9.10. OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 9.7

Select the correct answer or fill up the blanks in the following questions :

1. $y_n = A\,2^n + B\,3^n$, is the solution of the difference equation ..... .
2. The solution of $(E-1)^3 u_n = 0$ is ....
3. The solution of the difference equation $u_{n+3} - 2u_{n+2} - 5u_{n+1} + 6u_n = 0$ is ..... .
4. The solution of $y_{n+1} - y_n = 2^n$ is ....
5. The difference equation $y_{n+1} - 2y_n = n$ has $y_n =$ .... as its solution.
6. The difference equation corresponding to the family of curves $y = ax^2 + bx$ is ..... .
7. The particular integral of the equation $(E-2)\,y_n = 1$.
8. The solution of $4y_n = y_{n+2}$ such that $y_0 = 0, y_1 = 2$, is ....
9. The equation $\Delta^2 u_{n-1} + \frac{1}{2}\Delta^2 u_n = 0$ is of order ....
10. The difference equation satisfied by $y = a + b/x$ is ....
11. The order of the difference equation $y_{n+2} - 2y_{n+1} + y_n = 0$ is ....
12. The solution of $y_{n+2} - 4y_{n+1} + 4y_n = 0$ is ....
13. The particular integral of $u_{x+2} - 6u_{x+1} + 9u_x = 3$ is...... .
14. The difference equation generated by $u_n = (a + bn)\,3^n$ is .....
15. Solution of $6y_{n+2} + 5y_{n+1} - 6y_n = 2^n$ is $y_n = A(2/3)^n + B(-3/2)^n + 2^n/28$. (True or False)

10

NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

10.1. (1) INTRODUCTION

A number of problems in science and technology can be formulated into differential equations. The analytical methods of solving differential equations are applicable only to a limited class of equations. Quite often differential equations appearing in physical problems do not belong to any of these familiar types and one is obliged to resort to numerical methods. These methods are of even greater importance when we realise that computing machines are now readily available which reduce numerical work considerably.

(2) Solution of a differential equation. The solution of an ordinary differential equation means finding an explicit expression for y in terms of a finite number of elementary functions of x. Such a solution of a differential equation is known as the *closed* or *finite form of solution*. In the absence of such a solution, we have recourse to numerical methods of solution.

Let us consider the first order differential equation

$$dy/dx = f(x, y), \text{ given } y(x_0) = y_0, \qquad \ldots(1)$$

to study the various numerical methods of solving such equations. In most of these methods, we replace the differential equation by a difference equation and then solve it. These methods yield solutions *either* as a power series in x from which the values of y can be found by direct substitution, *or* a set of values of x and y. The methods of Picard and Taylor series belong to the former class of solutions. In these methods, y in (1) is approximated by a truncated series, each term of which is a function of x. The information about the curve at one point is utilized and the solution is not iterated. As such, these are referred to as *single-step methods*. The methods of Euler, Runge-Kutta, Milne, Adams-Bashforth etc. belong to the latter class of solutions. In these methods, the next point on the curve is evaluated in short steps ahead, by performing iterations till sufficient accuracy is achieved. As such, these methods are called *step-by-step methods*.

Euler and Runga-Kutta methods are used for computing y over a limited range of x-values whereas Milne and Adams methods may be applied for finding y over a wider range of x-values. Therefore Milne and Adams methods require starting values which are found by Picard's Taylor series or Runge-Kutta methods.

(3) Initial and boundary conditions. An ordinary differential equation of the nth order is of the form

$$F(x, y, dy/dx, d^2y/dx^2, \ldots\ldots, d^ny/dx^n) = 0 \qquad \ldots(2)$$

Its general solution contains n arbitrary constants and is of the form

$$\phi(x, y, c_1, c_2, \ldots\ldots, c_n) = 0 \qquad \ldots(3)$$

To obtain its particular solution, n conditions must be given so that the constants c_1, c_2, ..., c_n can be determined. If these conditions are prescribed at one point only (say : x_0), then the differential equation together with the conditions constitute an *initial value problem* of the nth order. If the conditions are prescribed at two or more points, then the problem is termed as *boundary value problem*.

In this chapter, we shall first describe methods for solving initial value problems and then explain **finite difference method** and **shooting method** for solving boundary value problems.

10.2. PICARD'S METHOD

Consider the first order equation $\dfrac{dy}{dx} = f(x, y)$ ...(1)

It is required to find that particular solution of (1) which assumes the value y_0 when $x = x_0$. Integrating (1) between limits, we get

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)\,dx \quad \text{or} \quad y = y_0 + \int_{x_0}^{x} f(x, y)\,dx \qquad \ldots(2)$$

This is an integral equation equivalent to (1), for it contains the unknown y under the integral sign.

As a first approximation y_1 to the solution, we put $y = y_0$ in $f(x, y)$ and integrate (2), giving

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

For a second approximation y_2, we put $y = y_1$ in $f(x, y)$ and integrate (2), giving

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx.$$

Similarly, *a third approximation is*

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\,dx.$$

Continuing this process, we obtain $y_4, y_5, \ldots\ldots y_n$ where

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx$$

Hence this method gives a sequence of approximations y_1, y_2, y_3 ...... each giving a better result than the preceding one.

Obs. Picard's method is of considerable theoretical value, but can be applied only to a limited class of equations in which the successive integrations can be performed easily. The method can be extended to simultaneous equations and equations of higher order (See § 10.11 and § 10.12).

Example 10.1. *Using Picard's process of successive approximations, obtain a solution upto the fifth approximation of the equation $dy/dx = y + x$, such that $y = 1$ when $x = 0$. Check your answer by finding the exact particular solution.*

Sol. (*i*) We have $y = 1 + \int_0^x (y + x)\,dx$.

First approximation. Put $y = 1$ in $y + x$, giving

$$y_1 = 1 + \int_0^x (1 + x)\,dx = 1 + x + x^2/2.$$

Second approximation. Put $y = 1 + x + x^2/2$ in $y + x$, giving

$$y_2 = 1 + \int_0^x (1 + 2x + x^2/2)\,dx = 1 + x + x^2 + x^3/6.$$

Third approximation. Put $y = 1 + x + x^2 + x^3/6$ in $y + x$, giving

$$y_3 = 1 + \int_0^x (1 + 2x + x^2 + x^3/6)\,dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}.$$

Fourth approximation. Put $y = y_3$ in $y + x$, giving

$$y_4 = 1 + \int_0^x \left(1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}\right) dx$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}.$$

Fifth approximation. Put $y = y_4$ in $y + x$, giving

$$y_5 = 1 + \int_0^x \left(1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}\right) dx$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{720} \qquad \ldots(1)$$

(*ii*) Given equation :

$\dfrac{dy}{dx} - y = x$ is a Leibnitzs linear in x.

Its I.F. being e^{-x}, the solution is

$$ye^{-x} = \int xe^{-x}dx + c \qquad \text{[Integrate by parts}$$

$$= -xe^{-x} - \int(-e^{-x})\,dx + c = -xe^{-x} - e^{-x} + c$$

$$\therefore \qquad y = ce^{x} - x - 1.$$

Since $y = 1$, when $x = 0$, $\therefore c = 2$.

Thus the desired particular solution is

$$y = 2e^{x} - x - 1 \qquad \text{...(2)}$$

Or using the series : $e^{x} = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \ldots\ldots\infty,$

we get

$$y = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{360} + \ldots\ldots\infty \qquad \text{...(3)}$$

Comparing (1) and (3), it is clear that (1), approximates to the exact particular solution (3) upto the term in x^5.

Obs. At $x = 1$, the fourth approximation $y_4 = 3.433$ and the fifth approximation $y_5 = 3.434$ whereas exact value is 3.44.

Example 10.2. *Find the value of y for x = 0.1 by Picard's method, given that*

$$\frac{dy}{dx} = \frac{y-x}{y+x},\ y(0) = 1. \qquad \textit{(P.T.U. B.E., 2002)}$$

Sol. We have $\quad y = 1 + \displaystyle\int_0^x \frac{y-x}{y+x}\,dx$

First approximation. Put $y = 1$ in the integrand, giving

$$y_1 = 1 + \int_0^x \frac{1-x}{1+x}\,dx = 1 + \int_0^x \left(-1 + \frac{2}{1+x}\right)dx$$

$$= 1 + [-x + 2\log(1+x)]_0^x = 1 - x + 2\log(1+x) \qquad \text{...}(i)$$

Second approximation. Put $y = 1 - x + 2\log(1+x)$ in the integrand, giving

$$y_2 = 1 + \int_0^x \frac{1 - x + 2\log(1+x) - x}{1 - x + 2\log(1+x) + x}\,dx$$

$$= 1 + \int_0^x \left[1 - \frac{2x}{1 + 2\log(1+x)}\right]dx$$

which is very difficult to integrate.

Hence we use the first approximation and taking $x = 0.1$ in (i) we obtain

$$y(0.1) = 1 - (0.1) + 2\log 1.1 = 0.9828.$$

10.3. TAYLOR'S SERIES METHOD

Consider the first order equation $\dfrac{dy}{dx} = f(x, y)$ ...(1)

Differentiating (1), we have

$$\frac{d^2y}{dx^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \quad i.e. \quad y'' = f_x + f_y f' \qquad \text{...(2)}$$

Differentiating this successively, we can get y''', y^{iv} etc. Putting $x = x_0$ and $y = 0$, the values of $(y')_0$, $(y'')_0$, $(y''')_0$ can be obtained. Hence the Taylor's series

$$\boxed{y = y_0 + (x - x_0)(y')_0 + \frac{(x-x_0)^2}{2!}(y'')_0 + \frac{(x-x_0)^3}{3!}(y''')_0 + \ldots} \qquad \text{...(3)}$$

gives the values of y for every value of x for which (3) converges.

On finding the value y_1 for $x = x_1$ from (3), y', y'' etc. can be evaluated at $x = x_1$ by means of (1), (2) etc. Then y can be expanded about $x = x_1$. In this way, the solution can be extended beyond the range of convergence of series (3).

Obs. This is a single step method and works well so long as the successive derivatives can be calculated easily. If (x, y) is somewhat complicated and the calculation of higher order derivatives becomes tedious, then Taylor's method cannot be used gainfully. This is the main drawback of this method and therefore, has little application for computer programmes. However, it is useful for finding starting values for the application of powerful methods like Runga-Kutta, Milne and Adams-Bashforth which will be described in the subsequent sections.

Example 10.3. *Solve $y' = x + y$, $y(0) = 1$ by Taylor's series method. Hence find the values of y at x = 0.1 and x = 0.2.* *(Anna, B. Tech., 2005)*

$x=0$

Sol. Differentiating successively, we get

$$\begin{aligned} y' &= x + y & y'(0) &= 1 & [\because\ y(0) = 1 \\ y'' &= 1 + y' & y''(0) &= 2 & \\ y''' &= y'' & y'''(0) &= 2 & \\ y^{iv} &= y''' & y^{iv}(0) &= 2, \text{ etc.} & \end{aligned}$$

Taylor's series is

$$y = y_0 + (x - x_0)(y')_0 + \frac{(x-x_0)^2}{2!}(y'')_0 + \frac{(x-x_0)^3}{3!}(y''')_0 + \ldots\ldots,$$

Here $x_0 = 0$, $y_0 = 1$.

$$\therefore \qquad y = 1 + x(1) + \frac{x^2}{2}(2) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(4) + \ldots\ldots$$

Thus
$$y(0.1) = 1 + 0.1 + (0.1)^2 + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{6} + \ldots\ldots$$
$$= 1.1103$$

and
$$y(0.2) = 1 + 0.2 + (0.2)^2 + \frac{(0.2)^2}{3} + \frac{(0.2)^4}{6} + \ldots\ldots$$
$$= 1.2427.$$

Example 10.4. *Find by Taylor's series method, the values of y at $x = 0.1$ and $x = 0.2$ to five places of decimals from $dy/dx = x^2y - 1$, $y(0) = 1$.* (V.T.U., B. Tech., 2009)

Sol. Differentiating successively, we get $[\because \ y(0) = 1]$

$$y' = x^2y - 1, \qquad (y')_0 = -1$$
$$y'' = 2xy + x^2y', \qquad (y'')_0 = 0$$
$$y''' = 2y + 4xy' + x^2y'', \qquad (y''')_0 = 2$$
$$y^{iv} = 6y' + 6xy'' + x^2y''', \qquad (y^{iv})_0 = -6, \text{ etc.}$$

Putting these values in the Taylor's series, we have

$$y = 1 + x(-1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \ldots$$
$$= 1 - x + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

Hence $y(0.1) = 0.90033$ and $y(0.2) = 0.80227$.

Example 10.5. *Employ Taylor's method to obtain approximate value of y at $x = 0.2$ for the differential equation $dy/dx = 2y + 3e^x$, $y(0) = 0$. Compare the numerical solution obtained with the exact solution.* (V.T.U., B.E., 2009)

Sol. (a) We have $y' = 2y + 3e^x$; $y'(0) = 2y(0) + 3e^0 = 3$.

Differentiating successively and substituting $x = 0, y = 0$ we get

$$y'' = 2y' + 3e^x, \qquad y''(0) = 2y'(0) + 3 = 9$$
$$y''' = 2y'' + 3e^x, \qquad y'''(0) = 2y''(0) + 3 = 21$$
$$y^{iv} = 2y''' + 3e^x, \qquad y^{iv}(0) = 2y'''(0) + 3 = 45 \text{ etc.}$$

Putting these values in the Taylor's series, we have

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{iv}(0) + \ldots\ldots$$
$$= 0 + 3x + \frac{9}{2}x^2 + \frac{21}{6}x^3 + \frac{45}{24}x^4 + \ldots\ldots$$
$$= 3x + \frac{9}{2}x^2 + \frac{7}{2}x^3 + \frac{15}{8}x^4 + \ldots\ldots$$

Hence $y(0.2) = 3(0.2) + 4.5(0.2)^2 + 3.5(0.2)^3 + 1.875(0.4)^4 + \ldots\ldots = 0.8110$ ...(i)

(b) Now $\dfrac{dy}{dx} - 2y = 3e^x$ is a Leibnitz's linear in x.

Its I.F. being e^{-2x}, the solution is

$$ye^{-2x} = \int 3e^x \cdot e^{-2x}\,dx + c = -3e^{-x} + c \quad \text{or} \quad y = -3e^x + ce^{2x}$$

Since $y = 0$ when $x = 0$, $\therefore c = 3$.

Thus the exact solution is $y = 3(e^{2x} - e^x)$

When $x = 0.2$, $y = 3(e^{0.4} - e^{0.2}) = 0.8112$ ...(ii)

Comparing (i) and (ii), it is clear that (i) approximates to the exact value upto 3 decimal places.

Example 10.6. *Solve by Taylor series method of third order the equation $\dfrac{dy}{dx} = \dfrac{x^3 + xy^2}{e^x}$, $y(0) = 1$ for y at $x = 0.1$, $x = 0.2$ and $x = 0.3$*

Sol. We have $y' = (x^3 + xy^2)e^{-x}$; $y'(0) = 0$.

Differentiating successively and substituting $x = 0, y = 1$.

$$y'' = (x^3 + xy^2)(-e^{-x}) + (3x^2 + y^2 + x \cdot 2y \cdot y')e^{-x}$$
$$= (-x^3 - xy^2 + 3x^2 + y^2 + 2xyy')e^{-x}; \qquad y''(0) = 1$$
$$y''' = (-x^3 - xy^2 + 3x^2 + y^2 + 2xyy')(-e^{-x}) + \{-3x^2 - (y^2 + x \cdot 2y \cdot y') + 6x + 2yy'$$
$$+ 2[yy' + x(y'^2 + yy'')]\}e^{-x} \qquad y'''(0) = -2$$

Substituting these values in the Taylor's series, we have

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \ldots\ldots$$
$$= 1 + x(0) + \frac{x^2}{2}(1) + \frac{x^3}{6}(-2) + \ldots\ldots$$
$$= 1 + \frac{x^2}{2} - \frac{x^3}{3} + \ldots\ldots$$

Hence $\quad y(0.1) = 1 + \dfrac{1}{2}(0.1)^2 - \dfrac{1}{3}(0.1)^3 = 1.005$

$$y(0.2) = 1 + \frac{1}{2}(0.2)^2 - \frac{1}{3}(0.2)^3 = 1.017$$
$$y(0.3) = 1 + \frac{1}{2}(0.3)^2 - \frac{1}{3}(0.3)^3 = 1.036.$$

Example 10.7. *Solve by Taylor's series method the equation $\dfrac{dy}{dx} = \log(xy)$ for $y(1.1)$ and $y(1.2)$, given $y(1) = 2$* (Hazaribagh, B.E., 2009)

Sol. We have $y' = \log x + \log y$; $\qquad y'(1) = \log 2$

Differentiating w.r.t. x and substituting $x = 1, y = 2$, we get

$$y'' = \frac{1}{x} + \frac{1}{y}y'; \qquad y''(1) = 1 + \frac{1}{2}\log 2$$
$$y''' = -\frac{1}{x^2} + \frac{1}{y}y'' + y'\left(-\frac{1}{y^2}\right)y'; \quad y'''(1) = 1 + \frac{1}{2}\left(1 + \frac{1}{2}\log 2\right) - \frac{1}{4}(\log 2)^2$$

Substituting these values in the Taylor's series about $x = 1$, we have

$$y(x) = y(1) + (x-1)y'(1) + \frac{(x-1)^2}{2!}y''(1) + \frac{(x-1)^3}{3!}y'''(1) + \ldots\ldots$$

$$= 2 + (x-1)\log 2 + \frac{1}{2}(x-1)^2\left(1+\frac{1}{2}\log 2\right) + \frac{1}{6}(x-1)^3\left[-\frac{1}{2}+\frac{1}{4}\log 2 - \frac{1}{4}(\log 2)^2\right]$$

$\therefore$
$$y(1.1) = 2 + (0.1)\log 2 + \frac{(0.1)^2}{2}\left(1+\frac{1}{2}\log 2\right) + \frac{(0.1)^3}{6}\left[-\frac{1}{2}+\frac{1}{4}\log 2 - \frac{1}{4}(\log 2)^2\right]$$

$$= 2.036$$

$$y(1.2) = 2 + (0.2)\log 2 + \frac{(0.2)^2}{2}\left(1+\frac{1}{2}\log 2\right) + \frac{(0.2)^3}{6}\left[-\frac{1}{2}+\frac{1}{4}\log 2 - \frac{1}{4}(\log 2)^2\right]$$

$$= 2.081.$$

PROBLEMS 10.1

1. Using Picard's method, solve $dy/dx = -xy$ with $x_0 = 0$, $y_0 = 1$ upto third approximation. *(Mumbai, B. Tech., 2005)*
2. Employ Picard's method to obtain, correct to four places of decimal, solution of the differential equation $dy/dx = x^2 + y^2$ for $x = 0.4$, given that $y = 0$ when $x = 0$. *(J.N.T.U., B. Tech., 2009)*
3. Obtain Picard's second approximate solution of the initial value problem
$$y' = x^2/(y^2+1),\ y(0) = 0.$$
4. Find an approximate value of y when $x = 0.1$, if $dy/dx = x - y^2$ and $y = 1$ at $x = 0$, using (*a*) Picard's method, (*b*) Taylor's series. *(Madras, B.E., 2006)*
5. Solve $y' = x + y$ given $y(1) = 0$. Find $y(1.1)$ and $y(1.2)$ by Taylor's method. Compare the result with its exact value. *(J.N.T.U., B. Tech., 2008)*
6. Using Taylor's series method, compute $y(0.2)$ to three places of decimal from $\dfrac{dy}{dx} = 1 - 2xy$ given that $y(0) = 0$.
7. Evaluate $y(0.1)$ correct to six places of decimals by Taylor's series method if $y(x)$ satisfies
$$y' = xy + 1,\ y(0) = 1.$$
8. Solve $y' = y^2 + x$, $y(0) = 1$ using Taylor's series method and compute $y(0.1)$ and $y(0.2)$. *(J.N.T.U., B. Tech., 2006)*
9. Evaluate $y(0.1)$ correct to four decimal places using Taylor's series methods if
$$dy/dx = x^2 + y^2,\ y(0) = 1.$$ *(V.T.U., B.E., 2006)*
10. Using Taylor series method, find $y(0.1)$ correct to 3-decimal places given that
$$dy/dx = e^x - y^2,\ y(0) = 1.$$

10.4. EULER'S METHOD

Consider the equation $$\frac{dy}{dx} = f(x, y) \qquad ...(1)$$

given that $y(x_0) = y_0$. Its curve of solution through $P(x_0, y_0)$ is shown dotted in Fig. 10.1. Now we have to find the ordinate of any other point Q on this curve.

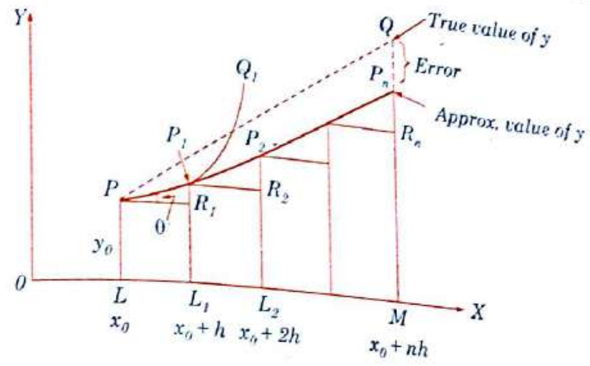


Fig. 10.1

Let us divide LM into n sub-intervals each of width h at L_1, L_2 ...... so that h is quite small.

In the interval LL_1, we approximate the curve by the tangent at P. If the ordinate through L_1 meets this tangent in $P_1(x_0 + h, y_1)$, then

$$y_1 = L_1P_1 = LP + R_1P_1 = y_0 + PR_1 \tan\theta$$

$$= y_0 + h\left(\frac{dy}{dx}\right)_P = y_0 + h\, f(x_0, y_0)$$

Let P_1Q_1 be the curve of solution of (1) through P_1 and let its tangent at P_1 meet the ordinate through L_2 in $P_2(x_0 + 2h, y_2)$. Then

$$y_2 = y_1 + hf(x_0 + h, y_1) \qquad ...(1)$$

Repeating this process n times, we finally reach on an approximation MP_n of MQ given by

$$y_n = y_{n-1} + hf(x_0 + \overline{n-1}\,h, y_{n-1}).$$

This is *Euler's method* of finding an approximate solution of (1).

Obs. In Euler's method, we approximate the curve of solution by the tangent in each interval, *i.e.* by a sequence of short lines. Unless h is small, the error is bound to be quite significant. This sequence of lines may also deviate considerably from the curve of solution. As such, the method is very slow and hence there is a modification of this method which is given in the next section.

Example 10.8. *Using Euler's method, find an approximate value of y corresponding to x = 1, given that dy/dx = x + y and y = 1 when x = 0.* *(Anna, B. Tech., 2005)*

Sol. We take $n = 10$ and $h = 0.1$ which is sufficiently small. The various calculations are arranged as follows :

x	y	$x + y = dy/dx$	*Old y + 0.1 (dy/dx) = new y*
0.0	1.00	1.00	1.00 + 0.1 (1.00) = 1.10
0.1	1.10	1.20	1.1 + 0.1 (1.20) = 1.22
0.2	1.22	1.42	1.22 + 0.1 (1.42) = 1.36
0.3	1.36	1.66	1.36 + 0.1 (1.66) = 1.53
0.4	1.53	1.93	1.53 + 0.1 (1.93) = 1.72
0.5	1.72	2.22	1.72 + 0.1 (2.22) = 1.94
0.6	1.94	2.54	1.94 + 0.1 (2.54) = 2.19
0.7	2.19	2.89	2.19 + 0.1 (2.89) = 2.48
0.8	2.48	3.29	2.48 + 0.1 (3.29) = 2.81
0.9	2.81	3.71	2.81 + 0.1 (3.71) = 3.18
1.0	3.18		

Thus the required approximate value of $y = 3.18$.

Obs. In example 10.1 (Obs.), we obtained the true values of y from its exact solution to be 3.44 whereas by Euler's method $y = 3.18$ and by Picard's method $y = 3.434$. In above solution, had we chosen $n = 20$, the accuracy would have been considerably increased but at the expense of double the labour of computation. Euler's method is no doubt very simple but cannot be considered as one of the best.

Example 10.9. *Given* $\dfrac{dy}{dx} = \dfrac{y - x}{y + x}$ *with initial condition $y = 1$ at $x = 0$; find y for $x = 0.1$ by Euler's method.* *(P.T.U., B.E., 2001)*

Sol. We divide the interval (0, 0.1) into five steps *i.e.* we take $n = 5$ and $h = 0.02$. The various calculations are arranged as follows :

x	y	dy/dx	*Old y + 0.02 (dy/dx) = new y*
0.00	1.0000	1.0000	1.0000 + 0.02(1.0000) = 1.0200
0.02	1.0200	0.9615	1.0200 + 0.02(0.9615) = 1.0392
0.04	1.0392	0.926	1.0392 + 0.02(0.926) = 1.0577
0.06	1.0577	0.893	1.0577 + 0.02(0.893) = 1.0756
0.08	1.0756	0.862	1.0756 + 0.02(0.862) = 1.0928
0.10	1.0928		

Hence the required approximate value of $y = 1.0928$.

10.5. MODIFIED EULER'S METHOD

In the Euler's method, the curve of solution in the interval LL_1 is approximated by the tangent at P (Fig. 10.1) such that at P_1, we have

$$y_1 = y_0 + hf(x_0, y_0) \qquad ...(1)$$

Then the slope of the curve of solution through P_1

[*i.e.* $(dy/dx)P_1 = f(x_0 + h, y_1)$]

is computed and the tangent at P_1 to P_1Q_1 is drawn meeting the ordinate through L_2 in $P_2(x_0 + 2h, y_2)$.

Now we find a better approximation $y_1^{(1)}$ of $y(x_0 + h)$ by taking the slope of the curve as the mean of the slopes of the tangents at P and P_1, *i.e.*

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_0 + h, y_1)] \qquad ...(2)$$

As the slope of the tangent at P_1 is not known, we take y_1 as found in (1) by Euler's method and insert it on R.H.S. of (2) to obtain the first modified value $y_1^{(1)}$.

Again (2) is applied and we find a still better value $y_1^{(2)}$ corresponding to L_1 as

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_0 + h, y_1^{(1)})]$$

We repeat this step, till two consecutive values of y agree. This is then taken as the starting point for the next interval L_1L_2.

Once y_1 is obtained to desired degree of accuracy, y corresponding to L_2 is found from (1).

$$y_2 = y_1 + hf(x_0 + h, y_1)$$

and a better approximation $y_2^{(1)}$ is obtained from (2)

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_0 + h, y_1) + f(x_0 + 2h, y_2)].$$

We repeat this step until y_2 becomes stationary. Then we proceed to calculate y_3 as above and so on.

This is the *modified Euler's method* which gives great improvement in accuracy over the original method.

Example 10.10. *Using modified Euler's method, find an approximate value of y when $x = 0.3$, given that $dy/dx = x + y$ and $y = 1$ when $x = 0$.* *(Rohtak, B. Tech., 2005)*

Sol. The various calculations are arranged as follows taking $h = 0.1$:

x	$x + y = y'$	*Mean slope*	*Old y + 0.1 (mean slope) = new y*
0.0	0 + 1	—	1.00 + 0.1 (1.00) = 1.10
0.1	0.1 + 1.1	$\frac{1}{2}$(1 + 1.2)	1.00 + 0.1 (1.1) = 1.11
0.1	0.1 + 1.11	$\frac{1}{2}$(1 + 1.21)	1.00 + 0.1 (1.105) = 1.1105
0.1	0.1 + 1.1105	$\frac{1}{2}$(1 + 1.2105)	1.00 + 0.1 (1.1052) = 1.1105
Since last two values are equal, we take $y(0.1) = 1.1105$.			
0.1	1.2105	—	1.1105 + 0.1 (1.2105) = 1.2316
0.2	0.2 + 1.2316	$\frac{1}{2}$(1.12105 + 1.4316)	1.1105 + 0.1 (1.3211) = 1.2426
0.2	0.2 + 1.2426	$\frac{1}{2}$(1.2105 + 1.4426)	1.1105 + 0.1 (1.3266) = 1.2432
0.2	0.2 + 1.2432	$\frac{1}{2}$(1.2105 + 1.4432)	1.1105 + 0.1 (1.3268) = 1.2432

Since last two values are equal, we take $y(0.2) = 1.2432$

0.2	1.4432	—	1.2432 + 0.1 (1.4432) = 1.3875
0.3	0.3 + 1.3875	$\frac{1}{2}$ (1.4432 + 1.6875)	1.2432 + 0.1 (1.5654) = 1.3997
0.3	0.3 + 1.3997	$\frac{1}{2}$ (1.4432 + 1.6997)	1.2432 .1 (1.5715) = 1.4003
0.3	0.3 + 1.4003	$\frac{1}{2}$ (1.4432 + 1.7003)	1.2432 + 0.1 (1.5718) = 1.4004
0.3	0.3 + 1.4004	$\frac{1}{2}$ (1.4432 + 1.7004)	1.2432 + 0.1 (1.5718) = 1.4004
Since last two values are equal, we take $y(0.3) = 1.4004$.			

Hence $y(0.3) = 1.4004$ approximately.

Obs. In example 10.8, we obtained the approximate value of y for $x = 0.3$ to be 1.53 whereas by modified Euler's method the corresponding value is 1.4003 which is nearer its true value 1.3997, obtained from its exact solution $y = 2e^x - x - 1$ by putting $x = 0.3$.

Example 10.11. *Using modified Euler's method, find $y(0.2)$ and $y(0.4)$ given*

$$y' = y + e^x, \; y(0) = 0.$$

(J.N.T.U., B. Tech., 2009)

Sol. We have $y' = y + e^x = f(x, y)$; $x = 0$, $y = 0$ and $h = 0.2$

The various calculations are arranged as under :

To calculate $y(0.2)$:

x	$y + e^x = y'$	*Mean slope*	*Old y + h (Mean slope) = new y*
0.0	1	—	0 + 0.2 (1) = 0.2
0.2	$0.2 + e^{0.2}$ = 1.4214	$\frac{1}{2}$ (1 + 1.4214) = 1.2107	0 + 0.2 (1.2107) = 0.2421
0.2	$0.2421 + e^{0.2}$ = 1.4635	$\frac{1}{2}$ (1 + 1.4635) =1.2317	0 + 0.2 (1.2317) = 0.2463
0.2	$0.2463 + e^{0.2}$ = 1.4677	$\frac{1}{2}$ (1 + 1.4677) = 1.2338	0 + 0.2 (1.2338) = 0.2468
0.2	$0.2468 + e^{0.2}$ = 1.4682	$\frac{1}{2}$ (1 + 1.4682) = 1.2341	0 + 0.2 (1.2341) = 0.2468

Since the last two values of y are equal, we take $y(0.2) = 0.2468$.

To calculate $y(0.4)$:

x	$y + e^x$	*Mean slope*	*Old y + 0.2 (mean slope) new y*
0.2	$0.2468 + e^{0.2}$ = 1.4682	–	0.2468 + 0.2 (1.4682) = 0.5404
0.4	$0.5404 + e^{0.4}$ = 2.0322	$\frac{1}{2}$ (1.4682 + 2.0322) = 1.7502	0.2468 + 0.2 (1.7502) = 0.5968
0.4	$0.5968 + e^{0.4}$ = 2.0887	$\frac{1}{2}$ (1.4682 + 2.0887) = 1.7784	0.2468 + 0.2 (1.7784) = 0.6025
0.4	$0.6025 + e^{0.4}$ = 2.0943	$\frac{1}{2}$ (1.4682 + 2.0943) = 1.78125	0.2468 + 0.2 (1.78125) = 0.6030
0.4	$0.6030 + e^{0.4}$ = 2.0949	$\frac{1}{2}$ (1.4682 + 2.0949) = 1.7815	0.2468 + 0.2 (1.7815) = 0.6031
0.4	$0.6031 + e^{0.4}$ = 2.0949	$\frac{1}{2}$ (1.4682 + 2.0949) = 1.7816	0.2468 + 0.2 (1.7815) = 0.6031

Since the last two value of y are equal, we take $y(0.4) = 0.6031$

Hence $y(0.2) = 0.2468$ and $y(0.4) = 0.6031$ approximately.

Example 10.12. *Solve the following by Euler's modified method :*

$$\frac{dy}{dx} = \log (x + y), \; y(0) = 2$$

at $x = 1.2$ and 1.4 with $h = 0.2$.

(Bhopal, B.E., 2009)

Sol. The various calculations are arranged as follows :

x	*log (x + y) = y′*	*Mean slope*	*Old y + 0.2 (mean slope) = new y*
0.0	log (0 + 2)	—	2 + 0.2(0.301) = 2.0602
0.2	log (0.2 + 2.0602)	$\frac{1}{2}$ (0.310 = 0.3541)	2 + 0.2 (0.3276) = 2.0655
0.2	log (0.2 + 2.0655)	$\frac{1}{2}$ (0.301 + 0.3552)	2 + 0.2 (0.3281) = 2.0656
0.2	0.3552	—	2.0656 + 0.2 (0.3552) = 2.1366
0.4	log (0.4 + 2.1366)	$\frac{1}{2}$ (0.3552 + 0.4042)	2.0656 + 0.2 (0.3797) = 2.1415
0.4	log (0.4 + 2.1415)	$\frac{1}{2}$ (0.3552 + 0.4051)	2.0656 + 0.2 (0.3801) = 2.1416
0.4	0.4051	—	2.1416+ 0.2 (0.4051) = 2.2226
0.6	log (0.6 + 2.2226)	$\frac{1}{2}$ (0.4051 + 0.4506)	2.1416 + 0.2 (0.4279) = 2.2272
0.6	log (0.6 + 2.2272)	$\frac{1}{2}$ (0.4051 + 0.4514)	2.1416 + 0.2 (0.4282) = 2.2272
0.6	0.4514	—	2.2272 + 0.2 (0.4514) = 2.3175
0.8	log (0.8 + 2.3175)	$\frac{1}{2}$ (0.4514 + 0.4938)	2.2272 + 0.2 (0.4726) = 2.3217
0.8	log (0.8 + 2.3217)	$\frac{1}{2}$ (0.4514 + 0.4943)	2.2272 + 0.2 (0.4727) = 2.3217

0.8	0.4943	—	2.3217 + 0.2 (0.4943) = 2.4206
1.0	log (1 + 2.4206)	$\frac{1}{2}$ (0.4943 + 0.5341)	2.3217 + 0.2 (0.5142) = 2.4245
1.0	log (1 + 2.4245)	$\frac{1}{2}$ (0.4943 + 0.5346)	2.3217 + 0.2 (0.5144) = 2.4245
1.0	0.5346	—	2.4245 + 0.2 (0.5346) = 2.5314
1.2	log (1.2 + 2.5314)	$\frac{1}{2}$ (0.5346 + 0.5719)	2.4245 + 0.2 (0.5532) = 2.5351
1.2	log (1.2 + 2.5351)	$\frac{1}{2}$ (0.5346 + 0.5723)	2.4245 + 0.2 (0.5534) = 2.5351
1.2	0.5723	—	2.5351 + 0.2 (0.5723) = 2.6496
1.4	log (1.4 + 2.6496)	$\frac{1}{2}$ (0.5723 + 0.6074)	2.5351 + 0.2 (0.5898) = 2.6531
1.4	log (1.4 + 2.6531)	$\frac{1}{2}$ (0.5723 + 0.6078)	2.5351 + 0.2 (0.5900) = 2.6531

Hence $y(1.2) = 2.5351$ and $y(1.4) = 2.6531$ approximately.

Example 10.13. *Using Euler's modified method, obtain a solution of the equation*

$$dy/dx = x + |\sqrt{y}|,$$

with initial conditions $y = 1$ at $x = 0$, for the range $0 \le x \le 0.6$ in steps of 0.2. *(V.T.U., B. Tech., 2007)*

Sol. The various calculations are arranged as follows :

x	$x + \lvert\sqrt{y}\rvert = y'$	*Mean slope*	*Old y + 0.2 (mean slope) = new y*
0.0	0 + 1 = 1	—	1 + 0.2 (1) = 1.2
0.2	$0.2 + \lvert\sqrt{(1.2)}\rvert$ = 1.2954	$\frac{1}{2}$(1 + 1.2954) = 1.1477	1 + 0.2 (1.1477) = 1.2295
0.2	$0.2 + \lvert\sqrt{(1.2295)}\rvert$ = 1.3088	$\frac{1}{2}$ (1 + 1.3088) = 1.1544	1 + 0.2 (1.1544) = 1.2309
0.2	$0.2 + \lvert\sqrt{(1.2309)}\rvert$ = 1.3094	$\frac{1}{2}$(1 + 1.3094) = 1.1547	1 + 0.2 (1.1547) = 1.2309
0.2	1.3094	—	1.2309 + 0.2 (1.3094) = 1.4927
0.4	$0.4 + \lvert\sqrt{(1.4927)}\rvert$ = 1.6218	$\frac{1}{2}$(1.3094 + 1.6218) = 1.4654	1.2309 + 0.2 (1.4654) = 1.5240
0.4	$0.4 + \lvert\sqrt{(1.524)}\rvert$ = 1.6345	$\frac{1}{2}$(1.3094 + 1.6345) = 1.4718	1.2309 + 0.2 (1.4718) = 1.5253
0.4	$0.4 + \lvert\sqrt{(1.5253)}\rvert$ = 1.6350	$\frac{1}{2}$ (1.3094 + 1.6350) = 1.4721	1.2309 + 0.2 (1.4721) = 1.5253

0.4	1.6350	—	1.5253 + 0.2 (1.635) = 1.8523
0.6	$0.6 + \lvert\sqrt{(1.8523)}\rvert$ = 1.9610	$\frac{1}{2}$(1.635 + 1.961) = 1.798	1.5253 + 0.2 (1.798) = 1.8849
0.6	$0.6 + \lvert\sqrt{(1.8849)}\rvert$ = 1.9729	$\frac{1}{2}$(1.635 + 1.9729) = 1.8040	1.5253 + 0.2 (1.804) = 1.8861
0.6	$0.6 + \lvert\sqrt{(1.8861)}\rvert$ = 1.9734	$\frac{1}{2}$ (1.635 + 1.9734) = 1.8042	1.5253 + 0.2 (1.8042) = 1.8861

Hence $y(0.6) = 1.8861$ approximately.

PROBLEMS 10.2

1. Apply Euler's method to solve $y' = x + y$, $y(0) = 0$, choosing the step length = 0.2. (Carry out 6 steps). *(Kottayam, B.E., 2005)*
2. Using Euler's method, find approximate value of y when $x = 0.6$ of $dy/dx = 1 - 2xy$, given that $y = 0$ when $x = 0$ (take $h = 0.2$).
3. Using simple Euler's method solve for y at $x = 0.1$ from
$$dy/dx = x + y + xy,\ y(0) = 1, \text{ taking step size } h = 0.025.$$
4. Solve $y' = 1 - y$, $y(0) = 0$ by modified Euler's method and obtain y at $x = 0.1, 0.2, 0.3$. *(Anna, B. Tech., 2005)*
5. Given that $dy/dx = x + y^2$ and $y = 1$ at $x = 0$. Find an approximate value of y at $x = 0.5$ by modified Euler's method. *(Bhopal, B.E., 2002)*
6. Given $y' = x + \sin y$, $y(0) = 1$. Compute $y(0.2)$ and $y(0.4)$ with $h = 0.2$ using Euler's modified method. *(J.N.T.U., B. Tech., 2007)*
7. Given $\frac{dy}{dx} = \frac{y - x}{y + x}$ with boundary conditions $y = 1$ when $x = 0$, find approximately y for $x = 0.1$, by Euler's modified method (5 steps). *(V.T.U., B. Tech., 2007)*
8. Given that $dy/dx = 2 + \sqrt{(xy)}$ and $y = 1$ when $x = 1$. Find approximate value of y at $x = 2$ in steps of 0.2, using Euler's modified method. *(Anna, B.E., 2004)*

10.6. RUNGE'S METHOD*

Consider the differential equation, $\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0$ ...(1)

Clearly the slope of the curve through $P(x_0, y_0)$ is $f(x_0, y_0)$ (Fig. 10.2).

Integrating both sides of (1) from (x_0, y_0) to $(x_0 + h, y_0 + k)$, we have

$$\int_{y_0}^{y_0 + k} dy = \int_{x_0}^{x_0 + h} f(x, y)\, dx \quad ...(2)$$

*Called after the German mathematician *Carl Runge* (1856—1927).

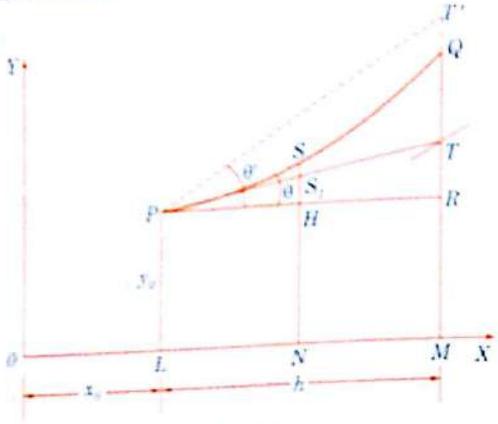


Fig. 10.2

To evaluate the integral on the right, we take N as the mid-point of LM and find the values of $f(x, y)$ (*i.e.* dy/dx) at the points x_0, $x_0 + h/2$, $x_0 + h$. For this purpose, we first determine the values of y at these points.

Let the ordinate through N cut the curve PQ in S and the tangent PT in S_1. The value of y_S is given by the point S_1.

$$\therefore \quad y_S = NS_1 = LP + HS_1 = y_0 + PH \,.\, \tan\theta$$

$$= y_0 + h\,(dy/dx)_P = y_0 + \frac{h}{2} f(x_0, y_0) \qquad \text{...(3)}$$

Also $\quad y_T = MT = LP + RT = y_0 + PR \,.\, \tan\theta = y_0 + hf(x_0, y_0)$.

Now the value of y_Q at $x_0 + h$ is given by the point T' where the line through P draw with slope at $T(x_0 + h, y_T)$ meets MQ.

$\therefore$ Slope at $\quad T = \tan\theta' = f(x_0 + h, y_T) = f[x_0 + h, y_0 + hf(x_0, y_0)]$

$$\therefore \quad y_Q = MR + RT' = y_0 + PR \,.\, \tan\theta' = y_0 + hf[x_0 + h, y_0 + hf(x_0, y_0)] \qquad \text{...(4)}$$

Thus the value of $f(x, y)$ at $P = f(x_0, y_0)$,

the value of $f(x, y)$ at $\quad S = f(x_0 + h/2, y_S)$

and the value of $f(x, y)$ at $\quad Q = f(x_0 + h, y_Q)$

where y_S and y_Q are given by (3) and (4).

Hence from (2), we obtain

$$k = \int_{x_0}^{x_0+h} f(x, y)\,dx = \frac{h}{6}\,[f_P + 4f_S + f_Q] \quad \text{by Simpson's rule (p. 191)}$$

$$= \frac{h}{6}\,[f(x_0, y_0) + 4f(x_0 + h/2, y_S) + (x_0 + h, y_Q)] \qquad \text{...(5)}$$

which gives a sufficiently accurate value of k and also of $y = y_0 + k$

The repeated application of (5) gives the values of y for equi-spaced points.

Working rule *to solve (1) by Runge's method :*

Calculate successively

$$k_1 = hf(x_0, y_0), \; k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$$

$$k' = hf(x_0 + h, y_0 + k_1) \quad \text{and} \quad k_3 = hf(x_0 + h, y_0 + k')$$

Finally compute, $k = \dfrac{1}{6}\,(k_1 + 4k_2 + k_3)$.

(Note that k is the weighted mean of k_1, k_2 and k_3).

Example 10.14. *Apply Runge's method to find an approximate value of y when $x = 0.2$, given that $dy/dx = x + y$ and $y = 1$ when $x = 0$.*

Sol. Here we have $x_0 = 0$, $y_0 = 1$, $h = 0.2$, $f(x_0, y_0) = 1$

$$\therefore \quad k_1 = hf(x_0, y_0) = 0.2(1) = 0.200$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = 0.2\,f(0.1, 1.1) = 0.240$$

$$k' = hf(x_0 + h, y_0 + k_1) = 0.2\,f(0.2, 1.2) = 0.280$$

$$\text{and} \quad k_3 = hf(x_0 + h, y_0 + k') = 0.2\,f(0.1, 1.28) = 0.296$$

$$\therefore \quad k = \frac{1}{6}(k_1 + 4k_2 + k_3) = \frac{1}{6}(0.200 + 0.960 + 0.296) = 0.2426$$

Hence the required approximate value of y is 1.2426.

10.7. RUNGE-KUTTA METHOD*

The Taylor's series method of solving differential equations numerically is restricted by the labour involved in finding the higher order derivatives. However there is a class of methods known as Runge-Kutta methods which do not require the calculations of higher order derivatives and give greater accuracy. The Runge-Kutta formulae possess the advantage of requiring only the function values at some selected points. These methods agree with Taylor's series solution upto the term in h^r where r differs from method to method and is called the *order of that method.*

(i) First order R-K method. We have seen that Euler's method (§ 10.4) gives

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + hy_0' \qquad [\because \; y' = f(x, y)$$

Expanding by Taylor's series

$$y_1 = y(x_0 + h) = y_0 + hy_0' + \frac{h^2}{2}\,y_0'' + \ldots$$

It follows that the Euler's method agrees with the Taylor's series solution upto the term in h.

Hence, *Euler's method is the Runge-Kutta method of the first order.*

(ii) Second order R-K method. The modified Euler's method gives

$$y_1 = y + \frac{h}{2}\,[f(x_0, y_0) + f(x_0 + h, y_1)] \qquad \text{...(1)}$$

---

*$O(h^2)$ means 'terms containing second and higher powers of h' and is read as *order of* h^2.

Substituting $y_1 = y_0 + hf(x_0, y_0)$ on the right hand side of (1), we obtain

$$y_1 = y_0 + \frac{h}{2}[f_0 + f(x_0 + h, y_0 + hf_0)] \quad \text{where } f_0 = f(x_0, y_0) \qquad ...(2)$$

Expanding L.H.S. by Taylor's series, we get

$$y_1 = y(x_0 + h) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + ... \qquad ...(3)$$

Expanding $f(x_0 + h, y_0 + hf_0)$ by Taylor's series for a function of two variables, (2) gives

$$y_1 = y_0 + \frac{h}{2}\left[f_0 + \left\{f(x_0, y_0) + h\left(\frac{\partial f}{\partial x}\right)_0 + hf_0\left(\frac{\partial f}{\partial y}\right)_0 + O(h^2)*\right\}\right]$$

$$= y_0 + \frac{1}{2}\left[hf_0 + hf_0 + h^2\left\{\left(\frac{\partial f}{\partial x}\right)_0 + f\left(\frac{\partial f}{\partial y}\right)_0\right\} + O(h^3)\right]$$

$$= y_0 + hf_0 + \frac{h^2}{2}f_0' + O(h^3) \qquad \left[\because \frac{df(x,y)}{dx} = \frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right]$$

$$= y_0 + hy_0' + \frac{h^2}{2!}y_0'' + O(h^3) \qquad ...(4)$$

Comparing (3) and (4), it follows that the modified Euler's method agrees with the Taylor's series solution upto the term in h^2.

Hence the modified Euler's method is the Runge-Kutta method of the second order.

$\therefore$ *The second order Runge-Kutta formula is*

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

where $k_1 = hf(x_0, y_0)$ and $k_2 = hf(x_0 + h, y_0 + k_1)$.

(iii) Third order R-K method. Similarly, it can be seen that Runge's method (§10.6) agrees with the Taylor's series solution upto the term in h^3.

As such, *Runge's method is the Runge-Kutta method of the third order.*

The third order Runge-Kutta formula is

$$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

where $k_1 = hf(x_0, y_0)$, $k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$

and $k_3 = hf(x_0 + h, y_0 + k')$, where $k' = hf(x_0 + h, y_0 + k_1)$.

(iv) Fourth order R-K method. This method is most commonly used and is often referred to as *Runge-Kutta method* only.

Working rule *for finding the increment k of y corresponding to an increment h of x by Runge-Kutta method from*

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0$$

is as follows :

Calculate successively $\boxed{k_1 = hf(x_0, y_0)}$, $\boxed{k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)}$

$$\boxed{k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right)} \text{ and } \boxed{k_4 = hf(x_0 + h, y_0 + k_3)}$$

Finally compute $\boxed{k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$

which gives the required approximate value as $y_1 = y_0 + k$.

(Note that k is the weighted mean of k_1, k_2, k_3 and k_4).

Obs. One of the advantages of these methods is that the operation is identical whether the differential equation is linear or non-linear.

Example 10.15. *Apply Runge-Kutta fourth order method to find an approximate value of y when x = 0.2 given that dy/dx = x + y and y = 1 when x = 0.* (V.T.U., B. Tech., 2009)

Sol. Here $x_0 = 0$, $y_0 = 1$, $h = 0.2$, $f(x_0, y_0) = 1$

$$\therefore \quad k_1 = hf(x_0, y_0) = 0.2 \times 1 = 0.2000$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = 0.2 \times f(0.1, 1.1) = 0.2400$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = 0.2 \times f(0.1, 1.12) = 0.2440$$

and $$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2 \times f(0.2, 1.244) = 0.2888$$

$$\therefore \quad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \frac{1}{6}(0.2000 + 0.4800 + 0.4880 + 0.2888)$$

$$= \frac{1}{6} \times (1.4568) = 0.2428.$$

Hence the required approximate value of y is 1.2428.

Example 10.16. *Using Runge-Kutta method of fourth order, solve* $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ *with y(0) = 1 at x = 0.2, 0.4.* (J.N.T.U., B. Tech., 2009)

Sol. We have $f(x, y) = \dfrac{y^2 - x^2}{y^2 + x^2}$

To find y(0.2) :

Here $x_0 = 0$, $y_0 = 1$, $h = 0.2$

$$k_1 = h\,f(x_0, y_0) = 0.2\,f(0, 1) = 0.2000$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = 0.2\,f(0.1, 1.1) = 0.19672$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = 0.2f(0.1, 1.09836) \quad = 0.1967$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2f(0.2, 1.1967) \quad = 0.1891$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[0.2 + 2(0.19672) + 2(0.1967) + 0.1891] \quad = 0.19599$$

Hence $y(0.2) = y_0 + k = 1.196$.

To find y(0.4) :

Here $x_1 = 0.2, y_1 = 1.196, h = 0.2$.

$$k_1 = h\,f(x_1, y_1) \quad = 0.1891$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = 0.2\,f(0.3, 1.2906) \quad = 0.1795$$

$$k_3 = h\,f\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = 0.2\,f(0.3, 1.2858) \quad = 0.1793$$

$$k_4 = h\,f(x_1 + h, y_1 + k_3) = 0.2\,f(0.4, 1.3753) \quad = 0.1688$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[0.1891 + 2(0.1795) + 2(0.1793) + 0.1688] \quad = 0.1792$$

Hence $y(0.4) = y_1 + k = 1.196 + 0.1792 = 1.3752$.

Example 10.17. *Apply Runge-Kutta method to find approximate value of y for x = 0.2, in steps of 0.1, if $dy/dx = x + y^2$, given that y = 1 where x = 0.* (V.T.U., B.E., 2009)

Sol. *Given* $f(x, y) = x + y^2$.

Here we take $h = 0.1$ and carry out the calculations in two steps.

Step I. $x_0 = 0, y_0 = 1, h = 0.1$

$$\therefore \quad k_1 = hf(x_0, y_0) = 0.1\,f(0, 1) \quad = 0.1000$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = 0.1\,f(0.05, 1.1) \quad = 0.1152$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = 0.1\,f(0.05, 1.1152) \quad = 0.1168$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1\,f(0.1, 1.1168) \quad = 0.1347$$

$$\therefore \quad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.1000 + 0.2304 + 0.2336 + 0.1347) \quad = 0.1165$$

giving $y(0.1) = y_0 + k = 1.1165$.

Step II. $x_1 = x_0 + h = 0.1, y_1 = 1.1165, h = 0.1$

$$\therefore \quad k_1 = hf(x_1, y_1) = 0.1\,f(0.1, 1.1165) \quad = 0.1347$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = 0.1\,f(0.15, 1.1838) \quad = 0.1551$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = 0.1\,f(0.15, 1.194) \quad = 0.1576$$

$$k_4 = hf(x_1 + h, y_2 + k_3) = 0.1\,f(0.2, 1.1576) \quad = 0.1823$$

$$\therefore \quad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad = 0.1571$$

Hence $y(0.2) = y_1 + k = 1.2736$.

Example 10.18. *Using Runge-Kutta method of fourth order, solve for y at x = 1.2, 1.4 from* $\dfrac{dy}{dx} = \dfrac{2xy + e^x}{x^2 + xe^x}$ *given* $x_0 = 1, y_0 = 0$.

Sol. We have $f(x, y) = \dfrac{2xy + e^x}{x^2 + xe^x}$

To find y(1.2):

Here $x_0 = 1, y_0 = 0, h = 0.2$

$$\therefore \quad k_1 = h\,f(x_0, y_0) = 0.2\,\frac{0 + e'}{1 + e'} = 0.1462.$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2\left\{\frac{2(1+0.1)(0+0.073) + e^{1+0.1}}{(1+0.1)^2 + (1+0.1)e^{1+0.1}}\right\}$$
$$= 0.1402$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2\left\{\frac{2(1+0.1)(0+0.07) + e^{1.1}}{(1+0.1)^2 + (1+0.1)e^{1.1}}\right\}$$
$$= 0.1399$$

$$k_4 = hf(x_0 + k, y_0 + k_3) = 0.2\left\{\frac{2(1.2)(0.1399) + e^{1.2}}{(1.2)^2 + (1.2)e^{1.2}}\right\}$$
$$= 0.1348$$

and $$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \frac{1}{6}[0.1462 + 0.2804 + 0.2798 + 0.1348]$$
$$= 0.1402.$$

Hence $y(1.2) = y_0 + k = 0 + 0.1402 = 0.1402$.

To find y (1.4) :

Here $x_1 = 1.2, y_1 = 0.1402, h = 0.2$.

$k_1 = hf(x_1, y_1) = 0.2\,f(1.2, 0) = 0.1348$

$k_2 = hf(x_1 + h/2, y_1 + k_1/2) = 0.2\,f(1.3, 0.2076) = 0.1303$

$k_3 = hf(x_1 + h/2, y_1 + k_2/2) = 0.2\,f(1.3, 0.2053) = 0.1301$

$k_4 = hf(x_1 + h, y_1 + k_3) = 0.2\,f(1.3, 0.2703) = 0.1260$

$$\therefore \quad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \frac{1}{6}[0.1348 + 0.2606 + 0.2602 + 0.1260]$$
$$= 0.1303$$

Hence $y(1.4) = y_1 + k = 0.1402 + 0.1303 = 0.2705$

PROBLEMS 10.3

1. Use Runge's method to approximate y when $x = 1.1$, given that $y = 1.2$ when $x = 1$ and $dy/dx = 3x + y^2$.
2. Using Runge-Kutta method of order 4, find $y(0.2)$ given that $dy/dx = 3x + \frac{1}{2}y$, $y(0) = 1$, taking $h = 0.1$. (*V.T.U., B.Tech., 2004*)
3. Using Runge-Kutta method of order 4, compute $y(0.2)$ and $y(0.4)$ from $10\frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$, taking $h = 0.1$. (*Rohtak, B.Tech., 2003*)
4. Using fourth order Runge Kutta method, find the approximate solution at $x = 1.2$, $x = 1.4$ of the initial value problem $y' = xy$, $y(1) = 2$. (*Bombay, B. Tech., 2004*)
5. Given $dy/dx = x^3 + y$, $y(0) = 2$. Compute $y(0.2)$, $y(0.4)$ and $y(0.6)$ by Runge-Kutta method of fourth order. (*Anna, B.E., 2004*)
6. Find $y(0.1)$ and $y(0.2)$ using Runge-Kutta 4th order formula, given that $y' = x^2 - y$ and $y(0) = 1$. (*J.N.T.U., B. Tech., 2006*)
7. Using 4th order Runge-Kutta method, solve the following equation, taking each step of $h = 0.1$, given $y(0) = 3$. $dy/dx\ (4x/y - xy)$. Calculate y for $x = 0.1$ and 0.2. (*Anna, B.E., 2007*)
8. Find by Runge-Kutta method an approximate value of y for $x = 0.8$, given that $y = 0.41$ when $x = 0.4$ and $dy/dx = \sqrt{(x + y)}$. (*S.V.T.U., B. Tech., 2007 S*)
9. Using Runge-Kutta method of order 4, find $y(0.2)$ for the equation $\frac{dy}{dx} = \frac{y - x}{y + x}$, $y(0) = 1$. Take $h = 0.2$. (*V.T.U., B.E., 2003*)
10. Using 4th order Runge-Kutta method, integrate $y' = -2x^3 + 12x^2 - 20x + 8.5$, using a step size of 0.5 and initial condition of $y = 1$ at $x = 0$.
11. Using fourth order Runge-Kutta method, find y at $x = 0.1$ given that $dy/dx = 3e^x + 2y$, $y(0) = 0$ and $h = 0.1$. (*V.T.U., B. Tech., 2006*)

10.8. PREDICTOR-CORRECTOR METHODS

If x_{i-1} and x_i be two consecutive mesh points, we have $x_i = x_{i-1} + h$. In the Euler's method (§ 10.4), we have

$$y_i = y_{i-1} + hf(x_0 + \overline{i-1}h, y_{i-1})\ ;\ i = 1, 2, 3, \ldots.. \qquad \ldots(1)$$

The modified Euler's method (§ 10.5), gives

$$y_i = y_{i-1} + \frac{h}{2}\left[f(x_{i-1}, y_{i-1}) + f(x_i, y_i)\right] \qquad \ldots(2)$$

The value of y_i is first estimated by using (1), then this value is inserted on the right side of (2), giving a better approximation of y_i. This value of y_i is again substituted in (2) to find a still better approximation of y_i. This step is repeated till two consecutive values of y_i agree. *This technique of refining an initially crude estimate of y_i by means of a more accurate formula is known as* **predictor-corrector method.** The equation (1) is therefore called the *predictor* while (2) serves as a *corrector* of y_i.

In the methods so far described to solve a differential equation over an interval, only the value of y at the beginning of the interval was required. In the predictor-corrector methods, four prior values are needed for finding the value of y at x_i. Though slightly complex, these methods have the advantage of giving an estimate of error from successive approximations to y_i.

We now describe two such methods, namely : Milne's method and Adams-Bashforth method.

10.9. MILNE'S METHOD

Given $dy/dx = f(x, y)$ and $y = y_0$, $x = x_0$; to find an approximate value of y for $x = x_0 + nh$ by Milne's method, we proceed as follows :

The value $y_0 = y(x_0)$ being given, we compute

$$y_1 = y(x_0 + h),\ y_2 = y(x_0 + 2h),\ y_3 = y(x_0 + 3h),$$

by Picard's or Taylor's series method.

Next we calculate,

$$f_0 = f(x_0, y_0),\ f_1 = f(x_0 + h, y_1),\ f_2 = f(x_0 + 2h, y_2),\ f_3 = f(x_0 + 3h, y_3)$$

Then to find $y_4 = y(x_0 + 4h)$, we substitute Newton's forward interpolation formula

$$f(x, y) = f_0 + n\Delta f_0 + \frac{n(n-1)}{2}\Delta^2 f_0 + \frac{n(n-1)(n-2)}{6}\Delta^3 f_0 + \ldots$$

in the relation

$$y_4 = y_0 + \int_{x_0}^{x_0+4h} f(x, y)\, dx.$$

$$\therefore \quad y_4 = y_0 + \int_{x_0}^{x_0+4h}\left(f_0 + n\Delta f_0 + \frac{n(n-1)}{2}\Delta^2 f_0 + \ldots\ldots\right)dx$$

[Put $x = x_0 + nh$, $dx = hdn$]

$$= y_0 + h\int_0^4\left(f_0 + n\Delta f_0 + \frac{n(n-1)}{2}\Delta^2 f_0 + \ldots\ldots\right)dn$$

$$= y_0 + h\left(4f_0 + 8\Delta f_0 + \frac{20}{3}\Delta^2 f_0 + \frac{8}{3}\Delta^3 f_0 + \ldots\ldots\right)$$

Neglecting fourth and higher order differences and expressing Δf_0, $\Delta^2 f_0$ and $\Delta^3 f_0$ in terms of the function values, we get

$$y_4^{(p)} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

which is called a *predictor.*

Having found y_4, we obtain a first approximation to

$$f_4 = f(x_0 + 4h, y_4).$$

Then a better value of y_4 is found by Simpson's rule (p. 211) as

$$y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$$

which is called a *corrector.*

Then an improved value of f_4 is computed and again the corrector is applied to find a still better value of y_4. We repeat this step until y_4 remains unchanged.

Once y_4 and f_4 are obtained to desired degree of accuracy, $y_5 = y(x_0 + 5h)$ is found from the *predictor* as

$$y_5^{(p)} = y_1 + \frac{4h}{3}(2f_2 - 4f_3 + 2f_4)$$

and $f_5 = f(x_0 + 5h, y_5)$ is calculated. Then a better approximation to the value of y_5 is obtained from the *corrector* as

$$y_5^{(c)} = y_3 + \frac{h}{3}(f_3 + 4f_4 + f_5).$$

We repeat this step till y_5 becomes stationary and we, then proceed to calculate y_6 as before.

This is *Milne's predictor-corrector method*. To insure greater accuracy, we must first improve the accuracy of the starting values and then sub-divide the intervals.

Example 10.19. *Apply Milne's method, to find a solution of the differential equation $y' = x - y^2$ in the range $0 \le x \le 1$ for the boundary condition $y = 0$ at $x = 0$.*

(*V.T.U., B.E., 2009*)

Sol. Using Picard's method, we have

$$y = y(0) + \int_0^x f(x, y)\,dx, \text{ where } f(x, y) = x - y^2.$$

To get the first approximation, we put $y = 0$ in $f(x, y)$,

giving $$y_1 = 0 + \int_0^x x\,dx = \frac{x^2}{2}.$$

To find the second approximation, we put $y = x^2/2$ in $f(x, y)$,

giving $$y_2 = \int_0^x \left(x - \frac{x^4}{4}\right)dx = \frac{x^2}{2} - \frac{x^5}{20}$$

Similarly, the third approximation is

$$y_3 = \int_0^x \left[x - \left(\frac{x^2}{2} - \frac{x^5}{20}\right)^2\right]dx = \frac{x^2}{2} - \frac{x^5}{20} + \frac{x^8}{160} - \frac{x^{11}}{4400} \quad ...(i)$$

Now let us determine the starting values of the Milne's method from (1), by choosing $h = 0.2$.

$\therefore$

$x_0 = 0.0,$	$y_0 = 0.0000,$	$f_0 = 0.0000$
$x_1 = 0.2,$	$y_1 = 0.020,$	$f_1 = 0.1996$
$x_2 = 0.4,$	$y_2 = 0.0795,$	$f_2 = 0.3937$
$x_3 = 0.6,$	$y_3 = 0.1762,$	$f_3 = 0.5689$

Using the *predictor*, $y_4^{(p)} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$

$x = 0.8,\quad y_4^{(p)} = 0.3049,\quad f_4 = 0.7070$

and the *corrector*, $y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$, yields

$$y_4^{(c)} = 0.3046,\quad f_4 = 0.7072 \quad ...(ii)$$

Again using the *corrector*,

$y_4^{(c)} = 0.3046$, which is same as in (*ii*).

Now using the *predictor*,

$$y_5^{(p)} = y_1 + \frac{4h}{3}(2f_2 - f_3 + 2f_4),$$

$x = 1.0,\quad y_5^{(p)} = 0.4554,\quad f_5 = 0.7926$

and the *corrector* $y_5^{(c)} = y_3 + \frac{h}{3}(f_3 + 4f_4 + f_5)$, gives

$y_5^{(c)} = 0.4555,\quad f_5 = 0.7925$

Again using the *corrector*,

$y_5^{(c)} = 0.4555$, a value which is the same as before.

Hence $y(1) = 0.4555$.

Example 10.20. *Using Milne's method find $y(4.4)$ given $5xy' + y^2 - 2 = 0$ given $y(4) = 1$, $y(4.1) = 1.0049$, $y(4.2) = 1.0097$, $y(4.3) = 1.0143$; $y(4.4) = 1.0187$.* (*Anna, B.E., 2007*)

Sol. We have $y' = (2 - y^2)/5x = f(x)$ [say]

Then the starting values of the Milne's method are

$$x_0 = 4,\quad y_0 = 1,\quad f_0 = \frac{2 - 1^2}{5 \times 4} = 0.05$$

$x_1 = 4.1,$	$y_1 = 1.0049,$	$f_1 = 0.0485$
$x_2 = 4.2,$	$y_2 = 1.0097,$	$f_2 = 0.0467$
$x_3 = 4.3,$	$y_3 = 1.0143,$	$f_3 = 0.0452$
$x_4 = 4.4,$	$y_4 = 1.0187,$	$f_4 = 0.0437$

Since y_5 is required, we use the *predictor*

$$y_5^{(p)} = y_1 + \frac{4h}{3}(2f_2 - f_3 + 2f'_4) \quad (h = 0.1)$$

$$x = 4.5,\quad y_5^{(p)} = 1.0049 + \frac{4(0.1)}{3}(2 \times 2.0467 - 0.0452 + 2 \times 0.0437) = 1.023$$

$$f_5 = \frac{2 - y_5^2}{5x_5} = \frac{2 - (1.023)^2}{5 \times 4.5} = 0.0424$$

Now using the *corrector* $y_5^{(c)} = y_3 + \frac{h}{3}(f_3 + 4f_4 + f_5)$, we get

$$y_5^{(c)} = 1.0143 + \frac{0.1}{3}(0.0452 + 4 \times 0.0437 + 0.0424) = 1.023.$$

Hence $y(4.5) = 1.023$.

Example 10.21. *Given $y' = x(x^2 + y^2)e^{-x}$, $y(0) = 1$, find y at $x = 0.1, 0.2$ and 0.3 by Taylor's series method and compute $y(0.4)$ by Milne's method.* (Anna, B.E., 2007)

Sol. Given $y(0) = 1$ and $h = 0.1$

We have $y'(x) = x(x^2 + y^2)e^{-x}$; $\quad y'(0) = 0$

$$\therefore \quad y''(x) = [(x^3 + xy^2)(-e^{-x}) + (3x^2 + y^2 + x(2y)y')]e^{-x}$$

$$= e^{-x}[-x^3 - xy^2 + 3x^2 + y^2 + 2xyy'] ; \quad y''(0) = 1$$

$$y'''(x) = -e^{-x}[-x^3 - xy^2 + 3x^2 + y^2 + 2xyy' + 3x^2 + y^2 + 2xyy' - 6x - 2yy' - 2xy'^2 - 2xyy'']$$

$$y'''(0) = -2$$

Substituting these values in the Taylor's series,

$$y(x) = y(0) + \frac{x}{1!}y'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \ldots\ldots$$

$$y(0.1) = 1 + (0.1)(0) + \frac{1}{2}(0.1)^2(1) + \frac{1}{6}(0.1)^3(-2) + \ldots\ldots$$

$$= 1 + 0.005 - 0.0003 = 1.0047 \text{ i.e., } 1.005$$

Now taking $x = 0.1$, $y(0.1) = 1.005$, $h = 0.1$

$$y'(0.1) = 0.092,\ y''(0.1) = 0.849 ;\ y'''(0.1) = -1.247$$

Substituting these values in the Taylor's series about $x = 0.1$,

$$y(0.2) = y(0.1) + \frac{0.1}{1!}y'(0.1) + \frac{(0.1)^2}{2!}y''(0.1) + \frac{(0.1)^3}{3!}y'''(0.1) + \ldots\ldots$$

$$= 1.005 + (0.1)(0.092) + \frac{(0.1)^2}{2}(0.849) + \frac{(0.1)^3}{6}(-1247) + \ldots\ldots$$

$$= 1.018$$

Now taking $x = 0.2$, $y(0.2) = 1.018$, $h = 0.1$

$$y'(0.2) = 0.176,\ y''(0.2) = 0.77,\ y'''(0.2) = 0.819$$

Substituting these values in the Taylors series

$$y(0.3) = y(0.2) + \frac{0.1}{1!}y''(0.2) + \frac{(0.1)^2}{2!}y''(0.2) + \frac{(0.1)^3}{3!}y'''(0.2) + \ldots\ldots$$

$$= 1.018 + 0.0176 + 0.0039 + 0.0001$$

$$= 1.04$$

Thus the starting values of the Milne's method with $h = 0.1$ are

$x_0 = 0.0$	$y_0 = 1$	$f_0 = y'_0 = 0$
$x_1 = 0.1$	$y_1 = 1.005$	$f_1 = 0.092$
$x_2 = 0.2$	$y_2 = 1.018$	$f_2 = 0.176$
$x_3 = 0.3$	$y_3 = 1.04$	$f_3 = 0.26$

Using the predictor, $y_4^{(p)} = y_0 + \dfrac{4h}{3}(2f_1 - f_2 + 2f_3)$

$$= 1 + \frac{4(0.1)}{3}[2(0.092) - 0.176 + 2(0.26)]$$

$$= 1.09.$$

$\therefore \quad x = 0.4 \qquad y_4^{(p)} = 1.09 \qquad f_4 = y'(0.4) = 0.362$

Using the corrector, $y_4^{(c)} = y_2 + \dfrac{h}{3}(f_2 + 4f_3 + f_4)$

$$\therefore \quad y_4^{(c)} = 0.018 + \frac{0.1}{3}(0.176 + 4(0.26) + 0.362) = 1.071$$

Hence $y(0.4) = 1.071.$

Example 10.22. *Using Runge-Kutta method of order 4, find y for $x = 0.1, 0.2, 0.3$ given that $dy/dx = xy + y^2$, $y(0) = 1$. Continue the solution at $x = 0.4$ using Milne's method.* (S.V.T.U., B. Tech., 2007)

Sol. We have $f(x, y) = xy + y^2.$

To find $y(0.1)$:

Here $x_0 = 0$, $y_0 = 1$, $h = 0.1$.

$$\therefore \quad k_1 = h\,f(x_0, y_0) = (0.1)f(0, 1) \qquad = 0.1000$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = (0.1)\,f(0.05, 1.05) \qquad = 0.1155$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = (0.1)\,f(0.05, 1.0577) \qquad = 0.1172$$

$$k_4 = h\,f(x_0 + h, y_0 + k_3) = (0.1)\,f(0.1, 1.1172) \qquad = 0.13598$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.1 + 0.231 + 0.2343 + 0.13598) \qquad = 0.11687$$

Thus $y(0.1) = y_1 = y_0 + k = 1.1169.$

To find $y(0.2)$:

Here $x_1 = 0.1$, $y_1 = 1.1169$, $h = 0.1$.

$$k_1 = h\,f(x_1, y_1) = (0.1)\,f(0.1, 1.1169) \qquad = 0.1359$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = (0.1)\,f(0.15, 1.1848) \qquad = 0.1581$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = (0.1)\,f(0.15, 1.1959) \qquad = 0.1609$$

$$k_4 = h\,f(x_1 + h, y_1 + k_3) = (0.1)\,f(0.2, 1.2778) \qquad = 0.1888$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad = 0.1605$$

Thus $y(0.2) = y_2 = y_1 + k = 1.2773.$

To find $y(0.3)$:

Here $x_2 = 0.2$, $y_2 = 1.2773$, $h = 0.1$.

$$k_1 = h\,f(x_2, y_2) = (0.1)\,f(0.2, 1.2773) \qquad = 0.1887$$

$$k_2 = hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_1\right) = (0.1)\,f(0.25, 1.3716) \qquad = 0.2224$$

$$k_3 = hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_2\right) = (0.1) f(0.25, 1.3885) \qquad = 0.2275$$

$$k_4 = h\, f(x_2 + h, y_2 + k_3) = (0.1) f(0.3, 1.5048 \qquad = 0.2716$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad = 0.2267$$

Thus $y(0.3) = y_3 = y_2 + k = 1.504$

Now the starting values for the Milne's method are :

$x_0 = 0.0$	$y_0 = 1.0000$	$f_0 = 1.0000$
$x_1 = 0.1$	$y_1 = 1.1169$	$f_1 = 1.3591$
$x_2 = 0.2$	$y_2 = 1.2773$	$f_2 = 1.8869$
$x_3 = 0.3$	$y_3 = 1.5049$	$f_3 = 2.7132$

Using the *predictor*,

$$y_4^{(p)} = y_0 = \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$$x_4 = 0.4 \qquad y_4^{(p)} = 1.8344 \qquad f_4 = 4.0988$$

and the *corrector*,

$$y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4) \text{ yields}$$

$$y_4^{(c)} = 1.2773 + \frac{0.1}{3}[1.8869 + 4(2.7132) + 4.098]$$

$$= 1.8386 \qquad f_4 = 4.1159.$$

Again using the *corrector*,

$$y_4^{(c)} = 1.2773 + \frac{0.1}{3}[1.8869 + 4(2.7132) + 4.1159]$$

$$= 1.8391 \qquad f_4 = 4.1182 \qquad ...(i)$$

Again using the *corrector*

$$y_4^{(c)} = 1.2773 + \frac{0.1}{3}[1.8869 + 4(2.7132) + 4.1182]$$

$$= 1.8392 \text{ which is same as } (i).$$

Hence $y(0.4) = 1.8392$.

PROBLEMS 10.4

1. Given $\frac{dy}{dx} = x^3 + y, y(0) = 2$. The values of $y(0.2) = 2.073$, $y(0.4) = 2.452$, and $y(0.6) = 3.023$ are got by R.K. method of 4th order. Find $y(0.8)$ by Milne's predictor – corrector method taking $h = 0.2$ *(Anna, B.E., 2004)*
2. Given $2\, dy/dx = (1 + x^2)y^2$ and $y(0) = 1$, $y(0.1) = 1.06$, $y(0.2) = 1.12$, $y(0.3) = 1.21$, evaluate $y(0.4)$ by Milne's predictor corrected method. *(Madras, B.E., 2003)*

3. Solve that initial value problem
$$\frac{dy}{dx} = 1 + xy^2, y(0) = 1$$
for $x = 0.4$ by using Milne's method, when it is given that

x :	0.1	0.2	0.3
y :	1.105	1.223	1.355.

4. Use Milne's method to find $y(0.3)$ from $y' = x^2 + y^2$, $y(0) = 1$. Find the initial values $y(-0.1)$, $y(0.1)$ and $y(0.2)$ from the Taylor's series method.
5. Using Tayler's series method, solve $\frac{dy}{dx} = xy + x^2$, $y(0) = 1$; at $x = 0.1, 0.2, 0.3$. Continue the solution at $x = 0.4$ by Milne's predictor-corrector method.
6. Given that $\frac{dy}{dx} = \frac{1}{x^2} - \frac{y}{x}$, $y(1) = 1$, $y(1.1) = 0.996$, $y(1.2) = 0.986$ and $y(0.3) = 0.972$, find the values of $y(1.4)$ and $y(1.5)$ using Milne's predictor-corrector method.
7. Using Runge-Kutta method, calculate $y(0.1)$, $y(0.2)$, and $y(0.3)$ given that $\frac{dy}{dx} - \frac{2xy}{1+x^2} = 1$, $y(0) = 0$. Taking these values as starting values, find $y(0.4)$ by Milne's method.

10.10. ADAMS-BASHFORTH METHOD

Given $\frac{dy}{dx} = f(x, y)$ and $y_0 = y(x_0)$, we compute

$$y_{-1} = y(x_0 - h), y_{-2} = y(x_0 - 2h), y_{-3} = y(x_0 - 3h)$$

by Taylor's series or Euler's method or Runge-Kutta method.

Next we calculate

$$f_{-1} = f(x_0 - h, y_{-1}), f_{-2} = f(x_0 - 2h, y_{-2}), f_{-3} = f(x_0 - 3h, y_{-3}).$$

Then to find y_1, we substitute Newton's backward interpolation formula

$$f(x, y) = f_0 + n\nabla f_0 + \frac{n(n+1)}{2}\nabla^2 f_0 + \frac{n(n+1)(n+2)}{6}\nabla^3 f_0 + ...$$

in

$$y_1 = y_0 + \int_{x_0}^{x_0+h} f(x, y)\, dx \qquad ...(1)$$

$$\therefore \quad y_1 = y_0 + \int_{x_0}^{x_1}\left(f_0 + n\nabla f_0 + \frac{n(n+1)}{2}\nabla^2 f_0 + ...\right) dx$$

[Put $x = x_0 + nh$, $dx = hdn$

$$= y_0 + h\int_0^1\left(f_0 + n\nabla f_0 + \frac{n(n+1)}{2}\nabla^2 f_0 + ...\right) dn$$

$$= y_0 + h\left(f_0 + \frac{1}{2}\nabla f_0 + \frac{5}{12}\nabla^2 f_0 + \frac{3}{8}\nabla^3 f_0 + ...\right)$$

Neglecting fourth and higher order differences and expressing ∇f_0, $\nabla^2 f_0$ and $\nabla^3 f_0$ in terms of function values, we get

$$y_1^{(p)} = y_0 + \frac{h}{24}(55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3}) \qquad ...(2)$$

This is called *Adams-Bashforth predictor formula*.

Having found y_1, we find $f_1 = f(x_0 + h_1, y_1)$

Then to find a better value of y_1, we derive a *corrector formula* by substituting Newton's backward formula at f_1 *i.e.*

$$f(x, y) = f_1 + n\nabla f_1 + \frac{n(n+1)}{2}\nabla^2 f_1 + \frac{n(n+1)(n+2)}{6}\nabla^3 f_1 + ...$$

in (1).

$$\therefore \qquad y_1 = y_0 + \int_{x_0}^{x_1}\left(f_1 + n\nabla f_1 + \frac{n(n+1)}{2}\nabla^2 f_1 + ...\right)dx$$

[Put $x = x_1 + nh$, $dx = h\,dn$

$$= y_0 + \int_{-1}^{0}\left(f_1 + n\nabla f_1 + \frac{n(n+1)}{2}\nabla^2 f_1 + ...\right)dn$$

$$= y_0 + h\left(f_1 - \frac{1}{2}\nabla f_1 - \frac{1}{12}\nabla^2 f_1 - \frac{1}{24}\nabla^3 f_1 - ......\right)$$

Neglecting fourth and higher order differences and expressing ∇f_1, $\nabla^2 f_1$ and $\nabla^3 f_1$ in terms of function values, we obtain

$$y_1^{(c)} = y_0 + \frac{h}{24}(9f_1 + 19f_0 - 5f_{-1} + f_{-2}) \qquad ...(3)$$

which is called *Adams-Moulton corrector formula*.

Then an improved value of f_1 is calculated and again the corrector (3) is applied to find a still better value y_1. This step is repeated till y_1 remains unchanged and then we proceed to calculate y_2 as above.

Obs. To apply both Milne and Adams-Bashforth methods, we require four starting values of y which are calculated by means of Picard's method or Taylor's series method or Euler's method or Runge-Kutta method. In practice, the Adams formulae (2) and (3) above together with fourth order Runge-Kutta formulae have been found to be most useful.

Example 10.23. *Given* $\frac{dy}{dx} = x^2(1 + y)$ *and* $y(1) = 1$, $y(1.1) = 1.233$, $y(1.2) = 1.548$, $y(1.3) = 1.979$, *evaluate* $y(1.4)$ *by Adams-Bashforth method.* *(J.N.T.U., B.Tech., 2009)*

Sol. Here $f(x, y) = x^2(1 + y)$.

Starting values of the Adams-Bashforth method with $h = 0.1$ are

$x = 1.0, y_{-3} = 1.000,$ $\quad f_{-3} = (1.0)^2(1 + 1.000) = 2.000$

$x = 1.1, y_{-2} = 1.233,$ $\quad f_{-2} = 2.702$

$x = 1.2, y_{-1} = 1.548,$ $\quad f_{-1} = 3.669$

$x = 1.3, y_0 = 1.979,$ $\quad f_0 = 5.035$

Using the *predictor*,

$$y_1^{(p)} = y_0 + \frac{h}{24}(55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3})$$

$$x = 1.4, \quad y_1^{(p)} = 2.573, \quad f_1 = 7.004$$

Using the *corrector*,

$$y_1^{(c)} = y_0 + \frac{h}{24}(9f_1 + 19f_0 - 5f_{-1} + f_{-2})$$

$$y_1^{(c)} = 1.979 + \frac{0.1}{24}(9 \times 7.004 + 19 \times 5.035 - 5 \times 3.669 + 2.702) = 2.575$$

Hence $y(1.4) = 2.575$.

Example 10.24. *If* $\frac{dy}{dx} = 2e^x y$, $y(0) = 2$, *find* $y(4)$ *using Adams predictor corrector formula by calculating* $y(1)$, $y(2)$ *and* $y(3)$ *using Euler's modified formula.* *(J.N.T.U., B. Tech., 2006)*

Sol. We have $f(x, y) = 2e^x y$

To find 0.1 :

x	$2e^x y = y'$	*Mean slope*	*Old* $y + h$ *(mean slop) = new* y
0	4	—	$2 + 0.1(4) = 2.4$
0.1	$2e^{0.1}(2.4) = 5.305$	$\frac{1}{2}(4 + 5.305) = 4.6524$	$2 + 0.1(4.6524) = 2.465$
0.1	$2e^{0.1}(2.465) = 5.449$	$\frac{1}{2}(4 + 5.449) = 4.7244$	$2 + 0.1(4.7244) = 2.472$
0.1	$2e^{0.1}(2.4724) = 5.465$	$\frac{1}{2}(4 + 5.465) = 4.7324$	$2 + 0.1(4.7324) = 2.473$
0.1	$2e^{0.1}(2.473) = 5.467$	$\frac{1}{2}(4 + 5.467) = 4.7333$	$2 + 0.1(4.7333) = 2.473$
0.1	5.467	—	$2 + 0.1(5.467) = 3.0199$
0.2	$2e^{0.2}(3.0199) = 7.377$	$\frac{1}{2}(5.467 + 7.377) = 6.422$	$2.473 + 0.1(6.422) = 3.1155$
0.2	7.611	$\frac{1}{2}(5.467 + 7.611) = 6.539$	$2.473 + 0.1(6.539) = 3.127$
0.2	7.639	$\frac{1}{2}(5.467 + 7.639) = 6.553$	$2.473 + 0.1(6.553) = 3.129$

0.2	7.643	$\frac{1}{2}(5.467 + 7.643) = 6.555$	$2.473 + 0.1\,(6.555) = 3.129$
0.2	7.643	—	$3.129 + 0.1\,(7.643) = 3.893$
0.3	$2e^{0.3}\,(3.893) = 10.51$	$\frac{1}{2}(7.643 + 10.51) = 9.076$	$3.129 + 0.1\,(9.076) = 4.036$
0.3	10.897	$\frac{1}{2}(7.643 + 10.897) = 9.266$	$3.129 + 0.1\,(9.2696) = 4.056$
0.3	10.949	$\frac{1}{2}(7.643 + 10.949) = 9.296$	$3.129 + 0.1\,(9.296) = 4.058$
0.3	10.956	$\frac{1}{2}(7.643 + 10.956) = 9.299$	$3.129 + 0.1\,(9.299) = 4.0586$

To find y(0.4) by Adam's method, the starting values with $h = 0.1$ are

$$
\begin{array}{lll}
x = 0.0 & y_{-3} = 2.4 & f_{-3} = 4 \\
x = 0.1 & y_{-2} = 2.473 & f_{-2} = 5.467 \\
x = 0.2 & y_{-1} = 3.129 & f_{-1} = 7.643 \\
x = 0.3 & y_0 = 4.059 & f_0 = 10.956
\end{array}
$$

Using the predictor formula

$$y_1^{(p)} = y_0 + \frac{h}{24}(55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3})$$

$$= 4.059 + \frac{0.1}{24}(55 \times 10.957 - 59 \times 7.643 + 37 \times 5.467 - 9 \times 4)$$

$$= 5.383$$

Now $x = 0.4$ $\quad y_1 = 5.383 \quad f_1 = 2e^{0.4}\,(5.383) = 16.061$

Using the corrector formula,

$$y_1^{(c)} = y_0 + \frac{h}{24}(9f_1 + 19f_0 - 5f_{-1} + f_{-2})$$

$$= 4.0586 + \frac{0.1}{24}(9 \times 16.061 + 19 \times 10.956 - 5 \times 7.643 + 5.467) = 5.392$$

Hence $\quad y(0.4) = 5.392.$

Example 10.25. *Solve the initial value problem $dy/dx = x - y^2$, $y(0) = 1$ to find $y(0.4)$ by Adam's method. Starting solutions required are to be obtained using Runge-Kutta method of order 4 using step value $h = 0.1$.* *(P.T.U., B.E., 2003)*

Sol. We have $f(x, y) = x - y^2$.

To find y(0.1) :

Here $x_0 = 0, y_0 = 1, h = 0.1$.

$\therefore$
$$k_1 = h\,f(x_0, y_0) = (0.1)\,f(0, 1) \qquad = -0.1000$$

$$k_2 = h\,f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = (0.1)\,f(0.05, 0.95) \qquad = -0.08525$$

$$k_3 = h\,f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = (0.1)\,f(0.05, 0.9574) \qquad = -0.0867$$

$$k_4 = h\,f(x_0 + h, y_0 + k_3) = (0.1)\,f(0.1, 0.9137) \qquad = -0.07341$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad = -0.0883$$

Thus $\quad y(0.1) = y_1 = y_0 + k = 1 - 0.0883 \qquad = 0.9117$

To find y(0.2) :

Here $x_1 = 0.1, y_1 = 0.9117, h = 0.1$.

$\therefore$
$$k_1 = h\,f(x_1, y_1) = (0.1)\,f(0.1, 0.9117) \qquad = -0.0731$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = (0.1)\,f(0.15, 0.8751) \qquad = -0.0616$$

$$k_3 = h\,f\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = (0.1)\,f(0.15, 0.8809) \qquad = -0.0626$$

$$k_4 = h\,f(x_1 + h, y_1 + k_3) = (0.1)\,f(0.2, 0.8491) \qquad = -0.0521$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad = -0.0623$$

Thus $\quad y(0.2) = y_2 = y_1 + k = 0.8494.$

To find y(0.3) :

Here $\quad x_2 = 0.2, y_2 = 0.8494, y = 0.1$

$$k_1 = h\,f(x_2, y_2) = (0.1)\,f(0.2, 0.8494) \qquad = -0.0521$$

$$k_2 = h\,f\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_1\right) = (0.1)\,f(0.25, 0.8233) \qquad = -0.0428$$

$$k_3 = h\,f\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_2\right) = (0.1)\,f(0.25, 0.828) \qquad = -0.0436$$

$$k_4 = h\,f(x_2 + h, y_2 + k_3) = (0.1)\,f(0.3, 0.8058) \qquad = -0.0349$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad = -0.0438$$

Thus $\quad y(0.3) = y_3 = y_2 + k = 0.8061$

Now the starting values of Adam's method with $h = 0.1$ are :

$$
\begin{array}{llll}
x = 0.0 & y_{-3} = 1.0000 & f_{-3} = 0.0 - (1.0)^2 & = -1.0000 \\
x = 0.1 & y_{-2} = 0.9117 & f_{-2} = 0.1 - (0.9117)^2 & = -0.7312 \\
x = 0.2 & y_{-1} = 0.8494 & f_{-1} = 0.2 - (0.8494)^2 & = -0.5215 \\
x = 0.3 & y_0 = 0.8061 & f_0 = 0.3 - (0.8061)^2 & = -0.3498
\end{array}
$$

Using the *predictor*,

$$y_1^{(p)} = y_0 + \frac{h}{24}(55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3})$$

$x = 0.4$ $\quad y_1^{(p)} = 0.8061 + \frac{0.1}{24}[55(-0.3498) - 59(-0.5215) + 37(-0.7312) - 9(-1)]$

$= 0.7789$ $\qquad f_1 = -0.2067$

Using the *corrector*,

$$y_1^{(c)} = y_0 + \frac{h}{24}(9f_1 + 19f_0 - 5f_{-1} + f_{-2})$$

$$y_1^{(c)} = 0.8061 + \frac{0.1}{24}[9(-0.2067) + 19(-0.3498) - 5(-0.5215) - 0.7312] = 0.7785$$

Hence $y(0.4) = 0.7785$.

PROBLEMS 10.5

1. Using Adams-Bashforth method, obtain the solution of $dy/dx = x - y^2$ at $x = 0.8$, given the values

x :	0	0.2	0.4	0.6
y :	0	0.0200	0.0795	0.1762

2. Using Adams-Bashforth formulae, determine $y(0.4)$ given the differential equation $dy/dx = \frac{1}{2}xy$ and the data :

x :	0	0.1	0.2	0.3
y :	1	1.0025	1.0101	1.0228.

3. Using Adams-Bashforth method, obtain the solution of $dy/dx = x - y^2$ at $x = 0.8$, given the values

x :	0	0.2	0.4	0.6
y :	0	0.0200	0.0795	0.1762

(*Bhopal, B.E., 2002*)

4. Given $y' = x^2 - y$, $y(0) = 1$ and the starting values $y(0.1) = 0.90516$, $y(0.2) = 0.82127$, $y(0.3) = 0.74918$, evaluate $y(0.4)$ using Adams-Bashforth method. (*S.V.T.U., B.E., 2007*)

5. Using Adams-Bashforth method, find $y(4.4)$ given $5xy' + y^2 = 2$, $y(4) = 1$, $y(4.1) = 1.0049$, $y(4.2) = 1.0097$ and $y(4.3) = 1.0143$.

6. Given the differential equation $dy/dx = x^2y + x^2$ and the data :

x :	1	1.1	1.2	1.3
y :	1	1.233	1.548488	1.978921,

determine $y(1.4)$ by any numerical method. (*Indore, B.E., 2003 S*)

7. Using Adams-Bashforth method, evaluate $y(1.4)$; if y satisfies $dy/dx + y/x = 1/x^2$ and $y(1) = 1$, $y(1.1) = 0.996$, $y(1.2) = 0.986$, $y(1.3) = 0.972$. (*Madras, B.E., 2003*)

10.11. SIMULTANEOUS FIRST ORDER DIFFERENTIAL EQUATIONS

The simultaneous differential equations of the type

$$\frac{dy}{dx} = f(x, y, z) \qquad \ldots(1)$$

and

$$\frac{dz}{dx} = \phi(x, y, z) \qquad \ldots(2)$$

with initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$ can be solved by the methods discussed in the preceding sections, especially Picard's or Runge-Kutta methods.

(*i*) *Picard's method gives*

$$y_1 = y_0 + \int f(x, y_0, z_0)\, dx,\ z_1 = z_0 + \int \phi(x, y_0, z_0)\, dx$$

$$y_2 = y_0 + \int f(x, y_1, z_1)\, dx,\ z_2 = z_0 + \int \phi(x, y_1, z_1)\, dx$$

$$y_3 = y_0 + \int f(x, y_2, z_2)\, dx,\ z_3 = z_0 + \int \phi(x, y_2, z_2)\, dx$$

and so on.

(*ii*) *Taylor's series method* is used as follows :

If h be the step-size, $y_1 = y(x_0 + h)$ and $z_1 = z(x_0 + h)$. Then Taylor's algorithm for (1) and (2) gives

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \ldots \qquad \ldots(3)$$

$$z_1 = z_0 + hz_0' + \frac{h^2}{2!}z_0'' + \frac{h^3}{3!}z_0''' + \ldots \qquad \ldots(4)$$

Differentiating (1) and (2) successively, we get y'', z'', etc. So the values y_0', y_0'', y_0''' ... and z_0', z_0'', z_0''' ... are known. Substituting these in (3) and (4), we obtain y_1, z_1 for the next step.

Similarly, we have the algorithms

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \ldots \qquad \ldots(5)$$

$$z_2 = z_1 + hz_1' + \frac{h^2}{2!}z_1'' + \frac{h^3}{3!}z_1''' + \ldots \qquad \ldots(6)$$

Since y_1 and z_1 are known, we can calculate y_1', y_1'', ... and z_1', z_1'',... . Substituting these in (5) and (6), we get y_2 and z_2.

Proceeding further, we can calculate the other values of y and z step by step.

(*iii*) *Runge-Kutta method* is applied as follows :

Starting at (x_0, y_0, z_0) and taking the step-sizes for x, y, z to be h, k, l respectively, the Runge-Kutta method gives,

$$k_1 = hf(x_0, y_0, z_0)$$
$$l_1 = h\phi(x_0, y_0, z_0)$$
$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$$

$$l_2 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right)$$

$$l_3 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$$
$$l_4 = h\phi(x_0 + h, y_0 + k_3, z_0 + l_3)$$

Hence
$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

and
$$z_1 = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

To compute y_2 and z_2, we simply replace x_0, y_0, z_0 by x_1, y_1, z_1 in the above formulae.

Example 10.26. *Using Picard's method, find approximate values of y and z corresponding to $x = 0.1$, given that $y(0) = 2$, $z(0) = 1$ and*

$$dy/dx = x + z, \; dz/dx = x - y^2.$$

Sol. Here $x_0 = 0, y_0 = 2, z_0 = 1$.

and
$$\frac{dy}{dx} = f(x, y, z) = x + z$$

$$\frac{dz}{dx} = \phi(x, y, z) = x - y^2$$

$$\therefore \quad y = y_0 + \int_{x_0}^{x} f(x, y, z)\, dx \quad \text{and} \quad z = z_0 + \int_{x_0}^{x} \phi(x, y, z)\, dx.$$

First approximations

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0, z_0)\, dx = 2 + \int_0^x (x + 1)\, dx = 2 + x + \frac{1}{2}x^2$$

$$z_1 = z_0 + \int_{x_0}^{x} \phi(x, y, z_0)\, dx = 1 + \int_0^x (x - 4)\, dx = 1 - 4x + \frac{1}{2}x^2.$$

Second approximations

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1, z_1)\, dx = 2 + \int_0^x \left(x + 1 - 4x + \frac{1}{2}x^2\right) dx$$

$$= 2 + x - \frac{3}{2}x^2 + \frac{x^3}{6}$$

$$z_2 = z_0 + \int_{x_0}^{x} \phi(x, y_1, z_1)\, dx = 1 + \int_0^x \left[x - \left(2 + x + \frac{1}{2}x^2\right)^2\right] dx$$

$$= 1 - 4x + \frac{3}{2}x^2 - x^3 - \frac{x^4}{4} - \frac{x^5}{20}.$$

Third approximations

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2, z_2)\, dx = 2 + x - \frac{3}{2}x^2 - \frac{1}{2}x^3 - \frac{1}{4}x^4 - \frac{1}{20}x^5 - \frac{1}{120}x^6$$

$$z_3 = z_0 + \int_{x_0}^{x} \phi(x, y_2, z_2)\, dx$$

$$= 1 - 4x - \frac{3}{2}x^2 + \frac{5}{3}x^3 + \frac{7}{12}x^4 - \frac{31}{60}x^5 + \frac{1}{12}x^6 - \frac{1}{252}x^7$$

and so on.

When $x = 0.1$

$y_1 = 2.105$, $\quad y_2 = 2.08517$, $\quad y_3 = 2.08447$

$z_1 = 0.605$, $\quad z_2 = 0.58397$, $\quad z_3 = 0.58672$.

Hence $y(0.1) = 2.0845$, $\quad z(0.1) = 0.5867$

correct to four decimal places.

Example 10.27. *Find an approximate series solution of the simultaneous equations $dx/dt = xy + 2t$, $dy/dt = 2ty + x$ subject to the initial conditions $x = 1$, $y = -1$, $t = 0$.*

Sol. x and y both being functions of t, Taylor's series gives

$$\left.\begin{aligned} x(t) &= x_0 + tx_0' + \frac{t^2}{2!}x_0'' + \frac{t^3}{3!}x_0''' + \ldots\ldots \\ \text{and} \quad y(t) &= y_0 + ty_0' + \frac{t^2}{2!}y_0'' + \frac{t^3}{3!}y_0''' + \ldots\ldots \end{aligned}\right\} \quad \ldots(i)$$

Differentiating the given equations

$$x' = xy + 2t \quad \ldots(ii) \qquad y' = 2ty + x \quad \ldots(iii)$$

w.r.t. t, we get

$$\left.\begin{array}{l|l} x'' = xy' + x'y + 2 & y'' = 2ty' + 2y + x' \\ x''' = (xy'' + x'y') + x''y + x'y' & y''' = 2ty'' + 2y' + 2y' + x'' \\ \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \quad \ldots(iv)$$

Putting $x_0 = 1, y_0 = -1, t_0 = 0$ in (ii), (iii) and (iv), we obtain

$$\begin{array}{l|l} x_0' = -1 + 2(0) = -1 & y_0' = 1 \\ x_0'' = x_0y_0' + x_0'y_0 + 2 & y_0'' = 0 + 2y_0 + x_0' \\ \quad = 1.1 + (-1)(-1) + 2 = 4 & \quad = 2(-1) + (-1) = -3 \\ x_0''' = -3 + (-1)(1) + 4(-1) - 1 = -9 & y_0''' = 2 + 2 + 4 = 8 \text{ etc} \\ \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

Substituting these values in (i), we get

$$x(t) = 1 - t + 4\frac{t^2}{2!} + (-9)\frac{t^3}{3!} + \ldots\ldots = 1 - t + 2t^2 - \frac{3}{2}t^3 + \ldots\ldots$$

$$y(t) = 1 + t - 3\frac{t^2}{2!} + 8\frac{t^3}{3!} + \ldots\ldots = 1 + t - \frac{3}{2}t^2 + \frac{4}{3}t^3 + \ldots\ldots$$

Example 10.28. *Solve the differential equations*

$$\frac{dy}{dx} = 1 + xz, \ \frac{dz}{dx} = -xy \text{ for } x = 0.3.$$

using fourth order Runge-Kuta method. Intial values are $x = 0$, $y = 0$, $z = 1$.

Sol. Here $f(x, y, z) = 1 + xz$, $\phi(x, y, z) = -xy$

$x_0 = 0, y_0 = 0, z_0 = 1$. Let us take $h = 0.3$.

$\therefore$

$$k_1 = h\,f(x_0, y_0, z_0) = 0.3\,f(0, 0, 1) = 0.3\,(1 + 0) = 0.3.$$

$$l_1 = h\,\phi(x_0, y_0, z_0) = 0.3\,(-0 \times 0) = 0$$

$$k_2 = h\,f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$$

$$= (0.3)\,f(0.15, 0.15, 1) = 0.3\,(1 + 0.15) = 0.345$$

$$l_2 = h\,\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$$

$$= 0.3\,[-(0.15)\,(0.15)] = -0.00675.$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$= (0.3)\,f(0.15, 0.1725, 0.996625)$$

$$= 0.3\,[1 + 0.996625 \times 0.15] = 0.34485$$

$$l_3 = h\,\phi\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$= 0.3\,[-(0.15)\,(0.1725)] = -0.007762$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$= 0.3\,f(0.3, 0.34485, 0.99224) = 0.3893$$

$$l_4 = h\,\phi\,(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$= 0.3\,[-(0.3)\,(0.34485)] = -0.03104$$

Hence $\quad y\,(x_0 + h) = y_0 + \frac{1}{6}\,(k_1 + 2k_2 + 2k_3 + k_4)$

i.e., $\quad y\,(0.3) = 0 + \frac{1}{6}\,[0.3 + 2\,(0.345) + 2\,(0.34485) + 0.3893] = 0.34483$

and $\quad z\,(x + h) = z_0 + \frac{1}{6}\,[l_1 + 2l_2 + 2l_3 + l_4]$

i.e., $\quad z\,(0.3) = 1 + \frac{1}{6}\,[0 + 2\,(-0.00675) + 2\,(-0.0077625) + (-0.03104)] = 0.98999$

10.12. SECOND ORDER DIFFERENTIAL EQUATIONS

Consider the second order differential equation

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right)$$

By writing $dy/dx = z$, it can be reduced to two first order simultaneous differential equations

$$\frac{dy}{dx} = z, \ \frac{dz}{dx} = f(x, y, z)$$

These equations can be solved as explained above.

Example 10.29. *Find the value of $y(1.1)$ and $y(1.2)$ from $y'' + y^2y' = x^3$; $y(1) = 1$, $y'(1) = 1$, using Taylor series method.*

Sol. Let $\quad y' = z$ so that $y'' = z'$

Then the given equation becomes $z' + y^2z = x^3$

$\therefore$

$$y' = z \qquad ...(i)$$

$$z' = x^3 - y^2z \qquad ...(ii)$$

such that $\quad y(1) = 1, z(1) = 1, h = 0.1.$

Now from (*i*) $\quad y' = z, y'' = z', y''' = z'' \qquad ...(iii)$

and from (*ii*) $\quad z' = x^3 - y^2z, \ z'' = 3x^2 - y^2z' - 2yz^2 \quad (\because \ y' = z)$

$$z''' = 6x - (y^2z'' + 2yy'z') - 2\,(y'z^2 + y^2zz') \qquad ...(iv)$$

$$= 6x - (y^2z'' + 2yz'^2) - 2\,(z^3 + 2yzz')$$

Taylor's series for $y(1.1)$ is

$$y(1.1) = y(1) + hy'(1) + \frac{h^2}{2!}\,y''(1) + \frac{h^3}{3!}\,y'''(1) + ...$$

Also $\quad y(1) = 1, y'(1) = 1, y''(1) = z'(1) = 0, y'''(1) = z''(1) = 1 \qquad$ [From (*iii*)]

$\therefore \quad y(1.1) = 1 + 0.1(1) + \frac{(0.1)^2}{2}\,(0) + \frac{(0.1)^3}{6}\,(1) = 1.1002.$

Taylor's series for $z(1.1)$ is

$$z(1.1) = z(1) + hz'(1) + \frac{h^2}{2!}\,z''(1) + \frac{h^3}{3!}\,z'''(1) + ...$$

Here $\quad z(1) = 1, z'(1) = 0, z''(1) = 1, z'''(1) = 3. \qquad$ [From (*iv*)]

$\therefore \quad z(1.1) = 1 + 0.1(0) + \frac{(0.1)^2}{2}\,(1) + \frac{(0.1)^3}{6}\,(3) = 1.0055$

Hence $\quad y(1.1) = 1.1002$ and $z(1.1) = 1.0055.$

Example 10.30. *Using Runge-Kutta method, solve $y'' = xy'^2 - y^2$ for $x = 0.2$ correct to 4 decimal places. Initial conditions are $x = 0$, $y = 1$, $y' = 0$.* (Delhi, B.E., 2002)

Sol. Let $\quad dy/dx = z = f(x, y, z)$

Then $\quad \frac{dz}{dx} = xz^2 - y^2 = \phi(x, y, z)$

We have $x_0 = 0, y_0 = 1, z_0 = 0, h = 0.2.$

$\therefore$ Runge-Kutta formulae become

$$k_1 = hf(x_0, y_0, z_0) = 0.2(0) = 0$$
$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right) = 0.2(-0.1) = -0.02$$
$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right) = 0.2(-0.0999) = -0.02$$
$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3) = 0.2(-0.1958) = -0.0392$$
$$\therefore \quad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = -0.0199$$

$$l_1 = h\phi(x_0, y_0, z_0) = 0.2(-1) = -0.2$$
$$l_2 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right) = 0.2(-0.999) = -0.1998$$
$$l_3 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right) = 0.2(-0.9791) = -0.1958$$
$$l_4 = h\phi(x_0 + h, y_0 + k_3, z_0 + l_3) = 0.2(0.9527) = -0.1905$$
$$l = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = -0.1970$$

Hence at $x = 0.2$,

$$y = y_0 + k = 1 - 0.0199 = 0.9801$$

and $$y' = z = z_0 + l = 0 - 0.1970 = -0.1970.$$

■ **Example 10.31.** *Given $y'' + xy' + y = 0$, $y(0) = 1$, $y'(0) = 0$, obtain y for $x = 0(0.1)\,0.3$ by any method. Further, continue the solution by Milne's method to calculate $y(0.4)$.*

(Anna, B.E., 2004)

Sol. Putting $y' = z$, the given equation reduces to the simultaneous equations

$$z' + xz + y = 0, \; y' = z \qquad \ldots(1)$$

We employ Taylor's series method to find y.

Differentiating the given equation n times, we get

$$y_{n+2} + xy_{n+1} + ny_n + y_n = 0$$

At $\quad x = 0$, $(y_{n+2})_0 = -(n+1)(y_n)_0$

$\therefore$ $y(0) = 1$, gives $y_2(0) = -1, y_4(0) = 3, y_6(0) = -5 \times 3, \ldots\ldots$

and $y_1(0) = 0$ yields $y_3(0) = y_5(0) = \ldots\ldots = 0.$

Expanding $y(x)$ by Taylor's series, we have

$$y(x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \ldots\ldots$$

$$\therefore \quad y(x) = 1 - \frac{x^2}{2!} + \frac{3}{4!}x^4 - \frac{5 \times 3}{6!}x^6 + \ldots\ldots \qquad \ldots(2)$$

and $$z(x) = y'(x) = -x + \frac{1}{2}x^3 - \frac{1}{8}x^5 + \ldots\ldots = -xy, \qquad \ldots(3)$$

From (2), we have

$$y(0.1) = 1 - \frac{(0.1)^2}{2} + \frac{1}{8}(0.1)^4 - \ldots\ldots = 0.995$$
$$y(0.2) = 1 - \frac{(0.2)^2}{2} + \frac{(0.2)^4}{8} - \ldots\ldots = 0.9802$$
$$y(0.3) = 1 - \frac{(0.3)^2}{2} + \frac{(0.3)^4}{8} - \frac{(0.3)^6}{48} + \ldots\ldots = 0.956$$

From (3), we have

$$z(0.1) = -0.0995, \; z(0.2) = -0.196, \; z(0.3) = -0.2863.$$

Also from (1), $z'(x) = -(xz + y)$

$$\therefore \quad z'(0.1) = 0.985, \; z'(0.2) = -0.941, \; z'(0.3) = -0.87.$$

Applying Milne's predictor formula, first to z and then to y, we obtain

$$z(0.4) = z(0) + \frac{4}{3}(0.1)\{2z'(0.1) - z'(0.2) + 2z'(0.3)\}$$
$$= 0 + \left(\frac{0.4}{3}\right)\{-1.79 + 0.941 - 1.74\} = -0.3692$$

and $$y(0.4) = y(0) + \frac{4}{3}(0.1)\{2y'(0.1) - y'(0.2) + 2y'(0.3)\}$$
$$= 0 + \left(\frac{0.4}{3}\right)\{-0.199 + 0.196 - 0.5736\} = 0.9231 \qquad [\because \; y' = z]$$

Also $$z'(0.4) = -\{x(0.4)\,z(0.4) + y(0.4)\}$$
$$= -\{0.4(-0.3692) + 0.9231\} = -0.7754.$$

Now applying Milne's corrector formula, we get

$$z(0.4) = z(0.2) + \frac{h}{3}\{z'(0.2) + 4z'(0.3) + z'(0.4)\}$$
$$= -0.196 + \left(\frac{0.1}{3}\right)\{-0.941 - 3.48 - 0.7754\} = -0.3692$$

and $$y(0.4) = y(0.2) + \frac{h}{3}\{y'(0.2) + 4y'(0.3) + y'(0.4)\}$$
$$= 0.9802 + \left(\frac{0.1}{3}\right)\{-0.196 - 1.1452 - 0.3692\} = 0.9232$$

Hence $\quad y(0.4) = 0.9232$ and $z(0.4) = -0.3692.$

PROBLEMS 10.6

1. Apply Picard's method to find the third approximations to the values of y and z, given that $dy/dx = z$, $dz/dx = x^3(y + z)$, given $y = 1$, $z = \frac{1}{2}$ when $x = 0$.
2. Using Picard's method, obtain the second approximation to the solution of
$$\frac{d^2y}{dx^2} = x^3\frac{dy}{dx} + x^3y \quad \text{so that} \quad y(0) = 1, \; y'(0) = \frac{1}{2}.$$
3. Use Picard's method to approximate y when $x = 0.1$, given that $\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + y = 0$ and $y = 0.5$, $\frac{dy}{dx} = 0.1$, when $x = 0$.
4. Using Taylor's series method, find the values of x and y for $t = 0.4$, satisfying the differential equations
$$dx/dt = x + y + t, \; d^2y/dt^2 = x - t \;\text{ with initial conditions } x = 0, y = 1, dy/dt = -1 \text{ at } t = 0.$$

5. Solve the following simultaneous differential equations, using Taylor series method of the 4th order, for $x = 0.1$ and 0.2 :
$$\frac{dy}{dx} = xz + 1;\ \frac{dz}{dy} = -xy\ ;\ y(0) = 0,\ \text{and } z(0) = 1.$$
6. Find $y(0.1)$, $z(0.1)$, $y(0.2)$ and $z(0.2)$ from the system of equations: $y' = x + z$, $z' = x - y^2$ given $y(0) = 0$, $z(0) = 1$ using Runge-Kutta method of fourth order. *(J.N.T.U., B. Tech., 2009)*
7. Find $y(0.2)$ from the differential equation $y'' + 3xy' - 6y = 0$ where $y(0) = 1$, $y'(0) = 0.1$, using Taylor series method.
8. Using Runge-Kutta method of order 4, solve $y'' = y + xy'$, $y(0) = 1$, $y'(0) = 0$ to find $y(0.2)$ and $y'(0.2)$.
9. Consider the second order initial value problem $y'' - 2y' + 2y = e^{2t} \sin t$ with $y(0) = -0.4$ and $y'(0) = -0.6$. Using fourth order Runge-Kutta method, find $y(0.2)$. *(Anna B. Tech., 2003)*
10. The angular displacement θ of a simple pendulum is given by the equation
$$\frac{d^2\theta}{dt^2} + \frac{g}{l} \sin \theta = 0$$
where $l = 98$ cm and $g = 980$ cm/sec². If $\theta = 0$ and $d\theta/dt = 4.472$ at $t = 0$, use Runge-Kutta method to find θ and $d\theta/dt$ when $t = 0.2$ sec.
11. In a *L-R-C* circuit the voltage $v(t)$ across the capacitor is given by the equation
$$LC\frac{d^2v}{dt^2} + RC\frac{dv}{dt} + v = 0,$$
subject to the conditions $t = 0$, $v = v_0$, $dv/dt = 0$.
Taking $h = 0.02$ sec, use Runge-Kutta method to calculate v and dv/dt when $t = 0.02$, for the data $v_0 = 10$ volts, $C = 0.1$ farad, $L = 0.5$ henry and $R = 10$ ohms.

10.13. ERROR ANALYSIS

The numerical solutions of differential equations certainly differ from their exact solutions. *The difference between the computed value y_i and the true value $y(x_i)$ at any stage is known as the* **total error.** *The total error at any stage is comprised of* **truncation error** *and* **round-off error.**

The most important aspect of numerical methods is to minimize the errors and obtain the solutions with the least errors. It is usually not possible to follow error development quite closely. We can make only rough estimates. That is why, our treatment of error analysis at times, has to be somewhat intuitive.

In any method, the truncation error can be reduced by taking smaller sub-intervals. The round-off error cannot be controlled easily unless the computer used has the double precision arithmetic facility. In fact, this error has proved to be more elusive than the truncation error.

The truncation error in Euler's method is $\frac{1}{2} h^2 y_n''$ *i.e.* of $O(h^2)$ while that of modified Euler's method is $\frac{1}{2} h^3 y_n'''$ *i.e.* of $O(h^3)$.

Similarly in the fourth order Runge-Kutta method, the truncation error is of $O(h^5)$.

In the Milne's method, the truncation error

due to predictor formula $= \frac{14}{45} y_n^{v} h^5$

and due to corrector formula $= -\frac{1}{90} y_n^{v} h^5$.

i.e. the truncation error in Milne's method is also of $O(h^5)$.

Similarly the error in Adams-Bashforth method is of the fifth order. Also the predictor error T_p and the corrector error T_c are so related that $19T_p = -251\ T_c$.

The **relative error** *of an approximate solution is the ratio of the total error to the exact value.* It is of greater importance than the error itself for if the true value becomes larger, then a larger error may be acceptable. If the true value diminishes, then the error must also diminish otherwise the computed results may be absurd.

Example 10.32. *Applying Euler's method to the differential equation $dy/dx = f(x, y)$, $y(x_0) = y_0$, estimate the total error ? When $f(x, y) = -y$, $y(0) = 1$, compute this error neglecting the round-off error.*

Sol. We know that the Euler's solution of the given differential equation is
$$y_{n+1} = y_n + hf(x_n, y_n) \text{ where } x_n = x_0 + nh.$$
i.e.
$$y_{n+1} = y_n + hy_n' \qquad ...(1)$$
Denoting the exact solution of the given equation at $x = x_n$ by $y(x_n)$ and expanding $y(x_{n+1})$ by Taylor's series, we obtain
$$y(x_{n+1}) = y(x_n) + hy'(x_n) + \frac{h^2}{2!} y''(\xi_n),\ x_n \le \xi_n \le x_{n+1} \qquad ...(2)$$
$\therefore$ The truncation error $T_{n+1} = y(x_{n+1}) - y_{n+1} = \frac{1}{2} h^2 y''(\xi_n)$

Thus the truncation error is of $O(h^2)$ as $h \to 0$.

To include the effect of round-off error R_n, we introduce a new approximation $\bar{y}_n$ which is defined by the same procedure allowing for the round-off error.
$$\bar{y}_{n+1} = \bar{y}_n + hf(x_n, \bar{y}_n) - R_{n+1} \qquad ...(3)$$
$\therefore$ The total error is defined by
$$E_{n+1} = y(x_{n+1}) - \bar{y}_{n+1} \qquad [(2)-(3)]$$
$$= y(x_n) + hy'(x_n) + \frac{h^2}{2!} y''(\xi_n) - \{\bar{y}_n + hf(x_n, y_n) - R_{n+1}\}$$
$$= [y(x_n) - \bar{y}_n] + h[h'(x_n) - f(x_n, y_n)] + T_{n+1} + R_{n+1} \qquad ...(4)$$
Assuming continuity of $\partial f/\partial y$ and using Mean-Value theorem, we have
$$f[x_n, y(x_n)] - f(x_n, y_n) = [y(x_n) - y_n]\ f_y(x_n, \xi_n),\ \text{ where } \xi_n \text{ lies between } y(x_n) \text{ and } y_n.$$
$\therefore$ (4) takes the form
$$E_{n+1} = [y(x_n) - \bar{y}_n]\ [1 + hf_y(x_n, \xi_n)] + T_{n+1} + R_{n+1}$$
or
$$E_{n+1} = E_n\ [1 + hf_y(x_n, \xi_n)] + T_{n+1} + R_{n+1} \qquad ...(5)$$

This is the *recurrence formula* for finding the total error. The first terms on the right hand side is the *inherited error* i.e. the propagation of the error from the previous step y_n to y_{n+1}.

(*b*) We have $dy/dx = -y$, $y(0) = 1$.

Taking $h = 0.01$ and applying (1) successively, we obtain

$$y(0.01) = 1 + 0.01(-1) = 0.99$$
$$y(0.02) = 0.99 + 0.01(-0.99) = 0.9801$$
$$y(0.03) = 0.9703,\ y(0.04) = 0.9606$$

∴ The truncation error

$$T_{n+1} = \frac{1}{2}h^2 y''(\xi) = 0.00005\, y(\xi) \le 5 \times 10^{-5}\, y(x_n) \qquad [\because dy/dx \text{ is } -\text{ve}]$$

i.e.
$$T_1 \le 5 \times 10^{-5}\, y(0) = 5 \times 10^{-5}$$
$$T_2 \le 5 \times 10^{-5}\, y(0.01) = 5 \times 10^{-5}\,(0.99) < 5 \times 10^{-5}$$
$$T_3 \le 5 \times 10^{-5}\, y(0.02) = 5 \times 10^{-5}\,(0.9801) < 5 \times 10^{-5}$$
$$T_4 \le 5 \times 10^{-5}\, y(0.03) = 5 \times 10^{-5}\,(0.9703) < 5 \times 10^{-5} \text{ etc.}$$

Also $1 + hf_0(x_n, y_n) = 1 + 0.01(-1) = 0.99$.

Neglecting the round-off error and using the above results, (5) gives

$$E_0 = 0,\ E_1 = E_0(0.99) + T_1 \le 5 \times 10^{-5} = 0.00005$$
$$E_2 = E_1(0.99) + T_2 < 5 \times 10^{-5} + 5 \times 10^{-5} = 0.0001$$
$$E_3 = E_2(0.99) + T_3 < 10^{-4} + 5 \times 10^{-5} = 0.00015$$
$$E_4 = E_3(0.99) + T_4 < 1.5 \times 10^{-4} + 5 \times 10^{-5} = 0.0002 \text{ etc.}$$

Obs. The exact solution is $y = e^{-x}$.

∴ Actual error in $y(0.03) = e^{-0.03} - 0.9703 = 0.00014$

and actual error in $y(0.04) = e^{-0.04} - 0.9606 = 0.00019$.

Clearly the total error E_4 agrees with the actual error in $y(0.04)$.

10.14. CONVERGENCE OF A METHOD

Any numerical method for solving a differential equation is said to be convergent if the approximate solution y_n approaches the exact solution $y(x_n)$ as h tends to zero provided the rounding errors arising from the initial conditions approach zero. This means that as a method is continually refined by taking smaller and smaller step-sizes, the sequence of approximate solutions must converge to the exact solution.

Taylor's series method is convergent provided $f(x, y)$ possesses enough continuous derivatives. The Runge-Kutta methods are also convergent under similar conditions. Predictor-corrector methods are convergent if $f(x, y)$ satisfies *Lipschitz condition i.e.*

$$|\, f(x, y) - f(x, \bar{y})\,| \le k\, |\, y - \bar{y}\,|,$$

k being a constant, then the sequence of approximations to the numerical solution converges to the exact solution.

10.15. STABILITY ANALYSIS

There is a limit to which the step-size h can be reduced for controlling the truncation error, beyond which a further reduction in h will result in the increase of round-off error and hence increase in the total error. This behaviour of the error bound is shown in Fig. 10.3.

In such situations, we have to use stable methods so that an error introduced at any stage does not get magnified.

A method is said to be **stable** *if it produces a bounded solution which imitates the exact solution. Otherwise it is said to be* **unstable.** If a method is stable for all values of the parameter, it is said to be *absolutely* or *unconditionally stable.* If it is stable for some values of the parameter, it is said to be *conditionally stable.*

The Taylor's method and Adams-Bashforth method prove to be relatively stable. Euler's method and Runge-Kutta method are conditionally stable as will be seen from Example 10.23. The Milne's method is however, unstable since when the parameter is negative, each of the errors is magnified while the exact solution decays.

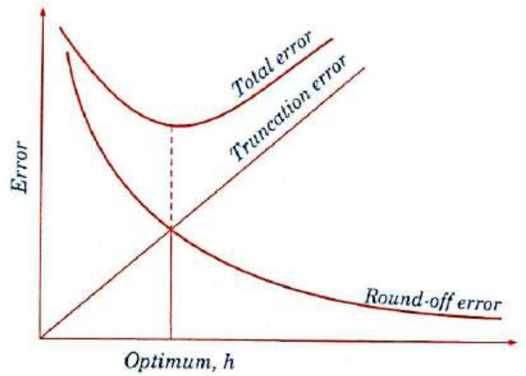


Fig. 10.3

Example 10.33. *Applying Euler's method to the equation*

$$dy/dx = \lambda y, \text{ given } y(x_0) = y_0,$$

determine its stability zone ? What would be the range of stability when $\lambda = -1$?

(U.P.T.U., B.Tech., 2006)

Sol. We have $y' = \lambda y,\ y(x_0) = y_0$ ...(1)

By Euler's method,

$$y_n = y_{n-1} + hy'_{n-1} = y_{n-1} + \lambda h y_{n-1} = (1 + \lambda h)\, y_{n-1} \qquad [\text{by } (1)$$

∴
$$y_{n-1} = (1 + \lambda h)\, y_{n-2}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$y_2 = (1 + \lambda h)\, y_1$$
$$y_1 = (1 + \lambda h)\, y_0$$

Multiplying all these equations, we obtain

$$y_n = (1 + \lambda h)^n\, y_0 \qquad ...(2)$$

Integrating (1), we get $y = ce^{\lambda x}$

Using $\quad y(x_0) = y_0, y_0 = ce^{\lambda x_0} \quad \therefore \quad y = y_0 e^{\lambda(x - x_0)}$

In particular, the exact solution through (x_n, y_n) is

$$y_n = y_0 e^{\lambda(x_n - x_0)} = y_0 e^{\lambda nh} \qquad [\because x_n = x_0 + nh]$$

or
$$y_n = y_0 (e^{\lambda h})^n = y_0 \left[1 + \lambda h + \frac{(\lambda h)^2}{2} + \ldots\ldots\right]^n \qquad \ldots(3)$$

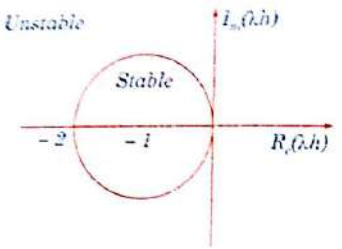


Fig. 10.4

Clearly the numerical solution (2) agrees with exact solution (3) for small values of h. The solution (2) increases if $|1 + \lambda h| > 1$.

Hence $|1 + \lambda h| < 1$ defines a stable zone.

When λ is real, then the method is stable if $|1 + \lambda h| < 1$ *i.e.* $-2 < \lambda h < 0$

When λ is complex ($= a + ib$), then it is stable if

$$|1 + (a + ib)h| < 1 \quad i.e. \quad (1 + ah)^2 + (bh)^2 < 1$$

i.e. $\quad (x + 1)^2 + y^2 < 1, \qquad$ [where $x = ah, y = bh$.

i.e. λh lies within the unit circle shown in Fig. 10.4.

When λ is imaginary ($= ib$), $|1 + \lambda h| = 1$, then we have a *periodic-stability*.

Hence Euler's method is absolutely stable if and only if

(*i*) real λ : $-2 < \lambda h = 0$.

(*ii*) complex λ : λh lies within the unit circle (Fig. 10.4) *i.e.* Euler's method is conditionally convergent.

When $\lambda = -1$, the solution is stable in the range $-2 < -h < 0$ *i.e.* $0 < h < 2$.

PROBLEMS 10.7

1. Show that the approximate values y_n, obtained from $y' = y$ with $y(0) = 1$ by Taylor's series method, converge to the exact solution for h tending to zero.
2. Show that the modified Euler's method is convergent.
3. Starting with the equation $y' = \lambda y$, show that the modified Euler's method is relatively stable.

4. Apply fourth order Runge-Kutta method to the equation $dy/dx = \mu y, y(x_0) = y_0$ and show that the range of absolute stability is
$-2.78 < \mu h < 0.$
5. Find the range of absolute stability of the equation
$y' + 10y = 0, y(0) = 1$, using
(*a*) Euler's method. (*b*) Runge-Kutta method.
6. Show that the local truncation errors in the Milne's predictor and corrector formulae are $\frac{14}{45} h^5 y^v$ and $-\frac{1}{90} h^5 y^v$ respectively.

10.16. BOUNDARY VALUE PROBLEMS

Such a problem requires the solution of a differential equation in a region R subject to the various conditions on the boundary of R. Practical applications give rise to many such problems. We shall discuss two-point linear boundary value problems of the following types :

(*i*) $\dfrac{d^2 y}{dx^2} + \lambda(x) \dfrac{dy}{dx} + \mu(x) y = \gamma(x)$ with the conditions $y(x_0) = a, y(x_n) = b$.

(*ii*) $\dfrac{d^4 y}{dx^4} + \lambda(x) y = \mu(x)$ with the conditions $y(x_0) = y'(x_0) = a$ and $y(x_n) = y'(x_n) = b$.

There exist two numerical methods for solving such boundary value problems. The first one is known as the *finite difference method* which makes use of finite difference equivalents of derivatives. The second one is called the *shooting method* which makes use of the techniques for solving initial value problems.

10.17. FINITE-DIFFERENCE METHOD

In this method, the derivatives appearing in the differential equation and the boundary conditions are replaced by their finite-difference approximations and the resulting linear system of equations are solved by any standard procedure. These roots are the values of the required solution at the pivotal points.

The finite-difference approximations to the various derivatives are derived as under :

If $y(x)$ and its derivatives are single-valued continuous functions of x then by Taylor's expansion, we have

$$y(x + h) = y(x) + hy'(x) + \frac{h^2}{2!} y''(x) + \frac{h^3}{3!} y'''(x) + \ldots\ldots \qquad \ldots(1)$$

and
$$y(x - h) = y(x) - hy'(x) + \frac{h^2}{2!} y''(x) - \frac{h^3}{3!} y'''(x) + \ldots\ldots \qquad \ldots(2)$$

Equation (1) gives

$$y'(x) = \frac{1}{h} [y(x + h) - y(x)] - \frac{h}{2} y''(x) - \ldots\ldots$$

i.e.
$$y'(x) = \frac{1}{h} [y(x + h) - y(x)] + O(h)$$

which is the *forward difference approximation of $y'(x)$* with an error of the order h.

Similarly (2) gives

$$y'(x) = \frac{1}{h}[y(x) - y(x-h)] + O(h)$$

which is the *backward difference approximation of* $y'(x)$ with an error of the order h.

Subtracting (2) from (1), we obtain

$$y'(x) = \frac{1}{2h}[y(x+h) - y(x-h)] + O(h^2)$$

which is the *central-difference approximation of* $y'(x)$ with an error of the order h^2. Clearly this central difference approximation to $y'(x)$ is better than the forward or backward difference approximations and hence should be preferred.

Adding (1) and (2), we get

$$y''(x) = \frac{1}{h^2}[y(x+h) - 2y(x) + y(x-h)] + O(h^2)$$

which is the *central difference approximation of* $y''(x)$. Similarly we can derive central difference approximations to higher derivatives.

Hence *the working expressions for the central difference approximations to the first four derivatives of* y_i *are as under :*

$$y_i' = \frac{1}{2h}(y_{i+1} - y_{i-1}) \qquad ...(3)$$

$$y_i'' = \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1}) \qquad ...(4)$$

$$y_i''' = \frac{1}{2h^3}(y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}) \qquad ...(5)$$

$$y_i^{iv} = \frac{1}{h^4}(y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}) \qquad ...(6)$$

Obs. The accuracy of this method depends on the size of the sub-interval h and also on the order of approximation. As we reduce h, the accuracy improves but the number of equations to be solved also increases.

Example 10.34. *Solve the equation $y'' = x + y$ with the boundary conditions $y(0) = y(1) = 0$.*

Sol. We divide the interval (0, 1) into four sub-intervals so that $h = 1/4$ and the pivot points are at $x_0 = 0, x_1 = 1/4, x_2 = 1/2, x_3 = 3/4$ and $x_4 = 1$.

Then the differential equation is approximated as

$$\frac{1}{h^2}[y_{i+1} - 2y_i + y_{i-1}] = x_i + y_i$$

or

$$16y_{i+1} - 33y_i + 16y_{i-1} = x_i,\ i = 1, 2, 3.$$

Using $y_0 = y_4 = 0$, we get the system of equations

$$16y_2 - 33y_1 = \frac{1}{4}\ ;\ 16y_3 - 33y_2 + 16y_1 = \frac{1}{2}\ ;\ -33y_3 + 16y_2 = \frac{3}{4}$$

Their solution gives

$$y_1 = -0.03488,\ y_2 = -0.05632,\ y_3 = -0.05003.$$

Obs. The exact solution being $y(x) = \frac{\sinh x}{\sinh 1} - x$, the error at each nodal point is given in the table below :

x	*Computed value y(x)*	*Exact value y(x)*	*Error*
0.25	− 0.03488	− 0.03505	0.00017
0.5	− 0.05632	− 0.05659	0.00027
0.75	− 0.05003	− 0.05028	0.00025

Example 10.35. *Using the finite difference method, find y(0.25), y(0.5) and y(0.75) satisfying the differential equation $\frac{d^2y}{dx^2} + y = x$, subject to the boundary conditions $y(0) = 0$, $y(1) = 2$.* *(Anna, B.E., 2004)*

Sol. Dividing the interval (0, 1) into four sub-intervals so that $h = 0.25$ and the pivot points are at $x_0 = 0, x_1 = 0.25, x_2 = 0.5, x_3 = 0.75$ and $x_4 = 1$.

The given equation $y''(x) + y(x) = x$, is approximated as

$$\frac{1}{h^2}[y_{i+1} - 2y_i + y_{i-1}] + y_i = x_i$$

or

$$16y_{i+1} - 31y_i + 16y_{i-1} = x_i \qquad ...(i)$$

Using $y_0 = 0$ and $y_4 = 2$, (*i*) gives the system of equation,

$(i = 1)$ $16y_2 - 31y_1 = 0.25$; ...(*ii*)

$(i = 2)$ $16y_3 - 31y_2 + 16y_1 = 0.5$ ...(*iii*)

$(i = 3)$ $32 - 31y_3 + 16y_2 = 0.75$ *i.e.*, $-31y_3 + 16y_2 = -31.25$ ...(*iv*)

Solving the equations (*ii*), (*iii*) and (*iv*), we get

$$y_1 = 0.5443,\ y_2 = 1.0701,\ y_3 = 1.5604$$

Hence $y(0.25) = 0.5443$, $y(0.5) = 1.0701$, $y(0.75) = 1.5604$

Example 10.36. *Determine values of y at the pivotal points of the interval (0, 1) if y satisfies the boundary value problem $y^{iv} + 81y = 81x^2$, $y(0) = y(1) = y''(0) = y''(1) = 0$. (Take $n = 3$).*

Sol. Here $h = 1/3$ and the pivotal points are $x_0 = 0$, $x_1 = 1/3$, $x_2 = 2/3$, $x_3 = 1$. The corresponding y-values are $y_0 (= 0), y_1, y_2, y_3 (= 0)$.

Replacing y^{iv} by its central difference approximation, the differential equation becomes

$$\frac{1}{h^4}(y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}) + 81y_i = 81x_i^2$$

or

$$y_{i+2} - 4y_{i+1} + 7y_i - 4y_{i-1} + y_{i-2} = x_i^2,\ i = 1, 2$$

At $i = 1$, $y_3 - 4y_2 + 7y_1 - 4y_0 + y_{-1} = 1/9$

At $i = 2$, $y_4 - 4y_3 + 7y_2 - 4y_1 + y_0 = 4/9$

Using $y_0 = y_3 = 0$, we get $-4y_2 + 7y_1 + y_{-1} = 1/9$ ...(*i*)

$y_4 + 7y_2 - 4y_1 = 4/9$ ...(*ii*)

Regarding the conditions $y_0'' = y_3'' = 0$, we know that

$$y_i'' = \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1})$$

At $i = 0$, $\quad y_0'' = 9(y_1 - 2y_0 + y_{-1})$ or $y_{-1} = -y_1$ $\quad [\because y_0 = y_0'' = 0]$ ...(iii)

At $i = 3$, $\quad y_3'' = 9(y_4 - 2y_3 + y_2)$ or $y_4 = -y_2$ $\quad [\because y_3 = y_3'' = 0]$ ...(iv)

Using (iii), the equation (i) becomes

$$-4y_2 + 6y_1 = 1/9 \quad \text{...(v)}$$

Using (iv), the equation (ii) reduces to

$$6y_2 - 4y_1 = 4/9 \quad \text{...(vi)}$$

Solving (v) and (vi), we obtain

$$y_1 = 11/90 \quad \text{and} \quad y_2 = 7/45.$$

Hence $y(1/3) = 0.1222$ and $y(2/3) = 0.1556$.

Example 10.37. *The deflection of a beam is governed by the equation*

$$\frac{d^4y}{dx^4} + 81y = \phi(x), \text{ where } \phi(x) \text{ is given by the table}$$

x :	1/3	2/3	1
$\phi(x)$:	81	162	243

and boundary condition $y(0) = y'(0) = y''(1) = y'''(1) = 0$. Evaluate the deflection at the pivotal points of the beam using three sub-intervals.

Sol. Here $h = 1/3$ and the pivotal points are $x_0 = 0$, $x_1 = 1/3$, $x_2 = 2/3$, $x_3 = 1$. The corresponding y-values are $y_0 (= 0)$, y_1, y_2, y_3.

The given differential equation is approximated to

$$\frac{1}{h^4}(y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}) + 81y_i = \phi(x_i)$$

At $i = 1$, $\quad y_3 - 4y_2 + 7y_1 - 4y_0 + y_{-1} = 1$ ...(i)

At $i = 2$, $\quad y_4 - 4y_3 + 7y_2 - 4y_1 + y_0 = 2$ ...(ii)

At $i = 3$, $\quad y_5 - 4y_4 + 7y_3 - 4y_2 + y_1 = 3$ ...(iii)

We have $\quad y_0 = 0$ ...(iv)

Since $\quad y_i' = \frac{1}{2h}(y_{i+1} - y_{i-1})$

$\therefore$ for $i = 0$, $\quad 0 = y_0' = \frac{1}{2h}(y_1 - y_{-1})$ *i.e.* $y_{-1} = y_1$ ...(v)

Since $\quad y_i'' = \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1})$

$\therefore$ for $i = 3$, $\quad 0 = y_3'' = \frac{1}{h^2}(y_4 - 2y_3 + y_2)$, *i.e.* $y_4 = 2y_3 - y_2$ ...(vi)

Also $\quad y_i''' = \frac{1}{2h^3}(y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2})$

$\therefore$ for $i = 3$, $\quad 0 = y_3''' = \frac{1}{2h^3}(y_5 - 2y_4 + 2y_2 - y_1)$

i.e. $\quad y_5 = 2y_4 - 2y_2 + y_1$ ...(vii)

Using (iv) and (v), the equation (i) reduces to

$$y_3 - 4y_2 + 8y_1 = 1 \quad \text{...(viii)}$$

Using (iv) and (vi), the equation (ii) becomes

$$-y_3 + 3y_2 - 2y_1 = 1 \quad \text{...(ix)}$$

Using (vi) and (vii), the equation (iii) reduces to

$$3y_3 - 4y_2 + 2y_1 = 3 \quad \text{...(x)}$$

Solving (viii), (ix) and (x), we get

$$y_1 = 8/13,\ y_2 = 22/13,\ y_3 = 37/13.$$

Hence $\quad y(1/3) = 0.6154$, $y(2/3) = 1.6923$, $y(1) = 2.8462$.

10.18. SHOOTING METHOD

In this method, the given boundary value problem is first transformed to an initial value problem. Then this initial value problem is solved by Taylor's series method or Runge-Kutta method etc. Finally the given boundary value problem is solved. The approach in this method is quite simple.

Consider the boundary value problem

$$y''(x) = y(x),\ y(x) = A,\ y(b) = B \quad \text{...(1)}$$

One condition is $y(a) = A$ and let us assume that $y'(a) = m$ which represents the slope. We start with two initial guesses for m, then find the corresponding value of $y(b)$ using any initial value method.

Let the two guesses be m_0, m_1 so that the corresponding values of $y(b)$ are $y(m_0, b)$ and $y(m_1, b)$. Assuming that the values of m and $y(b)$ are linearly related, we obtain a better approximation m_2 for m from the relation :

$$\frac{m_2 - m_1}{y(b) - y(m_1, b)} = \frac{m_1 - m_0}{y(m_1, b) - y(m_0, b)}$$

This gives $\quad m_2 = m_1 - (m_1 - m_0)\dfrac{y(m_1, b) - y(b)}{y(m_1, b) - y(m_0, b)}$ ...(2)

We now solve the initial value problem

$$y''(x) = y(x),\quad y(a) = A,\ y'(a) = m_2$$

and obtain the solution $y(m_2, b)$.

To obtain a better approximation m_3 for m, we again use the linear relation (2) with $[m_1, y(m_1, b)]$ and $[m_2, y(m_2, b)]$. This process is repeated until the value of $y(m_i, b)$ agrees with $y(b)$ to desired accuracy.

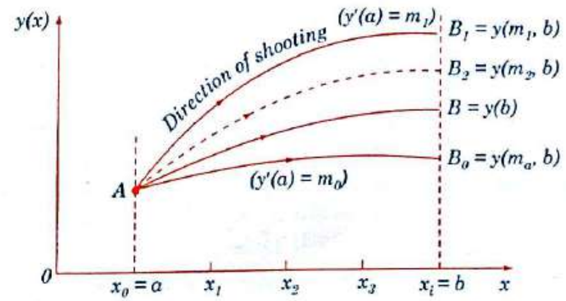


Fig. 10.5

Obs. This method resembles an artillery problem and as such is called the *shooting method* (Fig. 10.5). The speed of convergence in this method depends on our initial choice of two guesses for m. However, the shooting method is quite slow in practice. Also this method is quite tedius to apply to higher order boundary value problems.

Example 10.38. *Using the shooting method, solve the boundary value problem :*

$$y''(x) = y(x),\ y(0) = 0 \quad \text{and} \quad y(1) = 1.17.$$

Sol. Let the initial guesses for $y'(0) = m$ be $m_0 = 0.8$ and $m_1 = 0.9$. Then $y''(x) = y(x)$, $y(0) = 0$ gives

$$\begin{aligned} & & y''(0) &= y(0) = 0 \\ y'(0) &= m & y^{iv}(0) &= y''(0) = 0 \\ y'''(0) &= y'(0) = m, & y^{vi}(0) &= y^{iv}(0) = 0 \\ y^{v}(0) &= y'''(0) = m, & & \end{aligned}$$

and so on.

Putting these values in the Taylor's series, we have

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \ldots\ldots$$

$$= m\left(x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \ldots\ldots\right)$$

$\therefore \quad y(1) = m(1 + 0.1667 + 0.0083 + 0.0002 + \ldots\ldots) = m\ (1.175)$

For $m_0 = 0.8$, $y(m_0, 1) = 0.8 \times 1.175 = 0.94$

For $m_1 = 0.9$, $y(m_1, 1) = 0.9 \times 1.175 = 1.057$

Hence a better approximation for m *i.e.* m_2 is given by

$$m_2 = m_1 - (m_1 - m_0)\frac{y(m_1, 1) - y(1)}{y(m_1, 1) - y(m_0, 1)}$$

$$= 0.9 - (0.1)\frac{1.057 - 1.175}{1.057 - 0.94} = 0.9 + 0.10085 = 1.00085$$

which is closer to the exact value of $y'(0) = 0.996$

We now solve the initial value problem

$$y''(x) = y(x),\ y(0) = 0,\ y'(0) = m_2.$$

Taylor's series solution is given by

$$y(m_2, 1) = m_2\,(1.175) = 1.1759$$

Hence the solution at $x = 1$ is $y = 1.176$ which is close to the exact value of $y(1) = 1.17$.

PROBLEMS 10.8

1. Solve the boundary value problem for $x = 0.5$:
$$\frac{d^2y}{dx^2} + y + 1 = 0,\ y(0) = y(1) = 0, \qquad \text{(Take } n = 4)$$
2. Find an approximate solution of the boundary value problem :
$$y'' + 8(\sin^2 \pi y)\,y = 0,\ 0 \le x \le 1,\ y(0) = y(1) = 1. \qquad \text{(Take } n = 4)$$

3. Solve the boundary value problem :
$$xy'' + y = 0,\ y(1) = 1,\ y(2) = 2. \qquad \text{(Take } n = 4)$$
4. Solve the equation $y'' - 4y' + 4y = e^{3x}$, with the conditions $y(0) = 0$, $y(1) = -2$, taking $n = 4$.
5. Solve the boundary value problem $y'' - 64y + 10 = 0$ with $y(0) = y(1) = 0$ by the finite difference method. Compute the value of $y(0.5)$ and compare with the true value.
6. Solve the boundary value problem
$$y'' + xy' + y = 3x^2 + 2,\ y(0) = 0,\ y(1) = 1.$$
7. The boundary value problem governing the deflection of a beam of length 3 meters is given by
$$\frac{d^4y}{dx^4} + 2y = \frac{1}{9}x^2 + \frac{2}{3}x + 4,\ y(0) = y'(0) = y(3) = y'(3) = 0.$$
The beam is built-in at the left end ($x = 0$) and simply supported at the right end ($x = 3$). Determine y at the pivotal points $x = 1$ and $x = 2$.
8. Solve the boundary value problem,
$$\frac{d^4y}{dx^4} + 81y = 729x^2,\ y(0) = y'(0) = y''(1) = y'''(1) = 0. \text{ Use } n = 3.$$
9. Solve the equation $y^{iv} - y''' + y = x^2$,
subject to the boundary conditions
$$y(0) = y'(0) = 0 \quad \text{and} \quad y(1) = 2,\ y'(1) = 0. \qquad \text{(Take } n = 5).$$
10. Apply shooting method to solve the boundary value problem
$$\frac{d^2y}{dx^2} = y,\quad y(0) = 0 \quad \text{and} \quad y(1) = 1.1752.$$
11. Using shooting method, solve the boundary value problem
$$\frac{d^2y}{dx^2} = 6y^2,\quad y(0) = 1,\ y(0.5) = 0.44.$$

10.19. OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 10.9

Select the correct answer or fill up the blanks in the following questions :

1. Which of the following is a step by step method :
(*a*) Taylor's (*b*) Adams-Bashforth
(*c*) Picard's (*d*) None.
2. The finite difference scheme for the equation $2y'' + y = 5$ is ....
3. If $y' = x + y$, $y(0) = 1$ and $y^{(1)} = 1 + x + x^2/2$, then by Picard's method, the value of $y^{(2)}(x)$ is ......
4. The iterative formula of Euler's method for solving $y' = f(x, y)$ with $y(x_0) = y_0$, is ....
5. Taylor's series for solution of first order ordinary differential equations is ....
6. The disadvantage of Picard's method is ....
7. Given y_0, y_1, y_2, y_3, Milne's corrector formula to find y_4 for $dy/dx = f(x, y)$, is ....
8. The second order Runge-Kutta formula is ....

9. Adams-Bashforth predictor formula to solve $y' = f(x, y)$, given $y_0 = y(x_0)$ is .... .
10. Runge-Kutta method is better than Taylor's series method because ....
11. To predict Adam's method atleast ...... values of y, prior to the desired value, are required.
12. Taylor's series solution of $y' - xy = 0$, $y(0) = 1$ upto x^4 is ....
13. If dy/dx is a function of x alone, the 4th order Runge-Kutta method reduces to ......
14. Milne's Predictor formula is ....
15. Adam's Corrector formula is ....
16. Using Euler's method, $dy/dx = (y - 2x)/y$, $y(0) = 1$; gives $y\ (0.1)$ = ...
17. $\dfrac{d^2y}{dx^2} + y^2\dfrac{dy}{dx} + y = 0$ is equivalent to a set of two first order differential equations ...... and ....
18. The formula for the 4th order Runge-Kutta method is ....
19. Taylor's series method will be useful to give some ...... of Milne's method.
20. The names of two self-starting methods to solve $y' = f(x, y)$ given $y(x_0) = y_0$ are ....
21. In the derivation of fourth order Runge-Kutta formula, it is called fourth order because ....
22. If $y' = x - y$, $y(0) = 1$ then by Picard's method, the value of $y^{(1)}\ (1)$ is ....
    (*a*) 0.915 (*b*) 0.905
    (*c*) 1.091 (*d*) none.
23. The second order Runge-Kutta method is ......
24. If $y' = -y$, $y(0) = 1$, then by Euler's method, the value of $y(1)$ is
    (*a*) 0.99 (*b*) 0.999
    (*c*) 0.981 (*d*) none.
25. Write down the difference between initial value problem and boundary value problem ....
26. Which of the following methods is the best for solving initial value problems:
    (*a*) Taylor's series method (*b*) Euler's method
    (*c*) Runge-Kutta method of 4th order (*d*) Modified Euler's method.
27. The finite difference scheme of the differential equation $y'' + 2y = 0$ is .....
28. Using modified Euler's method, the value of $y(0.1)$ for $\dfrac{dy}{dx} = x - y, y(0) = 1$ is
    (*a*) 0.809 (*b*) 0.909
    (*c*) 0.0809 (*c*) none.
29. The multi-step methods available for solving ordinary differential equations are ....
30. In Euler's method, if h is small the method is too slow, if h is large, it gives inaccurate value.
31. Runge-Kutta method is a self-starting method. (True or False)
32. Predictor-corrector methods are self-starting methods. (True or False)
33. Using Runge Kutta method of order four, the value of $y(0.1)$ for $y' = x - 2y$, $y(0) = 1$, taking $h = 0.1$, is ......
    (*a*) 0.813 (*b*) 0.825
    (*c*) 0.0825 (*c*) none.

11

NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

11.1. INTRODUCTION

Partial differential equations arise in the study of many branches of applied mathematics *e.g.* in fluid dynamics, heat transfer, boundary layer flow, elasticity, quantum mechanics and electro-magnetic theory. Only a few of these equations can be solved by analytical methods which are also complicated requiring use of advanced mathematical techniques. In most of the cases, it is easier to develop approximate solutions by numerical methods. Of all the numerical methods available for the solution of partial differential equations, the method of finite differences is most commonly used. In this method, the derivatives appearing in the equation and the boundary conditions are replaced by their finite difference approximations. Then the given equation is changed to a system of linear equations which are solved by iterative procedures. This process is slow but produces good results in many boundary value problems. An added advantage of this method is that the computation can be carried by electronic computers. To accelerate the solution, sometimes the method of relaxation proves quite effective.

Besides discussing finite difference method, we shall briefly describe the relaxation method also in this chapter.

11.2. CLASSIFICATION OF SECOND ORDER EQUATIONS

The general linear partial differential equation of the second order in two independent variables is of the form

$$A(x, y)\frac{\partial^2 u}{\partial x^2} + B(x, y)\frac{\partial^2 u}{\partial x \partial y} + C(x, y)\frac{\partial^2 u}{\partial y^2} + F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0 \quad ...(1)$$

Such a partial differential equation is said to be

(*i*) **elliptic** *if* $B^2 - 4AC < 0$. (*ii*) **parabolic** *if* $B^2 - 4AC = 0$,

and (*iii*) **hyperbolic** *if* $B^2 - 4AC > 0$.

Obs. *A partial equation is classified according to the region in which it is desired to be solved.*

For instance, the partial differential equation $f_{xx} + f_{yy} = 0$ is elliptic if $y > 0$, parabolic if $y = 0$; hyperbolic if $y < 0$.

Example 11.1. *Classify the following equations :*

(*i*) $\dfrac{\partial^2 u}{\partial x^2} + 4\dfrac{\partial^2 u}{\partial x \partial y} + 4\dfrac{\partial^2 u}{\partial y^2} - \dfrac{\partial u}{\partial x} + 2\dfrac{\partial u}{\partial y} = 0$

(*ii*) $x^2 \dfrac{\partial^2 u}{\partial x^2} + (1 - y^2)\dfrac{\partial^2 u}{\partial y^2} = 0, -\infty < x \infty, -1 < y < 1$

(*iii*) $(1 + x^2)\dfrac{\partial^2 u}{\partial x} + (5 + 2x^2)\dfrac{\partial^2 u}{\partial x \partial t} + (4 + x^2)\dfrac{\partial^2 u}{\partial t} = 0.$

Sol. (*i*) Comparing this equation with (1) above, we find that

$$A = 1, B = 4, C = 4$$

$$\therefore \quad B^2 - 4AC = (4)^2 - 4 \times 1 \times 4 = 0$$

So the equation is parabolic.

(*ii*) Here $A = x^2$, $B = 0$, $C = 1 - y^2$

$$\therefore \quad B^2 - 4AC = 0 - 4x^2(1 - y^2) = 4x^2(y^2 - 1)$$

For all x between $-\infty$ and ∞, x^2 is positive

For all y between -1 and 1, $y^2 < 1$

$$B^2 - 4AC < 0$$

Hence the equation is elliptic.

(*iii*) Here $A = 1 + x^2$, $B = 5 + 2x^2$, $C = 4 + x^2$

$$\therefore \quad B^2 - 4AC = (5 + 2x^2)^2 - 4(1 + x^2)(4 + x^2) = 9 \text{ i.e. } > 0$$

So the equation is hyperbolic.

PROBLEMS 11.1

1. What is the classification of the equation $f_{xx} + 2f_{xy} + f_{yy} = 0$. (*Madras B.E., 2001*)
2. Determine whether the following equation is elliptic or hyperbolic ?
$$(x + 1)u_{xx} - 2(x + 2)u_{xy} + (x + 3)u_{yy} = 0.$$
3. Classify the equation (*i*) $y^2 \dfrac{\partial^2 u}{\partial x^2} - 2y\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} - \dfrac{\partial u}{\partial y} = 8y$.
(*ii*) $y^2 u_{xx} - 2xyu_{xy} + x^2 u_{yy} + 2u_x - 3u = 0$. (*Madras, B.E., 2003*)
(*iii*) $y^2 u_{xx} + u_{yy} + u_x^2 + u^2 y + 7 = 0$. (*Madras, B.E., 2003 S*)
4. In which parts of the (x, y) plane is the following equation elliptic ?
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + (x^2 + 4y^2)\frac{\partial^2 u}{\partial y^2} = 2 \sin (xy).$$

11.3. FINITE DIFFERENCE APPROXIMATIONS TO PARTIAL DERIVATIVES

Consider a rectangular region R in the x, y plane. Divide this region into a rectangular network of sides $\Delta x = h$ and $\Delta y = k$ as shown in Fig. 11.1. The points of intersection of the dividing lines are called *mesh points, nodal points or grid points.*

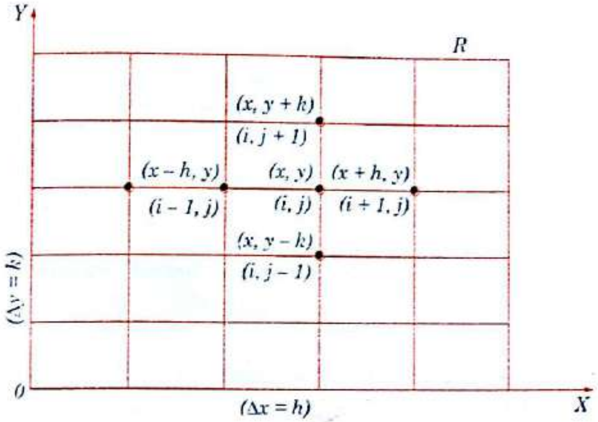


Fig. 11.1

Then we have the finite difference approximations for the partial derivatives in x-direction (§ 10.17) :

$$\frac{\partial u}{\partial x} = \frac{u(x + h, y) - u(x, y)}{h} + O(h) = \frac{u(x, y) - u(x - h, y)}{h} + O(h)$$

$$= \frac{u(x + h, y) - u(x - h, y)}{2h} + O(h^2)$$

and
$$\frac{\partial^2 u}{\partial x^2} = \frac{u(x - h, y) - 2u(x, y) + u(x + h, y)}{h^2} + O(h^2)$$

Writing $u(x, y) = u(ih, jk)$ as simply $u_{i,j}$, the above approximations become

$$u_x = \frac{u_{i+1,j} - u_{i,j}}{h} + O(h) \quad ...(1)$$

$$= \frac{u_{i,j} - u_{i-1,j}}{h} + O(h) \quad ...(2)$$

$$= \frac{u_{i+1,j} - u_{i-1,j}}{2h} + O(h^2) \quad ...(3)$$

and
$$u_{xx} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + O(h^2) \quad ...(4)$$

Similarly we have the approximations for the derivatives w.r.t. y :

$$u_y = \frac{u_{i,j+1} - u_{i,j}}{k} + O(k) \quad ...(5)$$

$$= \frac{u_{i,j} - u_{i,j-1}}{k} + O(k) \qquad \text{...(6)}$$

$$= \frac{u_{i,j+1} - u_{i,j-1}}{2k} + O(k^2) \qquad \text{...(7)}$$

and $$u_{yy} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} + O(k^2) \qquad \text{...(8)}$$

Replacing the derivatives in any partial differential equation by their corresponding difference approximations (1) to (8), we obtain the finite-difference analogues of the given equation.

11.4. ELLIPTIC EQUATIONS

The *Laplace equation* $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and the *Poisson's equation* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$ are examples of elliptic partial differential equations. Laplace equation arises in steady-state flow and potential problems. Poisson's equation arises in fluid mechanics, electricity and magnetism and torsion problems.

The solution of these equations is a function $u(x, y)$ which is satisfied at every point of a region R subject to certain boundary conditions specified on the closed curve C (Fig. 11.2).

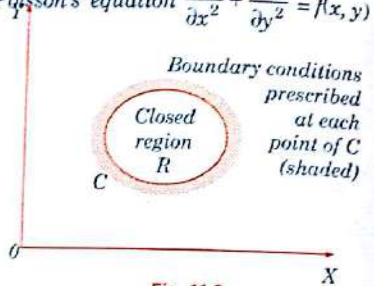


Fig. 11.2

In general, problems concerning steady viscous flow, equilibrium stresses in elastic structures etc., lead to elliptic type of equations.

11.5. SOLUTION OF LAPLACE EQUATION

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \nabla^2 U = 0 \qquad \text{...(1)}$$

Consider a rectangular region R for which $u(x, y)$ is known at the boundary. Divide this region into a network of square mesh of side h, as shown in Fig. 11.3 (assuming that an exact sub-division of R is possible). Replacing the derivatives in (1) by their difference approximations, we have

$$\frac{1}{h^2}[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}] + \frac{1}{h^2}[u_{i,j-1} - 2u_{i,j} + u_{i,j+1}] = 0$$

or $$u_{i,j} = \frac{1}{4}[u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1}] \qquad \text{...(2)}$$

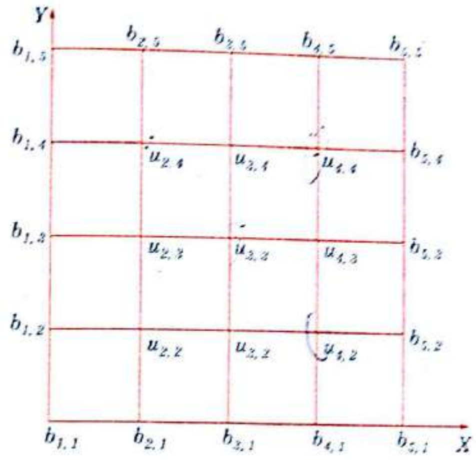


Fig. 11.3

This shows that the value of u at any interior mesh point is the average of its values at four neighbouring points to the left, right, above and below. (2) is called the **standard 5-point formula** which is exhibited in Fig. 11.4.

Sometimes a formula similar to (2) is used which is given by

$$u_{i,j} = \frac{1}{4}(u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1}) \qquad \text{...(3)}$$

This shows that the value of $u_{i,j}$ is the average of its values at the four neighbouring diagonal mesh points. (3) is called the **diagonal 5-point formula** which is represented in Fig. 11.5. Although (3) is less accurate than (2), yet it serves as a reasonably good approximation for obtaining the starting values at the mesh points.

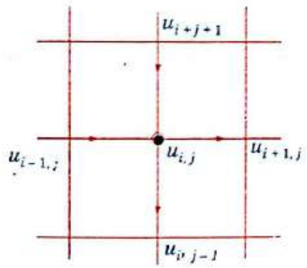


Fig. 11.4

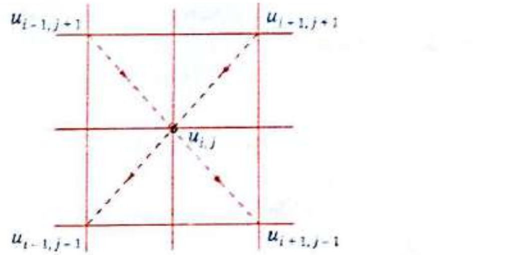


Fig. 11.5

Now to find the initial values of u at the interior mesh points, we first use the diagonal five point formula (3) and compute $u_{3,3}$, $u_{2,4}$, $u_{4,4}$, $u_{4,2}$ and $u_{2,2}$, in this order. Thus we get,

$$u_{3,3} = \frac{1}{4}(b_{1,5} + b_{5,1} + b_{5,5} + b_{1,1})\ ;\ u_{2,4} = \frac{1}{4}(b_{1,5} + u_{3,3} + b_{3,5} + b_{1,3})$$

$$u_{4,4} = \frac{1}{4}(b_{3,5} + b_{5,3} + b_{3,5} + u_{3,3})\ ;\ u_{4,2} = \frac{1}{4}(u_{3,3} + b_{5,1} + b_{3,1} + b_{5,3})$$

$$u_{2,2} = \frac{1}{4}(b_{1,3} + b_{3,1} + u_{3,3} + b_{1,1})$$

The values at the remaining interior points *i.e.* $u_{2,3}, u_{3,4}, u_4$ and $u_{3,2}$ are computed by the standard five-point formula (2). Thus, we obtain

$$u_{2,3} = \frac{1}{4}(b_{1,3} + u_{3,3} + u_{2,4} + u_{2,2}); u_{3,4} = \frac{1}{4}(u_{2,4} + u_{4,4} + b_{3,5} + u_{3,3})$$

$$u_{4,3} = \frac{1}{4}(u_{3,3} + b_{5,3} + u_{4,4} + u_{4,2}), u_{3,2} = \frac{1}{4}(u_{2,2} + u_{4,2} + u_{3,3} + u_{3,1})$$

Having found all the nine values of $u_{i,j}$ once, their accuracy is improved by either of the following iterative methods. In each case, the method is repeated till the difference between two consecutive iterates becomes negligible.

(*i*) **Jacobi's method.** Denoting the nth iterative value of $u_{i,j}$ by $u^{(n)}_{i,j}$, the iterative formula to solve (2) is

$$u^{(n+1)}_{i,j} = \frac{1}{4}[u^{(n)}_{i-1,j} + u^{(n)}_{i+1,j} + u^{(n)}_{i,j+1} + u^{(n)}_{i,j-1}] \quad ...(4)$$

It gives improved values of $u_{i,j}$ at the interior mesh points and is called the *point Jacobi's formula*.

(*ii*) **Gauss-Seidal method.** In this method, the iteration formula is

$$u^{(n+1)}_{i,j} = \frac{1}{4}[u^{(n+1)}_{i-1,j} + u^{(n)}_{i+1,j} + u^{(n+1)}_{i,j+1} + u^{(n)}_{i,j-1}]$$

It utilises the latest iterative value available and scans the mesh points symmetrically from left to right along successive rows.

Obs. Gauss-Seidal method is simple and can be adapted to computer calculations. Its convergence being slow, the working is somewhat lengthy. It can however, be shown that the Gauss-Seidal scheme converges twice as fast as Jacobi's scheme.

The accuracy of calculations depends on the mesh-size *i.e.* smaller the h, better the accuracy. But if h is too small, it may increase rounding-off errors and also increases the labour of computation.

Example 11.2. *Solve the elliptic equation $u_{xx} + u_{yy} = 0$ for the following square mesh with boundary values as shown in Fig. 11.6.* (V.T.U., B. Tech., 2006)

Sol. Let $u_1, u_2, ..., u_9$ be the values of u at the interior mesh-points. Since the boundary values of u are symmetrical about AB, $\therefore\ u_7 = u_1, u_8 = u_2, u_9 = u_3$.

Also the values of u being symmetrical about CD, $u_3 = u_1, u_6 = u_4, u_9 = u_7$.

Thus it is sufficient to find the values u_1, u_2, u_4 and u_5.

Now we find their initial values in the following order :

$$u_5 = \frac{1}{4}(2000 + 2000 + 1000 + 1000) = 1500 \quad \text{(Std. formula)}$$

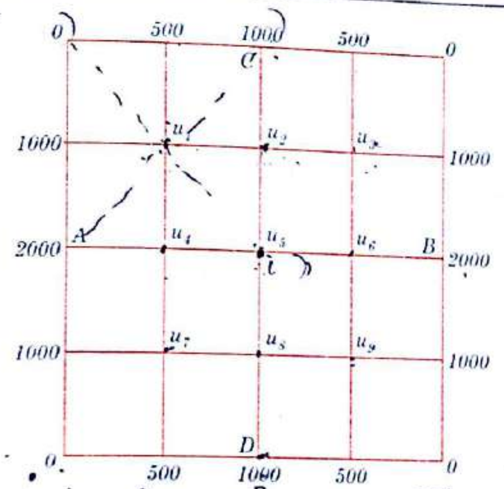
$$u_1 = \frac{1}{4}(0 + 1500 + 1000 + 2000) = 1125 \quad \text{(Diag. formula)}$$

5

↑ 4

J 3

2



i 1 → 2 3 4 5

Fig. 11.6

$$u_2 = \frac{1}{4}(1125 + 1125 + 1000 + 1500) \approx 1188 \quad \text{(Std. formula)}$$

$$u_4 = \frac{1}{4}(2000 + 1500 + 1125 + 1125) \approx 1438 \quad \text{(Std. formula)}$$

Now we carry out the iteration process using the standard formulae :

$$u_1^{(n+1)} = \frac{1}{4}[1000 + u_2^{(n)} + 500 + u_4^{(n)}]$$

$$u_2^{(n+1)} = \frac{1}{4}[u_1^{(n+1)} + u_3^{(n)} + 1000 + u_5^{(n)}]$$

$$u_4^{(n+1)} = \frac{1}{4}[2000 + u_5^{(n)} + u_1^{(n+1)} + u_1^{(n)}]$$

$$u_5^{(n+1)} = \frac{1}{4}[u_4^{(n+1)} + u_4^{(n)} + u_2^{(n+1)} + u_2^{(n)}]$$

First iteration : (put $n = 0$ in the above results)

$$u_1^{(1)} = \frac{1}{4}(1000 + 1188 + 500 + 1438) \approx 1032$$

$$u_2^{(1)} = \frac{1}{4}(1032 + 1125 + 1000 + 1500) = 1164$$

$$u_4^{(1)} = \frac{1}{4}(2000 + 1500 + 1032 + 1125) = 1414$$

$$u_5^{(1)} = \frac{1}{4}(1414 + 1438 + 1164 + 1188) = 1301$$

Second iteration : (put $n = 1$)

$$u_1^{(2)} = \frac{1}{4}(1000 + \boxed{1164} + 500 + 1414) = 1020$$

$$u_2^{(2)} = \frac{1}{4}(1020 + 1032 + 1000 + 1301) = 1088$$

$$u_4^{(2)} = \frac{1}{4}(2000 + 1301 + 1020 + 1032) = 1338$$

$$u_5^{(2)} = \frac{1}{4}(1338 + 1414 + 1088 + 1164) = 1251$$

Third iteration :

$$u_1^{(3)} = \frac{1}{4}(1000 + 1088 + 500 + 1338) = 982$$

$$u_2^{(3)} = \frac{1}{4}(982 + 1020 + 1000 + 1251) = 1063$$

$$u_4^{(3)} = \frac{1}{4}(2000 + 1251 + 982 + 1020) = 1313$$

$$u_5^{(3)} = \frac{1}{4}(1313 + 1338 + 1063 + 1088) = 1201$$

Fourth iteration :

$$u_1^{(4)} = \frac{1}{4}(1000 + 1063 + 500 + 1313) = 969$$

$$u_2^{(4)} = \frac{1}{4}(969 + 982 + 1000 + 1201) = 1038$$

$$u_4^{(4)} = \frac{1}{4}(2000 + 1201 + 969 + 982) = 1288$$

$$u_5^{(4)} = \frac{1}{4}(1288 + 1313 + 1038 + 1063) = 1176$$

Fifth iteration :

$$u_1^{(5)} = \frac{1}{4}(1000 + 1038 + 500 + 1288) = 957$$

$$u_2^{(5)} = \frac{1}{4}(957 + 969 + 1000 + 1176) \approx 1026$$

$$u_4^{(5)} = \frac{1}{4}(2000 + 1176 + 957 + 969) \approx 1276$$

$$u_5^{(5)} = \frac{1}{4}(1276 + 1288 + 1026 + 1038) = 1157$$

Similarly,

$u_1^{(6)} = 951,\ u_2^{(6)} = 1016,\ u_4^{(6)} = 1266,\ u_5^{(6)} = 1146$

$u_1^{(7)} = 946,\ u_2^{(7)} = 1011,\ u_4^{(7)} = 1260,\ u_5^{(7)} = 1138$

$u_1^{(8)} = 943,\ u_2^{(8)} = 1007,\ u_4^{(8)} = 1257,\ u_5^{(8)} = 1134$

$u_1^{(9)} = 941,\ u_2^{(9)} = 1005,\ u_4^{(9)} = 1255,\ u_5^{(9)} = 1131$

$u_1^{(10)} = 940,\ u_2^{(10)} = 1003,\ u_4^{(10)} = 1253,\ u_5^{(10)} = 1129$

$u_1^{(11)} = 939,\ u_2^{(11)} = 1002,\ u_4^{(11)} = 1252,\ u_5^{(11)} = 1128$

$u_1^{(12)} \approx 939,\ u_2^{(12)} \approx 1001,\ u_4^{(12)} \approx 1251,\ u_5^{(12)} = 1126$

There is negligible difference between the values obtained in the 11th and 12th iterations.

Hence $u_1 = 939,\ u_2 = 1001,\ u_4 = 1251$ and $u_5 = 1126$.

Example 11.3. *Given the values of $u(x, y)$ on the boundary of the square in the Fig. 11.7, evaluate the function $u(x, y)$ satisfying the Laplace equation $\nabla^2 u = 0$ at the pivotal points of this figure by*

(a) Jacobi's method.

(b) Gauss-Seidal method. *(Bhopal, B. Tech., 2009)*

Sol. To get the initial values of u_1, u_2, u_3, u_4, we assume that $u_4 = 0$. Then

$$u_1 = \frac{1}{4}(1000 + 0 + 1000 + 2000) = 1000 \qquad \text{(Diag. formula)}$$

$$u_2 = \frac{1}{4}(1000 + 500 + 1000 + 0) = 625 \qquad \text{(Std. formula)}$$

$$u_3 = \frac{1}{4}(2000 + 0 + 1000 + 500) = 875 \qquad \text{(Std. formula)}$$

$$u_4 = \frac{1}{4}(875 + 0 + 625 + 0) = 375 \qquad \text{(Std. formula)}$$

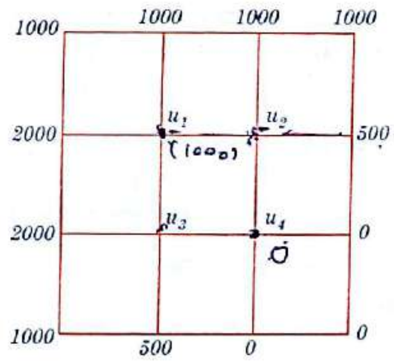


Fig. 11.7

(a) We carry out the successive iterations, using Jacobi's formulae :

$$u_1^{(n+1)} = \frac{1}{4}[2000 + u_2^{(n)} + 1000 + u_3^{(n)}]$$

$$u_2^{(n+1)} = \frac{1}{4}[u_1^{(n)} + 500 + 1000 + u_4^{(n)}]$$

$$u_3^{(n+1)} = \frac{1}{4}[2000 + u_4^{(n)} + u_1^{(n)} + 500]$$

$$u_4^{(n+1)} = \frac{1}{4}[u_3^{(n)} + 0 + u_2^{(n)} + 0]$$

First iteration : (put $n = 0$ in the above results)

$$u_1^{(1)} = \frac{1}{4}(2000 + 625 + 1000 + 875) = 1125$$

$$u_2^{(1)} = \frac{1}{4}(1000 + 500 + 1000 + 375) \approx 719$$

$$u_3^{(1)} = \frac{1}{4}(2000 + 375 + 1000 + 500) \approx 969$$

$$u_4^{(1)} = \frac{1}{4}(875 + 0 + 625 + 0) = 375$$

Second iteration : (put $n = 1$)

$$u_1^{(2)} = \frac{1}{4}(2000 + 719 + 1000 + 969) = 1172$$

$$u_2^{(2)} = \frac{1}{4}(1125 + 500 + 1000 + 375) = 750$$

$$u_3^{(2)} = \frac{1}{4}(2000 + 375 + 1125 + 500) = 1000$$

$$u_4^{(2)} = \frac{1}{4}(969 + 0 + 719 + 0) = 422$$

Similarly, $u_1^{(3)} \approx 1188$, $u_2^{(3)} \approx 774$, $u_3^{(3)} \approx 1024$, $u_4^{(3)} \approx 438$
$u_1^{(4)} \approx 1200$, $u_2^{(4)} \approx 782$, $u_3^{(4)} \approx 1032$, $u_4^{(4)} \approx 450$
$u_1^{(5)} \approx 1204$, $u_2^{(5)} \approx 788$, $u_3^{(5)} \approx 1038$, $u_4^{(5)} \approx 454$
$u_1^{(6)} \approx 1206.5$, $u_2^{(6)} \approx 790$, $u_3^{(6)} \approx 1040$, $u_4^{(6)} \approx 456.5$
$u_1^{(7)} \approx 1208$, $u_2^{(7)} \approx 791$, $u_3^{(7)} \approx 1041$, $u_4^{(7)} \approx 458$
and $u_1^{(8)} \approx 1208$, $u_2^{(8)} \approx 791.5$, $u_3^{(8)} \approx 1041.5$, $u_4^{(8)} \approx 458.$

There is no significant difference between the seventh and eighth iteration values.

Hence $u_1 = 1208$, $u_2 = 792$, $u_3 = 1042$ and $u_4 = 458$.

(*b*) We carry out the successive iterations, using Gauss-Seidal formulae

$$u_1^{(n+1)} = \frac{1}{4}[2000 + u_2^{(n)} + 1000 + u_3^{(n)}]$$

$$u_2^{(n+1)} = \frac{1}{4}[u_1^{(n+1)} = 500 + 1000 + u_4^{(n)}]$$

$$u_3^{(n+1)} = \frac{1}{4}[2000 + u_4^{(n)} + u_1^{(n+1)} = 500]$$

$$u_4^{(n+1)} = \frac{1}{4}[u_3^{(n+1)} + 0 + u_2^{(n+1)} + 0]$$

First iteration : (put $n = 0$ in the above results)

$$u_1^{(1)} = \frac{1}{4}(2000 + 625 + 1000 + 875) = 1125$$

$$u_2^{(1)} = \frac{1}{4}(1125 + 500 + 1000 + 375) = 750$$

$$u_3^{(1)} = \frac{1}{4}(2000 + 375 + 1125 + 500) = 1000$$

$$u_4^{(1)} = \frac{1}{4}(1000 + 0 + 750 + 0) \approx 438$$

Second iteration : (put $n = 1$)

$$u_1^{(2)} = \frac{1}{4}(2000 + 750 + 1000 + 1000) \approx 1188$$

$$u_2^{(2)} = \frac{1}{4}(1188 + 500 + 1000 + 438) \approx 782$$

$$u_3^{(2)} = \frac{1}{4}(2000 + 438 + 1188 + 500) \approx 1032$$

$$u_4^{(2)} = \frac{1}{4}(1032 + 0 + 782 + 0) \approx 454$$

Similarly $u_1^{(3)} \approx 1204$, $u_2^{(3)} \approx 789$, $u_3^{(3)} \approx 1040$, $u_4^{(3)} \approx 458$
$u_1^{(4)} \approx 1207$, $u_2^{(4)} \approx 791$, $u_3^{(4)} \approx 1041$, $u_4^{(4)} = 458$
and $u_1^{(5)} = 1208$, $u_2^{(5)} \approx 791.5$, $u_3^{(5)} \approx 1041.5$, $u_4^{(5)} \approx 458.25$

Thus there is no significant difference between the fourth and fifth iteration values.

Hence $u_1 = 1208$, $u_2 = 792$, $u_3 = 1042$ and $u_4 = 458$.

Example 11.4. *Solve the Laplace equation $u_{xx} + u_{yy} = 0$ given that*

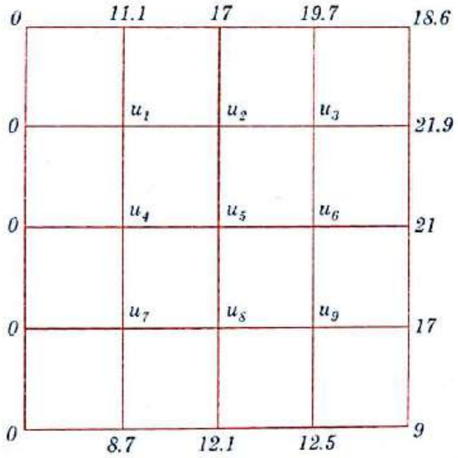


Fig. 11.8

(*V.T.U., B. Tech., 2009*)

Sol. We first find the initial values in the following order:

$$u_5 = \frac{1}{4}(0 + 17 + 21 + 12.1) = 12.5 \qquad \text{(Std. formula)}$$

$$u_1 = \frac{1}{4}(0 + 12.5 + 0 + 17) = 7.4 \qquad \text{(Diag. formula)}$$

$$u_3 = \frac{1}{4}(12.5 + 18.6 + 17 + 21) = 17.28 \qquad \text{(Diag. formula)}$$

$$y_7 = \frac{1}{4}(12.5 + 0 + 0 + 12.1) = 6.15 \qquad \text{(Diag. formula)}$$

$$u_9 = \frac{1}{4}(12.5 + 9 + 21 + 12.1) = 13.65 \qquad \text{(Diag. formula)}$$

$$u_2 = \frac{1}{4}(17 + 12.5 + 7.4 + 17.3) = 13.55 \qquad \text{(Std. formula)}$$

$$u_4 = \frac{1}{4}(7.4 + 6.2 + 0 + 12.5) = 6.52 \qquad \text{(Std. formula)}$$

$$u_6 = \frac{1}{4}(17.3 + 13.7 + 12.5 + 21) = 16.12 \qquad \text{(Std. formula)}$$

$$u_8 = \frac{1}{4}(12.5 + 12.1 + 6.2 + 13.7) = 11.12 \qquad \text{(Std. formula)}$$

Now we carry out the iteration process using the Standard formula:

$$u_1^{(n+1)} = \frac{1}{4}\left[0 + 11.1 + u_4^{(n)} + u_2^{(n)}\right]$$

$$u_2^{(n+1)} = \frac{1}{4}\left[u_1^{(n+1)} + 17 + u_5^{(n)} + u_3^{(n)}\right]$$

$$u_3^{(n+1)} = \frac{1}{4}\left[u_2^{(n+1)} + 19.7 + u_6^{(n)} + 21.9\right]$$

$$u_4^{(n+1)} = \frac{1}{4}\left[0 + u_1^{(n+1)} + u_7^{(n)} + u_5^{(n)}\right],$$

$$u_5^{(n+1)} = \frac{1}{4}\left[u_4^{(n+1)} + u_2^{(n+1)} + u_8^{(n)} + u_6^{(n)}\right]$$

$$u_6^{(n+1)} = \frac{1}{4}\left[u_5^{(n+1)} + u_3^{(n+1)} + u_9^{(n)} + 21\right]$$

$$u_7^{(n+1)} = \frac{1}{4}\left[0 + u_4^{(n+1)} + 8.7 + u_8^{(n)}\right]$$

$$u_8^{(n+1)} = \frac{1}{4}\left[u_7^{(n+1)} \; u_5^{(n+1)} + 12.1 + u_9^{(n)}\right]$$

$$u_9^{(n+1)} = \frac{1}{4}\left[u_8^{(n+1)} + u_6^{(n+1)} + 12.8 + 17\right]$$

First iteration (put $n = 0$, in the above results)

$$u_1^{(1)} = \frac{1}{4}\left(0 + 11.1 + u_4^{(0)} + u_2^{(0)}\right)$$

$$= \frac{1}{4}(0 + 11.1 + 6.52 + 13.55) = 7.79$$

$$u_2^{(1)} = \frac{1}{4}(7.79 + 17 + 12.5 + 17.28) = 13.64$$

$$u_3^{(1)} = \frac{1}{4}(13.64 + 19.7 + 16.12 + 21.9) = 12.84$$

$$u_4^{(1)} = \frac{1}{4}(0 + 7.79 + 6.15 + 12.5) = 6.61$$

$$u_5^{(1)} = \frac{1}{4}(6.61 + 13.64 + 11.12 + 16.12) = 11.88$$

$$u_6^{(1)} = \frac{1}{4}(11.88 + 17.84 + 13.65 + 21) = 16.09$$

$$u_7^{(1)} = \frac{1}{4}(0 + 6.61 + 8.7 + 11.12) = 6.61$$

$$u_8^{(1)} = \frac{1}{4}(6.61 + 11.88 + 12.1 + 13.65) = 11.06$$

$$u_9^{(1)} = \frac{1}{4}(11.06 + 16.09 + 12.8 + 17) = 12.238$$

Second iteration (put $n = 1$)

$$u_1^{(2)} = \frac{1}{4}(0 + 11.1 + 6.61 + 13.64) = 7.84$$

$$u_2^{(2)} = \frac{1}{4}(7.84 + 17 + 11.88 + 17.84) = 16.64$$

$$u_3^{(2)} = \frac{1}{4}(13.64 + 19.7 + 16.09 + 21.9) = 17.83$$

$$u_4^{(2)} = \frac{1}{4}(0 + 7.84 + 6.61 + 11.88) = 6.58$$

$$u_5^{(2)} = \frac{1}{4}(6.58 + 13.64 + 11.06 + 16.09) = 11.84$$

$$u_6^{(2)} = \frac{1}{4}(11.84 + 17.83 + 14.24 + 21) = 16.23$$

$$u_7^{(2)} = \frac{1}{4}(0 + 6.58 + 8.7 + 11.06) = 6.58$$

$$u_8^{(2)} = \frac{1}{4}(6.58 + 11.84 + 12.1 + 14.24) = 11.19$$

$$u_9^{(2)} = \frac{1}{4}(11.19 + 16.23 + 12.8 + 17) = 14.30$$

Third iteration (put $n = 2$)

$$u_1^{(3)} = \frac{1}{4}(0 + 11.1 + 6.58 + 13.64) = 7.83$$

$$u_2^{(3)} = \frac{1}{4}(7.83 + 17 + 11.84 + 17.83) = 13.637$$

$$u_3^{(4)} = \frac{1}{4}(13.63 + 19.7 + 16.23 + 21.9) = 17.86$$

$$u_4^{(3)} = \frac{1}{4}(0 + 7.83 + 6.58 + 11.84) = 6.56$$

$$u_5^{(3)} = \frac{1}{4}(6.56 + 13.63 + 11.19 + 16.23) = 11.90$$

$$u_6^{(3)} = \frac{1}{4}(11.90 + 17.86 + 14.30 + 21) = 16.27$$

$$u_7^{(3)} = \frac{1}{4}(0 + 6.56 + 8.7 + 11.19) = 6.61$$

$$u_8^{(3)} = \frac{1}{4}(6.61 + 11.90 + 12.1 + 14.30) = 11.23$$

$$u_9^{(3)} = \frac{1}{4}(11.23 + 16.27 + 12.8 + 17) = 14.32$$

Similarly $u_1^{(4)} = 7.82$, $u_2^{(4)} = 13.65$, $u_3^{(4)} = 17.88$, $u_4^{(4)} = 6.58$, $u_5^{(4)} = 11.94$, $u_6^{(4)} = 16.28$,

$u_7^{(4)} = 6.63$, $u_8^{(4)} = 11.25$, $u_9^{(4)} = 14.33$

$u_1^{(5)} = 7.83$, $u_2^{(5)} = 13.66$, $u_3^{(5)} = 17.89$, $u_4^{(5)} = 6.50$, $u_5^{(5)} = 11.95$, $u_6^{(5)} = 16.29$, $u_7^{(5)} = 6.64$,

$u_8^{(5)} = 11.25$, $u_9^{(5)} = 14.34$

There is no significant difference between the fourth and fifth iteration values.

Hence $u_1 = 7.83$, $u_2 = 13.66$, $u_3 = 17.89$, $u_4 = 6.6$, $u_5 = 11.95$, $u_6 = 16.29$, $u_7 = 6.64$, $u_8 = 11.25$, $u_9 = 14.34$.

11.6. SOLUTION OF POISSON'S EQUATION

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \qquad ...(1)$$

Its method of solution is similar to that of the Laplace equation. Here the standard 5-point formula for (1) takes the form

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f(ih, jh) \qquad ...(2)$$

By applying (2) at each interior mesh point, we arrive at linear equations in the nodal values $u_{i,j}$. These equations can be solved by Gauss-Seidal method.

Obs. The error in replacing u_{xx} by the finite difference approximation is of the order $O(h^2)$. Since $k = h$, the error in replacing u_{yy} by the difference approximation is also of the order $O(h^2)$. Hence the error in solving Laplace and Poisson's equations by finite difference method is of the order $O(h^2)$.

Example 11.5. *Solve the Poisson equation $u_{xx} + u_{yy} = -81xy$, $0 < x < 1$, $0 < y < 1$ given that $u(0, y) = 0$, $u(x, 0) = 0$, $u(1, y) = 100$, $u(x, 1) = 100$ and $h = 1/3$.* (Anna, B. Tech., 2005)

Sol. Here $h = 1/3$.

The standard 5-point formula for the given equation is

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f(ih, jh) = h^2[-81(ih \cdot jh)] = h^4(-81)\,ij = -ij \qquad ...(i)$$

For u_1 $(i = 1, j = 2)$, (i) gives $0 + u_2 + u_3 + 100 - 4u_1 = -2$

i.e. $\qquad -4u_1 + u_2 + u_3 = -102 \qquad ...(ii)$

For u_2 $(i = 2, j = 2)$, (i) gives $u_1 + 100 + u_4 + 100 - 4u_2 = -4$

i.e. $\qquad u_1 - 4u_2 + u_4 = -204 \qquad ...(iii)$

For u_3 $(i = 1, j = 1)$, (i) gives $0 + u_4 + 0 + u_1 - 4u_3 = -1$

i.e. $\qquad u_1 - 4u_3 + u_4 = -1 \qquad ...(iv)$

For u_4 $(i = 2, j = 1)$ gives $u_3 + 100 + u_2 - 4u_4 = -2$

i.e. $\qquad u_2 + u_3 - 4u_4 = -102 \qquad ...(v)$

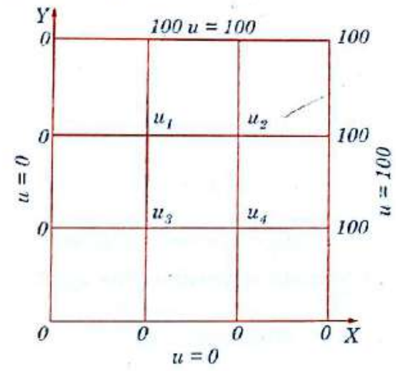


Fig. 11.9

Subtracting (v) from (ii), $-4u_1 + 4u_4 = 0$ i.e. $u_1 = u_4$

Then (iii) becomes $2u_1 - 4u_2 = -204 \qquad ...(vi)$

and (iv) becomes $\qquad 2u_1 - 4u_3 = -1 \qquad ...(vii)$

Now (4) × (ii) + (vi) gives $-14u_1 + 4u_3 = -612 \qquad ...(viii)$

(vii) + (viii) gives $-12u_1 = -613$

Thus $\qquad u_1 = 613/12 = 51.0833 = u_4.$

From (vi), $\qquad u_2 = \frac{1}{2}(u_1 + 102) = 76.5477$

From (vii), $\qquad u_3 = \frac{1}{2}\left(u_1 + \frac{1}{2}\right) = 25.7916$

Example 11.6. *Solve the equation $\nabla^2 u = -10(x^2 + y^2 + 10)$ over the square with sides $x = 0 = y$, $x = 3 = y$ with $u = 0$ on the boundary and mesh length = 1.* (Anna, B.E., 2007)

Sol. Here $h = 1$.

∴ The standard 5-point formula for the given equation is

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = -10(i^2 + j^2 + 10) \qquad ...(i)$$

For u_1 $(i = 1, j = 2)$, (i) gives $0 + u_2 + 0 + u_3 - 4u_1 = -10(1 + 4 + 10)$

i.e. $\qquad u_1 = \frac{1}{4}(u_2 + u_3 + 150) \qquad ...(ii)$

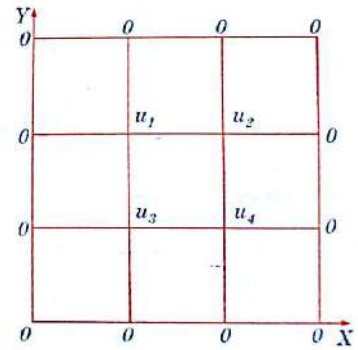


Fig. 11.10

Scanned by CamScanner

For u_2 $(i = 2, j = 2)$, (i) gives $\quad u_2 = \frac{1}{4}(u_1 + u_4 + 180)$ ...(iii)

For u_3 $(i = 1, j = 1)$, we have $\quad u_3 = \frac{1}{4}(u_1 + u_4 + 120)$ ...(iv)

For u_4 $(i = 2, j = 1)$, we have $\quad u_4 = \frac{1}{4}(u_2 + u_3 + 150)$ ...(v)

Equations (ii) and (v) show that $u_4 = u_1$. Thus the above equations reduce to

$$u_1 = \frac{1}{4}(u_2 + u_3 + 150),\ u_2 = \frac{1}{4}(u_2 + 90),\ u_3 = \frac{1}{4}(u_1 + 60)$$

Now let us solve these equations by Gauss-Seidal iteration method.

First iteration : Starting from the approximations $u_2 = 0$, $u_3 = 0$, we obtain $u_1^{(1)} = 37.5$

Then $\quad u_2^{(1)} = \frac{1}{2}(37.5 + 90) = 64$

$$u_2^{(1)} = \frac{1}{2}(37.5 + 60) = 49$$

Second iteration : $\quad u_1^{(2)} = \frac{1}{4}(64 + 49 + 150) = 66,\ u_2^{(2)} = \frac{1}{2}(66 + 90) = 78$

$$u_3^{(2)} = \frac{1}{2}(66 + 60) = 63$$

Third iteration : $\quad u_1^{(3)} = \frac{1}{4}(78 + 63 + 150) = 73,\ u_2^{(3)} = \frac{1}{2}(73 + 90) = 82$

$$u_3^{(3)} = \frac{1}{2}(73 + 60) = 67$$

Fourth iteration : $\quad u_1^{(4)} = \frac{1}{2}(82 + 67 + 150) = 75,\ u_2^{(4)} = \frac{1}{2}(75 + 90) = 82.5$

$$u_3^{(4)} = \frac{1}{2}(75 + 60) = 67.5$$

Fifth iteration : $\quad u_1^{(5)} = \frac{1}{4}(82.5 + 67.5 + 150) = 75,\ u_2^{(5)} = \frac{1}{2}(75 + 90) = 82.5$

$$u_3^{(5)} = \frac{1}{2}(75 + 90) = 67.5$$

Since these values are the same as those of fourth iteration, we have $u_1 = 75$, $u_2 = 82.5$, $u_3 = 67.5$ and $u_4 = 75$.

PROBLEMS 11.2

1. Solve the equation $u_{xx} + u_{yy} = 0$ for the square mesh with the boundary values as shown in Fig. 11.11. (Delhi, B.E., 2002)

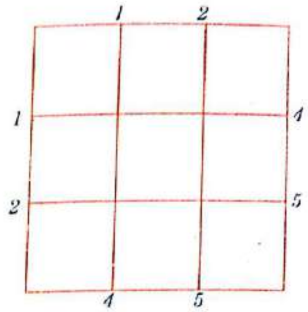


Fig. 11.11

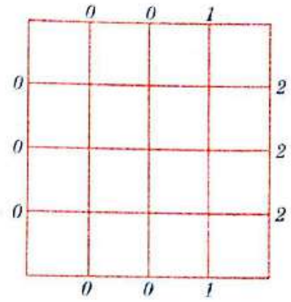


Fig. 11.12

2. Solve $u_{xx} + u_{yy} = 0$ over the square mesh of side 4 units satisfying the following boundary conditions: $u(0, y) = 0$ for $0 \le y \le 4$, $u(4, y) = 12 + y$ for $0 \le y \le 4$; $u(x, 0) = 3x$ for $0 \le x \le 4$, $u(x, 4) = x^2$ for $0 \le x \le 4$. (Cusat, B. Tech., 2008)

3. Solve the elliptic equation $u_{xx} + u_{yy} = 0$ for the square mesh with boundary values as shown in Fig. 11.12. Iterate until the maximum difference between successive values at any point is less than 0.005.

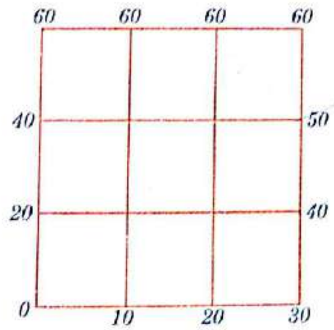


Fig. 11.13

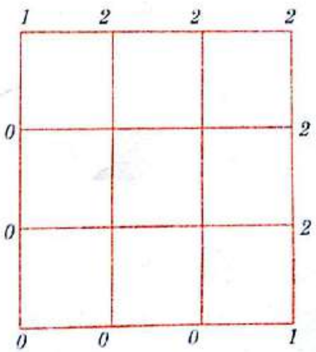


Fig. 11.14

4. Using central-difference approximation solve $\nabla^2 u = 0$ at the nodal points of the square grid of Fig. 11.13 using the boundary values indicated. (Nagarjuna, B. Tech., 2003 S)

5. Solve $u_{xx} + u_{yy} = 0$ for the square mesh with boundary values as shown in Fig. 11.14. Iterate till the mesh values are correct to two decimal places.

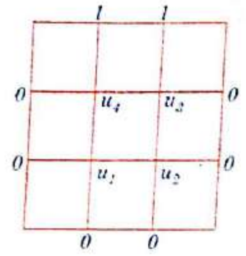


Fig. 11.15

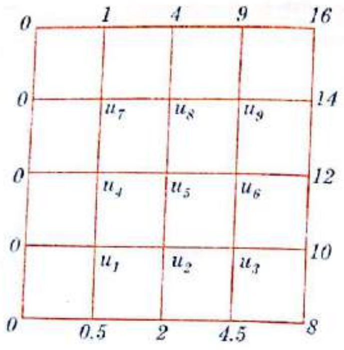


Fig. 11.16

6. Solve the Laplace's equation $u_{xx} + u_{yy} = 0$ in the domain of Fig. 11.15 by (*a*) Jacobi's method, (*b*) Gauss-Seidal method.
7. Solve the Laplace's equation $\nabla^2 u = 0$ in the domain of the Fig. 11.16.
8. Solve the Poisson's equation $\nabla^2 u = 8x^2y^2$ for the square mesh of Fig. 11.17 with $u(x, y) = 0$ on the boundary and mesh length = 1. (*J.N.T.U., B. Tech., 2004 S*)

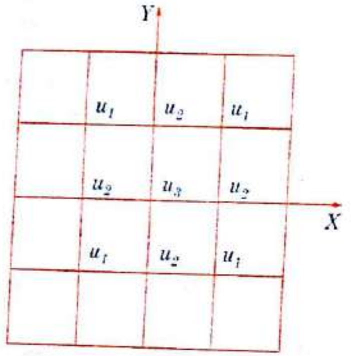


Fig. 11.17

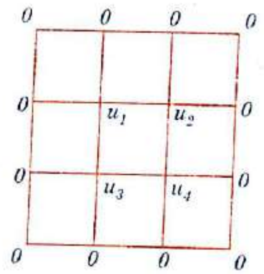


Fig. 11.18

9. Solve the Poisson's equation $\nabla^2 u = x^3 + y^3$ over the square region of Fig. 10.18. Taking $h = 1$.

11.7. (1) SOLUTION OF ELLIPTIC EQUATIONS BY RELAXATION METHOD

If the equations for all the mesh points are written using (2) of § 11.6, we get a system of equations which can be solved by any method. For this purpose, the method of relaxation is particularly well-suited. Here we shall describe this method in relation to elliptic equations.

Consider the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \ldots(1)$$

We take a square region and divide it into a square net of mesh size h. Let the value of u at A be u_0 and its values at the four adjacent points be u_1, u_2, u_3, u_4 (Fig. 11.19). Then

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_1 + u_3 - 2u_0}{h^2} \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} \approx \frac{u_2 + u_4 - 2u_0}{h^2}$$

If (1) is satisfied at A, then

$$\frac{u_1 + u_3 - 2u_0}{h^2} + \frac{u_2 + u_4 - 2u_0}{h^2} \approx 0$$

or

$$u_1 + u_2 + u_3 + u_4 - 4u_0 \approx 0$$

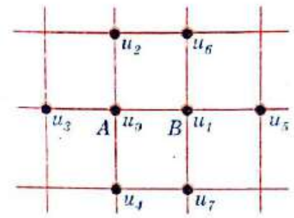


Fig. 11.19

If r_0 be the residual (discrepency) at the mesh point A, then

$$r_0 = u_1 + u_2 + u_3 + u_4 - 4u_0 \qquad \ldots(2)$$

Similarly the residual at the point B, is given by

$$r_1 = u_0 + u_5 + u_6 + u_7 - 4u_1 \text{ and so on} \qquad \ldots(3)$$

The main aim of the relaxation process is to reduce all the residuals to zero by making them as small as possible step by step. We, therefore, try to adjust the value of u at an internal mesh point so as to make the residual there at zero. But when the value of u is changing at a mesh point, the values of the residuals at the neighbouring interior points will also be changed. If u_0 is given an increment 1, then

(*i*) (2) shows that r_0 is changed by -4.

(*ii*) (3) shows that r_1 is changed by 1.

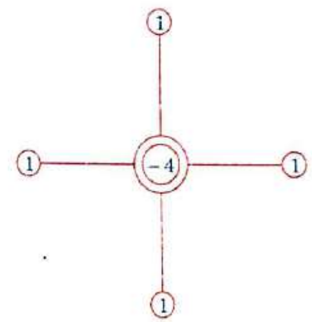


Fig. 11.20

i.e. if the value of the function is increased by 1 at a mesh point (shown by a double ring), *then the residual at that point is decreased by 4 while the residuals at the adjacent interior points* (shown by a single ring), *get increased each by 1.* This relaxation pattern is shown in Fig. 11.20.

(2) Working procedure *to solve an equation by relaxation method :*

I. *Write down by trial, the initial values of u at the interior mesh points by diagonal-averaging or cross-averaging.*

II. *Calculate the residuals at each of these points by (2) above.* If we apply this formula at a point near the boundary, one or more end points get chopped off since there are no residuals at the boundary.

III. *Write the residuals at a mesh-point on the right of this point and the value of u on its left.*

IV. *Obtain the solution by reducing the residuals to zero, one by one, by giving suitable increments to u and using Fig. 11.20. At each step, we reduce the numerically largest residual to zero and record the increment of u on the left* (below the earlier value thereat) *and the modified residual on the right* (below the earlier residual).

V. *When a round of relaxation is completed, the value of u and its increments are added at each point. Using these values, calculate all the residuals afresh. If some of the recalculated residuals are large, liquidate these again.*

VI. *Stop the relaxation process, when the current values of the residuals are quite small. The solution will be the current value of u at each of the nodes.*

Obs. Relaxation method combines simplicity with the speed of convergence. *Its only drawback is its unsuitability for computer calculations.*

Example 11.7. *Solve by relaxation method, the Laplace equation* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, *inside the square bounded by the lines* $x = 0, x = 4, y = 0, y = 4$, *given that* $u = x^2y^2$ *on the boundary.*

Sol. Taking $h = 1$, we find u on the boundary from $u = x^2y^2$. The initial values of u at the nine mesh-points are estimated to be 24, 56, 104 ; 16, 32, 56 ; 8, 16, 24 as shown on the left of the points in Fig. 11.21.

$\therefore$ Residual at A *i.e.* $r_A = 0 + 56 + 16 + 16 - 4 \times 24 = -8$

Similarly $r_B = 0, r_C = -16, r_D = 0, r_E = 16, r_F = 0, r_G = 0, r_H = 0, r_I = -8$.

(*i*) The numerically largest residual is 16 at E. To liquidate it, we increase u by 4 so that the residual becomes zero and the residuals at neighbouring nodes get increased by 4.

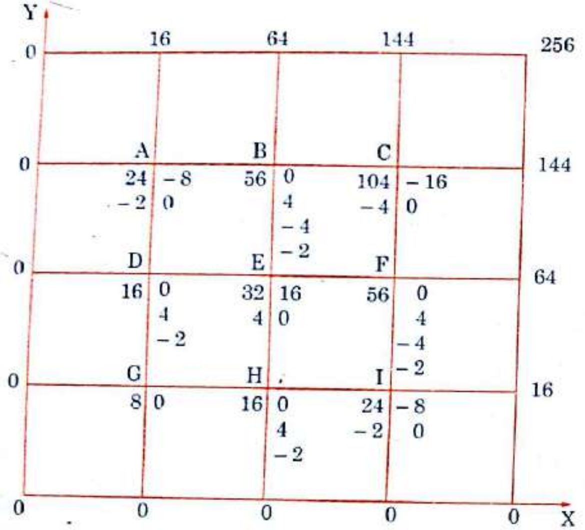


Fig. 11.21

(*ii*) Next, the numerically largest residual is – 16 at C. To reduce it to zero, we increase u by – 4 so that the residuals at the adjacent nodes are increased by – 4.

(*iii*) Now, the numerically largest residual is – 8 at A. To liquidate it, we increase u by – 2 so that the residuals at the adjacent nodes are increased by – 2.

(*iv*) Finally, the largest residual is – 8 at I. To liquidate it, we increase u by – 2 so that the residuals at the adjacent points are increased by – 2.

(*v*) The numerically largest current residual being 2, we stop the relaxation process. Hence the final values of u are

$$u_A = 22, \quad u_B = 56, \quad u_C = 100 ;$$
$$u_D = 16, \quad u_E = 36, \quad u_F = 56 ;$$
$$u_G = 8, \quad u_H = 16, \quad u_I = 22.$$

Example 11.8. *Solve by relaxation method Example 11.3.*

Sol. (*i*) The initial values of u at A, B, C and D are estimated to be 1000, 625, 875 and 375 [Fig. 11.22 (*i*)].

$\therefore \quad r_A = 500, r_B = 375, r_C = 375, r_D = 0$

To liquidate r_A, increase u by 125

To liquidate r_B, increase u by 94

To liquidate r_C, increase u by 94

(*ii*) Modified values of u are 1125, 719, 969, 375 [Fig. (*ii*)]

$\therefore \quad r_A = 188, r_B = 124, r_C = 124, r_D = 188.$

To liquidate r_A, r_D, r_B, r_C increase u by 47, 47, 31, 31 in turn.

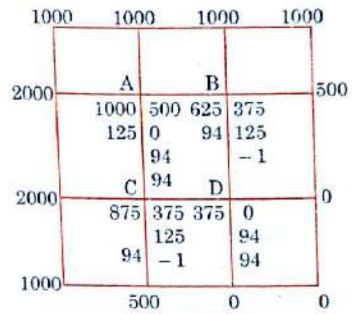
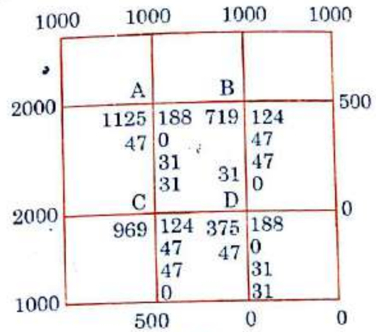
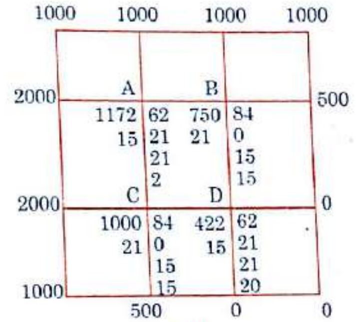


Fig. 11.22 (*i*)



(*ii*)



(*iii*)

(*iii*) Revised values of u are 1172, 750, 1000, 422 [Fig. (*iii*)]

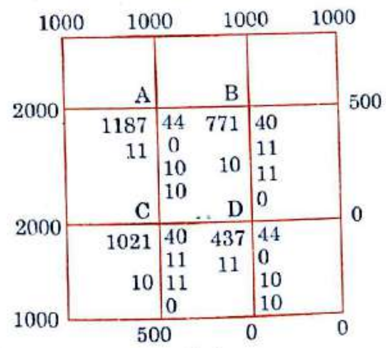
$\therefore \quad r_A = 62, r_B = 84, r_C = 84, r_D = 62$

To liquidate r_B, r_C, r_A, r_D increase u by 21, 21, 15, 15 respectively.

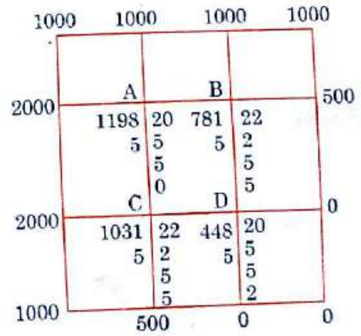
(*iv*) Improved values of u are 1187, 771, 1021, 437 [Fig. (*iv*)]

$\therefore \quad r_A = 44, r_B = 40, r_C = 40, r_D = 44.$

To liquidate r_A, r_D, r_B, r_C increase u by 11, 11, 10, 10 respectively



(*iv*)



(*v*)

(*v*) Modified values of u are 1198, 781, 1031, 448 [Fig. (*v*)]

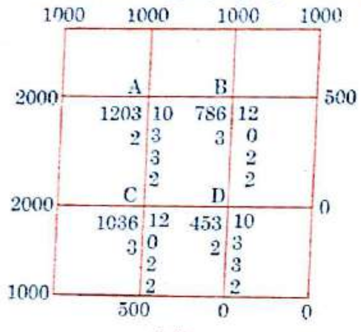
$\therefore$ $r_A = 20, r_B = 22, r_C = 22, r_D = 20.$

To liquidate r_B, r_C, r_A, r_D increase u by 5, 5, 5, 5 respectively

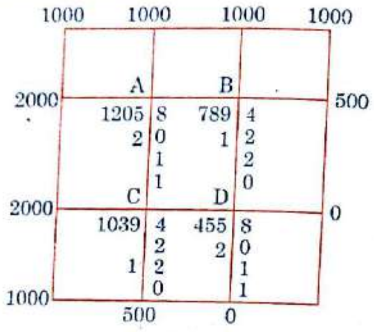
(*vi*) Revised values of u are 1203, 786, 1036, 453 [Fig. (*vi*)]

$\therefore$ $r_A = 10, r_B = 12, r_C = 12, r_D = 10$

To liquidate r_B, r_C, r_A, r_D increase u by 3, 3, 2, 2 respectively.



(*vi*)



(*vi*)

(*vii*) Improved values of u are 1205, 789, 1039, 455 [Fig. (*vii*)]

$\therefore$ $r_A = 8, r_B = 4, r_C = 4, r_D = 8.$

To liquidate r_A, r_D, r_B, r_C increase u by 2, 2, 1, 1.

(*viii*) Finally the current residuals being 1, 0, 0, 1, we stop the relaxation process.

Hence the values of u at A, B, C, D are 1207, 790, 1040, 457.

PROBLEMS 11.3

1. Given that $u(x, y)$ satisfies the equation $\nabla^2 u = 0$ and the boundary conditions are $u(0, y) = 0$, $u(4, y) = 8 + 2y$, $u(x, 0) = \frac{1}{2}x^2$, $u(x, 4) = x^2$, find the values $u(i, j)$, $i = 1, 2, 3$; $j = 1, 2, 3$ by relaxation method.
2. Apply relaxation method to solve the equation $\nabla^2 u = -400$, when the region of u is the square bounded by $x = 0, y = 0, x = 4$ and $y = 4$ and u is zero on the boundary of the square.
3. Solve by relaxation method, the equation $\nabla^2 u = 0$ in the square region with square meshes (Fig. 11.23) starting with the initial values $u_1 = u_2 = u_3 = u_4 = 1$.

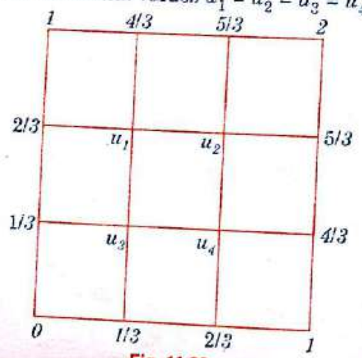


Fig. 11.23

11.8. PARABOLIC EQUATIONS

The one-dimensional heat conduction equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ is a well-known example of parabolic partial differential equations. The solution of this equation is a temperature function $u(x, t)$ which is defined for values of x from 0 to l and for values of time t from 0 to ∞. The solution is not defined in a closed domain but advances in an open-ended region from initial values, satisfying the prescribed boundary conditions (Fig. 11.24).

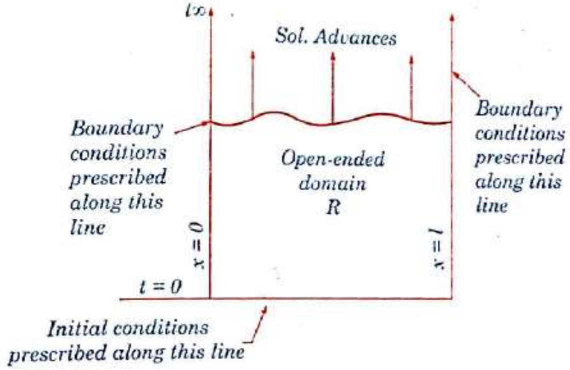


Fig. 11.24

In general, the study of pressure waves in a fluid, propagation of heat and unsteady state problems lead to parabolic type of equations.

11.9. SOLUTION OF ONE DIMENSIONAL HEAT EQUATION

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{...(1)}$$

where $c^2 = k/s\rho$ is the diffusivity of the substance (cm²/sec.)

(1) Schmidt method. Consider a rectangular mesh in the x-t plane with spacing h along x direction and k along time t direction. Denoting a mesh point $(x, t) = (ih, jk)$ as simply i, j, we have

$$\frac{\partial u}{\partial t} = \frac{u_{i, j+1} - u_{i, j}}{k} \quad \text{[by (5) § 11.3]}$$

and

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1, j} - 2u_{i, j} + u_{i+1, j}}{h^2} \quad \text{[by (4) § 11.3]}$$

Substituting these in (1), we obtain $u_{i, j+1} - u_{i, j} = \frac{kc^2}{h^2}[u_{i-1, j} - 2u_{i, j} + u_{i+1, j}]$

or

$$u_{i, j+1} = \alpha u_{i-1, j} + (1 - 2\alpha) u_{i, j} + \alpha u_{i+1, j} \quad \text{...(2)}$$

where $\alpha = kc^2/h^2$ is the mesh ratio parameter.

This formula enables us to determine the value of u at the $(i, j+1)$th mesh point in terms of the known function values at the points x_{i-1}, x_i and x_{i+1} at the instant t_j. It is a relation between the function values at the two time levels $j+1$ and j and is therefore, called a *2-level formula*. In schematic form (2) is shown in Fig. 11.25.

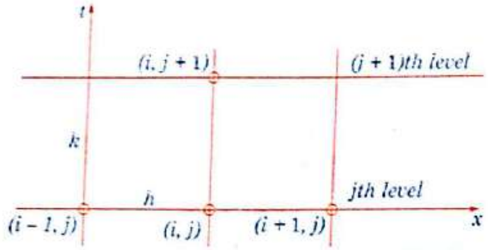


Fig. 11.25

Hence (2) is called the *Schmidt explicit formula* which is valid only for $0 < \alpha \le \frac{1}{2}$.

Obs. In particular when $\alpha = \frac{1}{2}$, (2) reduces to

$$u_{i,j+1} = \frac{1}{2}(u_{i-1,j} + u_{i+1,j}) \qquad ...(3)$$

which shows that *the value of u at x_i at time t_{j+1} is the mean of the u-values at x_{i-1} and x_{i+1} at time t_j*. This relation, known as *Bendre-Schmidt recurrence relation*, gives the values of u at the internal mesh points with the help of boundary conditions.

(2) Crank-Nicolson method. We have seen that the Schmidt scheme is computationally simple and for convergent results $\alpha \le \frac{1}{2}$ *i.e.* $k \le h^2/2c^2$. To obtain more accurate results, h should be small *i.e.* k is necessarily very small. This makes the computations exceptionally lengthy as more time-levels would be required to cover the region. A method that does not restrict α and also reduces the volume of calculations was proposed by Crank and Nicolson in 1947.

According to this method, $\partial^2 u/\partial x^2$ is replaced by the average of its central-difference approximations on the jth and $(j+1)$th time rows. Thus (1) is reduced to

$$\frac{u_{i,j+1} - u_{i,j}}{k} = c^2 \cdot \frac{1}{2}\left\{\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}\right\} + \left\{\frac{u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}}{h^2}\right\}$$

or

$$-\alpha u_{i-1,j+1} + (2+2\alpha)u_{i,j+1} - \alpha u_{i+1,j+1} = \alpha u_{i-1,j} + (2-2\alpha)u_{i,j} + \alpha u_{i+1,j} \qquad ...(4)$$

where $\alpha = kc^2/h^2$.

Clearly the left side of (4) contains three unknown values of u at the $(j+1)$th level while all the three values on the right are known values at the jth level. Thus (4) is a *2-level implicit relation* and is known as *Crank-Nicolson formula*. It is convergent for all finite values of α. Its computational model is given in Fig. 11.26.

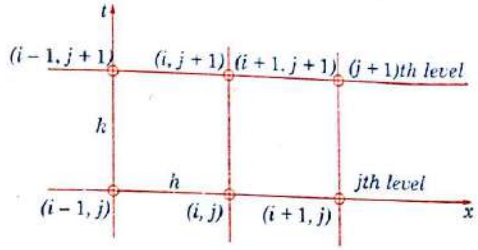


Fig. 11.26

If there are n internal mesh points on each row, then the relation (4) gives n simultaneous equations for the n unknown values in terms of the known boundary values. These equations can be solved to obtain the values at these mesh points. Similarly, the values at the internal mesh points on all rows can be found. A method such as this in which the calculation of an unknown mesh value necessitates the solution of a set of simultaneous equations, is known as an *implicit scheme*.

(3) Iterative methods of solution *for an implicit scheme.*

From (4), we have

$$(1+\alpha)\,u_{i,j+1} = \frac{1}{2}\alpha(u_{i-1,j+1} + u_{i+1,j+1}) + u_{i,j} + \frac{1}{2}\alpha(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) \qquad ...(5)$$

Here only $u_{i,j+1}$, $u_{i-1,j+1}$ and $u_{i+1,j+1}$ are unknown while all others are known since these were already computed in the jth step.

Writing $\quad b_i = u_{i,j} + \frac{\alpha}{2}(u_{i-1,j} - 2u_{i,j} + u_{i+1,j})$

and dropping j's (5) becomes $u_i = \dfrac{\alpha}{2(1+\alpha)}(u_{i-1} + u_{i+1}) + \dfrac{b_i}{1+\alpha}$

This gives the iteration formula

$$u_i^{(n+1)} = \frac{\alpha}{2(1+\alpha)}\{u_{i-1}^{(n)} + u_{i+1}^{(n)}\} + \frac{b_i}{1+\alpha} \qquad ...(6)$$

which expresses the $(n+1)$th iterates in terms of the nth iterates only. This is known as the *Jacobi's iteration formula*.

As the latest value of u_{i-1} *i.e.* $u_{i-1}^{(n+1)}$ is already available, the convergence of the iteration formula (6) can be improved by replacing $u_{i-1}^{(n)}$ by $u_{i-1}^{(n+1)}$. Accordingly (6) may be written as

$$u_i^{(n+1)} = \frac{\alpha}{2(1+\alpha)}\{u_{i-1}^{(n+1)} + u_{i+1}^{(n)}\} + \frac{b_i}{1+\alpha} \qquad ...(7)$$

which is known as *Gauss-Seidal iteration formula*.

Obs. Gauss-Seidal iteration scheme is valid for all finite values of α and converges twice as fast as Jacobi's scheme.

(4) Du Fort and Frankel method. If we replace the derivatives in (1) by the central-difference approximations,

$$\frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j-1}}{2k}$$ [From (7) § 11.3

and $$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$ [From (4) § 11.3

we obtain $$u_{i,j+1} - u_{i,j-1} = \frac{2kc^2}{h^2}\,[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}]$$

i.e. $$u_{i,j+1} = u_{i,j-1} + 2\alpha\,[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}] \quad ...(7)$$

where $\alpha = kc^2/h^2$. This difference equation is called the *Richardson scheme* which is a *3-level method*.

If we replace $u_{i,j}$ by the mean of the values $u_{i,j-1}$ and $u_{i,j+1}$ i.e. $u_{i,j} = \frac{1}{2}(u_{i,j-1} + u_{i,j+1})$ in (7), then we get

$$u_{i,j+1} = u_{i,j-1} + 2\alpha[u_{i-1,j} - (u_{i,j-1} + u_{i,j+1}) + u_{i+1,j}]$$

On simplification, it can be written as

$$u_{i,j+1} = \frac{1-2\alpha}{1+2\alpha}\,u_{i,j-1} + \frac{2\alpha}{1+2\alpha}\,\{u_{i-1,j} + u_{i+1,j}\} \quad ...(8)$$

This difference scheme is called *Du Fort-Frankel method* which is a *3-level explicit method*. Its computational model is given in Fig. 11.27.

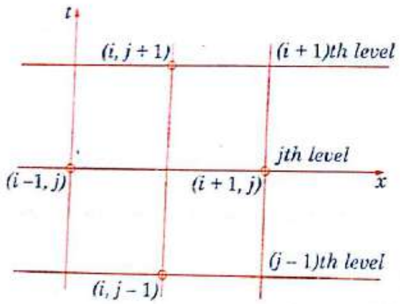


Fig. 11.27

Example 11.9. Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ *in $0 < x < 5$, $t \geq 0$ given that $u(x, 0) = 20$, $u(0, t) = 0$, $u(5, t) = 100$. Compute u for the time-step with $h = 1$ by Crank-Nicholson method.*

(Anna, B. Tech., 2006)

Sol. Here $c^2 = 1$ and $h = 1$.

Taking α (i.e. c^2k/h) = 1, we get $k = 1$.

Also we have

j \ i	0	1	2	3	4	5
0	0	20	20	20	20	100
1	0	u_1	u_2	u_3	u_4	100

Then Crank-Nicholson formula becomes

$$4u_{i,j+1} = u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}$$

$\therefore$ $4u_1 = 0 + 20 + 0 + u_2$ i.e. $4u_1 - u_2 = 20$ ...(1)

$4u_2 = 20 + 20 + u_1 + u_3$ i.e. $u_1 - 4u_2 + u_3 = -40$ ...(2)

$4u_3 = 20 + 20 + u_2 + u_4$ i.e. $u_2 - 4u_3 + u_4 = -40$ ...(3)

$4u_4 = 20 + 100 + u_3 + 100$ i.e. $u_3 - 4u_4 = -220$ ...(4)

Now (1) – 4(2) gives $15u_2 - 4u_3 = 180$ ...(5)

4(3) + (4) gives $4u_2 - 15u_3 = -380$ ...(6)

Then 15(5) – 4(6) gives $209\,u_2 = 4220$ i.e. $u_2 = 20.2$

From (5), we get $4u_3 = 15 \times 20.2 - 180$ i.e. $u_3 = 30.75$

From (1), $4u_1 = 20 + 20.2$ i.e. $u_1 = 10.05$

From (4), $4u_4 = 220 + 30.75$ i.e. $u_4 = 62.69$

Thus the required values are 10.05, 20.2, 30.75 and 62.68.

Example 11.10. *Solve the boundary value problem $u_t = u_{xx}$ under the conditions $u(0, t) = u(1, t) = 0$ and $u(x, 0) = \sin \pi x$, $0 \leq x \leq 1$ using Schmidt method (Take $h = 0.2$ and $\alpha = 1/2$).*

(Rohtak, B. Tech., 2003)

Sol. Since $h = 0.2$ and $\alpha = 1/2$

$\therefore$ $\alpha = \frac{k}{h^2}$ gives $k = 0.02$

Since $\alpha = 1/2$, we use Bendre-Schmidt relation

$$u_{i,j+1} = \frac{1}{2}\,(i_{i-1,j} + u_{i+1,j}) \quad ...(i)$$

We have $u(0, 0) = 0$, $u(0.2, 0) = \sin \pi/5 = 0.5875$

$u(0.4, 0) = \sin 2\pi/5 = 0.9511$, $u(0.6, 0) = \sin 3\pi/5 = 0.9511$

$u(0.8, 0) = \sin 4\pi/5 = 0.5875$, $u(1, 0) = \sin \pi = 0$

The values of u at the mesh points can be obtained by using the recurrence relation (i) as shown in table below :

$x \longrightarrow$		0	0.2	0.4	0.6	0.8	1.0
$t \downarrow$	j \ i	0	1	2	3	4	5
0	0	0	0.5878	0.9511	0.9511	0.5878	0
0.02	1	0	0.4756	0.7695	0.7695	0.4756	0
0.04	2	0	0.3848	0.6225	0.6225	0.3848	0
0.06	3	0	0.3113	0.5036	0.5036	0.3113	0
0.08	4	0	0.2518	0.4074	0.4074	0.2518	0
0.1	5	0	0.2037	0.3296	0.3296	0.2037	0

Example 11.11. *Find the values of $u(x, t)$ satisfying the parabolic equation $\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$ and the boundary conditions $u(0, t) = 0 = u(8, t)$ and $u(x, 0) = 4x - \frac{1}{2}x^2$ at the points $x = i$: $i = 0, 1, 2, ..., 7$ and $t = \frac{1}{8}j : j = 0, 1, 2, ..., 5$.*

Sol. Here $c^2 = 4$, $h = 1$ and $k = 1/8$. Then $\alpha = c^2k/h^2 = 1/2$.

$\therefore$ We have Bendre-Schmidt's recurrence relation $u_{i,j+1} = \frac{1}{2}(u_{i-1,j} + u_{i+1,j})$ ...(i)

Now since $u(0, t) = 0 = u(8, t)$

$\therefore$ $u_{0,i} = 0$ and $u_{8,j} = 0$ for all values of j, *i.e.* the entries in the first and last column are zero.

Since $u(x, 0) = 4x - \frac{1}{2}x^2$

$\therefore$ $u_{i,0} = 4i - \frac{1}{2}i^2$

$= 0, 3.5, 6, 7.5, 8, 7.5, 6, 3.5$ for $i = 0, 1, 2, 3, 4, 5, 6, 7$ at $t = 0$

These are the entries of the first row.

Putting $j = 0$ in (i), we have $u_{i,1} = \frac{1}{2}(u_{i-1,0} + u_{i+1,0})$

Taking $i = 1, 2, ..., 7$ successively, we get

$$u_{1,1} = \frac{1}{2}(u_{0,0} + u_{2,0}) = \frac{1}{2}(0 + 6) = 3$$

$$u_{2,1} = \frac{1}{2}(u_{1,0} + u_{3,0}) = \frac{1}{2}(3.5 + 7.5) = 5.5$$

$$u_{3,1} = \frac{1}{2}(u_{2,0} + u_{4,0}) = \frac{1}{3}(6 + 8) = 7$$

$$u_{4,1} = 7.5, u_{5,1} = 7, u_{6,1} = 5.5, u_{7,1} = 3.$$

These are the entries in the second row.

Putting $j = 1$ in (i), the entries of the third row are given by

$$u_{1,2} = \frac{1}{2}(u_{i-1,1} + u_{i+1,1})$$

Similarly putting $j = 2, 3, 4$ successively in (i), the entries of the fourth, fifth and sixth rows are obtained.

Hence the values of $u_{i,j}$ are as given in the following table :

j \ i	0	1	2	3	4	5	6	7	8
0	0	3.5	6	7.5	8	7.5	6	3.5	0
1	0	3	5.5	7	7.5	7	5.5	3	0
2	0	2.75	5	6.5	7	6.5	5	2.75	0
3	0	2.5	4.625	6	6.5	6	4.625	2.5	0
4	0	2.3125	4.25	5.5625	6	5.5625	4.25	2.3125	0
5	0	2.125	3.9375	5.125	5.5625	5.125	3.9375	2.125	0

Example 11.12. *Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ subject to the conditions $u(x, 0) = \sin \pi x$, $0 \le x \le 1$; $u(0, t) = u(1, t) = 0$, using (a) Schmidt method, (b) Crank-Nicolson method, (c) Du Fort-Frankel method. Carryout computations for two levels, taking $h = 1/3$, $k = 1/36$.* (V.T.U., B. Tech., 2005)

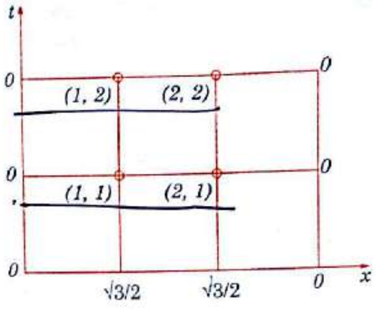


Fig. 11.28

Sol. Here $c^2 = 1$, $h = 1/3$, $k = 1/36$ so that $\alpha = kc^2/h^2 = 1/4$.

Also $u_{1,0} = \sin \pi/3 = \sqrt{3}/2$, $u_{2,0} = \sin 2\pi/3 = \sqrt{3}/2$

and all boundary values are zero as shown in Fig. 11.28.

(a) *Schmidt's formula* [(2) of § 11.9]

$$u_{i,j+1} = \alpha\, u_{i-1,j} + (1 - 2\alpha)\, u_{i,j} + \alpha\, u_{i+1,j}$$

becomes $u_{i,j+1} = \frac{1}{4}[u_{i-1,j} + 2u_{i,j} + u_{i+1,j}]$

For $i = 1, 2$; $j = 0$:

$$u_{1,1} = \frac{1}{4}[u_{0,0} + 2u_{1,0} + u_{2,0}] = \frac{1}{4}(0 + 2 \times \sqrt{3}/2 + 3/2) = 0.65$$

$$u_{2,1} = \frac{1}{2}[u_{1,0} + 2u_{2,0} + u_{3,0}] = \frac{1}{4}(\sqrt{3}/2 + 2 \times \sqrt{3}/2 + 0) = 0.65$$

For $i = 1, 2, j = 1$:

$$u_{1,2} = \frac{1}{4}(u_{0,1} + 2u_{1,1} + u_{2,1}) = 0.49$$

$$u_{2,2} = \frac{1}{4}(u_{1,1} + 2u_{2,1} + u_{3,1}) = 0.49$$

(b) *Crank-Nicolson formula* [(4) of § 11.9] becomes

$$-\frac{1}{4}u_{i-1,j+1} + \frac{5}{2}u_{i,j+1} - \frac{1}{4}u_{i+1,j+1} = \frac{1}{4}u_{i-1,j} + \frac{3}{2}u_{i,j} + \frac{1}{4}u_{i+1,j}$$

For $i = 1, 2$; $j = 0$:

$$-u_{0,1} + 10u_{1,1} - u_{2,1} = u_{0,0} + 6u_{1,0} + u_{2,0}$$

i.e. $10\,u_{1,1} - u_{2,1} = 7\sqrt{3}/2$

$$-u_{1,1} + 10u_{2,1} - u_{3,1} = u_{1,0} + 6u_{2,0} + u_{3,0}$$

i.e. $-u_{1,1} + 10u_{2,1} = 7\sqrt{3}/2$

Solving these equations, we find

$$u_{1,1} = u_{2,1} = 0.67$$

For $i = 1, 2$; $j = 1$:

$$-u_{0,2} + 10u_{1,2} - u_{2,2} = u_{0,1} + 6u_{1,1} + u_{2,1}$$

i.e. $10u_{1,2} - u_{2,2} = 4.69$

$$-u_{1,2} + 10u_{2,2} - u_{3,2} = u_{1,1} + 6u_{2,1} + u_{3,1}$$

i.e. $$-u_{1,2} + 10u_{2,2} = 4.69$$

Solving these equations, we get $u_{1,2} = u_{2,2} = 0.52$.

(c) *Du Fort-Frankel formula* [(8) of § 11.9] becomes $u_{i,j+1} = \frac{1}{3}(u_{i,j-1} + u_{i-1,j} + u_{i+1,j})$

To start the calculations, we need $u_{1,1}$ and $u_{2,1}$.

We may take $u_{1,1} = u_{2,1} = 0.65$ from Schmidt method.

For $i = 1, 2; j = 1$:

$$u_{1,2} = \frac{1}{3}(u_{1,0} + u_{0,1} + u_{2,1}) = \frac{1}{3}(\sqrt{3}/2 + 0 + 0.65) = 0.5$$

$$u_{2,2} = \frac{1}{3}(u_{2,0} + u_{1,1} + u_{3,1}) = \frac{1}{3}(\sqrt{3}/2 + 0.65 + 0) = 0.5.$$

11.10. SOLUTION OF TWO DIMENSIONAL HEAT EQUATION

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad ...(1)$$

The methods employed for the solution of one dimensional heat equation can be readily extended to the solution of (1).

Consider a square region $0 \le x \le y \le a$ and assume that u is known at all points within and on the boundary of this square.

If h be the step-size then a mesh point $(x, y, t) = (ih, jh, nl)$ may be denoted as simply (i, j, n).

Replacing the derivatives in (1) by their finite difference approximations, we get

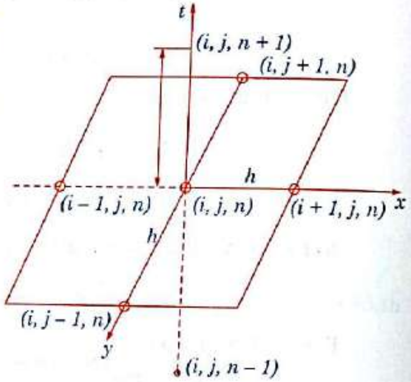


Fig. 11.29

$$\frac{u_{i,j,n+1} - u_{i,j,n}}{l}$$

$$= \frac{c^2}{h^2}[(u_{i-1,j,n} - 2u_{i,j,n} + u_{i+1,j,n})$$

$$+ (u_{i,j-1,n} - 2u_{i,j,n} + u_{i,j+1,n})]$$

i.e. $$u_{i,j,n+1} = u_{i,j,n} + \alpha(u_{i-1,j,n} + u_{i+1,j,n} + u_{i,j+1,n} + u_{i,j-1,n} - 4u_{i,j,n}) \quad ...(2)$$

where $\alpha = lc^2/h^2$. This equation needs the five points available on the nth plane (Fig. 11.29).

The computation process consists of point-by-point evaluation in the $(n + 1)$th plane using the points on the nth plane. It is followed by plane-by-plane evaluation. This method is known as *ADE (Alternating Direction Explicit) method*.

Example 11.13. *Solve the equation* $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ *subject to the initial conditions* $u(x, y, 0) = \sin 2\pi x \sin 2\pi y$, $0 \le x, y \le 1$, *and the conditions* $u(x, y, t) = 0$, $t > 0$ *on the boundaries, using ADE method with* $h = \frac{1}{3}$ *and* $\alpha = \frac{1}{8}$. *(Calculate the results for one time level).*

Sol. The equation (2) above becomes

$$u_{i,j,n+1} = u_{i,j,n} + \frac{1}{8}(u_{i-1,j,n} + u_{i+1,j,n} + u_{i,j+1,n} + u_{i,j-1,n} - 4u_{i,j,n})$$

i.e. $$u_{i,j,n+1} = \frac{1}{2}u_{i,j,n} + \frac{1}{8}(u_{i-1,j,n} + u_{i+1,j,n} + u_{i,j+1,n} + u_{i,j-1,n}) \quad ...(1)$$

The mesh points and the computational model is given in Fig. 11.30.

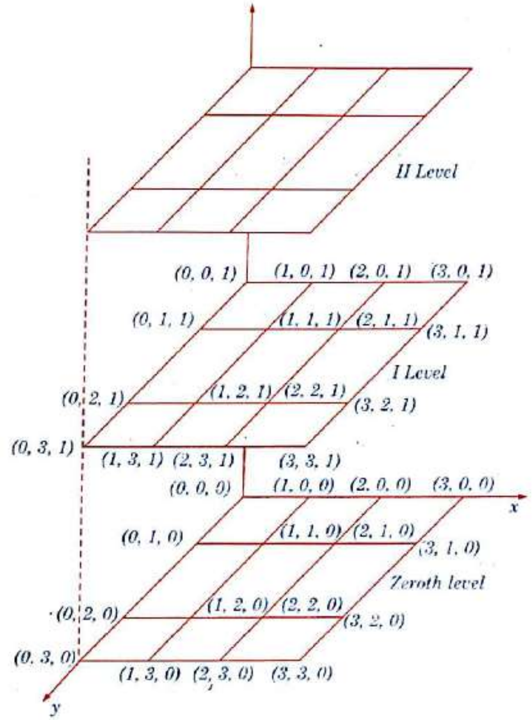


Fig. 11.30

At the zeroth level $(n = 0)$, the initial and boundary conditions are

$$u_{i,j,0} = \sin \frac{2\pi i}{3} \sin \frac{2\pi j}{3}$$

and
$$u_{i,0,0} = u_{0,j,0} = u_{3,j,0} = u_{i,3,0} = 0\ ;\ i, j = 0, 1, 2, 3.$$

Now we calculate the mesh values at the first level :

For $n = 0$, (1) gives

$$u_{i,j,1} = \tfrac{1}{2} u_{i,j,0} + \tfrac{1}{8} (u_{i-1,j,0} + u_{i+1,j,0} + u_{i,j+1,0} + u_{i,j-1,0}) \qquad ...(2)$$

(*i*) Put $i = j = 1$ in (2) :

$$u_{1,1,1} = \tfrac{1}{2} u_{1,1,0} + \tfrac{1}{8} (u_{0,1,0} + u_{2,1,0} + u_{1,2,0} + u_{1,0,0})$$

$$= \frac{1}{2}\left(\sin \frac{2\pi}{3}\right)^2 + \frac{1}{8}\left(0 + \sin \frac{4\pi}{3} \sin \frac{2\pi}{3} + \sin \frac{2\pi}{3} \sin \frac{4\pi}{3} + 0\right)$$

$$= \frac{3}{8} + \frac{1}{8}\left(-\frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2}\right) = \frac{3}{16}$$

(*ii*) Put $i = 2, j = 1$ in (2)

$$u_{2,1,1} = \tfrac{1}{2} u_{2,1,0} + \tfrac{1}{8} (u_{1,1,0} + u_{3,1,0} + u_{2,2,0} + u_{2,0,0})$$

$$= \frac{1}{2} \sin \frac{4\pi}{3} \sin \frac{2\pi}{3} + \frac{1}{8}\left\{\left(\sin \frac{2\pi}{3}\right)^2 + 0 + \left(\sin \frac{4\pi}{3}\right)^2 + 0\right\}$$

$$= -\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{8}\left\{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2\right\} = -\frac{3}{16}.$$

(*iii*) Put $i = 1, j = 2$ in (2) :

$$u_{1,2,1} = \tfrac{1}{2} u_{1,2,0} + \tfrac{1}{8} (u_{0,2,0} + u_{2,2,0} + u_{1,1,0})$$

$$= \frac{1}{2} \sin \frac{2\pi}{3} \sin \frac{4\pi}{3} + \frac{1}{8}\left\{0 + \left(\sin \frac{4\pi}{3}\right)^2 + 0 + \left(\sin \frac{2\pi}{3}\right)^2\right\}$$

$$= -\frac{3}{8} + \frac{1}{8}\left(\frac{3}{4} + \frac{3}{4}\right) = -\frac{3}{16}$$

(*iv*) Put $i = 2, j = 2$ in (2) :

$$u_{2,2,1} = \tfrac{1}{2} u_{2,2,0} + \tfrac{1}{8} (u_{1,2,0} + u_{3,2,0} + u_{2,3,0} + u_{2,1,0})$$

$$= \frac{1}{2}\left(\sin \frac{4\pi}{3}\right)^2 + \frac{1}{8}\left(\sin \frac{2\pi}{3} \sin \frac{4\pi}{3} + 0 + 0 + \sin \frac{4\pi}{3} \sin \frac{2\pi}{3}\right)$$

$$= \frac{3}{8} + \frac{1}{8}\left(-\frac{3}{4} - \frac{3}{4}\right) = -\frac{3}{16}$$

Similarly the mesh values at the second and higher levels can be calculated.

PROBLEMS 11.4

1. Find the solution of the parabolic equation $u_{xx} = 2u_t$ when $u(0, t) = u(4, t) = 0$ and $u(x, 0) = x(4 - x)$, taking $h = 1$. Find the values upto $t = 5$. *(Madras, B.E., 2001)*

2. Solve the equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$ with the conditions $u(0, t) = 0$, $u(x, 0) = x(1 - x)$ and $u(1, t) = 0$. Assume $h = 0.1$. Tabulate u for $t = k$, $2k$ and $3k$ choosing an appropriate value of k. *(Anna, B.E., 2004)*

3. Given $\dfrac{\partial^2 f}{\partial x^2} - \dfrac{\partial f}{\partial t} = 0$; $f(0, t) = f(5, t) = 0$, $f(x, 0) = x^2(25 - x^2)$; find the values of f for $x = ih$ $(i = 0, 1, ..., 5)$ and $t = jk$ $(j = 0, 1, ..., 6)$ with $h = 1$ and $k = \frac{1}{2}$, using the explicit method.

4. Given $\partial u/\partial t = \partial^2 u/\partial t^2$, $u(0, t) = 0$, $u(4, t) = 0$ and $u(x, 0) = \dfrac{x}{3}(16 - x^2)$. Obtain $u_{i,j}$ for $10 = 1, 2, 3, 4$ and $j = 1, 2$ using Crank-Nicholson's method.

5. Solve the heat equation $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ subject to the conditions $u(0, t) = u(1, t) = 0$ and
$$u(x, 0) = \begin{cases} 2x & \text{for } 0 \le x \le 1/2 \\ 2(1 - x) & \text{for } 1/2 \le x \le 1 \end{cases}$$
Take $h = 1/4$ and k according to Bandre-Schmidt equation.

6. Solve the 2-dimensional heat equation $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$ satisfying the initial condition : $u(x, y, 0) = \sin \pi x \sin \pi y$, $0 \le x, y \le 1$ and the boundary conditions : $u = 0$ at $x = 0$ and $x = 1$ for $t > 0$. Obtain the solution upto two time levels with $h = \frac{1}{3}$ and $\alpha = \frac{1}{8}$.

11.11. HYPERBOLIC EQUATIONS

The wave equation $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ is the simplest example of hyperbolic partial differential equations. Its solution is the displacement function $u(x, t)$ defined for values of x from 0 to l and for t from 0 to ∞, satisfying the initial and boundary conditions. The solution, as for parabolic equations, advances in an open-ended region (Fig. 11.24). In the case of hyperbolic equations however, we have two initial conditions and two boundary conditions.

Such equations arise from convective type of problems in vibrations, wave mechanics and gas dynamics.

11.12. SOLUTION OF WAVE EQUATION

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad ...(1)$$

subject to the initial conditions : $u = f(x)$, $\dfrac{\partial u}{\partial t} = g(x)$, $0 \le x \le 1$ at $t = 0$ ...(2)

and the boundary conditions : $u(0, t) = \phi(t)$, $u(1, t) = \psi(t)$ ...(3)

Consider a rectangular mesh in the x-t plane spacing h along x direction and k along time direction. Denoting a mesh point $(x, t) = (ih, jk)$ as simply i, j, we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} \quad \text{and} \quad \frac{\partial^2 u}{\partial t^2} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h^2}$$

Replacing the derivatives in (1) by their above approximations, we obtain

$$u_{i,j-1} - 2u_{i,j} + u_{i,j+1} = \frac{c^2k^2}{h^2}(u_{i-1,j} - 2u_{i,j} + u_{i,j+1})$$

or
$$u_{i,j+1} = 2(1 - \alpha^2c^2)\,u_{i,j} + \alpha^2c^2(u_{i-1,j} + u_{i+1,j}) - u_{i,j-1} \qquad ...(4)$$

where $\alpha = k/h$.

Now replacing the derivative in (2) by its central difference approximation, we get

$$\frac{u_{i,j+1} - u_{i,j-1}}{2k} = \frac{\partial u}{\partial t} = g(x)$$

or $\quad u_{i,j+1} = u_{i,j-1} + 2kg(x)$ at $t = 0$

i.e. $\quad u_{i,1} = u_{i,-1} + 2kg(x)$ for $j = 0$ ...(5)

Also initial condition $u = f(x)$ at $t = 0$ becomes $u_{i,-1} = f(x)$ ...(6)

Combining (5) and (6), we have $u_{i,1} = f(x) + 2kg(x)$ ...(7)

Also (3) gives $u_{0,j} = \phi(t)$ and $u_{1,j} = \psi(t)$.

Hence the explicit form (4) gives the values of $u_{i,j+1}$ at the $(j+1)$th level when the nodal values at $(j-1)$th and jth levels are known from (6) and (7) as shown in Fig 11.31.

Obs. 1. The coefficient of $u_{i,j}$ in (4) will vanish if $\alpha c = 1$ or $k = h/c$. Then (4) reduces to the simple form

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1} \qquad ...(8)$$

2. For $\alpha = 1/c$, the solution of (4) is stable and coincides with the solution of (1).

For $\alpha < 1/c$, the solution is stable but inaccurate.

For $\alpha > 1/c$, the solution is unstable.

3. The formula (4) converges for $\alpha \le 1$ i.e. $k \le h$.

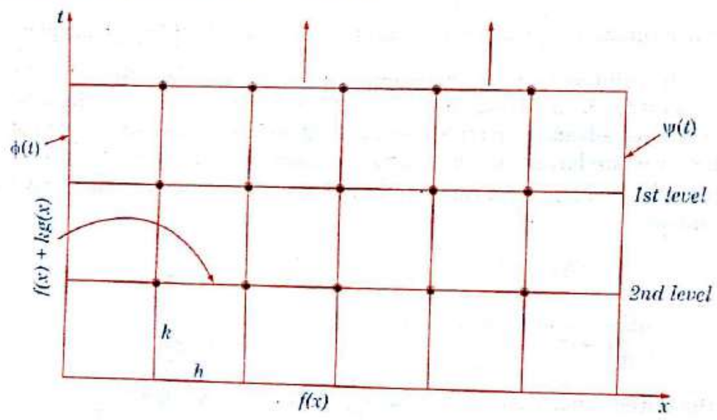


Fig. 11.31

Example 11.14. *Evaluate the pivotal values of the equation $u_{tt} = 16u_{xx}$, taking $\Delta x = 1$ upto $t = 1.25$. The boundary conditions are $u(0, t) = u(5, t) = 0$, $u_t(x, 0) = 0$ and $u(x, 0) = x^2(5 - x)$.* (Madras, B.E., 2006)

Sol. Here $c^2 = 16$.

∴ The difference equation for the given equation is

$$u_{i,j+1} = 2(1 - 16\alpha^2)\,u_{i,j} + 16\alpha^2(u_{i-1,j} + u_{i+1,j}) - u_{i,j-1} \qquad ...(i)$$

where $\alpha = k/h$.

Taking $h = 1$ and choosing k so that the coefficient of $u_{i,j}$ vanishes, we have $16\alpha^2 = 1$, i.e. $k = h/4 = 1/4$.

∴ (1) reduces to $u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$ ...(ii)

which gives a convergent solution (since $k/h < 1$). Its solution coincides with the solution of the given differential equation.

Now since $u(0, t) = u(5, t) = 0$, ∴ $u_{0,j} = 0$ and $u_{5,j} = 0$ for all values of j

i.e. the entries in the first and last columns are zero.

Since $\quad u_{(x,0)} = x^2(5 - x)$

∴ $\quad u_{i,0} = i^2(5 - i) = 4, 12, 18, 16$ for $i = 1, 2, 3, 4$ at $t = 0$.

These are the entries for the first row.

Finally since $u_{t(x,0)} = 0$ becomes

∴ $$\frac{u_{i,j+1} - u_{i,j-1}}{2k} = 0, \text{ when } j = 0, \text{ giving } u_{i,1} = u_{i,-1} \qquad ...(iii)$$

Thus the entries of the second row are the same as those of the first row.

Putting $j = 0$ in (ii), $\quad u_{i,1} = u_{i-1,0} + u_{i+1,0} - u_{i,-1} = u_{i-1,0} + u_{i+1,0} - u_{i,1}$, using (iii)

or $$u_{i,1} = \frac{1}{2}(u_{i-1,0} + u_{i+1,0}) \qquad ...(iv)$$

Taking $i = 1, 2, 3, 4$ successively, we obtain

$$u_{1,1} = \frac{1}{2}(u_{0,0} + u_{2,0}) = \frac{1}{2}(0 + 12) = 6$$
$$u_{2,1} = \frac{1}{2}(u_{1,0} + u_{3,0}) = \frac{1}{2}(4 + 18) = 11$$
$$u_{3,1} = \frac{1}{2}(u_{2,0} + u_{4,0}) = \frac{1}{2}(12 + 16) = 14$$
$$u_{4,1} = \frac{1}{2}(u_{3,0} + u_{5,0}) = \frac{1}{2}(18 + 0) = 9$$

These are the entries of the *second row*.

Putting $j = 1$ in (ii), we get $u_{i,2} = u_{i-1,1} + u_{i+1,1} - u_{i,0}$

Taking $i = 1, 2, 3, 4$ successively, we obtain

$$u_{1,2} = u_{0,1} + u_{2,1} - u_{1,0} = 0 + 11 - 4 = 7$$
$$u_{2,2} = u_{1,1} + u_{3,1} - u_{2,0} = 6 + 14 - 12 = 8$$
$$u_{3,2} = u_{2,1} + u_{4,1} - u_{3,0} = 11 + 9 - 18 = 2$$
$$u_{4,2} = u_{3,1} + u_{5,1} - u_{4,0} = 14 + 0 - 16 = -2$$

These are the entries of the *third row*.

Similarly putting $j = 2, 3, 4$ successively in (ii), the entries of the fourth, fifth and sixth rows are obtained.

Hence the values of $u_{i,j}$ are as shown in the table below :

j \ i	0	1	2	3	4	5
0	0	4	12	18	16	0
1	0	6	11	14	9	0
2	0	7	8	2	– 2	0
3	0	2	– 2	– 8	– 7	0
4	0	– 9	– 14	– 11	– 6	0
5	0	– 16	– 18	– 12	– 4	0

Example 11.15. *Solve $y_{tt} = y_{xx}$ upto $t = 0.5$ with a spacing of 0.1 subject to $y(0, t) = 0$, $y(1, t) = 0$, $y_t(x, 0) = 0$ and $y(x, 0) = 10 + x(1 - x)$.* *(Anna, B.E., 2004)*

Sol. As $c^2 = 1$, $h = 0.1$, $k = \frac{h}{c} = 0.1$; we use the formula

$$u_{i,j+1} = y_{i-1,j} + y_{i+1,j} - y_{i,j-1} \qquad ...(i)$$

Since $y(0, t) = 0, y(1, t) = 0$, $\therefore$ $y_{0,j} = 0, y_{1,j} = 0$ for all values of i.

i.e., all the entries in the first and last columns are zero.

Since $y(x, 0) = 10 + x(1 - x)$, $\therefore$ $y_{i,0} = 10 + i(1 - i)$

$\therefore$ $y_{0.1,0} = 10.09, y_{0.2,0} = 10.16, y_{0.3,0} = 10.21, y_{0.4,0} = 10.24$

$\therefore$ $y_{0.5,0} = 10.25, y_{0.6,0} = 10.24, y_{0.7,0} = 10.21, y_{0.8,0} = 10.16, y_{0.9,0} = 10.09$

These are the entries of the first row.

Since $y_t(x, 0) = 0$, we have $\frac{1}{2}(y_{i,j+1} - y_{i,j-1}) = 0$ ...(ii)

When $j = 0, y_{i,1} = y_{i,-1}$

Putting $j = 0$ in (i), $y_{i,1} = y_{i-1,0} + y_{i+1,0} - y_{i,-1}$

Using (ii), $y_{i,1} = \frac{1}{2}(y_{i-1,0} + y_{i+1,0})$

Taking $i = 1, 2, 3$ ......, 9 successively, we obtain the entries of the second row.

Putting $j = 1$ in (i), $y_{i,2} = y_{i-1,1} + y_{i+1,1} - y_{i,0}$

Taking $i = 1, 2, 3$, ..., 9 successively, we get the entries of the third row.

Similarly putting $j = 2, 3$, ..., 7 successively in (i), the entries of the fourth to ninth row are obtained. Hence the values of $u_{i,j}$ are as given in the table below :

j \ i	0	1	2	3	4	5	6	7	8	9	10
0	0	10.19	10.16	10.21	10.24	10.25	10.24	10.21	10.16	10.09	0
1	0	5.08	10.15	10.20	10.23	10.24	10.23	10.20	10.15	5.08	0
2	0	0.06	5.12	10.17	10.20	10.21	10.20	10.17	10.12	0.06	0
3	0	0.04	0.08	5.12	10.15	10.16	10.15	10.12	10.08	0.04	0
4	0	0.02	0.04	0.06	5.08	10.09	10.08	10.06	10.04	0.02	0
5	0	0	0	0	0	0	0	0	0	– 0.02	0

Example 11.16. *The transverse displacement u of a point at a distance x from one end and at any time t of a vibrating string satisfies the equation $\partial^2 u/\partial t^2 = 4\partial^2 u/\partial x^2$, with boundary conditions $u = 0$ at $x = 0, t > 0$ and $u = 0$ at $x = 4, t > 0$ and initial conditions $u = x(4 - x)$ and $\partial u/\partial t = 0$, $0 \le x \le 4$. Solve this equation numerically for one half period of vibration, taking $h = 1$ and $k = 1/2$.*

Sol. Here, $h/k = 2 = c$.

$\therefore$ The difference equation for the given equation is

$$u_{i,j+1} = u_{i-1,j}\, u_{i+1,j} - u_{i,j-1} \qquad ...(i)$$

which gives a convergent solution (since $k < h$).

Now since $u(0, t) = u(4, t) = 0$,

$\therefore$ $u_{0,j} = 0$ and $u_{4,j} = 0$ for all values of j.

i.e. the entries in the first and last columns are zero.

Since $u_{(x,0)} = x(4 - x)$,

$\therefore$ $u_{i,0} = i(4 - i) = 3, 4, 3$ for $i = 1, 2, 3$ at $t = 0$.

These are the entries of the *first row*.

Also $u_t(x, 0) = 0$ becomes

$$\frac{u_{i,j+1} - u_{i,j-1}}{2k} = 0 \text{ when } j = 0, \text{ giving } u_{i,1} = u_{i,-1} \qquad ...(ii)$$

Putting j = 0 in (i), $u_{i,1} = u_{i-1,0} + u_{i+1,0} - u_{i,-1} = u_{i-1,0} + u_{i+1,0} - u_{i,1}$, using (ii)

or $u_{i,1} = \frac{1}{2}(u_{i-1,0} + u_{i+1,0})$ ...(iii)

Taking $i = 1, 2, 3$ successively, we obtain

$u_{1,1} = \frac{1}{2}(u_{0,0} + u_{1,0}) = 2$; $u_{2,1} = \frac{1}{2}(u_{1,0} + u_{3,0}) = 3$, $u_{3,1} = \frac{1}{2}(u_{2,0} + u_{4,0}) = 2$

These are the entries of the *2nd row*.

Putting j = 1 in (i), $u_{i,2} = u_{i-1,1} + u_{i+1,1} - u_{i,0}$

Taking $i = 1, 2, 3$, successively, we get

$$u_{1,2} = u_{0,1} + u_{2,1} - u_{1,0} = 0 + 3 - 3 = 0$$
$$u_{2,2} = u_{1,1} + u_{3,1} - u_{2,0} = 2 + 2 - 4 = 0$$
$$u_{3,2} = u_{2,1} + u_{4,1} - u_{3,0} = 3 + 0 - 3 = 0$$

These are the entries of the *3rd row* and so on.

Now the equation of the vibrating string of length l is $u_{tt} = c^2 u_{xx}$.

$\therefore$ Its period of vibration $= \frac{2l}{c} = \frac{2 \times 4}{2} = 4$ sec. $[\because\ l = 4$ and $c = 2]$

This shows that we have to compute $u_{(x,t)}$ upto $t = 2$

i.e. Similarly we obtain the values of $u_{i,2}$ (4th row) and $u_{i,3}$ (5th row).

Hence the values of $u_{i,j}$ are as shown in the table below :

j \ i	0	1	2	3	4
0	0	3	4	3	0
1	0	2	3	2	0
2	0	0	0	0	0
3	0	– 2	– 3	– 2	0
4	0	– 3	– 4	– 3	0

Example 11.17. *Find the solution of the initial boundary value problem :*

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},\ 0 \le x \le 1 \text{ : subject to the initial conditions } u(x, 0) = \sin \pi x,\ 0 \le x \le 1,$$

$\left(\frac{\partial u}{\partial t}\right)(x, 0) = 0,\ 0 \le x \le 1$ *and the boundary conditions* $u(0, t) = 0$, $u(1, t) = 0$, $t > 0$ *; by using in the (a) the explicit scheme (b) the implicit scheme.* (Anna, B.E., 2007)

Sol. *(a) Explicit scheme*

Take $h = 0.2$, $k = \frac{h}{c} = 0.2$ $[\because\ c = 1]$

$\therefore$ We use the formula $u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$ ...(*i*)

Since $u(0, t) = 0$, $u(1, t) = 0$, $u_{0,j} = 0$, $u_{1,j} = 0$ for all values of j

i.e., the entries in the first and last columns are zero.

Since $u(x, 0) = \sin \pi x$, $u_{i,0} = \sin \pi x$

$\therefore\ u_{1,0} = 0$, $u_{2,0} = \sin(.2\pi) = 0.5878$, $u_{3,0} = \sin(.4\pi) = 0.9511$,

$$u_{4,0} = \sin(.6\pi) = 0.5878.$$

These are the entries of the first row.

Since $u_t(x, 0) = 0$ we have $\frac{1}{2}(u_{i,j+1} - u_{i,j-1}) = 0$, when $j = 0$

i.e., $$u_{i,1} = u_{i,-1} \quad ...(ii)$$

Putting $j = 0$ in (*i*), $u_{i,1} = u_{i-1,0} + u_{i+1,0} - u_{i,-1}$

Using (*ii*) $u_{i,1} = \frac{1}{2}(u_{i-1,0} + u_{i+1,0})$

Taking $i = 1$ 2, 3, 4 successively, we obtain the enteries of the second row.

Putting $j = 1$ in (*i*), $u_{i,2} = u_{i-1,1} + u_{i+1,1} - u_{i,0}$

Now taking $i = 1, 2, 3, 4$ successively, we get the entries of the third row.

Similarly taking $j = 2, j = 3, j = 4$ successively, we obtain the entries of the fourth, fifth and sixth rows respectively.

Hence the values of $u_{i,j}$ are as given in the table below:

j \ i	0	1	2	3	4	5
0	0	0.5878	0.9511	.9511	0.5878	0
1	0	0.4756	0.7695	0.9511	0.7695	0
2	0	0.1817	0.4756	0.5878	0.3633	0
3	0	0	0.0001	– 0.1122	– 0.1816	0
4	0	– 0.1816	– 0.5878	– 0.7694	0.4755	0
5	0	– 0.5878	– 0.9511	– 0.9511	– 0.5878	0

(b) Implicit scheme

We have the formula:

$u_{i,j+1} = 2(1 - \alpha^2 c^2)\, u_{i,j} + \alpha^2 c^2 (u_{i-1,j} + u_{i+1,j}) - u_{i,j-1}$, where $\alpha = k/h$. ...(*i*)

Here $c^2 = 1$, Take $h = 0.25$ and $k = 0.5$ so that $\alpha = k/h = 2$.

$\therefore$ (*i*) reduces to

$$u_{i,j+1} = -6u_{i,j} + 4(u_{i-1,j} + u_{i+1,j}) - u_{i,j-1} \quad ...(ii)$$

Since $\quad u_{(i,0)} = \sin \pi x$

$\therefore \quad u_{(1,0)} = 0.7071$, $u_{(2,0)} = 0.5$, $u_{(3,0)} = 0.7071$

There are the entries of the first row.

Since $u_t(x, 0) = 0$, we have $\frac{1}{2}(y_{i,i+1} - y_{i,i-1}) = 0$, where $j = 0$

$$\therefore \quad y_{i,1} = y_{i,-1} \quad ...(ii)$$

Putting $j = 0$ and using (*iii*), (*ii*) reduces to

$$u_{i,1} = -3\,u_{i,0} + 2(u_{i-1,0} + u_{i+1,0})$$

Now taking $\quad i = 1$, $u_{1,1} = -3\,u_{1,0} + 2(u_{0,0} + u_{2,0}) = -0.1213$

$i = 2$, $u_{2,1} = -3\,u_{2,0} + 2(u_{1,0} + u_{3,0}) = -0.1716$

$i = 3$, $u_{3,1} = -3u_{3,0} + 2(u_{2,0} + u_{4,0}) = -0.1213$

These are the entries of the second row.

Putting $j = 1$, (*ii*) reduces to

$$u_{i,2} = -6u_{i,1} + 4(u_{i-1,1} + u_{i+1,1})$$

Now taking $\quad i = 1$, $u_{1,2} = -6u_{1,1} + 4(u_{0,1} + u_{2,1}) = 0.414$

$i = 2$, $u_{2,2} = -6u_{2,1} + 4(u_{1,1} + u_{3,1}) = 0.0592$

$i = 3$, $u_{3,2} = -6u_{3,1} + 4(u_{2,1} + u_{4,1}) = 0.0414$

These are the entries of the third row.

Putting $j = 2$, (*ii*) reduces to

$$u_{i,3} = -6u_{i,2} + 4(u_{i-1,2} + u_{i+1,2}) - u_{i,1}$$

Now taking $i = 1, u_{1,3} = -6u_{1,2} + 4(u_{0,2} + u_{2,2}) - u_{1,1} = 0.1097$

$i = 2, u_{2,3} = -6u_{2,2} + 4(u_{1,2} + u_{3,2}) - u_{2,1} = 0.1476$

$i = 3, u_{3,3} = -6u_{3,2} + 4(u_{2,2} + u_{4,2}) - u_{3,1} = 0.1097$

These are the entries of the fourth row.

Hence the values of $u_{i,j}$ are as tabulated below :

j \ i	0	1	2	3	4
0	0	0.7071	0.5	0.7071	0
1	0	– 0.1213	– 0.1716	– 0.1213	0
2	0	0.0414	0.0592	0.0414	0

PROBLEMS 11.5

1. Solve the boundary value problem $u_{tt} = u_{xx}$ with the conditions $u(0, t) = u(1, t) = 0$, $u(x, 0) = \frac{1}{2} x(1 - x)$ and $u_t(x, 0) = 0$, taking $h = k = 0.1$ for $0 \le t \le 0.4$. Compare your solution with the exact solution at $x = 0.5$ and $t = 0.3$. *(V.T.U., B.E., 2000)*

2. The transverse displacement of a point at a distance x from one end and at any time t of a vibrating string satisfies the equation $\frac{\partial^2 u}{\partial t^2} = 25 \frac{\partial^2 u}{\partial x^2}$, with the boundary conditions $u(0, t) = u(5, t) = 0$ and the initial conditions $u(x, 0) = \begin{cases} 20x & \text{for } 0 \le x < 1 \\ 5(5 - x) & \text{for } 1 \le x < 5 \end{cases}$ and $u_t(x, 0) = 0$. Solve this equation numerically for one half period of vibration, taking $h = 1$, $k = 0.2$.

3. The function u satisfies the equation $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ and the conditions : $u(x, 0) = \frac{1}{8} \sin \pi x$, $u_t(x, 0) = 0$ for $0 \le x \le 1$, $u(0, t) = u(1, t) = 0$ for $t \ge 0$.
Use the explicit scheme to calculate u for $x = 0(0.1)\,1$ and $t = 0(0.1)\,0.5$.

4. Solve $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$, $0 < x < 1, t > 0$, given $u(x, 0) = u_t(x, 0) = u(0, 1) = 0$ and $u(1, t) = 100 \sin \pi t$.
Compute u for 4 times with $h = 0.25$. *(Anna, B.E., 2003)*

11.13. OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 11.6

1. Which of the following equations is parabolic :
(a) $f_{xy} - f_x = 0$ (b) $f_{xx} + 2f_{xy} + f_{yy} = 0$
(c) $f_{xx} + 2f_{xy} + 4f_{yy} = 0$.
2. $u_{ij} = \frac{1}{4}(u_{i+1,j} - u_{i-1,j} + u_{i,j+1} - u_{i,j-1})$ is Leibmann's five point formula. (True or False)
3. $u_{xx} + 3u_{xy} + u_{yy} = 0$ is classified as ....
4. $\nabla^2 u = f(x, y)$ is known as ....
5. The simplest formula to solve $u_{tt} = \alpha^2 u_{xx}$ is ....
6. The finite difference form of $\partial^2 u/\partial x^2$ is ....
7. Schmidt's finite difference scheme to solve $u_t = c^2 u_{xx}$ is ....
8. The 5-point diagonal formula gives u_{ij} = ...
9. The partial differential equation $(x + 1) u_{xx} - 2(x + 2) u_{xy} + (x + 3) u_{yy} = 0$ is classified as ....
10. $u_{i,j+1} = \frac{1}{2}(u_{i+1,j} + u_{i-1,j})$ is called .... recurrence relation.
11. In terms of difference quotients $4u_{xx} = u_{tt}$ is ....
12. Bendre-Schmidt recurrence relation for one dimensional heat equation is ..... .
13. The diagonal 5-point formula to solve the Laplace equation $u_{xx} + u_{yy} = 0$ is ..... .
14. Crank-Nicholson formula to solve $u_{xx} = au_t$ when $k = ah^2$, is ..... .
15. In the parabolic equation $u_t = \alpha^2 u_{xx}$ if $\lambda = k\alpha^2/h^2$, where $k = \Delta t$, and $h = \Delta x$, then explicit method is stable if λ = ...
16. Bendre-Schmidt recurrence scheme is useful to solve ..... equation.
17. The two methods of solving one-dimensional diffusion (heat) equation are ..... .
18. The finite difference scheme to solve $\nabla^2 u = f(x,y)$ is ..... .
19. The order of error in solving Laplace and Poisson's equations by finite difference method is ....
20. The boundary conditions of one-dimensional wave equation are ....
21. The explicit formula for one-dimensional wave equation with $1 - \lambda^2\alpha^2 = 0$ and $\lambda = k/h$ is ....
22. The general form of Poisson's equation in partial derivatives is ......
23. If u satisfies Laplace equation and $u = 100$ on the boundary of a square, the value of u at an interior grid point is ....
24. The Laplace equation $u_{xx} + u_{yy} = 0$ in difference quotients is ....
25. The equation $yu_{xx} + u_{yy} = 0$ is hyperbolic in the region ....
26. To solve $\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}$ by Bendre-Schmidt method with $h = 1$, the value of k is ....
27. Crank Nicholson's scheme is called an implicit scheme because ....