Information Flow over Wireless Networks: A Deterministic Approach

Ankita Pasad

Guide:
Prof. Nikhil Karamchandani and Prof. Vinod Prabhakaran

EE 765: Course Project

November 27, 2015

Overview

Introduction

- The Problem under Analysis
  To find a maximum rate of achievable information flow in a wireless network with a single source, a single destination and an arbitrary number of relay nodes
- Distinguishing features of wireless communication are:
  1. *Broadcast*
  2. *Superposition*
- A two-step approach to tackle this problem:
  1. A deterministic channel model
  2. *Quantize-map-forward* scheme for Gaussian Networks
- Major motivation for this scheme comes from the fact that wireless networks operate in the interference-limited regime.

Deterministic Modeling of Wireless Channel

$$y = hx + z \tag{1}$$

where, $z \sim \mathcal{N}(0,1)$, and transmit power and noise power are both normalized to unity, and $|h| = \sqrt{SNR}$. Binary expansion of 1 gives

$$y = 2^{\frac{1}{2}log(SNR)} \sum_{i=1}^{\infty} x(i)2^{-i} + \sum_{i=1}^{\infty} z(i)2^{-i}$$

$$y \approx 2^{n} \sum_{i=1}^{\infty} x(i)2^{-i} + \sum_{i=1}^{\infty} z(i)2^{-i}$$

$$y \approx 2^{n} \sum_{i=1}^{n} x(i)2^{-i} + \sum_{i=1}^{\infty} (x(i+n) + z(i))2^{-i}$$

where, $n = [\frac{1}{2}log(SNR)]^{+}$

Deterministic Modeling of Wireless Channel (contd.)

This input-output relationship can algebraically written as $y = \mathbf{S}^{q-n}x$, where $\mathbf{S}$ is a q X q shift matrix:

$$\mathbf{S}^1 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

For example, say $n = 4$ and $q = 5$,

$$y = \mathbf{S}^{5-4}x = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x1 \\ x2 \\ x3 \\ x4 \\ x5 \end{bmatrix} = \begin{bmatrix} 0 \\ x1 \\ x2 \\ x3 \\ x4 \end{bmatrix}$$

Deterministic Modeling of Wireless Channel (contd.)

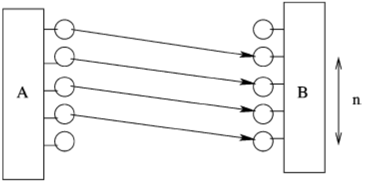

Figure : Pictorial representation of deterministic model for point-to-point channel (taken from [1])

Modeling Broadcast

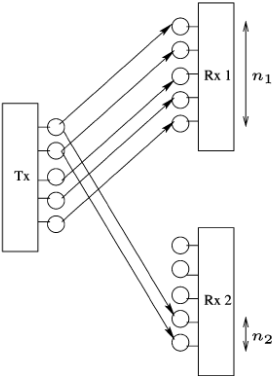

Figure : Pictorial representation of deterministic model for Gaussian BC with $(SNR)_1 \geq (SNR)_2$ and $n_1 = 5$, $n_2 = 2$ (taken from [1])

Modeling Superposition

$y = h_1 x_2 + h_2 x_2 + z$[1], where $h_i = \sqrt{SNR_i}, i = 1, 2.$ Assume $SNR_2 \leq SNR_1$, and $n_i = [\frac{1}{2} log(SNR)_i]^+$

$$y = 2^{\frac{1}{2} log(SNR_1)} \sum_{i=1}^{\infty} x_1(i) 2^{-i} + 2^{\frac{1}{2} log(SNR_2)} \sum_{i=1}^{\infty} x_2(i) 2^{-i} + \sum_{i=1}^{\infty} z(i) 2^{-i}$$

$$y \approx 2^{n_1} \sum_{i=1}^{n_1 - n_2} x_1(i) 2^{-i} + 2^{n_2} \sum_{i=1}^{n_2} (x_1(i + n_1 - n_2) + x_2(i)) 2^{-i}$$

$$+ \sum_{i=1}^{\infty} (x_1(i + n_1) + x_2(i + n_2) + z(i)) 2^{-i}$$

[1] assuming x_1, x_2 and z are all positive real numbers smaller than one

Modeling Superposition (contd.)

Different operations are done on following 3 types of bits

- The part of x_1, that is above SNR_2 (i.e., $x_1(i), 1 \leq i \leq n_1 - n_2$)
- The remaining part of x_1 that is above SNR_1, and that part of x_2 that is above SNR_2
- And the remaining part of x_1 and x_2, along with noise bits

Algebraically , we can write above operations as

$$y = \mathbf{S}^{q-n_1} x_1 \oplus \mathbf{S}^{q-n_2} x_2$$

Modeling Superposition (contd.)

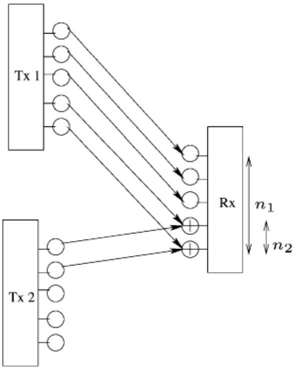

Figure : Pictorial representation of deterministic model for Gaussian MAC with $(SNR)_1 \geq (SNR)_2$ and $n_1 = 5$, $n_2 = 2$ (taken from [1])

Comparison with Gaussian BC and Gaussian MAC

It has been verified that the worst case gap between the deterministic rate regions and the corresponding rate regions for Gaussian case are within one bit gap. For instance,

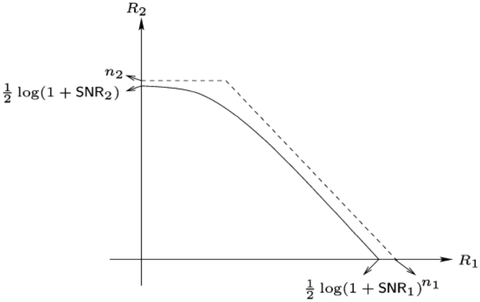

Figure : Capacity region for Gaussian and deterministic BC (taken from [1])

Linear Deterministic Model

Characteristics of a relay network

- Defined over a set of vertices $\mathcal{V}$
- A non-negative integer n_{ij} is associated with communication link from node i to node j
- A node i receives vector $\mathbf{y}_i[t] \in \mathbb{F}_p^q$ and transmits vector $\mathbf{x}_i[t] \in \mathbb{F}_p^q$, where p is positive integer indicating field size and $q = max_{i,j}(n_{ij})$
- The received signal at each node is a linear, deterministic function of transmitted signals at all other nodes, i.e., if n is the total number of transmitting nodes,

$$\mathbf{y}_j[t] = \sum_{i=1}^{n} \mathbf{S}^{q-n_{ij}} \mathbf{x}_i[t]^2 \tag{2}$$

[2] all the summations and multiplications are in $\mathbb{F}_2$ (i.e. $p = 2$) throughout our discussion

Motivation

Whenever a network is assumed to be synchronized, i.e., all transmissions occur on a common clock, the relay is allowed to do any causal processing. And, for any such network, there is a natural information-theoretic cut-set bound, which upper bounds the reliable transmission rate R. Such a bound on relay network is proved to be

$$\bar{C} = \max_{p(\{x_j\}_{j \in \mathcal{V}})} \min_{\Omega \in \Lambda_D} I(\mathbf{y}_{\Omega^c}; \mathbf{x}_{\Omega} | \mathbf{x}_{\Omega^c}) \tag{3}$$

where $\Lambda_D = \{\Omega : S \in \Omega, D \in \Omega^c\}$

Single Relay

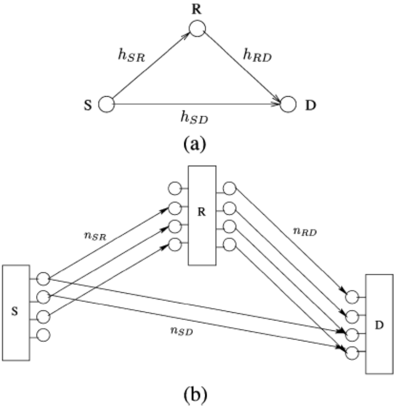

Figure : Example network with single relay: (a) Gaussian model; (b) linear finite-field deterministic model (taken from [1])

Single Relay (contd.)

$$C^d_{relay} \leq min(max(n_{SR}, n_{SD}), max(n_{RD}, n_{SD}))$$

Thus, we can see that, this is in fact the cut-set upper bound for the linear deterministic network.
Also, it has been verified that this scheme achieves within 1 bit of the cut-set bound of single relay Gaussian network, for all channel gains.

Diamond Network

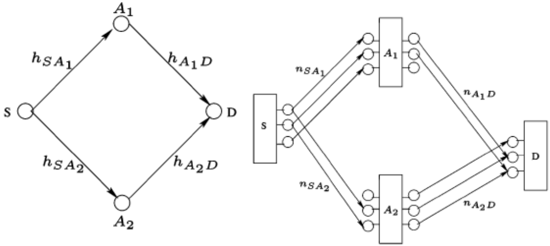

Figure : Diamond network: (a) Gaussian model; (b) linear finite-field deterministic model (taken from [1])

Diamond Network (contd.)

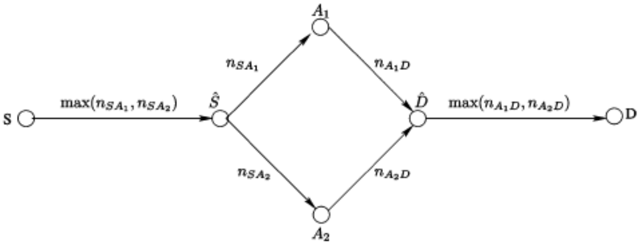

Figure : Diamond network: Wired model (taken from [1])

Diamond Network (contd.)

We can say that

$$C^{d}_{diamond} \leq C^{w}_{diamond}$$

$$= min\{max(n_{SA_1}, n_{SA_2}), max(n_{A_1D}, n_{A_2D}), n_{SA_1}+n_{A_2D}, n_{SA_2}+n_{A_1D}\}$$

Because capacity of the wired network is achieved by a routing solution, we can definitely mimic the routing scheme in linear deterministic network, and send the same amount of information.

Approach

A layered structure has been analyzed first, as it allows us to use a block of symbols for encoding sequence of messages, and these blocks do not interact as they pass through the relay nodes.

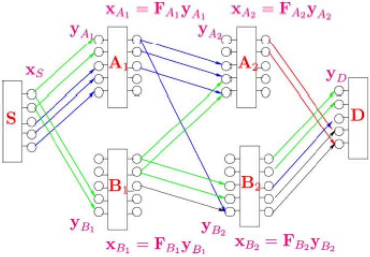

Figure : An example of a layered network (taken from [1])

Approach (contd.)

Encoding Strategy:
A sequence of messages: $\omega_k \in \{1, 2, ..., 2^{TR}\}$, $k = 1, 2, ...$
Each message is encoded into a signal over T transmission symbols
So, for linear deterministic model, following mapping is used at relay j

$$x_j = \mathbf{F}_j y_j$$

where, $x_j = [x_j[1], ..., x_j[T]]^t$ and $y_j = [y_j[1], ..., y_j[T]]^t$ represent the transmit and received units over T time units
and $\mathbf{F}_j$ is chosen uniformly at random over all matrices in $\mathbb{F}_2^{qTXqT}$

Approach (contd.)

Analysis of probability of error:
WLOG, consider message $\omega = \omega_1$, transmitted at block $k = 1$
Error occurs when, at some relay, for $\omega \neq \omega'$ the relay transmits same output $\mathbf{y}_j(\omega) = \mathbf{y}_j(\omega')$

Thus, the P_e can be upper bounded as

$$P_e \leq 2^{RT} \mathbb{P}\{\mathbf{y}_D(\omega) = \mathbf{y}_D(\omega')\} \tag{4}$$

Here, all the randomness is induced just because of encoder maps.

$$\mathbb{P}\{\mathbf{y}_D(\omega) = \mathbf{y}_D(\omega')\}$$

$$= \sum_{\Omega \in \Lambda_D} \mathbb{P}\{\text{Nodes in } \Omega \text{ can distinguish } \omega, \omega' \text{ and nodes in } \Omega^c \text{ cannot}\} \tag{5}$$

Approach (contd.)

For example, lets take a cut $\Omega = \{S, A_1, B_1\}$, so for Ω to be a distinguishability set, $\mathbf{y}_{A_2}(\omega) = \mathbf{y}_{A_2}(\omega')$, $\mathbf{y}_{B_2}(\omega) = \mathbf{y}_{B_2}(\omega')$, and $\mathbf{y}_D(\omega) = \mathbf{y}_D(\omega')$ Let us define following events:
$\mathcal{X}$ = event that ω and ω' are undistinguished at node X (i.e., $\mathbf{y}_X(\omega) = \mathbf{y}_X(\omega')$),
where $\mathcal{X} = \{\mathcal{A}_i, \mathcal{B}_i, \mathcal{D}\}$, $i = 1, 2$
Let $\mathcal{P}$ be one of the terms in summation 5 corresponding to Ω,

$$\mathcal{P} = \mathbb{P}\{\mathcal{A}_2, \mathcal{B}_2, \mathcal{D}, \mathcal{A}_1^c, \mathcal{B}_1^c\}$$

$$= \mathbb{P}\{\mathcal{A}_2\} \text{ X } \mathbb{P}\{\mathcal{B}_2, \mathcal{A}_1^c | \mathcal{A}_2\} \text{ X } \mathbb{P}\{\mathcal{D}, \mathcal{B}_1^c | \mathcal{A}_2, \mathcal{B}_2, \mathcal{A}_1^c\}$$

$$\leq \mathbb{P}\{\mathcal{A}_2\} \text{ X } \mathbb{P}\{\mathcal{B}_2, \mathcal{A}_1^c | \mathcal{A}_2\} \text{ X } \mathbb{P}\{\mathcal{D} | \mathcal{B}_1^c, \mathcal{A}_2, \mathcal{B}_2, \mathcal{A}_1^c\}$$

$$= \mathbb{P}\{\mathcal{A}_2\} \text{ X } \mathbb{P}\{\mathcal{B}_2, \mathcal{A}_1^c | \mathcal{A}_2\} \text{ X } \mathbb{P}\{\mathcal{D} | \mathcal{B}_1^c, \mathcal{B}_2\}$$

$$X_S \rightarrow (Y_{A_1}, Y_{A_2}) \rightarrow (Y_{B_1}, Y_{B_2}) \rightarrow Y_D$$

Approach (contd.)

Now, analyzing step-by-step

$$\mathbb{P}\{\mathcal{A}_2\} = \mathbb{P}\{(\mathbf{I}_T \otimes \mathbf{G}_{S,A_2})(\mathbf{x}_S(\omega) - \mathbf{x}_S(\omega')) = 0\}$$

$$= 2^{-Trank(\mathbf{G}_{S,A_2})}$$

And similarly we have,

$$\mathbb{P}\{\mathcal{B}_2, \mathcal{A}_1^c | \mathcal{A}_2\} = 2^{-Trank(\mathbf{G}_{A_1,B_2})}$$

$$\mathbb{P}\{\mathcal{D} | \mathcal{B}_1^c, \mathcal{B}_2\} = 2^{-Trank(\mathbf{G}_{B_1,D})}$$

Approach (contd.)

And putting them all together, we get

$$\mathcal{P} \leq 2^{-T rank(\mathbf{G}_{\Omega,\Omega^c})}$$

as, $\mathbf{G}_{\Omega,\Omega^c}$ is

$$\begin{bmatrix} \mathbf{G}_{S,A_2} & 0 & 0 \\ 0 & \mathbf{G}_{A_1,B_2} & 0 \\ 0 & 0 & \mathbf{G}_{B_1,D} \end{bmatrix}$$

And, substituting this back in 4, we get

$$\mathcal{P} \leq 2^{RT} |\Lambda_D| 2^{-T \min_{\Omega \in \Lambda_D} rank(\mathbf{G}_{\Omega,\Omega^c})}$$

which can be made as small as desired if $R \leq \min_{\Omega \in \Lambda_D} rank(\mathbf{G}_{\Omega,\Omega^c})$

Theorem

Upper-limit on achievable rate of a linear finite-field deterministic relay network

The capacity C of a linear finite-field deterministic relay networks is given by

$$C = \min_{\Omega \in \Lambda_d} rank(\mathbf{G}_{\Omega,\Omega^c}) \tag{6}$$

Approach (contd.)

And the proof for *General Layered Networks* follows the same principle equations 4 and 5.
$L_l(\Omega)$: the nodes that are in Ω and are at layer l ($S \in L_1(\Omega)$)
$R_l(\Omega)$: the nodes that are in Ω^c and are at layer l ($D \in R_{l_D}(\Omega)$)
And probability events under consideration are:
$\mathcal{L}_l$: Event that the nodes in L_l can distinguish between ω and ω' ($\mathbf{y}_{L_l}(\omega) \neq \mathbf{y}_{L_l}(\omega')$)
$\mathcal{R}_l$: Event that the nodes in R_l cannot distinguish between ω and ω' ($\mathbf{y}_{R_l}(\omega) = \mathbf{y}_{R_l}(\omega')$)

And, the aim is to find an upper bound on

$$\mathcal{P} = \mathbb{P}\{\mathcal{R}_l, \mathcal{L}_{l-1}, l = 2, ..., l_D\}$$

Theorem

Upper-limit on achievable rate of a Gaussian relay network

The capacity of C of a Gaussian relay network satisfies

$$\bar{C} - \kappa \leq C \leq \bar{C} \tag{7}$$

where $\bar{C}$ is the cutset upper bound on the capacity of $\mathcal{G}$ as described in 3, and κ is a constant which is upper bounded by a function of M_i and N_i, which are the number of transmit and receive antennas for each node.

Approach

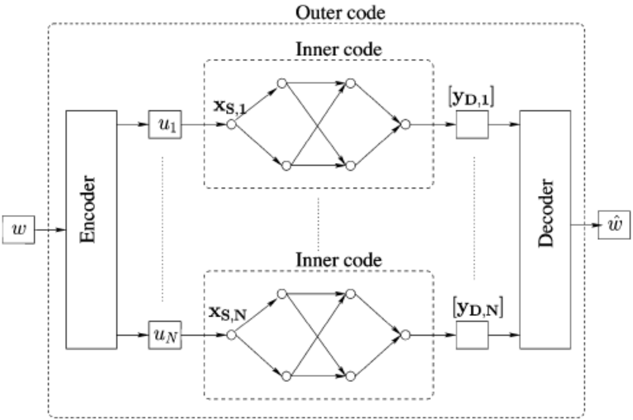

Figure : Proof Illustration (taken from [1])

Approach (contd.)

- Inner code
  - Introduces a relaying scheme similar to the deterministic case, and provides a bound on end-to-end mutual information.
  - All the symbols transmitted by the source, and the quantized received symbols at each relay are all mapped i.i.d to complex Gaussian codewords of length $T \sim \mathcal{CN}(0,1)$
- Outer code
  - Maps every message to multiple inner code symbols and sends them to destination.
  - It acts like a *error-corrector block*
- Aim of the proof is to lower-bound $\frac{1}{T}I(u;[\mathbf{y}_D]|\mathcal{F}_{\mathcal{V}})$ and it eventually results into

$$\frac{1}{T}I(u;[\mathbf{y}_D]|\mathcal{F}_{\mathcal{V}}) \geq \bar{C} - 15|\mathcal{V}| - \delta$$

Approach (contd.)

Thus, there exists a mapping that provides an end-to-end mutual information close to $\bar{C} - 15|\mathcal{V}|$, and a good outer code can be used to reliably send a message over N channel-uses at any rate upto $\bar{C} - 15|\mathcal{V}|$.

Limitations

- The linear deterministic model does not work very well for all the Gaussian channels, for instance, it fails for the case of MIMO channels, where the gap shoots to infinity as a function of gain.

References

Wireless Network Information Flow: A Deterministic Approach
A. Salman Avestimehr, Suhas N. Diggavi, David N. C. Tse
IEEE Transactions on Information Theory. Vol. 57, no. 4, April 2011

Selected chapters from Network Information Theory
Abbas El Gamal, Young-Han Kim

Thank You