Probability and Random Processes

Problems

Gaussian random vectors

1. Let $Z = S + N$ where $S \sim \mathsf{N}(m_S, \sigma_S^2)$ and $N \sim \mathsf{N}(0, \sigma_N^2)$ are independent. Find the joint characteristic function of Z and S.

2. Prove that the standard normal distribution $\mathsf{N}(0,1)$ has all its moments of odd order equal to 0 while the moments of even order are given by

$$\mu_{2j} = \frac{(2j)!}{2^j (j!)} \quad j = 0, 1, 2, \ldots$$

3. Let $X \sim \mathsf{N}(0,1)$ be a standard normal random variable.
   (a) Find the density of $Y = X^2$.
   (b) The characteristic function of X^2 is $(1 - 2i\omega)^{-1/2}$. Find the characteristic function of $Z = X_1^2 + X_2^2 + \cdots + X_n^2$ where $X_1, X_2, \ldots, X_n$ are all standard normal and independent.

4. Let X_1, X_2 be gaussian random variables with mean 0 and covariance matrix

$$\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$$

   If $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$, find the characteristic function of (Y_1, Y_2) and the marginal density of Y_1.

5. *(Midterm exam, November 2014)* Find the covariance matrix of the jointly gaussian random variables X, Y if we know that $(2X - Y)/3$ and $(X + Y)/3$ are $\mathrm{N}(0,1)$ independent random variables.

6. *(Midterm exam, November 2013)* Let X_1, X_2 and X_3 be jointly gaussian random variables with mean 1 and covariance matrix

$$\begin{pmatrix} 1 & 1/2 & -1/2 \\ 1/2 & 1 & 1/2 \\ -1/2 & 1/2 & 1 \end{pmatrix}$$

   If $Y_1 = X_1 + X_2 - X_3$ and $Y_2 = X_1 - 2X_3$, find the covariance matrix of (Y_1, Y_2), the marginal densities of Y_1 and Y_2, and $\mathrm{P}(X_3 < X_1 + X_2)$.

7. Let X and Y be continuous random variables and let $Z = aX + bY$ where $a, b \in \mathbb{R}$.
   (a) Prove that if the density f_Z is known for all $a, b \in \mathbb{R}$, then the joint density f_{XY} is univocally determined.
   (b) Prove that if the random variable Z is gaussian for all $a, b \in \mathbb{R}$, then X and Y are jointly gaussian.

8. Let Y_1 and Y_2 be jointly gaussian random variables such that $\mathrm{E}(Y_1) = 1$, $\mathrm{E}(Y_2) = -1$, $\mathrm{Var}(Y_1) = 4$, $\mathrm{Var}(Y_2) = 1$ and $\rho = 1/2$. Let N be also gaussian with $\mathrm{E}(N) = 0$, $\mathrm{Var}(N) = 2$ and independent of (Y_1, Y_2). If $X = Y_1 - Y_2 + N$, prove that (X, Y_1) is a gaussian vector and compute its parameters.

9. (a) Let X, Y be random variables with characteristic functions M_X and M_Y respectively, and joint characteristic function M_{XY}. A necessary and sufficient condition for X and Y to be independent is $M_{XY}(\omega_1, \omega_2) = M_X(\omega_1) M_Y(\omega_2)$. Prove the necessity of the condition.

(b) Let X and Y be independent gaussian $\mathsf{N}(0,1)$ random variables. Use characteristic functions to prove that $S = X + Y$ and $R = X - Y$ are independent. (Check that the above sufficient condition holds for S and R.)

10. The independent r.v. X_i $(i \geq 1)$ are all gaussian with mean 1 and variance 2. Find

$$\mathrm{P}(X_{n+1} > X_1 + X_2 + \cdots + X_n).$$

Express the result in terms of $F_{\mathsf{N}(0,1)}$.

11. Let X, Y, Z be jointly gaussian random variables such that $m_X = m_Y = 0$, $m_Z = -1$, $\sigma_X^2 = \sigma_Y^2 = \sigma_Z^2 = 1$. Moreover, we know that X and Y are uncorrelated but the correlation coefficient of X and Z is $1/2$ and the correlation coefficient of Y and Z is $3/4$. Calculate the variance of $X + 2Y - 3Z$.

12. Let X be a gaussian random variable with expectation vector $m = (1, -1, 2)^t$ and covariance matrix

$$K = \begin{pmatrix} 4 & 0 & -1 \\ 0 & 5 & 0 \\ -1 & 0 & 2 \end{pmatrix}$$

Which of the following random variables are independent?: X_1 and X_2; X_1 and X_3; X_2 and X_3; X_1 and $X_1 + 3X_2 - 2X_3$.

13. *(Midterm exam, November 2016)* Let $X = (X_1, X_2, X_3)^t$ have a three-dimensional normal distribution.

(a) Show that if X_1 and $X_2 + X_3$ are uncorrelated, X_2 and $X_1 + X_3$ are uncorrelated, and X_3 and $X_1 + X_2$ are uncorrelated, then X_1, X_2, and X_3 are independent.

(b) Give and example for which X_1 and $X_2 + X_3$ are uncorrelated, X_2 and $X_1 + X_3$ are uncorrelated, and X_3 and $X_1 - X_2$ are uncorrelated, but X_1, X_2, and X_3 are not independent.