Probability and Random Processes

Problems

Generating and characteristic functions

1. $G_{X+Y}(s) = \dfrac{1}{6}s + \dfrac{1}{12}s^2 + \dfrac{3}{8}s^3 + \dfrac{5}{24}s^4 + \dfrac{1}{12}s^5 + \dfrac{1}{12}s^6$

   Hence, $\mathrm{P}(Z=0)=0$, $\mathrm{P}(Z=1)=\dfrac{1}{6}$, $\mathrm{P}(Z=2)=\dfrac{1}{12}$, $\mathrm{P}(Z=3)=\dfrac{3}{8}$, $\mathrm{P}(Z=4)=\dfrac{5}{24}$, $\mathrm{P}(Z=5)=\dfrac{1}{12}$, $\mathrm{P}(Z=6)=\dfrac{1}{12}$

2. (c) $\mathrm{E}(X)=3$

3. (b) $G_Y(s) = (1-p^2+p^2 s)^n$. Thus, $Y \sim \mathsf{Bin}(n, p^2)$.

4. $G_X(s) = \dfrac{2+3s}{5(4-3s^2)}, \quad |s| < \dfrac{2}{\sqrt{3}}$

5. If X is the number of cars and Y the number of persons in each car, then $G_N(s) = G_X(G_Y(s)) = \exp\left(\lambda(e^{\alpha(s-1)}-1)\right)$. $\quad \mathrm{E}(N)=\lambda\alpha$, $\mathrm{Var}(N)=\lambda\alpha(\alpha+1)$.

6. (a) $G_M(t) = e^{\lambda(-2+t+t^2)/3}$. (b) $\mathrm{E}(M)=300$, $\mathrm{Var}(M)=500$.

7. $\phi_{X-Y}(t) = \dfrac{1}{16} + \dfrac{3e^{-3t}}{8} + \dfrac{3e^{-2t}}{16} + \dfrac{5e^{-t}}{16} + \dfrac{e^t}{16}$

8. $np + 14\dbinom{n}{2}p^2 + 36\dbinom{n}{3}p^3 + 24\dbinom{n}{4}p^4$

9. $X \sim \mathsf{Unif}[0,3]$

10.

11. $\phi_Z(t) = \dfrac{a^2}{a^2-t^2}, \quad |t|<a. \qquad \mathrm{E}(Z^n) = \begin{cases} \dfrac{n!}{a^n}, & n \text{ even,} \\ 0, & n \text{ odd.} \end{cases}$

12.

13. $\mathrm{E}(X)=2; \quad \mathrm{Var}(X)=5$

14. $M_X(\omega) = \dfrac{1}{10}\left(3+6e^{i\omega}+e^{i2\omega}\right); \quad M_Y(\omega) = \dfrac{1}{100}\left(3+6e^{i\omega}+e^{i2\omega}\right)^2$

15. $M_{X_nY_n}(\omega_1,\omega_2) = \dfrac{1}{2^n}(\cos\omega_1+\cos\omega_2)^n$

16. $M_X(\omega) = \dfrac{1}{3}\left(\dfrac{\sin\omega}{\omega} + \dfrac{\sin^2\omega}{\omega^2} + \dfrac{\sin^3\omega}{\omega^3}\right)$

17. $\phi_S(t) = \exp\left(\lambda\left(pe^t+qe^{2t}-1\right)\right); \quad \mathrm{E}(S)=\lambda(1+q); \quad \mathrm{Var}(S)=\lambda(1+3q)$

18. $|s|<n; \quad \mathrm{E}(X) = 1+\frac{n}{n-1}; \quad \mathrm{P}(X=k) = \frac{3}{64}\left(\frac{1}{4}\right)^{k-4} - \frac{3}{8}\left(\frac{1}{2}\right)^{k-4} + \frac{27}{64}\left(\frac{3}{4}\right)^{k-4}, \quad k \geq 4.$