Network Models

Learning Objectives

After completing this chapter, students will be able to:

1. Connect all points of a network while minimizing total distance using the minimal-spanning tree technique.
2. Determine the maximum flow through a network using the maximal-flow technique and linear programming.
3. Find the shortest path through a network using the shortest-route technique and linear programming.
4. Understand the important role of software in solving network problems.

Chapter Outline

Introduction

- This chapter covers three network models that can be used to solve a variety of problems.
- The *minimal-spanning tree technique* determines a path through a network that connects all the points while minimizing the total distance.
- The *maximal-flow technique* finds the maximum flow of any quantity or substance through a network.
- The *shortest-route technique* can find the shortest path through a network.

Introduction

- Large scale problems may require hundreds or thousands of iterations making efficient computer programs a necessity.
- All types of networks use a common terminology.
- The points on a network are called *nodes* and may be represented as circles of squares.
- The lines connecting the nodes are called *arcs.*

Minimal-Spanning Tree Technique

- The minimal-spanning tree technique involves connecting all the points of a network together while minimizing the distance between them.
- The Lauderdale Construction Company is developing a housing project.
- It wants to determine the least expensive way to provide water and power to each house.
- There are eight houses in the project and the distance between them is shown in Figure 11.1.

Network for Lauderdale Construction

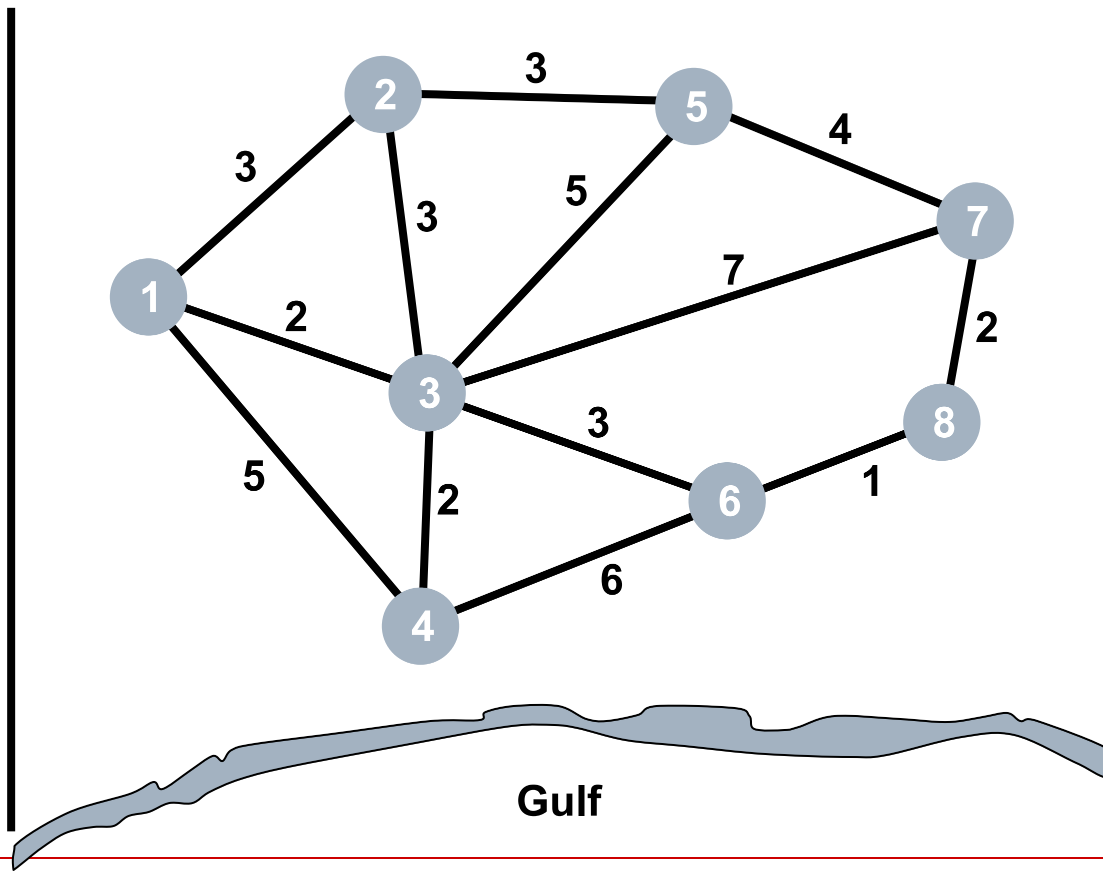


Figure 11.1

Steps for the Minimal-Spanning Tree Technique

1. Select any node in the network.
2. Connect this node to the nearest node that minimizes the total distance.
3. Considering all the nodes that are now connected, find and connect the nearest node that is not connected. If there is a tie, select one arbitrarily. A tie suggests there may be more than one optimal solution.
4. Repeat the third step until all nodes are connected.

Lauderdale Construction Company

- Start by arbitrarily selecting node 1.
- The nearest node is node 3 at a distance of 2 (200 feet) and we connect those nodes.
- Considering nodes 1 and 3, we look for the next nearest node.
- This is node 4, the closest to node 3.
- We connect those nodes.
- We now look for the nearest unconnected node to nodes 1, 3, and 4.
- This is either node 2 or node 6.
- We pick node 2 and connect it to node 3.

Minimal-Spanning Tree Technique

- Following this same process we connect from node 2 to node 5.
- We then connect node 3 to node 6.
- Node 6 will connect to node 8.
- The last connection to be made is node 8 to node 7.
- The total distance is found by adding up the distances in the arcs used in the spanning tree:

  2 + 2 + 3 + 3 + 3 + 1 + 2 = 16 (or 1,600 feet)

Minimal-Spanning Tree Technique

First Iteration for Lauderdale Construction

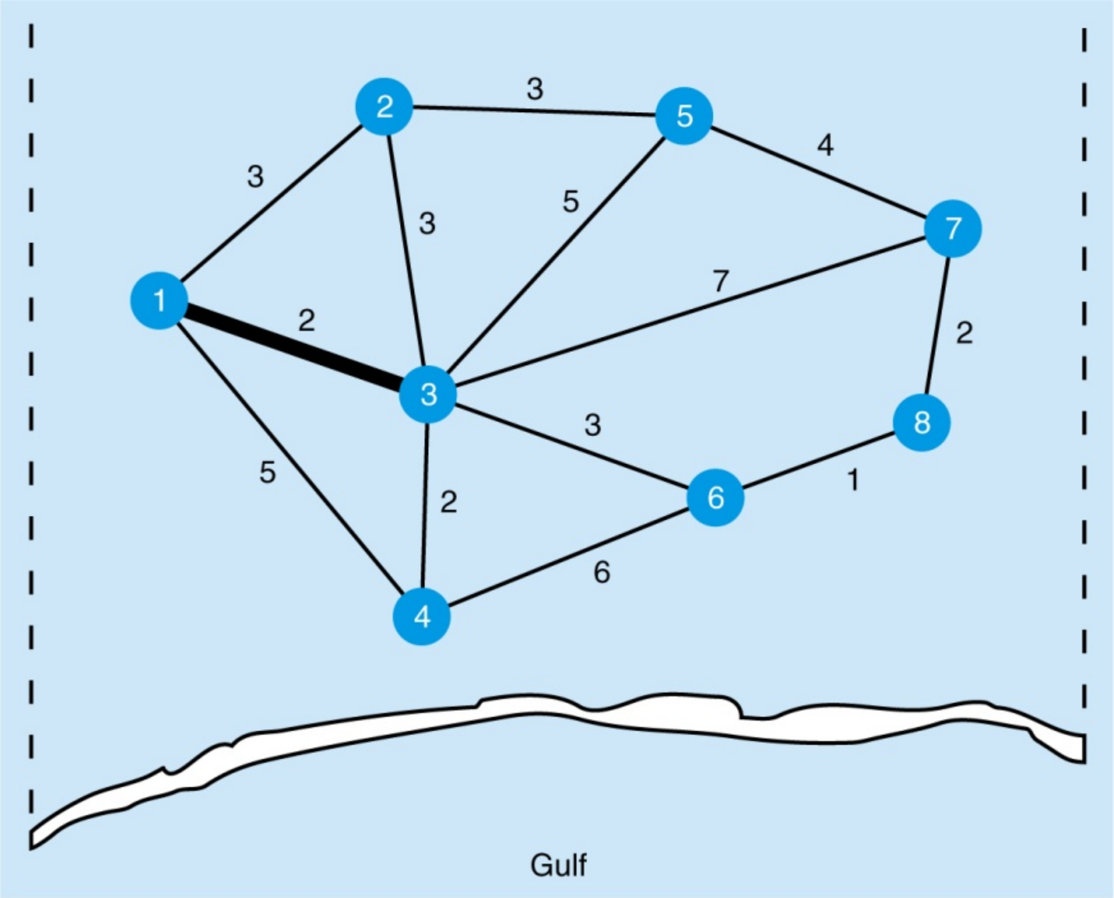


Figure 11.2

Minimal-Spanning Tree Technique

Second and Third Iterations for Lauderdale Construction

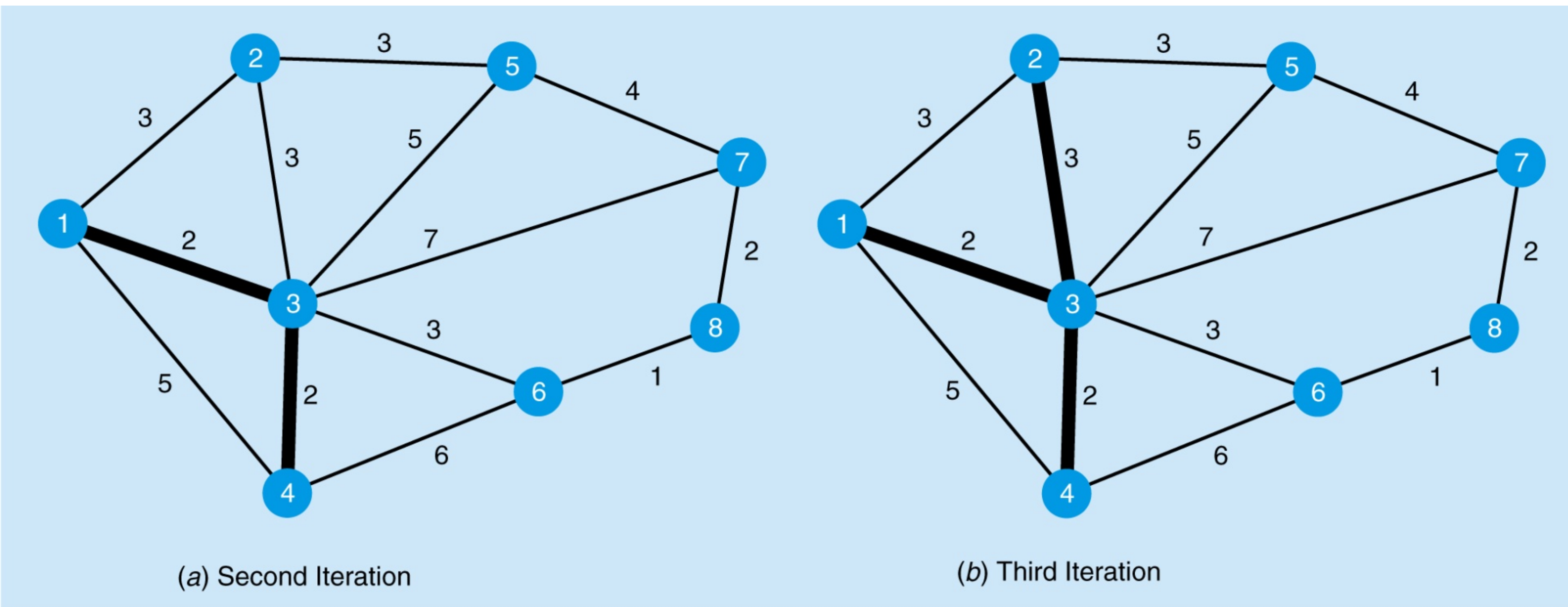
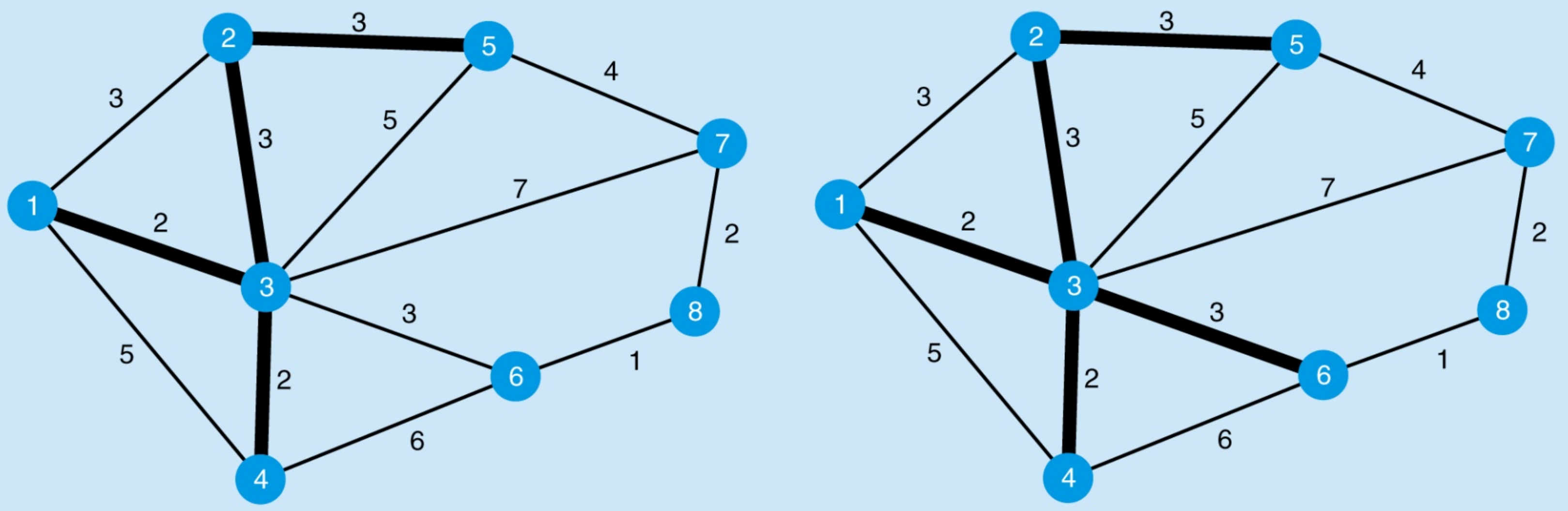


Figure 11.3

Minimal-Spanning Tree Technique

Fourth and Fifth Iterations for Lauderdale Construction



(*a*) Fourth Iteration

(*b*) Fifth Iteration

Figure 11.4

Minimal-Spanning Tree Technique

Sixth and Seventh (Final) Iterations for Lauderdale Construction

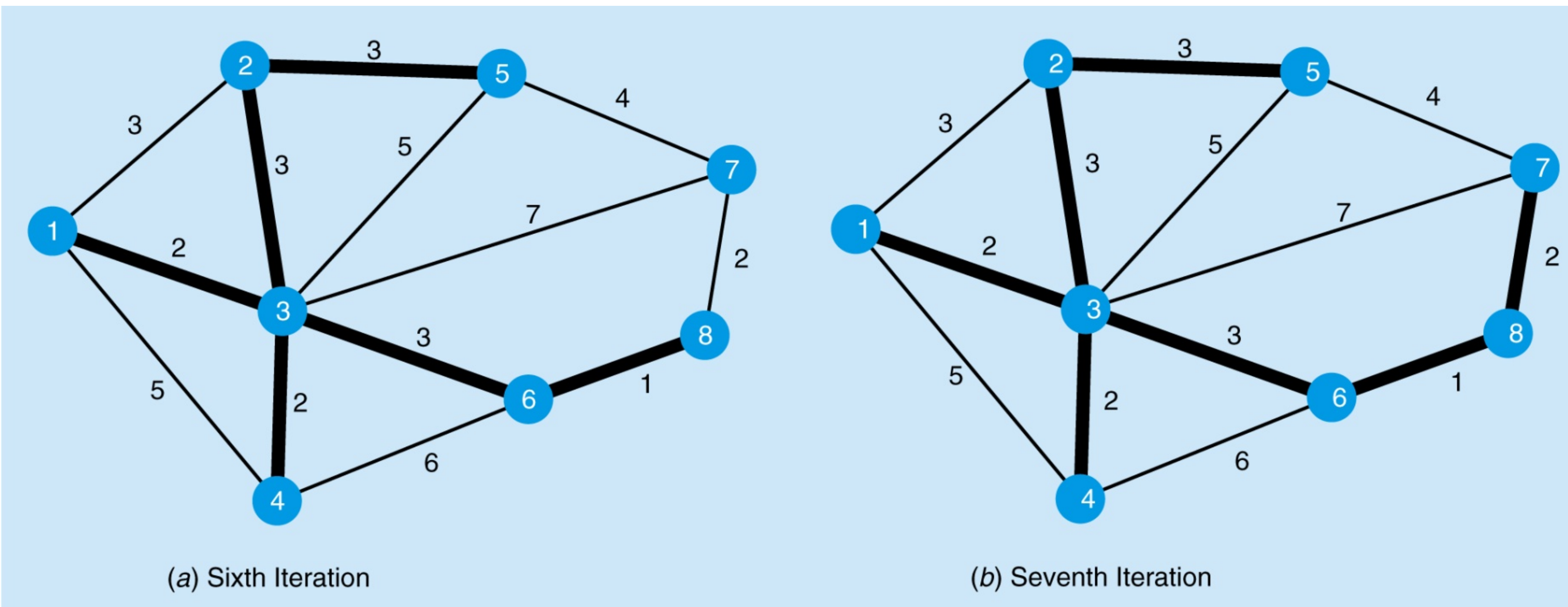


Figure 11.5

Summary of Steps in Lauderdale Construction Minimal-Spanning Tree Problem

Step	Connected Nodes	Unconnected Nodes	Closest Un-connected Node	Arc Selected	Arc Length	Total Distance
1	**1**	**2,3,4,5,6,7,8**	**3**	**1-3**	**2**	**2**
2	**1,3**	**2,4,5,6,7,8**	**4**	**3-4**	**2**	**4**
3	**1,3,4**	**2,5,6,7,8**	**2 or 6**	**2-3**	**3**	**7**
4	**1,2,3,4**	**5,6,7,8**	**5 or 6**	**2-5**	**3**	**10**
5	**1,2,3,4,5**	**6,7,8**	**6**	**3-6**	**3**	**13**
6	**1,2,3,4,5,6**	**7,8**	**8**	**6-8**	**1**	**14**
7	**1,2,3,4,5,6,8**	**7**	**7**	**7-8**	**2**	**16**

Table 11.1

QM for Windows Solution for Lauderdale Construction Company Minimal Spanning Tree Problem

Starting node for iterations: 1

Note: Multiple optimal solutions exist

Networks Results

Lauderdale Construction Company Solution

Branch name	Start node	End node	Cost	Include	Cost
Branch 1	1	2	3	Y	3
Branch 2	1	3	2	Y	2
Branch 3	1	4	5		
Branch 4	2	3	3		
Branch 5	2	5	3	Y	3
Branch 6	3	4	2	Y	2
Branch 7	3	5	5		
Branch 8	3	6	3	Y	3
Branch 9	3	7	7		
Branch 10	4	6	6		
Branch 11	5	7	4		
Branch 12	6	8	1	Y	1
Branch 13	7	8	2	Y	2
Total					16

Program 11.1

Maximal-Flow Technique

- The maximal-flow technique allows us to determine the maximum amount of a material that can flow through a network.
- Waukesha, Wisconsin is in the process of developing a road system for the downtown area.
- Town leaders want to determine the maximum number of cars that can flow through the town from west to east.
- The road network is shown in Figure 11.6.
- The numbers by the nodes indicate the number of cars that can flow *from* the node.

Maximal-Flow Technique

Road network for Waukesha

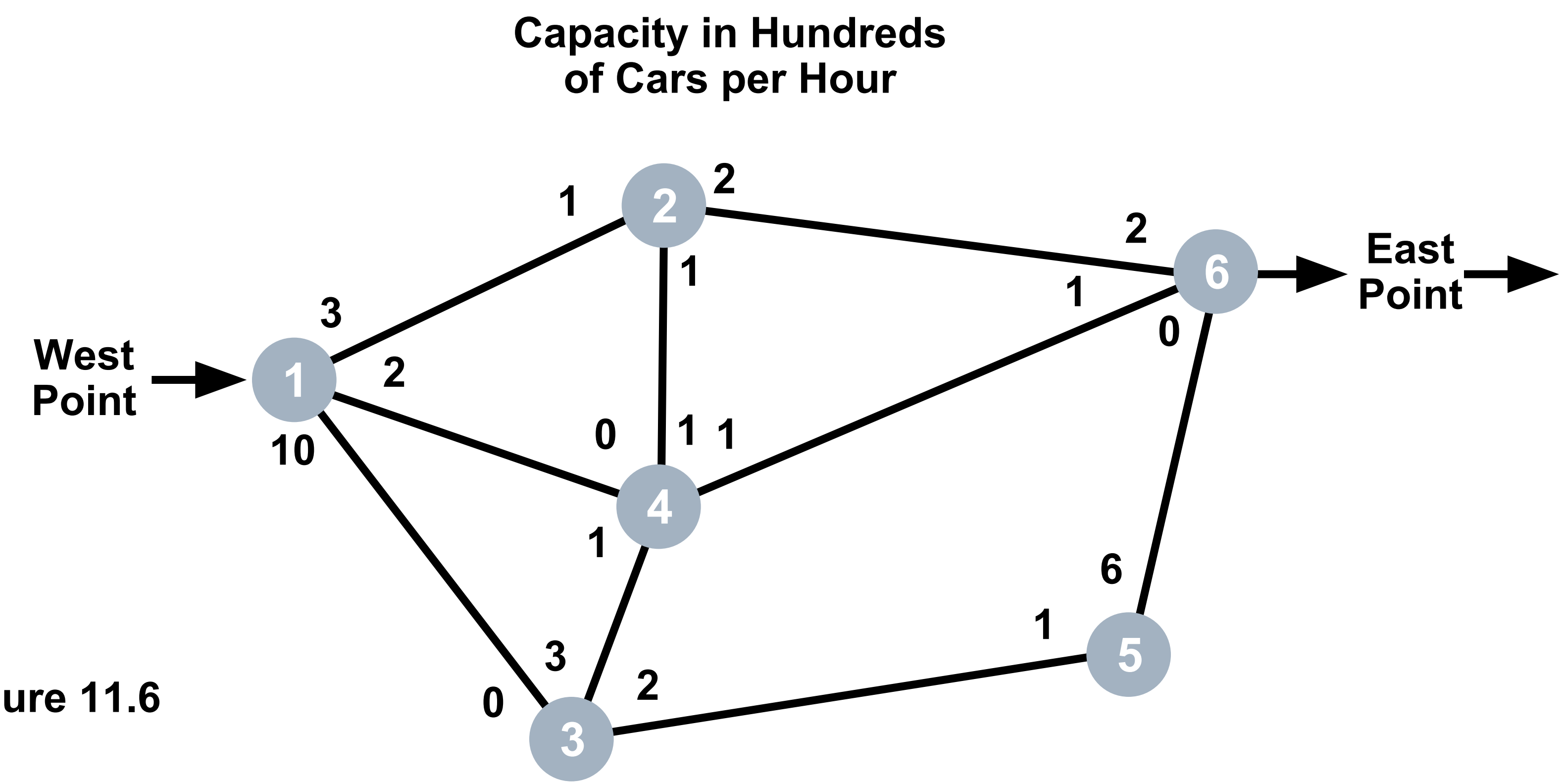


Figure 11.6

Maximal-Flow Technique

Four steps of the Maximal-Flow Technique

1. Pick any path from the start (*source*) to the finish (*sink*) with some flow. If no path with flow exists, then the optimal solution has been found.
2. Find the arc on this path with the smallest flow capacity available. Call this capacity C. This represents the maximum additional capacity that can be allocated to this route.

Maximal-Flow Technique

Four steps of the Maximal-Flow Technique

3. For each node on this path, decrease the flow capacity in the direction of flow by the amount C. For each node on the path, increase the flow capacity in the reverse direction by the amount C.
4. Repeat these steps until an increase in flow is no longer possible.

Maximal-Flow Technique

- We start by arbitrarily picking the path 1–2–6 which is at the top of the network.
- The maximum flow is 2 units from node 2 to node 6.
- The path capacity is adjusted by adding 2 to the westbound flows and subtracting 2 from the eastbound flows.
- The result is the new path in Figure 11.7 which shows the new relative capacity of the path at this stage.

Maximal-Flow Technique

Capacity Adjustment for Path 1–2–6 Iteration 1

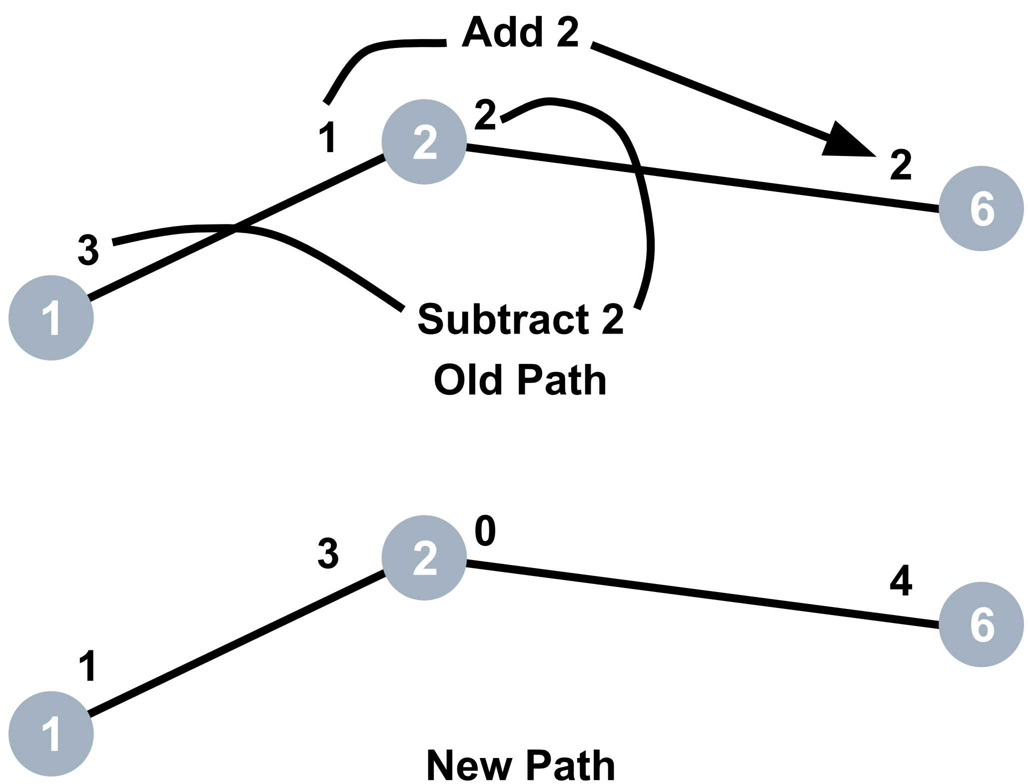


Figure 11.7

Maximal-Flow Technique

- We repeat this process by picking the path 1–2–4–6.
- The maximum capacity along this path is 1.
- The path capacity is adjusted by adding 1 to the westbound flows and subtracting 1 from the eastbound flows.
- The result is the new path in Figure 11.8.
- We repeat this process by picking the path 1–3–5–6.
- The maximum capacity along this path is 2.
- Figure 11.9 shows this adjusted path.

Maximal-Flow Technique

Second Iteration for Waukesha Road System

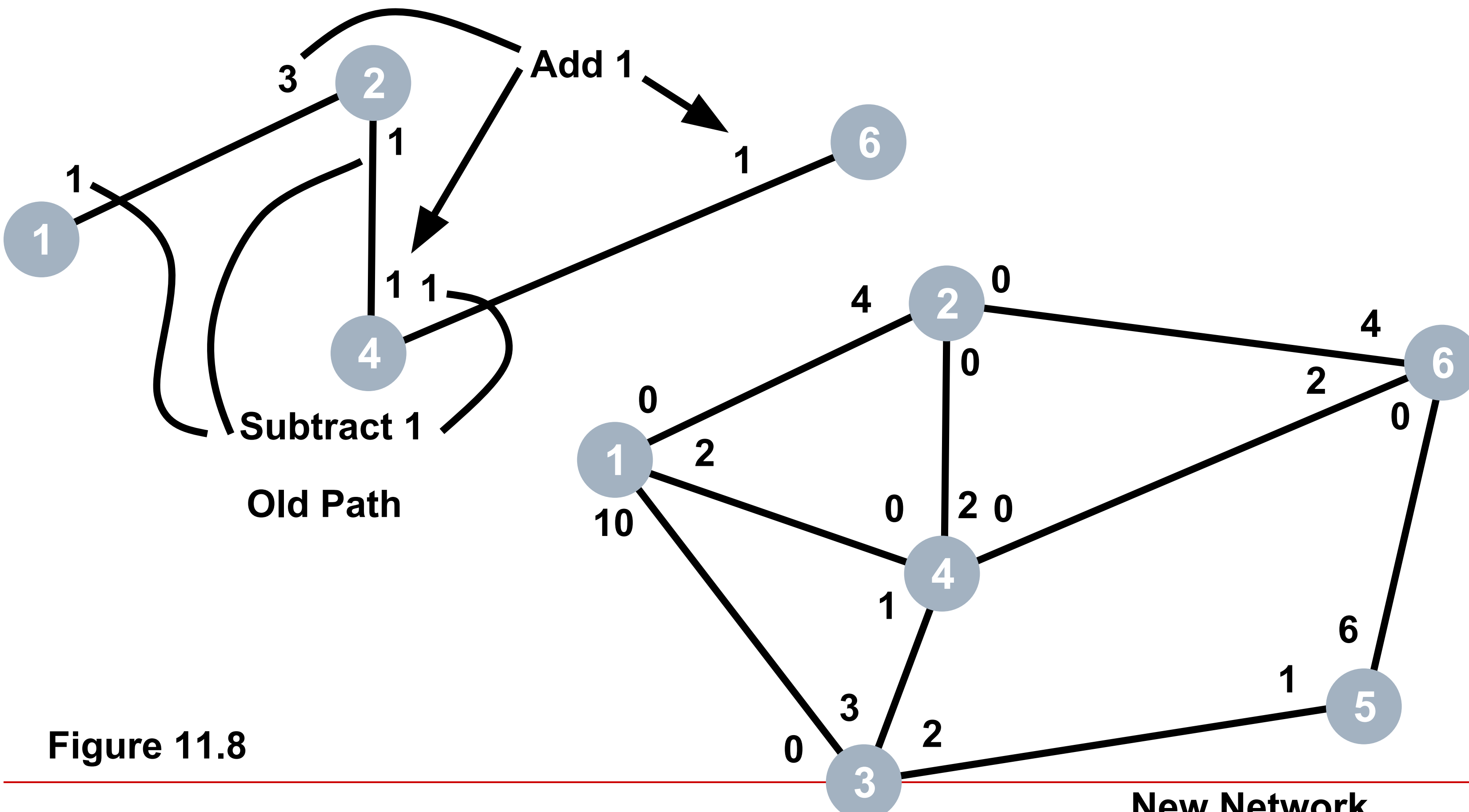


Figure 11.8

Maximal-Flow Technique

Third and Final Iteration for Waukesha Road System

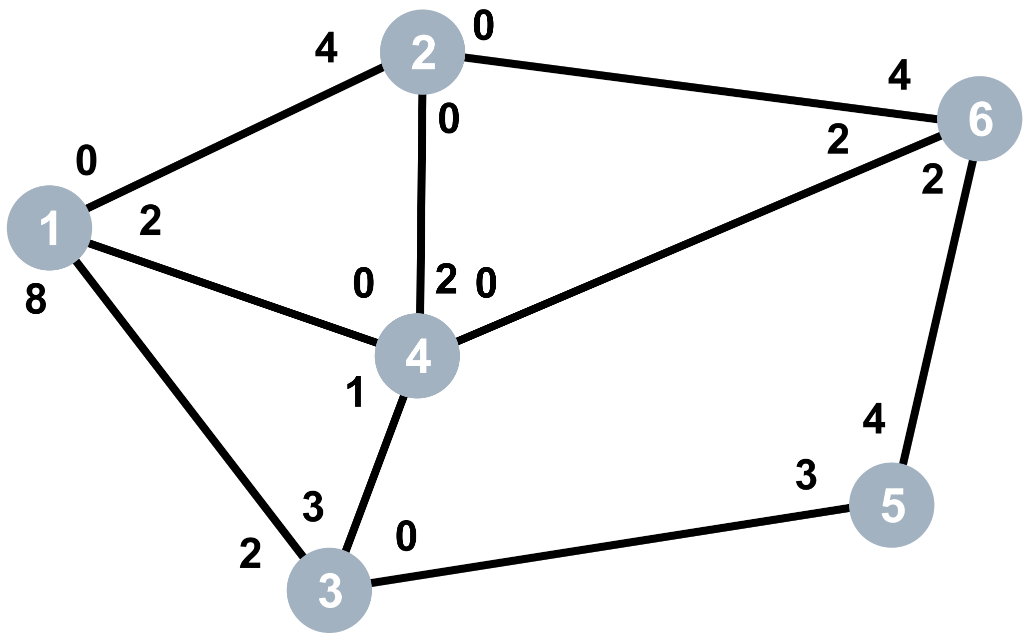


Figure 11.9

Maximal-Flow Technique

- There are no more paths from nodes 1 to 6 with unused capacity so this represents a final iteration.
- The maximum flow through this network is 500 cars.

PATH		FLOW (CARS PER HOUR)
1–2–6		200
1–2–4–6		100
1–3–5–6		200
	Total	500

Linear Programming for Maximal Flow

- Define the variables as:
  - *X_{ij} = flow from node i to node j.*

- *Goal: Maximize Flow = X_{61}*

Linear Programming for Maximal Flow

Constraints

$X_{12} \leq 3$	$X_{13} \leq 10$	$X_{14} \leq 2$
$X_{21} \leq 1$	$X_{24} \leq 1$	$X_{26} \leq 2$
$X_{34} \leq 3$	$X_{35} \leq 2$	$X_{42} \leq 1$
$X_{43} \leq 1$	$X_{46} \leq 1$	$X_{53} \leq 1$
$X_{56} \leq 1$	$X_{62} \leq 2$	$X_{64} \leq 1$

Linear Program for Maximal Flow

Constraints continued:

$X_{61} = X_{12} + X_{13} + X_{14}$ or $X_{61} - X_{12} - X_{13} - X_{14} = 0$

$X_{12}+X_{42}+X_{62} = X_{21}+X_{24}+X_{26}$ or $X_{12}+X_{42}+X_{62}-X_{21}-X_{24}-X_{26} = 0$

$X_{13}+X_{43}+X_{53} = X_{34} + X_{35}$ or $X_{13}+X_{43}+X_{53}-X_{34}-X_{35} = 0$

$X_{14}+X_{24}+X_{34}+X_{64} = X_{42} + X_{43} + X_{46}$ or $X_{14}+X_{24} + X_{34}+X_{64} - X_{42} - X_{43} - X_{53} = 0$

$X_{35} = X_{56} + X_{53}$ or $X_{35} - X_{53} - X_{56} = 0$

$X_{26}+X_{46}+X_{56} = X_{61}$ or $X_{26} + X_{46} + X_{56} - X_{61} = 0$

$X_{ij} \geq 0$ and integer

This problems can now be solved in QM for Windows or using Excel Solver.

QM for Windows Solution for Waukesha Road Network Maximal Flow Problem

Source 1 | Sink 6

Networks Results

Waukesha Road Network Solution

Branch name	Start node	End node	Capacity	Reverse capacity	Flow
Maximal Network Flow	5				
Branch 1	1	2	3	1	3
Branch 2	1	3	10	0	2
Branch 3	1	4	2	0	0
Branch 4	2	4	1	1	1
Branch 5	2	6	2	2	2
Branch 6	3	4	3	1	0
Branch 7	3	5	2	1	2
Branch 8	4	6	1	1	1
Branch 9	5	6	6	0	2

Program 11.2

Shortest-Route Problem

- The *shortest-route technique* identifies how a person or item can travel from one location to another while minimizing the total distance traveled.
- It finds the shortest route to a series of destinations.
- Ray Design, Inc. transports beds, chairs, and other furniture from the factory to the warehouse.
- The company would like to find the route with the shortest distance.
- The road network is shown in Figure 11.10.

Shortest-Route Problem

Roads from Ray's Plant to Warehouse

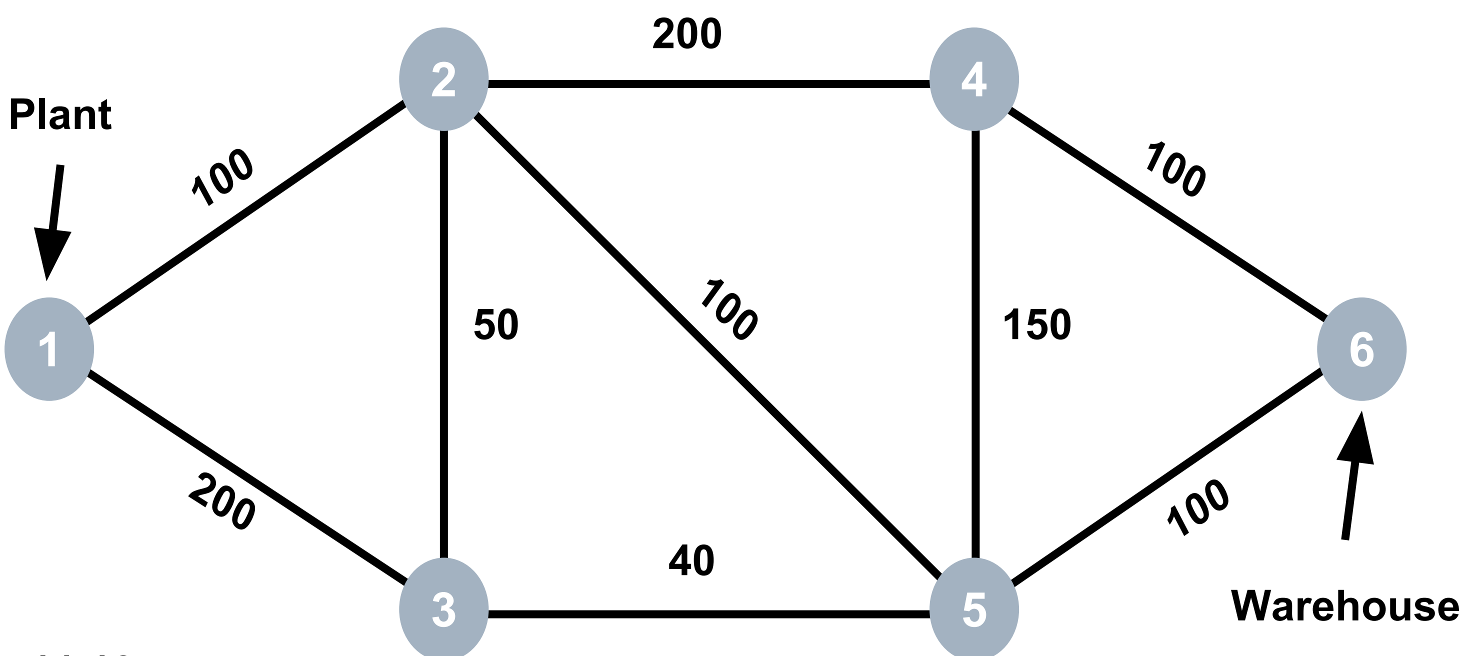


Figure 11.10

Shortest-Route Problem

Steps of the shortest-route technique:

1. Find the nearest node to the origin (plant). Put the distance in a box by the node.
2. Find the next-nearest node to the origin and put the distance in a box by the node. Several paths may have to be checked to find the nearest node.
3. Repeat this process until you have gone through the entire network. The last distance at the ending node will be the distance of the shortest route.

Shortest-Route Technique

- We can see that the nearest node to the plant is node 2.
- We connect these two nodes.
- After investigation, we find node 3 is the next nearest node but there are two possible paths.
- The shortest path is 1–2–3 with a distance of 150.
- We repeat the process and find the next node is node 5 by going through node 3.
- The next nearest node is either 4 or 6 and 6 turns out to be closer.
- The shortest path is 1–2–3–5–6 with a distance of 290 miles.

Shortest-Route Problem

First Iteration for Ray Design

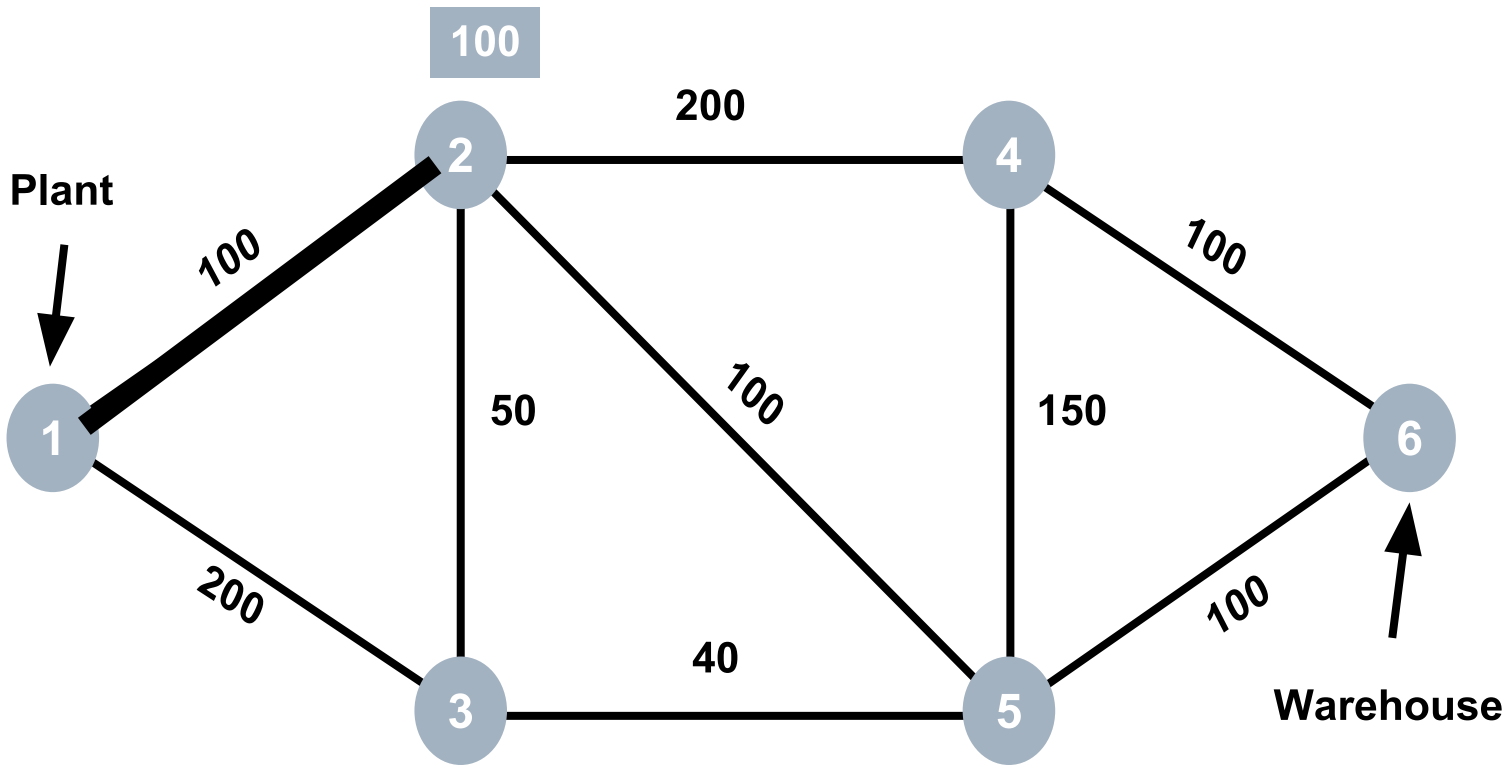


Figure 11.11

Shortest-Route Technique

Second Iteration for Ray Design

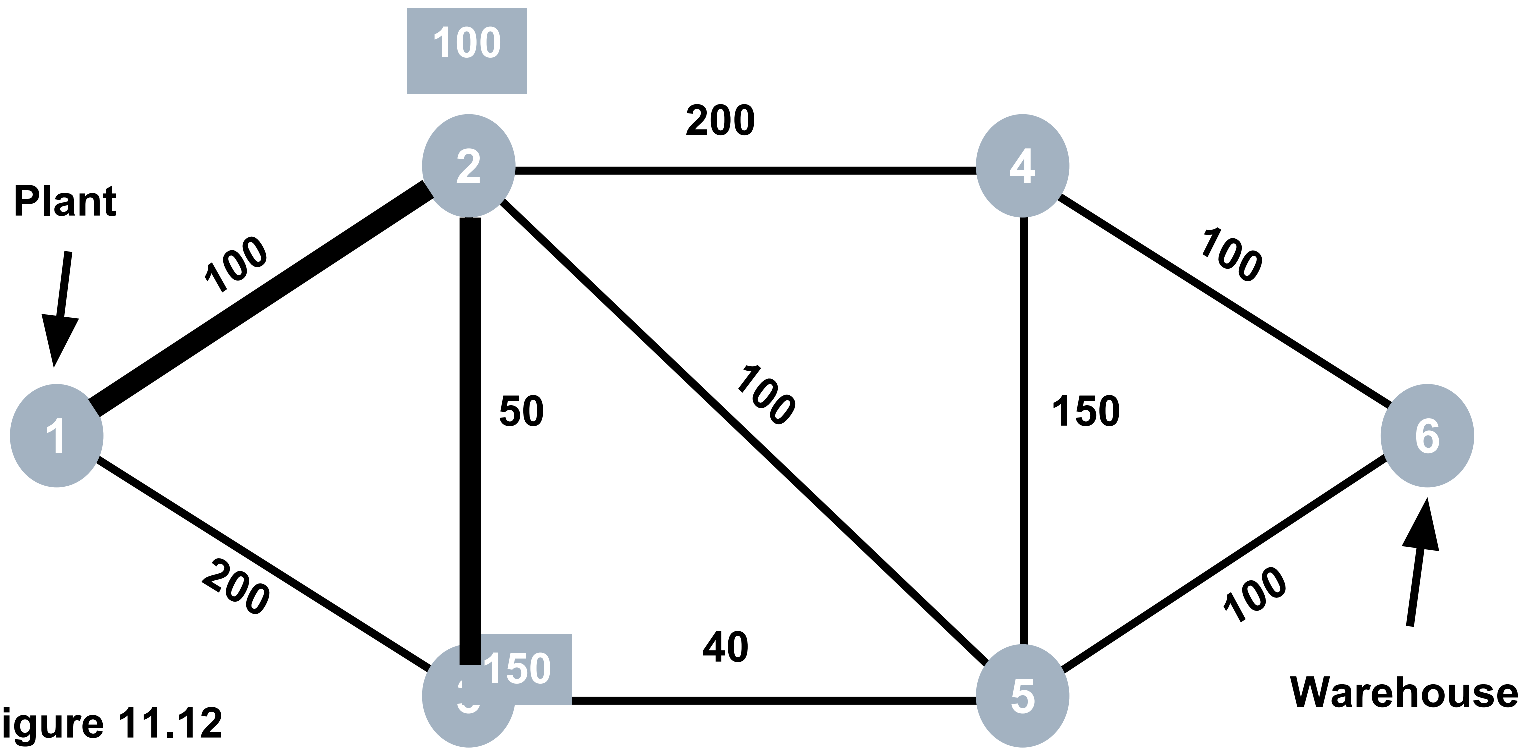


Figure 11.12

Shortest-Route Technique

Third Iteration for Ray Design

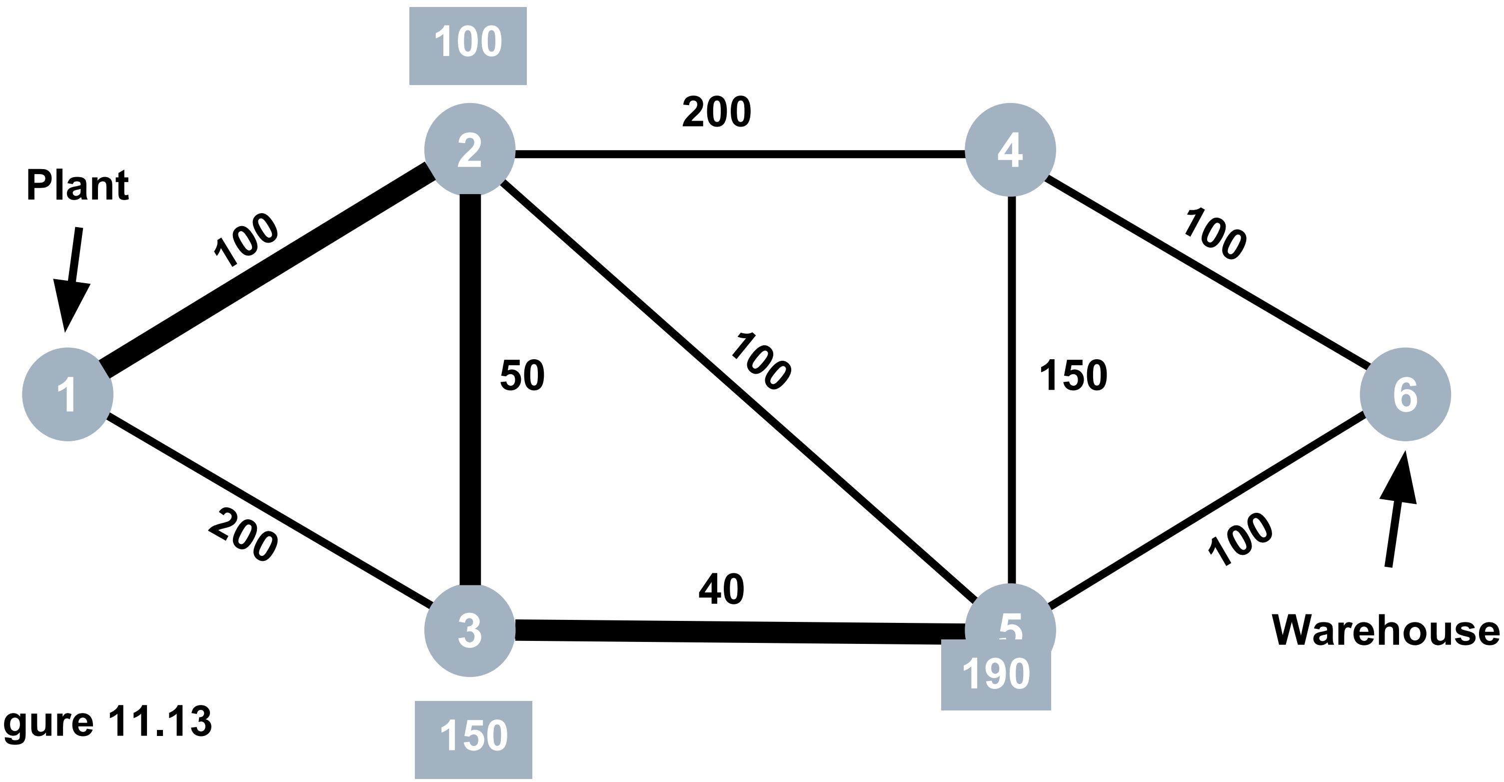


Figure 11.13

Shortest-Route Technique

Fourth and Final Iteration for Ray Design

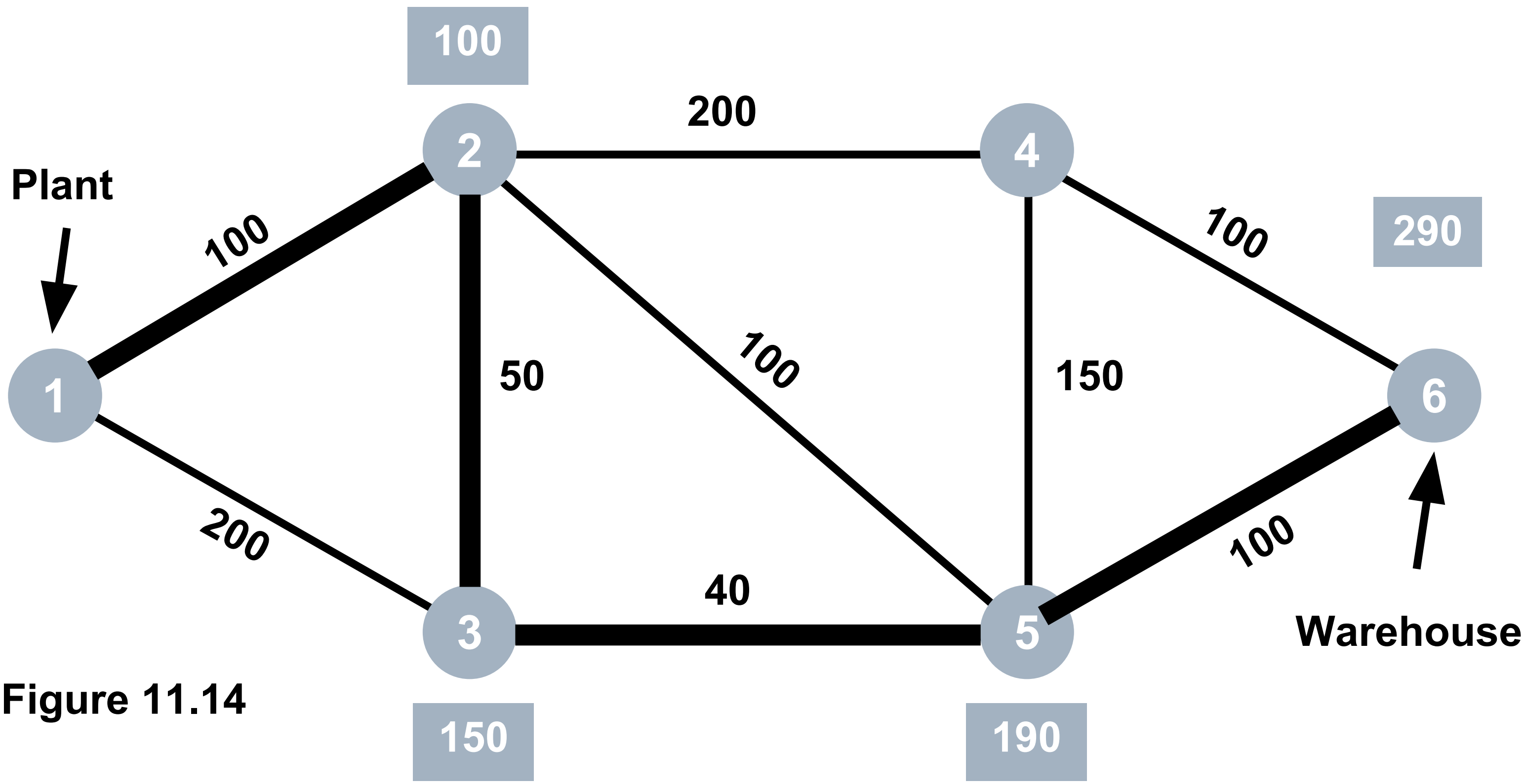


Figure 11.14

Linear Program for Shortest-Route Problem

- Objective is to minimize the total distance (cost) from the start to finish.
- Variables:

  X_{ij} = 1 if arc from node *i* to node *j* is selected
  = 0 otherwise.
- It is helpful to view this as a transshipment problem.

Linear Program for Shortest-Route Problem

Minimize distance =

$$100X_{12} + 200X_{13} + 50X_{23} + 50X_{32} + 200X_{24} + 200X_{42} + 100X_{25} + 100X_{52} + 40X_{35} + 40X_{53} + 150X_{45} + 150X_{54} + 100X_{46} + 100X_{56}$$

Subject to:

$X_{12} + X_{13}$	$= 1$	Node 1
$X_{12} + X_{32} - X_{23} - X_{24} - X_{25}$	$= 0$	Node 2
$X_{13} + X_{23} - X_{32} - X_{35}$	$= 0$	Node 3
$X_{24} + X_{54} - X_{42} - X_{45} - X_{46}$	$= 0$	Node 4
$X_{25} + X_{35} + X_{45} - X_{52} - X_{53} - X_{54} - X_{56}$	$= 0$	Node 5
$X_{46} + X_{56}$	$= 1$	Node 6
All variables	$= 0$ or 1	

This problems can now be solved in QM for Windows or using Excel Solver.

QM for Windows Input Screen for Ray Design, Inc., Shortest-Route Problem

Network type: (•) Undirected ( ) Directed

Origin: 1

Destination: 6

Ray Design, Inc.	Start node	End node	Distance
Branch 1	1	2	100
Branch 2	1	3	200
Branch 3	2	3	50
Branch 4	2	4	200
Branch 5	2	5	100
Branch 6	3	5	40
Branch 7	4	5	150
Branch 8	4	6	100
Branch 9	5	6	100

Program 11.3A

QM for Windows Solution Screen for Ray Design, Inc., Shortest-Route Problem

Network type
- (•) Undirected
- ( ) Directed

Origin: 1

Destination: 6

Networks Results

Ray Design, Inc. Solution

Total distance = 290	Start node	End node	Distance	Cumulative Distance
Branch 1	1	2	100	100
Branch 3	2	3	50	150
Branch 6	3	5	40	190
Branch 9	5	6	100	290

Program 11.3B

Source and Reference

- Textbook: Barry Render and Ralph M. Stair
- Instructor support material from pHI (CD accompanying textbook / Online pHI support)