Cryptography (CS-501)

Mathematics of Cryptography
Part III: Primes and Related Congruence Equations

Dr Samayveer Singh
Assistant Professor
Department of Computer Science & Engineering
National Institute Technology Jalandhar, Punjab, India
samays@nitj.ac.in

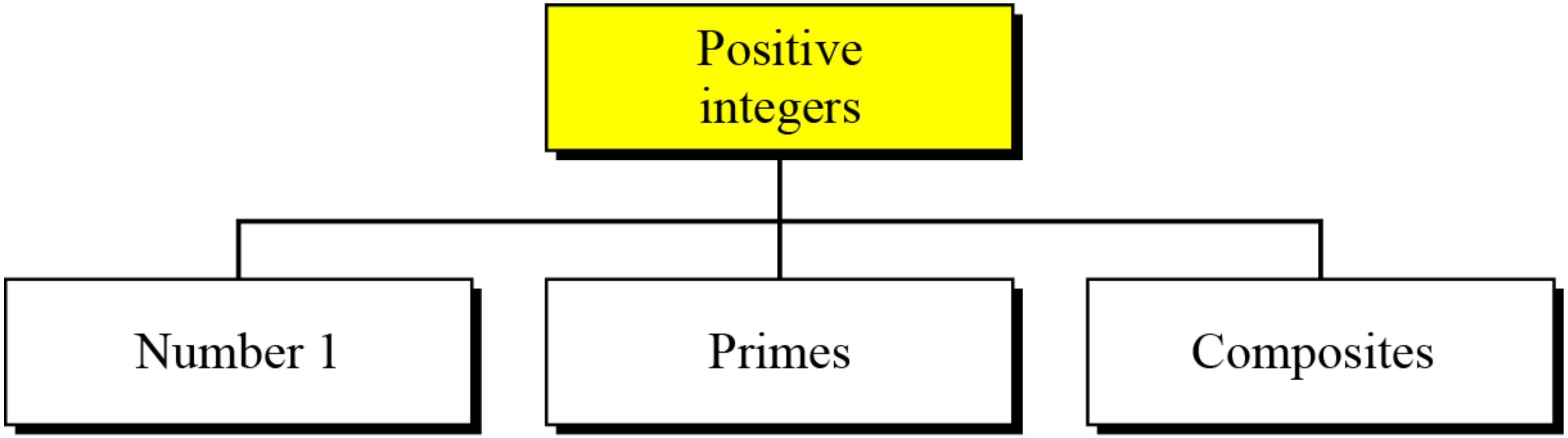
9-1 PRIMES

Asymmetric-key cryptography uses primes extensively. The topic of primes is a large part of the number theory in the Asymmetric-key cryptography .

Topics discussed in this section:

9.1.1 Definition

Figure 9.1 ***Three groups of positive integers***



Exactly one divisor | **Exactly two divisors** | **More than two divisors**

Note

A prime is divisible only by itself and 1.

9.1.1 Continued

Example 9.1

What is the smallest prime?

Solution

The smallest prime is 2, which is divisible by 2 (itself) and 1.

Example 9.2

List the primes smaller than 10.

Solution

There are four primes less than 10: 2, 3, 5, and 7. It is interesting to note that the percentage of primes in the range 1 to 10 is 40%. The percentage decreases as the range increases.

9.1.2 Cardinality of Primes

Infinite Number of Primes

Note

There is an infinite number of primes.

Example 9.3

As a trivial example, assume that the only primes are in the set {2, 3, 5, 7, 11, 13, 17}. Here P = 510510 and P + 1 = 510511. However, 510511 = 19 × 97 × 277; none of these primes were in the original list. Therefore, there are three primes greater than 17.

9.1.2 Continued

Number of Primes

$$[n / (\ln n)] \quad < \quad \pi(n) \quad < \quad [n/(\ln n - 1.08366)]$$

Example 9.4

Find the number of primes less than 1,000,000.

Solution

The approximation gives the range 72,383 to 78,543. The actual number of primes is 78,498.

9.1.3 Checking for Primeness

Given a number n, how can we determine if n is a prime? The answer is that we need to see if the number is divisible by all primes less than

$$\sqrt{n}$$

We know that this method is inefficient, but it is a good start.

9.1.3 Continued

Example 9.5

Is 97 a prime?

Solution

The floor of $\sqrt{97}$ = 9. The primes less than 9 are 2, 3, 5, and 7. We need to see if 97 is divisible by any of these numbers. It is not, so 97 is a prime.

Example 9.6

Is 301 a prime?

Solution

The floor of $\sqrt{301}$ = 17. We need to check 2, 3, 5, 7, 11, 13, and 17. The numbers 2, 3, and 5 do not divide 301, but 7 does. Therefore 301 is not a prime.

9.1.3 Continued

Sieve of Eratosthenes

Table 9.1 *Sieve of Eratosthenes*

	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	79	~~80~~
~~81~~	~~82~~	83	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~
~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~

9.1.4 Euler's Phi-Function

Euler's phi-function, $\phi(n)$, which is sometimes called the Euler's totient function plays a very important role in cryptography.

1. $\phi(1) = 0.$
2. $\phi(p) = p - 1$ if p is a prime.
3. $\phi(m \times n) = \phi(m) \times \phi(n)$ if m and n are relatively prime.
4. $\phi(p^e) = p^e - p^{e-1}$ if p is a prime.

9.1.4 Continued

Example 9.7

What is the value of $\phi(13)$?

Solution

Because 13 is a prime, $\phi(13) = (13 - 1) = 12$.

Example 9.8

What is the value of $\phi(10)$?

Solution

We can use the third rule: $\phi(10) = \phi(2) \times \phi(5) = 1 \times 4 = 4$, because 2 and 5 are primes.

9.1.4 Continued

Example 9.9

What is the value of $\phi(240)$?

Solution

We can write $240 = 2^4 \times 3^1 \times 5^1$. Then

$$\phi(240) = (2^4 - 2^3) \times (3^1 - 3^0) \times (5^1 - 5^0) = 64$$

Example 9.10

Can we say that $\phi(49) = \phi(7) \times \phi(7) = 6 \times 6 = 36$?

Solution

No. The third rule applies when m and n are relatively prime. Here $49 = 7^2$. We need to use the fourth rule: $\phi(49) = 7^2 - 7^1 = 42$.

9.1.4 Continued

Example 9.11

What is the number of elements in Z_{14}*?

Solution

The answer is $\phi(14) = \phi(7) \times \phi(2) = 6 \times 1 = 6$. The members are 1, 3, 5, 9, 11, and 13.

Note

Interesting point: If $n > 2$, the value of $\phi(n)$ is even.

9.1.5 Fermat's Little Theorem

First Version: ***if p is prime and a is positive integer where a is not divisible by p, then***

$$\boxed{a^{p-1} \equiv 1 \bmod p}$$

Second Version: ***if p is prime and a is positive integer, then***

$$\boxed{a^{p} \equiv a \bmod p}$$

9.1.5 Continued

Example 9.12

Find the result of 7^{18} mod 19.

Solution

We have 7^{18} mod 19 = 1. This is the first version of Fermat's little theorem where p = 19.

$$
\begin{aligned}
& a = 7, p = 19 \\
& 7^2 = 49 \equiv 11 \pmod{19} \\
& 7^4 \equiv 121 \equiv 7 \pmod{19} \\
& 7^8 \equiv 49 \equiv 11 \pmod{19} \\
& 7^{16} \equiv 121 \equiv 7 \pmod{19} \\
& a^{p-1} = 7^{18} = 7^{16} \times 7^2 \equiv 7 \times 11 \equiv 1 \pmod{19}
\end{aligned}
$$

Example 9.13

Find the result of 3^{12} mod 11.

Solution

Here the exponent (12) and the modulus (11) are not the same. With substitution this can be solved using Fermat's little theorem.

$$3^{12} \bmod 11 = (3^{11} \times 3) \bmod 11 = (3^{11} \bmod 11)\,(3 \bmod 11) = (3 \times 3) \bmod 11 = 9$$

9.1.5 Continued

Multiplicative Inverses

$$\boxed{a^{-1} \bmod p = a^{p-2} \bmod p}$$

Example 9.14

The answers to multiplicative inverses modulo a prime can be found without using the extended Euclidean algorithm:

a. $8^{-1} \bmod 17 = 8^{17-2} \bmod 17 = 8^{15} \bmod 17 = 15 \bmod 17$

b. $5^{-1} \bmod 23 = 5^{23-2} \bmod 23 = 5^{21} \bmod 23 = 14 \bmod 23$

c. $60^{-1} \bmod 101 = 60^{101-2} \bmod 101 = 60^{99} \bmod 101 = 32 \bmod 101$

d. $22^{-1} \bmod 211 = 22^{211-2} \bmod 211 = 22^{209} \bmod 211 = 48 \bmod 211$

9.1.6 Euler's Theorem

$$\boxed{a^{\phi(n)} \equiv 1 \pmod{n}}$$

Example 9.15

Find the result of 6^{24} mod 35.

Solution

We have 6^{24} mod 35 = $6^{\phi(35)}$ mod 35 = 1.

9.1.6 Continued

Multiplicative Inverses

Euler's theorem can be used to find multiplicative inverses modulo a composite.

$$\boxed{a^{-1} \bmod n = a^{\phi(n)-1} \bmod n}$$

9.1.5 Continued

Example 9.17

The answers to multiplicative inverses modulo a composite can be found without using the extended Euclidean algorithm if we know the factorization of the composite:

a. $8^{-1} \bmod 77 = 8^{\phi(77)-1} \bmod 77 = 8^{59} \bmod 77 = 29 \bmod 77$

b. $7^{-1} \bmod 15 = 7^{\phi(15)-1} \bmod 15 = 7^{7} \bmod 15 = 13 \bmod 15$

c. $60^{-1} \bmod 187 = 60^{\phi(187)-1} \bmod 187 = 60^{159} \bmod 187 = 53 \bmod 187$

d. $71^{-1} \bmod 100 = 71^{\phi(100)-1} \bmod 100 = 71^{39} \bmod 100 = 31 \bmod 100$

9-2 PRIMALITY TESTING

Finding an algorithm to correctly and efficiently test a very large integer and output a prime or a composite has always been a challenge in number theory, and consequently in cryptography. However, recent developments look very promising.

Topics discussed in this section:

9.2.1 Miller-Rabin Algorithms

9.2.2 Continued

```
TEST (n)
1. Find integers k, q, with k > 0, q odd, so that
   (n − 1 = 2^k q);
2. Select a random integer a, 1 < a < n − 1;
3. if a^q mod n = 1 then return("inconclusive");
4. for j = 0 to k − 1 do
5. if a^(2^j q) mod n = n − 1 then return("inconclusive");
6. return("composite");
```

Note

The Miller-Rabin test needs from step 0 to step $k - 1$.

9.2.2 Continued

Example 9.25

Let us apply the test to the prime number $n = 29$. We have $(n - 1) = 28 = 2^2(7) = 2^k q$. First, let us try $a = 10$. We compute $10^7 \bmod 29 = 17$, which is neither 1 nor 28, so we continue the test. The next calculation finds that $(10^7)^2 \bmod 29 = 28$, and the test returns `inconclusive` (i.e., 29 may be prime). Let's try again with $a = 2$. We have the following calculations: $2^7 \bmod 29 = 12$; $2^{14} \bmod 29 = 28$; and the test again returns `inconclusive`. If we perform the test for all integers a in the range 1 through 28, we get the same `inconclusive` result, which is compatible with n being a prime number.

Now let us apply the test to the composite number $n = 13 \times 17 = 221$. Then $(n - 1) = 220 = 2^2(55) = 2^k q$. Let us try $a = 5$. Then we have $5^{55} \bmod 221 = 112$, which is neither 1 nor 220 $(5^{55})^2 \bmod 221 = 168$. Because we have used all values of j (i.e., $j = 0$ and $j = 1$) in line 4 of the TEST algorithm, the test returns `composite`, indicating that 221 is definitely a composite number. But suppose we had selected $a = 21$. Then we have $21^{55} \bmod 221 = 200$; $(21^{55})^2 \bmod 221 = 220$; and the test returns `inconclusive`, indicating that 221 may be prime. In fact, of the 218 integers from 2 through 219, four of these will return an inconclusive result, namely 21, 47, 174, and 200.

9-4 CHINESE REMAINDER THEOREM

The Chinese remainder theorem (CRT) is used to solve a set of congruent equations with one variable but different moduli, which are relatively prime, as shown below:

$$
\begin{aligned}
x &\equiv a_1 \pmod{m_1} \\
x &\equiv a_2 \pmod{m_2} \\
&\cdots \\
x &\equiv a_k \pmod{m_k}
\end{aligned}
$$

9-4 Continued

Example 9.35

The following is an example of a set of equations with different moduli:

$$
\begin{aligned}
x &\equiv 2 \pmod{3}\\
x &\equiv 3 \pmod{5}\\
x &\equiv 2 \pmod{7}
\end{aligned}
$$

The solution to this set of equations is given in the next section; for the moment, note that the answer to this set of equations is $x = 23$. This value satisfies all equations: $23 \equiv 2 \pmod{3}$, $23 \equiv 3 \pmod{5}$, and $23 \equiv 2 \pmod{7}$.

9-4 Continued

Solution To Chinese Remainder Theorem

1. Find $M = m_1 \times m_2 \times \ldots \times m_k$. This is the common modulus.
2. Find $M_1 = M/m_1$, $M_2 = M/m_2$, …, $M_k = M/m_k$.
3. Find the multiplicative inverse of M_1, M_2, …, M_k using the corresponding moduli (m_1, m_2, …, m_k). Call the inverses M_1^{-1}, M_2^{-1}, …, M_k^{-1}.
4. The solution to the simultaneous equations is

$$\boxed{x = (a_1 \times M_1 \times M_1^{-1} + a_2 \times M_2 \times M_2^{-1} + \cdots + a_k \times M_k \times M_k^{-1}) \bmod M}$$

9-4 Continued

Example 9.36

Find the solution to the simultaneous equations:

$$x \equiv 2 \pmod 3$$
$$x \equiv 3 \pmod 5$$
$$x \equiv 2 \pmod 7$$

Solution

We follow the four steps.

1. $M = 3 \times 5 \times 7 = 105$

2. $M_1 = 105 / 3 = 35$, $M_2 = 105 / 5 = 21$, $M_3 = 105 / 7 = 15$

3. The inverses are $M_1^{-1} = 2$, $M_2^{-1} = 1$, $M_3^{-1} = 1$

4. $x = (2 \times 35 \times 2 + 3 \times 21 \times 1 + 2 \times 15 \times 1) \bmod 105 = 23 \bmod 105$

9-4 Continued

Example 9.37

Find an integer that has a remainder of 3 when divided by 7 and 13, but is divisible by 12.

Solution

This is a CRT problem. We can form three equations and solve them to find the value of x.

$$x = 3 \bmod 7$$
$$x = 3 \bmod 13$$
$$x = 0 \bmod 12$$

If we follow the four steps, we find x = 276. We can check that 276 = 3 mod 7, 276 = 3 mod 13 and 276 is divisible by 12 (the quotient is 23 and the remainder is zero).