Probability and Stochastic Processes

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1 Introduction

Problem 5:
(A. Gut, first ed., III.15) The number of cars passing a road crossing during a day follows a Poisson distribution with parameter λ. The number of persons in each car is a Poisson(α) random variable. Find the probability generating function of the total number of persons, N, passing the road crossing during a day. Find the mean and variance of N.

Consider that, If X is the number of cars and Y the number of persons in each car, then the number of total persons is equal to the product of the total number of cars (X) and the total number of persons in each car (Y). So, if we denote the total number of persons by N, then we will have the following-

$$G_N(s) = G_X(G_Y(s))$$

where X and Y are the total number of cars and total number of persons in each of the cars. Now,

$$G_N(s) = G_X(G_Y(s)) \implies e^{\lambda((e^{\alpha(s-1)})-1)}$$

Now, our objective is to find the expected number of persons crossing the road on a given day. So, we define the expectation of a random variable X as follows-

$$E(X) = G'(1)$$

and,

$$G'(N) = e^{\lambda((e^{\alpha(s-1)})-1)}\lambda\frac{d(e^{\alpha(s-1)})}{ds}$$

or,

$$G'(N) = e^{\lambda((e^{\alpha(s-1)})-1)}\lambda(e^{\alpha(s-1)})\alpha$$

Similarly,

$$G''(N) = \lambda\alpha\{e^{\lambda((e^{\alpha(s-1)})-1)}\lambda(e^{\alpha(s-1)})\alpha + (e^{\alpha(s-1)}))\alpha\}$$

or,

$$G''(N) = (\lambda\alpha)^2(e^{\lambda((e^{\alpha(s-1)})-1)})(e^{\alpha(s-1)}) + \lambda\alpha^2(e^{\alpha(s-1)})$$

So, now when we have the $G'(N)$, we can proceed towards finding the value of $G'(1)$, that is,

$$G'(1) = \lambda\alpha$$

So, we can say that the mean number of person crossing the road is equal to the $\lambda\alpha$. Now, it would be a great idea to estimate the variance which is define as follows-

$$Var(N) = E[(N - E(N))^2] = E(N^2) - (E(N))^2$$

But,

$$E(N(N-1)) = G''(1)$$

or,

$$E(N^2) = G''(1) + E(N)$$

or,

$$E(N^2) = \lambda\alpha^2(\lambda + 1) + \lambda\alpha$$

or,

$$E(N^2) = \lambda\alpha(\lambda\alpha + \alpha + 1)$$

so,

$$Var(N) = E(N^2) - (E(N))^2 = (\lambda\alpha)^2 + \lambda\alpha^2 + \lambda\alpha - (\lambda\alpha)^2$$

or,

$$Var(N) = \lambda\alpha(\alpha + 1)$$