2.Stationary Processes
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Outline

- Weakly Stationary Time Series
- Stationary process
- AR, MA, ARMA models

Properties

1. mean value μ_t is constant and it does not depend on time t
2. $\gamma_x(s,t) = cov(x_s, x_t) = E[(x_s - \mu_s)(x_t - \mu_t)]$

   Note that: $\gamma_x(s,t) = \gamma_x(t,s)$
3. $\gamma_x(t,t) == E[(x_t - \mu_t)^2] = V(x_t)$

Autocovariance of white noise (ACV)

$$z_t \equiv \text{"}\textit{white noise}\text{"}$$

$$E(z_t) = 0$$

$$\gamma_z(s,t) = cov(z_s, z_t) = \begin{cases} \sigma_z^2 & , s\text{=t} \\ 0 & , s \neq \text{t} \end{cases}$$

Autocorrelation function (ACF)

$$\rho(s,t) = \frac{\gamma(s,t)}{\sqrt{\gamma(s,s)\gamma(t,t)}} = \frac{E[(x_s-\mu_s)(x_t-\mu_t)]}{\sqrt{E[(x_s-\mu_s)^2]E[(x_t-\mu_t)^2]}}$$

$$-1 \leq \rho \leq 1$$

Note: ρ does not have units

Stationary process (Weakly stationary)

Properties

1. $E(x_t) = \mu$
2. $V(x_t) = \sigma^2$
3. $\gamma(t+h, t) = cov(x_{t+h}, x_t) = cov(x_h, x_0) = \gamma(h, 0)$

Stationary process (Weakly stationary)

Autocovariance of stationary time series

$$\gamma(h) = cov(x_{t+h}, x_t) = E[(x_{t+h} - \mu)(x_t - \mu)]$$

Particular case: Autocovariance of white noise

$$\gamma_z(h) = cov(z_{t+h}, z_t) = \begin{cases} \sigma_z^2 & , \ h=0 \\ 0 & , \ h \neq 0 \end{cases}$$

Autocorrelation function

$$\rho(h) = \frac{\gamma(t+h,t)}{\sqrt{\gamma(t+h,t+h)\gamma(t,t)}} = \frac{\gamma(h)}{\gamma(0)}$$

And it is *stationary*

$$-1 \leq \rho(h) \leq 1$$

Partial Autocorrelation function

Ordinary Least Squares:

$x_t = \phi_{\mathbf{1,1}} x_{t-1} + Z_t$

$x_t = \phi_{1,2} x_{t-1} + \phi_{\mathbf{2,2}} x_{t-2} + Z_t$

$x_t = \phi_{1,3} x_{t-1} + \phi_{2,3} x_{t-2} + \phi_{\mathbf{3,3}} x_{t-3} + Z_t$

$$\vdots$$

$$x_t = \phi_{1,h} x_{t-1} + \phi_{2,h} x_{t-2} + \phi_{3,h} x_{t-3} + \ldots + \phi_{\mathbf{h,h}} x_{t-h} + Z_t$$

$$\vdots$$

PACF: $\{\phi_{\mathbf{1,1}}, \phi_{\mathbf{2,2}}, \ldots, \phi_{\mathbf{h,h}}, \ldots\}$

Stationary process (Weakly stationary)

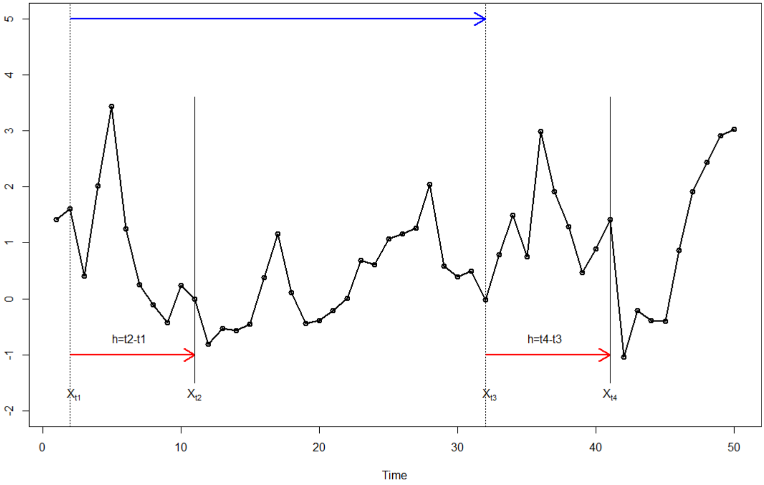


Figure: Example of an stationary process

Gaussian process $\{\ x_t\ \}$, $t = 1, \cdots, n$

$$\{x_t\} \sim \text{Multivariate Normal/Gaussian distribution}$$

$$E(x) = \mu = (\mu_1, \cdots, \mu_n)^{'}$$

Covariance matrix (nxn) is

$$V(x) = \Gamma = \{\gamma(t_i, t_j) | t, j = 1, \cdots, n\}$$

And the multivariate Normal density function can be written as

$$f(x) = (2\pi)^{-n/2} |\Gamma|^{-1/2} exp\{-1/2(x - \mu)^{'} \Gamma^{-1}(x - \mu)\}$$

where $|\Gamma| \equiv$ determinant

General Stochastic model for a time series:

$$x_t = G(x_{t-1}, x_{t-2}, ..., z_{t-1}, z_{t-2}, ...) + z_t \quad z_t \sim WN(0, \sigma_z)$$

Linear Stochastic model for a time series:

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + ... + \theta_1 z_{t-1} + \theta_2 z_{t-2} + ... + z_t \quad z_t \sim WN(0, \sigma_z)$$

$$\phi_1, \phi_2, ..., \theta_1, \theta_2, ... \in \mathbf{R}$$

The variable at time t is a linear combination of past observations and disturbances plus a new disturbance independent form the past

AR(p) models

A **pth-order autoregressive model**, or AR(p), takes the form:

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \cdots + \phi_p x_{t-p} + z_t$$

$$(1 - \phi_1 B - \phi_2 B^2 - \cdots + \phi_p B^p) x_t = z_t$$

where

x_t is stationary and $\phi_1, \cdots, \phi_p$ are the parameters (constants)

z_t is a Gaussian white noise ($E(z_t) = 0$, $V(z_t) = \sigma_z^2$)

B is the Backshift operator: $Bx_t = x_{t-1}$

p is the lag of the farthest observation included

An **AR(p) model** is a regression model with lagged values of the dependent variable in the independent variable positions, hence the name **Auto-Regressive model**.

- If $\mu \neq 0$, then

$$(x_t - \mu) = \phi_1(x_{t-1} - \mu) + \cdots + \phi_p(x_{t-p} - \mu) + z_t$$

or

$$x_t = \alpha + \phi_1 x_{t-1} + \cdots + \phi_p x_{t-p} + z_t$$

where

$$\alpha = \mu(1 - \phi_1 - \cdots - \phi_p)$$

Or more concisely: If $\phi(B) = (1 - \phi_1 B - \cdots - \phi_p B^p)$ (characteristic autoregressive polynomial)

$$\phi(B)x_t = z_t \tag{1}$$

AR(1): $(1-\phi B)x_t = z_t$

Considering $\mu = 0$, then

$$x_t = \phi x_{t-1} + z_t \quad z_t \sim N(0, \sigma_z^2)$$

Autocovariance function:

$$\gamma(0) = E[x_t^2] = E[(\phi x_{t-1} + z_t)^2] = \phi^2\gamma(0) + \sigma_z^2$$

$$\gamma(0) = \frac{\sigma_z^2}{1-\phi^2}$$

$$\gamma(1) = E[x_t x_{t-1}] = E[(\phi x_{t-1} + z_t)x_{t-1}] = \phi\gamma(0)$$

$$\vdots$$

$$\gamma(h) = E[x_t x_{t-h}] = E[(\phi x_{t-1} + z_t)x_{t-h}] = \phi\gamma(h-1) = \phi^h\gamma(0)$$

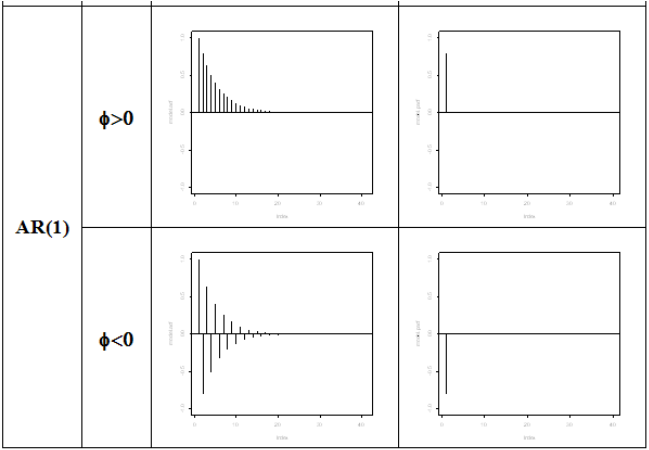
AR(1): $(1-\phi B)x_t = z_t$
Autocorrelation function:

$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \phi$$
$$\vdots$$
$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \phi^h$$

Recursion: $\rho(h) = \phi\rho(h-1)$
Partial Autocorrelation function:

$$\phi_{1,1} = \phi$$
$$\vdots$$
$$\phi_{h,h} = 0 \quad h > 1$$

AR(1): $(1 - \phi B)x_t = z_t$



AR(2): $(1-\phi_1 B - \phi_2 B^2)x_t = z_t$

Considering $\mu = 0$, then

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-1} + z_t \quad z_t \sim N(0, \sigma_z^2)$$

Autocovariance function:

$$\gamma(0) = E[x_t^2] = (\phi_1^2 + \phi_2^2)\gamma(0) + 2\phi_1\phi_2\gamma(1) + \sigma_z^2$$

$$\gamma(1) = E[x_t x_{t-1}] = \phi_1\gamma(0) + \phi_2\gamma(1)$$

$$\vdots$$

$$\gamma(h) = E[x_t x_{t-h}] = \phi_1\gamma(h-1) + \phi_2\gamma(h-2) \quad h > 1$$

MA(q) models

A **qth-order moving average model**, or MA(q), takes the form:

$$x_t = z_t + \theta_1 z_{t-1} + \theta_2 z_{t-2} + \cdots + \theta_q z_{t-q}$$

In other words,

$$x_t = \theta(B) z_t$$

The moving average operator is

$$\theta(B) = 1 + \theta_1 B + \theta_2 B^2 + \cdots + \theta_q B^q$$

An **MA(q) mode**l is a regression model with the dependent variable, x_t, depending on previous values of the errors rather than on the variable itself.

MA(q) models

MA(1): $x_t = (1 + \theta B) z_t$

Considering $\mu = 0$, then

$$x_t = z_t + \theta z_{t-1} \quad z_t \sim N(0, \sigma_z^2)$$

Autocovariance function:

$$\gamma(0) = E[x_t^2] = E[(z_t + \theta z_{t-1})^2] = (1 + \theta^2)\sigma_z^2$$

$$\gamma(1) = E[x_t x_{t-1}] = E[(z_t + \theta z_{t-1})(z_{t-1} + \theta z_{t-2})] = \theta \sigma_z^2$$

$$\vdots$$

$$\gamma(h) = E[x_t x_{t-h}] = 0 \quad h > 1$$

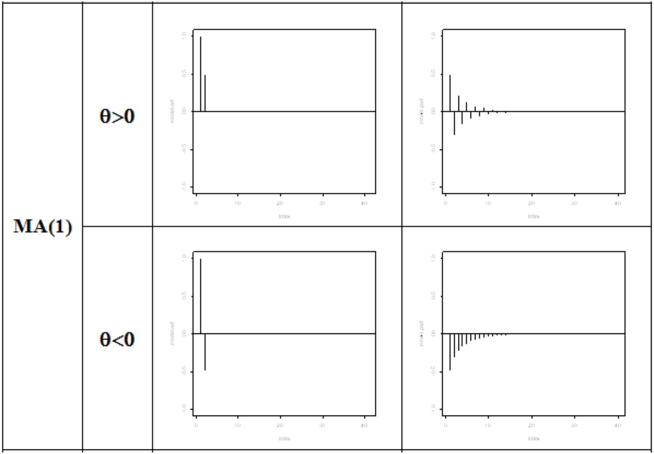
MA(q) models

MA(1): $x_t = (1 + \theta B) z_t$
Autocorrelation function:

$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{\theta}{1 + \theta^2}$$
$$\vdots$$
$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = 0 \quad h > 1$$

MA(q) models

MA(1): $x_t = (1 + \theta B) z_t$



ARMA(p,q) models

A times series $\{x_t; t = 0, \pm 1, \pm 2, \cdots\}$ is an **AutoRegressive Moving Average model**, ARMA (p,q), if it is stationary and

$$x_t = \phi_1 x_{t-1} + \cdots + \phi_p x_{t-p} + z_t + \theta_1 z_{t-1} + \cdots + \theta_q z_{t-q}$$

with,
$\phi_p \neq 0$, $\theta_q \neq 0$ and $\sigma_z^2 > 0$

The parameters p and q are called the autoregressive and the moving average orders, respectively.

ARMA(p,q) models

If x_t has a nonzero mean μ and $\alpha = \mu(1 - \phi_1 - \cdots - \phi_p)$, the model del will be:

$$x_t = \alpha + \phi_1 x_{t-1} + \cdots + \phi_p x_{t-p} + z_t + \theta_1 z_{t-1} + \cdots + \theta_q z_{t-q}$$

When $q = 0$, the model is an AR(p).
When $p = 0$, the model is a MA(q)

ARMA(p,q) model can be written in concise form as,

$$\Phi(B) x_t = \theta(B) z_t$$

Under certain conditions of stationarity, the ARMA(p, q) models can be expressed as an AR(∞) or MA(∞)model:

$$\phi(B)x_t = \theta(B)z_t$$

Expression as an AR(∞):

$$\frac{\phi(B)}{\theta(B)}x_t = \pi(B)x_t = z_t$$

Expression as an MA(∞):

$$x_t = \frac{\theta(B)}{\phi(B)}z_t = \psi(B)z_t$$