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IPCV Assignment 6

1] 1-D Derivative Filters

Given,

- 1D discrete signal $f = (f_i)$

ⓐ Derive the linear system of equations using f_{i-1}, f_i, f_{i+1} & f_{i+2}

- It is mentioned that f is sampled from a smooth function $f(x)$ on a uniform grid with step size h

- For this problem, we need 4 coefficients & 4 equations, if we use less equations, the co-effs may not be unique & if we use more equations it may lead to higher consistency (because of higher order). WKT Higher order approximation does not give explicit access to the implicit function.

i.e
$$\boxed{f(x+h) = \sum_{k=0}^{n} \frac{h^k}{k!} \cdot f^{(k)}(x) + O(h^{n+1})} \quad \text{—} ①$$

using ①:

- $f_{i-1} = f_i - hf_i' + \frac{h^2}{2} f_i'' - \frac{h^3}{6} f_i''' + \frac{h^4}{24} f_i'''' - \frac{h^5}{120} f_i''''' + O(h^6)$

- $f_i = f_i$

- $f_{i+1} = f_i + hf_i' + \frac{h^2}{2} \cdot f_i'' + \frac{h^3}{6} \cdot f_i''' + \frac{h^4}{24} \cdot f_i'''' + \frac{h^5}{120} \cdot f_i''''' + O(h^6)$

- $f_{i+2} = f_i + 2h \cdot f_i' + \frac{2^2 h^2}{2} \cdot f_i'' + \frac{2^3 h^3}{6} \cdot f_i''' + \frac{2^4 h^4}{24} \cdot f_i'''' + \frac{2^5 h^5}{120} f_i''''' + O(h^6)$

(P.T.O)

$$0 \cdot f_i + 1 \cdot f_i' + 0 \cdot f_i'' + 0 \cdot f_i''' = \alpha_{-1} f_{i-1} + \alpha_0 f_i + \alpha_1 f_{i+1} + \alpha_2 f_{i+2} \quad (2)$$

The above equation can be expanded further to obtain a system of equations i.e (R.H.S)

$$= \alpha_{-1}\left(f_i - h f_i' + \frac{h^2}{2} f_i'' - \frac{h^3}{6} f_i''' + \frac{h^4}{24} f_i'''' - \frac{h^5}{120} f_i''''' + O(h^6)\right)$$

$$+ \alpha_0 (f_i) + \alpha_{+1}\left(f_i + h f_i' + \frac{h^2}{2} f_i'' + \frac{h^3}{6} f_i''' + \frac{h^4}{24} f_i'''' + \frac{h^5}{120} f_i'''''\right.$$

$$\left. + O(h^6)\right) + \alpha_2\left(f_i + 2h f_i' + 2h^2 f_i'' + \frac{4}{3} \cdot h^3 f_i''' + \right.$$

$$\left. \frac{2}{3} h^4 \cdot f_i'''' + \frac{4}{15} h^5 \cdot f_i''''' + O(h^6)\right)$$

- Rewriting (2) as:

$$0 \cdot f_i + 1 \cdot f_i' = (\alpha_{-1} + \alpha_0 + \alpha_1 + \alpha_2) f_i + h(-\alpha_{-1} + \alpha_1 + 2\alpha_2) f_i'$$

$$+ \frac{h^2}{2} (\alpha_{-1} + \alpha_1 + 4\alpha_2) f_i'' + \frac{h^3}{6} (-\alpha_{-1} + \alpha_1 + 8\alpha_2)$$

$$f_i''' + O(h^4)$$

$(\alpha_{-1} + \alpha_0 + \alpha_1 + \alpha_2)$

Separating terms for every Derivative f_i' to f_i''''

$$\alpha_{-1} + \alpha_0 + \alpha_1 + \alpha_2 = 0 \quad \text{--- (3)}$$

$$-\alpha_{-1} + 0\alpha_0 + \alpha_1 + 2\alpha_2 = 1/h \quad \text{--- (4)}$$

$$\alpha_{-1} + 0\alpha_0 + \alpha_1 + 4\alpha_2 = 0 \quad \text{--- (5)}$$

$$-\alpha_{-1} + 0\alpha_0 + \alpha_1 + 8\alpha_2 = 0 \quad \text{--- (6)}$$

$\therefore$ The final linear system of equations for unknown weights α_{-1}, α_0, α_1 & α_2 is given as:

(P.T.O)

(Continue)

[1(a)]

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \\ -1 & 0 & 1 & 8 \end{bmatrix} \begin{bmatrix} \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/h \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ It is verified that we need 4 equations

(b) Derive the order of consistency

Given,

$$\boxed{f'_i \approx \frac{-f_{i+2} + 6f_{i+1} - 3f_i - 2f_{i-1}}{6h}} \quad — (6)$$

From (a):

using Eqn (4) + (5)

$$\Rightarrow (-\alpha_{-1} + 0 + \alpha_1 + 2\alpha_2) + (\alpha_{-1} + 0 + \alpha_1 + 4\alpha_2) = 1/h + 0$$

$$\Rightarrow 2\alpha_1 + 6\alpha_2 = 1/h \quad — (7)$$

- eqn (5) + (6)

$$\Rightarrow \alpha_{-1} + 0 + \alpha_1 + 4\alpha_2 + (-\alpha_{-1} + 0 + \alpha_1 + 8\alpha_2) = 0 + 0$$

$$\Rightarrow 2\alpha_1 + 12\alpha_2 = 0 \quad — (8)$$

- eqn (8) − (7)

$$\Rightarrow -2\alpha_1 - 6\alpha_2 + 2\alpha_1 + 12 = 0 - 1/h$$

$$\Rightarrow -1/h = 6\alpha_2$$

$$\alpha_2 = -1/6h$$

substitute α_2 in (8)

$$2\alpha_1 + 12(-1/6h) = 0$$

$$\alpha_1 = \frac{1}{h}$$

- Now again substitute α_1 & α_2 in (6)

$$\Rightarrow -\alpha_{-1} + 0\alpha_0 + \alpha_1 + 8\alpha_2 = 0$$

$$\Rightarrow -\alpha_{-1} + 1/h + 8(-1/6h) = 0$$

$$\alpha_{-1} = -\frac{4}{3h} + \frac{1}{h}$$

$$= \frac{-1}{3h}$$

Illy, from (3)

$$\Rightarrow \frac{-1}{3h} + \alpha_0 + \frac{1}{h} - \frac{1}{6h} = 0$$

$$\Rightarrow \alpha_0 = \frac{1}{6h} + \frac{1}{3h} - \frac{1}{h}$$

$$= \frac{-3}{6h}$$

$$= \frac{-1}{2h}$$

- substituting these values in (*)

i.e

$$\frac{-f_{i+2} + 6f_{i+1} - 3f_i - 2f_{i-1}}{6h} = f_i' + \frac{1}{6h}\Big(\frac{-2^4h^4}{2^4} \cdot f_i'''' - \frac{2^5h^5}{120} \cdot f_i''''' - O(h^6)\Big) + \frac{6h^4}{2^4} \cdot f_i'''' + \frac{6h^5}{120} \cdot f_i''''' + O(h^6)$$

$$- \frac{2h^4}{24} \cdot f_i'''' + \frac{2h^5}{120} f_i''''' + O(h^6)\Big)$$

$$= f_i' + \frac{1}{6h}\left(-\frac{1}{2} \cdot h^4 f'''' + \left(\frac{\cancel{-24}}{\cancel{120}}\right) \cdot h^5 \cdot f_i''''' + O(h^6)\right)$$

∴ The consistency Order is ~~4~~ 3 (since leading error term has degree 4) (✓)

ⓒ If a Derivative of order d is approximated with n points then we would get (n-1) Derivatives

• Lower bound of order of consistency would be

$p = n - 1 + \cancel{1} \rightarrow$ (no of derivatives + order of derivative)

$= n$

$= 4$ (no of ways) f

(P.T.O)

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2] 2-D Derivative Filter

→ Given,

$$g_{i,j} = \frac{1}{4h^2} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} f_{i,j}$$

(a) which derivative $Df(x,y)$ is approximated by $g(x,y)$

→ we can rewrite the stencil to find different co-efficients by writing it as the product of 2 separate derivative approximations i.e

$$g_{i,j} = \frac{1}{2h} [1 \quad 0 \quad -1] * \frac{1}{2h} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} * f_{i,j}$$

co-efficients

here, $\partial x = \frac{1}{2h} \cdot [1 \quad 0 \quad -1]$ & $\partial y = \frac{1}{2h} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

- $g(x,y)$ approximates a 2^{nd} order mixed partial derivative of $Df(x,y)$

i.e $\boxed{g_{i,j} \approx \frac{\partial^2 f_{i,j}}{\partial y \cdot \partial x}}$ ✓

(b) We can use 2-D Taylor expansion on the laplacian depending on the above pixels by:

$$\boxed{u(x+h) = \sum_{k=0}^{n} \frac{1}{k!} (h \cdot \nabla)^k u(x) + O(|h|^{n+1})}$$ — (kmp)

On computing taylor expansions for different pixel values $u_{i,j}$ we know that the Order of Consistency is 2 ✓

ⓒ Compute the Fourier Transform of $g(x,y)$

→ WKT from ⓐ

$$g(i,j) = \frac{\partial^2 f_{i,j}}{\partial x . \partial y}$$

$$\Downarrow$$

$$g(x,y) = \frac{\partial^2 f(x,y)}{\partial x . \partial y}$$

$$= \frac{\partial}{\partial y} . \left(\frac{\partial f}{\partial x} \right)$$

• $$F\left(\frac{\partial^2 f}{\partial y \partial x} \right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi i (ux+vy)} \frac{\partial^2 f}{\partial y \partial x} . \partial x . \partial y$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-2\pi i (ux+vy)} \frac{\partial^2 f}{\partial y \partial x} . \partial y \right) . \partial x$$

using: $F[f'] = 2\pi i u . F[f]$

$$= \int_{-\infty}^{\infty} e^{-2\pi i u x} . \int_{-\infty}^{\infty} e^{-2\pi i v y} . \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} \right) . \partial y . \partial x$$

$$= \int_{-\infty}^{\infty} e^{-2\pi i u x} (2\pi i v) \int_{-\infty}^{\infty} e^{-2\pi i v y} . \frac{\partial f}{\partial x} . \partial y . \partial x$$

$$= 2\pi i v \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi i v y} . e^{-2\pi i u x} . \frac{\partial f}{\partial x} . \partial x \, \partial y$$

$$= 2\pi i v \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi i v y} (2\pi i u)\, e^{-2\pi i u x} \cdot f(x,y) \cdot \partial x\, \partial y$$

$$= 2\pi i v\,(2\pi i u) \cdot \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi i (ux+vy)} \cdot f(x,y)\, \partial x\, \partial y$$

$$= \underbrace{-u\pi^2(u,v)}_{w(u,v)} \cdot \underbrace{F[f(x,y)](u,v)}_{F[f](u,v)} \quad (\checkmark)$$

ⓔ The fourier analysis gives frequency dependent results on the approximation quality (error)
- For a fixed frequency, the approximation error remains constant w.r.t h ✓
- When the frequency increases, the error terms also increases linearly $O(u)$ f

$\partial_{xy} f \neq \theta$ (Consider) Z

ⓓ Compute $F[g](u,v) - F[Df](u,v) = F[Z](u,v)$

WKT, $F[g](u,v) = -u\pi^2 u.v \cdot F[f(x,y)](u,v)$

Therefore. $F[Z](u,v) = -u\pi^2 uv \cdot F[f(x,y)](u,v) + \left(\frac{\partial^2 f(x,y)}{\partial x\, \partial y}\right) \cdot u\pi^2 uv.\ F[f(x,y)](u,v)$

applying taylor expansion eqn around 0 (eqn ⊛⊛)

$= -u\pi^2 uv \cdot F[f(x,y)](u,v) - u\pi^2 uv \cdot F[f'(x,y)](u,v) - u\pi^2 u.v\ F[f''(x,y)](u,v) - u\pi^2 uv\ F[f'''(x,y)](u,v) + g'(x,y) \cdot u\pi^2 u.v \cdot F[f'(x,y)](u,v) + \ldots + u\pi^2 u.v\ F[f'''(x,y)](u,v)$

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[on further simplification...]

∴ The final order of consistency is : 3

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