Lengthbashing

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Thanks to Luke Robitaille for reading this and giving lots of helpful suggestions. Also evan.sty <3.

Contents

1 Philosophy

In some sense, geometry problems are often about trying to "understand" points. It's been a while since I wasn't in the mindset of lengthbashing problems, but if my memory serves me correctly, the go-to ways of thinking about points are:

- What lines and circles does it lie on?
- What angles does it form with other points?
- What are its cross ratios with other points?

People generally know lots of ways of working with the above types of information. The main goal of this handout is to help you deal with the following types of information:

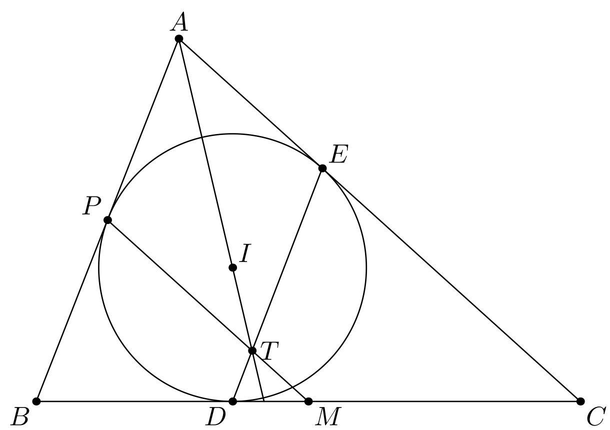
- What is its distance to other points?
- What is the ratio of its distances to other points?

These techniques are often powerful because, if the construction of something "isn't that bad," we can often easily deal with it. You can view some sections of this handout as picking a particular way of thinking about points, then finding theorems that make this way of thinking about points viable.

Ok so here's an example of what I mean.

Example 1.1 (Iran Lemma)

Let ABC be a triangle. Let its incircle with center I touch CB, CA at D, E, and let the midpoints of BC and BA be M and P. Prove that AI, DE, and MP concur.



These lines aren't related in any super obvious ways. However, if you look at it from the perspective of A, none of them are actually that bad:

- For AI - walk at angle $A/2$ to AB and AC.
- For MP - start $c/2$ along AB, and walk at an angle of A away from AB.
- For DE - start $s-a$ along AC, and walk at an angle of $90 - C/2$ away from AC.

Hence, we should probably be able to deal with the distance from A to the concurrence point. This intuition translates to the following proof:

Proof. Let MP meet AI at T, and let DE meet AI at T'. By the Law of Sines,

$$AT = AP \cdot \frac{\sin(\angle APT)}{\sin(\angle ATP)} = \frac{c}{2} \cdot \frac{\sin(A)}{\sin(A/2)}$$
$$AT' = AE \cdot \frac{\sin(\angle AET')}{\sin(\angle AT'E)} = (s-a) \cdot \frac{\cos(C/2)}{\sin(B/2)}$$

It now suffices to check that these are equal. At this point, we can convert everything in terms of half angles. We have

$$s - a = r\cot(A/2) \implies c = (s-a) + (s-b) = r\left(\frac{\cos(A/2)}{\sin(A/2)} + \frac{\cos(B/2)}{\sin(B/2)}\right) = r \cdot \frac{\sin(A/2 + B/2)}{\sin(A/2)\sin(B/2)}.$$

Plugging these back in for AT and AT' gives that they're both $r \cdot \frac{\cos(A/2)\cos(C/2)}{\sin(A/2)\sin(B/2)}$, as desired. $\square$

A key takeaway from this proof is that throughout the proof, we try to "delete" more and more things. With the length computations, we reduce the problem to something only referencing the triangle. Then, we try converting all of our lengths to just r and trig functions of half angles.

The final computation may seem a bit tricky, but it's worth noting that:

- There's actually almost nothing going on. We already know that the two expressions have to be equal, and none of the terms are unusual.
- Surprisingly, you won't have to do things like that often.
- For a better finish, you can use the fact that $AET' \sim AIB$. The intuition here is "if trig ratios are surprisingly nice, you can often try to make them show up naturally." Being able to see this more easily probably just comes from practice.

2 Disclaimer-ish Things

2.1 Directed Angles and Lengths

This handout is going to use a lot of trig. The issue is that, as far as I know, there's no canonical fix to the following fact:

> If we want to use directed angles, we need to consider angles mod 180°. However, $\sin(x) \neq \sin(x + 180^\circ)$.

I have a way of fixing this, see this handout, but its pretty annoying. I'm not going to direct angles and lengths and trig in this handout since:

- I want this handout to be comprehensible without reading the directed trig and lengths handout.
- Honestly, it's pretty likely best to try WLOG-ing away (or just ignoring if it's obvious enough) configuration issues for actual writeups.

But I still think the fact that we can get stuff to work with directed angles is pretty cool, and also works surprisingly well. For example, we end up getting injectivity statements like the following:

- BP/CP is a bijection between line BC and $\mathbb{R} \cup \infty$ (directing lengths just on BC suffices for this).
- For the definition of sin in the handout, if $\sin \alpha = \sin \beta$, then $\alpha \equiv \beta \pmod{180^\circ}$.

2.2 Citing Stuff

Lots of claims on this handout aren't citable. I think the law of sines, Ceva, Menelaus, (normal) ratio lemma, and the statement of linearity of power of apoint are all safely citable. For other things, you're probably best off writing out the statement of the fact you're trying to cite and its proof.

2.3 Maybe Don't Bash Too Hard Or Something

Synthetic is good. Bashing is also good. The issue is that bash solutions have pretty unbounded ability to get time-consuming, and synthetic solutions don't. A few corollaries of this are:

- You should eventually get an intuitive grasp on how hard things are to compute. If something's too hard, it may be a good idea to look for better solutions (including better bashes).
- Trying bash on every problem before getting pretty good at angle chasing is dangerous.

2.4 For Some Definition of Bashing

Most of the "bashing" in this handout is very very clean. You won't even have to add numbers that often. Calling it bashing is kind of questionable but I'm too used to it to stop.

3 Lengths and the (Normal) Ratio Lemma

The first basic tool we use is the Law of Sines:

Theorem 3.1 (Law of Sines)
Let ABC be a triangle. Then, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Without actually going through any computations, we can think of this as:

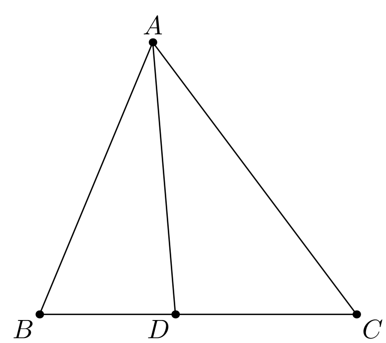
- If we can deal with the angles, we can deal with the ratios of the side lengths.
- If we can deal with the ratio of two of the angles, we can deal with the ratio of the corresponding sides.
- If we can deal with the ratio of two of the sides, we can deal with the ratio of the corresponding angles.

In our proof of the Iran Lemma, before actually computing AT and AT', we know that they have to be easy to deal with. We can deal with all the angles of APT and AET', and we can deal with AP and AE.

We'll now find ways to deal with length and sine ratios turn, which out to be very useful.

Theorem 3.2 (Ratio Lemma)
Let $\triangle ABC$ be a triangle, and let D be a point on BC. Then, we have

$$\frac{BD}{CD} = \frac{BA}{CA} \cdot \frac{\sin \angle BAD}{\sin \angle CAD}.$$



The proof is left as an exercise. It's also written out in the third appendix. Intuitively, what this means is:

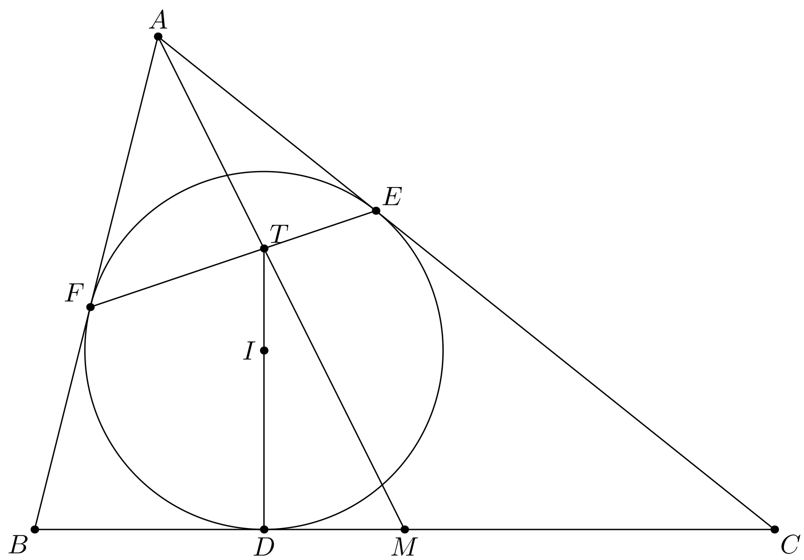
- If we can deal with BA/CA and BD/CD, we can find the "direction" of AD in $\angle BAC$.
- If we can deal with BA/CA and the "direction" of AD in $\angle BAC$, then we can deal with BD/CD.

We're thinking about the trig ratio as the direction since it only depends on angles at A.

Here's an example problem:

Example 3.3

Let ABC be a triangle with incenter I. Let its incircle meet BC, CA, AB at D, E, F, and let M be the midpoint of BC. Then, AM, DI, and EF concur.



Without actually computing anything, here's how we're going to deal with everything:

- Since ABC is well-behaved, and $BM : CM$ is just 1, we can deal with the direction of AM in $\angle BAC$.
- Since we can deal with the direction of AM, and AEF is well-behaved, we can find the ratio that AM splits EF into.
- Since all the angles between D, E, F, I are nice, we can deal with all the length ratios and directions, so we can find the ratio that DI splits EF into.

The above should intuitively be enough to solve the problem, perhaps up to some conversion at the end. Here's the solution once we actually go through the computation:

Proof. Let AM and DI meet EF at T and T'. Our first step is:

$$\frac{\sin(BAM)}{\sin(CAM)} = \frac{BM}{CM} \cdot \frac{AC}{AB} = \frac{AC}{AB}.$$

Our second step is:

$$\frac{ET}{FT} = \frac{\sin(EAM)}{\sin(FAM)} \cdot \frac{EA}{FA} = \frac{AB}{CB}.$$

Our last step is:

$$\frac{ET'}{FT'} = \frac{ED}{FD} \cdot \frac{\sin(EDI)}{\sin(FDI)} = \frac{\cos(C/2)}{\cos(B/2)} \cdot \frac{\sin(C/2)}{\sin(B/2)} = \frac{\sin(C)}{\sin(B)}.$$

Hence, $\frac{ET}{FT} = \frac{ET'}{FT'}$, so they are the same point, as desired. □

For points B, C, the function mapping T to BT/CT for points T on segment BC is injective, making the last line of the proof above valid. If I was properly directing lengths (or even just directing lengths on line BC), it would in fact be a bijection. This means we can actually just think about points T on line BC by thinking about the signed value of BT/CT, which is nice.

Problem 3.4. Prove the Ratio Lemma.

Problem 3.5 (Steiner's Theorem)**.** Let ABC be a triangle. Let P, Q be points on segment BC such that $\angle BAP = \angle CAQ$. Prove that $\frac{BP}{CP} \cdot \frac{BQ}{CQ} = \frac{BA^2}{CA^2}$

Problem 3.6 (IMO 2018)**.** Let Γ be the circumcircle of acute triangle ABC. Points D and E are on segments AB and AC respectively such that $AD = AE$. The perpendicular bisectors of BD and CE intersect minor arcs AB and AC of Γ at points F and G respectively. Prove that lines DE and FG are either parallel or they are the same line.

Problem 3.7 (USAMTS Year 32)**.** Let ABC be a triangle with $AB < AC$. T is the point on $\overline{BC}$ such that $\overline{AT}$ is tangent to the circumcircle of $\triangle ABC$. Additionally, H and O are the orthocenter and circumcenter of $\triangle ABC$, respectively. Suppose that $\overline{CH}$ passes through the midpoint of $\overline{AT}$. Prove that $\overline{AO}$ bisects $\overline{CH}$.

Problem 3.8 (XVII Olimpíada Matemática Rioplatense (2008))**.** In triangle ABC, where $AB < AC$, let X, Y, Z denote the points where the incircle is tangent to BC, CA, AB, respectively. On the circumcircle of ABC, let U denote the midpoint of the arc BC that contains the point A. The line UX meets the circumcircle again at the point K. Let T denote the point of intersection of AK and YZ. Prove that XT is perpendicular to YZ.

Problem 3.9 (USAMO 2013)**.** In triangle ABC, points P, Q, R lie on sides BC, CA, AB respectively. Let ω_A, ω_B, ω_C denote the circumcircles of triangles AQR, BRP, CPQ, respectively. Given the fact that segment AP intersects ω_A, ω_B, ω_C again at X, Y, Z, respectively, prove that $YX/XZ = BP/PC$.

Problem 3.10 (IMO Shortlist 2019)**.** Let ABC be an acute-angled triangle and let D, E, and F be the feet of altitudes from A, B, and C to sides BC, CA, and AB, respectively. Denote by ω_B and ω_C the incircles of triangles BDF and CDE, and let these circles be tangent to segments DF and DE at M and N, respectively. Let line MN meet circles ω_B and ω_C again at $P \neq M$ and $Q \neq N$, respectively. Prove that $MP = NQ$.

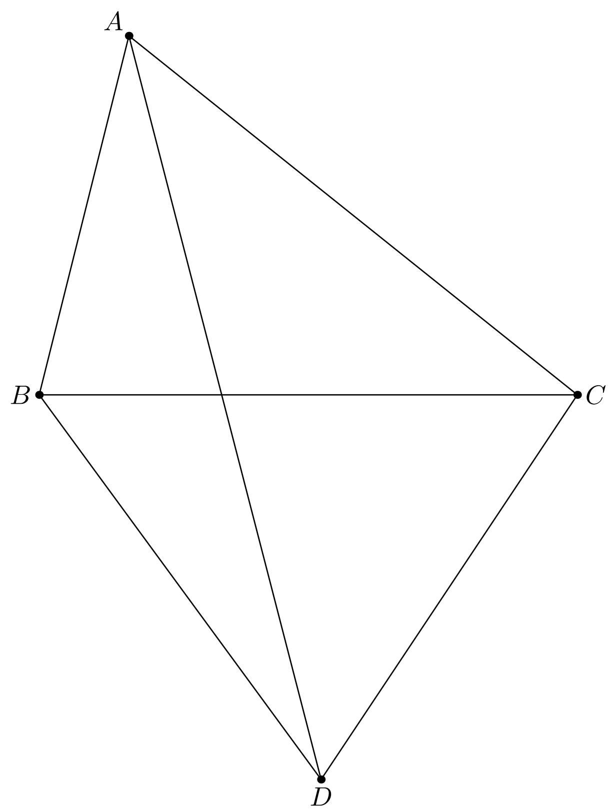
4 Quadrilateral Ratios, Ceva and Menelaus

We now extend our method of computing the "direction" of AD in $\angle BAC$ to cases where D isn't on line BC.

Theorem 4.1 (Quadrilateral Ratio Formula)

Let ABC be a triangle, and let D be a point. Then, we have

$$\frac{\sin(BAD)}{\sin(CAD)} = \frac{\sin(ABD)}{\sin(ACD)} \cdot \frac{BD}{CD} = \frac{\sin(ABD)}{\sin(ACD)} \cdot \frac{\sin(BCD)}{\sin(CBD)}.$$



Proof is left as an exercise. It's also written out in the third appendix. Although this is really similar to Ceva's theorem (the latter equality is equivalent), it makes sense to think about it separately, since we're looking "from A". Intuitively, this theorem states:

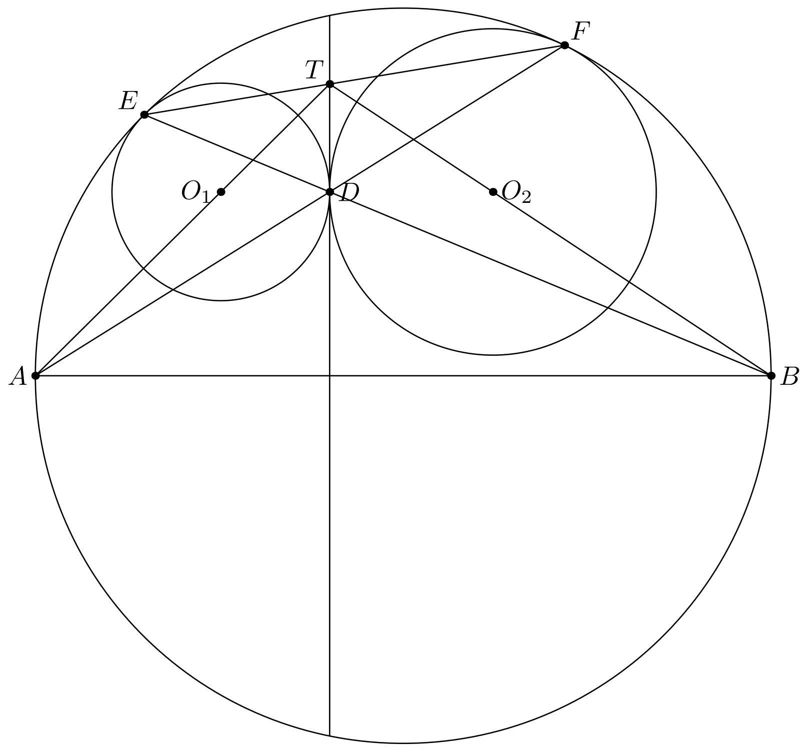
> If rays BD and CD aren't too bad, and we can either deal with BD/CD or the angles between BC and DB, DC, we can find the direction of AD in $\angle BAC$.

Having this theorem greatly expands the set of ratios we can deal with. Here's an example:

Example 4.2 (IMO Shortlist 2006)

Circles w_1 and w_2 with centres O_1 and O_2 are externally tangent at point D and internally tangent to a circle w at points E and F respectively. Line t is the common tangent of w_1 and w_2 at D. Let AB be the diameter of w perpendicular to t, so that A, E, O_1 are on the same side of t. Prove that lines AO_1, BO_2, EF and t are concurrent.

By homothety at E, F, we have that ED and FD pass through B and A, respectively.



Now, without explicitly doing any computations, we check if we can deal with ET/FT, where T is the desired concurrence point. A pretty natural way of constructing the diagram is starting with A, B, E, F and then constructing everything else, so we'll try finding things in terms of those points. We have:

- $\angle AEO_1$, $\angle ADO_1$, AD, AE, and $EO_1 : DO_1$ are all reasonable. Hence, we can compute the direction of AO_1 in $\angle EAF$, so we can compute the ratio AO_1 splits EF into.
- The above holds for BO_2.
- All the angles between D, E, F and the common tangent are nice, so we can compute the ratio the common tangent splits EF into.

It looks like we have enough info to prove the result. Filling in the details gives this proof:

Proof. We claim that they concur at the point T on EF such that $\frac{ET}{FT} = \frac{AE \cdot BE}{AF \cdot BF}$. We have

$$\frac{\sin(EAO_1)}{\sin(FAO_1)} = \frac{\sin(AEO_1)}{\sin(ADO_1)} \cdot \frac{DO_1}{EO_1} = \frac{\sin(BAE)}{\sin(FAB)} = \frac{BE}{BF} \implies \frac{AE}{AF}\frac{\sin(EAO_1)}{\sin(FAO_1)} = \frac{AE \cdot BE}{AF \cdot BF}.$$

Similarly, BO_2 passes through that point. Finally, if T meets EF at T', we have

$$\frac{ET'}{FT'} = \frac{ED}{FD} \cdot \frac{\sin(EDT')}{\sin(FDT')} = \frac{EA}{FB} \cdot \frac{\sin(BAE)}{\sin(ABF)} = \frac{EA}{FB} \cdot \frac{BE}{AF} = \frac{ET}{FT},$$

as desired. □

Finally, we'll see how to convert common goals of geo problems to ratios:

Theorem 4.3 (Ceva's Theorem)
Let ABC be a triangle, and let D, E, F be points. Then, AD, BE, and CF concur iff

$$\frac{\sin(BAD)}{\sin(CAD)} \cdot \frac{\sin(CBE)}{\sin(ABE)} \cdot \frac{\sin(ACF)}{\sin(BCF)} = 1.$$

Theorem 4.4 (Menelaus Theorem)
Let ABC be a triangle, and let D, E, F be points on BC, CA, and AB. Then, DEF are collinear iff

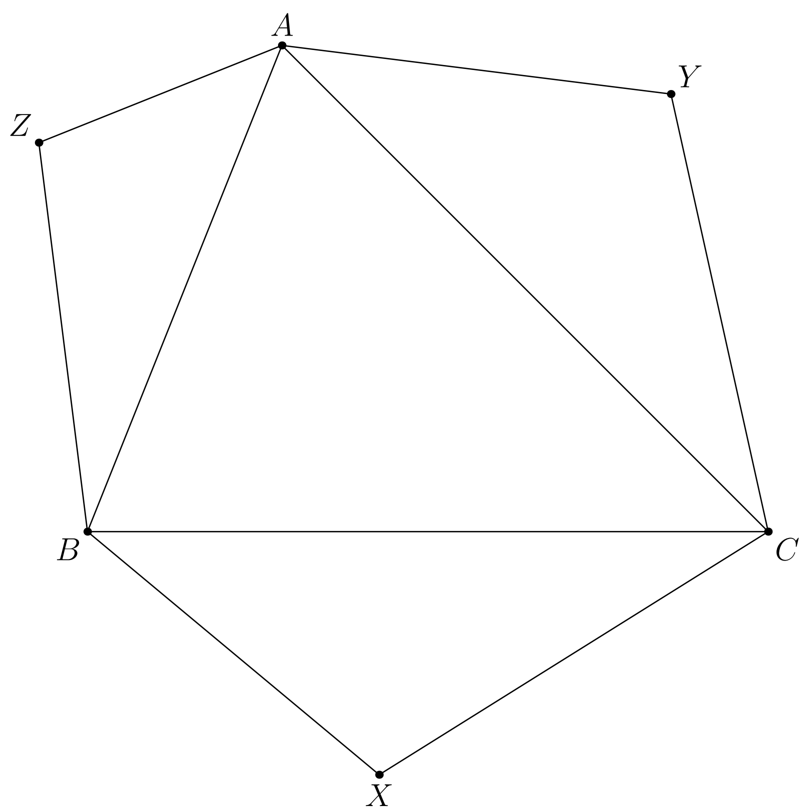
$$\frac{BD}{CD} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} = 1.$$

Thinking with the tools we've developed, Ceva's theorem states:

> If we want to prove AD, BE, CF concur, it suffices to find the directions of AD in $\angle BAC$ and similar for the others.

Menelaus can be thought of similarly, though it's restricted to the case where D, E, F are on the opposite sides.

Example 4.5 (Jacobi's Theorem)
Let ABC be a triangle. Points X, Y, Z are chosen such that $\angle BAZ = \angle CAY = \alpha$, for some α, and there exist β and γ with similar properties. Show that AX, BY, and CZ concur.



The key observation here is that since X is defined by given angles off of AB, AC, and BC, the direction of AX should be manageable. Similarly, the directions of BY and CZ should be manageable. By Ceva's theorem, we know that this is enough to check concurrence.

Now, we can just go through the computations.

Proof. By our Quadrilateral Ratio Formula (in its second form),

$$\frac{\sin(BAX)}{\sin(CAX)} = \frac{\sin(BCX)}{\sin(CXB)} \cdot \frac{\sin(ABX)}{\sin(ACX)} = \frac{\sin\gamma}{\sin\beta} \cdot \frac{\sin(B+\beta)}{\sin(C+\gamma)}.$$

The cyclic product of this is 1, so by Ceva's theorem, we are done. □

Problem 4.6. Prove the Quadrilateral Ratio Formula

Problem 4.7. Let ABC be a triangle. Let the feet of the altitudes from B and C to AC and AB be E and F, respectively. Let D be the intersection of the tangents to the circumcircle of ABC at B and C. Prove that AD bisects EF.

Problem 4.8 (Simson Lines)**.** Let $ABCD$ be a cyclic quadrilateral. Prove that the feet of the altitudes from D to the sides of ABC are collinear.

Problem 4.9 (Steinbart's Theorem)**.** Let ABC be a triangle with incircle ω. Let ω meet BC, CA, and AB at D, E, and F. Suppose X, Y, and Z are chosen on ω such that DX, EY, and CZ concur. Show that AX, BY, and CZ concur.

Problem 4.10 (Fall SDPC 2019-2020)**.** Let $\triangle ABC$ be an acute, scalene triangle with orthocenter H, and let AH meet the circumcircle of $\triangle ABC$ at a point $D \neq A$. Points E and F are chosen on AC and AB such that $DE \perp AC$ and $DF \perp AB$. Show that BE, CF, and the line through H parallel to EF concur.

Problem 4.11. Let ABC be a triangle. Suppose the Euler line of ABC meets BC at P. In terms of the angles of ABC, compute $\frac{BP}{CP}$.

Problem 4.12 (APMO 2021)**.** Let $ABCD$ be a cyclic convex quadrilateral and Γ be its circumcircle. Let E be the intersection of the diagonals of AC and BD. Let L be the center of the circle tangent to sides AB, BC, and CD, and let M be the midpoint of the arc BC of Γ not containing A and D. Prove that the excenter of triangle BCE opposite E lies on the line LM.

Problem 4.13 (ELMO Shortlist 2019)**.** Let ABC be an acute triangle with orthocenter H and circumcircle Γ. Let BH intersect AC at E, and let CH intersect AB at F. Let AH intersect Γ again at $P \neq A$. Let PE intersect Γ again at $Q \neq P$. Prove that BQ bisects segment $\overline{EF}$.

Problem 4.14 (TSTST 2020)**.** Let A, B, C, D be four points such that no three are collinear and D is not the orthocenter of ABC. Let P, Q, R be the orthocenters of $\triangle BCD$, $\triangle CAD$, $\triangle ABD$, respectively. Suppose that the lines AP, BQ, CR are pairwise distinct and are concurrent. Show that the four points A, B, C, D lie on a circle.

Problem 4.15 (Balkan MO 2019)**.** Cevians AP and AQ of a triangle ABC are symmetric with respect to its bisector. Let X, Y be the projections of B to AP and AQ respectively, and N, M be the projections of C to AP and AQ respectively. Prove that XM and NY meet on BC.

5 Ratios on a Circle

The exact set of content I'd want to include here (which is probably too big for one section anyways) is included in my ratio lemma handout. The handout is not too well titled. What it actually does is, given points on a line ℓ or circle ω through points B, C:

1. Shows how to get between BP/CP, BQ/CQ, and BR/CR for collinear points P, Q, R with P, Q on ω and R on ℓ
2. Shows how to, given only the ratios above, check if $PQRS$ is cyclic, where we have that $BCPQ$ and $BCRS$ are both either collinear or concyclic.
3. Given a triangle ABC and points E, F, P on $AC, AB, (ABC)$ such that $AEFP$ is cyclic, shows how to compute BP/CP

The tools we developed in the previous 2 sections are really useful for this; think about, for example, problem 3.8. I'll also add the following remark:

Theorem 5.1
Let ABC be a triangle, and let P be any point on its circumcircle. Then, you can go between $AP : BP$, $BP : CP$, and $CP : AP$ by Ptolemy's theorem.

This is useful since it allows us to go between ratios to different points on a circle.

For those of you who've already done the handout, here are some more problems.

Problem 5.2 (Sharygin 2022)**.** Let $ABCD$ be a cyclic quadrilateral, $E = AC \cap BD$, $F = AD \cap BC$. The bisectors of angles AFB and AEB meet CD at points X, Y. Prove that A, B, X, Y are concyclic.

Problem 5.3 (Jerabek's Theorem)**.** Let ABC be a triangle, and let P, Q be points in its interior. Suppose AP and AQ meet the circumcircle of ABC again at P_1 and Q_1, and let P_1Q_1 meet BC at A'. Define B' and C' similarly. Prove that A', B', and C' are collinear.

Problem 5.4 (AoPS)**.** Let $\triangle ABC$ have orthocenter H, circumcenter O, and let O' be the reflection of O through BC. Let E and F be the intersections of BO' and CO' with AC and AB, respectively. Prove that the circle with diameter AH, the circumcircle of ABC, and the circumcircle of AEF are coaxial.

Problem 5.5 (USAMTS Year 30)**.** Acute scalene triangle $\triangle ABC$ has circumcenter O and orthocenter H. Points X and Y, distinct from B and C, lie on the circumcircle of $\triangle ABC$ such that $\angle BXH = \angle CYH = 90°$. Show that if lines XY, AH, and BC are concurrent, then $\overline{OH}$ is parallel to $\overline{BC}$.

Problem 5.6. Let ABC be a triangle with incenter I, let the external angle bisector of $\angle A$ meet the circumcircle at M, and let A' be the antipode of A in the circumcircle of ABC. Prove that A, the second intersection of $A'I$ and the circumcircle, the midpoint of BC, and the intersection of MI and BC are concyclic.

Problem 5.7 (Sharygin 2020)**.** Let H be the orthocenter of a nonisosceles triangle ABC. The bisector of angle BHC meets AB and AC at points P and Q respectively. The perpendiculars to AB and AC from P and Q meet at K. Prove that KH bisects the segment BC.

6 Linearity of Power of a Point

Let $\text{Pow}(P, \omega)$ denote the power from P to ω. Linearity of pop refers to the following observation:

Theorem 6.1

For any two circles ω_1, ω_2, the function $f(P) = \text{Pow}(P, \omega_1) - \text{Pow}(P, \omega_2)$ is a linear function of P (i.e. linear in the coordinates of P).

Proof. If ω has equation $x^2 + y^2 + ax + by + c = 0$, the power from $P = (x_0, y_0)$ to ω is $x_0^2 + y_0^2 + ax_0 + by_0 + c$. Subtracting two such expressions gives a linear function in x_0 and y_0, as desired. □

One can think of this as "$\text{Pow}(P, \omega)$ is a linear function plus some term that's universal for all circles."

One common use of linearity of pop is computing powers at random points. In particular, we have:

> If $Z = xP + yQ + zR$, then $f(Z) = xf(P) + yf(Q) + zf(R)$. Hence, if we know the differences of powers at 3 points, we can compute that difference everywhere.

Since there are good and relatively known handouts regarding this (see Kagebaka's, for example), I'll focus on different uses of linearity of pop. It's a really versatile tool, and probably the hardest technique on this handout to get used to. Loosely, "it generally helps you deal with power-of-a-point information."

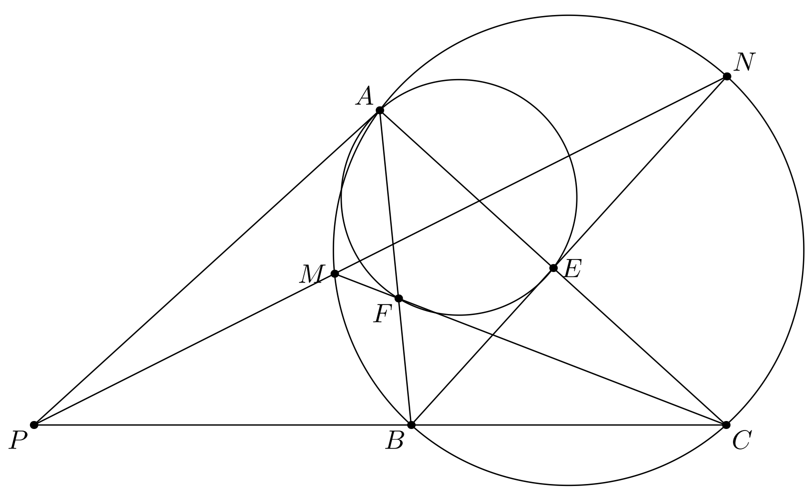
The following special case is very useful, and works well with previous sections of this handout:

> We can go between power differences and length ratios on a line! If P, Q, T are on a line, then
> $$\frac{PT}{QT} = \frac{f(P) - f(T)}{f(Q) - f(T)}.$$

Here's an example of a problem in which this is useful:

Example 6.2 (Brazil Olympic Revenge 2017 (sorta))

Let ABC be an acute triangle and I its incenter. Let E and F be the intersections of BI and CI with sides AC and AB. Let M and N be the midpoints of minor arcs AB and AC and let P be a point on BC such that AP is tangent to AEF. Prove that M, N, and P are collinear.



Relabel P to be on BC such that AP is tangent to (AEF). Line MN is pretty well behaved, so the hard part of this problem is to get some grasp on P. The key idea is the following characterization:

> P is the point on BC such that PA^2 is equal to the power from P to (AEF).

The motivation here is that A and (AEF) are both reasonable to access from B and C. Now, defining $f(X)$ to be $XA^2 - \operatorname{Pow}(X, (ABC))$, we have

$$\frac{BP}{CP} = \frac{f(B)}{f(C)} = \frac{AB \cdot AF}{AC \cdot AE},$$

which is just something in terms of the side lengths of ABC. We can also compute the ratio MN splits BC into by, say, ratio lemma. This should be enough to solve the problem.

Proof. Defining $f(X)$ to be $XA^2 - \operatorname{Pow}(X, (AEF))$, we have

$$\frac{BP}{CP} = \frac{f(B)}{f(C)} = \frac{AB \cdot AF}{AC \cdot AE} = \frac{c}{b} \cdot \frac{a+c}{a+b}.$$

It suffices to check that this is equal to $\frac{BM}{CM} \cdot \frac{BN}{CN}$. By Ptolemy, we have

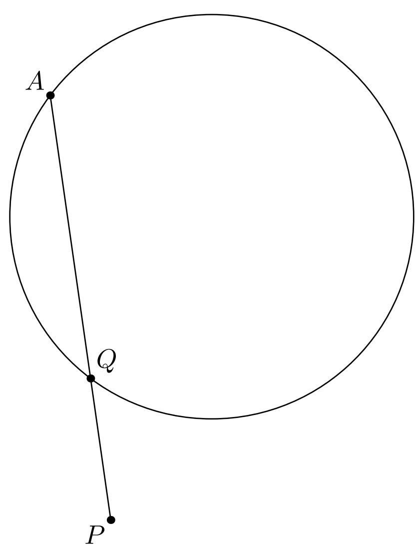
$$c \cdot CM = BM \cdot b + AM \cdot a = (a+b)BM \implies \frac{BM}{CM} = \frac{c}{a+b},$$

and similar for $\frac{BM}{CN}$, giving $\frac{BM}{CM} \cdot \frac{BN}{CN} = \frac{BP}{CP}$, as desired. $\square$

If the choice of f still feels too unmotivated, it may be worth noting that the choice of function "difference between power to a circle and power to a point on the circle" is, in general, very useful. We have the following:

Corollary 6.3

Let A be a point on a circle ω. Let P be a (non-fixed) point, and let AP meet ω at a point Q other than A. Then, $AP \cdot AQ$ is a linear function of P.

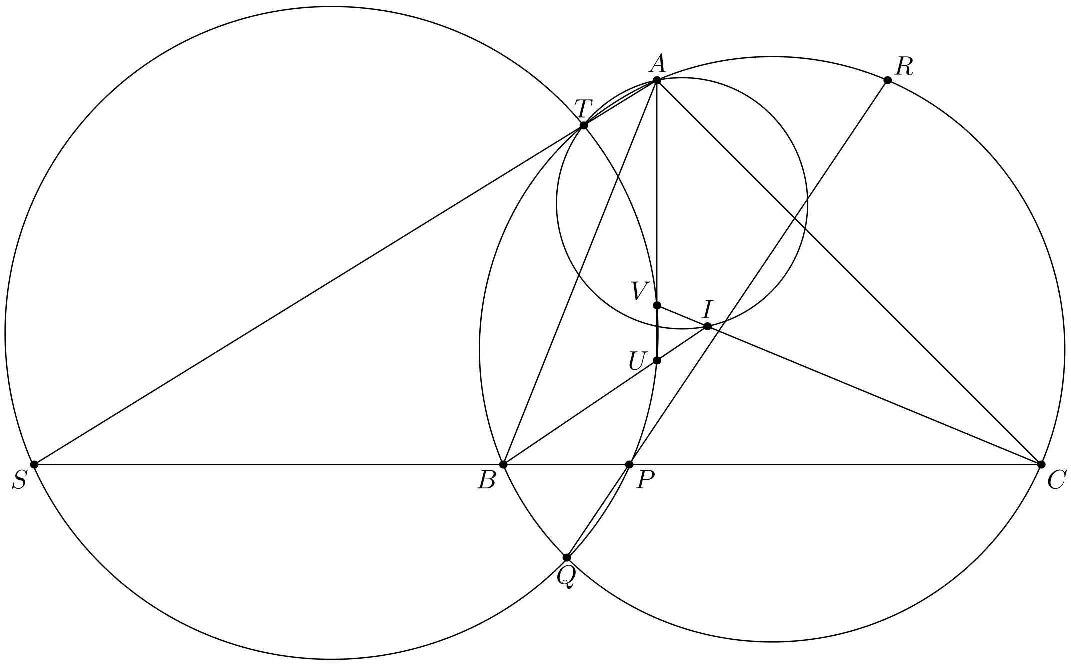


Proof. Rewrite $AP \cdot AQ$ as $AP^2 - PQ \cdot PA = AP^2 - \operatorname{Pow}(P, \omega)^2$. Conclude by linearity of pop. $\square$

We'll now see another problem that uses a similar application of linearity of power.

Example 6.4 (IMOC Shortlist 2021)

Let I be the incenter of the acute triangle $\triangle ABC$, and BI, CI intersect the altitude of $\triangle ABC$ through A at U, V, respectively. The circle with AI as a diameter intersects $\odot(ABC)$ again at T, and $\odot(TUV)$ intersects the segment BC and $\odot(ABC)$ at P, Q, respectively. Let R be another intersection of PQ and $\odot(ABC)$. Show that $AR \parallel BC$.



By Reim's, $AR||PQ$ is equivalent to $AT \cap BC$ also lying on (TUV). Let $S = AT \cap BC$. By LOS, we should be able to deal with $AU \cdot AV$, so by power of a point, it suffices to compute $AT \cdot AS$. Now, note that:

- By spiral similarity, we should be able to deal with $BT : CT$
- By Ratio Lemma, we should be able to deal with $BS : CS$
- By our previous corollary of linearity of power of a point, since B, C have reasonable powers to (AI), we should be able to compute $AT \cdot AS$.

Actually going through with these computations turns out to be slightly more annoying than usual, but still not bad. Here's the resulting solution:

Proof. Let E, F be the B and C intouch points. By spiral similarity,

$$\frac{BT}{CT} = \frac{BF}{CE} = \frac{s-b}{s-c}.$$

Now, by ratio lemma,

$$\frac{BS}{CS} = \frac{BT}{CT} \cdot \frac{BA}{CA} = \frac{c(s-b)}{b(s-c)}.$$

Let $f(Z)$ denote the function $AZ' \cdot AZ$, where Z' is the second intersection of AZ and (AI). By the corollary, this is linear in Z. Now,

$$\frac{c(s-b)}{b(s-c)} = \frac{BS}{CS} = \frac{f(S)-f(B)}{f(S)-f(C)} = \frac{f(S)-(s-a)c}{f(S)-(s-a)b} \implies f(S)s(b-c) = (s-a)bc(b-c).$$

Now, we have

$$f(S) = \frac{bc(s-a)}{s}.$$

We need to check that this is equal to $AU \cdot AV$, and do this via the law of sines:

$$AU = AB \cdot \frac{\sin(B/2)}{\sin(90^\circ + B/2)} = AB \cot B/2 \implies AU \cdot AV = bc \cot B/2 \cot C/2.$$

To deal with $\cot B/2$, note that it's $\frac{IF}{BF} = \frac{r}{s-b}$, giving

$$AU \cdot AV = \frac{bc}{(s-b)(s-c)} r^2 = \frac{bc}{(s-b)(s-c)} \cdot \frac{[ABC]^2}{s^2} = \frac{bc(s-a)}{s},$$

where the last equality is by Heron's, as desired. $\square$

Problem 6.5. Let $BPQC$ be on a line in that order, and pick R on the line such that $RB \cdot RC = RP \cdot RQ$. Prove, in a way different from the ratio lemma handout, that $\frac{BR}{CR} = \frac{BP}{CP} \cdot \frac{BQ}{CQ}$.

Problem 6.6 (USAMO 2013)**.** In triangle ABC, points P, Q, R lie on sides BC, CA, AB respectively. Let ω_A, ω_B, ω_C denote the circumcircles of triangles AQR, BRP, CPQ, respectively. Given the fact that segment AP intersects ω_A, ω_B, ω_C again at X, Y, Z, respectively, prove that $YX/XZ = BP/PC$.

Problem 6.7 (AoPS)**.** Let $\triangle ABC$ be a triangle with circumcenter O. The perpendicular bisectors of the segments OA, OB and OC intersect the lines BC, CA and AB at D, E and F, respectively. Prove that D, E, F are collinear.

Problem 6.8. Let ABC be a triangle, and let ω be the circle of points T with $BT : CT = \lambda$. Prove that the power from A to ω is $\frac{AC^2 \cdot \lambda^2 - AB^2}{\lambda^2 - 1}$.

Problem 6.9 (IMO 2017, Reworded)**.** Let $ABCD$ be a trapezoid with $AB || CD$, and let M be the midpoint of AC. Show that AD is tangent to the circumcircle of ABM if and only if BC is tangent to the circumcircle of CDM.

Problem 6.10 (GGG1)**.** Let ABC be an acute triangle, and let D, E, F be the feet of the altitudes from A, B, C, respectively. Let $\overleftrightarrow{EF}$ meet the circumcircle of ABC at points S_1 and S_2. Let P be the intersection of $\overline{BE}$ and $\overline{DF}$, and let Q be the intersection of $\overline{DE}$ and $\overline{CF}$.

Prove that the circumcircles of triangles DS_1S_2 and DPQ are tangent to each other.

Problem 6.11 (USA TST 2019)**.** Let ABC be a triangle and let M and N denote the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. Let X be a point such that $\overline{AX}$ is tangent to the circumcircle of triangle ABC. Denote by ω_B the circle through M and B tangent to $\overline{MX}$, and by ω_C the circle through N and C tangent to $\overline{NX}$. Show that ω_B and ω_C intersect on line BC.

Problem 6.12 (India TST 2019)**.** Let ABC be an acute triangle with circumcircle Ω and altitudes $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ meeting at H. Let ω be the circumcircle of $\triangle DEF$. Point $S \neq A$ lies on Ω such that $DS = DA$. Line $\overline{AD}$ meets $\overline{EF}$ at Q, and meets ω at $L \neq D$. Point M is chosen such that $\overline{DM}$ is a diameter of ω. Point P lies on $\overline{EF}$ with $\overline{DP} \perp \overline{EF}$. Prove that lines SH, MQ, PL are concurrent.

7 Two Length Conditions

You can view a lot of this handout as proving theorems that we "should" have. For example, in the "Ratios on a Circle" section, we think about points in terms of the ratio they form with distances to B and C. Since this uniquely determines points on a fixed circle through B, C or line BC, we "should" be able to check when things are collinear or concyclic, and find ways to do this in that section. This section answers the question:

> Say I fixed a point O, and gave you points P_1, P_2, P_3 by telling you the distance from O and the angle they leave O at. Can you tell me if they're collinear? If they're concyclic?

Here are the two length conditions:

Theorem 7.1 (Concyclicity Length Condition)
Let O be a fixed point and ℓ_1, ℓ_2, ℓ_3 be fixed lines through it. Let θ_1, θ_2, and θ_3 be the angles between ℓ_2 and ℓ_3, ℓ_3 and ℓ_1, and ℓ_1 and ℓ_2. Points P_1, P_2, P_3 are chosen on each of these lines. Then, $OP_1P_2P_3$ is a cyclic quadrilateral if and only if

$$\sin(\theta_1)OP_1 + \sin(\theta_2)OP_2 + \sin(\theta_3)OP_3 = 0$$

with signs chosen appropriately.

Theorem 7.2 (Collinearity Length Condition)
Let O be a fixed point and ℓ_1, ℓ_2, ℓ_3 be fixed lines through it. Let θ_1, θ_2, and θ_3 be the angles between ℓ_2 and ℓ_3, ℓ_3 and ℓ_1, and ℓ_1 and ℓ_2. Points P_1, P_2, P_3 are chosen on each of these lines. Then, P_1, P_2, and P_3 are collinear if and only if

$$\frac{\sin(\theta_1)}{OP_1} + \frac{\sin(\theta_2)}{OP_2} + \frac{\sin(\theta_3)}{OP_3} = 0$$

with signs chosen appropriately.

Being able to state the theorem without the "with signs chosen appropriately" thing requires actually learning how to do trig with directed angles.

There are plenty of proofs of these conditions. I'll just outline of 1.5 proofs for each:

- Concyclicity: apply Ptolemy's.
- Collinearity: compute $[P_1P_2P_3]$.
- The conditions are equivalent: Invert at P.

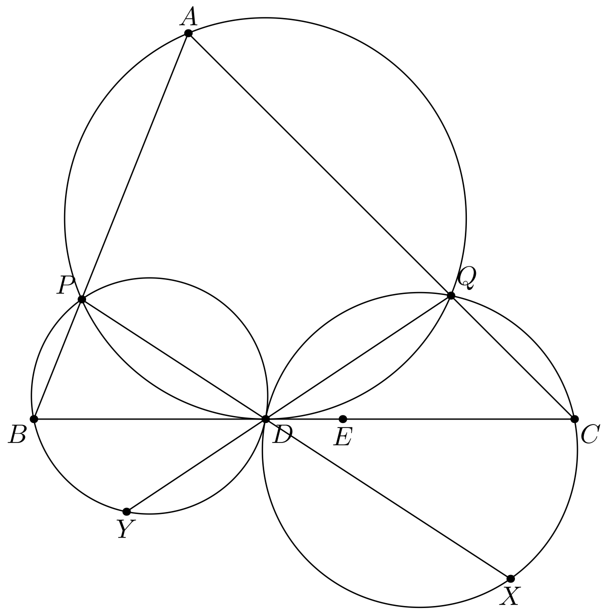
The proofs are also fully written out in the third appendix.

The philosophy behind applying these conditions is simple - if we can understand the directions and lengths well enough from the perspective of a certain point, you can check collinearity and concyclicity at that point. I'll give an example.

Example 7.3 (ELMO 2021)

In $\triangle ABC$, points P and Q lie on sides AB and AC, respectively, such that the circumcircle of $\triangle APQ$ is tangent to BC at D. Let E lie on side BC such that $BD = EC$. Line DP intersects the circumcircle of $\triangle CDQ$ again at X, and line DQ intersects the circumcircle of $\triangle BDP$ again at Y. Prove that D, E, X, and Y are concyclic.

Observe that (BDP) and (CDQ) are tangent. Since homothety at D maps these circles to each other, we have $DPBY \sim DXCQ$. Now, we have a pretty good grasp on E, X, Y from the perspective of D, and we can finish using the concyclicity criterion.



Proof. Assume $AB < AC$, so DX is contained in $\angle YDE$. We want to prove

$$DE\sin(XDY) + DY\sin(XDE) - DX\sin(YDE) = 0.$$

For the angles, we have $\angle XDY = 180 - \angle A$, $\angle XDE = \angle BDP = \angle BAD$, and similar for the last remaining one. For the lengths,

$$DX = \frac{DC}{DB}DP = \frac{DC}{DB} \cdot \frac{BD}{BA} \cdot AD = \frac{DC}{BA} \cdot AD,$$

where we use $BDP \sim BAD$ for the second to last step. Hence, we want to show

$$(DC - DB)\sin(A) + \frac{DB}{CA} \cdot AD\sin(BAD) - \frac{DC}{BA} \cdot AD\sin(CAD) = 0.$$

To get rid of the annoying sines, note that by ratio lemma,

$$\frac{\sin(BAC)}{\sin(DAC)} = \frac{DA}{BA} \cdot \frac{BC}{DC}.$$

Hence, dividing through by $\sin(A)$, we want to check

$$(DC - DB) + \frac{DB^2}{BC} - \frac{DC^2}{BC} = 0$$

which is clear, as desired. $\square$

Before going into practice problems, I'll mention a couple ideas used to make computation with these conditions easier. There are a couple interesting connections between the concyclicity length condition and other theorems:

- Corollary 6.3 applied to three collinear points P is pretty similar. For instance, in the above solution, we have $\angle XDE = \angle BAD$, $\angle EDY = 180^\circ - \angle DAC$. Therefore, we can translate and rotate D, E, X, Y so D goes to A, line DX goes to line AB, line DE goes to line AD, and line DY goes to line AC. Now, by Corollary 6.3, the concyclicity is equivalent to there being a line passing through the points $(0, DX \cdot AB)$, $(BD, DE \cdot AD)$, and $(BC, -DY \cdot AC)$.
- Say P_i, Q_i are chosen on ℓ_i for $i = 2, 3$, and we have that $OP_1P_2P_3$ and $OP_1Q_2Q_3$ are cyclic quadrilaterals. Then, if we subtract the concyclicity length conditions, we get
$$\frac{P_2Q_2}{P_3Q_3} = \frac{\sin(P_1OP_2)}{\sin(P_1OP_3)}.$$
This can also be proved by using spiral similarity at P_1.

Also, for either length condition, if we have a triangle with angles θ_1, θ_2, and θ_3, we can replace the sines in the length condition with the side lengths of that triangle, which is often useful.

Problem 7.4 (AoPS)**.** Let ABC be a triangle with orthocenter H. Let X, Y, Z be points on AH, BH, and CH. Show that $[BXC] + [CYA] + [AZB] = [ABC]$ if and only if $HXYZ$ is cyclic.

Problem 7.5 (APMO 2017)**.** Let ABC be a triangle with $AB < AC$. Let D be the intersection point of the internal bisector of angle BAC and the circumcircle of ABC. Let Z be the intersection point of the perpendicular bisector of AC with the external bisector of angle $\angle BAC$. Prove that the midpoint of the segment AB lies on the circumcircle of triangle ADZ.

Problem 7.6 (Sharygin 2017)**.** Let ABC be a triangle with incenter I. Show that the orthocenter of AIB, the orthocenter of AIC, and the A-intouch point are collinear.

Problem 7.7 (IMO Shortlist 2018)**.** Let ABC be a triangle with $AB = AC$, and let M be the midpoint of BC. Let P be a point such that $PB < PC$ and PA is parallel to BC. Let X and Y be points on the lines PB and PC, respectively, so that B lies on the segment PX, C lies on the segment PY, and $\angle PXM = \angle PYM$. Prove that the quadrilateral $APXY$ is cyclic.

Problem 7.8 (IMO Shortlist 2015)**.** Let ABC be an acute triangle and let M be the midpoint of AC. A circle ω passing through B and M meets the sides AB and BC at points P and Q respectively. Let T be the point such that $BPTQ$ is a parallelogram. Suppose that T lies on the circumcircle of ABC. Determine all possible values of $\frac{BT}{BM}$.

Problem 7.9 (Fake USAJMO 2020)**.** Let $\triangle ABC$ be a triangle. Points D, E, and F are placed on sides $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ respectively such that $EF \parallel BC$. The line DE meets the circumcircle of $\triangle ADC$ again at $X \neq D$. Similarly, the line DF meets the circumcircle of $\triangle ADB$ again at $Y \neq D$. If D_1 is the reflection of D across the midpoint of $\overline{BC}$, prove that the four points D, D_1, X, and Y lie on a circle.

Problem 7.10 (IMO Shortlist 2018)**.** Let ABC be a triangle with circumcircle Ω and incentre I. A line ℓ intersects the lines AI, BI, and CI at points D, E, and F, respectively, distinct from the points A, B, C, and I. The perpendicular bisectors x, y, and z of the segments AD, BE, and CF, respectively determine a triangle Θ. Show that the circumcircle of the triangle Θ is tangent to Ω.

Problem 7.11 (ELMO Shortlist 2010)**.** Let ABC be a triangle with circumcircle Ω. X and Y are points on Ω such that XY meets AB and AC at D and E, respectively. Show that the midpoints of XY, BE, CD, and DE are concyclic.

Problem 7.12 (USA TSTST 2019)**.** Let ABC be an acute triangle with orthocenter H and circumcircle Γ. A line through H intersects segments AB and AC at E and F, respectively. Let K be the circumcenter of $\triangle AEF$, and suppose line AK intersects Γ again at a point D. Prove that line HK and the line through D perpendicular to $\overline{BC}$ meet on Γ.

8 Appendix 1: General Advice and Random Things not in the Handout

Here are some things I didn't get to include, if you're interested. Luke heavily helped this section.

1. The method of using linearity to compute power at random points was omitted, maybe read Kagebaka's handout.
2. As mentioned after the Iran Lemma, if you want an angle (or ratio) to be nice, you can try to force it to synthetically show up.
3. I don't use the Forgotten Coaxiality Lemma often enough to write about it, but it's pretty strong.
4. If P, Q are points and ω is a circle, P is on Q's polar if and only if $PQ^2 = \text{Pow}(P, \omega) + \text{Pow}(Q, \omega)$. RMM 2013 #3, for example, can be solved with this.
5. It's often useful to apply Menelaus or Cevas to weird triangles. You can often motivate these weird applications by thinking about the information you have about points. This AoPS post is an example of a weird Menelaus application that I remember.
6. If you have two linear functions of a point, you basically have a coordinate system. This is already the idea behind barycentric coordinates, but more random coordinate systems can be useful. Examples:
   - EGMO 2022 #1 with "component along AB and AC." It suffices to check that the ordered pairs (component along AB, component along AC) of the three desired points lie on a line.
   - See the first problem of section 3.3 of Patrik Bak's Master's Thesis (GeoGen!). The point is, for general points D, E on BC, $FB = FC$ is equivalent to some linear relationship between D and E.
   - There's some old ISL problem but I can't remember it. The interested reader should solve every ISL problem from the 21st century since I'm pretty sure it was after 2000.
7. You can often think about whether or not points are "algebraically distinguishable." A common example is if a circle and a line (or two circles) intersect at two points P and Q in some configuration, one of the following is happening:
   - P and Q are somehow distinguishable points. For example, there could be some special line that P lies on and Q doesn't lie on. If you want to deal with these points you'll generally need to find these better characterizations, since you need the points to have different properties.
   - P and Q are indistinguishable. In this case, there are a few things you may want to do:
     - If the the problem is symmetric in P, Q, it's often helpful to work with symmetric properties of P, Q. Stuff like properties of line PQ, the midpoint of PQ, powers of points on line PQ, etc. This is analogous to Vietas problems on computational contests but with more possibilities.
     - If the problem isn't symmetric in P, Q, and another pair of algebraically indistinguishable points R, S comes up, there may be some way roots of this quadratic correspond to each other. For a simple example:

       > Let P be a point and ω be a circle with center O. Lines ℓ_1 and ℓ_2 through P are symmetric about PO, and they meet ω at Q, R and S, T, with $PQ < PR$ and $PS < PT$. Prove that $PQOT$ is cyclic.

       Q, R are, on their own, algebraically indistinguishable, and so are S, T, but we don't have stuff like "$PQOS$ is cyclic". This is because the $PQ < PR$ and $PS < PT$ conditions are actually there to ensure reflection over PO maps Q to S and R to T.

8. It's useful to think about "as some point moves, how do other points move?" Even if there aren't free points, this can be helpful in figuring out what you care about. For example, in the second bullet point of (3), we thought about "how do D, E move if F moves along the perpendicular bisector?"

   If you know any moving points ("/degree counting"), intuition from that is probably useful here. Yayups' moving points tutorial is pretty good for this.

9. I'll end on this one cuz its the corniest. You shouldn't let yourself be restricted lengthbashing along methods in this handout! I think one of the important takeaways from this handout is that trying to bash things that you feel like you should be able to deal with. The existence of this section implies that there are a ton of more ad-hoc lengthbashing methods out there to use or something.

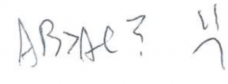
I considered adding an additional problems section, but this would distract from the fact that you can probably go out and do a random geo problem, and be able to use lengthbash with decent probability.

9 Appendix 2: My IMO 2021/3 Thought Process

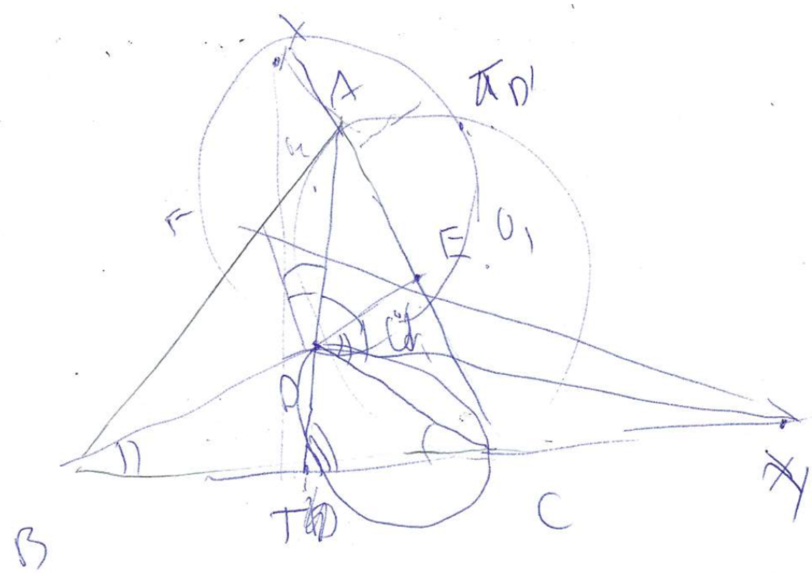
I'm including this for a couple reasons:

- It's a good representative of the techniques and intuition from this handout, but isn't direct from it.
- You scan all of your scratch work on the IMO since coordination exists, so I have a relatively good idea of what was going through my head during the contest.
- Some parts of this solution are pretty funny and I want to show off.

Problem 9.1 (IMO 2021 #3)**.** Let D be in acute triangle ABC with $AB > AC$ so that $\angle DAB = \angle CAD$. Point E on segment AC satisfies $\angle ADE = \angle BCD$, point F on segment AB satisfies $\angle FDA = \angle DBC$, and point X on the line AC satisfies $CX = BX$. Let O_1 and O_2 be the circumcenters of the triangles ADC and EXD, respectively. Prove that the lines BC, EF, and O_1O_2 are concurrent.



The first step of any geometry problem is drawing an accurate, to-scale diagram:



The first natural goal here is to figure out "what" point Y is, in terms of (triangle ABC and) D. Because of Menelaus, it suffices to compute $\frac{AE}{CE}$ and $\frac{AF}{BF}$.

The first observation I made was that DE is tangent to (CDT), and similar for DF, since angle conditions are scary. Now, linearity of PoP on $\text{Pow}(P, (DCT)) - PD^2$ gives $\frac{AE}{CE}$, and we can compute $\frac{BY}{CY} = \left(\frac{BD}{CD}\right)^2$. I wrote this down in my scratch paper as "YD is tangent to (BDC)".

Remark: The linearity of PoP here is kinda silly, ratio lemma just works.

Now, since there are lots of circles through D, inversion at D looks nice. I tried that for a bit and gave up once I realized X was impossible to deal with after inversion. I also introduced $D' = (DEX) \cap (DAC)$ at some point in this process

After giving up, I observed that the original problem, the collinearity is equivalent to the circle centered at Y passing through D also passing through D', which is just the D-Apollonius circle. Since we know where (DEX) and (DAC) meet line AC, I decided to reduce to the following problem:

> Given P, Q on AC with $\frac{BD}{CD} = \frac{BP}{CP} = \frac{BQ}{CQ}$, compute $\prod \frac{AP}{CP}$.

For ratio lemma reasons (or the forgotten coaxiality lemma), it suffices to check that what we get is equal to $\frac{AX}{CX} \cdot \frac{AE}{CE}$. This also gets rid of pretty much all the weird points in the problem.

this can't be that bad

AP/CP and AQ/CQ should be roots of some quadratic. Stewart's theorem is really the only theorem relating BP to the location of P on AC, so that should be enough to literally be able to find the quadratic. Indeed, we can make Stewart's homogenous in AP, BP, CP as follows:

$$CP \cdot AP \cdot AC + BP^2 \cdot AC = AB^2 \cdot CP + BC^2 \cdot AP$$

$$CP \cdot AP \cdot AC^2 + BP^2 \cdot AC^2 = AB^2 \cdot CP^2 + (AB^2 + BC^2) CP \cdot AP + BC^2 \cdot AP^2$$

$$\lambda \cdot AC^2 + \frac{BP^2}{CP^2} \cdot AC^2 = AB^2 + (AB^2 + BC^2)\lambda + BC^2 \cdot \lambda^2$$

product of λ is $\dfrac{\frac{BP^2}{CP^2} \cdot AC^2 - AB^2}{\cancel{AC^2 - 2BC^2}\ BC^2}$

where $\lambda = \frac{AP}{CP}$ and the last step is Vieta's.

Remark: Exercise 6.8 gives a better way of doing this step, the intended solution is around 1 line.

To compute $\frac{AX}{CX}$, do linearity of $CP^2 - BP^2$. We computed $\frac{AE}{CE}$ earlier. After actually writing out the computations, we want to check:

$$\frac{AX}{CX} \cdot \frac{AE}{CE} = \frac{AB^2 - AC^2}{BC^2} \cdot \frac{AD \cdot DT}{CD^2} \quad \text{equals} \quad \frac{AP}{CP} \cdot \frac{AQ}{CQ} = \frac{\frac{BD^2}{CD^2} AC^2 - AB^2}{BC^2}.$$

Unfortunately, this isn't trivial. But its relatively small so it can't be that bad. After cross multiplying, both sides are quadratics in D, so you can just check three cases and win. I ended up checking $D = A, T$ and the arc midpoint, all of which work! GG. ■

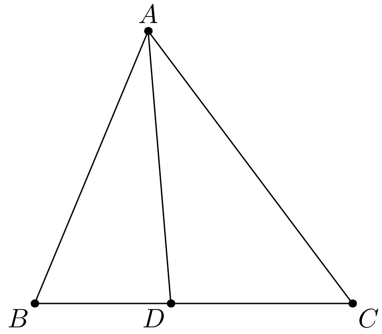
Remark: The last step also follows from exercise 6.8 on D and the A-Apollonius circle.

10 Appendix 3: Actually Writing Out Proofs

Ok now I'll finally write out the full proofs of the theorems I skipped over earlier.

Theorem 10.1 (Ratio Lemma)
Let $\triangle ABC$ be a triangle, and let D be a point on BC. Then, we have

$$\frac{BD}{CD} = \frac{BA}{CA} \cdot \frac{\sin \angle BAD}{\sin \angle CAD}.$$



Proof. By the law of sines on ABD and ACD, we have

$$\frac{BA}{BD} = \frac{\sin(BAD)}{\sin(ADB)} \quad \text{and} \quad \frac{CA}{CD} = \frac{\sin(CAD)}{\sin(ADC)}.$$

Since $\angle ADB + \angle ADC = 180^\circ$, we have

$$\sin(BAD) \cdot \frac{BD}{BA} = \sin(ADB) = \sin(ADC) = \frac{CD}{CA}\sin(CAD)$$

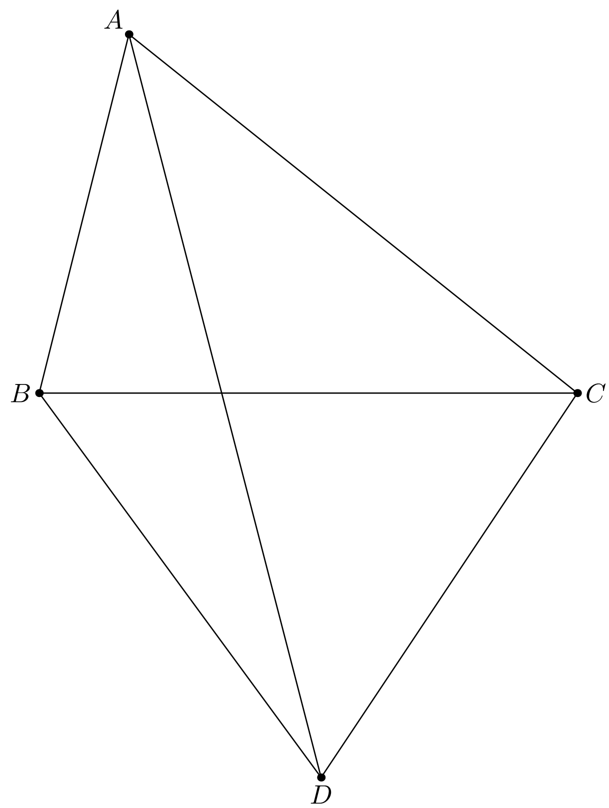
which rearranges to

$$\frac{BD}{CD} = \frac{BA}{CA} \cdot \frac{\sin \angle BAD}{\sin \angle CAD},$$

as desired. □

Theorem 10.2 (Quadrilateral Ratio Formula)
Let ABC be a triangle, and let D be a point. Then, we have

$$\frac{\sin(BAD)}{\sin(CAD)} = \frac{\sin(ABD)}{\sin(ACD)} \cdot \frac{BD}{CD} = \frac{\sin(ABD)}{\sin(ACD)} \cdot \frac{\sin(BCD)}{\sin(CBD)}.$$



Proof. By the law of sines on ABD and ACD, we have

$$AB \cdot \frac{\sin(BAD)}{\sin(ABD)} = AD = AC \cdot \frac{\sin(CAD)}{\sin(ACD)}.$$

This gives the first equality. The second equality follows from the law of sines of BCD. □

Theorem 10.3 (Concyclicity Length Condition)
Let O be a fixed point and ℓ_1, ℓ_2, ℓ_3 be fixed lines through it. Let θ_1, θ_2, and θ_3 be the angles between ℓ_2 and ℓ_3, ℓ_3 and ℓ_1, and ℓ_1 and ℓ_2. Points P_1, P_2, P_3 are chosen on each of these lines. Then, $OP_1P_2P_3$ is a cyclic quadrilateral if and only if

$$\sin(\theta_1)OP_1 + \sin(\theta_2)OP_2 + \sin(\theta_3)OP_3 = 0$$

with signs chosen appropriately.

Proof. We show that, assuming P_2 lies within $\angle P_3OP_1$, we have that $OP_1P_2P_3$ is a cyclic quadrilateral if and only if

$$\sin(\theta_1)OP_1 + \sin(\theta_3)OP_3 = \sin(\theta_2)OP_2.$$

Let P_2' be the point where OP_2 meets the circumcircle of OP_1P_3. Letting R be the circumradius of OP_1P_3, we have, by Ptolemy's theorem,

$$OP_1 \cdot \frac{P_2'P_3}{2R} + OP_3 \cdot \frac{P_2'P_1}{2R} = OP_2 \cdot \frac{P_1P_3}{2R}.$$

By the extended law of sines, this is

$$\sin(\theta_1)OP_1 + \sin(\theta_3)OP_3 = \sin(\theta_2)OP_2'.$$

Now, both the length condition from earlier and P_2 being on (OP_1P_3) are equivalent to $OP_2' = OP_2$, as desired. □

Theorem 10.4 (Collinearity Length Condition)

Let O be a fixed point and ℓ_1, ℓ_2, ℓ_3 be fixed lines through it. Let θ_1, θ_2, and θ_3 be the angles between ℓ_2 and ℓ_3, ℓ_3 and ℓ_1, and ℓ_1 and ℓ_2. Points P_1, P_2, P_3 are chosen on each of these lines. Then, P_1, P_2, and P_3 are collinear if and only if

$$\frac{\sin(\theta_1)}{OP_1} + \frac{\sin(\theta_2)}{OP_2} + \frac{\sin(\theta_3)}{OP_3} = 0$$

with signs chosen appropriately.

Proof. We show that, assuming P_2 lies within $\angle P_3OP_1$, we have that P_1, P_2, and P_3 are collinear if and only if

$$\frac{\sin(\theta_1)}{OP_1} + \frac{\sin(\theta_3)}{OP_3} = \frac{\sin(\theta_2)}{OP_2}.$$

Upon multiplying both sides by $\frac{1}{2}OP_1 \cdot OP_2 \cdot OP_3$, we have that it is equivalent to

$$\frac{1}{2}OP_2 \cdot OP_3 \sin(\theta_1) + \frac{1}{2}OP_2 \cdot OP_1 \sin(\theta_3) = \frac{1}{2}OP_1OP_3 \sin(\theta_2).$$

By the $\frac{1}{2}ab\sin C$ area formula, this is equivalent to $[OP_2P_3] + [OP_2P_1] = [OP_1P_3]$, which is equivalent to $[P_1P_2P_3]$ having area 0, as desired. □