IPCV - Assignment 1 (H1)

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Problem 1 (Peak-Signal-to-Noise Ratio)

Let the image $f = (f_{i,j})$ be a noisy version of $g = (g_{i,j})$ degraded by additive noise $n = (n_{i,j})$ with zero mean:

$$f_{i,j} = g_{i,j} + n_{i,j}.$$

In the lecture the peak-signal-to noise ratio (PSNR) is defined as a measure of quality.

(a) In which case do we have $PSNR(f,g) = 0$?

e.g. grey value range [0,255] $\Rightarrow$ max grey value = 255

$$PSNR(f,g) := 10 \log_{10} \left(\frac{(\text{max grey value})^2}{MSE(f,g)} \right)$$

$$MSE(f,g) := \frac{1}{MN} \sum_{i=1}^{M} \sum_{j=1}^{N} (f_{ij} - g_{ij})^2.$$

$$PSNR(f,g) = 0$$

$$10 \log_{10} (x) = 0$$

$$x \overset{!}{=} 1$$

$$\Rightarrow \frac{(\text{max grey value})^2}{MSE(f,g)} \overset{!}{=} 1$$

$$\Rightarrow MSE(f,g) \overset{!}{=} (\text{max grey value})^2$$

This is the case if $(f_{i,j} - g_{i,j})^2$ averages to $(\text{max grey value})^2$,

i.e. $((g_{i,j} - n_{i,j}) - g_{i,j})^2$ averages to $(\text{max grey value})^2$

$\Rightarrow n_{ij} \to$ max grey value

(b) Let a filtered version u of f be given such that

$$PSNR(u,g) = PSNR(f,g) + 20\,dB$$

How has the MSE of f and g changed during the filtering

$$10 \log_{10} \left(\frac{(\text{max grey value})^2}{MSE(u,g)} \right) = 10 \log_{10} \left(\frac{(\text{max grey value})^2}{MSE(f,g)} \right) + 20\,dB$$

$$\Leftrightarrow 10 \log_{10} \left(\frac{(\text{max grey value})^2}{MSE(u,g)} \right) - 10 \log_{10} \left(\frac{(\text{max grey value})^2}{MSE(f,g)} \right) = 20\,dB$$

by the rule $\log_b(x/y) = \log_b(x) - \log_b(y)$:

$$10 \log_{10} \left(\frac{\cancel{(\text{max grey value})^2}}{MSE(u,g)} \cdot \frac{MSE(f,g)}{\cancel{(\text{max grey value})^2}} \right) = 20\,dB$$

$$\Leftrightarrow 10 \log_{10} \left(\frac{MSE(f,g)}{MSE(u,g)} \right) = 20\,db \quad | : 10$$

$$\Leftrightarrow \log_{10} \left(\frac{MSE(f,g)}{MSE(u,g)} \right) = 2\,dB$$

$$\Leftrightarrow \frac{MSE(f,g)}{MSE(u,g)} = 10^2 = 100$$

$\Rightarrow$ the new MSE after filtering $MSE(u,g) = \frac{\overset{\text{MSE before filtering}}{MSE(f,g)}}{100}$ ✓

(c) Assume that $n \to 0$. Calculate $PSNR(f,g)$. How can you interpret this

$$\underset{n \to 0}{PSNR(f,g)} = 10 \log_{10} \left(\frac{(\text{max grey value})^2}{\underset{n \to 0}{MSE(f,g)}} \right)$$

$$\underset{n \to 0}{MSE(f,g)} = \frac{1}{MN} \sum_{i=1}^{M} \sum_{j=1}^{N} (f_{i,j} - g_{i,j})$$

$$= \frac{1}{MN} \sum_{i=1}^{M} \sum_{j=1}^{N} \underbrace{((g_{ij} - \underbrace{n_{i,j}}_{=0}) - g_{i,j})}_{=0}$$

$$= 0 \quad (\checkmark)$$

$$\Rightarrow \underset{n \to 0}{PSNR(f,g)} = 10 \log_{10} \left(\frac{(\text{max grey value})^2}{0} \right) ↯$$

UNDEFINED f

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2] Problem 2 : Continoous Convolution

(a) $f(x) = \begin{cases} 1/2 & (-1 \le x \le 1), \\ 0 & \text{else} \end{cases}$

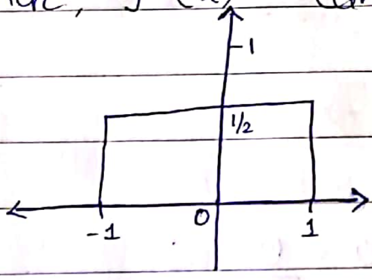
Find : • $h_1 = f * f$

• $h_2 = f * f * f$

→ WKT,

$$(g * f)\,x = \int_{-\infty}^{\infty} g(z) \cdot f(x-z) \cdot dz$$

where, $f(x)$ can be visualized as follows



The signal is in the range (-1 to 1)

• Given, $f(x-z) = 1/2$ [Integral]

Now, tracing the continuous signal in the range $f(x-z)$

i.e • Between $(-1 \le x - z \le 1)$

$$(f * f)\,x = \int_{(x-1)}^{(x+1)} \underbrace{1/2}_{f(z)} \cdot g(z) \cdot dz$$

• Between $(-2 \le x \le 0)$

$$(f * f)\,x = \underbrace{\int_{x-1}^{-1} 1/2 \; f(z) \cdot dz}_{\text{outside the range so, 0}} + \int_{-1}^{x+1} 1/2 \cdot f(z) \cdot dz$$

• Between $(0 \le x \le 2)$

$$(f * f)\,x = \int_{x-1}^{-1} 1/2 \; f(z) \cdot dz + \underbrace{\int_{-1}^{x+1} 1/2 \cdot f(z) \cdot dz}_{\text{outside range so, eliminate}}$$

Finally,

For $h_1 = (f * f)x$

> when $-1 \le x \le 1$

$$\int_{x-1}^{x+1} \frac{1}{2} \cdot f(z) \cdot dz$$

> when $-2 \le x \le 0$

$$\int_{-1}^{x+1} \frac{1}{2} \times \underbrace{\frac{1}{2}}_{f(z)} \cdot dz = \frac{2+x}{4}$$

> when $x \le -2$

Out of ranges, so 0

> when $x \ge 2$

out of range, so 0

$$\therefore h_1 = \begin{cases} \frac{2+x}{4} & (-2 \le x \le 0) \\ 0 & (x \le -2) \\ \frac{2-x}{4} & (0 \le x \le 2) \\ 0 & (x > 2) \end{cases}$$

Now,

$$h_2 = (f * f * f)x$$

$$= \int_{x-1}^{x+1} \frac{1}{2} \cdot \underbrace{(f * f)z}_{h_1} \cdot dz$$

(Associative property of convolution)

Again, tracing the signal across the range.

- Between $(-1 \le x \le 1)$

$$(f * f * f)x = \int_{x-1}^{0} \frac{2+z}{8} \cdot dz + \int_{0}^{x+1} \frac{2-z}{8} \cdot dz$$

$$= \int_{x-1}^{0} 1/4 \, dz + \int_{x-1}^{0} \frac{z}{8} \cdot dz + \int_{0}^{x+1} \frac{1}{4} \cdot dz - \int_{0}^{x+1} \frac{z}{8} \, dz$$

eg. $\frac{1}{4}[z]_{x-1}^{0}$

$$= 1/4 \, (z)_{x-1}^{0} + 1/16 \, (z^2)_{x-1}^{0} + 1/4 \, (z)_{0}^{x+1} - 1/16 \, (z^2)_{0}^{x+1}$$

$$= 1/16 \, [4(-(x-1)) - (x-1)^2 + 4(x+1) - (x+1)^2]$$

$$= 1/16 \, [6 - 2x^2]$$

✓

- Between $(1 \le x \le 3)$

$$(f * f * f)x = \int_{x-1}^{2} \frac{2-z}{8} \cdot dz$$

$$= \int_{x-1}^{2} \frac{1}{4} \cdot dz - \int_{x-1}^{2} \frac{z}{8} \cdot dz$$

$$= 1/4 \cdot (z)_{x-1}^{2} - 1/16 \, (z^2)_{x-1}^{2}$$

$$= \frac{x^2 - 6x + 9}{16}$$

✓

- Between $(-3 \le x \le -1)$

$$(f * f * f)x = \int_{-2}^{x+1} \frac{2+z}{8} \cdot dz = \int_{-2}^{x+1} \frac{1}{4} \cdot dz + \int_{-2}^{x+1} \frac{z}{8} \cdot dz$$

$$= \frac{1}{4}\left(z\right)_{-2}^{x+1} + \frac{1}{16}\left(z^2\right)_{-2}^{x+1}$$

$$= \frac{(x^2+9+6x)}{16}$$

- Between $(x \geq 3)$

$(f * f * f)x = 0$

- Betwen $(x \leq -3)$

$(f * f * f)x = 0$

$$\therefore h_2 = \begin{cases} 0 & (x \leq -3) \\ \frac{x^2+9+6x}{16} & (-3 \leq x \leq -1) \\ \frac{6-2x^2}{16} & (-1 \leq x \leq 1) \\ \frac{x^2+9-6x}{16} & (1 \leq x \leq 3) \\ 0 & (x \geq 3) \end{cases}$$

(b) We have already evaluated functions h_1 & h_2 in (a), let us summarize at the discrete points rather than summarizing in the range.

* $h_1 = (f * f)x$

For $x = -2$ (or) $x \leq -2$?

$(f * f)x = 0$

For $x = -1$ (or) $-2 \leq x \leq 0$?

$$(f * f)x = \frac{x+2}{4} = \frac{-1+2}{4} = \frac{1}{4}$$

For $x = 0$,
$(f * f)x = \frac{2-x}{4} = \frac{1}{2}$

For $x = 1$,
$(f * f)x = 1/4$

For $x = 2$
$(f * f)x = 0$

* $h_2 = (f * f * f)x$

For $x = -3$
$(f * f * f)x = 0$

For $x = -2$,
$(f * f * f)(-2) = \frac{(x^2 + 9 + 6x)}{16} = \frac{(2^2 + 9 + 6(-2))}{16} = 1/16$

For $x = -1$,
$\Rightarrow \frac{6 - 2x^2}{16} = \frac{6 - 2(-1)^2}{16} = 1/4$

For $x = 0$,
$\Rightarrow \frac{6 - 2(0)^2}{16} = \frac{6}{16}$

For $x = 1$,
$\Rightarrow \frac{(x)^2 + 9 - 6(x)}{16} = 1/4$

For $x = 2$,
$\Rightarrow \frac{(2)^2 + 9 - 6(2)}{16} = \frac{1}{16}$

For $x = 3$,
$\Rightarrow 0$

(c) Comparision of results b/n Continuous convolution & Discrete Convolution from P1

→ Differences:

1. In the functions of discrete convolution, we don't see time(t), which is a continuous parameter

2. In Discrete convolution, the parameter t is replaced by n (sample)

3. when we compare h_2 of both discrete & continuous convolution, we notice that in a continuous signal, the amplitude is 0 at -3, peaks at 0 & then then vanishes again at +3. whereas, in discrete, it is 0 for all values below 0 & then amplitude gradually increases ∀ positive values.

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Similarity:

1. Continuous convolution would appear discrete if we sample the signal / replace time (t) by sample (n).

4. In discrete, the amplitude is consistently in the form $\frac{x}{F}$ [here: x is any integer]

whereas, in continuous, it is always of the form $\frac{1}{F}$ f

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3] Properties of Convolution [Continuous]

→ • Commutativity Property

$f * g = g * f$

Proof :- WKT,

$$f(t) * g(t) = \int_{-\infty}^{\infty} \underbrace{f(\tau) . g(t-\tau) . d\tau}_{\text{Integral form of continuous signal}}$$

$(f*g)(t)$

Let, $z = t - \tau$

$$\Rightarrow - \int_{\infty}^{-\infty} f(t-z) . g(z) . dz$$

$$\Rightarrow \int_{-\infty}^{\infty} g(z) . f(t-z) . dz$$

$$= g(t) * f(t)$$

• Associativity Property [Ref: Univ of Tennessee]

$(f * g) * h = f * (g * h)$

Proof: WKT for a continuous convolution

$$(f * g)(t) = \int_{0}^{t} f(z) . g(t-z) . dz$$

?

Now with 3 functions f, g & h

Thus,

$$((f*g)*h)(t) = \int_{0}^{t} \underbrace{(f*g)(z)}_{\text{Integrating this part b/n } (y=0 \ \& \ z)} . h(t-z) . dz$$

$$= \int_{z=0}^{t} \left[\int_{y=0}^{z} f(y) g(z-y) . dy \right] . h(t-z) . dz$$

combining both the integrals

$$:= \int_{z=0}^{t}\int_{y=0}^{z} f(y).\,g(z-y).\,h(t-z).\,dy.\,dz$$

→ Rearranging, the integrals to t.

$$= \int_{y=0}^{t}\int_{z=y}^{t} f(y).\,g(z-y).\,h(t-z).\,dz.\,dy$$

$$= \int_{y=0}^{t}\int_{z=0}^{t-y} f(y).\,g(z).\,h(t-z-y).\,dz.\,dy$$

$$= \int_{y=0}^{t} f(y).\left[\int_{z=0}^{t-y} g(z).\,h(t-y-z).\,dz\right].\,dy$$

$$= \int_{y=0}^{t} f(y)\,(g * h)(t-y).\,dy$$

$$= (f * (g * h))(t) \quad (\checkmark)$$

• Distributive Property

$(f+g) * h = f * h + g * h$ &

$f * (g+h) = f * g + f * h$

Proof :-

R.H.S ⇒ $f * g + f * h$

Applying the conditions of convolution

⇒ $f(x) * g(x) + f(x) * h(x)$ $(f*g)(x)$

Let, $y_1(x) = f(x) * g(x)$ &

$y_2(x) = f(x) * h(x)$

$\therefore$ Adding both $y_1(x)$ & $y_2(x)$ together

i.e $y(x) = y_1(x) + y_2(x)$

$$= f(x) * g(x) + f(x) * h(x)$$

$$= \sum_{z=-\infty}^{\infty} f(z) . g(x-z) + \sum_{z=-\infty}^{\infty} f(z) . h(x-z)$$

$\rightarrow$ Taking $f(z)$ common

$$= \sum_{K=-\infty}^{\infty} f(z) \left[g(x-z) + h(x-z)\right]$$

$$= f(z) * (g(x) + h(x))$$

$$= f * (g+h) \quad (L.H.S)$$

• Differentiability Property [Ref: Univ of Tennessee]

if $f \in C^0(\mathbb{R})$ & $g \in C^n(\mathbb{R})$

then, $(f*g) \in C^n(\mathbb{R})$

Proof:

Let $(f*g)t$ be the convolution of $f(t)$ with $g(t)$

so,

$$(f*g)(t) = \int_{-\infty}^{\infty} f(\tau) . g(t-\tau) . d\tau$$

Differentiating b.s w.r.t t

$$(f*g)'(t) = \int_{-\infty}^{\infty} f(\tau) . g'(t-\tau) . d\tau$$

$$(f*g)'(t) = f(t) . g'(t)$$

[Applying commutative property]

$$(f*g)'(t) = f'(t) * g(t) \qquad | (f*g) \in C^n$$

- **Linearity Property**

$(\alpha f + \beta g) * h = \alpha (f * h) + \beta (g * h)$

Proof:

Let, $\gamma(x) = \alpha f(x) + \beta g(x)$

Now, substitute $\gamma(x)$ in L.H.S

$h(\gamma(x)) = h(\alpha f(x) + \beta g(x))$? $(h*f) \neq h(f)$

Applying Associativity,

$h(\alpha f(x) + \beta . g(x)) = \alpha h(f(x)) + \beta h(g(x))$

$= \alpha (f * h) + \beta (g * h)$

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- **Shift Invariance**

$(T_b f) * g = T_b (f * g)$

$\forall$ translations T_b with $(T_b f)(x) = f(x-b)$

Proof:- WKT,

$(T_b f) * g = \int_{-\infty}^{\infty} T_b . f(t) . g(x-t) . dt$

Let, $t = t - b$ over (t) | as, $x = x - b$

$\Rightarrow \int_{-\infty}^{\infty} f(t-b) . g[x-t] dt$ | so, $t = t - b$

Transform to Laplace domain & perform integration

Before that, let's assume : $s = t - b$

$\Rightarrow t = s + b$

$\Rightarrow \int_{-\infty}^{\infty} f(s) . g[x - [(s+b)]] . ds$

$= \int_{-\infty}^{\infty} f(s) . g[(x-b) - s] . ds$

f $(f*g)(x-b)$

$\Rightarrow \cancel{(f*g)(s)}$ → Transform back from laplacian domain.

$\Rightarrow$ [redacted] $[T_b (f * g)] (x)$ (✓)

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