Type your text

Theory of Computation

Dr Samayveer Singh

Context-Free Language

Context-Free Grammar

- A grammar is context-free if every production is of the form

$$A \rightarrow \alpha$$

where $A \ \mathcal{E} \ V_N$ *and* $\alpha \ \mathcal{E} \ (V_N \ U \sum)^*$.

Simplification of Context-free Grammars

- In the simplification of CFG, we try to eliminate those symbols and productions in G which are not useful for the derivation of sentences.

$S \to AB$

- Consider, for example,

$G = (\{S, A, B, C, E\}, \{a, b, c\}, P, S)$

where, $P = \{S \to AB,\ A \to a,\ B \to b,\ B \to C,\ E \to c \mid \wedge\}$

- It is easy to see that $L(G) = \{ab\}$. Let $G' = (\{S, A. B\}, \{a, b\}, p', S)$, where P' consists of $S \to AB,\ A \to a,\ B \to b$.

$L(G) = L(G')$.

- We have eliminated the symbols C, E and c and the productions $B \to C,\ E \to c \mid \wedge$.

- We note the following points regarding the symbols and productions which are eliminated:

  (i) C does not derive any terminal string.

  (ii) *E and c do not appear in any sentential form.*

  (iii) $E \rightarrow \wedge$ *is a null production.*

  (iv) $B \rightarrow C$ *simply replaces B by C.*

- In this section, we give the construction to eliminate (i) variables not deriving terminal strings, (ii) symbols not appearing in any sentential form, (iii) null productions. and (iv) productions of the form $A \rightarrow B$.

Construction of Reduced Grammars

$A \to w \quad C \to B$

$A \to a \quad \Sigma^*$

$3, b$

Theorem: If G is a CFG such that $L(G) \neq \Phi$, *we can find an equivalent* grammar G' such that each variable in G' derives some terminal string.

Proof: Let $G = (V_N, \Sigma, P, S)$. We define $G' = (V'_N, \Sigma, G', S)$ as follows: P

(a) *Construction of V'_N :*

ABabc

We define $Wi \subseteq V_N$ *by recursion:*

$W_1 = \{A \in V_N \mid$ there exists a production $A \to w$ where $w \in \Sigma^*\}$. *(If $W_1 = \Phi$. some variable will remain after the application of any production, and* so $L(G) = \Phi$.)

$W_{i+1} = W_i \cup \{A \in V_N \mid$ there exists some production $A \to \alpha$ **with** ***$\alpha \in (\Sigma \cup Wi)^*\}$***

By the definition of Wi, $Wi \subseteq Wi+1$ *for all i. As* V_N *has only a finite number* of variables, *$Wk = Wk+1$ for some $k <= |V_N|$. Therefore, $W_k = W_{k+j}$ for $j>=1$.*

We define $V'_N = Wk$

(b) *Construction of p':*

$P' = \{A \to \alpha \mid A, \quad \alpha \in (V'_N \cup \Sigma)^*$

We can define $G' = (V'N, \Sigma, G', S)$. *S is in V_N. (We are going to prove that* every variable in V_N derives some terminal string. So if S does not belongs to V_N, $L(G) = \Phi$. But $L(G) \neq \Phi$.)

Let G =({S, A, B, C, E}, {a, b, c}, P, S), where P consists of S → AB, A → a, B → b, B→C, E → c. Find G′ such that every variable in G′ derives some terminal string.

(a) Construction of V_N'

$W_1 = \{A, B, E\}$

$W_2 = \{A, B, E\} \cup \{S\} = W_2 = \{S, A, B, E\}$

$W_3 = \{S, A, B, E\} \cup \{\phi\} = \underbrace{\{S, A, B, E\}}_{V_N'}$ $\quad W_2 = W_3$

(b) P':

$S \to AB, \; A \to a, \; B \to b$

$E \to c$

$G' = (V_N', \Sigma, P', S)$

Construction of Reduced Grammars

Theorem : For every CFG G = ***(V_N, Σ, P, S), we can construct an*** equivalent grammar G' $=(V_N, \Sigma', p', S)$ *such that every symbol in $V_N \cup \Sigma'$* appears in some sentential form.

Proof: We construct G' = (V'_N, Σ', P', S) as follows:

(a) *Construction of W_i for $i >= 1$:*

$k_i \in V'_N$ $V_N \cup \Sigma$

(i) $W_1 = \{S\}$.

(ii) $W_{i+1} = W_i \cup \{X \in V_N \cup \Sigma \mid$ *there exists a production $A \rightarrow a$ with $A \in W_i$ and a ~~containing~~ the symbol $X\}$.*

We may note that $W_i \subseteq V_N \cup \Sigma$ *and* $W_i \subseteq W_{i+1}$. *As we have only a finite* number of elements in $V_N \cup \Sigma$, $W_k = W_{k+1}$ *for some k. This means that* $W_k = W_{k+j}$ *for all* $j >= 0$.

(b) *Construction of* V'_N, Σ' *and* p':

We define

$V'_N = V_N \cap W_k$,

$\Sigma' = \Sigma \cap W_k$

$P' = \{A \rightarrow a \mid A \in W_k\}$.

$a \in (V_N' \cup \Sigma')$

Consider G =*({S, A, B, E}, {a, b, c}, P, S), where P consists of S → AB, A → a, B → b, E → c.*

(a) $W_1 = \{S\}$

$W_2 = \{S\} \cup \{A, B\} = \{S, A, B\}$

$W_3 = \{S, A, B\} \cup \{a, b\} = \{S, A, B, a, b\}$

$W_4 = \{S, A, B, a, b\} \cup \{\phi\} = \{S, A, B, a, b\}$

$W_3 = W_4$

(b) $V'_N = \{S, A, B\}$

$\Sigma' = \{a, b\}$

$P' = \{S \rightarrow AB, A \rightarrow a, B \rightarrow b\}$

Construct a reduced grammar equivalent to the grammar $S \to aAa$, $A \to Sb \mid bCC \mid DaA$, $C \to abb \mid DD$, $E \to aC$, $D \to \cancel{aDA}$

Step ① ⓐ $W_1 = \{C\}$

$W_2 = \{C\} \cup \{A, E\} = \{A, C, E\}$

$W_3 = \{A, C, E\} \cup \{S\} = \{S, A, C, E\}$

$W_4 = \{S, A, C, E\} \cup \{\phi\} \Rightarrow W_3 = W_4$

ⓑ P': $S \to aAa$, $A \to Sb \mid bCC$, $C \to abb$, $E \to aC$

Step 2 ⓐ $W_1 = \{S\}$

$W_2 = \{S\} \cup \{a, A\} = \{S, A, a\}$

$W_3 = \{S, A, a\} \cup \{S, b, C\} = \{S, A, C, a, b\}$

$W_4 = \{S, A, C, a, b\} \cup \{a, b\} = \{S, A, C, a, b\} \Rightarrow W_3 = W_4$

ⓑ P': $V_N' = \{S, A, C\}$, $\Sigma' = \{a, b\}$,

P': $S \to aAa$, $S \to Sb \mid bCC$, $C \to abb$

Elimination of Null Productions

Theorem: If G =(V_N, Σ, *P. S) is a context-free grammar, then we can* find a context-free grammar G_1 having no null productions such that $L(G_1) = L(G) - \{\Lambda\}$.

Proof: We construct $G_1 = (V_N, \Sigma, p', S)$ as follows:

Step 1 *Construction of the set of nullable variables:*

We find the nullable variables recursively:

(i) $W_i = \{A \in V_N \mid A \rightarrow \Lambda \text{ is in } P\}$

(ii) $W_{i+1} = W_i \cup \{A \in V_N \mid \text{there exists a production } A \rightarrow \alpha \text{ with } \alpha \in W_i^*\}$.

By definition of W_i, $W_i \subseteq W_{i+1}$ *for all i. As* V_N *is finite.* $W_{k+1} = W_k$ *for some* $k <= |V_N|$. *So,* $W_{k+j} = W_k$ *for all j. Let* $W = W_k$. *W is the set of all nullable* variables.

Step 2 (i) *Construction of p':*

Any production whose R.H.S. does not have any nullable variable is included in *p'*.

(ii) If $A \rightarrow X_1, X_2 \ldots X_k$ *is in P, the productions of the form* $A \rightarrow \alpha_1\alpha_2 \ldots\ldots \alpha_k$ *are included in p', where* $\alpha_j = X_i$ *if* $X_i \notin W$. $\alpha_i = X_i$ *or* Λ *of* $X \in W$ *and* $\alpha_1\alpha_2 \ldots\ldots \alpha_k \neq \Lambda$.

Actually, (ii) gives several productions in P'. The productions are obtained either by not erasing any nullable variable on the R.H.S. of $A \rightarrow X_1X_2 \cdots X_k$ *or by erasing some or all nullable variables* provided some symbol appears on the R.H.S. after erasing.

Let $G_1 = (V_N, \Sigma, \boldsymbol{p'}, S)$. *$G_1$ has no null productions.*

Consider the grammar G whose productions are S → *aS* | *AB*, *A* → ∧, *B* → ∧, *D* →
b. Construct a grammar G without null productions generating L(G) - {∧}.

Step(1) Construction of nullable set

$S \to AB$
$\to \wedge\wedge$
$S \to \wedge$

$W_1 = \{A, B\}$

$W_2 = \{A, B\} \cup \{S\} = W_2 = \{A, B, S\}$

$W_3 = \{A, B, S\} \cup \{\phi\} \Rightarrow W_2 = W_3$

Step(2) Construction of P':

(i) $\underline{D \to b}$ is included in P'

(ii) $S \to aS$ gives rise to $\underline{S \to aS}$ and $\underline{S \to a}$

(iii) $S \to AB$ gives rise to $\underline{S \to AB}$, $\underline{S \to A}$ and $\underline{S \to B}$

$P' = \{ D \to a,\ S \to aS,\ S \to a,\ S \to AB$
$S \to A,\ S \to B$

Elimination of Unit Productions

Definition: A unit production in a context-free grammar G is a production of the form $A \rightarrow B$. *where A and B are variables* in G.

Theorem: If G is a context-free grammar, we can find a context-free grammar G_1 which has no null productions or unit productions such that $L(G_1) = L(G)$.

Proof: We can apply elimination of null productions theorem to grammar G to get a grammar $G' = (V_N, \sum, P. S)$ *without null productions such that* $L(G') = L(G)$. Let *A be any variable in* V_N.

Step 1 ***Construction of the set of variables derivable from A:***

Define $Wi(A)$ *recursively as follows:*

$Wo(A) = \{A\}$

$W_{i+1}(A) = W_i(A) \cup \{B \in V_N \mid C \rightarrow B \text{ is in } P \text{ with } C \in Wi(A)\}$

By definition of $W_1(A)$, $W_i(A) \subseteq W_{i+1}(A)$. *As* V_N *is finite.* $W_{k+j}(A) = W_k(A)$ for some $k <= |V_N|$. *So,* $W_{k+i}(A) = W_l(A)$ *for all* $j >= 0$. *Let* $W(A) = Wk(A)$. *Then* $W(A)$ *is the set of all variables derivable from A*

Step 2 ***Construction of A-productions in*** G_1 **:**

The A-productions in G_1 are either (i) the non-unit production in G' or (ii) $A \rightarrow a$ *whenever* $B \rightarrow a$ *is in G with* $B \in W(A)$ *and* $a \notin V_N$. (Actually, (ii) covers (i) as $A \in W(A)$). *Now, we define* $G_1 = (V_N, \sum, P_1, S)$, where P_1 *is constructed using step 2 for every* $A \in V_N$.

Let G be S → *AB*, *A* → *a*, *B* → *C* | *b*, *C* → *D*, *D* → *E and E* → *a*. *Eliminate* unit productions and get an equivalent grammar.

Step① $W_0(S) = \{S\}$

$W_1(S) = \{S\} \cup \{\phi\} = \{S\}$ ✓

$W_0(A) = \{A\}$

$W_1(A) = \{A\} \cup \{\phi\} = \{A\}$ ✓

$W_0(B) = \{B\}$

$W_1(B) = \{B\} \cup \{C\} = \{B, C\}$

$W_2(B) = \{B, C\} \cup \{D\} = \{B, C, D\}$

$W_3(B) = \{B, C, D\} \cup \{E\} = \{B, C, D, E\}$ ✓

$W_3(C) = \{C, D, E\}$ — $C \to E$, $C \to a$

$W_2(D) = \{D, E\}$ — $D \to E$, $E \to a \Rightarrow D \to a$

$B \to E$

$B \to a$

Step② P': $S \to AB$, $A \to a$, $B \to b$, $E \to a$

$B \to a$, $C \to a$, $D \to a$