MATH 15a: Linear Algebra
Practice Final Exam, Solutions

1. (a) (4 points) Complete the definition: Vectors $\vec{v}_1, \ldots, \vec{v}_k$ are *linearly independent* if...

   Answer: ...no v_i is in the span of $v_1, \ldots, v_{i-1}$.
   OR ...they do not satisfy any nontrivial linear relation.

   (b) (4 points) Suppose A is an $m \times n$ matrix whose columns are linearly independent. What is the nullity of A?

   Answer: If the columns are linearly independent, the rank of A is equal to n (the dimension of the span of the columns), so the nullity equals zero.

2. Let A denote the matrix

$$A = \begin{bmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}.$$

Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $T(\vec{x}) = A\vec{x}$.

   (a) (5 points) Describe T geometrically.

   Answer: For any angle θ, the matrix for counterclockwise rotation by θ is $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.
   Since $\cos(\frac{5\pi}{6}) = -\frac{\sqrt{3}}{2}$ and $\sin(\frac{5\pi}{6}) = \frac{1}{2}$, we see that T is rotation by $\frac{5\pi}{6} = 150°$.

   (b) (5 points) Find the characteristic polynomial of A, and use it to find all eigenvalues of A or to show that none exist. Explain why your answer makes sense geometrically.

   Answer: The characteristic polynomial of A is:

$$\begin{aligned} \det \begin{bmatrix} -\frac{\sqrt{3}}{2} - t & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} - t \end{bmatrix} &= (-\frac{\sqrt{3}}{2} - t)^2 + \frac{1}{4} \\ &= t^2 + \sqrt{3}t + 1. \end{aligned}$$

   In the quadratic formula, the quantity $b^2 - 4ac$ is $(\sqrt{3})^2 - 4 \cdot 1 \cdot 1 = -1$, so this polynomial has no real roots, and hence A has no eigenvalues.
   An eigenvector of A would be a vector $\vec{v}$ such that $T(\vec{v})$ is on the same line as $\vec{v}$. However, a rotation by an angle other than 0 or π cannot send any line to itself.

   (c) (5 points) Compute A^{2011}. (*Hint:* What power of A is equal to the identity?)

   Answer: Because T is a 5/12 rotation, doing T twelve times is the same as the identity. Thus $A^{12} = I_n$. Since $2011 = 167 \cdot 12 + 7$, we have $A^{2011} = A^7$. Also, doing T six times is a 1/2 rotation (i.e. rotation by π or 180°), which is minus the identity. Thus,

$$A^7 = -A = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

3. (a) (5 points) Compute the inverse of the matrix

$$\begin{bmatrix} -4 & 0 & 5 \\ -3 & 3 & 5 \\ -1 & 2 & 2 \end{bmatrix}$$

Answer: We apply Gauss-Jordan elimination to $[A|I_3]$ until we get $[I_3|A^{-1}]$:

$$\left[\begin{array}{ccc|ccc} -4 & 0 & 5 & 1 & 0 & 0 \\ -3 & 3 & 5 & 0 & 1 & 0 \\ -1 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & -2 & 0 & 0 & -1 \\ -3 & 3 & 5 & 0 & 1 & 0 \\ -4 & 0 & 5 & 1 & 0 & 0 \end{array}\right] \Longrightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & -2 & 0 & 0 & -1 \\ 0 & -3 & -1 & 0 & 1 & -3 \\ 0 & -8 & -3 & 1 & 0 & -4 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & -2 & 0 & 0 & -1 \\ 0 & 1 & \frac{1}{3} & 0 & -\frac{1}{3} & 1 \\ 0 & -8 & -3 & 1 & 0 & -4 \end{array}\right] \Longrightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & -2 & 0 & 0 & -1 \\ 0 & 1 & \frac{1}{3} & 0 & -\frac{1}{3} & 1 \\ 0 & 0 & -\frac{1}{3} & 1 & -\frac{8}{3} & 4 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & -2 & 0 & 0 & -1 \\ 0 & 1 & \frac{1}{3} & 0 & -\frac{1}{3} & 1 \\ 0 & 0 & 1 & -3 & 8 & -12 \end{array}\right] \Longrightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 0 & -6 & 16 & -25 \\ 0 & 1 & 0 & 1 & -3 & 5 \\ 0 & 0 & 1 & -3 & 8 & -12 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 10 & -15 \\ 0 & 1 & 0 & 1 & -3 & 5 \\ 0 & 0 & 1 & -3 & 8 & -12 \end{array}\right]$$

Thus,

$$A^{-1} = \begin{bmatrix} -4 & 10 & -15 \\ 1 & -3 & 5 \\ -3 & 8 & -12 \end{bmatrix}.$$

(b) (5 points) Find all solutions to the system of linear equations

$$\begin{aligned} -4x + 5z &= -2 \\ -3x - 3y + 5z &= 3 \\ -x + 2y + 2z &= -1 \end{aligned}$$

Answer: This system is $A\vec{x} = \vec{b}$, where A is as in the previous part and $\vec{b} = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$. Hence

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\vec{b} = \begin{bmatrix} -4 & 10 & -15 \\ 1 & -3 & 5 \\ -3 & 8 & -12 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 53 \\ -16 \\ 42 \end{bmatrix}.$$

4. (a) (4 points) Using Gaussian elimination, find all solutions to the following system of linear equations:

$$\begin{aligned} 2x_2 + 3x_3 + 4x_4 &= 1 \\ x_1 - 3x_2 + 4x_3 + 5x_4 &= 2 \\ -3x_1 + 10x_2 - 6x_3 - 7x_4 &= -4 \end{aligned}$$

Answer:

$$\left[\begin{array}{cccc|c} 0 & 2 & 3 & 4 & 1 \\ 1 & -3 & 4 & 5 & 2 \\ -3 & 10 & -6 & -7 & -4 \end{array}\right] \Longrightarrow \left[\begin{array}{cccc|c} 1 & -3 & 4 & 5 & 2 \\ 0 & 2 & 3 & 4 & 1 \\ -3 & 10 & -6 & -7 & -4 \end{array}\right] \Longrightarrow$$

$$\left[\begin{array}{cccc|c} 1 & -3 & 4 & 5 & 2 \\ 0 & 2 & 3 & 4 & 1 \\ 0 & 1 & 6 & 8 & 2 \end{array}\right] \Longrightarrow \left[\begin{array}{cccc|c} 1 & -3 & 4 & 5 & 2 \\ 0 & 1 & 6 & 8 & 2 \\ 0 & 2 & 3 & 4 & 1 \end{array}\right] \Longrightarrow \left[\begin{array}{cccc|c} 1 & -3 & 4 & 5 & 2 \\ 0 & 1 & 6 & 8 & 2 \\ 0 & 0 & -9 & -12 & -3 \end{array}\right] \Longrightarrow$$

$$\left[\begin{array}{cccc|c} 1 & -3 & 4 & 5 & 2 \\ 0 & 1 & 6 & 8 & 2 \\ 0 & 0 & 1 & \frac{4}{3} & \frac{1}{3} \end{array}\right] \Longrightarrow \left[\begin{array}{cccc|c} 1 & -3 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{4}{3} & \frac{1}{3} \end{array}\right] \Longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{4}{3} & \frac{1}{3} \end{array}\right]$$

So the general solution is:

$$\begin{aligned} x_1 &= \tfrac{2}{3} - \tfrac{1}{3}x_4 \\ x_2 &= 0 \\ x_3 &= \tfrac{1}{3} - \tfrac{4}{3}x_4 \\ x_4 &= \text{free.} \end{aligned}$$

(b) (4 points) Let V be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} -3 \\ 10 \\ -6 \\ -7 \end{bmatrix}$$

Find a basis for $V^\perp$.

Answer: $V^\perp$ is the kernel of the matrix

$$A = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 1 & -3 & 4 & 5 \\ -3 & 10 & -6 & -7 \end{bmatrix}.$$

We saw in the previous part that

$$\operatorname{rref}(A) = \left[\begin{array}{cccc|} 1 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{4}{3} \end{array}\right]$$

Therefore, the kernel is given by the general solution

$$\begin{aligned} x_1 &= -\tfrac{1}{3}x_4 \\ x_2 &= 0 \\ x_3 &= -\tfrac{4}{3}x_4 \\ x_4 &= \text{free,} \end{aligned}$$

so it is spanned by the vector

$$\begin{bmatrix} -\frac{1}{3} \\ 0 \\ -\frac{4}{3} \\ 1 \end{bmatrix}.$$

(c) (4 points) What is the rank of the linear transformation $T\colon \mathbb{R}^4 \to \mathbb{R}^3$ with $T(\vec{e}_1) = \vec{e}_2 - 3\vec{e}_3$, $T(\vec{e}_2) = 2\vec{e}_1 + -3\vec{e}_2 + 10\vec{e}_3$, $T(\vec{e}_3) = 3\vec{e}_1 + 4\vec{e}_2 - 6\vec{e}_3$, and $T(\vec{e}_4) = 4\vec{e}_1 + 5\vec{e}_2 - 7\vec{e}_3$?

Answer: The matrix for T is exactly A (given above). Since $\operatorname{rref}(A)$ has three pivots, we see that the rank of T is 3.

(*Hint:* The answers to all three parts are related!)

5. Let A denote the matrix

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 5 & 0 \\ -2 & 0 & 4 \end{bmatrix}$$

(a) (4 points) Find the eigenvalues of A.

Answer: The characteristic polynomial of A is

$$\begin{aligned} \det \begin{bmatrix} 1-t & 0 & -2 \\ 0 & 5-t & 0 \\ -2 & 0 & 4-t \end{bmatrix} &= (1-t)(5-t)(4-t) - (-2)(5-t)(-2) \\ &= (5-t)(t^2 - 5t + 4 - 4) \\ &= -t(t-5)^2 \end{aligned}$$

So the eigenvalues are 0 (with multiplicity 1) and 5 (with multiplicity 2). (*Note:* Even if I don't ask explicitly, you should always give the algebraic multiplicities of eigenvalues.)

(b) (6 points) Find an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors for A.

Answer: For the eigenvalue 0, we row-reduce $A - 0I_3 = A$:

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 5 & 0 \\ -2 & 0 & 4 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The solution to the system of equations is

$$\begin{aligned} x_1 &= 2x_3 \\ x_2 &= 0 \\ x_3 &= \text{free} \end{aligned}$$

so the kernel is spanned by $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. We normalize to get a unit vector:

$$\vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

For the eigenvalue 5, we row-reduce $A - 5I_3$:

$$\begin{bmatrix} -4 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The solution to the system of equations is

$$\begin{aligned} x_1 &= -\frac{1}{2}x_3 \\ x_2 &= \text{free} \\ x_3 &= \text{free} \end{aligned}$$

So the kernel is spanned by the vectors $\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Normalizing, we get:

$$\vec{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \quad \text{and} \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

(It happens that these vectors are already orthogonal to each other. If they weren't, we would have to use Gram-Schmidt to find an orthonormal basis for E_5.) Combining the bases for E_0 and E_5, we get an orthonormal basis for $\mathbb{R}^3$.

(c) (3 points) Find a 3×3 orthogonal matrix S and a 3×3 diagonal matrix D such that $A = SDS^T$.

Answer: S is gotten by putting the three basis vectors together in a matrix:

$$S = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \end{bmatrix}$$

Since S is orthogonal, $S^{-1} = S^T$.
D is gotten by listing the eigenvalues down the diagonal (in the same order as we wrote the corresponding eigenvectors in S):

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

(If you had put the eigenvectors in a different order, you would need a different D).

(d) (4 points) For any integer t, write an explicit formula for A^t.

Answer:

$$
\begin{aligned}
A^t &= (SDS^{-1})^t = SD^tS^{-1} = SD^tS^T \\
&= \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 5^t & 0 \\ 0 & 0 & 5^t \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{\sqrt{5}} \cdot 5^t & 0 & \frac{2}{\sqrt{5}} \cdot 5^t \\ 0 & 5^t & 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{5} \cdot 5^t & 0 & -\frac{2}{5} \cdot 5^t \\ 0 & 5^t & 0 \\ -\frac{2}{5} \cdot 5^t & 0 & \frac{4}{5}5^t \end{bmatrix} \\
&= \begin{bmatrix} 5^{t-1} & 0 & -2 \cdot 5^{t-1} \\ 0 & 5^t & 0 \\ -2 \cdot 5^{t-1} & 0 & 4 \cdot 5^{t-1} \end{bmatrix}
\end{aligned}
$$

(Either of the last two lines is acceptable.) As a sanity check, you should verify that plugging in $t = 1$ gives A.

(e) (3 points) Let Q denote the quadratic form on $\mathbb{R}^3$ given by

$$Q(x, y, z) = x^2 - 4xz + 5y^2 + 4z^2.$$

Is Q positive definite, positive semidefinite, negative definite, negative semidefinite, or indefinite? Explain.

Answer: This quadratic form corresponds to the matrix A considered above. Since the eigenvalues of A are nonnegative and at least one of them is 0, Q is positive semidefinite.

6. Let $\vec{v}_1$ and $\vec{v}_2$ denote the following vectors in $\mathbb{R}^3$:

$$\vec{v}_1 = \begin{bmatrix} 2/3 \\ -1/3 \\ -2/3 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} -\sqrt{2}/2 \\ 0 \\ -\sqrt{2}/2 \end{bmatrix}$$

(a) (3 points) Find a vector $\vec{v}_3$ so that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ form an orthonormal basis $\mathfrak{B}$ for $\mathbb{R}^3$. How many choices are there for the answer?

Answer: Let V be the span of $\vec{v}_1$ and $\vec{v}_2$. The orthogonal complement of V, which consists of all vectors orthogonal to both $\vec{v}_1$ and $\vec{v}_2$, is the kernel of the matrix

$$\begin{bmatrix} 2/3 & -1/3 & -2/3 \\ -\sqrt{2}/2 & 0 & -\sqrt{2}/2 \end{bmatrix}.$$

Row, reducing this, we get:

$$\begin{bmatrix} 2/3 & -1/3 & -2/3 \\ -\sqrt{2}/2 & 0 & -\sqrt{2}/2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & -1/2 & -1 \\ 1 & 0 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & -1/2 & -1 \\ 0 & 1/2 & 2 \end{bmatrix} \Longrightarrow$$

$$\begin{bmatrix} 1 & -1/2 & -1 \\ 0 & 1 & 4 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \end{bmatrix}$$

The solution to the system is

$$\begin{aligned} x_1 &= -x_3 \\ x_2 &= -4x_3 \\ x_4 &= \text{free} \end{aligned}$$

so any vector in $V^\perp$ is of the form $\begin{bmatrix} -t \\ -4t \\ t \end{bmatrix}$. Since the norm of this must be 1, we get $t^2 + 16t^2 + t^2 = 18t^2 = 1$, so $t = \pm\frac{1}{\sqrt{18}} = \pm\frac{1}{3\sqrt{2}} = \pm\frac{\sqrt{2}}{6}$. Thus, there are two possible answers. Let us choose $t = \frac{\sqrt{2}}{6}$, so that

$$\vec{v}_3 = \begin{bmatrix} -\sqrt{2}/6 \\ -2\sqrt{2}/3 \\ 2\sqrt{2}/6 \end{bmatrix}.$$

(b) (4 points) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ denote the linear transformation that interchanges $\vec{v}_1$ and $\vec{v}_3$ and has $\vec{v}_2$ as an eigenvector with eigenvalue -5. Write down $[T]_{\mathfrak{B}}$, the matrix of T with respect to $\mathfrak{B}$.

Answer: The matrix $[T]_{\mathfrak{B}}$ is gotten by writing down $T(\vec{v}_1)$, $T(\vec{v}_2)$, and $T(\vec{v}_3)$ in $\mathfrak{B}$ coordinates and putting them as the columns of a matrix. *You do not know what $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are.* Specifically, since

$$\begin{aligned} T(\vec{v}_1) &= \vec{v}_3 \\ T(\vec{v}_2) &= -5\vec{v}_2 \\ T(\vec{v}_3) &= \vec{v}_1, \end{aligned}$$

we have

$$[T]_{\mathfrak{B}} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -5 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

(c) (3 points) Write down a product of matrices that equals the standard matrix of T. (*Note:* Your answer will depend on your choice of $\vec{v}_3$ in part (a).)

Answer: The change-of-basis matrix $S_{\mathfrak{B}}$ is

$$S_{\mathfrak{B}} = \begin{bmatrix} 2/3 & -\sqrt{2}/2 & -\sqrt{2}/6 \\ -1/3 & 0 & -2\sqrt{2}/3 \\ -2/3 & -\sqrt{2}/2 & 2\sqrt{2}/6 \end{bmatrix}.$$

Since $S_{\mathfrak{B}}$ is orthogonal, we have $S_{\mathfrak{B}}^{-1} = S_{\mathfrak{B}}$. The standard matrix for T is then given by

$$A = S_{\mathfrak{B}}[T]_{\mathfrak{B}}S_{\mathfrak{B}}^{-1} = \begin{bmatrix} 2/3 & -\sqrt{2}/2 & -\sqrt{2}/6 \\ -1/3 & 0 & -2\sqrt{2}/3 \\ -2/3 & -\sqrt{2}/2 & 2\sqrt{2}/6 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & -5 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2/3 & -1/3 & -2/3 \\ -\sqrt{2}/2 & 0 & -\sqrt{2}/6 \\ -2\sqrt{2}/3 & -\sqrt{2}/2 & 2\sqrt{2}/6 \end{bmatrix}$$

(d) (3 points) What is the determinant of the standard matrix of T?

Answer: Since A and $[T]_{\mathfrak{B}}$ are similar, they have the same determinant:

$$\det A = \det[T]_{\mathfrak{B}} = -(-5) = 5.$$

7. Compute the determinant of each of the following matrices. Indicate clearly the method being used.

(a) (5 points)

$$\begin{bmatrix} 2 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 0 & 7 \\ -3 & 2 & 0 & -1 & -6 \\ 2 & -2 & -1 & 1 & 4 \\ 0 & 0 & 0 & 4 & 3 \end{bmatrix}$$

Answer:

$$\begin{aligned} \det A &= -7 \det \begin{bmatrix} 2 & 0 & 0 & -3 \\ -3 & 2 & 0 & -1 \\ 2 & -2 & -1 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix} && \text{(along the second row)} \\ &= (-7)(4) \det \begin{bmatrix} 2 & 0 & 0 \\ -3 & 2 & 0 \\ 2 & -2 & -1 \end{bmatrix} && \text{(along the fourth row)} \\ &= (-7)(4)(2) \det \begin{bmatrix} 2 & 0 \\ -2 & -1 \end{bmatrix} && \text{(along the first row)} \\ &= (-7)(4)(2)(-2) = 56 \end{aligned}$$

(b) (5 points)

$$\begin{bmatrix} 1 & b & b^2 \\ b & b^2 & b^3 \\ b^2 & b^3 & b^4 \end{bmatrix}$$

(where b is any real number).

Answer: We can do row operations: subtract b times the first row from the second row and b^2 times the first row from the second row. This gives

$$\begin{bmatrix} 1 & b & b^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, the original matrix is not invertible, so its determinant is zero.

8. Determine whether each statement is true or false, and provide a justification or a counterexample.

(a) (3 points) If A is an $n \times n$ matrix such that $\|A\vec{v}\| = \|\vec{v}\|$ for every $\vec{v} \in \mathbb{R}^n$, then there exists some $\vec{b} \in \mathbb{R}^n$ such that the system of linear equations $A\vec{x} = \vec{b}$ has infinitely many solutions.

Answer: *False.* If $\|A\vec{v}\| = \|\vec{v}\|$ for every $\vec{v} \in \mathbb{R}^n$, then A is orthogonal, hence invertible (since $A^T A = I_n$). Thus, $A\vec{x} = \vec{b}$ has a unique solution for every $\vec{b}$.

(b) (3 points) If A is an $m \times n$ matrix and $B = \mathrm{rref}(A)$, then $\ker(A) = \ker(B)$.

Answer: *True.* In general, doing row operations on $[A|\vec{b}]$ doesn't change the solution set. Here, if $\vec{b} = \vec{0}$, the operations leave the right-hand side unchanged. So if two matrices are related by row operations, they have the same kernel.

(c) (3 points) If A is an orthogonal matrix, there exists an orthonormal basis of eigenvectors for A.

Answer: *False.* The matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is orthogonal, but it doesn't have any eigenvectors at all, let alone an orthonormal basis of them.

(d) (3 points) If A and B are matrices whose eigenvalues, counted with their algebraic multiplicities, are the same, then A and B are similar.

Answer: *False.* The matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has 1 as an eigenvalue with eigenvalue 2, as does $B = I_2$, but A is not similar to B. (If it were, we would have $A = SBS^{-1} = SI_2S^{-1} = SS^{-1} = I_2$, which isn't true.)