Probability and Random Processes

Problems

Branching Processes. Distributions with random parameters.

1. *(Midterm exam November 2013)* Consider a branching process with one ancestor. Suppose that the probability function of the offspring distribution is

$$\mathrm{P}(X=0)=\frac{1}{4},\quad \mathrm{P}(X=1)=\frac{1}{2},\quad \mathrm{P}(X=2)=\frac{1}{8},\quad \mathrm{P}(X=3)=\frac{1}{8}.$$

What is the probability

(a) that the process is extinct in generation number 2 (i.e., the ancestor does not have any grandchildren)?

(b) of extinction?

Solution: The probability generating function of X is

$$G_X(s)=\sum_{k\geq 0}\mathrm{P}(X=k)s^k=\frac{1}{4}+\frac{1}{2}s+\frac{1}{8}s^2+\frac{1}{8}s^3$$

Thus

$$d_0=0,\quad d_1=G(0)=\frac{1}{4},\quad d_2=G\left(\frac{1}{4}\right)=\frac{197}{512}\approx 0,385.$$

To find the probability of ultimate extinction we have to solve the equation $G_X(s)=s$; that is, we have to find the roots of the polynomial

$$h(s)=G_X(s)-s=\frac{1}{4}-\frac{1}{2}s+\frac{1}{8}s^2+\frac{1}{8}s^3.$$

We know that $G_X(1)=1$. Hence

$$h(s)=(s-1)(as^2+bs+c).$$

So we have

$$G_X(s)=as^3+(b-a)s^2+(c-b)s-c.$$

By identifying the coefficients of the same degree we deduce $a=1/8$, $b-a=1/8$, $c-b=-1/2$, and $-c=1/4$. So we conclude that $a=1/8$, $b=1/4$, and $c=-1/4$. Therefore

$$h(s)=\frac{1}{8}\left(s-1\right)\left(s^2+2s-2\right)=\frac{1}{8}\left(s-1\right)\left(s+1+\sqrt{3}\right)\left(s+1-\sqrt{3}\right).$$

The roots of $h(s)$ are 1, $-1-\sqrt{3}$ and $-1+\sqrt{3}$, and the probability of extinction d is the smallest positive root. Then

$$d=-1+\sqrt{3}\approx 0,732.$$

2. *(Midterm exam October 2014)* At each unit of time a cell either dies with probability p or subdivides with probability $1-p$. Let X_n be the number of cells at time n.

(a) If $X_0=1$, what is the maximum value of p for which the probability of extinction is at most $1/4$.

(b) If $X_0=k$ and $p=1/4$, what is the minimum value of k such that the probability of extinction is at most $1/4$.

Solution:

(a) The sequence $X_0, X_1, \dots X_n \dots$ is a branching process with respect to Z, where

$$\mathrm{P}(Z = 0) = p, \quad \mathrm{P}(Z = 2) = 1 - p.$$

The probability generating function of Z is

$$G_Z(s) = p + (1 - p)s^2.$$

The equation

$$s = G_Z(s)$$

has solutions

$$s = 1 \text{ and } s = p/(1 - p).$$

Therefore the probability of extinction is at most $1/4$ if and only if $p/(1 - p) \leq 1/4$; if and only if $p \leq 1/5$.

(b) If α is the probability of extinction when $X_0 = 1$, then the probability of extinction when $X_0 = k$ is α^k, because it requires that all the independent branching processes started at each individual cell become extinct.

It has been shown that $\alpha = p/(1 - p)$. Thus we require that $(p/(1 - p))^k < 1/4$ or, with $p = 1/4$, that $3^k > 4$. Therefore $k \geq 2$.

3. *(Final exam January 2015)* Consider a branching process with one ancestor. The generating function of the offspring distribution is

$$G(s) = \frac{1}{2 - s^2}.$$

(a) What is the probability that the first generation has 4 individuals.

(b) Compute the probability of extinction.

(c) Compute the probability that the process is extinct in the third generation.

Solution:

(a) Let X denote the offspring random variable. By expanding $G(s) = G_X(s)$ in power series we have

$$G(s) = \frac{1}{2 - s^2} = \frac{1/2}{1 - s^2/2} = \frac{1}{2}\left(1 + \frac{s^2}{2} + \frac{s^4}{4} + \frac{s^6}{8} + \cdots\right),$$

where the coefficient of s^n is $\mathrm{P}(X = n)$. The probability distribution of the first generation coincides with the distribution of X. Therefore, the probability that the first generation has 4 individuals is the coefficient of s^4, that is $1/8$.

(b) The probability of extinction is the smallest positive root of the equation

$$G(s) = s.$$

In the present case, we are led to find the roots of the polynomial

$$h(s) = 1 - s(2 - s^2) = s^3 - 2s + 1.$$

We know that $G(1) = 1$. Therefore,

$$h(s) = (s - 1)(s^2 + s - 1) = (s - 1)\left(s - \frac{-1 + \sqrt{5}}{2}\right)\left(s - \frac{-1 - \sqrt{5}}{2}\right).$$

It follows that the probability of extinction is

$$\frac{-1 + \sqrt{5}}{2} \approx 0.618.$$

(c) Let d_k be the probability that the process is extinct in the k-th generation. We know that $d_k = G(d_{k-1})$. To compute d_3 we have

$$d_0 = 0, \quad d_1 = G(0) = \frac{1}{2}, \quad d_2 = G\left(\frac{1}{2}\right) = \frac{4}{7}, \quad d_3 = G\left(\frac{4}{7}\right) = \frac{49}{82} \approx 0.598.$$

4. *(A. Gut's book)* Consider a branching process with one ancestor. Suppose that the generating function of the offspring distribution is

$$G(s) = \frac{p^2}{(1-qs)^2},$$

where $0 < p = 1 - q < 1$. What is the probability

(a) of extinction?

(b) that the process is extinct in generation number 2 (i.e., the ancestor does not have any grandchildren)?

Answer:

(a) If $0 < p < 2/3$, then $d = \frac{1}{q} - \frac{1}{2} - \sqrt{\frac{1}{q} - \frac{3}{4}}$. If $p \geq 2/3$, then $d = 1$

(b) $d_2 = p^2/(1 - p^2 q)^2$

5. *(A. Gut's book)* The following model can be used to describe the number of women (mothers and daughters) in a given area. The number of mothers is a random variable $X \sim \mathsf{Po}(\lambda)$. Independently of the others, every mother gives birth to a $\mathsf{Po}(\mu)$-distributed number of daughters. Let Y be the total number of daughters and hence $Z = X + Y$ be the total number of women in the area.

(a) Find the generating function of Z.

(b) Compute $\mathrm{E}(Z)$ and $\mathrm{Var}(Z)$.

Answer:

(a) $G_Z(s) = \exp\left(\lambda\left(se^{\mu(s-1)} - 1\right)\right)$

(b) $\mathrm{E}(Z) = \lambda(1+\mu)$; $\mathrm{Var}(Z) = \lambda(1 + 3\mu + \mu^2)$

6.

7. Let M be an exponential random variable with parameter 1. We know that X given $\{M = m\}$ follows the Poisson distribution with parameter m. Compute the moment generating function of X and deduce that $\mathrm{P}(X = k) = (1/2)^{k+1}$, $k = 0, 1, 2, \ldots$

Answer: $\phi_X(t) = \phi_M(e^t - 1) = \dfrac{1}{2 - e^t}$; $\quad G_X(s) = \dfrac{1}{2-s}$; $\quad \mathrm{P}(X = k) = \dfrac{1}{2^{k+1}}$.

8. *(A. Gut's book)* Let X be the number of coin tosses until heads is obtained. Suppose that the probability of heads is unknown in the sense that we consider it to be a random variable $Y \sim \mathsf{U}(0, 1)$. Find the distribution of X.

Solution:

Answer: $\mathrm{P}(X = k) = \dfrac{1}{k(k+1)}$

9. *(Midterm exam November 2015)* At each time unit a virus replicates X copies of itself where $\Pr(X = k) = p^k$, $k \geq 1$, $0 < p \leq 1/2$. (So the probability of no replica is $1 - p/q$, $q = 1 - p$.) Assume that at time 0 there is a single originating virus.

(a) Prove that the probability generating function of X is $G_X(s) = 1/(1 - ps) - (p/q)$.

(b) Show that the probability of extinction is $1/p - 1/q$ if $p > (3 - \sqrt{5})/2$ and 1 otherwise.

(c) If $p = 1/4$, what is the expected total number of viruses at the end?

Solution:

(a) The sequence $Z_0, Z_1, \ldots, Z_n, \ldots$ is a branching process with respect to X, where

$$\Pr(X = k) = p^k, \; k \geq 1.$$

Observe that $\sum_{k \geq 1} p^k = p/(1-p) = p/q$, so that $\Pr(X = 0) = 1 - p/q$.
The generating function of X is

$$G_X(s) = 1 - \frac{p}{q} + \sum_{k \geq 1} p^k s^k = \sum_{k \geq 0} (ps)^k - \frac{p}{q} = \frac{1}{1 - ps} - \frac{p}{q}.$$

(b) The probability of extinction of the branching process is the smallest root of the equation

$$G_X(s) = s,$$

in the interval $[0, 1]$. This equation is

$$s^2 + \left(\frac{p}{q} - \frac{1}{p}\right) s + \left(\frac{1}{p} - \frac{1}{q}\right) = 0,$$

which, since $s = 1$ is always a root, it can be written as $(s-1)(s-a) = 0$ and, identifying coefficients, we deduce that the second root is

$$a = \frac{1}{p} - \frac{1}{q}.$$

We can check that $1/p - 1/q = 1/p - 1/(1-p) < 1$ is equivalent to $p^2 - 3p + 1 < 0$, that is, $p > p^\star = (3 - \sqrt{5})/2 \approx 0.382$. Thus the probability of extinction is 1 if $p \leq p^\star$ and $1/p - 1/q$ if $p^\star < p \leq 1/2$. (In particular, the probability of extinction is 0 if $p = 1/2$.)

(c) If $p = 1/4$, then, according to the previous result, the population stops growing with probability one. The expected number of individuals at time n is

$$\mathrm{E}(Z_n) = \mathrm{E}\left(\mathrm{E}(Z_n | Z_{n-1})\right) = \mathrm{E}(Z_{n-1}\mathrm{E}(X)) = \mathrm{E}(X)\mathrm{E}(Z_{n-1}) = \mathrm{E}(X)^n.$$

On the other hand

$$\mathrm{E}(X) = G'_X(1) = \frac{p}{(1-p)^2} = \frac{p}{q^2}.$$

Therefore, if U is the total population at the end, then

$$\mathrm{E}(U) = \sum_{n \geq 0} \mathrm{E}(Z_n) = \mathrm{E}(Z_0) + \sum_{n \geq 1} \left(\frac{p}{q^2}\right)^n = 1 + \sum_{n \geq 1} \left(\frac{p}{q^2}\right)^n = \frac{1}{1 - (p/q^2)}.$$

For $p = 1/4$ this gives $\mathrm{E}(U) = 9/5$.