Fourier Series :

"Any periodic signal can be represented as a linear combination of harmonically related complex exponentials."

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k t}}$$

$\omega_0 = 2\pi F_0$

$= \frac{2\pi}{T_0}$

T_0 : fundamental Period

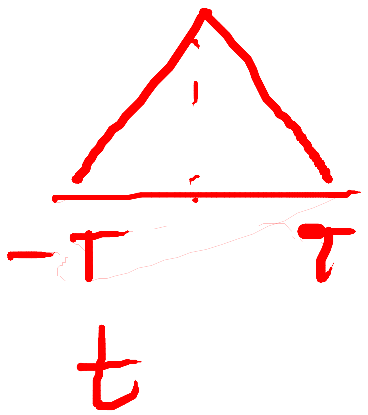
Fourier Series Coefficients

$$a_k = \frac{1}{T}\int_0^T x(t)\, e^{-j\omega_0 k t}\, dt$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\, dt$$

(d.c component)

Example : $x(t) = \cos(\omega_0 t)$



$$a_k = \frac{1}{T}\int_T x(t)\, e^{-j\omega_0 k t}\, dt$$

$$a_k = \frac{1}{T}\int_0^T \left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right] e^{-j\omega_0 k t}\, dt$$

$$= \frac{1}{2T}\left[\underbrace{\int_0^T e^{j\omega_0(1-k)t}\, dt}_{A} + \underbrace{\int_0^T e^{-j\omega_0(1+k)t}\, dt}_{B}\right]$$

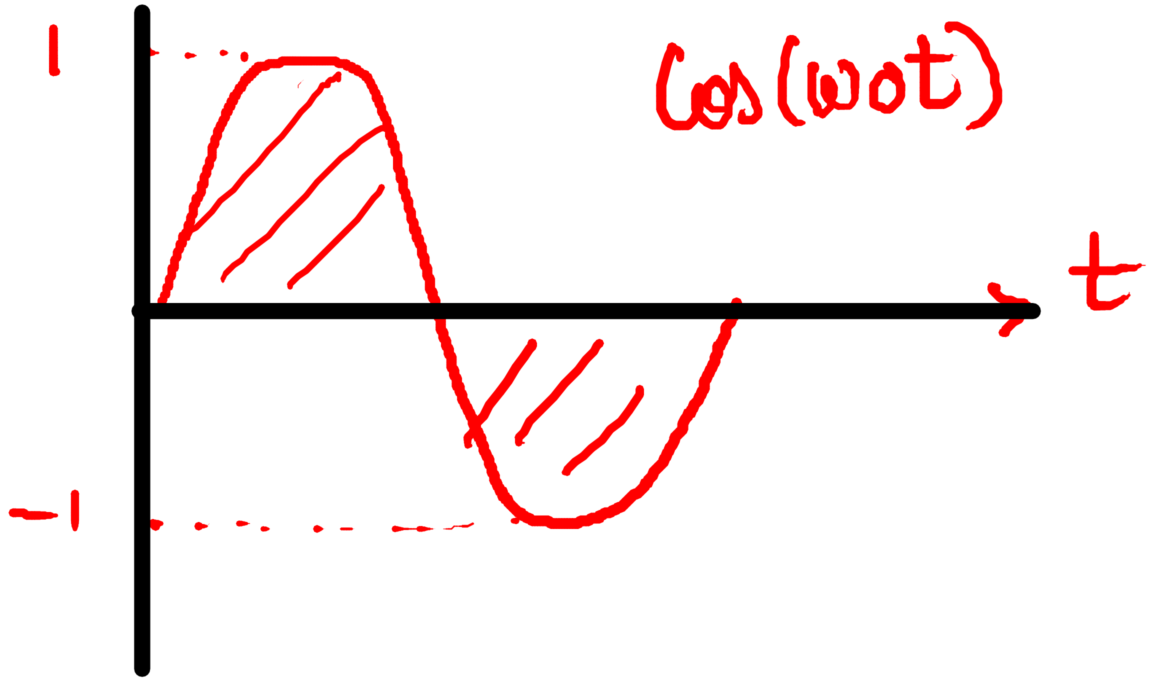
$$\int_0^T e^{j\omega_0(1-k)t}\, dt = T \; ; \; k=1$$
$$= 0 \; ; \; k \neq 1$$

$$\int_0^T e^{j\omega_0(1+k)t}\, dt = T \; ; \; k=-1$$
$$= 0 \; ; \; k \neq -1$$

$k = 1$; $a_1 = \frac{1}{2}$ ✓

$k = -1$; $a_{-1} = \frac{1}{2}$ ✓

$k = 0$; 0 ✓



For, $|k| > 1$ $a_k = 0$

e.g.: $\cos(\omega_0 t)$

$$= \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

$$= \left(\frac{1}{2}\right) \underbrace{e^{j\omega_0 t (1)}}_{k=1} + \left(\frac{1}{2}\right) \underbrace{e^{-j\omega_0 t (1)}}_{k=-1}$$

Example :

$$x(t) = 1 + \sin(\omega_0 t)$$

$$= \underbrace{1e^{j\omega_0 0 t}}_{k=0} + \underbrace{\frac{e^{j\omega_0 t}}{2j}}_{k=1} - \underbrace{\frac{e^{-j\omega_0 t}}{2j}}_{k=-1}$$

$k=0$; $a_0 = 1$

$k=1$; $a_1 = \frac{1}{2j}$

$k=-1$; $a_{-1} = \frac{-1}{2j}$

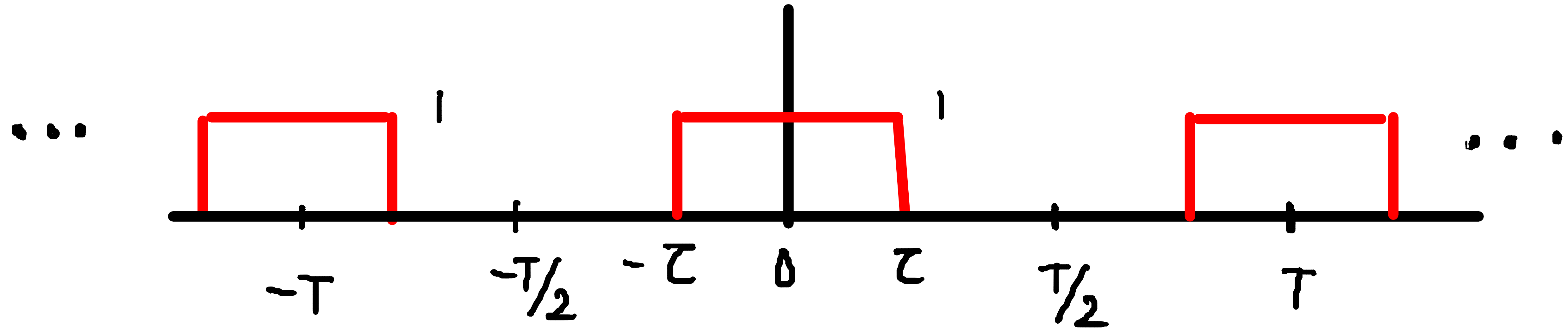
$$x(t) = \sum_k a_k e^{j\omega_0 k t}$$

$$= a_0 e^{j\omega_0 0 t}$$

$$+ a_1 e^{j\omega_0 1 t}$$

$$+ a_{-1} e^{-j\omega_0 1 t} \ldots$$

Example :



Periodic signal

$$x(t) = \begin{cases} 1 & ; \quad t < |\tau| \\ 0 & ; \quad \text{otherwise.} \end{cases}$$

T is the period

$$a_k = \frac{1}{T}\int_{\langle T \rangle} x(t)\, e^{-j\omega_0 k t}\, dt = \frac{1}{T}\int_{-\tau}^{\tau} e^{-j\omega_0 k t}\, dt$$

$$= \frac{1}{T}\, \frac{e^{-j\omega_0 k t}}{-j\omega_0 k}\Bigg|_{-\tau}^{\tau} = \frac{1}{-Tj\omega_0 k}\left[e^{-j\omega_0 k \tau} - e^{j\omega_0 k \tau}\right]$$

$$= \frac{1}{\cancel{-}T\cancel{j}\omega_0 k}\left[\cancel{-}2\cancel{j}\sin(\omega_0 k \tau)\right] = \frac{1}{\frac{\cancel{2}\pi}{\cancel{T}}\cancel{T}k}\left[\cancel{2}\sin(\omega_0 k \tau)\right]$$

$$\boxed{\begin{aligned} a_k &= \frac{\sin(\omega_0 k \tau)}{k\pi} \quad ; \quad k \neq 0 \\ &= \frac{2\tau}{T} \quad ; \quad k = 0 \end{aligned}}$$

Euler

Langrange.