How to Trig with Directed Angles

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This is probably too inconvenient for people to use in-contest, but I still think its cool. Thanks to Luke Robitaille for help! Also this is in evan.sty.

Contents

1 Directed Angles Review

1.1 Definitions

The reference I'm using for directed angles is Evan Chen's Directed Angles handout. If you haven't read it (or perhaps the corresponding part of EGMO), you probably should. Here's roughly how it goes:

There's a problem with the way angles usually work. You can see a couple examples in the original handout, but this one's enough to motivate the rest of our handout:

> Let P and Q be points, and let ℓ be a fixed line through Q. Let R be a variable point on ℓ. We want to be able to say what $\angle PQR$ is, but it changes depending on what side of Q we put R on.

We'll fix the issue by making angles only depend on the lines.

Definition 1. Suppose ℓ and m are lines. There are many angles θ such that if we rotate ℓ by θ, counter-clockwise, then we get a line parallel to m. In particular, these values of θ form a residue class modulo π. We let $\measuredangle(\ell, m)$ denote this value mod π.

We can now also define $\measuredangle POQ$ to be $\measuredangle(PO, OQ)$. This notion of angles now satisfies pretty much all the properties we'd want it to!

1.2 Arguments of Lines

For the rest of this handout, we'll work over $\mathbb{C}$. An important piece of intuition for this section, which I learnt from Luke Robitaille, is the following:

> We can treat each line as having a certain "argument" modulo π. The angles between two lines comes from subtracting their arguments.

We formalize this intuition and make it easier to algebraically work with.

Definition 2. For any line ℓ, define $\arg \ell = \angle(r, \ell)$, where r is the real line.

Indeed, we can now see that

$$\angle(\ell, m) = \angle(\ell, r) + \angle(r, m) = \arg m - \arg \ell.$$

Thinking about angles in this way makes angle chasing nicer in a bunch of ways. For example,

Theorem 3 (Cyclic Quadrilateral Condition)**.** *Points A, B, C, D lie on a circle if and only if* $\arg(AB) + \arg(CD) = \arg(AD) + \arg(BC)$.

More relevantly for future parts of this handout, we can go from any complex number to its corresponding angle as follows:

Proposition 4. *If z_ℓ is the difference between two points on a line ℓ, then $e^{2i \arg \ell} = z_\ell / \overline{z_\ell}$. More generally, if z_ℓ and z_m are differences between two points on ℓ and two points on m, $e^{2i\angle(\ell,m)} = z_m/\overline{z_m} \div z_\ell/\overline{z_\ell}$.*

Proof. For the former, write $z_\ell = re^{i\theta}$ for some real r. Now,

$$\overline{z_\ell} = r \cdot e^{-i\theta} \implies z_\ell/\overline{z_\ell} = e^{2i\theta} = e^{2i \arg \ell}.$$

For the latter, note that $e^{2i\angle(\ell,m)} = e^{2i(\arg m - \arg \ell)} = e^{2i \arg m} \div e^{2i \arg \ell}$ $\square$

2 The Problem

Now, the issue with doing trig is that the following statements are both true:

- If we're working with directed angles, we have $\theta = \pi + \theta = 2\pi + \theta = \cdots$.
- $\sin(\theta)$ is not equal to $\sin(\pi + \theta)$ (in particular, $\sin(\pi + \theta) = -\sin(\theta)$).

This means we literally can't take the sin of directed angles. As far as I can tell, there isn't really a fix for this that people use, other than not using directed angles. This is kinda sad for a couple reasons:

- It isn't always easy to WLOG away configuration issues.
- Directed angles are really clean, and it's sad that nothing like that works for trig.
- It's going to turn out that the fix in this handout actually makes some things slightly easier!

3 Lengths via Complex Numbers

3.1 Basic Properties

Instead of thinking about length as real numbers and directing each line, we'll think about the lengths as complex numbers:

Definition 5. Let AB denote the complex number $B - A$.

Now, $|AB|$ denotes the usual nonnegative real length of AB. This maintains the properties of directed lengths on lines:

Proposition 6. *Let A, B, and C be points on a (directed) line. Let $\ell(AB)$ denote the directed length of AB on this line. Then,*

$$\frac{AB}{AC} = \frac{\ell(AB)}{\ell(AC)}.$$

In particular, with our new notion of length, the statements of Ceva and Menelaus are unchanged. We also have the following:

Proposition 7. *Let A, B, C, D be points on a circle or a line. Then,*

$$(AB; CD) = \frac{AC}{AD} \div \frac{BC}{BD}$$

The proof of the above statements, and in general proofs for this section, will follow from the corresponding theorems with normal lengths, along with some boring sign analysis. Alternatively, repeating the derivation of the original proof also often works.

Since complex length encompasses both direction and angle, some things become easier to state:

Theorem 8. *(SAS Similarity) Triangles $A_1B_1C_1$ and $A_2B_2C_2$ are similar with the same orientation or with opposite orientation if and only if*

$$\frac{A_1B_1}{A_1C_1} = \frac{A_2B_2}{A_2C_2} \quad or \quad \frac{A_1B_1}{A_1C_1} = \overline{\left(\frac{A_2B_2}{A_2C_2}\right)}$$

respectively.

Theorem 9. *(Ptolemy) Let A, B, C, and D be any(!) points. Then, we have $AB \cdot CD + AC \cdot DB + AD \cdot BC = 0$. When $ABCD$ is cyclic, each term in this equation has the same argument.*

Other statements are harder. Some general pieces of intuition are the following:

- Avoid stuff like $|PQ|$, unless we're squaring it; $|PQ|^2 = PQ \cdot \overline{PQ}$ is pretty good! This is because $|PQ|$ and $-|PQ|$ are in some sense algebraically indistinguishable (I'm not sure how to make this rigorous).
- It should usually suffice to take the original theorem and, with some choice of signs, rotate each term such that every term has the same argument.

Here are some more conversions:

Theorem 10 (Power of a Point)**.** *Let ω be a fixed circle with center O, and let R be (any) radius of ω. Let P, Q be points on ω, and let A be a point on line PQ. We have*

$$|AO|^2 - |R|^2 = AP \cdot \overline{AQ}.$$

Theorem 11. *(Stewart's Theorem) Let ABC be a triangle, and let D be any point on line BC. Then,*

$$|AC|^2 \cdot BD + |AB|^2 \cdot DC = BD \cdot DC \cdot \overline{BC} + |AD|^2 \cdot BC$$

Note that every term in the above formula has the same argument as BC.

Theorem 12. *(Apollonius Circles) Let ABC be a triangle. The locus of points P with*

$$\frac{|BP|^2}{|CP|^2} = \frac{|BA|^2}{|CA|^2}$$

is a circle passing through A, known as the Apollonius Circle of A with respect to BC.

3.2 Angle Bisectors

You may note that the above conversions don't really give a way to compare the lengths of two complex numbers if they don't have the same argument. There's one important example of a case where we can actually do this:

Claim 13. Let OPQ be a triangle where $\angle OPQ = -\angle OQP$. We have

$$\frac{OQ}{OP} = e^{2i(\angle OPQ + \pi/2)}.$$

This is annoying, since we want to be able to say it's just $e^{i\angle POQ}$, but we're thinking about angles mod π. We'll proceed by standard complex bashing techniques.

Proof. Assume, without loss of generality, that $O = 0$ and P, Q are on the unit circle. Then, we have

$$e^{2i(\angle OPQ + \pi/2)} = -\left(\frac{Q-P}{P}\right) \div \overline{\left(\frac{Q-P}{P}\right)} = -\frac{Q-P}{P} \div \frac{\frac{1}{Q} - \frac{1}{P}}{\frac{1}{P}} = \frac{Q}{P},$$

as desired. $\square$

We can essentially think about this as saying that if, for three points XYZ, we can "naturally rotate" XY onto XZ if $\angle YXZ$ is twice of some angle that occurs in the diagram. This claim just happens to be an easy way of extracting that information. This situation also occurs naturally when we have angle bisectors show up. Here's an example of how we can work with this:

Theorem 14. *Let ABC be a triangle with incenter I. Suppose the incircle meets BC, CA, and AB at D, E, and F. Then, we have*

$$AE = \frac{1}{2}\left(AC + ABe^{2i\angle BAI} - BCe^{2i\angle BCI}\right).$$

The proof is just the normal proof, except we have to worry about random argument terms.

Proof. We have $EF \perp AI$, so for our analogue of $|AF| = |AE|$, we get

$$\frac{AF}{AE} = e^{2i(\angle AEF + 90^\circ)} = e^{2i\angle EAI} = e^{2i\angle CAI}$$

and similar for other intouch lengths. We now just expand the RHS into intouch lengths:

$$AC + ABe^{2i\angle BAI} - BCe^{2i\angle BCI} = AE + EC + (AF + FB)e^{2i\angle BAI} - (BD + DC)e^{2i\angle BCI}.$$

We have $AFe^{2i\angle BAI} = AE$, $FBe^{2i\angle BAI} = BDe^{2i\angle BCI}$, and $EC = DCe^{2i\angle BCI}$, giving the desired result. $\square$

4 Basics of Trig Functions

4.1 Redefining

We define new and improved trig functions:

$$\operatorname{sinr}\theta \overset{\text{def}}{=} ie^{i\theta}\sin\theta = \frac{e^{2i\theta} - 1}{2} \quad \text{and} \quad \operatorname{cosr}\theta \overset{\text{def}}{=} e^{i\theta}\cos\theta = \frac{e^{2i\theta} + 1}{2}.$$

From an algebraic perspective, this makes sense because:

- We also have $|\sin\theta| = |\operatorname{sinr}\theta|$, and similar for cos. Since the magnitudes of sin and cos never gave us configuration issues, this is good.
- We've previously shown that there's a natural correspondence between the line consisting of $r \cdot e^{i\theta}$ for real r and the unit complex number $e^{2i\theta}$
- The definitions are now injective functions of $e^{2i\theta}$, and are actually π-periodic like we want them to be!

Later, we'll see why this makes sense from a geometric standpoint. For now, we'll recover the analogues of basic properties of sin and cos for sinr and cosr.

I'll also note that we can define $\operatorname{tanr}\theta \overset{\text{def}}{=} \frac{\operatorname{sinr}\theta}{\operatorname{cosr}\theta}$, but since this ends up being equal to $i\tan\theta$, I won't really bother rewriting properties for it. tan already had period π!

4.2 Rederiving Algebraic Properties

The following are direct from the definition and corresponding identities for sin and cos:

- $\operatorname{sinr}(-\theta) = \overline{\operatorname{sinr}(\theta)}$ and $\operatorname{cosr}(-\theta) = \overline{\operatorname{cosr}(\theta)}$
- $\operatorname{sinr}(\theta + \pi/2) = -\operatorname{cosr}(\theta)$, and (equivalently) $\operatorname{cosr}(\theta + \pi/2) = -\operatorname{sinr}(\theta)$.
- $\operatorname{sinr}(\alpha + \beta) = \operatorname{sinr}(\alpha)\operatorname{cosr}(\beta) + \operatorname{cosr}(\alpha)\operatorname{sinr}(\beta)$ and $\operatorname{cosr}(\alpha + \beta) = \operatorname{cosr}(\alpha)\operatorname{cosr}(\beta) + \operatorname{sinr}(\alpha)\operatorname{sinr}(\beta)$.
- $|\operatorname{sinr}(\theta)|^2 + |\operatorname{cosr}(\theta)|^2 = 1$.

Although this list isn't exhaustive, it's just generally true that since sinr and cosr are defined as slight changes to sin and cos, rederiving any new properties you need should be easy.

A really nice aspect of this new definition is that it's now easier to go from trig relationships to angle relationships! In particular, with all angle equalities in modulo π,

- If $\operatorname{sinr}\alpha = \operatorname{sinr}\beta$, then $\alpha = \beta$.
- If $\alpha + \beta = \gamma + \delta$ and $\operatorname{sinr}\alpha \ \operatorname{sinr}\beta = \operatorname{sinr}\gamma \ \operatorname{sinr}\delta$, then α, β are γ, δ in some order.

  Remark: Only having $\operatorname{sinr}\alpha \ \operatorname{sinr}\beta = \operatorname{sinr}\gamma \ \operatorname{sinr}\delta$ gives $\alpha + \beta = \gamma + \delta$ for free by considering the argument of both sides. It's probably best to think about this fact as a sanity check for identities involving sinr.
- For $\theta \neq 0$, if $\frac{\operatorname{sinr}(\alpha+\theta)}{\operatorname{sinr}(\alpha)} = \frac{\operatorname{sinr}(\beta+\theta)}{\operatorname{sinr}(\beta)}$, then $\alpha = \beta$.
- Using $\operatorname{sinr}(\theta + \pi/2) = -\operatorname{cosr}(\theta)$, one can obtain similar statements for cosr

For all of these, the only idea needed in the proof is that $\theta \pmod{\pi}$ is uniquely determined by $e^{2i\theta}$, and then one can treat the functions in θ as functions of $e^{2i\theta}$.

4.3 Basic Geometric Properties

The main reason one should expect these redefinitions of trig to satisfy all of our geometric needs is:

Proposition 15. *(SOHCAH) Let ABC be a triangle with $\angle B = \pi/2$. Then,*

$$\operatorname{sinr}\angle CAB = \frac{CB}{CA} = \frac{BC}{AC} \quad \text{and} \quad \operatorname{cosr}\angle CAB = \frac{BA}{CA} = \frac{AB}{AC}.$$

Just as directed $\angle CAB$ denotes the angle of rotation from AC to AB, $\operatorname{cosr}\angle CAB$ intuitively denotes the rotation from AC to AB, along with the appropriate scaling. By this proposition, from a geometric standpoint, you can think of sinr as "opposite over hypotenuse, except we also include the rotation."

We present another characterization of the sinr and cosr between lines. This doesn't directly geometrically correspond with anything I can think of, but makes some theorems far easier to prove. Let w and z be parallel to ℓ_w and ℓ_z. Then,

$$\operatorname{sinr}(\ell_w, \ell_z) = \frac{1}{2}\left(z/\overline{z} \div w/\overline{w} - 1\right) \text{ and } \operatorname{cosr}(\ell_w, \ell_z) = \frac{1}{2}\left(z/\overline{z} \div w/\overline{w} + 1\right).$$

These are pretty helpful, and are also clearly independent of the choice of z, w along the lines.

4.4 Vague Notes on Rederiving Geometric Properties

The previous couple sections give pretty much all the basic properties of sinr and cosr that we'd want. Now, repeating the usual proofs of theorems pretty much just works. Except, one annoying thing happens. Here's an example:

Theorem 16. *(Area by Trig) In triangle ABC, using signed areas,*

$$[ABC] = \frac{1}{2i}\overline{AB} \cdot AC \sin(CAB) = \frac{1}{2i} AB \cdot AC \sin(CAB) \cdot e^{-2\arg AB}.$$

Basically, for a lot of theorems, there are going to be either annoying conjugations or random argument terms. This work also ends up necessarily leaking into solutions, and I don't know of a way around this. In fact a lot of this fix boils down to "if we treat lengths as complex numbers and track arguments throughout the solution, config issues disappear. With this in mind, we'll toss in this fact now:

Theorem 17. *We have* $-\operatorname{sinr}(-\theta)e^{2i\theta} = \operatorname{sinr}(\theta)$*, and similar for* cosr.

5 Doing Geometry (going through the lengthbash handout)

We want to go through and convert theorems involving sin or cos to ones involving sinr and cosr. To give an idea of how to do this in general, we'll go through the process of doing it for a decent sized collection of theorems, for which we'll use the theorems in my lengthbashing handout.

5.1 The Law of Sines and Its Extensions

In the first few sections, all that's really assumed is the law of sines. The standard derivation of the law of sines just works here.

Theorem 18. *(Law of Sines) In a triangle ABC, we have* $AB\operatorname{sinr}(ABC) = AC\operatorname{sinr}(ACB)$.

Proof. Letting D be the foot of the A-altitude, we have $AD = AB\operatorname{sinr}(ABC)$ and $AD = AC\operatorname{sinr}(ACB)$. □

We also want an extended law of sines. However, we can't have something like

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = R,$$

because $\sin A$ could be either of $\sin(BAC)$ or $\sin(CAB)$, and how do you even define R? This is where we're first have to start throwing in annoying argument terms.

Theorem 19. *(Extended Law of Sines) Let P, Q, and R be points on a circle with center O. We have*

$$RQ = 2OP\operatorname{sinr}(\angle QPR)e^{2i\angle PRQ}.$$

In the above, $\angle QPR$ and $\angle PRQ$ can be replaced with any $\angle QXR$ and $\angle QYR$, where X, Y are also on the circle. They just need to be half the measure of arcs QR and PQ. The proof is also not very different from the normal proof.

Proof. Let M be the midpoint of QR. We have

$$RQ = 2MQ = 2OQ\operatorname{sinr}(QOM) = 2OQ\operatorname{sinr}(QPR).$$

We also have $OQ = e^{2i\angle PRQ}OP$ by the earlier claim about isosceles triangles, giving the desired result. □

The main theorems that follow from here are the ratio lemma, quadrilateral ratio formula, and trig Ceva. All of these follow from repeated applications of the law of sines, so derivation should be reasonable. The main difference is that we now need to be careful about writing the angles in the statement.

Theorem 20. *(Ratio Lemma) Let ABC be a triangle, and let D be a point on line BC. We have*

$$\frac{BD}{CD} = \frac{BA}{CA}\cdot\frac{\operatorname{sinr}(BAD)}{\operatorname{sinr}(CAD)}$$

Proof. By the law of sines on triangles ABD and ACD, we have

$$\frac{BA\,\mathrm{sinr}(BAD)}{BD} = \mathrm{sinr}(BDA) = \mathrm{sinr}(BDC) = \frac{CA\,\mathrm{sinr}(CAD)}{CD},$$

giving the desired result. □

One way of sanity checking these statements is taking the argument of both sides. Since $\mathrm{sinr}\,\theta$ has argument θ, the right hand side of the statement of ratio lemma has argument

$$\arg BA - \arg CA + \angle BAD - \angle CAD = 0,$$

and the left hand side also has argument zero. Hence, the arguments of both sides match up, so this theorem looks reasonable. You can try sanity checking the rest yourself:

Theorem 21 (Quadrilateral Ratio Formula and Trig Ceva)**.** *Let ABC be a triangle, and let D be some point not on any of its sides. We have*

$$\frac{\mathrm{sinr}(BAD)}{\mathrm{sinr}(CAD)} = \frac{\mathrm{sinr}(ABD)}{\mathrm{sinr}(ACD)} \cdot \overline{\left(\frac{BD}{CD}\right)} = \frac{\mathrm{sinr}(ABD)}{\mathrm{sinr}(ACD)} \cdot \frac{\mathrm{sinr}(BCD)}{\mathrm{sinr}(CBD)}$$

Proof. By the law of sines in ABD and ACD, we have

$$\frac{BD\,\mathrm{sinr}(DBA)}{\mathrm{sinr}(DAB)} = AD = \frac{CD\,\mathrm{sinr}(DCA)}{\mathrm{sinr}(DAC)}.$$

Using $\mathrm{sinr}(\theta) = \overline{\mathrm{sinr}(-\theta)}$, taking the conjugate gives the first equality. By the law of sines on BCD,

$$\frac{BD}{CD} = \frac{\mathrm{sinr}(DCB)}{\mathrm{sinr}(DBC)} \implies \overline{\left(\frac{BD}{CD}\right)} = \frac{\mathrm{sinr}(BCD)}{\mathrm{sinr}(CBD)},$$

giving the second equality. □

It's worth noting here that the $\overline{BD}$ term is just one of many ways of making the arguments match; we can just write $BDe^{-2i\arg BD}$ if we want to.

5.2 Ratios on a Circle

Recall that the central function we study in the Ratios on a Circle handout is, for points X and Y, the function $f(Z) = \pm\frac{|XZ|}{|YZ|}$, where we choose sign to be negative for Z on segment XY and below line XY, and positive otherwise. We generally think about values of f on a circle through X and Y, or perhaps just line XY. It turns out that this works really well with using complex numbers for lengths. Let $g(Z) = \frac{XZ}{YZ}$.

Claim 22. (Ratios have Nice Arguments) We have $\arg(g(Z)) = \angle YZX$.

Proof. This follows from $\arg(XZ/YZ) = \arg(XZ) - \arg(YZ) = \angle YZX$. □

As a corollary, when looking at f and g on a fixed circle, they only differ by some constant unit factor. At this point, we can convert all theorems from using f to using g as follows:

Claim 23. Let Z_1, Z_2 be points such that X, Y, Z_1, Z_2 are either collinear or concyclic. Then, we have

$$f(Z_1)f(Z_2) = g(Z_1)\overline{g(Z_2)} = g(Z_1)g(Z_2)e^{-2\angle XZ_1Y}.$$

Proof. We have that if M is the midpoint of arc XY such that $f(M) = 1$, then we have $g(Z) = f(Z)g(M)$. The result follows from $g(M)^2 = e^{-2\angle XZ_1Y} = e^{-2\angle XMY}$. □

It also may be worth noting that using spiral similarity and Ptolemy's is a lot easier with g than it is with f.

5.3 Linearity of Power of a Point

Linearity of power of a point actually works pretty well! We've already previously mentioned how to redefine power of a point. Here's what the derivation now looks like:

Theorem 24. *(Linearity of Power of a Point) Let ω_1 and ω_2 be two circles. The function* $\operatorname{Pow}(z,\omega_1) - \operatorname{Pow}(z,\omega_2)$ *is a linear function in z and $\overline{z}$.*

Proof. The power from z to a circle with center O and radius R is

$$|Oz|^2 - |R|^2 = (O-z)\overline{(O-z)} - |R|^2 = |Z|^2 - O\overline{z} - \overline{O}z + |O|^2 - |R|^2 .$$

Subtracting two expressions of this form gives a linear function in z and $\overline{z}$, as desired. □

We also still have that if f is linear in z and $\overline{z}$, then

$$\frac{PT}{QT} = \frac{f(P) - f(T)}{f(Q) - f(T)}$$

for collinear P, Q, T. In general, nothing actually looks too different, since $\operatorname{Pow}(z,\omega)$ is going to have argument 0 for any point z and circle ω.

5.4 Collinearity and Concyclicity Length Conditions

These are a bit annoying. The derivations are the same but the argument terms are extra annoying.

Theorem 25. *(Concyclicity Criterion) Let O, P_1, P_2, and P_3 be points. We have that $OP_1P_2P_3$ is cyclic if and only if*

$$\sum_i OP_i \operatorname{sinr}(\angle P_{i+1}OP_{i+2}) e^{2i\arg(OP_{i+1})},$$

where indices are taken modulo 3.

The $e^{2i\arg(OP_{i+1})}$ is, as usual, only really there because arguments need to be the same.

Proof. We only show the only if direction, as the if direction follows by phantom points. Let the circumcenter of $OP_1P_2P_3$ be X. We have, letting R be OX, by the extended law of sines,

$$P_{i+1}P_{i+2} = 2R\operatorname{sinr}(\angle P_{i+1}OP_{i+2}) e^{2i\angle(OO,OP_{i+1})} = 2R\operatorname{sinr}(\angle P_{i+1}OP_{i+2}) e^{2i(\arg(OP_{i+1}) - \arg(OO))},$$

where OO denotes the tangent to the circumcircle at O. Now, the result follows from Ptolemy's theorem. □

Theorem 26. *(Collinearity Criterion) Let O, P_1, P_2, and P_3 be points. We have that P_1, P_2, and P_3 are collinear if and only if*

$$\sum_i \frac{\operatorname{sinr}(\angle P_{i+1}OP_{i+2}) e^{-2i\arg(OP_{i+2})}}{OP_i},$$

where indices are taken modulo 3.

Proof. We only show the only if direction, as the if direction follows by phantom points. We have

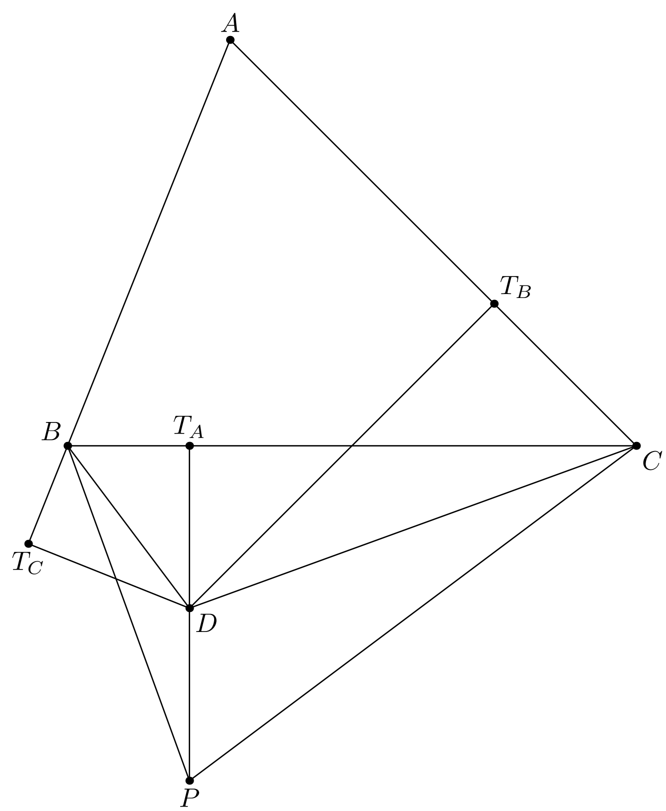
$$[P_1P_2P_3] = \sum_i [OP_{i+1}P_{i+2}] = \sum_i \frac{1}{2} OP_{i+1} \cdot OP_{i+2} \cdot \operatorname{sinr}(\angle P_{i+1}OP_{i+2}) e^{-2i\arg(OP_{i+2})},$$

is 0 if and only if P_1, P_2, and P_3 are collinear, as desired. □

6 Examples

We'll go through 3 examples - TSTST 2020/6, Brazil Olympic Revenge 2017/2, and ELMO 2021/1. These were selected since they sort of cover all the stuff previously discussed, and generally are examples that sort of show the necessity of many aspects of this handout. For each, I'll first present a solution with normal trig (disregarding configuration issues), and then present the fix.

Example 27 (TSTST 2020/6)**.** Let A, B, C, D be four points such that no three are collinear and D is not the orthocenter of ABC. Let P, Q, R be the orthocenters of $\triangle BCD$, $\triangle CAD$, $\triangle ABD$, respectively. Suppose that the lines AP, BQ, CR are pairwise distinct and are concurrent. Show that the four points A, B, C, D lie on a circle.



Non-Directed Proof. Let T_A be the foot of the altitude from D to BC. We have

$$\frac{BT_A}{CT_A} = \frac{BD}{CD}\frac{\cos(CBD)}{\cos(BCD)}.$$

We also have, by trig Ceva on P,

$$\frac{\sin(BAP)}{\sin(CAP)} = \frac{\sin(ABP)}{\sin(CBP)} \cdot \frac{\sin(BCP)}{\sin(ACP)} = \frac{\cos(AB, CD)}{\cos(AC, BD)} \cdot \frac{\cos(CBD)}{\cos(BCD)}$$

By Menelaus and Trig Ceva (respectively), we have that T_A, T_B, T_C are collinear if and only if AP, BQ, and CR concur, as both are equivalent to the cyclic product of $\frac{\cos(CBD)}{\cos(BCD)}$ being 1. Hence, D has a Simson line with respect to triangle ABC, so D lies on the circumcircle, as desired. $\square$

The fix for this one's pretty easy. A big motivation for including this one is that it's a problem where it isn't easy to WLOG away configuration issues.

Fixed Proof. Let T_A be the foot of the altitude from D to BC. We have

$$\frac{BT_A}{CT_A} = \frac{BD}{CD} \cdot \frac{\operatorname{cosr}(DBC)}{\operatorname{cosr}(DCB)} = \frac{BD}{CD}\overline{\left(\frac{\operatorname{cosr}(CBD)}{\operatorname{cosr}(BCD)}\right)}.$$

We also have, by trig Ceva on P,

$$\frac{\operatorname{sinr}(BAP)}{\operatorname{sinr}(CAP)} = \frac{\operatorname{sinr}(ABP)}{\operatorname{sinr}(CBP)} \cdot \frac{\operatorname{sinr}(BCP)}{\operatorname{sinr}(ACP)} = \frac{\operatorname{cosr}(AB, CD)}{\operatorname{cosr}(AC, BD)} \cdot \frac{\operatorname{cosr}(CBD)}{\operatorname{cosr}(BCD)}.$$

By Menelaus and trig Ceva (respectively), we have that T_A, T_B, T_C are collinear if and only if AP, BQ, and CR concur, as both are equivalent to the cyclic product of $\frac{\operatorname{cosr}(CBD)}{\operatorname{cosr}(BCD)}$ being 1. Hence, D has a Simson line with respect to triangle ABC, so D lies on the circumcircle, as desired. □

We'll now present an alternate finish without adding the points T_A. Although this argument can be executed without directing angles, it's a bit easier to see how exactly to do it with directed angles.

Let α, β, γ be $e^{2i\angle BAD}, e^{2i\angle CBD}, e^{2i\angle ACD}$, and α', β', γ' be $e^{2i\angle CAD}, e^{2i\angle ABD}, e^{2i\angle BCD}$. We have $\alpha\beta\gamma = \alpha'\beta'\gamma'$. Trig Ceva on the concurrence of AD, BD, and CD gives

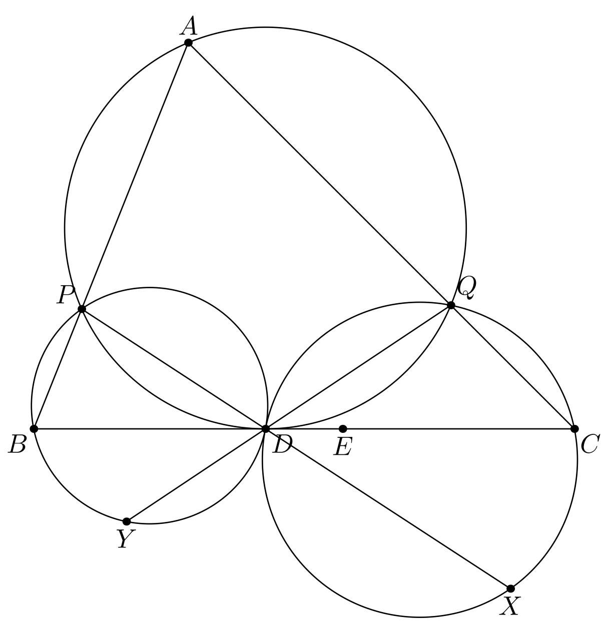
$$(\alpha - 1)(\beta - 1)(\gamma - 1) = (\alpha' - 1)(\beta' - 1)(\gamma' - 1),$$

and the cosine product being 1 gives

$$(\alpha + 1)(\beta + 1)(\gamma + 1) = (\alpha' + 1)(\beta' + 1)(\gamma' + 1).$$

This gives that $(x-\alpha)(x-\beta)(x-\gamma)$ and $(x-\alpha')(x-\beta')(x-\gamma')$ are the same cubic, so $\{\alpha, \beta, \gamma\} = \{\alpha', \beta', \gamma'\}$. From here, some casework gives that D must lie on the circumcircle of ABC, as desired.

Example 28 (ELMO 2021/1)**.** In $\triangle ABC$, points P and Q lie on sides AB and AC, respectively, such that the circumcircle of $\triangle APQ$ is tangent to BC at D. Let E lie on side BC such that $BD = EC$. Line DP intersects the circumcircle of $\triangle CDQ$ again at X, and line DQ intersects the circumcircle of $\triangle BDP$ again at Y. Prove that D, E, X, and Y are concyclic.



Non-Directed Proof. Observe that (BDP) and (CDQ) are tangent. Since homothety at D maps these circles to each other, we have $DPBY \sim DXCQ$. By the concyclicity criterion at D, it suffices to prove

$$DE\sin(XDY) + DY\sin(XDE) - DX\sin(YDE) = 0.$$

For the angles, we have $\angle XDY = 180 - \angle A$, $\angle XDE = \angle BDP = \angle BAD$, and similar for the last remaining one. For the lengths,

$$DX = \frac{DC}{DB}DP = \frac{DC}{DB}\cdot\frac{BD}{BA}\cdot AD = \frac{DC}{BA}\cdot AD,$$

where we use $BDP \sim BAD$ for the second to last step. Hence, we want to show

$$(DC - DB)\sin(A) + \frac{DB}{CA}\cdot AD\sin(BAD) - \frac{DC}{BA}\cdot AD\sin(CAD) = 0.$$

To get rid of the sines, note that by ratio lemma,

$$\frac{\sin(BAC)}{\sin(DAC)} = \frac{DA}{BA}\cdot\frac{BC}{DC}.$$

Hence, dividing through by $\sin(A)$, we want to check

$$(DC - DB) + \frac{DB^2}{BC} - \frac{DC^2}{BC} = 0$$

which is clear, as desired. □

The main point of this fix is that we're going to have to track the arguments of everything. Similarly, the length condition is going to have random argument terms. Besides that, it's all the same.

Fixed Proof. As in the previous proof, we have $DPBY \sim DXCQ$. The concyclicity criterion at D is now

$$DE\,\mathrm{sinr}(YDX)e^{2i\arg DY} + DY\,\mathrm{sinr}(XDE)e^{2i\arg DX} + DX\,\mathrm{sinr}(EDY)e^{2i\arg DE} = 0.$$

For the angles, we have $\angle YDX = \angle CAB$, $\angle XDE = \angle BAD$, and $\angle EDY = \angle DAC$. For the lengths,

$$DX = \frac{DC}{DB}\cdot DP = \frac{DC}{DB}\cdot DB\cdot\overline{\left(\frac{AD}{AB}\right)} = DC\cdot\frac{AD}{AB}\cdot e^{2i\angle DAB},$$

and similar for DY. We can deal with the argument terms by dividing through by $e^{2i\arg DY}$, so the argument terms become angles instead. We want to show:

$$\begin{aligned}(DC + DB)\sin(CAB) + DB\cdot\frac{AD}{AC}e^{2i\angle DAC}\,\mathrm{sinr}(BAD)e^{2i\angle CAB}\\ +DC\cdot\frac{AD}{AB}e^{2i\angle DAB}\,\mathrm{sinr}(DAC)e^{2i\angle CAD} = 0.\end{aligned}$$

To get rid of the sines, note that by ratio lemma,

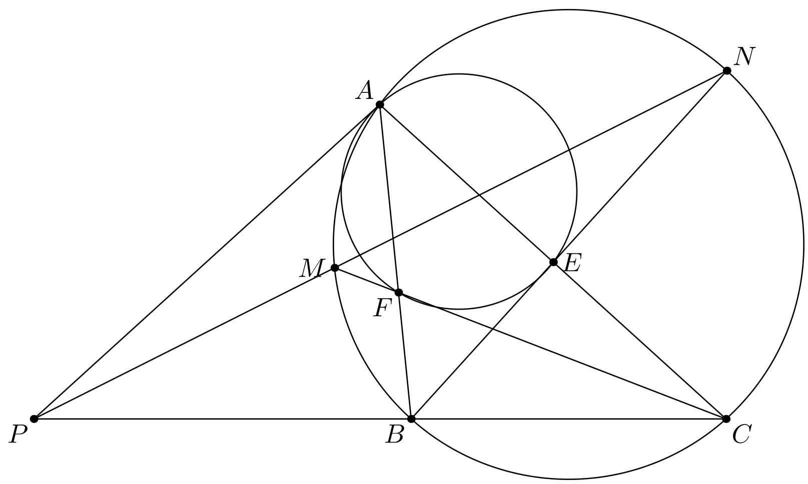
$$\frac{\mathrm{sinr}(BAD)e^{2i\angle DAB}}{\mathrm{sinr}(CAB)} = -\frac{\mathrm{sinr}(DAB)}{\mathrm{sinr}(CAB)} = -\frac{AC}{AD}\cdot\frac{BD}{BC},$$

and similar for $\mathrm{sinr}(DAC)$. Hence, dividing through by $\mathrm{sinr}(CAB)$, we want to check

$$(DB + DC) - DB\cdot\frac{AD}{AC}\cdot\frac{AC}{AD}\cdot\frac{BD}{BC} - DC\cdot\frac{AD}{AB}\cdot\frac{AB}{AD}\cdot\frac{CD}{CB} = 0,$$

which is easy. □

Example 29 (Brazil Olympic Revenge 2017/2 (sorta))**.** Let ABC be an acute triangle and I its incenter. Let E and F be the intersections of BI and CI with sides AC and AB. Let M and N be the midpoints of minor arcs AB and AC and let P be a point on BC such that AP is tangent to AEF. Prove that M, N, and P are collinear.



Non-Directed Proof. Defining $f(X)$ to be $XA^2 - \operatorname{Pow}(X, (AEF)$, we have

$$\frac{BP}{CP} = \frac{f(B)}{f(C)} = \frac{AB \cdot AF}{AC \cdot AE} = \frac{c}{b} \cdot \frac{a+c}{a+b}.$$

It suffices to check that this is equal to $\frac{BM}{CM} \cdot \frac{BN}{CN}$. By Ptolemy, we have

$$c \cdot CM = BM \cdot b + AM \cdot a = (a+b)BM \implies \frac{BM}{CM} = \frac{c}{a+b},$$

and similar for $\frac{BM}{CN}$, giving $\frac{BM}{CM} \cdot \frac{BN}{CN} = \frac{BP}{CP}$, as desired. □

This fix is going to bring lots of things together. We have to be a bit careful with everything because of the angle bisectors (for example, all the $a + b$ terms are gonna look like $BC + BA \cdot e^{2i\angle ABI}$ or something). This solution also includes applications of the ratio lemma and linearity of power of a point, which we fix as described earlier.

Fixed Proof. We'll start by having to rederive the angle bisector theorem; by the ratio lemma,

$$\frac{AE}{CE} = \frac{BA}{BC} \cdot \frac{\operatorname{sinr}(ABI)}{\operatorname{sinr}(CBI)} = -\frac{AB \cdot e^{2i\angle ABI}}{CB} \implies AE = AC \cdot \frac{AB \cdot e^{2i\angle ABI}}{AB \cdot e^{2i\angle ABI} + CB},$$

and similar for F. Now, defining $f(X) = |XA|^2 - \operatorname{Pow}(X, (AEF))$, we have

$$\frac{BP}{CP} = \frac{f(B) - f(P)}{f(C) - f(P)} = \frac{AB \cdot AF \cdot e^{-2i \arg AB}}{AC \cdot AE \cdot e^{-2i \arg AC}} = \frac{AB}{AC} \cdot \frac{AB \cdot e^{2i\angle ABI} + CB}{AC \cdot e^{2i\angle ABI} + BC} e^{2(\angle ACI - \angle ABI + \angle BAC)}.$$

It suffices to check that this is equal to $\frac{BM}{CM} \cdot \frac{BN}{CN} \cdot e^{2\angle BAC}$. By Ptolemy's Theorem,

$$0 = AM \cdot BC + BM \cdot CA + CM \cdot AB = CM \cdot AB + BM \cdot \left(CA + BC \cdot e^{2i(\angle MBA + 90^\circ)}\right).$$

Since $e^{2i(\angle MBA + 90^\circ)} = -e^{-2i\angle ACI}$, this gives

$$\frac{BM}{CM} = -\frac{AB}{AC \cdot e^{2i\angle ACI} + BC} \cdot e^{2i\angle ACI}.$$

Hence, we have $\frac{BP}{CP} = \frac{BM}{CM} \cdot \frac{BN}{CN} \cdot e^{2\angle BAC}$, as desired. □