Probability and Random Processes

Problems

Branching Processes. Distributions with random parameters.

1. *(Midterm exam November 2013)* Consider a branching process with one ancestor. Suppose that the probability function of the offspring distribution is
$$\mathrm{P}(X=0)=\frac{1}{4},\quad \mathrm{P}(X=1)=\frac{1}{2},\quad \mathrm{P}(X=2)=\frac{1}{8},\quad \mathrm{P}(X=3)=\frac{1}{8}.$$
What is the probability
   (a) that the process is extinct in generation number 2 (i.e., the ancestor does not have any grandchildren)?
   (b) of extinction?

   Answer:
   (a) $197/512 \approx 0.385$
   (b) $-1+\sqrt{3} \approx 0.732$

2. *(Midterm exam October 2014)* At each unit of time a cell either dies with probability p or subdivides with probability $1-p$. Let X_n be the number of cells at time n.
   (a) If $X_0 = 1$, what is the maximum value of p for which the probability of extinction is at most $1/4$.
   (b) If $X_0 = k$ and $p = 1/4$, what is the minimum value of k such that the probability of extinction is at most $1/4$.

   Answer:
   (a) $p \leq 1/5$
   (b) $k \geq 2$

3. *(Final exam January 2015)* Consider a branching process with one ancestor. The generating function of the offspring distribution is
$$G(s)=\frac{1}{2-s^2}.$$
   (a) What is the probability that the first generation has 4 individuals.
   (b) Compute the probability of extinction.
   (c) Compute the probability that the process is extinct in the third generation.

   Answer:
   (a) $1/8$
   (b) $(-1+\sqrt{5})/2 \approx 0.618$
   (c) $49/82 \approx 0.598$

4. *(A. Gut's book)* Consider a branching process with one ancestor. Suppose that the generating function of the offspring distribution is
$$G(s)=\frac{p^2}{(1-qs)^2},$$
where $0 < p = 1-q < 1$. What is the probability

(a) of extinction?

(b) that the process is extinct in generation number 2 (i.e., the ancestor does not have any grandchildren)?

Answer:

(a) If $0 < p < 2/3$, then $d = \frac{1}{q} - \frac{1}{2} - \sqrt{\frac{1}{q} - \frac{3}{4}}$. If $p \geq 2/3$, then $d = 1$

(b) $d_2 = p^2/(1 - p^2 q)^2$

5. *(A. Gut's book)* The following model can be used to describe the number of women (mothers and daughters) in a given area. The number of mothers is a random variable $X \sim \mathsf{Po}(\lambda)$. Independently of the others, every mother gives birth to a $\mathsf{Po}(\mu)$-distributed number of daughters. Let Y be the total number of daughters and hence $Z = X + Y$ be the total number of women in the area.

   (a) Find the generating function of Z.

   (b) Compute $\mathrm{E}(Z)$ and $\mathrm{Var}(Z)$.

   Answer:
   (a) $G_Z(s) = \exp\left(\lambda\left(se^{\mu(s-1)} - 1\right)\right)$
   (b) $\mathrm{E}(Z) = \lambda(1 + \mu)$; $\mathrm{Var}(Z) = \lambda(1 + 3\mu + \mu^2)$

6. *(A. Gut's book)* Suppose that the offspring distribution in a branching process is the $\mathsf{Ge}(p)$-distribution (i.e. $\mathrm{P}(X = k) = q^k p$, $k = 0, 1, 2, \ldots$), and let $X(n)$ be the number of individuals in generation n, $n = 0, 1, 2, \ldots$

   (a) What is the probability of extinction?

   Now suppose that $p = 1/2$ and set $G_n(s) = G_{X(n)}(s)$.

   (b) Show that
$$G_n(s) = \frac{n - (n-1)s}{(n+1) - ns}, \quad n = 1, 2, \ldots$$

   (c) Show that
$$\mathrm{P}(X(n) = k) = \begin{cases} \dfrac{n}{n+1} & \text{for } k = 0 \\ \dfrac{n^{k-1}}{(n+1)^{k+1}} & \text{for } k = 1, 2, \ldots \end{cases}$$

   (d) Calculate $\mathrm{P}(X(n) = k \mid X(n) > 0)$.

   Finally, suppose that the population becomes extinct at generation number N.

   (e) Show that
$$\mathrm{P}(N = n) = G_{n-1}\left(\frac{1}{2}\right) - G_{n-1}(0), \quad n = 1, 2 \ldots$$

   (f) Show that $\mathrm{P}(N = n) = 1/(n(n+1))$, $n = 1, 2, \ldots$ (and hence that $P(N < \infty) = 1$, i.e., the probability of ultimate extinction is 1).

   (g) Compute $\mathrm{E}(N)$.

   Answer:
   (a) If $p < q$ then $d = p/q < 1$. If $p \geq q$, then $d = 1$
   (d) $(1/(n+1))(n/(n+1))^{k-1}, \quad k = 1, 2, \ldots$
   (g) $\mathrm{E}(N) = \infty$

7. Let M be an exponential random variable with parameter 1. We know that X given $\{M = m\}$ follows the Poisson distribution with parameter m. Compute the moment generating function of X and deduce that $\mathrm{P}(X = k) = (1/2)^{k+1}$, $k = 0, 1, 2, \ldots$

8. *(A. Gut's book)* Let X be the number of coin tosses until heads is obtained. Suppose that the probability of heads is unknown in the sense that we consider it to be a random variable $Y \sim \mathsf{U}(0, 1)$. Find the distribution of X.

   Answer: $\mathrm{P}(X = k) = \dfrac{1}{k(k+1)}$

9. *(Midterm exam November 2015)* At each time unit a virus replicates X copies of itself where $\Pr(X = k) = p^k$, $k \geq 1$, $0 < p \leq 1/2$. (So the probability of no replica is $1 - p/q$, $q = 1 - p$.) Assume that at time 0 there is a single originating virus.

   (a) Prove that the probability generating function of X is $G_X(s) = 1/(1 - ps) - (p/q)$.

   (b) Show that the probability of extinction is $1/p - 1/q$ if $p > (3 - \sqrt{5})/2$ and 1 otherwise.

   (c) If $p = 1/4$, what is the expected total number of viruses at the end?

   Answer: (c) 9/5