Projective and whatever Ratio Lemma is

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1 Projective Review

2 A Perspective on Conics

One of the most important properties of conics in geometry is that we can define cross ratios on them. Translating this slightly using the previous section, we have the following:

Theorem 2.1

Let Γ be a conic. Then, we can parameterize Γ by complex numbers and infinity, say using the function $\gamma\colon \Gamma \to \mathbb{CP}$, such that for all A, B, C, D on Γ, we have

$$(\gamma(A), \gamma(B); \gamma(C), \gamma(D)) = (A, B; C, D).$$

This is a pretty powerful way of viewing the well-defined-ness of cross ratio on conics, since now we have a good way of thinking about points on conics as points on a line, or just as complex numbers. This is really abstract and hard to put into words, so I'll just try an example of this philosoophy.

Example 2.2 (Pascal's Theorem)

Let A, B, C, D, E, F be points on a conic. Let $P = AE \cap BD$, $Q = BF \cap CE$, and $R = CD \cap AF$. Then, P, Q, and R are collinear.

Proof. We claim that this theorem is equivalent to the following:

> Let a, b, c, d, e, f be elements of $\mathbb{CP}$. Let f_P be a cross ratio preserving function from $\mathbb{CP}$ to itself that swaps a with e and swaps b with d. Similarly, suppose f_Q swaps (b, f) and (c, e) and f_R swaps (c, d) and (a, f). Then, there are elements x, y of $\mathbb{CP}$ such that all of f_P, f_Q, and f_R swap x and y.

After we show this equivalence, the rest is pretty natural. We have that f_P swaps x and y if and only if $(a, d; x, y) = (e, b; y, x) = (b, e; x, y)$, and similar things hold for f_Q and f_R. Now, if we pick x, y to be points swapped by both f_P and f_Q (by, say, taking the two fixed points of $f_P \circ f_Q$), it's simple to see that f_R must swap x and y, as desired.

EXPLAIN EQUIVALENCE $\square$

3 Projectivized Ratio Lemma

3.1 Projectived Ratio Lemma

First, we'll aim to give a better definition than $f(P) = \pm\frac{BP}{CP}$. There are a few motivations for this:

1. Having signs take cases based on where P is is weird from a geometric perspective.
2. We proved Pascal's in the case where the conic is a circle, which is a purely projective theorem.
3. We only got Pascal's when the line intersects the circle. If we move to $\mathbb{CP}^2$, all circles intersect, but we can't define lengths anymore.

Basically, f behaves way too well with projective stuff for $\pm BP/CP$ to be the best definition. This motivates:

Definition 3.1. Let Γ be a conic and ℓ be a line. Suppose they meet at P_1 and P_2. Let P_Γ be an arbitrary point on Γ, and let the tangent to Γ at P_Γ meet ℓ at P_ℓ. Define the function f on $(\Gamma \cup \ell) \setminus \{B, C\}$ as follows:

- For X on Γ, define $f(X) = (XP_\Gamma; P_1P_2)$
- For X on ℓ, define $f(X) = (XP_\ell; P_1P_2)$.

This definition completely works, since in $\mathbb{CP}^2$, all lines and conics intersect. The case where a line and circle is tangent seems cursed and is probably best dealt with by continuity.

The proofs of the following facts, which recover most of the ratio lemma handout, are omitted (for now).

Theorem 3.2 ("Ratio Lemma")

Suppose we pick X, Y, Z on Γ, Γ, ℓ. Then, XYZ are collinear if and only if $f(X)f(Y) = f(Z)$.

Theorem 3.3

If X, Y, T are on ℓ, $TX \cdot TY = TP_1 \cdot TP_2$ if and only if $f(X)f(Y) = f(\infty_\ell)f(T)$.

Theorem 3.4 (Desargues Involution Theorem)

Let $PQRS$ be a quadrilateral with vertices on Γ. Let T_{PQ} denote $PQ \cap \ell$, and define other line intersections similarly. Then, there is an involution on ℓ swapping (P_1, P_2), (T_{PQ}, T_{RS}), (T_{PR}, T_{QS}), and (T_{PS}, T_{QR}).

Theorem 3.5 (Conic Condition)

If P, Q, R, S are on Γ and X, Y are on ℓ, then $PQRSXY$ lie on a common conic if and only if

$$f(P)f(Q)f(R)f(S) = f(X)f(Y)$$

4 Why the Ratio Lemma is Strong

With this new definition, we can see that f is a really good choice of parameterization of the conic, as well as a good parameterization of the line. In some sense, knowing $f(P)$ for $P \in \Gamma$ is almost as good as knowing directed length OP for P on given a line through O. What I'm actually saying is:

Theorem 4.1

For any points P, Q, R, S on Γ, we have $(PQ; RS) = (f(P)f(Q); f(R)f(S))$. Equivalently, f is a cross ratio preserving map from Γ to $\mathbb{CP}$. The same property holds on ℓ.

TODOOOOO

Theorem 4.2 (Swapping Ratios)

Let f_1 and f_2 be cross ratio type functions on a conic Γ. Then, for some fixed constants a, b, c, d with $ad - bc \neq 0$,

$$f_2(X) \equiv \frac{af_1(X) + b}{cf_1(X) + d}.$$

Theorem 4.3 (General Projective Maps)

Let g be any cross ratio preserving map from Γ to itself. Then, for some fixed constants a, b, c, d with $ad - bc \neq 0$,

$$Q = g(P) \iff f(Q) = \frac{af(P) + b}{cf(P) + d}$$

Theorem 4.4 (General Collinearity)

For some point T, let g be the map from $P \in \Gamma$ to $TP \cap \Gamma$ other than P. Then, for some fixed constants a, b, c, d with $a = -d$, $ad - bc \neq 0$, we have

$$Q = g(P) \iff f(Q) = \frac{af(P) + b}{cf(P) + d}.$$

This is because a non identity projective map from Γ to itself is a second intersection map if and only if $g(g(P)) = P$ for all P.

Theorems from this section will almost never be applicable on a contest. I just think they're cool and maybe give some more insight on why Ratio Lemma considerations actually work. I've used them once though:

Example 4.5 (2012 IMO Shortlist G8)

Let ABC be a triangle with circumcircle ω and ℓ a line without common points with ω. Denote by P the foot of the perpendicular from the center of ω to ℓ. The side-lines BC, CA, AB intersect ℓ at the points X, Y, Z different from P. Prove that the circumcircles of the triangles AXP, BYP and CZP have a common point different from P or are mutually tangent at P.

[DIAGRAMMM] Work in $\mathbb{CP}^2$, so we don't need to care about the "ω and ℓ don't intersect" condition. It's probably just there so we avoid degenerate cases like ℓ passing through A. Take $\ell \cap \omega = \{P_1, P_2\}$, let P_Γ be the "topmost" point on (ABC) if we draw ℓ horizontally, so P_ℓ is ∞, and define $f(T)$ as we did in section 9.1.

Let (APX) meet ω again at A'. By the Radical Axis Theorem, it suffices to show that AA', BB', and CC' are concurrent. By theorem 9.9, it suffices to show that for some constants p, q, r,

$$\begin{aligned} pf(A)f(A') + q(f(A) + f(A')) + r = 0 \\ pf(B)f(B') + q(f(B) + f(B')) + r = 0 \\ pf(C)f(C') + q(f(C) + f(C')) + r = 0. \end{aligned}$$

Hence, by some linear algebra, it suffices to check that the following determinant is 0:

$$\begin{vmatrix} f(A)f(A') & f(A) + f(A') & 1 \\ f(B)f(B') & f(B) + f(B') & 1 \\ f(C)f(C') & f(C) + f(C') & 1 \end{vmatrix}.$$

Letting $a = f(A)$ and defining b, c similarly, we have $f(X) = bc$ and $f(P) = -1$, so $f(A') = -bc/a$. Now, we'll just go through and calculate the determinant by doing row operations:

$$\begin{aligned} \begin{vmatrix} f(A)f(A') & f(A) + f(A') & 1 \\ f(B)f(B') & f(B) + f(B') & 1 \\ f(C)f(C') & f(C) + f(C') & 1 \end{vmatrix} &= \begin{vmatrix} -bc & a - \frac{bc}{a} & 1 \\ -ca & b - \frac{ca}{b} & 1 \\ -ab & c - \frac{ab}{c} & 1 \end{vmatrix} \\ &= \begin{vmatrix} -bc & a - \frac{bc}{a} & 1 \\ bc - ca & a - \frac{bc}{a} - \left(b - \frac{ca}{b}\right) & 0 \\ bc - ab & a - \frac{bc}{a} - \left(c - \frac{ab}{c}\right) & 0 \end{vmatrix} \\ &= \begin{vmatrix} -bc & a - \frac{bc}{a} & 1 \\ c(b-a) & (a-b)\left(1 + \frac{c(a+b)}{ab}\right) & 0 \\ b(c-a) & (a-c)\left(1 + \frac{b(c+a)}{ca}\right) & 0 \end{vmatrix}. \\ &= \begin{vmatrix} -bc & a - \frac{bc}{a} & 1 \\ c(b-a) & (a-b)\left(\frac{ab+bc+ca}{ab}\right) & 0 \\ b(c-a) & (a-c)\left(\frac{ab+bc+ca}{ca}\right) & 0 \end{vmatrix}. \end{aligned}$$

Now, the middle row times $\frac{b}{c} \cdot \frac{c-a}{b-a}$ is 0, so the determinant is 0, implying the result.