Transportation Problem (TP) and Assignment Problem (AP)

(special cases of Linear Programming)

1. Transportation Problem (TP)

Distributing any commodity from any group of supply centers, called *sources*, to any group of receiving centers, called *destinations*, in such a way as to minimize the total distribution cost (shipping cost).

1. Transportation Problem (TP)

Total supply must equal total demand.

If total supply exceeds total demand, a *dummy destination*, whose demand equals the difference between the total supply and total demand is created. Similarly if total supply is less than total demand, a *dummy source* is created, whose supply equals the difference.

All unit shipping costs into a dummy destination or out of a dummy source are 0.

Example 1:

		DESTINATIONS				Supply
		D_1	D_2	D_3	D_4	
	S_1	50	75	30	45	12
Sources	S_2	65	80	40	60	17
	S_3	40	70	50	55	11
Demand		10	10	10	10	

Example 2:

	Destination				Supply
	D1	D2	D3	D4	
S1	50	75	35	75	12
Source S2	65	80	60	65	17
S3	40	70	45	55	11
(D)	0	0	0	0	10
Demand	15	10	15	10	

Transportation Tableau:

	D_1	D_2	D_3	D_4	Supply
S_1	50	75	30	45	12
S_2	65	80	40	60	17
S_3	40	70	50	55	11
Demand	10	10	10	10	

Shipment will be placed here

Unit Shipping Cost

Initial Solution Procedure:

1. Northwest Corner Starting Procedure

1. Select the *remaining* variable in the upper left (northwest) corner and note the supply remaining in the row, s, and the demand remaining in the column, d.

2. Allocate the minimum of s or d to this variable. If this minimum is s, eliminate all variables in its row from future consideration and reduce the demand in its column by s; if the minimum is d, eliminate all variables in the column from future consideration and reduce the supply in its row by d.

REPEAT THESE STEPS UNTIL ALL SUPPLIES HAVE BEEN ALLOCATED.

	D_1	D_2	D_3	D_4	Supply
S_1	50 10	75 2	30	45	12
S_2	65	80 8	40 9	60	17
S_3	40	70	50 1	55 10	11
Demand	10	10	10	10	

Total sipping cost = 2250

2. Least Cost Starting Procedure

1. For the remaining variable with the lowest unit cost, determine the remaining supply left in its row, s, and the remaining demand left in its column, d (break ties arbitrarily).

2. Allocate the minimum of s or d to this variable. If this minimum is s, eliminate all variables in its row from future consideration and reduce the demand in its column by s; if the minimum is d, eliminate all variables in the column from future consideration and reduce the supply in its row by d.

REPEAT THESE STEPS UNTIL ALL SUPPLIES HAVE BEEN ALLOCATED.

Least Cost Cell	Remaining Supply in the Row (s)	Remaining Demand in the Column (d)	Allocate	Modifications
1. X_{13} Cost = 30	12	10	$X_{13} = 10$	Eliminate column 3; reduce row 1 supply to $12 - 10 = 2$
2. X_{31} Cost = 40	11	10	$X_{31} = 10$	Eliminate column 1; reduce row 3 supply to $11 - 10 = 1$
3. X_{14} Cost = 45	2	10	$X_{14} = 2$	Eliminate row 1; reduce column 4 demand to $10 - 2 = 8$
4. X_{34} Cost = 55	1	8	$X_{34} = 1$	Eliminate row 3; reduce column 4 demand to $8 - 1 = 7$

	D_1	D_2	D_3	D_4	Supply
S_1	50	75	30 10	45 2	12
S_2	65	80 10	40	60 7	17
S_3	40 10	70	50	55 1	11
Demand	10	10	10	10	

Total sipping cost = 2065

3. Vogel's Approximation Method Starting Procedure

1. For each remaining row and column, determine the difference between the lowest two *remaining* costs; these are called the *row and column penalties*.

2. Select the row or column with the largest penalty found in step 1 and note the supply remaining for its row, s, and the demand remaining in its column, d.

3. Allocate the minimum of s or d to the variable in the selected row or column with the lowest remaining unit cost. If this minimum is s, eliminate all variables in its row from future consideration and reduce the demand in its column by s; if the minimum is d, eliminate all variables in the column from future consideration and reduce the supply in its row by d.

REPEAT THESE STEPS UNTIL ALL SUPPLIES HAVE BEEN ALLOCATED.

	D_1	D_2	D_3	D_4	Supply
S_1	50	75 2	30	45 10	12
S_2	65	80 7	40 10	60	17
S_3	40 10	70 1	50	55	11
Demand	10	10	10	10	

Total sipping cost = 2030

Solving TP – Transportation Simplex Method

1. Find the current $C_{ij}–Z_{ij}$ values for each nonbasic variable and select the one with the most negative $C_{ij}–Z_{ij}$ value as the entering variable; if all $C_{ij}–Z_{ij}$ values are nonnegative, the current solution is optimal.

2. Determine which basic variable reaches 0 first when the entering variable is increased.

3. Determine a new basic solution and repeat the steps.

Step 1: Determine the C_{ij}–Z_{ij} Values for the Nonbasic Variables

1. If U_i is the dual variable associated with the i-th supply constraint, and V_j is the dual variable associated with the j-th demand constraint, then for shipments from node i to node j, one can find the corresponding Z_{ij} value by $Z_{ij} = U_i - V_j$. Thus the C_{ij}–Z_{ij} value for variable X_{ij} is found by

$$C_{ij} - Z_{ij} = C_{ij} - (U_i - V_j) = C_{ij} - U_i + V_j$$

2. Given that there is a redundant equation among the m n constraints (and any of the m n constraints can be considered the redundant one), one can show that the U_i or V_j associated with the redundant equation is 0. Thus one U_i or V_j can arbitrarily be selected and set to 0. Arbitrarily choose $U_1 = 0$.

3. Since the C_{ij}–Z_{ij} values for *basic* variables are 0 (i.e., $C_{ij} - U_i + V_j = 0$ for basic variables), we can easily solve for the remaining values of the U_i's and V_j's from the m + n - 1 equations for the basic variables.

4. Once the U_i's and V_j's have been determined, the $C_{ij}–Z_{ij}$ values for the nonbasic variables can be calculated by

$$C_{ij} - Z_{ij} = C_{ij} - U_i + V_j$$

Basic Cell	Basic C_{ij}–Z_{ij} Values = 0	Substitute	Implies
X_{12}	$C_{12} - U_1 + V_2 = 0$	$75 - 0 + V_2 = 0$	$V_2 = -75$
X_{14}	$C_{14} - U_1 + V_4 = 0$	$45 - 0 + V_4 = 0$	$V_4 = -45$
X_{22}	$C_{22} - U_2 + V_2 = 0$	$80 - U_2 + (-75) = 0$	$U_2 = \quad 5$
X_{23}	$C_{23} - U_2 + V_3 = 0$	$40 - 5 + V_3 = 0$	$V_3 = -35$
X_{32}	$C_{32} - U_3 + V_2 = 0$	$70 - U_3 + (-75) = 0$	$U_3 = - \ 5$
X_{31}	$C_{31} - U_3 + V_1 = 0$	$40 - (-5) + V_1 = 0$	$V_1 = -45$

Non-basic cells:

Variable Cell	C_{ij}–Z_{ij} Calculation
X_{11}	$C_{11} - U_1 + V_1 = 50 - 0 - 45 = 5$
X_{13}	$C_{13} - U_1 + V_3 = 30 - 0 - 35 = -5$
X_{21}	$C_{21} - U_2 + V_1 = 65 - 5 - 45 = 15$
X_{24}	$C_{24} - U_2 + V_4 = 60 - 5 - 45 = 10$
X_{33}	$C_{33} - U_3 + V_3 = 50 -(-5) - 35 = 20$
X_{34}	$C_{34} - U_3 + V_4 = 55 -(-5) - 45 = 15$

Note: X_{13} is the entering variable.

Step 2: Determine Which Current Basic Variable Reaches 0 First

	D_1	D_2	D_3	D_4	Supply
S_1	50	− 75 2	+ 30	45 10	12
S_2	65	+ 80 7	− 40 10	60	17
S_3	40 10	70 1	50	55	11
Demand	10	10	10	10	

Note: 1. Cycle property

2. X_{12} is the leaving variable

Step 3: Determine the Next Transportation Tableau

	D_1	D_2	D_3	D_4	Supply
S_1	50	75	30 2	45 10	12
S_2	65	80 9	40 8	60	17
S_3	40 10	70 1	50	55	11
Demand	10	10	10	10	

Total shipping cost = 2020

{Improvement = 2 (-5) = 10 }

Transportation Simplex Method

Find an initial basic feasible solution by some starting procedure. Then,

1. Set U_1 0. Solve for the other U_i's and V_j's by:

$C_{ij} – U_i + V_j = 0$ for basic variables.

Then calculate the $C_{ij}–Z_{ij}$ values for nonbasic variables by:

$C_{ij} – Z_{ij} = C_{ij} – U_i + V_j$

Choose the nonbasic variable with the most negative $C_{ij}–Z_{ij}$ value as the entering variable. If all $C_{ij}–Z_{ij}$ values are nonnegative, STOP; the current solution is optimal.

2. Find the cycle that includes the entering variable and some of the BASIC variables. Alternating positive and negative changes on the cycle, determine the "change amount" as the smallest allocation on the cycle at which a subtraction will be made.

3. Modify the allocations to the variables of the cycle found in step 2 by the "change amount" and return to step 1.

Note: **there must be m + n - 1 basic variables for the transportation simplex method to work!**

=> Add dummy source or dummy destination, if necessary

(m=# of sources and n=# of destinations)

2. The Assignment Problem (AP) —

a special case of TP with m=n and $s_i=d_j$ for all i, and j.

The Hungarian Algorithm

=> solving the assignment problem of a least cost assignment of m workers to m jobs

Assumptions:

1. There is a cost assignment matrix for the m "people" to be assigned to m "tasks." (If necessary dummy rows or columns consisting of all 0's are added so that the numbers of people and tasks are the same.)

2. All costs are nonnegative.

3. The problem is a minimization problem.

The Hungarian Algorithm

Initialization

1. For each row, subtract the minimum number from all numbers in that row.

2. In the resulting matrix, subtract the minimum number in each column from all numbers in the column.

Iterative Steps

1. Make as many 0 cost assignments as possible. If all workers are assigned, STOP; this is the minimum cost assignment. Otherwise draw the minimum number of horizontal and vertical lines necessary to cover all 0's in the matrix. (A method for making the maximum number of 0 cost assignments and drawing the minimum number of lines to cover all 0's follows.)

2. Find the smallest value not covered by the lines; this number is the *reduction value*.

3. Subtract the reduction value from all numbers not covered by any lines. Add the reduction value to any number covered by both a horizontal and vertical line.

GO TO STEP 1.

For small problems, one can usually determine the maximum number of zero cost assignments by observation. For larger problems, the following procedure can be used:

Determining the Maximum Number of Zero-Cost Assignments

1. For each row, if only one 0 remains in the row, make that assignment and eliminate the *row* and *column* from consideration in the steps below.

2. For each column, if only one 0 remains, make that assignment and eliminate that *row* and *column* from consideration.

3. Repeat steps 1 and 2 until no more assignments can be made. (If 0's remain, this means that there are at least two 0's in each remaining row and column. Make an arbitrary assignment to one of these 0's and repeat steps 1 and 2.)

Again, for small problems, the minimum number of lines required to cover all the 0's can usually be determined by observation. The following procedure, based on network flow arguments, can be used for larger problems:

Drawing the Minimum Number of Lines to Cover All 0's

1. Mark all rows with no *assignments* (with a "·").

2. For each row just marked, mark each column that *has a* 0 in that row (with a "·").

3. For each column just marked, mark each row that *has an assignment* in that column (with a "·").

4. Repeat steps 2 and 3 until no more marks can be made.

5. Draw lines through *unmarked rows* and *marked columns*.

Example:

	A	B	C	D	E
1	10	4	6	10	12
2	11	7	7	9	14
3	13	8	12	14	15
4	14	16	13	17	17
5	17	11	17	20	19

Row Subtraction

	A	B	C	D	E
1	6	0	2	6	8
2	4	0	0	2	7
3	5	0	4	6	7
4	1	3	0	4	4
5	6	0	6	9	8

Column Subtraction

	A	● B	C	D	E
● 1	5	$\boxed{0}$	2	4	4
2	3	0	$\boxed{0}$	0	3
● 3	4	0	4	4	3
4	$\boxed{0}$	3	0	2	0
● 5	5	0	6	7	4

Minimum uncovered number

	A	B	C	D	E
1	3	0	$\boxed{0}$	2	2
2	3	2	0	$\boxed{0}$	3
3	2	$\boxed{0}$	2	2	1
4	$\boxed{0}$	5	0	2	0
5	3	0	4	5	2

Minimum uncovered number

	A	B	C	D	E
1	3	1	$\boxed{0}$	2	2
2	3	3	0	$\boxed{0}$	3
3	1	0	1	1	$\boxed{0}$
4	$\boxed{0}$	6	0	2	0
5	2	$\boxed{0}$	3	4	1

Optimal Solution

1–C	6
2–D	9
3–E	15
4–A	14
5–B	11
	65

CONVERSION OF A MAXIMIZATION PROBLEM TO A MINIMIZATION PROBLEM

The Hungarian algorithm works only if the matrix is a cost matrix. A maximization assignment problem can be converted to a minimization problem by creating a lost opportunity matrix. The problem then is to minimize the total lost opportunity.

Profit Matrix:

	J1	J2	J3	J4
W1	67	58	90	55
W2	58	88	89	56
W3	74	99	80	22
(D)	0	0	0	0

The lost opportunity matrix given below is derived by subtracting each number in the J1 column from 74, each number in the J2 column from 99, each number in the J3 column from 90, and each number in the J4 from 56.

	J1	**J2**	**J3**	**J4**
W1	**7**	**41**	**0**	**1**
W2	**16**	**11**	**1**	**0**
W3	**0**	**0**	**10**	**34**
(D)	**74**	**99**	**90**	**56**

The Hungarian algorithm can now be applied to this lost opportunity matrix to determine the maximum profit set of assignments.