Sum of a random number of independent random variables

February 2, 2018

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Let $X_1, X_2, \ldots X_n, \ldots$, and N be jointly independent random variables such that

- $X_1, X_2, \ldots X_n, \ldots$ are identically distributed.
- N is discrete taking the values $0, 1, 2, 3, \ldots$.

Let us consider the sum (with a random number of terms)

$$S = X_1 + X_2 + \cdots + X_N$$

(We take $S = 0$ if $\{N = 0\}$ happens.)

Sum of a random number of independent random variables

The characteristic function of S can be obtained as follows:

$$M_S(\omega) = \mathrm{E}\left(e^{i\omega S}\right) = \sum_{k \geq 0} \mathrm{E}\left(e^{i\omega S} \mid N = k\right) \mathrm{P}(N = k)$$

We have $E\left(e^{i\omega S} \mid N = 0\right) = E\left(e^{i\omega 0}\right) = 1$. Moreover, for $k \geq 1$,

$$\begin{aligned} E\left(e^{i\omega S} \mid N = k\right) &= E\left(e^{i\omega(X_1 + X_2 + \cdots + X_N)} \mid N = k\right) \\ &= E\left(e^{i\omega(X_1 + X_2 + \cdots + X_k)} \mid N = k\right) = E\left(e^{i\omega(X_1 + X_2 + \cdots + X_k)}\right) \\ &= \mathrm{E}\left(e^{i\omega X_1}\right) \mathrm{E}\left(e^{i\omega X_2}\right) \cdots \mathrm{E}\left(e^{i\omega X_k}\right) = (M_X(\omega))^k \end{aligned}$$

Sum of a random number of independent random variables

Therefore, if

$$G_N(z) = \sum_{k \geq 0} z^k \, \mathrm{P}(N = k)$$

is the probability generating function of N, then

$$M_S(\omega) = \sum_{k \geq 0} (M_X(\omega))^k \; \mathrm{P}(N = k) = G_N\left(M_X(\omega)\right)$$

- Notice that the composition $G_N\left(M_X(\omega)\right)$ is well-defined, because $|M_X(\omega)| \leq 1$ for all $\omega \in \mathbb{R}$ and $G_N(z)$ converges for all $z \in \mathbb{C}$ such that $|z| \leq 1$.

Example 1

- Suppose that the number N of costumers that arrive at a service point is a Poiss(λ) random variable.
- Let p be the probability that a customer who arrives at the service point will be actually served.

Let us calculate the probability distribution of the number S of served customers.

Example 1

We have

$$S = X_1 + X_2 + \cdots + X_N,$$

where X_i is the indicator of the event "the i-th arrival has been served"

Moreover,

$$M_{X_i}(\omega) = M_X(\omega) = pe^{i\omega} + q, \quad \omega \in \mathbb{R}$$

$$G_N(z) = e^{\lambda(z-1)}, \quad z \in \mathbb{C},$$

Example 1

Therefore

$$M_S(\omega) = G_N\left(M_X(\omega)\right) = e^{\lambda(z-1)}\Big|_{z=pe^{i\omega}+q}$$

$$= e^{\lambda(pe^{i\omega}+q-1)} = e^{\lambda p(e^{i\omega}-1)}$$

This is the characteristic function of a Poisson random variable with parameter λp.

Thus

$$S \sim \text{Poiss}(\lambda p)$$

Example 1

Analogously, if R denotes the number of non-served customers,

$$R \sim \text{Poiss}(\lambda q)$$

- It can be proved that S and R are independent random variables
- Notice how the convolution theorem applies. Indeed, $S + R = N$ and, thus,

$$\begin{aligned} M_S(\omega)\, M_R(\omega) &= e^{\lambda p (e^{i\omega}-1)}\, e^{\lambda q (e^{i\omega}-1)} \\ &= e^{\lambda (p+q)(e^{i\omega}-1)} = e^{\lambda (e^{i\omega}-1)} = M_N(\omega) \end{aligned}$$

Example 2

Suppose now that each $X_i \sim \text{Exp}(\mu)$, that $N \sim \text{Geom}(p)$, and let

$$S = X_1 + X_2 + \cdots + X_N$$

Then

$$M_X(\omega) = \frac{\mu}{\mu - i\omega},$$

$$G_N(z) = \frac{zp}{1 - zq}$$

Example 2

Therefore

$$M_S(\omega) = G_N\left(M_X(\omega)\right) = \left.\frac{zp}{1 - zq}\right|_{z=\mu/(\mu - i\omega)}$$

$$= \frac{\frac{\mu}{\mu - i\omega}\, p}{1 - \frac{\mu}{\mu - i\omega}\, q} = \frac{\mu p}{\mu p - i\omega}$$

This is the characteristic function of a exponential r.v. with parameter μp.

Hence,

$$S \sim \text{Exp}(\mu p)$$

Expected values

If

$$S = X_1 + X_2 + \cdots + X_N,$$

we can obtain the expected value of S calculating first the conditional expected value given N.

$$\text{E}(S) = \text{E}\left(\text{E}(S \mid N)\right)$$

Expected values

But

$$\begin{aligned} \mathrm{E}(S \mid N = k) &= \mathrm{E}(X_1 + X_2 + \cdots + X_k \mid N = k) \\ &= \mathrm{E}(X_1 + X_2 + \cdots + X_k) = m\,k, \end{aligned}$$

where m denotes the mean value of each X_i.

Therefore

$$\mathrm{E}(S \mid N) = mN$$

and, so,

$$E(S) = \mathrm{E}\,(\mathrm{E}(S \mid N)) = \mathrm{E}(m\,N) = m\,\mathrm{E}(N) = \mathrm{E}(N)\,\mathrm{E}(X)$$

Expected values

- In Example 1, $X \sim \mathrm{B}(p)$ and $N \sim \mathrm{Poiss}(\lambda)$. Then $S \sim \mathrm{Poiss}(\lambda p)$. Thus,

$$\mathrm{E}(S) = \lambda p = \mathrm{E}(N)\,\mathrm{E}(X)$$

- In Example 2, $X \sim \mathrm{Exp}(\mu)$, $N \sim \mathrm{Geom}(p)$. Then $S \sim \mathrm{Exp}(\mu p)$. Hence

$$\mathrm{E}(S) = \frac{1}{\mu p} = \frac{1}{p} \cdot \frac{1}{\mu} = \mathrm{E}(N)\,\mathrm{E}(X)$$