<u>**Application of LPP in Game Theory:**</u>

<u>**Two person zero sum game:**</u>

In a game, if the algebraic sum of gains and losses of all the players is zero, called as "Zero sum game". A game with two players, where a gain of one player equals the loss to the other is known "two person zero sum game". In a conflict, each of two **players** (opponents) has a (finite or infinite) number of alternatives or **strategies**. Associated with each pair of strategies is the **payoff** one player receives from the other.

The characteristics of two person zero sum game are: **There are only two players with exactly the opposite interests. Each player has finite number of strategies. Each specific strategy results in a pay-off. Sum of pay-offs at the end of each play is zero.**

Designating the two players as *A* and *B* with *m* and *n* strategies, respectively, the game is usually presented in terms of the payoff matrix to player *A* as

$$
\begin{array}{c|cccc|}
 & B_1 & B_2 & \ldots & B_n \\
\hline
A_1 & a_{11} & a_{12} & \ldots & a_{1m} \\
A_2 & a_{21} & a_{22} & \ldots & a_{2m} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
A_m & a_{m1} & a_{m1} & \ldots & a_{mn} \\
\hline
\end{array}
$$

This representation indicates that if *A* uses strategy *i* and *B* uses strategy *j*, the payoff to *A* is a_{ij}, and the payoff to *B* is $-a_{ij}$.

Optimal Solution of Game:

Objective function for Players for above representation

for A: maximize its minimum gain (maxmin u)

for B: minimize its maximize loss (minmax v)

The optimal solution can be in the form of a single pure strategy or several strategies mixed randomly.

Pure Strategy game:

Ex**:** Two companies, *A* and *B*, sell two brands of flu medicine. Company *A* advertises in radio (A_1), television (A_2), and newspapers (A_3). Company *B*, in addition to using radio (B_1), television (B_2), and newspapers (B_3), also mails brochures (B_4). Depending on the effectiveness of each advertising campaign, one company can capture a portion of the market from the other. The following matrix summarizes the percentage of the market captured or lost by company A:

$$
\begin{array}{c|cccc|}
 & B_1 & B_2 & B_3 & B_4 \\
\hline
A_1 & 8 & -2 & 9 & -3 \\
A_2 & 6 & \mathbf{5} & 6 & 8 \\
A_3 & -2 & 4 & -9 & 5 \\
\hline
\end{array}
$$

Solution: The solution of the game is based on the principle of securing the *best of the worst or maximum of minimum gain,* for each player.

$$
\begin{array}{c|cccc|l}
 & B_1 & B_2 & B_3 & B_4 & \text{Row min} \\
\hline
A_1 & 8 & -2 & 9 & -3 & -3 \\
A_2 & 6 & \mathbf{5} & 6 & 8 & \mathbf{5} \leftarrow \text{Maximin} \\
A_3 & -2 & 4 & -9 & 5 & -9 \\
\hline
\text{Column max} & 8 & \mathbf{5} & 9 & 8 & \\
 & & \uparrow & & & \\
 & & \text{Minimax} & & & \\
\end{array}
$$

If A chooses A_1(=8) strategy with maximum value of row then B tries to minimize its loss and play with B_4 strategy, thus A loses 3% of the market share to B.

Similarly, with strategy A_2, the minimum value is for A to capture 5% from B, and for strategy A_3, the minimum is for A to lose 9% to B. These results are listed under *row min*. To achieve the maximum of the *minimum*, Company A chooses strategy A_2 because it corresponds to the maximin value.

Next, for Company B, the given payoff matrix is for A and B's *best of the worst* solution is based on the minimax value. The result is that Company B will select strategy B_2. The optimal solution of the game calls for selecting strategies A_2 and B_2, which means that both companies should use television advertising. The payoff will be in favor of company A, because its market share will increase by 5%. In this case, we say that the **value of the game** is 5% and that A and B are using a **pure strategy saddle-point** solution.

Saddle Point

Saddle point in a pay-off matrix is the position in the matrix **where the maximum of row minimum (maximin) coincides with the minimum of column maximum (minimax).** The pay-off at the saddle point is called the **value of the game which equals minimax of maximin value.** The optimum solution can be had for the game by applying minimax and maximin. Thus, the solution of the game is: Best strategy for A is A_2; Best strategy for B is B_2; Value of the game is 5.

Steps to find out saddle point

1. Find out the minimum element of each row of the pay-off matrix
2. Find out the maximum element in each column of the pay-off matrix
3. If these two values are same, that value is the saddle point of the matrix.

In pure strategy game both the players choose only one strategy to optimize the solution. The saddle-point solution precludes the selection of a better strategy by either company. If B moves to another strategy (B_1, B_3, or B_4), Company A can stay with strategy A_2, ensuring worse loss for B (6% or 8%). By the same token, A would not seek a different strategy because B can change to B_3 to realize a 9% market gain if A_1 is used and 3% if A_3 is used.

TWO PERSON ZERO SUM GAME WITH MIXED STRATEGIES (WITHOUT SADDLE POINT)

Two players, A and B, play the coin-tossing game. Each player, unbeknownst to the other, chooses a head (H) or a tail (T). Both players would reveal their choices simultaneously. If they match (HH or TT), player A receives \$1 from B. Otherwise, A pays B \$1.

The following payoff matrix for player A gives the row-min and the column-max values corresponding to A's and B's strategies, respectively:

	B_H	B_T	Row min
A_H	1	−1	−1
A_T	−1	1	−1
Column max	1	1	

The maximin and the minimax values of the games are - \$1 and \$1, respectively, and the game does not have a pure strategy solution because the two values are not equal.

Optimal Solution by LPP method:

Graphical Method: Games with mixed strategies can be solved either graphically or by linear programming. The graphical solution is suitable for games with exactly two pure strategies for one or both players. Linear programming, on the hand, can solve any two-person zero-sum game.

The graphical method consists of two graphs.

(i) the pay-off (gains) available to player 'A' against his strategies and options.

(ii) the pay-off (losses) faced by player 'B' against his strategies and options.
Let pay off table for A is

$$\begin{array}{rc|cccc|}
 & & y_1 & y_2 & \dots & y_n \\
 & & B_1 & B_2 & \dots & B_n \\
\hline
x_1: & A_1 & a_{11} & a_{12} & \dots & a_{1m} \\
1 - x_1: & A_2 & a_{21} & a_{22} & \dots & a_{2m} \\
\hline
\end{array}$$

here x_1 is the probability or proportion of strategy A_1 of A and for A_2 it is 1-x_1 (since for A_1 and A_2, sum of probabilities is 1). Similarly for B, $\sum_i y_i = 1$.

Illustration of the graphical method

The pay-off matrix is given below.

		Player B 1	2	3
Player A	1	4	2	3
	2	2	5	6

The problem has no saddle point.

$\therefore$ There are no pure strategies and mixed strategies are to be adopted.

Player 'A' adopts the probabilities x_1 and x_2 for strategies 1 and 2. for player 'B' the mixed strategies with probabilities are y_1, y_2 and y_3 respectively. At the optimal level with the value of the game, as V the following relationship can be established.

$$x_1 + x_2 = 1 \qquad ...(1)$$
$$y_1 + y_2 + y_3 = 1 \qquad ...(2)$$
$$4x_1 + 2x_2 \geq V \qquad ...(3)$$

$$
\begin{aligned}
2x_1 + 5x_2 &\geq V & \quad ...(4)\\
3x_1 + 6x_2 &\geq V & \quad ...(5)\\
4y_1 + 5y_2 + 6y_3 &\leq V & \quad ...(6)\\
2y_1 + 5y_2 + 6y_3 &\leq V & \quad ...(7)
\end{aligned}
$$

The above equations can be written in terms of the player having two strategies. *i.e.* in terms of player A.

$$\therefore \qquad x_2 = 1 - x_1$$

Substituting the value of x_2 in equations (3), (4) and (5). we get, Equation 93) can be written as:

$$
\begin{aligned}
4x_1 + 2\,(1 - x_1) &\geq V\\
2x_1 + 2 &\geq V\\
V - 2x_1 &\leq 2 & \quad ...(8)
\end{aligned}
$$

equation 4 can be written as

$$
\begin{aligned}
2\,x_1 + 5\,(1 - x_1) &\geq V\\
3x_1 + 5 &\geq V\\
V + 3x_1 &\leq 5 & \quad ...(9)
\end{aligned}
$$

equation 5 can be written as

$$
\begin{aligned}
3x_1 + 6\,(1 - x_2) &\geq V\\
-\,3x_1 + 6 &\geq V\\
V + 3x_1 &\leq 6 & \quad ...(10)
\end{aligned}
$$

Player A's objective is to maximise the value of 'V' and to find the combination of x_1 and x_2 which gives the maximum value.

The graph of x_1 versus V can be drawn with the relationships in equations (8), (9) and (10) by plotting x_1 on x-axis and 'V' on y-axis. The range of x_1 is between 0 and 1, and so we plot the graph within 0 and 1 of x_1.

by equation (8)

When $\qquad x_1 = 0, V = 2$

$\qquad\qquad x_1 = 1, V = 4.$

from equation (9)

$$x_1 = 0, V = 5$$

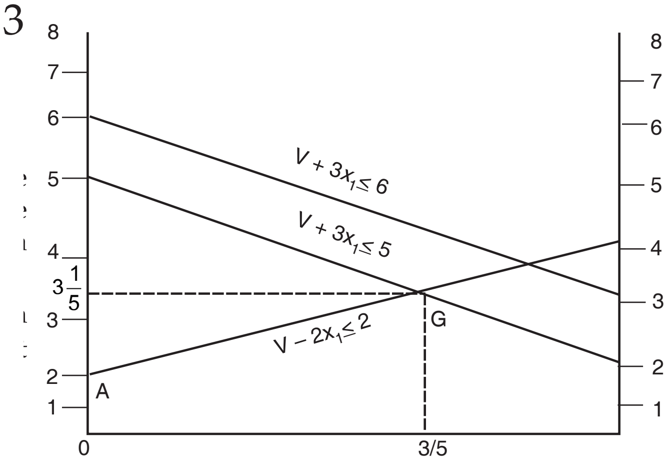
$$x_1 = 1, V = 2$$

from equation (10), when

$$x_1 = 0, V = 6$$
$$x_1 = 1, V = 3$$

The maximum value of 'V' in this region is given at point G. At this point, $V = 3\frac{1}{5}$ and

$$x_1 = \frac{3}{5}, x_2 = 1 - x_1 = 2/5$$



This diagram reveals that B will never play y_3 in which case his loss will be more than what it is by playing with strategy y_2 as $V + 3x_1 \leq 6$ (representing strategy y_3) which is above $V + 3x_1 \leq 5$ (representing strategy y_2). Eliminating this strategy for player B, we can plot the graphs for player B.

Equations (2), (6) and (7) can be rewritten as

$$y_1 + y_2 = 1 \quad ...(11)$$
$$4y_1 + 2y_2 \leq V \quad ...(12)$$
$$2y_1 + 5y_2 \leq V \quad ...(13)$$

$\therefore \quad y_2 = 1 - y_1$ Substituting for y_2 in the equations (12) becomes

$4y_1 + 2(1 - y_1) \leq v$ *i.e.* $2y_1 + 2 \leq V$

i.e. $\quad V - 2y_1 \geq 2 \quad ...(14)$

Equation (13) becomes

$$2y_1 + 5(1 - y_1) \leq V, \text{ i.e. } -3y_1 + 5 \leq V$$

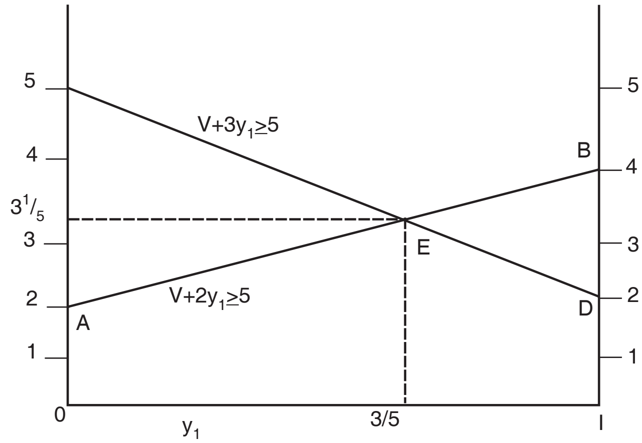
i.e. $\quad V + 3y_1 \geq 5 \quad ...(15)$

From equation (14), when $y_1 = 0, V = 2$

$y_1 = 1, V = 4$

From equation (15), when $y_1 = 0, V = 5$

$y_1 = 1, V = 2$



B's objective is to minimise the value of *V*, this arises in the feasible region at point *E*. Giving the values as:

$$V = 3\frac{1}{5}, y_1 = \frac{3}{5}, y_2 = 1 - y_1 = \frac{2}{5},$$

Solution by Simplex Method:

			Player B		
		1	2	3	Maximin
	1	3	-4	2	-4
Player A	2	1	-3	-7	-7
	3	-2	4	7	**-2**
	Minimax	**3**	4	7	

Minimax is not equal to Maximin

Value of Game lies between -2 and 3.

For simplex method add upper bound of value of function to each value of table.

Let x_1, x_2 and x_3 are probabilities for strategies 1, 2, and 3 of player A and y_1, y_2 and y_3 are for player B.

Player A's optimal probabilities, x_1, x_2, and x_m, can be determined by solving the following maximin problem:

$$\max_{x_i}\left\{\min\left(\sum_{i=l}^{m} a_{i1}x_i, \sum_{i=1}^{m} a_{i2}x_i, \ldots, \sum_{i=1}^{m} a_{in}x_i\right)\right\}$$

$$x_1 + x_2 + \cdots + x_m = 1$$

$$x_i \geq 0, i = 1, 2, \ldots, m$$

Let

$$v = \min\left\{\sum_{i=1}^{m} a_{i1}x_i, \sum_{i=1}^{m} a_{i2}x_i, \ldots, \sum_{i=1}^{m} a_{in}x_i\right\}$$

The equation implies that

$$\sum_{i=1}^{m} a_{ij}x_i \geq v, j = 1, 2, \ldots, n$$

Player A's problem thus can be written as

$$\text{Maximize } z = v$$

subject to

$$v - \sum_{i=1}^{m} a_{ij}x_i \leq 0, j = 1, 2, \ldots, n$$

$$x_1 + x_2 + \cdots + x_m = 1$$

$$x_i \geq 0, i = 1, 2, \ldots, m$$

$$v \text{ unrestricted}$$

Note that the value of the game, v, is unrestricted in sign.

Player B's optimal strategies, $y_1, y_2, \ldots,$ and y_n, are determined by solving the problem

$$\min_{y_j}\left\{\max\left(\sum_{j=1}^{n} a_{1j}y_j, \sum_{j=1}^{n} a_{2j}y_j, \ldots, \sum_{j=1}^{n} a_{mj}y_j\right)\right\}$$

$$y_1 + y_2 + \cdots + y_n = 1$$

$$y_j \geq 0, j = 1, 2, \ldots, n$$

Using a procedure similar to that of player A, B's problem reduces to

$$\text{Minimize} \quad w = v$$

subject to

$$v - \sum_{j=1}^{n} a_{ij}y_j \geq 0, i = 1, 2, \ldots, m$$

$$y_1 + y_2 + \cdots + y_n = 1$$

$$y_j \geq 0, j = 1, 2, \ldots, n$$

$$v \text{ unrestricted}$$

			Player B		
		1	2	3	
	1	6	-1	5	
Player A	2	4	0	-4	
	3	1	7	10	

Thus for problem objective function is

Minimize v

subject to

$$6y_1 - y_2 + 5y_3 \leq v \qquad (1)$$

$$4y_1 + y_2 - 4y_3 \leq v \qquad (2)$$

$$y_1 + 7y_2 + 10y_3 \leq v \qquad (3)$$

$$y_1 + y_2 + y_3 = 1 \qquad (4)$$

divide by v

$$6\frac{y_1}{v} - \frac{y_2}{v} + 5\frac{y_3}{v} \leq 1 \qquad (5)$$

$$4\frac{y_1}{v} + \frac{y_2}{v} - 4\frac{y_3}{v} \leq 1 \qquad (6)$$

$$\frac{y_1}{v} + 7\frac{y_2}{v} + 10\frac{y_3}{v} \leq 1 \qquad (7)$$

$$\frac{y_1}{v} + \frac{y_2}{v} + \frac{y_3}{v} \leq \frac{1}{v} \qquad (8)$$

Let $\frac{y_1}{v} = k_1, \frac{y_2}{v} = k_2, \frac{y_3}{v} = k_3$ and $\frac{1}{v} = k_1 + k_2 + k_3$

Now Minimize v or Maximize $\frac{1}{v}$ i.e. Maximize $k_1 + k_2 + k_3$

subject to

$6k_1 - k_2 + 5k_3 \leq 1$ (9)

$4k_1 + k_2 - 4k_3 \leq 1$ (10)

$k_1 + 7k_2 + 10k_3 \leq 1$ (11)

$k_1 \geq 0, k_2 \geq 0, k_3 \geq 0$

Basis	X1	X2	X3	X4	X5	X6	RHS
x_4	6	-1	5	1	0	0	1
x_5	4	0	-4	0	1	0	1
x_6	1	7	10	0	0	1	1
	-1	-1	-1	0	0	0	0
x_1	1	$-\frac{1}{6}$	$\frac{5}{6}$	$\frac{1}{6}$	0	0	$\frac{1}{6}$
x_5	0	$\frac{2}{3}$	$-\frac{22}{3}$	$-\frac{2}{3}$	1	0	$\frac{1}{3}$
x_6	0	$\frac{43}{6}$	$\frac{55}{6}$	$-\frac{1}{6}$	0	1	$\frac{5}{6}$
	0	$-\frac{7}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	0	0	$\frac{1}{6}$
x_1	1	0	$\frac{45}{43}$	$\frac{7}{43}$	0	$\frac{1}{43}$	$\frac{8}{43}$
x_5	0	0	$-\frac{352}{43}$	$-\frac{28}{43}$	1	$-\frac{4}{43}$	$\frac{11}{43}$
x_2	0	1	$\frac{55}{43}$	$-\frac{1}{43}$	0	$\frac{6}{43}$	$\frac{5}{43}$
	0	0	$\frac{57}{43}$	$\frac{6}{43}$	0	$\frac{7}{43}$	$\frac{13}{43}$

$k_1 + k_2 + k_3$=8/43+5/43+0=13/43=1/v

thus v=43/13

$$k_1 * v = \frac{8}{43} * \frac{43}{13} = \frac{8}{13}, k_2 * v = \frac{5}{43} * \frac{43}{13} = \frac{5}{13} \text{ and } k_3 * v = 0 * \frac{43}{13} = 0$$

Value of game is $v^* = v - 3 = \frac{43}{13} - 3 = \frac{4}{13}$