Probability and Random Processes

Problems

Generating and characteristic functions

1. Let X and Y be independent discrete r.v. such that $\mathrm{P}(X=0)=1/2$, $\mathrm{P}(X=1)=1/4$, $\mathrm{P}(X=2)=1/8$, $\mathrm{P}(X=3)=1/8$, $\mathrm{P}(Y=1)=1/3$, and $\mathrm{P}(Y=3)=2/3$. Use generating functions to find the probability function of $X+Y$.

   Solution:

$$\begin{aligned} G_{X+Y}(s) &= G_X(s)G_Y(s) \\ &= \left(\frac{1}{2}+\frac{1}{4}s+\frac{1}{8}s^2+\frac{1}{8}s^3\right)\left(\frac{1}{3}s+\frac{2}{3}s^3\right) \\ &= \frac{1}{6}s+\frac{1}{12}s^2+\frac{3}{8}s^3+\frac{5}{24}s^4+\frac{1}{12}s^5+\frac{1}{12}s^6. \end{aligned}$$

   Hence we have

$$\mathrm{P}(Z=0)=0, \quad \mathrm{P}(Z=1)=\frac{1}{6}, \quad \mathrm{P}(Z=2)=\frac{1}{12}, \quad \mathrm{P}(Z=3)=\frac{3}{8},$$

$$\mathrm{P}(Z=4)=\frac{5}{24}, \quad \mathrm{P}(Z=5)=\frac{1}{12}, \quad \mathrm{P}(Z=6)=\frac{1}{12}.$$

2. *(Midterm October 2014)* You choose a ball at random in a urn with three balls labeled $0, 1$ and 2. If the ball is 0 you are free. Otherwise, if the ball is $i \in \{1,2\}$, you have to pay i Euros, replace the ball and withdraw one again. Let X be the amount X of Euros that you pay before being free.

   (a) Show that, for $k \geq 2$, we have

$$\mathrm{P}(X=k)=\frac{1}{3}\left(\mathrm{P}(X=k-1)+\mathrm{P}(X=k-2)\right).$$

   (b) Deduce from the above that the probability generating function $G_X(t)$ of X is

$$G_X(t)=\frac{1}{3-t-t^2},$$

   (c) Compute the expectation of X.

   Solution:

   (a) Let Y be the label of the ball in the first withdrawal. For each $k \geq 0$ we have

$$\mathrm{P}(X=k)=\mathrm{P}(X=k|Y=0)\mathrm{P}(Y=0)+\mathrm{P}(X=k|Y=1)\mathrm{P}(Y=1)+\mathrm{P}(X=k|Y=2)\mathrm{P}(Y=2).$$

   We have

$$\mathrm{P}(X=k|Y=0)=\begin{cases} 1 & k=0 \\ 0 & k>0 \end{cases},$$

   and

$$\mathrm{P}(X=k|Y=1)=\begin{cases} 0 & k=0 \\ \mathrm{P}(X=k-1) & k>0 \end{cases},$$

and

$$\mathrm{P}(X=k|Y=2)=\begin{cases} 0 & k\le 1 \\ \mathrm{P}(X=k-2) & k>1 \end{cases}.$$

Hence,

$$\mathrm{P}(X=k)=\begin{cases} 1/3 & k=0 \\ 1/9 & k=1 \\ (1/3)(\mathrm{P}(X=k-1)+\mathrm{P}(X=k-2)) & k\ge 2 \end{cases}$$

(b) The generating function $G_X(t)$ satisfies

$$\begin{aligned} G_X(t) &= \sum_{k\ge 0}\mathrm{P}(X=k)t^k \\ &= \frac{1}{3}+\frac{1}{9}t+\frac{1}{3}\sum_{k\ge 2}\left(\mathrm{P}(X=k-1)+\mathrm{P}(X=k-2)\right)t^k \\ &= \frac{1}{3}+\frac{1}{9}t+\frac{1}{3}\left(\sum_{k\ge 2}\mathrm{P}(X=k-1)t^k)+\sum_{k\ge 2}\mathrm{P}(X=k-2)t^k\right) \\ &= \frac{1}{3}+\frac{1}{9}t+\frac{1}{3}\left(t(G_X(t)-1/3)+t^2G_X(t)\right) \\ &= \frac{1}{3}+\frac{1}{3}G_X(t)(t+t^2), \end{aligned}$$

which implies

$$G_X(t)=\frac{1}{3-t-t^2}.$$

(In the above equalities we have taken into account that

$$\sum_{k\ge 2}\mathrm{P}(X=k-1)t^k=t\sum_{k\ge 2}\mathrm{P}(X=k-1)t^{k-1}=t\sum_{k\ge 1}\mathrm{P}(X=k)t^k=t\left(G_X(t)-\frac{1}{3}\right),$$

and that

$$\sum_{k\ge 2}\mathrm{P}(X=k-2)t^k=t^2\sum_{k\ge 2}\mathrm{P}(X=k-2)t^{k-2}=t^2\sum_{k\ge 0}\mathrm{P}(X=k)t^k=t^2G_X(t).$$

)

(c) We have

$$\mathrm{E}(X)=G_X'(1)=\left.\frac{2t+1}{(3-t-t^2)^2}\right|_{t=1}=3.$$

(d) We can expand $G_X(t)$ in power series to find explicit expressions for $\Pr(X=k)$. The first terms of the power series expansion of $G_X(t)$ are

$$G_X(t)=\frac{1}{3-t-t^2}=\frac{1}{3}+\frac{1}{9}t+\frac{4}{27}t^2+\frac{7}{81}t^3+\frac{19}{243}t^4+\cdots$$

To find explicit formulas for the coefficients of the expansion we can decompose $G_X(t)$ into partial fractions. We have the factorization

$$3-t-t^2=(t-\phi)(\hat{\phi}-t),$$

with $\phi=(-1+\sqrt{13})/2$ and $\hat{\phi}=(-1-\sqrt{13})/2$.

Thus

$$G_X(t)=\frac{1}{(t-\phi)(\hat{\phi}-t)}=\frac{\alpha}{t-\phi}+\frac{\beta}{\hat{\phi}-t}.$$

We deduce that

$$(\beta - \alpha)t + (\alpha\hat{\phi} - \beta\phi) \equiv 1,$$

and hence

$$\alpha = \beta = \frac{1}{\hat{\phi} - \phi}.$$

By using the decomoposition into partial fractions we get

$$\begin{aligned} G_X(t) &= \frac{1}{\hat{\phi} - \phi}\left(\frac{1}{t - \phi} + \frac{1}{\hat{\phi} - t}\right) \\ &= \frac{1}{\hat{\phi} - \phi}\left(\frac{-1/\phi}{1 - (t/\phi)} + \frac{1/\hat{\phi}}{1 - (t/\hat{\phi})}\right) \\ &= \frac{1}{\hat{\phi} - \phi}\left(-\frac{1}{\phi}\sum_{n\geq 0}(t/\phi)^n + \frac{1}{\hat{\phi}}\sum_{n\geq 0}(t/\hat{\phi})^n\right) \\ &= \frac{1}{\hat{\phi} - \phi}\sum_{n\geq 0}\left((1/\hat{\phi})^{n+1} - (1/\phi)^{n+1}\right)t^n. \end{aligned}$$

Since $|\phi| \approx 1.3028$ and $|\hat{\phi}| \approx 2.3028$, the above expansion of $G_X(t)$ is valid for $|t| < |\phi|$. Moreover, the coefficent of t^n gives us the probability $\mathrm{P}(X = n)$. Therefore the exact expression of this probability is given for any $n \geq 0$ by the formula

$$\mathrm{P}(X = n) = \frac{1}{\hat{\phi} - \phi}\left(\left(\frac{1}{\hat{\phi}}\right)^{n+1} - \left(\frac{1}{\phi}\right)^{n+1}\right) = \frac{1}{\sqrt{13}}\left(\left(\frac{2}{-1 + \sqrt{13}}\right)^{n+1} - \left(\frac{-2}{1 + \sqrt{13}}\right)^{n+1}\right).$$

3. *(Midterm exam October 2015)*

   (a) Let X be a $\mathsf{Bin}(n, p)$ random variable. Prove that the probability generating function of X, $G_X(s)$, is $(1 - p + ps)^n$.

   (b) We toss n coins such that $\mathrm{P}(\text{heads}) = p$ for each one. Those coins showing heads are tossed again. Let Y be the number of heads obtained in this second round of tosses. Find the probability generating function of Y.

Solution:

(a) Since $\mathrm{P}(X = j) = \binom{n}{j}p^j(1 - p)^{n-j}$, $0 \leq j \leq n$, we have

$$G_X(s) = \mathrm{E}\left(s^X\right) = \sum_{j=0}^{n} s^j\,\mathrm{P}(X = j) = \sum_{j=0}^{n} s^j\binom{n}{j}p^j(1 - p)^{n-j}$$

$$= \sum_{j=0}^{n}\binom{n}{j}(ps)^j(1 - p)^{n-j} = (1 - p + ps)^n.$$

We can also obtain $G_X(s)$ by noticing that $X = X_1 + \cdots + X_n$, where the variables $X_i \sim \mathsf{B}(p)$ are independent. We have $G_{X_i}(s) = (1 - p) + ps$, $1 \leq i \leq n$, and thus, by applying the convolution theorem, we obtain

$$G_X(s) = G_{X_1}(s)\cdots G_{X_n}(s) = ((1 - p) + ps)^n.$$

(b) Let X be the number of heads in the first round of tosses. We know that $X \sim \mathsf{Bin}(n, p)$. Moreover, given the event $\{X = k\}$, the random variable Y follows a $\mathsf{Bin}(k, p)$ distribution. Therefore

$$\mathrm{E}(s^Y \mid X = k) = (1 - p + ps)^k.$$

Then,

$$G_Y(s) = \mathrm{E}\left(s^Y\right) = \sum_{k=0}^{n} \mathrm{E}\left(s^Y \mid X = k\right) \mathrm{P}(X = k) = \sum_{k=0}^{n} (1 - p + ps)^k \binom{n}{k} p^k (1-p)^{n-k}$$
$$= \sum_{k=0}^{n} \binom{n}{k} (p - p^2 + p^2 s)^k (1-p)^{n-k} = (p - p^2 + p^2 s + 1 - p)^n = (1 - p^2 + p^2 s)^n.$$

Thus, $Y \sim \mathsf{Bin}(n, p^2)$.

An equivalent way to obtain $G_Y(s)$ is as follows:

$$G_Y(s) = \mathrm{E}\left(s^Y\right) = \mathrm{E}\left(\mathrm{E}(s^Y|X)\right) = \mathrm{E}\left((1-p+ps)^X\right) = G_X(1-p+ps)$$
$$= (1-p+ps)^n|_{s=1-p+ps} = (1-p+p(1-p+ps))^n = (1-p^2+p^2s)^n.$$

4. Let X be a non-negative, integer-valued random variable satisfying:

$$\mathrm{P}(X=0) = \frac{2}{3}\,\mathrm{P}(X=1),$$
$$\mathrm{P}(X=2n) = \frac{1}{2}\,\mathrm{P}(X=2n-1) = \frac{2}{3}\,\mathrm{P}(X=2n+1), \quad n \geq 1.$$

Compute its probability generating function.

Answer: $G_X(s) = \dfrac{2+3s}{5(4-3s^2)}, \quad |s| < \dfrac{2}{\sqrt{3}}$

5. The number of cars passing a road crossing during a day follows a Poisson distribution with parameter λ. The number of persons in each car is a $\mathsf{Poiss}(\alpha)$ random variable. Find the probability generating function of the total number of persons, N, passing the road crossing during a day. Find the mean and variance of N.

Answer: If X is the number of cars and Y the number of persons in each car, then $G_N(s) = G_X(G_Y(s)) = \exp\left(\lambda(e^{\alpha(s-1)} - 1)\right)$. Then $\mathrm{E}(N) = \lambda\alpha$ and $\mathrm{Var}(N) = \lambda\alpha(\alpha+1)$.

6. *(Final exam January 2016)* The number N of passengers in a flight is a Poisson random variable with expected value $\mathrm{E}(N) = 300$. Each passenger carries a number of bags X independently of other passengers where $\Pr(X=0) = \Pr(X=1) = \Pr(X=2) = 1/3$. Let M be the total number of bags in the plane.

(a) Find the probability generating function of M. What is the probability that $M = 0$?

(b) Find the expected number $\mathrm{E}(M)$ of bags in the plane and its variance $\mathrm{Var}(M)$.

(c) The air company managing the flight provides room for 400 bags in the plane. By using Chebyshev's inequality show that the probability that this is not enough is not greater than 0.05.

Solution:

(a) We have $M = X_1 + \cdots + X_N$, so that

$$G_M(s) = \mathrm{E}(s^M) = \mathrm{E}(\mathrm{E}(s^M|N)) = \mathrm{E}\left(\left(\mathrm{E}(s^X)\right)^N\right) = \mathrm{E}\left((G_X(s))^N\right) = G_N(G_X(s)).$$

In the above calculation we have $\mathrm{E}(s^M|N) = \mathrm{E}(s^X)^N = (G_X(s))^N$, because

$$\mathrm{E}(s^M|N=n) = \mathrm{E}(s^{X_1+\cdots+X_n}|N=n) = \mathrm{E}(s^{X_1+\cdots+X_n})$$
$$= \mathrm{E}(s^{X_1})\cdots\mathrm{E}(s^{X_n}) = \left(\mathrm{E}(s^X)\right)^n = (G_X(s))^n.$$

From $G_X(s) = \frac{1}{3}(1+s+s^2)$ and $G_N(s) = \sum_{k\geq 0} \frac{\lambda^k e^{-\lambda}}{k!} s^k = e^{\lambda(s-1)}$, where $\lambda = 300$, we get

$$G_M(s) = e^{\lambda((1+s+s^2)/3-1)} = e^{\lambda(-2+s+s^2)/3}.$$

In particular, $\mathrm{P}(M=0) = G_M(0) = e^{-2\lambda/3}$.

(b) The expected value can be computed as

$$\mathrm{E}(M) = G_M'(1) = \lambda \frac{1+2s}{3} e^{\lambda(-2+s+s^2)/3)} \mid_{s=1} = \lambda = 300.$$

In order to compute the variance one can use:

$$\mathrm{E}(M(M-1)) = G_M''(1) = \left(\frac{2\lambda}{3} + \frac{\lambda^2}{9}(1+2s)^2\right) e^{\lambda(-2+s+s^2)/3)} \mid_{s=1} = \frac{2\lambda}{3} + \lambda^2.$$

Therefore, $\mathrm{E}(M^2) = 5\lambda/3 + \lambda^2$ and

$$\mathrm{Var}(M) = \mathrm{E}(M^2) - (\mathrm{E}(M))^2 = 5\lambda/3 = 500.$$

(c) According to the above computations,

$$\mathrm{P}(M > 400) \leq \mathrm{P}(M - 300 \geq 100) \leq \Pr(|M - \mathrm{E}(M)| \geq 100) \leq \frac{500}{100^2} = 0.05.$$

7. Let X and Y be independent discrete r.v. such that $\mathrm{P}(X=0) = 1/2$, $P(X=1) = 1/4$, $\mathrm{P}(X=2) = 1/4$, $\mathrm{P}(Y=1) = 1/4$, $\mathrm{P}(Y=3) = 3/4$. Find the moment generating function of $X - Y$.

   Answer:
   $\phi_{X-Y}(t) = \dfrac{1}{16} + \dfrac{3e^{-3t}}{8} + \dfrac{3e^{-2t}}{16} + \dfrac{5e^{-t}}{16} + \dfrac{e^t}{16}$

8. Use the moment generating function to compute $\mathrm{E}(X^4)$, where $X \sim \mathsf{Bin}(n,p)$.

   Answer: $np + 14\binom{n}{2}p^2 + 36\binom{n}{3}p^3 + 24\binom{n}{4}p^4$

9. The random variable X has the property that $\mathrm{E}(X^n) = 3^n/(n+1)$, $n = 1, 2, \ldots$ Find the unique distribution of X having these moments.

   Solution: We have

$$\phi_X(t) = \sum_{n\geq 0} \frac{\mathrm{E}(X^n)}{n!} t^n = \sum_{n\geq 0} \frac{3^n}{(n+1)n!} t^n = \frac{1}{3t} \sum_{n\geq 0} \frac{(3t)^{n+1}}{(n+1)!} = \frac{1}{3t} \sum_{n\geq 1} \frac{(3t)^n}{n!} = \frac{e^{3t}-1}{3t}, \quad t \neq 0.$$

   For $t = 0$ we have $\phi_X(0) = \mathrm{E}(1) = 1$. Hence $\phi_X(t)$ exists for any $t \in \mathbb{R}$.

   This moment generating function corresponds to a random variable X uniformly distributed in $[0,3]$. Indeed, if $X \sim \mathsf{Unif}[0,3]$, then

$$\phi_X(t) = \mathrm{E}\left(e^{tX}\right) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx = \frac{1}{3}\int_0^3 e^{tx}\, dx = \frac{e^{3t}-1}{3t}.$$

10. Use moment generating functions to prove that, if a random variable X has density function

$$f_X(x) = \frac{1}{2}e^{-|x|}, \quad -\infty < x < \infty,$$

   then X can be written as $X = Y - Z$, where Y and Z are independent, exponentially distributed random variables.

11. Let X and Y be independent random variables exponentially distributed with parameter a. Find the moment generating function of $Z = X - Y$ and calculate $\mathrm{E}(Z^n)$ for $n = 1, 2, \ldots$

   Solution: The common moment generating function of X and Y is $\phi_X(t) = \phi_Y(t) = a/(a-t)$ for $t < a$. On the other hand,

$$\phi_{-Y}(t) = \mathrm{E}\left(e^{t(-Y)}\right) = \mathrm{E}\left(e^{(-t)Y}\right) = \phi_Y(-t) = \frac{a}{a+t}, \quad t > -a.$$

Since X and Y are independent we have, by the convolution theorem,

$$\phi_Z(t) = \phi_X(t)\phi_{-Y}(t) = \frac{a^2}{a^2 - t^2}, \quad |t| < a.$$

Consider the power series expansion of $\phi_Z(t)$,

$$\phi_Z(t) = \frac{a^2}{a^2 - t^2} = \frac{1}{1 - (t/a)^2} = 1 + \frac{t^2}{a^2} + \frac{t^4}{a^4} + \frac{t^6}{a^6} + \cdots,$$

where we have taken into account that $1/(1+z) = 1 + z + z^2 + z^3 + \cdots$, $|z| < 1$. The coefficient of t^n in the above expansion is $\mathrm{E}(Z^n)/n!$. Hence, we deduce

$$\mathrm{E}(Z^n) = \begin{cases} \dfrac{n!}{a^n}, & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

12. A random variable X is called symmetric if X and $-X$ have the same probability distribution. Prove that X is symmetric if and only if the imaginary part of its characteristic function is 0.

13. (a) Calculate the characteristic function of an exponential r.v. with parameter λ.

    (b) Prove that
    $$M_X(\omega) = \frac{e^{3i\omega - 2\omega^2}}{1 + i\omega}$$
    is a characteristic function and find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

Answer: $\mathrm{E}(X) = 2; \quad \mathrm{Var}(X) = 5$

14. Two balls are picked at random from an urn that contains five balls, two of which are white and the rest are black. Let X be the number of white balls in the selection.

    (a) Find the characteristic function of X and calculate $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

    (b) Two additional balls are picked from another identical urn. Find the characteristic function of the total number Y of white balls in the two extractions.

Answer: $M_X(\omega) = \frac{1}{10}\left(3 + 6e^{i\omega} + e^{i2\omega}\right); \quad M_Y(\omega) = \frac{1}{100}\left(3 + 6e^{i\omega} + e^{i2\omega}\right)^2$

15. A 2-dimensional symmetric random walk is a sequence of points $\{(X_n, Y_n) : n \geq 0\}$ defined in the following way. If $(X_n, Y_n) = (x, y)$, then (X_{n+1}, Y_{n+1}) is, with equal probability, one of the four points $(x+1, y)$, $(x-1, y)$, $(x, y+1)$, $(x, y-1)$. We assume that $(X_0, Y_0) = (0, 0)$.

    (a) Prove that $E(X_n^2 + Y_n^2) = n$.

    (b) Find the characteristic function of (X_n, Y_n).

Solution:

(a) The total horizontal displacement X_n can be expressed as $X_n = \sum_{1=1}^n H_i$, where H_i denotes the i-th horizontal step. The random variables H_i, $1 \leq i \leq n$, are independent, identically distributed, and such that $\mathrm{P}(H_i = -1) = 1/4$, $\mathrm{P}(H_i = 0) = 1/2$, and $\mathrm{P}(H_i = 1) = 1/4$. Analogously, the total vertical displacement Y_n is $Y_n = \sum_{1=1}^n V_i$, where V_i is the i-th vertical step. The variables V_i are also independent and distributed as H_i. Moreover, if $i \neq j$, then H_i and V_j are independent random variables. (Observe, however, that for a given step i, the variables H_i and V_i are not independent.)

We have $\mathrm{E}(H_i) = 0$, $\mathrm{E}(H_i^2) = 1/2$, and $\mathrm{E}(H_i H_j) = \mathrm{E}(H_i)\mathrm{E}(H_j) = 0$ if $i \neq j$. Therefore

$$\mathrm{E}(X_n^2) = \mathrm{E}\left(\left(\sum_{i_1}^n H_i\right)^2\right) = \mathrm{E}\left(\sum_{i=1}^n H_i^2 + 2\sum_{i<j} H_i H_j\right) = \sum_{i=1}^n \mathrm{E}\left(H_i^2\right) = \frac{n}{2}.$$

Analogously, $\mathrm{E}(Y_n^2) = n/2$. Hence $E(X_n^2 + Y_n^2) = E(X_n^2) + \mathrm{E}(Y_n^2) = n$, as we wanted to prove.

(b) We can write $(X_n, Y_n) = \sum_{i=1}^{n}(H_i, V_i)$, where the 2-dimensional random variables (H_i, V_i), $1 \le i \le n$, are independent and identically distributed. Therefore the joint characteristic function $M_{X_nY_n}(\omega_1, \omega_2)$ of X_n and Y_n can be calculated as

$$\begin{aligned}
M_{X_nY_n}(\omega_1,\omega_2) &= \mathrm{E}\left(e^{i(\omega_1 X_n + \omega_2 Y_n)}\right) = \mathrm{E}\left(e^{\sum_{i=1}^{n} i(\omega_1 H_i + \omega_2 V_i)}\right) \\
&= \mathrm{E}\left(e^{i(\omega_1 H_1 + \omega_2 V_1)} e^{i(\omega_1 H_2 + \omega_2 V_2)} \cdots e^{i(\omega_1 H_n + \omega_2 V_n)}\right) \\
&= \mathrm{E}\left(e^{i(\omega_1 H_1 + \omega_2 V_1)}\right) \mathrm{E}\left(e^{i(\omega_1 H_2 + \omega_2 V_2)}\right) \cdots \mathrm{E}\left(e^{i(\omega_1 H_n + \omega_2 V_n)}\right) = \left(M_{H_1V_1}(\omega_1,\omega_2)\right)^n.
\end{aligned}$$

Moreover, since (H_1, V_1) takes the values $(1, 0)$, $(-1, 0)$, $(0, 1)$, and $(0, -1)$ with probability $1/4$ each one, we have

$$\begin{aligned}
M_{H_1V_1}(\omega_1,\omega_2) &= \mathrm{E}\left(e^{i(\omega_1 H_1 + \omega_2 V_1)}\right) = \frac{1}{4}\left(e^{i\omega_1} + e^{-i\omega_1} + e^{i\omega_2} + e^{-i\omega_2}\right) \\
&= \frac{1}{2}\left(\frac{e^{i\omega_1} + e^{-i\omega_1}}{2} + \frac{e^{i\omega_2} + e^{-i\omega_2}}{2}\right) = \frac{1}{2}\left(\cos\omega_1 + \cos\omega_2\right).
\end{aligned}$$

Then,

$$M_{X_nY_n}(\omega_1,\omega_2) = \frac{1}{2^n}\left(\cos\omega_1 + \cos\omega_2\right)^n.$$

16. Let X_1, X_2 and X_3 be independent r.v. uniform on $[-1, 1]$. Find the characteristic function of $X = \sum_{i=1}^{N} X_i$ where N is a r.v. independent of X_1, X_2, X_3, such that $\mathrm{P}(N = 1) = \mathrm{P}(N = 2) = \mathrm{P}(N = 3) = 1/3$.

    Solution:

    First let us find the characteristic function of a random variable U uniformly distributed in $[-1, 1]$. If $U \sim \mathsf{Unif}[0, 1]$, then

$$\begin{aligned}
M_U(\omega) &= \mathrm{E}\left(e^{i\omega U}\right) = \int_{-\infty}^{\infty} e^{i\omega x} f_U(x)\, dx = \frac{1}{2}\int_{-1}^{1} e^{i\omega x}\, dx = \frac{1}{2}\int_{-1}^{1} \left(\cos(\omega x) + i\sin(\omega x)\right) dx \\
&= \frac{1}{2}\int_{-1}^{1} \cos(\omega x) dx + i\,\frac{1}{2}\int_{-1}^{1} \sin(\omega x) dx = \int_{0}^{1} \cos(\omega x) dx = \frac{\sin\omega}{\omega}, \quad \omega \neq 0.
\end{aligned}$$

    For $\omega = 0$ we get $M_U(0) = \mathrm{E}\left(e^{i\cdot 0\cdot U}\right) = \mathrm{E}(1) = 1$.

    Now, the characteristic function of $X = \sum_{i=1}^{N} X_i$ can be calculated as

$$\begin{aligned}
M_X(\omega) &= \mathrm{E}\left(e^{i\omega X}\right) = \frac{1}{3}\left(\mathrm{E}\left(e^{i\omega X}|N = 1\right) + \mathrm{E}\left(e^{i\omega X}|N = 2\right) + \mathrm{E}\left(e^{i\omega X}|N = 3\right)\right) \\
&= \frac{1}{3}\left(\mathrm{E}\left(e^{i\omega X_1}\right) + \mathrm{E}\left(e^{i\omega(X_1+X_2)}\right) + \mathrm{E}\left(e^{i\omega(X_1+X_2+X_3)}\right)\right) \\
&= \frac{1}{3}\left(M_U(\omega) + \left(M_U(\omega)\right)^2 + \left(M_U(\omega)\right)^3\right) \\
&= \frac{1}{3}\left(\frac{\sin\omega}{\omega} + \left(\frac{\sin\omega}{\omega}\right)^2 + \left(\frac{\sin\omega}{\omega}\right)^3\right),
\end{aligned}$$

    where in the above calculations we have taken into account that N is independent of the random variables X_i, that $X_i \sim \mathsf{Unif}[0, 1]$, and we have applied the convolution theorem.

17. *(Midterm exam November 2016)* Let N and $X_1X_2, \ldots, X_n, \ldots$ be independent random variables, where $N \sim \mathsf{Poiss}(\lambda)$ and, for each X_i, $\mathrm{P}(X_i = 1) = p$ and $\mathrm{P}(X_i = 2) = 1-p$, $0 < p < 1$. If $S = X_1+X_2+\cdots+X_N$, find the moment generating function of S and use it to compute the mean and variance of this random variable.

Solution: The moment generating function of S can be computed as

$$\phi_S(t) = \mathrm{E}\left(e^{tS}\right) = \sum_{k\geq 0} \mathrm{E}\left(e^{tS} \,|\, N = k\right) \mathrm{P}(N = k),$$

where

$$\begin{aligned}\mathrm{E}\left(e^{tS} \,|\, N = k\right) &= \mathrm{E}\left(e^{t(X_1+\cdots+X_N)} \,|\, N = k\right) \\ &= \mathrm{E}\left(e^{t(X_1+\cdots+X_k)}\right) = \mathrm{E}\left(e^{tX_1}\right)\cdots \mathrm{E}\left(e^{tX_k}\right) = \left(pe^t + qe^{2t}\right)^k,\end{aligned}$$

because N is independent of $X_1 + \cdots + X_k$, $\mathrm{E}\left(e^{tX_1}\right) = \cdots = \mathrm{E}\left(e^{tX_k}\right)$, and

$$\mathrm{E}\left(e^{tX_1}\right) = e^t\,\mathrm{P}(X_i = 1) + e^{2t}\,\mathrm{P}(X_i = 2) = pe^t + qe^{2t},$$

where $q = 1 - p$. Therefore,

$$\phi_S(t) = \sum_{k\geq 0} \left(pe^t + qe^{2t}\right)^k \mathrm{P}(N = k) = G_N\left(pe^t + qe^{2t}\right) = \exp\left(\lambda\left(pe^t + qe^{2t} - 1\right)\right).$$

The mean of S is given by $\phi'_S(0)$. Therefore,

$$\mathrm{E}(S) = \phi'_S(0) = \lambda\left(pe^t + 2qe^{2t}\right)\exp\left(\lambda\left(pe^t + qe^{2t} - 1\right)\right)\Big|_{t=0} = \lambda(1+q).$$

Moreover,

$$\phi''_S(t) = \lambda^2\left(pe^t + 2qe^{2t}\right)^2 \exp\left(\lambda\left(pe^t + qe^{2t} - 1\right)\right) + \lambda\left(pe^t + 4qe^{2t}\right)\exp\left(\lambda\left(pe^t + qe^{2t} - 1\right)\right).$$

Hence,

$$\mathrm{E}\left(S^2\right) = \phi''_S(0) = \lambda^2(1+q)^2 + \lambda(1+3q),$$

and

$$\mathrm{Var}(S) = \lambda(1+3q).$$

18. *(Final exam January 2017)* A collection has n different types of coupons, $n \geq 2$. A collector sequentially buys coupons, and each time a coupon is bought it can be of any of the types with equal probability.

    (a) Let X be the number of coupons bought by the collector until he has coupons of two different types. Prove that the probability generating function of X is

    $$G_X(s) = \frac{(n-1)s^2}{n-s}.$$

    Give the interval of convergence of $G_X(s)$ and use this generating function to compute $\mathrm{E}(X)$. [*Hint*: Consider the probability law of $X - 1$.]

    (b) Let N be the total number of coupons bought by the collector until he has one coupon of each type. Prove that the probability generating function of N is

    $$G_N(s) = \frac{(n-1)!\,s^n}{(n-s)(n-2s)\cdots(n-(n-1)s)}.$$

    (c) For $n = 4$ find $\mathrm{P}(N = k)$. [*Hint*: write $G_N(s)$ as $s^4\left(A/(4-s) + B/(4-2s) + C/(4-3s)\right)$ and use the geometric series $1/(1-z) = \sum_{k=0}^{\infty} z^k$.]

Solution:

(a) The number $R = X - 1$ of coupons bought after the first and until one of a different type is obtained follows a geometric distribution with parameter $p = (n-1)/n$. Hence, $X = 1 + R$ and

$$\begin{aligned} G_X(s) &= \mathrm{E}\left(s^X\right) = \mathrm{E}\left(s^{1+R}\right) = s\,\mathrm{E}\left(s^R\right) = s\,G_R(s) \\ &= s \cdot \frac{ps}{1-(1-p)s} = s \cdot \frac{(n-1)s/n}{1-s/n} = \frac{(n-1)s^2}{n-s}, \quad |s| < \frac{1}{1-p} = n. \end{aligned}$$

The expectation of X is

$$\mathrm{E}(X) = G_X'(1) = \left.\frac{2(n-1)s(n-s) + (n-1)s^2}{(n-s)^2}\right|_{s=1} = 1 + \frac{n}{n-1}.$$

Notice that $\mathrm{E}(X) = 1 + \mathrm{E}(R)$ as it should be.

(b) Now $N = 1 + R_2 + \cdots + R_n$ where $R_i \sim \mathsf{Ge}((n-i)/n)$ is the number of coupons bought after the collector has just obtained $i-1$ different types of coupons and until he gets one of a new type. The random variables R_i, $2 \le i \le n$, are independent. Then,

$$\begin{aligned} G_N(s) &= \mathrm{E}\left(s^N\right) = \mathrm{E}\left(s^{1+R_2+R_3+\cdots+R_n}\right) = s\,\mathrm{E}\left(s^{R_2}\right)\mathrm{E}\left(s^{R_3}\right)\cdots\mathrm{E}\left(s^{R_n}\right) \\ &= s\,G_{R_2}(s)G_{R_3}(s)\cdots G_{R_n}(s) = s \cdot \frac{(n-1)s/n}{1-s/n} \cdot \frac{(n-2)s/n}{1-2s/n} \cdots \frac{s/n}{1-(n-1)s/n} \\ &= \frac{(n-1)!\,s^n}{(n-s)(n-2s)\cdots(n-(n-1)s)}, \qquad |s| < \frac{n}{n-1}. \end{aligned}$$

(c) For $n = 4$ the probability generating function is

$$G_N(s) = \frac{6s^4}{(4-s)(4-2s)(4-3s)}. \tag{1}$$

Decomposing the above expression into partial fractions $G_N(s)$ can be expressed as

$$G_N(s) = s^4\left(\frac{A}{4-s} + \frac{B}{4-2s} + \frac{C}{4-3s}\right). \tag{2}$$

Equating (1) and (2) we have the identity:

$$6 = A(4-2s)(4-3s) + B(4-s)(4-3s) + C(4-s)(4-2s).$$

For $s = 4$ we have $6 = A \cdot (-4)(-8)$, that is $A = 3/16$. For $s = 2$ we have $6 = B \cdot (2)(-2)$, that is $B = -3/2$. Finally, for $s = 4/3$ we obtain $6 = C \cdot (8/3)(4/3)$ which imply $C = 27/16$. Therefore,

$$G_N(s) = s^4\left(\frac{3/64}{1-s/4} - \frac{3/8}{1-s/2} + \frac{27/64}{1-3s/4}\right). \tag{3}$$

Expanding each term of (3) into powers of s using the geometric series $1/(1-z) = \sum_{k=0}^{\infty} z^k$ and identifying the coefficient of s^k we deduce that

$$\mathrm{P}(X = k) = \frac{3}{64}\left(\frac{1}{4}\right)^{k-4} - \frac{3}{8}\left(\frac{1}{2}\right)^{k-4} + \frac{27}{64}\left(\frac{3}{4}\right)^{k-4}, \quad k \ge 4.$$

For instance, $\mathrm{P}(X = 4) = 3/64 - 3/8 + 27/64 = 3/32$ which is the probability $(3/4)(2/4)(1/4)$ that the first four coupons are all of different types.