Probability and Random Processes

Problems

Generating and characteristic functions

1. Let X and Y be independent discrete r.v. such that $\mathrm{P}(X=0)=1/2$, $\mathrm{P}(X=1)=1/4$, $\mathrm{P}(X=2)=1/8$, $\mathrm{P}(X=3)=1/8$, $\mathrm{P}(Y=1)=1/3$, and $\mathrm{P}(Y=3)=2/3$. Use generating functions to find the probability function of $X+Y$.

2. *(Midterm exam October 2014)* You choose a ball at random in a urn with three balls labeled $0, 1$ and 2. If the ball is 0 you are free. Otherwise, if the ball is $i \in \{1,2\}$, you have to pay i Euros, replace the ball and withdraw one again. Let X be the amount X of Euros that you pay before being free.

   (a) Show that, for $k \geq 2$, we have
   $$\mathrm{P}(X=k)=\frac{1}{3}\left(\mathrm{P}(X=k-1)+\mathrm{P}(X=k-2)\right).$$

   (b) Deduce from the above that the probability generating function $G_X(t)$ of X is
   $$G_X(t)=\frac{1}{3-t-t^2},$$

   (c) Compute the expectation of X.

3. *(Midterm exam October 2015)*

   (a) Let X be a $\mathsf{Bin}(n,p)$ random variable. Prove that the probability generating function of X, $G_X(s)$, is $(1-p+ps)^n$.

   (b) We toss n coins such that $\mathrm{P}(\text{heads})=p$ for each one. Those coins showing heads are tossed again. Let Y be the number of heads obtained in this second round of tosses. Find the probability generating function of Y.

4. *(A. Gut, first ed., III.2)* Let X be a non-negative, integer-valued random variable satisfying:
   $$\mathrm{P}(X=0)=\frac{2}{3}\,\mathrm{P}(X=1),$$
   $$\mathrm{P}(X=2n)=\frac{1}{2}\,\mathrm{P}(X=2n-1)=\frac{2}{3}\,\mathrm{P}(X=2n+1), \quad n\geq 1.$$
   Compute its probability generating function.

5. *(A. Gut, first ed., III.15)* The number of cars passing a road crossing during a day follows a Poisson distribution with pararmeter λ. The number of persons in each car is a $\mathsf{Poiss}(\alpha)$ random variable. Find the probability generating function of the total number of persons, N, passing the road crossing during a day. Find the mean and variance of N.

6. *(Final exam January 2016)* The number N of passengers in a flight is a Poisson random variable with expected value $\mathrm{E}(N)=300$. Each passenger carries a number of bags X independently of other passengers where $\Pr(X=0)=\Pr(X=1)=\Pr(X=2)=1/3$. Let M be the total number of bags in the plane.

   (a) Find the probability generating function of M. What is the probability that $M=0$?

   (b) Find the expected number $\mathrm{E}(M)$ of bags in the plane and its variance $\mathrm{Var}(M)$.

   (c) The air company managing the flight provides room for 400 bags in the plane. By using Chebyshev's inequality show that the probability that this is not enough is not greater than 0.05.

7. Let X and Y be independent discrete r.v. such that $\mathrm{P}(X=0)=1/2$, $P(X=1)=1/4$, $\mathrm{P}(X=2)=1/4$, $\mathrm{P}(Y=1)=1/4$, $\mathrm{P}(Y=3)=3/4$. Find the moment generating function of $X-Y$.

8. *(A. Gut, first ed., III.3)* Use the moment generating function to compute $\mathrm{E}(X^4)$, where $X\sim \mathsf{Bin}(n,p)$.

9. *(A. Gut, first ed., III.5)* The random variable X has the property that $\mathrm{E}(X^n)=3^n/(n+1)$, $n=1,2,\ldots$ Find the unique distribution of X having these moments.

10. *(A. Gut, first ed., III.7)* Use moment generating functions to prove that, if a random variable X has density function
$$f_X(x)=\frac{1}{2}e^{-|x|},\quad -\infty<x<\infty,$$
then X can be written as $X=Y-Z$, where Y and Z are independent, exponentially distributed random variables.

11. *(Midterm exam November 2013)* Let X and Y be independent random variables exponentially distributed with parameter a. Find the moment generating function of $Z=X-Y$ and calculate $\mathrm{E}(Z^n)$ for $n=1,2,\ldots$

12. A random variable X is called symmetric if X and $-X$ have the same probability distribution. Prove that X is symmetric if and only if the imaginary part of its characteristic function is 0.

13. (a) Calculate the characteristic function of an exponential r.v. with parameter λ.
    (b) Prove that
$$M_X(\omega)=\frac{e^{3i\omega-2\omega^2}}{1+i\omega}$$
    is a characteristic function and find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

14. Two balls are picked at random from an urn that contains five balls, two of which are white and the rest are black. Let X be the number of white balls in the selection.
    (a) Find the characteristic function of X and calculate $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.
    (b) Two additional balls are picked from another identical urn. Find the characteristic function of the total number Y of white balls in the two extractions.

15. A 2-dimensional symmetric random walk is a sequence of points $\{(X_n,Y_n):\ n\geq 0\}$ defined in the following way. If $(X_n,Y_n)=(x,y)$, then (X_{n+1},Y_{n+1}) is, with equal probability, one of the four points $(x+1,y)$, $(x-1,y)$, $(x,y+1)$, $(x,y-1)$. We assume that $(X_0,Y_0)=(0,0)$.
    (a) Prove that $E(X_n^2+Y_n^2)=n$.
    (b) Find the characteristic function of (X_n,Y_n).

16. Let X_1, X_2 and X_3 be independent r.v. uniform on $[-1,1]$. Find the characteristic function of $X=\sum_{i=1}^N X_i$ where N is a r.v. independent of X_1, X_2, X_3, such that $\mathrm{P}(N=1)=\mathrm{P}(N=2)=\mathrm{P}(N=3)=1/3$.

17. *(Midterm exam November 2016)* Let N and $X_1X_2,\ldots,X_n,\ldots$ be independent random variables, where $N\sim\mathsf{Poiss}(\lambda)$ and, for each X_i, $\mathrm{P}(X_i=1)=p$ and $\mathrm{P}(X_i=2)=1-p$, $0<p<1$. If $S=X_1+X_2+\cdots+X_N$, find the moment generating function of S and use it to compute the mean and variance of this random variable.

18. *(Final exam January 2017)* A collection has n different types of coupons, $n\geq 2$. A collector sequentially buys coupons, and each time a coupon is bought it can be of any of the types with equal probability.
    (a) Let X be the number of coupons bought by the collector until he has coupons of two different types. Prove that the probability generating function of X is
$$G_X(s)=\frac{(n-1)s^2}{n-s}.$$
    Give the interval of convergence of $G_X(s)$ and use this generating function to compute $\mathrm{E}(X)$. [*Hint*: Consider the probability law of $X-1$.]

(b) Let N be the total number of coupons bought by the collector until he has one coupon of each type. Prove that the probability generating function of N is

$$G_N(s) = \frac{(n-1)!\, s^n}{(n-s)(n-2s)\cdots(n-(n-1)s)}.$$

(c) For $n = 4$ find $\mathrm{P}(N = k)$.
[*Hint*: write $G_N(s)$ as $s^4\,(A/(4-s) + B/(4-2s) + C/(4-3s))$ and use the geometric series $1/(1-z) = \sum_{k=0}^{\infty} z^k$.]