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IPCV Assignment - 3

1] Discrete Fourier Transforms

→ Given:

Noisy discrete signal : $f = (6, 4, 6, 4, 16, 14, 16, 14)^T$

W.K.T the formula of DFT to transform/represent a Time signal into Frequency signal is given by:

$$\boxed{x(k) = \frac{1}{\sqrt{m}} \sum_{n=0}^{N-1} x(n) \cdot e^{-j \cdot \frac{2\pi n k}{N}}}$$

[Formula referred from Berkeley/Caltech DSP courses]

here,

$N = 8$

- when $k=0$

$$X(0) = \sum_{n=0}^{7} x(n) \cdot e^{-j \cdot 0}$$

$$= \sum_{n=0}^{7} x(n) \cdot 1$$

$$= x(0)+x(1)+x(2)+x(3)+x(4)+x(5)+x(6)+x(7)$$

$$= 6+4+6+4+16+14+16+14$$

$$= \frac{80}{\sqrt{8}} + 0j$$

- when $k=1$

$$x(1) = \sum_{n=0}^{7} x(n) \cdot e^{-j \cdot \frac{2\pi n}{8}}$$

$$= \sum_{n=0}^{7} x(n) \cdot e^{-j\frac{n\pi}{4}}$$

$$= x(0)+x(1)+x(2)+x(3)+x(4)+x(5)+x(6)+x(7)$$

$$= \left(6 \cdot e^{-j\frac{6\pi}{2}}\right) + \left(4 \cdot e^{-j\frac{4\pi}{4}}\right) + \left(6 \cdot e^{-j\frac{6\pi}{4}}\right) + \ldots + $$

$$= \frac{-10}{\sqrt{8}} + \frac{24.142}{\sqrt{8}}j$$

- when $K = 2$

$$X(2) = \sum_{n=0}^{7} x(n) \cdot e^{-j\frac{2\pi n(2)}{8}}$$

$$= \sum_{n=0}^{7} x(n) \cdot e^{-j\frac{n\pi}{2}}$$

$$= \underline{0 + 0j}$$

- when $K = 3$

$$X(3) = \sum_{n=0}^{7} x(n) \cdot e^{-j\frac{2\pi n(3)}{8}}$$

$$= \sum_{n=0}^{7} x(n) \cdot e^{-j\frac{3n\pi}{4}}$$

$$= \frac{-10}{\sqrt{8}} + \frac{4.142j}{\sqrt{8}}$$

- when $K = 4$

$$X(4) = \sum_{n=0}^{7} x(n) \cdot e^{-j\frac{2\pi n(4)}{8}}$$

$$= \sum_{n=0}^{7} x(n) \cdot \underbrace{e^{-jn\pi}}_{(-1)^n}$$

$$= \frac{8}{\sqrt{8}} + 0j$$

- when $K = 5$

$$X(5) = \sum_{n=0}^{7} x(n) \cdot e^{-j\frac{2\pi n(5)}{8}}$$

$$= \frac{-10}{\sqrt{8}} - \frac{4.142j}{\sqrt{8}}$$

• when $k=6$

$$x(6) = \sum_{n=0}^{7} x(n) \cdot e^{\frac{-j2\pi n(6)}{8}}$$

$$= \underline{0+0j}$$

• when $k=7$

$$x(7) = \sum_{n=0}^{7} x(n) \cdot e^{\frac{-j2\pi n(7)}{8}}$$

$$= \frac{-10}{\sqrt{8}} - \frac{24.142j}{\sqrt{8}}$$

(b) The coefficient $\hat{f}_4(16)$ has the highest frequency $\frac{(80+0j)}{\sqrt{8}}$

(c) Set the co-effs of highest frequencies to 0. & perform backtransformation.

$$\boxed{\hat{f}_m = \frac{1}{\sqrt{m}} \sum_{p=0}^{m-1} \hat{f}_p \cdot \exp\left(\frac{i2\pi pm}{m}\right)}$$

(-1)

• For $\hat{f}_0 = \frac{1}{\sqrt{8}} \cdot \sum_{p=0}^{7} \hat{f}_0 \cdot \exp\left(\frac{i2\pi 0m}{m}\right)$

$$= \frac{1}{\sqrt{8}}\left(80 \cdot e^{\frac{i2\pi 00}{8}} + \ldots\right)$$

$$= \underline{13}$$

• For $\hat{f}_1 = \frac{1}{\sqrt{8}} \cdot \sum_{p=0}^{7} \hat{f}_1 \cdot \exp\left(\frac{i2\pi 0m}{m}\right)$

$$= \underline{16.30}$$

IIIly

For $\hat{f}_2 = 15.92$

$\hat{f}_3 = 12$

$\hat{f}_4 = 7$

$\hat{f}_5 = 3.72$

$\hat{f}_6 = 4.30$

$\hat{f}_7 = 8$

$\therefore$ The time signal is:

$(13, 16.30, 15.92, 12, 7, 3.72, 4.30, 8)^T$

(8) If an image has high frequency over all the pixels, then it would appear sharp.

If we suppress such high frequencies we end up with smooth images.

While sampling we tend to suppress high frequencies to prevent artifacts (overlap). These artifacts are created because of the influence of n^{th} term elements of the signal over $n-1^{th}$ term (so usually appears at the corners / boundaries).

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2] Relation b/n Discrete Fourier Transform & a Discrete cosine Transform

→ To show:

DCT of a discrete signal of length M, $\boxed{f = (f_m)_{m=0}^{M-1}}$ is related to DFT of a shifted signal of length $2M$, $\boxed{g = (g_m)_{m=0}^{2M-1}}$

• The above relation can be defined as:-

$$g_m = \begin{cases} f_m, & (0 \le m \le M-1) \\ f_{2M-1-m}, & (M \le m \le 2M-1) \end{cases}$$

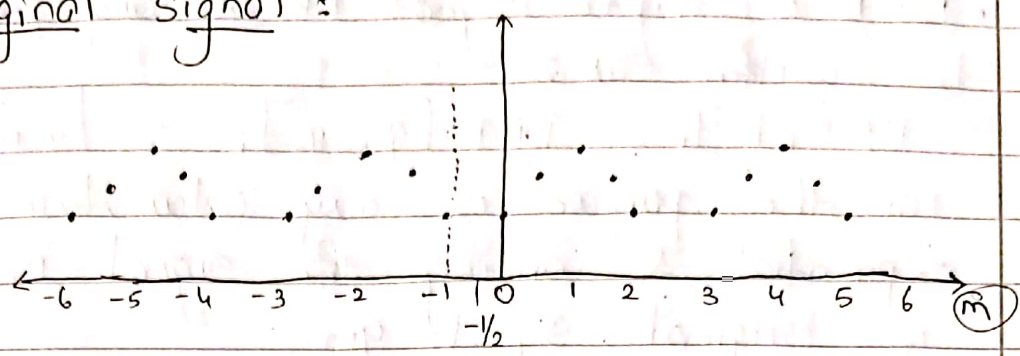
(a) Using hint (a) provided in the question.
Since the signal g is symmetric w.r.t the point $-1/2$ instead of achieving symmetry at 0 we have to perform index shift on the signal by $+1/2$ towards right.
i.e $\boxed{K = m + 1/2}$

Also, discrete signal g is defined as
$\Rightarrow g = g_{2M-m-1}$.

Now, after performing the Index shift, let the resultant signal be represented as h which can be written as, $\boxed{h_K = g_{K-1/2}}$
where, K is an Index & a subset of Z.
If we notice, the index shifted signal h_K is represented over K offshifted by $1/2$

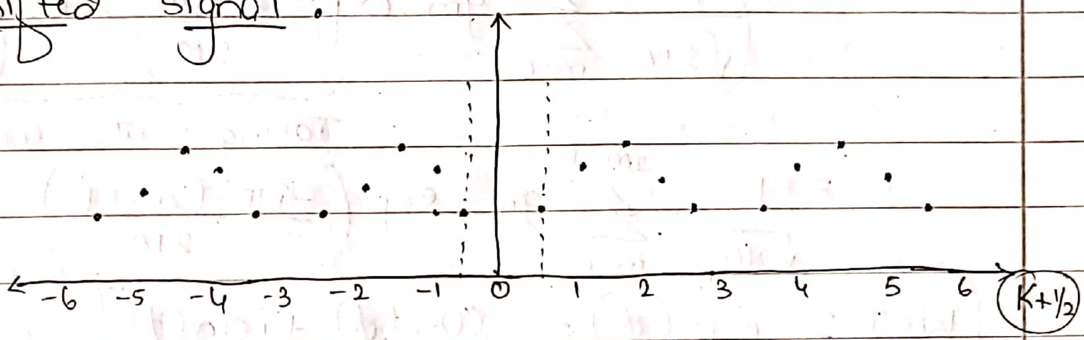
(P.T.O)

Original signal :



⇓ Index shift along right (+1/2)

Shifted signal :



(b) DFT of the shifted signal : we can now get the DCT of the original signal by computing DFT of the shifted signal g at a point w.r.t 0 & 2m. But, it is also given that we have defined the DFT of origniol signal at Intiger indices whereas, we previously defined the shifted signal along other points, $K = m + \frac{1}{2}$.

$\therefore$ we have to represent the shifted signal h_K as the original/unshifted signal g.

so rearranging, $K = m + \frac{1}{2}$

we get : $\boxed{m = K - 1/2}$ — (1)

Original points

shifted points

• DFT of shifted signal h_k at any point j can be written as:

$$DFT[h_k]_j = DFT[g_{k-1/2}]_j \quad \text{from } ①$$

For the purpose of easy calculation using exponential & writing the signal in terms of original signal g_m.

i.e $= \underbrace{DFT(g_m)_j}_{\text{(Limits given in the function)}} \cdot \exp\left(\frac{-i 2\pi \frac{1}{2} j}{2M}\right)$

$$= \left(\frac{1}{\sqrt{2M}} \sum_{m=0}^{2M-1} g_m \cdot \exp\left(\frac{-i2\pi j m}{2M}\right)\right) \cdot \exp\left(\frac{-i\pi j}{2M}\right)$$

Taking $(-i\pi j)$ common

$$= \frac{1}{\sqrt{2M}} \sum_{m=0}^{2M-1} g_m \cdot \exp\left(\frac{-i\pi j(2m+1)}{2M}\right)$$

[WKT: $\exp(\phi) = \cos(\phi) + i\sin(\phi)$]

$$= \underbrace{\frac{1}{\sqrt{2M}} \sum_{m=0}^{2M-1}}_{} g_m \cdot \left[\cos\left(\frac{\pi j(2m+1)}{2M}\right) - i\sin\left(\frac{\pi j(2m+1)}{2M}\right)\right]$$

[multiplying this part with both cos & sin]

$$= \left(\frac{1}{\sqrt{2M}} \sum_{m=0}^{2M-1} g_m \cdot \cos\left(\frac{\pi j(2m+1)}{2M}\right)\right) - \cancel{\left(\frac{1}{\sqrt{2M}} \sum_{m=0}^{2M-1} g_m \, i \sin\left(\frac{\pi j(2m+1)}{2M}\right)\right)} \to 0$$

Now, we can eliminate the imaginary part of the signal i.e:

$$= \frac{1}{\sqrt{2M}} \sum_{m=0}^{2M-1} g_m \cdot \cos\left(\frac{\pi j(2m+1)}{2M}\right) \quad — ②$$

(P.T.O)

© Now we have to further simplify eqn ② such that the sum only contains entries of f instead of g_m & the sum lies b/n 0 to $M-1$

(because the limits of the Σ has changed)

$$= 2\left(\frac{1}{\sqrt{2M}} \cdot \sum_{m=0}^{M-1} f_m \cdot \cos\left(\frac{\pi j(2m+1)}{2M}\right)\right)$$

$$= \frac{\sqrt{2}}{\sqrt{M}} \cdot \sum_{m=0}^{M-1} f_m \cdot \cos\left(\frac{\pi j(2m+1)}{2M}\right)$$

→ Now let us obtain the sum b/n 0 to $M-1$. Before we can do that, we have to represent our discrete signal in terms of positional vectors.

i.e $m = 0, 1, 2, 3, 4, \ldots, M-1$

$$V_j = \frac{\sqrt{2}}{\sqrt{M}}\left(\cos\left(\frac{\pi j}{2M}\right), \cos\left(\frac{\pi j(2(1)+1)}{2M}\right), \cos\left(\frac{\pi j((2)(2)+1)}{2M}\right), \ldots\ldots, \cos\left(\frac{\pi j(2(M-1)+1)}{2M}\right)\right)^T$$

④ Applying the discrete orthogonal property (which is the foundation of DFT) on the above vectors reveals that, these vectors are orthogonal to each other.

i.e on normalizing the above expression we get:

$$V_{j\,norm} = \sqrt{\frac{2}{M} \sum_{m=0}^{M-1} \cos\left(\frac{\pi j(2m+1)}{2M}\right)}$$

$$= \sqrt{\frac{2}{M} \sum_{m=0}^{M-1} \cos^2\left(\frac{\pi j(2m+1)}{2M}\right)} \quad — ③$$

Scanned with CamScanner

Using the identity in (3)

$$\rightarrow \boxed{\cos(x) = \frac{\exp(ix) + \exp(-ix)}{2}}$$

$$= \sqrt{\frac{2}{m} \cdot \sum_{m=0}^{m-1} \left(\frac{\exp(i\pi j(2m+1))/2m) + \exp(-i\pi j(2m+1)/2m)}{2} \right)^2}$$

On simplifying further, we see that the sum of normalized vector would be $\sqrt{2}$ when $j=0$ ~~&~~ +1 when j is in the range 1 to m-1.

∴ It is evident that we can obtain the DCT (Discrete signal) by the DFT (shifted signal)

✓

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Problem 3: Signal Pyramid

1D signal $f = (7, 11, 4, 8, 5, 0, 8, 8)^T$

3a Gaussian Pyramid:

$$\text{length} = 8 = 2^3$$

$$\therefore \underline{N = 3}$$

now,

$\rightarrow$ $v^3 = v^N = f = (7, 11, 4, 8, 5, 0, 8, 8)^T$ ✓

$v^{k-1} = R_k^{k-1} v^k$

$v^{3-1} = v^2 = R_3^{3-1} v^3$

$R \rightarrow 2^{k-1} \times 2^k$ matrix

$\Rightarrow \dfrac{2^2 \times 2^3}{4 \times 8}$

$$R_3^{3-1} = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}$$

now,

$$v^2 = R_3^{3-1} v^3 \Rightarrow \left(\frac{7}{2} + \frac{11}{2}, \frac{4}{2} + \frac{8}{2}, \frac{5}{2} + \frac{0}{2}, \frac{8}{2} + \frac{8}{2}\right)$$

$$\therefore v^2 = (9, 6, 2.5, 8)$$ ✓

now,

$R \rightarrow 2^1 \times 2^2$

$\Rightarrow 2 \times 4$

$=$

$$v^1 = v^{2-1} = R_2^{2-1} v^2$$

$$\Rightarrow \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix} [9, 6, 2.5, 8]$$

$$v^1 \Rightarrow (7.5, \cancel{2.5})$$ 5.25 ✓

now,

$$\underline{v^0} = R_1^0 v^1 \Rightarrow (1/2 \ \ 1/2)(7.5, 2.5) = \underline{6.375}$$ ✓

∴ Gaussian Pyramid :-

$v^3 \Rightarrow$ 7 11 4 8 5 0 8 8

$v^2 \Rightarrow$ 9 6 2.5 8

$v^1 \Rightarrow$ 7.5 5.25

$v^0 \Rightarrow$ 6.375.

3b Laplacian Pyramids :-

$$w^3 = w^k = v^k - P^k_{k-1} v^{k-1}$$

($P \rightarrow$ interpolation operator)

$P^k_{k-1} \Rightarrow 2^k \times 2^{k-1}$ Matrix

$$\Rightarrow P^k_{k-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & & 0 \\ 0 & 1 & & & \\ 0 & 1 & & 0 & 0 \\ \vdots & 0 & & 1 & 0 \\ \vdots & 0 & & 1 & 0 \\ 0 & 0 & & 0 & 1 \\ 0 & 0 & & 0 & 1 \end{bmatrix}$$

$$w^k = (7, 11, 4, 8, 5, 0, 8, 8)$$
$$- (9, 9, 6, 6, 2.5, 2.5, 8, 8)$$

$$w^3 = w^k \Rightarrow (-2, 2, -2, 2, 2.5, -2.5, 0, 0)$$

$$w^2 = v^2 - P^2_1 v^1$$

$$\Rightarrow (9, 6, 2.5, 8) - (7.5, 7.5, 5.25, 5.25)$$

$$w^2 \Rightarrow (1.5, -1.5, -2.75, 2.75)$$

$$w^1 = v^1 - P^1_0 v^0$$

$$\Rightarrow (7.5, 5.25) - (6.375, 6.375)$$

$$w^1 \Rightarrow (1.125, -1.125)$$

$$w^0 = v^0 = 6.375.$$

∴ laplacian Pyramid:

$w^3 \rightarrow \quad -2 \quad 2 \quad -2 \quad 2 \quad 2.5 \quad -2.5 \quad 0 \quad 0$

$w^2 \Rightarrow \quad 1.5 \quad -1.5 \quad -2.75 \quad 2.75$

$w^1 \Rightarrow \quad 1.125 \quad -1.125$

$w^0 = \quad 6.375.$

—

3c Reconstruct initial signal from laplacian pyramid.

from laplacian we can reconstruct Gaussian pyramid, and we know Nth level of Gaussian pyramid is the original image itself.

∴

$$\underset{\text{Gaussian}}{v^0} = \underset{\text{(laplacian)}}{w^0} = 6.375.$$

$P^k_{k-1} = 2^k \times 2^{k-1}$ matrix

$$\Rightarrow \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \cdots \quad \cdots$$

we have,

$$\boxed{v^k = w^k + P^k_{k-1} v^{k-1}}$$

or, $v^1 = w^1 + P^1_0 v^0$

$$v^1 = (1.125, -1.125) + (6.375, 6.375)$$

$$v^1 \Rightarrow (7.5, 5.25)$$

—

$$v^2 = w^2 + P^2_1 v^1$$

$$\Rightarrow (1.5, -1.5, -2.75, 2.75) + (7.5, 7.5, 5.25, 5.25)$$

$$\Rightarrow (9, 6, 2.5, 8)$$

$v^3 = w^3 + P_2^3 \cdot v^2$

$= (-2, 2, -2, 2, 2.5, -2.5, 0, 0)$

$+ (9, 9, 6, 6, 2.5, 2.5, 8, 8)$

$\Rightarrow (7, 11, 4, 8, 5, 0, 8, 8)$.

$\therefore$ $v^3 = u = f =$ original image.

$\therefore$ we are able to reconstruct original image also goussion pyromid using loplocion pyromid.

3d Loplocion pyromid follows bondposs, i.e it does not down somple at poths of high frequency, which results in an additional 1/3 size. compared to wovelet.

loplocion pyromid is only localised in frequency but not in space. It does frequency decomposition but has size more thon original image.

It loplocion we have several frequency bonds, i.e good localisation at fine scales but bod localisation at coarse scoles, therefore it has redundant data as it lossless and we can reconstruct original image.

$\therefore$ loploce pyromid are redundant hence we use wovelet for general purpose.

(M)

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4a)

We need to set the coefficient with a high contribution to the spectrum to 0 to successfully remove line artifacts.

For the given question we have given horizontal artifacts, **hence we use lowpass filter** as we want to let low frequency pass through but sample high frequency so that we don't get overlap between frequency bands.

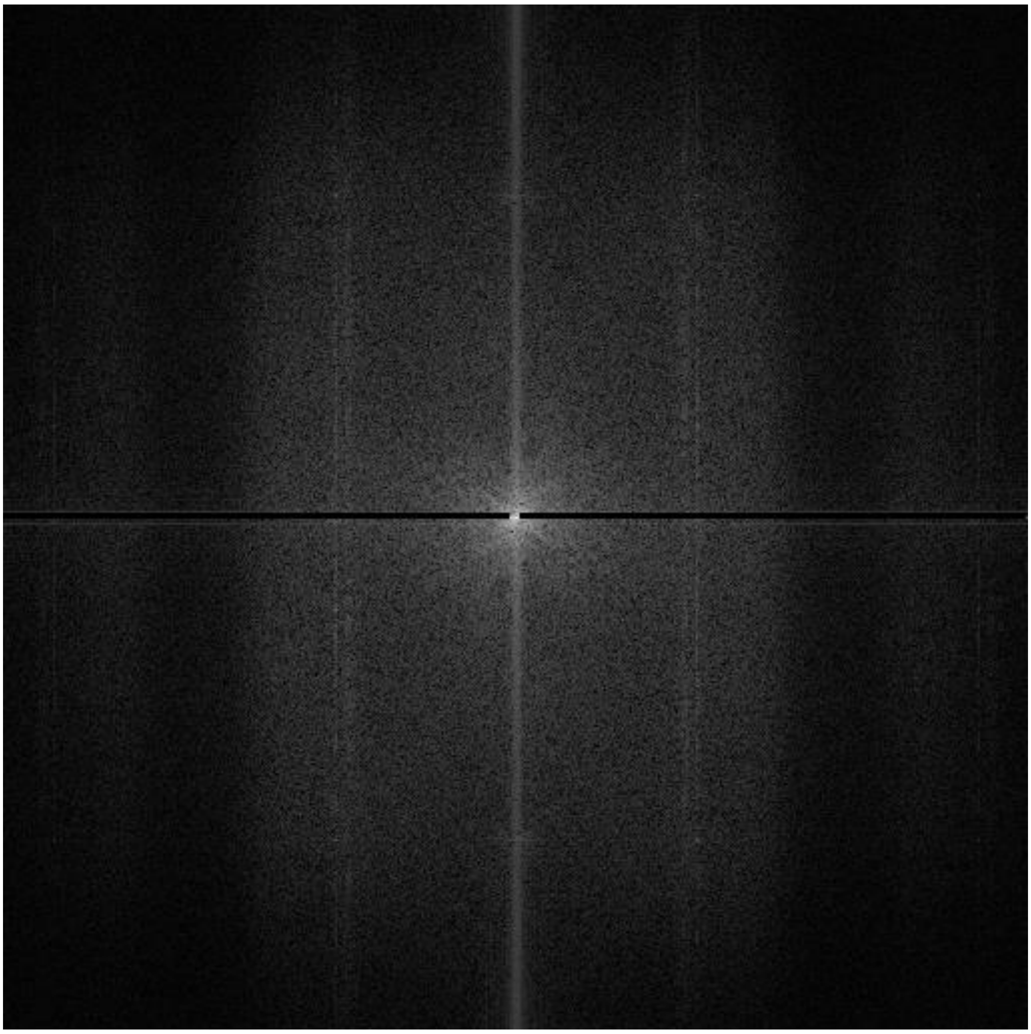
When we get a frequency band that is more distant to centre in x-dimension and less distant to centre in y-dimension we set its pixel to 0 for both real and imaginary part. **So we make the coefficients resulting to high frequency 0.**

Original Smoke Image:

Image after filtering with inverse DFT:

Here we can see that the artifacts has been removed.

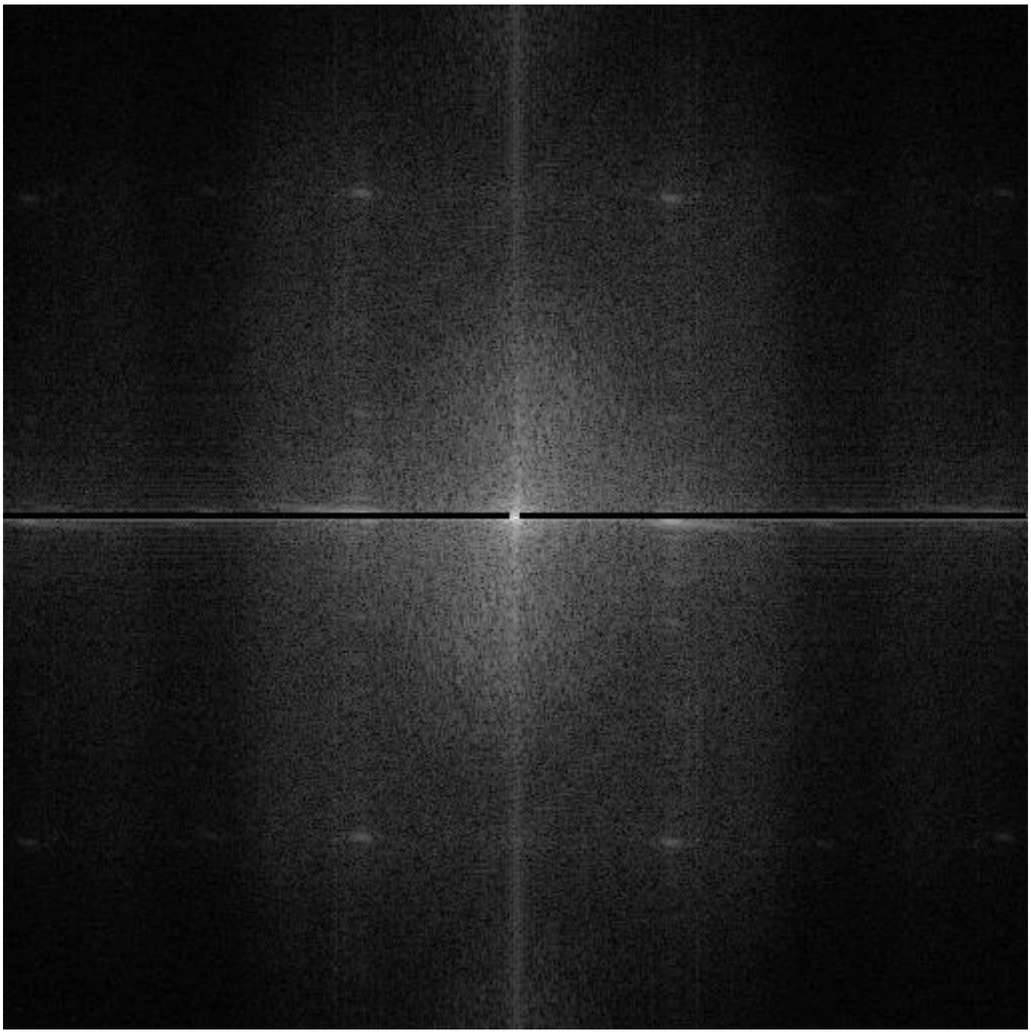
Log Spectrum after ~~filter and inverse~~ DFT:



4b)

Image after applying filter and inverse DFT:

Log spectrum after filter for fire image:



Explanation:

In the original fire image, we can see some vertical textures in the iron gate.

These vertical structures has almost similar frequencies to the artifacts, so when we try to filter for the artifacts these patterns also gets impacted, and the image get deteriorated as we can see in the above image. The original structure is deteriorated.

✓

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