 1

		C_J	0	0	0	0	0	1	
C_B	Basis Variable	X_B	X_1	X_2	X_3	S_1	S_2	A_1	θ
1	A_1	20	2	1	-6	0	0	1	10
0	S_1	76	6	5	10	1	0	0	76/6
0	S_2	50	8	-3	6	0	1	0	50/8
		Z_J	2	1	6	0	0	1	
		C_J- Z_J	-2	-1	6	0	0	0	

X1 is entering variable and S2 is leaving variable

Table 2

- New R1= Old R1 - New R3*2
- New R2= Old R2 - NewR3*6

	C_J		0	0	0	0	0	1	
C_B	Basis Variable	X_B	X_1	X_2	X_3	S_1	S_2	A_1	θ
1	A_1	7.5	0	1.75	-7.5	0	-1/4	1	4.28 ←
0	S_1	77/2	0	29/4	11/2	1	-0.75	0	5.31
0	X_1	50/8	1	-3/8	6/8	0	1/8	0	---
		Z_J	0	1.75	-7.5	0	-1/4	1	
		C_J- Z_J	0	-1.75 ↑	7.5	0	1/4	0	

X2 is entering variable and A1 is leaving variable

Table 3

- New R2= Old R2 - New R1*29/4
- New R3= Old R3 + NewR1*(3/8)

C_B	Basis Variable	X_B	X_1	X_2	X_3	S_1	S_2
	C_J		0	0	0	0	0
0	X_2	4.28	0	1	-4.28	0	-0.14
0	S_1	7.47	0	0	36.53	0	0.765
0	X_1	7.85	1	0	-0.86	0	0.073
		Z_J	0	0	0	0	0
		C_J-Z_J	0	0	0	0	0

As there is no artificial variable in the basis go to Phase II

Phase II

$$Max\,Z = 5x_1 - 4x_2 + 3x_3 + 0S_1 + 0S_2$$

		C_J	5	-4	3	0	0
C_B	Basis Variable	X_B	X_1	X_2	X_3	S_1	S_2
-4	X_2	4.28	0	1	-4.28	0	-0.14
0	S_1	7.47	0	0	36.53	0	0.765
5	X_1	7.85	1	0	-0.86	0	0.073
		Z_J	5	-4	12.82	0	0.925
		C_J-Z_J	0	0	-9.82	0	-0.925

As the given problem is of maximization and all the values in C_J-Z_J row are either zero or negative, an optimal solution is reached and is given by

X_1=7.855

X_2=4.28 and

$Z=5X_1-4X_2+3X_3$

$Z=5(7.855)-4(4.28)+3(0)$

$= 22.15$

Problem

Solve by Two-Phase Simplex Method

$$Max\,Z = -4x_1 - 3x_2 - 9x_3$$

$$\text{Subject to} \quad 2x_1 + 4x_2 + 6x_3 \geq 15$$

$$6x_1 + x_2 + 6x_3 \geq 12$$

$$x_1, x_2, x_3 \geq 0$$

$X_1 = 3/2$

$X_3 = 3$

Transportation Problem

Module 2

INTRODUCTION

The **basic transportation problem was developed in 1941 by F.I.** Hitchaxic. However it could be solved for optimally as an answer to complex business problem only in 1951,when Geroge B. Dantzig applied the concept of Linear Programming in solving the Transportation models.

Transportation problems are primarily concerned with the optimal (best possible) way in which a product produced at different factories or plants (called supply origins) can be transported to a number of warehouses (called demand destinations)

AIM OF TRANSPORTATION PROBLEM

- To find out the **optimum** transportation schedule keeping in mind cost of Transportation to be **minimized**.
- The **origin** of a transportation problem is the location from which shipments are dispatched.
- The **destination** of a transportation problem is the location to which shipments are transported.
- The **unit transportation cost** is the cost of transporting one unit of the consignment from an origin to a destination

WHAT IS A TRANSPORTATION PROBLEM?

- The transportation problem is a special type of LPP where the objective is to minimize the cost of distributing a product from a number of sources or origins to a number of destinations.
- Because of its special structure the usual simplex method is not suitable for solving transportation problems. These problems require special method of solution.

THE TRANSPORTATION PROBLEM

- The problem of finding the **minimum-cost** distribution of a given commodity from a group of supply centers (**sources**) $i=1,\ldots,m$ to a group of receiving centers (**destinations**) $j=1,\ldots,n$
- Each source has a certain supply (s_i)
- Each destination has a certain demand (d_j)
- The cost of shipping from a source to a destination is directly proportional to the number of units shipped

APPLICATION OF TRANSPORTATION PROBLEM

- It is used to compute transportation routes in such a way as to minimize transportation cost for finding out locations of warehouses.
- It is used to find out locations of transportation corporations depots where insignificant total cost difference may not matter.
- Minimize shipping costs from factories to warehouses(or from warehouses to retail outlets).
- Determine lowest cost location for new factory, warehouse, office ,or other outlet facility.
- Find minimum cost production schedule that satisfies firms demand and production limitations.

TYPES OF TRANSPORTATION PROBLEM

- **Balanced Transportation Problem:** where the total supply equals to the total demand
- **Unbalanced Transportation Problem:** where the total supply is not equal to the total demand

PHASES OF SOLUTION OF TRANSPORTATION PROBLEM

- **Phase I**- obtains the initial basic feasible solution
- **Phase II**-obtains the optimal basic solution

Optimality condition:

$$\boxed{m+n-1}$$

Where m= number of rows and n= number of columns

INITIAL BASIC FEASIBLE SOLUTION

The following methods are to find the initial basic feasible solution for the transportation problems.

- North West Corner Rule (NWCR)
- Least Cost Method
- Vogle's Approximation Method (VAM)

NORTH-WEST CORNER RULE (NWCR)

DEFINATION:

The North-West Corner Rule is a method adopted to compute the Initial Feasible Solution of the transportation problem. The name North-West Corner is given to this method because the basic variables are selected from the extreme left corner.

- It is most systematic and easiest method for obtaining Initial Feasible Basic Solution.

STEPS IN NORTH-WEST CORNER METHOD

Steps

1. Select the north west (upper left-hand) corner cell of the transportation table and allocate as many units as possible equal to the minimum between available supply and demand requirements, i.e., min (s1, d1).
2. Adjust the supply and demand numbers in the respective rows and columns allocation.
3. If the supply for the first row is exhausted then move down to the first cell in the second row.
4. If the demand for the first cell is satisfied then move horizontally to the next cell in the second column.
5. If for any cell supply equals demand then the next allocation can be made in cell either in the next row or column.
6. Continue the procedure until the total available quantity is fully allocated to the cells as required.

- Ex :1. In the table, three sources A,B and C with the Production Capacity of 50units, 40units, 60 units of product respectively is given. Every day the demand of three retailers D,E,F is to be furnished with at least 20units, 95units and 35units of product respectively. The transcriptation costs are also given in the matrix.

Source	D	E	F	Supply
A	5	8	4	50
B	6	6	3	40
C	3	9	6	60
Demand	20	95	35	150/150

Source	D	E	F	Supply
A	~~5~~ $\boxed{20}$	8	4	~~50~~ $\boxed{30}$
B	~~6~~	6	3	40
C	~~3~~	9	6	60
Demand	20 $\boxed{0}$	95	35	150/150

Source	D	E	F	Supply
A	~~5~~ $\boxed{20}$	~~8~~ $\boxed{30}$	~~4~~	~~50~~ $\boxed{0}$
B	~~6~~	6	3	40
C	~~3~~	9	6	60
Demand	~~20~~ $\boxed{0}$	~~95~~ $\boxed{65}$	35	150/150

Source	D	E	F	Supply
A	5 20	8 30	4	~~50~~ 0
B	6	6 40	3	~~40~~ 0
C	3	9	6	60
Demand	~~20~~ 0	~~95~~ 25	35	150/150

Source	D	E	F	Supply
A	5 20	8 30	4	~~50~~ 0
B	6	6 40	3	~~40~~ 0
C	3	9 25	6	~~60~~ 35
Demand	~~20~~ 0	~~95~~ 0	35	150/150

Source	D	E	F	Supply
A	5 20	8 30	4	50 0
B	6	6 40	3	40 0
C	3 0	9 25 0	6 35 0	60 0
Demand	20	95	35	150/150

INTIAL TRANSPORTATION COST

Total Cost = (20*5)+(30*8)+(40*6)+(25*9)+(35*6)= Rs 1015.

OPTIMALITY CONDITION

Total allocated cell = 6 i.e) m+n-1= 3+3-1=6

Problem

1. Solve the following Transportation problem using North West Corner Method.

Source	Destination D1	D2	D3	D4	Supply
O1	3	1	7	4	300
O2	2	6	5	9	400
O3	8	3	3	2	500
Demand:	250	350	400	200	1200

LEAST COST METHOD

Step 1:

Select the cell having minimum unit cost c_{ij} and allocate as much as possible, i.e. $\min(s_i, d_j)$

Step 2:

a. Subtract this min value from supply s_i and demand d_j.

b. If the supply s_i is 0, then cross (strike) that row and If the demand d_j is 0 then cross (strike) that column.

c. If min unit cost cell is not unique, then select the cell where maximum allocation can be possible

Step 3:

Repeat this steps for all uncrossed (unstricked) rows and columns until all supply and demand values are 0

Ex.2. Find the initial basic feasible solution using Least cost method

	D1	D2	D3	D4	Supply
S1	19	30	50	10	7
S2	70	30	40	60	9
S3	40	8	70	20	18
Demand	5	8	7	14	34/34

- The smallest transportation cost is 8 in cell $S3D2$

  The allocation to this cell is min(18,8) = **8**.
  This satisfies the entire demand of $D2$ and leaves 18 - 8 = 10 units with $S3$

	D1	***D2***	***D3***	***D4***	**Supply**
$S1$	19	30	50	10	7
$S2$	70	30	40	60	9
$S3$	40	8**(8)**	70	20	10
Demand	5	0	7	14	

- The smallest transportation cost is 10 in cell $S1D4$

  The allocation to this cell is min(7,14) = **7**.
  This exhausts the capacity of $S1$ and leaves 14 - 7 = 7 units with $D4$

	D1	***D2***	***D3***	***D4***	**Supply**
$S1$	~~19~~	~~30~~	~~50~~	~~10~~**(7)**	0
$S2$	70	~~30~~	40	60	9
$S3$	40	8**(8)**	70	20	10
Demand	5	0	7	7	

- The smallest transportation cost is 20 in cell *S*3*D*4

  The allocation to this cell is min(10,7) = **7**.
  This satisfies the entire demand of *D*4 and leaves 10 - 7 = 3 units with *S*3

	***D*1**	***D*2**	***D*3**	***D*4**	**Supply**
*S*1	19	30	50	10**(7)**	0
*S*2	70	30	40	60	9
*S*3	40	8**(8)**	70	20**(7)**	3
Demand	5	0	7	0	

- The smallest transportation cost is 40 in cell *S*2*D*3

  The allocation to this cell is min(9,7) = **7**.
  This satisfies the entire demand of *D*3 and leaves 9 - 7 = 2 units with *S*2

	***D*1**	***D*2**	***D*3**	***D*4**	**Supply**
*S*1	~~19~~	~~30~~	~~50~~	~~10~~**(7)**	0
*S*2	70	~~30~~	40**(7)**	~~60~~	2
*S*3	40	~~8~~**(8)**	~~70~~	~~20~~**(7)**	3
Demand	5	0	0	0	

- The smallest transportation cost is 70 in cell $S2D1$

  The allocation to this cell is min(2,2) = **2**

	***D*1**	***D*2**	***D*3**	***D*4**	**Supply**
$S1$	19	30	50	10(**7**)	0
$S2$	70(**2**)	30	40(**7**)	60	0
$S3$	40(**3**)	8(**8**)	70	20(**7**)	0
Demand	0	0	0	0	

- The minimum total transportation cost =10*7+70*2+40*7+40*3+8*8+20*7=814

Here, the number of allocated cells = 6 is equal to m + n - 1 = 3 + 4 - 1 = 6

Initial Feasible Solution is

	***D*1**	***D*2**	***D*3**	***D*4**	**Supply**
*S*1	19	30	50	10 **(7)**	7
*S*2	70 **(2)**	30	40 **(7)**	60	9
*S*3	40 **(3)**	8 **(8)**	70	20 **(7)**	18
Demand	5	8	7	14	34/34

Problem

1. Solve the following Transportation problem using Least Cost Method.

Source	D1	D2	D3	D4	Supply
O1	3	1	7	4	300
O2	2	6	5	9	400
O3	8	3	3	2	500
Demand:	250	350	400	200	1200

VOGEL'S APPROXIMATION METHOD (VAM METHOD)

The **Vogel Approximation Method** is an improved version of the Minimum Cell Cost Method and the Northwest Corner Method that in general produces better initial basic feasible solution, that report a smaller value in the objective (minimization) function of a balanced Transportation Problem. (**sum of the supply = sum of the demand).**

Applying the Vogel Approximation Method requires the following **STEPS**:

Step 1:

Determine a ***penalty cost*** for each row (column) by subtracting the lowest unit cell cost in the row (column)from the *next lowest unit cell cost* in the same row(column).

Step 2:

Identify the row or column with the greatest penalty cost. Break the ties arbitrarily (if there are any). Allocate as much as possible to the variable with the lowest unit cost in the selected row or column. Adjust the supply and demand and cross out the row or column that is already satisfied. If a row and column are satisfied simultaneously, only cross out one of the two and allocate a supply or demand of zero to the one that remains.

Step 3:

If there is exactly one row or column left with a supply or demand of zero, stop. If there is one row (column) left with a *positive* supply (demand), determine the basic variables in the row (column) using the Least Cost Method. Stop. If all of the rows and columns that were not crossed out have zero supply and demand (remaining), determine the basic *zero* variables using the Least Cost Cost Method. Stop.

In any other case, continue with Step1

Ex.3. Solve the following transportation problem using VAM Method

Source	D1	D2	D3	D4	Supply
O1	3	1	7	4	300
O2	2	6	5	9	400
O3	8	3	3	2	500
Demand:	250	350	400	200	1200

Solution:

For each row find the least value and then the second least value and take the absolute difference of these two least values and write it in the corresponding row difference as shown in the image below. In row **O1**, **1** is the least value and **3** is the second least value and their absolute difference is **2**. Similarly, for row **O2** and **O3**, the absolute differences are **3** and **1** respectively.

For each column find the least value and then the second least value and take the absolute difference of these two least values then write it in the corresponding column difference as shown in the figure. In column **D1**, **2** is the least value and **3** is the second least value and their absolute difference is **1**. Similarly, for column **D2**, **D3** and **D3**, the absolute differences are **2**, **2** and **2** respectively.

	Destination					
Source	D1	D2	D3	D4	Supply	Row Difference
O1	3	1	7	4	300	2
O2	2	6	5	9	400	3
O3	8	3	3	2	500	1
Demand:	250	350	400	200	1200	
Column Difference:	1	2	2	2		

- These value of row difference and column difference are also called as penalty. Now select the maximum penalty. The maximum penalty is **3** i.e. row **O2**. Now find the cell with the least cost in row **O2** and allocate the minimum among the supply of the respective row and the demand of the respective column. Demand is smaller than the supply so allocate the column's demand i.e. **250** to the cell. Then cancel the column **D1**.

Destination

Source	D1	D2	D3	D4	Supply	Row Difference
O1	3	1	7	4	300	2
O2	250 2	6	5	9	~~400~~ 150	3
O3	8	3	3	2	500	1
Demand:	~~250~~ 0	350	400	200	1200	
Column Difference:	1	2	2	2		

- From the remaining cells, find out the row difference and column difference.

	Destination D1	D2	D3	D4	Supply	Row Difference	
Source O1	3	1	7	4	300	2	3
O2	250 / 2	6	5	9	~~400~~ 150	3	1
O3	8	3	3	2	500	1	1
Demand:	~~250~~ 0	350	400	200	1200		
Column Difference:	1	2	2	2			
	-	2	2	2			

- Again select the maximum penalty which is **3** corresponding to row **O1**. The least-cost cell in row **O1** is **(O1, D2)** with cost **1**. Allocate the minimum among supply and demand from the respective row and column to the cell. Cancel the row or column with zero value.

Source	D1	D2	D3	D4	Supply	Row Difference	
O1	3	300 1	7	4	~~300~~ 0	2	3
O2	250 2	6	5	9	~~400~~ 150	3	1
O3	8	3	3	2	500	1	1
Demand:	~~250~~ 0	~~350~~ 50	400	200	1200		
Column Difference:	1	2	2	2			
	-	2	2	2			

- Now find the row difference and column difference from the remaining cells

Source	D1	D2	D3	D4	Supply	Row Difference		
O1	3	300 1	7	4	~~300~~ 0	2	3	-
O2	250 2	6	5	9	~~400~~ 150	3	1	1
O3	8	3	3	2	500	1	1	1
Demand:	~~250~~ 0	~~350~~ 50	400	200	1200			
Column Difference:	1	2	2	2				
	-	2	2	2				
	-	3	2	7				

- Now select the maximum penalty which is **7** corresponding to column **D4**. The least cost cell in column **D4** is **(O3, D4)** with cost **2**. The demand is smaller than the supply for cell **(O3, D4)**. Allocate **200** to the cell and cancel the column.

Source	D1	D2	D3	D4	Supply	Row Difference		
O1	3	[300] 1	7	4	~~300~~ 0	2	3	-
O2	[250] 2	6	5	9	~~400~~ 150	3	1	1
O3	8	3	3	[200] 2	~~500~~ 300	1	1	1
Demand:	~~250~~ 0	~~350~~ 50	400	~~200~~ 0	1200			
Column Difference:	1	2	2	2				
	-	2	2	2				
	-	3	2	7				

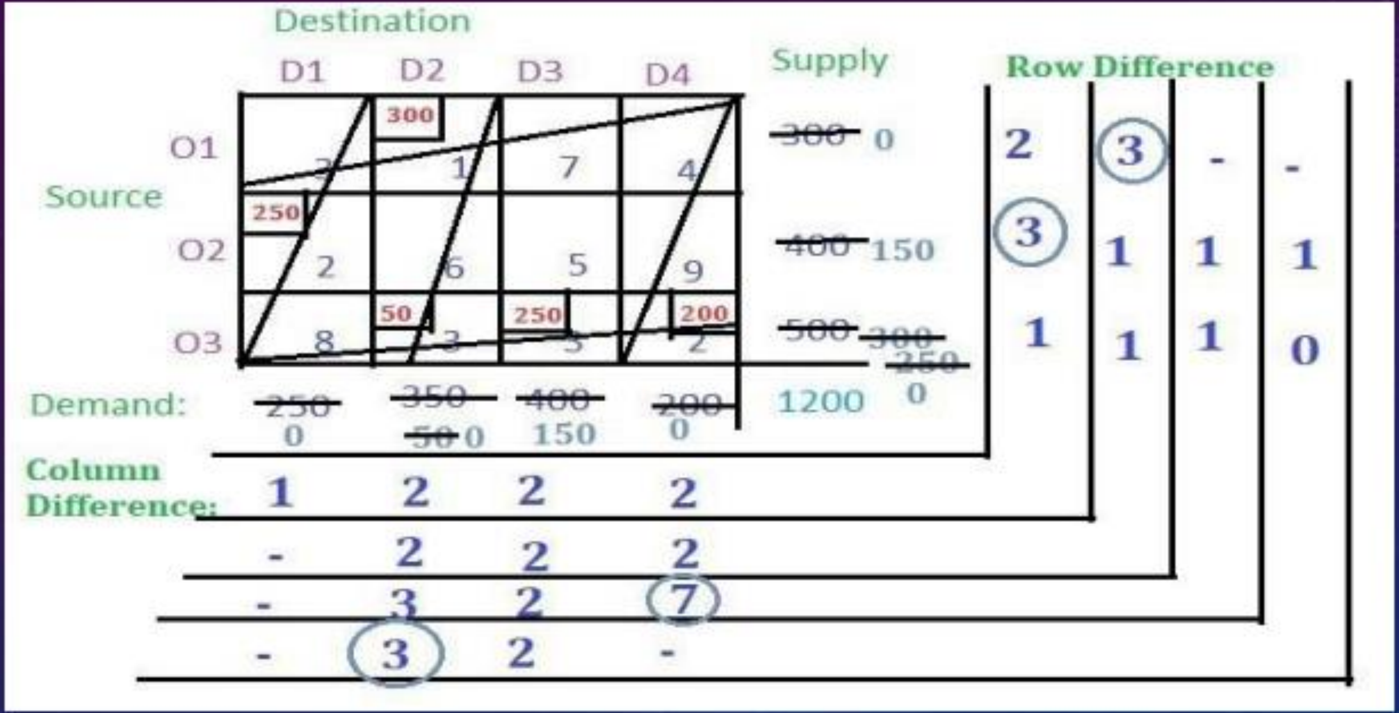
- Find the row difference and the column difference from the remaining cells.

	Destination D1	D2	D3	D4	Supply	Row Difference			
Source O1	3	300 / 1	7	4	~~300~~ 0	2	3	-	-
O2	250 / 2	6	5	9	~~400~~ 150	3	1	1	1
O3	8	3	3	200 / 2	~~500~~ 300	1	1	1	0
Demand:	~~250~~ 0	~~350~~ 50	400	~~200~~ 0	1200				
Column Difference:	1	2	2	2					
	-	2	2	2					
	-	3	2	7					
	-	3	2	-					

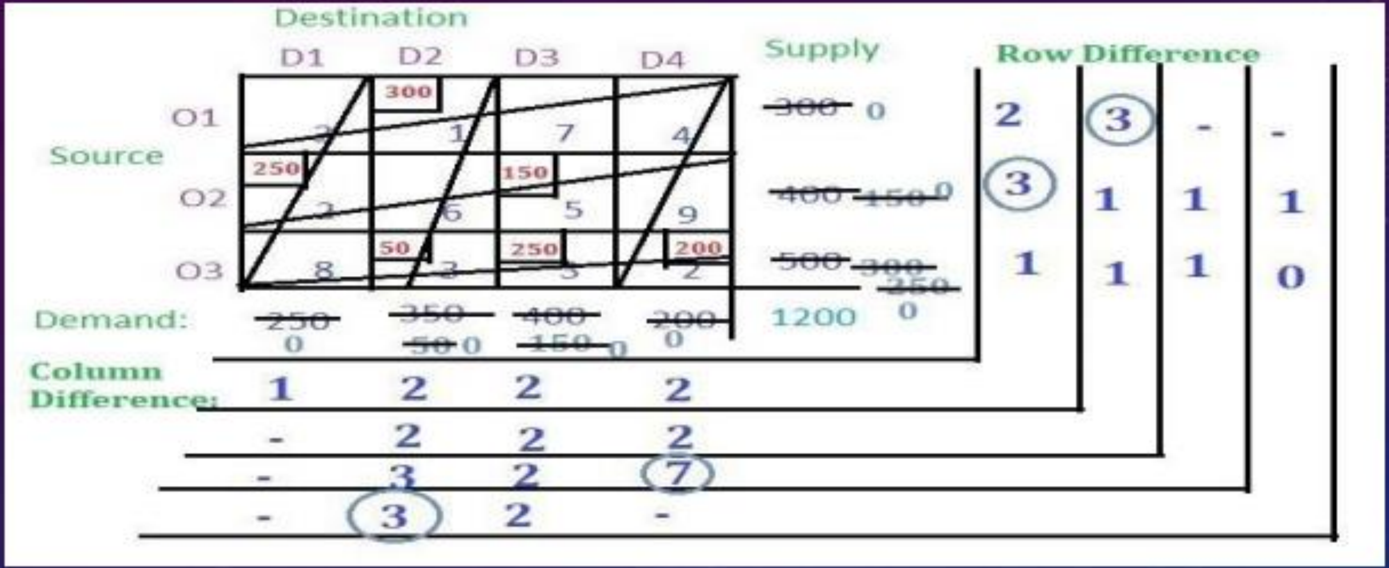
- Now the maximum penalty is **3** corresponding to the column **D2**. The cell with the least value in **D2** is **(O3, D2)**. Allocate the minimum of supply and demand and cancel the column.

Source	D1	D2	D3	D4	Supply	Row Difference			
O1	3	300 1	7	4	~~300~~ 0	2	3	-	-
O2	250 2	6	5	9	~~400~~ 150	3	1	1	1
O3	8	50 3	3	200 2	~~500~~ ~~300~~ 250	1	1	1	0
Demand:	~~250~~ 0	~~350~~ ~~50~~ 0	400	~~200~~ 0	1200				
Column Difference:	1	2	2	2					
	-	2	2	2					
	-	3	2	7					
	-	3	2	-					

- Now there is only one column so select the cell with the least cost and allocate the value.



- Now there is only one cell so allocate the remaining demand or supply to the cell



- No balance remains. So multiply the allocated value of the cells with their corresponding cell cost and add all to get the final cost i.e.
- **(300 * 1) + (250 * 2) + (50 * 3) + (250 * 3) + (200 * 2) + (150 * 5) = 2850**

Problem

A mobile phone manufacturing company has three branches located in three different regions, say Jaipur, Udaipur and Mumbai. The company has to transport mobile phones to three destinations, say Kanpur, Pune and Delhi. The availability from Jaipur, Udaipur and Mumbai is 40, 60 and 70 units respectively. The demand at Kanpur, Pune and Delhi are 70, 40 and 60 respectively. The transportation cost is shown in the matrix below (in Rs). Use the Vogel's Approximation Method to find a basic feasible solution (BFS).

		Destinations			
		Kanpur	Pune	Delhi	Supply
sources	Jaipur	4	5	1	40
	Udaipur	3	4	3	60
	Mumbai	6	2	8	70
	Demand	70	40	60	170

UNBALANCED TRANSPORTATION PROBLEM

A transportation problem is said to be **unbalanced if the** supply and demand are not equal. Two situations are possible:-

1. If Supply < demand, a dummy supply variable is introduced in the equation to make it equal to demand.
2. If demand < supply, a dummy demand variable is introduced in the equation to make it equal to supply.

Example of Unbalanced Transportation Problems:

Source	Destination				Supply
	1	2	3	4	
1	10	16	9	12	200
2	12	12	13	5	300
3	14	8	13	4	300
Demand	100	200	450	250	**1000/800**

Source	Destination			Supply
	1	2	3	
1	25	45	10	200
2	30	65	15	100
3	15	40	55	400
Demand	200	100	300	

Convert Unbalanced Transportation Problems into Balanced Transportation Problems:

Source	Destination				
	1	2	3	4	Supply
1	10	16	9	12	200
2	12	12	13	5	300
3	14	8	13	4	300
4	0	0	0	0	200
Demand	100	200	450	250	1000/1000

Source	Destination				Supply
	1	2	3	4	
1	25	45	10	0	200
2	30	65	15	0	100
3	15	40	55	0	400
Demand	200	100	300	100	700/700

Optimum Solution by 'MODI' Method

MODI Method Steps (Rules)

Step 1: Find an initial basic feasible solution using any one of the three methods NWCM, LCM or VAM

Step 2: Find u_i and v_j for rows and columns. To start

a. assign 0 to u_i or v_j where maximum number of allocation in a row or column respectively.

b. Calculate other u_i's and v_j's using $c_{ij}=u_i+v_j$, for all occupied cells.

Step 3: For all unoccupied cells, calculate $d_{ij}=(u_i+v_j) - c_{ij}$.

- **Step 4**: Check the sign of *dij*

  a. If *dij*<0, then current basic feasible solution is optimal and stop this procedure.

  b. If *dij*=0 then alternative soluion exists, with different set allocation and same transportation cost. Now stop this procedure.

  c. If *dij*>0, then the given solution is not an optimal solution and further improvement in the solution is possible.

- **Step 5**: Select the unoccupied cell with the largest negative value of *dij*, and included in the next solution.

- **Step 6**: Draw a closed path (or loop) from the unoccupied cell (selected in the previous step). The right angle turn in this path is allowed only at occupied cells and at the original unoccupied cell. Mark (+) and (-) sign alternatively at each corner, starting from the original unoccupied cell.

- **Step 7**:

   1. Select the minimum value from cells marked with (-) sign of the closed path.

   2. Assign this value to selected unoccupied cell (So unoccupied cell becomes occupied cell).

   3. Add this value to the other occupied cells marked with (+) sign.

   4. Subtract this value to the other occupied cells marked with (-) sign.

Step 8:

Repeat Step-2 to step-7 until optimal solution is obtained. This procedure stops when all $d_{ij} < 0$ for unoccupied cells.

Ex:4. Solve the following Transportation Problem by 'MODI' Method

Destination

Source	D1	D2	D3	D4	Supply(S_i)
O1	3	1	7	4	250
O2	2	6	5	9	350
O3	8	3	3	2	400
Demand(D_j):	200	300	350	150	

Solution:

Step 1: Check whether the problem is balanced or not. if the total sum of all the supply from sources **o1**, **o2**, and **o3** is equal to the total sum of all the demands for destinations **d1**, **d2**, **d3** and **d4** then the transportation problem is a balanced transportation problem.

	Destination				
Source	D1	D2	D3	D4	Supply(S_i)
O1	3	1	7	4	250
O2	2	6	5	9	350
O3	8	3	3	2	400
Demand(D_j):	200	300	350	150	1000

Step 2: Finding the initial basic feasible solution. Any of the three aforementioned methods can be used to find the initial basic feasible solution. Here, North-West Corner method will be used. and according to the northwest corner method this is the final initial basic feasible solution:

Destination

Source	D1	D2	D3	D4	Supply(S_i)
O1	200 3	50 1	7	4	~~250~~ ~~50~~ 0
O2	2	250 6	100 5	9	~~350~~ ~~100~~ 0
O3	8	3	250 3	150 2	~~400~~ ~~150~~ 0
Demand(D_j):	~~200~~ 0	~~300~~ ~~250~~ 0	~~350~~ ~~250~~ 0	~~150~~ 0	1000

Now, the total cost of transportation will be

(200 * 3) + (50 * 1) + (250 * 6) + (100 * 5) + (250 * 3) + (150 * 2) = 3700.

Step 3: U-V method to Optimize the initial basic feasible solution.
the following is the initial basic feasible solution:

200 3	50 1	7	4
2	250 6	100 5	9
8	3	250 3	150 2

For U-V method the values $\mathbf{u_i}$ and $\mathbf{v_j}$ have to be found for the rows and the columns respectively. As there are three rows so three $\mathbf{u_i}$ values have to be found i.e, $\mathbf{u_1}$ for the first row, $\mathbf{u_2}$ for the second row and $\mathbf{u_3}$ for the third row. Similarly, for four columns four $\mathbf{v_j}$ values have to be found I.E. $\mathbf{v_1}$, $\mathbf{v_2}$, $\mathbf{v_3}$ and $\mathbf{v_4}$. Check the image below:

	V1=	V2=	V3=	V4=
u1 =	200 3	50 1	7	4
u2 =	2	250 6	100 5	9
u3 =	8	3	250 3	150 2

- There is a separate formula to find $\mathbf{u_i}$ and $\mathbf{v_j}$,
  $\mathbf{u_i + v_j = C_{ij}}$ where $\mathbf{C_{ij}}$ is the cost value only for the allocated cell. Read more about it here.
- Before applying the above formula we need to check whether **m + n – 1 is equal to the total number of allocated cells** or not where **m** is the total number of rows and **n** is the total number of columns.
  In this case m = 3, n = 4 and total number of allocated cells is 6 so m + n – 1 = 6. The case when m + n – 1 is not equal to the total number of allocated cells will be discussed in the later posts.
- Now to find the value for u and v we assign any of the three u or any of the four v as 0. Let we assign $\mathbf{u_1 = 0}$ in this case. Then using the above formula we will get $\mathbf{v_1 = 3}$ as $\mathbf{u_1 + v_1 = 3}$ (i.e. $\mathbf{C_{11}}$) and $\mathbf{v_2 = 1}$ as $\mathbf{u_1 + v_2 = 1}$ (i.e. $\mathbf{C_{12}}$). Similarly, we have got the value for $\mathbf{v_2 = 3}$ so we get the value for $\mathbf{u_2 = 5}$ which implies $\mathbf{v_3 = 0}$. From the value of $\mathbf{v_3 = 0}$ we get $\mathbf{u_3 = 3}$ which implies $\mathbf{v_4 = -1}$. See the image below:

	$v_1 = 3$	$v_2 = 1$	$v_3 = 0$	$v_4 = -1$
$u_1 = 0$	200 / 3	250 / 1	7	4
$u_2 = 5$	2	50 / 6	100 / 5	9
$u_3 = 3$	8	3	250 / 3	150 / 2

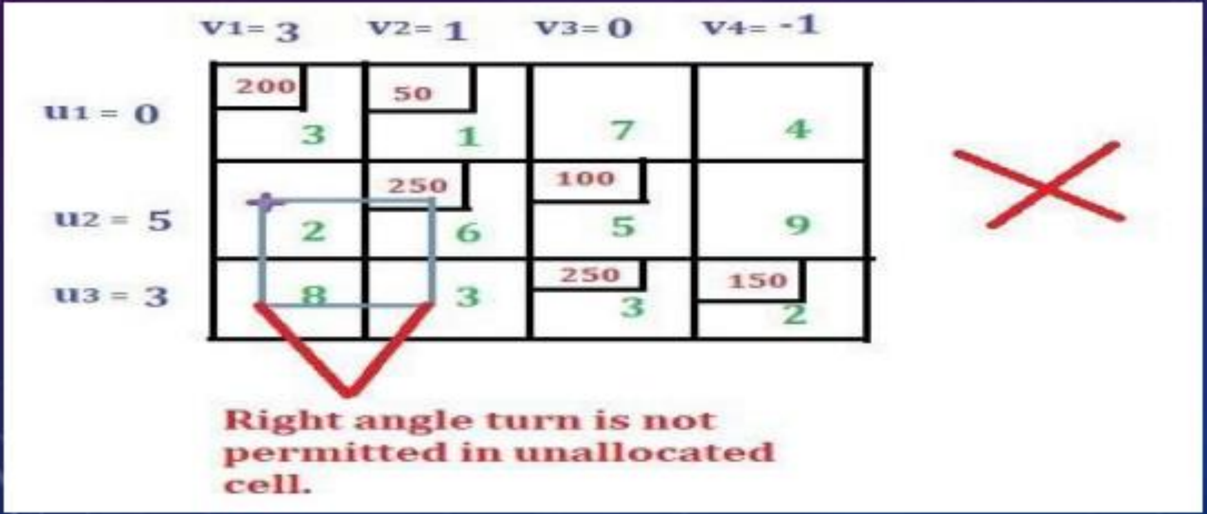
- Now, compute penalties using the formula $\mathbf{P_{ij} = u_i + v_j - C_{ij}}$ only for unallocated cells. We have two unallocated cells in the first row, two in the second row and two in the third row. Lets compute this one by one.
- For $\mathbf{C_{13}}$, $\mathbf{P_{13} = 0 + 0 - 7 = -7}$ (here $\mathbf{C_{13} = 7}$, $\mathbf{u_1 = 0}$ and $\mathbf{v_3 = 0}$)
- For $\mathbf{C_{14}}$, $\mathbf{P_{14} = 0 + (-1) -4 = -5}$
- For $\mathbf{C_{21}}$, $\mathbf{P_{21} = 5 + 3 - 2 = 6}$
- For $\mathbf{C_{24}}$, $\mathbf{P_{24} = 5 + (-1) - 9 = -5}$
- For $\mathbf{C_{31}}$, $\mathbf{P_{31} = 3 + 3 - 8 = -2}$
- For $\mathbf{C_{32}}$, $\mathbf{P_{32} = 3 + 1 - 3 = 1}$

The Rule: If we get all the penalties value as Zero or negative values that mean the optimality is reached and this answer is the final answer. But if we get any positive value means we need to proceed with the sum in the next step.

Now find the maximum positive penalty. Here the maximum value is 6 which corresponds to $\mathbf{C_{21}}$ cell. Now this cell is new basic cell. This cell will also be included in the solution.

	v1= 3	v2= 1	v3= 0	v4= -1
u1 = 0	200 / 3	50 / 1	7	4
u2 = 5	+ 2	250 / 6	100 / 5	9
u3 = 3	8	3	250 / 3	150 / 2

The rule for drawing closed-path or loop. Starting from the new basic cell draw a closed-path in such a way that the right angle turn is done only at the allocated cell or at the new basic cell. See the below images:



	V1= 3	V2= 1	V3= 0	V4= -1
u1 = 0	200 3	50 1	7	4
u2 = 5	+ 2	250 6	100 5	9
u3 = 3	8	3	250 3	150 2

- Assign alternate plus-minus sign to all the cells with right angle turn (or the corner) in the loop with plus sign assigned at the new basic cell.

	$v_1 = 3$	$v_2 = 1$	$v_3 = 0$	$v_4 = -1$
$u_1 = 0$	200 - 3	50 1 +	7	4
$u_2 = 5$	+ 2	250 6 -	100 5	9
$u_3 = 3$	8	3	250 3	150 2

- Consider the cells with a negative sign. Compare the allocated value (i.e. 200 and 250 in this case) and select the minimum (i.e. select 200 in this case). Now subtract 200 from the cells with a minus sign and add 200 to the cells with a plus sign. And draw a new iteration. The work of the loop is over and the new solution looks as shown below.

3	1 (250)	7	4
2 (200)	6 (50)	5 (100)	9
8	3	3 (250)	2 (150)

- Check the total number of allocated cells is equal to (m + n – 1). Again find u values and v values using the formula $\mathbf{u_i} + \mathbf{v_j} = \mathbf{C_{ij}}$ where $\mathbf{C_{ij}}$ is the cost value only for allocated cell. Assign $\mathbf{u_2 = 0}$ then we get $\mathbf{v_1 = 2}$ Similarly, we will get following values for $\mathbf{u_i}$ and $\mathbf{v_j}$.

	$v_1=2$	$v_2=6$	$v_3=5$	$v_4=4$
$u_1=-1$	3	250 1	7	4
$u_2=0$	200 2	50 6	100 5	9
$u_3=-2$	8	3	250 3	150 2

- Find the penalties for all the unallocated cells using the formula $P_{ij} = u_i + v_j - C_{ij}$.
- For C_{11}, P_{11} = -2
- For C_{13}, P_{13} = – 3
- For C_{14}, P_{14} = -1
- For C_{24}, P_{24} = -5
- For C_{31}, P_{31} = – 8
- For C_{32}, P_{32} = 1
- There is one positive value i.e. 1 for C_{32}. Now this cell becomes new basic cell.

	V1= 2	V2= 6	V3=5	V4= 4
u1=-1	3	250 / 1	7	4
u2=0	200 / 2	50 / 6	100 / 5	9
u3=-2	8	+ 3	250 / 3	150 / 2

- Now draw a loop starting from the new basic cell. Assign alternate plus and minus sign with new basic cell assigned as a plus sign.

	V1= 2	V2= 6	V3=5	V4= 4
u1=-1	3	250 1	7	4
u2=0	200 2	50 - 6	100 +5	9
u3=-2	8	+ 3	250 - 3	150 2

- Select the minimum value from allocated values to the cell with a minus sign. Subtract this value from the cell with a minus sign and add to the cell with a plus sign. Now the solution looks as shown in the image below:

3	250 1	7	4
200 2	6	150 5	9
8	50 3	200 3	150 2

- Check if the total number of allocated cells is equal to (m + n – 1). Find u and v values as above.

	$v_1=-2$	$v_2=1$	$v_3=1$	$v_4=0$
$u_1=0$	3	250 1	7	4
$u_2=4$	200 2	6	150 5	9
$u_3=2$	8	50 3	200 3	150 2

- Now again find the penalties for the unallocated cells as above.

For $\mathbf{P_{11} = 0 + (-2) - 3 = -5}$

For $\mathbf{P_{13} = 0 + 1 - 7 = -6}$

For $\mathbf{P_{14} = 0 + 0 - 4 = -4}$

For $\mathbf{P_{22} = 4 + 1 - 6 = -1}$

For $\mathbf{P_{24} = 4 + 0 - 9 = -5}$

For $\mathbf{P_{31} = 2 + (-2) - 8 = -8}$

All the penalty values are negative values. So the optimality is reached.

Now, find the total cost i.e.

(250 * 1)+(200 * 2)+(150 * 5)+(50 * 3)+(200 * 3)+(150 * 2) = 2450

Problem

The transportation cost per unit of a product is given below. Find out optimal transportation cost using MODI method.

S/D	D1	D2	D3	Supply
S1	8	9	7	40
S2	4	3	5	25
S3	8	5	6	35
Demand	30	30	40	100/100

Assignment Problem & Travelling Salesman Problem

Module 3

Syllabus

Assignment Problem & Travelling Salesman Problem: Definition of assignment Problem : Hungarian method (minimization and maximization), Travelling Salesman Problem : Hungarian method.

Self Learning Topics: Simple applications in daily life

Meaning of Assignment Problem:

An assignment problem is a particular case of transportation problem where the objective is to assign a number of resources to an equal number of activities so as to minimize total cost or maximize total profit of allocation.

The problem of assignment arises because available resources such as men, machines etc. have varying degrees of efficiency for performing different activities, therefore, cost, profit or loss of performing the different activities is different.

Thus, the problem is "How should the assignments be made so as to optimize the given objective". Some of the problem where the assignment technique may be useful are assignment of workers to machines, salesman to different sales areas.

The assignment problem deals with assigning machines to tasks, workers to jobs, soccer players to positions, and so on. The goal is to determine the optimum assignment that, for example, minimizes the total cost or maximizes the team effectiveness.

The Hungarian algorithm is an easy to understand and easy to use algorithm that solves the assignment problem.

The Assignment Problem

The assignment problem is a fundamental problem in the area of combinatorial optimization.

Assume for example that we have four jobs that need to be executed by four workers. Because each worker has different skills, the time required to perform a job depends on the worker who is assigned to it.

The matrix below shows the time required (in minutes) for each combination of a worker and a job. The jobs are denoted by J1, J2, J3, and J4, the workers by W1, W2, W3, and W4.

	J1	J2	J3	J4
W1	82	83	69	92
W2	77	37	49	92
W3	11	69	5	86
W4	8	9	98	23

The Assignment Problem

Each worker should perform exactly one job and the objective is to minimize the total time required to perform all jobs.

It turns out to be optimal to assign worker 1 to job 3, worker 2 to job 2, worker 3 to job 1 and worker 4 to job 4. The total time required is then 69 + 37 + 11 + 23 = 140 minutes. All other assignments lead to a larger amount of time required.

HUNGARIAN METHOD

The Hungarian mathematician D. Konig developed simpler & more efficient method of solving assignment problem which is known as Hungarian techniques or method.

Steps to follow

1. Prepare cost matrix, if it is not square matrix it means problem is unbalanced, so to convert it into balanced matrix add a dummy row with zero.
2. Starting with this smallest the first row, locate the smallest cost element in each row of the cost table. Now subtract this smallest element from each element in that row. As a result there shall be at least one zero in each row of this new table.
3. In the reduced cost table obtained in step 2, consider each column and locate the smallest element in it . Subtract the smallest element in each column from every element of that column.

HUNGARIAN METHOD

4. Make the assignment for the reduced matrix obtained from step no 2 & 3 in the following way.
    - Examine the rows successively, until a row with exactly one zero is found make an assignment to this single zero by putting square around it and cross out all other zeros appearing in the corresponding column as they will not be used to male any other assignment in that column. Proceed in this manner until all rows have been examined.
    - Examine the columns successively, until a column with exactly one zero is found make an assignment to this single zero by putting square around it and cross out all other zeros appearing in the corresponding column as they will not be used to make any other assignment in that row. proceed in this manner until all columns have been examined.
    - Repeat step 4(i) and 4(ii) until all zeros in the rows and columns are either marked or crossed out. If the no. of assignment made are equal to no. of rows / columns. Then it is an optimum solution and there is exactly one assignment in each row and in each column.

HUNGARIAN METHOD

5. Draw the minimum number of horizontal and vertical lines necessary to cover all the zeros in the reduced matrix obtained from step 4 in the following way
    - Mark all rows that do not have any assignment
    - Mark all columns that have zero in marked rows [step 5(a)]
    - Mark all rows that have assignment in marked columns. [step 5(b)]
    - Repeat step 5(a) and 5(b) until no more rows and columns can be marked
    - Draw straight lines through all unmarked rows and marked columns
6. If the number of lines drawn are equal to 'n' that is equal to number of rows or columns then it is an optimum solution, otherwise go to step 7

HUNGARIAN METHOD

7. Select the smallest element among all the uncovered elements. Subtract this smallest element from all the uncovered elements and add it to the element which lies at the intersection of two lines. Then we obtain another reduced matrix for fresh assignment
8. Go to step 4 and repeat the procedure until the number of assignment become equal to the number of rows and columns. In such a case we shall observe that every row and column has an assignment. Then the current solution is the optimum solution.

Example

Solve the following assignment problem shown in Table using Hungarian method. The matrix entries are processing time of each man in hours.

$$
\begin{array}{ccc}
 & & \textbf{Men} \\
 & & \begin{array}{ccccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} \end{array} \\
\textbf{Job} & \begin{array}{c} \mathbf{I} \\ \mathbf{II} \\ \mathbf{III} \\ \mathbf{IV} \\ \mathbf{V} \end{array} & \begin{pmatrix} 20 & 15 & 18 & 20 & 25 \\ 18 & 20 & 12 & 14 & 15 \\ 21 & 23 & 25 & 27 & 25 \\ 17 & 18 & 21 & 23 & 20 \\ 18 & 18 & 16 & 19 & 20 \end{pmatrix}
\end{array}
$$

Solution: The row-wise reductions are shown in Table

Row-wise Reduction Matrix

$$
\begin{array}{ccc}
 & & \textbf{Men} \\
 & & \begin{array}{ccccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} \end{array} \\
\textbf{Job} & \begin{array}{c} \mathbf{I} \\ \mathbf{II} \\ \mathbf{III} \\ \mathbf{IV} \\ \mathbf{V} \end{array} & \begin{pmatrix} 5 & 0 & 3 & 5 & 10 \\ 6 & 8 & 0 & 2 & 3 \\ 0 & 2 & 4 & 6 & 4 \\ 0 & 1 & 4 & 6 & 3 \\ 2 & 2 & 0 & 3 & 4 \end{pmatrix}
\end{array}
$$

Column-wise Reduction Matrix

$$
\begin{array}{cc}
 & \textbf{Men} \\
 & \begin{array}{ccccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} \end{array} \\
\textbf{Job} \begin{array}{c} \mathbf{I} \\ \mathbf{II} \\ \mathbf{III} \\ \mathbf{IV} \\ \mathbf{V} \end{array} &
\begin{pmatrix}
5 & 0 & 3 & 3 & 7 \\
6 & 8 & 0 & 0 & 0 \\
0 & 2 & 4 & 4 & 1 \\
0 & 1 & 4 & 4 & 0 \\
2 & 2 & 0 & 1 & 1
\end{pmatrix}
\end{array}
$$

Optimal Assignment

$$
\text{Job}\quad
\begin{array}{c}
 \\ \text{I} \\ \text{II} \\ \text{III} \\ \text{IV} \\ \text{V}
\end{array}
\begin{array}{c}
\text{Men} \\
\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{pmatrix}
5 & \boxed{0} & 3 & 3 & 7 \\
6 & 8 & \cancel{0} & \boxed{0} & \cancel{0} \\
\boxed{0} & 2 & 4 & 4 & 1 \\
\cancel{0} & 1 & 4 & 4 & \boxed{0} \\
2 & 2 & \boxed{0} & 1 & 1
\end{pmatrix}
\end{array}
$$

Therefore, the optimal solution is:

Job	Men	Time
I	2	15
II	4	14
III	1	21
IV	5	20
V	3	16
	Total time	= 86 hours

Example 2 -Maximization

At the head office of a company there are five registration counters. Five persons are available for service. How should the counters be assigned to persons so as to maximize the profit?

	Person				
Counter	A	B	C	D	E
1	30	37	40	28	40
2	40	24	27	21	36
3	40	32	33	30	35
4	25	38	40	36	36
5	29	62	41	34	39

Solution.

Here, the highest value is 62. So we subtract each value from 62, to convert the problem into minimization.The conversion is shown in the following table.

Counter	**Person**				
	A	B	C	D	E
1	32	25	22	34	22
2	22	38	35	41	26
3	22	30	29	32	27
4	37	24	22	26	26
5	33	0	21	28	23

Now the above problem can be easily solved by *Hungarian method*. After applying steps 1 to 3 of the Hungarian method, we get the following matrix.

Table

	Person				
Counter	A	B	C	D	E
1	10	3	$\boxed{0}$	8	$\cancel{0}$
2	$\boxed{0}$	16	13	15	4
3	$\cancel{0}$	8	7	6	5
4	15	2	$\cancel{0}$	$\boxed{0}$	4
5	33	$\boxed{0}$	21	24	23

Draw the minimum number of vertical and horizontal lines necessary to cover all the zeros in the reduced matrix.

Table

Person					
Counter	A	B	C	D	E
1	10	3	0	8	0
2	0	16	13	15	4
3	0	8	7	6	5
4	15	2	0	0	4
5	33	0	21	24	23

Select the smallest element from all the uncovered elements, i.e., 4. Subtract this element from all the uncovered elements and add it to the elements, which lie at the intersection of two lines. Thus, we obtain another reduced matrix for fresh assignment. Repeating step 3, we obtain a solution which is shown in the following table.

Final Table: Maximization Problem

	Person				
Counter	A	B	C	D	E
1	14	3	$\boxed{0}$	8	~~0~~
2	~~0~~	12	9	11	$\boxed{0}$
3	$\boxed{0}$	4	3	2	1
4	19	2	~~0~~	$\boxed{0}$	4
5	37	$\boxed{0}$	21	24	23

Optimal Assignment Root= 1-C , 2-E, 3A + 4D + 5B

Substituting values from original table:

40 + 36 + 40 + 36 + 62 = 214.

Unbalanced Assignment Problem

If number of rows is not equal to number of columns then it is called Unbalanced Assignment Problem.

So to solve this problem, we have to add dummy rows or columns with cost 0, to make it a square matrix.

Example Find Solution of Assignment problem using Hungarian method (MIN case)

Work\Job	I	II	III	IV
A	9	14	19	15
B	7	17	20	19
C	9	18	21	18
D	10	12	18	19
E	10	15	21	16

Solution:

The number of rows = 5 and columns = 4. Here given problem is unbalanced and add 1 new column to convert it into a balance.

	I	II	III	IV	J5
A	9	14	19	15	0
B	7	17	20	19	0
C	9	18	21	18	0
D	10	12	18	19	0
E	10	15	21	16	0

Step-1: Find out the each row minimum element and subtract it from that row

	I	*II*	*III*	*IV*	*Js*	
A	9 9=9-0	14 14=14-0	19 19=19-0	15 15=15-0	0 0=0-0	(-0) Minimum element of 1st row
B	7 7=7-0	17 17=17-0	20 20=20-0	19 19=19-0	0 0=0-0	(-0) Minimum element of 2nd row
C	9 9=9-0	18 18=18-0	21 21=21-0	18 18=18-0	0 0=0-0	(-0) Minimum element of 3rd row
D	10 10=10-0	12 12=12-0	18 18=18-0	19 19=19-0	0 0=0-0	(-0) Minimum

Step-2: Find out the each column minimum element and subtract it from that column.

	I	*II*	*III*	*IV*	*J5*	
A	2 2=9-7	2 2=14-12	1 1=19-18	0 0=15-15	0 0=0-0	
B	0 0=7-7	5 5=17-12	2 2=20-18	4 4=19-15	0 0=0-0	
C	2 2=9-7	6 6=18-12	3 3=21-18	3 3=18-15	0 0=0-0	
D	3 3=10-7	0 0=12-12	0 0=18-18	4 4=19-15	0 0=0-0	
E	3 3=10-7	3 3=15-12	3 3=21-18	1 1=16-15	0 0=0-0	
	(-7) Minimum	(-12) Minimum	(-18) Minimum	(-15) Minimum	(-0) Minimum	

Step-3: Make assignment in the opporunity cost table

(1) Rowwise cell (C,J5) is assigned, so columnwise cell (A,J5),(B,J5),(D,J5),(E,J5) crossed off.

(2) Columnwise cell (B,I) is assigned

(3) Columnwise cell (D,II) is assigned, so rowwise cell (D,III) crossed off.

(4) Columnwise cell (A,IV) is assigned

Rowwise & columnwise assignment shown in table

	I	*II*	*III*	*IV*	J_5
A	2	2	1	[0]	$\cancel{0}$
B	[0]	5	2	4	$\cancel{0}$
C	2	6	3	3	[0]
D	3	[0]	$\cancel{0}$	4	$\cancel{0}$
E	3	3	3	1	$\cancel{0}$

Step-4: Number of assignments = 4, number of rows = 5
Which is not equal, so solution is not optimal.

Step-5: Cover the 0 with minimum number of lines

(1) Mark(√) row E since it has no assignment

(2) Mark(√) column J5 since row E has 0 in this column

(3) Mark(√) row C since column J5 has an assignment in this row C.

(4) Since no other rows or columns can be marked, therefore draw straight lines through the unmarked rows A,B,D and marked columns J5

	I	II	III	IV	J_5
A	2	2	1	[0]	~~0~~
B	[0]	5	2	4	~~0~~
C	2	6	3	3	[0]
D	3	[0]	~~0~~	4	~~0~~
E	3	3	3	1	~~0~~

Step-6:

Develop the new revised table by selecting the smallest element, among the cells not covered by any line (say k = 1)

Subtract k = 1 from every element in the cell not covered by a line.

Add k = 1 to every elment in the intersection cell of two lines.

	I	*II*	*III*	*IV*	J_5
A	2	2	1	0	1
B	0	5	2	4	1
C	1	5	2	2	0
D	3	0	0	4	1
E	2	2	2	0	0

Repeat stpes 3 to 6 until an optimal solution is obtained.

Step-3: Make assignment in the opporunity cost table

(1) Rowwise cell (A,IV) is assigned, so columnwise cell (E,IV) crossed off.

(2) Rowwise cell (B,I) is assigned

(3) Rowwise cell (C,J5) is assigned, so columnwise cell (E,J5) crossed off.

(4) Columnwise cell (D,II) is assigned, so rowwise cell (D,III) crossed off.

Rowwise & columnwise assignment shown in table

	I	II	III	IV	J_5
A	2	2	1	[0]	1
B	[0]	5	2	4	1
C	1	5	2	2	[0]
D	3	[0]	$\cancel{0}$	4	1
E	2	2	2	$\cancel{0}$	$\cancel{0}$

Step-4: Number of assignments = 4, number of rows = 5
Which is not equal, so solution is not optimal.

Step-5: Cover the 0 with minimum number of lines
(1) Mark(✓) row E since it has no assignment

(2) Mark(✓) column IV since row E has 0 in this column

(3) Mark(✓) column J5 since row E has 0 in this column

(4) Mark(✓) row A since column IV has an assignment in this row A.

(5) Mark(✓) row C since column J5 has an assignment in this row C.

(6) Since no other rows or columns can be marked, therefore draw straight lines through the unmarked rows B,D and marked columns IV,J5

Tick mark not allocated rows and allocated columns

	I	II	III	IV	J_5	
A	2	2	1	[0]	1	✓(4)
B	[0]	5	2	4	1	
C	1	5	2	2	[0]	✓(5)
D	3	[0]	$\cancel{0}$	4	1	
E	2	2	2	$\cancel{0}$	$\cancel{0}$	✓(1)
				✓ (2)	✓ (3)	

Step-6:

Develop the new revised table by selecting the smallest element, among the cells not covered by any line (say k = 1)

Subtract k = 1 from every element in the cell not covered by a line.

Add k = 1 to every elment in the intersection cell of two lines.

	I	*II*	*III*	*IV*	J_5	
A	1	1	0	0	1	
B	0	5	2	5	2	
C	0	4	1	2	0	
D	3	0	0	5	2	
E	1	1	1	0	0	

Repeat stpes 3 to 6 until an optimal solution is obtained.

Step-3: Make assignment in the opporunity cost table
(1) Rowwise cell (B,I) is assigned, so columnwise cell (C,I) crossed off.

(2) Rowwise cell (C,J5) is assigned, so columnwise cell (E,J5) crossed off.

(3) Rowwise cell (E,IV) is assigned, so columnwise cell (A,IV) crossed off.

(4) Columnwise cell (D,II) is assigned, so rowwise cell (D,III) crossed off.

(5) Columnwise cell (A,III) is assigned

Rowwise & columnwise assignment shown in table

	I	II	III	IV	J_5	
A	1	1	[0]	$\xcancel{0}$	1	
B	[0]	5	2	5	2	
C	$\xcancel{0}$	4	1	2	[0]	
D	3	[0]	$\xcancel{0}$	5	2	
E	1	1	1	[0]	$\xcancel{0}$	

Step-4: Number of assignments = 5, number of rows = 5
Which is equal, so solution is optimal

Optimal assignments are

	I	*II*	*III*	*IV*	J_5
A	1	1	[0]	$\cancel{0}$	1
B	[0]	5	2	5	2
C	$\cancel{0}$	4	1	2	[0]
D	3	[0]	$\cancel{0}$	5	2
E	1	1	1	[0]	$\cancel{0}$

Optimal solution is

Work	Job	Cost
A	III	19
B	I	7
C	J_5	0
D	II	12
E	IV	16
	Total	54

Problem 1

Solve the following assignment problem shown in Table using Hungarian method so that all the jobs can be completed in minimum time.

		machines			
		I	II	III	IV
jobs	*A*	10	12	19	11
	B	5	10	7	8
	C	12	14	13	11
	D	8	15	11	9

Problem 2

Find Solution of Assignment problem using Hungarian method (MAX case)

Work\Job	I	II	III	IV
A	42	35	28	21
B	30	25	20	15
C	30	25	20	15
D	24	20	16	12

Travelling Salesman Problem

There are ‘n’ cities. A salesman wants to visit every city **exactly** once and then should be **back to the starting city**. The time/cost of travelling from a city every other city is known. We want to find a route such that total travelling time/cost is minimum. We will solve the problem using Hungarian method.

Travelling Salesman Problem – Hungarian Method

Steps

- Select minimum elements of a row. Subtract it from each element of the row.
- Select minimum elements of a column. Subtract it from each element of the column.
- Use only 0's to find the solution
- If not possible with only 0's use the next higher value i.e. 1 along with 0's to get the optimal solution
- Repeat the above step with next higher element with lower element till you get the proper solution

	L1	L2	L3	L4
L1	0	1	5	6
L2	3	0	2	7
L3	4	3	0	5
L4	2	5	3	0

Example

A travelling salesman has to tour 5 cities. The travelling cost in thousand of rupees from one city to another is given in the table below. Which sequence to travelling the cities will minimize the cost?

CITY	A	B	C	D	E
A	-	3	6	8	2
B	7	-	4	9	3
C	9	8	-	5	8
D	3	5	7	-	6
E	2	4	3	9	-

Step 1- Row Reduction

CITY	A	B	C	D	E
A	-	1	4	6	0
B	4	-	1	6	0
C	4	3	-	0	3
D	0	2	4	-	3
E	0	2	1	7	-

Step 2- Column Reduction

CITY	A	B	C	D	E
A	-	0	3	6	0
B	4	-	0	6	0
C	4	2	-	0	3
D	0	1	3	-	3
E	0	1	0	7	-

CITY	A	B	C	D	E
A	-	$\boxed{0}$	3	6	~~0~~
B	4	-	~~0~~	6	$\boxed{0}$
C	4	2	-	$\boxed{0}$	3
D	$\boxed{0}$	1	3	-	3
E	~~0~~	1	$\boxed{0}$	7	-

A->B, B->E, E->C, C->D, D->A
= 3 + 3+3+5+3= 17 thousand is the minimum cost for the salesman to travel all cities exactly once and come back to the starting city.

Example

A Travelling salesman has to visit five cities. He wishes to start from a particular city, visit each city once and then return to his starting point. The travelling cost (in Rs.) of each city from a particular city is given below. What should be the sequence of the salesman's visit, so that the cost is minimum?

$$
\begin{array}{ccc}
 & & \textbf{To city} \\
 & & \begin{array}{ccccc} \mathbf{A} & \mathbf{B} & \mathbf{C} & \mathbf{D} & \mathbf{E} \end{array} \\
\textbf{From city} & \begin{array}{c} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \\ \mathbf{D} \\ \mathbf{E} \end{array} & \begin{pmatrix} \alpha & 2 & 5 & 7 & 1 \\ 6 & \alpha & 3 & 8 & 2 \\ 8 & 7 & \alpha & 4 & 7 \\ 12 & 4 & 6 & \alpha & 5 \\ 1 & 3 & 2 & 8 & \alpha \end{pmatrix}
\end{array}
$$

Solution: The problem is solved as an assignment problem using Hungarian method; an optimal solution is reached as shown in Table.
After performing row reduction and column reduction.

$$
\textbf{From city}\quad
\begin{array}{c}
\\ \\ \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \\ \mathbf{D} \\ \mathbf{E}
\end{array}
\begin{array}{c}
\textbf{To city} \\
\begin{array}{ccccc}
\mathbf{A} & \mathbf{B} & \mathbf{C} & \mathbf{D} & \mathbf{E}
\end{array} \\
\begin{pmatrix}
\alpha & 1 & 3 & 6 & \boxed{0} \\
4 & \alpha & \boxed{0} & 6 & \cancel{0} \\
4 & 3 & \alpha & \boxed{0} & 3 \\
8 & \boxed{0} & 1 & \alpha & 1 \\
\boxed{0} & 2 & \cancel{0} & 7 & \alpha
\end{pmatrix}
\end{array}
$$

In this assignment, it means that the travelling salesman will start from city A, then go to city E and return to city A without visiting the other cities. The cycle is not complete. To overcome this situation, the next highest element can be assigned to start with. In this case it is 1, and there are three 1’s. Therefore, consider all these 1’s one by one and find the route which completes the cycle.

***Case 1*:** Make the assignment for the cell (A, B) which has the value 1. Now, make the assignments for zeros in the usual manner. The resulting assignments are shown in table.

Resulting Assignment

$$
\begin{array}{cc}
 & \textbf{To city} \\
 & \begin{array}{ccccc} \textbf{A} & \textbf{B} & \textbf{C} & \textbf{D} & \textbf{E} \end{array} \\
\textbf{From city} \quad \begin{array}{c} \textbf{A} \\ \textbf{B} \\ \textbf{C} \\ \textbf{D} \\ \textbf{E} \end{array} & \begin{pmatrix} \alpha & \boxed{1} & 3 & 6 & \cancel{0} \\ 4 & \alpha & \boxed{0} & 6 & \cancel{0} \\ 4 & 3 & \alpha & \boxed{0} & 3 \\ 8 & \cancel{0} & 1 & \alpha & \boxed{1} \\ \boxed{0} & 2 & \cancel{0} & 7 & \alpha \end{pmatrix}
\end{array}
$$

The assignment shown in Table gives the route sequence

A → B, B → C, C → D, D →E and E → A.

The travelling cost to this solution is

= 2000 + 3000 + 4000 + 5000 + 1000

= Rs.15,000.00

Problem

You work as a manager for a chip manufacturer, and you currently have 3 people on the road meeting clients. Your salespeople are in Jaipur, Pune and Bangalore, and you want them to fly to three other cities: Delhi, Mumbai and Kerala. The table below shows the cost of airline tickets in INR between the cities:

	Delhi	Kerala	Mumbai
Jaipur	2500	4000	3500
Pune	4000	6000	3500
Bangalore	2000	4000	2500

Game Theory & Decision Making

Module 4

Syllabus

Rules of Game Theory, Two person zero sum game, solving simple games (2x2 games), solving simple games (3x3 games) Decision making under certainty, under uncertainty,Maximax Criterion,Maximin Criterion, Savage Minimax Regret criterion,Laplace criterion of equal Likelihoods, Hurwicz criterion of Realism

Game Theory

A **game** is a generic term, involving conflict situations of particular sort.

Game Theory is a set of tools and techniques for decisions under uncertainty involving two or more intelligent opponents in which each opponent aspires to optimize his own decision at the expense of the other opponents. In game theory, an opponent is referred to as player. Each player has a number of choices, finite or infinite, called **strategies**. The **outcomes** or **payoffs** of a game are summarized as functions of the different strategies for each player.

Game Theory

Competitive Game- A competitive situation will be called as “Competitive Game” if it has 6 properties.

1. There are finite number of participants i.e n>=2. If n=2 then it is called a 2-person game. If n>2 then it is called a n- person game.
2. Each participant has finite number of possible course of action.
3. Each participant must know all the courses of action available to others but must not know which will be chosen.
4. A play of the game is said to occur when each player choose one course of action. The choice are made simultaneously so that no participant knows the choice of the other participant until he has decided his own.
5. After all participants have chosen a course of action, the games are finite.
6. The game of the participants depends on the participant’s action as well as of the other participants.

Terminologies

Player - A competitor in the game is known as a Player.

Strategy - A set of alternative course of action available to the player based on knowledge is known as strategy.

Pure Strategy - If the player selects the same strategy each time and neglects the remaining strategy, then it is called as Pure Strategy

Mixed Strategy - If the player selects his course of action in accordance with some fixed probability.

Optimum Strategy - A course of action which put the player in most preferred position.

Terminologies

Saddle point

A saddle point is an element of the matrix that is both the smallest element in its row and the largest element in its column. Furthermore, saddle point is also regarded as an equilibrium point in the theory of games.

Pay-off

The outcome of playing the game is called pay-off.

Pay-off Matrix

It is a table showing the outcomes or payoffs of different strategies of the game.

Value of the Game

It refers to the expected outcome per play, when players follow their optimal strategy. It is generally denoted by V.

2- Person Zero Sum Game

A game with two players where the gain of one player is equal to the loss of other players is known as the 2-person zero sum game.

They are also called rectangular game because the payoff matrix is in rectangular form.

Solutions of Game Theory

1. Saddle Point
2. Dominance rule
3. Arithmetic method
4. Matrix method

Saddle Point

In a payoff matrix of a 2-person zero sum game always look for a saddle point.

If maximum of minimum of rows in a payoff matrix is equal to minimum of maximum of columns , then the game is said to have a saddle point.

E.g

Player B

Player A

	I	II	III	IV	V
I	1	3	2	7	4
II	3	4	1	5	6
III	6	5	7	6	5
IV	2	0	6	3	1

Maximum of minimum of rows= minimum of maximum of columns. Hence Saddle point exist
Strategy of Player A = { 0,0,5,0} , Strategy of Player B = {0,5,0,0,0} and Value of the game = 5

Example

Player B

Player A

	I	II	III	IV	V
I	-2	0	0	5	3
II	3	2	1	2	2
III	-4	-3	0	-2	6
IV	5	3	-4	2	-6

Problem

Find Solution of game theory problem using saddle point.

a.

Player A \ Player B	B1	B2	B3	B4
A1	20	15	12	35
A2	25	14	8	10
A3	40	2	10	5
A4	-5	4	11	0

b.

Company B

$$
\text{Company A}\quad
\begin{array}{c c}
 & \begin{array}{ccc} \mathbf{1} & \mathbf{2} & \mathbf{3} \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 2 & 4 & 2 \\ 1 & -5 & -4 \\ 2 & 6 & -2 \end{pmatrix}
\end{array}
$$

Dominance Rule

1. If smaller row is dominated delete it.
2. If greater column is dominated delete it.
3. We may use average of two or more rows/ two or more columns to delete.

e.g

Player B

Player A

	I	II	III	IV	V
I	1	3	2	7	4
II	3	4	1	5	6
III	6	5	7	6	5
IV	2	0	6	3	1

Since row IV is dominated by row III delete row IV.

Player B

Player A

	I	II	III	IV	V
I	1	3	2	7	4
II	3	4	1	5	6
III	6	5	7	6	5

Since column IV is dominated by column II delete column IV.

Player B

Player A

	I	II	III	V
I	1	3	2	4
II	3	4	1	6
III	6	5	7	5

Since column V is dominated by column II delete column V.

Player B

Player A

	I	II	III
I	1	3	2
II	3	4	1
III	6	5	7

Since row I is dominated by row III delete row I.

Player B

Player A

	I	II	III
II	3	4	1
III	6	5	7

Since row II is dominated by row III delete row II.

	Player B			
		I	II	III
Player A	III	6	5	7

Since column I & II is dominated by column II delete column I & III.

	Player B	
Player A		II
	III	5

Strategy of Player A = { 0,0,5,0} ,
Strategy of Player B = {0,5,0,0,0}
and Value of the game = 5

Problem

Find the solution of the game theory problem using dominance rule.

Player B

Player A

	I	II	III	IV	V
I	3	5	4	9	6
II	5	6	3	7	8
III	8	7	9	8	7
IV	4	4	8	5	3

Arithmetic Method

1. Use dominance rule and reduce the Matrix to 2x2.
2.

Player B

Player A		B1	B2
	A1	a11	a12
	A2	a21	a22

3. Probability that the Player A uses strategy A1 is

   $x1 = (a22 - a21)/(a11+a22 - (a12 + a21))$ and A uses strategy A2 is x2, $x2 = 1 - x1$
4. Probability that the Player B uses strategy B1 is

   $y1 = (a22 - a12)/(a11+a22 - (a12 + a21))$ and B uses strategy B2 is y2, $y2 = 1 - y1$
5. Value of the game $V = (a11 \times a22 - a12 \times a21)/(a11+a22 - (a12 + a21))$

Example

Find the optimum strategy for Player A & B and value of the game where payoff matrix is given as follows

Player B

Player A

	B1	B2	B3
A1	2	6	1
A2	8	4	6
A3	1	2	1

Solution : First check for saddle point

Since max of min of row ≠ min of max of col

Therefore saddle point does not exist

Using dominance rule.

Player B

Player A

	B1	B2	B3
A1	2	6	1
A2	8	4	6
A3	1	2	1

Since row A3 is dominated by row A2 , delete row A3

Player B

Player A

	B1	B2	B3
A1	2	6	1
A2	8	4	6

Since col B1 is dominated by col B3 , delete col B1

Player B

Player A

	B2	B3
A1	6	1
A2	4	6

Probability that the Player A uses strategy A1 is

x1 =(a22 - a21)/(a11+a22 - (a12 +a21)) and A uses strategy A2 is x2, x2=1-x1

x1 = (6- 4)/(6 + 6 - (1+ 4) =2/7 x2 = 1- 2/7 = 5/7

Probability that the Player B uses strategy B1 is

y1 =(a22 - a12)/(a11+a22 - (a12 +a21)) and B uses strategy B2 is y2, y2=1-y1

y1 = (6 - 1)/(6+6 -(1+4)) = 5/7 y2 = 1- 5/7 =2/7

Value of the game

V= (a11 x a22 - a12 x a21)/(a11+a22 - (a12 +a21))

V= (6 x 6 - 4 x1)/ (6 + 6 - (1+ 4) = 32/7

Player A’s optimal strategy { 2/7, 5/7, 0}

Player B’s optimal strategy { 0 , 5/7, 2/7}

Value of the game = 32/7

Problem

Two companies are competing for business under the condition that one company's gain is another company's loss. The payoff matrix is as follows

Company B

Company A

	B1	B2	B3
A1	10	5	-2
A2	13	12	15
A3	16	14	10

Find the value of the game.

Matrix Method

1. Find row differences(r1-r2) & (r2-r3)
2. Find column differences(c1-c2) & (c2-c3)
3. Find oddments of row differences and column differences
4. If sum of oddments of player A = sum of oddments of player B then the game can be solved.
5. Player A’s optimal strategy = oddments of A / sum of oddments of A

   Player B’s optimal strategy = oddments of B / sum of oddments of B

6. Value of the game = any column x oddments of A / sum of oddments of A OR

   Value of the game = any row x oddments of B / sum of oddments of B

Example

Find the optimum strategy for Player A & B and value of the game where payoff matrix is given as follows

Player B

Player A

	B1	B2	B3
A1	7	3	1
A2	1	7	3
A3	0	1	7

Solution: Check for saddle point.Since max of min of row ≠ min of max of col
Therefore saddle point does not exist

Using dominance rule

Cannot be solved using dominance rule since there is no dominant row or column.

Using Arithmetic method

Since dominance rule cannot be used to reduce the matrix to 2x2 , hence arithmetic method cannot be used.

Using Matrix method

	B1	B2	B3	c1- c2	c2-c3	Oddments of A
A1	7	3	1	4	2	40
A2	1	7	3	-6	4	22
A3	0	1	7	-1	-6	28
r1-r2	6	-4	-2			
r2-r3	1	6	-4			
Oddments of B	28	22	40			

sum of oddments of player A = sum of oddments of player B

Player A’s optimal strategy = oddments of A / sum of oddments of A

A1 = 40/90   A2 = 22/90   A3= 28/90   {40/90,22/90,28/90}

Player B’s optimal strategy = oddments of B / sum of oddments of B

B1 = 28/90   B2 = 22/90   B3= 40/90   {28/90,22/90,40/90}

Value of the game = any row x oddments of B / sum of oddments of B

V= (0 x 28 + 1x 22 + 7 x 40)/90 = 302/90

Problem

Find the optimum strategy for Player A & B and value of the game where payoff matrix is given as follows

Player B

Player A

	B1	B2	B3
A1	3	-2	4
A2	-1	4	2
A3	2	2	6

Decision Theory

Decision theory is primarily concerned with helping people and organizations in making decisions. It provides a meaningful conceptual framework for important decision making. The decision making refers to the selection of an act from amongst various alternatives, the one which is judged to be the best under given circumstances.

A decision is simply a selection from two or more courses of action. Decision making may be defined as - " a process of best selection from a set of alternative courses of action, that course of action which is supposed to meet objectives upto satisfaction of the decision maker." The knowledge of statistical techniques helps to select the best action.

The decision maker: The decision maker refers to individual or a group of individual responsible for making the choice of an appropriate course of action amongst the available courses of action.

Acts (or courses of action): Decision making problems deals with the selection of a single act from a set of alternative acts.

Events (or States of nature): The events identify the occurrences, which are outside of the decision maker' s control and which determine the level of success for a given act. These events are often called ' States of nature' or outcomes.

Decision Theory

Let the acts or action be $a_1, a_2, a_3, \ldots$ then the totality of all these actions is known as action space denoted by A. For three actions a_1, a_2 a_3; A = action space = (a_1, a_2, a_3) or A = (A_1, A_2, A_3).

A set of states of nature may be represented in any one of the following ways:

$S = \{S_1, S_2, \ldots, S_n\}$ or $E = \{E_1, E_2, \ldots, E_n\}$

Pay-off: The result of combinations of an act with each of the states of nature is the outcome and monetary gain or loss of each such outcome is the pay-off. This means that the expression pay-off should be in quantitative form.

In general, if there are k alternatives and n states of nature, there will be $k \times n$ outcomes or pay-offs. These k × n payoffs can be very conveniently represented in the form of a $k \times n$ pay -off table.

States of nature	Decision alternative			
	A_1	A_2	....	A_k
E_1	a_{11}	a_{12}	....	a_{1k}
E_2	a_{21}	a_{22}	....	a_{2k}
.	.	.	....	.
.	.	.	....	.
	.	.	....	.
E_n	a_{n1}	a_{n2}	....	a_{nk}

Decision Theory

Types of decision making: Decisions are made based upon the information data available about the occurrence of events as well as the decision situation. There are two types of decision making situations: certainty and uncertainty-Y-*

- **Decision making under certainty**
- **Decision making under uncertainty**
- **Decision making under risk**
- **Decision making under ignorance**

Decision making under certainty

In this case the decision maker has the complete knowledge of consequence of every decision choice with certainty. In this decision model, assumed certainty means that only one possible state of nature exists. The outcome of decision alternative is known. The complete certainty about future is there hence best decision can be taken easily.

In these circumstances, it is possible to foresee (if not control) the facts and the results

Considering a simple example, every farmer has knowledge of the time periods for growing crops. Therefore, they make definite decisions within the relevant time frame. That is, they have prior knowledge of the decision they make.

Decision making under certainty

For example; a farmer wants to decide which crop should plant among three crops, on his 100-acre farm.
The payoff from each is dependent on the rainfall during the growing seasons. The farmer has categorized the amount of rainfall. It was high rainfall, medium rainfall and low rainfall. his estimated profit for each is given in the table below.

crops	Estimated conditional profits ($) , under rainfall		
	High	Medium	Low
Crop A	8000	4500	2000
Crop B	3500	4500	5000
Crop C	5000	5000	4000

Considering the above information, the farmer decided which crop should be planted under the amount of rainfall as high, medium and low.

- When the rainfall is high, the farmer decided to plant crop A because Crop A profit is higher than others under the amount of rainfall as high.
- When the rainfall is medium, the farmer decided to plant crop C because crop C is higher than others under the amount of rainfall as the medium.
- When the rainfall is low, the farmer decided to plant crop B because crop B profit is 5000. It is higher than others.

According to this situation decision maker has complete knowledge about outcome therefore he could be able to take an effective decision with maximum payoff.

Decision making under uncertainty

Under conditions of uncertainty, only pay-offs are known and nothing is known about the lilkelihood of each state of nature. Such situations arise when a new product is introduced in the market or a new plant is set up. The number of different decision criteria available under the condition of uncertainty is given below.

- Maximin criteria
- Maximax criteria
- Hurwicz alpha criteria
- Minimax regret criteria
- Laplace criteria and maximization of expected value

Decision making under uncertainty

There is a newspaper boy and he is thinking of selling, a special one-time edition of a sports magazine to his regular newspaper customers. Based on his knowledge of his customer, he believes that he can sell between 9 to 12 copies.

The magazines can be purchased at 8$ each and can be sold for 12$ each. Magazines that are not sold can be returned to the publisher for a refund of 50%.

If he sells a magazine, he gets four rupees as a profit. If he is unable to sell a magazine he can return it to the publisher but he loss 50% purchasing price. It means he is losing four dollars.

He thinks that the demand can be either 9 or 10 or 11 or 12 copies. So, in other terms basically, we have four states of nature that are N1 =9, N2=10, N3=11, N4 =12. In order to this demand which is varying between 9 to 12, the newspaper boy has four options or four strategies that he can adopt that is either he can buy 9 magazines 10 or 11 or 12. Strategies are shown as S1 =9, S2=10, S3=11, S4 =12. Pay off metric is given below regarding the above information.

Strategy (stock)	Events (demand)				Minimum strategy(m)	Maximum strategy(M)
	N_1=9	N_2=10	N_3=11	N_4=12		
S_1=9	36	36	36	36	36	36
S_2=10	32	40	40	32	32	40
S_3=11	28	36	44	28	28	44
S_4=12	24	32	48	24	24	48

Payoff Matrix

Maximin Criterion

The maximin criteria is called the criterion of pessimism. This implies that the worst possible outcomes for each action. The decision-maker should choose the best of the worst by selecting minimum payoff considering the above metrics.

The working method is:

(i) Determine the lowest outcome for each alternative.

(ii) Choose the alternative associated with the maximum of these.

Strategy (stock)	Events (demand)				Minimum strategy(m)	Maximum strategy(M)
	$N_1=9$	$N_2=10$	$N_3=11$	$N_4=12$		
$S_1=9$	36	36	36	36	36	36
$S_2=10$	32	40	40	32	32	40
$S_3=11$	28	36	44	28	28	44
$S_4=12$	24	32	48	24	24	48

Considering the above table, 36 is the maximin value, so choose strategy S1. It means that he is selling 9 newspaper. This action maximizes the minimum payoff. This decision is well-suited to firms whose very survival is at stake because of losses.

Maximax Criterion

The Maximax criteria is known as the criteria of optimism. These criteria are well-suited to those who are extreme risk-takers. This also implies that the 'best of the best' decision.

This criterion is the decision to take the course of action which maximizes the maximum possible pay-off. Since this decision criterion locates the alternative strategy that has the greatest possible gain. The working method is:

1. Determine the highest outcome for each alternative.
2. Choose the alternative associated with the maximum of these.

Strategy (stock)	Events (demand)				Minimum strategy(m)	Maximum strategy(M)
	$N_1=9$	$N_2=10$	$N_3=11$	$N_4=12$		
$S_1=9$	36	36	36	36	36	36
$S_2=10$	32	40	40	32	32	40
$S_3=11$	28	36	44	28	28	44
$S_4=12$	24	32	48	24	24	48

In our example best of the best decision is strategy S4. because the maximum value is 48. It means that Selling 12 newspapers gives maximum payoff.

Savage Minimax Regret criterion

These criteria suggest that the decision-maker should attempt to minimize his maximum regret. Regret represents an opportunity lost so for each of the demands.
If we did not choose the best strategy then how much would be the regret or opportunity lost for all the other strategies. for that particular event or demand that is the regret. so once we calculate all the regrets what these criteria say is to find the maximum of the regrets.
So, find the maximum of the regrets for each of the strategies and then find the strategy which has the minimum of all the maximum regrets so then the second is to find the minimum amongst those regrets.

Strategy (stock)	Events				Maximum regrets	Minimax regret
	N_1	N_2	N_3	N_4		
$S_1=9$	0	4	8	12	12	
$S_2=10$	4	0	4	8	8	8
$S_3=11$	8	4	0	4	8	8
$S_4=12$	12	8	4	0	12	

Basically, we want to minimize the regret and find the one which is the minimum because that then will be the best strategy.
So, minimax so out of these 8 is the minimum and it is the regret for both S2 and S3. So, the optimal strategy as per the Maximax regrets criteria is that we can choose S2 or S3.

Hurwicz criterion of Realism

Hurwicz's Criterion, or the realism criterion is a technique used to make decisions under uncertainty. The setting is for a decision make to be faced to uncertain states of nature and a number of decision alternatives that can be chosen. The decision made and the final state of nature (which the decision maker does not know beforehand) determines the payoff. Under this the decision maker calculates a weighted average between the best and worst possible payoff for each decision alternative (among all possible states of nature, for that specific alternative), and then she chooses the decision that has the maximum weighted average. So, this method is considering that things will go somewhere in the middle between going well and badly. In these criteria, we have to find out the expected monetary value for each of the strategies.

In this example alpha= 0.4 and the calculating formula is,

Strategy (stock)	Events				Minimum strategy(m)	Maximum strategy(M)	EMV
	N_1	N_2	N_3	N_4			
S_1=9	36	36	36	36	36	36	36
S_2=10	32	40	40	32	32	40	35.2
S_3=11	28	36	44	28	28	44	34.8
S_4=12	24	32	48	24	24	48	33.6

EMV=α * maximum payoff(M) + (1- α) * minimum payoff(m)

After Calculating, EMV (expected monetary value) can be stated as follows.Based on the EMV, strategy S1 has the highest value of EMV. So, our optimal strategy will be to choose S1 that is by nine magazines.

Laplace criterion of equal Likelihoods

Laplace criteria suggest that if we do not know of any reason for one event to occur more than the other, we should assume that all events have an equal chance of occurrence. In this criterion, we assign the same probability to each of the events. That is we assume that each event is equal- probable.

Here there are four events and each has to assume an equal probability of occurrence the probability of each event to occur is one -fourth.So, each strategy should calculate a monetary value which is nothing but the addition of the products of the probability with the payoffs so for s1 the EMV will be one -fourth multiplied by 36 plus.

Strategy (stock)	Events				EMV
	N_1	N_2	N_3	N_4	
S_1=9	36	36	36	36	36
S_2=10	32	40	40	32	36
S_3=11	28	36	44	28	34
S_4=12	24	32	48	24	32

Maximum EMV is obtained by strategies S1 and S2 so based on Laplace criteria again decision could be either choose S1 or S2.

Problems

1. The research department of Hindustan Ltd. has recommended to pay marketing department to launch a shampoo of three different types. The marketing types of shampoo to be launched under the following estimated pay-offs for various level of sales. What will be the marketing manager's decision if (i) Maximin and (ii) Maximax principle (iii) minimax regret criterion applied?

Types of shampoo	Estimated Sales (in Units)		
	15000	10000	5000
Egg shampoo	30	10	10
Clinic Shampoo	40	15	5
Deluxe Shampoo	55	20	3

2. Following payoff matrix, which is the optimal decision under each of the following rule (i) maxmin (ii) maximax

Act	States of nature			
	S_1	S_2	S_3	S_4
A_1	14	9	10	5
A_2	11	10	8	7
A_3	9	10	10	11
A_4	8	10	11	13

Queuing Models

Module 5

Syllabus

Queuing Models: Essential features of queuing systems, operating characteristics of queuing system, probability distribution in queuing systems, M/M/1 : N/FCFS.

Queuing Theory

Queuing theory is the mathematical study of the congestion and delays of waiting in line.

Queuing theory examines every component of waiting in line to be served, including the arrival process, service process, number of servers, number of system places, and the number of customers—which might be people, data packets, cars, etc.

As a branch of operations research, queuing theory can help users make informed business decisions on how to build efficient and cost-effective workflow systems. Real-life applications of queuing theory cover a wide range of applications, such as how to provide faster customer service, improve traffic flow, efficiently ship orders from a warehouse, and design of telecommunications systems, from data networks to call centers.

Introduction

Waiting line problems arise either because

(*i*) there is too much demand on the facilities so that we say that there is an excess of waiting time or inadequate number of service facilities.

(*ii*) there is too less demand, in which case there is too much idle facility time or too many facilities.

In either case, the problem is to either *schedule arrivals* or *provide proper number of facilities or both* so as to obtain an optimum balance between the costs associated with waiting time and idle time.

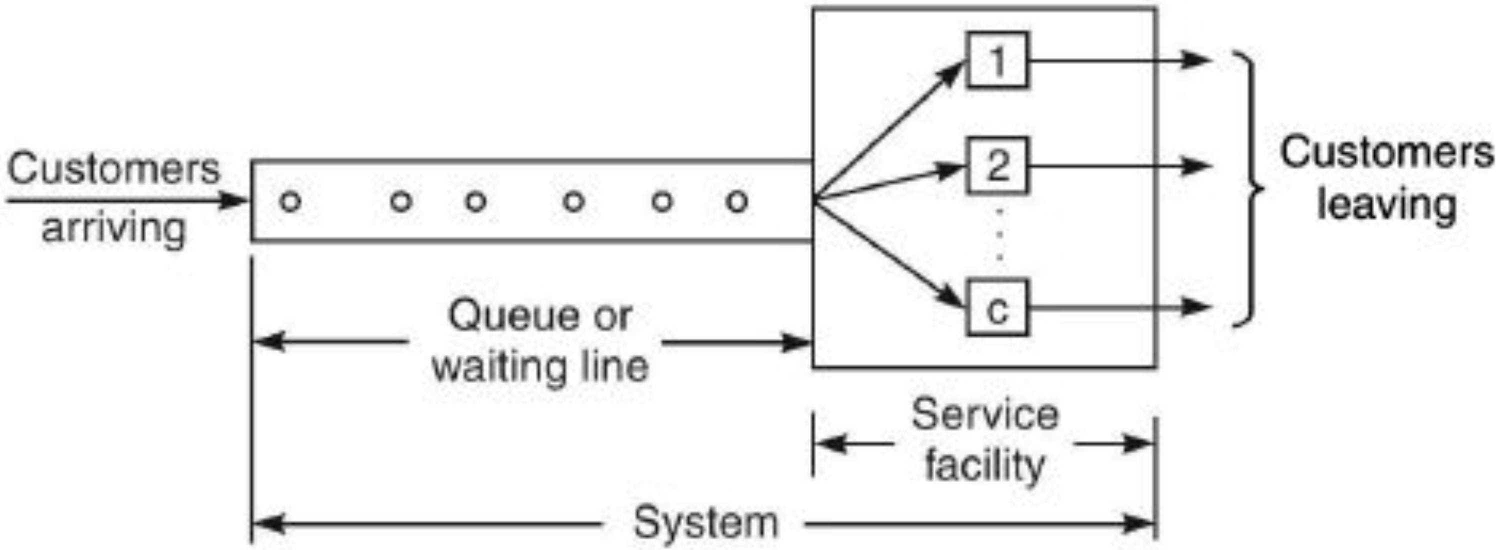
Contd..

Units arrive, at regular or irregular intervals of time, at a given point called the service centre. All these units are called *entries* or *arrivals of customers*.

One or more *service channels* or *service stations* or *service facilities* (ticket windows,

are assembled at the service centre.

Queuing System



Contd..

1. **Customer:** The arriving unit that requires some service to be performed. As already described, the customers may be persons, machines, vehicles, parts, etc.
2. **Queue (Waiting line):** The number of customers waiting to be serviced. The queue does not include the customer(s) currently being serviced.
3. **Service Channel:** The process or facility which is performing the services to the customer. This may be single or multi-channel. The number of service channels is denoted by the symbol c.

Elements of queuing system

A queuing system is specified completely by seven main elements:

1. Input or arrival (inter-arrival) distribution
2. Output or departure (service) distribution
3. Service channels
4. Service discipline
5. Maximum number of customers allowed in the system
6. Calling source or population
7. Customer's behaviour.

Customer’s Behaviour

- **Balking** - customers deciding not to join the **queue** if it is too long)
- **Reneging** - customers leave the **queue** if they have waited too long for service
- **Jockeying** -customers switch between queues if they think they will get served faster by so doing

Operating characteristics

Analysis of a queuing system involves a study of its different operating characteristics. Some of them are

1. *Queue length* (L_q) – the average number of customers in the queue waiting to get service. This excludes the customer(*s*) being served.
2. *System length* (L_s) – the average number of customers in the system including those waiting as well as those being served.
3. *Waiting time in the queue* (W_q) – the average time for which a customer has to wait in the queue to get service.
4. *Total time in the system* (W_s) – the average total time spent by a customer in the system from the moment he arrives till he leaves the system. It is taken to be the waiting time plus the service time.
5. *Utilization factor* (ρ) – it is the proportion of time a server actually spends with the customers. It is also called *traffic intensity*.

Transient and Steady State of the system

- **Transient State:**
  - the operating characteristics vary with time
  - Early stages of operation-transient state

- **Steady State**
  - Behaviour becomes independent of its initial conditions and of the elapsed time
  - Long-run behaviour

- **Avg arrival rate < Avg service rate** and constant then eventually settles to **steady state**
- If the rates are **not constant**, system will not reach a steady state and could remain **unstable**
- **Avg arrival rate > Avg service rate**, system cannot attain a steady state (queue length increases steadily and may reach to infinity……**explosive state of system**

Arrival time

The Poisson distribution involves the probability of occurrence of an arrival. Poisson assumption is quite restrictive in some cases. It assumes that arrivals are random and independent of all other operating conditions. The mean arrival rate (*i.e.*, the number of arrivals per unit of time) λ is assumed to be constant over time and is independent of the number of units already serviced, queue length or any other random property of the queue.

$$f(x) = P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} \text{ for x} = 0,1,2\ldots.\infty$$

Since the mean arrival rate is constant over time, it follows that the probability of an arrival between time t and $t + dt$ is $\lambda \, . \, dt$.

Thus probability of an arrival in time $dt = \lambda \, dt$. ...(10.1)

The following characteristics of Poisson distribution are written here without proof :

$$\text{Probability of } n \text{ arrivals in time } t = \frac{(\lambda t)^n \, . \, e^{-\lambda t^*}}{n\,!}, \; n = 0, 1, 2, \ldots.., \qquad \ldots(10.2)$$

Probability density function of inter-arrival time (time interval between two consecutive arrivals)

$$= \lambda \, . \, e^{-\lambda t}. \qquad \ldots(10.3)$$

Finally, Poisson distribution assumes that the time period dt is very small so that $(dt)^2$, $(dt)^3$, etc. $\rightarrow 0$ and can be ignored.

Service time

Service time is the time required for completion of a service *i.e.*, it is the time interval between beginning of a service and its completion. The mean service rate is the number of customers served per unit of time (assuming the service to be continuous throughout the entire time unit), while the average service time $1/\mu$ is the time required to serve one customer. The most common type of distribution used for service times is exponential distribution. It involves the probability of completion of a service. It should be noted that Poisson distribution cannot be applied to servicing because of the possibility of the service facility remaining idle for some time. Poisson distribution assumes fixed time interval of continuous servicing, which can never be assured in all services.

Exponential Distribution Formula

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$f(x;\lambda)$ = probability density function

λ = rate parameter

x = random variable

Mean service rate μ is also assumed to be constant over time and independent of number of units already serviced, queue length or any other random property of the system. Thus probability that a service is completed between t and $t + dt$, provided that the service is continuous

$$= \mu dt.$$

Under the condition of continuous service, the following characteristics of exponential distribution are written, without proof :

$$\text{Probability of } n \text{ complete services in time } t = \frac{(\mu t)^n \, . \, e^{-\mu t}}{n\,!}. \qquad \text{...(10.4)}$$

Probability density function (*p.d.f*) of interservice time, *i.e.*, time between two consecutive services = $\mu \, . \, e^{-\mu t}$. ...(10.5)

Probability that a customer shall be serviced in more than time $t = e^{-\mu t}$. ...(10.6)

Contd…

The symbols *e* and *f* represent a finite (N) or infinite (∞) number of customers in the system and calling source respectively. For instance, $(M/E_k/1) : (FCFS/N/\infty)$ represents Poisson arrival (exponential interarrival), Erlangian departure, single server, 'first come, first served' discipline, maximum allowable customers N in the system and infinite population model.

Notations:

n = number of customers in the system (waiting line + service facility) at time t.

λ = mean arrival rate (number of arrivals per unit of time).

μ = mean service rate per busy server (number of customers served per unit of time).

L_q = expected (average) number of customers in the queue.

L_s = expected number of customers in the system (waiting + being served).

W_q = expected waiting time per customer in the queue (expected time a customer keeps waiting in line).

W_s = expected time a customer spends in the system. (in waiting + being served)

L_n = expected number of customers waiting in line excluding those *times when the line is empty i.e.,* expected number in *non-empty queue* (expected number of customers in a *queue that is formed from time to time)*.

W_n = expected time a customer waits in line if *he has to wait at all i.e.,* expected time in the queue for *non-empty queue*.

p_n = steady state probability of exactly n customers in the system.

Model I. Single-Channel Poisson Arrivals with Exponential Service, Infinite Population Model [(M/M/I) : (FCFS/∞/∞)]

1. *Expected number of units in the system (waiting + being served)*, L_s is obtained by using the definition of an expected value:

$$E(x) = \sum_{i=0}^{i=\infty} x_i \, p_i$$

$$\therefore \qquad L_s = \sum_{n=0}^{n=\infty} np_n$$

$$\therefore \qquad L_s = \left(1 - \frac{\lambda}{\mu}\right)\left[\frac{\lambda/\mu}{(1-\lambda/\mu)^2}\right] = \frac{\lambda/\mu}{1-\lambda/\mu} = \frac{\lambda}{\mu-\lambda}. \qquad \text{...(10.14)}$$

2. *Expected number of units in the queue*, L_q = Expected number of units in the system – Expected number in service (single server).

$$\therefore \qquad L_q = L_s - \frac{\lambda}{\mu} = \frac{\lambda}{\mu-\lambda} - \frac{\lambda}{\mu} = \lambda\left[\frac{\mu-\mu+\lambda}{\mu(\mu-\lambda)}\right] = \frac{\lambda}{\mu}\cdot\frac{\lambda}{\mu-\lambda}. \qquad \text{...(10.15)}$$

Note that the expected number in service is 1 times the probability that the service channel is busy *i.e.*, $1 \cdot \frac{\lambda}{\mu}$.

3. *Expected time per unit in the system (expected time a unit spends in the system)*,

$$W_s = \frac{\text{Expected number of units in the system}}{\text{Arrival rate}} = \frac{L_s}{\lambda} = \frac{\lambda}{(\mu - \lambda) \, . \, \lambda} = \frac{1}{\mu - \lambda} . \quad ...($$

4. *Expected waiting time per unit in the queue, W_q* = Expected time in system – time in service.

$$\therefore \qquad W_q = W_s - \frac{1}{\mu} = \frac{1}{\mu - \lambda} - \frac{1}{\mu} = \frac{\lambda}{\mu} . \frac{1}{\mu - \lambda} . \qquad ...(10.17)$$

∴ Variance of queue length

$$= \frac{\lambda / \mu\,(1 + \lambda / \mu)}{(1 - \lambda / \mu)^2} - \frac{\dfrac{\lambda^2}{\mu^2}}{\left(1 - \dfrac{\lambda}{\mu}\right)^2} = \frac{\dfrac{\lambda}{\mu}}{\left(1 - \dfrac{\lambda}{\mu}\right)^2}. \qquad \text{...(10.18)}$$

6. *Average length of non-empty queue (length of queue that is formed from time to time),* L_n : For a non-empty queue, the number of units in the system should be at least 2 (one in service and the other in the queue). Probability of a non-empty queue

$$= \sum_{n=0}^{\infty} p_n - (p_0 + p_1) = 1 - \left(p_0 + \frac{\lambda}{\mu}\, p_0\right) = 1 - p_0 \left(1 + \frac{\lambda}{\mu}\right)$$

$$= 1 - \left(1 - \frac{\lambda}{\mu}\right)\left(1 + \frac{\lambda}{\mu}\right) = \left(\frac{\lambda}{\mu}\right)^2.$$

average length of non-empty queue,

$$L_n = \frac{\text{Average length of queue}}{\text{Probability of non-empty queue}} = \frac{\frac{\lambda}{\mu}\cdot\frac{\lambda}{\mu-\lambda}}{\left(\frac{\lambda}{\mu}\right)^2} = \frac{\mu}{\mu-\lambda}.$$

7. *Average waiting time in non-empty queue (average waiting time of an arrival who waits), or expected waiting time per busy period,*

$$W_n = \frac{1}{\mu-\lambda}. \qquad \text{...(10.20)}$$

8. *Probability of queue being greater than or equal to k,* $= \left(\frac{\lambda}{\mu}\right)^k.$

9. *Probability of queue being greater than k,*

$$p\,(>k) = \left(\frac{\lambda}{\mu}\right)^{k+1}.$$

10. *Probability that the queue is non-empty,*

$$p(n>1) = 1 - p_0 - p_1 = 1 - \left(1 - \frac{\lambda}{\mu}\right) - \frac{\lambda}{\mu}\left(1 - \frac{\lambda}{\mu}\right) = \left(\frac{\lambda}{\mu}\right)^2.$$

11. *Probability density function of waiting time (excluding service) distribution*

$$= \begin{cases} \dfrac{\lambda}{\mu}(\mu - \lambda)\,.\,e^{-(\mu-\lambda)t} & , \quad t > 0 \\[2ex] \dfrac{\lambda}{\mu}(\mu - \lambda) & , \quad t = 0. \end{cases}$$

12. *Probability density function of (waiting + service) time distribution*

$$= (\mu - \lambda)\,.\,e^{-(\mu-\lambda)t}.$$

Ex:1

A self-service store employs one cashier at its counter. Nine customers arrive on an average every 5 minutes while the cashier can serve 10 customers in 5 minutes. Assuming Poisson distribution for arrival rate and exponential distribution for service time, find

1. *Average number of customers in the system.*
2. *Average number of customers in the queue or average queue length.*
3. *Average time a customer spends in the system.*
4. *Average time a customer waits before being served.*

Arrival rate $\lambda = 9/5 = 1.8$ customers/minute,

service rate $\mu = 10/5 = 2$ customers/minute.

1. Average number of customers in the system,

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{1.8}{2 - 1.8} = 9.$$

2. Average number of customers in the queue,

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{\lambda}{\mu} \cdot \frac{\lambda}{(\mu - \lambda)} = \frac{1.8}{2} \times \frac{1.8}{2 - 1.8} = 8.1.$$

3. Average time a customer spends in the system,

$$W_s = \frac{1}{\mu - \lambda} = \frac{1}{2 - 1.8} = 5 \text{ minutes.}$$

4. Average time a costumer waits in the queue,

$$W_q = \frac{\lambda}{\mu}\left(\frac{1}{\mu - \lambda}\right) = \frac{1.8}{2}\left(\frac{1}{2 - 1.8}\right) = 4.5 \text{ minutes.}$$

Ex 2:

A person repairing radios finds that the time spent on the radio sets has exponential distribution with mean 20 minutes. If the radios are repaired in the order in which they come in and their arrival is approximately Poisson with an average rate of 15 for 8-hour day, what is the repairman's expected idle time each day ? How many jobs are ahead of the average set just brought in ?

Solution

$$\text{Arrival rate } \lambda = \frac{15}{8 \times 60} = \frac{1}{32} \text{ units/minute,}$$

$$\text{service rate } \mu = \frac{1}{20} \text{ units/minute.}$$

Number of jobs ahead of the set brought in = Average number of jobs in the system,

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{1/32}{1/20 - 1/32} = \frac{5}{3}.$$

Number of hours for which the repairman remains busy in an 8-hour day

$$= 8\ \frac{\lambda}{\mu} = 8 \times \frac{1/32}{1/20} = 8 \times \frac{20}{32} = 5 \text{ hours.}$$

$\therefore$ Time for which repairman remains idle in an 8-hour day

$$= 8 - 5 = 3 \text{ hours.}$$

Model (M/M/1:FCFS/N/∞) Finite length Model

This model differs from model I in that the maximum number of customers in the system is limited to N and hence the difference equations of model I are valid only so long as $n < N$.

Thus in this model, $\lambda_n = \lambda,\ \mu_n = \mu$ for $n < N$,

$\lambda_n = 0,\ \mu_n = \mu$ for $n \geq N$,

because when the number of customers in the system becomes N, no new arrivals can be accommodated.

Characteristics:

1. Average number of customers in the system,

$$L_s = \sum_{n=0}^{N} n p_n = \sum_{n=0}^{N} n \cdot \left(\frac{1-\rho}{1-\rho^{N+1}}\right) \rho^n$$

$$= \frac{1-\rho}{1-\rho^{N+1}} \cdot \sum_{n=0}^{N} n\rho^n$$

$$= \frac{1-\rho}{1-\rho^{N+1}} \cdot [0 + \rho + 2\rho^2 + 3\rho^3 + \ldots + N\,\rho^N].$$

$$L_s = \frac{\rho\,[1-(1+N)\,\rho^N + N\,\rho^{N+1}]}{(1-\rho)\,(1-\rho^{N+1})}.$$

2. Average number of customers in the queue,

$$
\begin{aligned}
L_q &= \sum_{n=1}^{N} (n-1)\, p_n = \sum_{n=1}^{N} n p_n - \sum_{n=1}^{N} p_n \\
&= \sum_{n=1}^{N} (0 + n p_n) - \left[\sum_{n=0}^{N} p_n - p_0 \right] \\
&= \sum_{n=0}^{N} n p_n - \sum_{n=0}^{N} p_n + p_0 = L_s - 1 + p_0 \\
&= \frac{\rho\,[1 - (1+N)\rho^{N} + N\rho^{N+1}]}{(1-\rho)\,(1-\rho^{N+1})} - 1 + \frac{1-\rho}{1-\rho^{N+1}} \\
&= \frac{\{\rho - (1+N)\rho^{N+1} + N\rho^{N+2}\} - \{1 - \rho - \rho^{N+1} + \rho^{N+2}\} + \{1 - 2\rho + \rho^{2}\}}{(1-\rho)\,(1-\rho^{N+1})} \\
&= \frac{\rho^{2} - N\rho^{N+1} + (N-1)\rho^{N+2}}{(1-\rho)\,(1-\rho^{N+1})} \\
&= \frac{1 - N\rho^{N-1} + (N-1)\rho^{N}}{(1-\rho)\,(1-\rho^{N+1})} \cdot \rho^{2}.
\end{aligned}
$$

3. Average time a customer spends in the system,

$$W_s = \frac{L_s}{\lambda'}, \qquad \text{where } \lambda' = \lambda\,(1 - p_N).$$

4. Average waiting time in the queue,

$$W_q = \frac{L_q}{\lambda'}.$$

Ex 1:

Trains arrive at the yard every 15 minutes and the service time is 33 minutes. If the line capacity of the yard is limited to 4 trains, find

(i) the probability that the yard is empty,

(ii) the average number of trains in the system.

Solution

$$\lambda = 1/15 \text{ per minute,}$$
$$\mu = 1/33 \text{ per minute.}$$
$$\therefore \qquad \rho = \lambda/\mu = 33/15 = 2.2,$$
$$N = 4.$$

(*i*)
$$p_0 = \frac{1-\rho}{1-\rho^{N+1}} = \frac{\rho - 1}{\rho^{N+1} - 1} = \frac{2.2 - 1}{2.2^5 - 1} = \frac{1.2}{51.5 - 1} = 0.0237.$$

(*ii*) Average number of trains in the system,

$$L_s = \sum_{n=0}^{N} np_n = 0 + p_1 + 2p_2 + 3p_3 + 4p_4$$
$$= p_0(\rho + 2\rho^2 + 3\rho^3 + 4\rho^4) \qquad (\because\ p_n = \rho^n \cdot p_0.)$$
$$= 0.0237[2.2 + 2 \times 2.2^2 + 3 \times 2.2^3 + 4 \times 2.2^4]$$
$$= 0.0237[2.2 + 9.68 + 31.94 + 93.70] = 3.26.$$

Ex 2:

If for a period of 2 hours in a day (8 A.M. to 10 A.M.) trains arrive at the yard every 20 minutes but the service time is 36 minutes, calculate for this period

(a) the probability that the yard is empty,

(b) the average number of trains at the yard.

Line capacity of the yard is limited to 4 trains only.

Solution

Here, $\lambda = \frac{60}{20} = 3$ trains/hour, $\mu = \frac{60}{36} = \frac{5}{3}$ trains/hour.

$$\therefore \qquad \rho = \frac{\lambda}{\mu} = \frac{3 \times 3}{5} = 1.8.$$

$$(a) \qquad p_0 = \frac{1-\rho}{1-\rho^{N+1}} = \frac{\rho - 1}{\rho^{N+1} - 1} = \frac{1.8 - 1}{1.8^5 - 1} = \frac{0.8}{18.9 - 1} = 0.045.$$

$$\therefore \qquad L_s = p_0 \sum_{n=0}^{4} n \cdot p_n = p_0 [0 + \rho + 2\rho^2 + 3\rho^3 + 4\rho^4] \; (\because p_n = \rho^n . p_0 .)$$

$$= 0.045 \times 1.8 [1 + 2 \times 1.8 + 3 \times 1.8^2 + 4 \times 1.8^3]$$

$$= 0.081 \; [1 + 3.6 + 9.72 + 23.328] = 3.05 \text{ trains.}$$

Ex 3:

At a railway station only one train is handled at a time. The railway yard is sufficient only for two trains to wait while the other is given signal to leave the station. Trains arrive at the station at an average rate of 6 per hour and the railway station can handle them on an average of 12 per hour. Assuming Poisson arrivals and exponential service distribution, find the steady state probabilities for the various number of trains in the system. Also find the average number of trains at the railway station and the average waiting time of a new train coming into the yard.

Solution

Here, $\lambda = 6$ trains/hour, $\mu = 12$ trains/hour, $\rho = \dfrac{\lambda}{\mu} = \dfrac{6}{12} = 0.5.$

Since the maximum queue length is 2, the maximum number of trains in the system is 3 (= N).

Now, probability that there is no train in the system, $p_0 = \dfrac{1-\rho}{1-\rho^{N+1}}$.

$$\therefore \qquad p_0 = \frac{1-0.5}{1-(0.5)^{3+1}} = \frac{0.5}{1-(0.5)^4} = 0.53.$$

Since $p_n = p_0 \,.\, \rho^n$, $p_1 = 0.53 \times (.5)^1 = 0.265$, $p_2 = 0.53\ (.5)^2 = 0.13$, $p_3 = 0.53\ (.5)^3 = 0.066$.

$\therefore$ Average number of trains in the system,

$$L_s = 1p_1 + 2p_2 + 3p_3$$

or

$$L_s = 1 \times 0.265 + 2 \times 0.13 + 3 \times 0.066 = 0.723.$$

To find W_q

$$L_q = \frac{1 - N\rho^{N-1} + (N-1)\rho^N}{(1-\rho)(1-\rho^{N+1})} \,.\, \rho^2 = \frac{1 - 3 \times 0.5^2 + 2 \times 0.5^3}{(1-0.5)[1-(0.5)^4]} \times (0.5)^2$$

$$= \frac{1 - 0.75 + 0.25}{0.5 \times 0.9375} \times 0.25 = 0.267.$$

$$\lambda' = \lambda\ (1 - p_N) = 6\ (1 - p_3) = 6\ (1 - 0.066) = 5.604.$$

$\therefore$

$$W_q = \frac{L_q}{\lambda'} = \frac{0.267}{5.604} = 0.0476 \text{ hours} = 2.85 \text{ minutes.}$$

Simulation

Module6

Syllabus

Introduction to simulation, steps in simulation, advantages of simulation, limitations of simulation, applications of simulation, Monte-Carlo method: simple examples, single server queue model.

Introduction to Simulation

- A *simulation* is the imitation of the operation of real-world process or system over time.
  - Generation of artificial history and observation of that observation history
- A *model* construct a conceptual framework that describes a system
- The behavior of a system that evolves over time is studied by developing a simulation *model*.
- The model takes a set of expressed assumptions:
  - Mathematical, logical
  - Symbolic relationship between the *entities*

Goal of modeling and simulation

- A model can be used to investigate a wide verity of “what if” questions about real-world system.
  - Potential changes to the system can be simulated and predicate their impact on the system.
  - Find adequate parameters before implementation
- So simulation can be used as
  - Analysis tool for predicating the effect of changes
  - Design tool to predicate the performance of new system
- It is better to do simulation before Implementation.

How a model can be developed?

- Mathematical Methods
  - Probability theory, algebraic method ,…
  - Their results are accurate
  - They have a few Number of parameters
  - It is impossible for complex systems
- Numerical computer-based simulation
  - It is simple
  - It is useful for complex system

When Simulation Is the Appropriate Tool

- Simulation enable the study of internal interaction of a subsystem with complex system
- Informational, organizational and environmental changes can be simulated and find their effects
- A simulation model help us to gain knowledge about improvement of system
- Finding important input parameters with changing simulation inputs
- Simulation can be used with new design and policies before implementation
- Simulating different capabilities for a machine can help determine the requirement
- Simulation models designed for training make learning possible without the cost disruption
- A plan can be visualized with animated simulation
- The modern system (factory, wafer fabrication plant, service organization) is too complex that its internal interaction can be treated only by simulation

When Simulation Is Not Appropriate

- When the problem can be solved by common sense.
- When the problem can be solved analytically.
- If it is easier to perform direct experiments.
- If cost exceed savings.
- If resource or time are not available.
- If system behavior is too complex.
  - Like human behavior

Advantages and disadvantages of simulation

- In contrast to optimization models, simulation models are “run” rather than solved.
  - Given as a set of inputs and model characteristics the model is run and the simulated behavior is observed

Advantages of simulation

- New policies, operating procedures, information flows and son on can be explored without disrupting ongoing operation of the real system.
- New hardware designs, physical layouts, transportation systems and … can be tested without committing resources for their acquisition.
- Time can be compressed or expanded to allow for a speed-up or slow-down of the phenomenon(clock is self-control).
- Insight can be obtained about interaction of variables and important variables to the performance.
- Bottleneck analysis can be performed to discover where work in process, the system is delayed.
- A simulation study can help in understanding how the system operates.
- "What if" questions can be answered.

Disadvantages of simulation

- Model building requires special training.
  - Vendors of simulation software have been actively developing packages that contain models that only need input (templates).
- Simulation results can be difficult to interpret.
- Simulation modeling and analysis can be time consuming and expensive.
  - Many simulation software have output-analysis.

Areas of application

- Manufacturing Applications
- Semiconductor Manufacturing
- Construction Engineering and project management
- Military application
- Logistics, Supply chain and distribution application
- Transportation modes and Traffic
- Business Process Simulation
- Health Care
- Automated Material Handling System (AMHS)
  - Test beds for functional testing of control-system software
- Risk analysis
  - Insurance, portfolio,...
- Computer Simulation
  - CPU, Memory,…
- Network simulation
  - Internet backbone, LAN (Switch/Router), Wireless, PSTN (call center),...

Systems and System Environment

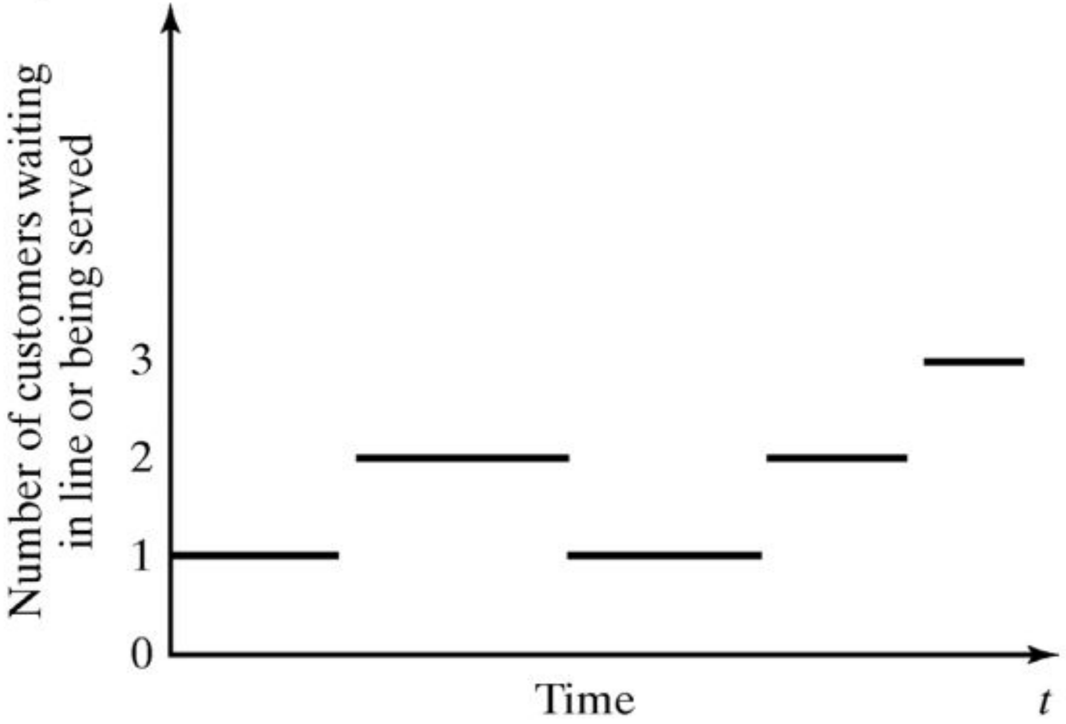
- A *system* is defined as a groups of objects that are joined together in some regular interaction toward the accomplishment of some purpose.
  - An automobile factory: Machines, components parts and workers operate jointly along assembly line
- A system is often affected by changes occurring outside the system: system environment.
  - Factory : Arrival orders
    - Effect of supply on demand : relationship between factory output and arrival (activity of system)
  - Banks : arrival of customers

Components of system

- Entity
  - An object of interest in the system : Machines in factory
- Attribute
  - The property of an entity : speed, capacity
- Activity
  - A time period of specified length :welding, stamping
- State
  - A collection of variables that describe the system in any time : status of machine (busy, idle, down,…)
- Event
  - A instantaneous occurrence that might change the state of the system: breakdown
- Endogenous
  - Activities and events occurring with the system
- Exogenous
  - Activities and events occurring with the environment

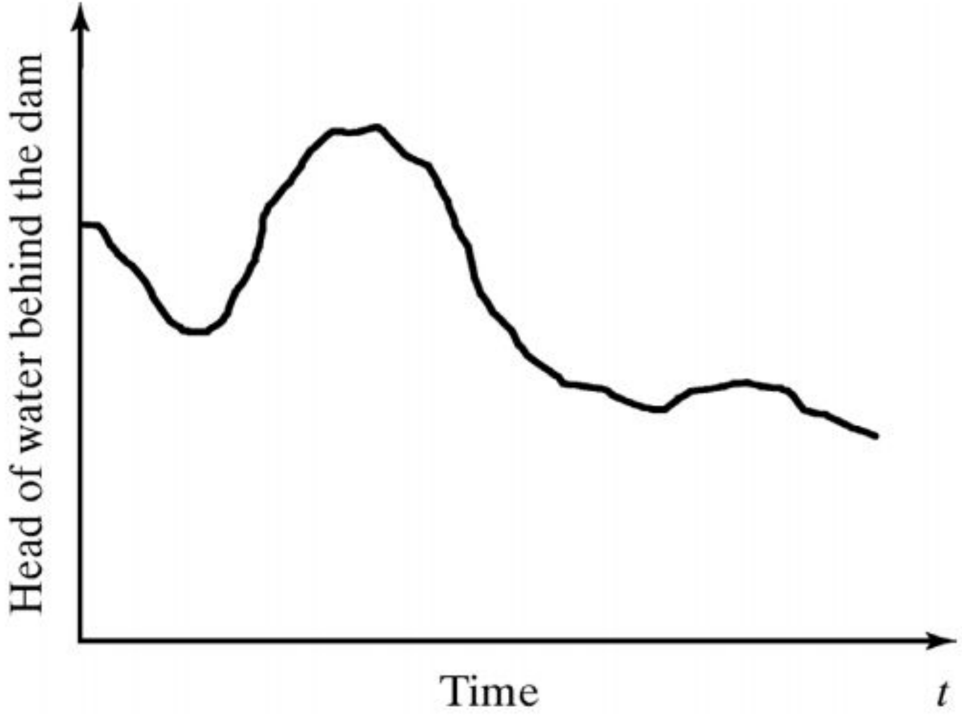
Discrete and Continues Systems

- A discrete system is one in which the state variables change only at a discrete set of points in time : Bank example



Discrete and Continues Systems (cont.)

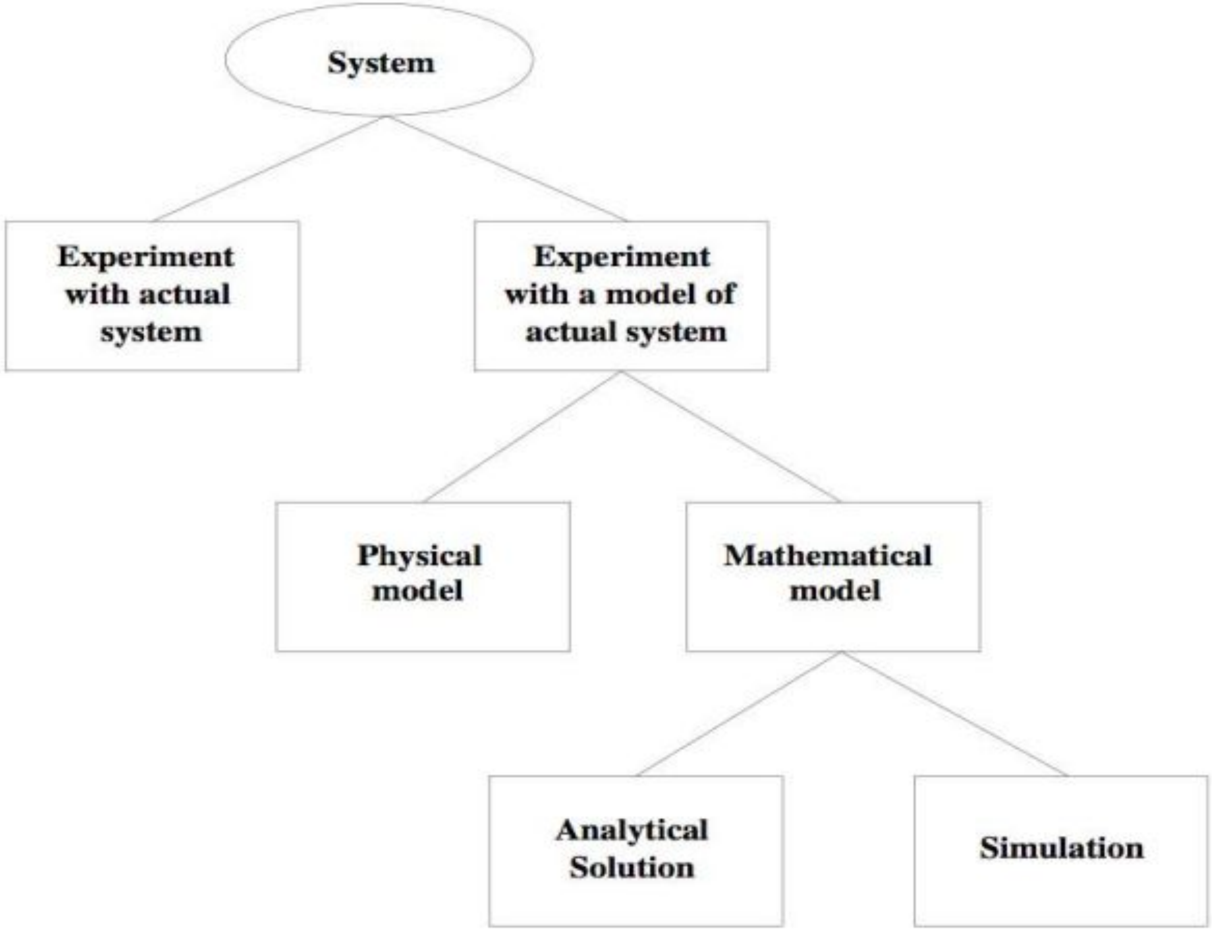
- A continues system is one in which the state variables change continuously over time: Head of water behind the dam



Model of a System

- To study the system
  - it is sometimes possible to experiments with system
    - This is not always possible (bank, factory,…)
    - A new system may not yet exist
- Model: construct a conceptual framework that describes a system
  - It is necessary to consider those accepts of systems that affect the problem under investigation (unnecessary details must remove)

Types of Models



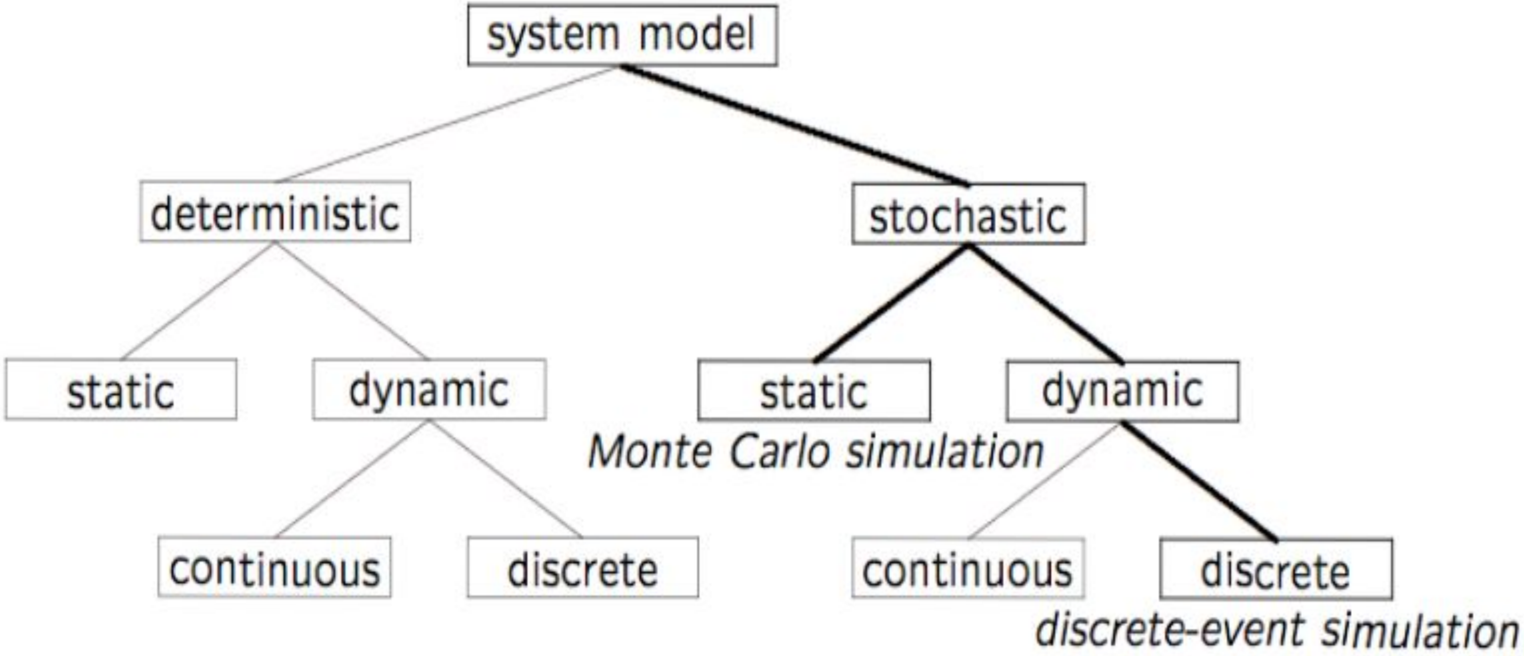
Characterizing a Simulation Model

- Deterministic or Stochastic
  - Does the model contain stochastic components?
  - Randomness is easy to add to a DES
- Static or Dynamic
  - Is time a significant variable?
- Continuous or Discrete
  - Does the system state evolve continuously or only at discrete points in time?
  - Continuous: classical mechanics
  - Discrete: queuing, inventory, machine shop models

Discrete-Event Simulation Model

- Stochastic: some state variables are random
- *Dynamic*: time evolution is important
- *Discrete-Event*: significant changes occur at discrete time instances

Model Taxonomy



DES Model Development

How to develop a model:

1) Determine the goals and objectives
2) Build a ***conceptual*** model
3) Convert into a ***specification*** model
4) Convert into a ***computational*** model
5) Verify
6) Validate

Typically an iterative process

Three Model Levels

- Conceptual
  - Very high level
  - How comprehensive should the model be?
  - What are the *state variables,* which are dynamic, and which are important?
- Specification
  - On paper
  - May involve equations, pseudocode, etc.
  - How will the model receive input?
- Computational
  - A computer program
  - General-purpose PL or simulation language?

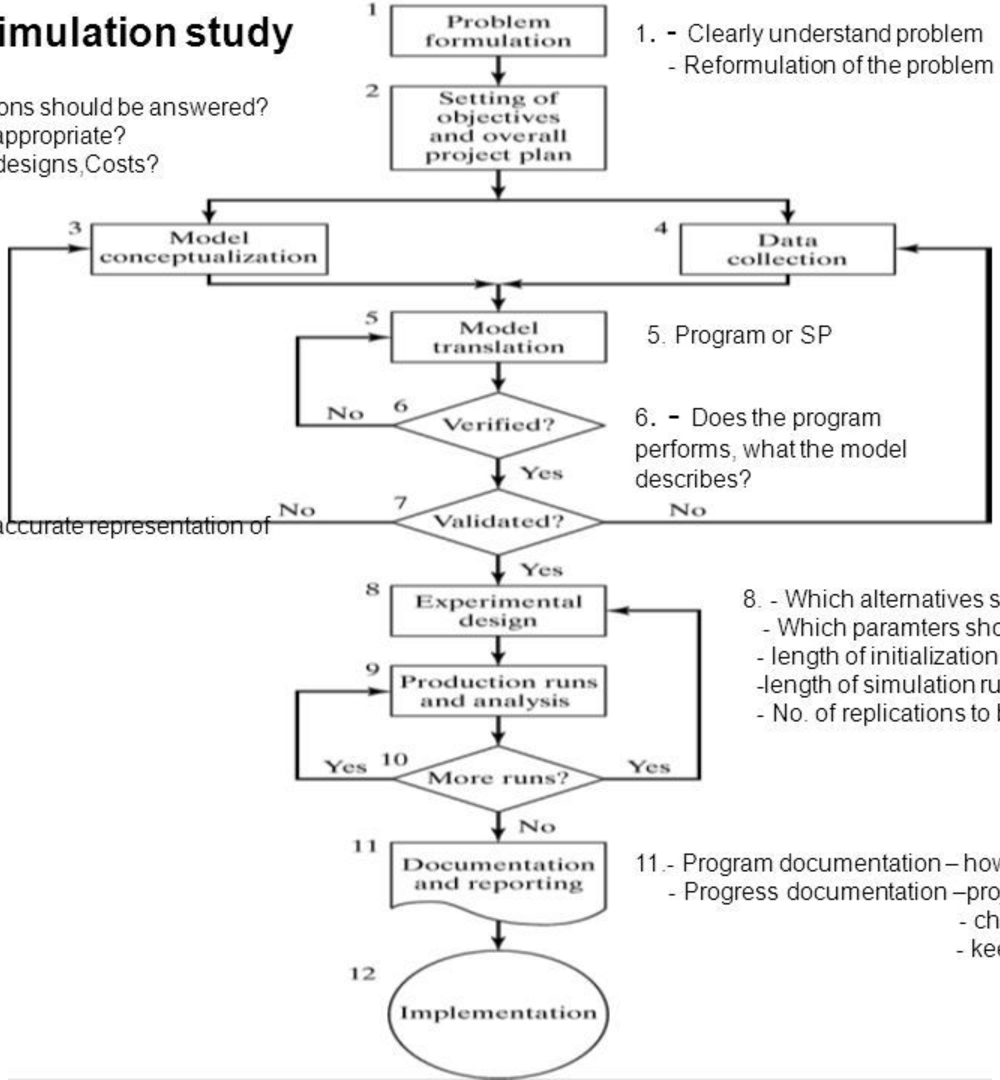
Verification vs. Validation

- *Verification*
  - Computational model should be consistent with specification model
  - Did we build the model right?
- *Validation*
  - Computational model should be consistent with the system being analyzed
  - Did we build the right model?
  - Can an expert distinguish simulation output from system output?
- Interactive graphics can prove valuable

Steps in simulation study

2. - Which questions should be answered?
 - Is simulation appropriate?
 - Alternate sys designs,Costs?

3. -complexity
 - involve Model user



1. - Clearly understand problem
 - Reformulation of the problem

5. Program or SP

6. - Does the program performs, what the model describes?

7. –Is the model accurate representation of real system

8. - Which alternatives should be run?
 - Which paramters should be varied?
 - length of initialization period
 -length of simulation run
 - No. of replications to be made of each run.

11.- Program documentation – how does the prog work
 - Progress documentation –project history
 - chronology of the work
 - keep project on course

Monte Carlo Simulation

https://www.youtube.com/watch?v=7ESK5SaP-bc

Simulation Examples

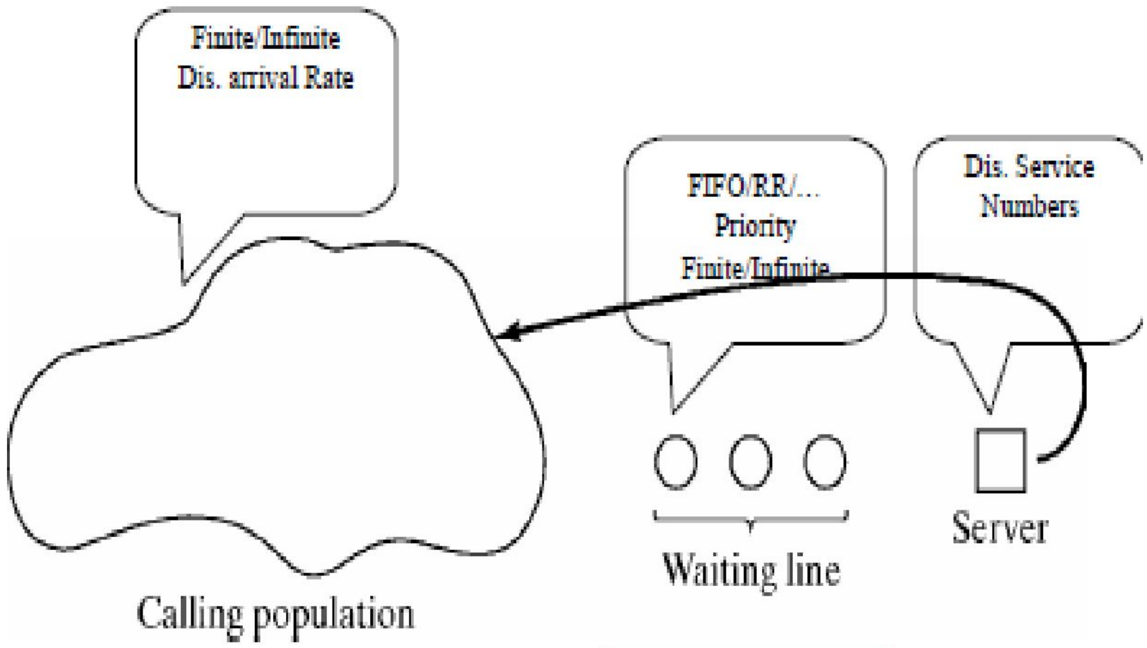
Simulation steps using Simulation Table

1. **Determine the characteristics of each of the inputs to the simulation (probability distributions).**
2. **Construct a simulation table (repetition 1).**
3. **For each repetition *i*, generate a value for the inputs, and evaluate function, calculating a value of response yi.**

Simulation Table

	Inputs				Response
Repetition	X1	x2	….	xp	yi
1					
2					
:					
n					

Simulation of Queuing System (Details in pre. unit)

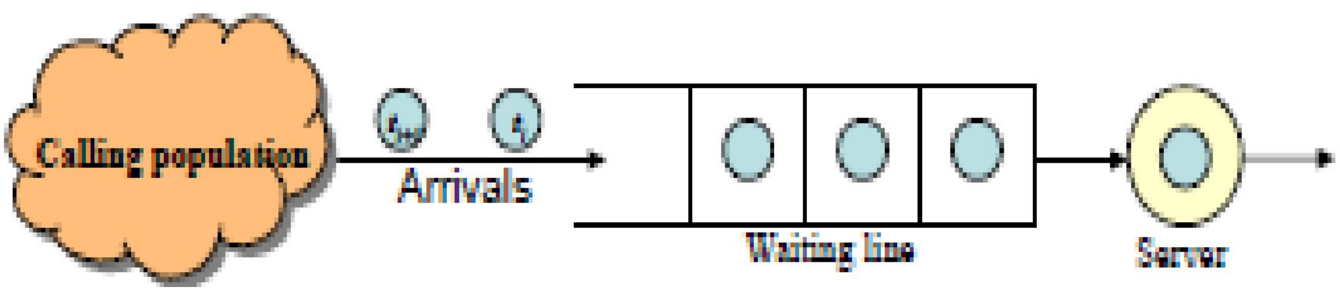


Attributes:

1. **Calling Population**
2. **Nature of arrivals**
3. **Service Mechanism**
4. **System capacity**
5. **Queuing Discipline**

- **Single server queue:**
  - **Calling population is infinite-Arrival rate does not change**
  - **Units are served according FIFO**
  - **Arrivals are defined by the distribution of the time between arrivals - inter-arrival time**
  - **Service times are according to distribution**
  - **Arrival rate must be less than service rate- stable system**

Otherwise waiting line will grow unbounded.



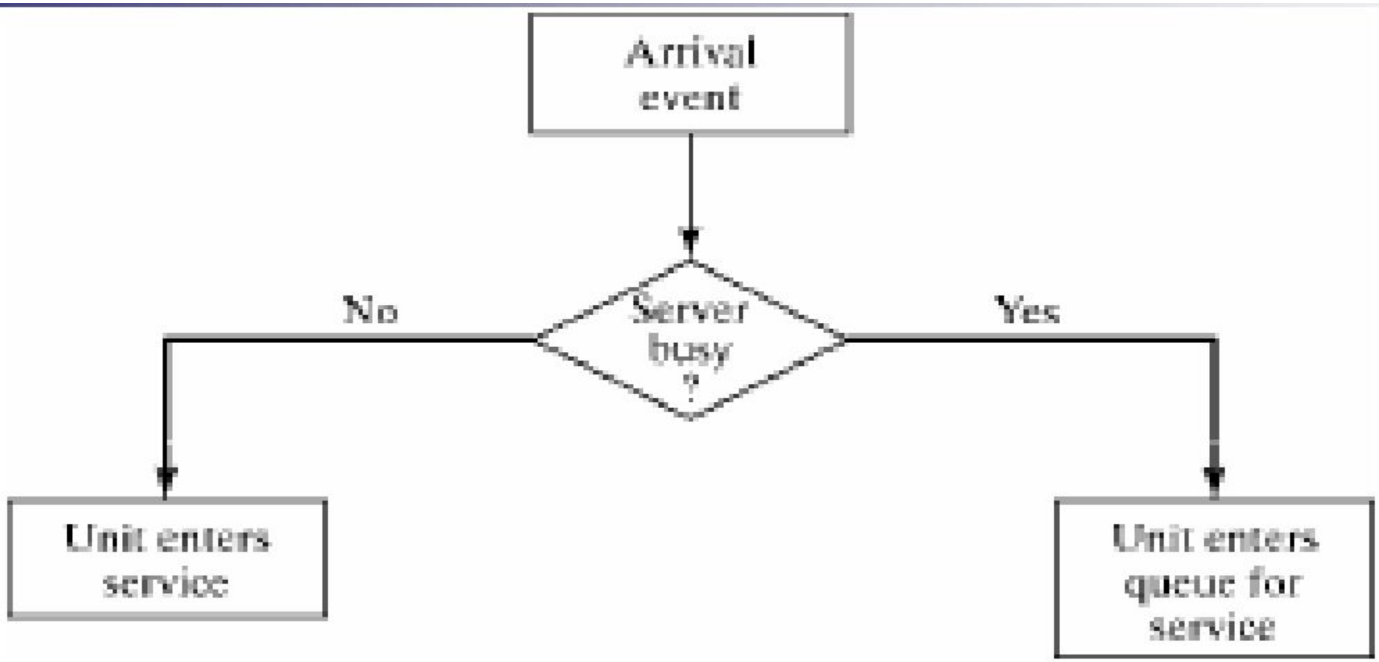
- **Queuing system :**

  - *System*
    - **Server**
    - **Units (in queue or being served)**
    - **Clock**

  - *State of the system*
    - **Number of units in the system**
    - **Status of server (idle, busy)**

  - *Events*
    - **Arrival of a unit**
    - **Departure of a unit**

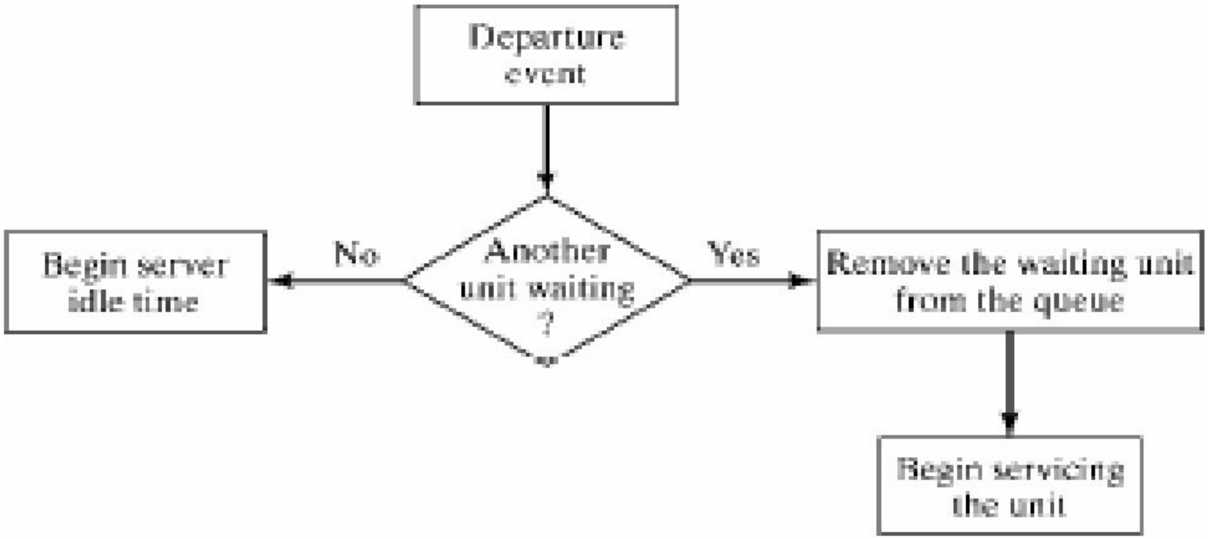
- **Arrival Event**
  - **If server is idle customer gets service, otherwise customer enters queue.**



		Queue status	
		Not empty	Empty
Server status	Busy	Enter queue	Enter queue
	Idle	Impossible	Enter service

Departure Event

If queue is not empty then server begin servicing next unit, otherwise server will be idle.



		Queue status	
		Not empty	Empty
Server status	Busy		Impossible
	Idle	Impossible	

Grocery Store Example(Ex 1)

• Producing Random Numbers from Random Digits

• Select randomly a number, e.g. 99219
- One digit: 0.9
- Two digits: 0.19
- Three digits: 0.219

• Proceed in a systematic direction,
e.g.
- first down then right
- first up then left

The interarrival and service times are taken from distributions!

Customer	Interarrival Time	Arrival Time on Clock	Service Time
1	-	0	2
2	2	2	1
3	4	6	3
4	1	7	2
5	2	9	1
6	6	15	4

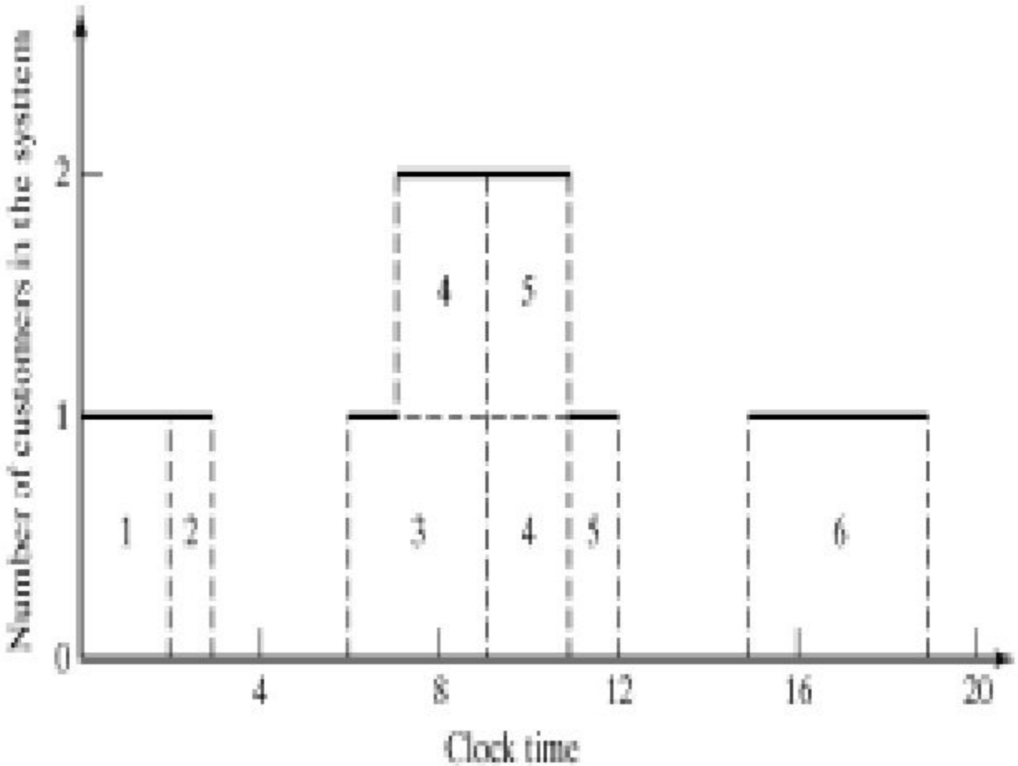
The simulation run is build by meshing clock, arrival and service times!

Customer Number	Arrival Time [Clock]	Time Service Begins [Clock]	Service Time [Duration]	Time Service Ends [Clock]
1	0	0	2	2
2	2	2	1	3
3	6	6	3	9
4	7	9	2	11
5	9	11	1	12
6	15	15	4	19

Chronological ordering of events

Clock Time	Customer Number	Event Type	Number of customers
0	1	Arrival	1
2	1	Departure	0
2	2	Arrival	1
3	2	Departure	0
6	3	Arrival	1
7	4	Arrival	2
9	3	Departure	1
9	5	Arrival	2
11	4	Departure	1
12	5	Departure	0
16	6	Arrival	1
19	6	Departure	0

Number of customers in the system



• Example1: A Grocery Store

- **Analysis of a small grocery store**
  - One checkout counter
  - Customers arrive at random times from {1,2,…,8} minutes
  - Service times vary from {1,2,…,6} minutes
  - Consider the system for 100 customers
- **Problems/Simplifications**
  - Sample size is too small to be able to draw reliable conclusions
  - Initial condition is not considered

Interarrival Time	Probability	Cumulative Probability
1	0.125	0.125
2	0.125	0.250
3	0.125	0.375
4	0.125	0.500
5	0.125	0.625
6	0.125	0.750
7	0.125	0.875
8	0.125	1.000

Service Time	Probability	Cumulative Probability
1	0.10	0.10
2	0.20	0.30
3	0.30	0.60
4	0.25	0.85
5	0.10	0.95
6	0.05	1.00

Customer	Interarrival Time [Minutes]	Arrival Time [Clock]	Service Time [Minutes]	Time Service Begins [Clock]	Time Service Ends [Clock]	Waiting Time in Queue [Minutes]	Time Customer in System [Minutes]	Idle Time of Server [Minutes]
1	-	0	4	0	4	0	4	0
2	1	1	2	4	6	3	5	0
3	1	2	5	6	11	4	9	0
4	6	8	4	11	15	3	7	0
5	3	11	1	15	16	4	5	0
6	7	18	5	18	23	0	5	2
				...				
100	5	415	2	416	418	1	3	0
Total	415		317			174	491	101

- **Average waiting time**

$$\bar{w} = \frac{\sum \text{Waiting time in queue}}{\text{Number of customers}} = \frac{174}{100} = 1.74\,\text{min}$$

- **Probability that a customer has to w**

$$p(\text{wait}) = \frac{\text{Number of customer who wait}}{\text{Number of customers}} = \frac{46}{100} = 0.46$$

- **Proportion of server idle time**

$$p(\text{idle server}) = \frac{\sum \text{Idle time of server}}{\text{Simulation run time}} = \frac{101}{418} = 0.24$$

- **Average service time**

$$\bar{s} = \frac{\sum \text{Service time}}{\text{Number of customers}} = \frac{317}{100} = 3.17\,\text{min}$$

$$E(s) = \sum_{s=0}^{\infty} s \cdot p(s) = 0.1 \cdot 10 + 0.2 \cdot 20 + \cdots + 0.05 \cdot 6 = 3.2\,\text{min}$$

- **Average time between arrivals**

$$\bar{\lambda} = \frac{\sum \text{Times between arrivals}}{\text{Number of arrivals - 1}} = \frac{415}{99} = 4.19 \text{ min}$$

$$E(\lambda) = \frac{a+b}{2} = \frac{1+8}{2} = 4.5 \text{ min}$$

- **Average waiting time of those who wait**

$$\bar{w}_{waited} = \frac{\sum \text{Waiting time in queue}}{\text{Number of customers that wait}} = \frac{174}{54} = 3.22 \text{ min}$$

- **Average time a customer spends in system**

$$\bar{t} = \frac{\sum \text{Time customers spend in system}}{\text{Number of customers}} = \frac{491}{100} = 4.91 \text{ min}$$

$$\bar{t} = \bar{w} + \bar{s} = 1.74 + 3.17 = 4.91 \text{ min}$$

Example 2

Simulation Problem

Problem : A dentist schedule her patients for 30 min appointments. Some take more or less time than 30 min depending the type of dental work. Following table shows Probabilities and the actual time taken to complete the job.

Category	Time required	No. of patients	Probability
Filling	45 min	40	0.40
Crown	60 min	15	0.15
Cleaning	15 min	15	0.15
Extracting	45 min	10	0.10
Check up	15 min	20	0.20

Simulate the doctor's clinic for 4 hrs.Find the average waiting time for patients and idle time for doctor. Assuming all patients show up at their scheduled time, arrival time starting at 8 am. Use following random numbers to solve the problem
40, 82, 11, 34, 25, 66, 17, 79

Simulating for 8 patients.
Cumulative distribution table.

Table 1

Category	Probability	Cumulative Probability	Random number
Filling	0.40	0.40	0- 39
Crown	0.15	0.55	40-54
Cleaning	0.15	0.70	55-69
Extracting	0.10	0.80	70-79
Check up	0.20	1.00	80-99

Setting random number intervals

Table 2

Patient	Arrival time	Random number	category	Service time needed
1	8:00	40	Crown	60
2	8:30	82	Checkup	15
3	9:00	11	Filling	45
4	9:30	34	Filling	45
5	10:00	25	Filling	45
6	10:30	66	Cleaning	15
7	11:00	17	Filling	45
8	11:30	79	Extracting	45

Simulation table

Table 3

Patient	Arrival time	Service start	service duration	Service ends	Waiting time	Idle time
1	8:00	8:00	60	9:00	0	0
2	8:30	9:00	15	9:15	30	0
3	9:00	9:15	45	10:00	15	0
4	9:30	10:00	45	10:45	30	0
5	10:00	10:45	45	11:30	45	0
6	10:30	11:30	15	11:45	60	0
7	11:00	11:45	45	12:30	45	0
8	11:30	12:30	45	1:15	60	0
Total					285	

Average waiting time = 285 /8 = 35.62 minutes

Idle time for doctor =0

Monte Carlo Simulation

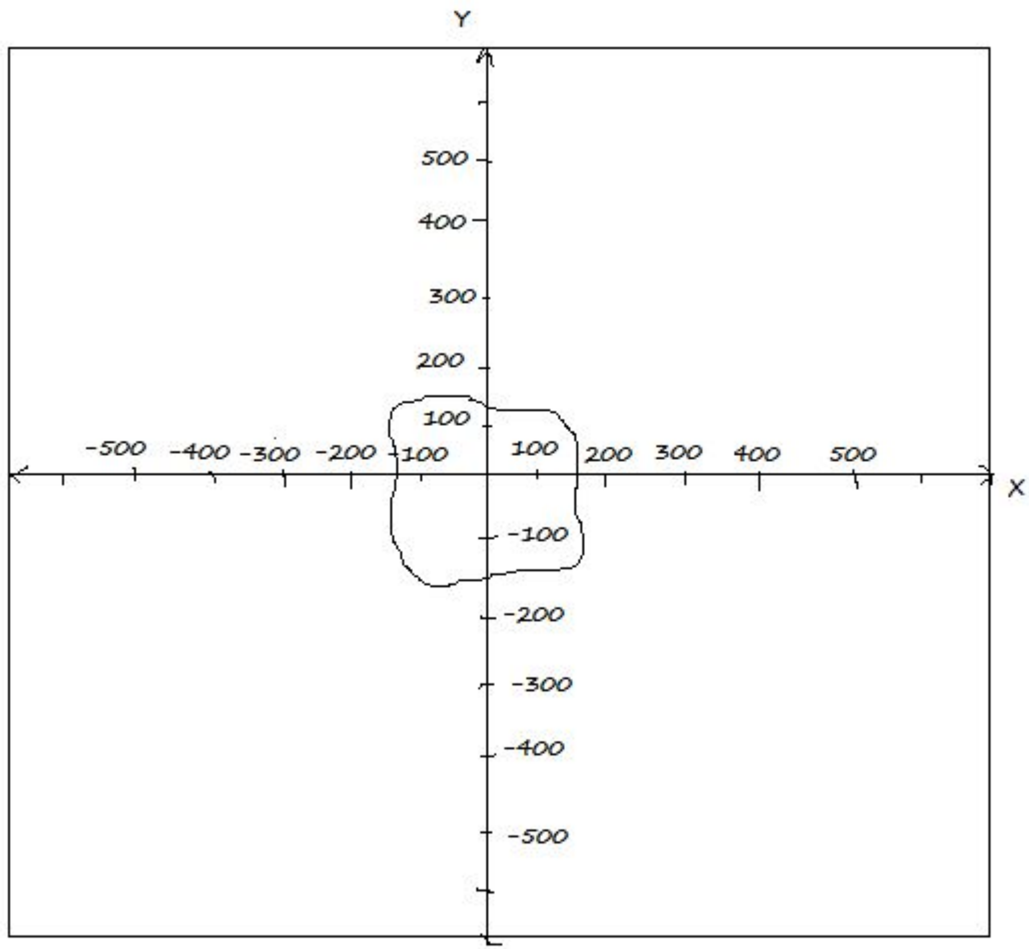
Monte Carlo simulation is a computerized mathematical technique that allows people to account for risk in quantitative analysis and decision making. The technique is used by professionals in such widely disparate fields as finance, project management, energy, manufacturing, engineering, research and development, insurance, oil & gas, transportation, and the environment.

Monte Carlo simulation furnishes the decision-maker with a range of possible outcomes and the probabilities they will occur for any choice of action. It shows the extreme possibilities—the outcomes of going for broke and for the most conservative decision—along with all possible consequences for middle-of-the-road decisions.

The technique was first used by scientists working on the atom bomb; it was named for Monte Carlo, the Monaco resort town renowned for its casinos. Since its introduction in World War II, Monte Carlo simulation has been used to model a variety of physical and conceptual systems.

Example :

A squad of bombers are attempting to destroy an ammunition depot as shown in figure. If the bomb lands anywhere on the depot it's hit otherwise bomb is a miss. The aiming point is the dot located at the heart of the ammunition depot. The point of impact is assumed to be normally distributed around the aiming point with a standard deviation of 600 mtrs in horizontal direction and 300 in vertical ditrection. The problem is to simulate the operation and make a statement about the number of bombs on the target.



Ammunition Depot

Random numbers for x- coordinate -0.84,1.03,0.92,-1.82,-0.16,-1.78,2.04 , 1.08,-1,50,-0.42
Random numbers for y- coordinate 0.66,-0.13,0.06,-1.40,0.23 , 1.33,0.69,-1.10,-0.72,-0.60

20th / 80

In this eg the aiming point is considered as (0,0) hence mean in horizontal direction & vertical direction is assumed to be zero.

$$z = \frac{x - \mu}{\sigma}$$

$$z = \frac{x - \mu}{\sigma}$$

$$z\sigma = x - \mu$$

$$z\sigma + \mu = x$$

$$x = z\sigma + \mu$$

$$\therefore \underline{\mu = 0}$$

$$\therefore \underset{y}{x} = z\underset{600}{\sigma}$$

$$\therefore y = z\sigma$$

σ = 600 m (horizontal) 300 m (vertical)

$$\therefore \cancel{x = z\,600}$$

$$\cancel{x = z300}$$

$$x = 600z$$

$$y = 300z$$

Bomber	Random digits (x)	X co-ordinate (600z)	Random digits (y)	Y co-ordinate (300z)	Result
1	-0.84	$=0.84 \times 600 =$ -504	0.66	$0.66 \times 300 =$ 198	miss.
2	1.03	$1.03 \times 600 =$ 618	-0.13	-39	miss
3	0.92	552	0.06	18	miss
4	-1.82	-1092	-1.40	-420	miss
5	-0.16	-96	0.23	69	Hit
6	-1.78	-1068	1.33	399	miss
7	2.04	1224	0.69	207	miss
8	1.08	648	-1.10	-330	miss
9	-1.50	-900	-0.72	-216	miss
10	-0.42	-252	-0.60	-180	hit.

~~So~~ The result is that 10 bombers had 2 hits & 8 misses. This is an eg of Monty Carlo or Static simulation because time is not a factor in the solution.

https://www.youtube.com/watch?v=7ESK5SaP-bc