Ratio Lemma

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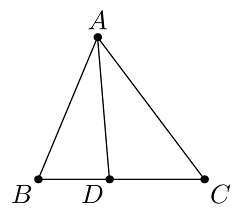
1 Introduction

Most people encounter the ratio lemma in the following form:

Theorem 1.1 (Boring Ratio Lemma)
Let $\triangle ABC$ be a triangle, and let D be a point on BC. Then, we have

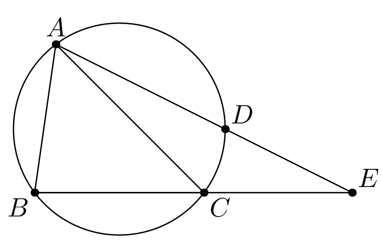
$$\frac{BD}{CD} = \frac{BA}{CA} \cdot \frac{\sin \angle BAD}{\sin \angle CAD}.$$



This is really useful for computing lengths in a triangle. Though this is useful, we'll introduce an extremely powerful equivalent theorem. In this handout, we'll be focusing on the following version of the Ratio Lemma:

Theorem 1.2 (Cool Ratio Lemma)
Let ω be a circle through B and C, and let a line meet ω at points A, D and BC at E. Then,

$$\frac{BE}{CE} = \frac{BA}{CA} \cdot \frac{BD}{CD}.$$



Proof.

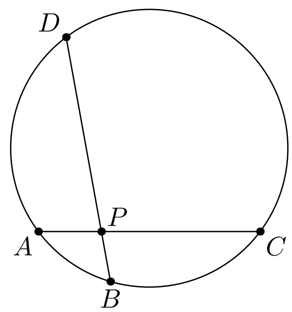
$$\frac{BE}{CE} = \frac{d(B, AD)}{d(C, AD)} = \frac{[ABD]}{[ACD]} = \frac{\frac{1}{2}\sin(\angle ABD)BA \cdot BD}{\frac{1}{2}\sin(\angle ACD)CA \cdot CD} = \frac{BA}{CA} \cdot \frac{BD}{CD}$$

□

Here's a simple direct application:

Example 1.3 (2017 HMMT G1)

Let A, B, C, D be four points on a circle in that order. Also, $AB = 3$, $BC = 5$. $CD = 6$, and $DA = 4$. Let the diagonals AC and BD intersect at P. Compute $\frac{AP}{CP}$.



Proof. By the Ratio Lemma,

$$\frac{AP}{CP} = \frac{AD}{CD} \cdot \frac{AB}{CB} = \frac{4}{6} \cdot \frac{3}{5} = \boxed{\frac{2}{5}}$$

□

As far as I know, nothing after this point is citable.

2 Powerful!

This fact is surprisingly powerful! Here's the typical approach to using ratio lemma on a problem:

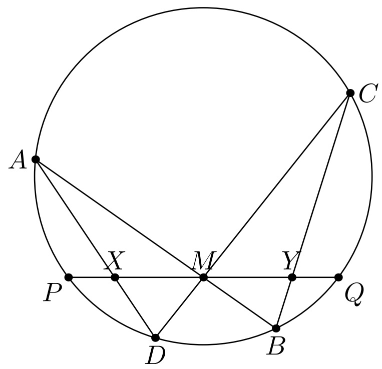
- Pick points X and Y such that most other points in the problem either lie on a circle through X, Y or on line XY. In a problem centered around $\triangle ABC$ and its circumcircle, this is often (but not always) B and C.
- Consider the function $\frac{XP}{YP}$ for points P either on a circle through X, Y or on a line through X, Y.
- Try to write the values of this function for many point in terms of a few (usually corresponding to the amount of points that can 'move' in the problem).

In essence, this is nice because it allows us to track a function of any point defined by either the intersection of a chord of a circle through X and Y with line XY, as well as intersections of similar lines with circles.

Here's an example of this strategy:

Example 2.1 (Butterfly Theorem)

Let M be the midpoint of a chord PQ of a circle, through which two other chords AB and CD are drawn; AD and BC intersect chord PQ at X and Y correspondingly. Then M is the midpoint of XY.



Proof. Suppose that A and C are on the same side of PQ, the other case can be dealt with similarly. Let $g(Z) = \frac{PZ}{QZ}$ for all points Z. Since we can construct all points from A, C, M, we'll try writing $g(B), g(D), g(X), g(Y)$ in terms of $g(A)$, $g(C)$, and $g(M) = 1$.

By the ratio lemma, we have:

$$g(A)g(B) = g(M) = 1, \text{ and similarly, } g(C)g(D) = 1$$

$$\implies g(X) = g(A)g(D) = \frac{g(A)}{g(C)} \text{ and } g(Y) = g(B)g(C) = \frac{g(C)}{g(A)}.$$

Now, since X, Y are both on segment PQ, and $g(X)g(Y) = 1$, it can be seen that X and Y must be reflections of each other over the midpoint of PQ, as desired. □

To avoid working with which side of XY something is on, we'll generally consider defining $f(P) = \pm\frac{XP}{YP}$ to be negative when P is on segment XY or below line XY, and positive otherwise. This is helpful because it satisfies a couple convenient claims:

Lemma 2.2 (Properties of f)
Let X, Y be points in the plane, and define $f(P) = \pm\frac{XP}{YP}$ to be negative when P is on segment XY or below line XY, and positive otherwise. Then,

- For A, B on either a common circle or common line through X and Y, $f(A) = f(B)$ if and only if $A = B$.
- If $ABXY$ is cyclic and AB meets XY at C, then $f(A)f(B) = f(C)$.

Proof.

- f is either increasing or decreasing along any arc, and opposite arcs have opposite signs. The line case is similar.
- Casework gives that that both sides have the same sign, and they have the same magnitude by ratio lemma.

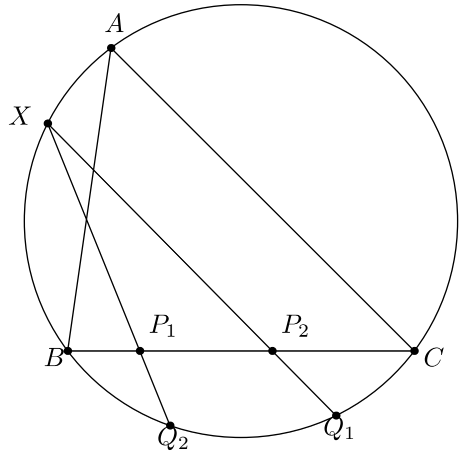
□

These are nice because we can uniquely identify points based on their f and the circle/line through X, Y they're on.

Now, we'll apply this version of the function to a few problems.

Example 2.3

Let $\triangle ABC$ be a triangle, and let P_1 and Q_1 be points on BC and the circumcirle of $\triangle ABC$, respectively, such that $\angle BAP_1 = \angle CAQ_1$. Points P_2 and Q_2 have a similar property. Prove that P_1Q_2 and P_2Q_1 meet on the circumcircle of $\triangle ABC$.



Proof. For all points Z, define $f(P) = \pm\frac{BP}{CP}$ to be negative when P is on segment BC or below line BC, and to be positive otherwise.

Suppose that we construct the point Q_1' on the circumcircle of ABC such that $Q_1Q_1'||BC$. Since $\angle BAP_1 = \angle CAQ_1 = \angle BAQ_1'$, AP_1Q_1' is a line, so

$$f(P_1) = f(A)f(Q_1') = \frac{f(A)}{f(Q_1)}.$$

Similarly, we obtain $f(P_2) = \frac{f(A)}{f(Q_2)}$. Thus, if P_1Q_2 and P_2Q_1 meet the circumcircle at X_1 and X_2, we have

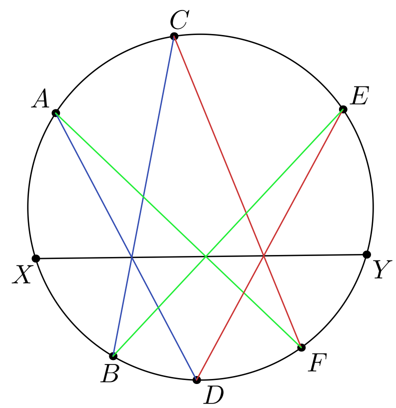
$$f(X_1) = \frac{f(P_1)}{f(Q_2)} = \frac{1}{f(Q_1)f(Q_2)} = \frac{f(P_2)}{f(Q_1)} = f(X_2),$$

and since both X_1 and X_2 are on the circumcircle of ABC, this implies that they are the same point, as desired. $\square$

As a final demonstration of the power of the Ratio Lemma, we'll prove Pascal's theorem for six points of a circle (in the case that the Pascal line intersects the circle):

Example 2.4 (Pascal on a circle, sorta)

Let A, B, C, D, E, and F be six points on a circle, and let XY be a chord of the circle. Suppose that AD and BC meet on XY, and CF and DE meet on XY. Prove that AF and BE meet on XY.



Proof. Define $f(Z) = \pm\frac{XZ}{YZ}$ to be negative for Z on segment XY and below line XY, and positive otherwise.

Note that AD and BC meeting on XY is equivalent to

$$f(A)f(D) = f(B)f(C) \iff \frac{f(A)}{f(C)} = \frac{f(B)}{f(D)}.$$

Similarly, CF and DE meeting on XY is equivalent to

$$\frac{f(C)}{f(E)} = \frac{f(D)}{f(F)},$$

and what we want to show is equivalent to

$$\frac{f(A)}{f(E)} = \frac{f(B)}{f(D)}.$$

This just follows from multiplying the two equations. □

Just to recap, the steps are:

- Figure out points X and Y to define the function f on.
- Write all the terms you can in terms of a few points.
- Try using these to solve the problem.

Now try these on some problems!

Problem 2.5. Distinct points B, C, X_1, X_2, Y_1, Y_2 lie on circle ω. Show that $\frac{BX_1}{CX_1} = \frac{BX_2}{CX_2}$ and $\frac{BY_1}{CY_1} = \frac{BY_2}{CY_2}$ if and only if $X_1Y_1 \cap X_2Y_2$ and $X_1Y_2 \cap X_2Y_1$ lie on BC.

Problem 2.6 (AoPS Post)**.** Points D and E are selected on the circumcircle of $\triangle ABC$. AD and AE meet BC at X and Y. Let D' and E' be the reflections of D and E over the perpendicular bisector of BC. Prove that $D'Y$ and $E'X$ intersect on circumcircle of $\triangle ABC$.

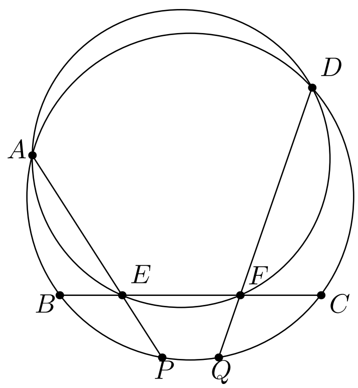
Problem 2.7. Let ABC be a triangle with circumcircle ω, let E, F be points on segments AC and AB, and let M be the foot of the altitude from the circumcenter of ABC to EF. Let AM meet ω at $D \neq A$, and suppose that EF, BC, and the tangent to ω at D concur. Show that M is the midpoint of EF.

Problem 2.8 (Based on POGCHAMP #1)**.** Let ABC be a triangle with orthocenter H. Let AH, BH, CH, meet the opposite sides at D, E, F, and let D' me the reflection of D over the midpoint of BC. Show that if a point P on EF satisfies $AP||BC$, then HP and AD' intersect on the circumcircle of AEF.

3 Dealing with Circles

One of the major limitations of the ratio lemma is that we can only apply it to lines. In this section, we find similar facts for circles intersecting circles and/or lines through X and Y.

Theorem 3.1 (Condition for Circle and Line)
Let $ABCD$ be a cyclic quadrilateral, and let E and F be points on line BC. For all points Z, define $f(Z) = \pm\frac{BZ}{CZ}$ to be negative when Z is on segment BC or below line BC, and positive otherwise. Then, $ADEF$ is cyclic if and only if $f(A)f(D) = f(E)f(F)$.



Proof. Let AE and DF meet the circumcircle of $ABCD$ again at P and Q. Note that by Reim's theorem, or by

$$\angle DQP = 180^\circ - \angle DAP \implies (\angle DFE + \angle DAE = 180^\circ \iff \angle DFE = \angle DQP),$$

$ADEF$ is cyclic if and only if $PQ||BC$. This is equivalent to

$$1 = f(P)f(Q) = \frac{f(E)}{f(A)} \cdot \frac{f(F)}{f(D)} \iff f(A)f(D) = f(E)f(F),$$

as desired. □

Corollary 3.2
Let points P, Q, R be on line BC. Then,

$$PQ \cdot PR = PB \cdot PC \iff \frac{BP}{CP} = \frac{BQ}{CQ} \cdot \frac{BR}{CR},$$

where lengths are directed.

Proof. Define f as usual. If a circle through Q, R and a circle through B, C meet at X and Y, by Theorem 3.1,

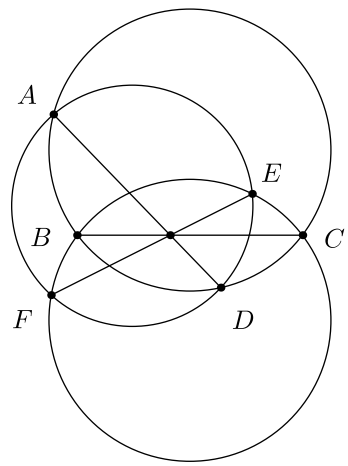
$$f(P) = f(Q)f(R) = f(X)f(Y).$$

Since $BCXY$ is cyclic, by the ratio lemma, XY passes through P. Thus, P lies on the radical axis of the circles, giving $PQ \cdot PR = PB \cdot PC$. The reverse direction is similar. □

The case for circles is a bit more straightforward.

Theorem 3.3 (Condition for Two Circles)

Let $ABCD$ and $BCEF$ also cyclic quadrilaterals with different circumcircles.. For all points Z, define $f(Z) = \pm\frac{BZ}{CZ}$ to be negative when Z is on segment BC or below line BC, and positive otherwise. Then, $ADEF$ is cyclic if and only if $f(A)f(D) = f(E)f(F)$.

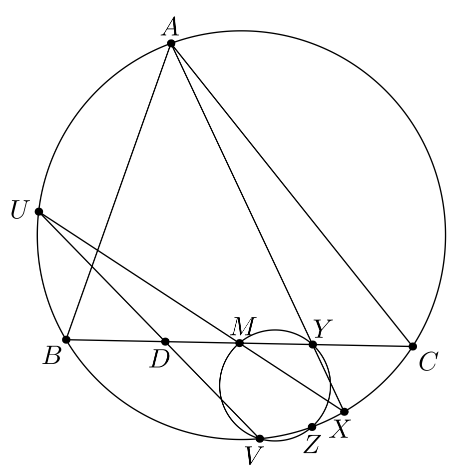


Proof. By the radical axis theorem, $ADEF$ is cyclic if and only AD and EF they meet on BC, which is true if and only if $f(A)f(D) = f(E)f(F)$ by the ratio lemma. $\square$

Now, we have the power to deal with circles as well as lines! Let's use this power!

Example 3.4 (AQGO 2020 #1)

Let ABC be a triangle with altitude AD. Let V be a variable point on (ABC) and $\overline{VD}$ intersect (ABC) at U. Let $\overline{UM}$ intersect (ABC) at X where M is the midpoint of BC. Let $\overline{AX} \cap \overline{BC} = Y$. Let $\odot(YVM)$ intersect (ABC) at a point Z. Let Z' be the reflection of Z over $\overline{OH}$. Finally let $(Z'OH)$ intersect (ABC) at $T \neq Z'$. Then prove that TZ passes through M.



Proof. We show that Z is the A-antipode, after which the rest will be an exercise in non-ratio-lemma geometry. As usual, define $f(P) = \pm\frac{BP}{CP}$ to be negative if P is on segment BC or below line BC, and positive otherwise. We aim to write f in terms of $f(V)$ and properties of ABC.

We'll write out the f of each point in order of construction using ratio lemma and theorems 3.1, 3.3:

$$f(U) = \frac{f(D)}{f(V)}$$
$$f(X) = \frac{f(M)}{f(U)} = -\frac{1}{f(U)} = -\frac{f(V)}{f(D)}$$
$$f(Y) = f(A)f(X) = -\frac{f(A)f(V)}{f(D)}$$
$$f(V)f(Z) = f(M)f(Y) \implies f(Z) = \frac{f(M)f(Y)}{f(V)} = -\frac{f(Y)}{f(V)} = -\frac{-\frac{f(A)f(V)}{f(D)}}{f(V)} = \frac{f(A)}{f(D)}.$$

Now, we can recognize that if D' is the intersection of AD and the circumcircle,

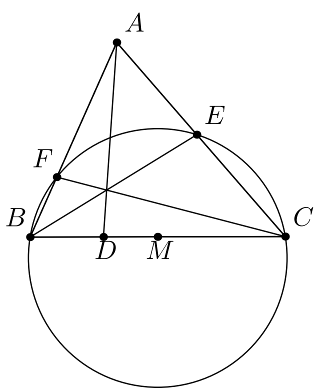
$$f(D') = \frac{f(D)}{f(A)} \implies f(D')f(Z) = 1 \implies D'Z||BC.$$

Now, $\angle AD'Z = \angle(AD, BC) = 90°$, implying the result. ☐

Outside of tracking constructed points, these theorems can also just be used as a way to check cyclic quadrilaterals:

Example 3.5

In triangle ABC, pick points D, E, and F on segments BC, CA, and AB, respectively, such that AD, BE, CF concur and $BCEF$ is cyclic. Prove that the circumcircle of DEF passes through the midpoint of BC.



Proof. Define $f(Z) = \pm\frac{BZ}{CZ}$ to be negative on segment BC or below line BC, and positive otherwise. It suffices to show

$$f(E)f(F) = f(D)f(M) = -f(D).$$

It's simple to see that both sides have the same sign, and using $AEB \sim AFC$,

$$\frac{BF}{CF} \cdot \frac{BE}{CE} = \frac{BF}{AF} \cdot \frac{AE}{EC} = \frac{BD}{CD}$$

where the last line follows from Ceva's theorem, implying the result. ☐

Problem 3.6. Show that in example 2.3, AP_1P_2X is cyclic.

Problem 3.7 (Sharygin 2017)**.** Let AA_1, CC_1 be the altitudes of triangle ABC, B_0 the common point of the altitude from B and the circumcircle of ABC; and Q the common point of the circumcircles of ABC and $A_1C_1B_0$, distinct from B_0. Prove that BQ is the symmedian of ABC (that is, $\frac{AQ}{CQ} = \frac{AB}{CB}$).

Problem 3.8 (Sharygin 2020)**.** Let $ABCD$ be a cyclic quadrilateral. Consider such pairs of points P, Q of diagonal AC that the rays BP and BQ are symmetric with respect the bisector of angle B. Prove that the center of the circumcircle of PDQ is on a fixed line.

Problem 3.9. Let $BCDEF$ be the vertices of a cyclic pentagon in some order. Suppose DE and DF meet BC at X and Y, and let the circumcircle of DXY meet the circumcircle of the pentagon at Z. Show that E, F, the reflection of Z across BC, and the reflection of D across the midpoint of BC are concyclic.

Problem 3.10 (DIT on a cyclic quadrilateral, sorta)**.** Let $ABCD$ be a cyclic quadrilateral, and let X, Y be points on its circumcircle. Suppose that AB, CD meet XY at P_1, P_2, AC, BD meet XY at R_1, R_2, and AD, BC meet XY at S_1, S_2, show that for some point T on XY,

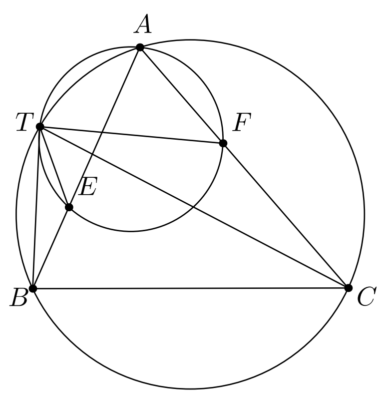
$$TX \cdot TY = TP_1 \cdot TP_2 = TR_1 \cdot TR_2 = TS_1 \cdot TS_2$$

4 Spiral Similarity (more circles!)

We'll present one more useful tool for applying ratio lemma..

Lemma 4.1 (Spiral Similarity)

Let ABC be a triangle, let E and F be points on AB and AC, and let $T \neq A$ be the intersection of the circumcircle of ABC and AEF. Then, $\triangle TEB \sim \triangle TFC$.



Proof. We have

$$\angle TBE = \angle TBA = \angle TCA = \angle TCF,$$

and similarly, $\angle TEB = \angle TFC$. □

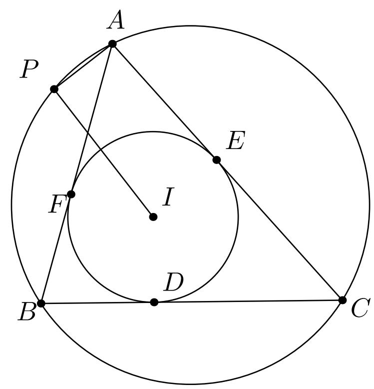
Corollary 4.2

With the same points, if we define f as usual with B and C, and direct lengths on AB and AC such that $f(A) = \frac{BA}{CA} > 0$, then

$$f(T) = \frac{BE}{CF}.$$

This is really useful because it allows us to compute the usual function for points on a circle through A.

Example 4.3 (Sharky Devil Point)
In triangle ABC with incenter I, the incircle touches BC at D. Point P on the circumcircle of ABC satisfies $\angle API = 90°$. Prove that $\angle BPD = \angle CPD$.

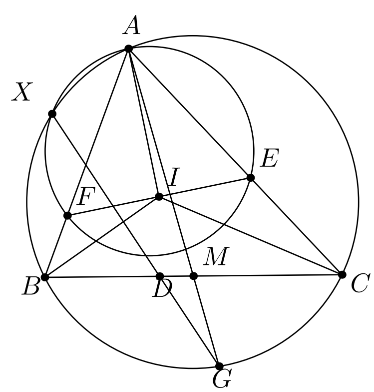


Proof. Note that $APIEF$ is cyclic. Thus, if we define $f(X) = \pm\frac{BX}{CX}$ as usual, we have

$$f(P) = \frac{BF}{CE} = \frac{BD}{DC} = -f(D),$$

so we are done by the angle bisector theorem. □

Example 4.4 (2016 ISL G2)
Let ABC be a triangle with circumcircle Γ and incenter I and let M be the midpoint of $\overline{BC}$. The points D, E, F are selected on sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ such that $\overline{ID} \perp \overline{BC}$, $\overline{IE} \perp \overline{AI}$, and $\overline{IF} \perp \overline{AI}$. Suppose that the circumcircle of $\triangle AEF$ intersects Γ at a point X other than A. Prove that lines XD and AM meet on Γ.



Proof. Define $f(P) = \pm\frac{BP}{CP}$ to have the same signs as usual. By the ratio lemma, this is equivalent to

$$\frac{f(D)}{f(X)} = \frac{f(M)}{f(A)} \iff f(A)f(D) = -f(X) = -\frac{BF}{CE}.$$

At this point, we realize we can bash out both sides with trig, so we just do that:

$$BF = IF \cdot \frac{\sin(\angle FIB)}{\sin(\angle FBI)} \text{ and } CE = IE \cdot \frac{\sin(\angle EIC)}{\sin(\angle ECI)}$$

$$\implies -\frac{BF}{CE} = -\frac{\sin(\angle FIB)}{\sin(\angle FBI)} \cdot \frac{\sin(\angle ECI)}{\sin(\angle EIC)} = -\frac{\sin(C/2)\sin(C/2)}{\sin(B/2)\sin(C/2)}$$

and

$$f(A)f(D) = \frac{\sin(C)}{\sin(B)} \cdot -\frac{\tan(B/2)}{\tan(C/2)} = -\frac{2\sin(C/2)\cos(C/2)\frac{\sin(C/2)}{\cos(C/2)}}{2\sin(B/2)\cos(B/2)\frac{\sin(B/2)}{\cos(B/2)}},$$

which are equal, as desired. □

Problem 4.5 (Adapted from this AoPS post)**.** Let ABC be a triangle, and let N satisfy $NBA \sim NAC$. Show that if the circle with diameter AN meets (ABC) at D and AC, AB at E, F, then AD, EF, and the tangent to (AN) at N concur.

Problem 4.6 (2019 EGMO #3)**.** Let ABC be a triangle such that $\angle CAB > \angle ABC$, and let I be its incentre. Let D be the point on segment BC such that $\angle CAD = \angle ABC$. Let ω be the circle tangent to AC at A and passing through I. Let X be the second point of intersection of ω and the circumcircle of ABC. Prove that the angle bisectors of $\angle DAB$ and $\angle CXB$ intersect at a point on line BC.

Problem 4.7 (This AoPS post after inversion)**.** Let AXY be an isosceles triangle, let M be the midpoint of XY, let A' be the reflection of A over XY, pick points B, C on XY, and let BA', CA' meet AX, AY at D, E. Then, show that (AXY), (ABC), (ADE) are coaxial.

Problem 4.8 (Generalized POGCHAMP #1)**.** Let ABC be a triangle, D, E be points on BC, F be point on (ABC), let (DEF) meet AD again at G and let (ADE) meet (ABC) again at H, and let (AFG) meet AB and AC at I and J. Then, show that AH, IJ, and GF concur.