Generating and Characteristic Functions

January 31, 2018

Generating and characteristic functions

Probability generating function

Let X be a nonnegative integer-valued random variable.

The probability generating function of X is defined to be

$$G_X(s) \equiv \mathrm{E}(s^X) = \sum_{k\geq 0} s^k\, \mathrm{P}(X = k)$$

- For each value of the argument s, we compute the expectation of the random variable s^X.

Probability generating function

If X takes a finite number of values, then $G_X(s)$ is a polynomial:

$$\begin{aligned} G_X(s) &= \sum_{k=1}^{n} s^k\, \mathrm{P}(X = k) \\ &= \mathrm{P}(X = 0) + \mathrm{P}(X = 1)\, s + \cdots + \mathrm{P}(X = n)\, s^n \end{aligned}$$

Probability generating function

Otherwise, if X takes a countable number of values, then

$$G_X(s) = \sum_{k\geq 0} s^k \, \mathrm{P}(X=k)$$
$$= \mathrm{P}(X=0) + \mathrm{P}(X=1)\, s + \cdots + \mathrm{P}(X=k)\, s^k + \cdots$$

is a series that converges at least for $s \in [-1,1]$, because if $|s| \leq 1$, then

$$\sum_{k\geq 0} |s|^k \, \mathrm{P}(X=k) \leq \sum_{k\geq 0} \mathrm{P}(X=k) = 1$$

Examples

Let X be a Bernoulli random variable, $X \sim \mathrm{B}(p)$.

$$\mathrm{P}(X=0) = q, \qquad \mathrm{P}(X=1) = p$$

We have

$$G_X(s) = \sum_{k\geq 0} s^k \, \mathrm{P}(X=k) = q + sp, \quad s \in \mathbb{R}$$

Examples

Let $X \sim \mathrm{Bin}(n,p)$.

$$\mathrm{P}(X=k) = \binom{n}{k} p^k q^{n-k}, \quad k = 0,1,\ldots,n$$

Then

$$G_X(s) = \sum_{k\geq 0} s^k \, \mathrm{P}(X=k) = \sum_{k=0}^{n} s^k \binom{n}{k} p^k q^{n-k}$$
$$= \sum_{k=0}^{n} \binom{n}{k} (sp)^k q^{n-k} = (q+sp)^n, \quad s \in \mathbb{R}$$

Examples

$X \sim \mathrm{Poiss}(\lambda)$.

$$\mathrm{P}(X=k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0,1,\ldots$$

Then

$$G_X(s) = \sum_{k\geq 0} s^k \, \mathrm{P}(X=k) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(s\lambda)^k}{k!}$$
$$= e^{-\lambda} e^{s\lambda} = e^{\lambda(s-1)}, \quad s \in \mathbb{R}$$

Examples

$X \sim \mathrm{Geom}(p)$

$$P(X = k) = q^{k-1}p, \quad k = 1, 2, \ldots, \quad 0 < p < 1$$

We have

$$G_X(s) = \sum_{k\geq 0} s^k \, P(X = k) = \sum_{k=1}^{\infty} s^k q^{k-1} p$$
$$= sp \sum_{k=1}^{\infty} (sq)^{k-1} = \frac{sp}{1 - sq}, \quad |s| < \frac{1}{q}$$

Unicity

If two nonnegative integer-valued random variables have the same generating function, then they follow the same probability law.

Theorem

Let X and Y be nonnegative integer-valued random variables such that

$$G_X(s) = G_Y(s).$$

Then

$$P(X = k) = P(Y = k) \quad \textit{for all } k \geq 0.$$

This result is a special case of the uniqueness theorem for power series

Convolution theorem

Theorem (Convolution)

Let X and Y be independent, nonnegative, integer-valued random variables, and let $Z = X + Y$. Then

$$G_Z(s) = G_X(s)\, G_Y(s)$$

Proof:

$$G_Z(s) = E(s^Z) = E(s^{X+Y})$$
$$= E(s^X s^Y) = E(s^X)E(s^Y) = G_X(s)G_Y(s)$$

Example

Let $X \sim \mathrm{Bin}(n, p)$, $Y \sim \mathrm{Bin}(m, p)$ be independent and let

$$Z = X + Y$$

We have

$$G_Z(s) = G_X(s)G_Y(s) = (q + sp)^n(q + sp)^m = (q + sp)^{n+m}$$

$G_Z(s)$ is the probability generating function of a $\mathrm{Bin}(n + m, p)$ random variable. Therefore, by the unicity theorem,

$$X + Y \sim \mathrm{Bin}(n + m, p)$$

Convolution theorem

More generally,

Theorem

Let $X_1, X_2, \ldots, X_n$ be independent, nonnegative, integer-valued random variables and set

$$S = X_1 + X_2 + \cdots + X_n.$$

Then

$$G_S(s) = \prod_{k=1}^{n} G_{X_k}(s).$$

Convolution theorem

A case of particular importance is:

Corollary

If, in addition, $X_1, X_2, \ldots, X_n$ are equidistributed, with common probability generating function $G_X(s)$, then

$$G_S(s) = (G_X(s))^n .$$

Example: Negative binomial probability law

A biased coin such that $\mathrm{P}(\text{heads}) = p$ is repeatedly tossed until a total amount of k heads has been obtained.

Let X be the number of tosses.

Notice that

$$X = X_1 + X_2 + \cdots + X_k,$$

where

$$X_i \sim \mathrm{Geom}(p)$$

is the number of tosses between the $(i-1)$-th and the i-th head.

Example: Negative binomial probability law

As $X_1, \ldots, X_k$ are independent and identically distributed we can apply the convolution theorem. Thus,

$$\begin{aligned} G_X(s) &= G_{X_1}(s) G_{X_2}(s) \cdots G_{X_k}(s) \\ &= (G_{X_1}(s))^k = \left(\frac{sp}{1 - sq} \right)^k, \quad |s| < \frac{1}{q} \end{aligned}$$

Example: Negative binomial probability law

Recall that if $\alpha \in \mathbb{R}$, then the Taylor series expansion about 0 of the function $(1+x)^\alpha$ is, for $x \in (-1,1)$,

$$1 + \alpha\, x + \frac{\alpha(\alpha-1)}{2}\, x^2 + \cdots + \frac{\alpha(\alpha-1)\dots(\alpha-r+1)}{r!}\, x^r + \cdots$$

We can write

$$(1+x)^\alpha = \sum_{r\geq 0} \binom{\alpha}{r} x^r$$

where

$$\binom{\alpha}{r} \equiv \frac{\alpha(\alpha-1)\dots(\alpha-r+1)}{r!}$$

Example: Negative binomial probability law

Consider the series expansion of $G_X(s)$:

$$G_X(s) = (sp)^k (1-sq)^{-k} = (sp)^k \sum_{r=0}^{\infty} \binom{-k}{r} (-sq)^r$$

where

$$\binom{-k}{r} = \frac{-k(-k-1)\cdots(-k-r+1)}{r!}$$
$$= (-1)^r \binom{k+r-1}{k-1}$$

Example: Negative binomial probability law

Therefore,

$$G_X(s) = \sum_{r=0}^{\infty} \binom{k+r-1}{k-1} p^k q^r s^{k+r} = \sum_{n=k}^{\infty} \binom{n-1}{k-1} p^k q^{n-k} s^n$$

Hence,

$$P(X=n) = \begin{cases} 0, & n < k \\ \binom{n-1}{k-1} p^k q^{n-k}, & n = k, k+1, \cdots \end{cases}$$

This is the negative binomial probability law.

Properties

- $G_X(0) = P(X=0)$
- $G_X(1) = 1$

  Indeed, we have

$$G_X(1) = \sum_{k\geq 0} s^k P(X=k)\Big|_{s=1} = \sum_{k\geq 0} P(X=k) = 1$$

Properties

Proposition

Let R be the radius of covergence of $G_X(s)$. If $R > 1$, then

$$\mathrm{E}(X) = G_X'(1)$$

Indeed,

$$G_X'(s) = \frac{d}{ds} \sum_{k \geq 0} s^k \mathrm{P}(X = k) = \sum_{k \geq 1} k\, s^{k-1} \mathrm{P}(X = k)$$

Hence,

$$G_X'(1) = \sum_{k \geq 1} k\, \mathrm{P}(X = k) = \mathrm{E}(X)$$

Properties

More generally,

Proposition

- $\mathrm{E}(X) = G_X'(1) \equiv \lim_{s \to 1^-} G_X'(s)$
- $\mathrm{E}(X(X-1)\cdots(X-k+1)) = G_X^{(k)}(1) \equiv \lim_{s \to 1^-} G_X^{(k)}(s)$

Examples

Let $X \sim \mathrm{Bin}(n, p)$.

$$\mathrm{E}(X) = G_X'(1) = \frac{d}{ds} (q + sp)^n \Big|_{s=1}$$

$$= np\, (q + sp)^{n-1} \Big|_{s=1} = np\, (q + p)^{n-1} = np$$

Examples

Let $X \sim \mathrm{Poiss}(\lambda)$.

$$\mathrm{E}(X) = G_X'(1) = \frac{d}{ds} e^{\lambda(s-1)} \Big|_{s=1} = \lambda\, e^{\lambda(s-1)} \Big|_{s=1} = \lambda$$

Analogously,

$$\mathrm{E}(X(X-1)) = G_X''(1) = \lambda^2 e^{\lambda(s-1)} \Big|_{s=1} = \lambda^2$$

Hence,

$$E(X^2) = \lambda^2 + \lambda, \qquad \mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = \lambda$$

Examples

$X \sim \text{Geom}(p)$.

$$\text{E}(X) = G'_X(1) = \frac{d}{ds}\frac{sp}{(1-sq)}\bigg|_{s=1} = \frac{p}{(1-sq)^2}\bigg|_{s=1} = \frac{1}{p}$$

Analogously,

$$\text{E}(X(X-1)) = G''_X(1) = \frac{2pq}{(1-sq)^3}\bigg|_{s=1} = \frac{2q}{p^2}$$

Therefore

$$E(X^2) = \frac{2q}{p^2} + \frac{1}{p}, \qquad \text{Var}(X) = \text{E}(X^2) - (\text{E}(X))^2 = \frac{q}{p^2}$$

Examples

Let X be a negative binomial random variable.

$$\text{E}(X) = G'_X(1) = \frac{d}{ds}\left(\frac{sp}{1-sq}\right)^k\bigg|_{s=1}$$

$$= k\left(\frac{sp}{1-sq}\right)^{k-1}\frac{p}{(1-sq)^2}\bigg|_{s=1} = \frac{k}{p}$$

This result can also be obtained from $X = X_1 + \cdots + X_k$, with each $X_i \sim \text{Geom}(p)$.

$$\text{E}(X) = \sum_{i=1}^{k}\text{E}(X_i) = \frac{k}{p}$$

Moment generating function

The moment generating function of a random variable X is defined as

$$\Phi_X(t) \equiv \text{E}\left(e^{tX}\right) = \begin{cases} \displaystyle\sum_i e^{tx_i}\,\text{P}(X = x_i), & \text{if } X \text{ is discrete} \\ \displaystyle\int_{-\infty}^{\infty} e^{tx}\, f_X(x)\, dx, & \text{if } X \text{ is continuous} \end{cases}$$

provided that the sum or the integral converges.

Unicity

The moment generating function specifies uniquely the probability distribution.

Theorem

Let X and Y be random variables. If there exists $h > 0$ such that $\Phi_X(t) = \Phi_Y(t)$ for $|t| < h$, then X and Y are identically distributed.

Examples

Let $X \sim \text{Bin}(n, p)$.

$$\Phi_X(t) = \sum_{k=0}^{n} e^{tk}\, \mathrm{P}(X = k)$$

$$= \sum_{k=0}^{n} \binom{n}{k} \left(pe^t\right)^k q^{n-k} = \left(q + pe^t\right)^n, \quad t \in \mathbb{R}$$

Examples

Let $X \sim \text{Poiss}(\lambda)$.

$$\Phi_X(t) = \sum_{k=0}^{\infty} e^{tk}\, \mathrm{P}(X = k) = \sum_{k=0}^{\infty} e^{tk}\, e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(e^t\lambda)^k}{k!} = e^{\lambda(e^t - 1)}, \quad t \in \mathbb{R}$$

Examples

Let $X \sim \text{Exp}(\mu)$.

$$\Phi_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx$$

$$= \int_0^{\infty} \mu e^{-(\mu - t)x}\, dx = \frac{\mu}{\mu - t}, \quad t < \mu$$

For continuos random variables, $\Phi_X(t)$ is related to the Laplace transform of the probability density function $f_X(x)$.

Examples

Let $Z \sim \mathrm{N}(0,1)$.

$$\Phi_Z(t) = \int_{-\infty}^{\infty} e^{tz} f_Z(z)\, dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2-2tz}{2}}\, dz$$

$$= e^{\frac{t^2}{2}} \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(z-t)^2}{2}}\, dz}_{1} = e^{\frac{t^2}{2}}, \quad t \in \mathbb{R}$$

because

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(z-t)^2}{2}}\, dz = \mathrm{P}(-\infty < \mathrm{N}(t,1) < \infty) = 1$$

Examples

More generally, if

$$X = \sigma Z + m,$$

then $X \sim \mathrm{N}(m, \sigma^2)$.

We have

$$\Phi_X(t) = \mathrm{E}\left(e^{tX}\right) = \mathrm{E}\left(e^{t(\sigma Z + m)}\right)$$

$$= e^{tm}\, \mathrm{E}\left(e^{t\sigma Z}\right) = e^{tm}\, \Phi_Z(\sigma t) = e^{\frac{\sigma^2 t^2}{2} + tm}$$

Power series expansion

Notice that

$$\Phi'_X(t) = \frac{d}{dt}\mathrm{E}\left(e^{tX}\right) = \mathrm{E}\left(\frac{d}{dt} e^{tX}\right) = \mathrm{E}\left(X\, e^{tX}\right)$$

Therefore,

$$\Phi'_X(0) = \mathrm{E}(X)$$

Analogously,

$$\Phi''_X(t) = \frac{d}{dt}\Phi'_X(t) = \frac{d}{dt}\mathrm{E}\left(X\, e^{tX}\right) = \mathrm{E}\left(X^2 e^{tX}\right)$$

Thus,

$$\Phi''_X(0) = \mathrm{E}(X^2)$$

Power series expansion

For instance, if $X \sim \mathrm{Exp}(\mu)$,

$$\Phi_X(t) = \frac{\mu}{\mu - t}, \quad t < \mu$$

and

$$\mathrm{E}(X) = \Phi'_X(0) = \frac{d}{dt}\left(\frac{\mu}{\mu - t}\right)\Big|_{t=0} = \frac{\mu}{(\mu - t)^2}\Big|_{t=0} = 1/\mu$$

Analogously,

$$\mathrm{E}(X^2) = \Phi''_X(0) = \frac{d}{dt}\left(\frac{\mu}{(\mu - t)^2}\right)\Big|_{t=0} = \frac{2\mu}{(\mu - t)^3}\Big|_{t=0} = 2/\mu^2$$

Power series expansion

More generally,

$$\Phi_X(t) = \mathrm{E}\left(e^{tX}\right) = \mathrm{E}\left(1 + tX + \frac{(tX)^2}{2!} + \cdots + \frac{(tX)^k}{k!} + \cdots\right)$$

$$= 1 + \mathrm{E}(X)\, t + \frac{\mathrm{E}(X^2)}{2!}\, t^2 + \cdots + \frac{\mathrm{E}(X^k)}{k!}\, t^k + \cdots$$

This is the power series expansion of $\Phi_X(t)$,

$$\Phi_X(t) = \sum_{k=0}^{\infty} \frac{\Phi_X^{(k)}(0)}{k!}\, t^k$$

Power series expansion

Theorem

If $\Phi_X(t)$ converges on some open interval containing the origin $t = 0$, then X has moments of any order,

$$\mathrm{E}\left(X^k\right) = \Phi_X^{(k)}(0),$$

and

$$\Phi_X(t) = \sum_{k=0}^{\infty} \frac{\mathrm{E}(X^k)}{k!}\, t^k$$

Power series expansion

For instance, let $X \sim \mathrm{Exp}(\mu)$. If $|t| < \mu$ we have

$$\Phi_X(t) = \frac{\mu}{\mu - t}$$

$$= \frac{1}{1 - (t/\mu)} = 1 + \frac{t}{\mu} + \left(\frac{t}{\mu}\right)^2 + \cdots + \left(\frac{t}{\mu}\right)^n + \cdots$$

Hence,

$$\frac{\mathrm{E}\left(X^n\right)}{n!} = \frac{1}{\mu^n}$$

Therefore,

$$\mathrm{E}\left(X^n\right) = \frac{n!}{\mu^n}$$

Convolution theorem

The convolution theorem applies also to moment generating functions.

Theorem

Let $X_1, X_2, \ldots, X_n$ be independent random variables and let $S = X_1 + X_2 + \cdots + X_n$. Then,

$$\Phi_S(t) = \prod_{k=1}^{n} \Phi_{X_k}(t)$$

Examples

$X \sim \text{Poiss}(\lambda_X)$, $Y \sim \text{Poiss}(\lambda_Y)$, independent.

Let $Z = X + Y$.

We have

$$\begin{aligned}\Phi_Z(t) &= \Phi_X(t)\Phi_Y(t)\\ &= e^{\lambda_X(e^t-1)}\, e^{\lambda_Y(e^t-1)} = e^{(\lambda_X+\lambda_Y)(e^t-1)}\end{aligned}$$

Hence,

$$Z \sim \text{Poiss}(\lambda_X + \lambda_Y)$$

Examples

$X \sim \text{N}(m_X, \sigma_X^2)$, $Y \sim \text{N}(m_Y, \sigma_Y^2)$, independent.

Let $Z = X + Y$.

We have

$$\begin{aligned}\Phi_Z(t) &= \Phi_X(t)\Phi_Y(t)\\ &= e^{\frac{\sigma_X^2 t^2}{2} + t m_X}\, e^{\frac{\sigma_Y^2 t^2}{2} + t m_Y} = e^{\frac{(\sigma_X^2+\sigma_Y^2)t^2}{2} + t(m_X+m_Y)}\end{aligned}$$

Therefore

$$Z \sim \text{N}(m_X + m_Y, \sigma_X^2 + \sigma_Y^2)$$

Characteristic function

The characteristic function of a random variable X is the complex-valued function of the real argument ω

$$M_X : \mathbb{R} \to \mathbb{C}$$

$$\omega \mapsto M_X(\omega)$$

defined as

$$M_X(\omega) \equiv \text{E}\left(e^{i\omega X}\right) = \text{E}\left(\cos\left(\omega X\right)\right) + i\,\text{E}\left(\sin\left(\omega X\right)\right)$$

Characteristic function

Therefore,

$$M_X(\omega) = \begin{cases} \displaystyle\sum_k e^{i\,\omega x_k}\,\mathrm{P}(X = x_k), & \text{if } X \text{ is discrete} \\ \displaystyle\int_{-\infty}^{\infty} e^{i\,\omega x}\, f_X(x)\, dx, & \text{if } X \text{ is continuous} \end{cases}$$

- The characteristic function exists for all $\omega \in \mathbb{R}$ and for all random variables.
- If X is continuous, $M_X(\omega)$ is the Fourier transform of the probability density $f_X(x)$.
  (Notice the change of sign from the usual definition.)
- If X is discrete, $M_X(\omega)$ is related to Fourier series.

Properties

- $|M_X(\omega)| \leq M_X(0) = 1$ for all $\omega \in \mathbb{R}$.

  Indeed,

$$|M_X(\omega)| = \left|\mathrm{E}\left(e^{i\,\omega X}\right)\right| \leq \mathrm{E}\left(\left|e^{i\,\omega X}\right|\right) = \mathrm{E}(1) = 1$$

  Moreover,

$$M_X(0) = \mathrm{E}\left(e^{i \cdot 0 \cdot X}\right) = \mathrm{E}(1) = 1$$

Properties

- $\overline{M_X(\omega)} = M_X(-\omega)$.

$$\begin{aligned} \overline{M_X(\omega)} &= \overline{\mathrm{E}\left(e^{i\,\omega X}\right)} = \mathrm{E}\left(\overline{e^{i\,\omega X}}\right) = \mathrm{E}\left(e^{-i\,\omega X}\right) \\ &= \mathrm{E}\left(\cos\left(\omega X\right)\right) - i\,\mathrm{E}\left(\sin\left(\omega X\right)\right) \\ &= \mathrm{E}\left(\cos\left(-\omega X\right)\right) + i\,\mathrm{E}\left(\sin\left(-\omega X\right)\right) = M_X(-\omega) \end{aligned}$$

- $M_X(\omega)$ is uniformly continuous in $\mathbb{R}$.

Examples

Let $X \sim \mathrm{Binom}(n, p)$. Then,

$$M_X(\omega) = \left(p\, e^{i\omega} + q\right)^n$$

If $X \sim \mathrm{Poiss}(\lambda)$, then

$$M_X(\omega) = e^{\lambda\left(e^{i\omega} - 1\right)}$$

If $X \sim \mathrm{N}(m, \sigma^2)$, then

$$M_X(\omega) = e^{i\omega m - \frac{1}{2}\sigma^2\omega^2}$$

Characteristic function and moments

Theorem

If $\mathrm{E}(X^n) < \infty$ *for some* $n = 1, 2, \ldots,$ *then*

$$M_X(\omega) = \sum_{k=0}^{n} \frac{\mathrm{E}(X^k)}{k!}(i\,\omega)^k + o(|\omega|^n) \quad as \quad \omega \to 0.$$

So,

$$\mathrm{E}(X^k) = \frac{M_X^{(k)}(0)}{i^k} \quad for \quad k = 1, 2, \ldots, n.$$

In particular, if $\mathrm{E}(X) = 0$ and $\mathrm{Var}(X) = \sigma^2$, then

$$M_X(\omega) = 1 - \frac{1}{2}\sigma^2\omega^2 + o(t^2) \quad \text{as} \quad \omega \to 0.$$

Characteristic function and moments

Indeed,

$$M_X(\omega) = \mathrm{E}\left(e^{i\omega X}\right) = \mathrm{E}\left(\sum_{k=0}^{\infty} \frac{(i\,\omega X)^k}{k!}\right) = \sum_{k=0}^{\infty} \frac{i^k\,\mathrm{E}(X^k)}{k!}\,\omega^k$$

But this is the Taylor's series expansion of $M_X(\omega)$:

$$M_X(\omega) = \sum_{k=0}^{\infty} \frac{M_X^{(k)}(0)}{k!}\,\omega^k$$

Therefore

$$i^k\,\mathrm{E}(X^k) = M_X^{(k)}(0)$$

Convolution theorem

Theorem

Let $X_1, X_2, \ldots, X_n$ *be independent random variables and let*

$$S = X_1 + X_2 + \cdots + X_n.$$

Then

$$M_S(\omega) = \prod_{k=1}^{n} M_{X_k}(\omega)$$

Convolution theorem

$$\begin{aligned} M_S(\omega) &= \mathrm{E}\left(e^{i\omega S}\right) = \mathrm{E}\left(e^{i\omega(X_1+X_2+\cdots+X_n)}\right) \\ &= \mathrm{E}\left(e^{i\omega X_1}e^{i\omega X_2}\cdots e^{i\omega X_n}\right) \\ &= \mathrm{E}\left(e^{i\omega X_1}\right)\mathrm{E}\left(e^{i\omega X_2}\right)\cdots\mathrm{E}\left(e^{i\omega X_n}\right) \\ &= M_{X_1}(\omega)M_{X_2}(\omega)\cdots M_{X_n}(\omega) \end{aligned}$$

Convolution theorem

In the case $n = 2$ we have essentially the convolution theorem for Fourier transforms.

If X and Y are continuous and independent random variables and $Z = X + Y$, then

$$f_Z = f_X * f_Y$$

This implies

$$\mathcal{F}(f_Z) = \mathcal{F}(f_X) \cdot \mathcal{F}(f_Y),$$

that is,

$$M_Z(\omega) = M_X(\omega) M_Y(\omega)$$

Unicity

Theorem

Let X have probability distribution function F_X and characteristic function M_X. Let $\overline{F}_X(x) = (F_X(x) + F_X(x^-))/2$.

Then

$$\overline{F}_X(b) - \overline{F}_X(a) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ia\omega} - e^{-ib\omega}}{i\omega} M_X(\omega)\, d\omega.$$

M_X specifies uniquely the probability law of X. Two random variables have the same characteristic function if and only if they have the same distribution function.

Inversion

Theorem (Inversion of the Fourier transform)

Let X be a continuous r.v. with density f_X and characteristic function M_X. Then

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} M_X(\omega)\, d\omega$$

at every point x at which f_X is differentiable.

Inversion

In the discrete case, $M_X(\omega)$ is related to Fourier series.

Theorem

If X is an integer-valued random variable, then

$$\mathrm{P}(X = k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ik\omega} M_X(\omega)\, d\omega$$

Joint characteristic functions

The joint characteristic function of the random variables X_1, X_2, ..., X_n is defined to be

$$M_X(\omega_1, \omega_2, \dots, \omega_n) = \mathrm{E}\left(e^{i(\omega_1 X_1 + \omega_2 X_2 + \cdots + \omega_n X_n)}\right)$$

Using vectorial notation one can write

$$\omega = (\omega_1, \omega_2, \cdots, \omega_n)^t, \quad X = (X_1, X_2, \cdots, X_n)^t$$

and

$$M_X(\omega^t) = \mathrm{E}\left(e^{i\omega^t X}\right)$$

Joint moments

The joint characteristic function allows as to calculate joint moments.

For instance, given X, Y:

$$m_{kl} = \mathrm{E}\left(X^k Y^l\right) = \frac{1}{i^{k+l}} \left. \frac{\partial^{k+l} M_{XY}(\omega_1, \omega_2)}{\partial^k \omega_1 \, \partial^l \omega_2} \right|_{(\omega_1, \omega_2) = (0,0)}$$

Marginal characteristic functions

Marginal characteristic functions are easily derived from the joint characteristic function.

For instance, given X, Y:

$$\begin{aligned} M_X(\omega) &= \mathrm{E}\left(e^{i\omega X}\right) \\ &= \mathrm{E}\left(e^{i(\omega_1 X + \omega_2 Y)}\right)\Big|_{(\omega_1 = \omega, \omega_2 = 0)} = M_{XY}(\omega, 0) \end{aligned}$$

Analogously,

$$M_Y(\omega) = M_{XY}(0, \omega)$$

Independent random variables

Theorem

The random variables $X_1, X_2, \dots, X_n$ are independent if and only if

$$M_X(\omega_1, \omega_2, \dots, \omega_n) = M_{X_1}(\omega_1) M_{X_2}(\omega_2) \cdots M_{X_n}(\omega_n)$$

If the random variables are independent, then

$$\begin{aligned} & M_X(\omega_1, \omega_2, \dots, \omega_n) \\ &\quad = \mathrm{E}\left(e^{i(\omega_1 X_1 + \omega_2 X_2 + \cdots + \omega_n X_n)}\right) = \mathrm{E}\left(e^{i\omega_1 X_1} \, e^{i\omega_2 X_2} \, \cdots \, e^{i\omega_n X_n}\right) \\ &\quad = \mathrm{E}\left(e^{i\omega_1 X_1}\right) \, \mathrm{E}\left(e^{i\omega_2 X_2}\right) \cdots \mathrm{E}\left(e^{i\omega_n X_n}\right) \\ &\quad = M_{X_1}(\omega_1) M_{X_2}(\omega_2) \cdots M_{X_n}(\omega_n) \end{aligned}$$