✱ Finite difference approximations for partial derivatives

$$u_x = \frac{u_{i+1,j} - u_{i,j}}{h} \quad [F.D]$$

$$= \frac{u_{i,j} - u_{i-1,j}}{h} \quad [B.D]$$

$$= \frac{u_{i+1,j} - u_{i-1,j}}{2h} \quad [C.D]$$

for,

$$u_y = \frac{u_{i,j+1} - u_{i,j}}{k} \quad [F.D]$$

$$= \frac{u_{i,j} - u_{i,j-1}}{k} \quad [B.D]$$

$$= \frac{u_{i,j+1} - u_{i,j-1}}{2k} \quad [C.D]$$

$$u_{xx} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} \quad [C.D]$$

$$u_{yy} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} \quad [C.D]$$

(*) let the (x, y) plane be devided into a network of rectangles of sides $\Delta x = h$ & $\Delta y = k$ by drawing the sets of lines,

$$x = ih \quad , \quad i = 0, 1, 2, \dots$$
$$y = jk \quad , \quad j = 0, 1, 2, \dots$$

→ For elliptic :— laplace & poisson's equation are elliptic.

(1) Solution for elliptic eq^n :—

$$u_{xx} + u_{yy} = 0 \quad \text{—— (1)}$$

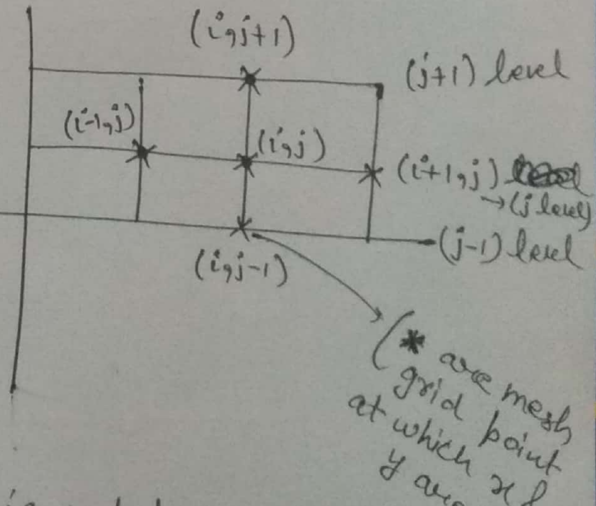
Using above rules,

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = 0 .$$

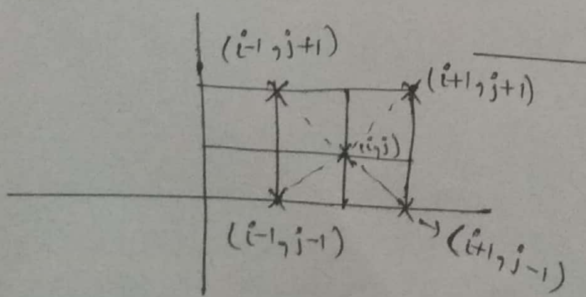
By taking a square mesh putting $h = k$. We get,

$$\boxed{u_{ij} = \frac{1}{4}\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}\right)}$$

→ This is standard five point formula.



OR.



leplace eq^n is invarient when it is rotated by $45°$.

$$\Rightarrow \boxed{u_{ij} = \frac{1}{4}\left(u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j-1} + u_{i+1,j-1}\right)}$$

This is diagonal five point formula.

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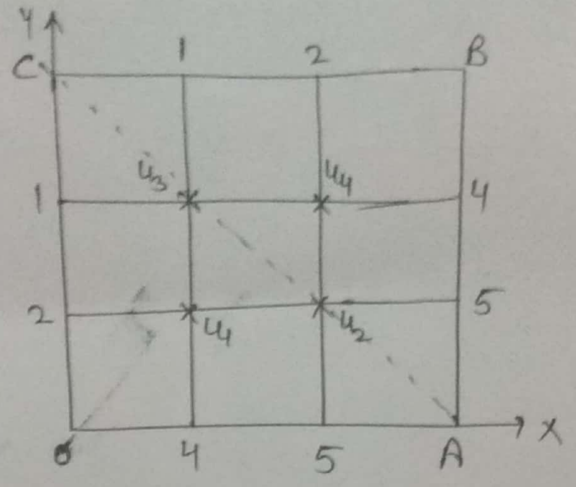
After finding the values of $u_{i,j}$ once, their accuracy is improved by Gauss-Seidal Method.

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j+1}^{(n+1)} + u_{i,j-1}^{(n)}\right]$$

Que:- Solve Laplace's equation for the square region shown in Fig 9·4, the boundary values being as indicated.



Sol:- From figure, It is shown that the boundary Values are symmetric about the diagonal AC.
Hence, $u_1 = u_4$ & We need find only u_1, u_2 & u_3.
The S.F.P.F → Standard five point formula,

$$u_1 = \frac{1}{4}(u_2+u_3+2+4) = \frac{1}{4}(u_2+u_3+6)$$

⇒ Iteration formula is therefore,
(~~Gauss~~ Seidal Method)

$$\boxed{u_4^{(n+1)} = \frac{1}{4}\left[u_2^{(n)} + u_3^{(n)} + 6\right]}$$

& $$u_2 = \frac{1}{4}\left[5+5+u_1+u_4\right] = \frac{1}{4}\left[10+2u_4\right] \quad [\because u_1 = u_4]$$

⇒ $$u_2 = \frac{5}{2} + \frac{u_4}{2}$$

⇒ $$\boxed{u_2^{(n+1)} = \frac{5}{2} + \frac{u_4^{(n+1)}}{2}}$$

& $$u_3 = \frac{1}{4}\left[1+1+2u_4\right] = \frac{1}{2} + \frac{u_4}{2}$$

⇒ $$\boxed{u_3^{(n+1)} = \frac{1}{2}u_4^{(n+1)} + \frac{1}{2}}$$

For first iteration :—

let $u_2^{(0)} = 5$ & $u_3^{(0)} = 1$ [$\because$ it is nearer to the value $u=5$ & $u=1$]

$u_1^{(1)} = \frac{1}{4}(5+1+6) = 3$.

$u_2^{(1)} = \frac{1}{2}(3) + \frac{5}{2} = 4$.

$u_3^{(1)} = \frac{1}{2}(3) + \frac{1}{2} = 2$.

For second iteration :— $u_4^{(2)} = \frac{1}{4}(4+2+6) = 3$

$u_2^{(2)} = \frac{1}{2}(3) + \frac{5}{2} = 4$

& $u_3^{(2)} = \frac{1}{2}(3) + \frac{1}{2} = 2$.

$\Rightarrow u_1 = 3$, $u_2 = 4$, $u_3 = 2$ & $u_4 = 3$.

(2) Poisson's Equation :— $\nabla^2 u = f(x,y)$

$\Rightarrow u_{xx} + u_{yy} = f(x,y)$ —— (1)

Sol: — Put $x = ih$, $y = jk$

$\Rightarrow (x,y) = (ih, jk)$

for square $(x,y) = (ih, jh)$ [$\because h = k$]

Replace (1) by finite differences to solve this numerically,

$\Rightarrow \frac{1}{h^2}\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right] + \frac{1}{h^2}\left[u_{i,j+1} - 2u_{i,j} + u_{i,j+1}\right] = f(ih, jh)$

$\Rightarrow \boxed{u_{i-1,j} + u_{i+1,j} + u_{i,j+1} - 4u_{i,j} + u_{i,j-1} = h^2 f(ih, jh)}$

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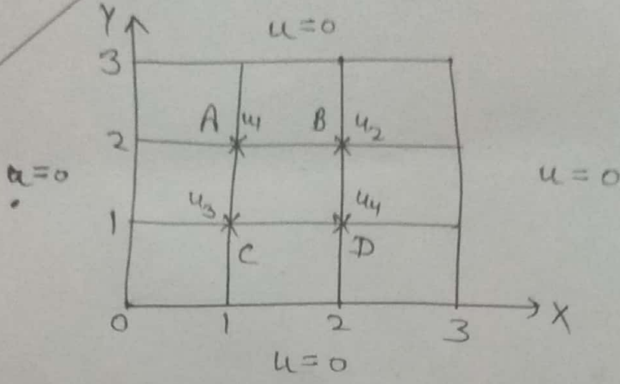
Que:- Solve the eqn, $\nabla^2 u = -10(x^2+y^2+10)$ over the square mesh with sides, $x=0$, $y=0$, $x=3$, $y=3$ with $u=0$ on the boundary and mesh length is 1.

Soln



Given, $\nabla^2 u = -10(x^2+y^2+10)$

⇒ $u_{xx}+u_{yy} = -10(x^2+y^2+10)$

Replace $u_{xx}+u_{yy}$ be finite difference & let $h=1$.

⇒ $$\boxed{u_{i-1,j}+u_{i+1,j}+u_{i,j-1}+u_{i,j+1}-4u_{i,j} = f(i,j) = -10(i^2+j^2+10)} \quad [\because h=k=1] \quad \text{— (1)}$$

$x=ih$, $y=jh$ & $h=1$

⇒ $x=i$, $y=j$ & $i,j=0,1,2,3$.

At the point C, $i=1$, $j=1$. ⇒ ~~Using Standard five point formula~~

~~$u_1 = \frac{1}{4}(\ldots)$~~ Put, $i=1$, $j=1$ in eqn (1).

$$u_{0,1}+u_{2,1}+u_{1,0}+u_{1,2}-4u_{1,1} = -10(1+1+10)$$

$$0+u_4+0+u_1-4u_3 = -120$$

⇒ $$u_3 = \frac{1}{4}[u_1+u_4+120] \quad \text{— (2)}$$

At A, $i=1$, $j=2$.

$$u_1 = \frac{1}{4}[u_2+u_3+150] \quad \text{— (3)}$$

At B, $i=2$, $j=2$

$$u_2 = \frac{1}{4}[u_1+u_4+180] \quad \text{— (4)}$$

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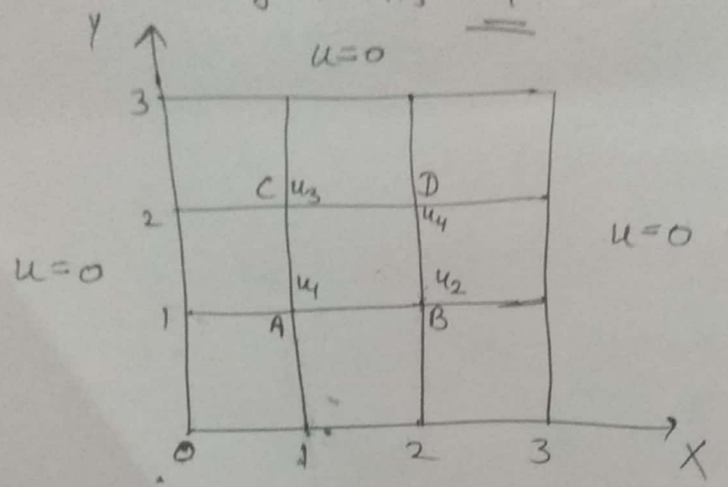
Que:- Solve the poisson equation (4)

$u_{xx} + u_{yy} = -10(x^2+y^2+10)$ over the square mesh with sides,

$x=0,\ y=0,\ x=3,\ y=3$ with $u=0$ on the boundary and mesh length is 1.

Sol:-



Let $x = ih,\ y = jh,\ \&\ h=1,\ i,j = 0,1,2,3$.

At A, $i=1, j=1$. So, by standard five-point formula,

$$u_1 = \frac{1}{4}(u_2+u_3+120) \quad \left[\because u_2+u_3+0+0-4u_1 = -10(1+1+10)\right]$$

$$\left[\because u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = f(i,j) = -10(i^2+j^2+10)\right.$$

$\Rightarrow$ Put $i=1$ & $j=1$, $u_{0,1} + u_{2,1} + u_{1,0} + u_{1,2} - 4u_{1,1} = -10(1+1+10)$

$$\Rightarrow 0 + u_2 + 0 + u_3 - 4u_1 = -120$$

$$\Rightarrow u_1 = \frac{1}{4}(u_2+u_3+120) \text{ --- i)}$$

Similarly,

$$u_2 = \frac{1}{4}(u_1+u_4+150) \text{ --- ii)}$$

$$u_3 = \frac{1}{4}(u_1+u_4+150) \text{ --- iii)}$$

$$u_4 = \frac{1}{4}(u_2+u_3+180) \text{ --- iv)}$$

From ii) & iii) $u_2 = u_3$.

So, Iteration formula is, [Gauss-Iteration use]

$$\boxed{u_1^{(n+1)} = \frac{1}{4}(2u_2^{(n)}+120) = \frac{1}{2}u_2^{(n)} + 30}$$

$$u_2^{(n+1)} = \frac{1}{4}\left[u_1^{(n+1)} + u_4^{(n)} + 150\right]$$

$$u_4^{(n+1)} = \frac{1}{2} u_2^{(n+1)} + 45 .$$

For the first iteration, we assume that

$u_2^{(0)} = u_4^{(0)} = 0$. Hence,

$$u_1^{(1)} = 30$$

$$u_2^{(1)} = \frac{1}{4}(30+0+150) = 45$$

$$u_4^{(1)} = \frac{1}{2}(45) + 45 = 67.5$$

For the second iteration,

$$u_1^{(2)} = \frac{1}{2} u_2^{(1)} + 30 = \frac{1}{2}(45) + 30 = 52.5$$

$$u_2^{(2)} = \frac{1}{4}\left[u_1^{(2)} + u_4^{(1)} + 150\right] = \frac{1}{4}\left[52.5 + 67.5 + 150\right] = 67.5$$

$$u_4^{(2)} = \frac{1}{2}\left[u_2^{(2)}\right] + 45 = 78.75$$

$\vdots$

$u_1 = 67$, $u_2 = u_3 = 75$ & $u_4 = 83$.

Parabolic Equation :-

Heat eqⁿ in 1-D, solⁿ → $u(x,t)$

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{--- (1)}$$ is parabolic in nature

$$B^2 - 4AC = 0$$

It represents 2nd order P.D.E in parabolic nature.

→ Replace $\frac{\partial u}{\partial t}$ & $\frac{\partial^2 u}{\partial x^2}$ by finite differences & ~~...~~, ~~...~~, & $c^2 = \frac{k}{s\rho}$, ρ = density, k = thermal conductivity, s = specific heat.

Method :-

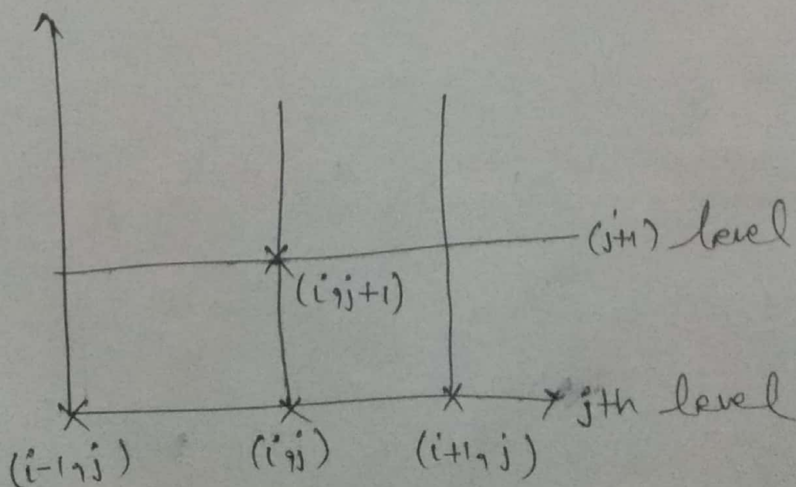
(1) **Bender-Schmidt Method** :- Again we take rectangular diagram in which $x = ih$ & $t = jk$. Replace (1) by finite difference. We get, (forward difference) (This is valid when $0 < \alpha \le \frac{1}{2}$)

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{c^2}{h^2}\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right]$$

Now,

$$\boxed{u_{i,j+1} = \alpha u_{i-1,j} + (1-2\alpha)u_{i,j} + \alpha u_{i+1,j}}$$

$\alpha = \frac{c^2 k}{h^2}$ → Mesh ratio parameter.



Note:— In particular when, $\alpha = \frac{1}{2}$

So, $u_{i,j+1} = \frac{1}{2}\left[u_{i-1,j} + u_{i+1,j}\right]$

→ Bender Schmidth Method.

Que:— $u_t = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}$, $\quad u(0,t)=0$, $u(4,t)=0$

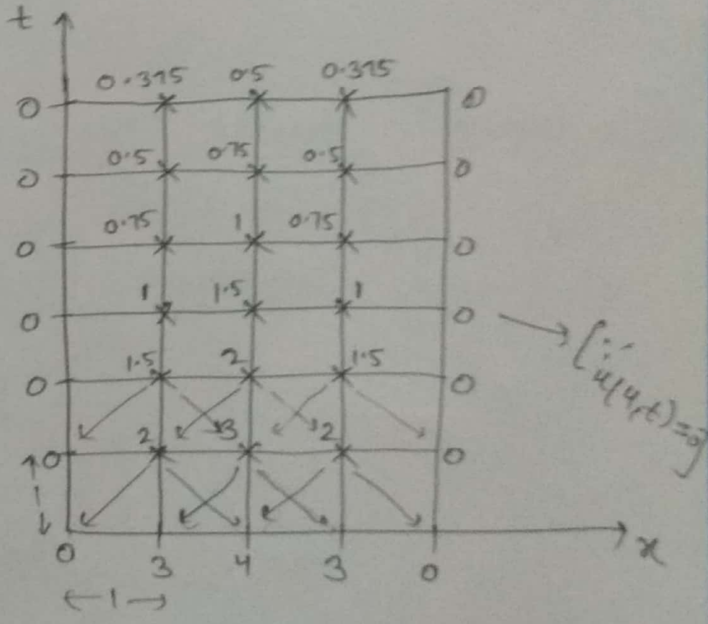
continue the solⁿ through 10 steps. $\quad u(x,0) = x(4-x)$, $h=1$

Sol:— Here, $c^2 = \frac{1}{2}$ & $\alpha = \frac{1}{2}$ (given)

$$\alpha = \frac{kc^2}{h^2} \Rightarrow \frac{1}{2} = k \times \frac{1}{2} \Rightarrow \boxed{k=1}$$

Now,

$u(x,0) = u(i,0) = i(4-i)$
$\quad i = 0,1,2,3,4$



$i=0$, $u(0,0)=0$
$i=1$, $u(1,0)=3$
$i=2$, $u(2,0)=4$
$i=3$, $u(3,0)=3$
$i=4$, $u(4,0)=0$

& $u(0,t)=0$ & $u(4,t)=0$

Now, using Bender smith.

$$u_{i,j+1} = \frac{1}{2}\left[u_{i-1,j} + u_{i+1,j}\right]$$

Now, $i=1, j=0$, $\quad u_{1,1} = \frac{1}{2}\left[u_{0,0} + u_{2,0}\right] = \frac{1}{2}[0+4] = 2$

$i=2, j=0$, $\quad u_{2,1} = \frac{1}{2}\left[u_{1,0} + u_{3,0}\right] = \frac{6}{2} = 3$

⋮

② Crank - Nicolson Method :— (Solⁿ of Parabolic eqⁿ) ⑥

Now, $u_t = c^2 u_{xx}$

According to this method, u_{xx} is replaced by the averge of its central difference approximations on the jth & (j+1)th time rows.

∴ $u_t = c^2 u_{xx}$ becomes,

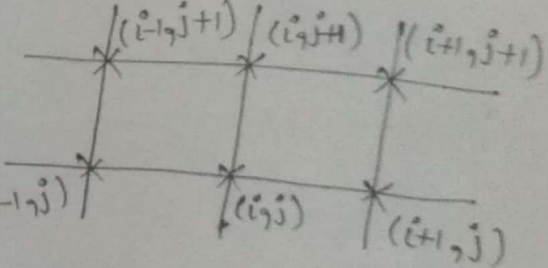
$$\frac{u_{i,j+1} - u_{i,j}}{k} = c^2 \left[\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j} + u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}}{h^2} \right]$$

It becomes.

$$\boxed{-\frac{\alpha}{2} u_{i-1,j+1} + (1+\alpha) u_{i,j+1} - \frac{\alpha}{2} u_{i+1,j+1} = \frac{\alpha}{2} u_{i-1,j} + (1-\alpha) u_{i,j} + \frac{\alpha}{2} u_{i+1,j}}$$

$$\alpha = \frac{kc^2}{h^2}$$

This is crank - Nicolson formula.



Que :— Solve the equation,

$$u_t = u_{xx} \quad \text{s.t.},$$

$u(x,0) = \sin \pi x$, $0 \le x \le 1$,

$u(0,t) = u(1,t) = 0$

Using crank Nicolson Method. Do two time step with $h = \frac{1}{3}$, $k = \frac{1}{36}$.

Sol :— compairing it with, $u_t = c^2 u_{xx}$, $c^2 = 1$

$$\Rightarrow \alpha = \frac{kc^2}{h^2} = \frac{1}{4}$$

Now, Using Crank Nicolson relation,

$$-\frac{1}{8}u_{i-1,j+1}+\left(\frac{5}{4}\right)u_{i,j+1}-\frac{1}{8}u_{i+1,j+1}=\frac{1}{8}u_{i-1,j}+\left(\frac{3}{4}\right)u_{i,j}+\frac{1}{8}u_{i+1,j}$$

$$\boxed{\Rightarrow -u_{i-1,j+1}+10u_{i,j+1}-u_{i+1,j+1}=u_{i-1,j}+6u_{i,j}+u_{i+1,j} \quad (1)}$$

Now, $u(x,0)=u(ih,0)=u\left(\frac{i}{3},0\right)$, $i=0,1,2,3$

$$=\sin\frac{\pi i}{3}$$

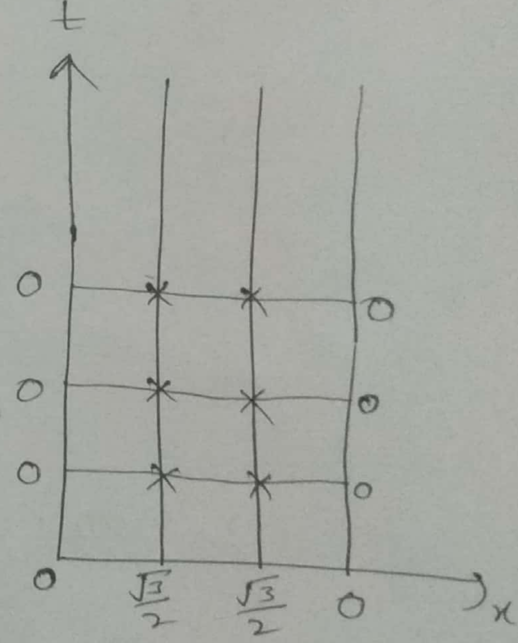
Put, $i=0$, $u\left(\frac{i}{3},0\right)=0$

$i=1$, $u\left(\frac{1}{3},0\right)=\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$

$i=2$, $u\left(\frac{2}{3},0\right)=\frac{\sqrt{3}}{2}$

$i=3$, $u(1,0)=0.$

$x=1.$



Using Crank - Nicolson formula.

One-time step :- $j=0$, $i=1,2$

eq^n (1) becomes,

for $i=1$ & $j=0$

$$-u_{0,1}+10u_{1,1}-u_{2,1}=u_{0,0}+6u_{1,0}+u_{2,0}$$

$$\Rightarrow 10u_{1,1}-u_{2,1}=6\times\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}$$

$$=6.06218 \quad (2)$$

take, $i=2$ & $j=0$

$$\Rightarrow -u_{1,1}+10u_{2,1}=6.06218 \quad (3)$$

from (2) & (3)

$$u_{1,1}=u_{2,1}=\frac{6.06218}{9}=0.673.$$

→ Second time step, $j=1$ & $i=1,2$

for $j=1$ & $i=1$

$$u_{1,2} = u_{2,2} = 0.52.$$

(*) Hyperbolic eqⁿ :— $u_{tt} = c^2 u_{xx}$ (1) (one-dimension Wave-eqⁿ)

with g.c, $u(x,0) = f'(x)$ & $\frac{\partial u}{\partial t}(x,0) = g(x)$

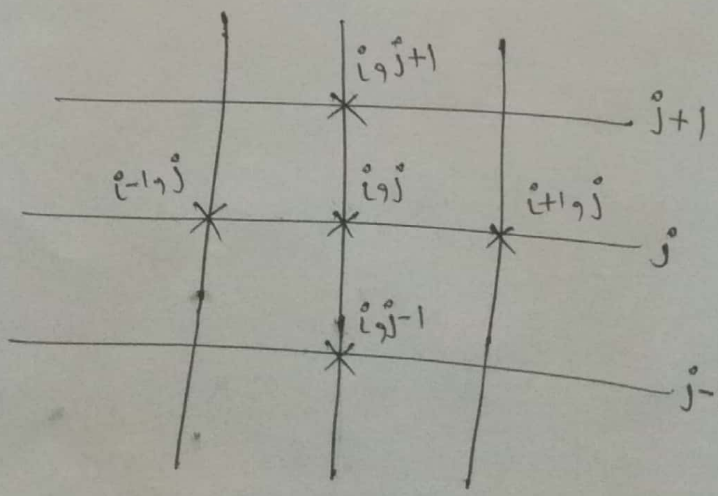
Again we take x-Horizontal & t-vertical solⁿ by Method of finite difference is,

$$x = ih \quad , \quad t = jk.$$

$$\frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} = c^2 \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

but, $\frac{kc}{h} = r$

$$\Rightarrow u_{i,j+1} = 2(1-r^2)u_{i,j} + r^2\left[u_{i+1,j} + u_{i-1,j}\right] - u_{i,j-1}, \quad (2)$$



→ $u(x,0) = f'(x)$, the initial condⁿ gives the solⁿ at all the nodal points on the initial line (level 0)

→ The values required on the level $t=k$ is obtained by writting approximations to the initial condition,

$$\frac{\partial u}{\partial t}(x,0) = g(x)$$

∴ by using finite difference, If we write the central difference appro. we get,

$$\frac{\partial u}{\partial t}(x,0) = \frac{1}{2k}\left(u_{i,j+1} - u_{i,j-1}\right)$$

$$\Rightarrow \quad g(x) = \frac{1}{2k}\left(u_{i,j+1} - u_{i,j-1}\right)$$

$$\Rightarrow \quad u_{i,j+1} = u_{i,j-1} + 2k\,g(x) \quad \text{at } t=k \text{ level.}$$

→ for $j=0$ we have,

$$u_{i,1} = u_{i,-1} + 2k\,g(x) \quad \text{— (3)}, \quad j=0$$

$$\Rightarrow \quad u_{i,-1} = u_{i,1} - 2k\,g(x) \quad \text{— (4)}$$

Now, we use the eq^n (2) at the nodes on the level $t=k$ i.e $j=0$.

Put $j=0$ in (2)

$$2u_{i,1} = 2(1-r^2)u_{i,0} + r^2\left[u_{i+1,0} + u_{i-1,0}\right] + 2k\,g(x_i) \quad \text{— (5)}$$

This gives the value at all nodes points on the level $t=k$.

<u>Note</u> :- (1) If the initial condition is prescribed at,

$$\frac{\partial u}{\partial t}(x,0) = 0 \quad \text{then, from (4)}$$

we have, $\boxed{u_{i,1} = u_{i,-1}}$ (6) & (5) becomes,

$$u_{i,1} = (1-r^2)u_{i,0} + \frac{r^2}{2}\left[u_{i+1,0} + u_{i-1,0}\right] \quad \text{— (7)}$$

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(2) for $r=1$, (7) becomes,

$$u_{i,1} = \frac{1}{2}\left[u_{i+1,0} + u_{i-1,0}\right] \quad \text{—— (8)}$$

(3) Again for $r=1$, eqn (2) becomes,

$$\boxed{u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1}} \quad \text{—— (9)}$$

Que: Solve, $u_{tt} = 4u_{xx}$ with Boundary condition,

$u(0,t) = 0 = u(4,t)$

$u_t(x,0) = 0$ & $u(x,0) = x(4-x)$

Soln $B^2 - 4AC = 4 > 0 \Rightarrow$ Hyperbolic

→ Compare it with stand. eqn we get $c^2 = 4$.

$\Rightarrow$ the value of stepsize is not describe. So, let $h=1$, let $r=1$ $\therefore k = \frac{1}{2}$ $\left[\because r = \frac{kc}{h}\right]$

So, from eqn (9) $[\because r=1]$

$$u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1}$$

Now, $u(x,0) = x(4-x)$

if,

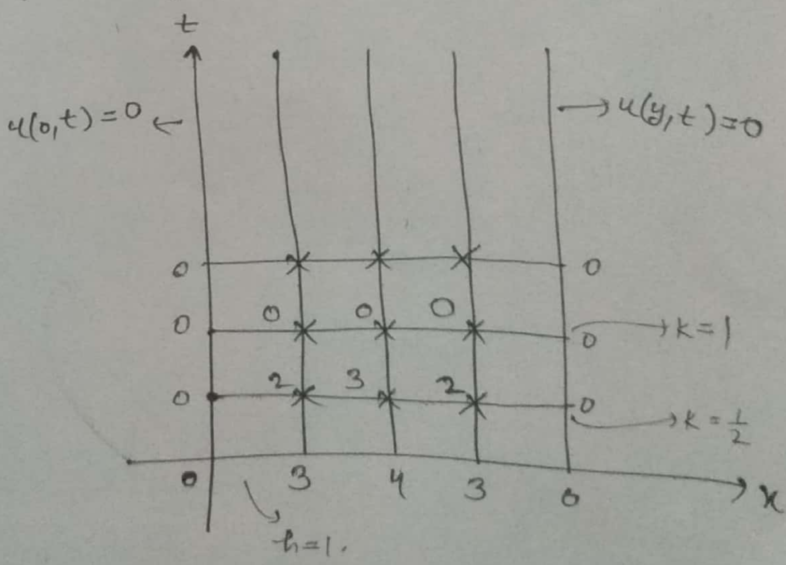
$u(i,0) = i(4-i)$ $[\because h=1]$

$\Rightarrow u(1,0) = 3$

$u(2,0) = 4$

$u(3,0) = 3$

$u(4,0) = 0$



Now, the condition,

$u_t(x,0) = 0$

$\Rightarrow$ from (6) & (8) $\boxed{u_{i,1} = u_{i,-1}}$ &

& $\boxed{u_{i,1} = \frac{1}{2}\left[u_{i+1,0} + u_{i-1,0}\right]}$

Put, $i=1$

$$u_{1,1} = \frac{1}{2}\left[u_{2,0} + u_{0,0}\right] = \frac{1}{2}\left[4+0\right] = \underline{\underline{2}}.$$

Put $i=2$,

$$u_{2,1} = \frac{1}{2}\left[u_{3,0} + u_{1,0}\right] = \frac{1}{2}\left[3+3\right] = 3$$

Put $i=3$,

$$u_{3,1} = \frac{1}{2}\left[u_{4,0} + u_{2,0}\right] = \frac{1}{2}\times 4 = \underline{\underline{2}}.$$

These are the solⁿ at the interior points on time level $= 0.5$

→ for $j=1$,

Using $u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1}$ → from (8)

$$\Rightarrow \quad u_{i,2} = u_{i+1,1} + u_{i-1,1} - u_{i,0}$$

Again varying, $i=1,2,3$

Now, $i=1$,

$$u_{1,2} = u_{2,1} + u_{0,1} - u_{1,0}$$

$$u_{1,2} = 3+0-3$$

$$u_{1,2} = 0$$

→ $i=2$,

$$u_{2,2} = 0 \quad . \quad \& \quad \rightarrow i=3 \quad u_{3,2} = 0$$