Periodicity in Taylor series coefficients of complex analytic functions

Ankit Baraskar
ankit.baraskar@atidiv.com

Rahul Madhavan
rahul.madhavan@atidiv.com

Atidiv Labs

Abstract

Taylor series are one of the most ubiquitously used tools in representing infinitely differentiable functions in terms of their derivatives, and form an important part of real and complex analysis. In this paper, we explore periodicity in Taylor series coefficients and the connection with a certain class of linear differential equations. We surprisingly (or not) find that the periodicity in these coefficients is linked to periodicity of the function itself. We formalize a notation for representing such functions, and construct a methodology to decompose them in the context of a vector space generated by the eigen-solutions to the corresponding differential equations. Finally, we define an inner product which imposes a euclidean norm in this function space, opening up avenues for further work.

1 Introduction: An alternative notation for Taylor series

We consider the class of $\mathbb{C}^\infty$ functions $f(z)$, $f : \mathbb{C} \to \mathbb{C}$ where all the derivatives at 0 are well defined, and consequently so is the Taylor series centered at 0. Thus the function can be represented as:

$$f(z) = \sum_{r=0}^{\infty} \frac{f^{(r)}(0)z^r}{r!} \tag{1}$$

Here $f^{(r)}$ is the r^{th} derivative, and the 0^{th} derivative is the value of the function itself. In some sense, these functions are defined by a countable number of complex numbers (similar to continuous real valued functions, which can be defined by their values at all rationals). Thus, we can represent them as a sequence of complex numbers as:

$$f \sim (a_0, a_1, a_2, ...) \tag{2}$$

Let's try writing some of the well known functions in this notation:

$$e^z \sim (1, 1, 1, ...)$$
$$sin(z) \sim (0, 1, 0, -1, 0, 1, 0, -1, ...)$$
$$cos(z) \sim (1, 0, -1, 0, 1, 0, -1, 0, ...)$$
$$z^2 \sim (0, 0, 2, 0, ...)$$

It is interesting to note that the representations for polynomials will always terminate, and that the representatons for the trigonometric and the exponential functions are periodic. Is there an inherent connection between periodic sequences of Taylor coefficients and periodicity of the function itself? Let's try to analyze this in the next section.

2 Analysis: Taylor series with periodic coefficients

Here we introduce another notation - if the sequence of coefficients is eventually repeating with period n, and the repetition starts at index m ($m \in \mathbb{Z}_0^+$, $n \in \mathbb{N}$) we represent the function as:

$$f_{m,n} \sim (a_0, a_1, a_2, ..., a_{m-1}, \overline{a_m, a_{m+1}, ..., a_{m+n-1}}) \tag{3}$$

For simplicity's sake, let's start with functions where $m = 0$, i.e. the repetition starts at the first coefficient.

2.1 The differentiation operator

It is quite easy to see that the differentiation operator acts as a left shift operator for the series, namely:

$$D(a_0, a_1, a_2, ...) = (a_1, a_2, a_3, ...) \tag{4}$$

This is perfectly valid mathematically as well, since a term-wise derivative of a differentiable power series has the same radius of convergence as the original series. This leads to a very natural conclusion, namely that if the sequence is a function of the form $f_{0,n}$ (the sequence starts at 0 and is periodic with period n), it satisfies the following differential equation:

$$f^{(n)}(z) = f(z) \tag{5}$$

The corresponding auxiliary equation is

$$x^n - 1 = 0 \tag{6}$$

the roots of which are simply the n^{th} roots of unity. Let

$$\omega_n = e^{2\pi i/n} = cos(2\pi/n) + isin(2\pi/n) \tag{7}$$

where $i = \sqrt{-1}$. Then the general solution to equation (4) is of the form

$$f(z) = \sum_{r=1}^{n} A_r e^{\omega_n^r z} \tag{8}$$

Where A_r are constants that can be solved for given boundary conditions, which are the Taylor series coefficients. Thus we see that any function of the form $f_{0,n}$ is a solution to the equation (4)

2.1.1 Note: Choice of ω_n

We note that our choice of representation uses $w_n = e^{2\pi i/n}$ to compute the solutions - this could have been any of the $\phi(n)$ primitive n^{th} roots of unity. This does not change the basis or the form of the solution.

2.2 Linear systems of equations

Let us look at some special instances of this solution. For instance, let us consider the case $n = 4, m = 0$. In this case, we have repeating sequences of length 4. Thus any function of the form $f_{0,4}$ can be written as

$$f_{0,4} = A_1 e^{iz} + A_2 e^{-z} + A_3 e^{-iz} + A_4 e^z$$

This can also be re-written as:

$$f_{0,4} = A_1^* cos(z) + A_2 e^{-z} + A_3^* sin(z) + A_4 e^z$$

We note that this is just a change of basis in the space of $f_{0,4}$ functions. If we think of these functions in terms of their sequence representations, this can be rewritten as:

$$f_{0,4} = A_1^*(\overline{1, 0, -1, 0}) + A_2(\overline{1, -1, 1, -1}) + A_3^*(\overline{0, 1, 0, -1}) + A_4(\overline{1, 1, 1, 1})$$

which can then be simplified to a matrix equation

$$f_{0,4} = \mathbf{\Theta} u$$

where

$$\mathbf{\Theta} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & -1 & -1 & 1 \end{bmatrix}, u = [A_1^* \quad A_2 \quad A_3^* \quad A_4]$$

Thus, knowing the four repeating coefficients of any f of the form $f_{0,4}$, we can determine the coefficients A_i. This can be extended to the case $f_{0,n}$.

2.3 Pure oscillations

As we have seen, functions of the form $f_{0,n}$ can be decomposed into exponential growth/decay and sinusoidal oscillations. Here we derive a condition for n where pure sinusoidal oscillations without any decay or growth are possible. To re-iterate, the general basis function for the solution space has the form $e^{\omega_n^r z}$. If this is periodic with period $\alpha \in \mathbb{R}$

$$\begin{aligned} e^{\omega_n^r z} &= e^{\omega_n^r (z+\alpha)} \\ \Rightarrow \omega_n^r \alpha &= 2\pi i n, n \in \mathbb{N} \\ &\Rightarrow Re(\omega_n^r) = 0 \end{aligned}$$

Thus for the case of pure oscillations we need ω_n^r to be purely imaginary, i.e.

$$\begin{aligned} cos(2\pi r/n) &= 0 \\ \Rightarrow 2\pi r/n &= (2k+1)\pi/2 \\ \Rightarrow r &= (2k+1)n/4 \\ &\Rightarrow 4|n \end{aligned}$$

Thus we have arrived at a necessary (though not sufficient) condition for the case where a Taylor series with periodic coefficients represents a purely sinusoidal function with a real frequency α, namely that the period has to be a multiple of 4.

3 Generalization: The case where $m > 0$

In this case, the function is represented as

$$f_{m,n} \sim (a_0, a_1, a_2, ..., a_{m-1}, \overline{a_m, a_{m+1}, ..., a_{m+n-1}})$$

We can apply the differentiation operator m times in succession to obtain:

$$D^{(m)}(a_0, a_1, a_2, ..., a_{m-1}, \overline{a_m, a_{m+1}, ..., a_{m+n-1}}) = (\overline{a_m, a_{m+1}, ..., a_{m+n-1}})$$

Now, differentiating the above equation n more times, we get the equation

$$f^{(m+n)}(z) = f^{(m)}(z) \tag{9}$$

In this case, the auxiliary equation becomes

$$x^m(x^n - 1) = 0 \tag{10}$$

This admits a solution of the form

$$f(z) = \sum_{s=0}^{m-1} B_s z^s + \sum_{r=1}^{n} A_r e^{\omega_n^r z} \tag{11}$$

where the coefficients B_s and A_r can be determined using the boundary conditions, i.e. the coefficients of the Taylor series for $f(z)$.

4 Representation: Vector Spaces

Every set of functions of the class $f_{m,n}$ constitute a vector space of dimension $m + n$. We further observe that the sum of two such functions with periodic Taylor coefficients has periodic Taylor coefficients as well. This can be verified with a quick calculation in our sequence representation of a Taylor series. The nature of the space containing such a sum can be determined as follows

$$f_{m_1,n_1} + f_{m_2,n_2} \in f_{max(m_1,m_2),lcm[n_1,n_2]} \tag{12}$$

where lcm stands for the least common multiple of two positive integers, and where the sum of two classes is the set of all sums of two functions, the first belonging to f_{m_1,n_1} and the second belonging to f_{m_2,n_2}.

Based on the previous section, the inner product on the space of functions $f_{m,n}$ arises naturally in terms of the eigenfunction representation. For this, recall the example from section 2.2 where we decomposed $f_{0,4}$ as a linear combination of the individual exponential functions. In general, when the function f is represented as in equation 11, we can define the inner product between two functions $f_{m,n}$, $g_{m,n}$ as

$$\langle f_{m,n}, g_{m,n}\rangle = \sum_{s=0}^{m-1} B_{s,f}B_{s,g} + \sum_{r=1}^{n} A_{r,f}A_{r,g} \tag{13}$$

The induced norm is calculated as

$$||f_{m,n}|| = \sqrt{\sum_{s=0}^{m-1} B_s^2 + \sum_{r=1}^{n} A_r^2} \tag{14}$$

In case we want to compare f_{m_1,n_1} and f_{m_2,n_2}, we can represent them both in terms of the eigenfunctions of $f_{max(m_1,m_2),lcm[n_1,n_2]}$. In general, comparison between any finite number of such functions represented in the same space is possible.

5 Conclusion

This paper might be a re-working of a pre-existing theory, but to the best of the authors' knowledge, it is not. Here we extend the interpretation of Taylor series from being just a power series to take advantage of its intrinsic structure to study periodicity in the series coefficients. We formalized a notation and related periodic coefficient sequences to solutions of a class of linear differential equations, which in turn enabled us to decompose these series into a combination of exponential growth/decay functions and sinusoidal oscillations. It should be noted that though existence of a periodic sequence in Taylor coefficients implies a periodic component, the converse is not true. (Think of $e^{\alpha x}$, $\alpha \neq 1$).

Further work would involve exploring the behavior of this class of functions as a subspace of the space of all $\mathbb{C}^{\infty}$ functions.

References

[1]Rudin W. (1976) Principles of Mathematical Analysis, *Third Edition*