Probability and Stochastic Processes

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Problem 4:
With $P_n = P(X = n)$, let $G_E(x) = p_0 + p_2(x^2) + p_4(x^4) + \dots$
and
$G_O = p_1x + p_3(x^3) + p_5(x^5) + \dots$ bet the even and odd components of the probability generating function $G(x)$.
One of the given conditions was $p_{2n} = \frac{2}{3}p_{2n+1}$ which gives,

$$G_E(x) = \frac{2}{3}p_1 + \frac{2}{3}p_3x^2 + \frac{2}{3}p_5x^4 + \dots = \frac{2}{3}G_O(x)/x \tag{1}$$

On the other hand, $p_{2n} = \frac{1}{2}p_{2n-1}$ for all n greater than 1. It gives us that,

$$G_E(x) = p_0 + \frac{1}{2}p_1(x^2) + \frac{1}{2}p_3x^4 + \dots = p + \frac{1}{2}xG_O(x) \tag{2}$$

On combining 1 and 2, we get,

$$p_0x + \frac{1}{2}x^2G_O(x) = \frac{2}{3}G_O(x) \tag{3}$$

$$\implies G_O(x) = \frac{p_0x}{\frac{2}{3} - \frac{1}{2}x^2} = \frac{6p_0x}{4 - 3x^2} \tag{4}$$

and therefore $G_E(x) = \frac{4p_0}{4-3x^2}$, so that

$$G(x) = G_E(x) + G_O(x) = \frac{p_0(6x + 4)}{4 - 3x^2} = \frac{1}{5}\frac{3x + 2}{4 - 3x^2} \tag{5}$$

Because any PGF must satisfy $G(x) = 1$, we get, $p_0 = \frac{1}{10}$

Problem 5:
(A. Gut, first ed., III.15) The number of cars passing a road crossing during a day follows a Poisson distribution with parameter λ. The number of persons in each car is a Poisson(α) random variable. Find the probability generating function of the total number of persons, N, passing the road crossing during a day. Find the mean and variance of N.

Consider that, If X is the number of cars and Y the number of persons in each car, then the number of total persons is equal to the product of the total number of cars (X) and the total number of persons in each car (Y). So, if we denote the total number of persons by N, then we will have the following-

$$G_N(s) = G_X(G_Y(s))$$

where X and Y are the total number of cars and total number of persons in each of the cars. Now,

$$G_N(s) = G_X(G_Y(s)) \implies e^{\lambda((e^{\alpha(s-1)})-1)}$$

Now, our objective is to find the expected number of persons crossing the road on a given day. So, we define the expectation of a random variable X as follows-

$$E(X) = G'(1)$$

and,

$$G'(N) = e^{\lambda((e^{\alpha(s-1)})-1)}\lambda\frac{d(e^{\alpha(s-1)})}{ds}$$

or,

$$G'(N) = e^{\lambda((e^{\alpha(s-1)})-1)}\lambda(e^{\alpha(s-1)})\alpha$$

Similarly,

$$G''(N) = \lambda\alpha\{e^{\lambda((e^{\alpha(s-1)})-1)}\lambda(e^{\alpha(s-1)})\alpha + (e^{\alpha(s-1)}))\alpha\}$$

or,

$$G''(N) = (\lambda\alpha)^2(e^{\lambda((e^{\alpha(s-1)})-1)})(e^{\alpha(s-1)}) + \lambda\alpha^2(e^{\alpha(s-1)})$$

So, now when we have the $G'(N)$, we can proceed towards finding the value of $G'(1)$, that is,

$$G'(1) = \lambda\alpha$$

So, we can say that the mean number of person crossing the road is equal to the $\lambda\alpha$. Now, it would be a great idea to estimate the variance which is define as follows-

$$Var(N) = E[(N - E(N))^2] = E(N^2) - (E(N))^2$$

But,

$$E(N(N-1)) = G''(1)$$

or,

$$E(N^2) = G''(1) + E(N)$$

or,

$$E(N^2) = \lambda\alpha^2(\lambda + 1) + \lambda\alpha$$

or,

$$E(N^2) = \lambda\alpha(\lambda\alpha + \alpha + 1)$$

so,

$$Var(N) = E(N^2) - (E(N))^2 = (\lambda\alpha)^2 + \lambda\alpha^2 + \lambda\alpha - (\lambda\alpha)^2$$

or,

$$Var(N) = \lambda\alpha(\alpha + 1)$$

Problem 10: Using a moment generating function to prove that if a random variable X has density function

$$f_X(x) = \frac{1}{2}e^{-|x|}, \quad -\infty < x <, \tag{6}$$

Bf then X can be written as X = YZ where Y and Z are independent random variables exponentially distributed.

We know that if $Y \sim Exp(\mu)$, then:

$$\Phi_Y(t) = \int_{-\infty}^{\infty} e^{ty} f_Y(y)dy = \int_0^{\infty} \mu e^{-(\mu-t)} dy = \frac{\mu}{\mu - t}, \quad t < \mu$$

Moreover, if $Z \sim Exp(\lambda)$, consider the moment generating function of $-Z$:

$$\Phi{-}Z(t) = \int_{-\infty}^{\infty} e^{-tz} f_Z(z)dz = \int_0^{\infty} e^{-tz}\lambda e^{-\lambda z} dz = \int_0^{\infty} \lambda e^{-z(t+\lambda)} dz = \int_0^{\infty} \lambda e^{-z(t+\lambda)} dz$$

$$= [\frac{-\lambda}{t+\lambda}.e^{-z(t+\lambda)}]_0^{\infty}$$

and if $t > -\lambda$:

$$= \frac{\lambda}{t+\lambda}$$

By the convolution theorem (seen in class), we know that if $S = X_1 + \cdots + X_n$, then:

$$\Phi_S(t) = \prod_{k=1}^{n} \phi x_k(t)$$

Then, in our case, we show that if $-\lambda < t < \mu$:

$$\Phi_X(t) = \phi YZ(t) = \phi Y(t)\phi{-}Z(t) = \frac{\mu}{\mu - t}\frac{\lambda}{t+\lambda}$$

On the other hand, let's see what the generating function of moments of our variable X is:

$$\Phi_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)dx = \int_{-\infty}^{\infty} e^{tx}\frac{1}{2}e^{-|x|}dx = \frac{1}{2}\left(\int_{-\infty}^{0} e^{tx}e^{x}dx + \int_0^{\infty} e^{tx}e^{-x}dx\right)$$

$$= \frac{1}{2}\left(\int_{-\infty}^{0} e^{x(t+1)}dx + \int_0^{\infty} e^{x(t-1)dx}\right) = \frac{1}{2}\left(\left[\frac{1}{t+1}e^{x(t+1)}\right]_{-\infty}^{0} + \left[\frac{1}{t-1}e^{x}(t-1)\right]_0^{\infty}\right) =$$

$$= \frac{1}{2}\left(\frac{1}{t+1} - \lim_{x\to-\infty}\frac{1}{t+1}e^{x(t+1)}\right)_{(1)} + \frac{1}{2}\left(\lim_{x\to\infty}\frac{1}{t-1}e^{x(t-1)} - \frac{1}{t-1}\right)_{(2)}$$

Evaluated separately (1) and (2). If $t > -1$:

$$(1) = \frac{1}{2}\left(\frac{1}{t+1} - \lim_{x\to-\infty}\frac{1}{t+1}e^{x(t+1)}\right) = \frac{1}{2(t+1)}$$

Moreover, if $t < 1$:

$$(2) = \frac{1}{2}\left(\lim_{x\to\infty}\frac{1}{t-1}e^{x(t-1)} - \frac{1}{t-1}\right) = \frac{-1}{2(t-1)} = \frac{1}{2(1-t)}$$

. Thus if $-1 < t < 1$, then:

$$\Phi_X(t) = \frac{1}{2}\left(\frac{1}{t+1} + \frac{1}{1-t}\right) = \frac{1}{2}\left(\frac{1-t+t+1}{(t+1)(1-t)}\right) = \frac{1}{2}\left(\frac{2}{(t+1)(1-t)}\right) = \frac{1}{(t+1)(1-t)}$$

If we recall the expression that we said before we wanted to get there:

$$\Phi_X(t) = \frac{\mu}{\mu - t}\frac{\lambda}{t+\lambda}, \quad -\lambda < t < \mu$$

Then we are considering $lambda = 1$ and $mu = 1$:

$$\Phi_X(t) = \frac{\mu}{\mu - t}\frac{\lambda}{t+\lambda} = \frac{1}{1-t}\frac{1}{t+1}$$

And so we have shown that X can be written as $X = YZ$, where Y and Z are exponential.

4. (A. Gut's book) Consider a branching process with one ancestor. Suppose that the generating function of the offspring distribution is

$$G(s) = \frac{p^2}{(1-qs)^2}$$

,where $\theta < p = 1 - q < 1$. What is the probability
a) Extinction
(b) that the process is extinct in generation number 2 (i.e., the ancestor does not have any grandchildren)?

to see the probability we have to solve the equation

$$s = G(s) = \frac{p^2}{(1-qs)^2} \implies h(x) = q^2s^3 - 2qs^2 + s - p^2 = 0$$

As $s = 1$ is always a solution, we have

$$h(s) = (s-1)(q^2s^2 + (q^2 - 2q)s + (q-1)^2)$$

This implies that,

$$s = \frac{2q - q^2 \pm \sqrt{(q^2-2q)^2 - 4q^2(q^2-2q+1)}}{2q^2}$$

$$= \frac{2q - q^2 \pm \sqrt{(q^2(4q-3q^2))}}{2q^2}$$

$$= \frac{2 - q \pm \sqrt{(4q-3q^2)}}{2q}$$

So, we can see that the roots of $h(s)$ are

$$s_1(q) = \frac{2 - q + \sqrt{(4q-3q^2)}}{2q}$$

and

$$s_2(q) = \frac{2 - q - \sqrt{(4q-3q^2)}}{2q}$$

As $0 < q < 1$, it is easier to see that $s_2(q) \leq s_1(q)$, then the root we are looking for is

$$s_2(q) = \frac{2 - q - \sqrt{(4q-3q^2)}}{2q} = \frac{1}{q} - \frac{1}{2}\frac{\sqrt{(4q-3q^2)}}{4q^2}$$

or,

$$s_2(q) = \frac{1}{q} - \frac{1}{2} - \sqrt{\frac{1}{q} - \frac{3}{4}}$$

or, since $p = 1 - q$,

$$s_2(q) = \frac{1}{1-p} - \frac{1}{2} - \sqrt{\frac{1}{1-p} - \frac{3}{4}}$$

Now, if $s_2 = 1$

$$\frac{1}{q} - \frac{1}{2} - \sqrt{\frac{1}{q} - \frac{3}{4}} = 1 \implies 3q^2 - 4q + 1 = 0$$

which gives, $q = \frac{1}{3}$ or $q = 1$. As the statement tells us $0 < p < 1$, and if $q = \frac{1}{3}$ then $p = \frac{2}{3}$. Finally, we can conclude that if $0 < p < \frac{2}{3}$, d= $\frac{1}{q} - \frac{1}{2} - \sqrt{\frac{1}{q} - \frac{3}{4}}$ and if $p \geq \frac{2}{3}$ then d=1.
Now we will see the probability that the process is extinct in generation number 2, for this we have to calculate d_2. Starting from $d_0 = 0$, knowing that $d_i = G_X(d_{i-1})$, we will calculate first d_1 and then d_2.

$$d_1 = G_x(d_0) = G_X(0) = \frac{p^2}{(1-qs)^2} = p^2$$

$$d_2 = G_x(d_1) = G_X(p^2) = \frac{p^2}{(1-qp^2)^2}$$

Problem 5: (A. Gut's book) The following model can be used to describe the number of women (mothers and daughters) in a given area. The number of mothers is a random variable X $\sim$ Poisson(λ). Independently of the others, every mother gives birth to a Poisson(μ)-distributed number of daughters. Let Y be the total number of daughters and hence Z = X + Y be the total number of women in the area.
(a) Find the generating function of Z.
(b) Compute E(Z) and Var(Z)

Let Y_k be the count of daughters for mother k so that

$$Y = \sum_{k=1}^{X} Y_k \tag{7}$$

The generating function of a sum of random variables is equal to the product of their generating functions only when the random variables are independent $Y = \sum_{k=1}^{X} Y_k$ means that Y is rather dependent on X, so the step, $G_{(X+Y)}(s) = G_X(s) + G_Y(s)$
Instead, return to the definition of probability generation, and use the identical distribution of all Y_k and the total independence of all Yk and X.

$$G_Z(s) = E(s^Z) = E(E(s^{X+\sum_{k=1}^{X} Y_k}|X)) \tag{8}$$

$$\implies G_Z(s) = G_X(sG_{Y_1}(s)) = exp(\lambda(se^{(\mu(s-1))} - 1)) \tag{9}$$

Differentiating it with respect to Z and substituting 1, we get, $G'_Z(s) = \exp(\lambda(se^{(\mu(s-1))} - 1)).\lambda\ (s.e^{(\mu(s-1))}\mu + e^{(\mu(s-1))})$
Setting, s=1, we get,

$$E(Z) = G'(1) = \lambda(\mu + 1)$$

Similarly, we differentiate G'(Z) to get G"(Z)

$$G''(Z) = \lambda[exp(\lambda(se^{(\mu(s-1))} - 1)).\lambda(s.e^{(\mu(s-1))}\mu + e^{(\mu(s-1))})(s.e^{(\mu(s-1))}\mu + e^{(\mu(s-1))})$$

$$+(s.(e^{(\mu(s-1)})\mu^2 + \mu.e^{(\mu(s-1))} + e^{(\mu(s-1))}\mu).(exp(\lambda(se^{(\mu(s-1))} - 1)))$$

Setting $s = 1$, in $G''(Z)$, we get

$$= \lambda[\lambda(\mu + 1)(\mu + 1) + (\mu^2 + \mu + \mu)]$$

$$= \lambda[\lambda(\mu^2 + 2\mu + 1) + (\mu^2 + 2\mu)]$$

$$= \lambda^2\mu^2 + 2\lambda^2\mu + \lambda^2 + \lambda\mu^2 + 2\lambda\mu$$

Then,

$$E(N^2) = G''(1) + E(N) = \lambda^2\mu^2 + 2\lambda^2\mu + \lambda^2 + \lambda\mu^2 + 2\lambda\mu + \lambda\mu + \lambda$$

or,

$$Var(Z) = E(N^2) - (E(N))^2$$

or,

$$Var(Z) = [\lambda^2\mu^2 + 2\lambda^2\mu + \lambda^2 + \lambda\mu^2 + 2\lambda\mu + \lambda\mu + \lambda] - [\lambda^2\mu^2 + 2\lambda^2\mu + \lambda^2]$$

on solving, we get,

$$Var(Z) = \lambda(1 + 3\mu + \mu^2)$$