An interpolation of the velocity field from data

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A method of interpolating a velocity field from the data measured at a few points in a region and at all points on its boundary is proposed. The interpolated field has zero divergence and differs from the linear interpolation in the sense of the least-squares error.

Key words: mathematical model, velocity field, interpolated velocity field

Introduction

The velocity field in an aquatic system is frequently determined by computation from the Navier–Stokes equations. For the computation of a steady flow, it is necessary to know boundary data and forces, whereas for the computation of an unsteady flow, initial data and forces have to be known. However, problems arise if forces in the first case or initial data in the second case are unknown. We offer a solution to these problems given that velocities are measured at a few points of the squatic system considered.

We consider two-dimensional and bounded regions. The corresponding theory of velocity field based on the Navier–Stokes equations can be found in the work of Ladyzhenskaya[1] and Temam.[2] In our case, the Navier–Stokes equations are of no use because we suppose that forces are unknown and only the boundary data and the data at a few points in the region considered are known. There is no unique velocity field fitting the given boundary data and the data in the region. In this situation, it is convenient to use linear interpolation. However, the field determined by means of linear interpolation does not satisfy the law of mass conservation. Hence, it can best be determined by using an interpolation which satisfies the law of mass conservation and differs minimally in the sense of the squares error from the field calculated using linear interpolation. This simple programme is carried out in the present work.

Representation of the velocity field

Let Ω be a bounded domain in R^2 (not necessarily simply connected). The boundary of Ω is denoted by Γ and the points of Ω by $\boldsymbol{x} = (x_1, x_2)$. An illustration of Ω is given in *Figure 1*. To remove pathological cases of a mathematical nature, we assume that Ω is a Lipschitzian domain. Hence a polyhedron (*Figure 2*) can be taken as an example of Ω. The outer normal $\boldsymbol{n}$ and the tangent $\boldsymbol{t}$ at the point $\boldsymbol{x} \in \Gamma$ have the orientation as drawn in *Figure 1*.

In our considerations we use the Hilbert space $L_2(\Omega)$ and the Sobolev spaces $H^1(\Omega) = \{u \in L_2(\Omega): \partial u/\partial x_i \in L_2(\Omega)\}$, $H^2(\Omega) = \{u \in H^1(\Omega): \partial^2 u/\partial x_i\, \partial x_j \in L_2(\Omega)\}$ and $H_0^1(\Omega) = \{u \in H^1(\Omega): u | \Gamma = 0\}$. Both components of the velocity field are assumed to belong to $L_2(\Omega)$. This can be written as $\boldsymbol{v} = (v_1, v_2) \in L_2(\Omega)$. A pair (v_1, v_2) is sometimes written in the form of a column.

For a velocity field $\boldsymbol{v}$ on Ω we have:

$$\operatorname{div} \boldsymbol{v} = 0$$

$$\operatorname{rot} \boldsymbol{v} = \frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1} = Q \qquad (1)$$

where we suppose $Q \in L_2(\Omega)$. Let us consider velocity field of the form:

$$\boldsymbol{v} = \left(-\frac{\partial \Psi}{\partial x_2}, \frac{\partial \Psi}{\partial x_1}\right) \qquad \Psi \in H_0^1(\Omega) \qquad (2)$$

The velocity fields of the form (2) determine a closed subspace of $L_2(\Omega)$ and we obtain the following result.

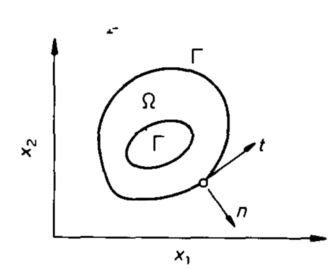

Figure 1 Region Ω occupied by fluid

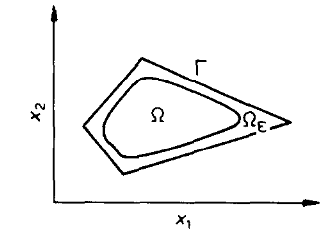

Figure 2 Simply connected region

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Every velocity field $v \in L_2(\Omega)$ *has a representation:*

$$v = \begin{bmatrix} \dfrac{\partial \Phi}{\partial x_1} \\ \dfrac{\partial \Phi}{\partial x_2} \end{bmatrix} + \begin{bmatrix} -\dfrac{\partial \Psi}{\partial x_2} \\ \dfrac{\partial \Psi}{\partial x_1} \end{bmatrix} \quad (3)$$

where:

$$\begin{aligned} &\Delta\Phi = 0 \qquad \Phi \in H^1(\Omega) \\ &\Delta\Psi \in L_2(\Omega) \quad \Psi \in H_0^1 \end{aligned} \quad (4)$$

If v is a vector at $x \in \Gamma$, then v_n, v_t are a normal and a tangential component, respectively, of v. Similarly, for a function u we have $\partial u/\partial n = n\nabla u$, $\partial u/\partial t = t\nabla u$.

To avoid meaningless expressions in the following considerations, we assume an additional property of Ω. The eigenvalue problem:

$$\Delta u = \lambda u$$

$$\frac{\partial u}{\partial n}\Big| \Gamma = 0$$

has solutions $u \in H^2(\Omega)$. Because of this additional property, solutions of various Neumann problems given in this paper belong to $H^2(\Omega)$.[3] For example, this property holds for a domain with the boundary of the class C^2, or a piecewise smooth boundary with vertices determining only sharp angles in Ω.[4]

Data measured at every point

Let v_m be a field on Ω obtained by measuring velocities at every point $x \in \Omega$. In general, $\operatorname{div} v_m \neq 0$ and v_m cannot be accepted as a velocity field. Our aim is to determine a velocity field $v \in L_2(\Omega)$ which is the best approximation to v_m in the sense of the least-squares error. Therefore we define:

$$\| v - v_m \|^2 = \int_\Omega |v(x) - v_m(x)|^2 \, d^2x \quad (5)$$

and our problem is formulated as an optimal control problem:

$$\|v - v_m\| \to \inf \quad \text{together with equations (3) and} \quad (6)$$

and Q varies over $L_2(\Omega)$, If the field v_m is of the class C^2 (it suffices to be of the class $H^2(\Omega)$), then necessary conditions for the optimal control problem (6) can be derived easily. Let $f = \operatorname{div} v_m$, $Q = \operatorname{rot} v_m$. For the given function f there exists the unique function $f_1 \in H^1(\Omega)$ such that the equality

$$\int_\Omega fu \, d^2x = \int_\Omega [f_1 u + \nabla f_1 \nabla u] \, d^2x$$

holds for all $u \in H^1(\Omega)$. All harmonic functions in $H^1(\Omega)$ determine the closed subspace $\mathscr{H}$ of $H^1(\Omega)$. Let f_1 have the component ρ_0 in $\mathscr{H}$. Then the necessary conditions are:

$$\begin{aligned} &\Delta\Phi = 0 \qquad \Phi \in H^1(\Omega) \\ &\Delta\Psi = Q \qquad \Psi \in H_0^1(\Omega) \\ &\frac{\partial \Phi}{\partial n} = v_n - \lambda\rho_0 \\ &\lambda = \left[\int_\Omega f\rho_0 \, d^2x\right]\left[\int_\Gamma \rho_0^2 \, d^2x\right]^{-1} \end{aligned} \quad (7)$$

Note that the velocity field v, which solves equation (7), coincides with v_m if $f = \operatorname{div} v_m = 0$.

Data measured at a few points

Let us suppose that the boundary values of a velocity field v are known together with velocities $v_1, v_2, \ldots, v_M$ at M points $x_1, x_2, \ldots, x_M$, inside Ω. Suppose furthermore that the vortices of this velocity field are unknown. In this case there is no unique velocity field v satisfying the boundary data and having the prescribed values at the points $x_1, x_2, \ldots, x_M$.

In such cases, one tries to determine a velocity field by using interpolation. Suppose that $v = \mathbf{0}$ on the boundary. Then v is represented in the form $v = \operatorname{rot} \Psi$. Let us connect the points x_i and certain points on the boundary by smooth curves to obtain a system of sets Ω_i covering Ω and the corresponding finite elements ϕ_k having supports in Ω_i and vertices at x_i. There are $2M$ such finite elements. The velocity field v can be uniquely approximated in the form:

$$v_{\text{ap}}(x) = \sum_1^{2M} \alpha_k \phi_k(x) \quad (8)$$

The field v_{ap} has the prescribed values at the boundary and the points x_i. However, the field (8) cannot be accepted as a velocity field because $\operatorname{div} v_{\text{ap}} \neq 0$. Next, we have to look for the best approximation to v_{ap} in the sense of problem (6). The unique solution v of problem (6), obtained in the preceding section, differs from the prescribed values at x_i. The difference $|v - v_{\text{ap}}|$ at x_i may be 60% of v_{ap} if the vector v_i differs enough from the vectors v_j at the surrounding points x_j.

It is now easy to change the finite elements ϕ_k into new ones w_k which satisfy $\operatorname{div} w_k = 0$. Let z_k be the best approximation to ϕ_k in the sense of problem (6). Then z_k, $k = 1, 2, \ldots, 2M$, are linearly independent and one easily obtains w_k such that $\operatorname{div} w_k = 0$, $w_1(x_1) = (1, 0)$, $w_2(x_1) = (0, 1)$, $w_3(x_2) = (1, 0)$, etc. It would be preferable to look for solutions of problem (6) with an additional property, namely, the supports of z_{2i-1} and z_{2i} are in $\bar{\Omega}_i$.

Let us now consider the general case $v|\Gamma \neq \mathbf{0}$. Our aim is to reduce this case to the preceding one. Let Ω_ϵ be a neighbourhood of the boundary Γ, as illustrated in *Figure 2*. We define the minimal velocity:

$$v_{\min} = \nabla\Phi + \operatorname{rot} \Psi \quad (9)$$

where, in accordance with equation (7), the function Φ is the solution of:

$$\begin{aligned} &\Delta\Phi = 0 \\ &\frac{\partial \Phi}{\partial n} = v_n \end{aligned} \quad (10)$$

and the function Ψ is the unique solution of the optimal control problem:

$$\begin{aligned} &\|Q\|^2 = \int_\Omega Q^2(x) \, d^2x \to \inf \\ &\Delta\Psi = Q \qquad \Psi \in H_0^1(\Omega) \\ &\frac{\partial \Psi}{\partial n} = \frac{\partial \Phi}{\partial t} - v_t \end{aligned} \quad (11)$$

Q varies and its support is always in $\bar{\Omega}_\epsilon$.

Knowing the minimal velocity field $v_{\min}$, one determines the desired velocity field v as:

$$v = v_{\min} + v_{\text{vor}} \tag{12}$$

where the vorticity field v_{vor} is defined by:

$$v_{\text{vor}} = \Sigma \alpha_k w_k \tag{13}$$

as in the first part of this section.

Computation of the vorticity field

Let us consider briefly the computation of the fields z_k which are solutions of the optimal control problem (6) with $v_{\text{ap}} = \phi_k$. If z_k are supposed to have supports in $\bar{\Omega}_i$, as suggested in the preceding section, the system (7) cannot help us and therefore the original problem (6) has to be solved.

The number of finite elements z_k is determined by the number of data. Therefore we call z_k global finite elements. As $z_k = \operatorname{rot} \Psi_k$, where $\bar{\Omega}_i$ is the support of Ψ_k, we wish to determine Ψ_k by using finite elements $u_1, u_2, \ldots, u_N$, with supports in $\bar{\Omega}_i$. We call them local finite elements. Hence an approximation is:

$$z_k = \Sigma \beta_j \operatorname{rot} u_j$$

where β_j have to be computed from the minimizing problem:

$$\left\| \sum_{i}^{N} \beta_j \operatorname{rot} u_j - \phi_k \right\| \to \inf \tag{14}$$

Let us define the numbers:

$$A_{ij} = \int_{\Omega} \operatorname{rot} u_i \cdot \operatorname{rot} u_j \, \mathrm{d}^2 x$$

$$b_i = \int_{\Omega} \phi_k \cdot \operatorname{rot} u_i \, \mathrm{d}^2 x$$

the corresponding $N \times N$ matrix [A] and column [b]. Then the unique solution of problem (14) is the solution of the system:

$$[\mathrm{A}][\alpha] = [\mathrm{b}]$$

where $[\alpha]$ is a column having α_i, $i = 1, 2, \ldots, N$, as elements. An extensive description of the local finite elements $\operatorname{rot} u_j$ can be found in Temam's monograph.[2]

An example

We consider a two-dimensional model of the northern region of Rijeka Bay as drawn in *Figure 3*. The dimensions

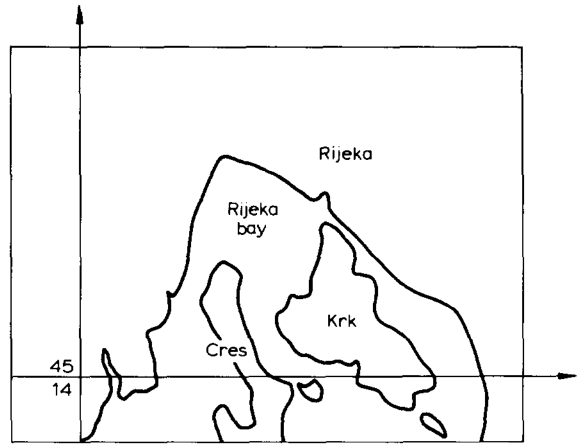

Figure 3 Northern region of Rijeka Bay

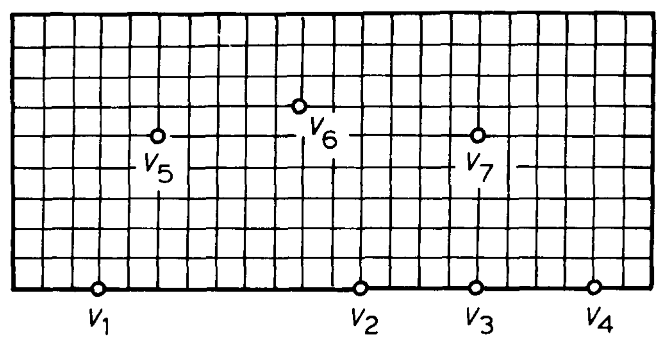

Figure 4 Mesh for northern region of Rijeka Bay

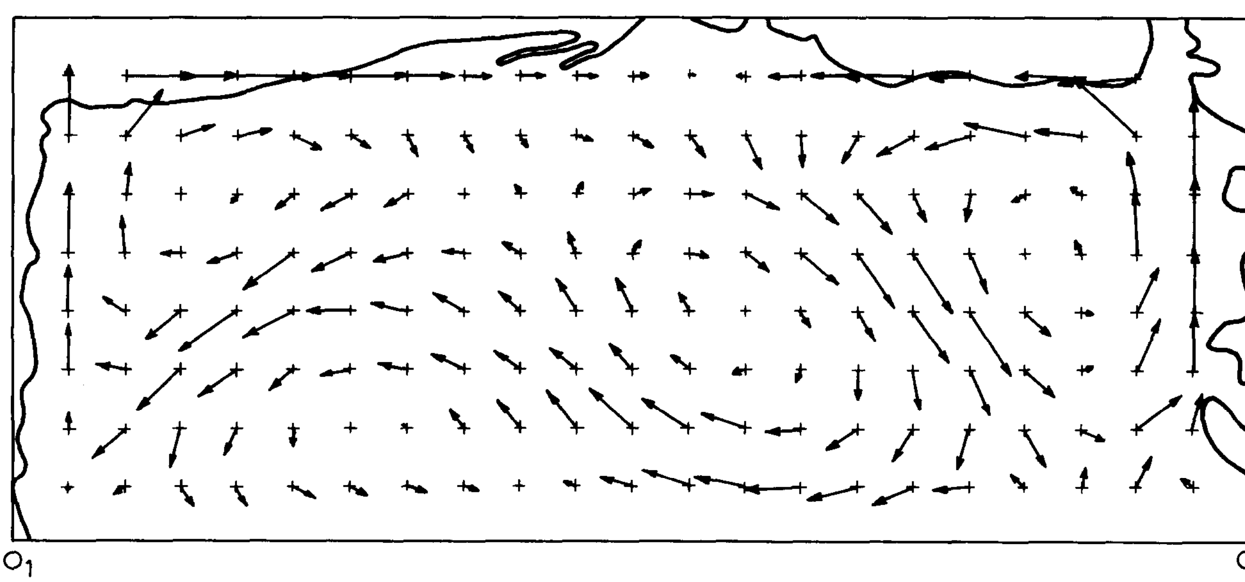

Figure 5 Flow in northern region of Rijeka Bay

of the region Ωv are 22 km x 9 km and the depth throughout the region is about 60 m. The velocities v are measured at seven points denoted by V in *Figure 4* (the data are taken from ref. 5). The data measured at four points at the part O_1O_2 of the boundary are used to obtain the boundary data of v along O_1O_2 by using linear interpolation. The boundary data at the remaining part of the boundary Γ are defined by $v = \mathbf{0}$. The mesh of *Figure 4* has a step size of 1 km.

We used a finite-difference method to determine the velocity field, although finite-element methods are more appropriate. The mesh knots are denoted by integers $i = 1, 2, \ldots, 230$. A knot i is surrounded by four knots $i - 10$, $i - 1, i + 10$ and $i + 1$. Let Φ_i be an approximation of Φ at the mesh knot i. Then the difference scheme is defined by:

$$-\Delta\Phi = h^{-2}[4\Phi_i - \Phi_{i-10} - \Phi_{i-1} - \Phi_{i+10} - \Phi_{i+1}]$$

at every interior point i:

$$\frac{\partial\Phi}{\partial n} = h^{-1}[\Phi_{i+1} - \Phi_i]$$

at every boundary point $i \in O_1O_2$, etc.

To determine the minimal velocity field $v_{\min}$ of equation (9), we define Ω_ϵ as a set of points in Ω not farther than 1.5 km from the boundary Γ. Then the optimal control problem (11), after the discretization, can be solved simply because only one Q satisfies the boundary data of equation (10). Note that the values of $v_{\min}$ calculated at the points V_5 and V_7 of *Figure 4* are 10 times smaller than the measured values. This suggests that the vorticity field v_{vor} in the total velocity field v of equation (12) is the dominant part.

The vorticity field v_{vor} is a linear combination of six elements w_k which are determined from six fields z_k. We should say something more about the local elements u_j with the help of which z_k are computed. Let i be a mesh knot of the form $i = 23 + (l - 1) + 10(k - 1), l = 1, 2, \ldots, 6$, $k = 1, 2, \ldots, 19$. There are 114 knots of this type. Their set is denoted by $\mathscr{L}$. To every $n \in \mathscr{L}$ there corresponds a local element u_n defined by:

$$u_n = \text{rot}\,\Psi = \begin{bmatrix} -(2h)^{-1} & (\Psi_{i+1} - \Psi_{i-1}) \\ (2h)^{-1} & (\Psi_{i+10} - \Psi_{i-10}) \end{bmatrix}$$

where the mesh function Ψ is defined by $\Psi_i = 1$ for $i = n$ and $\Psi_i = 0$ otherwise. The elements u_n are linearly dependent and if we remove one of them, we obtain a sequence of linearly independent elements. We have removed the last one. Then the fields z_k, w_k and v_{vor} are calculated as described in the preceding sections. The velocity field v thus obtained is drawn in *Figure 5*.

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