Spectral Compressive Sensing

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Overview

Problem Statement

Develop an algorithm for recovery of the compressed version of general frequency-sparse signals with non-integral frequencies. Signals of the form:

$$x[n] = \sum_{k=1}^{K} a_k e^{-jw_k n} \tag{1}$$

Motivation

- Signals of the form (1) are k-sparse in frequency domain, as the coefficient vector corresponding to the Fourier basis of the signal will have only k non-zero elements corresponding to w_k; ie,
  DTFT of $\mathbf{x}$:

$$X[w] = \sum_{k=1}^{K} a_k \delta(w - w_k) \tag{2}$$

$$\mathbf{x} = \Psi\theta \tag{3}$$

Where, Ψ is the Fourier basis and θ is the DFT of $\mathbf{x}$

Motivation (contd.)

- But, an N-point DFT, which is DTFT sampled at an interval of $2\pi/N$, might not have peaks at exact locations if w_k is not an integral multiple[1] of f_s/N
- So, we require modifications in the original algorithms for compression and recovery of sparse-signals.

[1] Hereon, 'Integral Frequency' will be used interchangeably with 'Integral multiple of f_s/N'

Compressive Sensing: Basics

- A signal $\mathbf{x}$ is said to be k-sparse in basis of frame Ψ if there exists a vector θ with $||\theta||_0 = k$ such that $\mathbf{x} = \Psi\theta$
- So, compressive sensing aims at representing $\mathbf{x}$ with a lower dimension signal (say $\mathbf{y}$) such that $\mathbf{y}_{mx1} = \Phi_{mxn}\mathbf{x}_{nx1}$ $(m < n)$, where Φ is known as the measurement matrix and $\mathbf{y}$ is said to be the signal measurement.
- Recovery of signal $\mathbf{x}$, knowing $\mathbf{y}$ and Φ is also a part of theory of compressive sensing.

$$\mathbf{y} = \Phi\Psi\theta$$

Compressive Sensing (contd.)

- The recovery problem is framed as

$$min||\theta||_1 \text{ such that } ||y - A\theta||_2^2 \leq \epsilon \tag{4}$$

$$A = \Phi\Psi$$

- The main condition that is required to be satisfied for a successful recovery is:
  The measurement matrix Φ should be poorly correlated with the signal basis matrix Ψ, ie the rows of Φ (sensing waveforms) should have a very dense representation in the signal basis, so that it carries as much information possible about $\mathbf{x}$
- As $\mathbf{y} = \Phi\mathbf{x}$ is an under-determined system, not all Φ will prove to be a good measurement matrix.

Compressive Sensing: Results

Result 1

If $m \geq Clog(n/\delta)||\theta||_0\mu^2(\Psi, \Phi)$ then the solution to (4) is exact with probability $1 - \delta$, where m is the number of measurements.

Result 2

As $\mathbf{y} = (\Phi\Psi)_{mxn}\theta$ is an under-determined system, there should be some restrictions on the choice of Φ such that the null-space of $A = \Phi\Psi$ does not have any k-sparse vectors (Two different sparse signals should not give same measurement matrix). This restriction condition on A is known as *Restricted Isometry Property*

It is observed that the measurement matrix with i.i.d Gaussian entries follows *RIP* provided the number of rows is freater than a certain function of n and k

Other Methods

- Periodogram-based methods, issues:
  - Spectral leakage because of frequencies which are non-integral multiples.
  - Windowing (tapered window) can be done to reduce spectral leakage, but that will degrade spectral resolution (depending on main-lobe width).
- Line-Spectrum Estimation, issues:
  - In the pseudospectrum of the signal, non-integral multiples would not show-up.
- But when Root MUSIC algorithm is implemented in spectral compressive sensing framework (using SIHT[2] for recovery), it gives very good results.

[2] Spectral Iterative Hard Thresholding - Recovery Algorithm for Compressive Sensing framework discussed here

Redundant DFT frame

- In order to overcome the spectral leakage problem as discussed before, a simple approach using *redundant DFT frame* is employed
- A frequency sampling interval, $\Delta = \frac{2\pi}{cN}$ is used instead of $\Delta = \frac{2\pi}{N}$, where c is the *frequency redundancy factor*

$$\mathbf{e}(\omega) := \frac{1}{\sqrt{N}}[1 e^{j\omega} e^{j2\omega} ... e^{j(N-1)\omega}]^T \tag{5}$$

- Then the redundant frame with the redundancy factor c is defined as

$$\Psi(c) := [\mathbf{e}(0)\mathbf{e}(\Delta)...\mathbf{e}(2\pi - \Delta)] \tag{6}$$

So, $\theta = \Psi(c)^H \mathbf{x}$ given cN equispaced samples of signal's DTFT.

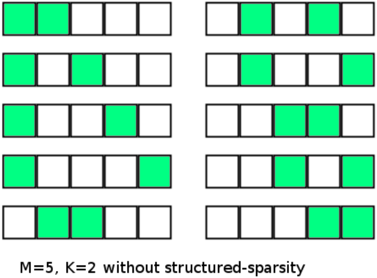
Redundant DFT frame (contd.)

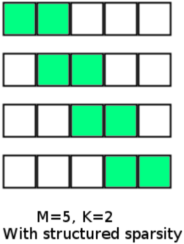
- Thus, as c increases, K-sparse approximation provided by $\Phi(c)$ becomes highly accurate.
- But, as c increases, coherence/correlation between between frame vectors increases (for c $= 1$, they are all orthonormal).
- Because of this trade-off, there is an upper limit on c for given reconstruction error tolerance

Structured Sparsity

- Instead of looking at Σ_k, set of all length-N, K-sparse signals, ie $\binom{N}{k}$ k-dimensional subspaces, we restrict the model which allows only m_k supports chosen on the basis of some additional structure to the signal $\mathbf{x}$
- Signal model M_k is defined by the set of m_k allowed supports $\{\Omega_1, ..., \Omega_{m_k}\}$
- Signals that are well-approximated as K-structured sparse are called *structured compressible*
- This apriori knowledge reduces the number of measurements (m) with high probability.

Structured Sparsity: Visualization





Model and Assumption

- Assumption:
  K-frequency sparse signal $\mathbf{x}$ has its component frequencies in the over-sampled grid of redundant frame $\Psi(c)$.
- Model:

$$T_{K,c,v} = \Bigg\{ \sum_{k=1}^{K} a_k \mathbf{e}(d_k \Delta), \text{ s.t } d_k \in \{0, ..., cN-1\},$$

$$|\langle \mathbf{e}(d_k\Delta), \mathbf{e}(d_j\Delta)\rangle| \leq v, 1 \leq k \neq j \leq K \Bigg\}$$

  where, $v \in [0,1]$ is the max coherence allowed and $\Delta = \frac{2\pi}{cN}$
- $T_{K,c,v}$ corresponds to all linear combinations of K elements from DFT frame $\Psi(c)$ (as in 6) that are pairwise sufficiently incoherent.

Approximation Algorithm :**T**(**x**,K,c,v)

Coherence-Inhibiting Model

Given: Signal vector (**x**), target sparsity (K), redundancy factor (c), maximum coherence (v)

To find: K-structured sparse representation for **x** ie, $\hat{\theta}$

Initialization:

$\theta_{cNx1} = \Psi(c)^H_{cNxN}\mathbf{x}_{Nx1}$, actual components corresponding to **x**

$c_\theta[i] = |\theta[i]|^2$, energy equivalent

Limitations:

Let $\mathbf{s} \in [0, 1]^{cN}$ and the locations of 1's in **s** correspond to the frames from $\Psi(c)$ being chosen.

- **s** should have just K number of ones.
- Frames at all the locations chosen by **s** should be pairwise sufficiently incoherent.

Approximation Algorithm

Final aim:
Find K elements in θ with the maximum possible energy and following the above two conditions.
Equivalent to finding $\mathbf{s}$ which maximizes $c_\theta^T \mathbf{s}$

Recovery Algorithm (SIHT)

Given: Measurement matrix (Φ), measurement vector ($\mathbf{y}$), Structured Sparsity Approximation Algorithm ($\mathbf{T}(\mathbf{x}, K, c, v)$)
To find: K-sparse signal extimate $\hat{\mathbf{x}}$
Concept: An iterative algorithm is used, everytime minimizing the residual function $\mathbf{r}(\hat{\mathbf{x}}, \mathbf{y})$ using the signal extimate $\hat{\mathbf{x}}$ updated in last iteration, until a satisfactory criterion is reached
Initialization:

$$\mathbf{r} = \mathbf{y}$$

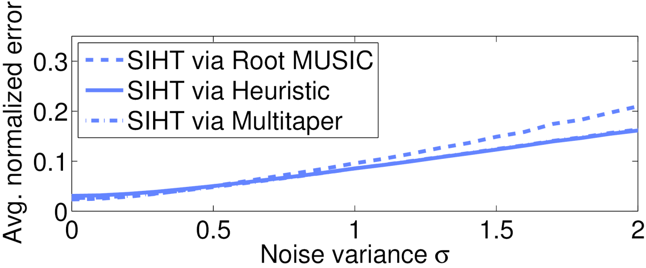
$$\hat{\mathbf{x}}_0 = 0$$

In every iteration, $\mathbf{x}$ is updated with the nearest *k-structured sparse representation*[3] of ($\mathbf{x}_{old}$ + projection of residual on Φ)
and the residual is updated with ($\mathbf{y}$ - projection of $\mathbf{x}_{new}$ on Φ)

[3] k-structured sparse representation is obtained using ($\mathbf{T}(\mathbf{x}, K, c, v)$)

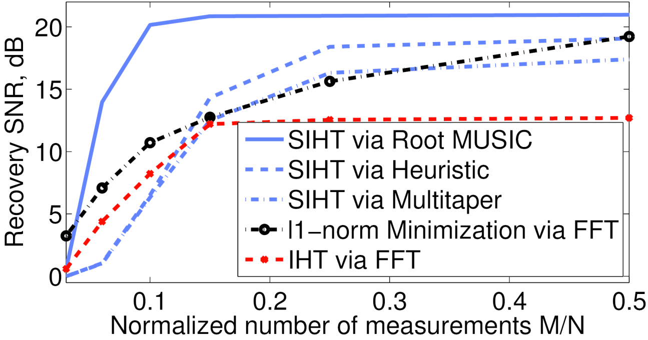
Obervations (as in paper)

Robustness to additive noise



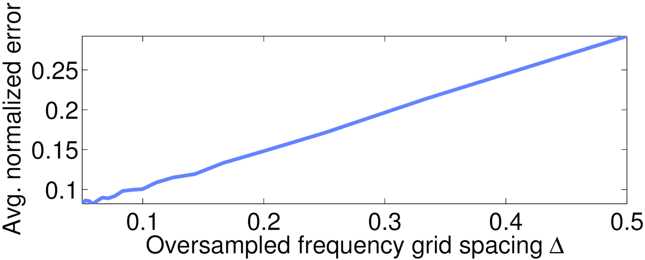
Obervations (as in paper)

Performance with change in number of measurements



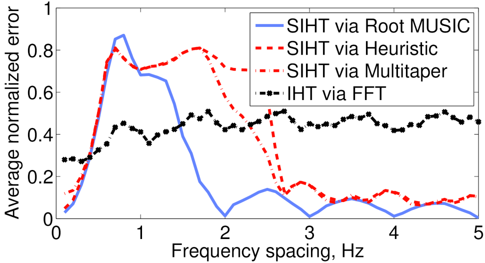
Obervations (as in paper)

Trade-off between redundancy factor (computational complexity) and the recovery performance



Observations (as in paper)

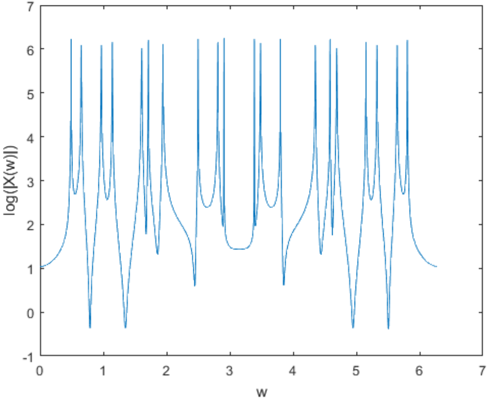
Limitations when it comes to resolution



SCS algorithm does not work well for closely-spaced frequencies, as the coherence then is too high (as larger redundancy factor will have to be used for the frequencies to lie on the over-sampled grid) to combat even using a coherence-inhibiting model.

Spectrum Plot

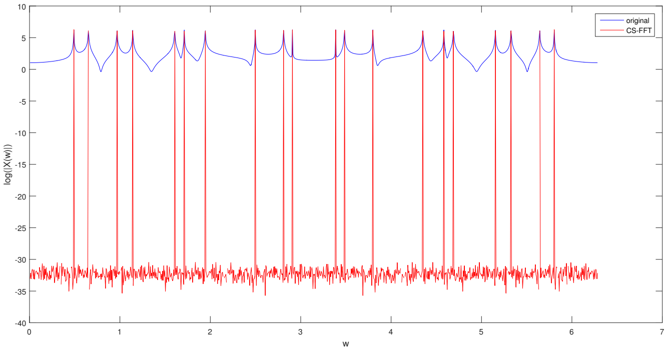
Original Signal Length : 1024 Tones = 10



Reconstructed Spectrum

Standard Compressive Sensing using DFT Basis

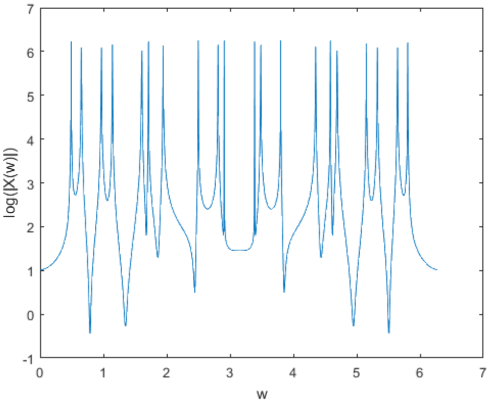
Number of measurements = 300, K = 10



Reconstructed Spectrum

Spectral Compressive sensing using Periodogram (using SIHT)

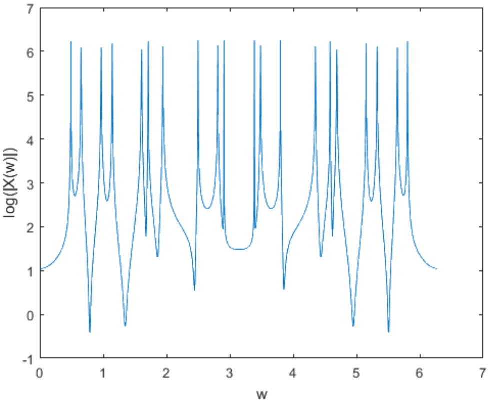
Number of measurements = 300, K = 10



Reconstructed Spectrum

Spectral Compressive sensing using ROOT MUSIC (using SIHT)

Number of measurements = 300, K = 10



Conclusion

- The original signal has 10 frequency tones (randomly generated) which have both integral and non-integral frequencies.
- The FFT plot for original signal does not have distinct spikes because of non-integral frequencies, but has 10 distinct peaks (the spectrum is somewhat distorted).
- As can be seen from the plots, using DFT basis for *Compressive Sensing* itself is not sufficient to reconstruct the plot accurately; it just detects the integral frequency approximation.
- On the contrary, using *Spectral Compressive Sensing* framework (SIHT) with Periodogram and Root MUSIC gives almost perfect reconstruction of the signal.

References

Marco F. Duarte, Richard G. Baraniuk
Spectral Compressive Sensing
Appl. Comput. Harmon. Anal. 35 (2013) 111-129

Link to the MATLAB toolbox having implementation for SCS

- `http://dsp.rice.edu/scs`

Links to lectures on *Compressive Sensing* by *Prof. Ajit Rajwade*, CSE dept, IIT Bombay

- `http://www.cse.iitb.ac.in/~ajitvr/CS763_Spring2015/CS/CS_Algorithms.pdf`
- `http://www.cse.iitb.ac.in/~ajitvr/CS763_Spring2015/CS/CS_Theory.pdf`

Thank You