IPCV Assignment H2

Team members (Tutorial T1)

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Problem 1 (Properties of the Continuous Fourier Transform)

(a) Linearity: $\mathcal{F}[a\,f(x) + b\,g(x)](u) = \langle af(x) + bg(x), b_u \rangle$

$$= \int_{-\infty}^{+\infty} (af(x) + bg(x))\, e^{-i2\pi ux}\, dx$$

factor rule & sum rule $$= a \cdot \int_{-\infty}^{+\infty} f(x)\, e^{-i2\pi ux}\, dx + b \cdot \int_{-\infty}^{+\infty} g(x)\, e^{-i2\pi ux}\, dx$$

$$= a \cdot \langle f, b_u \rangle + b \cdot \langle g, b_u \rangle$$

$$= a \cdot \mathcal{F}[f](u) + b \cdot \mathcal{F}[g](u) \quad \square$$

(b) Spatial Shift: $\mathcal{F}[f(x-a)](u) = \langle f(x-a), b_u \rangle$

$$= \int_{-\infty}^{+\infty} f(x-a)\, e^{-i2\pi ux}\, dx$$

$z = x - a$
$\rightarrow x = z + a$

$$= \int_{-\infty}^{+\infty} f(z)\, e^{-i2\pi u(z+a)}\, dz$$

$$= \int_{-\infty}^{+\infty} f(z)\, e^{-i2\pi uz + (-i2\pi ua)}\, dz$$

$$= \int_{-\infty}^{+\infty} f(z)\, e^{-i2\pi uz} \cdot \underbrace{e^{-i2\pi ua}}_{\text{constant}}\, dz$$

factor rule $$= e^{-i2\pi ua} \int_{-\infty}^{+\infty} f(z)\, e^{-i2\pi uz}\, dz$$

$$= e^{-i2\pi ua} \cdot \langle f, b_u \rangle = e^{-i2\pi ua}\, \mathcal{F}[f](u) \quad \square$$

(c) Frequency Shift: $\mathcal{F}[f(x) \cdot e^{-i2\pi u_0 x}](u) = \langle f(x) \cdot e^{-i2\pi u_0 x}, b_u \rangle$

$$= \int_{-\infty}^{+\infty} f(x)\, e^{-i2\pi u_0 x}\, e^{-i2\pi ux}\, dx$$

$$= \int_{-\infty}^{+\infty} f(x)\, e^{-i2\pi x(u_0 + u)}\, dx$$

$$= \langle f, b_{u_0+u} \rangle$$

$$= \mathcal{F}[f](u_0 + u)$$

(d) Scaling: $\mathcal{F}[f(ax)](u) = \langle f(ax), b_u \rangle$

$$= \int_{-\infty}^{+\infty} f(ax) e^{-i2\pi ux} \, dx$$

$z = ax$
$\rightarrow x = \frac{z}{a}$

$$= \int_{-\infty}^{+\infty} f(z) e^{-i2\pi u \frac{z}{a}} \frac{1}{|a|} dz$$

factor rule

$$= \frac{1}{|a|} \int_{-\infty}^{+\infty} f(z) e^{-i2\pi \frac{u}{a} z} dz$$

to make up for the flip of the integration limit, if a is negative

$$= \frac{1}{|a|} \langle f, b_{\frac{u}{a}} \rangle = \frac{1}{|a|} \mathcal{F}[f]\left(\frac{u}{a}\right)$$

✓

(e) Convolution: $\mathcal{F}[(f*g)(x)](u) = \langle (f*g)(x), b_u \rangle$

Convolution is defined as:

$$(f*g)(x) := \int_{-\infty}^{+\infty} f(x') g(x-x') \, dx'$$

$$= \int_{-\infty}^{+\infty} (f*g)(x) \, e^{-i2\pi ux} \, dx$$

$$= \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(x') g(x-x') dx' \right) e^{-i2\pi ux} dx$$

Change order of integration

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x') g(x-x') e^{-i2\pi ux} \, dx \, dx'$$

$$= \int_{-\infty}^{+\infty} f(x') \Big(\underbrace{\int_{-\infty}^{+\infty} g(x-x') e^{-i2\pi ux} dx}_{\mathcal{F}[g(x-x')](u)} \Big) dx'$$

Spatial Shift Property

$$= \int_{-\infty}^{+\infty} f(x') e^{i2\pi ux'} dx' \; \mathcal{F}[g](u)$$

$$= \langle f, b_u \rangle \, \mathcal{F}[g](u) = \mathcal{F}[f](u) \, \mathcal{F}[g](u) \quad \square$$

✓

(f) Differentiation:

inverse F.T.

$$f'(x) = \frac{d f(x)}{dx} = \frac{d \, \mathcal{F}^{-1}[\hat{f}]x}{dx} = \frac{d \int \hat{f}(u) \, e^{i2\pi ux} du}{dx}$$

$$= \int_{-\infty}^{+\infty} i2\pi u \, \hat{f}(u) e^{i2\pi ux} \, du = i2\pi u \underbrace{\int_{-\infty}^{+\infty} \mathcal{F}[f](u) \, e^{i2\pi ux} \, du}_{\mathcal{F}^{-1}[\mathcal{F}[f](u)]}$$

Linearity

$$= \mathcal{F}^{-1}[i2\pi u \cdot \mathcal{F}[f](u)]$$

f

$$\Rightarrow \mathcal{F}[f'](u) = \cancel{\mathcal{F}[\mathcal{F}^{-1}}[i2\pi u \cdot \mathcal{F}[f](u)]] = i2\pi u \cdot \mathcal{F}[f](u) \quad \square$$

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2] Continuous Fourier Transforms

→ Given,

$$f(x) = \begin{cases} 0, & (x \le -3) \\ \frac{x^2+6x+9}{16}, & (-3 < x \le -1), \\ \frac{6-2x^2}{16}, & (-1 < x \le 1). \\ \frac{x^2-6x+9}{16}, & (1 < x \le 3), \\ 0, & (x > 3). \end{cases}$$

Solution :

$$f(x) = h(x) * h(x) * h(x)$$

WKT,

$$h(x) = \begin{cases} 1/2, & (-1 \le x \le 1) \\ 0, & \text{other} \end{cases}$$

$$\hat{f}(u) = \int_{-\infty}^{\infty} h(x) . e^{-2\pi i u x} . dx$$

$$= \int_{-1}^{1} 1/2 . e^{-2\pi i u x} . dx$$

$$= \frac{-1}{4i\pi u} . e^{-2\pi i u x} \Big|_{-1}^{1}$$

$$= \frac{-1}{4i\pi u} . \left(e^{-2\pi i u} - e^{2\pi i u}\right)$$

[Rearronge]

$$= \frac{1}{2\pi u} . \left(\frac{e^{2\pi i u} - e^{-2\pi i u}}{2i}\right)$$

$$= \frac{1}{2\pi u} . \sin(2\pi u)$$

Using convolution property of Fourier transforms

$$\therefore \boxed{\hat{f}(u) = \frac{\sin^3 2\pi u}{8\pi^3 u^3}}$$

✓

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3] Symmetry and Anti-Symmetry of Fourier Transforms :

→ (a) Given, f : 1-D real signal

Show that,

$\hat{f}(u) = \mathcal{F}[f]$ has both real & imaginary parts

WKT, we can write the above expression as

$$= \int_{-\infty}^{\infty} f(x) \cdot e^{-i 2\pi u x} \cdot \partial x \quad \text{—} \ (*)$$

Now, let's use the hint

i.e Using Euler's Rule : $e^{i\phi} = \cos\phi + i\sin\phi$ — ①

Let $\phi = -2\pi ux$

Then ① becomes

$$e^{i(-2\pi ux)} = \cos(-2\pi ux) + i\sin(-2\pi ux)$$

Using this eqn (*)

i.e $\hat{f}(u) = \int_{-\infty}^{\infty} f(x) \cdot [\underbrace{\cos(-2\pi xu)}_{\text{real}} + \underbrace{i\sin(-2\pi ux)}_{\text{imaginary}}] \cdot \partial x$

Splitting the expression

i.e $\hat{f}(u) = \int_{-\infty}^{\infty} f(x) \cdot \cos(-2\pi ux) \cdot \partial x + i \int_{-\infty}^{\infty} f(x) \cdot \sin(-2\pi ux) \cdot \partial x$

Thus, using Euler's formula we can write as.

- Real : $\hat{f}(u) = \int_{-\infty}^{\infty} f(x) \cdot \cos(-2\pi ux) \cdot \partial x$ ✓

- Imaginary : $\hat{f}(u) = \int_{-\infty}^{\infty} f(x) \cdot \sin(-2\pi ux) \cdot \partial x$ ✓

(P.T.O)

Now, to show that FT is symmetric (Real) around origin, Let us take $u = -u$.
(i.e the function is symmetric along both the axis)

$\Rightarrow$ Real $[\hat{f}(-u) = \mathcal{F}[f](-u)$.

$$= \int_{-\infty}^{\infty} f(x) \cdot \cos(-2\pi(-u)x) \cdot dx$$

$$= \int_{-\infty}^{\infty} f(x) \cdot \cos(2\pi u x) \cdot dx$$

$= \mathcal{F}[f](u)$ ✓

From Euler's we know that,

$\boxed{\cos\phi = \frac{e^{i\phi} + e^{-i\phi}}{2}}$ & $\boxed{\phi = 2\pi u x}$

$$= \int_{-\infty}^{\infty} f(x) \cdot \frac{e^{i2\pi u x} + e^{-i2\pi u x}}{2} \cdot dx$$

$[u \to -u]$

$$= \int_{-\infty}^{\infty} f(x) \cdot \frac{e^{i2\pi(-u)x} + e^{-i2\pi(-u)x}}{2} \cdot dx$$

$$= \int_{-\infty}^{\infty} f(x) \cdot \underbrace{\frac{e^{-i2\pi u x} + e^{i2\pi u x}}{2}} \cdot dx$$

$$\hat{f}(-u) = \int_{-\infty}^{\infty} f(x) \cdot \cos(-2\pi u x) \cdot dx$$

$\therefore$ Real $[\hat{f}(u)] \Leftrightarrow$ Real $[\hat{f}(-u)]$ ✓

i.e For both u & $-u$, it is equal. so real part of the fourier transform is symmetric!

(P.T.O) ✓

Now, let's show that the imaginary part of the FT is anti-symmetric.

We already know that:

$$\text{Imaginary}\,[\hat{f}(u)] = \int_{-\infty}^{\infty} f(x) . \sin(-2\pi u x) . dx \quad — ②$$

- Just to prove this, let $\boxed{u = -u}$ in ②

$$\text{Imaginary}\,[\hat{f}(-u)] = \int_{-\infty}^{\infty} f(x) . \sin(-2\pi(-u).x) . dx$$

$$= \int_{-\infty}^{\infty} f(x) . \sin(2\pi u x) . dx$$

Again, from Euler's we know that:

$$\boxed{\sin\phi = \frac{e^{i\phi} - e^{-i\phi}}{2i}} \quad \& \quad \boxed{\phi = 2\pi u x}$$

$$= \int_{-\infty}^{\infty} f(x) . \frac{e^{i2\pi u x} - e^{-i2\pi u x}}{2i} . dx$$

$(u \to -u)$

$$= \int_{-\infty}^{\infty} f(x) . \underbrace{\frac{e^{-i2\pi u x} - e^{i2\pi u x}}{2i}}_{\text{Take '-' common.}} . dx$$

$$= \int_{-\infty}^{\infty} f(x) . (-\sin(-2\pi u x)) . dx$$

$$= - \int_{-\infty}^{\infty} f(x) . \sin(-2\pi u x) . dx$$

$\therefore$ $\text{Img}[\hat{f}(-u)] \neq \text{Img}\,(\hat{f}(u)]$ f

i.e. Imaginary part of FT is anti-symmetric

(b) FT decomposes a signal into frequencies. What's the meaning of a negative freq?

→ WKT,

$$\cos\theta = \frac{e^{i\phi} + e^{-i\phi}}{2} = \underbrace{\frac{e^{i\phi}}{2}}_{\text{'+ve' freq}} + \underbrace{\frac{e^{-i\phi}}{2}}_{\text{'-'ve freq}}$$

$$= f(\phi) + f(-\phi)$$

From the above expression it is rather clear that Fourier Transform's basis function is a real-valued sine wave i.e it generates results in 2 phases, thus we represent it as '+ve' & '-'ve frequency. [Reference: Lecture 4/slide 23]

There are always 2 equal magnitude complex exponential functions & each of them are rotating in opposite directions. As a result of which, the imaginary parts gets canceled & the real parts are combined. Thus the resultant signal is a pure/real sine wave.

Ultimately, when we see this real sine wave, we notice 2 phases: one (+ve) frequency & one (-ve) frequency.

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