CHAPTER 8

Vector Calculus and Its Applications

1. Differentiation of vectors. 2. Curves in space. 3. Velocity and acceleration, Tangential and normal acceleration, Relative velocity and acceleration. 4. Scalar and vector point functions—Vector operator del. 5. Del applied to scalar point functions—Gradient. 6. Del applied to vector point functions—Divergence and Curl. 7. Physical interpretations of div **F** and curl **F**. 8. Del applied twice to point functions. 9. Del applied to products of point functions. 10. Integration of vectors. 11. Line integral—Circulation—Work. 12. Surface integral—Flux. 13. Green's theorem in the plane. 14. Stoke's theorem. 15. Volume integral. 16. Divergence theorem. 17. Green's theorem. 18. Irrotational and Solenoidal fields. 19. Orthogonal curvilinear coordinates, Del applied to functions in orthogonal curvilinear coordinates. 20. Cylindrical coordinates. 21. Spherical polar coordinates. 22. Objective Type of Questions.

8.1 (1) DIFFERENTIATION OF VECTORS

If a vector **R** varies continuously as a scalar variable t changes, then **R** is said to be a function of t and is written as $\mathbf{R} = \mathbf{F}(t)$.

Just as in scalar calculus, *we define* **derivative** *of a vector function* $\mathbf{R} = \mathbf{F}(t)$ *as*

$$\underset{\delta t \to 0}{\text{Lt}} \frac{\mathbf{F}(t+\delta t) - \mathbf{F}(t)}{\delta t} \text{ and write it as } \frac{d\mathbf{R}}{dt} \text{ or } \frac{d\mathbf{F}}{dt} \text{ or } \mathbf{F}'(t).$$

(2) **General rules of differentiation** are similar to those of ordinary calculus *provided the order of factors in vector products is maintained.* Thus, if ϕ, **F, G, H** are scalar and vector functions of a scalar variable t, we have

(*i*) $\dfrac{d}{dt}(\mathbf{F} + \mathbf{G} - \mathbf{H}) = \dfrac{d\mathbf{F}}{dt} + \dfrac{d\mathbf{G}}{dt} - \dfrac{d\mathbf{H}}{dt}$

(*ii*) $\dfrac{d}{dt}(\mathbf{F}\phi) = \mathbf{F}\dfrac{d\phi}{dt} + \dfrac{d\mathbf{F}}{dt}\phi$

(*iii*) $\dfrac{d}{dt}(\mathbf{F} \cdot \mathbf{G}) = \mathbf{F} \cdot \dfrac{d\mathbf{G}}{dt} + \dfrac{d\mathbf{F}}{dt} \cdot \mathbf{G}$

(*iv*) $\dfrac{d}{dt}(\mathbf{F} \times \mathbf{G}) = \mathbf{F} \times \dfrac{d\mathbf{G}}{dt} + \dfrac{d\mathbf{F}}{dt} \times \mathbf{G}$

(*v*) $\dfrac{d}{dt}[\mathbf{FGH}] = \left[\dfrac{d\mathbf{F}}{dt}\mathbf{GH}\right] + \left[\mathbf{F}\dfrac{d\mathbf{G}}{dt}\mathbf{H}\right] + \left[\mathbf{FG}\dfrac{d\mathbf{H}}{dt}\right]$

(*vi*) $\dfrac{d}{dt}[(\mathbf{F} \times \mathbf{G}) \times \mathbf{H}] = \left(\dfrac{d\mathbf{F}}{dt} \times \mathbf{G}\right) \times \mathbf{H} + \left(\mathbf{F} \times \dfrac{d\mathbf{G}}{dt}\right) \times \mathbf{H} + (\mathbf{F} \times \mathbf{G}) \times \dfrac{d\mathbf{H}}{dt}$

As an illustration, let us prove (*iv*), while the others can be proved similarly :

$$\frac{d}{dt}(\mathbf{F} \times \mathbf{G}) = \underset{\delta t \to 0}{\text{Lt}} \frac{(\mathbf{F} + \delta\mathbf{F}) \times (\mathbf{G} + \delta\mathbf{G}) - \mathbf{F} \times \mathbf{G}}{\delta t} = \underset{\delta t \to 0}{\text{Lt}} \frac{\mathbf{F} \times \delta\mathbf{G} + \delta\mathbf{F} \times \mathbf{G} + \delta\mathbf{F} \times \delta\mathbf{G}}{\delta t}$$

$$= \underset{\delta t \to 0}{\text{Lt}} \left[\mathbf{F} \times \frac{\delta\mathbf{G}}{\delta t} + \frac{\delta\mathbf{F}}{\delta t} \times \mathbf{G} + \frac{\delta\mathbf{F}}{\delta t} \times \delta\mathbf{G}\right] = \mathbf{F} \times \frac{d\mathbf{G}}{dt} + \frac{d\mathbf{F}}{dt} \times G \qquad [\because \ \delta\mathbf{G} \to 0 \text{ as } \delta t \to 0]$$

Obs. 1. *If* $\mathbf{F}(t)$ *has a constant magnitude, then* $\mathbf{F} \cdot \dfrac{d\mathbf{F}}{dt} = 0$

For $\mathbf{F}(t) \cdot \mathbf{F}(t) = [\mathbf{F}(t)]^2 =$ constant

$$\therefore \quad \mathbf{F} \cdot \frac{d\mathbf{F}}{dt} = 0, \quad i.e., \quad \frac{d\mathbf{F}}{dt} \perp \mathbf{F}.$$

Obs. 2. *If* $\mathbf{F}(t)$ *has constant (fixed) direction, then* $\mathbf{F} \times \dfrac{d\mathbf{F}}{dt} = \mathbf{0}$

Let $\mathbf{G}(t)$ be a unit vector in the direction of $\mathbf{F}(t)$ so that

$$\mathbf{F}(t) = f(t)\,\mathbf{G}(t) \quad \text{where} \quad f(t) = |\,\mathbf{F}(t)\,|.$$

$$\therefore \quad \frac{d\mathbf{F}}{dt} = f\frac{d\mathbf{G}}{dt} + \frac{df}{dt}\mathbf{G} \quad \text{and} \quad \mathbf{F} \times \frac{d\mathbf{F}}{dt} = f\,\mathbf{G} \times \left[f\frac{d\mathbf{G}}{dt} + \frac{df}{dt}\mathbf{G}\right]$$

$$= f^2\,\mathbf{G} \times \frac{d\mathbf{G}}{dt} = \mathbf{0}. \qquad [\text{since } \mathbf{G} \text{ is constant, } d\mathbf{G}/dt = \mathbf{0}.]$$

Example 8.1. *If* $\mathbf{A} = 5t^2\mathbf{I} + t\mathbf{J} - t^3\mathbf{K}$, $\mathbf{B} = \sin t\mathbf{I} - \cos t\mathbf{J}$, *find (i)* $\dfrac{d}{dt}(\mathbf{A} \cdot \mathbf{B})$; *(ii)* $\dfrac{d}{dt}(\mathbf{A} \times \mathbf{B})$.

Solution. (*i*) $\dfrac{d}{dt}(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \cdot \dfrac{d\mathbf{B}}{dt} + \dfrac{d\mathbf{A}}{dt} \cdot \mathbf{B}$

$$= (5t^2\mathbf{I} + t\mathbf{J} - t^3\mathbf{K}) \cdot [\cos t\mathbf{I} - (-\sin t)\,\mathbf{J}] + (10t\mathbf{I} + \mathbf{J} - 3t^2\mathbf{K}) \cdot (\sin t\mathbf{I} - \cos t\mathbf{J})$$

$$= (5t^2 \cos t + t \sin t) + (10t \sin t - \cos t) = 5t^2 \cos t + 11t \sin t - \cos t.$$

(*ii*) $\dfrac{d}{dt}(\mathbf{A} \times \mathbf{B}) = \mathbf{A} \times \dfrac{d\mathbf{B}}{dt} + \dfrac{d\mathbf{A}}{dt} \times \mathbf{B}$

$$= (5t^2\mathbf{I} + t\mathbf{J} - t^3\mathbf{K}) \times (\cos t\mathbf{I} + \sin t\mathbf{J}) + (10\,t\mathbf{I} + \mathbf{J} - 3t^2\mathbf{K}) \times (\sin t\mathbf{I} - \cos t\mathbf{J})$$

$$= [5t^2 \sin t\mathbf{K} + r \cos t\,(-\mathbf{K}) - t^3 \cos t\mathbf{J} - t^3 \sin t\,(-\mathbf{I})]$$

$$+ [-10t \cos t\mathbf{K} + \sin t(-\mathbf{K}) - 3t^2 \sin t\mathbf{J} + 3t^2 \cos t\,(-\mathbf{I})]$$

$$= (t^3 \sin t - 3t^2 \cos t)\,\mathbf{I} - t^2(t \cos t + 3 \sin t)\,\mathbf{J} + [(5t^2 - 1) \sin t - 11t \cos t]\,\mathbf{K}.$$

8.2 CURVES IN SPACE

(1) **Tangent.** Let $\mathbf{R}(t) = x(t)\mathbf{I} + y(t)\mathbf{J} + z(t)\mathbf{K}$ be the position vector of a point P. Then as the scalar parameter t takes different values, the point P traces out a *curve in space* (Fig. 8.1). If the neighbouring point Q corresponds to $t + \delta t$, then $\delta\mathbf{R} = \mathbf{R}(t + \delta t) - \mathbf{R}(t)$ or $\delta\mathbf{R}/\delta t$ is directed along the chord PQ. As $\delta t \to 0$, Lt $\delta\mathbf{R}/\delta t$ becomes the tangent (vector) to the curve at P whenever it exists and is not zero.

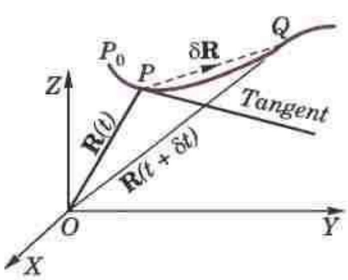

Fig. 8.1

Thus the vector $\mathbf{R}' = d\mathbf{R}/dt$ is a tangent to the space curve $\mathbf{R} = \mathbf{F}(t)$.

Let P_0 be a fixed point of this curve corresponding to $t = t_0$. If s be the length of the arc P_0P, then

$$\frac{\delta s}{\delta t} = \frac{\delta s}{|\,\delta\mathbf{R}\,|} \cdot \frac{|\,\delta\mathbf{R}\,|}{\delta t} = \frac{\text{arc } PQ}{\text{chord } PQ}\left|\frac{\delta\mathbf{R}}{\delta t}\right|$$

As $Q \to P$ along the curve QR *i.e.*, $\delta t \to 0$, then arc PQ/chord $PQ \to 1$ and

$$\frac{ds}{dt} = \left|\frac{d\mathbf{R}}{dt}\right| \quad \text{or} \quad |\,\mathbf{R}'(t)\,|.$$

If $\mathbf{R}'(t)$ is continuous, then arc P_0P is given by

$$s = \int_{t_0}^{t} |\,\mathbf{R}'\,|\,dt = \int_{t_0}^{t} \sqrt{(x')^2 + (y')^2 + (z')^2}\;dt$$

If we take s the parameter in place of t then the magnitude of the tangent vector, *i.e.*, $|\,d\mathbf{R}/ds\,| = 1$. Thus denoting the unit tangent vector by $\mathbf{T}$, we have

$$\mathbf{T} = \frac{d\mathbf{R}}{ds} \qquad \text{...(1)}$$

(2) **Principal normal.** Since $\mathbf{T}$ is a unit vector, we have

$$d\mathbf{T}/ds \cdot \mathbf{T} = 0$$

i.e., $d\mathbf{T}/ds$ is perpendicular to $\mathbf{T}$. Or else $d\mathbf{T}/ds = 0$, in which case $\mathbf{T}$ is a constant vector w.r.t. the arc length s and so has a fixed direction, *i.e.*, the curve is a straight line.

If we denote a unit normal vector to the curve at P by $\mathbf{N}$ then $d\mathbf{T}/ds$ is in the direction of $\mathbf{N}$ which is known as the *principal normal* to the space curve at P. The plane of $\mathbf{T}$ and $\mathbf{N}$ is called the *osculating plane* of the curve at P (Fig. 8.2).

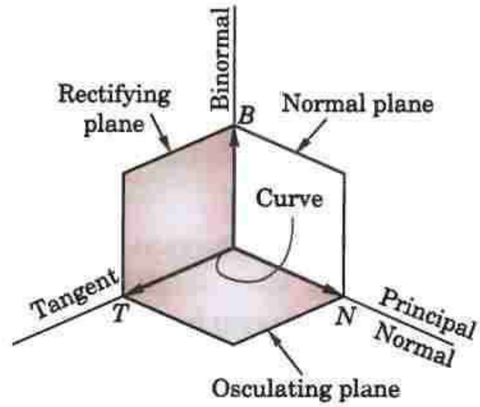

Fig. 8.2

(3) Binormal. A third unit vector $\mathbf{B}$ defined by $\mathbf{B} = \mathbf{T} \times \mathbf{N}$, is called the *binormal at P*. Since $\mathbf{T}$ and $\mathbf{N}$ are unit vectors, $\mathbf{B}$ is also a unit vector perpendicular to both $\mathbf{T}$ and $\mathbf{N}$ and hence normal to the *osculating plane at P*.

Thus at each point P of a space curve there are three mutually perpendicular unit vectors, $\mathbf{T}, \mathbf{N}, \mathbf{B}$ which form a moving trihedral such that

$$\mathbf{T} = \mathbf{N} \times \mathbf{B},\ \mathbf{N} = \mathbf{B} \times \mathbf{T},\ \mathbf{B} = \mathbf{T} \times \mathbf{N} \quad ...(2)$$

This moving trihedral determines the following three fundamental planes at each point of the curve :

(*i*) The osculating plane containing $\mathbf{T}$ and $\mathbf{N}$

(*ii*) The normal plane containing $\mathbf{N}$ and $\mathbf{B}$

(*iii*) The rectifying plane containing $\mathbf{B}$ and $\mathbf{T}$.

(4) Curvature. The arc rate of turning of the tangent (*i.e.*, the magnitude of $d\mathbf{T}/ds$) is called the *curvature* of the curve and is denoted by k.

Since $d\mathbf{T}/ds$ is in the direction of the principal normal $\mathbf{N}$, therefore,

$$\frac{d\mathbf{T}}{ds} = k\mathbf{N} \quad ...(3)$$

(5) Torsion. Since $\mathbf{B}$ is a unit vector, we have $\frac{d\mathbf{B}}{ds} \cdot \mathbf{B} = 0$

Also $\mathbf{B} \cdot \mathbf{T} = 0$, therefore $\frac{d\mathbf{B}}{ds} \cdot \mathbf{T} + \mathbf{B} \cdot \frac{d\mathbf{T}}{ds} = 0.$

or $$\frac{d\mathbf{B}}{ds} \cdot \mathbf{T} + \mathbf{B} \cdot (k\mathbf{N}) = 0, \qquad i.e., \quad \frac{d\mathbf{B}}{ds} \cdot \mathbf{T} = 0 \qquad [\because \mathbf{B} \cdot \mathbf{N} = 0]$$

Hence $d\mathbf{B}/ds$ is perpendicular to both $\mathbf{B}$ and $\mathbf{T}$ and is, therefore, parallel to $\mathbf{N}$.

The arc rate of turning of the binormal (*i.e.*, the magnitude of $d\mathbf{B}/ds$) is called *torsion* of the curve and is denoted by τ. We may, therefore, write

$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N} \quad ...(4)$$

(The negative sign indicates that for $\tau > 0$, $d\mathbf{B}/ds$ has direction of $-\mathbf{N}$).

Finally to find $d\mathbf{N}/ds$, we differentiate $\mathbf{N} = \mathbf{B} \times \mathbf{T}$.

$$\therefore \quad \frac{d\mathbf{N}}{ds} = \frac{d\mathbf{B}}{ds} \times \mathbf{T} + \mathbf{B} \times \frac{d\mathbf{T}}{ds} = -\tau\mathbf{N} \times \mathbf{T} + \mathbf{B} \times k\mathbf{N}$$

Using the relation (2), it reduces to $\frac{d\mathbf{N}}{ds} = \tau\mathbf{B} - k\mathbf{T}$ ...(5)

The equations (3), (4) and (5) constitute the well-known *Frenet formulae** for space curves.

Obs. 1. $\rho = 1/k$ and $\sigma = 1/\tau$ are respectively called the radii of curvature and torsion.

2. For a plane curve $\tau = 0$.

Example 8.2. *Find the angle between the tangents to the curve* $\mathbf{R} = t^2\mathbf{I} + 2t\mathbf{J} - t^3\mathbf{K}$ *at the point* $t = \pm 1$. *(V.T.U., 2010)*

Solution. The tangent at any point 't' is given by

$$\frac{d\mathbf{R}}{dt} = 2t\mathbf{I} + 2\mathbf{J} - 3t^2\mathbf{K}$$

$\therefore$ the tangents $\mathbf{T}_1, \mathbf{T}_2$ at $t = 1$ and $t = -1$ are respectively given by

$$\mathbf{T}_1 = 2\mathbf{I} + 2\mathbf{J} - 3\mathbf{K}\ ;\ \mathbf{T}_2 = -2\mathbf{I} + 2\mathbf{J} - 3\mathbf{K},$$

* Named after a French mathematician *Jean-Frederic Frenet* (1816–1900).

Then the required $\angle\theta$ is given by $T_1 T_2 \cos\theta = \mathbf{T}_1 \cdot \mathbf{T}_2 = 2(-2) + 2 \cdot 2 + (-3)(-3)$

i.e., $\sqrt{17}\sqrt{17}\cos\theta = 9 \quad \therefore \quad \theta = \cos^{-1}(9/17)$.

Example 8.3. *Find the curvature and torsion of the curve $x = a\cos t$, $y = a\sin t$, $z = bt$.*

(This curve is drawn on a circular cylinder cutting its generators at a constant angle and is known as a *circle helix*).

Solution. The vector equation of the curve is $\mathbf{R} = a\cos t\mathbf{I} + a\sin t\mathbf{J} + bt\mathbf{K}$

$\therefore$ $d\mathbf{R}/dt = -a\sin t\mathbf{I} + a\cos t\mathbf{J} + b\mathbf{K}$

Its arc length from P_0 ($t = 0$) to any point $P(t)$ (Fig. 8.3) is given by

$$s = \int_0^t |d\mathbf{R}/dt|\,dt = \sqrt{(a^2+b^2)}\,t$$

$\therefore$ $$\frac{ds}{dt} = \sqrt{(a^2+b^2)}$$

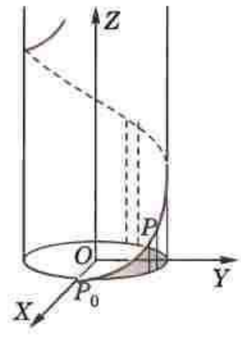

Fig. 8.3

Then $$\mathbf{T} = \frac{d\mathbf{R}}{ds} = \frac{d\mathbf{R}}{dt}\Big/\frac{ds}{dt} = \frac{-a\sin t\mathbf{I} + a\cos t\mathbf{J} + b\mathbf{K}}{\sqrt{(a^2+b^2)}}$$

and $$\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{dt}\Big/\frac{ds}{dt} = \frac{-a(\cos t\mathbf{I} + \sin t\mathbf{J})}{a^2+b^2}$$

Thus $$k = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{a}{a^2+b^2} \quad ...(i) \qquad \text{and} \quad \mathbf{N} = -(\cos t\mathbf{I} + \sin t\mathbf{J})$$

Also $$\mathbf{B} = \mathbf{T}\times\mathbf{N} = (b\sin t\mathbf{I} - b\cos t\mathbf{J} + a\mathbf{K})/\sqrt{(a^2+b^2)}$$

$\therefore$ $$\frac{d\mathbf{B}}{ds} = \frac{d\mathbf{B}}{dt}\Big/\frac{ds}{dt} = b(\cos t\mathbf{I} + \sin t\mathbf{J})/(a^2+b^2) = -\tau\mathbf{N} = \tau(\cos t\mathbf{I} + \sin t\mathbf{J})$$

Hence $$\tau = \frac{b}{a^2+b^2}. \qquad ...(ii)$$

PROBLEMS 8.1

1. Show that, if $\mathbf{R} = \mathbf{A}\sin\omega t + \mathbf{B}\cos\omega t$, where $\mathbf{A}$, $\mathbf{B}$, ω are constants, then (*i*) $\dfrac{d^2\mathbf{R}}{dt^2} = -\omega^2\mathbf{R}$ (*Bhopal, 2007 S*)

   (*ii*) $\mathbf{R}\times\dfrac{d\mathbf{R}}{dt} = -\omega\mathbf{A}\times\mathbf{B}$.
2. Given $\mathbf{R} = t^m\mathbf{A} + t^n\mathbf{B}$, where $\mathbf{A}$, $\mathbf{B}$ are constant vectors, show that, if $\mathbf{R}$ and $d^2\mathbf{R}/dt^2$ are parallel vectors, then $m + n = 1$, unless $m = n$.
3. If $\mathbf{P} = 5t^2\mathbf{I} + t^3\mathbf{J} - t\mathbf{K}$ and $\mathbf{Q} = 2\mathbf{I}\sin t - \mathbf{J}\cos t + 5t\mathbf{K}$, find (*i*) $\dfrac{d}{dt}(\mathbf{P}\cdot\mathbf{Q})$; (*ii*) $\dfrac{d}{dt}(\mathbf{P}\times\mathbf{Q})$.
4. If $\dfrac{d\mathbf{U}}{dt} = \mathbf{W}\times\mathbf{U}$ and $\dfrac{d\mathbf{V}}{dt} = \mathbf{W}\times\mathbf{V}$, prove that $\dfrac{d}{dt}(\mathbf{U}\times\mathbf{V}) = \mathbf{W}\times(\mathbf{U}\times\mathbf{V})$. (*Mumbai, 2009*)
5. If $\mathbf{A} = x^2yz\mathbf{I} - 2xz^3\mathbf{J} + xz^2\mathbf{K}$ and $\mathbf{B} = 2z\mathbf{I} + y\mathbf{J} - x^2\mathbf{K}$, find $\dfrac{\partial^2}{\partial x\partial y}(\mathbf{A}\times\mathbf{B})$ at $(1, 0, -2)$.
6. If $\mathbf{R} = (a\cos t)\mathbf{I} + (a\sin t)\mathbf{J} + (at\tan\alpha)\mathbf{K}$, find the value of

   (*i*) $\left|\dfrac{d\mathbf{R}}{dt}\times\dfrac{d^2\mathbf{R}}{dt^2}\right|$ (*ii*) $\left|\dfrac{d\mathbf{R}}{dt}, \dfrac{d^2\mathbf{R}}{dt^2}, \dfrac{d^3\mathbf{R}}{dt^3}\right|$ (*Rohtak, 2005*)

   Also find the unit tangent vector at any point t of the curve.
7. Find the unit tangent vector at any point on the curve $x = t^2 + 2$, $y = 4t - 5$, $z = 2t^2 - 6t$, where t is any variable. Also determine the unit tangent vector at the point $t = 2$.
8. Find the equation of the tangent line to the curve $x = a\cos\theta$, $y = a\sin\theta$, $z = a\theta\tan\alpha$ at $\theta = \pi/4$.
9. Find the curvature of the (*i*) ellipse $\mathbf{R}(t) = a\cos t\mathbf{I} + b\sin t\mathbf{J}$; (*ii*) parabola $\mathbf{R}(t) = 2t\mathbf{I} + t^2\mathbf{J}$ at the point $t = 1$.

10. Find the education of the osculating plane and binormal to the curve
(i) $x = 2\cosh(t/2), y = 2\sinh(t/2), z = 2t$ at $t = 0$; (ii) $x = e^t \cos t, y = e^t \sin t, z = e^t$ at $t = 0$.
11. A circular helix is given by the equation $\mathbf{R}(t) = (2\cos t)\mathbf{I} + (2\sin t)\mathbf{J} + \mathbf{K}$. Find the curvature and torsion of the curve at any point and show that they are constant.
12. Show that for the curve $\mathbf{R} = a(3t - t^3)\mathbf{I} + 3at^2\mathbf{J} + a(3t + t^2)\mathbf{K}$, the curvature equals torsion.

8.3 (1) VELOCITY AND ACCELERATION

Let the position of a particle P at time t on a path (curve) C be $\mathbf{R}(t)$. At time $t + \delta t$, let the particle be at Q (Fig. 8.1), then $\delta\mathbf{R} = \mathbf{R}(t + \delta t) - \mathbf{R}(t)$ or $\delta\mathbf{R}/\delta t$ is directed along PQ. As $Q \to P$ along C, the line PQ becomes the tangent at P to C.

$$\therefore \qquad \underset{\delta t \to 0}{\text{Lt}} \frac{\delta\mathbf{R}}{\delta t} = \frac{d\mathbf{R}}{dt} = \mathbf{V}$$

is the tangent vector of C at P which is the *velocity* (vector) $\mathbf{V}$ of the motion and its magnitude is the *speed* $v = ds/dt$, where s is the arc length of P from a fixed point P_0 $(s = 0)$ on C.

The derivative of the velocity vector $\mathbf{V}(t)$ is called the *acceleration* (vector) $\mathbf{A}(t)$, which is given by

$$\mathbf{A}(t) = \frac{d\mathbf{V}}{dt} = \frac{d^2\mathbf{R}}{dt^2}.$$

(2) Tangential and normal accelerations. It is important to note that the magnitude of acceleration is not always the rate of change of $|\mathbf{V}|$ because $\mathbf{A}(t)$ is not always tangential to the path C. Infact

$$\mathbf{V}(t) = \frac{d\mathbf{R}}{dt} = \frac{d\mathbf{R}}{ds}\cdot\frac{ds}{dt}. \text{ where } d\mathbf{R}/ds \text{ is a unit tangent vector of } C.$$

$$\therefore \qquad \mathbf{A}(t) = \frac{d\mathbf{V}}{dt} = \frac{d}{dt}\left[\frac{ds}{dt}\cdot\frac{d\mathbf{R}}{ds}\right] = \frac{d^2s}{dt^2}\cdot\frac{d\mathbf{R}}{ds} + \left(\frac{ds}{dt}\right)^2\frac{d^2\mathbf{R}}{ds^2}$$

Now since $d\mathbf{R}/dt \,.\, d^2\mathbf{R}/dt^2 = 0$, $d^2\mathbf{R}/dt^2$ is perpendicular to $d\mathbf{R}/dt$. Hence the acceleration $\mathbf{A}(t)$ is comprised of (*i*) the tangential component $d^2s/dt^2 \,.\, d\mathbf{R}/ds$, called the *tangential acceleration, and*

(*ii*) the normal component $(ds/dt)^2 \,.\, d^2\mathbf{R}/ds^2$, called the *normal acceleration.*

Obs. The acceleration is the time rate change of $|\mathbf{V}| = ds/dt$, if the normal acceleration is zero, for then

$$|A| = \left|\frac{d^2s}{dt^2}\right|\cdot\left|\frac{d\mathbf{R}}{ds}\right| = \left|\frac{d^2s}{dt^2}\right|.$$

(3) Relative velocity and acceleration. Let two particles P_1 and P_2 moving along the curves C_1 and C_2 have position vectors $\mathbf{R}_1$ and $\mathbf{R}_2$ at time t, (Fig. 8.4), so that $\mathbf{R} = \overrightarrow{P_1P_2} = \mathbf{R}_2 - \mathbf{R}_1$

Differentiating w.r.t. t, we get $\dfrac{d\mathbf{R}}{dt} = \dfrac{d\mathbf{R}_2}{dt} - \dfrac{d\mathbf{R}_1}{dt}$ ...(*iii*)

This defines the *relative velocity* (vector) of P_2 w.r.t. P_1 and states that the *velocity (vector) of P_2 relative to P_1 = velocity (vector) of P_2 – velocity (vector) of P_1.*

Again differentiating (*iii*), we have $\dfrac{d^2\mathbf{R}}{dt^2} = \dfrac{d^2\mathbf{R}_2}{dt^2} - \dfrac{d^2\mathbf{R}_1}{dt^2}$ ...(*iv*)

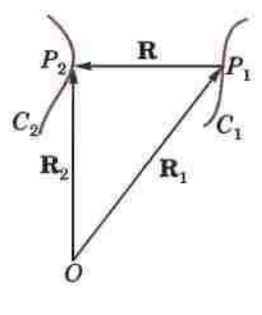

Fig. 8.4

i.e., acceleration (vector) of P_2 relative to P_1 = acceleration (vector) of P_2 – acceleration (vector) of P_1.

Example 8.4. *A particle moves along the curve $x = t^3 + 1$, $y = t^2$, $z = 2t + 3$ where t is the time. Find the components of its velocity and acceleration at $t = 1$ in the direction* $\mathbf{I} + \mathbf{J} + 3\mathbf{K}$.

Solution. *Velocity* $= \dfrac{d\mathbf{R}}{dt} = \dfrac{d}{dt}[(t^3 + 1)\mathbf{I} + t^2\mathbf{J} + (2t + 3)\mathbf{K}]$
$= 3t^2\mathbf{I} + 2t\mathbf{J} + 2\mathbf{K} = 3\mathbf{I} + 2\mathbf{J} + 2\mathbf{K}$ at $t = 1$

and *acceleration* $= \dfrac{d^2\mathbf{R}}{dt^2} = 6t\mathbf{I} + 2\mathbf{J} + 0\mathbf{K} = 6\mathbf{I} + 2\mathbf{J}$ at $t = 1$.

Now unit vector in the direction of **I** + **J** + 3**K** is $\dfrac{\mathbf{I}+\mathbf{J}+3\mathbf{K}}{\sqrt{(1^2+1^2+3^2)}} = \dfrac{1}{\sqrt{11}}\,(\mathbf{I}+\mathbf{J}+3\mathbf{K})$.

∴ component of velocity at $t = 1$ in the direction **I** + **J** + 3**K**

$$= \frac{(3\mathbf{I}+2\mathbf{J}+2\mathbf{K}).(\mathbf{I}+\mathbf{J}+3\mathbf{K})}{\sqrt{(11)}} = \frac{3+2+6}{\sqrt{(11)}} = \sqrt{(11)}$$

and component of acceleration at $t = 1$ in the direction

$$\mathbf{I}+\mathbf{J}+3\mathbf{K} = (6\mathbf{I}+2\mathbf{J}).(\mathbf{I}+\mathbf{J}+3\mathbf{K})/\sqrt{(11)} = \frac{6+2}{\sqrt{(11)}} = \frac{8}{\sqrt{(11)}}.$$

Example 8.5. *A particle moves along the curve* $\mathbf{R} = (t^3-4t)\,\mathbf{I} + (t^2+4t)\mathbf{J} + (8t^2-3t^3)\mathbf{K}$ *where t denotes time. Find the magnitudes of acceleration along the tangent and normal at time t = 2.* (V.T.U., 2003 S)

Solution. Velocity $\dfrac{d\mathbf{R}}{dt} = (3t^2-4)\mathbf{I} + (2t+4)\mathbf{J} + (16t-9t^2)\mathbf{K}$

and acceleration $\dfrac{d^2\mathbf{R}}{dt^2} = 6t\mathbf{I} + 2\mathbf{J} + (16-18t)\mathbf{K}$

∴ at $t = 2$, velocity $\mathbf{V} = 8\mathbf{I} + 8\mathbf{J} - 4\mathbf{K}$ and acceleration $\mathbf{A} = 12\mathbf{I} + 2\mathbf{J} - 20\mathbf{K}$.

Since the velocity is along the tangent to the curve, therefore, the component of **A** along the tangent

$$= \mathbf{A}.\frac{\mathbf{V}}{|\mathbf{V}|} = (12\mathbf{I}+2\mathbf{J}-20\mathbf{K}).\frac{8\mathbf{I}+8\mathbf{J}-4\mathbf{K}}{\sqrt{(64+64+16)}}$$

$$= \frac{12\times 8 + 2\times 8 + (-20)\times(-4)}{12} = 16.$$

Now the component of **A** along the normal

$$= |\,\mathbf{A} - \text{Resolved part of } \mathbf{A} \text{ along the tangent}\,|$$

$$= \left|\,12\mathbf{I}+2\mathbf{J}-20\mathbf{K}-16\,\frac{8\mathbf{I}+8\mathbf{J}-4\mathbf{K}}{12}\,\right| = \frac{1}{3}\,|\,4\mathbf{I}-26\mathbf{J}-44\mathbf{K}\,| = 2\sqrt{73}.$$

Example 8.6. *The position vector of a particle at time t is* $\mathbf{R} = \cos(t-1)\,\mathbf{I} + \sinh(t-1)\,\mathbf{J} + \alpha t^3\mathbf{K}$. *Find the condition imposed on α by requiring that at time t = 1, the acceleration is normal to the position vector.*

Solution. *Velocity* $= \dfrac{d\mathbf{R}}{dt} = -\sin(t-1)\,\mathbf{I} + \cosh(t-1)\,\mathbf{J} + 3\alpha t^2\mathbf{K}$

Acceleration $= \dfrac{d^2\mathbf{R}}{dt^2} = -\cos(t-1)\,\mathbf{I} + \sinh(t-1)\,\mathbf{J} + 6\alpha t\mathbf{K} = -\mathbf{I} + 6\alpha\mathbf{K}$ at $t = 1$.

Also $\mathbf{R} = \mathbf{I} + \alpha\mathbf{K}$ at $t = 1$.

If **R** and acceleration at $t = 1$ are normal, then their scalar product is zero.

∴ $(-\mathbf{I}+6\alpha\mathbf{K}).(\mathbf{I}+\alpha\mathbf{K}) = 0$ or $-1+6\alpha^2 = 0$

or $\alpha^2 = 1/6$ or $\alpha = 1/\sqrt{6}$.

Example 8.7. *Find the radial and transverse acceleration of a particle moving in a plane curve.* (*Kurukshetra, 2006 ; Rajasthan, 2006*)

Solution. At any time t, let the position vector of the moving particle $P(r, \theta)$ be **R** (Fig. 8.5) so that

$$\mathbf{R} = r\hat{\mathbf{R}} = r(\cos\theta\mathbf{I} + \sin\theta\mathbf{J})$$

∴ its velocity $\mathbf{V} = \dfrac{d\mathbf{R}}{dt} = \dfrac{dr}{dt}\hat{\mathbf{R}} + r\dfrac{d\hat{\mathbf{R}}}{dt}$ ...(*i*)

As $\hat{\mathbf{R}} = \cos\theta\mathbf{I} + \sin\theta\mathbf{J}$

and $\dfrac{d\hat{\mathbf{R}}}{dt} = (-\sin\theta\mathbf{I} + \cos\theta\mathbf{J})\,\dfrac{d\theta}{dt}$

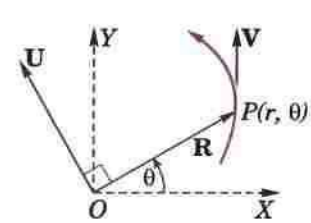

Fig. 8.5

$\therefore \qquad \dfrac{d\hat{\mathbf{R}}}{dt} \perp \hat{\mathbf{R}}$ and $\left|\dfrac{d\hat{\mathbf{R}}}{dt}\right| = \dfrac{d\theta}{dt}$, *i.e.*, if **U** is a unit vector $\perp$ **R**, then

$$\frac{d\hat{\mathbf{R}}}{dt} = \frac{d\theta}{dt}\mathbf{U}$$

$\therefore$ (*i*) becomes, $\quad \mathbf{V} = \dfrac{dr}{dt}\hat{\mathbf{R}} + r\dfrac{d\theta}{dt}\mathbf{U}$ ...(*ii*)

Thus the radial and transverse components of the velocity are dr/dt and $r\, d\theta/dt$.

Also
$$\mathbf{A} = \frac{d\mathbf{V}}{dt} = \frac{d^2r}{dt^2}\hat{\mathbf{R}} + \frac{dr}{dt}\frac{d\hat{\mathbf{R}}}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\mathbf{U} + r\frac{d^2\theta}{dt^2}\mathbf{U} + r\frac{d\theta}{dt}\frac{d\mathbf{U}}{dt}$$

$$= \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{\mathbf{R}} + \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\mathbf{U} \qquad \left[\because\ \mathbf{U} = -\sin\theta\mathbf{I} + \cos\theta\mathbf{J} \text{ gives } \frac{d\mathbf{U}}{dt} = -\frac{d\theta}{dt}\hat{\mathbf{R}}\right]$$

Thus the radial and transverse components of the acceleration are

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \text{ and } 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}.$$

Example 8.8. *A person going eastwards with a velocity of 4 km per hour, finds that the wind appears to blow directly from the north. He doubles his speed and the wind seems to come from north-east. Find the actual velocity of the wind.*

Solution. Let the actual velocity of the wind be $x\mathbf{I} + y\mathbf{J}$, where **I**, **J** represent velocities of 1 km per hour towards the east and north respectively. As the person is going eastwards with a velocity of 4 km per hour, his actual velocity is 4**I**.

Then the velocity of the wind relative to the man is $(x\mathbf{I} + y\mathbf{J}) - 4\mathbf{I}$, which is parallel to $-\mathbf{J}$, as it appears to blow from the north. Hence $x = 4$. ...(*i*)

When the velocity of the person becomes 8**I**, the velocity of the wind relative to man is $(x\mathbf{I} + y\mathbf{J}) - 8\mathbf{I}$. But this is parallel to $-(\mathbf{I} + \mathbf{J})$.

$\therefore \qquad (x - 8)/y = 1$, which by (*i*) gives $y = -4$.

Hence the actual velocity of the wind is $4(\mathbf{I} - \mathbf{J})$, *i.e.*, $4\sqrt{2}$ km. per hour towards the south-east.

PROBLEMS 8.2

1. A particle moves along a curve $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$, where t is the time variable. Determine its velocity and acceleration vectors and also the magnitudes of velocity and acceleration at $t = 0$.
(*P.T.U., 2003 ; V.T.U., 2003 S*)
2. The position vector of a particle at time t is $\mathbf{R} = \cos(t-1)\,\mathbf{I} + \sinh(t-1)\,\mathbf{J} + at^3\mathbf{K}$. Find the condition imposed on a by requiring that at time $t = 1$, the acceleration is normal to the position vector.
3. A particle moves on the curve $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$, where t is the time. Find the components of velocity and acceleration at time $t = 1$ in the direction $\mathbf{I} - 3\mathbf{J} + 2\mathbf{K}$. (*V.T.U., 2008*)
4. A particle moves so that its position vector is given by $\mathbf{R} = \mathbf{I}\cos\omega t + \mathbf{J}\sin\omega t$. Show that the velocity **V** of the particle is perpendicular to **R** and $\mathbf{R} \times \mathbf{V}$ is a constant vector.
5. A particle (position vector **R**) is moving in a circle with constant angular velocity ω. Show by vector methods, that the acceleration is equal to $-\omega^2\mathbf{R}$.
6. (*a*) Find the tangential and normal accelerations of a point moving in a plane curve. (*Rajasthan, 2005*)
(*b*) The position vector of a moving particle at a time t is $\mathbf{R} = 3\cos t\mathbf{I} + 3\sin t\mathbf{J} + 4t\mathbf{K}$. Find the tangent and normal components of its acceleration at $t = 1$. (*Marathwada, 2008*)
7. The velocity of a boat relative to water is represented by $3\mathbf{I} + 4\mathbf{J}$ and that of wate relative to earth is $\mathbf{I} - 3\mathbf{J}$. What is the velocity of the boat relative to the earth if **I** and **J** represent one km an hour east and north respectively.
8. A vessel A is sailing with a velocity of 11 knots per hour in the direction S.E. and a second vessel B is sailing with a velocity of 13 knots per hour in a direction 30°E of N. Find the velocity of A relative to B.
9. A person travelling towards the north-east with a velocity of 6 km per hour finds that the wind appears to blow from the north, but when he doubles his speed it seems to come from a direction inclined at an angle $\tan^{-1} 2$ to the north of east. Show that the actual velocity of the wind is $3\sqrt{2}$ km per hour towards the east.

8.4 SCALAR AND VECTOR POINT FUNCTIONS

(1) *If to each point P(**R**) of a region E in space there corresponds a definite scalar denoted by f (**R**), then f (**R**) is called a* **scalar point function** *in E. The region E so defined is called a* **scalar field.**

The temperature at any instant, density of a body and potential due to gravitational matter are all examples of scalar point functions.

(2) *If to each point P(**R**) of a region E in space there corresponds a definite vector denoted by **F**(**R**), then it is called the* **vector point function** *in E. The region E so defined is called a* **vector field.**

The velocity of a moving fluid at any instant, the gravitational intensity of force are examples of vector point functions.

Differentiation of vector point functions follows the same rules as those of ordinary calculus. Thus if $\mathbf{F}(x, y, z)$ be a vector point function, then

$$\frac{d\mathbf{F}}{dt} = \frac{\partial \mathbf{F}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathbf{F}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{F}}{\partial z}\frac{dz}{dt}$$ (See (*iii*) p. 203]

and
$$d\mathbf{F} = \frac{\partial \mathbf{F}}{\partial x}dx + \frac{\partial \mathbf{F}}{\partial y}dy + \frac{\partial \mathbf{F}}{\partial z}dz = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy + \frac{\partial}{\partial z}dz\right)\mathbf{F} \qquad \ldots(i)$$

(3) **Vector operator del.** The operator on the right side of the equation (*i*) is in the form of a scalar product of $\mathbf{I}\frac{\partial}{\partial x} + \mathbf{J}\frac{\partial}{\partial y} + \mathbf{K}\frac{\partial}{\partial z}$ and $\mathbf{I}dx + \mathbf{J}dy + \mathbf{K}dz$.

If ∇ (*read as del*) *be defined by the equation* $\nabla = \mathbf{I}\frac{\partial}{\partial x} + \mathbf{J}\frac{\partial}{\partial y} + \mathbf{K}\frac{\partial}{\partial z}$ ...(*ii*)

then (*i*) may be written as $d\mathbf{F} = (\nabla \,.\, d\mathbf{R})\,\mathbf{F}$ for when $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$, $d\mathbf{R} = \mathbf{I}dx + \mathbf{J}dy + \mathbf{K}dz$.

8.5 DEL APPLIED TO SCALAR POINT FUNCTIONS—GRADIENT

(1) **Def.** *The vector function ∇f is defined as the gradient of the scalar point function f and is written as grad f.*

Thus
$$\text{grad } f = \nabla f = \mathbf{I}\frac{\partial f}{\partial x} + \mathbf{J}\frac{\partial f}{\partial y} + \mathbf{K}\frac{\partial f}{\partial z}$$

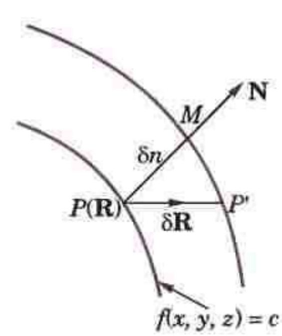

Fig. 8.6

(2) **Geometrical interpretation.** Consider the scalar point function $f(\mathbf{R})$, where $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$.

If a surface $f(x, y, z) = c$ be drawn through any point $P(\mathbf{R})$ such that at each point on it, the function has the same value as at P, then such a surface is called a *level surface* of the function f through P, *e.g.*, equipotential or isothermal surface (Fig. 8.6).

Let $P'(\mathbf{R} + \delta\mathbf{R})$ be a point on a neighbouring level surface $f + \delta f$. Then

$$\nabla f .\, \delta\mathbf{R} = \left[\mathbf{I}\frac{\partial f}{\partial x} + \mathbf{J}\frac{\partial f}{\partial y} + \mathbf{K}\frac{\partial f}{\partial z}\right] . (\mathbf{I}\delta x + \mathbf{J}\delta y + \mathbf{K}\delta z)$$

$$= \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z = \delta f.$$

Now if P' lies on the same level surface as P, then $\delta f = 0$, *i.e.*, $\nabla f \,.\, \delta\mathbf{R} = 0$. This means that ∇f is perpendicular to every $\delta\mathbf{R}$ lying on this surface. Thus ∇f *is normal to the surface* $f(x, y, z) = c$.

$$\therefore \qquad \nabla f = |\nabla f|\,\mathbf{N}$$

where **N** is a unit vector normal to this surface. If the perpendicular distance PM between the surfaces through P and P' be δn, then the rate of change of f normal to the surface through P

$$= \frac{\partial f}{\partial n} = \underset{\delta n \to 0}{\text{Lt}} \frac{\delta f}{\delta n} = \underset{\delta n \to 0}{\text{Lt}} \nabla f \,.\, \frac{\delta\mathbf{R}}{\delta n}$$

$$= |\nabla f| \underset{\delta n \to 0}{\text{Lt}} \frac{\mathbf{N} \,.\, \delta\mathbf{R}}{\delta n} = |\nabla f|. \qquad [\because \ \mathbf{N} \,.\, \delta\mathbf{R} = |\delta\mathbf{R}| \cos\theta = \delta n]$$

Hence the magnitude of $\nabla f = \partial f/\partial n$.

Thus *grad f is a vector normal to the surface f = constant and has a magnitude equal to the rate of change of f along this normal.*

(3) Directional derivative. If δr denotes the length PP' and $\mathbf{N}'$ is a unit vector in the direction PP', then the limiting value of $\delta f/\delta r$ as $\delta r \to 0$ (*i.e.*, $\partial f/\partial r$) is known as the *directional derivative of f at P along the direction PP'*.

Since $\qquad \delta r = \delta n/\cos\alpha = \delta n/\mathbf{N}\,.\,\mathbf{N}'$

$$\therefore \qquad \frac{\partial f}{\partial r} = \underset{\delta r \to 0}{\mathrm{Lt}}\left[\mathbf{N}\,.\,\mathbf{N}'\,\frac{\delta f}{\delta n}\right] = \mathbf{N}'\,.\,\frac{\partial f}{\partial n}\,\mathbf{N} = \mathbf{N}'\,.\,\nabla f$$

Thus the directional derivation of f in the direction of $\mathbf{N}'$ is the resolved part of ∇f in the direction $\mathbf{N}'$.

Since $\qquad \nabla f\,.\,\mathbf{N}' = |\,\nabla f\,|\,\cos\alpha \le |\,\nabla f\,|$

It follows that ∇f *gives the maximum rate of change of f.*

Example 8.9. *Prove that* $\nabla r^n = nr^{n-2}\,\mathbf{R}$, *where* $\mathbf{R} + x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$.
(*Bhopal, 2007 ; Anna, 2003 S ; V.T.U., 2000*)

Solution. We have $\quad f(x, y, z) = r^n = (x^2 + y^2 + z^2)^{n/2}$

$$\therefore \qquad \frac{\partial f}{\partial x} = \frac{n}{2}(x^2 + y^2 + z^2)^{n/2-1}.\,2x = nxr^{n-2}.\ \text{Similarly, } \frac{\partial f}{\partial y} = nyr^{n-2} \text{ and } \frac{\partial f}{\partial z} = nzr^{n-2}$$

$$\text{Thus} \qquad \nabla r^n = \mathbf{I}\frac{\partial f}{\partial x} + \mathbf{J}\frac{\partial f}{\partial y} + \mathbf{K}\frac{\partial f}{\partial z} = nr^{n-2}(x\mathbf{I} + y\mathbf{J} + z\mathbf{K}) = nr^{n-2}\,\mathbf{R}.$$

Otherwise : The level surfaces for f = constant, *i.e.*, r^n = constant are concentric spheres with centre O and hence unit normal $\mathbf{N}$ to the level surface through P is along the radius $\mathbf{R}$

i.e., $\qquad \mathbf{N} = \hat{\mathbf{R}}$.

$$\therefore \qquad \nabla f = \frac{\partial f}{\partial n}\,.\,\mathbf{N} = \frac{df}{dr}\hat{\mathbf{R}} = nr^{n-1}\,\hat{\mathbf{R}} \qquad [\because\ f = r^n]$$

$$= nr^{n-1}(\mathbf{R}/r) = nr^{n-2}\mathbf{R}.$$

Example 8.10. *If* $\nabla u = 2r^4\,\mathbf{R}$, *find u.* (*Mumbai, 2008*)

Solution. We have $\nabla u = 2(x^2 + y^2 + z^2)^2\,\mathbf{R}$ $\qquad [\because\ r = \sqrt{(x^2 + y^2 + z^2)}\,]$

$$= 2(x^2 + y^2 + z^2)^2\,(x\mathbf{I} + y\mathbf{J} + z\mathbf{K}) \qquad ...(i)$$

$$\text{But} \qquad \nabla u = \frac{\partial u}{\partial x}\mathbf{I} + \frac{\partial u}{\partial y}\mathbf{J} + \frac{\partial u}{\partial z}\mathbf{K} \qquad ...(ii)$$

Comparing (*i*) and (*ii*), we get

$$\frac{\partial u}{\partial x} = 2x(x^2 + y^2 + z^2)^2,\ \frac{\partial u}{\partial y} = 2y(x^2 + y^2 + z^2)^2\ ;\ \frac{\partial u}{\partial z} = 2z(x^2 + y^2 + z^2)$$

$$\text{Also} \qquad du(x, y, z) = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 2(x^2 + y^2 + z^2)^2\,(xdx + ydy + zdz)$$

$$= 2t^2\,.\,\frac{dt}{2},\ \text{taking } x^2 + y^2 + z^2 = t \quad \text{and} \quad 2(xdx + ydy + zdz) = dt$$

$$\text{Integrating both sides, } u = \int t^2\,dt + c = \frac{1}{3}t^3 + c = \frac{1}{3}(x^2 + y^2 + z^2)^3 + c$$

$$\text{Hence} \qquad u = \frac{1}{3}r^{3/2} + c.$$

Example 8.11. *If* $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = yz + zx + xy$, *prove that grad u, grad v and grad w are coplanar.* (*U.T.U., 2010 ; U.P.T.U., 2002*)

Solution. $\qquad \text{grad } u = \left(\mathbf{I}\frac{\partial}{\partial x} + \mathbf{J}\frac{\partial}{\partial y} + \mathbf{K}\frac{\partial}{\partial z}\right)(x + y + z) = \mathbf{I} + \mathbf{J} + \mathbf{K}$

$$\text{grad } v = 2x\mathbf{I} + 2y\mathbf{J} + 2z\mathbf{K},\ \text{grad } w = (y + z)\,\mathbf{I} + (z + x)\,\mathbf{J} + (z + x)\,\mathbf{K}$$

We know that three vectors are coplanar if their scalar triple product is zero.

Here [grad u, grad v, grad w]

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & z+x & x+y \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 1 \\ x+y+z & y+z+x & z+x+y \\ y+z & z+x & x+y \end{vmatrix} \qquad [\text{Operate } R_2 + R_3]$$

$$= 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix} = 0.$$

Hence grad u, grad v and grad w are coplanar.

Example 8.12. *Find a unit vector normal to the surface $xy^3z^2 = 4$ at the point $(-1, -1, 2)$.* *(Mumbai, 2008)*

Solution. A vector normal to the given surface is $\nabla\,(xy^3z^2)$

$$= \mathbf{I}\frac{\partial}{\partial x}(xy^3z^2) + \mathbf{J}\,\frac{\partial}{\partial y}(xy^3z^2) + \mathbf{K}\frac{\partial}{\partial z}(xy^3z^2) = \mathbf{I}(y^3\,z^2) + \mathbf{J}(3xy^2z^2) + \mathbf{K}(2xy^3z)$$

$$= -\,4\mathbf{I} - 12\mathbf{J} + 4\mathbf{K} \text{ at the point } (-1, -1, 2).$$

Hence the desired unit normal to the surface

$$= \frac{-\,4\mathbf{I} - 12\mathbf{J} + 4\mathbf{K}}{\sqrt{[(-4)^2 + (-12)^2 + 4^2]}} = -\frac{1}{\sqrt{11}}(\mathbf{I} + 3\mathbf{J} - \mathbf{K}).$$

Example 8.13. *Find the directional derivative of $f(x, y, z) = xy^3 + yz^3$ at the point $(2, -1, 1)$ in the direction of vector $\mathbf{I} + 2\mathbf{J} + 2\mathbf{K}$.* *(Bhopal, 2008 ; Kurukshetra, 2006 ; Rohtak, 2003)*

Solution. Here $\nabla f = \mathbf{I}(y^2) + \mathbf{J}(2xy + z^3) + \mathbf{K}(3yz^2) = \mathbf{I} - 3\mathbf{J} - 3\mathbf{K}$ at the point $(2, -1, 1)$.

$\therefore$ directional derivative of f in the direction $\mathbf{I} + 2\mathbf{J} + 2\mathbf{K}$

$$= (\mathbf{I} - 3\mathbf{J} - 3\mathbf{K})\,.\,\frac{\mathbf{I} + 2\mathbf{J} + 2\mathbf{K}}{\sqrt{(1^2 + 2^2 + 2^2)}} = (1\,.\,1 - 3\,.\,2 - 3\,.\,2)/3 = -\,3\frac{2}{3}.$$

Example 8.14. *Find the directional derivative of $f = x^2 - y^2 + 2z^2$ at the point $P\,(1, 2, 3)$ in the direction of the line PQ where Q is the point $(5, 0, 4)$. Also calculate the magnitude of the maximum directional derivative.*

Solution. We have $\quad \nabla f = \left(\mathbf{I}\,\frac{\partial}{\partial x} + \mathbf{J}\,\frac{\partial}{\partial y} + \mathbf{K}\,\frac{\partial}{\partial z}\right)(x^2 - y^2 + 2z^2) = 2x\mathbf{I} - 2y\mathbf{J} + 4z\mathbf{K}$

$$= 2\mathbf{I} - 4\mathbf{J} + 12\mathbf{K} \text{ at } P(1, 2, 3)$$

Also $\qquad \vec{PQ} = \vec{OQ} - \vec{OP} = (5\mathbf{I} + 0\mathbf{J} + 4\mathbf{K}) - (\mathbf{I} + 2\mathbf{J} + 3\mathbf{K}) = 4\mathbf{I} - 2\mathbf{J} + \mathbf{K} = \mathbf{A}$ (say)

$\therefore \qquad$ unit vector of $\mathbf{A} = \hat{A} = \dfrac{\mathbf{A}}{a} = \dfrac{4\mathbf{I} - 2\mathbf{J} + \mathbf{K}}{\sqrt{(16 + 4 + 1)}} = \dfrac{4\mathbf{I} + 2\mathbf{J} + \mathbf{K}}{\sqrt{21}}$

Thus the directional derivative of f in the direction of $\vec{PQ}$

$$\nabla f\,.\,\hat{A} = (2\mathbf{I} - 4\mathbf{J} + 12\mathbf{K})\,.\,(4\mathbf{I} - 2\mathbf{J} + \mathbf{K})/\sqrt{21}$$

$$= (8 + 8 + 12)/\sqrt{21} = 28/\sqrt{21}$$

The directional derivative of its maximum in the direction of the normal to the surface *i.e.*, in the direction of ∇f.

Hence maximum value of this directional derivative

$$= |\,\nabla f\,| = |\;2\mathbf{I} - 4\mathbf{J} + 12\mathbf{K}\;| = (4 + 16 + 144) = \sqrt{164}\,.$$

Example 8.15. *Find the directional derivative of $\phi = 5x^2y - 5y^2z + 2.5z^2x$ at the point $P(1, 1, 1)$ in the direction of the line $\frac{x-1}{2} = \frac{y-3}{-2} = z$.* (Bhopal, 2008 ; U.P.T.U., 2004)

Solution. We have $\nabla\phi = \mathbf{I}\frac{\partial\phi}{\partial x} + \mathbf{J}\frac{\partial\phi}{\partial y} + \mathbf{K}\frac{\partial\phi}{\partial z}$

$$= (10xy + 2.5z^2)\,\mathbf{I} + (5x^2 - 10yz)\,\mathbf{J} + (-5y^2 + 5zx)\,\mathbf{K}$$
$$= 12.5\mathbf{I} - 5\mathbf{J} \text{ at } P(1, 1, 1)$$

Also direction of the given line is $\hat{A} = \frac{2\mathbf{I} - 2\mathbf{J} + \mathbf{K}}{3}$

Hence the required directional derivative

$$= \nabla\phi \,.\, \hat{A} = (12.5\mathbf{I} - 5\mathbf{J}) \,.\, (2\mathbf{I} - 2\mathbf{J} + \mathbf{K})/3 = (25 + 10)/3 = 11\frac{2}{3}.$$

Example 8.16. *Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$.* (V.T.U., 2010 ; Kottayam, 2005 ; U.P.T.U., 2003)

Solution. Let $f_1 = x^2 + y^2 + z^2 - 9 = 0$ and $f_2 = x^2 + y^2 - z - 3 = 0$

Then $N_1 = \nabla f_1$ at $(2, -1, 2) = (2x\mathbf{I} + 2y\mathbf{J} + 2z\mathbf{K})$ at $(2, -1, 2) = 4\mathbf{I} - 2\mathbf{J} + 4\mathbf{K}$

and $N_2 = \nabla f_2$ at $(2, -1, 2) = (2x\mathbf{I} + 2y\mathbf{J} - \mathbf{K})$ at $(2, -1, 2) = 4\mathbf{I} - 2\mathbf{J} - \mathbf{K}$

Since the angle θ between the two surfaces at a point is the angle between their normals at that point and $\mathbf{N}_1, \mathbf{N}_2$ are the normals at $(2, -1, 2)$ to the given surfaces, therefore

$$\cos\theta = \frac{\mathbf{N}_1 \,.\, \mathbf{N}_2}{n_1 n_2} = \frac{(4I - 2J + 4K) \,.\, (4I - 2J - K)}{\sqrt{(16 + 4 + 16)}\,\sqrt{(16 + 4 + 1)}}$$
$$= \frac{4(4) + (-2)(-2) + 4(-1)}{6\sqrt{21}} = \frac{16}{6\sqrt{21}}$$

Hence the required angle $\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$.

Example 8.17. *Find the values of a and b such that the surface $ax^2 - byz = (a + 2)\,x$ and $4x^2y + z^3 = 4$ cut orthogonally at $(1, -1, 2)$.* (Madras, 2004)

Solution. Let $f_1 = ax^2 - byz - (a + 2)\,x = 0$ ...(*i*)

and $f_2 = 4x^2y + z^3 - 4 = 0$ ...(*ii*)

Then $\nabla f_1 = (2ax - a - 2)\mathbf{I} - 4z\mathbf{J} - by\mathbf{K} = (a - 2)\,\mathbf{I} - 2b\mathbf{J} + b\mathbf{K}$ at $(1, -1, 2)$.

$\nabla f_2 = 8xy\mathbf{I} + 4x^2\mathbf{J} + 3z^2\mathbf{K} = -8\mathbf{I} + 4\mathbf{J} + 12\mathbf{K}$ at $(1, -1, 2)$.

The surfaces (*i*) and (*ii*) will cut orthogonally if $\nabla f_1 \,.\, \nabla f_2 = 0$, *i.e.*, $-8\,(a - 2) - 8b + 12b = 0$

or $-2a + b + 4 = 0$ ...(*iii*)

Also since the point $(1, -1, 2)$ lies on (*i*) and (*ii*),

$\therefore$ $a + 2b - (a + 2) = 0$ or $b = 1$

From (*iii*), $-2a + 5 = 0$ or $a = 5/2$.

Hence $a = 5/2$ and $b = 1$.

PROBLEMS 8.3

1. (*a*) Find $\nabla\phi$, if $\phi = \log\,(x^2 + y^2 + z^2)$. (*b*) Show that grad $(1/r) = -\mathbf{R}/r^3$.
2. Find a unit vector normal to the surface $x^3 + y^3 + 3xyz = 3$ at the point $(1, 2, -1)$. (P.T.U., 1999)
3. Find the directional derivative of $\phi = x^2yz + 4xz^2$ at the point $(1, -2, 1)$ in the direction of the vector $2\mathbf{I} - \mathbf{J} - 2\mathbf{K}$. (V.T.U., 2007 ; Rohtak 2006 S ; J.N.T.U., 2006 ; U.P.T.U., 2006)
4. What is the directional derivative of $\phi = xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of the normal to the surface $x \log z - y^2 = -4$ at $(-1, 2, 1)$? (S.V.T.U., 2009)

5. Find the values of constants a, b, c so that the directional derivative of $p = axy^2 + byz + cz^2x^3$ at $(1, 2, -1)$ has a maximum magnitude 64 in the direction parallel to the z-axis. *(Rajasthan, 2006)*
6. Find the directional derivative of $\phi = x^4 + y^4 + z^4$ at the point $A(1, -2, 1)$ in the direction AB where B is $(2, 6, -1)$. Also find the maximum directional derivative of ϕ at $(1, -2, 1)$. *(Mumbai, 2009)*
7. If the directional derivative of $\phi = ax^2y + by^2z + cz^2x$ at the point $(1, 1, 1)$ has maximum magnitude 15 in the direction parallel to the line $\dfrac{x-1}{2} = \dfrac{y-3}{-2} = z$, find the values of a, b and c. *(U.P.T.U., 2002)*
8. In what direction from $(3, 1, -2)$ is the directional derivative of $\phi = x^2y^2z^4$ maximum ? Find also the magnitude of this maximum. *(Rohtak, 2003)*
9. What is the greatest rate of increase of $u = xyz^2$ at the point $(1, 0, 3)$? *(Bhopal, 2008)*
10. The temperature of points in space is given by $T(x, y, z) = x^2 + y^2 - z$. A mosquito located at $(1, 1, 2)$ desires to fly in such a direction that it will get warm as soon as possible. In what direction should it move ?
11. Calculate the angle between the normals to the surface $xy = z^2$ at the points $(4, 1, 2)$ and $(3, 3, -3)$.
12. Find the angle between the tangent planes to the surfaces $x \log z = y^2 - 1, x^2y = 2 - z$ at the point $(1, 1, 1)$. *(Hissar, 2005 S ; J.N.T.U., 2003)*
13. Find the values of a and b so that the surface $5x^2 - 2yz - 9z = 0$ may cut the surface $ax^2 + by^3 = 4$ orthogonally at $(1, -1, 2)$. *(Nagpur, 2009)*
14. If f and $\mathbf{G}$ are point functions, prove that the components of the latter normal and tangential to the surface $f = 0$ are
$$\frac{(\mathbf{G} \cdot \nabla f)\nabla f}{(\nabla f)^2} \text{ and } \frac{\nabla f \times (\mathbf{G} \times \nabla f)}{(\nabla f)^2}$$ *[Cf. Ex. 3.24]*

8.6 DEL APPLIED TO VECTOR POINT FUNCTIONS

(1) Divergence. *The divergence of a continuously differentiable vector point function* $\mathbf{F}$ *is denoted by div* $\mathbf{F}$ *and is defined by the equation*

$$\textbf{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \mathbf{I} \cdot \frac{\partial \mathbf{F}}{\partial x} + \mathbf{J} \cdot \frac{\partial \mathbf{F}}{\partial y} + \mathbf{K} \cdot \frac{\partial \mathbf{F}}{\partial z}$$

If $\qquad \mathbf{F} = f\mathbf{I} + \phi\mathbf{J} + \psi\mathbf{K}$

then $$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \left(\mathbf{I}\frac{\partial}{\partial x} + \mathbf{J}\frac{\partial}{\partial y} + \mathbf{K}\frac{\partial}{\partial z}\right) \cdot (f\mathbf{I} + \phi\mathbf{J} + \psi\mathbf{K}) = \frac{\partial f}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial z}$$

(2) Curl. *The curl of a continuously differentiable vector point function* $\mathbf{F}$ *is defined by the equation*

$$\textbf{curl } \mathbf{F} = \nabla \times \mathbf{F} = \mathbf{I} \times \frac{\partial \mathbf{F}}{\partial x} + \mathbf{J} \times \frac{\partial \mathbf{F}}{\partial y} + \mathbf{K} \times \frac{\partial \mathbf{F}}{\partial z}$$

If $\mathbf{F} = f\mathbf{I} + \phi\mathbf{J} + \psi\mathbf{K}$ then curl $\mathbf{F} = \nabla \times \mathbf{F} = \left(\mathbf{I}\frac{\partial}{\partial x} + \mathbf{J}\frac{\partial}{\partial y} + \mathbf{K}\frac{\partial}{\partial z}\right) \times (f\mathbf{I} + \phi\mathbf{J} + \psi\mathbf{K})$

$$= \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f & \phi & \psi \end{vmatrix} = \mathbf{I}\left(\frac{\partial \psi}{\partial y} - \frac{\partial \phi}{\partial z}\right) + \mathbf{J}\left(\frac{\partial f}{\partial z} - \frac{\partial \psi}{\partial x}\right) + \mathbf{K}\left(\frac{\partial \phi}{\partial x} - \frac{\partial f}{\partial y}\right).$$

Example 8.18. *If* $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$, *show that*
(i) $\nabla \cdot \mathbf{R} = 3$ $\qquad$ *(ii)* $\nabla \times \mathbf{R} = 0$. *(V.T.U. 2008 ; P.T.U., 2006 ; U.P.T.U., 2006)*

Solution. *(i)* $\quad \nabla \cdot \mathbf{R} = \dfrac{\partial}{\partial x}(x) + \dfrac{\partial}{\partial y}(y) + \dfrac{\partial}{\partial z}(z) = 3.$

(ii) $$\nabla \times \mathbf{R} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \mathbf{I}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) - \mathbf{J}\left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right) + \mathbf{K}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)$$

$$= \mathbf{I}(0 - 0) - \mathbf{J}(0 - 0) + \mathbf{K}(0 - 0) = \mathbf{0}.$$

[Remember : *div* $\mathbf{R} = \mathbf{3}$; *curl* $\mathbf{R} = \mathbf{0}$**]**

Example 8.19. *Find div* **F** *and curl* **F**, *where* **F** = *grad* $(x^3 + y^3 + z^3 - 3xyz)$.

(*V.T.U.*, 2008 ; *Kurukshetra*, 2006 ; *Burdwan*, 2003)

Solution. If $u = x^3 + y^3 + z^3 - 3xyz$, then

$$\mathbf{F} = \nabla u = \mathbf{I}\frac{\partial u}{\partial x} + \mathbf{J}\frac{\partial u}{\partial y} + \mathbf{K}\frac{\partial u}{\partial z} = \mathbf{I}(3x^2 - 3yz) + \mathbf{J}(3y^2 - 3zx) + \mathbf{K}(3z^2 - 3xy)$$

$$\therefore \qquad \text{div } \mathbf{F} = \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3zx) + \frac{\partial}{\partial z}(3z^2 - 3xy) = 6(x + y + z)$$

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3(x^2 - yz) & 3(y^2 - zx) & 3(z^2 - xy) \end{vmatrix}$$

$$= \mathbf{I}(-3x + 3x) - \mathbf{J}(-3y + 3y) + \mathbf{K}(-3z + 3z) = \mathbf{0}.$$

8.7 (1) PHYSICAL INTERPRETATION OF DIVERGENCE

Consider the motion of the fluid having velocity $\mathbf{V} = v_x\mathbf{I} + v_y\mathbf{J} + v_z\mathbf{K}$ at a point $P(x, y, z)$. Consider a small parallelopiped with edges δx, δy, δz parallel to the axes in the mass of fluid, with one of its corners at P (Fig. 8.7).

$\therefore$ the amount of fluid entering the face PB' in unit time $= v_y \delta z \delta x$ and the amount of fluid leaving the face $P'B$ in unit time

$$= v_{y+\delta y}\delta z \delta x = \left(v_y + \frac{\partial v_y}{\partial y}\delta y\right)\delta z \delta x \quad \text{nearly}$$

$\therefore$ the net decrease of the amount of fluid due to flow across these two faces $= \dfrac{\partial v_y}{\partial y}\delta x \delta y \delta z$.

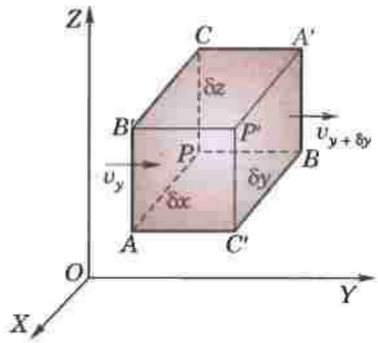

Fig. 8.7

Finding similarly the contributions of other two pairs of faces, we have the total decrease of amount of fluid inside the parallelopiped per unit time $= \left(\dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} + \dfrac{\partial v_z}{\partial z}\right)\delta x \delta y \delta z$.

Thus the rate of loss of fluid per unit volume

$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \text{div } \mathbf{V}.$$

Hence *div* **V** *gives the rate at which fluid is originating at a point per unit volume.*

Similarly, if V represents an electric flux, div **V** is the amount of flux which diverges per unit volume in unit time. If **V** represents heat flux, div **V** is the rate at which heat is issuing from a point per unit volume. In general, the divergence of a vector point function representing any physical quantity gives at each point, the rate per unit volume at which the physical quantity is issuing from that point. This explains the justification for the name *divergence of a vector point function.*

If the fluid is incompressible, there can be no gain or loss in the volume element. Hence div **V** = 0, which is known in Hydrodynamics as the **equation of continuity** for incompressible fluids.

Def. *If the flux entering any element of space is the same as that leaving it, i.e., div* **V** *= 0 everywhere then such a point function is called a* **solenoidal vector function.**

(2) Physical interpretation of curl. Consider the motion of a rigid body rotating about a fixed axis through O. If Ω be its angular velocity, then the velocity **V** of any particle $P(\mathbf{R})$ of the body is given by $\mathbf{V} = \Omega \times \mathbf{R}$.

[See p. 91]

If $\qquad \Omega = \omega_1\mathbf{I} + \omega_2\mathbf{J} + \omega_3\mathbf{K}$ and $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$

then
$$\mathbf{V} = \Omega \times \mathbf{R} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix} = \mathbf{I}(\omega_2 z - \omega_3 y) + \mathbf{J}(\omega_3 x - \omega_1 z) + \mathbf{C}(\omega_1 y - \omega_2 x)$$

$$\therefore \qquad \text{curl } \mathbf{V} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y, & \omega_3 x - \omega_1 z, & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= \mathbf{I}(\omega_1 + \omega_1) + \mathbf{J}(\omega_2 + \omega_2) + \mathbf{K}(\omega_3 + \omega_3) \qquad [\because \ \omega_1, \omega_2, \omega_3 \text{ are constants.}]$$

$$= 2(\omega_1 \mathbf{I} + \omega_2 \mathbf{J} + \omega_3 \mathbf{K}) = 2\Omega. \quad \text{Hence} \quad \Omega = \frac{1}{2} \text{ curl } \mathbf{V}$$

Thus the angular velocity of rotation at any point is equal to half the curl of the velocity vector which justifies the name *rotation* used for curl.

In general, the curl of any vector point function gives the measure of the angular velocity at any point of the vector field.

Def. *Any motion in which the curl of the velocity vector is zero is said to be* **irrotational**, *otherwise* **rotational.**

Example 8.20. *Prove that div* $(r^n\mathbf{R}) = (n+3)\, r^n$. *Hence show that* $\mathbf{R}/r^3$ *is solenoidal.* *(V.T.U., 2006 ; U.P.T.U., 2006 ; P.T.U., 2005)*

Solution. We have $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$ and $r = \sqrt{(x^2 + y^2 + z^2)}$

$$\therefore \qquad div\,(r^n\mathbf{R}) = \nabla \cdot (x^2 + y^2 + z^2)^{n/2}\,(x\mathbf{I} + y\mathbf{J} + z\mathbf{K})$$

$$= \frac{\partial}{\partial x}[x(x^2 + y^2 + z^2)^{n/2}] + \frac{\partial}{\partial y}[y(x^2 + y^2 + z^2)^{n/2}] + \frac{\partial}{\partial z}[z(x^2 + y^2 + z^2)^{n/2}]$$

$$= \Sigma \left\{ 1 \,.\, (x^2 + y^2 + z^2)^{n/2} + x \,.\, \frac{n}{2}(x^2 + y^2 + z^2)^{\frac{n}{2} - 1} \,.\, 2x \right\}$$

$$= \Sigma r^n + n\, \Sigma x^2 (x^2 + y^2 + z^2)^{\frac{n}{2} - 1} = 3r^n + nr^2 \,.\, r^{n-2}$$

Thus $div\,(r^n\mathbf{R}) = (n+3)\, r^n$

When $n = -3$, $div\,(\mathbf{R}/r^3) = 0$ *i.e.*, $\mathbf{R}/r^3$ is solenoidal.

Example 8.21. *Show that* $r^\alpha \mathbf{R}$ *is any irrotational vector for any value of* α *but is solenoidal if* $\alpha + 3 = 0$ *where* $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$ *and r is the magnitude of* $\mathbf{R}$. *(V.T.U., 2006 ; Kottayam, 2005)*

Solution. Let $\mathbf{A} = r^\alpha \mathbf{R} = (x^2 + y^2 + z^2)^{\alpha/2}\,(x\mathbf{I} + y\mathbf{I} + z\mathbf{K}) = \Sigma x\,(x^2 + y^2 + z^2)^{\alpha/2}\,\mathbf{I}$

$$\therefore \qquad \text{curl } \mathbf{A} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x(x^2 + y^2 + z^2)^{\alpha/2} & y(x^2 + y^2 + z^2)^{\alpha/2} & z(x^2 + y^2 + z^2)^{\alpha/2} \end{vmatrix}$$

$$= \Sigma\, \mathbf{I} \left\{ \frac{\alpha z}{2}(x^2 + y^2 + z^2)^{\alpha/2 - 1}\,(2y) - \frac{\alpha y}{2}(x^2 + y^2 + z^2)^{\alpha/2 - 1} \,.\, 2z \right\} = 0$$

Hence $\mathbf{A}$ is irrotational for any value of α.

But $\qquad div\, \mathbf{A} = \nabla \cdot (r^\alpha\, \mathbf{R}) = (\alpha + 3)\, r^\alpha$

which is zero for $\alpha + 3 = 0$, *i.e.*, $\mathbf{A}$ is solenoidal if $\alpha + 3 = 0$.

8.8 DEL APPLIED TWICE TO POINT FUNCTIONS

∇f and $\nabla \times \mathbf{F}$ being vector point functions, we can form their divergence and curl whereas $\nabla . \mathbf{F}$ being a scalar point function, we can have its gradients only. Thus we have the following five formulae :

(1) *div grad* $f = \nabla^2 f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$

(2) *curl grad* $f = \nabla \times \nabla f = \mathbf{0}$

(3) *div curl* $\mathbf{F} = \nabla \cdot \nabla \times \mathbf{F} = 0$

(4) *curl curl* $\mathbf{F}$ = *grad div* $\mathbf{F} - \nabla^2\mathbf{F}$, *i.e.*, $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$

(5) *grad div* $\mathbf{F}$ = *curl curl* $\mathbf{F} + \nabla^2\mathbf{F}$, *i.e.*, $\nabla(\nabla \cdot \mathbf{F}) = \nabla \times (\nabla \times \mathbf{F}) + \nabla^2\mathbf{F}$.

Proofs. **(1)** $\nabla^2 f = \nabla \cdot \nabla f = \nabla \cdot \left(\mathbf{I}\frac{\partial f}{\partial x} + \mathbf{J}\frac{\partial f}{\partial y} + \mathbf{K}\frac{\partial f}{\partial z}\right)$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f$$

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called the *Laplacian operator* and $\nabla^2 f = 0$ is called the *Laplace's equation.*

(2) $$\nabla \times \nabla f = \nabla \times \left(\mathbf{I}\frac{\partial f}{\partial x} + \mathbf{J}\frac{\partial f}{\partial y} + \mathbf{K}\frac{\partial f}{\partial z}\right) = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \Sigma \mathbf{I}\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right) = \mathbf{0}$$ *(V.T.U., 2007)*

(3) $$\nabla \cdot \nabla \times \mathbf{F} = \left(\Sigma \mathbf{I}\frac{\partial}{\partial x}\right) \cdot \left(\mathbf{I} \times \frac{\partial \mathbf{F}}{\partial x} + \mathbf{J} \times \frac{\partial \mathbf{F}}{\partial y} + \mathbf{K} \times \frac{\partial \mathbf{F}}{\partial z}\right)$$

$$= \Sigma \mathbf{I} \cdot \left(\mathbf{I} \times \frac{\partial^2 \mathbf{F}}{\partial x^2} + \mathbf{J} \times \frac{\partial^2 \mathbf{F}}{\partial x \partial y} + \mathbf{K} \times \frac{\partial^2 \mathbf{F}}{\partial x \partial z}\right)$$

$$= \Sigma\left(\mathbf{I} \times \mathbf{I} \cdot \frac{\partial^2 \mathbf{F}}{\partial x^2} + \mathbf{I} \times \mathbf{J} \cdot \frac{\partial^2 \mathbf{F}}{\partial x \partial y} + \mathbf{I} \times \mathbf{K} \cdot \frac{\partial^2 \mathbf{F}}{\partial x \partial z}\right) = \Sigma\left(\mathbf{K} \cdot \frac{\partial^2 \mathbf{F}}{\partial x \partial y} - \mathbf{J} \cdot \frac{\partial^2 \mathbf{F}}{\partial x \partial z}\right) = 0.$$

(4) $$\nabla \times (\nabla \times \mathbf{F}) = \left(\Sigma \mathbf{I}\frac{\partial}{\partial x}\right) \times \left(\mathbf{I} \times \frac{\partial \mathbf{F}}{\partial x} + \mathbf{J} \times \frac{\partial \mathbf{F}}{\partial y} + \mathbf{K} \times \frac{\partial \mathbf{F}}{\partial z}\right)$$

$$= \Sigma \mathbf{I} \times \left(\mathbf{I} \times \frac{\partial^2 \mathbf{F}}{\partial x^2} + \mathbf{J} \times \frac{\partial^2 \mathbf{F}}{\partial x \partial y} + \mathbf{K} \times \frac{\partial^2 \mathbf{F}}{\partial x \partial z}\right)$$

$$= \Sigma\left[\left\{\left(\mathbf{I} \cdot \frac{\partial^2 \mathbf{F}}{\partial x^2}\right)\mathbf{I} - (\mathbf{I} \cdot \mathbf{I})\frac{\partial^2 \mathbf{F}}{\partial x^2}\right\} + \left\{\left(\mathbf{I} \cdot \frac{\partial^2 \mathbf{F}}{\partial x \partial y}\right)\mathbf{J} - (\mathbf{I} \cdot \mathbf{J})\frac{\partial^2 \mathbf{F}}{\partial x \partial y}\right\} + \left\{\left(\mathbf{I} \cdot \frac{\partial^2 \mathbf{F}}{\partial x \partial z}\right)\mathbf{K} - (\mathbf{I} \cdot \mathbf{K})\frac{\partial^2 \mathbf{F}}{\partial x \partial z}\right\}\right]$$

$$= \Sigma\left[\left(\mathbf{I} \cdot \frac{\partial^2 \mathbf{F}}{\partial x^2}\right)\mathbf{I} + \left(\mathbf{I} \cdot \frac{\partial^2 \mathbf{F}}{\partial x \partial y}\right)\mathbf{J} + \left(\mathbf{I} \cdot \frac{\partial^2 \mathbf{F}}{\partial x \partial z}\right)\mathbf{K}\right] - \Sigma\frac{\partial^2 \mathbf{F}}{\partial x^2}$$

$$= \Sigma \mathbf{I}\frac{\partial}{\partial x}\left(\mathbf{I} \cdot \frac{\partial \mathbf{F}}{\partial x} + \mathbf{J} \cdot \frac{\partial \mathbf{F}}{\partial y} + \mathbf{K} \cdot \frac{\partial \mathbf{F}}{\partial z}\right) - \Sigma\frac{\partial^2 \mathbf{F}}{\partial x^2} = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}.$$ *(Madras, 2006)*

(5) is just another way of writing (4) above.

Obs. *Interpretation of* ∇ *as a vector according to rules of vector products leads to correct results so far so the repeated application of* ∇ *is concerned.*

e.g., **1.** $\nabla \cdot \nabla f = \nabla^2 f$ $(\because \nabla \cdot \nabla = \nabla^2)$

2. $\nabla \times \nabla f = \mathbf{0}$ $(\because \nabla \times \nabla = \mathbf{0})$

3. $\nabla \cdot \nabla \times \mathbf{F} = \mathbf{0}$ $(\because [\nabla\nabla\mathbf{F}] = 0)$

4. $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$ by expanding it as a vector triple product.

8.9 DEL APPLIED TO PRODUCTS OF POINT FUNCTIONS

To prove that

(1) grad $(fg) = f$ (grad g) + g(grad f) *i.e.* $\nabla(fg) = f\nabla g + g\nabla f$.

(2) *div* $(f\mathbf{G})$ = (grad f) . $\mathbf{G} + f$ (div $\mathbf{G}$) *i.e.* $\nabla(f\mathbf{G}) = \nabla f \cdot \mathbf{G} + f\nabla \cdot \mathbf{G}$

(3) *curl* $(f\mathbf{G})$ = (*grad* f) $\times \mathbf{G} + f$ (*curl* $\mathbf{G}$) *i.e.* $\nabla \times (f\mathbf{G}) = \nabla f \times \mathbf{G} + f\nabla \times \mathbf{G}$

(4) *grad* $(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} + \mathbf{F} \times$ *curl* $\mathbf{G} + \mathbf{G} \times$ *curl* $\mathbf{F}$

i.e., $\nabla(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} + \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F})$

(5) *div* (**F** × **G**) = **G** . (*curl* **F**) − **F** . (*curl* **G**) *i.e.,* $\nabla . (\mathbf{F} \times \mathbf{G}) = \mathbf{G} . (\nabla \times \mathbf{F}) - \mathbf{F} . (\nabla \times \mathbf{G})$

(6) *curl* (**F** × **G**) = **F** (*div* **G**) − **G** (*div* **F**) + (**G** . ∇) **F** − (**F** . ∇) **G**

i.e., $\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\nabla . \mathbf{G}) - \mathbf{G}(\nabla . \mathbf{F}) + (\mathbf{G} . \nabla)\mathbf{F} - (\mathbf{F} . \nabla)\mathbf{G}$

Proofs (1) $\nabla (fg) = \Sigma \mathbf{I} . \dfrac{\partial}{\partial x}(fg) = \Sigma \mathbf{I}\left(f\dfrac{\partial g}{\partial x} + g\dfrac{\partial f}{\partial x}\right)$

$$= f\Sigma \mathbf{I}\frac{\partial g}{\partial x} + g\,\Sigma \mathbf{I}\frac{\partial f}{\partial x} = f\nabla g + g\nabla f$$

(2) $\nabla . (f\mathbf{G}) = \Sigma \mathbf{I} . \dfrac{\partial}{\partial x}(f\mathbf{G}) = \Sigma \mathbf{I} . \left(\dfrac{\partial f}{\partial x}\mathbf{G} + f\dfrac{\partial \mathbf{G}}{\partial x}\right)$

$$= \left(\Sigma \mathbf{I}\frac{\partial f}{\partial x}\right) . \mathbf{G} + f\left(\Sigma \mathbf{I} . \frac{\partial \mathbf{G}}{\partial x}\right) = \nabla f . \mathbf{G} + f\nabla . \mathbf{G}$$ (*V.T.U., 2011*)

(3) $\nabla \times (f\mathbf{G}) = \Sigma \mathbf{I} \times \dfrac{\partial}{\partial x}(f\mathbf{G}) = \Sigma \mathbf{I} \times \left(f\dfrac{\partial \mathbf{G}}{\partial x} + \dfrac{\partial f}{\partial x}\mathbf{G}\right)$

$$= f\Sigma \mathbf{I} \times f\frac{\partial \mathbf{G}}{\partial x} + \Sigma \mathbf{I}\frac{\partial f}{\partial x} \times \mathbf{G} = f\nabla \times \mathbf{G} + \nabla f \times \mathbf{G}$$ (*V.T.U. 2008*)

(4) $$\nabla(\mathbf{F} . \mathbf{G}) = \Sigma \mathbf{I}\frac{\partial}{\partial x}(\mathbf{F} . \mathbf{G}) = \Sigma \mathbf{I}\left(\frac{\partial \mathbf{F}}{\partial x} . \mathbf{G} + \mathbf{F} . \frac{\partial \mathbf{G}}{\partial x}\right) = \Sigma \mathbf{I}\frac{\partial \mathbf{F}}{\partial x} . \mathbf{G} + \Sigma \mathbf{I}\left(\mathbf{F} . \frac{\partial \mathbf{G}}{\partial x}\right) \quad ...(i)$$

Now $\mathbf{G} \times \left(\mathbf{I} \times \dfrac{\partial \mathbf{F}}{\partial x}\right) = \left(\mathbf{G} . \dfrac{\partial \mathbf{F}}{\partial x}\right)\mathbf{I} - (\mathbf{G} . \mathbf{I})\dfrac{\partial \mathbf{F}}{\partial x}$

or $$\left(\mathbf{G} . \frac{\partial \mathbf{F}}{\partial x}\right)\mathbf{I} = \mathbf{G} \times \left(\mathbf{I} \times \frac{\partial \mathbf{F}}{\partial x}\right) + (\mathbf{G} . \mathbf{I})\frac{\partial \mathbf{F}}{\partial x}$$

$$\therefore \quad \Sigma\left(\mathbf{G} . \frac{\partial \mathbf{F}}{\partial x}\right)\mathbf{I} = \mathbf{G} \times \Sigma \mathbf{I} \times \frac{\partial \mathbf{F}}{\partial x} + \Sigma(\mathbf{G} . \mathbf{I})\frac{\partial \mathbf{F}}{\partial x} = \mathbf{G} \times (\nabla \times \mathbf{F}) + (\mathbf{G} . \nabla)\mathbf{F} \quad ...(ii)$$

Interchanging **F** and **G**, $\Sigma\left(\mathbf{F} . \dfrac{\partial \mathbf{G}}{\partial x}\right)\mathbf{I} = \mathbf{F} \times (\nabla \times \mathbf{G}) + (\mathbf{F} . \nabla)\mathbf{G}$ ...(*iii*)

Substituting in (*i*) from (*ii*) and (*iii*), we get

$$\nabla(\mathbf{F} . \mathbf{G}) = (\mathbf{F} . \nabla)\mathbf{G} + (\mathbf{G} . \nabla)\mathbf{F} + \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F})$$

(5) $$\nabla . (\mathbf{F} \times \mathbf{G}) = \Sigma \mathbf{I} . \frac{\partial}{\partial x}(\mathbf{F} \times \mathbf{G}) = \Sigma \mathbf{I}\left(\frac{\partial \mathbf{F}}{\partial x} \times \mathbf{G} + F \times \frac{\partial \mathbf{G}}{\partial x}\right) = \Sigma \mathbf{I} . \frac{\partial \mathbf{F}}{\partial x} \times \mathbf{G} - \Sigma \mathbf{I} . \left(\frac{\partial \mathbf{G}}{\partial x} \times \mathbf{F}\right)$$

$$= \Sigma\left(I \times \frac{\partial \mathbf{F}}{\partial x}\right) . \mathbf{G} - \Sigma\left(I \times \frac{\partial \mathbf{G}}{\partial x}\right) . \mathbf{F}$$ $[\because \ \mathbf{A} . (\mathbf{B} \times \mathbf{C}) = (A \times B) . C]$

$$= \mathbf{G} . (\nabla \times \mathbf{F}) - \mathbf{F} . (\nabla \times \mathbf{G})$$

(6) $\nabla \times (\mathbf{F} \times \mathbf{G}) = \Sigma \mathbf{I} \times \dfrac{\partial}{\partial x}(\mathbf{F} \times \mathbf{G}) = \Sigma \mathbf{I} \times \left(\dfrac{\partial \mathbf{F}}{\partial x} \times \mathbf{G} + \mathbf{F} \times \dfrac{\partial \mathbf{G}}{\partial x}\right)$

$$= \Sigma\left[(\mathbf{I} . \mathbf{G})\frac{\partial \mathbf{F}}{\partial x} - \left(\mathbf{I} . \frac{\partial \mathbf{F}}{\partial x}\right)\mathbf{G}\right] + \Sigma\left[\left(\mathbf{I} . \frac{\partial \mathbf{G}}{\partial x}\right)\mathbf{F} - (\mathbf{I} . \mathbf{F})\frac{\partial \mathbf{G}}{\partial x}\right]$$

$$= \Sigma(\mathbf{G} . \mathbf{I})\frac{\partial \mathbf{F}}{\partial x} - \mathbf{G}\Sigma \mathbf{I} . \frac{\partial \mathbf{F}}{\partial x} + \mathbf{F}\Sigma \mathbf{I} . \frac{\partial \mathbf{G}}{\partial x} - \Sigma(\mathbf{F} . \mathbf{I})\frac{\partial \mathbf{G}}{\partial x}$$

$$= \mathbf{F}\left(\Sigma \mathbf{I} . \frac{\partial \mathbf{G}}{\partial x}\right) - \mathbf{G}\Sigma\left(\mathbf{I} . \frac{\partial \mathbf{F}}{\partial x}\right) + \Sigma(\mathbf{G} . \mathbf{I})\frac{\partial \mathbf{F}}{\partial x} - \Sigma(\mathbf{F} . \mathbf{I})\frac{\partial \mathbf{G}}{\partial x}$$

$$= \mathbf{F}(\nabla . \mathbf{G}) - \mathbf{G}(\nabla . \mathbf{F}) + (\mathbf{G} . \nabla)\mathbf{F} - (\mathbf{F} . \nabla)\mathbf{G}$$

Rule to reproduce the above formulae easily :

(*i*) *Treating each of the factors as constants separately, expresss the results of ∇-operation as a sum of the two terms.*

(*ii*) *Transform each of the two terms, noting that ∇ always appears before a function and keeping in mind whether the result of operation is a scalar or a vector.* To carry out the simplification, we may sometimes, employ the properties of triple products.

(*iii*) *Restore the change of treating the functions as constants.*

Let us illustrate the application of this rule to **(2)**, **(4)** and **(6)** above :

(2) $\nabla . (f\mathbf{G}) = \nabla . (f_c\,\mathbf{G} + f\,\mathbf{G}_c) = f_c\,\nabla . \mathbf{G} + \mathbf{G}_c . \nabla f = f\,\nabla . \mathbf{G} + \mathbf{G} . \nabla f$

(4) $\nabla(\mathbf{F} . \mathbf{G}) = \nabla(\mathbf{F}_c . \mathbf{G}) + \nabla(\mathbf{F} . \mathbf{G}_c)$

$= [\mathbf{F}_c \times (\nabla \times \mathbf{G}) + (\mathbf{F}_c . \nabla)\mathbf{G}] + [\mathbf{G}_c \times (\nabla \times \mathbf{F}) + (\mathbf{G}_c . \nabla)\mathbf{F}]$

$= \mathbf{F} \times (\nabla \times \mathbf{G}) + (\mathbf{F} . \nabla)\,\mathbf{G} + \mathbf{G} \times (\nabla \times \mathbf{F}) + (\mathbf{G} . \nabla)\mathbf{F}$

(6) $\nabla \times (\mathbf{F} \times \mathbf{G}) = \nabla \times (\mathbf{F}_c \times \mathbf{G}) + \nabla \times (\mathbf{F} \times \mathbf{G}_c) = [\nabla . \mathbf{G}\mathbf{F}_c - (\mathbf{F}_c . \nabla)\mathbf{G}] + (\mathbf{G}_c . \nabla)\,\mathbf{F} - \nabla . \mathbf{F}\mathbf{G}_c]$

$= \mathbf{F}(\nabla . \mathbf{G}) - (\mathbf{F} . \nabla)\,\mathbf{G} + (\mathbf{G} . \nabla)\mathbf{F} - \mathbf{G}(\nabla . \mathbf{F}).$

Example 8.22. *Show that* $\nabla^2(r^n) = n(n+1)r^{n-2}$ *(S.V.T.U., 2006 ; J.N.T.U., 2006 ; U.P.T.U., 2005)*

Solution. $\nabla^2 r^n = \nabla . (\nabla r^n)$

$$= \nabla . \left(nr^{n-1}\frac{\mathbf{R}}{r}\right) = n\nabla . (r^{n-2}\mathbf{R}) = n[(\nabla r^{n-2}) . \mathbf{R} + r^{n-2}(\nabla . \mathbf{R})] \qquad \text{[By § 8.9 (2)]}$$

$$= n\left[(n-2)r^{n-3}\frac{\mathbf{R}}{r} . \mathbf{R} + r^{n-2}\,(3)\right] \qquad \text{[Using Ex. 8.18 (}i\text{)]}$$

$$= n[(n-2)r^{n-4}(r^2) + 3r^{n-2}] = n(n+1)\,r^{n-2} \qquad [\because \ \mathbf{R} . \mathbf{R} = r^2]$$

Otherwise : $$\nabla^2(r^n) = \frac{\partial^2(r^n)}{\partial x^2} + \frac{\partial^2(r^n)}{\partial y^2} + \frac{\partial^2(r^n)}{\partial z^2} \qquad \text{[By § 8.8 (1)]} \quad ...(i)$$

Now $$\frac{\partial(r^n)}{\partial x} = nr^{n-1}\frac{\partial r}{\partial x} = nr^{n-1}\frac{x}{r} = nr^{n-2}x \qquad [\because \ r^2 = x^2 + y^2 + z^2]$$

$$\therefore \quad \frac{\partial^2(r^n)}{\partial x^2} = n\left[r^{n-2} + (n-2)r^{n-3}\frac{\partial r}{\partial x}x\right] = n\left[r^{n-2} + (n-2)r^{n-3}\frac{x}{r}x\right]$$

$$= n\left[r^{n-2} + (n-2)r^{n-4}\,x^2\right] \qquad ...(ii)$$

Similarly, $$\frac{\partial^2(r^n)}{\partial y^2} = n\left[r^{n-2} + (n-2)r^{n-4}\,y^2\right] \qquad ...(iii)$$

$$\frac{\partial^2(r^n)}{\partial z^2} = n\left[r^{n-2} + (n-2)r^{n-4}\,z^2\right] \qquad ...(iv)$$

Adding (*ii*), (*iii*) and (*iv*), (*i*) gives

$$\nabla^2(r^n) = n\,[3r^{n-2} + (n-2)r^{n-4}(x^2 + y^2 + z^2)]$$
$$= n\,[3r^{n-2} + (n-2)r^{n-4}\,r^2] = n(n+1)r^{n-2}.$$

In particular $\nabla^2(1/r) = 0$. *(U.P.T.U., 2003 ; P.T.U., 2003)*

Example 8.23. *If* $u\mathbf{F} = \nabla v$, *where* u, v *are scalar fields and* $\mathbf{F}$ *is a vector field, show that* $\mathbf{F}$. *curl* $\mathbf{F} = 0$.

Solution. Since $$\mathbf{F} = \frac{1}{u}\nabla v \quad \therefore \quad \text{curl}\,\mathbf{F} = \nabla \times \left(\frac{1}{u}\nabla v\right)$$

or $$\text{curl}\,\mathbf{F} = \nabla\frac{1}{u} \times \nabla v + \frac{1}{u}\nabla \times (\nabla v) \qquad \text{[By § 8.9 (3)]}$$

$$= \nabla\frac{1}{u} \times \nabla v \qquad [\because \ \nabla \times \nabla v = 0$$

Hence $\mathbf{F}$. curl $\mathbf{F} = \frac{1}{u}\nabla v . \left(\nabla\frac{1}{u} \times \nabla v\right) = 0$, for it is a scalar triple product in which two factors are equal.

Example 8.24. *If* r *and* $\mathbf{R}$ *have their usual meanings and* $\mathbf{A}$ *is a constant vector, prove that* $\nabla \times \left(\frac{\mathbf{A} \times \mathbf{R}}{r^n}\right) = \frac{2-n}{r^n}\mathbf{A} + \frac{n(\mathbf{A} . \mathbf{R})}{r^{n+2}}\mathbf{R}.$ *(Mumbai, 2009 ; Kurukshetra, 2006 ; J.N.T.U., 2005)*

Solution. $\nabla \times [r^{-n}(\mathbf{A} \times \mathbf{R})] = r^{-n}[\nabla \times (\mathbf{A} \times \mathbf{R})] + \nabla r^{-n} \times (\mathbf{A} \times \mathbf{R})]$ [By § 8.9 (3)]

$= r^{-n}[(\nabla . \mathbf{R})\,\mathbf{A} - (\mathbf{A} . \nabla)\,\mathbf{R}] + (-nr^{-(n+1)}\,\mathbf{R}/r) \times (\mathbf{A} \times \mathbf{R})$

$$= r^{-n}(3\mathbf{A} - \mathbf{A}) - nr^{-(n+2)}\,\mathbf{R} \times (\mathbf{A} \times \mathbf{R}) \qquad [\because\ \nabla . \mathbf{R} = 3, (\mathbf{A} . \nabla)\,\mathbf{R} = \mathbf{A}]$$

$$= 2\mathbf{A}r^{-n} - nr^{-(n+2)}\,[(\mathbf{R} . \mathbf{R})\,\mathbf{A} - (\mathbf{A} . \mathbf{R})\,\mathbf{R}]$$

$$= \frac{2\mathbf{A}}{r^n} - \frac{n}{r^{n+2}}\,[r^2\mathbf{A} - (\mathbf{A} . \mathbf{R})\,\mathbf{R}] = \frac{2-n}{r^n}\,\mathbf{A} + \frac{n(\mathbf{A} . \mathbf{R})}{r^{n+2}}\,\mathbf{R}.$$

Example 8.25. *If r is the distance of a point (x, y, z) from the origin, prove that curl* $\left(\mathbf{K} \times grad\,\frac{1}{r}\right) + grad\left(\mathbf{K} . grad\,\frac{1}{r}\right) = 0$, *where* **K** *is the unit vector in the direction OZ.* (U.P.T.U., 2001)

Solution. $grad\,\dfrac{1}{r} = \left(\mathbf{I}\dfrac{\partial}{\partial x} + \mathbf{J}\dfrac{\partial}{\partial y} + \mathbf{K}\dfrac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)^{-1/2}$ $\qquad [\because\ r = \sqrt{(x^2 + y^2 + z^2)}\,]$

$$= -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}(2x\mathbf{I} + 2y\mathbf{J} + 2z\mathbf{K})$$

$$= -(x^2 + y^2 + z^2)^{-3/2}(x\mathbf{I} + y\mathbf{J} + z\mathbf{K})$$

$$curl\left(\mathbf{K} \times grad\frac{1}{r}\right) = \nabla \times [-(x^2 + y^2 + z^2)^{-3/2}(x\mathbf{J} - y\mathbf{I})]$$

$$= \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y/(x^2 + y^2 + z^2)^{3/2} & -x/(x^2 + y^2 + z^2)^{3/2} & 0 \end{vmatrix}$$

$$= \mathbf{I}\,\frac{\partial}{\partial z}\left\{\frac{x}{(x^2 + y^2 + z^2)^{3/2}}\right\} + \mathbf{J}\,\frac{\partial}{\partial z}\left\{\frac{y}{(x^2 + y^2 + z^2)^{3/2}}\right\}$$

$$- \mathbf{K}\left\{\frac{\partial}{\partial x}\left[\frac{x}{(x^2 + y^2 + z^2)^{3/2}}\right] + \frac{\partial}{\partial y}\left[\frac{y}{(x^2 + y^2 + z^2)^{3/2}}\right]\right\}$$

$$= \frac{-3xz\mathbf{I} - 3yz\mathbf{J} + (x^2 + y^2 - 2z^2)\mathbf{K}}{(x^2 + y^2 + z^2)^{5/2}} \qquad ...(i)$$

$$\text{grad}\left(\mathbf{K} . \text{grad}\,\frac{1}{r}\right) = \nabla\left\{-\mathbf{K} . \frac{(x\mathbf{I} + y\mathbf{J} + z\mathbf{K})}{(x^2 + y^2 + z^2)^{3/2}}\right\}$$

$$= \left(\mathbf{I}\frac{\partial}{\partial x} + \mathbf{J}\frac{\partial}{\partial y} + \mathbf{K}\frac{\partial}{\partial z}\right)\left\{\frac{-z}{(x^2 + y^2 + z^2)^{3/2}}\right\}$$

$$= \frac{3xz\mathbf{I}}{(x^2 + y^2 + z^2)^{5/2}} + \frac{3yz\mathbf{J}}{(x^2 + y^2 + z^2)^{5/2}} + \frac{(3z^2 - x^2 - y^2 - z^2)\mathbf{K}}{(x^2 + y^2 + z^2)^{5/2}}$$

$$= \frac{3xz\mathbf{I} + 3yz\mathbf{J} - (x^2 + y^2 - 2z^2)\mathbf{K}}{(x^2 + y^2 + z^2)^{5/2}} \qquad ...(ii)$$

Adding (*i*) and (*ii*), we get

$$\text{curl}\left(\mathbf{K} \times \text{grad}\,\frac{1}{r}\right) + \text{grad}\left(\mathbf{K} . \text{grad}\,\frac{1}{r}\right) = \mathbf{0}.$$

Example 8.26. *In electromagnetic theory, we have* $\nabla . \mathbf{D} = \rho$, $\nabla . \mathbf{H} = 0$, $\nabla \times D = -\dfrac{1}{c}\dfrac{\partial \mathbf{H}}{\partial t}$, $\nabla \times \mathbf{H} = \dfrac{1}{c}\left(\rho\mathbf{V} + \dfrac{\partial \mathbf{D}}{\partial t}\right)$. *Prove that*

(*i*) $\nabla^2\mathbf{D} - \dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{D}}{\partial t^2} = \nabla\rho + \dfrac{1}{c^2}\dfrac{\partial}{\partial t}(\rho\mathbf{V})$ $\qquad$ (*ii*) $\nabla^2\mathbf{H} - \dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{H}}{\partial t^2} = -\dfrac{1}{c}\,\nabla \times \rho\mathbf{V}$

Solution. (*i*) We have $\dfrac{1}{c^2}\left\{\dfrac{\partial^2 \mathbf{D}}{\partial t^2}+\dfrac{\partial}{\partial t}(\rho\mathbf{V})\right\}=\dfrac{1}{c^2}\dfrac{\partial}{\partial t}\left(\dfrac{\partial \mathbf{D}}{\partial t}+\rho\mathbf{V}\right)$

$$= \frac{1}{c}\frac{\partial}{\partial t}(\nabla\times\mathbf{H}) = \frac{1}{c}\nabla\times\frac{\partial \mathbf{H}}{\partial t} \qquad \left[\because\ \nabla\times\mathbf{H}=\frac{1}{c}\left(\rho\mathbf{V}+\frac{\partial\mathbf{D}}{\partial t}\right)\right]$$

$$= -\nabla\times(\nabla\times\mathbf{D}) \qquad \left[\because\ \nabla\times\mathbf{D}=-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}\right]$$

$$= -[\nabla(\nabla\cdot\mathbf{D})-\nabla^2\mathbf{D}] \qquad [\text{Using § 8.8 (4)}]$$

$$= -\nabla\rho+\nabla^2\mathbf{D}$$

Hence $\nabla^2\mathbf{D}-\dfrac{1}{c^2}\dfrac{\partial^2\mathbf{D}}{\partial t^2}=\nabla\rho+\dfrac{1}{c^2}\dfrac{\partial}{\partial t}(\rho\mathbf{V})$

(*ii*) $\quad$ L.H.S. $=\nabla^2\mathrm{H}-\dfrac{1}{c^2}\dfrac{\partial^2\mathbf{H}}{\partial t^2}=\nabla^2\mathbf{H}+\dfrac{1}{c}\dfrac{\partial}{\partial t}\left(-\dfrac{1}{c}\dfrac{\partial\mathbf{H}}{\partial t}\right)$

$$= \nabla^2\mathbf{H}+\frac{1}{c}\frac{\partial}{\partial t}(\nabla\times\mathbf{D}) \qquad \left[\because\ \nabla\times\mathbf{D}=-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}\right.$$

$$= \nabla^2\mathbf{H}+\frac{1}{c}\left(\nabla\times\frac{\partial\mathbf{D}}{\partial t}\right) \qquad \left[\because\ \nabla\times\mathbf{H}=\frac{1}{c}\left(\rho V+\frac{\partial\mathbf{D}}{\partial t}\right)\right.$$

$$= \nabla^2\mathbf{H}+\nabla\times\left(\nabla\times\mathbf{H}-\frac{1}{C}\rho\mathbf{V}\right) = \nabla^2\mathbf{H}+\nabla\times(\nabla\times\mathbf{H})-\frac{1}{c}\nabla\times(\rho\mathbf{V})$$

$$= \nabla^2\mathbf{H}+\nabla(\nabla\cdot\mathbf{H})-\nabla^2\mathbf{H}-\frac{1}{c}\nabla\times(\rho\mathbf{V}), \qquad [\text{Using § 8.9 (4)}]$$

$$= \nabla(\nabla\cdot\mathbf{H})-\frac{1}{c}\nabla\times(\rho\mathbf{V}) \qquad [\because\ \nabla\cdot\mathbf{H}=0]$$

$$= -\frac{1}{c}\nabla\times\rho V = \text{R.H.S.}$$

PROBLEMS 8.4

1. Evaluate div **F** and curl **F** at the point (1, 2, 3) given (*i*) $\mathbf{F}=x^2yz\mathbf{I}+xy^2z\mathbf{J}+xyz^2\mathbf{K}$. *(B.P.T.U., 2005)*
   (*ii*) $\mathbf{F}=3x^2\mathbf{I}+5xy^2\mathbf{J}+5xyz^3\mathbf{K}$. *(S.V.T.U., 2009)*
   (*iii*) $\mathbf{F}=\text{grad}\,[x^3y+y^3z+z^3x-x^2y^2z^2]$ *(V.T.U., 2007)*
2. If $\mathbf{V}=(x\mathbf{I}+y\mathbf{J}+z\mathbf{K})/\sqrt{(x^2+y^2+z^2)}$, show that $\nabla.\mathbf{V}=2/\sqrt{(x^2+y^2+z^2)}$ and $\nabla\times\mathbf{V}=\mathbf{0}$. *(Osmania, 2002)*
3. If $\mathbf{F}=(x+y+1)\,\mathbf{I}+\mathbf{J}-(x+y)\mathbf{K}$, show that **F**. curl **F** = 0. *(V.T.U., 2000 S)*
4. Find the value of a if the vector $(ax^2y+yz)\,\mathbf{I}+(xy^2-xz^2)\,\mathbf{J}+(2xyz-2x^2y^2)\,\mathbf{K}$ has zero divergence. Find the curl of the above vector which has zero divergence.
5. Show that each of following vectors are solenoidal :
   (*i*) $(-x^2+yz)\,\mathbf{I}+(4y-z^2x)\,\mathbf{J}+(2xz-4z)\,\mathbf{K}$ *(Delhi, 2002)*
   (*ii*) $3y^4z^2\mathbf{I}+4x^3z^2\mathbf{J}+3x^2y^2\mathbf{K}$ $\qquad$ (*iii*) $\nabla\phi\times\nabla\psi$.
6. If **A** and **B** are irrotational, prove that $\mathbf{A}\times\mathbf{B}$ is solenoidal. *(Madras, 2003 ; V.T.U., 2001)*
7. If $u=x^2+y^2+z^2$ and $\mathbf{V}=x\mathbf{I}+y\mathbf{J}+z\mathbf{K}$, show that div $(u\mathbf{V})=5u$.
8. If $\mathbf{R}=x\mathbf{I}+y\mathbf{J}+z\mathbf{K}$ and $r\neq 0$, show that (*i*) $\nabla/(1/r^2)=-2\mathbf{R}/r^4$; $\nabla\cdot(\mathbf{R}/r^2)=1/r^2$
   (*ii*) div $(r^n\mathbf{R})=(n+3)r^n$; curl $(r^n\mathbf{R})=\mathbf{0}$ *(P.T.U., 2006 ; Kottayam, 2005)*
   (*iii*) *grad* $\left(\text{div}\,\dfrac{\mathbf{R}}{r}\right)=-\dfrac{2\mathbf{R}}{r^3}$. *(V.T.U., 2010 S)*
9. If $\mathbf{V}_1$ and $\mathbf{V}_2$ be the vectors joining the fixed points (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively to a variable point (x, y, z), prove that
   (*i*) div $(\mathbf{V}_1\times\mathbf{V}_2)=0$, $\qquad$ (*ii*) grad $(\mathbf{V}_1\cdot\mathbf{V}_2)=\mathbf{V}_1+\mathbf{V}_2$,
   (*iii*) curl $(\mathbf{V}_1\times\mathbf{V}_2)=2(\mathbf{V}_1-\mathbf{V}_2)$

10. Show that (*i*) $\nabla \cdot \left[\dfrac{f(r)}{r}\mathbf{R}\right] = \dfrac{1}{r^2}\dfrac{d}{dr}[r^2 f(r)]$ (*Mumbai, 2008*)

(*ii*) $\nabla^2 f(r) = f''(r) + \dfrac{2}{r}f'(r)$ (*U.T.U., 2010 ; Bhopal, 2008 ; S.V.T.U., 2008 ; V.T.U., 2006*)

(*iii*) $\nabla \cdot (\phi\nabla\psi - \psi\nabla\phi) = \phi\nabla^2\psi - \psi\nabla^2\phi$.

11. If **A** is a constant vector and $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$, prove that

(*i*) grad $(\mathbf{A} \cdot \mathbf{R}) = \mathbf{A}$ (*Delhi, 2002*) (*ii*) div $(\mathbf{A} \times \mathbf{R}) = 0$ (*Burdwan, 2003*)

(*iii*) curl $(\mathbf{A} \times \mathbf{R}) = 2\mathbf{A}$ (*V.T.U., 2010 S*) (*iv*) curl $[(\mathbf{A} \cdot \mathbf{R})\mathbf{R}] = \mathbf{A} \times \mathbf{R}$. (*Kurukshetra, 2009 S*)

12. Prove that (*i*) $\nabla A^2 = 2(\mathbf{A} \cdot \nabla)\,\mathbf{A} + 2\mathbf{A} \times (\nabla \times \mathbf{A})$, where **A** is a constant vector.

(*ii*) $\nabla \times (\mathbf{R} \times \mathbf{U}) = \mathbf{R}(\nabla \cdot \mathbf{U}) - 2\mathbf{U} - (\mathbf{R} \cdot \nabla)\mathbf{U}$.

13. Calculate (*i*) curl (grad f), given $f(x, y, z) = x^2 + y^2 - z$. (*B.P.T.U., 2006*)

(*ii*) curl (curl A) given $A = x^2y\mathbf{I} + y^2z\mathbf{J} + z^2y\mathbf{K}$ (*V.T.U., 2003*)

14. (*a*) If $f = (x^2 + y^2 + z^2)^{-n}$, find div grad f and determine n if div grad $f = 0$. (*S.V.T.U., 2009 ; J.N.T.U. 2003*)

(*b*) Show that div (grad r^n) $= n(n + 1)r^{n-2}$ where $r^2 = x^2 + y^2 + z^2$. (*Bhopal, 2008 ; U.P.T.U., 2003*)

15. For a solenoidal vector **F**, show that curl curl curl curl $\mathbf{F} = \nabla^4\mathbf{F}$.

16. If $u = x^2yz$, $v = xy - 3z^2$, find (*i*) $\nabla(\nabla u \cdot \nabla v)$; (*ii*) $\nabla \cdot (\nabla u \times \nabla v)$.

17. Find the directional derivative of $\nabla \cdot (\nabla\phi)$ at the point $(1, -2, 1)$ in the direction of the normal to the surface $xy^2z = 3x + z^2$, where $\phi = 2x^3y^2z^4$. (*Raipur, 2005*)

18. Prove that $\mathbf{A} \cdot \nabla\left(\mathbf{B} \cdot \nabla\dfrac{1}{r}\right) = \dfrac{3(\mathbf{A} \cdot \mathbf{R})(\mathbf{B} \cdot \mathbf{R})}{r^5} - \dfrac{\mathbf{A} \cdot \mathbf{B}}{r^3}$ where **A** and **B** are constant vectors.

8.10 INTEGRATION OF VECTORS

If two vector functions $\mathbf{F}(t)$ and $\mathbf{G}(t)$ be such that

$$\frac{d\mathbf{G}(t)}{dt} = \mathbf{F}(t),$$

then $\mathbf{G}(t)$ is called an integral of $\mathbf{F}(t)$ with respect to the scalar variable t and we write

$$\int \mathbf{F}(t)\,dt = \mathbf{G}(t).$$

If **C** be an arbitrary constant vector, we have

$$\mathbf{F}(t) = \frac{d\mathbf{G}(t)}{dt} = \frac{d}{dt}\,[\mathbf{G}(t) + \mathbf{C}] \quad \text{then} \quad \int \mathbf{F}(t)\,dt = \mathbf{G}(t) + \mathbf{C}$$

This is called the *indefinite integral of* $\mathbf{F}(t)$ and its *definite integral is*

$$\int_a^b \mathbf{F}(t)\,dt = \left[\mathbf{G}(t) + \mathbf{C}\right]_a^b = \mathbf{G}(b) - \mathbf{G}(a).$$

Example 8.27. *Given* $\mathbf{R}(t) = 3t^2\,\mathbf{I} + t\mathbf{J} - t^3\mathbf{K}$, *evaluate* $\int_0^1 (R \times d^2\mathbf{R}/dt^2)\,dt$.

Solution. $$\frac{d}{dt}\left(\mathbf{R} \times \frac{d\mathbf{R}}{dt}\right) = \frac{d\mathbf{R}}{dt} \times \frac{d\mathbf{R}}{dt} + \mathbf{R} \times \frac{d^2\mathbf{R}}{dt^2} = \mathbf{R} \times \frac{d^2\mathbf{R}}{dt^2}$$

$$\therefore \quad \int \left(\mathbf{R} \times \frac{d^2\mathbf{R}}{dt^2}\right) dt = \mathbf{R} \times \frac{d\mathbf{R}}{dt}$$

$$= (3t^2\mathbf{I} + t\mathbf{J} - t^3\mathbf{K}) \times (6t\mathbf{I} + \mathbf{J} - 3t^2\mathbf{K})$$

$$= \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ 3t^2 & t & -t^3 \\ 6t & 1 & -3t^2 \end{vmatrix} = -2t^3\mathbf{I} + 3t^4\mathbf{J} - 3t^2\mathbf{K}$$

Thus $$\int_0^1 \left(\mathbf{R} \times \frac{d^2\mathbf{R}}{dt^2}\right) dt = \left| -2t^3\mathbf{I} + 3t^4\mathbf{J} - 3t^2\mathbf{K} \right|_0^1$$

$$= -2\mathbf{I} + 3\mathbf{J} - 3\mathbf{K}$$

PROBLEMS 8.5

1. Given $\mathbf{F}(t) = (5t^2 - 3t)\mathbf{I} + 6t^3\mathbf{J} - 7t\,\mathbf{K}$, evaluate $\int_{t=2}^{t=4} \mathbf{F}(t)\, dt$.

2. If $\frac{d^2\mathbf{P}}{dt^2} = 6t\mathbf{I} - 12t^2\mathbf{J} + 4\cos t\mathbf{K}$, find $\mathbf{P}$, Given that $\frac{d\mathbf{P}}{dt} = -1 - 3\mathbf{K}$ and $\mathbf{P} = 2\mathbf{I} + \mathbf{J}$ when $t = 0$.

3. The acceleration of a particle at any time $t \geq 0$ is given by $12\cos 2t\mathbf{I} - 8\sin 2t\mathbf{J} + 16t\mathbf{K}$, the velocity and acceleration are initially zero. Find the velocity and displacement at any time.

4. If $\mathbf{R}(t) = \begin{cases} 2\mathbf{I} - \mathbf{J} + 2\mathbf{K} & \text{when } t = 1 \\ 3\mathbf{I} - 2\mathbf{J} + 4\mathbf{K} & \text{when } t = 2, \end{cases}$

   show that $\int_1^2 \left(\mathbf{R} \cdot \frac{d\mathbf{R}}{dt}\right) dt = 10.$

8.11 (1) LINE INTEGRAL

Consider a continuous vector function $\mathbf{F}(\mathbf{R})$ which is defined at each point of curve C in space. Divide C into n parts at the points $A = P_0, P_1, \ldots, P_{i-1}, P_i \ldots, P_n = B$ (Fig. 8.8). Let their position vectors be $\mathbf{R}_0, \mathbf{R}_1, \ldots, \mathbf{R}_{i-1}, \mathbf{R}_i, \ldots, \mathbf{R}_n$. Let $\mathbf{U}_i$, be the position vector of any point on the arc $P_{i-1}P_i$.

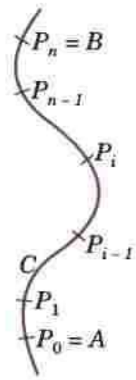

Fig. 8.8

Now consider the sum $S = \sum_{i=0}^{n} \mathbf{F}(\mathbf{U}_i) \cdot \delta\mathbf{R}_i$, *where* $d\mathbf{R}_i = \mathbf{R}_i - \mathbf{R}_{i-1}$.

The limit of this sum as $n \to \infty$ *in such a way that* $|\delta\mathbf{R}_i| \to 0$, *provided it exists, is called the* **tangential line integral** *of* $\mathbf{F}(\mathbf{R})$ *along* C and is symbolically written as

$$\int_C \mathbf{F}(\mathbf{R}) \cdot d\mathbf{R} \quad \text{or} \quad \int_C \mathbf{F} \cdot \frac{d\mathbf{R}}{dt}\, dt.$$

When the path of integration is a closed curve, this fact is denoted by using $\oint$ in place of $\int$.

If $\quad \mathbf{F}(\mathbf{R}) = \mathbf{I}f(x, y, z) + \mathbf{J}\phi(x, y, z) + \mathbf{K}\psi(x, y, z)$

and $\quad d\mathbf{R} = \mathbf{I}dx + \mathbf{J}dy + \mathbf{K}dz$

then $\quad \int_C \mathbf{F}(\mathbf{R}) \cdot d\mathbf{R} = \int_C (f\,dx + \phi dy + \psi dz).$

Two other types of line integrals are $\int_C \mathbf{F} \times d\mathbf{R}$ and $\int_C f d\mathbf{R}$ which are both vectors.

(2) **Circulation.** *If* $\mathbf{F}$ *represents the velocity of a fluid particle then the line integral* $\int_C \mathbf{F} \cdot d\mathbf{R}$ *is called the* **circulation of F** *around the curve. When the circulation of* $\mathbf{F}$ *around every closed curve in a region E vanishes,* $\mathbf{F}$ *is said to be* **irrotational** *in E.*

(3) **Work.** If $\mathbf{F}$ represents the force acting on a particle moving along an arc AB then the work done during the small displacement $\delta\mathbf{R} = \mathbf{F} \cdot \delta\mathbf{R}$.

$\therefore$ *the total work done by* $\mathbf{F}$ *during the displacement from A to B is given by the line integral* $\int_A^B \mathbf{F} \cdot d\mathbf{R}$.

Example 8.28. *If* $\mathbf{F} = 3xy\mathbf{I} - y^2\mathbf{J}$, *evaluate* $\int \mathbf{F} \cdot d\mathbf{R}$, *where C is the curve in the xy-plane* $y = 2x^2$ *from (0, 0) to (1, 2).* *(V.T.U., 2010)*

Solution. Since the particle moves in the xy-plane ($z = 0$), we take $\mathbf{R} = x\mathbf{I} + y\mathbf{J}$. Then $\int_C \mathbf{F} \cdot d\mathbf{R}$, where C is the parabola $y = 2x^2$

$$= \int_C (3xy\mathbf{I} - y^2\mathbf{J}) \cdot (dx\mathbf{I} + dy\mathbf{J}) = \int_C (3xy\,dx - y^2\,dy) \qquad \ldots(i)$$

Substituting $y = 2x^2$, where x goes from 0 to 1, (i) becomes

$$= \int_{x=0}^{1} [3x(2x^2)\,dx - (2x^2)^2\,d(2x^2)] = \int_0^1 (6x^3 - 16x^5)\,dx = -7/6.$$

Otherwise, let $x = t$ in $y = 2x^2$. Then the parametric equation of C are $x = t$, $y = 2t^2$. The points (0, 0) and (1, 2) correspond to $t = 0$ and $t = 1$ respectively. Then (i) becomes

$$= \int_{t=0}^{1} [3t(2t^2)\,dt - (2t^2)^2 d(2t^2)] = \int_0^1 (6t^3 - 16t^5)\,dt = -7/6.$$

Example 8.29. *A vector field is given by* $\mathbf{F} = \sin y\mathbf{I} + x(1 + \cos y)\,\mathbf{J}$. *Evaluate the line integral over a circular path given by* $x^2 + y^2 = a^2$, $z = 0$. *(Rohtak, 2006 S ; P.T.U., 2003)*

Solution. As the particle moves in xy-plane ($z = 0$), let $\mathbf{R} = x\mathbf{I} + y\mathbf{J}$ so that $d\mathbf{R} = dx\mathbf{I} + dy\mathbf{J}$. Also the circular path is $x = a\cos t$, $y = a \sin t$, $z = 0$ where t varies from 0 to 2π.

$$\therefore \quad \oint_c \mathbf{F}\cdot d\mathbf{R} = \oint_c [\sin y\mathbf{I} + x(1 + \cos y)\mathbf{J}]\cdot(dx\mathbf{I} + dy\mathbf{J})$$

$$= \oint_c [\sin y\,dx + x(1 + \cos y)\,dy] = \oint_c [(\sin y\,dx + x\cos y\,dy) + x dy]$$

$$= \oint [d(x \sin y) + x\,dy] = \int_0^{2\pi} [d\{a\cos t \sin(a \sin t)\} + a^2\cos^2 t\,dt]$$

$$= \Big| a\cos t \sin(a\sin t)\Big|_0^{2\pi} + \frac{a^2}{2}\int_0^{2\pi} (1 + \cos 2t)\,dt = \frac{a^2}{2}\left| t + \frac{\sin 2t}{2}\right|_0^{2\pi} = \pi a^2.$$

Example 8.30. *Find the work done in moving a particle in the force field* $\mathbf{F} = 3x^2\mathbf{I} + (2xz - y)\,\mathbf{J} + z\mathbf{K}$, *along*

(a) the straight line from (0, 0, 0) to (2, 1, 3). *(S.V.T.U., 2007 ; J.N.T.U., 2002)*

(b) the curve defined by $x^2 = 4y$, $3x^3 = 8z$ *from* $x = 0$ *to* $x = 2$. *(Delhi, 2002)*

Solution. $$\int_C \mathbf{F}\cdot d\mathbf{R} = \int_C [3x^2\mathbf{I} + (2xz - y)\mathbf{J} + z\mathbf{K}]\cdot(dx\mathbf{I} + dy\mathbf{J} + dz\mathbf{K})$$

$$= \int_C [3x^2dx + (2xz - y)dy + zdz] \qquad ...(i)$$

(a) The equations of the straight line from (0, 0, 0) to (2, 1, 3) are $x/2 = y/1 = z/3 = t$ (say)

$\therefore$ $x = 2t, y = t, z = 3t$ are its parametric equations. The points (0, 0, 0) and (2, 1, 3) correspond to $t = 0$ and $t = 1$, respectively

$$\therefore \qquad \text{work done} = \int_C \mathbf{F}\cdot d\mathbf{R} = \int_0^1 [3(2t)^2\,2dt + \{(4t)(3t) - t\}dt + (3t)\,3dt]$$

$$= \int_0^1 (36t^2 + 8t)\,dt = 16.$$

(b) Let $x = t$ in $x^2 = 4y$, $3x^3 = 8z$. Then the parametric equations of C are $x = t$, $y = t^2/4$, $z = 3t^3/8$ and t varies from 0 to 2.

$$\therefore \qquad \text{work done} = \int_C \mathbf{F}\cdot d\mathbf{R} = \int_0^2 \left[3t^2\,dt + \left\{2t\left(\frac{3t^3}{8}\right) - \frac{t^2}{4}\right\} d\left(\frac{t^2}{4}\right) + \frac{3t^3}{8}\,d\left(\frac{3t^2}{8}\right)\right]$$

$$= \int_0^2 \left(3t^2 - \frac{t^3}{8} + \frac{51}{64}t^5\right) dt = \left| t^3 - \frac{t^4}{32} + \frac{17}{128}t^6\right|_0^2 = 16.$$

PROBLEMS 8.6

1. Evaluate the line integral $\int_C [(x^2 + xy)dx + (x^2 + y^2)dy]$ where C is the square formed by the lines $y = \pm 1$ and $x = \pm 1$. *(Delhi, 2002)*

2. If $\mathbf{F} = (5xy - 6x^2)\mathbf{I} + (2y - 4x)\mathbf{J}$, evaluate $\int_C \mathbf{F} \cdot d\mathbf{R}$ along the curve C in the xy-plane, $y = x^3$ from the point (1, 1) to (2, 8). *(J.N.T.U., 2006)*
3. Compute the line integral $\int_C (y^2 dx - x^2 dy)$ about the triangle whose vertices are (1, 0), (0, 1) and (– 1, 0).
4. If $A = (3x^2 + 6y)\mathbf{I} - 14yz\mathbf{J} + 20xz^2\mathbf{K}$, evaluate $\int \mathbf{A} \cdot d\mathbf{R}$ from (0, 0, 0) to (1, 1, 1) along the path $x = t, y = t^2, z = t^3$. *(V.T.U., 2001)*
5. Evaluate $\int_C (xy + z^2)\, ds$ where C is the arc of the helix $x = \cos t, y = \sin t, z = t$ which joins the points (1, 0, 0) and $(-1, 0, \pi)$.
6. Find the total work done by the force $\mathbf{F} = 3xy\mathbf{I} - y\mathbf{J} + 2zx\mathbf{K}$ in moving a particle around the circle $x^2 + y^2 = 4$. *(V.T.U., 2010)*
7. Find the total work done in moving a particle in a force field given by $\mathbf{F} = 3xy\mathbf{I} - 5z\mathbf{J} + 10x\mathbf{K}$ along the curve $x = t^2 + 1, y = 2t^2, z = t^3$ from $t = 1$ to $t = 2$. *(Bhopal, 2008)*
8. Using the line integral, compute the work done by the force $\mathbf{F} = (2y + 3)\mathbf{I} + xz\mathbf{J} + (yz - x)\mathbf{K}$ when it moves a particle from the point (0, 0, 0) to the point (2, 1, 1) along the curve $x = 2t^2, y = t, z = t^3$. *(Madras, 2000)*
9. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{R}$, where $\mathbf{F} = [2z, x, -y]$ and C is $\mathbf{R} = [\cos t, \sin t, 2t]$ from (1, 0, 0) to $(1, 0, 4\pi)$. *(B.P.T.U., 2006)*
10. If $\mathbf{F} = 2y\mathbf{I} - z\mathbf{J} + x\mathbf{K}$, evaluate $\int_C \mathbf{F} \times d\mathbf{R}$ along the curve $x = \cos t, y = \sin t, z = 2\cos t$ from $t = 0$ to $t = \pi/2$.

8.12 (1) SURFACES

As seen in § 5.8, a surface S may be represented by $F(x, y, z) = 0$.

The *parametric representation of* S is of the form $\mathbf{R}(u, v) = x(u, v)\mathbf{I} + y(u, v)\mathbf{J} + z(u, v)\mathbf{K}$ and the continuous functions $u = \phi(t)$ and $v = \psi(t)$ of a real parameter t represent a curve C on this surface S.

For example, the parametric representation of the circular cylinder $x^2 + y^2 = a^2, -1 \le z \le 1$, (radius a and height 2), is

$$\mathbf{R}(u, v) = a \cos u\mathbf{I} + a \sin u\mathbf{J} + v\mathbf{K}$$

where the parameters u and v vary in the rectangle $0 \le u \le 2\pi$ and $-1 \le v \le 1$. Also $u = t, v = bt$ represent a *circular helix* (Fig. 8.3) on this circular cylinder. The equation of the circular helix is $\mathbf{R} = a \cos t\mathbf{I} + a \sin t\mathbf{J} + bt\mathbf{K}$.

Differentiating $\mathbf{R} = \mathbf{R}(u, v)$, w.r.t. t, we get $\dfrac{d\mathbf{R}}{dt} = \dfrac{\partial \mathbf{R}}{\partial u} \cdot \dfrac{du}{dt} + \dfrac{\partial \mathbf{R}}{\partial v} \cdot \dfrac{dv}{dt}$

The vectors $\partial\mathbf{R}/\partial u$ and $\partial\mathbf{R}/\partial v$ are tangential to S at P and determine the tangent plane of S at P. $\mathbf{N} = \partial\mathbf{R}/\partial u \times \partial\mathbf{R}/\partial v \; (\ne 0)$ gives a normal vector $\mathbf{N}$ of S at P.

Def. *If S has a unique normal at each of its points whose direction depends continuously on the points of S, then the surface S is called a* **smooth surface**. If S is not smooth but can be divided into finitely many smooth portions, then it is called a **piecewise smooth surface.**

For instance, the surface of a sphere is *smooth* while the surface of a cube is *piecewise smooth.*

Def. *A surface S is said to be* **orientable or two sided** *if the positive normal direction at any point P of S can be continued in a unique and continuous way to the entire surface.* If the positive direction of the normal is reversed as we move around a curve on S passing through P, then the surface is **non-orientable** (*i.e.*, **one-sided**).

An example of a non-orientable surface is the *Mobius strip**. If we take a long rectangular strip of paper and giving a half-twist join the shorter sides so that the two points A and the two points B in Fig. 8.9 coincide, then the surface generated is non-orientable. Such a surface is a model of a Mobius strip.

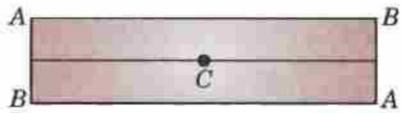

Fig. 8.9

(2) Surface integral. Consider a continuous function $\mathbf{F}(\mathbf{R})$ and a surface S. Divide S into a finite number of sub-surfaces. Let the surface element surrounding any point $P(\mathbf{R})$ be $\delta\mathbf{S}$ which can be regarded as a vector ; its magnitude being the area and its direction that of the outward normal to the element.

*Named after a German mathematician *August Ferdinand Möbius (1790–1868)* who was a student of Gauss and professor of astronomy at Leipzig. His important contributions are in projective geometry, theory of surfaces and mechanics.

Consider the sum $\Sigma\mathbf{F}(\mathbf{R}) \,.\, \delta\mathbf{S}$, *where the summation extends over all the sub-surfaces. The limit of this sum as the number of sub-surfaces tends to infinity and the area of each sub-surface tends to zero, is called the* **normal surface integral** *of* $\mathbf{F}(\mathbf{R})$ *over S and is denoted by*

$$\int_S \mathbf{F} \,.\, d\mathbf{S} \quad \text{or} \quad \int_S \mathbf{F} \,.\, \mathbf{N} ds \text{ where } \mathbf{N} \text{ is a unit outward normal at } P \text{ to } S.$$

Other types of surface integrals are $\int_S \mathbf{F} \times d\mathbf{S}$ and $\int_S f\, d\mathbf{S}$ which are both vectors.

Notation : Only one integrals sign is used when there is one differential (say $d\mathbf{R}$ or $d\mathbf{S}$) and two (or three) signs when there are two (or three) differentials.

(3) **Flux across a surface.** *If* **F** *represent the velocity of a fluid particle then the total outward flux of* **F** *across a closed surface S is the surface integral* $\int_S \mathbf{F} \,.\, d\mathbf{S}$.

When the flux of **F** *across every closed surface S in a region E vanishes,* **F** *is said to be a* **solenoidal vector point function** *in E.*

It may be noted that **F** could equally well be taken as any other physical quantity *e.g.*, gravitational force, electric force and magnetic force.

Example 8.31. *Evaluate* $\int_S \mathbf{F} \,.\, \mathbf{N} ds$ *where* $\mathbf{F} = 2x^2y\mathbf{I} - y^2\mathbf{J} + 4xz^2\mathbf{K}$ *and S is the closed surface of the region in the first octant bounded by the cylinder* $y^2 + z^2 = 9$ *and the planes* $x = 0, x = 2, y = 0$ *and* $z = 0$.

Solution. The given closed surface S is piecewise smooth and is comprised of S_1 – the rectangular face $OAEB$ in xy-plane ; S_2–the rectangular face $OADC$ in xz-plane ; S_3–the circular quadrant ABC in yz-plane, S_4–the circular quadrant AED and S_5–the curved surface $BCDE$ of the cylinder in the first octant (Fig. 8.10).

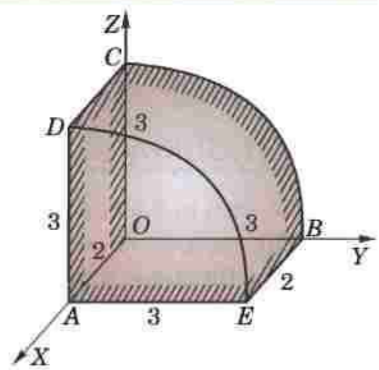

Fig. 8.10

$$\therefore \quad \int_S \mathbf{F} \,.\, \mathbf{N} ds = \int_{S_1} \mathbf{F} \,.\, \mathbf{N} ds + \int_{S_2} \mathbf{F} \,.\, \mathbf{N} ds + \int_{S_3} \mathbf{F} \,.\, \mathbf{N} ds + \int_{S_4} \mathbf{F} \,.\, \mathbf{N} ds + \int_{S_5} \mathbf{F} \,.\, \mathbf{N} ds \quad ...(i)$$

Now $\int_{S_1} \mathbf{F} \,.\, \mathbf{N} ds = \int_{S_1} (2x^2y\mathbf{I} - y^2\mathbf{J} + 4xz^2\mathbf{K}) \,.\, (-\mathbf{K})\, ds$

$$= -4\int_{S_1} xz^2\, ds = 0 \qquad [\because \; z = 0 \text{ in the } xy\text{–plane}]$$

Similarly, $\int_{S_2} \mathbf{F} \,.\, \mathbf{N} ds = 0$ and $\int_{S_3} \mathbf{F} \,.\, \mathbf{N} ds = 0$

$$\int_{S_4} \mathbf{F} \,.\, \mathbf{N} ds = \int_{S_4} (2x^2y\mathbf{I} - y^2\,\mathbf{J} + 4xz^2\,\mathbf{K}) \,.\, \mathbf{I}\, ds$$

$$= \int_{S_4} 2x^2y\, ds = \int_0^3 \int_0^{\sqrt{9-z^2}} 8y\, dydz = 4\int_0^3 (9 - z^2)\, dz = 72$$

To find **N** in S_5, we note that $\nabla(y^2 + z^2) = 2y\mathbf{J} + 2z\mathbf{K}$

$$\therefore \qquad \mathbf{N} = \frac{2y\mathbf{J} + 2z\mathbf{K}}{\sqrt{(4y^2 + 4z^2)}} = \frac{y\mathbf{J} + z\mathbf{K}}{3} \qquad [\because \; y^2 + z^2 = 9]$$

and $\qquad |\,\mathbf{N} \,.\, \mathbf{K}\,| = z/3 \qquad$ so that $ds = dxdy/(z/3)$

Thus
$$\int_{S_5} \mathbf{F} \,.\, \mathbf{N}\, ds = \int_0^2 \int_0^3 \frac{(-y^3 + 4xz^3)}{3} \,.\, dydx/(z/3) = \int_0^2 \int_0^3 \left(\frac{-y^3}{z} + 4xz^2\right) dydx$$

$$\begin{bmatrix} \text{Put } y = 3\sin\theta,\ z = 3\cos\theta \\ \therefore \;\; dy = 3\cos\theta\, d\theta \end{bmatrix}$$

$$= \int_0^2 \int_0^{\pi/2} \left[\frac{-27\sin^3\theta}{3\cos\theta} + 4x(9\cos^2\theta\right] 3\cos\theta\, d\theta dx = \int_0^2 \left[-27 \times \frac{2}{3} + 108x \times \frac{2}{3}\right] dx = 108$$

Hence (*i*) gives $\int_S \mathbf{F} \,.\, \mathbf{N}\, ds = 0 + 0 + 0 + 72 + 108 = 180.$

PROBLEMS 8.7

1. If velocity vector is $\mathbf{F} = y\mathbf{I} + 2\mathbf{J} + xz\mathbf{K}$ m/sec., show that the flux of water through the parabolic cylinder $y = x^2$, $0 \le x \le 3$, $0 \le z \le 2$ is 69 m³/sec.
2. Evaluate $\int_S \mathbf{F}\,.\,d\mathbf{S}$ where $\mathbf{F} = x\mathbf{I} + (z^2 - zx)\,\mathbf{J} - xy\mathbf{K}$ and S is the triangular surface with vertices (2, 0, 0), (0, 2, 0) and (0, 0, 4).
3. Evaluate $\int_S \mathbf{F}\,.\,\mathbf{N}\,ds$ where $\mathbf{F} = 6z\mathbf{I} - 4\mathbf{J} + y\mathbf{K}$ and S is the portion of the plane $2x + 3y + 6z = 12$ in the first octant.
4. If $\mathbf{F} = 2y\mathbf{I} - 3\mathbf{J} + x^2\,\mathbf{K}$ and S is the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$ and $z = 6$, show that $\int_S \mathbf{F}\,.\,\mathbf{N}\,ds = 132$.

8.13 GREEN'S THEOREM IN THE PLANE*

If $\phi(x, y)$, $\psi(x, y)$, ϕ_y and ψ_x be continuous in a region E of the xy-plane bounded by a closed curve C, then

$$\int_C (\phi dx + \psi dy) = \iint_E \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dxdy \qquad ...(1)$$

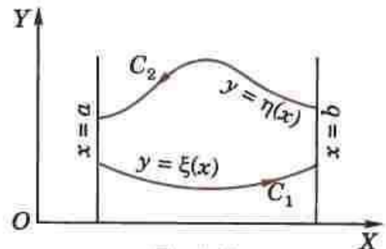

Fig. 8.11

Consider the region E bounded by a single closed curve C which is cut by any line parallel to the axes at the most in two points.

Let E be bounded by $x = a$, $y = \xi(x)$, $x = b$ and $y = \eta(x)$, where $\eta \ge \xi$, so that C is divided into curves C_1 and C_2 (Fig. 8.11).

$$\therefore \qquad \iint_E \frac{\partial \phi}{\partial y} dxdy = \int_a^b dx \left[\int_\xi^\eta \frac{\partial \phi}{\partial y} dy\right] = \int_a^b dx \,|\, \phi \,|_\xi^\eta$$

$$= \int_a^b [\phi(x, \eta) - \phi(x, \xi)]\, dx = -\int_{C_2} \phi(x, y)\, dx - \int_{C_1} \phi(x, y)\, dx$$

$$= -\int_C \phi(x, y)\, dx \qquad ...(2)$$

Similarly, it can be shown that

$$\iint_E \frac{\partial \psi}{\partial x} dxdy = \int_C \psi(x, y)\, dy \qquad ...(3)$$

On subtracting (2) from (3), we get (1).

This result can be extended to regions which may be divided into a finite number of sub-regions such that the boundary of each is cut at the most in two points by any line parallel to either axis. Applying (1) to each of these sub-regions and adding the results, the surface integrals combine into an integral over the whole region ; the line integrals over the common boundaries cancel (for each is covered twice but in opposite directions), whereas the remaining line integrals combine into the line integral over the external curve C.

Obs. This theorem converts a line integral around a closed curve into a double integral and is a special case of Stoke's theorem. (See Cor. p. 342)

Example 8.32. *Verify Green's theorem for $\int_C [(xy + y^2)\, dx + x^2 dy]$, where C is bounded by $y = x$ and $y = x^2$.*

(V.T.U., 2011 ; S.V.T.U., 2009 ; Rohtak, 2003)

Solution. Here $\qquad \phi = xy + y^2$ and $\psi = x^2$

$$\therefore \qquad \int_C (\phi\, dx + \psi\, dy) = \int_{C_1} + \int_{C_2}$$

*Named after the English mathematician *George Green* (1793–1841) who taught at Cambridge and is known for his work on potential theory in connection with waves, vibrations, elasticity, electricity and magnetism.

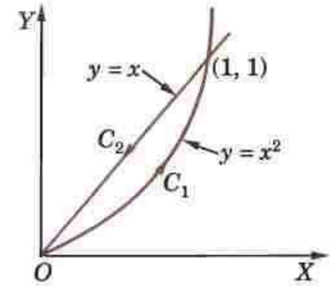

Fig. 8.12

Along C_1, $y = x^2$ and x varies from 0 to 1 (Fig. 8.12)

$$\therefore \quad \int_{C_1} = \int_0^1 [\{x(x)^2 + (x^2)^2\}] \, dx + x^2 d(x^2)]$$

$$= \int_0^1 (3x^3 + x^4) \, dx = \frac{19}{20}$$

Along C_2, $y = x$ and x varies from 1 to 0.

$$\therefore \quad \int_{C_2} = \int_1^0 [\{x(x) + (x)^2\} \, dx + x^2 \, d(x)] = \int_1^0 3x^2 \, dx = -1.$$

Thus $\int_C (\phi \, dx + \psi \, dy) = \frac{19}{20} - 1 = -\frac{1}{20}$ ...(*i*)

Also $\iint_E \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dxdy = \iint_E \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy + y^2)\right] dxdy$

$$= \int_0^1 \int_{x^2}^x (2x - x - 2y) \, dydx = \int_0^1 \left[xy - y^2\right]_{x^2}^x dx = \int_0^1 (x^4 - x^3) \, dx = -\frac{1}{20} \quad \text{...(ii)}$$

Hence, Green theorem is verified from the equality of (*i*) and (*ii*).

Example 8.33. *If C is a simple closed curve in the xy-plane not enclosing the origin, show that*

$$\int_C \mathbf{F} \, . \, d\mathbf{R} = 0, \quad \text{where} \quad \mathbf{F} = \frac{y\mathbf{I} - x\mathbf{J}}{x^2 + y^2}$$ *(P.T.U., 2005)*

Solution. $\int_C \mathbf{F} \, . \, d\mathbf{R} = \int_C \frac{y\mathbf{I} - x\mathbf{J}}{x^2 + y^2}(dx\mathbf{I} + dy\mathbf{J})$ $[\because \quad \mathbf{R} = x\mathbf{I} + y\mathbf{J}]$

$$= \int_C \frac{ydx - xdy}{x^2 + y^2} = \int_C (\phi dx + \psi dy) \text{ where } \phi = \frac{y}{x^2 + y^2}, \psi = \frac{-x}{x^2 y^2}$$

$$= \iint_S \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dxdy$$ [By Green's theorem]

$$= \iint_S \left[\frac{-(x^2 + y^2) + x(2x)}{(x^2 + y^2)^2} - \frac{(x^2 + y^2) - y(2y)}{(x^2 + y^2)^2}\right] dxdy$$

$$= \iint \left[\frac{x^2 - y^2}{(x^2 + y^2)^2} - \frac{x^2 - y^2}{(x^2 + y^2)^2}\right] dxdy = 0.$$

Example 8.34. *Using Green's theorem, evaluate* $\int_C [(y - \sin x) \, dx + \cos x \, dy]$ *where C is the plane triangle enclosed by the lines* $y = 0$, $x = \pi/2$ *and* $y = \frac{2}{\pi} x$. *(J.N.T.U., 2005 ; Anna, 2003)*

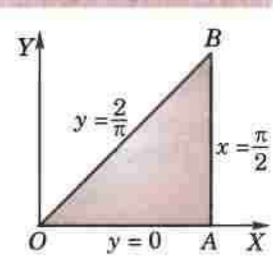

Fig. 8.13

Solution. Here $\phi = y - \sin x$ and $\psi = \cos x$.

By Green's theorem $\int_C [(y - \sin x) \, dx + \cos x \, dy]$

$$= \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx \, dy$$

$$= \int_{x=0}^{x=\pi/2} \int_{y=0}^{y=2x/\pi} (-\sin x - 1) \, dy \, dx = -\int_0^{\pi/2} (\sin x + 1) \left| \, y \, \right|_0^{2x/\pi} dx$$

$$= -\frac{2}{\pi} \int_0^{\pi/2} x(\sin x + 1) \, dx = -\frac{2}{\pi}\left\{\left| \, x(-\cos x + x) \right|_0^{\pi/2} - \int_0^{\pi/2} 1 \, . \, (-\cos x + x) \, dx\right\}$$

$$= -\frac{2}{\pi}\left\{\frac{\pi^2}{4} - \left| -\sin x + \frac{x^2}{2} \right|_0^{\pi/2}\right\} = -\frac{\pi}{2} + \frac{2}{\pi}\left(-1 + \frac{\pi^2}{8}\right) = -\left(\frac{\pi}{4} + \frac{2}{\pi}\right)$$

Example 8.35. *Apply Green's theorem to evaluate* $\int_C [(2x^2 - y^2)dx + (x^2 + y^2)dy]$, *where C is the boundary of the area enclosed by the x-axis and the upper-half of the circle* $x^2 + y^2 = a^2$. (U.P.T.U., 2005)

Solution. By Green's theorem

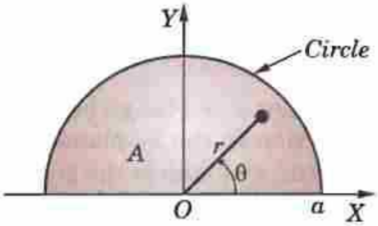

Fig. 8.14

$$\int_C [(2x^2 - y^2)dx + (x^2 + y^2)dy]$$

$$= \iint_A \left[\frac{\partial}{\partial x}(x^2 + y^2) - \frac{\partial}{\partial y}(2x^2 - y^2)\right] dxdy$$

$$= 2\iint_A (x + y)\, dxdy, \text{ where } A \text{ is the region of Fig. 8.14}$$

$$= 2\int_0^a \int_0^\pi r\ (\cos\theta + \sin\theta)\cdot r d\theta\, dr$$

[Changing to polar coordinates (r, θ), r varies from 0 to a and θ varies from 0 to π]

$$= 2\int_0^a r^2\, dr \int_0^\pi (\cos\theta + \sin\theta)\, d\theta = 2\cdot\frac{a^3}{3}\cdot(1+1) = \frac{4a^3}{3}.$$

PROBLEMS 8.8

1. Verify Green's theorem for $\int_C [(3x - 8y^2)\, dx + (4y - 6xy)\, dy]$ where C is the boundary of the region bounded by $x = 0, y = 0$ and $x + y = 1$. (Nagpur, 2008 ; Kerala, 2005 ; Anna, 2003 S)
2. Verify Green's theorem for $\int_C [(x^2 - \cosh y)dx + (y + \sin x)dy]$ where C is the rectangle with vertices $(0, 0)$, $(\pi, 0)$, $(\pi, 1)$, $(0, 1)$. (Nagpur, 2009 ; P.T.U., 2006)
3. Verify Green's theorem for $\int (x^2 ydx + x^2 dy)$ where C is the boundary described counter clockwise of triangle with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$. (U.T.U., 2010)
4. Apply Green's theorem to prove that the area enclosed by a plane curve is $\frac{1}{2}\int_C (xdy - ydx)$.

   Hence find the area of an ellipse whose semi-major and semi-minor axes are of lengths a and b. (Kerala, 2005 ; V.T.U., 2000 S)
5. Find the area of a circle of radius a using Green's theorem. (Madras, 2003)
6. Evaluate $\int_C [(x^2 + xy)dx + (x^2 + y^2)dy]$, where C is the square formed by the lines $x = \pm 1$, $y = \pm 1$. (S.V.T.U., 2008 ; Marathwada, 2008)
7. Evaluate $\int_C [(x^2 - 2xy)dx + (x^2y + 3)dy]$, around the boundary of the region defined by $y^2 = 8x$ and $x = 2$.
8. Evaluate by Green's theorem $\int_C \mathbf{F}\cdot d\mathbf{R}$ where $\mathbf{F} = -xy(x\mathbf{I} - y\mathbf{J})$ and C is $r = a(1 + \cos\theta)$. (Mumbai, 2006)

8.14 STOKE'S THEOREM* (*Relation between line and surface integrals*)

If S be an open surface bounded by a closed curve C and $\mathbf{F} = f_1\mathbf{I} + f_2\mathbf{J} + f_3\mathbf{K}$ *be any continuously differentiable vector point function, then*

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \int_S curl\ \mathbf{F}\cdot\mathbf{N}ds$$

where $\mathbf{N} = \cos\alpha\mathbf{I} + \cos\beta\mathbf{J} + \cos\gamma\mathbf{K}$ *is a unit external normal at any point of S.*

* Named after an Irish mathematician *Sir George Gabriel Stokes* (1819–1903) who became professor in Cambridge. His important contributions are to infinite series, geodesy and theory of viscous fluids.

Writing $d\mathbf{R} = dx\mathbf{I} + dy\mathbf{J} + dz\mathbf{K}$, it may be reduced to the form

$$\int_C (f_1dx + f_2dy + f_3dz) = \int_S \left[\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\cos\beta + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\cos\gamma\right] ds \quad ...(1)$$

Let us first prove that

$$\oint_C f_1 dx = \int_S \left(\frac{\partial f_1}{\partial z}\cos\beta - \frac{\partial f_1}{\partial y}\cos\gamma\right) ds \quad ...(2)$$

Let $z = g(x, y)$ be the equation of the surface S whose projection on the xy-plane is the region E. Then the projection of C on the xy-plane is the curve C' enclosing region E.

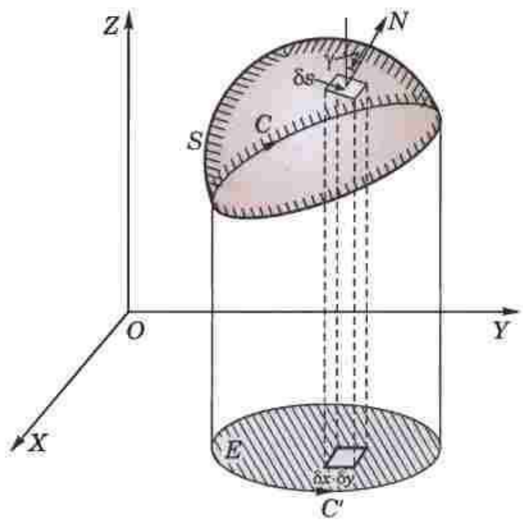

Fig. 8.15

$$\therefore \quad \int_C f_1(x, y, z)dx = \int_{C'} f_1\{x, y, g(x, y)\}\,dx$$

$$= -\iint_E \frac{\partial}{\partial y} f_1(x, y, g)\,dxdy, \text{ by Green's theorem}$$

$$= -\iint_E \left(\frac{\partial f_1}{\partial y} + \frac{\partial f_1}{\partial z}\frac{\partial g}{\partial y}\right) dxdy \quad ...(3)$$

The direction cosines of the normal to the surface $z = g(x, y)$ are given by

$$\frac{\cos\alpha}{-\partial g/\partial x} = \frac{\cos\beta}{-\partial g/\partial y} = \frac{\cos\gamma}{1} \quad \text{(See p. 219)} \quad ...(4)$$

Moreover

$dxdy$ = projection of ds on the xy-plane
$= ds\cos\gamma$, *i.e.*, $ds = dxdy/\cos\gamma$.

$\therefore$ right side of (2)

$$= \iint_E \left(\frac{\partial f_1}{\partial z}\frac{\cos\beta}{\cos\gamma} - \frac{\partial f_1}{\partial y}\right) dxdy = -\iint \left(\frac{\partial f_1}{\partial y} + \frac{\partial f_1}{\partial z}\cdot\frac{\partial g}{\partial y}\right) dxdy \quad \left[\because \frac{\cos\beta}{\cos\gamma} = -\frac{\partial g}{\partial y} \text{ by (4)}\right]$$

= Left side of (2), by (3).

Thus we have proved (2). Similarly, we can prove the other corresponding relations for f_2 and f_3. Adding these three results, we get (1).

Cor. Green's theorem in a plane as a special case of Stokes theorem. Let $\mathbf{F} = \phi\mathbf{I} + \psi\mathbf{J}$ be a vector function which is continuously differentiable in a region S of the xy-plane bounded by a closed curve C. Then

$$\int_C \mathbf{F}\,.\,dR = \int_C (\phi\mathbf{I} + \psi\mathbf{J})\cdot(dx\,\mathbf{I} + dy\,\mathbf{J}) = \int_C (\phi\,dx + \psi\,dy)$$

and
$$\text{curl}\,\mathbf{F}\cdot\mathbf{N} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \partial/\partial x & \partial/\partial y & 0 \\ \phi & \psi & 0 \end{vmatrix}\cdot\mathbf{K} = \frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}$$

Hence *Stoke's theorem* takes the form $\int_C (\phi\,dx + \psi\,dy) = \int_C \left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right) dxdy$ which is *Green's theorem* in a plane.

Example 8.36. *Verify Stoke's theorem for* $\mathbf{F} = (x^2 + y^2)\mathbf{I} - 2xy\mathbf{J}$ *taken around the rectangle bounded by the lines* $x = \pm a$, $y = 0$, $y = b$. *(Bhopal, 2008 S ; V.T.U., 2007 ; J.N.T.U., 2003 ; U.P.T.U., 2003)*

Solution. Let $ABCD$ be the given rectangle as shown in Fig. 8.16.

$$\int_{ABCD} \mathbf{F}\cdot d\mathbf{R} = \int_{AB} \mathbf{F}\cdot d\mathbf{R} + \int_{BC} \mathbf{F}\cdot d\mathbf{R} + \int_{CD} \mathbf{F}\cdot d\mathbf{R} + \int_{DA} \mathbf{F}\cdot d\mathbf{R}$$

and
$$\mathbf{F}\cdot d\mathbf{R} = [(x^2 + y^2)\mathbf{I} - 2xy\mathbf{J}]\cdot(\mathbf{I}dx + \mathbf{J}dy) = (x^2 + y^2)dx - 2xydy$$

Along AB, $x = a$ (*i.e.*, $dx = 0$) and y varies from 0 to b.

$$\therefore \quad \int_{AB} \mathbf{F}\cdot d\mathbf{R} = -2a\int_0^b y\,dy = -2a\cdot\frac{b^2}{2} = -ab^2.$$

Similarly, $\int_{BC} \mathbf{F} \cdot d\mathbf{R} = \int_a^{-a} (x^2 + b^2)dx = -\frac{2a^3}{3} - 2ab^2 \cdot$

$\int_{CD} \mathbf{F} \cdot d\mathbf{R} = 2a \int_b^0 y\, dy = -ab^2$

and $\int_{DA} \mathbf{F} \cdot d\mathbf{R} = \int_{-a}^{a} x^2\, dx = \frac{2a^3}{3} \cdot$

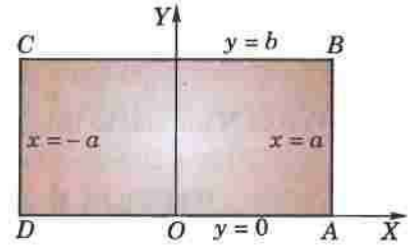

Fig. 8.16

Thus $\int_{ABCD} \mathbf{F} \cdot d\mathbf{R} = -4ab^2$ ...(i)

Also since curl $\mathbf{F} = -4\mathbf{K}y$

$$\therefore \quad \int_S \text{curl } \mathbf{F} \cdot \mathbf{N}\, ds = \int_0^b \int_{-a}^{a} -4\mathbf{K}y \cdot \mathbf{K}\, dxdy = -4 \int_0^b \int_{-a}^{a} y\, dxdy$$

$$= -4 \int_0^b |x|_{-a}^{a}\, y\, dy = -8a \left| \frac{y^2}{2} \right|_0^b = -4ab^2 \quad ...(ii)$$

Hence Stoke's theorem is verified from the equality of (i) and (ii).

Example 8.37. *Verify Stoke's theorem for the vector field $F = (2x - y)\,\mathbf{I} - yz^2\,\mathbf{J} - y^2z\,\mathbf{K}$ over the upper half surface of $x^2 + y^2 + z^2 = 1$, bounded by its projection on the xy-plane.*

(*Bhopal, 2008 ; Madras, 2006 ; S.V.T.U., 2006*)

Solution. The projection of the upper half of given sphere on the xy-plane ($z = 0$) is the circle $c[x^2 + y^2 = 1]$ (Fig. 8.17).

$$\oint_c \mathbf{F}.\, d\mathbf{R} = \oint_c [(2x - y)dx - yz^2\, dy - y^2\, zdz] = \oint_c (2x - y)dx \qquad [z = 0 \text{ in the } xy\text{-plane}]$$

$$= \int_{\theta=0}^{2\pi} (2 \cos\theta - \sin\theta)(-\sin\theta\, d\theta) \qquad [\text{Putting } x = \cos\theta,\ y = \sin\theta]$$

$$= \int_0^{2\pi} (-\sin 2\theta + \sin^2\theta)\, d\theta = \times 0 + 4\int_0^{\pi/2} \sin^2\theta\, d\theta = \pi. \quad ...(i)$$

Now
$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2z \end{vmatrix}$$

$$= (-2yz + 2yz)\,\mathbf{I} + 0\,\mathbf{J} + \mathbf{K} = \mathbf{K}$$

$$\therefore \quad \int \text{curl } \mathbf{F}.\, Nds = \int_S K.\mathbf{N}\, ds = \int_A \mathbf{K}\,.\,\mathbf{N} \frac{dxdy}{|\mathbf{N}\,.\,\mathbf{K}|}$$

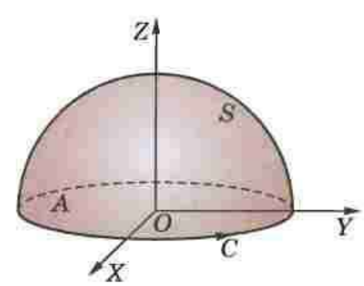

Fig. 8.17

where A is the projection of S on xy-plane and $ds = dxdy/\mathbf{N}\,.\,\mathbf{K}$

$$= \int_A dx\, dy = \text{area of circle } C = \pi \quad ...(ii)$$

Hence, the Stokes theorem is verified from the equality of (i) and (ii).

Example 8.38. *Uses Stoke's theorem evaluate $\int_C [(x + y)dx + (2x - z)dy + (y + z)dz]$ where C is the boundary of the triangle with vertices (2, 0, 0), (0, 3, 0) and (0, 0, 6).*

(*Nagpur, 2009 ; Kurukshetra, 2009 S ; Kerala, 2005*)

Solution. Here $\mathbf{F} = (x + y)\,\mathbf{I} + (2x - z)\,\mathbf{J} + (y + z)\,\mathbf{K}$

$$\therefore \quad \text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + y & 2x - z & y + z \end{vmatrix} = 2\mathbf{I} + \mathbf{K}$$

Also equation of the plane through A, B, C (Fig. 8.18) is

$$\frac{x}{2}+\frac{y}{3}+\frac{z}{6}=1 \text{ or } 3x+2y+z=6$$

Vector **N** normal to this plane is

$$\nabla(3x+2y+z-6)=3\mathbf{I}+2\mathbf{J}+\mathbf{K}$$

$$\therefore \quad \hat{\mathbf{N}}=\frac{3\mathbf{I}+2\mathbf{J}+\mathbf{K}}{\sqrt{(9+4+1)}}=\frac{1}{\sqrt{14}}(3\mathbf{I}+2\mathbf{J}+\mathbf{K})$$

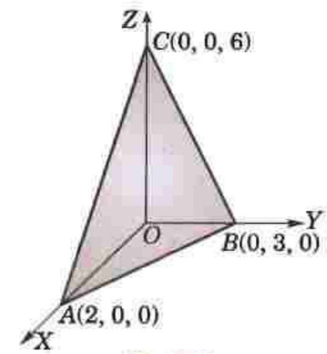

Fig. 8.18

Hence $\int_C [(x+y)dx+(2x-z)dy+(y+z)dz]=\int_C \mathbf{F}\,.\,d\mathbf{R}$

$$=\int_S \text{curl } \mathbf{F}\,.\,\hat{\mathbf{N}}\,ds \qquad \text{where } S \text{ is the triangle } ABC$$

$$=\int_S (2\mathbf{I}+\mathbf{K})\,.\left(\frac{3\mathbf{I}+2\mathbf{J}+\mathbf{K}}{\sqrt{14}}\right)ds=\frac{1}{\sqrt{14}}(6+1)\int_S ds$$

$$=\frac{7}{\sqrt{14}}(\text{Area of } \Delta ABC)=\frac{7}{\sqrt{14}}\,.\,3\sqrt{14}=21.$$

Example 8.39. *If $\mathbf{F}=3y\mathbf{I}-xz\mathbf{J}+yz^2\mathbf{K}$ and S is the surface of the paraboloid $2z=x^2+y^2$ bounded by $z=2$, evaluate $\iint_S (\nabla\times\mathbf{F})\,.\,d\mathbf{S}$ using Stoke's theorem.*

Solution. By Stokes theorem, $I=\iint_S (\nabla\times\mathbf{F})\,.\,d\mathbf{S}=\oint_C \mathbf{F}\,.\,d\mathbf{R}$

where S is the surface $2z=x^2+y^2$ bounded by $z=2$.

$$\therefore \quad I=\oint_C \mathbf{F}\,.\,d\mathbf{R}=\oint_C (3yI-xzJ+yz^2K)\,.\,(dxI+dyJ+dzK)$$

$$=\oint_C (3ydx-xzdy+yz^2dz)$$

$\because\ S\equiv x^2+y^2=4,\ z=2$
$\therefore$ Put $x=2\cos\theta,\ y=2\sin\theta$
$C\equiv x^2+y^2=4,\ \theta=0$ to 2π.

$$=\int_0^{2\pi}[6\sin\theta(-2\cos\theta\,d\theta)-4\cos\theta(2\cos\theta\,d\theta)+8\sin\theta(0)]$$

$$=-4\int_0^{2\pi}(12\sin^2\theta+8\cos^2\theta)\,d\theta$$

$$=-4\left(12\,.\,\frac{1}{2}\frac{\pi}{2}+8\,.\,\frac{1}{2}\,.\,\frac{\pi}{2}\right)=-20\pi.$$

Example 8.40. *Apply Stoke's theorem to evaluate $\int_C (ydx+zdy+xdz)$ where C is the curve of intersection of $x^2+y^2+z^2=a^2$ and $x+z=a$.* (Bhopal, 2008)

Solution. The curve C is evidently a circle lying in the plane $x+z=a$, and having $A(a, 0, 0)$, $B(0, 0, a)$ as the extremities of the diameter (Fig. 8.19).

$$\therefore \quad \int_C (y\,dx+z\,dy+x\,dz)=\int_C (y\mathbf{I}+z\mathbf{J}+x\mathbf{K})\,.\,d\mathbf{R}$$

$$=\int_S \text{curl } (y\mathbf{I}+z\mathbf{J}+x\mathbf{K})\,.\,\mathbf{N}ds$$

where S is the circle on AB as diameter and $\mathbf{N}=\frac{1}{\sqrt{2}}\mathbf{I}+\frac{1}{\sqrt{2}}\mathbf{K}$

$$=\int_S -(\mathbf{I}+\mathbf{J}+\mathbf{K})\,.\left(\frac{1}{\sqrt{2}}\mathbf{I}+\frac{1}{\sqrt{2}}\mathbf{K}\right)ds$$

$$=-\frac{2}{\sqrt{2}}\int_S ds=-\frac{2}{\sqrt{2}}\pi\left(\frac{a}{\sqrt{2}}\right)^2=-\frac{\pi a^2}{\sqrt{2}}.$$

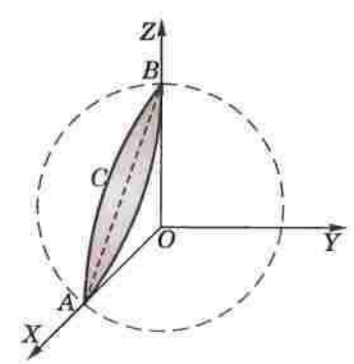

Fig. 8.19

Example 8.41. *If S be any closed surface, prove that* $\int_S$ *curl* $\mathbf{F} . d\mathbf{S} = 0$.

Solution. Cut open the surface S by any plane and let S_1, S_2 denote its upper and lower portions. Let C be the common curve bounding both these portions.

$$\therefore \qquad \int_S \text{curl } \mathbf{F} . d\mathbf{S} = \int_{S_1} \text{curl } \mathbf{F} . d\mathbf{S} + \int_{S_2} \text{curl } \mathbf{F} . d\mathbf{S} = \int_C \mathbf{F} . d\mathbf{R} - \int_C \mathbf{F} . d\mathbf{R} = 0,$$

on applying Stoke's theorem. The second integral is negative because it is traversed in a direction opposite to that of the first.

PROBLEMS 8.9

1. Verify Stoke's theorem for the vector field (*i*) $\mathbf{F} = (x^2 - y^2)\mathbf{I} + 2xy\mathbf{J}$ over the box bounded by the planes $x = 0, x = a; y = 0, y = b; z = 0, z = c$; if the face $z = 0$ is cut. (*B.P.T.U., 2006 ; Delhi, 2002*)
(*ii*) $\mathbf{F} = (z^2, 5x, 0)$ and $S : 0 \le x \le 1, 0 \le y \le 1, z = 1$.
2. Verify Stoke's theorem for a vector field defined by $\mathbf{F} = -y^3\mathbf{I} + x^3\mathbf{J}$, in the region $x^2 + y^2 \le 1, z = 0$.
3. Evaluate $\int_c \mathbf{F} . d\mathbf{R}$ where $\mathbf{F} = (x^2 + y^2)\mathbf{I} - 2xy\,\mathbf{J}$ and c is the rectangle in the xy-plane bounded by $y = 0, x = a, y = b, x = 0$. (*Mumbai, 2007*)
4. Verity Stoke's theorem for $\mathbf{F} = (y - z + 2)\mathbf{I} + (yz + 4)\mathbf{J} - xz\mathbf{K}$ where S is the surface of the cube $x = 0, y = 0, z = 0, x = 2, y = 2, z = 2$ above the xy-plane. (*Andhra, 2000*)
5. Evaluate $\int_C \mathbf{F} . d\mathbf{R}$ where $\mathbf{F} = y\mathbf{I} + xz^3\mathbf{J} - zy^3\mathbf{K}$, C is the circle $x^2 + y^2 = 4, z = 1.5$.
6. Evaluate by Stoke's theorem $\oint_C (yz\,dz + zx\,dy + zx\,dz)$ where C is the curve $x^2 + y^2 = 1, z = y^2$. (*J.N.T.U., 2005*)
7. If S be the surface of the sphere $x^2 + y^2 + z^2 = 1$, prove that $\int_S \text{curl } \mathbf{F} . d\mathbf{S} = 0$. (*J.N.T.U., 1999*)
8. Prove that $\int_C A \times \mathbf{R} . d\mathbf{R} = 2\mathbf{A} . \int d\mathbf{S}$, $\mathbf{A}$ being any constant vector, and deduce that $\oint_C \mathbf{R} \times d\mathbf{R}$ is twice the vector area of the surface enclosed by C.
9. If ϕ is a scalar point function, use Stoke's theorem to prove that (*i*) curl (grad ϕ) = 0. (*ii*) div curl $\mathbf{F} = 0$. (*Kerala, 2005*)
10. Evaluate $\oint_C (\sin z\,dx - \cos x\,dy + \sin y\,dz)$ where C is the boundary of the rectangle $0 \le x \le \pi, 0 \le y \le 1, z = 3$. (*Rohtak, 2005*)
11. Use Stoke's theorem to evaluate $(\nabla \times \mathbf{F}) . \mathbf{N}\,ds$, where $\mathbf{F} = y\mathbf{I} + (x - 2xz)\,\mathbf{J} - xy\mathbf{K}$ and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above the xy-plane. (*Kottayam, 2005*)
12. Evaluate $\int \nabla \times \mathbf{V} . d\mathbf{S}$ over the surface of the paraboloid $z = 1 - x^2 - y^2, z \ge 0$ where $\mathbf{V} = y\mathbf{I} + z\mathbf{J} + x\mathbf{K}$.

8.15 VOLUME INTEGRAL

Consider a continuous vector function $\mathbf{F}(\mathbf{R})$ and surface S enclosing the region E. Divide E into finite number of sub-regions $E_1, E_2, \ldots, E_n$. Let δv_i be the volume of the sub-region E_i enclosing any point whose position vector is $\mathbf{R}_i$.

Consider the sum $\qquad \mathbf{V} = \sum_{i=1}^{n} \mathbf{F}(\mathbf{R}_i)\,\delta v_i$

The limit of this sum as $n \to \infty$ *in such a way that* $\delta v_i \to 0$, *is called the volume integral of* $\mathbf{F}(\mathbf{R})$ *over E and is symbolically written as* $\int_E \mathbf{F}\,dv$.

If $\mathbf{F}(\mathbf{R}) = f(x, y, z)\mathbf{I} + \phi(x, y, z)\mathbf{J} + \psi(x, y, z)\mathbf{K}$ so that $\delta v = \delta x \delta y \delta z$, then

$$\int_E \mathbf{F}\,dv = \mathbf{I} \iiint_E f dxdydz + \mathbf{J} \iiint_E \phi dxdydz + \mathbf{K} \iiint_E \psi dxdydz.$$

8.16 GAUSS DIVERGENCE THEOREM* (*Relation between surface and volume integrals*)

If **F** is a continuously differentiable vector function in the region E bounded by the closed surface S, then

$$\int_S \mathbf{F}\,.\,\mathbf{N}ds = \int_E div\,\mathbf{F}\,dv$$

where **N** *is the unit external normal vector.*

If $\mathbf{F}(\mathbf{R}) = f(x, y, z)\mathbf{I} + \phi(x, y, z)\mathbf{J} + \psi(x, y, z)\mathbf{K}$

then it is required to prove that

$$\iint_S (fdydz + \phi dzdx + \psi dxdy)$$

$$= \iiint_E \left(\frac{\partial f}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial z} \right) dxdydz \qquad ...(1)$$

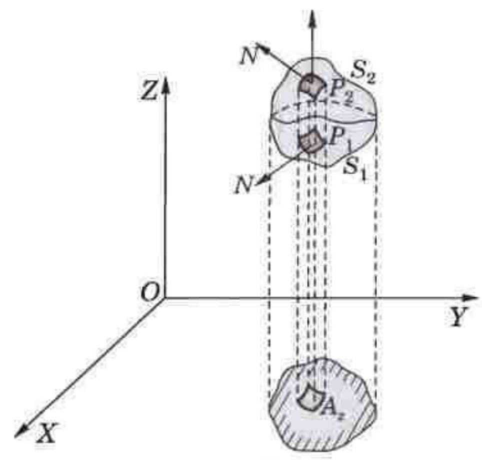

Fig. 8.20

Firstly consider such a surface S that a line parallel to z-axis cuts it in two points; say $P_1(x, y, z_1)$ *and* $P_2(x, y, z_2)$ $(z_1 \le z_2)$ (Fig. 8.20).

If S projects into the area A_z on the xy-plane, then

$$\iiint_E \frac{\partial \psi}{\partial z}\,dxdydz = \iint_{A_z} dxdy \int_{z_1}^{z_2} \frac{\partial \psi}{\partial z}\,dz$$

$$= \iint_{A_z} [\Psi(x, y, z_2) - \Psi(x, y, z_1)]\,dxdy = \iint_{A_z} \Psi(x, y, z_2)\,dxdy - \iint_{A_z} \Psi(x, y, z_1)\,dxdy \qquad ...(2)$$

Let S_1, S_2 be the lower and upper parts of the surface S corresponding to the points P_1 and P_2 respectively and **N** be the unit external normal vector at any point of S. As the external normal at any point of S_2 makes an acute angle with the positive direction of z-axis and that at any point of S_1 an obtuse angle, therefore

$$\iint_{A_z} \Psi(x, y, z_2)\,dxdy = \int_{S_2} \psi\,\mathbf{N}\,.\,\mathbf{K}ds \qquad ...(3)$$

$$\iint_{A_z} \Psi(x, y, z_1)\,dxdy = -\int_{S_1} \psi\mathbf{N}\,.\,\mathbf{K}ds \qquad ...(4)$$

Using (3) and (4), (2) now becomes

$$\iiint_E \frac{\partial \psi}{\partial z}\,dxdydz = \int_{S_2} \psi\,\mathbf{N}\,.\,\mathbf{K}ds + \int_{S_1} \psi\,\mathbf{N}\,.\,\mathbf{K}ds = \int_S \psi\,\mathbf{N}\,.\,\mathbf{K}ds \qquad ...(5)$$

Similarly, we have

$$\iiint_E \frac{\partial f}{\partial x}\,dxdydz = \int_S f\,\mathbf{N}\,.\,\mathbf{I}ds \qquad ...(6)$$

$$\iiint_E \frac{\partial \phi}{\partial y}\,dxdydz = \int_S \phi\mathbf{N}\,.\,\mathbf{J}ds \qquad ...(7)$$

Addition of (5), (6) and (7) gives

$$\iiint_E \left(\frac{\partial f}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial z} \right) dxdydz = \int_S (f\mathbf{I} + \phi\mathbf{J} + \psi\mathbf{K})\,.\,\mathbf{N}ds \text{ which is same as (1).}$$

Secondly, consider a general region E. Assume that it can be split up into a finite number of sub-regions each of which is met by a line parallel to any axis in only two points. Applying (1) to each of these sub-regions and adding the results, the volume integrals will combine to give the volume integral over the whole region E. Also the surface integrals over the common boundaries of two sub-regions cancel because each occurs twice and having corresponding normals in opposite directions whereas the remaining surface integrals combine to give the surface integral over the entire surface S.

Finally consider a region E bounded by two closed surfaces S_1, S_2 (S_1 *being within* S_2). Noting that outward normal at points of S_1 is directed inwards (*i.e.*, away from S_2) and introducing an additional surface cutting S_1, S_2 so that all parts of E are bounded by a single closed surface, the truth of the theorem follows as before. Thus theorem also holds for regions enclosed by several surfaces.

Hence the theorem is completely established.

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*See footnote p. 37.

Example 8.42. *Verify Divergence theorem for* $\mathbf{F} = (x^2 - yz)\mathbf{I} + (y^2 - zx)\mathbf{J} + (z^2 - xy)\mathbf{K}$ *taken over the rectangular parallelepiped* $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$. *(Rohtak, 2006 S ; Madras, 2000 S)*

Solution. As $\quad \text{div } \mathbf{F} = \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy)$

$$= 2(x + y + z)$$

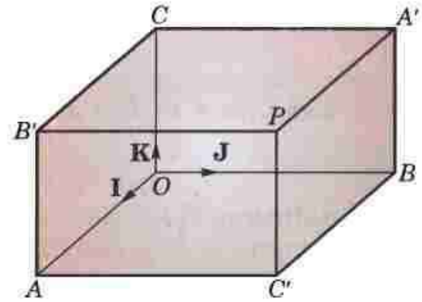

Fig. 8.21

$$\therefore \quad \int_R \text{div } \mathbf{F}\, dv = 2 \int_0^c \int_0^b \int_0^a (x + y + z)\, dxdydz$$

$$= 2 \int_0^c dz \int_0^b dy \left(\frac{a^2}{2} + ya + za \right)$$

$$= 2 \int_0^c dz \left(\frac{a^2}{2} b + \frac{ab^2}{2} + abz \right)$$

$$= 2 \left(\frac{a^2 b}{2} c + \frac{ab^2}{2} c + ab \frac{c^2}{2} \right)$$

$$= abc\,(a + b + c) \qquad \ldots(i)$$

Also $\quad \int_S \mathbf{F} \cdot \mathbf{N} ds = \int_{S_1} \mathbf{F} \cdot \mathbf{N} ds + \int_{S_2} \mathbf{F} \cdot \mathbf{N} ds + \ldots + \int_{S_6} \mathbf{F} \cdot \mathbf{N} ds$

where S_1 in the face $OAC'B$, S_2 the face $CB'PA'$, S_3 the face $OBA'C$, S_4 the face $AC'PB'$, S_5 the face $OCB'A$ and S_6 the face $BAP'C'$ (Fig. 8.21).

Now $\quad \int_{S_1} \mathbf{F} \cdot \mathbf{N} ds = \int_{S_1} \mathbf{F} \cdot (-\mathbf{K})\, ds = -\int_0^b \int_0^a (0 - xy)\, dxdy = \frac{a^2 b^2}{4}$

$$\int_{S_2} \mathbf{F} \cdot \mathbf{N} ds = \int_{S_2} \mathbf{F} \cdot \mathbf{K} ds = \int_0^b \int_0^a (c^2 - xy)\, dxdy = abc^2 - \frac{a^2 b^2}{4}$$

Similarly, $\quad \int_{S_3} \mathbf{F} \cdot \mathbf{N} ds = \frac{b^2 c^2}{4}, \int_{S_4} \mathbf{F} \cdot \mathbf{N} ds = a^2 bc - \frac{b^2 c^2}{4},$

$$\int_{S_5} \mathbf{F} \cdot \mathbf{N} ds = \frac{c^2 a^2}{4} \text{ and } \int_{S_6} \mathbf{F} \cdot \mathbf{N} ds = ab^2 c - \frac{c^2 a^2}{4}$$

Thus $\quad \int_S \mathbf{F} \cdot \mathbf{N} ds = abc(a + b + c) \qquad \ldots(ii)$

Hence the theorem is verified from the equality of (i) and (ii).

Example 8.43. *Evaluate* $\int_S \mathbf{F} \cdot d\mathbf{s}$ *where* $\mathbf{F} = 4x\mathbf{I} - 2y^2\mathbf{J} + z^2\mathbf{K}$ *and S is the surface bounding the region* $x^2 + y^2 = 4$, $z = 0$ *and* $z = 3$. *(S.V.T.U., 2007 S ; Mumbai, 2006 ; J.N.T.U., 2006)*

Solution. By divergence theorem,

$$\int_S \mathbf{F} \cdot d\mathbf{s} = \int_V div\, \mathbf{F}\, dv$$

$$= \int_V \left[\frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2) \right] dv$$

$$= \iiint_V ((4 - 4y + 2z)\, dxdydz$$

$$= \int_{-2}^{2} \int_{\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} \int_0^3 (4 - 4y + 2z)\, dzdydx$$

$$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} \left| 4z - 4yz + z^2 \right|_0^3 dydx$$

$$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} (12 - 12y + 9)\, dydx$$

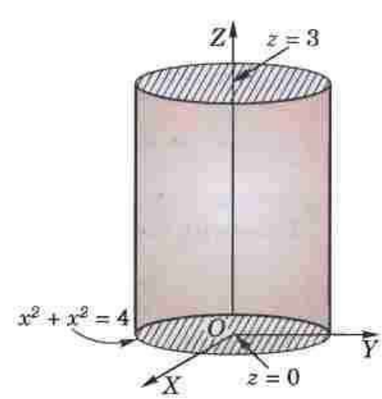

Fig. 8.22

$$= \int_{-2}^{2} \left| 21y - 6y^2 \right|_{-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} dx$$

$$= 42 \int_{-2}^{2} \sqrt{(4-x^2)}\, dx = 84 \int_0^2 \sqrt{(4-x^2)}\, dx = 84 \left| \frac{x\sqrt{(4-x^2)}}{2} + \frac{4}{2}\sin^{-1}\frac{x}{2} \right|_0^2 = 84\pi.$$

Example 8.44. *Evaluate* $\int_S (yz\mathbf{I} + zx\mathbf{J} + xy\mathbf{K}) . d\mathbf{S}$ *where S is the surface of the sphere* $x^2 + y^2 + z^2 = a^2$ *in the first octant.* (U.P.T.U., 2004 S)

Solution. The surface of the region V: $OABC$ is piecewise smooth (Fig. 8.23) and is comprised of four surfaces (*i*) S_1 – circular quadrant OBC in the yz-plane,

(*ii*) S_2 – circular quadrant OCA in the zx-plane,

(*iii*) S_3 – circular quadrant OAB in the xy-plane,

and (*iv*) S–surface ABC of the sphere in the first octant.

Also $\mathbf{F} = yz\mathbf{I} + zx\mathbf{J} + xy\mathbf{K}$

By Divergence theorem,

$$\int_V div\, \mathbf{F}\, dv = \int_{S_1} \mathbf{F} . d\mathbf{S} + \int_{S_2} \mathbf{F} . d\mathbf{S} + \int_{S_3} \mathbf{F} . d\mathbf{S} + \int_S \mathbf{F} . d\mathbf{S} \quad ...(1)$$

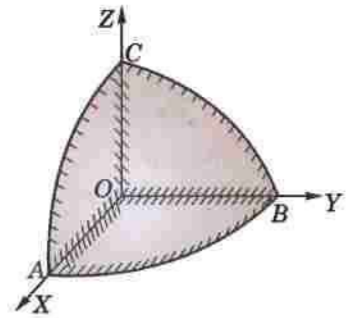

Fig. 8.23

Now $$div\, \mathbf{F} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(zx) + \frac{\partial}{\partial z}(xy) = 0.$$

For the surface S_1, $x = 0$

$$\therefore \quad \int_{S_1} \mathbf{F} . d\mathbf{S} = \int_0^a \int_0^{\sqrt{(a^2-y^2)}} (yz\mathbf{I}) . (-\, dydz\mathbf{I}) = -\int_0^a \int_0^{\sqrt{(a^2-y^2)}} yzdydz = = -\frac{a^4}{8}$$

Thus (1) becomes $0 = -\dfrac{3a^4}{8} + \int_S \mathbf{F} . d\mathbf{S}$ whence $\int_S \mathbf{F} . d\mathbf{S} = 3a^4/8$.

Example 8.45. *Apply divergence theorem to evaluate* $\int (lx^2 + my^2 + nz^2)\, ds$ *taken over the sphere* $(x-a)^2 + (y-b)^2 + (z-c)^2 = \rho^2$; *l, m, n being the direction cosines of the external normal to the sphere.*

Solution. The parametric equations of the sphere are $x = a + \rho \sin\theta \cos\phi$, $y = b + \rho \sin\theta \sin\phi$, $z = c + \rho\cos\theta$ and to cover the whole sphere, r varies from 0 to ρ, θ varies from 0 to π and ϕ from 0 to 2π.

$$\therefore \quad \int_S (lx^2 + my^2 + nz^2)\, ds = \int_S (x^2\mathbf{I} + y^2\mathbf{J} + z^2\mathbf{K}) . \mathbf{N}ds$$

$$= \int_V \text{div}\,(x^2\mathbf{I} + y^2\mathbf{J} + z^2\mathbf{K})dv = 2\int_V (x + y + z)dv$$

$$= 2 \int_0^{2\pi} \int_0^{\pi} \int_0^{\rho} [(a + b + c) + \rho(\sin\theta\cos\phi + \sin\theta\ \sin\phi + \cos\theta)] \times \rho^2 \sin\theta\, dr d\theta d\phi$$

$$= 2(a + b + c)\ \frac{\rho^3}{3} \left| -\cos\theta \right|_0^{\pi} . \ 2\pi = \frac{8\pi}{3}(a + b + c)\,\rho^3.$$

Example 8.46. *Evaluate* $\int_S (a^2x^2 + b^2y^2 + c^2z^2)^{-1/2}\ dS$, *where S is the surface of the ellipsoid*

$$ax^2 + by^2 + cz^2 = 1.$$

Solution. Taking $\phi = ax^2 + by^2 + cz^2 - 1 = 0$, $\nabla\phi = 2ax\mathbf{I} + 2by\mathbf{J} + 2cz\mathbf{K}$

$\therefore$ Unit vector normal to the ellipsoid $= \hat{N} = \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{ax\mathbf{I} + by\mathbf{J} + cz\mathbf{K}}{\sqrt{(a^2x^2 + b^2y^2 + c^2z^2)}}$

Since $\mathbf{F} . \hat{\mathbf{N}} = (a^2x^2 + b^2y^2 + c^2z^2)^{-1/2}$, $\quad \therefore\ \mathbf{F} . (ax\mathbf{I} + by\mathbf{J} + cz\mathbf{K}) = 1$

Obviously $\quad \mathbf{F} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$ $\quad [\because ax^2 + by^2 + cz^2 = 1$

$\therefore$ By Divergence theorem,

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_V div\, \mathbf{F} dv = \int_V \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) \right] dv = 3 \int_V dv = 3V$$

$$= 3 \cdot \frac{4\pi}{3} \frac{1}{\sqrt{(abc)}} = \frac{4\pi}{\sqrt{(abc)}}. \qquad \left[\because \text{ Vol. of ellipsoid} = \frac{4\pi}{3} \frac{1}{\sqrt{(abc)}} \right]$$

Example 8.47. *If the position vector of any point (x, y, z) within a closed surface S, be* **R** *measured from an origin O, then show that*

$$\iint_S \frac{\mathbf{R}}{r^3} \cdot \mathbf{N}\, ds = \begin{cases} 0, \text{ if } O \text{ lies outside } S \\ 4\pi, \text{ if } O \text{ lies inside } S \end{cases}$$

Solution. (*a*) *When O is outside S.* Here $\mathbf{F} = \mathbf{R}/r^3$ is continuously differentiable throughout the volume V enclosed by S. Hence by Divergence theorem, we have

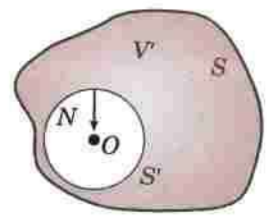

Fig. 8.24

$$\iint_S \frac{\mathbf{R}}{r^3} \cdot \mathbf{N}\, ds = \iiint_V \text{div}\left(\frac{\mathbf{R}}{r^3}\right) dV = 0 \qquad \left[\because \text{div}\left(\frac{\mathbf{R}}{r^3}\right) = 0 \right.$$

(*b*) *When O is inside S.* Hence $F = \mathbf{R}/r^3$ has a point of discontinuity at O and as such Divergence theorem cannot be applied to the region V enclosed by S. To remove this point of discontinuity, we enclose O by a small sphere S' of radius ρ.

Now **F** is continuously differentiable throughout the region V' enclosed between S and S'. Therefore applying Divergence theorem to region V', we get

$$\iint_S \frac{\mathbf{R}}{r^3} \cdot \mathbf{N}\, ds + \iint_{S'} \frac{\mathbf{R}}{r^3} \cdot \mathbf{N}\, ds' = \iiint_{V'} \text{div}\left(\frac{\mathbf{R}}{r^3}\right) dV' = 0 \qquad \left[\because \text{ div}\left(\frac{\mathbf{R}}{r^3}\right) = 0 \right]$$

$$\therefore \qquad \iint_S \frac{\mathbf{R}}{r^3} \cdot \mathbf{N}\, ds = -\iint_{S} \frac{\mathbf{R}}{r^3} \cdot \mathbf{N}\, ds' \qquad ...(i)$$

Now the outward normal **N** on the sphere S' is directed towards the centre O. Therefore $\mathbf{N} = -\mathbf{R}/\rho$ on S' (Fig. 8.24).

$$\therefore \qquad -\iint_{S'} \frac{\mathbf{R}}{r^3} \cdot \mathbf{N}\, ds' = -\iint_{S'} \frac{\mathbf{R}}{\rho^3} \cdot \left(-\frac{\mathbf{R}}{\rho}\right) ds' \qquad [\because \text{ on } S', r = \rho]$$

$$= \iint_{S'} \frac{r^2}{\rho^4} ds' = \iint_{S'} \frac{\rho^2}{\rho^4} ds' = \frac{1}{\rho^2} \iint_{S'} ds' = \frac{1}{\rho^2} \cdot 4\pi\rho^2 = 4\pi$$

Hence from (*i*), $\qquad \iint_S \frac{\mathbf{R}}{r^3} \cdot \mathbf{N}\, ds = 4\pi.$

8.17 GREEN'S THEOREM*

If ϕ and ψ are scalar point functions possessing continuous derivatives of first and second orders, then

$$\int_E (\phi\nabla^2\psi - \psi\nabla^2\phi) dv = \int_S \left(\phi \frac{\partial\psi}{\partial n} - \psi \frac{\partial\psi}{\partial n} \right) ds \qquad ...(1)$$

where $\partial/\partial n$ denotes differentiation in the direction of the external normal to the bounding surface S enclosing the region E.

Applying Divergence theorem : $\int_S \mathbf{F} \cdot \mathbf{N} ds = \int_S \nabla \cdot \mathbf{F} dv$ to the function $\phi\nabla\psi$, we get

$$\int_S \phi\nabla\psi \cdot \mathbf{N} ds = \int_E \nabla \cdot (\phi\nabla\psi) dv = \int_E (\nabla\phi \cdot \nabla\psi + \phi\nabla^2\psi) dv \qquad \text{[By (2) page 329]}$$

$$= \int_E \nabla\phi \cdot \nabla\psi dv + \int_E \phi\nabla^2\psi dv \qquad ...(2)$$

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*See footnote p. 339.

Interchanging ϕ and ψ, (*ii*) gives

$$\int_S \psi\nabla\phi \,.\, \mathbf{N}ds = \int_E \nabla\psi \,.\, \nabla\phi dv + \int_E \psi\nabla^2\phi dv \quad ...(3)$$

Subtracting (3) from (2), we have $\int_S (\phi\nabla\psi - \psi\nabla\phi) \,.\, \mathbf{N}ds = \int_E (\phi\nabla^2\psi - \psi\nabla^2\phi)dv$

But $\nabla\psi \,.\, \mathbf{N} = \partial\psi/\partial n$ the directional derivative of ψ along the external normal at any point of S. Hence

$$\int_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)ds = \int_E (\phi\nabla^2\psi - \psi\nabla^2\phi)\,dv$$ which is the required result (1).

Obs. Harmonic function. *A scalar point function ϕ satisfying the Laplace's equation $\nabla^2\phi = 0$ at every point of a region E, is called a harmonic function in E.*

If ϕ and ϕ be both harmonic functions in E, (1) gives

$$\int_S \phi\frac{\partial\psi}{\partial n}ds = \int_S \psi\frac{\partial\phi}{\partial n}ds$$ which is known as **Green's reciprocal theorem.**

PROBLEMS 8.10

1. Verify divergence theorem for $\mathbf{F}$ taken over the cube bounded by $x = 0, x = 1$; $y = 0, y = 1$; $z = 0, z = 1$ where
   (*i*) $\mathbf{F} = 4xz\mathbf{I} - y^2\mathbf{J} + yz\mathbf{K}$ (*Madras, 2006*) (*ii*) $x^2\mathbf{I} + z\mathbf{J} + yz\mathbf{K}$ (*Bhopal, 2008*)
2. Verify Gauss divergence theorem for the function $\mathbf{F} = y\mathbf{I} + x\mathbf{J} + z^2\mathbf{K}$ over the cylindrical region bounded by $x^2 + y^2 = 9$, $z = 0$ and $z = 2$.
3. Using divergence theorem, prove that
   (*i*) $\int_S \mathbf{R} \,.\, d\mathbf{S} = 3V$ (*ii*) $\int_S \nabla r^2 \,.\, d\mathbf{S} = 6V$ (*U.P.T.U., 2003*)
   where S is any closed surface enclosing a volume V and $r^2 = x^2 + y^2 + z^2$.
4. Using divergence theorem, evaluate $\int_S \mathbf{R} \,.\, \mathbf{N}ds$ where S is the surface of the sphere $x^2 + y^2 + z^2 = 9$.
5. If S is any closed surface enclosing a volume V and $\mathbf{F} = ax\mathbf{I} + by\mathbf{J} + cz\mathbf{K}$, prove that
   $$\int_S \mathbf{F} \,.\, \mathbf{N}ds = (a + b + c)V$$ (*Madras, 2003*)
6. For any closed surface S, prove that $\int [x(y - z)\mathbf{I} + y(z - x)\mathbf{J} + z(x - y)\mathbf{K}] \,.\, ds = 0$.
7. Use divergence theorem to evaluate $\int_S \mathbf{F} \,.\, d\mathbf{S}$, where
   (*i*) $\mathbf{F} = x^3\mathbf{I} + y^3\mathbf{J} + z^3\mathbf{K}$, and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$. (*V.T.U., 2008 ; P.T.U., 2005*)
   (*ii*) $\mathbf{F} = [e^x, e^y, e^z]$ and S is the surface of the cube $|x| \le 1$, $|y| \le 1$, $|z| \le 1$. (*B.P.T.U., 2005*)
8. Evaluate $\iint (xdydz + ydzdx + zdxdy)$ over the surface of a sphere of radius a. (*Kurukshetra, 2008 S*)
9. Evaluate $\int_S \mathbf{F} \,.\, d\mathbf{S}$ where $F = y^2z^2\mathbf{I} + z^2x^2\mathbf{J} + x^2y^2\mathbf{K}$ and S is the upper part of the sphere $x^2 + y^2 + z^2 = a^2$ above XOY plane.
10. By transforming to triple integral, evaluate $\iint_S (x^3dydz + x^2ydzdx + x^2zdxdy)$ where S is the closed surface consisting of the cylinder $x^2 + y^2 = a^2$ and the circular discs $z = 0$ and $z = b$. (*Burdwan, 2003*)
11. Evaluate $\int_S (\nabla \times \mathbf{F}) \,.\, d\mathbf{S}$, where S is the surface of the paraboloid $x^2 + y^2 + z = 4$ above the xy-plane, and $\mathbf{F} = (x^2 + y - 4)\mathbf{I} + 3xy\mathbf{J} + (2xz + z^2)\mathbf{K}$.
12. If $\mathbf{F} = (2x^2 - 3z)\mathbf{I} - 2xy\mathbf{J} - 4x\mathbf{K}$, then evaluate $\iiint_V \nabla \,.\, \mathbf{F}\, dv$, where V is bounded by $x = y = z = 0$ and $2x + 2y + z = 4$. (*Bhopal, 2008*)
13. If $\mathbf{F} = \text{grad}\,\phi$ and $\nabla^2\phi = -4\pi\rho$, prove that $\int_S \mathbf{F} \,.\, \mathbf{N}ds = -4\pi\rho \int_V dV$ where the symbol have their usual meanings.

8.18 (1) IRROTATIONAL FIELDS

An irrotational field **F** *is characterised by any one of the following conditions :*

(*i*) $\Delta \times \mathbf{F} = \mathbf{0}$. (*ii*) *Circulation* $\int \mathbf{F} \cdot d\mathbf{R}$ *along every closed surface is zero.*

(*iii*) $\mathbf{F} = \Delta\phi$, *if the domain is simply connected.**

If $\nabla \times \mathbf{F} = \mathbf{0}$, then by Stoke's theorem,

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \mathbf{0}, \text{ i.e., the circulation along every closed surface is zero.}$$

Again since $\nabla \times \nabla\phi = 0$

$\therefore$ in an irrotational field for which $\Delta \times \mathbf{F} = 0$, the vector **F** can always be expressed as the gradient of a scalar function ϕ provided the domain is simply connected. Thus

$$\mathbf{F} = \nabla\phi.$$

Such a scalar function ϕ is called the *potential*. In a rotational field, **F** cannot be expressed as the gradient of a scalar potential.

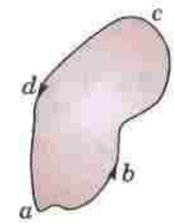

Fig. 8.25

Obs. 1. *In an irrotational field, the line integral* **F** *between two points is independent of the path of integration and is equal to the potential difference between these points.*

If *aced* be any closed contour in an irrotational field F (Fig. 8.25), then

$$\int_{abcd} \mathbf{F} \cdot d\mathbf{R} = \int_{abc} \mathbf{F} \cdot d\mathbf{R} + \int_{cda} \mathbf{F} \cdot d\mathbf{R} = 0$$

or
$$\int_{abc} \mathbf{F} \cdot d\mathbf{R} = \int_{abc} \mathbf{F} \cdot d\mathbf{R}$$

i.e. the value of the line integral is independent of the path joining the end points.

Further, substituting $\mathbf{F} = \nabla\phi$, we have

$$\int_a^c \mathbf{F} \cdot d\mathbf{R} = \int_a^c \nabla\phi \cdot d\mathbf{R} = \int_a^c \left(\mathbf{I}\frac{\partial\phi}{\partial x} + \mathbf{J}\frac{\partial\phi}{\partial y} + \mathbf{K}\frac{\partial\phi}{\partial z} \right) \cdot (\mathbf{I}dx + \mathbf{J}dy + \mathbf{K}dz)$$

$$= \int_a^c \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz \right) = \int_a^c d\phi = \phi_c - \phi_a.$$

Obs. 2. If **F** is a vector force acting on a particle, then $\oint_C \mathbf{F} \cdot d\mathbf{R}$ represents the work done in moving the particle around a closed path. [See p. 328]

When $\oint_C \mathbf{F} \cdot d\mathbf{R} = 0$, *the field is said to be* **conservative,** *i.e.*, no work is done in displacement from a point a to another point in the field and back to a and the mechanical energy is conserved.

Thus *every irrotational field is conservative.*

Obs. 3. *The well-known equations of the Poisson and Laplace hold good for every irrotational field.*

Suppose $\nabla \cdot \mathbf{F} = f(x, y, z)$. Then $\nabla \cdot \nabla\phi = f(x, y, z)$ *i.e.*, $\nabla^2\phi = f(x, y, z)$ ...(*i*)

which is known as *Poisson's equation*. Its solutions for electrostatic fields enable us to determine the potential ϕ as a function of the charge distribution $f(x, y, z)$.

If $f(x, y, z) = 0$ then (*i*) reduces to $\nabla^2\phi = 0$ which is the *Laplace's equation*. The solutions of this equation are of great importance in modern engineering and physics, some of which we'll study in § 18.11 and 18.12.

(2) **Solenoidal fields.** *A solenoidal field* **F** *is characterised by any one of the following conditions :*

(*i*) $\nabla \cdot \mathbf{F} = 0$. (*ii*) *flux* $\int \mathbf{F} \cdot \mathbf{N}\, ds$ *across every closed surface is zero.* (*iii*) $\mathbf{F} = \nabla \times \mathbf{V}$.

If $\nabla \cdot \mathbf{F} = 0$ then by the Divergence theorem,

$$\int_S \mathbf{F} \cdot \mathbf{N}ds = \int_V \nabla \cdot \mathbf{F}dv = 0, \text{ i.e., the flux across every closed surface is zero.}$$

Again since $\nabla \cdot \nabla \times \mathbf{V} = 0$.

$\therefore$ in a solenoidal field for which $\nabla \cdot \mathbf{F} = 0$, the vector **F** can always be expressed as the curl of a vector function **V**; thus $\mathbf{F} = \nabla \times \mathbf{V}$.

*A domain D is said to be *simply connected* if every closed curve in D can be shrunk to any point within D.

Example 8.48. *A vector field is given by* $\mathbf{F} = (x^2 - y^2 + x)\mathbf{I} - (2xy + y)\mathbf{J}$. *Show that the field is irrotational and find its scalar potential. Hence evaluate the line integral from (1, 2) to (2, 1).*

Solution. Since $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - y^2 + x & -(2xy + y) & 0 \end{vmatrix} = \mathbf{0}$

$\therefore$ this field is *irrotational* and the vector **F** can be expressed as the gradient of a scalar potential,

i.e., $(x^2 - y^2 + x)\mathbf{I} - (2xy + y)\mathbf{J} = \nabla\phi = \dfrac{\partial\phi}{\partial x}\mathbf{I} + \dfrac{\partial\phi}{\partial y}\mathbf{J}$

whence $\qquad \partial\phi/\partial x = x^2 - y^2 + x \qquad$ ...(*i*)

$\qquad \partial\phi/\partial y = -(2xy + y) \qquad$ ...(*ii*)

Integrating (*i*) w.r.t. x, keeping y constant, we get $\phi = \dfrac{x^3}{3} - y^2x + \dfrac{x^2}{2} + f(y)$ ...(*iii*)

Similarly integrating (*ii*) w.r.t. y, keeping x constant, we obtain $\phi = -xy^2 - \dfrac{y^2}{2} + g(x)$ ...(*iv*)

Equating (*iii*) and (*iv*), we get $\dfrac{x^3}{3} - y^2x + \dfrac{x^2}{2} + f(y) = -xy^2 - \dfrac{y^2}{2} + g(x)$

$\therefore \qquad f(y) = -\dfrac{y^2}{2}$ and $g(x) = \dfrac{x^3}{3} + \dfrac{x^2}{2}$

Hence $\qquad \phi = \dfrac{x^3}{3} - xy^2 + \dfrac{x^2}{2} - \dfrac{y^2}{2}$

Since the field is irrotational,

$\therefore \quad \int \mathbf{F} \, . \, d\mathbf{R}$ from (1, 2) to (2, 1) $= \phi_{1,2} - \phi_{2,1} = \left(\dfrac{1}{3} - 1 \times 4 + \dfrac{1}{2} - \dfrac{4}{2}\right) - \left(\dfrac{8}{3} - 2 \times 1 + \dfrac{4}{2} - \dfrac{1}{2}\right) = -7\dfrac{1}{3}$.

Example 8.49. *A fluid motion is given by* $\mathbf{V} = (y + z)\mathbf{I} + (z + x)\mathbf{J} + (x + y)\mathbf{K}$.
(*a*) *Is this motion irrotational? If so, find the velocity potential.* (*U.P.T.U., 2004*)
(*b*) *Is the motion possible for an incompressible fluid?*

Solution. We have $\nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y + z & z + x & x + y \end{vmatrix} = \mathbf{I}(1 - 1) - \mathbf{J}(1 - 1) + \mathbf{K}(1 - 1) = \mathbf{0}.$

$\therefore$ this motion is irrotational and if ϕ is the velocity potential then $\mathbf{V} = \nabla\phi$. [§ 20.6]

i.e., $\qquad (y + z)\mathbf{I} + (z + x)\mathbf{J} + (x + y)\mathbf{K} = \dfrac{\partial\phi}{\partial x}\mathbf{I} + \dfrac{\partial\phi}{\partial y}\mathbf{J} + \dfrac{\partial\phi}{\partial z}\mathbf{K}$

$\therefore \qquad \dfrac{\partial\phi}{\partial x} = y + z, \dfrac{\partial\phi}{\partial y} = z + x, \dfrac{\partial\phi}{\partial z} = x + y$

Integrating these, we get $\qquad \phi = (y + z)x + f_1(y, z) \qquad$ ...(*i*)

$\qquad \phi = (z + x)y + f_2(z, x) \qquad$ ...(*ii*)

and $\qquad \phi = (x + y)z + f_3(x, y) \qquad$ ...(*iii*)

Equality of (*i*), (*ii*) and (*iii*), requires that

$$f_1(y, z) = yz, f_2(z, x) = zx, f_3(x, y) = xy.$$

Hence $\qquad \phi = yz + zx + xy.$

(*b*) The fluid motion is possible if **V** satisfies the equation of continuity which for an incompressible fluid is $\nabla \, . \, \mathbf{V} = 0$. [See § 8.7 (1)

Here $$\nabla \cdot \mathbf{V} = \frac{\partial}{\partial x}(y+z) + \frac{\partial}{\partial y} = (z+x) + \frac{\partial}{\partial z}(x+y) = 0.$$

Hence, the fluid motion is possible.

Example 8.50. *Find whether* $\int_C [2xyz^2 dx + (x^2z^2 + z\cos yz)\,dy + (2x^2yz + y\cos yz)\,dz]$ *is independent of the path joining* $(0, \pi/2, 1)$ *and* $(1, 0, 1)$. *If so, evaluate this line integral.*

Solution. The line integral of **F** is independent of path of integration if $\nabla \times \mathbf{F} = \mathbf{0}$.

$$= \int_C [2xyz^2\,\mathbf{I} + (x^2z^2 + z\cos yz)\mathbf{J} + (2x^2yz + y\cos yz)\mathbf{K}] \cdot (\mathbf{I}dx + \mathbf{J}dy + \mathbf{K}dz) = \int_C \mathbf{F}\cdot d\mathbf{R}$$

and $$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^2 & x^2z^2 + z\cos yz & 2x^2yz + y\cos yz \end{vmatrix}$$

$$= \mathbf{I}\,[2x^2z + \cos yz - yz\sin yz - (2x^2z + \cos yz - yz\sin yz)]$$
$$- \mathbf{J}[4xyz - 4xyz] + \mathbf{K}[2xz^2 - 2xz^2] = \mathbf{0}$$

$\therefore$ the given integral is independent of the path C.

Now let $\mathbf{F} = \nabla\phi$

i.e., $$(2xyz^2)\mathbf{I} + (x^2z^2 + z\cos yz)\mathbf{J} + (2x^2yz + y\cos yz)\,\mathbf{K} = \mathbf{I}\frac{\partial\phi}{\partial x} + \mathbf{J}\frac{\partial\phi}{\partial y} + \mathbf{K}\frac{\partial\phi}{\partial z}$$

$$\therefore \quad 2xyz^2 = \frac{\partial\phi}{\partial x},\ x^2z^2 + z\cos yz = \frac{\partial\phi}{\partial y},\ 2x^2yz + y\cos yz = \frac{\partial\phi}{\partial z}$$

Integrating first w.r.t. x partially, we get

$$\phi = x^2y^2z^2 + \Psi_1(y, z) \quad ...(i)$$

Integrating second w.r.t. y partially, we get

$$\phi = x^2yz^2 + \sin yz + \Psi_2(z, x) \quad ...(ii)$$

Integrating third w.r.t. z partially, we get

$$\phi = x^2yz^2 + \sin yz + \Psi_3(x, y) \quad ...(iii)$$

Comparing (*i*), (*ii*), (*iii*), we have

$\Psi_1(y, z)$ = terms in ϕ independent of $x = \sin yz$

$\Psi_2(z, x)$ = terms in ϕ independent of $y = 0$

$\Psi_3(z, x)$ = terms in ϕ independent of $z = 0$

Thus $\phi = x^2yz^2 + \sin yz$

Hence the value of the given integral $= \left|\phi\right|_{(0,\pi/2,1)}^{(1,0,1)}$

$$= (0 + 0) - (0 + \sin \pi/2) = -1.$$

Example 8.51. *Determine whether* $\mathbf{F} = (y^2\cos x + z^3)\mathbf{I} + (2y\sin x - 4)\mathbf{J} + (3xz^2 + 2)\mathbf{K}$ *is a conservative vector field? If so find the scalar potential* ϕ. *Also compute the work done in moving the particle from* $(0, 1, -1)$ *to* $(\pi/2, -1, 2)$. *(Mumbai, 2006)*

Solution. **F** is a conservative vector field when curl $\mathbf{F} = \mathbf{0}$. Here

$$Curl = F = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2\cos x + z^3 & 2y\sin x - 4 & 3xz^2 + 2 \end{vmatrix}$$

$$= \mathbf{I}(0 - 0) - \mathbf{J}(3z^2 - 3z^2) + \mathbf{K}(2y\cos x - 2y\cos x) = \mathbf{0}$$

$\therefore$ **F** is a conservative field.

Now let $\mathbf{F} = \nabla\phi$

i.e., $$(y^2 \cos x + z^3)\, \mathbf{I} + (2y \sin x - 4)\, \mathbf{J} + (3xz^2 + 2)\, \mathbf{K} = \mathbf{I}\frac{\partial\phi}{\partial x} + \mathbf{J}\frac{\partial\phi}{\partial y} + \mathbf{K}\frac{\partial\phi}{\partial z}$$

$\therefore$ $$y^2 \cos x + z^3 = \frac{\partial\phi}{\partial x},\ 2y \sin x - 4 = \frac{\partial\phi}{\partial y},\ 3xz^2 + 2 = \frac{\partial\phi}{\partial z}$$

Integrating first w.r.t. x partially, we get

$$\phi = y^2 \sin x + xz^3 + \psi_1(y, z) \qquad ...(i)$$

Integrating second w.r.t. y partially, we get

$$\phi = y^2 \sin x - 4y + \Psi_2(z, x) \qquad ...(ii)$$

Integrating third w.r.t. z partially, we obtain

$$\phi = xz^3 + 2z + \Psi_3(x, y) \qquad ...(iii)$$

Comparing (*i*), (*ii*), (*iii*), we get

$$\Psi_1(y, z) = \text{terms in } \phi \text{ independent of } x = -4y + 2z$$
$$\Psi_2(z, x) = \text{terms in } \phi \text{ independent of } y = xz^3 + 2z$$
$$\Psi_3(z, x) = \text{terms in } \phi \text{ independent of } z = y^2 \sin x - 4y$$

Thus $$\phi = xz^3 + y^2 \sin x - 4y + 2z$$

In a conservative field, the work done $= \phi_B - \phi_A$

$$= \phi\left(\frac{\pi}{2}, -1, 2\right) - \phi(0, 1, -1)$$
$$= (4\pi + 1 + 4 + 4) - (-4 - 2) = 4\pi + 15.$$

PROBLEMS 8.11

1. If ϕ is a solution of the Laplace equation, prove that $\nabla\phi$ is both solenoidal and irrotational.
2. Show that the vector field defined by $\mathbf{F} = (x^2 + xy^2)\mathbf{I} + (y^2 + x^2y)\mathbf{J}$ is conservative and find the scalar potential. Hence evaluate $\int \mathbf{F} \cdot d\mathbf{R}$ from (0, 1) to (1, 2).
3. Find the work done by the variable force $\mathbf{F} = 2y\mathbf{I} + xy\mathbf{J}$ on a particle when it is displaced from the origin to the point $\mathbf{R} = 4\mathbf{I} + 2\mathbf{J}$ along the parabola $y^2 = x$.
4. Show that the vector field given by $\mathbf{A} = 3x^2y\mathbf{I} + (x^3 - 2yz^2)\mathbf{J} + (3z^2 - 2y^2z)\mathbf{K}$ is irrotational but not solenoidal. Also find $\phi(x, y, z)$ such that $\nabla\phi = \mathbf{A}$.
5. Show that the following vectors are irrotational and find the scalar potential in each case :
   (*i*) $(x^2 - yz)\mathbf{I} + (y^2 - zx)\mathbf{J} + (z^2 - xy)\mathbf{K}$
   (*ii*) $2xy\mathbf{I} + (x^2 + 2yz)\mathbf{J} + (y^2 + 1)\mathbf{K}$ (*V.T.U., 2007*)
   (*iii*) $(6xy + z^3)\mathbf{I} + (3x^2 - z)\mathbf{J} + (3xz^2 - y)\mathbf{K}$ (*Raipur, 2005 ; V.T.U., 2003 S*)
   (*iv*) $(2xy^2 + yz)\mathbf{I} + (2x^2y + xz + 2yz^2)\mathbf{J} + (2y^2z + xy)\mathbf{K}$. (*V.T.U., 2010*)
6. Fluid motion is given by $\mathbf{V} = ax\mathbf{I} + ay\mathbf{J} - 2az\mathbf{K}$.
   (*i*) Is it possible to find out the velocity potential ? If so, find it.
   (*ii*) Is the motion possible for an incompressible fluid ?
7. Show that the vector field defined by $\mathbf{F} = (y \sin z - \sin x)\mathbf{I} + (x \sin z + 2yz)\mathbf{J} + (xy \cos z + y^2)\mathbf{K}$ is irrotational and find its velocity potential. (*Kottayam, 2005*)
8. Show that $\mathbf{F} = (2xy + z^3)\mathbf{I} + x^2\mathbf{J} + 3xz^2\mathbf{K}$ is a conservative vector field and find a function ϕ such that $\mathbf{F} = \nabla\phi$. Also find the work done in moving an object in this field from (1, – 2, 1) to (3, 1, 4). (*Nagpur, 2009*)
9. If $\mathbf{F} = (x + y + az)\mathbf{I} + (bx + 2y - z)\mathbf{J} + (x + cy + 2z)\mathbf{K}$, find a, b, c such that curl $\mathbf{F} = \mathbf{0}$, then find ϕ such that $\mathbf{F} = \nabla\phi$. (*V.T.U., 2000*)
10. Find the constant a so that **V** is a conservative vector field, where
    $$\mathbf{V} = (axy - z^3)\mathbf{I} + (a - 2)x^2\mathbf{J} + (1 - a)xz^2\mathbf{K}.$$
    Calculate its scalar potential and work done in moving a particle from (1, 2, – 3) to (1, – 4, 2) in the field. (*Mumbai, 2006 ; Rajasthan, 2006*)

8.19 (1) ORTHOGONAL CURVILINEAR COORDINATES

Let the rectangular coordinates (x, y, z) of any point be expressed as functions of u, v, w so that

$$x = x(u, v, w), y = y(u, v, w), z = z(u, v, w) \quad \text{...(1)}$$

Suppose that (1) can be solved for u, v, w in terms of x, y, z, so that

$$u = u(x, y, z), v = v(x, y, z), w = w(x, y, z) \quad \text{...(2)}$$

We assume that the functions in (1) and (2) are single-valued and have continuous partial derivatives so that the correspondence between (x, y, z) and (u, v, w) is unique. Then (u, v, w) are called *curvilinear coordinates* of (x, y, z).

Each of u, v, w has a level surface through an arbitrary point. The surfaces $u = u_0, v = v_0, w = w_0$ are called *coordinate surfaces* through $P(u_0, v_0, w_0)$. Each pair of these coordinate surfaces intersect in curves called the *coordinate curves*. The curve of intersection of $u = u_0$ and $v = v_0$ will be called the w-curve, for only w changes along this curve. Similarly we define u and v-curves.

In vector notation, (1) can be written as $\mathbf{R} = x(u, v, w)\mathbf{I} + y(u, v, w)\mathbf{J} + z(u, v, w)\mathbf{K}$

$$\therefore \quad d\mathbf{R} = \frac{\partial \mathbf{R}}{\partial u} du + \frac{\partial \mathbf{R}}{\partial v} dv + \frac{\partial \mathbf{R}}{\partial w} dw \quad \text{...(3)}$$

Then $\partial\mathbf{R}/\partial u$ is a tangent vector to the u-curve at P. If $\mathbf{T}_u$ is a unit vector at P in this direction, then $\partial\mathbf{R}/\partial u = h_1\mathbf{T}_v$ where $h_1 = |\partial\mathbf{R}/\partial u|$.

Similarly if $\mathbf{T}_v$ and $\mathbf{T}_w$ be unit tangent vectors to v- and w-curves at P, then

$$\frac{\partial \mathbf{R}}{\partial v} = h_2\mathbf{T}_v \text{ and } \frac{\partial \mathbf{R}}{\partial w} = h_3\mathbf{T}_w$$

where $h_2 = |\partial\mathbf{R}/\partial v|$ and $h_3 = |\partial\mathbf{R}/\partial w|$. [$h_1, h_2, h_3$ are called scalar factors.]

Then (3) can be written as

$$d\mathbf{R} = h_1 du\, \mathbf{T}_u + h_2 dv\, \mathbf{T}_v + h_3\, dw\, \mathbf{T}_w \quad \text{...(4)}$$

Since ∇u is normal to the surface $u = u_0$ at P, therefore, a unit vector in this direction is given by $\mathbf{N}_u = \dfrac{\nabla u}{|\nabla u|}$.

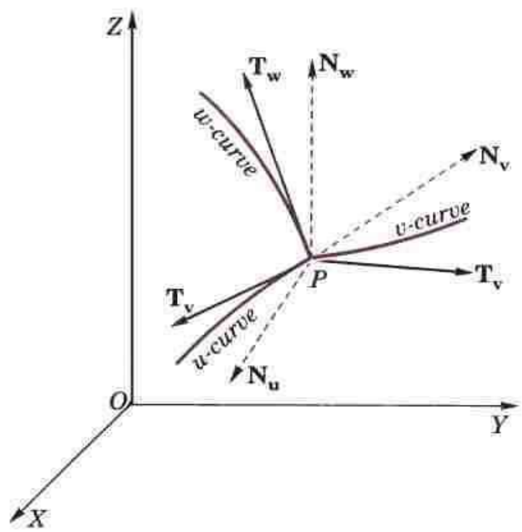

Fig. 8.26

Similarly, the unit vectors $\mathbf{N}_v = \dfrac{\nabla v}{|\nabla v|}$ and $\mathbf{N}_w = \dfrac{\nabla w}{|\nabla w|}$ are normal to the surfaces $v = v_0$ and $w = w_0$ at P respectively. Thus at each point P of a curvilinear coordinate system there exist two triads of unit vectors : $\mathbf{T}_u, \mathbf{T}_v, \mathbf{T}_w$ tangents to u, v, w-curves and $\mathbf{N}_u, \mathbf{N}_v, \mathbf{N}_w$ normals to the co-ordinates surfaces (Fig. 8.26).

In particular, when the coordinate surfaces intersect a right angles, the three coordinate curves are also mutually orthogonal and u, v, w are called the *orthogonal curvilinear coordinates*. In this case $\mathbf{T}_u, \mathbf{T}_v, \mathbf{T}_w$ and $\mathbf{N}_u, \mathbf{N}_v, \mathbf{N}_w$ are mutually perpendicular unit vector triads and hence become identical. Henceforth, we shall refer to orthogonal curvilinear coordinates only.

Multiplying (3) scalarly by ∇u, we get

$$\nabla u \,.\, d\mathbf{R} = du = \left(\nabla u \,.\, \frac{\partial \mathbf{R}}{\partial u}\right) du + \left(\nabla u \,.\, \frac{\partial \mathbf{R}}{\partial v}\right) dv + \left(\nabla u \,.\, \frac{\partial \mathbf{R}}{\partial w}\right) dw$$

whence $\quad \nabla u \,.\, \dfrac{\partial \mathbf{R}}{\partial u} = 1, \nabla u \,.\, \dfrac{\partial \mathbf{R}}{\partial v} = 0, \nabla u \,.\, \dfrac{\partial \mathbf{R}}{\partial w} = 0$

Similarly, $\quad \nabla v \,.\, \dfrac{\partial \mathbf{R}}{\partial u} = 0, \nabla v \,.\, \dfrac{\partial \mathbf{R}}{\partial v} = 1, \nabla v \,.\, \dfrac{\partial \mathbf{R}}{\partial w} = 0$

and $\quad \nabla w \,.\, \dfrac{\partial \mathbf{R}}{\partial u} = 0, \nabla w \,.\, \dfrac{\partial \mathbf{R}}{\partial v} = 0, \nabla w \,.\, \dfrac{\partial \mathbf{R}}{\partial w} = 1.$

These relations show that the sets $\partial\mathbf{R}/\partial u, \partial\mathbf{R}/\partial v, \partial\mathbf{R}/\partial w$ and $\nabla u, \nabla v, \nabla w$ constitute reciprocal system of vectors.

$$\therefore \qquad \nabla u = \frac{\frac{\partial \mathbf{R}}{\partial v} \times \frac{\partial \mathbf{R}}{\partial w}}{\left[\frac{\partial \mathbf{R}}{\partial u} \cdot \frac{\partial \mathbf{R}}{\partial v} \times \frac{\partial \mathbf{R}}{\partial w}\right]} = \frac{(h_2 \mathbf{T_v}) \times (h_3 \mathbf{T_w})}{[(h_1 \mathbf{T_u}) \cdot (h_2 \mathbf{T_v}) \times (h_3 T_\mathbf{w})]}$$

$$= \frac{h_2 h_3 \, \mathbf{T}_v \times \mathbf{T}_w}{h_1 h_2 h_3 [\mathbf{T}_u \mathbf{T}_v \mathbf{T}_w]} = \frac{\mathbf{T}_u}{h_1} \qquad [\because \ \mathbf{T}_u \mathbf{T}_v \mathbf{T}_w = 1]$$

or $\quad \mathbf{T}_v = h_1 \nabla u$

Similarly $\quad \mathbf{T}_v = h_2 \nabla v$ and $\mathbf{T}_w = h_3 \nabla w$ $\qquad$ ...(5)

Also $\quad = \mathbf{T}_v \times \mathbf{T}_w = h_2 h_3 \nabla v \times \nabla w$

Similarly $\quad \mathbf{T}_v = h_3 h_1 \nabla w \times \nabla u$ and $\mathbf{T}_w = h_1 h_2 \nabla u \times \nabla v$ $\qquad$ ...(6)

Arc, area and volume elements

(*i*) *Arc element*. The element of arc length ds is determined from (4).

$$\therefore \qquad ds^2 = d\mathbf{R} \cdot d\mathbf{R} = h_1^2 du^2 + h_2^2 dv^2 + h_3^2 dw^2 \qquad \text{...(7)}$$

The arc length ds_1 along u-curve at P is $h_1 du$ for v and w are constants. Therefore the vector arc element along the u-curve is $d\mathbf{u} = h_1 du \mathbf{T}_u$. Similarly vector arc elements along v and w curves at P are $d\mathbf{v} = h_2 dv \ \mathbf{T}_v$ and $dw = h_3 dw \ \mathbf{T}_w$. The arc element ds therefore corresponds to the length of the diagonal of the rectangular parallelopiped of Fig. 8.27.

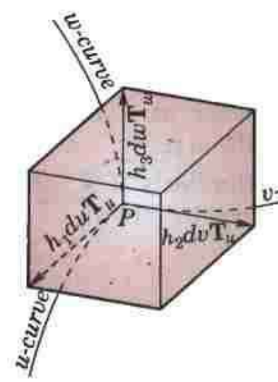

Fig. 8.27

(*ii*) *Area elements.* The area of the parallelogram formed by $d\mathbf{u}$ and $d\mathbf{v}$ is called the area element on the uv surface which is perpendicular to w-curve and we denote it by dS_w. Hence, $dS_w = |du \times dv| = h_1 h_2 du dv$. Similarly, $dS_u = h_2 h_3 dv dw$, $dS_v = h_3 h_1 dw du$.

(*iii*) *Volume element* is the volume of the parallelopiped formed by $d\mathbf{u}$, $d\mathbf{v}$, $d\mathbf{w}$.

$$\therefore \qquad dV = [h_1 du \mathbf{T}_u) \cdot (h_2 dv \mathbf{T}_v) \times (h_3 dw \mathbf{T}_w)]$$
$$= h_1 h_2 h_3 \, du dv dw \qquad \text{...(8)} \ [\because \ [\mathbf{T}_u \mathbf{T}_v \mathbf{T}_w] = 1$$

This can also be written as

$$dV = \frac{\partial \mathbf{R}}{\partial u} \cdot \frac{\partial \mathbf{R}}{\partial v} \times \frac{\partial \mathbf{R}}{\partial w} \ du dv dw = \frac{\partial(x, y, z)}{\partial(u, v, w)} du dv dw \qquad \text{...(9)}$$

where $\partial(x, y, z)/\partial(u, v, w)$ is called the *Jacobian of the transformation* from (x, y, z) to (u, v, w) coordinates.

(2) Del applied to Functions in Orthogonal Curvilinear coordinates

To prove that

(1) $\nabla f = \frac{\mathbf{T}_u}{h_1} \frac{\partial f}{\partial u} + \frac{\mathbf{T}_v}{h_2} \frac{\partial f}{\partial v} + \frac{\mathbf{T}_w}{h_3} \frac{\partial f}{\partial w}$

(2) $\nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u}(h_2 h_3 f_1) + \frac{\partial}{\partial v}(h_3 h_1 f_2) + \frac{\partial}{\partial w}(h_1 h_2 f_3) \right]$

(3) $\nabla \times \mathbf{F} = \begin{vmatrix} \frac{\mathbf{T}_u}{h_2 h_3} & \frac{\mathbf{T}_v}{h_3 h_1} & \frac{\mathbf{T}_w}{h_1 h_2} \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_1 f_1 & h_2 f_2 & h_3 f_3 \end{vmatrix}$ where $\mathbf{F} = f_1 \mathbf{T}_u + f_2 \mathbf{T}_v + f_3 \mathbf{T}_w$.

(1) Let $f(u, v, w)$ be any scalar point function in terms of u, v, w, the orthogonal curvilinear coordinates. Taking u, v, w as functions of x, y, z, we have

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} \qquad \text{...}(i)$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y} \qquad \text{...}(ii)$$

and

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial z} \qquad \text{...}(iii)$$

Multiplying (*i*) by **I**, (*ii*) by **J**, (*iii*) by **K** and adding, we have

$$\nabla f = \frac{\partial f}{\partial u}\nabla u + \frac{\partial f}{\partial v}\nabla v + \frac{\partial f}{\partial w}\nabla w \qquad ...(iv)$$

$$= \frac{\mathbf{T}_u}{h_1}\frac{\partial f}{\partial u} + \frac{\mathbf{T}_v}{h_2}\frac{\partial f}{\partial v} + \frac{\mathbf{T}_w}{h_3}\frac{\partial f}{\partial w} \qquad \text{[By (5) p. 356]}$$

which is the required result.

(2) Let $\mathbf{F}(u, v, w)$ be a vector point function such that

$$\mathbf{F} = f_1\mathbf{T}_u + f_2\mathbf{T}_v + f_3\mathbf{T}_w = \Sigma f_1 h_2 h_3\, \nabla v \times \nabla w \qquad \text{[By (6) p. 356]}$$

$$\therefore \quad \nabla \cdot \mathbf{F} = \Sigma \nabla \cdot \{(f_1h_2h_3)(\nabla v \times \nabla w)\}$$

$$= \Sigma[(f_1h_2h_3)\,\nabla \cdot (\nabla v \times \nabla w) + (\nabla v \times \nabla w)\,\nabla(f_1h_2h_3)] \qquad ...(v)$$

Now $\nabla \cdot (\nabla v \times \nabla w) = \nabla w \cdot \nabla \times (\nabla v) - \nabla v \cdot \nabla \times (\nabla w) = 0$ [By (5) p. 330]

and
$$\nabla(f_1h_2h_3) = \frac{\partial(f_1h_2h_3)}{\partial u}\nabla u + \frac{\partial(f_1h_2h_3)}{\partial v}\nabla v + \frac{\partial(f_1h_2h_3)}{\partial w}\nabla w \qquad \text{[By } (iv) \text{ above]}$$

$\therefore$ (*v*) now becomes

$$\nabla \cdot \mathbf{F} = \sum(\nabla v \times \nabla w) \cdot \left\{\frac{\partial(f_1h_2h_3)}{\partial u}\nabla u + \frac{\partial(f_1h_2h_3)}{\partial v}\nabla v + \frac{\partial(f_1h_2h_3)}{\partial w}\nabla w\right\}$$

$$= [\nabla u, \nabla v, \nabla w]\sum \frac{\partial(f_1h_2h_3)}{\partial u} = \frac{1}{h_1h_2h_3}\sum \frac{\partial(f_1h_2h_3)}{\partial u}$$ which is the required result.

Cor. Laplacian. $\nabla^2 f = \nabla \cdot (\nabla f)$

$$= \nabla \cdot \left(\frac{\mathbf{T}_u}{h_1}\frac{\partial f}{\partial u} + \frac{\mathbf{T}_v}{h_2}\frac{\partial f}{\partial v} + \frac{\mathbf{T}_w}{h_3}\frac{\partial f}{\partial w}\right) = \frac{1}{h_1h_2h_3}\sum \frac{\partial}{\partial u}\left(\frac{1}{h_1}\frac{\partial f}{\partial u}h_2h_3\right)$$

(3) Let $\mathbf{F}(u, v, w)$ be a vector point function such that

$$\mathbf{F} = f_1\mathbf{T}_u + f_2\mathbf{T}_v + f_3\mathbf{T}_w = f_1h_1\nabla u + f_2h_2\nabla v + f_3h_3\,\nabla w \qquad \text{[By (5) p. 356]}$$

$$\nabla \times \mathbf{F} = \Sigma \nabla \times (f_1h_1\nabla u) \qquad \text{[Using (3) p. 329]}$$

$$= \Sigma[\nabla(f_1h_1) \times \nabla u + f_1h_1 \cdot \nabla \times \nabla u] = \sum\left[\frac{\partial(f_1h_1)}{\partial u}\nabla u + \frac{\partial(f_1h_1)}{\partial v}\nabla v + \frac{\partial(f_1h_1)}{\partial w}\nabla w\right] \times \nabla u$$

$$= \sum\left[\frac{\partial(f_1h_1)}{\partial v}\nabla v \times \nabla u + \frac{\partial(f_1h_1)}{\partial w}\nabla w \times \nabla u\right]$$

$$= \sum\left[\frac{\partial(f_1h_1)}{\partial v}\left(-\frac{\mathbf{T}_u \times \mathbf{T}_v}{h_1h_2}\right) + \frac{\partial(f_1h_1)}{\partial w}\left(\frac{\mathbf{T}_w \times \mathbf{T}_u}{h_3h_1}\right)\right]$$

$$= -\frac{\partial(f_1h_1)}{\partial v}\frac{\mathbf{T}_w}{h_1h_2} + \frac{\partial(f_1h_1)}{\partial w}\frac{\mathbf{T}_v}{h_3h_1} - \frac{\partial(f_2h_2)}{\partial w}\frac{\mathbf{T}_u}{h_2h_3} + \frac{\partial(f_2h_2)}{\partial u}\frac{\mathbf{T}_w}{h_1h_2} - \frac{\partial(f_3h_3)}{\partial u}\frac{\mathbf{T}_v}{h_3h_1} + \frac{\partial(f_3h_3)}{\partial v}\frac{\mathbf{T}_u}{h_2h_3}$$

$$= \frac{\mathbf{T}_u}{h_2h_3}\left[\frac{\partial(f_3h_3)}{\partial v} - \frac{\partial(f_2h_2)}{\partial w}\right]$$ + two similar terms, whence follows the required result.

TWO SPECIAL CURVILINEAR SYSTEMS

8.20 (1) CYLINDRICAL COORDINATES

Any point $P(x, y, z)$ whose projection on the xy-plane is $Q(x, y)$ has the *cylindrical coordinates* (ρ, ϕ, z), where $\rho = OQ$, $\phi = \angle XOQ$ and $z = QP$.

The level surfaces $\rho = \rho_0$, $\phi = \phi_0$, $z = z_0$ are respectively cylinders about the Z-axis; planes through the Z-axis and planes perpendicular to the Z-axis.

The coordinate curves for ρ are rays perpendicular to the Z-axis; for ϕ, horizontal circles with centres on the Z-axis; for z, lines parallel to the Z-axis.

From Fig. 8.28, we have

$$x = \rho\cos\phi,\ y = \rho\sin\phi,\ z = z$$

(i) *Arc element.*

$\therefore \quad (ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 = (d\rho)^2 + \rho^2 (d\phi)^2 + (dz)^2$

so that the scale factors are $h_1 = 1$, $h_2 = \rho$, $h_3 = 1$.

(ii) *Area elements* $dS_\rho = \rho d\phi dz$, $dS_\phi = dz d\rho$, $dS_z = \rho d\rho d\phi$ where dS_ρ is the area element $\perp$ to ρ-direction, etc.

(iii) *Volume element* $dV = \rho d\rho d\phi dz$.

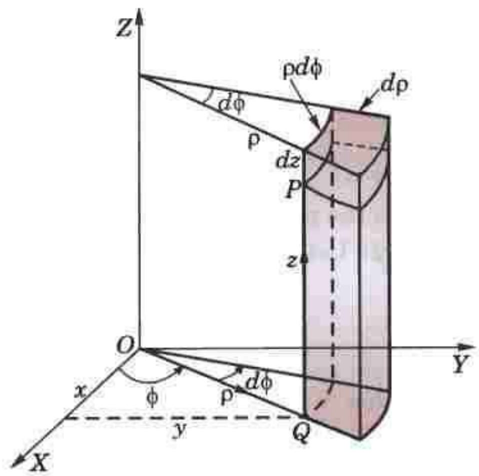

Fig. 8.28

(2) Cylindrical co-ordinate system is orthogonal

At any point P, we have $x = \rho \cos \phi$, $y = \rho \sin \phi$, $z = z$,

so that $\quad \mathbf{R} = \rho \cos \phi \mathbf{I} + \rho \sin \phi \mathbf{J} + z\mathbf{K}$

If $\mathbf{T}_\rho$, $\mathbf{T}_\phi$, $\mathbf{T}_z$ be the unit vectors at P in the directions of the tangents to the ρ, ϕ, z-curves respectively, then

$$\mathbf{T}_\rho = \frac{\partial \mathbf{R}/\partial \rho}{|\partial \mathbf{R}/\partial \rho|} = \frac{\cos \phi \mathbf{I} + \sin \phi \mathbf{J}}{\sqrt{(\cos^2 \phi + \sin^2 \phi)}} = \cos \phi \mathbf{I} + \sin \phi \mathbf{J}$$

$$\mathbf{T}_\phi = \frac{\partial \mathbf{R}/\partial \phi}{|\partial \mathbf{R}/\partial \phi|} = \frac{-\rho \sin \phi \mathbf{I} + \rho \cos \phi \mathbf{J}}{\sqrt{[(-\rho \sin \phi)^2 + (\rho \cos \phi)^2]}} = -\sin \phi \mathbf{I} + \cos \phi \mathbf{J}$$

and $$\mathbf{T}_z = \frac{\partial \mathbf{R}/\partial z}{|\partial \mathbf{R}/\partial z|} = \mathbf{K}$$

Now $\mathbf{T}_\rho \cdot \mathbf{T}_\phi = (\cos \phi \mathbf{I} + \sin \phi \mathbf{J}) \cdot (-\sin \phi \mathbf{I} + \cos \phi \mathbf{J}) = -\cos \phi \sin \phi + \sin \phi \cos \phi = 0$,

$\mathbf{T}_\phi \cdot \mathbf{T}_z = (-\sin \phi \mathbf{I} + \cos \phi \mathbf{J}) \cdot \mathbf{K} = 0$, and $\mathbf{T}_z \cdot \mathbf{T}_\rho = \mathbf{K} \cdot (\cos \phi \mathbf{I} + \sin \phi \mathbf{J}) = 0$.

Hence the cylindrical coordinate system is orthogonal.

Also $\quad \mathbf{T}_\rho \times \mathbf{T}_\phi = (\cos \phi \mathbf{I} + \sin \phi \mathbf{J}) \times (-\sin \phi \mathbf{I} + \cos \phi \mathbf{J}) = (\cos^2 \phi + \sin^2 \phi)\, \mathbf{I} \times \mathbf{J} = \mathbf{K} = T_z$

$\mathbf{T}_\phi \times \mathbf{T}_z = (-\sin \phi \mathbf{I} + \cos \phi \mathbf{J}) \times \mathbf{K} = \sin \phi \mathbf{J} + \cos \phi \mathbf{I} = \mathbf{T}_\rho$

$\mathbf{T}_z \times \mathbf{T}_\rho = \mathbf{K} \times (\cos \phi \mathbf{I} + \sin \phi \mathbf{J}) = \cos \phi \mathbf{J} - \sin \phi \mathbf{I} = \mathbf{T}_\phi$

These conditions satisfied by T_ρ, T_ϕ, and T_z, show that the *cylindrical coordinates system is a right handed orthogonal coordinate system.* (*V.T.U., 2008*)

(3) Del applied to functions in Cylindrical coordinates

We have $u = \rho$, $v = \phi$, $w = z$ and $h_1 = 1$, $h_2 = \rho$, $h_3 = 1$.

Let T_ρ, T_ϕ, T_z be the unit vectors in the directions of the tangents to the ρ, ϕ, z curves.

(i) *Expression for grad f.*

Since $$\nabla f = \frac{\mathbf{T}_u}{h_1}\frac{\partial f}{\partial u} + \frac{\mathbf{T}_v}{h_2}\frac{\partial f}{\partial v} + \frac{\mathbf{T}w}{h_3}\frac{\partial f}{\partial w}$$

$$\therefore \quad \nabla f = \frac{\partial f}{\partial \rho}\mathbf{T}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\mathbf{T}_\phi + \frac{\partial f}{\partial z}\mathbf{T}_z$$

(ii) *Expression for div* $\mathbf{F}$ *where* $\mathbf{F} = f_1\mathbf{T}_u + f_2\mathbf{T}_v + f_3\mathbf{T}_w$

Since $$\nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u}(h_2 h_3 f_1) + \frac{\partial f}{\partial v}(h_3 h_1 f_2) + \frac{\partial}{\partial w}(h_1 h_2 f_3)\right]$$

$$\therefore \quad \nabla \cdot \mathbf{F} = \frac{1}{\rho}\left\{\frac{\partial}{\partial \rho}(\rho f_1) + \frac{\partial f_2}{\partial \phi} + \frac{\partial}{\partial z}(\rho f_3)\right\}$$

(iii) *Expression for curl* $\mathbf{F}$ *where* $\mathbf{F} = f_1\mathbf{T}_u + f_2\mathbf{T}_v + f_3\mathbf{T}_w$

Since $$\nabla \times \mathbf{F} = \begin{vmatrix} \dfrac{\mathbf{T}_u}{h_2 h_3} & \dfrac{\mathbf{T}_v}{h_3 h_1} & \dfrac{\mathbf{T}_w}{h_1 h_2} \\ \dfrac{\partial}{\partial u} & \dfrac{\partial}{\partial v} & \dfrac{\partial}{\partial w} \\ h_1 f_1 & h_2 f_2 & h_3 f_3 \end{vmatrix} = \begin{vmatrix} \mathbf{T}_\rho/\rho & \mathbf{T}_\phi & \mathbf{T}_z/\rho \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ f_1 & \rho f_2 & f_3 \end{vmatrix}$$

$$= \mathbf{T}_\rho\left(\frac{1}{\rho}\frac{\partial f_3}{\partial \phi} - \frac{\partial f_2}{\partial z}\right) + \mathbf{T}_\phi\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial \rho}\right) + \mathbf{T}_z\left(\frac{\partial f_2}{\partial \rho} - \frac{1}{\rho}\frac{\partial f_1}{\partial \phi}\right)$$

(iv) Expression for $\nabla^2 f$

Since $$\nabla^2 f = \frac{1}{h_1 h_2 h_3}\left\{\frac{\partial}{\partial u}\left(\frac{1}{h_1}\frac{\partial f}{\partial u}h_2 h_3\right) + \frac{\partial}{\partial v}\left(\frac{1}{h_2}\frac{\partial f}{\partial v}h_3 h_1\right) + \frac{\partial}{\partial w}\left(\frac{1}{h_3}\frac{\partial f}{\partial w}h_1 h_2\right)\right\}$$

$\therefore$ $$\nabla^2 f = \frac{1}{\rho}\left\{\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\rho}\frac{\partial f}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(\rho\frac{\partial f}{\partial z}\right)\right\} = \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}.$$

Example 8.52. *Express the vector $z\mathbf{I} - 2x\mathbf{J} + y\mathbf{K}$ in cylindrical coordinates.* (V.T.U., 2010)

Solution. We have $x = \rho\cos\phi$, $y = \rho\sin\phi$ and $z = z$.

so that $$\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K} = \rho\cos\phi\mathbf{I} + \rho\sin\phi\mathbf{J} + z\mathbf{K}$$

If $\mathbf{T}_\rho, \mathbf{T}_\phi, \mathbf{T}_z$ be the unit vectors along the tangents to ρ, ϕ and z curves respectively, then

$$\mathbf{T}_\rho = \frac{\partial \mathbf{R}/\partial \rho}{|\partial \mathbf{R}/\partial \rho|} = \frac{\cos\phi\mathbf{I} + \sin\phi\mathbf{J}}{\sqrt{(\cos^2\phi + \sin^2\phi)}} = \cos\phi\mathbf{I} + \sin\phi\mathbf{J}$$

$$\mathbf{T}_\phi = \frac{\partial \mathbf{R}/\partial \phi}{|\partial \mathbf{R}/\partial \phi|} = \frac{-\rho\sin\phi\mathbf{I} + \rho\cos\phi\mathbf{J}}{\sqrt{[(-\rho\sin\phi)^2 + (\rho\cos\phi)^2]}} = -\sin\phi\mathbf{I} + \cos\phi\mathbf{J}$$

$$\mathbf{T}_z = \frac{\partial \mathbf{R}/\partial z}{|\partial \mathbf{R}/\partial z|} = \mathbf{K}$$

Let the expression for $\mathbf{F} = z\mathbf{I} - 2x\mathbf{J} + y\mathbf{K}$ in cylindrical coordinates be

$$\mathbf{F} = f_1\mathbf{T}_\rho + f_2\mathbf{T}_\phi + f_3\mathbf{T}_z \qquad ...(i)$$

Then $$f_1 = \mathbf{F}\cdot\mathbf{T}_\rho = z\cos\phi - 2x\sin\phi$$
$$f_2 = \mathbf{F}\cdot\mathbf{T}_\phi = -z\sin\phi - 2x\cos\phi$$
$$f_3 = \mathbf{F}\cdot\mathbf{T}_z = y$$

Substituting the values of f_1, f_2, f_3 in (*i*), we get

$$\mathbf{F} = (z\cos\phi - 2x\sin\phi)\,\mathbf{T}_\rho - (z\sin\phi + 2x\cos\phi)\,\mathbf{T}_\phi + y\mathbf{T}_z$$
$$= (z\cos\phi - \rho\sin 2\phi)\,\mathbf{T}_\rho - (z\sin\phi + 2\rho\cos^2\phi)\,\mathbf{T}_\phi + \rho\sin\phi\,\mathbf{T}_z$$

Example 8.53. *Show that $\nabla(\log\rho)$ and $\nabla\phi$, $\rho \neq 0$, $\phi \neq 0$ are solenoidal vectors.*

Solution. (*i*) $f = \log\rho$ is a function of ρ only. We have to prove that $\nabla\cdot(\nabla f)$, *i.e.*, $\nabla^2 f = 0$

$$\nabla^2 f = \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} = \frac{\partial^2}{\partial \rho^2}(\log\rho) + \frac{1}{\rho}\frac{\partial(\log\rho)}{\partial \rho} + 0 + 0 = -\frac{1}{\rho^2} + \frac{1}{\rho^2} = 0$$

Hence $\nabla(\log\rho)$ is a solenoidal vector.

(*ii*) $f = \nabla\phi$ is a function of ϕ only. We have to show that $\nabla\cdot(\nabla f)$, *i.e.*, $\nabla^2 f = 0$.

$$\nabla^2 f = \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} = 0 + 0 + \frac{1}{\rho^2}\frac{\partial^2 \phi}{\partial \phi^2} + 0 = 0.$$

Hence the result.

8.21 (1) SPHERICAL POLAR COORDINATES

Let $P(x, y, z)$ be any point whose projection on the XY-plane is $Q(x, y)$. Then the spherical polar co-ordinates of P are (r, θ, ϕ) such that $r = OP$, $\theta = \angle ZOP$ and $\phi = \angle XOQ$.

The level surfaces $r = r_0$, $\theta = \theta_0$, $\phi = \phi_0$ are respectively spheres about O, cones about the Z-axis with vertex at O and planes through the Z-axis.

The co-ordinate curves for r are rays from the origin; for θ, vertical circles with centre at O (called *meridians*); for ϕ, horizontal circles with centres on the Z-axis

From Fig. 8.29, we have

$$x = OQ\cos\phi = OP\cos(90° - \theta)\cos\phi = r\sin\theta\cos\phi,$$
$$y = OQ\sin\phi = r\sin\theta\sin\phi;\; z = r\cos\theta.$$

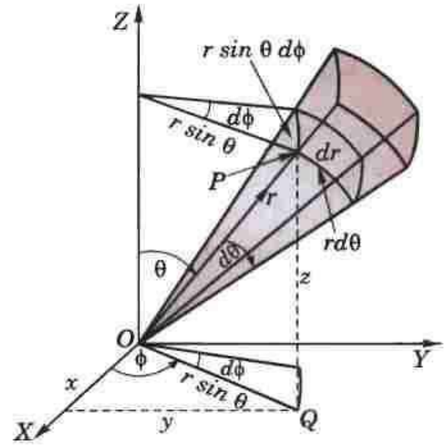

Fig. 8.29

(*i*) *Arc element*

$\therefore \quad (ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 = (dr)^2 + r^2 (d\theta)^2 + (r \sin\theta)^2 (d\phi)^2$

so that the scale factors are

$$h_1 = 1, h_2 = r, h_3 = r \sin\theta.$$

(*ii*) *Area elements*

$$dS_r = r^2 \sin\theta\, d\theta d\phi, dS_\theta = r \sin\theta d\phi dr, dS_\phi = r dr d\theta$$

where dS_r is the area element perpendicular to the r-direction, etc.

(*iii*) *Volume element* $dV = r^2 \sin\theta\, dr d\theta d\phi$.

(2) Spherical polar coordinate system is orthogonal

At any point P, we have $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$,

so that $\quad \mathbf{R} = r\sin\theta\cos\phi\mathbf{I} + r\sin\theta\sin\phi\mathbf{J} + r\cos\theta\mathbf{K}$

If $\mathbf{T}_r, \mathbf{T}_\theta, \mathbf{T}_\phi$ be the unit vectors at P in the directions of the tangents to the r, θ, ϕ-curves respectively, then

$$\mathbf{T}_r = \frac{\partial\mathbf{R}/\partial r}{|\partial\mathbf{R}/\partial r|} = \frac{\sin\theta\cos\phi\mathbf{I} + \sin\theta\sin\phi\mathbf{J} + \cos\theta\mathbf{K}}{\sqrt{(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta)}}$$

$$= \sin\theta\cos\phi\mathbf{I} + \sin\theta\sin\phi\mathbf{J} + \cos\theta\mathbf{K}$$

$$\mathbf{T}_\theta = \frac{\partial\mathbf{R}/\partial\theta}{|\partial\mathbf{R}/\partial\theta|} = \frac{r\cos\theta\cos\phi\mathbf{I} + r\cos\theta\sin\phi\mathbf{J} - r\sin\theta\mathbf{K}}{r\sqrt{(\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta)}}$$

$$= \cos\theta\cos\phi\mathbf{I} + \cos\theta\sin\phi\mathbf{J} - \sin\theta\mathbf{K}$$

and $$\mathbf{T}_\phi = \frac{\partial\mathbf{R}/\partial\phi}{|\partial\mathbf{R}/\partial\phi|} = \frac{-r\sin\theta\sin\phi\mathbf{I} + r\sin\theta\cos\phi\mathbf{J}}{r\sin\theta} = -\sin\phi\mathbf{I} + \cos\phi\mathbf{J}$$

Now $$\mathbf{T}_r \cdot \mathbf{T}_\theta = \sin\theta\cos\theta\cos^2\phi + \sin\theta\cos\theta\sin^2\phi - \sin\theta\cos\theta = 0$$

$$\mathbf{T}_\theta \cdot \mathbf{T}_\phi = -\cos\theta\cos\phi\sin\phi + \cos\theta\sin\phi\cos\phi = 0$$

$$\mathbf{T}_\phi \cdot \mathbf{T}_r = -\sin\theta\cos\phi\sin\phi + \sin\theta\sin\phi\cos\phi = 0$$

Also $$\mathbf{T}_r \times \mathbf{T}_\theta = \sin\theta\cos\phi\cos\theta\sin\phi\mathbf{K} + \sin^2\theta\cos\phi\mathbf{J} - \sin\theta\sin\phi\cos\theta\cos\phi\mathbf{K}$$

$$- \sin^2\theta\sin\phi\mathbf{I} + \cos^2\theta\cos\phi\mathbf{J} - \cos^2\theta\sin\phi\mathbf{I}$$

$$= -\sin\phi\mathbf{I} + \cos\phi\mathbf{J} = \mathbf{T}_\phi$$

$$\mathbf{T}_\theta \times \mathbf{T}_\phi = \cos\theta\cos^2\phi\mathbf{K} + \sin^2\phi\cos\theta\mathbf{K} + \sin\theta\sin\phi\mathbf{J} + \sin\theta\cos\phi\mathbf{I} = \mathbf{T}_r$$

and $$\mathbf{T}_\phi \times \mathbf{T}_r = -\sin\theta\sin^2\phi\mathbf{K} + \sin\phi\cos\theta\mathbf{J} - \sin\theta\cos^2\phi\mathbf{K} + \cos\phi\cos\theta\mathbf{I} = \mathbf{T}_\theta$$

The above conditions satisfied by $\mathbf{T}_r, \mathbf{T}_\theta$, and $\mathbf{T}_\phi$ show that *the spherical polar coordinate system is a right handed orthogonal coordinate system.* (*V.T.U., 2008*)

(3) Del applied to functions in spherical polar coordinates

We have $u = r, v = \theta, w = \phi$ and $h_1 = 1, h_2 = r, h_3 = r\sin\theta$.

Let $\mathbf{T}_r, \mathbf{T}_\theta, \mathbf{T}_\phi$ be the unit vectors in the directions of the tangents to the r, θ, ϕ-curves.

(*i*) *Expression for grad f*

Since $$\nabla f = \frac{\mathbf{T}_u}{h_1}\frac{\partial f}{\partial u} + \frac{\mathbf{T}_v}{h_2}\frac{\partial f}{\partial v} + \frac{\mathbf{T}_w}{h_3}\frac{\partial f}{\partial w}$$

$\therefore$ $$\nabla f = \frac{\partial f}{\partial r}\mathbf{T}_r + \frac{1}{r}\frac{\partial f}{\partial\theta}\mathbf{T}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\mathbf{T}_\phi$$

(*ii*) *Expression for div* $\mathbf{F}$ *where* $\mathbf{F} = f_1\mathbf{T}_u + f_2\mathbf{T}_v + f_3\mathbf{T}_w$

Since $$\nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u}(h_2 h_3 f_1) + \frac{\partial}{\partial v}(h_3 h_1 f_2) + \frac{\partial}{\partial w}(h_1 h_2 f_3)\right]$$

$\therefore$ $$\nabla \cdot \mathbf{F} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta\, f_1) + \frac{\partial}{\partial\theta}(r\sin\theta\, f_2) + \frac{\partial}{\partial\phi}(rf_3)\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(f_1 r^2) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(f_2\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial f_3}{\partial\phi}$$

(iii) Expression for curl $\mathbf{F}$ *where* $\mathbf{F} = f_1\mathbf{T}_u + f_2\mathbf{T}_v + f_3\mathbf{T}_w$

Since
$$\nabla \times \mathbf{F} = \begin{vmatrix} \dfrac{\mathbf{T}_u}{h_2h_3} & \dfrac{\mathbf{T}_v}{h_3h_1} & \dfrac{\mathbf{T}_w}{h_1h_2} \\ \dfrac{\partial}{\partial u} & \dfrac{\partial}{\partial v} & \dfrac{\partial}{\partial w} \\ h_1f_1 & h_2f_2 & h_3f_3 \end{vmatrix}$$

$\therefore$
$$\nabla \times \mathbf{F} = \begin{vmatrix} \dfrac{\mathbf{T}_r}{r^2 \sin\theta} & \dfrac{\mathbf{T}_\theta}{r\sin\theta} & \dfrac{\mathbf{T}\phi}{r} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ f_1 & rf_2 & r\sin\theta f_3 \end{vmatrix}$$

$$= \frac{\mathbf{T}_r}{r^2\sin\theta}\left\{\frac{\partial}{\partial\theta}(r\sin\theta f_3) - \frac{\partial}{\partial\phi}(rf_2)\right\} - \frac{\mathbf{T}_\theta}{r\sin\theta}\left\{\frac{\partial}{\partial r}(r\sin\theta f_3) - \frac{\partial f_1}{\partial\phi}\right\} + \frac{\mathbf{T}_\phi}{r}\left\{\frac{\partial}{\partial r}(rf_2) - \frac{\partial f_1}{\partial\theta}\right\}$$

$$= \frac{\mathbf{T}_r}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}(f_3\sin\theta) - \frac{\partial f_2}{\partial\phi}\right\} + \frac{\mathbf{T}_\theta}{r}\left\{\frac{1}{\sin\theta}\frac{\partial f_1}{\partial\phi} - \frac{\partial}{\partial r}(rf_3)\right\} + \frac{\mathbf{T}_\phi}{r}\left\{\frac{\partial}{\partial r}(rf_2) - \frac{\partial f_1}{\partial\theta}\right\}$$

(iv) Expression for $\nabla^2 f$.

Since
$$\nabla^2 f = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial u}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_3h_1}{h_2}\frac{\partial f}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial w}\right)\right\}$$

$\therefore$
$$\nabla^2 f = \frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{r\sin\theta}{r}\frac{\partial f}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{r}{r\sin\theta}\frac{\partial f}{\partial\phi}\right)\right\}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

$$= \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial f}{\partial\theta}.$$

Example 8.54. *Express the vector field* $2y\mathbf{I} - z\mathbf{J} + 3x\mathbf{K}$ *in spherical polar coordinate system.*

Solution. We have $x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$

so that $\mathbf{R} = r\sin\theta\cos\phi\mathbf{I} + r\sin\theta\sin\phi\mathbf{J} + r\cos\theta\mathbf{K}$.

If $\mathbf{T}_r$, $\mathbf{T}_\theta$, $\mathbf{T}_\phi$ be the unit vectors along the tangents to r, θ, ϕ, curves respectively, then

$$\mathbf{T}_r = \frac{\partial\mathbf{R}/\partial r}{|\partial\mathbf{R}/\partial r|} = \frac{\sin\theta\cos\phi\mathbf{I} + \sin\theta\sin\phi\mathbf{J} + \cos\theta\mathbf{K}}{\sqrt{[(\sin\theta\cos\phi)^2 + (\sin\theta\sin\phi)^2 + \cos^2\theta]}}$$

$$= \sin\theta\cos\phi\mathbf{I} + \sin\theta\sin\phi\mathbf{J} + \cos\theta\mathbf{K}$$

$$\mathbf{T}_\theta = \frac{\partial\mathbf{R}/\partial\theta}{|\partial\mathbf{R}/\partial\theta|} = \frac{r\cos\theta\cos\phi\mathbf{I} + r\cos\theta\sin\phi\mathbf{J} - r\sin\theta\mathbf{K}}{\sqrt{[(r\cos\theta\cos\phi)^2 + (r\cos\theta\sin\phi)^2 + (-r\sin\theta)^2]}}$$

$$= \cos\theta\cos\phi\mathbf{I} + \cos\theta\sin\phi\mathbf{J} - \sin\theta\mathbf{K}$$

$$\mathbf{T}_\phi = \frac{\partial\mathbf{R}/\partial\phi}{|\partial\mathbf{R}/\partial\phi|} = \frac{-r\sin\theta\sin\phi\mathbf{I} + r\sin\theta\cos\phi\mathbf{J}}{\sqrt{[(-r\sin\theta\sin\phi)^2 + (r\sin\theta\cos\phi)^2]}} = -\sin\phi\mathbf{I} + \cos\phi\mathbf{J}.$$

Let the expression for $\mathbf{F} = 2y\mathbf{I} - z\mathbf{J} + 3x\mathbf{K}$ in spherical polar coordinates be

$$\mathbf{F} = f_1\mathbf{T}_r + f_2\mathbf{T}_\theta + f_3\mathbf{T}_\phi \qquad ...(i)$$

Then $f_1 = \mathbf{F}\cdot\mathbf{T}_r = (2r\sin\theta\sin\phi\mathbf{I} - r\cos\theta\mathbf{J} + 3r\sin\theta\cos\phi\mathbf{K})\cdot(\sin\theta\cos\phi\mathbf{I} + \sin\theta\sin\phi\mathbf{J} + \cos\theta\mathbf{K})$

$= 2r\sin^2\theta\sin\phi\cos\phi - r\sin\theta\cos\theta\sin\phi + 3r\sin\theta\cos\theta\cos\phi$

$$f_2 = \mathbf{F}\,.\,\mathbf{T}_\theta = (2r\sin\theta\sin\phi\mathbf{I} - r\cos\theta\mathbf{J} + 3r\sin\theta\cos\phi\mathbf{K})\,.\,(\cos\theta\cos\phi\mathbf{I} + \cos\theta\sin\phi\mathbf{J} - \sin\theta\mathbf{K})$$
$$= 2r\sin\theta\cos\theta\sin\phi\cos\phi - r\cos^2\theta\sin\phi - 3r\sin^2\theta\cos\phi.$$

and
$$f_3 = \mathbf{F}\,.\,\mathbf{T}_\phi = (2r\sin\theta\sin\phi\mathbf{K} - r\cos\theta\mathbf{J} + 3r\sin\theta\cos\phi\mathbf{K}).\,(-\sin\phi\mathbf{I} + \cos\phi\mathbf{J})$$
$$= -2r\sin\theta\sin^2\phi - r\cos\theta\cos\phi$$

Substituting the values of f_1, f_2, f_3 in (*i*), we get the desired expression.

Example 8.55. *Prove that $\nabla(\cos\theta) \times \nabla\phi = \nabla(1/r)$, $r \neq 0$.*

Solution. In spherical polar coordinates,

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{T}_r + \frac{1}{r}\frac{\partial f}{\partial\theta}\mathbf{T}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\mathbf{T}_\phi$$

$$\therefore \quad \nabla(\cos\theta) = \frac{1}{r}\frac{\partial}{\partial\theta}(\cos\theta)\mathbf{T}_\theta = -\frac{1}{r}\sin\theta\,\mathbf{T}_\theta \quad \text{...(i)}$$

$$\nabla\phi = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\phi)\mathbf{T}_\phi = \frac{1}{r\sin\theta}\mathbf{T}_\phi \quad \text{...(ii)}$$

and
$$\nabla\left(\frac{1}{r}\right) = \frac{\partial}{\partial r}(r^{-1})\mathbf{T}_r = -\frac{1}{r^2}\mathbf{T}_r$$

Now from (*i*) and (*ii*), we get

$$\nabla(\cos\theta) \times \nabla\phi = -\frac{1}{r^2}\mathbf{T}_\theta \times \mathbf{T}_\phi = -\frac{1}{r^2}\mathbf{T}_r = \nabla\left(\frac{1}{r}\right).$$

Example 8.56. *If $F = r^2\cos\theta\,\mathbf{T}_r - \frac{1}{r}\mathbf{T}_\theta + \frac{1}{r\sin\theta}\mathbf{T}_\phi$, find the value of $\mathbf{F} \times$ curl $\mathbf{F}$.*

Solution. In spherical coordinates,

$$\text{curl }\mathbf{F} = \begin{vmatrix} \mathbf{T}_r/r^2\sin\theta & \mathbf{T}_\theta/r\sin\theta & \mathbf{T}_\phi/r \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ f_1 & rf_2 & r\sin\theta f_3 \end{vmatrix}$$

Here $f_1 = r^2\cos\theta$, $f_2 = -1/r$, $f_3 = 1/r\sin\theta$.

$$\therefore \quad \text{curl }\mathbf{F} = \frac{2}{r^2\sin\theta}\begin{vmatrix} \mathbf{T}_r & r\mathbf{T}_\theta & r\sin\theta\,\mathbf{T}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ r^2\cos\theta & -1 & 1 \end{vmatrix} = r\sin\theta\mathbf{T}_\phi$$

$$\therefore \quad \mathbf{F} \times \text{curl }\mathbf{F} = \left(r^2\cos\theta\mathbf{T}_r - \frac{1}{r}\mathbf{T}_\theta + \frac{1}{r\sin\theta}\mathbf{T}_\phi\right) \times (r\sin\theta\mathbf{T}_\phi) = -(r^3\sin\theta\cos\theta\mathbf{T}_\theta + \sin\theta\mathbf{T}_r).$$

PROBLEMS 8.12

1. Express the following vectors in cylindrical coordinates
   (*i*) $2y\mathbf{I} - z\mathbf{J} + 3x\mathbf{K}$ (*ii*) $2x\mathbf{I} - 3y^2\mathbf{J} + zx\mathbf{K}$ (*V.T.U., 2009*)
2. Express the following vectors in spherical polar coordinates
   (*i*) $x\mathbf{I} + 2y\mathbf{J} + yz\mathbf{K}$ (*ii*) $xy\mathbf{I} + yz\mathbf{J} + zx\mathbf{K}$
3. Evaluate $\nabla\phi = xyz$ in cylindrical coordinates.
4. Show that $\nabla(r/\sin\theta) \times \nabla\theta = \nabla\phi$.
5. Prove that $\mathbf{V} = \dfrac{\cos\theta}{r^3}(\mathbf{T}_r/\sin\theta - \mathbf{T}_\theta/\cos\theta + r^4\mathbf{T}_\phi)$ is solenoidal.
6. Show that (*i*) $\nabla^2(\log r) = 1/r^2$ (*ii*) $\nabla \times [(\cos\theta)(\nabla\phi)] = \nabla(1/r)$.

7. Prove that $\mathbf{V} = \rho z \sin 2\phi \left[\mathbf{T}_\rho + \cot 2\phi \mathbf{T}_\phi + \frac{\rho}{2z}\mathbf{T}_z\right]$ is irrotational.
8. If u, v, w are orthogonal curvilinear coordinates with h_1, h_2, h_3 as scale factors, prove that
$$\left[\frac{\partial \mathbf{R}}{\partial u}, \frac{\partial \mathbf{R}}{\partial v}, \frac{\partial \mathbf{R}}{\partial w}\right] = \frac{1}{[\nabla u, \nabla v, \nabla w]} = h_1 h_2 h_3.$$

8.22 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 8.13

Fill up the blanks or choose the correct answer from the following problems :

1. A unit tangent vector to the surface $x = t, y = t^2, z = t^3$ at $t = 1$ is ....
2. The equation of the normal to the surface $2x^2 + y^2 + 2z = 3$ at $(2, 1, -3)$ is ....
3. If $u = u(x, y)$ and $v = v(x, y)$, then the area-element $dudv$ is related to the area-element $dxdy$ by the relation ....
4. If $A = 2x^2\mathbf{I} - 3yz\mathbf{J} + xz^2\mathbf{K}$, then $\nabla . A$ = ...
5. div curl $\mathbf{F}$ = ...
6. Area bounded by a simple closed curve C is ....
7. If S is a closed surface enclosing a volume V and if $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$, then
$$\int_S \mathbf{R} . \mathbf{N}\, ds = ...$$
8. div $\mathbf{R}$ = ...; curl $\mathbf{R}$ = ...
9. If $\mathbf{A}$ is such that $\nabla \times \mathbf{A} = 0$, then $\mathbf{A}$ is called ....
10. If $\nabla . \mathbf{F} = 3$, then $\int_S \mathbf{F} . \mathbf{N} ds$ where S is a surface of a unit sphere, is ....
11. If $\nabla . \mathbf{F} = 0$, then $\mathbf{F}$ is called.......
12. The directional derivative of $\phi(x, y, z) = x^2yz + 4xz^2$ at the point $(1, -2, -1)$ in the direction PQ where $P = (1, 2, -1)$ and $Q = (-1, 2, 3)$ is ....
13. If $u = x^2yz$, $v = xy - 3z^2$, then $\nabla . (\nabla u \times \nabla v)$ =.......
14. curl $(xy\mathbf{I} + yz\mathbf{J} + zx\mathbf{K})$ = ...
15. If $\mathbf{F} = f_1\mathbf{I} + f_2\mathbf{J} + f_3\mathbf{K}$, then $\nabla . \mathbf{F}$ = ...
16. If $\mathbf{F}$ is a conservative force field then curl $\mathbf{F}$ is ....
17. If $\phi = 3x^2y - y^3z^2$, grad ϕ at the point $(1, -2, -1)$ is ....
18. curl $(x\mathbf{I} + y\mathbf{J} + z\mathbf{K})$ = ...
19. Workdone by a particle along the square formed by the lines $y = \pm 1$ and $x = \pm 1$ under the force
$$\mathbf{F} = (x^2 + xy)\mathbf{I} + (x^2 + y^2)\mathbf{J} \text{ is.......}$$
20. Curl (grad ϕ) = ...
21. If $\mathbf{A}$ is a constant vector, then *div* $(\mathbf{A} \times \mathbf{R})$ = ...
22. If $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$, then $\nabla \log r$ = ... ; $\nabla (r^n)$ = ...
23. A level surface is defined as ....
24. Unit normal vector to the surface $z = 2xy$ at the point $(2, 1, 4)$ is ....
25. If the directional derivative of $f = ax + by + cz$ at $(1, 1, 1)$ has maximum magnitude 4 in direction parallel to x-axis, then the values of a, b, c are ....
26. Maximum value of the directional derivative of $\phi = x^2 - 2y^2 + 4z^2$ at the point $(1, 1, -1)$ is ....
27. If $r^2 = x^2 + y^2 + z^2$, then $\nabla . (\mathbf{R}/r)$ = ...
28. Directional derivative of $f = xyz$ at the point $(1, -1, -2)$ in the direction of the vector $2\mathbf{I} - 2\mathbf{J} + \mathbf{K}$ is ....
29. If $\mathbf{V} = x^2\mathbf{I} + xye^x\mathbf{J} + \sin z\mathbf{K}$, then $\nabla . (\nabla \times F)$ = ...
30. If $f = \tan^{-1}(y/x)$ then div (grad f) is equal to
(*a*) 1 (*b*) -1 (*c*) 0 (*d*) 2.
31. The value of curl (grad f), where $f = 2x^2 - 3y^2 + 4z^2$ is
(*a*) $4x - 6y + 8z$, (*b*) $4x\mathbf{I} - 6y\mathbf{J} + 8z\mathbf{K}$ (*c*) 0 (*d*) 3.

32. The value of $\int \text{grad}\,(x+y-z)\,d\mathbf{R}$ from $(0, 1, -1)$ to $(1, 2, 0)$ is
(*a*) 0 (*b*) 3 (*c*) -1 (*d*) not obtainable.

33. If $\mathbf{F} = ax\mathbf{I} + by\mathbf{J} + cz\mathbf{K}$, then $\int_S \mathbf{F}\,.\,d\mathbf{S}$, S being the surface of a unit sphere, is
(*a*) $(4/3)\,\pi\,(a+b+c)^2$ (*b*) 0 (*c*) $4\pi/3\,(a+b+c)$ (*d*) none of these.

34. A necessary and sufficient condition that the line integral $\int_L \mathbf{F}\,.\,d\mathbf{R}$ for every closed C vanishes, is
(*a*) curl $\mathbf{F} = 0$ (*b*) div $\mathbf{F} = 0$ (*c*) curl $\mathbf{F} \neq 0$ (*d*) div $\mathbf{F} \neq 0$.

35. The value of $\iint_S (yzdydz + zxdz\,dx + xy\,dxdy)$, where S is the surface of unit sphere $x^2 + y^2 + z^2 = 1$ is
(*a*) 0 (*b*) 4π (*c*) $4\pi/3$ (*d*) 10π.

36. If $u = x^2 + y^2 + z^2$ and $\mathbf{V} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$, then $\nabla\,(u\mathbf{V}) =$ ....

37. For any scalar function ψ, $\nabla \times \nabla\,\psi =$ ....

38. $\int_C \mathbf{F}\,.\,d\mathbf{R}$ is independent of the path joining any two points if and only if it is ....

39. The value of the line integral $\int_C (y^2dx + x^2dy)$ where C is the boundary of the square $-1 \leq x \leq 1, -1 \leq y \leq 1$ is
(*a*) 0 (*b*) $2(x+y)$ (*c*) 4 (*d*) 4/3. (*V.T.U., 2010*)

40. If $\mathbf{V}$ is the instantaneous velocity vector of the moving fluid at a point P, then div $\mathbf{V}$ represents ....

41. The spherical coordinate system is
(*a*) Orthogonal (*b*) Coplanar (*c*) Non-coplanar (*d*) Not orthogonal. (*V.T.U., 2010*)

42. Physical interpretation of $\nabla\,\phi$ is that ....

43. The magnitude of the vector drawn perpendicular to the surface $x^2 + 2y^2 + z^2 = 7$ at the point $(1, -1, 2)$ is
(*a*) 2/3 (*b*) 3/2 (*c*) 3 (*d*) 6.

44. The value of λ so that the vector $(x+3y)\mathbf{I} + (y-2z)\mathbf{J} + (x+\lambda z)\mathbf{K}$ is a solenoidal vector, is
(*a*) -2 (*b*) 3 (*c*) 1 (*d*) none of these.

45. The work done by the force $\mathbf{F} = yz\mathbf{I} + zx\mathbf{J} + xy\mathbf{K}$, in moving a particle from the point $(1, 1, 1)$ to the point $(3, 3, 2)$ along the path c is
(*a*) 17 (*b*) 10 (*c*) 0 (*d*) cannot be found.

46. Value of $\int_c (y^2dx + x^2dy)$ where c is the boundary of the square $-1 \leq x \leq 1, -1 \leq y \leq 1$, is
(*a*) 4 (*b*) 0 (*c*) $2(x+y)$ (*d*) 4/3.

47. The directional derivative of $f(x, y) = (x^2 - y^2)/xy$ at $(1, 1)$ is zero along a ray making an angle with the positive direction of x-axis :
(*a*) 45° (*b*) 60° (*c*) 135° (*d*) none of these.

48. The vector $\mathbf{V} = e^x \sin y\mathbf{I} + e^x \cos y\mathbf{J}$, is
(*a*) solenoidal (*b*) irrotational (*c*) rotational.

49. If $u = 1/r$ where $r^2 = x^2 + y^2$, then $\nabla^2 u = 0$. (True or False)

50. $\mathbf{F} = (x+3y)\,\mathbf{I} + (z-3y)\mathbf{J} + (x+2z)\mathbf{K}$ is a solenoidal vector function. (True or False)

51. $\mathbf{F} = yz\mathbf{I} + zx\mathbf{J} + xy\mathbf{K}$ is irrotational. (True or False)

CHAPTER 9

Infinite Series

9.1 INTRODUCTION

Infinite series occur so frequently in all types of problems that the necessity of studying their convergence or divergence is very important. Unless a series employed in an investigation is convergent, it may lead to absurd conclusions. Hence it is essential that the students of engineering begin by acquiring an intelligent grasp of this subject.

9.2 SEQUENCES

(1) An ordered set of real numbers, $a_1, a_2, a_3, \ldots, a_n$ is called a *sequence* and is denoted by (a_n). If the number of terms is unlimited, then the sequence is said to be an *infinite sequence* and a_n is its *general term*.

For instance (*i*) $1, 3, 5, 7, \ldots, (2n-1), \ldots$, (*ii*) $1, 1/2, 1/3, \ldots, 1/n, \ldots$,

(*iii*) $1, -1, 1, -1, \ldots, (-1)^{n-1}, \ldots$ are infinite sequences.

(2) Limit. A sequence is said to tend to a limit l, if for every $\varepsilon > 0$, a value N of n can be found such that $|a_n - l| < \varepsilon$ for $n \geq N$.

We then write $\underset{n \to \infty}{\text{Lt}} (a_n) = l$ or simply $(a_n) \to l$ as $n \to \infty$.

(3) Convergence. *If a sequence (a_n) has a finite limit, it is called a* **convergent sequence.** If (a_n) is not convergent, it is said to be **divergent.**

In the above examples, (*ii*) is convergent, while (*i*) and (*iii*) are divergent.

(4) Bounded sequence. A sequence (a_n) is said to be bounded, if there exists a number k such that $a_n < k$ for every n.

(5) Monotonic sequence. The sequence (a_n) is said to increase steadily or to decrease steadily according as $a_{n+1} \geq a_n$ or $a_{n+1} \leq a_n$, for all values of n. Both increasing and decreasing sequences are called *monotonic sequences.*

A monotonic sequence always tends to a limit, finite or infinite. Thus, *a sequence which is monotonic and bounded is* **convergent.**

(6) Convergence, Divergence and Oscillation. If $\underset{n \to \infty}{\text{Lt}} (a_n) = l$ is finite and unique then the sequence is said to be *convergent*.

If $\underset{n \to \infty}{\text{Lt}} (a_n)$ is infinite ($\pm \infty$), the sequence is said to be *divergent*.

If $\underset{n \to \infty}{\text{Lt}} (a_n)$ is not unique, then (a_n) is said to be *oscillatory*.

Example 9.1. *Examine the following sequences for convergence :*

(i) $a_n = \dfrac{n^2 - 2n}{3n^2 + n}$. (ii) $a_n = 2^n$ (iii) $a_n = 3 + (-1)^n$.

Solution. (i) $\underset{n \to \infty}{\text{Lt}} \left(\dfrac{n^2 - 2n}{3n^2 + n}\right) = \underset{n \to \infty}{\text{Lt}} \dfrac{1 - 2/n}{3 + 1/n} = 1/3$ which is finite and unique. Hence the sequence (a_n) is convergent.

(ii) $\underset{n \to \infty}{\text{Lt}} (2^n) = \infty$. Hence the sequence (a_n) is divergent.

(iii) $\underset{n \to \infty}{\text{Lt}} [3 + (-1)^n] = 3 + 1 = 4$ when n is even

$= 3 - 1 = 2$, when n is odd

i.e., this sequence doesn't have a unique limit. Hence it oscillates.

PROBLEMS 9.1

Examine the convergence of the following sequences :

1. $a_n = \dfrac{3n - 1}{1 + 2n}$
2. $a_n = 1 + 2/n$
3. $a_n = [n + (-1)^n]^{-1}$
4. $a_n = \sin n$
5. $a_n = 1/2n$
6. $a_n = 1 + (-1)^n/n$
7. $\left(\dfrac{n}{n-1}\right)^2$
8. $a_n = 2n$.

9.3 SERIES

(1) Def. *If $u_1, u_2, u_3, \ldots, u_n, \ldots$ be an infinite sequence of real numbers, then*

$$u_1 + u_2 + u_3 + \ldots + u_n + \ldots \infty$$

is called an infinite series. An infinite series is denoted by Σu_n and the sum of its first n terms is denoted by s_n.

(2) Convergence, divergence and oscillation of a series.

Consider the infinite series $\Sigma u_n = u_1 + u_2 + u_3 + \ldots + u_n + \ldots \infty$

and let the sum of the first n terms be $s_n = u_1 + u_2 + u_3 + \ldots + u_n$

Clearly, s_n is a function of n and as n increases indefinitely three possibilities arise :

(i) If s_n tends to a finite limit as $n \to \infty$, the series Σu_n is said to be *convergent*.

(ii) If s_n tends to $\pm \infty$ as $n \to \infty$, the series Σu_n is said to be *divergent*.

(iii) If s_n does not tend to a unique limit as $n \to \infty$, then the series Σu_n is said to be *oscillatory* or *non-convergent*.

Example 9.2. *Examine for convergence the series (i) $1 + 2 + 3 + \ldots + n + \ldots \infty$.*
(ii) $5 - 4 - 1 + 5 - 4 - 1 + 5 - 4 - 1 + \ldots \infty$

Solution. (i) Here $s_n = 1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$

$\therefore$ $\underset{n \to \infty}{\text{Lt}} s_n = \dfrac{1}{2} \underset{n \to \infty}{\text{Lt}} n(n+1) \to \infty$. Hence this series is *divergent*.

(ii) Here $s_n = 5 - 4 - 1 + 5 - 4 - 1 + 5 - 4 - 1 + \ldots n$ terms

$= 0, 5$ or 1 according as the number of terms is $3m$, $3m + 1$, $3m + 2$.

Clearly in this case, s_n does not tend to a unique limit. Hence the series is *oscillatory*.

Examples 9.3. Geometric series. *Show that the series* $1 + r + r^2 + r^3 + \ldots \infty$ *(i) converges if* $|r| < 1$, *(ii) diverges if* $r \geq 1$, *and (iii) oscillates if* $r \leq -1$.

Solution. Let $$s_n = 1 + r + r^2 + \ldots + r^{n-1}$$

Case I. When $|r| < 1$, $\underset{n \to \infty}{\text{Lt}}\ r^n = 0$.

Also $$s_n = \frac{1 - r^n}{1 - r} = \frac{1}{1 - r} - \frac{r^n}{1 - r} \text{ so that } \underset{n \to \infty}{\text{Lt}}\ s_n = \frac{1}{1 - r}$$

$\therefore$ the series is convergent.

Case II. (*i*) When $r > 1$, $\underset{n \to \infty}{\text{Lt}}\ r^n \to \infty$.

Also $$s_n = \frac{r^n - 1}{r - 1} = \frac{r^n}{r - 1} - \frac{1}{r - 1} \text{ so that } \underset{n \to \infty}{\text{Lt}}\ s_n \to \infty$$

$\therefore$ the series is divergent.

(*ii*) When $r = 1$, then $s_n = 1 + 1 + 1 \ldots\ldots + 1 = n$

and $$\underset{n \to \infty}{\text{Lt}}\ s_n \to \infty \quad \therefore \text{ The series is divergent.}$$

Case III. (*i*) When $r = -1$, then the series becomes $1 - 1 + 1 - 1 + 1 - 1 \ldots$ which is an oscillatory series.

(*ii*) When $r < -1$, let $r = -\rho$ so that $\rho > 1$. Then $r^n = (-1)^n \rho^n$

and $$s_n = \frac{1 - r^n}{1 - r} = \frac{1 - (-1)^n \rho^n}{1 + \rho} \text{ as } \underset{n \to \infty}{\text{Lt}}\ \rho^n \to \infty.$$

$\therefore$ $\underset{n \to \infty}{\text{Lt}}\ s_n \to -\infty$ or $+\infty$ according as n is even or odd. Hence the series oscillates.

PROBLEMS 9.2

Examine the following series for convergence :

1. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \infty$.
2. $1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} + \frac{1}{3^4} - \ldots \infty$.
3. $6 - 10 + 4 + 6 - 10 + 4 + 6 - 10 + 4 + \ldots \infty$.
4. $\frac{1}{1 \cdot 2} + \frac{1}{2 . 3} + \frac{1}{3 . 4} + \ldots \infty$. (*V.T.U., 2006*)
5. A ball is dropped from a height h metres. Each time the ball hits the ground, it rebounds a distance r times the distance fallen where $0 < r < 1$. If $h = 3$ metres and $r = 2/3$, find the total distance travelled by the ball.

9.4 GENERAL PROPERTIES OF SERIES

The truth of the following properties is self-evident and these may be regarded as axioms :

1. *The convergence or divergence of an infinite series remains unaffected by the addition or removal of a finite number of its terms* ; for the sum of these terms being the finite quantity does not on addition or removal alter the nature of its sum.

2. *If a series in which all the terms are positive is convergent, the series remains convergent even when some or all of its terms are negative* ; for the sum is clearly the greatest when all the terms are positive.

3. *The convergence or divergence of an infinite series remains unaffected by multiplying each term by a finite number.*

9.5 SERIES OF POSITIVE TERMS

1. *An infinite series in which all the terms after some particular terms are positive, is a positive term series.* *e.g.*, $-7 - 5 - 2 + 2 + 7 + 13 + 20 + \ldots$ is a positive term series as all its terms after the third are positive.

2. *A series of positive terms either converges or diverges to* $+\infty$; for the sum of its first n terms, omitting the negative terms, tends to either a finite limit or $+\infty$.

3. **Necessary condition for convergence.** *If a positive term series Σu_n is convergent, then* $\underset{n\to\infty}{Lt}\ u_n = 0$.

(*P.T.U., 2009*)

Let $\quad s_n = u_1 + u_2 + u_3 + \ldots + u_n$. Since Σu_n is given to be convergent.

$\therefore \quad \underset{n\to\infty}{\text{Lt}}\ s_n =$ a finite quantity k (say). Also $\underset{n\to\infty}{\text{Lt}}\ s_{n-1} = k$

But $\quad u_n = s_n - s_{n-1} \quad \therefore \quad \underset{n\to\infty}{\text{Lt}}\ u_n = \underset{n\to\infty}{\text{Lt}}\ (s_n - s_{n-1}) = 0.$

Hence the result.

Obs. 1. *It is important to note that the converse of this result is not true.*

Consider, for instance, the series $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} + \ldots \infty$

Since the term go on descending,

$$\therefore \quad s_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \ldots + \frac{1}{\sqrt{n}} = \frac{n}{\sqrt{n}} \quad \text{i.e.,} \quad \sqrt{n}$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\ s_n = \underset{n\to\infty}{\text{Lt}}\ \sqrt{n} \to \infty$$

Thus the series is divergent even though $\underset{n\to\infty}{\text{Lt}}\ u_n = \underset{n\to\infty}{\text{Lt}}\ \frac{1}{\sqrt{n}} = 0$

Hence $\underset{n\to\infty}{Lt}\ u_n = 0$ *is a necessary but not sufficient condition for convergence of* Σn_n.

Obs. The above result leads to a simple test for divergence :

If $\underset{n\to\infty}{Lt}\ u_n \neq 0$, *the series* Σu_n *must be divergent.*

9.6 COMPARISON TESTS

I. If two positive term series Σu_n and Σv_n be such that

(*i*) *Σv_n converges,* (*ii*) *$u_n \le v_n$ for all values of n, then Σu_n also converges.*

Proof. Since Σv_n is convergent,

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\ (v_1 + v_2 + v_3 + \ldots + v_n) = \text{a finite quantity } k \text{ (say)}$$

Also since $u_1 \le v_1, u_2 \le v_2, \ldots, u_n \le v_n$

$\therefore$ Adding, $\quad u_1 + u_2 + \ldots + u_n \le v \ + v_2 + \ldots + v_n$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\ (u_1 + u_2 + \ldots + u_n) \le \underset{n\to\infty}{\text{Lt}}\ (v_1 + v_2 + \ldots + v_n) = k.$$

Hence the series Σu_n also converges.

Obs. If, however, the relation $u_n \le v_n$ holds for values of n greater than a fixed number m, then the first m terms of both the series can be ignored without affecting their convergence or divergence.

II. If two positive term series Σu_n and Σv_n be such that :

(*i*) *Σv_n diverges,* (*ii*) *$u_n \ge v_n$ for all values of n, then Σu_n also diverges.*

Its proof is similar to that of Test *I*.

III. **Limit form**

If two positive term series Σu_n and Σv_n be such that

$$\underset{n\to\infty}{Lt}\ \frac{u_n}{v_n} = \textit{finite quantity } (\neq 0), \textit{ then } \Sigma u_n \textit{ and } \Sigma v_n \textit{ converge or diverge together.}$$

Proof. Since $\underset{n\to\infty}{\text{Lt}}\ \frac{u_n}{v_n} = l$, a finite number $(\neq 0)$

By definition of a limit, there exists a positive number ε, however small, such that

$$\left| \frac{u_n}{v_n} - l \right| < \varepsilon \qquad \text{for } n \ge m$$

or $$-\varepsilon < \frac{u_n}{v_n} - l < \varepsilon \qquad \text{for } n \geq m$$

or $$l - \varepsilon < \frac{u_n}{v_n} < l + \varepsilon \qquad \text{for } n \geq m$$

Omitting the first m terms of both the series, we have

$$l - \varepsilon < \frac{u_n}{v_n} < l + \varepsilon \qquad \text{for all } n \qquad \text{...(1)}$$

Case I. When Σv_n is convergent, then

$$\underset{n \to \infty}{\text{Lt}} (v_1 + v_2 + \ldots + v_n) = k, \text{ a finite number} \qquad \text{...(2)}$$

Also from (1), $\frac{u_n}{v_n} < l + \varepsilon$, *i.e.*, $u_n < (l + \varepsilon)v_n$ for all n.

$$\therefore \quad \underset{n \to \infty}{\text{Lt}} (u_1 + u_2 + \ldots + u_n) < (l + \varepsilon) \underset{n \to \infty}{\text{Lt}} (v_1 + v_2 + \ldots + v_n) = (l + \varepsilon)k \qquad \text{[By (2)]}$$

Hence Σu_n is also convergent.

Case II. When Σv_n is divergent, then

$$\underset{n \to \infty}{\text{Lt}} (v_1 + v_2 + \ldots + v_n) \to \infty \qquad \text{...(3)}$$

Also from (1) $\quad l - \varepsilon < \frac{u_n}{v_n}$ or $u_n > (l - \varepsilon)v_n$ for all n

$$\therefore \quad \underset{n \to \infty}{\text{Lt}} (u_1 + u_2 + \ldots + u_n) > (l - \varepsilon) \underset{n \to \infty}{\text{Lt}} (v_1 + v_2 + \ldots + v_n) \to \infty \qquad \text{[By (3)]}$$

Hence Σu_n is also divergent.

9.7 INTEGRAL TEST

A positive term series $f(1) + f(2) + \ldots + f(n) + \ldots$, where $f(n)$ decreases as n increases, converges or diverges according as the integral

$$\int_1^\infty f(x)\,dx \qquad \text{...(1) is finite or infinite.}$$

The area under the curve $y = f(x)$, between any two ordinates lies between the set of inscribed and escribed rectangles formed by ordinates at $x = 1, 2, 3, \ldots$ as in Fig. 9.1. Then

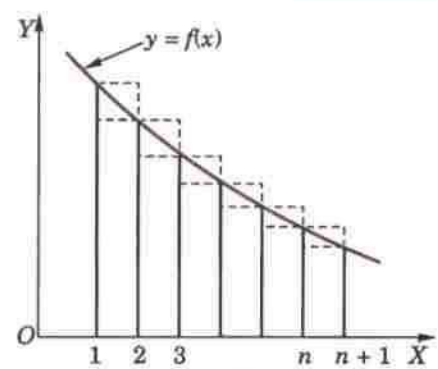

Fig. 9.1

$$f(1) + f(2) + \ldots + f(n) \geq \int_1^{n+1} f(x)\,dx \geq f(2) + f(3) + \ldots + f(n + 1)$$

or $$s_n \geq \int_1^{n+1} f(x)\,dx \geq s_{n+1} - f(1)$$

Taking limits as $n \to \infty$, we find from the second inequality that Lt $s_{n+1} \leq \int_1^\infty f(x)\,dx + f(1)$.

Hence if integral (1) is finite, so is Lt s_{n+1}. Similarly, from the first inequality, we see that if the integral (1) is infinite, so is Lt s_n. But the given series either converges or diverges to $+\infty$, *i.e.*, Lt s_n is either finite or infinite as $n \to \infty$.

Hence the result follows.

Example 9.4. Test for Comparison. *Show that the p-series*

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots \infty$$

(i) converges for $p > 1$ (ii) diverges for $p \leq 1$. *(P.T.U., 2009 ; V.T.U., 2006 ; Rohtak, 2003)*

Solution. By the above test, this series will converge or diverge according as $\int_1^\infty \frac{dx}{x^p}$ is finite or infinite.

If $p \neq 1$, $$\int_1^\infty \frac{dx}{x^p} = \underset{m \to \infty}{\text{Lt}} \int_1^m \frac{dx}{x^p} = \underset{m \to \infty}{\text{Lt}} \left(\frac{m^{1-p} - 1}{1-p} \right)$$

$$= \frac{1}{p-1}, \text{ i.e. finite } \quad \text{for } p > 1$$

$$\to \infty \quad \text{for } p < 1$$

If $p = 1$, $\int_1^\infty \frac{dx}{x} = \int_1^\infty \log x \to \infty$, this proves the result.

Obs. Application of comparison tests. Of all the above tests the 'limit form' is the most useful. To apply this comparison test to a given series Σu_n, the auxiliary series Σv_n must be so chosen that Lt (u_n/v_n) is non-zero and finite. To do this, we take v_n equal to that term of u_n which is of the highest degree in $1/n$ and the convergence or divergence of v_n is known with the help of the above series.

Example 9.5. *Test for convergence the series*

(i) $\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \ldots \infty$ (P.T.U., 2009)

(ii) $\frac{1}{4.7.10} + \frac{4}{7.10.13} + \frac{9}{10.13.16} + \ldots \infty$ (V.T.U., 2010)

(iii) $1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \ldots \infty$

Solution. (i) We have $$u_n = \frac{2n-1}{n(n+1)(n+2)} = \frac{1}{n^2} \frac{2 - 1/n}{(1+1/n)(1+2/n)}$$

Take $v_n = 1/n^2$; then

$$\underset{n \to \infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n \to \infty}{\text{Lt}} \frac{2 - 1/n}{(1+1/n)(1+2/n)} = \frac{2-0}{(1+0)(1+0)}$$

$= 2$, which is finite and non-zero

$\therefore$ both Σu_n and Σv_n converge or diverge together.

But $\Sigma v_n = \Sigma 1/n^2$ is known to be convergent.

Hence Σu_n is also convergent.

(ii) Here $$u_n = \frac{n^2}{(3n+1)(3n+4)(3n+7)} = \frac{1}{n\left(3+\frac{1}{n}\right)\left(3+\frac{4}{n}\right)\left(3+\frac{7}{n}\right)}$$

Taking $v_n = \frac{1}{n}$, we find that

$$\underset{n \to \infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n \to \infty}{\text{Lt}} \frac{1}{\left(3+\frac{1}{n}\right)\left(3+\frac{4}{n}\right)\left(3+\frac{7}{n}\right)} = \frac{1}{27} \neq 0$$

Now since Σv_n is divergent, therefore Σu_n is also divergent.

(iii) Here $$u_n = \frac{n^n}{(n+1)n+1} = \frac{1}{n+1} \cdot \left(\frac{n}{n+1}\right)^n,$$ ignoring the first term.

Taking $v_n = 1/n$, we have

$$\underset{n \to \infty}{\text{Lt}} \left(\frac{u_n}{v_n}\right) = \underset{n \to \infty}{\text{Lt}} \frac{n}{n+1} \cdot \underset{n \to \infty}{\text{Lt}} \left(\frac{n}{n+1}\right)^n$$

$$= \underset{n \to \infty}{\text{Lt}} \left(\frac{1}{1+1/n}\right) \cdot \underset{n \to \infty}{\text{Lt}} \frac{1}{(1+1/n)^n} = 1 \cdot \frac{1}{e} \neq 0$$

Now since Σv_n is divergent, therefore Σu_n is also divergent.

Example 9.6. *Test the convergence of the series :*

(i) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{(n+1)}}$ *(V.T.U., 2008)* *(ii)* $\sum_{n=1}^{\infty} \frac{1}{x^n + x^{-n}}$ *(iii)* $\sum_{n=1}^{\infty} \sqrt{\frac{3^n - 1}{2^n + 1}}$ *(V.T.U., 2000 S)*

Solution. (*i*) We have $u_n = \frac{\sqrt{(n+1)} - \sqrt{n}}{[\sqrt{(n+1)} + \sqrt{n}][\sqrt{(n+1)} - \sqrt{n}]} = \sqrt{(n+1)} - \sqrt{n}$

$$= \sqrt{n}\,[(1 + 1/n)^{1/2} - 1] \qquad \textbf{(Expanding by Binomial Theorem)}$$

$$= \sqrt{n}\left\{\left(1 + \frac{1}{2n} - \frac{1}{8n^2} + \dots\right) - 1\right\} = \sqrt{n}\left(\frac{1}{2n} - \frac{1}{8n^2} + \dots\right) = \frac{1}{\sqrt{n}}\left(\frac{1}{2} - \frac{1}{8n} + \dots\right)$$

Taking $v_n = 1/\sqrt{n}$, we have

$$\underset{u \to \infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n \to \infty}{\text{Lt}} \left(\frac{1}{2} - \frac{1}{8n} + \dots\right) = \frac{1}{2}, \text{ which is finite and non-zero.}$$

$\therefore$ both Σu_n and Σv_n converge or diverge together.

But $\Sigma v_n = \Sigma 1/\sqrt{n}$ is known to be divergent. Hence Σu_n is also divergent.

(*ii*) *When $x < 1$*, comparing the given series Σu_n with $\Sigma v_n = \Sigma x^n$,

we get $$\underset{n \to \infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n \to \infty}{\text{Lt}} \left(\frac{1}{x^n + x^{-n}} \cdot \frac{1}{x^n}\right) = \underset{n \to \infty}{\text{Lt}} \frac{1}{x^{2n} + 1} = 1 \qquad [\because \ x^{2n} \to 0 \text{ as } n \to \infty]$$

But Σv_n is convergent, so Σu_n is also convergent.

When $x > 1$, comparing Σu_n with $\Sigma w_n = \Sigma x^{-n}$, we get

$$\underset{n \to \infty}{\text{Lt}} \frac{u_n}{w_n} = \underset{n \to \infty}{\text{Lt}} \left(\frac{1}{x^n + x^{-n}} \cdot x^n\right) = \underset{n \to \infty}{\text{Lt}} \frac{1}{1 + x^{-2n}} = 1. \qquad [\because \ x^{-2n} \to 0 \text{ as } n \to \infty]$$

But Σw_n is convergent, so Σu_n is also convergent.

When $x = 1$, $\Sigma u_n = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots \infty$ which is divergent.

Hence, Σu_n converges for $x < 1$ and $x > 1$ but diverges for $x = 1$.

(*iii*) Here $$u_n = \sqrt{\frac{3^n - 1}{2^n + 1}} = \left(\frac{3}{2}\right)^{n/2} \sqrt{\left(\frac{1 - 1/3^n}{1 + 1/2^n}\right)}$$

Taking $$v_n = \left(\frac{3}{2}\right)^{n/2}, \text{ we get}$$

$$\underset{n \to \infty}{\text{Lt}} \left(\frac{u_n}{v_n}\right) = \underset{n \to \infty}{\text{Lt}} \sqrt{\left(\frac{1 - 1/3^n}{1 + 1/2^n}\right)} = 1 \neq 0$$

Also since $\Sigma v_n = r^n$ where $r = \sqrt{3/2}$ is a geometric series having $r > 1$, is divergent.

$\therefore$ Σu_n is also divergent.

Example 9.7. *Determine the nature of the series :*

(i) $\frac{\sqrt{2} - 1}{3^3 - 1} + \frac{\sqrt{3} - 1}{4^3 - 1} + \frac{\sqrt{4} - 1}{5^3 - 1} + \dots \infty$ *(ii)* $\sum \frac{1}{n} \sin \frac{1}{n}$

(iii) $\sum_{1}^{\infty} \frac{(\log n)^2}{n^{3/2}}$ *(iv)* $\sum_{2}^{\infty} \frac{1}{n (\log n)\, p} (p > 0)$ *(P.T.U., 2010)*

Solution. (*i*) We have $u_n = \frac{\sqrt{(n+1)} - 1}{(n+2)^3 - 1} = \frac{\sqrt{n}\,[(1 + 1/n) - 1/\sqrt{n}]}{n^3\,[(1 + 2/n)^3 - 1/n^3]}$

Taking $v_n = \dfrac{1}{n^{5/2}}$, we find that

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \frac{[\sqrt{(1+1/n)} - 1/\sqrt{n}]}{[(1+2/n)^3 - 1/n^3]} = 1 \neq 0$$

Since Σv_n is convergent, therefore Σu_n is also convergent.

(*ii*) Here $$u_n = \frac{1}{n}\sin\frac{1}{n} = \frac{1}{n}\left[\frac{1}{n} - \frac{1}{3!\,n^3} + \frac{1}{5!\,n^5} - \ldots\right] = \frac{1}{n^2}\left[1 - \frac{1}{3!\,n^2} + \frac{1}{5!\,n^4} - \ldots\right]$$

Taking $v_n = \dfrac{1}{n^2}$, we have

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \left[1 - \frac{1}{3!\,n^2} + \frac{1}{5!\,n^4}\ldots\right] = 1 \neq 0$$

Since Σv_n is convergent, therefore Σu_n is also convergent.

(*iii*) We have $\underset{n\to\infty}{\text{Lt}} \dfrac{(\log n)^2}{n^{1/4}} = 0$, *i.e.*, $\dfrac{(\log n)^2}{n^{1/4}} < 1$ or $(\log n)^2 < n^{1/4}$

$$\therefore \qquad u_n = \frac{(\log n)^2}{n^{3/2}} < \frac{n^{1/4}}{n^{3/2}} = \frac{1}{n^{5/4}}$$

Since $\Sigma 1/n^{5/4}$ converges by p-series. $(\because \quad p = 5/4 > 1)$

Hence by comparison test, Σu_n also converges.

(*iv*) Let $$f(n) = \frac{1}{n(\log n)^p} \text{ so that } f(x) = \frac{(\log x)^{-p}}{x}$$

$$\therefore \qquad f'(x) = \frac{-p}{x}(\log x)^{-p-1}\cdot\frac{1}{x} + (\log x)^{-p}\cdot\left(-\frac{1}{x^2}\right) = -\frac{1}{x^2}\left\{\frac{p}{(\log x)^{p+1}} + \frac{1}{(\log x)^p}\right\} < 0$$

i.e., $f(x)$ is a decreasing function.

Also $$\int_2^\infty f(x)\,dx = \int_2^\infty \frac{dx}{x(\log x)^p} = \left|\frac{(\log x)^{-p+1}}{-p+1}\right|_2^\infty$$

If $p > 1$, then $p - 1 = k$(say) > 0

$$\therefore \qquad \int_2^\infty f(x)\,dx = \left|\frac{(\log x)^{-k}}{-k}\right|_2^\infty = \frac{1}{k}[0 + (\log 2)^{-k}] \text{ which is finite}$$

Thus by integral test, *the given series converges for* $p > 1$.

If $p < 1$, then $1 - p > 0$ and $(\log x)^{1-p} \to \infty$ as $x \to \infty$.

$$\therefore \qquad \int_2^\infty f(x)\,dx \to \infty.$$

Thus *the given series diverges for* $p < 1$.

If $p = 1$, then $$\int_2^\infty f(x)\,dx = \int_2^\infty \frac{dx}{x\log x} = |\log(\log x)|_2^\infty \to \infty$$

Thus *the given series diverges for* $p = 1$.

PROBLEMS 9.3

Test the following series for convergence :

1. $1 - \dfrac{1}{3} + \dfrac{1}{3^2} - \dfrac{1}{3^3} + \dfrac{1}{3^4} - \ldots\infty$ (*J.N.T.U., 2000*)
2. $\dfrac{3}{5} + \dfrac{4}{5^2} + \dfrac{3}{5^3} + \dfrac{4}{5^4} + \ldots\infty$
3. $\dfrac{1}{1.2} + \dfrac{2}{3.4} + \dfrac{3}{5.6} + \ldots\infty$ (*Cochin, 2001*)
4. $\dfrac{1}{1.3} + \dfrac{2}{3.5} + \dfrac{3}{5.7} + \ldots\infty$ (*P.T.U., 2009*)

5. $1+\frac{2^2}{2!}+\frac{3^2}{3!}+\frac{4^2}{4!}+\dots\infty$

6. $\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\frac{5}{4^p}+\dots\infty$

7. $\frac{1}{1^2}+\frac{1+2}{1^2+2^2}+\frac{1+2+3}{1^2+2^2+3^2}+\dots\infty$

8. $\frac{1}{1.3.5}+\frac{2}{3.5.7}+\frac{3}{5.7.9}+\dots\infty$ (V.T.U., 2009 S)

9. $\frac{3}{1}+\frac{4}{8}+\frac{5}{27}+\frac{6}{64}+\dots\infty$

10. $\sum\frac{\sqrt{n}}{n^2+1}$ (Osmania, 2000 S)

11. $\sum_{n=0}^{\infty}\frac{2n^3+5}{4n^5+1}$

12. $\sum\frac{(n+1)(n+2)}{n^2\sqrt{n}}$ (J.N.T.U., 2006 S)

13. $\sum_{n=1}^{\infty}[\sqrt{(n^2+1)}-n]$ (V.T.U., 2010 ; P.T.U., 2009)

14. $\sum[\sqrt[3]{(n^3+1)}-n]$ (P.T.U., 2007 ; Rohtak 2003)

15. $\sum[\sqrt{(n^4+1)}-\sqrt{(n^4-1)}]$

16. $\sum\frac{1}{\sqrt{n}}\sin\frac{1}{n}$

17. $\sum_{n=2}^{\infty}\frac{1}{\sqrt{n\log n}}$

18. $\sum_{n=1}^{\infty}\frac{\sqrt{(n+1)}-1}{(n+2)^3-1}$ (J.N.T.U., 2003)

9.8 COMPARISON OF RATIOS

If Σu_n and Σv_n be two positive term series, then Σu_n converges if (i) Σv_n converges, and (ii) from and after some particular term,

$$\frac{u_{n+1}}{u_n}<\frac{u_{n+1}}{v_n}$$

Let the two series beginning from the particular term be $u_1+u_2+u_3+\dots$ and $v_1+v_2+v_3+\dots$

If $\frac{u_2}{u_1}<\frac{v_2}{v_1},\frac{u_3}{u_2}<\frac{v_3}{v_2},\dots$

then $$u_1+u_2+u_3+\dots=u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_1}+\dots\right)$$

$$=u_1\left(1+\frac{u_2}{u_1}+\frac{u_2}{u_1}\cdot\frac{u_3}{u_2}+\dots\right)<u_1\left(1+\frac{v_2}{v_1}+\frac{v_2}{v_1}\cdot\frac{v_3}{v_2}+\dots\right)<\frac{u_1}{v_1}(v_1+v_2+v_3+\dots).$$

Hence, if Σv_n converges, Σu_n also converges.

Obs. A more convenient form of the above test to apply is as follows :

Σu_n converges if (i) Σv_n converges and (ii) from and after a particular term $\frac{u_n}{u_{n+1}}>\frac{v_n}{v_{n+1}}$.

Similarly, Σu_n diverges, if (i) Σv_n diverges and (ii) from and after a particular term $\frac{u_n}{u_{n+1}}<\frac{v_n}{v_{n+1}}$.

9.9 D'ALEMBERT'S RATIO TEST*

In a positive term series Σu_n, if

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\lambda, \text{ then the series converges for } \lambda<1 \text{ and diverges for } \lambda>1.$$

***Case I.** When* $\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\lambda<1.$

*Called after the French mathematician *Jean le-Rond d'Alembert* (1717–1783), who also made important contributions to mechanics.

By definition of a limit, we can find a positive number r (< 1) such that $\frac{u_{n+1}}{u_n} < r$ for all $n > m$

Leaving out the first m terms, let the series be $u_1 + u_2 + u_3 + \dots$

so that $\frac{u_2}{u_1} < r, \frac{u_3}{u_2} < r, \frac{u_4}{u_3} < r, \dots$ and so on. Then $u_1 + u_2 + u_3 + \dots \infty$

$$= u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2}\cdot\frac{u_2}{u_1} + \frac{u_4}{u_3}\cdot\frac{u_3}{u_2}\cdot\frac{u_2}{u_1} + \dots \infty\right) < u_1(1 + r + r^2 + r^3 + \dots \infty)$$

$$= \frac{u_1}{1-r}, \text{ which is finite quantity. Hence } \Sigma u_n \text{ is convergent.} \qquad [\because \; r < 1]$$

***Case II.** When* $\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \lambda > 1$

By definition of limit, we can find m, such that $\frac{u_{n+1}}{u_n} \geq 1$ for all $n \geq m$.

Leaving out the first m terms, let the series be

$$u_1 + u_2 + u_3 + \dots \text{ so that } \frac{u_2}{u_1} \geq 1, \quad \frac{u_3}{u_2} \geq 1, \quad \frac{u_4}{u_3} \geq 1 \text{ and so on.}$$

$$\therefore \qquad u_1 + u_2 + u_3 + u_4 + \dots + u_n = u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2}\cdot\frac{u_2}{u_1} + \dots\right)$$

$$\geq u_1(1 + 1 + 1 + \dots \text{ to } n \text{ terms}) = nu_1$$

$$\therefore \qquad \underset{n\to\infty}{\text{Lt}} (u_1 + u_2 + \dots + u_n)$$

$$\geq \underset{n\to\infty}{\text{Lt}} (nu_1), \text{ which tends to infinity. Hence } \Sigma u_n \text{ is divergent.}$$

Obs. 1. *Ratio test fails when $\lambda = 1$. Consider, for instance, the series $\Sigma u_n = \Sigma 1/n^p$.*

Here $$\lambda = \underset{n\to\infty}{\text{Lt}} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{\text{Lt}} \left[\frac{1}{(n+1)^p}\cdot\frac{n^p}{1}\right] = \underset{n\to\infty}{\text{Lt}} \frac{1}{(1+1/n)^p} = 1.$$

Then for all values of p, $\lambda = 1$; whereas $\Sigma 1/n^p$ converges for $p > 1$ and diverges for $p < 1$.

Hence $\lambda = 1$ both for convergence and divergence of Σu_n, which is absurd.

Obs. 2. *It is important to note that this test makes no reference to the magnitude of u_{n+1}/u_n but concerns only with the limit of this ratio.*

For instance in the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} + \dots$, the ratio $\frac{u_{n+1}}{u_n} = \frac{n}{n+1} < 1$ for all finite values of n, but tends to unity as $n \to \infty$. Hence the Ratio test fails although this series is divergent.

Practical form of Ratio test. Taking reciprocals, the ratio test can be stated as follows :

In the positive term series Σu_n, if $\underset{n\to\infty}{Lt} \frac{u_n}{u_{n+1}} = k$, then the series converges for $k > 1$ and diverges for $k < 1$ but fails for $k = 1$.

Example. 9.8. *Test for convergence the series*

(*i*) $\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \dots \infty.$ *(P.T.U., 2005 ; V.T.U., 2003 ; I.S.M., 2001)*

(*ii*) $1 + \frac{2}{5}x + \frac{6}{9}x^2 + \frac{14}{17}x^3 + \dots + \frac{2^n - 2}{2^n + 1}x^{n-1} + \dots (x > 0).$ *(P.T.U., 2009 ; V.T.U., 2004)*

Solution. (*i*) We have $u_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}}$ and $u_{n+1} = \frac{x^{2n}}{(n+2)\sqrt{(n+1)}}$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{x^{2n-2}}{(n+1)\sqrt{n}} \cdot \frac{(n+2)\sqrt{(n+1)}}{x^{2n}}$$

$$= \underset{n\to\infty}{\text{Lt}} \left[\frac{n+2}{n+1}\left(\frac{n+1}{n}\right)^{1/2}\right] x^{-2} = \underset{n\to\infty}{\text{Lt}} \left[\frac{1+2/n}{1+1/n} \cdot \sqrt{(1+1/n)}\right] x^{-2} = x^{-2}.$$

Hence Σu_n converges if $x^{-2} > 1$, *i.e.*, for $x^2 < 1$ and diverges for $x^2 > 1$.

If $x^2 = 1$, then, $u_n = \dfrac{1}{(n+1)\sqrt{n}} = \dfrac{1}{n^{3/2}} \cdot \dfrac{1}{1+1/n}$

Taking $v_n = \dfrac{1}{n^{3/2}}$, we get $\underset{n\to\infty}{\text{Lt}} \dfrac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \dfrac{1}{1+1/n} = 1$, a finite quantity.

$\therefore$ Both Σu_n and Σv_n converge or diverge together. But $\Sigma v_n = \sum \dfrac{1}{n^{3/2}}$ is a convergent series.

$\therefore$ Σu_n is also convergent. Hence the given series converges if $x^2 \le 1$ and diverges if $x^2 > 1$.

(*ii*) Here
$$\frac{u_n}{u_{n+1}} = \frac{2^n - 2}{2^n + 1} x^{n-1} \cdot \frac{2^{n+1}+1}{2^{n+1}-2} \frac{1}{x^n} = \frac{1-\dfrac{2}{2^n}}{1+\dfrac{1}{2^n}} \cdot \frac{2+\dfrac{1}{2^n}}{2-\dfrac{2}{2^n}} \frac{1}{x}$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{1-0}{1+0} \cdot \frac{2+0}{2-0} \frac{1}{x} = \frac{1}{x}$$

Thus by Ratio test, Σu_n converges for $x^{-1} > 1$ *i.e.*, for $x < 1$ diverges for $x > 1$. But it fails for $x = 1$.

When $x = 1$,
$$\underset{n\to\infty}{\text{Lt}} u_n = \underset{n\to\infty}{\text{Lt}} \frac{2^n - 2}{2^n + 1} = \underset{n\to\infty}{\text{Lt}} \frac{1-\dfrac{2}{2^n}}{1+\dfrac{1}{2^n}} = 1 \ne 0$$

$\therefore$ Σu_n diverges for $x = 1$. Hence the given series converges for $x < 1$ and diverges for $x \ge 1$.

Example 9.9. *Discuss the convergence of the series*

(*i*) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(n^n)^2}$ (*P.T.U., 2010*) (*ii*) $1 + \dfrac{2!}{2^2} + \dfrac{3!}{3^3} + \dfrac{4!}{4^4} + \ldots \infty$ (*V.T.U., 2008 S*)

Solution. (*i*) We have $u_n = \dfrac{n!}{(n^n)^2}$ and $u_{n+1} = \dfrac{(n+1)!}{[(n+1)^{n+1}]^2}$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{n!}{n^{2n}} \times \frac{(n+1)^{2(n+1)}}{(n+1)!}$$

$$= \underset{n\to\infty}{\text{Lt}} \frac{(n+1)^{2n+1}}{n^{2n}} = \underset{n\to\infty}{\text{Lt}} \left(\frac{n+1}{n}\right)^{2n} .(n+1)$$

$$= \underset{n\to\infty}{\text{Lt}} \left[\left(1+\frac{1}{n}\right)^n\right]^2 .(n+1) = e. \underset{n\to\infty}{\text{Lt}} (n+1) \to \infty$$

Hence the given series is convergent.

(*ii*) Given series is $\Sigma u_n = \displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$. Here $\dfrac{u_n}{u_{n+1}} = \dfrac{n!}{n^n} \cdot \dfrac{(n+1)^{n+1}}{(n+1)!} = \dfrac{(n+1)^n}{n^n} = \left(1+\dfrac{1}{n}\right)^n$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \left(1+\frac{1}{n}\right)^n = e,$$ which is > 1. Hence the given series is convergent.

Example 9.10. *Examine the convergence of the series :*

(i) $\dfrac{x}{1+x}+\dfrac{x^2}{1+x^2}+\dfrac{x^3}{1+x^3}+\dots\infty$

(ii) $1+\dfrac{a+1}{b+1}+\dfrac{(a+1)(2a+1)}{(b+1)(2b+1)}+\dfrac{(a+1)(2a+1)(3a+1)}{(b+1)(2b+1)(3b+1)}+\dots\infty$

Solution. (*i*) Here $u_n=\dfrac{x^n}{1+x^n}$ and $u_{n+1}=\dfrac{x^{n+1}}{1+x^{n+1}}$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\frac{u_n}{u_{n+1}}=\underset{n\to\infty}{\text{Lt}}\left(\frac{x^n}{x^{n+1}}\cdot\frac{1+x^{n+1}}{1+x^n}\right)=\underset{n\to\infty}{\text{Lt}}\left(\frac{1+x^{n+1}}{x+x^{n+1}}\right)$$

$$=\frac{1}{x}, \text{ if } x<1. \qquad [\because \; x^{n+1}\to 0 \text{ as } n\to\infty]$$

Also $$\underset{n\to\infty}{\text{Lt}}\frac{u_n}{u_{n+1}}=\underset{n\to\infty}{\text{Lt}}\left(\frac{1+1/x^{n+1}}{1+x/x^{n+1}}\right)=1 \text{ if } x>1.$$

$\therefore$ by *Ratio test*, Σu_n converges for $x<1$ and fails for $x\geq 1$.

When $x=1$, $\Sigma u_n=\dfrac{1}{2}+\dfrac{1}{2}+\dfrac{1}{2}+\dots+\infty$, which is divergent.

Hence the given series converges for $x<1$ and diverges for $x\geq 1$.

(*ii*) Neglecting the first term, we have

$$u_{n+1}=u_n\cdot\frac{n_{a+1}}{n_{b+1}}$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\frac{u_n}{u_{n+1}}=\underset{n\to\infty}{\text{Lt}}\frac{n_{b+1}}{n_{a+1}}=\underset{n\to\infty}{\text{Lt}}\frac{b+1/n}{a+1/n}=\frac{b}{a}.$$

By *Ratio test*, Σu_n converges for $b/a>1$ or $a<b$, and diverges for $a>b$.

When $a=b$, the series becomes $1+1+1+\dots\infty$, which is divergent.

Hence the given series converges for $0<a<b$ and diverges for $0<b\leq a$.

PROBLEMS 9.4

Test for convergence the following series :

1. $x+\dfrac{x^2}{2}+\dfrac{x^3}{3}+\dots\infty.$

2. $\sqrt{\dfrac{1}{2}}x+\sqrt{\dfrac{2}{3}}x^2+\sqrt{\dfrac{3}{4}}x^3+\dots\infty$

3. $1+\dfrac{x}{2}+\dfrac{x^2}{5}+\dfrac{x^3}{10}+\dots+\dfrac{x^n}{n^2+1}\dots\infty$

4. $\displaystyle\sum_{n=2}^{\infty}\frac{x^n}{n(n-1)(n-2)}$ (*J.N.T.U., 2006*)

5. $1+\dfrac{2^p}{2!}+\dfrac{3^p}{3!}+\dfrac{4^p}{4!}+\dots\infty$ (*Kurukshetra, 2005*)

6. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n^2}{2^n}+\frac{1}{n^2}\right)$ (*Rohtak, 2005*)

7. $\displaystyle\sum_{n=1}^{\infty}\frac{n!\,3^n}{n^n}$ (*Kerala, 2005*)

8. $\displaystyle\sum_{n=1}^{\infty}\frac{n^3+a}{2^n+a}$

9. $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n}}{\sqrt{(n^2+1)}}x^n$ (*P.T.U., 2006*)

10. $\displaystyle\sum_{1}^{\infty}\frac{n^3-n+1}{n!}$ (*Madras, 2000*)

11. $\dfrac{2}{3.4}+\dfrac{2.4}{3.5.6}+\dfrac{2.4.6}{3.5.7.8}+\dots$ (*V.T.U., 2010*)

12. $\left(\dfrac{1}{3}\right)^2+\left(\dfrac{1.2}{3.5}\right)^2+\left(\dfrac{1.2.3}{3.5.7}\right)^2+\dots$

13. $1+\frac{1^2 \cdot 2^2}{1 \cdot 3 \cdot 5}+\frac{1^2 \cdot 2^2 \cdot 3^2}{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}+\ldots\infty$ (*Delhi, 2002*)

14. $\frac{4}{18}+\frac{4 \cdot 12}{18 \cdot 27}+\frac{4 \cdot 12 \cdot 20}{18 \cdot 27 \cdot 36}+\ldots\infty$ (*Madras, 2000*)

15. $\frac{1}{1^p}+\frac{x}{3^p}+\frac{x^2}{5^p}+\ldots+\frac{x^{n-1}}{(2n-1)^p}+\ldots\infty$ (*J.N.T.U., 2006*)

16. $\sum_{n=1}^{\infty} \frac{3 \cdot 6 \cdot 9 \ldots 3n}{4 \cdot 7 \cdot 10 \ldots (3n+1)} \cdot \frac{5^n}{3n+2}$ (*V.T.U., 2004*)

17. $1+\frac{1+\alpha}{1+\beta}+\frac{(1+\alpha)(1+2\alpha)}{(1+\beta)(1+2\beta)}+\frac{(1+\alpha)(1+2\alpha)(1+3\alpha)}{(1+\beta)(1+2\beta)(1+3\beta)}+\ldots$

9.10 FURTHER TESTS OF CONVERGENCE

When the Ratio test fails, we apply the following tests :

(1) Raabe's test*. *In the positive term series* Σu_n, if $\underset{n\to\infty}{\text{Lt}}\ n\left(\frac{u_n}{u_{n+1}}-1\right)=k$, *then the series converges for* $k>1$ *and diverges for* $k<1$, but the test fails for $k=1$.

When $k>1$, choose a number p such that $k>p>1$, and compare Σu_n with the series $\sum \frac{1}{n^p}$ which is convergent since $p>1$.

$\therefore$ Σu_n will converge, if from and after some term,

$$\frac{u_n}{u_{n+1}}>\frac{(n+1)^p}{n^p} \text{ or } \left(1+\frac{1}{n}\right)^p \qquad \text{or if,}\quad \frac{u_n}{u_{n+1}}>1+\frac{p}{n}+\frac{p(p-1)}{2n^2}+\ldots$$

or if, $$n\left(\frac{u_n}{u_{n+1}}-1\right)>p+\frac{p(p-1)}{2n}+\ldots \quad \text{or if,}\quad \underset{n\to\infty}{\text{Lt}}\ n\left(\frac{u_n}{u_{n+1}}-1\right)>\underset{n\to\infty}{\text{Lt}}\left[p+\frac{p(p-1)}{2n}+\ldots\right]$$

i.e., if $k>p$, which is true. Hence Σu_n is convergent.

The other case when $k<1$ can be proved similarly.

(2) Logarithmic test. *In the positive term series* Σu_n *if* $\underset{n\to\infty}{Lt}\left(n \log \frac{u_n}{u_{n+1}}\right)=k$, *then the series converges for* $k>1$, *and diverges for* $k<1$, but the test fails for $k=1$.

Its proof is similar to that of Raabe's test.

Obs. 1. *Logarithmic test is a substitute for Raabe's test and should be applied when* **either** *n occurs as an exponent in* u_n/u_{n+1}, **or** *evaluation of* $\underset{n\to\infty}{Lt}$ *becomes easier on taking logarithm of* u_n/u_{n+1}.

Obs. 2. If u_n/u_{n+1} does not involve n as an exponent or a logarithm, the series Σu_n diverges.

Example 9.11. *Test for convergence the series*

(*i*) $\sum \frac{4 \cdot 7 \ldots (3n+1)}{1 \cdot 2 \ldots n} x^n$ (*V.T.U., 2009 ; P.T.U., 2006 S*) (*ii*) $\sum \frac{(n!)^2}{(2n)!} x^{2n}$.

Solution. (*i*) Here $\frac{u_n}{u_{n+1}}=\frac{4 \cdot 7 \ldots (3n+1)}{1 \cdot 2 \ldots n} x^n \div \frac{4 \cdot 7 \ldots (3n+4)}{1 \cdot 2 \ldots (n+1)} x^{n+1}=\frac{n+1}{3n+4} \cdot \frac{1}{x}=\left[\frac{1+1/n}{3+4/n}\right]\frac{1}{x}$

$$\therefore \qquad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}}=\frac{1}{3x}.$$

*Called after the Swiss mathematician *Joseph Ludwig Raabe* (1801–1859).

Thus by *Ratio test*, the series converges for $\frac{1}{3x} > 1$, *i.e.*, for $x < \frac{1}{3}$ and diverges for $x > \frac{1}{3}$. But it fails for $x = \frac{1}{3}$. $\therefore$ Let us try the *Raabe's test.*

Now $$\frac{u_n}{u_{n+1}} = \left(1+\frac{1}{n}\right)\left(1+\frac{4}{3n}\right)^{-1}$$ [*Expand by Binomial Theorem*]

$$= \left(1+\frac{1}{n}\right)\left(1-\frac{4}{3n}+\frac{16}{9n^2}-\ldots\right) = 1-\frac{1}{3n}+\frac{4}{9n^2}+\ldots$$

$$\therefore \quad n\left(\frac{u_n}{u_{n+1}}-1\right) = -\frac{1}{3}+\frac{4}{9n}+\ldots \quad \therefore \quad \underset{n\to\infty}{\text{Lt}}\, n\left(\frac{u_n}{u_{n+1}}-1\right) = -\frac{1}{3} \text{ which} < 1.$$

Thus by *Raabe's test,* the series diverges.

Hence the given series converges for $x < \frac{1}{3}$ and diverges for $x \geq \frac{1}{3}$.

(*ii*) Here $$\frac{u_n}{u_{n+1}} = \left(\frac{n!}{(n+1)!}\right)^2 \frac{[2(n+1)]!}{(2n)!} \cdot \frac{x^{2n}}{x^{2(n+1)}} = \frac{(2n+1)(2n+2)}{(n+1)^2}\frac{1}{x^2} = \frac{2(2n+1)}{n+1}\frac{1}{x^2}$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{2(2+1/n)}{1+1/n} \cdot \frac{1}{x^2} = \frac{4}{x^2}$$

Thus by *Ratio Test,* the series converges for $x^2 < 4$ and diverges for $x^2 > 4$. But fails for $x^2 = 4$.

When $x^2 = 4$, $$n\left(\frac{u_n}{u_{n+1}}-1\right) = n\left(\frac{2n+1}{2n+2}-1\right) = -\frac{n}{2n+2}$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\, n\left(\frac{u_n}{u_{n+1}}-1\right) = -\frac{1}{2} < 1$$

Thus by *Raabe's test*, the series diverges.

Hence the given series converges for $x^2 < 4$ and diverges for $x^2 \geq 4$.

Example 9.12. *Discuss the convergence of the series*

$$x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \frac{4^4x^4}{4!} + \frac{5^5x^5}{5!} + \ldots \infty$$ (*P.T.U., 2008 ; Cochin, 2005 ; Rohtak, 2003*)

Solution. Here $$\frac{u_n}{u_{n+1}} = \frac{n^n x^n}{n!} \div \frac{(n+1)^{n+1} x^{n+1}}{(n+1)!} = \frac{n^n}{(n+1)^n x} = \frac{1}{(1+1/n)^n} \cdot \frac{1}{x}$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{1}{ex}.$$

Thus by Ratio test, the series converges for $x < 1/e$ and diverges for $x > 1/e$. But it fails for $x = 1/e$. Let us try the *log-test.*

Now $$\frac{u_n}{u_{n+1}} = \frac{e}{(1+1/n)^n}$$

$$\therefore \quad \log \frac{u_n}{u_{n+1}} = \log_e e - n \log\left(1+\frac{1}{n}\right) = 1 - n\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\ldots\right) = \frac{1}{2n}-\frac{1}{3n^2}+\ldots$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\left(n \log \frac{u_n}{u_{n+1}}\right) = \frac{1}{2},$$ which < 1. Thus by the *log-test*, the series diverges.

Hence the given series converges for $x < 1/e$ and diverges for $x \geq 1/e$.

Example 9.13. *Discuss the convergence of the* **hypergeometric series**

$$1 + \frac{\alpha.\beta}{1.\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}x^3 + \ldots \infty.$$ (*Kurukshetra, 2005*)

Solution. Neglecting the first term, we have

$$u_{n+1} = u_n \frac{(\alpha+n)(\beta+n)}{(n+1)(\gamma+n)} x$$

$$\therefore \qquad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)} \cdot \frac{1}{x} = \underset{n\to\infty}{\text{Lt}} \frac{(1+1/n)(1+\gamma/n)}{(1+\alpha/n)(1+\beta/n)} \cdot \frac{1}{x} = \frac{1}{x}$$

$\therefore$ by *Ratio test*, the series converges for $1/x > 1$, *i.e.*, for $x < 1$, and diverges for $x > 1$. But it fails for $x = 1$.

$\therefore$ let us try the *Raabe's test*.

$$\underset{n\to\infty}{\text{Lt}}\ n\left(\frac{u_n}{u_{n+1}} - 1\right) = \underset{n\to\infty}{\text{Lt}}\ n\left\{\frac{(n+1)(n+\gamma)}{(n+\alpha)(n+\beta)} - 1\right\} = \underset{n\to\infty}{\text{Lt}}\ n\left\{\frac{n(1+\gamma-\alpha-\beta)+\gamma-\alpha\beta}{n^2+n(\alpha+\beta)+\alpha\beta}\right\}$$

$$= \underset{n\to\infty}{\text{Lt}} \left\{\frac{(1+\gamma-\alpha-\beta)+(\gamma-\alpha\beta)\frac{1}{n}}{1+(\alpha+\beta)\frac{1}{n}+\alpha\beta\cdot\frac{1}{n^2}}\right\} = 1+\gamma-\alpha-\beta$$

Thus the series converges for $1+\gamma-\alpha-\beta > 1$, *i.e.*, for $\gamma > \alpha+\beta$ and diverges for $\gamma < \alpha+\beta$. But it fails for $\gamma = \alpha+\beta$. Since u_n/u_{n+1} does not involve n as an exponent or a logarithm, the series Σu_n diverges for $\gamma = \alpha+\beta$.

Hence the series converges for $x < 1$ and diverges for $x > 1$. When $x = 1$, the series converges for $\gamma > \alpha+\beta$ and diverges for $\gamma \le \alpha+\beta$.

PROBLEMS 9.5

Test the following series for convergence :

1. $\dfrac{x}{1.2} + \dfrac{x^2}{3.4} + \dfrac{x^3}{5.6} + \dfrac{x^4}{7.8} + \dots \infty\ (x > 0)$ *(Mumbai, 2009)*
2. $\dfrac{x}{1.2} + \dfrac{x^2}{2.3} + \dfrac{x^3}{3.4} + \dfrac{x^4}{4.5} + \dots \infty$ *(V.T.U., 2008 ; J.N.T.U., 2003)*
3. $1 + \dfrac{1}{2}x + \dfrac{1.3}{2.4}x^2 + \dfrac{1.3.5}{2.4.6}x^3 + \dots \infty\ (x > 0)$ *(Raipur, 2005)*
4. $1 + \dfrac{2}{3}x + \dfrac{2.3}{3.5}x^2 + \dfrac{2.3.4}{3.5.7}x^3 + \dots \infty$ *(V.T.U., 2009 S)*
5. $1 + \dfrac{x}{2} + \dfrac{2!}{3^2}x^2 + \dfrac{3!}{4^3}x^3 + \dfrac{4!}{5^4}x^4 + \dots \infty$
6. $1 + \dfrac{3}{7}x + \dfrac{3.6}{7.10}x^2 + \dfrac{3.6.9}{7.10.13}x^3 + \dfrac{3.6.9.12}{7.10.13.16}x^4 + \dots \infty$
7. $\dfrac{x}{1} + \dfrac{1}{2}\cdot\dfrac{x^3}{3} + \dfrac{1.3}{2.4}\cdot\dfrac{x^5}{5} + \dfrac{1.3.5}{2.4.6}\cdot\dfrac{x^7}{7} + \dots \infty,\ (x > 0)$ *(V.T.U., 2007 ; Raipur, 2005)*
8. $1 + \dfrac{1}{2}\cdot\dfrac{x^2}{4} + \dfrac{1.3.5}{2.4.6}\cdot\dfrac{x^4}{8} + \dfrac{1.3.5.7.9}{2.4.6.8.10}\cdot\dfrac{x^6}{12} + \dots \infty$ *(Rohtak, 2006 S ; Roorkee, 2000)*
9. $1 + \dfrac{(1!)^2}{2!}x^2 + \dfrac{(2!)^2}{4!}x^4 + \dfrac{(3!)^2}{6!}x^6 + \dots \infty\ (x > 0)$
10. $\dfrac{1^2}{4^2} + \dfrac{1^2.5^2}{4^2.8^2} + \dfrac{1^2.5^2.9^2}{4^2.8^2.12^2} + \dots \infty$
11. $\dfrac{a+x}{1!} + \dfrac{(a+2x)^2}{2!} + \dfrac{(a+3x)^3}{3!} + \dots \infty$
12. $x^2(\log 2)^q + x^3(\log 3)^q + x^4(\log 4)^q + \dots \infty$
13. $\dfrac{1}{1^2} + \dfrac{1+2}{1^2+2^2} + \dfrac{1+2+3}{1^2+2^2+3^2} + \dots$ *(V.T.U., 2000)*
14. $1 + \dfrac{a}{b}x + \dfrac{a(a+1)}{b(b+1)}x^2 + \dfrac{a(a+1)(a+2)}{b(b+1)(b+2)}x^3 + \dots \infty\ (a, b > 0, x > 0).$

9.11 CAUCHY'S ROOT TEST*

In a positive series Σu_n, if $\underset{n\to\infty}{\text{Lt}}\ (u_n)^{1/n} = \lambda$,

then the series converges for $\lambda < 1$, and diverges for $\lambda > 1$.

Case I. When $\underset{n\to\infty}{\text{Lt}}\ (u_n)^{1/n} = \lambda < 1$.

By definition of a limit, we can find a positive number r ($\lambda < r < 1$) such that

$$(u_n)^{1/n} < r \text{ for all } n > m, \quad \text{or} \quad u_n < r^n \text{ for all } n > m.$$

Since $r < 1$, the geometric series Σr^n is convergent. Hence, by comparison test, Σu_n is also convergent.

Case II. When Lt $(u_n)^{1/n} = \lambda > 1$.

By definition of a limit, we can find a number m, such that

$$(u_n)^{1/n} > 1 \text{ for all } n > m, \quad \text{or} \quad u_n > 1 \text{ for all } n > m.$$

Omitting the first m terms, let the series be $u_1 + u_2 + u_3 + \ldots$ so that $u_1 > 1, u_2 > 1, u_3 > 1$ and so on.

$$\therefore \qquad u_1 + u_2 + u_3 + \ldots + u_n > n \quad \text{and} \quad \underset{n\to\infty}{\text{Lt}}\ (u_1 + u_2 + \ldots + u_n) \to \infty$$

Hence the series Σu_n is divergent.

Obs. *Cauchy's root test fails when $\lambda = 1$.*

Example 9.14. *Test for convergence the series*

(i) $\Sigma \dfrac{n^3}{3^n}$ *(ii)* $\Sigma (\log n)^{-2n}$ *(iii)* $\Sigma (1 + 1/\sqrt{n})^{-n^{3/2}}$ *(P.T.U., 2009 ; Kurukshetra, 2005)*

Solution. (*i*) We have $u_n = n^3/3^n$.

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\ (u_n)^{1/n} = \underset{n\to\infty}{\text{Lt}} \left(\frac{n^{3/n}}{3}\right) = \underset{n\to\infty}{\text{Lt}}\ \frac{(n^{1/n})^3}{3} = \frac{1}{3}\ (< 1) \qquad \left[\because \underset{n\to\infty}{\text{Lt}}\ n^{1/n} = 1\right]$$

Hence the given series converges by Cauchy's root test.

(*ii*) Here $\qquad u_n = (\log n)^{-2n}$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\ (u_n)^{1/n} = \underset{n\to\infty}{\text{Lt}}\ (\log n)^{-2} = 0\ (< 1) \qquad [\because \underset{n\to\infty}{\text{Lt}}\ \log n = 0]$$

Hence, by Cauchy's root test, the given series converges.

(*iii*) Here $\qquad u_n = (1 + 1/\sqrt{n})^{-n^{3/2}}$

$$\therefore \quad (u_n)^{1/n} = \left[\frac{1}{(1 + 1/\sqrt{n})^{n^{3/2}}}\right]^{1/n} = \frac{1}{(1 + 1/\sqrt{n})^{\sqrt{n}}}$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\ (u_n)^{1/n} = \underset{n\to\infty}{\text{Lt}}\ \frac{1}{(1 + 1/\sqrt{n})^{\sqrt{n}}} = \frac{1}{e}$$, which is < 1. Hence the given series is convergent.

Example 9.15. *Discuss the nature of the following series :*

(i) $\dfrac{1}{2} + \dfrac{2}{3}x + \left(\dfrac{3}{4}\right)^2 x^2 + \left(\dfrac{4}{5}\right)^3 x^3 + \ldots \infty\ (x > 0)$ *(J.N.T.U., 2006)*

(ii) $\Sigma \dfrac{(n+1)^n x^n}{x^{n+1}}$

(iii) $\left(\dfrac{2^2}{1^2} - \dfrac{2}{1}\right)^{-1} + \left(\dfrac{3^3}{2^3} - \dfrac{3}{2}\right)^{-2} + \left(\dfrac{4^4}{3^4} - \dfrac{4}{3}\right)^{-3} + \ldots \infty$ *(V.T.U., 2006)*

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*See footnote p. 144.

Solution. (*i*) After leaving the first term, we find that $u_n = \left(\frac{n+1}{n+2}\right)^n x^n$, so that

$$\underset{n\to\infty}{\text{Lt}} (u_n)^{1/n} = \underset{n\to\infty}{\text{Lt}} \left(\frac{1+1/n}{1+2/n}\right) x = x$$

$\therefore$ By Cauchy's root test, the given series converges for $x < 1$ and diverges for $x > 1$.

When $x = 1$, $\quad u_n = \left(\frac{n+1}{n+2}\right)^n = \frac{1}{\left(1+\frac{1}{n+1}\right)^{n+1}}\left(1+\frac{1}{n+1}\right)$

$\therefore$ $\quad \underset{n\to\infty}{\text{Lt}} u_n = \frac{1}{e} \neq 0$. Since u_n does not tend to zero, Σu_n is divergent.

Thus the given series converges for $x < 1$ and diverges for $x \geq 1$.

(*ii*) Here $\quad (u_n^{1/n}) = \frac{n+1}{n^{1+1/n}} x$

$\therefore$ $\quad \underset{n\to\infty}{\text{Lt}} (u_n)^{1/n} = \underset{n\to\infty}{\text{Lt}} \frac{n+1}{n} \cdot \frac{1}{n^{1/n}} x = \underset{n\to\infty}{\text{Lt}} \left(1+\frac{1}{n}\right)\left(\frac{1}{n^{1/n}}\right) x = x \qquad \left[\because \underset{n\to\infty}{\text{Lt}} n/n = 1\right]$

$\therefore$ The given series converges for $x < 1$ and diverges for $x > 1$.

When $x = 1$, $\quad u_n = \frac{(n+1)^n}{n^{n+1}} = \frac{1}{n}\left(1+\frac{1}{n}\right)^n$

Taking $\qquad v_n = \frac{1}{n}$, $\underset{n\to\infty}{\text{Lt}} \left(\frac{u_n}{v_n}\right) = \underset{n\to\infty}{\text{Lt}} \left(1+\frac{1}{n}\right)^n = e \neq 0$ and finite.

$\therefore$ By comparison test both Σu_n and Σv_n behave alike.

But $\Sigma v_n = \Sigma \frac{1}{n}$ is divergent ($\because \quad p = 1$). $\quad \therefore \quad \Sigma u_n$ also diverges. Hence the given series converges for $x < 1$ and diverges for $x \geq 1$.

(*iii*) Here $\qquad u_n = \left[\frac{(n+1)^{n+1}}{n^{n+1}} - \frac{n+1}{n}\right]^{-n}$

$\therefore$ $\qquad (u_n)^{1/n} = \left(\frac{n+1}{n}\right)^{-1}\left[\left(\frac{n+1}{n}\right)^n - 1\right]^{-1} = \left(1+\frac{1}{n}\right)^{-1}\left[\left(1+\frac{1}{n}\right)^n - 1\right]^{-1}$

$\therefore$ $\quad \underset{n\to\infty}{\text{Lt}} (u_n)^{1/n} = 1 \cdot (e-1)^{-1} = \frac{1}{e-1} 9 < 1 \qquad [\because \quad e > 1]$

Thus the given series converges.

PROBLEMS 9.6

Discuss the convergence of the following series :

1. $\sum \frac{1}{n^n}$
2. $\sum \frac{1}{(\log n)^n}$ *(P.T.U., 2005)*
3. $\sum \left(\frac{n}{n+1}\right)^{n^2}$ *(P.T.U., 2010)*
4. $1 + \frac{x}{2} + \frac{x^2}{3^2} + \frac{x^3}{4^3} + \ldots + (x > 0)$
5. $\sum \left(\frac{n+2}{n+3}\right)^n x^n$
6. $\sum \frac{[(2n+1)\,x]^n}{n^{n+1}}$, $x > 0$
7. $\frac{3}{4} x + \left(\frac{4}{5}\right)^2 x^2 + \left(\frac{5}{6}\right)^3 x^3 + \ldots - \infty \; (x > 0)$ *(V.T.U., 2007)*

9.12 ALTERNATING SERIES

(1) Def. *A series in which the terms are alternately positive or negative is called an alternating series.*

(2) Leibnitz's series. *An alternating series $u_1 - u_2 + u_3 - u_4 + \ldots$ converges if (i) each term is numerically less than its preceding term, and (ii) $\underset{n\to\infty}{\text{Lt}} u_n = 0$.*

If $\underset{n\to\infty}{\text{Lt}} u_n \neq 0$, the given series is oscillatory.

The given series is $\quad u_1 - u_2 + u_3 - u_4 + \ldots$

Suppose
$$u_1 > u_2 > u_3 > u_4 \ldots > u_{n+1} \ldots \quad \ldots(1)$$
and
$$\underset{n\to\infty}{\text{Lt}} u_n = 0 \quad \ldots(2)$$

Consider the sum of $2n$ terms. It can be written as
$$s_{2n} = (u_1 - u_2) + (u_3 - u_4) + \ldots + (u_{2n-1} - u_{2n}) \quad \ldots(3)$$
or as
$$s_{2n} = u_1 - (u_2 - u_3) - (u_4 - u_5) \ldots - u_{2n} \quad \ldots(4)$$

By virtue of (1), the expressions within the brackets in (3) and (4) are all positive.

$\therefore$ It follows from (3) that s_{2n} is positive and increases with n.

Also from (4), we note that s_{2n} always remains less than u_1.

Hence s_{2n} must tend to a finite limit.

Moreover
$$\underset{n\to\infty}{\text{Lt}} s_{2n+1} = \underset{n\to\infty}{\text{Lt}} (s_{2n} + u_{2n+1}) = \underset{n\to\infty}{\text{Lt}} s_{2n} + 0 \quad \text{[by (2)}$$

Thus $\underset{n\to\infty}{\text{Lt}} s_n$ tends to the same finite limit whether n is even or odd.

Hence the given series is convergent.

When $\quad \underset{n\to\infty}{\text{Lt}} u_n \neq 0,\ \underset{n\to\infty}{\text{Lt}} s_{2n} \neq \underset{n\to\infty}{\text{Lt}} s_{2n+1}$. $\quad\therefore$ The given series is oscillatory.

Example 9.16. *Discuss the convergence of the series*

(i) $1 - \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} - \dfrac{1}{\sqrt{4}} + \ldots$ $\qquad$ *(ii)* $\dfrac{5}{2} - \dfrac{7}{4} + \dfrac{9}{6} - \dfrac{11}{8} + \ldots$

(iii) $\dfrac{1}{\log 2} - \dfrac{1}{\log 3} + \dfrac{1}{\log 4} - \dfrac{1}{\log 5} + \ldots$ $\qquad$ *(P.T.U., 2010)*

Solution. (*i*) The terms of the given series are alternately positive and negative ; each term is numerically less than its preceding term $\left[\because\ u_n - u_{n-1} = \dfrac{1}{\sqrt{n}} - \dfrac{1}{\sqrt{n-1}} < 0\right]$

Also $\underset{n\to\infty}{\text{Lt}} u_n = \underset{n\to\infty}{\text{Lt}} (1/\sqrt{n}) = 0$. Hence by Leibnitz's rule, the given series is convergent.

(*ii*) The terms of the given series are alternately positive and negative and
$$u_n - u_{n-1} = \frac{2n+3}{2n} - \frac{2n+1}{2n-2} = \frac{-6}{4n(n-1)} < 0 \text{ for } n > 1.$$
i.e.,
$$u_n < u_{n-1} \text{ for } n > 1. \text{ Also } \underset{n\to\infty}{\text{Lt}} u_n = \underset{n\to\infty}{\text{Lt}} \frac{2n+3}{2n} = 1 \neq 0$$

Hence by Leibnitz's rule, the given series is oscillatory.

(*iii*) The terms of the given series are alternately positive and negative.

Also $n + 2 > n + 1$, *i.e.*, $\log(n+2) > \log(n+1)$

i.e.,
$$\frac{1}{\log(n+2)} < \frac{1}{\log(n+1)}, \text{ i.e., } u_{n+1} < u_n.$$
and
$$\underset{n\to\infty}{\text{Lt}} u_n = \underset{n\to\infty}{\text{Lt}} \frac{1}{\log(n+1)} = 0$$

Hence the given series is convergent.

Example 9.17. *Examine the character of the series*

(i) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, n}{2n-1}$. *(ii)* $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}\, x^n}{n(n-1)}$, $0 < x < 1$.

Solution. *(i)* The terms of the given series are alternately positive and negative ; each term is numerically less than its preceding term.

$$\left[\because \quad u_n - u_{n-1} = \frac{n}{2n-1} - \frac{n-1}{2n-3} = \frac{-1}{(2n-1)(n-3)} < 0\right.$$

But $\quad \underset{n\to\infty}{\text{Lt}}\, u_n = \underset{n\to\infty}{\text{Lt}}\, \frac{n}{2n-1} = \underset{n\to\infty}{\text{Lt}}\, \frac{1}{2-1/n} = \frac{1}{2}$ which is not zero.

Hence the given series is oscillatory.

(ii) The terms of the given series are alternately positive and negative

$$u_n - u_{n-1} = \frac{x^n}{n(n-1)} - \frac{x^{n-1}}{(n-1)(n-2)} = \frac{x^{n-1}[(n-2)x - n]}{n(n-1)(n-2)} < 0 \quad \text{for} \quad n \geq 2, \qquad (\because \quad 0 < x < 1)$$

i.e., $\quad u_n < u_{n-1}$ for $n \geq 2$. Also $\underset{n\to\infty}{\text{Lt}}\, u_n = \underset{n\to\infty}{\text{Lt}}\, \frac{x^n}{n(n-1)} = 0 \qquad (\because \quad 0 < x < 1)$

Hence the given series is convergent.

PROBLEMS 9.7

Discuss the convergence of the following series :

1. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \infty.$ *(P.T.U., 2009)*
2. $1 - \frac{1}{5} + \frac{1}{9} - \frac{1}{13} + \ldots \infty.$ *(V.T.U., 2010)*
3. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ *(Delhi, 2002)*
4. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$.
5. $\frac{1}{1\cdot 2} - \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} - \frac{1}{7\cdot 8} + \ldots \infty$ *(Osmania, 2003)*
6. $\frac{1}{6} - \frac{2}{11} + \frac{3}{16} - \frac{4}{21} + \frac{5}{26} - \ldots \infty.$
7. $1 - 2x + 3x^2 - 4x^3 + \ldots + \infty, \left(x < \frac{1}{2}\right).$ *(Cochin, 2005)*
8. $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2+1}$.
9. $\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - \frac{x^4}{1+x^4} + \ldots \infty\ (0 < x < 1).$ *(V.T.U., 2004 ; Delhi, 2002)*
10. $\left(\frac{1}{2} - \frac{1}{\log 2}\right) - \left(\frac{1}{2} - \frac{1}{\log 3}\right) + \left(\frac{1}{2} - \frac{1}{\log 4}\right) - \left(\frac{1}{2} - \frac{1}{\log 5}\right) + \ldots \infty.$

9.13 SERIES OF POSITIVE AND NEGATIVE TERMS

The series of positive terms and the alternating series are special types of these series with arbitrary signs.

Def. (1) *If the series of arbitrary terms* $u_1 + u_2 + u_3 + \ldots + u_n + \ldots$
be such that the series $\quad |u_1| + |u_2| + |u_3| + \ldots + |u_n| + \ldots$
is convergent, then the series Σu_n *is said to be* **absolutely convergent.**

(2) *If* $\Sigma\, |u_n|$ *is divergent but* Σu_n *is convergent, then* Σu_n *is said to be* **conditionally convergent.**

For instance, the series $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \ldots$ *is absolutely convergent,* since the series

$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ldots$ is known to be convergent.

Again, since the alternating series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$ is convergent, and the series of absolute values $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$ is divergent, so the original series is conditionally convergent.

Obs. 1. *An absolutely convergent series is necessarily convergent but not conversely.*

Let Σu be an absolutely convergent series.

Clearly $u_1 + u_2 + u_3 + \ldots + u_n + \ldots$

$\leq |u_1| + |u_2| + |u_3| + \ldots + |u_n| + \ldots$ which is known to be convergent.

Hence the series Σu_n is also convergent.

Obs. 2. As the series $\Sigma |u_n|$ is of positive terms, the tests already established for positive term series can be applied to examine Σu_n for its absolute convergence. For instance, Ratio test can be restated as follows :

The series Σu_n *is absolutely convergent if* $\underset{n\to\infty}{\text{Lt}} \frac{|u_{n+1}|}{|u_n|} < 1$,

and is divergent if $\underset{n\to\infty}{\text{Lt}} \frac{|u_{n+1}|}{|u_n|} > 1$. This test fails when the limit is unity.

Example 9.18. *Examine the following series for convergence :*

(*i*) $1 + \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} - \frac{1}{8^2} + \ldots \infty$ (*V.T.U., 2006*)

(*ii*) $\frac{1}{2^3} - \frac{1}{3^3}(1+2) + \frac{1}{4^3}(1+2+3) - \frac{1}{5^3}(1+2+3+4) + \ldots \infty.$

Solution. (*i*) The series of absolute terms is $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \ldots$ which is, evidently convergent.

$\therefore$ the given series is absolutely convergent and hence it is convergent.

(*ii*) Here
$$u_n = (-1)^{n-1} \frac{(1+2+3+\ldots+n)}{(n+1)^3}$$
$$= (-1)^{n-1} \frac{n(n+1)}{2(n+1)^3} = (-1)^{n-1} \frac{n}{2(n+1)^2} = (-1)^{n-1} a_n \text{ (Say).}$$

Then
$$a_n - a_{n+1} = \frac{1}{2}\left[\frac{n}{(n+1)^2} - \frac{n+1}{(n+2)^2}\right] = \frac{1}{2} \frac{n^2+n-1}{(n+1)^2(n+2)^2} > 0.$$

i.e.,
$$a_{n+1} < a_n. \text{ Also } \underset{n\to\infty}{\text{Lt}} a_n = \frac{1}{2} \underset{n\to\infty}{\text{Lt}} \frac{n}{(n+1)^2} = 0.$$

Thus by Leibnitz's rule, Σa_n and therefore Σu_n is convergent.

Also
$$|u_n| = \frac{1}{2} \frac{n}{n^2+1}. \text{ Taking } v_n = \frac{1}{n}, \text{ we note that}$$
$$\underset{n\to\infty}{\text{Lt}} \frac{|u_n|}{v_n} = \frac{1}{2} \underset{n\to\infty}{\text{Lt}} \frac{n^2}{n^2+1} = \frac{1}{2} \neq 0$$

Since Σv_n is divergent, therefore $\Sigma |u_n|$ is also divergent.

i.e., Σu_n is convergent but $\Sigma |u_n|$ is divergent.

Thus the given series Σu_n is conditionally convergent.

Example 9.19. *Test whether the following series are absolutely convergent or not ?*

(*i*) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$ (*ii*) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\log n)^2}$.

Solution. (*i*) Given series is $\Sigma u_n = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \infty$

This is an alternating series of which terms go on decreasing and $\underset{n \to \infty}{\text{Lt}} u_n = \underset{n \to \infty}{\text{Lt}} \dfrac{1}{2n-1} = 0$

$\therefore$ by Leibnitz's rule, Σu_n converges.

The series of absolute terms is $1 + \dfrac{1}{3} + \dfrac{1}{5} + \dfrac{1}{7} + \dots \infty$

Here $u_n = \dfrac{1}{2n-1}$. Taking $v_n = \dfrac{1}{n}$, we have

$$\underset{n \to \infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n \to \infty}{\text{Lt}} \left(\frac{n}{2n-1}\right) = \underset{n \to \infty}{\text{Lt}} \left(\frac{1}{2 - \frac{1}{n}}\right) = \frac{1}{2} \neq 0 \text{ and finite.}$$

$\therefore$ by Comparison test, Σu_n diverges [$\because$ Σv_n diverges.

Hence the given series converges and the series of absolute terms diverges, therefore the given series converges conditionally.

(*ii*) The terms of given series are alternately positive and negative. Also each term is numerically less than the preceding term and $\underset{n \to \infty}{\text{Lt}} |u_n| = \underset{n \to \infty}{\text{Lt}} [1/n(\log n)^2] = 0$.

$\therefore$ by Leibnitz's rule, the given series converges.

Also $$\int_2^\infty \frac{dx}{x(\log x)^2} = \left| -\frac{1}{\log x} \right|_2^\infty = \frac{1}{\log 2} = 0 \text{ and finite.}$$

i.e., the series of absolute terms converges.

Hence, the given series converges absolutely.

9.14 POWER SERIES

(1) Def. *A series of the form* $a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$ ...(*i*)

where the a's are independent of x, is called a **power series** *in x.* Such a series may converge for some or all values of x.

(2) Interval of convergence

In the power series (*i*), $u_n = a_nx^n$.

$$\therefore \quad \underset{n \to \infty}{\text{Lt}} \frac{u_{n+1}}{u_n} = \underset{n \to \infty}{\text{Lt}} \frac{a_{n+1}x^{n+1}}{a_nx^n} = \underset{n \to \infty}{\text{Lt}} \left(\frac{a_{n+1}}{a_n}\right) . x$$

If $\underset{n \to \infty}{\text{Lt}} \left(\dfrac{a_{n+1}}{a_n}\right) = l$, then by Ratio test, the series (*i*) converges, when lx is numerically less than 1, *i.e.*, when $|x| < 1/l$ and diverges for other values.

Thus the power series (*i*) has an interval $-1/l < x < 1/l$ within which it converges and diverges for values of x outside this interval. Such an interval is called the *interval of convergence of the power series.*

Example 9.20. *State the values of x for which the following series converge :*

(*i*) $x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \dfrac{x^5}{5} - \dots \infty$, (*ii*) $\dfrac{1}{1-x} + \dfrac{1}{2(1-x)^2} + \dfrac{1}{3(1-x)^3} + \dots \infty$.

Solution. (*i*) Here $u_n = (-1)^{n-1} \dfrac{x^n}{n}$ and $u_{n+1} = (-1)^n \dfrac{x^{n+1}}{n+1}$

$$\therefore \quad \frac{u_{n+1}}{u_n} = -\frac{n}{n+1}x \quad \text{and} \quad \underset{n \to \infty}{\text{Lt}} \left|\frac{u_{n+1}}{u_n}\right| = \left(\underset{n \to \infty}{\text{Lt}} \frac{1}{1+1/n}\right) |x| = |x|$$

$\therefore$ by Ratio test the given series converges for $|x| < 1$ and diverges for $|x| > 1$.

Let us examine the series for $x = \pm 1$.

For $x = 1$, the series reduces to $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \ldots$

which is an alternating series and is convergent.

For $x = -1$, the series becomes $-\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots\right)$

which is a divergent series as can be seen by comparison with p-series when $p = 1$.

Hence the given series converges for $-1 < x \le 1$.

(*ii*) Here $\quad u_n = \dfrac{1}{n(1-x)^n}$

$$\underset{n\to\infty}{\text{Lt}}\left|\frac{u_{n+1}}{u_n}\right| = \underset{n\to\infty}{\text{Lt}}\left|\frac{1}{(n+1)(1-x)^{n+1}}\cdot n(1-x)^n\right| = \left|\frac{1}{1-x}\right|\underset{n\to\infty}{\text{Lt}}\frac{n}{n+1} = \left|\frac{1}{1-x}\right|$$

By Ratio test, Σu_n converges for $\left|\dfrac{1}{1-x}\right| < 1$, *i.e.*, $|1-x| > 1$

i.e., for $-1 > 1 - x > 1$ or $x < 0$ and $x > 2$.

Let us examine the series for $x = 0$ and $x = 2$.

For $x = 0$, the given series becomes $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n}$ which is a divergent harmonic series.

For $x = 2$, the given series becomes $-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \ldots + \frac{(-1)^n}{n} + \ldots$

It is an alternating series which is convergent by Leibnitz's rule

$$[\because \quad u_n < u_{n-1} \text{ for all } n \text{ and } \underset{n\to\infty}{\text{Lt}} u_n = 0.]$$

Hence the given series converges for $x < 0$ and $x \ge 2$.

Example 9.21. *Test the series* $\dfrac{x}{\sqrt{3}} - \dfrac{x^2}{\sqrt{5}} + \dfrac{x^3}{\sqrt{7}} - \ldots$ *for absolute convergence and conditional convergence.*

(V.T.U., 2010)

Solution. We have $\quad u_n = (-1)^{n-1}\dfrac{x^n}{\sqrt{(2n+1)}}$ and $u_{n+1} = \dfrac{(-1)^n x^{n+1}}{\sqrt{(2n+3)}}$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\left|\frac{u_{n+1}}{u_n}\right| = \underset{n\to\infty}{\text{Lt}}\left|\frac{\dfrac{(-1)^n x^{n+1}}{\sqrt{(2n+3)}}\cdot\sqrt{(2n+1)}}{(-1)^{n-1}x^n}\right| = \underset{n\to\infty}{\text{Lt}}\left|(-1)\sqrt{\left(\frac{2n+1}{2n+3}\right)}x\right|$$

$$= \underset{n\to\infty}{\text{Lt}}\left|\sqrt{\left(\frac{2+1/n}{2+3/n}\right)}x\right| = |x|$$

Hence *the given series is absolutely convergent for $|x| < 1$ and is divergent for $|x| > 1$* and the test fails for $|x| = 1$.

For $x = 1$, $\qquad u_n = \dfrac{(-1)^{n-1}}{\sqrt{(2n+1)}}$. Since $2n + 1 < 2n + 3$ or $(2n+1)^{-1/2} > (2n+3)^{-1/2}$

i.e., $\qquad u_n > u_{n+1}$. Also $\underset{n\to\infty}{\text{Lt}} u_n = \underset{n\to\infty}{\text{Lt}}\dfrac{1}{\sqrt{(2n+1)}} = 0.$

$\therefore$ the series is convergent by Leibnitz's test.

But $\qquad \dfrac{1}{\sqrt{3}} + \dfrac{1}{\sqrt{5}} + \dfrac{1}{\sqrt{7}} + \ldots$ has $u_n = \dfrac{1}{\sqrt{(2n+1)}} = \dfrac{1}{\sqrt{n}}\cdot\dfrac{1}{\sqrt{(2+1/n)}}$

On comparing it with $v_n = \frac{1}{\sqrt{n}}$, Σu_n is divergent.

Hence the *given series is conditionally convergent for x = 1.*

For x = – 1, the series becomes $-\left(\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{5}}+\frac{1}{\sqrt{7}}+\ldots\right)$

But we have seen that the series $\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{5}}+\frac{1}{\sqrt{7}}+\ldots$ is divergent.

Hence, *the given series is divergent when x = – 1.*

9.15 (1) CONVERGENCE OF EXPONENTIAL SERIES

The series $1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots \infty$ *is convergent for all values of x.* (J.N.T.U., 2006)

Here $\quad \underset{n\to\infty}{\text{Lt}} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{\text{Lt}} \left[\frac{x^n}{n!} \div \frac{x^{n-1}}{(n-1)!}\right] = \underset{n\to\infty}{\text{Lt}} \frac{x}{n} = 0$

Hence the series converges, whatever be the value of x.

(2) Convergence of logarithmic series

The series $x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n \frac{x^n}{n} + \ldots \infty$ *is convergent for* $-1 < x \le 1$.

Here $\quad \underset{n\to\infty}{\text{Lt}} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{\text{Lt}} \frac{(-1)^{n+1}x^{n+1}}{n+1} \cdot \frac{n}{(-1)^n x^n} = -x \underset{n\to\infty}{\text{Lt}} \frac{n}{n+1} = -x \underset{n\to\infty}{\text{Lt}} \left\{\frac{1}{1+1/n}\right\} = -x.$

Hence the series converges for $|x| < 1$ and diverges for $|x| > 1$.

When $x = 1$, the series being $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$, is convergent.

When $x = -1$, the series being $-\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\right)$, is divergent.

Hence the series converges for $-1 < x \le 1$.

(3) Convergence of binomial series

The series $1 + nx + \frac{n(n-1)}{2!} x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!} x^r + \ldots \infty$

converges for $|x| < 1$.

Here $\quad u_r = \frac{n(n-1)\ldots(n-r)}{(r-1)!} x^{r-1}$ and $u_{r+1} = \frac{n(n-1)\ldots(n-r+1)}{r!} x^r$

$\therefore \quad \underset{r\to\infty}{\text{Lt}} \frac{u_{r+1}}{u_r} = \underset{r\to\infty}{\text{Lt}} \frac{n-r+1}{r} x = \underset{r\to\infty}{\text{Lt}} \left(\frac{n+1}{r} - 1\right) x = -x$ for $r > n + 1$.

Hence, the series converges for $|x| < 1$.

PROBLEMS 9.8

1. Test the following series for conditional convergence : (*i*) $\sum \frac{(-1)^{n-1}}{\sqrt{n}}$ (*ii*) $\sum \frac{(-1)^{n-1} n}{n^2+1}$.
2. Prove that the series $\frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \ldots$ converges absolutely. (Rohtak, 2006 S)
3. Test the following series for conditional convergence :

   (*i*) $1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \ldots \infty$ (*ii*) $1 - \frac{1}{2} + \frac{1.3}{2.4} - \frac{1.3.5}{2.4.6} + \ldots \infty$

4. Discuss the absolute convergence of (*i*) $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$ (*Hissar, 2005 S*)

(*ii*) $x - \frac{x^2}{4} + \frac{x^3}{9} - \frac{x^4}{16} + \ldots \infty$ (*iii*) $\frac{1}{\sqrt{(1^3+1)}} - \frac{1}{\sqrt{(2^3+1)}}x + \frac{1}{\sqrt{(3^3+1)}}x^2 - \ldots \infty$

5. Find the nature of the series $\frac{x}{1 \cdot 2} - \frac{x^2}{2 \cdot 3} + \frac{x^3}{3 \cdot 4} - \frac{x^4}{4 \cdot 5} + \ldots \infty$ (*V.T.U., 2009*)

6. For what values of x are the following series convergent :

(*i*) $x - \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{3}} - \frac{x^4}{\sqrt{4}} + \ldots \infty$ (*P.T.U., 2009 S ; V.T.U., 2008*)

(*ii*) $x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \ldots \infty$

7. Find the radius of convergence of the series $\sum \frac{n!}{n^n} x^n$. (*Calicut, 2005*)

8. Prove that $\frac{1}{a} + \frac{1}{a+1} - \frac{1}{a+2} + \frac{1}{a+3} + \frac{1}{a+4} - \frac{1}{a+5} + \ldots$ is a divergent series.

9. Test the series $1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}}$ for

(*i*) absolute convergence and (*ii*) conditional convergence. (*V.T.U., 2007 ; Rohtak, 2005*)

9.16 PROCEDURE FOR TESTING A SERIES FOR CONVERGENCE

First see whether the given series is

(*i*) a series with terms alternately positive and negative ;

(*ii*) a series of positive terms excluding power series ;

or (*iii*) a power series.

For alternating series (*i*), apply the Leibnitz's rule (§ 9.12).

For series (*ii*), first find u_n and if possible evaluate Lt u_n. If Lt $u_n \neq 0$, the series is divergent. If Lt $u_n = 0$, compare Σu_n with $\Sigma 1/n^p$ and apply the comparison tests (§ 9.6).

If the comparison tests are not applicable, apply the Ratio test (§ 9.9). If Lt $u_n/u_{n+1} = 1$, *i.e.*, the ratio test fails, apply Raabe's teat (§ 9.10). If Raabe's test fails for a similar reason, apply Logarithmic test (§ 9.10). If this also fails, apply Cauchy's root test (§ 9.11).

For the power series (*iii*), apply the Ratio test as in § 9.14. If the Ratio test fails, examine the series as in case (*ii*) above.

PROBLEMS 9.9

Test the convergence of the following series :

1. $\sum_{n=1}^{\infty} \frac{2^n - 2}{2^n + 1} \cdot x^{n-1} (x > 0)$. (*Osmania, 1999*)

2. $\sum \left(\frac{1}{\sqrt{n}} - \sqrt{\frac{n}{n+1}}\right)$.

3. $1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \ldots$

4. $\sum_{n=1}^{\infty} \sqrt{\left\{\frac{2^n+1}{3^n+1}\right\}}$

5. $\frac{1}{1+\sqrt{2}} + \frac{2}{1+2\sqrt{3}} + \frac{3}{1+3\sqrt{4}} + \ldots \infty$.

6. $\frac{x}{1+\sqrt{1}} + \frac{x^2}{2+\sqrt{2}} + \frac{x^3}{3+\sqrt{3}} + \ldots \infty$.

7. $1 + \frac{2^2}{3^2}x + \frac{2^2 . 4^2}{3^2 . 5^2}x^2 + \frac{2^2 . 4^2 . 6^2}{3^2 . 5^2 . 7^2}x^3 + \ldots \infty$.

8. $\sum_{1}^{\infty} \frac{n x^n}{(n+1)(n+2)} (x > 0)$.

9. $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} x^n$.

10. $\sum_{n=1}^{\infty} \frac{x^n}{(2n-1)^2 2^n}$.

11. $\sum_{0}^{\infty} \frac{(3x+5)^n}{(n+1)!}$.

12. $\sum_{n=1}^{\infty} \frac{(x+2)^n}{3^n n}$.

13. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\log n}$.

14. $\sum \frac{(-1)^{n-1} \sin nx}{n^3}$.

15. $\frac{\sqrt{2}-1}{3^3-1} + \frac{\sqrt{3}-1}{4^3-1} + \frac{\sqrt{4}-1}{5^3-1} + \ldots \infty$. (V.T.U., 2003)

16. $\sum_{n=2}^{\infty} \frac{1}{(n \log n)(\log \log n)^p}$.

9.17 UNIFORM CONVERGENCE

Let $$u_1(x) + u_2(x) + \ldots \infty = \sum_{n=1}^{\infty} u_n(x) \qquad \ldots(1)$$

be an infinite series of functions each of which is defined in the interval (a, b). Let $s_n(x)$ be the sum of its first n terms, *i.e.*, $s_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x)$

At some point $x = x_1$, if $\underset{n \to \infty}{\text{Lt}}\; s_n(x_1) = s(x_1)$,

then the series (1) is said to converge to sum $s(x_1)$ at that point. This means at $x = x_1$ given a positive number ε, we can find a number N such that $|\, s(x_1) - s_n(x_1) \,| < \varepsilon$ for $n > N$ ...(2)

Evidently N will depend on ε but generally it will also depend on x_1. Now if we keep the same ε but take some other value x_2 of x for which (1) is convergent, then we may have to change N for the inequality (2) to hold. If we wish to approximate the sum $s(x)$ of the series by its partial sums $s_n(x)$, we shall require different partial sums at different points of the interval and the problem will become quite complicated. If, however, we choose an N which is independent of the values of x, the problem becomes simpler. Then the partial sum $s_n(x)$, $(n > N)$ approximates to $s(x)$ for all values of x in the interval (a, b) and ε is uniform throughout this interval. Thus we have

Definition. *The series $\Sigma u_n(x)$ is said to be uniformly convergent in the interval (a, b), if for a given $\varepsilon > 0$, a number N can be found* **independent of x,** *such that for every x in the interval (a, b),*

$$|\, s(x) - s_n(x) \,| < \varepsilon \text{ for all } n > N.$$

Example 9.21. *Examine the geometric series $1 + x + x^2 + \ldots + x^{n-1} + \ldots \infty$ for uniform convergence in the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$.*

Solution. We have $$s_n(x) = 1 + x + x^2 + \ldots + x^{n-1} = \frac{1-x^n}{1-x}.$$

and $$s(x) = \underset{n \to \infty}{\text{Lt}} \frac{1-x^n}{1-x} = \frac{1}{1-x} \text{ for } |\, x \,| < 1$$

$\therefore$ $$|\, s(x) - s_n(x) \,| = \left| \frac{x^n}{1-x} \right| = \frac{|\, x^n \,|}{1-x} = \frac{|\, x \,|^n}{1-x} \text{ which will be } < \varepsilon, \text{ if } |\, x \,|^n < \varepsilon\,(1-x).$$

Choose N such that $|\, x \,|^N = \varepsilon\,(1-x)$

or $$N = \log\,[\varepsilon\,(1-x)]/\log |\, x \,| \qquad \ldots(i)$$

Evidently N increases with the increase of $|\, x \,|$ and in the interval $-\frac{1}{2} \le x \le \frac{1}{2}$, it assumes a maximum value $N' = \log(\varepsilon/2)/\log \frac{1}{2}$ at $x = \frac{1}{2}$ for a given ε.

Thus $|\, s(x) - s_n(x) \,| < \varepsilon$ for all $n \ge N'$ for every value of x in the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

Hence the geometric series converges uniformly in the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

Obs. *The geometric series though convergent in the interval $(-1, 1)$, is not uniformly convergent in this interval,* since we cannot find a fixed number N for every x in this interval

($\because$ N given by (i) $\to \infty$ as $|\, x \,| \to 1$).

9.18 WEIERSTRASS'S M-TEST*

A series $\Sigma u_n(x)$ is uniformly convergent in an interval (a, b), if there exists a convergent series ΣM_n of positive constants such that $|u_n(x)| \le M_n$ for all values of x in (a, b).

Since ΣM_n is convergent, therefore, for a given $\varepsilon > 0$, we can find a number N, such that $|s - s_n| < \varepsilon$ for every $n > N$,

where $\quad s = M_1 + M_2 + \dots + M_n + M_{n+1} + \dots$ and $s_n = M_1 + M_2 + \dots + M_n$

This implies that $|M_{n+1} + M_{n+2} + \dots| < \varepsilon$ for every $n > N$.

Since $\quad |u_n(x)| \le M_n$

$$\therefore \quad |u_{n+1}(x)| + u_{n+2}(x) + \dots| \le |u_{n+1}(x)| + |u_{n+2}(x)| + \dots$$
$$\le M_{n+1} + M_{n+2} + \dots < \varepsilon \text{ for every } n > N.$$

i.e., $\quad |s(x) - s_n(x)| < \varepsilon$ for every $n > N$, where $s(x)$ is the sum of the series $\Sigma u_n(x)$.

Since N does not depend on x, the series $\Sigma u_n(x)$ converges uniformly in (a, b).

Obs. *$\Sigma u_n(x)$ is also absolutely convergent for every x, since $|u_n(x)| \le M_n$.*

Example 9.22. *Show that the following series converges uniformly in any interval :*

(i) $\sum \dfrac{\cos nx}{n^p}$ *(Andhra, 1999)* *(ii)* $\sum \dfrac{1}{n^3 + n^4x^2}$.

Solution. *(i)* $\left|\dfrac{\cos nx}{n^p}\right| = \dfrac{|\cos nx|}{n^p} \le \dfrac{1}{n^p}$ $(= M_n)$ for all values of x.

Since $\displaystyle\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for p > 1,

$\therefore$ By M-test, the given series converges uniformly for all real values of x and $p > 1$.

(ii) For all values of x, $n^3 + n^4x^2 > n^3$

$$\therefore \quad \left|\frac{1}{n^3 + n^4x^2}\right| < \frac{1}{n^3} \ (= M_n).$$ But ΣM_n being p-series with $p > 1$, is convergent.

$\therefore$ By M-test, the given series converges uniformly in any interval.

Example 9.23. *Examine the following series for uniform convergence :*

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(nx + x^2)}{n(n+2)}$ *(P.T.U., 2009)* *(ii)* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p + n^q x^2}$ *(P.T.U., 2005 S)*

Solution. *(i)* $\left|\dfrac{\sin(nx + x^2)}{n(n+2)}\right| = \dfrac{|\sin(nx + x^2)|}{n^2 + 2n} \le \dfrac{1}{n^2}$ $(= M_n)$ for all real x.

Since $\displaystyle\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, therefore, by M-test, the given series is uniformly convergent for all real values of x.

(ii) For all real values of x, $x^2 \ge 0$, *i.e.,* $n^q x^2 \ge 0$

i.e., $\quad n^p + n^q x^2 \ge n^p \quad$ or $\quad \dfrac{1}{n^p + n^q x^2} \le \dfrac{1}{n^p}$ $(= M_n)$

Since $\displaystyle\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for $p > 1$,

$\therefore$ by M-test, the given series is uniformly convergent for all real values of x and $p > 1$.

* Named after the great German mathematician *Karl Weierstrass* (1815–1897) who made basic contributions to Calculus, Approximation theory, Differential geometry and Calculus of variations. He was also one of the founders of Complex analysis.

9.19 PROPERTIES OF UNIFORMLY CONVERGENT SERIES

I. If the series $\Sigma u_n(x)$ converges uniformly to sum $s(x)$ in the interval (a, b) and each of the functions $u_n(x)$ is continuous in this interval, then the sum $s(x)$ is also continuous in (a, b).

II. If the series $\Sigma u_n(x)$ converges uniformly in the interval (a, b) and each of the functions $u_n(x)$ is continuous in this interval, then the series can be integrated term by term

i.e.,
$$\int_a^b [u_1(x) + u_2(x) + \ldots]\, dx = \int_a^b u_1(x)\, dx + \int_a^b u_2(x)\, dx + \ldots$$

III. If $\Sigma u_n(x)$ is a convergent series having continuous derivatives of its terms, and the series $\Sigma u_n'(x)$ converges uniformly, then the series can be differentiated term by term

$$\frac{d}{dx}[u_1(x) + u_2(x) + \ldots] = u_1'(x) + u_2'(x) + \ldots.$$

Example 9.24. *Prove that* $\int_0^1 \left(\sum \frac{x^n}{n^2}\right) dx = \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}$.

Solution. $|x^n| \le 1$ for $0 \le x \le 1$

$\therefore$ $\left|\frac{x^n}{n^2}\right| \le \frac{1}{n^2}$ $(= M_n)$ for $0 \le x \le 1$. But ΣM_n is a convergent series.

$\therefore$ by M-test, the series $\Sigma(x^n/n^2)$ is uniformly convergent in $0 \le x \le 1$. Also x^n/n^2 is continuous in this interval.

$\therefore$ the series $\Sigma(x^n/n^2)$ can be integrated term by term in the interval $0 \le x \le 1$.

i.e.,
$$\int_0^1 \left(\sum \frac{x^n}{n^2}\right) dx = \sum\left(\int_0^1 \frac{x^n}{n^2}\, dx\right) = \sum\left(\frac{1}{n^2}\int_0^1 x^n\, dx\right) = \sum \frac{1}{n^2(n+1)}.$$

Imp. Obs. *There is no relation between absolute and uniform convergence.* In fact, a series may converge absolutely but not uniformly while another series may converge uniformly but not absolutely.

For instance, the series

$\frac{1}{x^2+1} - \frac{1}{x^2+2} + \frac{1}{x^2+3} - \ldots$ can be seen to converge uniformly but not absolutely, while the series

$x^2 + \frac{x^2}{x^2+1} + \frac{x^2}{(x^2+1)^2} + \frac{x^2}{(x^2+1)^3} + \ldots$ can be shown to converge absolutely but not uniformly.

PROBLEMS 9.10

Test for uniform convergence the series :

1. $\sum_{n=1}^{\infty} \frac{x^n}{n^{3/2}}$.

2. $\sum \frac{\cos nx}{2^n}$.

3. $\frac{\sin x}{1^2} + \frac{\sin 2x}{2^2} + \frac{\sin 3x}{3^2} + \frac{\sin 4x}{4^2} + \ldots \infty$. (*P.T.U., 2003 ; Andhra, 2000*)

4. $\sin x - \frac{\sin 2x}{2\sqrt{2}} + \frac{\sin 3x}{3\sqrt{3}} - \frac{\sin 4x}{4\sqrt{4}} + \ldots \infty$.

5. $\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots \infty$.

6. $\frac{ax}{2} + \frac{a^2x^2}{5} + \frac{a^3x^3}{10} + \ldots + \frac{a^nx^n}{n^2+1} + \ldots \infty$.

7. Show that the series $\Sigma r^n \sin n\theta$ and $\Sigma r^n \cos n\theta$ converge uniformly for all real values of θ if $0 < r < 1$.

8. Show that $\frac{1}{1+x^2} - \frac{1}{2+x^2} + \frac{1}{3+x^2} - \frac{1}{4+x^2} + \ldots$ converges uniformly in the interval $x \ge 0$ but not absolutely.

9. Prove that $\sum \frac{x}{n(1+nx^2)}$ is uniformly convergent for all real values of x.

10. Examine the following series for uniform convergence :

(i) $\sum_{n=1}^{\infty} \frac{1}{n^4 + n^3 x^2}$ (ii) $\sum_{n=1}^{\infty} \frac{\cos(x^2 + n^2 x)}{n(n^2+2)}$.

11. Show that

(i) $\int_0^1 \left(\sum \frac{\sin x}{x}\right) dx = 1 - \frac{1}{3.3!} + \frac{1}{5.5!} - \frac{1}{7.7!} + \ldots \infty$; (ii) $\int_0^{\pi} \left(\sum \frac{\sin n\theta}{n^3}\right) d\theta = 2 \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}$.

9.20 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 9.11

Choose the correct answer or fill up the blanks in each of the following problems :

1. The series $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots$ converges if

(a) $p > 0$ (b) $p < 1$ (c) $p > 1$ (d) $p \le 1$.

2. The series $\sum_{n=0}^{\infty} (2x)^n$ converges if

(a) $-1 \le x \le 1$ (b) $-\frac{1}{2} < x < \frac{1}{2}$ (c) $-2 < x < 2$ (d) $-\frac{1}{2} \le x \le \frac{1}{2}$.

3. The series $\frac{2}{1^2} - \frac{3}{2^2} + \frac{4}{3^2} - \frac{5}{4^2} + \ldots$ is

(a) conditionally convergent (b) absolutely convergent
(c) divergent (d) none of the above.

4. Which one of the following series is *not* convergent ?

(a) $\frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \ldots \infty$ (b) $1\frac{1}{2} - 1\frac{1}{3} + 1\frac{1}{4} - 1\frac{1}{5} + \ldots \infty$

(c) $\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \ldots \infty$ (d) $x + x^2 + x^3 + x^4 + \ldots \infty$ where $|x| < 1$.

5. The sum of the alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ is

(a) zero (b) infinite (c) log 2
(d) not defined as the series is not convergent.

6. Let Σu_n be a series of positive terms. Given that Σu_n is convergent and also

$\underset{n\to\infty}{\text{Lt}} \frac{u_{n+1}}{u_n}$ exists, then the said limit is

(a) necessarily equal to 1 (b) necessarily greater than 1
(c) may be equal to 1 or less than 1 (d) necessarily less than 1.

7. $\sum \left(1 + \frac{1}{n}\right)^{-n^2}$ is

(a) convergent (b) oscillatory (c) divergent.

8. $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ldots$ is

(a) oscillatory (b) conditionally convergent
(c) divergent (d) absolutely convergent.

9. $1+\frac{1}{2^2}-\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}-\frac{1}{7^2}-\frac{1}{8^2}+\ldots\infty$ is

(a) conditionally convergent (b) convergent
(c) oscillatory (d) divergent.

10. $\int_0^1\left(\sum\frac{x^n}{n^2}\right)dx=$

(a) $\sum_{n=0}^{\infty}\frac{1}{n(n+1)}$ (b) $\sum_{n=1}^{\infty}\frac{1}{n^2(n-1)}$ (c) $\sum_{n=0}^{\infty}\frac{1}{n(n-1)}$ (d) $\sum_{n=1}^{\infty}\frac{1}{n^2(n+1)}$.

11. If Σu_n is a convergent series of positive terms, then $\underset{n\to\infty}{\text{Lt}}\ u_n$ is

(a) 1 (b) ± 1 (c) 0 (d) 0. *(V.T.U., 2010)*

12. Geometric series $1+x+x^2+\ldots+x^{n-1}+\ldots\infty$

(a) converges in the interval .... (b) converges uniformly in the interval ....

13. The series $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\infty$ converges in the interval ....

14. If $\underset{n\to\infty}{\text{Lt}}\frac{u_n}{u_{n+1}}=k$, then Σu_n converges for k ....

15. A sequence (a_n) is said to be bounded, if there exists a number k such that for every n, a_n is ....

16. The series $2-5+3+2-5+3-5+\ldots\infty$ is .... (Convergent etc.)

17. The series $1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots\infty$ converges for ....

18. If $\underset{n\to\infty}{\text{Lt}}\ n\left\{\frac{u_n}{u_{n+1}}-1\right\}=k$, then Σu_n diverges for k ....

19. A sequence which is monotonic and bounded is ....

20. The series $\frac{1}{1.2}+\frac{2}{3.4}+\frac{3}{5.6}+\ldots\infty$ is .... (Convergent etc.)

21. The series $\frac{2^p}{1^q}+\frac{3^p}{2^q}+\frac{4^p}{3^q}+\ldots\infty$ converges for ....

22. The series $\sqrt{\frac{1}{4}}+\sqrt{\frac{2}{6}}+\sqrt{\frac{3}{8}}+\ldots\infty$ is .... (Convergent etc.)

23. The series $\sqrt{\left(\frac{2^n-1}{3^n-1}\right)}$ is ... (Convergent etc.)

24. The series $1-\frac{1}{2}(x-2)+\frac{1}{4}(x-2)^2\ldots+\left(-\frac{1}{2}\right)^n(x-2)^n+\ldots\infty$ converges in the interval ......

25. Is the series $\sum_{n=1}^{\infty}\frac{8\tan^{-1}n}{1+n^2}$ convergent ?

26. The exponential series $1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}+\ldots\infty$ is absolutely convergent. (True/False)

27. The series $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots+\frac{1}{n(n+1)}+\ldots\infty$, is .... (Convergent/divergent/oscillatory)

28. Is the series $\sum n\tan 1/n$ convergent ?

29. The series $\sum\frac{1}{nx^n}$ converges for x ......

30. The series $\sum_{n=1}^{\infty}\frac{x^n}{n^3}$ converges uniformly when x lies in the interval ....

31. Every absolutely convergent series is necessarily
(*a*) divergent (*b*) convergent
(*c*) conditionally convergent (*d*) none of these. (*V.T.U., 2009*)

32. The convergence of the series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ is tested by
(*a*) Ratio test (*b*) Raabe's test (*c*) Leibnitz's (*d*) Cauchy root test. (*V.T.U., 2009*)

33. The series $\sum \frac{x^n}{(n+1)^n}, x > 0$ is
(*a*) divergent (*b*) convergent (*c*) oscillatory (*d*) none of these. (*V.T.U., 2010*)

34. $\sum \sin\left(\frac{1}{n}\right)$ is
(*a*) convergent (*b*) divergent (*c*) oscillatory (*d*) none of these.

35. $\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$ is convergent. (True or False)

CHAPTER 10

Fourier Series

1. Introduction. 2. Euler's Formulae. 3. Conditions for a Fourier expansion. 4. Functions having points of discontinuity. 5. Change of interval. 6. Odd and even function—Expansions of odd or even periodic functions. 7. Half-range series. 8. Typical wave-forms. 9. Parseval's formula. 10. Complex form of F-series. 11. Practical Harmonic Analysis. 12. Objective Type of Questions.

10.1 INTRODUCTION

In many engineering problems, especially in the study of periodic phenomenae* in conduction of heat, electro-dynamics and acoustics, it is necessary to express a function in a series of sines and cosines. Most of the single-valued functions which occur in applied mathematics can be expressed in the form.

$$\tfrac{1}{2}a_0\dagger + a_1 \cos x + a_2 \cos 2x + \ldots\ldots\dagger$$
$$+ b_1 \sin x + b_2 \sin 2x + \ldots\ldots$$

within a desired range of values of the variable. Such a series is known as the **Fourier series$**.

10.2 EULER'S FORMULAE

The Fourier series for the function f(x) in the interval $\alpha < x < \alpha + 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx \end{aligned}\right\} \qquad \ldots(1)$$

These values of a_0, a_n, b_n are known as *Euler's formulae***.

***Periodic functions.** If at equal intervals of abscissa x, the value of each ordinate $f(x)$ repeats itself, *i.e.*, $f(x) = f(x + a)$, for all x, then $y = f(x)$ is called a *periodic function* having **period** α, *e.g.*, $\sin x$, $\cos x$ are periodic functions having a period 2π.

† To write $a_0/2$ instead of a_0 is a conventional device to be able to get more symmetric formulae for the coefficients.

$ Named after the French mathematician and physicist *Jacques Fourier* (1768–1830) who was first to use Fourier series in his memorable work *'Theorie Analytique de la Chaleur'* in which he developed the theory of heat conduction. These series had a deep influence in the further development of mathematics and mathematical physics.

**See footnote p. 205.

To establish these formulae, the following definite integrals will be required :

1. $\int_{\alpha}^{\alpha+2\pi} \cos nx\, dx = \left| \frac{\sin nx}{n} \right|_{\alpha}^{\alpha+2\pi} = 0$ $\quad (n \neq 0)$

2. $\int_{\alpha}^{\alpha+2\pi} \sin nx\, dx = -\left| \frac{\cos nx}{n} \right|_{\alpha}^{\alpha+2\pi} = 0$ $\quad (n \neq 0)$

3. $\int_{\alpha}^{\alpha+2\pi} \cos mx \cos nx\, dx$ $\quad (n \neq 0)$

$$= \frac{1}{2} \int_{\alpha}^{\alpha+2\pi} [\cos (m+n)x + \cos (m-n)x]\, dx$$

$$= \frac{1}{2} \left| \frac{\sin (m+n)x}{m+n} + \frac{\sin (m-n)x}{m-n} \right|_{\alpha}^{\alpha+2\pi} = 0 \qquad (m \neq n)$$

4. $\int_{\alpha}^{\alpha+2\pi} \cos^2 nx\, dx = \left| \frac{x}{2} + \frac{\sin 2nx}{4n} \right|_{\alpha}^{\alpha+2\pi} = \pi$ $\quad (n \neq 0)$

5. $\int_{\alpha}^{\alpha+2\pi} \sin mx \cos nx\, dx = -\frac{1}{2}\left[\frac{\cos (m-n)x}{m-n} + \frac{\cos (m+n)x}{m+n} \right] = 0$ $\quad (m \neq n)$

6. $\int_{\alpha}^{\alpha+2\pi} \sin nx \cos nx\, dx = \left| \frac{\sin^2 nx}{2n} \right|_{\alpha}^{\alpha+2\pi} = 0$ $\quad (n \neq 0)$

7. $\int_{\alpha}^{\alpha+2\pi} \sin mx \sin nx\, dx = \frac{1}{2} \left| \frac{\sin (m-n)x}{m-n} - \frac{\sin (m+n)x}{m+n} \right|_{\alpha}^{\alpha+2\pi} = 0$ $\quad (m \neq n)$

8. $\int_{\alpha}^{\alpha+2\pi} \sin^2 nx\, dx = \left| \frac{x}{2} - \frac{\sin 2nx}{4n} \right|_{\alpha}^{\alpha+2\pi} = \pi.$ $\quad (n \neq 0)$

Proof. Let $f(x)$ be represented in the interval $(\alpha, \alpha + 2\pi)$ by the Fourier series :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad ...(i)$$

To find the coefficients a_0, a_n, b_n, we assume that the series (i) can be integrated term by term from $x = \alpha$ to $x = \alpha + 2\pi$.

To find a_0, integrate both sides of (i) from $x = \alpha$ to $x = \alpha + 2\pi$. Then

$$\int_{\alpha}^{\alpha+2\pi} f(x)\, dx = \frac{1}{2} a_0 \int_{\alpha}^{\alpha+2\pi} dx + \int_{\alpha}^{\alpha+2\pi} \left(\sum_{n=1}^{\infty} a_n \cos nx \right) dx + \int_{\alpha}^{\alpha+2\pi} \left(\sum_{n=1}^{\infty} b_n \sin nx \right) dx$$

$$= \frac{1}{2} a_0 (\alpha + 2\pi - \alpha) + 0 + 0 = a_0 \pi \qquad \text{[By integrals (1) and (2) above]}$$

Hence $\quad a_0 = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x)\, dx.$

To find a_n, multiply each side of (i) by $\cos nx$ and integrate from $x = \alpha$ to $x = \alpha + 2\pi$. Then

$$\int_{\alpha}^{\alpha+2\pi} f(x) \cos nx\, dx = \frac{1}{2} a_0 \int_{\alpha}^{\alpha+2\pi} \cos nx\, dx + \int_{\alpha}^{\alpha+2\pi} \left(\sum_{n=1}^{\infty} a_n \cos nx \right) \cos nx\, dx$$

$$+ \int_{\alpha}^{\alpha+2\pi} \left(\sum_{n=1}^{\infty} b_n \sin nx \right) \cos nx\, dx$$

$$= 0 + \pi a_n + 0 \qquad \text{[By integrals (1), (3), (4), (5) and (6)]}$$

Hence $\quad a_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cos nx\, dx.$

To find b_n, multiply each side of (i) by $\sin nx$ and integrate from $x = \alpha$ to $x = \alpha + 2\pi$. Then

$$\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx = \frac{1}{2}a_0 \int_{\alpha}^{\alpha+2\pi} \sin nx\,dx + \int_{\alpha}^{\alpha+2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)\sin nx\,dx$$

$$+ \int_{\alpha}^{\alpha+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)\sin nx\,dx$$

$$= 0 + 0 + \pi b_n \qquad \text{[By integrals (2), (5), (6), (7) and (8)]}$$

Hence $$b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx.$$

Cor. 1. Making $\alpha = 0$, *the interval becomes* $0 < x < 2\pi$, and the formulae (I) reduce to

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \end{aligned}\right\} \qquad \text{...(II)}$$

Cor. 2. Putting $\alpha = -\pi$, *the interval becomes* $-\pi < x < \pi$ and the formulae (I) take the form :

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \end{aligned}\right\} \qquad \text{...(III)}$$

Example 10.1. *Obtain the Fourier series for $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$.* *(S.V.T.U., 2007)*

Solution. Let $$e^{-x} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \text{...}(i)$$

Then $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx = \frac{1}{\pi}\Big| -e^{-x}\Big|_0^{2\pi} = \frac{1-e^{-2\pi}}{\pi}$$

and $$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$$

$$= \frac{1}{\pi(n^2+1)}\Big| e^{-x}(-\cos nx + n\sin nx)\Big|_0^{2\pi} = \left(\frac{1-e^{-2\pi}}{\pi}\right)\cdot\frac{1}{n^2+1}$$

$\therefore$ $$a_1 = \left(\frac{1-e^{-2\pi}}{\pi}\right)\frac{1}{2},\; a_2 = \left(\frac{1-e^{-2\pi}}{\pi}\right)\cdot\frac{1}{5} \text{ etc.}$$

Finally, $$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$$

$$= \frac{1}{\pi(n^2+1)}\Big| e^{-x}(-\sin nx - n\cos nx)\Big|_0^{2\pi} = \left(\frac{1-e^{-2\pi}}{\pi}\right)\cdot\frac{n}{n^2+1}$$

$\therefore$ $$b_1 = \frac{1-e^{-2\pi}}{\pi}\cdot\frac{1}{2},\; b_2 = \left(\frac{1-e^{-2\pi}}{\pi}\right)\cdot\frac{2}{5} \text{ etc.}$$

Substituting the values of a_0, a_n, b_n in (i), we get

$$e^{-x} = \frac{1-e^{-2\pi}}{\pi}\left\{\frac{1}{2} + \left(\frac{1}{2}\cos x + \frac{1}{5}\cos 2x + \frac{1}{10}\cos 3x + \ldots\right) + \left(\frac{1}{2}\sin x + \frac{2}{5}\sin 2x + \frac{3}{10}\sin 3x + \ldots\right)\right\}.$$

Example 10.2. *Find a Fourier series to represent $x - x^2$ from $x = -\pi$ to $x = \pi$.* *(V.T.U., 2011 ; Madras, 2006)*

Solution. Let $x - x^2 = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ ...(*i*)

Then $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} (x - x^2)\, dx = \dfrac{1}{\pi}\left|\dfrac{x^2}{2} - \dfrac{x^3}{3}\right|_{-\pi}^{\pi} = -\dfrac{2\pi^2}{3}.$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2) \cos nx\, dx^*$$

$$= \frac{1}{\pi}\left[(x - x^2)\frac{\sin nx}{n} - (1 - 2x)\times\left(-\frac{\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{-4(-1)^n}{n^2} \qquad [\because \cos n\pi = (-1)^n]$$

$\therefore$ $a_1 = 4/1^2, a_2 = -4/2^2, a_3 = 4/3^2, a_4 = -4/4^2$ etc.

Finally, $b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} (x - x^2) \sin nx\, dx$

$$= \frac{1}{\pi}\left[(x - x^2)\left(-\frac{\cos nx}{n}\right) - (1 - 2x)\times\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi} = -2(-1)^n/n$$

$\therefore$ $b_1 = 2/1, b_2 = -2/2, b_3 = 2/3, b_4 = -2/4$ etc.

Substituting the values of a's and b's in (*i*), we get

$$x - x^2 = -\frac{\pi^2}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \dots\right]$$
$$+ 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \dots\right]$$

Obs. Putting $x = 0$, we find another interesting series $0 = -\dfrac{\pi^2}{3} + 4\left(\dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \dots\right)$

i.e., $\dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \dots = \dfrac{\pi^2}{12}.$ *(V.T.U., 2011)*

Note. In the above example, we have used the results $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$

Also $\sin\left(n + \dfrac{1}{2}\right)\pi = (-1)^n$ and $\cos\left(n + \dfrac{1}{2}\right)\pi = 0$. *The reader should remember these results.*

Example 10.3. *Expand $f(x) = x \sin x$ as a Fourier series in the interval $0 < x < 2\pi$.* *(S.V.T.U., 2009 ; Bhopal, 2009 ; Rohtak, 2006)*

Solution. Let $f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ ...(*i*)

Then $a_0 = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} x \sin x\, dx = \dfrac{1}{\pi}\left| x(-\cos x) - 1.(-\sin x)\right|_0^{2\pi} = -2.$

---

* Apply the *general rule of integration by parts* which states that if u, v be two functions of x and dashes denote differentiations and suffixes integrations w.r.t. x, then

$$\int uv\, dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \dots$$

In other words : *Integral of the product of two functions*

= 1st function × integral of 2nd – go on differentiating 1st, integrating 2nd signs alternately +ve and –ve.

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos nx \, dx = \frac{1}{2\pi}\int_0^{2\pi} x\,(2\cos nx \sin x)\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\,[\sin (n+1)x - \sin (n-1)x]\, dx$$

$$= \frac{1}{2\pi}\left[x\left\{\frac{-\cos (n+1)x}{n+1} + \frac{\cos (n-1)x}{n-1}\right\} - 1\left\{-\frac{\sin (n+1)x}{(n+1)^2} + \frac{\sin (n-1)x}{(n-1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left\{-\frac{\cos 2(n+1)\pi}{n+1} + \frac{\cos 2(n-1)\pi}{n-1}\right\}\right] = \frac{2}{n^2-1}.\ (n \neq 1)$$

When $n = 1$, $a_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos x \, dx = \frac{1}{2\pi}\int_0^{2\pi} x \sin 2x \, dx$

$$= \frac{1}{2\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - 1.\left(-\frac{\sin 2x}{4}\right)\right]_0^{2\pi} = -\frac{1}{2}.$$

Finally, $b_n = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin nx \, dx = \frac{1}{2\pi}\int_0^{2\pi} x\,[\cos (n-1)x - \cos (n+1)x]dx$

$$= \frac{1}{2\pi}\left[x\left\{\frac{\sin (n-1)x}{n-1} - \frac{\sin (n+1)}{n+1}\right\} - 1.\left\{-\frac{\cos (n-1)x}{(n-1)^2} + \frac{\cos (n+1)x}{(n+1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\cos 2(n-1)\pi}{(n-1)^2} - \frac{\cos 2(n+1)\pi}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right] = 0\ (n \neq 1)$$

When $n = 1$, $b_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin x \, dx = \frac{1}{2\pi}\int_0^{2\pi} x\,(1 - \cos 2x)\, dx$

$$= \frac{1}{2\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - 1.\left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right]_0^{2\pi} = \pi$$

Substituting the values of a's and b's, in (*i*), we get

$$x \sin x = -1 + \pi \sin x - \frac{1}{2}\cos x + \frac{2}{2^2-1}\cos 2x + \frac{2}{3^2-1}\cos 3x + \ldots$$

Example 10.4. *Expand $f(x) = \sqrt{(1-\cos x)}$, $0 < x < 2\pi$ in a Fourier series. Hence evaluate*

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots\ldots \qquad \textit{(Mumbai, 2006 ; J.N.T.U., 2006)}$$

Solution. We have $f(x) = \sqrt{(1-\cos x)} = \sqrt{2} \sin x/2$.

Let $\qquad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots(i)$

Then $\qquad a_0 = \frac{1}{\pi}\int_0^{2\pi} \sqrt{2} \sin x/2 \, dx = \frac{\sqrt{2}}{\pi}\left| -2\cos\frac{\pi}{2}\right|_0^{2\pi} = \frac{4\sqrt{2}}{\pi}$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \sqrt{2} \sin\frac{x}{2} \cos nx \, dx = \frac{\sqrt{2}}{2\pi}\int_0^{2\pi} 2\cos nx \sin x/2 \, dx$$

$$= \frac{1}{\sqrt{2}\pi}\int_0^{2\pi} \sin\left(n + \frac{1}{2}\right)x - \sin\left(n - \frac{1}{2}\right)x\, \Big] dx$$

$$= \frac{1}{\sqrt{2}\,\pi}\left| -\frac{2}{2n+1}\cos\left(\frac{2n+1}{2}\right) + \frac{2}{2n-1}\cos\frac{2n-1}{2}x\right|_0^{2\pi}$$

$$= \frac{2}{\sqrt{2}\,\pi}\left\{-\frac{1}{2n+1}[\cos (2n+1)\pi - 1] + \frac{1}{2n-1}[\cos (2n-1)\pi - 1]\right\}$$

$$= \frac{\sqrt{2}}{\pi}\left(\frac{2}{2n+1} - \frac{2}{2n-1}\right) = -\frac{4\sqrt{2}}{\pi(4n^2-1)} \qquad [\because \cos(2n+1)\pi = \cos(2n-1)\pi = -1]$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\sin\frac{x}{2}\sin nx\,dx = \frac{\sqrt{2}}{2\pi}\int_0^{2\pi} 2\sin nx \sin x/2\,dx$$

$$= \frac{1}{\sqrt{2}\,\pi}\int_0^{2\pi}\left[\cos\left(n-\frac{1}{2}\right)x - \cos\left(n+\frac{1}{2}\right)x\right]dx$$

$$= \frac{1}{\sqrt{2}\,\pi}\left|\frac{2}{2n-1}\sin\left(\frac{2n-1}{2}\right)x - \frac{2}{2n+1}\sin\left(\frac{2n+1}{2}\right)x\right|_0^{2\pi}$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{1}{2n-1}\{\sin(2n-1)\pi - 0\} - \frac{1}{2n+1}\{\sin(2n+1)\pi - 0\}\right] = 0$$

Substituting the values of a's and b's in (i), we get

$$\sqrt{(1-\cos x)} = \frac{2\sqrt{2}}{\pi} - \sum_{n=1}^{\infty}\frac{4\sqrt{2}}{(4n^2-1)\pi}\cos nx$$

When $x = 0$, we have

$$0 = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)} \qquad \text{i.e.,} \quad \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots\ldots \infty = \frac{1}{2}.$$

PROBLEMS 10.1

1. Obtain a Fourier series to represent e^{-ax} from $x = -\pi$ to $x = \pi$. Hence derive series for $\pi/\sinh\pi$.
2. Prove that $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}$, $-\pi < x < \pi$. (*P.T.U., 2009 ; Bhopal, 2008 ; B.P.T.U., 2006*)

   Hence show that (*i*) $\sum\frac{1}{n^2} = \frac{\pi^2}{6}$. (*Anna, 2009 ; P.T.U., 2009 ; Osmania, 2003*)

   (*ii*) $\sum\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$ (*iii*) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}$ (*S.V.T.U., 2008*)

   (*iv*) $\sum\frac{1}{n^4} = \frac{\pi^4}{90}$. (*Bhopal, 2008*)
3. If $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in the range 0 to 2π, show that $f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$. (*Delhi, 2002 ; Madras, 2000*)
4. Prove that in the range $-\pi < x < \pi$, $\cosh ax = \frac{2a^2}{\pi}\sinh a\pi\left[\frac{1}{2a^2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+a^2}\cos nx\right]$.
5. $f(x) = x + x^2$ for $-\pi < x < \pi$ and $f(x) = \pi^2$ for $x = \pm\pi$. Expand $f(x)$ in Fourier series. (*Kurukshetra, 2005 ; U.P.T.U., 2003*)

   Hence show that $x + x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n\left\{\frac{4}{n^2}\cos nx - \frac{2}{n}\sin nx\right\}$

   and $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots \frac{\pi^2}{8}$ (*V.T.U., 2008*)

10.3 CONDITIONS FOR A FOURIER EXPANSION

The reader must not be misled by the belief that the Fourier expansion of $f(x)$ in each case shall be valid. The above discussion has merely shown that if $f(x)$ has an expansion, then the coefficients are given by Euler's formulae. The problems concerning the possibility of expressing a function by Fourier series and convergence of this series are many and cumbersome. Such questions should be left to the curiosity of a pure-mathematician. However, almost all engineering applications are covered by the following well-known **Dirichlet's conditions***:

Any function f(x) can be developed as a Fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ *where* a_0, a_n, b_n *are constants, provided :*

(i) f (x) is periodic, single-valued and finite;

(ii) f (x) has a finite number of discontinuties in any one period;

(iii) f (x) has at the most a finite number of maxima and minima. (Anna, 2009 ; P.T.U., 2009)

In fact the problem of expressing any function $f(x)$ as a Fourier series depends upon the evaluation of the integrals.

$$\frac{1}{\pi}\int f(x)\cos nx\, dx \,;\, \frac{1}{\pi}\int f(x)\sin nx\, dx$$

within the limits $(0, 2\pi)$, $(-\pi, \pi)$ or $(\alpha, \alpha + 2\pi)$ according as $f(x)$ is defined for every value of x in $(0, 2\pi)$, $(-\pi, \pi)$ or $(\alpha, \alpha + 2\pi)$.

PROBLEMS 10.2

State giving reasons whether the following functions can be expanded in Fourier series in the interval $-\pi \le x \le \pi$.

1. cosec x 2. $\sin 1/x$ (P.T.U., 2002)
3. $f(x) = (m+1)/m$, $\pi/(m+1) < |x| \le \pi/m$, $m = 1, 2, 3, \ldots \infty$.

10.4 FUNCTIONS HAVING POINTS OF DISCONTINUITY

In deriving the Euler's formulae for a_0, a_n, b_n, it was assumed that $f(x)$ was continuous. Instead a function may have a finite number of points of finite discontinuity *i.e.*, its graph may consist of a finite number of different curves given by different equations. Even then such a function is expressible as a Fourier series.

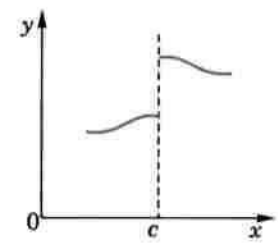


Fig. 10.1

For instance, if in the interval $(\alpha, \alpha + 2\pi)$, $f(x)$ is defined by

$$f(x) = \phi(x),\ \alpha < x < c.$$
$$= \psi(x),\ c < x < \alpha + 2\pi,$$ *i.e.*, c is the point of discontinuity, then

$$a_0 = \frac{1}{\pi}\left[\int_\alpha^c \phi(x)\,dx + \int_c^{\alpha+2\pi} \psi(x)\,dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_\alpha^c \phi(x)\cos nx\,dx + \int_c^{\alpha+2\pi} \psi(x)\cos nx\,dx\right]$$

and $$b_n = \frac{1}{\pi}\left[\int_\alpha^c \phi(x)\sin nx\,dx + \int_c^{\alpha+2\pi} \psi(x)\sin nx\,dx\right]$$

At a point of finite discontinuity $x = c$, there is a finite jump in the graph of function (Fig. 10.1). Both the limit on the left [*i.e.*, $f(c-0)$] and the limit on the right [*i.e.*, $f(c+0)$] exist and are different. At such a point, Fourier series gives the value of $f(x)$ as the arithmetic mean of these two limits,

i.e., at $x = c$, $f(x) = \frac{1}{2}[f(c-0) + f(c+0)]$.

Example 10.5. *Find the Fourier series expansion for f (x), if*

$$f(x) = -\pi,\ -\pi < x < 0$$
$$x,\ 0 < x < \pi.$$ (Bhopal, 2008 S)

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$. (Kottayam, 2005)

*See footnote p. 307.

Solution. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ ...(*i*)

Then $$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\, dx + \int_0^{\pi} x\, dx\right] = \frac{1}{\pi}\left[-\pi \left| x \right|_{-\pi}^{0} + \left| x^2/2 \right|_0^{\pi}\right] = \frac{1}{\pi}\left(-\pi^2 + \frac{\pi^2}{2}\right) = -\frac{\pi}{2};$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi) \cos nx\, dx + \int_0^{\pi} x \cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[-\pi \left| \frac{\sin nx}{n} \right|_{-\pi}^{0} + \left| \frac{x \sin nx}{n} + \frac{\cos nx}{n^2} \right|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{1}{n^2}\cos n\pi - \frac{1}{n^2}\right] = \frac{1}{\pi n^2}(\cos n\pi - 1)$$

$\therefore$ $$a_1 = \frac{-2}{\pi \,.\, 1^2},\ a_2 = 0,\ a_3 = -\frac{2}{\pi \,.\, 3^2},\ a_4 = 0,\ a_5 = -\frac{2}{\pi \,.\, 5^2} \text{ etc.}$$

Finally, $$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi) \sin nx\, dx + \int_0^{\pi} x \sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\left| \frac{\pi \cos nx}{n} \right|_{-\pi}^{0} + \left| -x \frac{\cos nx}{n} + \frac{\sin nx}{n^2} \right|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(1 - \cos n\pi) - \frac{\pi}{n}\cos n\pi\right] = \frac{1}{n}(1 - 2\cos n\pi)$$

$\therefore$ $$b_1 = 3,\ b_2 = -\frac{1}{2},\ b_3 = 1,\ b_4 = -\frac{1}{4}, \text{ etc.}$$

Hence substituting the values of a's and b's in (*i*), we get

$$f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\right) + 3\sin x - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} - \frac{\sin 4x}{4} + \ldots \quad \text{...(ii)}$$

which is the required result.

Putting $x = 0$ in (*ii*), we obtain $f(0) = -\frac{\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots \infty\right)$ ...(*iii*)

Now $f(x)$ is discontinuous at $x = 0$. As a matter of fact

$f(0 - 0) = -\pi$ and $f(0 + 0) = 0$ $\therefore$ $f(0) = \frac{1}{2}[f(0 - 0) + f(0 + 0)] = -\pi/2.$

Hence (*iii*) takes the form $-\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right]$ whence follows the result.

Example 10.6. *If* $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ \sin x, & 0 \le x \le \pi \end{cases}$, *prove that* $f(x) = \frac{1}{\pi} + \frac{\sin x}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2 - 1}$.

Hence show that $\frac{1}{1\,.\,3} - \frac{1}{3\,.\,5} + \frac{1}{5\,.\,7} - \ldots - \infty = \frac{1}{4}(\pi - 2)$ (*Bhopal, 2008 ; Mumbai, 2005 S ; Rohtak, 2005*)

Solution. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

Then $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\, dx + \int_0^{\pi} \sin x\, dx\right] = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,.\, dx + \int_0^{\pi} \sin x \cos nx\, dx\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi} [\sin (n+1)x - \sin (n-1)x]\, dx = \frac{1}{2\pi}\left[-\frac{\cos (n+1)x}{n+1} + \frac{\cos (n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right] \qquad (n \neq 1)$$

$$= \frac{1}{2\pi}\left\{\frac{1-(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1}-1}{n-1}\right\} = 0, \text{ when } n \text{ is odd}$$

$$= -\frac{2}{\pi(n^2-1)}, \text{ when } n \text{ is even.}$$

When $n = 1$, $a_1 = \frac{1}{\pi}\int_0^\pi \sin x \cos x\, dx = \frac{1}{2\pi}\int_0^\pi \sin 2x\, dx = \frac{1}{2\pi}\left| -\frac{\cos 2x}{2}\right|_0^\pi = 0$

Finally, $b_n = \frac{1}{\pi}\int_{-\pi}^\pi f(x) \sin nx\, dx = \frac{1}{\pi}\left[\int_{-\pi}^0 0 \,.\, dx + \int_0^\pi \sin x \sin nx\, dx\right]$

$$= \frac{1}{2\pi}\int_0^\pi [\cos \overline{n-1}x - \cos \overline{n+1}x]\, dx = \frac{1}{2\pi}\left[\frac{\sin \overline{n-1}x}{n-1} - \frac{\sin \overline{n+1}x}{n+1}\right]_0^\pi = 0 \quad (n \neq 1)$$

When $n = 1$, $b_1 = \frac{1}{\pi}\int_0^\pi \sin x \sin x\, dx = \frac{1}{2\pi}\int_0^\pi (1-\cos 2x)\, dx = \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^\pi = \frac{1}{2}$

Hence $f(x) = \frac{1}{\pi} - \frac{2}{\pi}\left[\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \ldots\right] + \frac{1}{2}\sin x \qquad \ldots(i)$

Putting $x = \frac{\pi}{2}$ in (*i*), we get $1 = \frac{1}{\pi} - \frac{2}{\pi}\left(-\frac{1}{1.3} + \frac{1}{3.5} - \frac{1}{5.7} + \ldots \infty\right) + \frac{1}{2}$

Whence $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots \infty = \frac{1}{4}(\pi - 2)$.

Example 10.7. *Find the Fourier series for the function*

$$f(t) = \begin{cases} -1 & \text{for } -\pi < t < -\pi/2 \\ 0 & \text{for } -\pi/2 < t < \pi/2 \\ 1 & \text{for } \pi/2 < t < \pi \end{cases}$$

Solution. Let $f(t) = \frac{a_0}{2} + \sum_{n=1}^\infty a_n \cos nt + \sum_{n=1}^\infty b_n \sin nt \qquad \ldots(i)$

Then $a_0 = \frac{1}{\pi}\left\{\int_{-\pi}^{-\pi/2} (-1) dt + \int_{-\pi/2}^{\pi/2} (0) dt + \int_{\pi/2}^{\pi} (1)\, dt\right\}$

$$= \frac{1}{\pi}\left\{\left| -x\right|_{-\pi}^{-\pi/2} + \left| x\right|_{\pi/2}^{\pi}\right\} = \frac{1}{\pi}(\pi/2 - \pi + \pi - \pi/2) = 0$$

$$a_n = \frac{1}{\pi}\left\{\int_{-\pi}^{-\pi/2} (-1)\cos nt\, dt + \int_{-\pi/2}^{\pi/2} (0)\cos nt\, dt + \int_{\pi/2}^{\pi} (1)\cos nt\, dt\right\}$$

$$= \frac{1}{\pi}\left\{\left| -\frac{\sin nt}{n}\right|_{-\pi}^{-\pi/2} + \left|\frac{\sin nt}{n}\right|_{\pi/2}^{\pi}\right\} = \frac{1}{n\pi}\left(\frac{\sin n\pi}{2} - \frac{\sin n\pi}{2}\right) = 0$$

and $b_n = \frac{1}{\pi}\left\{\int_{-\pi}^{-\pi/2} (-1)\sin nt\, dt + \int_{-\pi/2}^{\pi/2} (0)\sin nt\, dt + \int_{\pi/2}^{\pi} (1)\sin nt\, dt\right\}$

$$= \frac{1}{\pi}\left\{\left|\frac{\cos nt}{n}\right|_{-\pi}^{-\pi/2} + \left| -\frac{\cos nt}{n}\right|_{\pi/2}^{\pi}\right\} = \frac{2}{n\pi}\left(\cos\frac{n\pi}{2} - \cos n\pi\right)$$

$\therefore \qquad b_1 = \frac{2}{\pi}, b_2 = -\frac{2}{\pi}, b_3 = \frac{2}{3\pi}$ etc.

Hence substituting the values of a's and b's in (i), we get $f(t) = \frac{2}{\pi}\left(\sin t - \sin 2t + \frac{1}{3}\sin 3t + \dots\right)$.

PROBLEMS 10.3

1. Find the Fourier series to represent the function $f(x)$ given by

   $f(x) = x$ for $0 \le x \le \pi$, and $= 2\pi - x$ for $\pi \le x \le 2\pi$. *(S.V.T.U., 2008 ; B.P.T.U., 2005 S)*

   Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \infty = \frac{\pi^2}{8}$. *(Madras 2000 S ; V.T.U., 2000 S)*

2. An alternating current after passing through a rectifier has the form

   $$i = I_0 \sin x \quad \text{for } 0 \le x \le \pi$$
   $$= 0 \quad \text{for } \pi \le x \le 2\pi$$

   where I_0 is the maximum current and the period is 2π (Fig. 10.2). Express i as a Fourier series and evaluate

   $$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \dots \infty$$

   (V.T.U., 2007 ; Calicut, 2005)

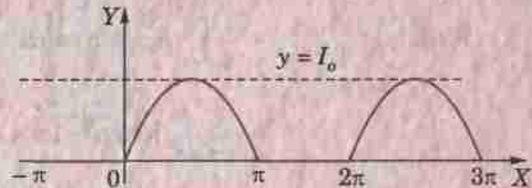


Fig. 10.2

3. Draw the graph of the function $f(x) = 0, -\pi < x < 0$

   $= x^2, 0 < x < \pi$.

   If $f(2\pi + x) = f(x)$, obtain Fourier series of $f(x)$.

4. Find the Fourier series of the following function :

   $$f(x) = x^2, \quad 0 \le x \le \pi,$$
   $$= -x^2, \quad -\pi \le x \le 0.$$

   (Mumbai, 2009) *(Hissar, 2007)*

5. Find a Fourier series for the function defined by

   $$f(x) = \begin{cases} -1, \text{ for } -\pi < x < 0 \\ 0, \text{ for } x = 0 \\ 1, \text{ for } 0 < x < \pi \end{cases}$$

   Hence prove that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \infty = \frac{\pi}{4}$. *(U.P.T.U., 2005)*

10.5 CHANGE OF INTERVAL

In many engineering problems, the period of the function required to be expanded is not 2π but some other interval, say : $2c$. In order to apply the foregoing discussion to functions of period $2c$, this interval must be converted to the length 2π. This involves only a proportional change in the scale.

Consider the periodic function $f(x)$ defined in $(\alpha, \alpha + 2c)$. To change the problem to period 2π

put $\quad z = \pi x/c \quad$ or $\quad x = cz/\pi \qquad$ ...(1)

so that when $\quad x = \alpha, \quad z = \alpha\pi/c = \beta$ (say)

when $\quad x = \alpha + 2c, \quad z = (\alpha + 2c)\pi/c = \beta + 2\pi$.

Thus the function $f(x)$ of period $2c$ in $(\alpha, \alpha + 2c)$ is transformed to the function $f(cz/\pi)$ [$= F(z)$ say] of period 2π in $(\beta, \beta + 2\pi)$. Hence $f(cz/\pi)$ can be expressed as the Fourier series

$$f\left(\frac{cz}{\pi}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nz + \sum_{n=1}^{\infty} b_n \sin nz \qquad \text{...(2)}$$

where

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_{\beta}^{\beta+2\pi} f\left(\frac{cz}{\pi}\right) dz \\ a_n &= \frac{1}{\pi}\int_{\beta}^{\beta+2\pi} f\left(\frac{cz}{\pi}\right) \cos nz\, dz \\ b_n &= \frac{1}{\pi}\int_{\beta}^{\beta+2\pi} f\left(\frac{cz}{\pi}\right) \sin nz\, dz \end{aligned}\right\} \qquad \text{...(3)}$$

Making the inverse substitutions $z = \pi x/c$, $dz = (\pi/c)\,dx$ in (2) and (3) *the Fourier expansion of $f(x)$ in the interval $(\alpha, \alpha + 2c)$ is given by*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c}$$

where
$$\left.\begin{aligned} a_0 &= \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\,dx \\ a_n &= \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\cos\frac{n\pi x}{c}\,dx \\ b_n &= \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\sin\frac{n\pi x}{c}\,dx \end{aligned}\right\} \qquad ...(4)$$

Cor. Putting $\alpha = 0$ in (4), we get the results for the interval $(0, 2c)$ and putting $\alpha = -c$ in (4), we get results for the interval $(-c, c)$.

Example 10.8. *Expand $f(x) = e^{-x}$ as a Fourier series in the interval $(-l, l)$.*

(Kerala, 2005 ; V.T.U., 2004)

Solution. The required series is of the form

$$e^{-x} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad ...(i)$$

Then $$a_0 = \frac{1}{l}\int_{-l}^{l} e^{-x}\,dx = \frac{1}{l}\left|-e^{-x}\right|_{-l}^{l} = \frac{1}{l}(e^{l} - e^{-l}) = \frac{2\sinh l}{l}$$

and $$a_n = \frac{1}{l}\int_{-l}^{l} e^{x}\cos\frac{n\pi x}{l}\,dx \qquad \left[\because \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\right]$$

$$= \frac{1}{l}\left|\frac{e^{-x}}{1+(n\pi/l)^2}\left(-\cos\frac{n\pi x}{l} + \frac{n\pi}{l}\sin\frac{n\pi x}{l}\right)\right|_{-l}^{l} = \frac{2l(-1)^n\sinh l}{l^2+(n\pi)^2} \qquad [\because \cos n\pi = (-1)^n$$

$$\therefore \quad a_1 = \frac{-2l\sinh l}{l^2+\pi^2},\ a_2 = \frac{2l\sinh l}{l^2+2^2\pi^2},\ a_3 = \frac{2l\sinh l}{l^2+3^2\pi^2} \text{ etc.}$$

Finally, $$b_n = \frac{1}{l}\int_{-l}^{l} e^{-x}\sin\frac{n\pi x}{l}\,dx \qquad \left[\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$$

$$= \frac{1}{l}\left|\frac{e^{-x}}{1+(n\pi/l)^2}\left(-\sin\frac{n\pi x}{l} - \frac{n\pi}{l}\cos\frac{n\pi x}{l}\right)\right|_{-l}^{l} = \frac{2n\pi(-1)^n\sinh l}{l^2+(n\pi)^2}$$

$$\therefore \quad b_1 = \frac{-2\pi\sinh l}{l^2+\pi^2},\ b_2 = \frac{4\pi\sinh l}{l^2+2^2\pi^2},\ b_2 = \frac{-6\pi\sinh l}{l^2+3^2\pi^2} \text{ etc.}$$

Substituting the values of a's and b's in (i), we get

$$e^{-x} = \sinh l\left\{\frac{1}{l} - 2l\left(\frac{1}{l^2+\pi^2}\cos\frac{\pi x}{l} - \frac{1}{l^2+2^2\pi^2}\cos\frac{2\pi x}{l} + \frac{1}{l^2+3^2\pi^2}\cos\frac{3\pi x}{l} - \ldots\right)\right.$$
$$\left. - 2\pi\left(\frac{1}{l^2+\pi^2}\sin\frac{n\pi}{l} - \frac{2}{l^2+2^2\pi^2}\sin\frac{2\pi x}{l} + \frac{3}{l^2+3^2\pi^2}\sin\frac{3\pi x}{l} - \ldots\right)\right\}$$

Example 10.9. *Find the Fourier series expansion of $f(x) = 2x - x^2$ in $(0, 3)$ and hence deduce that $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots - \infty = \frac{\pi}{12}$.*

(Mumbai, 2005)

Solution. The required series is of the form

$$2x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x \quad \text{where } l = 3/2. \qquad ...(i)$$

Then
$$a_0 = \frac{1}{l}\int_0^{2l}(2x - x^2)dx = \frac{2}{3}\left| x^2 - \frac{x^3}{3}\right|_0^3 = 0$$

$$a_n = \frac{1}{l}\int_0^{2l}(2x - x^2)\cos\frac{n\pi x}{l}dx = \frac{2}{3}\int_0^3 (2x - x^2)\cos\frac{2n\pi x}{3}dx$$

$$= \frac{2}{3}\left[(2x - x^2)\frac{\sin 2n\pi x/3}{2n\pi/3} - (2 - 2x)\frac{-\cos 2n\pi x/3}{(2n\pi/3)^2} + (-2)\frac{-\sin 2n\pi x/3}{(2n\pi/3)^3}\right]_0^3$$

$$= \frac{2}{3}\cdot\frac{9}{4n^2\pi^2}[(2-6)\cos 2n\pi - 2] = -\frac{9}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l}(2x - x^2)\sin\frac{n\pi x}{l}dx = \frac{2}{3}\int_0^3 (2x - x^2)\sin\frac{2n\pi x}{3}dx$$

$$= \frac{2}{3}\left[(2x - x^2)\frac{-\cos 2n\pi x/3}{2n\pi/3} - (2 - 2x)\frac{-\sin 2n\pi x/3}{(2n\pi/3)^2} + (-2)\frac{\cos 2n\pi x/3}{(2n\pi/3)^3}\right]_0^3$$

$$= \frac{2}{3}\left\{-\frac{6}{n^2\pi^2}\cos 2n\pi - \frac{27}{4n^3\pi^3}(\cos 2n\pi - 1)\right\} = \frac{3}{n\pi}$$

Substituting the values of a_0, a_n, b_n in (i), we get

$$2x - x^2 = -\sum_{n=1}^{\infty}\frac{9}{n^2\pi^2}\cos\frac{2n\pi x}{3} + \sum_{n=1}^{\infty}\frac{3}{n\pi}\sin\frac{2n\pi x}{3}$$

Putting $x = 3/2$, we get

$$3 - \frac{9}{4} = -\sum_{n=1}^{\infty}\frac{9}{n^2\pi^2}\cos n\pi \quad \text{or} \quad -\sum_{n=1}^{\infty}\frac{\cos n\pi}{n^2} = \frac{\pi^2}{9}\cdot\frac{3}{4}$$

or
$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \ldots \infty = \frac{\pi^2}{12}.$$

Example 10.10. *Obtain Fourier series for the function*

$$f(x) = \begin{cases} \pi x, & 0 \le x \le 1 \\ \pi(2 - x), & 1 \le x \le 2 \end{cases}$$

(V.T.U., 2011 ; Bhopal, 2008 ; Mumbai, 2007)

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \infty = \frac{\pi^2}{8}$.

Solution. The required series is of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

Then
$$a_0 = \int_0^1 \pi x dx + \int_1^2 \pi(2 - x)dx = \pi\left|\frac{x^2}{2}\right|_0^1 + \pi\left|2x - \frac{x^2}{2}\right|_1^2 = \pi\left(\frac{1}{2}\right) + \pi\left\{(4-2) - \left(2 - \frac{1}{2}\right)\right\} = \pi$$

$$a_n = \int_0^1 \pi x \cos n\pi x dx + \int_1^2 \pi(2 - x)\cos n\pi x dx$$

$$= \left|\pi x\cdot\frac{\sin n\pi x}{n\pi} - \pi\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_0^1 + \left|\pi(2-x)\frac{\sin n\pi x}{n\pi} - (-\pi)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_1^2$$

$$= \left(\frac{\cos n\pi}{n^2\pi} - \frac{1}{n^2\pi^2}\right) - \left(\frac{\cos 2n\pi}{n^2\pi} - \frac{\cos n\pi}{n^2\pi}\right) = \frac{2}{n^2\pi}[(-1)^n - 1]$$

$= 0$ when n is even ; $-\dfrac{4}{n^2\pi}$ when n is odd.

$$b_n = \int_0^1 \pi x \sin n\pi x \, dx + \int_1^2 \pi(2-x)\sin n\pi x \, dx$$

$$= \left| \pi x\left(-\frac{\cos n\pi x}{n\pi}\right) - \pi\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_0^1 + \left|\pi(2-x)\left(\frac{-\cos n\pi x}{n\pi}\right) - (-\pi)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_1^2$$

$$= \left(-\frac{\cos n\pi}{n}\right) + \left(\frac{\cos n\pi}{n}\right) = 0$$

Hence $\quad f(x) = \dfrac{\pi}{2} - \dfrac{4}{\pi}\left(\dfrac{\cos \pi x}{1^2} + \dfrac{\cos 3\pi x}{3^2} + \dfrac{\cos 5\pi x}{5^2} + \ldots \infty\right)$

Putting $x = 2$, $0 = \dfrac{\pi}{2} - \dfrac{4}{\pi}\left(\dfrac{\cos 2\pi}{1^2} + \dfrac{\cos 6\pi}{3^2} + \dfrac{\cos 10\pi}{5^2} + \ldots \infty\right)$

Whence $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots \infty = \dfrac{\pi^2}{8}$.

Example 10.11. *Find the Fourier series for*

$$f(t) = 0, \; -2 < t < -1$$
$$= 1 + t, \; -1 < t < 0$$
$$= 1 - t, \; 0 < t < 1$$
$$= 0, \quad 1 < t < 2.$$

Solution. Let $\quad f(t) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{2} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{2} \quad \ldots(i)$

[$\because \quad 2c = 2 - (-2)$ so that $c = 2$]

Then $\quad a_0 = \dfrac{1}{2}\left\{\displaystyle\int_{-2}^{-1}(0)dt + \int_{-1}^{0}(1+t)dt + \int_0^1 (1-t)dt + \int_1^2 (0)dt\right\} = \dfrac{1}{2}\left\{\left|t + \dfrac{t^2}{2}\right|_{-1}^0 + \left|t - \dfrac{t^2}{2}\right|_0^1\right\}$

$$= \frac{1}{2}\left\{-\left(-1+\frac{1}{2}\right) + \left(1 - \frac{1}{2}\right)\right\} = \frac{1}{2}$$

$$a_n = \frac{1}{2}\left\{\int_{-1}^0 (1+t)\cos\frac{n\pi t}{2}dt + \int_0^1 (1-t)\cos\frac{n\pi t}{2}dt\right\} \quad \text{[Integrate by parts]}$$

$$= \frac{1}{2}\left\{\left|(1+t)\left(\sin\frac{n\pi t}{2}\right)\frac{2}{n\pi} - (1)\left(-\cos\frac{n\pi t}{2}\right)\frac{4}{n^2\pi^2}\right|_{-1}^0\right.$$
$$\left. + \left|(1-t)\left(\sin\frac{n\pi t}{2}\right)\frac{2}{n\pi} - (-1)\left(-\cos\frac{n\pi t}{2}\right)\frac{4}{n^2\pi^2}\right|_0^1\right\}$$

$$= \frac{4}{n^2\pi^2}(1 - \cos n\pi/2)$$

$$b_n = \frac{1}{2}\left\{\int_{-1}^0 (1+t)\sin\frac{n\pi t}{2}dt + \int_0^1 (1-t)\sin\frac{n\pi t}{2}dt\right\}$$

$$= \frac{1}{2}\left\{\left|(1+t)\left(-\cos\frac{n\pi t}{2}\right)\frac{2}{n\pi} - 1\left(-\sin\frac{n\pi t}{2}\right)\frac{4}{n^2\pi^2}\right|_{-1}^0\right.$$
$$\left. + \left|(1-t)\left(-\cos\frac{n\pi t}{2}\right)\frac{2}{n\pi} - (-1)\left(-\sin\frac{n\pi t}{2}\right)\frac{4}{n^2\pi^2}\right|_0^1\right\}$$

$$= \frac{1}{2}\left\{\frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right) - \left(\frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right\} = 0$$

Substituting the values of a's and b's in (i), we get

$$f(t) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left(1 - \cos\frac{n\pi}{2}\right)\cos\frac{n\pi t}{2}.$$

PROBLEMS 10.4

1. Obtain the Fourier series for $f(x) = \pi x$ in $0 \le x \le 2$.
2. (i) Find the Fourier series to represent x^2 in the interval $(0, a)$. *(Mumbai, 2009)*
   (ii) Find a Fourier series for $f(t) = 1 - t^2$ when $-1 \le t \le 1$. *(Mumbai, 2006)*
3. If $f(x) = 2x - x^2$ in $0 \le x \le 2$, show that $f(x) = \frac{2}{3} - \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\cos n\pi x$. *(V.T.U., 2006)*
4. Find the Fourier series for $f(x) = \begin{cases} x & \text{in } 0 \le x \le 3 \\ 6 - x & \text{in } 3 \le x \le 6 \end{cases}$ *(Anna, 2008)*
5. A sinusoidal voltage $E \sin \omega t$ is passed through a half-wave rectifier which clips the negative portion of the wave. Develop the resulting periodic function
   $U(t) = 0$ when $-T/2 < t < 0$
   $= E \sin \omega t$ when $0 < t < T/2$,
   and $T = 2\pi/\omega$, in a Fourier series. *(Calicut, 1999)*
6. Find the Fourier series of the function $f(x) = \begin{cases} \pi x, & 0 < x < 1 \\ 0, & x = 1 \\ \pi(x - 2), & 1 < x < 2 \end{cases}$

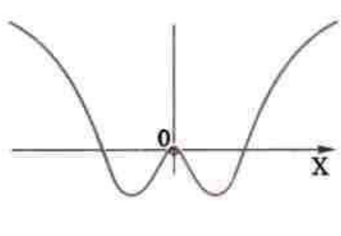
   Hence show that $\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ *(Mumbai, 2008)*

10.6 (1) EVEN AND ODD FUNCTIONS

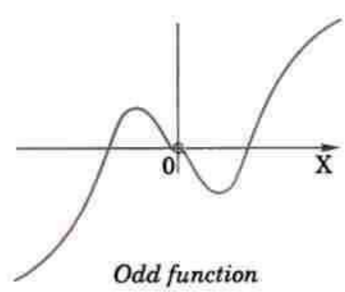
A function $f(x)$ is said to be **even** *if $f(-x) = f(x)$,*

e.g., $\cos x$, $\sec x$, x^2 are all even functions. Graphically an even function is symmetrical about the y-axis.

A function $f(x)$ is said to be **odd** *if $f(-x) = -f(x)$,*



Even function



Odd function

Fig. 10.3

e.g. $\sin x$, $\tan x$, x^3 are odd functions. Graphically, an odd function is symmetrical about the origin.

We shall be using the following property of definite integrals in the next paragraph :

$$\int_{-c}^{c} f(x)\,dx = 2\int_0^c f(x)\,dx, \text{ when } f(x) \text{ is an even function.}$$

$$= 0, \text{ when } f(x) \text{ is an odd function.}$$

(2) Expansions of even or odd periodic functions. We know that a periodic function $f(x)$ defined in $(-c, c)$ can be represented by the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c},$$

where $$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)dx,\ a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}dx,\ b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}dx.$$

Case I. *When f(x) is an even function* $a_0 = \frac{1}{c}\int_{-c}^{c} f(x)dx = \frac{2}{c}\int_0^c f(x)dx.$

Since $f(x)\cos\frac{n\pi x}{c}$ is also an even function,

$$\therefore \quad a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}dx = \frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{c}dx$$

Again since $f(x)\sin\frac{n\pi x}{c}$ is an odd function, $\therefore \quad b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}dx = 0.$

Hence, *if a periodic function f (x) is even, its Fourier expansion contains only cosine terms, and*

$$\left.\begin{aligned} a_0 &= \frac{2}{c}\int_0^c f(x)dx \\ a_n &= \frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{c}dx \end{aligned}\right\} \quad ...(1)$$

Case II. *When f(x) is an odd function,* $a_0 = \frac{1}{c}\int_{-c}^{c} f(x)dx = 0,$

Since $\cos\frac{n\pi x}{c}$ is an even function, therefore, $f(x)\cos\frac{n\pi x}{c}$ is an odd function.

$$\therefore \quad a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c} = 0$$

Again since $\sin\frac{n\pi x}{c}$ is an odd function, therefore, $f(x)\sin\frac{n\pi x}{c}$ is an even function.

$$\therefore \quad b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}dx = \frac{2}{c}\int_0^c f(x)\sin\frac{n\pi x}{c}dx$$

Thus, *if a periodic function f (x) is odd, its Fourier expansion contains only sine terms and*

$$b_n = \frac{2}{c}\int_0^c f(x)\sin\frac{n\pi x}{c}dx \quad ...(2)$$

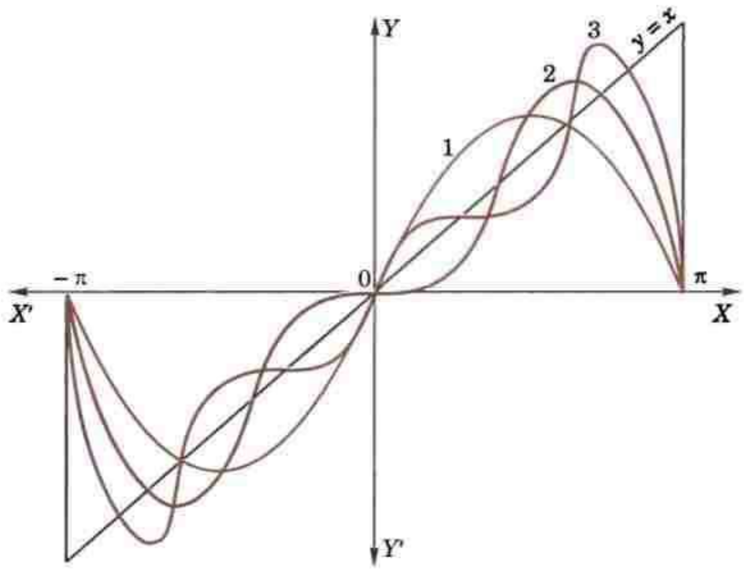


Fig. 10.4

Example 10.12. *Express* $f(x) = x/2$ *as a Fourier series in the interval* $-\pi < x < \pi$. (*J.N.T.U., 2006*)

Solution. Since $f(-x) = -x/2 = -f(x)$.

$\therefore$ $f(x)$ is an odd function and hence $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

where $$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} \frac{x}{2}\sin nx\,dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi} = -\frac{\cos n\pi}{n}.$$

$\therefore$ $b_1 = 1/1, b_2 = -1/2, b_3 = 1/3, b_4 = -1/4$, etc.

Hence the series is $x/2 = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \ldots$ ...(*i*)

Obs. The graphs of $y = 2\sin x$, $y = 2(\sin x - \frac{1}{2}\sin 2x)$ and $y = 2(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x)$ are shown in Fig. 10.4, by the curves 1, 2 and 3 respectively. These illustrate the manner in which the successive approximations to the series (*i*) approach more and more closely to $y = x$ for all values of x in $-\pi < x < \pi$, but not for $x = \pm\pi$.

As the series has a period 2π, it represents the discontinuous function, called *saw-toothed waveform*, shown in Fig. 10.5. It is important to note that the given function $y = x$ is continuous and each term of the series (*i*) is continuous, but the function represented by the series (*i*) has finite discontinuities at $x = \pm\pi, \pm 3\pi, \pm 5\pi$ etc.

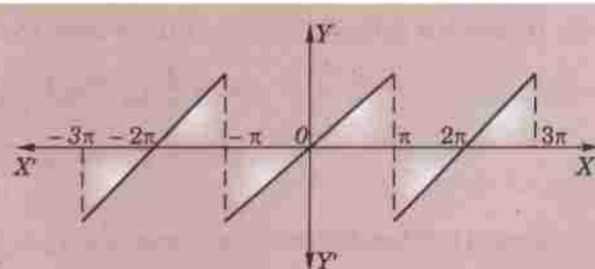


Fig. 10.5

Example 10.13. *Find a Fourier series to represent x^2 in the interval $(-l, l)$.* (S.V.T.U., 2008)

Solution. Since $f(x) = x^2$ is an even function in $(-l, l)$,

$$\therefore \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \qquad ...(i)$$

Then $$a_0 = \frac{2}{l}\int_0^l x^2\,dx = \frac{2}{l}\left|\frac{x^3}{3}\right|_0^l = \frac{2l^2}{3}$$

$$a_n = \int_0^l x^2 \cos\frac{n\pi x}{l}\,dx$$ [*See* footnote p. 398]

$$= \frac{2}{l}\left[x^2\left(\frac{\sin n\pi x/l}{n\pi/l}\right) - 2x\left(-\frac{\cos n\pi x/l}{n^2\pi^2/l^2}\right) + 2\left(-\frac{\sin n\pi x/l}{n^3\pi^3/l^3}\right)\right]_0^l$$

$$= 4l^2(-1)^n/n^2\pi^2$$ [$\because \cos n\pi = (-1)^n$]

$\therefore$ $a_1 = -4l^2/\pi^2, a_2 = 4l^2/2^2\pi^2, a_3 = -4l^2/3^2\pi^2, a_4 = 4l^2/4^2\pi^2$ etc.

Substituting these values in (*i*), we get

$$x^2 = \frac{l^2}{3} - \frac{4l^2}{\pi^2}\left(\frac{\cos \pi x/l}{1^2} - \frac{\cos 2\pi x/l}{2^2} + \frac{\cos 3\pi x/l}{3^2} - \frac{\cos 4\pi x/l}{4^2} + \ldots\right)$$

which is the required Fourier series.

Example 10.14. *If $f(x) = |\cos x|$, expand $f(x)$ as a Fourier series in the interval $(-\pi, \pi)$.*

Solution. As $f(-x) = |\cos(-x)| = |\cos x| = f(x)$, $|\cos x|$ is an even function.

$$\therefore \quad f(x) = \frac{a_0}{2} + \Sigma a_n \cos nx$$

where $$a_0 = \frac{2}{\pi}\int_0^{\pi} |\cos x|\,dx = \frac{2}{\pi}\int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi}(-\cos x)\,dx$$

[$\because$ $\cos x$ is $-$ve when $\pi/2 < x < \pi$]

$$= \frac{2}{\pi}\left\{\left| \sin x \right|_0^{\pi/2} - \left| \sin x \right|_{\pi/2}^{\pi}\right\} = \frac{2}{\pi}[(1-0)-(0-1)] = \frac{4}{\pi}$$

and $$a_n = \frac{2}{\pi}\int_0^{\pi} |\cos x| \cos nx \, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x \cos nx \, dx + \int_{\pi/2}^{\pi} (-\cos x) \cos nx \, dx\right]$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi/2} [\cos (n+1)x + \cos (n-1)x] \, dx - \int_{\pi/2}^{\pi} [\cos (n+1)x + \cos (n-1)x] \, dx\right\}$$

$$= \frac{1}{\pi}\left\{\left| \frac{\sin (n+1)x}{n+1} + \frac{\sin (n-1)x}{n-1} \right|_0^{\pi/2} - \left| \frac{\sin (n+1)x}{n+1} + \frac{\sin (n-1)x}{n-1} \right|_{\pi/2}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left[\left\{\frac{\sin (n+1)\pi/2}{n+1} + \frac{\sin (n-1)\pi/2}{n-1}\right\} + \left\{\frac{\sin (n+1)\pi/2}{n+1} + \frac{\sin (n-1)\pi/2}{n-1}\right\}\right]$$

$$= \frac{2}{\pi}\left(\frac{\cos n\pi/2}{n+1} - \frac{\cos n\pi/2}{n-1}\right) = \frac{-4 \cos n\pi/2}{\pi(n^2-1)} \; (n \neq 1)$$

In particular $a_1 = \frac{2}{\pi}\left[\int_0^{\pi/2} \cos^2 x \, dx - \int_{\pi/2}^{\pi} \cos^2 x \, dx\right] = 0$

Hence $|\cos x| = \frac{2}{\pi} + \frac{4}{\pi}\left\{\frac{1}{3}\cos 2x - \frac{1}{15}\cos 4x + \ldots\right\}$.

Example 10.15. *Obtain Fourier series for the function f(x) given by*

$$f(x) = 1 + 2x/\pi, \qquad -\pi \le x \le 0,$$
$$= 1 - 2x/\pi, \qquad 0 \le x \le \pi.$$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$. *(V.T.U., 2010 ; Mumbai, 2007)*

Solution. Since $f(-x) = 1 - \frac{2x}{\pi}$ in $(-\pi, 0) = f(x)$ in $(0, \pi)$

and $f(-x) = 1 + \frac{2x}{\pi}$ in $(0, \pi) = f(x)$ in $(-\pi, 0)$

$\therefore$ $f(x)$ is an even function in $(-\pi, \pi)$. This is also clear from its graph $A'BA$ (Fig. 10.6) which is symmetrical about the y-axis.

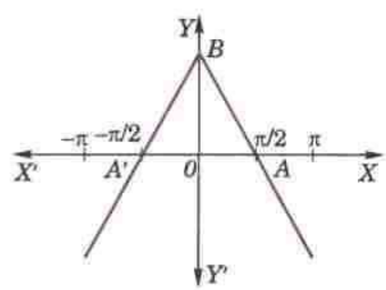


Fig. 10.6

$$\therefore \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad \ldots(i)$$

where $$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)dx = \frac{2}{\pi}\left(x - \frac{x^2}{\pi}\right)_0^{\pi} = 0$$

and $$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \, dx = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)\cos nx \, dx$$

$$= \frac{2}{\pi}\left|\left(1 - \frac{2x}{\pi}\right)\frac{\sin nx}{n} - \left(-\frac{2}{\pi}\right)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{\pi} = \frac{2}{\pi}\left(-\frac{2\cos n\pi}{\pi n^2} + \frac{2}{\pi n^2}\right) = \frac{4}{n^2\pi^2}[1 - (-1)^n]$$

$\therefore$ $a_1 = 8/\pi^2, a^3 = 8/3^2\pi^2, a_5 = 8/5^2\pi^2, \ldots$

and $a_2 = a_4 = a_6 = \ldots = 0.$

Thus substituting the values of a's in (i), we get

$$f(x) = \frac{8}{\pi^2}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right] \qquad \ldots(ii)$$

as the required Fourier expansion

Putting $x = 0$ in (*ii*), we get $1 = f(0) = \dfrac{8}{\pi^2}\left(\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots\right)$

whence follows the desired result.

PROBLEMS 10.5

1. Obtain the Fourier series expansion of $f(x) = x^2$ in $(0, a)$. Hence show that
$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$
(*Mumbai, 2009 ; S.V.T.U., 2008*)

2. Show that for $-\pi < x < \pi$, $\sin ax = \dfrac{2\sin a\pi}{\pi}\left(\dfrac{\sin x}{1^2 - a^2} - \dfrac{2\sin 2x}{2^2 - a^2} + \dfrac{3\sin 3x}{3^2 - a^2} - \ldots\right)$

3. Expand the function $f(x) = x \sin x$ as a Fourier series in the interval $-\pi \le x \le \pi$. (*V.T.U., 2008 ; Anna, 2003*)

   Deduce that $\dfrac{1}{1.3} - \dfrac{1}{3.5} + \dfrac{1}{5.7} - \dfrac{1}{7.9} + \ldots = \dfrac{1}{4}(\pi - 2)$. (*U.P.T.U., 2005*)

4. Prove that in the interval $-\pi < x < \pi$, $x \cos x = -\dfrac{1}{2}\sin x + 2\displaystyle\sum_{n=2}^{\infty} \frac{n(-1)^n}{n^2 - 1}\sin nx$. (*S.V.T.U., 2009*)

5. For a function $f(x)$ defined by $f(x) = |x|$, $-\pi < x < \pi$, obtain a Fourier series. (*Bhopal, 2007 ; V.T.U., 2004*)

   Deduce that $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \dfrac{1}{7^2} + \ldots = \dfrac{\pi^2}{8}$. (*S.V.T.U., 2009 ; Kerala, 2005 ; P.T.U., 2005*)

6. Find the Fourier series to represent the function

   (*i*) $f(x) = |\sin x|$, $-\pi < x < \pi$. (*Mumbai, 2008*)

   (*ii*) $f(x) = |\cos(\pi x/l)|$ in the interval $(-1, 1)$. (*P.T.U., 2009 S*)

7. Given $f(x) = \begin{cases} -x + 1 \text{ for } -\pi \le x \le 0, \\ x + 1 \text{ for } 0 \le x \le \pi. \end{cases}$

   Is the function even or odd? Find the Fourier series for $f(x)$ and deduce the value of
$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

8. Find the Fourier series of the periodic function $f(x)$: $f(x) = -k$ when $-\pi < x < 0$ and $f(x) = k$ when $0 < x < \pi$, and $f(x + 2\pi) = f(x)$. Sketch the graph of $f(x)$ and the two partial sums. (See Fig. 10.7)

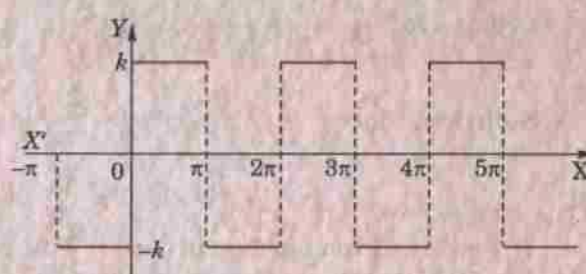


Fig. 10.7

   Deduce that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n - 1} = \frac{\pi}{4}$. (*Rohtak, 2005*)

9. A function is defined as follows :
$$f(x) = -x \text{ when } -\pi < x \le 0 = x \quad \text{when } 0 < x < \pi.$$
   Show that $f(x) = \dfrac{\pi}{2} - \dfrac{4}{\pi}\left[\dfrac{1}{1^2}\cos x + \dfrac{1}{3^2}\cos 3x + \dfrac{1}{5^2}\cos 5x + \ldots\right]$

   Deduce that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n - 1)^2} = \frac{\pi^2}{8}$.

10.7 HALF RANGE SERIES

Many a time is is required to obtain a Fourier expansion of a function $f(x)$ for the range $(0, c)$ which is half the period of the Fourier series. As it is immaterial whatever the function may be outside the range $0 < x < c$, we extend the function to cover the range $-c < x < c$ so that the new function may be odd or even. The Fourier expansion of such a function of half the period, therefore, consists of sine or cosine terms only. In such cases the graphs for the values of x in $(0, c)$ are the same but outside $(0, c)$ are different for odd or even functions. That is why we get different forms of series for the same function as is clear from the examples 10.16 and 10.17.

Sine series. If it be required to expand $f(x)$ as a sine series in $0 < x < c$; then we extend the function reflecting it in the origin, so that $f(x) = -f(-x)$.

Then the extended function is odd in $(-c, c)$ and the expansion will give the desired Fourier sine series :

$$\left.\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{c} \\ \text{where} \quad b_n &= \frac{2}{c}\int_0^c f(x) \sin \frac{n\pi x}{c}\, dx \end{aligned}\right\} \quad ...(1)$$

Cosine series. If it be required to express $f(x)$ as a cosine series in $0 < x < c$, we extend the function reflecting it in the y-axis, so that $f(-x) = f(x)$.

Then the extended function is even in $(-c, c)$ and its expansion will give the required Fourier cosine series :

$$\left.\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{c} \\ \text{where} \quad a_0 &= \frac{2}{c}\int_0^c f(x)\, dx \\ \text{and} \quad a_n &= \frac{2}{c}\int_0^c f(x) \cos \frac{n\pi x}{c}\, dx \end{aligned}\right\} \quad ...(2)$$

Example 10.16. *Express $f(x) = x$ as a half-range sine series in $0 < x < 2$.* *(U.P.T.U., 2004)*

Solution. The graph of $f(x) = x$ in $0 < x < 2$ is the line OA. Let us extend the function $f(x)$ in the interval $-2 < x < 0$ (shown by the line BO) so that the new function is symmetrical about the origin and, therefore, *represents an odd function in* $(-2, 2)$ (*Fig. 10.8*)

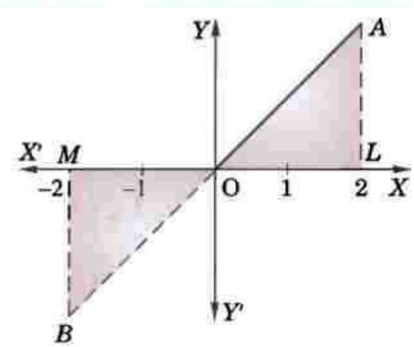


Fig. 10.8

Hence the Fourier series for $f(x)$ over the full period $(-2, 2)$ will contain only sine terms given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2}$$

where
$$b_n = \frac{2}{2}\int_0^2 f(x) \sin \frac{n\pi x}{2}\, dx = \int_0^2 x \sin \frac{n\pi x}{2}\, dx$$

$$= \left| -\frac{2x}{n\pi} \cos \frac{n\pi x}{2} + \frac{4}{n^2\pi^2} \sin \frac{n\pi x}{2} \right|_0^2 = -\frac{4(-1)^n}{n\pi}$$

Thus $\quad b_1 = 4/\pi,\ b_2 = -4/2\pi,\ b_3 = 4/3\pi,\ b_4 = -4/4\pi$ etc.

Hence the Fourier sine series for $f(x)$ over the half-range $(0, 2)$ is

$$f(x) = \frac{4}{\pi}\left(\sin \frac{\pi x}{2} - \frac{1}{2} \sin \frac{2\pi x}{2} + \frac{1}{3} \sin \frac{3\pi x}{2} - \frac{1}{4} \sin \frac{4\pi x}{2} + \ldots \right).$$

Example 10.17. *Express $f(x) = x$ as a half-range cosine series in $0 < x < 2$.*
(S.V.T.U., 2009 ; Bhopal, 2007 ; Mumbai, 2006)

Solution. The graph of $f(x) = x$ in $(0, 2)$ is the line OA. Let us extend the function $f(x)$ in the interval $(-2, 0)$ shown by the line OB' so that the new function is symmetrical about the y-axis and, therefore, *represents an even function in* $(-2, 2)$. (Fig. 10.9)

Hence the Fourier series for $f(x)$ over the full period $(-2, 2)$ will contain only cosine terms given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2}$$

where $$a_0 = \frac{2}{2}\int_0^2 f(x)dx = \int_0^2 x\,dx = 2$$

and $$a_n = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}dx = \int_0^2 x\cos\frac{n\pi x}{2}dx$$

$$= \left|\frac{2x}{n\pi}\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right|_0^2 = \frac{4}{n^2\pi^2}[(-1)^n - 1]$$

Thus $a_1 = -8/\pi^2, a_2 = 0, a_3 = -8/3^2\pi^2, a_4 = 0, a_5 = -8/5^2\pi^2$ etc.

Hence the desired Fourier series for $f(x)$ over the half-range (0, 2) is

$$f(x) = 1 - \frac{8}{\pi^2}\left[\frac{\cos \pi x/2}{1^2} + \frac{\cos 3\pi x/2}{3^2} + \frac{\cos 5\pi x/2}{5^2} + \dots\right]$$

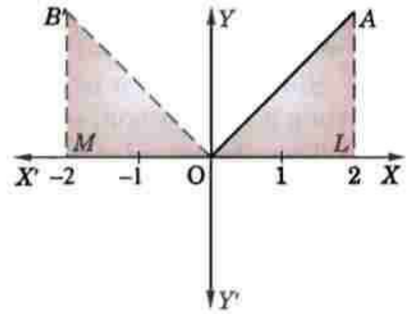


Fig. 10.9

Important Obs. It must be clearly understood that we expand a function in $0 < x < c$ as a series of sines or cosines, merely looking upon it as an odd or even function of period $2c$. It hardly matters whether the function is odd or even or neither.

Example 10.18. *Obtain the Fourier expansion of $x \sin x$ as a cosine series in $(0, \pi)$.* (V.T.U., 2003 ; U.P.T.U., 2002)

Hence show that $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots \infty = \frac{\pi - 2}{4}$. (Anna, 2001)

Solution. Let $x \sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

Then $$a_0 = \frac{2}{\pi}\int_0^\pi x\sin x\,dx = \frac{2}{\pi}\left| x(-\cos x) - 1(-\sin x)\right|_0^\pi = 2$$

$$a_n = \frac{2}{\pi}\int_0^\pi x\sin x\cos nx\,dx = \frac{1}{\pi}\int_0^\pi x(\sin\overline{n+1}\,x - \sin\overline{n-1}\,x)\,dx$$

$$= \frac{1}{\pi}\left[x\left\{\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - 1.\left\{\frac{-\sin(n+1)x}{(n+1)^2} - \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^\pi$$

$$= \frac{1}{\pi}\pi\left\{\frac{\cos(n-1)\pi}{n-1} - \frac{\cos(n+1)\pi}{n+1}\right\}(n \neq 1).$$

When $n = 1$, $$a_1 = \frac{2}{\pi}\int_0^\pi x\sin x\cos x\,dx = \frac{1}{\pi}\int_0^\pi x\sin 2x\,dx$$

$$= \frac{1}{\pi}\left| x\left(\frac{-\cos 2x}{2}\right) - 1\left(\frac{-\sin 2x}{2}\right)\right|_0^\pi = \frac{1}{\pi}\left(-\frac{\pi\cos 2\pi}{2}\right) = -\frac{1}{2}.$$

Hence $$x\sin x = 1 - \frac{1}{2}\cos x - 2\left\{\frac{\cos 2x}{1.3} - \frac{\cos 3x}{3.5} + \frac{\cos 4x}{5.7} - \dots\infty\right\}$$

Putting $x = \pi/2$, we obtain $$\pi/2 = 1 + 2\left\{\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots\infty\right\}$$

Hence $$\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots\infty = \frac{\pi - 2}{4}.$$

Example 10.19. *Obtain a half range cosine series for*

$$f(x) = \begin{cases} kx, & 0 \le x \le l/2 \\ k(l - x), & l/2 \le x \le l. \end{cases}$$ (Bhopal, 2008 ; V.T.U., 2008)

Deduce the sum of the series $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \infty$ (Rohtak, 2006 ; U.P.T.U., 2003)

Solution. Let the half-range cosine series be

$$f(x) = \frac{a_0}{2} + \sum_{n+1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

Then $$a_0 = \frac{2}{l}\left\{\int_0^{l/2} kx\,dx + \int_{l/2}^{l} k(l-x)\right\}dx = \frac{2k}{l}\left\{\left|\frac{x^2}{2}\right|_0^{l/2} - \left|\frac{(l-x)^2}{2}\right|_{l/2}^{l}\right\}$$

$$= \frac{2k}{l}\cdot\frac{1}{2}\left\{\frac{l^2}{4} - \left(0 - \frac{l^2}{4}\right)\right\} = \frac{kl}{2}$$

$$a_n = \frac{2}{l}\left\{\int_0^{l/2} kx\cos\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} k(l-x)\cos\frac{n\pi x}{l}\,dx\right\}$$

$$= \frac{2k}{l}\left|x\left(\frac{\sin n\pi x/l}{n\pi/l}\right) - 1\left\{-\cos\frac{n\pi x/l}{(n\pi/l)^2}\right\}\right|_0^{l/2}$$

$$+ \frac{2k}{l}\left|\left\{\frac{(l-x)\sin n\pi x/l}{n\pi/l}\right\} - (-1)\left(\frac{-\cos n\pi x/l}{(n\pi/l)^2}\right)\right|_{l/2}^{l}$$

$$= \frac{2k}{l}\left[\left(\frac{l^2}{2n\pi}\cdot\sin\frac{n\pi}{2}\right) + \frac{l^2}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - \cos 0\right)\right] + \frac{2k}{l}\left[\left(\frac{l}{n\pi}\left(-\frac{l}{2}\sin\frac{n\pi}{2}\right)\right.\right.$$

$$\left.\left. - \frac{l^2}{n^2\pi^2}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)\right)\right]$$

$$= \frac{2k}{l}\cdot\frac{l^2}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - 1 - (-1)^n\right] = \frac{2kl}{n^2\pi^2}\left\{2\cos\frac{n\pi}{2} - 1 - (-1)^n\right\}$$

Hence the required Fourier series is

$$f(x) = \frac{kl}{4} - \frac{8kl}{\pi^2}\left[\frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{6^2}\cos\frac{6\pi x}{l} + \frac{1}{10^2}\cos\frac{10\pi x}{l} + \dots\right]$$

Putting $x = l$, we get

$$0 = \frac{kl}{4} - \frac{8kl}{\pi^2}\left(\frac{1}{2^2} + \frac{1}{6^2} + \frac{1}{10^2} + \dots\infty\right)$$

Thus $$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\infty = \frac{\pi^2}{8}.$$

Example 10.20. *Expand* $f(x) = \frac{1}{4} - x$, *if* $0 < x < \frac{1}{2}$,

$$= x - \frac{3}{4},\ \text{if } \frac{1}{2} < x < 1,$$

as the Fourier series of sine terms. *(V.T.U., 2011; Andhra, 2000)*

Solution. Let $f(x)$ represent an odd function in $(-1, 1)$ so that $f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$

where $$b_n = \frac{2}{1}\int_0^1 f(x)\sin n\pi x\,dx$$

$$= 2\left[\int_0^{\frac{1}{2}}\left(\frac{1}{4} - x\right)\sin n\pi x\,dx + \int_{\frac{1}{2}}^{1}\left(x - \frac{3}{4}\right)\sin n\pi x\,dx\right]$$

$$= 2\left|-\left(\frac{1}{4} - x\right)\frac{\cos n\pi x}{n\pi} - \frac{\sin n\pi x}{n^2\pi^2}\right|_0^{\frac{1}{2}} + 2\left|-\left(x - \frac{3}{4}\right)\frac{\cos n\pi x}{n\pi} + \frac{\sin n\pi x}{n^2\pi^2}\right|_{\frac{1}{2}}^{1}$$

$$= 2\left[\frac{1}{4n\pi}\cos\frac{n\pi}{2} + \frac{1}{4n\pi} - \frac{\sin n\pi/2}{n^2\pi^2}\right] + 2\left[-\frac{1}{4n\pi}\cos n\pi - \frac{1}{4n\pi}\cos\frac{n\pi}{2} - \frac{\sin n\pi/2}{n^2\pi^2}\right]$$

$$= \frac{1}{2n\pi}[1 - (-1)^n] - \frac{4\sin n\pi/2}{n^2\pi^2}$$

Thus $$b_1 = \frac{1}{\pi} - \frac{4}{\pi^2};\ b_2 = 0$$
$$b_3 = \frac{1}{3\pi} + \frac{4}{3^2\pi^2};\ b_4 = 0$$
$$b_5 = \frac{1}{5\pi} - \frac{4}{5^2\pi^2};\ b_6 = 0 \text{ etc.}$$

Hence $$f(x) = \left(\frac{1}{\pi} - \frac{4}{\pi^2}\right)\sin \pi x + \left(\frac{1}{3\pi} + \frac{4}{3^2\pi^2}\right)\sin 3\pi x + \left(\frac{1}{5\pi} - \frac{4}{5^2\pi^2}\right)\sin 5\pi x + \cdots$$

PROBLEMS 10.6

1. Show that a constant c can be expanded in an infinite series $\frac{4c}{\pi}\left\{\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\right\}$ in the range $0 < x < \pi$. *(Marathwada, 2008 ; Kerala, 2005)*
2. Obtain cosine and sine series for $f(x) = x$ in the interval $0 \le x \le \pi$. Hence show that
$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}.$$ *(Osmania, 2003 S)*
3. Find the half-range cosine series for the function $f(x) = x^2$ in the range $0 \le x \le \pi$. *(B.P.T.U., 2005 ; Kottayam, 2005)*
4. Find the Fourier cosine series of the function $f(x) = \pi - x$ in $0 < x < \pi$. Hence show that
$$\sum_{r=0}^{\infty} \frac{1}{(2r+1)^2} = \frac{\pi^2}{8}$$ *(West Bengal, 2004)*
5. Find the half-range cosine series for the function $f(x) = (x-1)^2$ in the interval $0 < x < 1$. *(V.T.U., 2010 ; J.N.T.U., 2006)*

   Hence show that $\pi^2 = 8\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right)$ *(Anna, 2003)*
6. Find the half-range sine series for the function $f(t) = t - t^2$, $0 < t < 1$.
7. Represent $f(x) = \sin(\sin(\pi x/l))$, $0 < x < l$ by a half-range cosine series. *(Mumbai, 2009)*
8. Find the half range sine series for $f(x) = x \cos x$ in $(0, \pi)$. *(Anna, 2008 S)*
9. Obtain the half-range sine series for e^x in $0 < x < 1$.
10. Find the half range Fourier sine series of $f(x) = x(\pi - x)$, $0 \le x \le \pi$ and hence deduce that

    (*i*) $\sum \frac{1}{n^4} = \frac{\pi^4}{90}$ *(Anna, 2009)* (*ii*) $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} = \frac{\pi^6}{960}$ *(Mumbai, 2005)*
11. If $f(x) = x$, $0 < x < \pi/2$
    $= \pi - x$, $\pi/2 < x < \pi$,

    show that (*i*) $f(x) = \frac{4}{\pi}\left[\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \ldots\right]$ *(Mumbai, 2008 ; S.V.T.U., 2008 ; V.T.U., 2004)*

    (*ii*) $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2}\cos 2x + \frac{1}{3^2}\cos 6x + \frac{1}{5^2}\cos 10x + \ldots\right]$ *(V.T.U., 2011)*
12. Find the half-range cosine series expansion of the function $f(x) = \begin{cases} 0, & 0 \le x \le l/2 \\ l - x, & l/2 \le x \le l \end{cases}$. *(P.T.U., 2010)*
13. If $f(x) = \sin x$ for $0 \le x \le \pi/4$
    $= \cos x$ for $\pi/4 \le x \le \pi/2$, expand $f(x)$ in a series of sines.
14. For the function defined by the graph OAB in Fig. 10.10, find the half-range Fourier sine series.

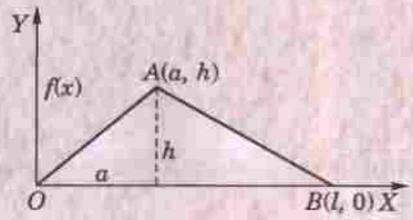


Fig. 10.10

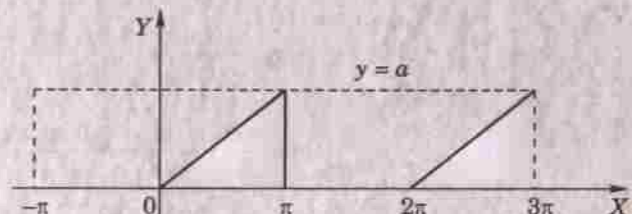


Fig 10.11

10.8 TYPICAL WAVEFORMS

We give below six typical waveforms usually met with in communication engineering :

(1) *Square waveform* (Fig. 10.7) is an extension of the function of Problem 8, page 412.

(2) *Saw-toothed waveform* (Fig. 10.5) is an extension of the function in Ex. 10.12, page 409.

(3) *Modified saw-toothed waveform* (Fig. 10.11) is extension of the function

$$f(x) = 0, \qquad -\pi < x \leq 0$$
$$= x, \qquad 0 \leq x < \pi,$$

Its Fourier expansion is

$$f(x) = \frac{a}{4} - \frac{2a}{\pi^2}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right) + \frac{\alpha}{\pi}\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \dots\right)$$

(4) *Triangular waveform* (Fig. 10-6) is an extension of the function of Ex. 10.15, page 411.

(5) *Half-wave rectifier* (Fig. 10.2) is an extension of the function of Problem 2, page 412.

(6) *Full-wave rectifier* (Fig. 10.12) is an extension of the function $f(x) = a \sin x, 0 \leq x \leq \pi$. Its Fourier expansion is

$$f(x) = \frac{4a}{\pi}\left\{\frac{1}{2} - \frac{1}{1.3}\cos 2x - \frac{1}{3.5}\cos 4x - \frac{1}{5.7}\cos 6x - \dots\right\}$$

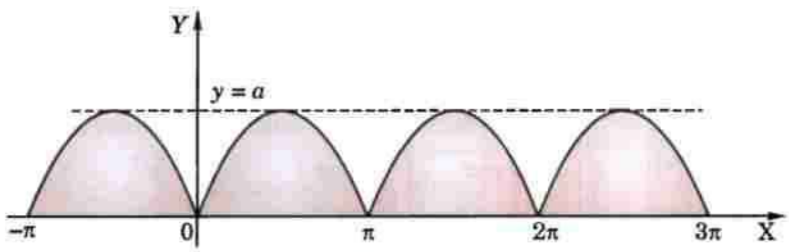


Fig. 10.12

10.9 (1) PARSEVAL'S FORMULA*

To prove that $\int_{-l}^{l} [f(x)]^2\, dx = l\left\{\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right\}$,

provided the Fourier series for f(x) converges uniformly in (– l, l).

The Fourier series for $f(x)$ in $(-l, l)$ is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$ ...(1)

Multiplying both sides of (1) by $f(x)$ and integrating term by term from $-l$ to l [which is justified as the series (1) is uniformly convergent –p. 389], we get

$$\int_{-l}^{l}[f(x)]^2\,dx = \frac{a_0}{2}\int_{-l}^{l} f(x)\,dx + \sum_{n=1}^{\infty}\left\{a_n\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx + b_n\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx\right\} \quad \text{...(2)}$$

Now $\int_{-l}^{l} f(x)\,dx = la_0$,

$\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx = la_n$ and $\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx = lb_n$, by (4) of p. 405

$\therefore$ (2) takes the form $\int_{-l}^{l}[f(x)]^2\,dx = l\left\{\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right\}$ ...(3)

which is the desired *Parseval's formula*. *(Mumbai, 2005 S)*

*Named after the French mathematician *Marc Antoine Parseval* (1755–1836).

Cor. 1. *If* $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$ *in* $(0, 2l)$, then

$$\int_0^{2l} [f(x)]^2 \, dx = l \left\{ \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right\} \qquad ...(4)$$

Cor. 2. *If the half-range cosine series is* $(0, l)$ *for* $f(x)$ *is*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l} \right), \text{ then}$$

$$\int_0^{l} [f(x)]^2 \, dx = \frac{l}{2} \left(\frac{a_0^2}{2} + a_1^2 + a_2^2 + a_3^2 + ... \infty \right) \qquad ...(5)$$

Cor. 3. *If the half-range sine series in* $(0, l)$ *for* $f(x)$ *is* $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l} \right)$, then

$$\int_0^{l} [f(x)]^2 = \frac{l}{2} (b_1^2 + b_2^2 + b_3^2 + ... \infty) \qquad ...(6)$$

(2) Root mean square (rms) value. *The root mean square value of the function* $f(x)$ *over an interval* (a, b) *is defined as*

$$[f(x)]_{\text{rms}} = \sqrt{\left\{ \frac{\int_a^b [f(x)]^2 \, dx}{b - a} \right\}} \qquad ...(7)$$

The use of root mean square value of a periodic function is frequently made in the theory of mechanical vibrations and in electric circuit theory. The r.m.s. value is also known as the effective value of the function.

Example 10.21. *Obtain the Fourier series for* $y = x^2$ *in* $-\pi < x < \pi$. *Using the two values of* y, *show that*

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + ... = \frac{\pi^4}{90}$$

Solution. Let $\quad y = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

We have $\quad a_0 = 2\frac{n^2}{3}, a_n = \frac{4}{n^2}(-1)^n, b_n = 0$ for all n (See problem 2, p. 400)

If $\bar{y}$ be the r.m.s. value of y in $(-\pi, \pi)$, then

$$(\bar{y})^2 = \frac{\pi}{2\pi} \left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right] \qquad \text{[By (3) and (7) §10.9]}$$

$$= \frac{1}{4} \left(\frac{2\pi^2}{3} \right)^2 + \frac{1}{2} \sum_{n=1}^{\infty} \left[\frac{16}{n^4} (-1)^{2n} + 0 \right] = \frac{\pi^4}{9} + 8 \sum_{1}^{\infty} \frac{1}{n^4}$$

Also by definition,

$$(\bar{y})^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} y^2 \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^4 \, dx = \frac{\pi^4}{5}$$

Equating the two values of $(\bar{y})^2$, we get

$$\frac{\pi^4}{9} + 8 \sum_{1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{5} \text{ i.e., } \sum_{1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

PROBLEMS 10.7

1. By using the sine series for $f(x) = 1$ in $0 < x < \pi$, show that $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots$

2. Prove that in $0 < x < l$, $x = \frac{l}{2} - \frac{4l}{\pi^2}\left(\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l} + \ldots\right)$

   and deduce that $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\ldots = \frac{\pi^4}{96}$.

3. If $\frac{a_0}{2} + \sum a_n \cos\frac{n\pi x}{l}$ is the half-range cosine series of $f(x)$ of period $2l$ in $(0, l)$, then show that the mean square value of $f(x)$ in $(0, l)$ is $\frac{l}{2}\left\{\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2\right\}$

   Use this result to evaluate $1^{-4} + 3^{-4} + 5^{-4} + \ldots\ldots$ from the half-range cosine series of the function $f(x)$ of period 4 defined in $(0, 2)$ by

$$f(x) = \begin{cases} \pi x, & 0 < x < 1 \\ \pi(2 - x), & 1 < x < 2 \end{cases}.$$

10.10 COMPLEX FORM OF FOURIER SERIES

The Fourier series of a periodic function $f(x)$ of period $2l$, is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right) \quad \ldots(1)$$

Since $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$,

therefore, we can express (1) as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n\left(\frac{e^{in\pi x/l} + e^{-in\pi x/l}}{2}\right) + b_n\left(\frac{e^{in\pi x/l} - e^{-in\pi x/l}}{2i}\right)\right\}$$

$$= c_0 + \sum_{n=1}^{\infty}\left\{c_n e^{in\pi x/l} + c_{-n} e^{-in\pi x/l}\right\} \quad \ldots(2)$$

where $c_0 = \frac{1}{2}a_0,\ c_n = \frac{1}{2}(a_n - ib_n),\ c_{-n} = \frac{1}{2}(a_n + ib_n)$

Now
$$c_n = \frac{1}{2l}\left\{\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx - i\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx\right\}$$

$$= \frac{1}{2l}\int_{-l}^{l} f(x)\left(\cos\frac{n\pi x}{l} - i\sin\frac{n\pi x}{l}\right)dx = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-in\pi x/l}\,dx$$

and
$$c_{-n} = \frac{1}{2l}\int_{-l}^{l} f(x)\left(\cos\frac{n\pi x}{l} - i\sin\frac{n\pi x}{l}\right)dx = \frac{1}{2l}\int_{-l}^{l} f(x)e^{in\pi x/l}\,dx$$

Combining these, we have $c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-in\pi x/l}\,dx$

where $n = 0, \pm 1, \pm 2, \pm 3, \ldots\ldots$ ...(3)

Then the series (2) can be compactly written as :

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}$$

which is the *complex form of Fourier series* and its coefficients are given by (3).

Obs. The complex form of a Fourier series is especially useful in problems on electrical circuits having impressed periodic voltage.

Example 10.22. *Find the complex form of the Fourier series of $f(x) = e^{-x}$ in $-1 \le x \le 1$.* (*Mumbai, 2005 S ; Madras, 2000 S*)

Solution. We have $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x}$ $(\because \ l = 1)$

where
$$c_n = \frac{1}{2}\int_{-1}^{1} e^{-x}\cdot e^{-in\pi x}\,dx = \frac{1}{2}\int_{-1}^{1} e^{-(1+in\pi)x}\,dx = \frac{1}{2}\left|\frac{e^{-(1+in\pi)x}}{-(1+in\pi)}\right|_{-1}^{1} = \frac{e^{1+in\pi} - e^{-(1+in\pi)}}{2(1+in\pi)}$$
$$= \frac{e(\cos n\pi + i\sin n\pi) - e^{-1}(\cos n\pi - i\sin n\pi)}{2(1+in\pi)} = \frac{e-e^{-1}}{2}(-1)^n\cdot\frac{1-in\pi}{1+n^2\pi^2}$$
$$= \frac{(-1)^n(1-in\pi)\sinh 1}{1+n^2\pi^2}$$

Hence
$$e^{-x} = \sum_{n=-\infty}^{\infty}\frac{(-1)^n(1-in\pi)}{1+n^2\pi^2}\sinh 1\cdot e^{in\pi x}.$$

PROBLEMS 10.8

Find the complex form of the Fourier series of the following periodic functions :

1. $f(x) = e^{ax}, -l < x < l$. (*Madras, 2003*)
2. $f(t) = \sin t, 0 < t < \pi$
3. $f(x) = \cos ax, -\pi < x < \pi$ (*Anna, 2009 ; Mumbai, 2009*)
4. $f(x) = \cosh 3x + \sinh 3x$ in $(-3, 3)$. (*Mumbai, 2008*)
5. $f(x) = \begin{cases} 0 \text{ when } 0 < x < l \\ a \text{ when } l < x < 2l \end{cases}$.

10.11 PRACTICAL HARMONIC ANALYSIS

We have discussed at length, the problem of expanding $y = f(x)$ as Fourier series :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \quad ...(1)$$

where
$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \end{aligned}\right\} \quad ...(2)$$

So far, the function has always been defined by an explicit function of an independent variable. In practice, however, the function is often given not by a formula but by a graph or by a table of corresponding values. In such cases, the integrals in (2) cannot be evaluated and instead, the following alternative forms of (2) are employed.

Since the mean value of a function $y = f(x)$ over the range (a, b) is $\frac{1}{b-a}\int_a^b f(x)\,dx$.

$\therefore$ the equations (2) give,

$$\left.\begin{aligned} \mathbf{a_0} &= 2\times\frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \mathbf{2[mean\ value\ of\ f(x)\ in\ (0, 2\pi)]} \\ \mathbf{a}_n &= 2\times\frac{1}{2\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \mathbf{2[mean\ value\ of\ f(x)\cos nx\ in\ (0, 2\pi)]} \\ \mathbf{b}_n &= 2\times\frac{1}{2\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \mathbf{2[mean\ value\ of\ f(x)\sin nx\ in\ (0, 2\pi)]} \end{aligned}\right\} \quad ...(3)$$

There are numerous other methods of finding the value of a_0, a_n, b_n which constitute the field of **harmonic analysis.**

In (1), *the term* $(a_1 \cos x + b_1 \sin x)$ *is called the* **fundamental or first harmonic,** *the term* $(a_2 \cos 2x + b_2 \sin 2x)$ **the second harmonic** *and so on.*

Example 10.23. *The displacement y of a part of a mechanism is tabulated with corresponding angular movement x° of the crank. Express y as a Fourier series neglecting the harmonic above the third :*

$x°$	0	30	60	90	120	150	180	210	240	270	300	330
y	1.80	1.10	0.30	0.16	1.50	1.30	2.16	1.25	1.30	1.52	1.76	2.00

Solution. Let the Fourier series upto the third harmonic representing y in $(0, 2\pi)$ be

$$y = \tfrac{1}{2}a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x \qquad \text{...(i)}$$

To evaluate the coefficients, we form the following table.

$x°$	$\sin x$	$\cos x$	$\sin 2x$	$\cos 2x$	$\sin 3x$	$\cos 3x$	y	$y \sin x$	$y \cos x$	$y \sin 2x$	$y \cos 2x$	$y \sin 3x$	$y \cos 3x$
0	0	1	0	1	0	1	1.80	0.00	1.80	0.00	1.80	0.00	1.80
30	0.50	0.87	0.87	0.50	1	0	1.10	0.55	0.96	0.96	0.55	1.10	0.00
60	0.87	0.50	0.87	–0.50	0	–1	0.30	0.26	0.15	0.26	–0.15	0.00	–0.30
90	1.00	0	0	–1.00	–1	0	0.16	0.16	0.00	0.00	–0.16	–0.16	0.00
120	0.87	–0.50	–0.87	–0.50	0	1	0.50	0.43	–0.25	–0.43	–0.25	0.00	0.50
150	0.50	–0.87	–0.87	–0.50	1	0	1.30	0.65	–1.13	–1.13	0.65	1.30	0.00
180	0	–1.00	0	1.00	0	–1	2.16	0.00	–2.16	–0.00	2.16	0.00	–2.16
210	–0.50	–0.87	0.87	0.50	–1	0	1.25	–0.63	–1.09	1.09	0.63	–1.25	0.00
240	–0.87	–0.50	0.87	–0.50	0	1	1.30	–1.13	–0.65	1.13	–0.65	0.00	1.30
270	–1.00	0	0	–1.00	1	0	1.52	–1.52	0.00	0.00	–1.52	1.52	0.00
300	–0.87	0.50	–0.87	–0.50	0	–1	1.76	–1.53	0.88	–1.53	–0.88	0.00	–1.76
330	–0.50	0.87	–0.87	0.50	–1	0	2.00	–1.00	1.74	–1.74	1.00	–2.00	0.00
						Σ =	15.15	–3.76	0.25	–1.39	3.18	0.51	–0.62

$$\therefore \quad a_0 = 2 \cdot \frac{\Sigma y}{12} = \frac{15.15}{6} = 2.53 \; ; \; a_1 = \frac{1}{6} \Sigma y \cos x = \frac{0.25}{6} = 0.04$$

$$a_2 = \frac{1}{6} \Sigma y \cos 2x = \frac{3.18}{6} = 0.53 \; ; \; a_3 = \frac{1}{6} \Sigma y \cos 3x = \frac{-0.62}{6} = -0.1$$

$$b_1 = \frac{1}{6} \Sigma y \sin x = \frac{-3.76}{6} = -0.63 \; ;$$

$$b_2 = \frac{1}{6} \Sigma y \sin 2x = \frac{-1.39}{6} = -0.23$$

$$b_3 = \frac{1}{6} \Sigma y \sin 3x = \frac{0.51}{6} = 0.085$$

Substituting the values of a's and b's in (i), we get

$$y = 1.26 + 0.04 \cos x + 0.53 \cos 2x - 0.1 \cos 3x - 0.63 \sin x - 0.23 \sin 2x + 0.085 \sin 3x.$$

Example 10.24. *The following table gives the variations of periodic current over a period.*

t sec	*0*	*T/6*	*T/3*	*T/2*	*2T/3*	*5T/6*	*T*
A amp.	*1.98*	*1.30*	*1.05*	*1.30*	*– 0.88*	*– 0.25*	*1.98*

Show that there is a direct current part of 0.75 amp in the variable current and obtain the amplitude of the first harmonic. (V.T.U., 2010 ; S.V.T.U., 2009)

Solution. Here length of the interval is T, *i.e.* $C = T/2$ (§ 10.5)

Then $$A = \frac{a_0}{2} + a_1 \cos \frac{2\pi t}{T} + b_1 \sin \frac{2\pi t}{T} + a_2 \cos \frac{4\pi t}{T} + b_2 \sin \frac{4\pi t}{T} + \ldots$$

The desired values are tabulated as follows :

t	$2\pi t/T$	$\cos 2\pi t/T$	$\sin 2\pi t/T$	A	$A \cos 2\pi t/T$	$A \sin 2\pi t/T$
0	0	1.0	0.000	1.98	1.980	0.000
$T/6$	$\pi/3$	0.5	0.866	1.30	0.650	1.126
$T/3$	$2\pi/3$	–0.5	0.866	1.05	–0.525	0.909
$T/2$	π	–1.0	0.000	1.30	–1.300	0.000
$2T/3$	$4\pi/3$	–0.5	–0.866	–0.88	0.440	0.762
$5T/6$	$5\pi/3$	0.5	–0.866	–0.25	–0.125	0.217
			$\Sigma =$	4.5	1.12	3.014

$$\therefore \quad a_0 = 2 \cdot \frac{1}{6} \Sigma A = \frac{1}{3}(4.5) = 1.5$$

$$a_1 = 2 \cdot \frac{1}{6} \Sigma A \cos \frac{2\pi t}{T} = \frac{1}{3}(1.12) = 0.373$$

$$b_1 = 2 \cdot \frac{1}{6} \Sigma A \sin \frac{2\pi t}{T} = \frac{1}{3}(3.014) = 1.005$$

Thus the direct current part in the variable current = $a_0/2 = 0.75$ and amplitude of the first harmonic

$$= \sqrt{(a_1^2 + b_1^2)} = \sqrt{[(0.373)^2 + (1.005)^2]} = 1.072$$

Example 10.25. *Obtain the first three coefficients in the Fourier cosine series for y, where y is given in the following table :*

x :	*0*	*1*	*2*	*3*	*4*	*5*
y :	*4*	*8*	*15*	*7*	*6*	*2*

(V.T.U., 2009 ; V.T.U., 2006 ; J.N.T.U., 2004 S)

Solution. Taking the interval as 60°, we have

$\theta =$	0°	60°	120°	180°	240°	300°
$x =$	0	1	2	3	4	5
$y =$	4	8	15	7	6	2

$\therefore$ Fourier cosine series in the intervals $(0, 2\pi)$ is

$$y = \frac{a_0}{2} + a_1 \cos \theta + a_2 \cos 2\theta + a_3 \cos 3\theta + \ldots$$

$\theta°$	$\cos \theta$	$\cos 2\theta$	$\cos 3\theta$	y	$y \cos \theta$	$y \cos 2\theta$	$y \cos 3\theta$
0°	1	1	1	4	4	4	4
60°	$\frac{1}{2}$	$-\frac{1}{2}$	–1	8	4	–4	–8
120°	$-\frac{1}{2}$	$-\frac{1}{2}$	1	15	–7.5	–7.5	15
180°	–1	1	–1	7	–7	7	–7
240°	$-\frac{1}{2}$	$-\frac{1}{2}$	1	6	–3	–3	6
300°	$\frac{1}{2}$	$-\frac{1}{2}$	–1	2	1	–1	–2
			$\Sigma =$	42	– 8.5	–4.5	8

Hence $\quad a_0 = 2 \cdot \dfrac{42}{6} = 14, a_1 = 2\left(\dfrac{-8.5}{6}\right) = -2.8, a_2 = 2\left(\dfrac{-4.5}{6}\right) = -1.5,$

$$a_3 = 2\left(\frac{8}{6}\right) = 2.7.$$

Example 10.26. *The turning moment T is given for a series of values of the crank angle θ° = 75°*

$\theta°$:	0	30	60	90	120	150	180
T :	0	5224	8097	7850	5499	2626	0

Obtain the first four terms in a series of sines to represent T and calculate T for θ = 75°.

Solution. Let the Fourier sine series to represent T in (0, 180) be

$T = b_1 \sin\theta + b_2 \sin 2\theta + b_3 \sin 3\theta + b_4 \sin 4\theta + \ldots$

To evaluate the coefficients, we form the following table :

$\theta°$	T	$\sin\theta$	$\sin 2\theta$	$\sin 3\theta$	$\sin 4\theta$
0	0	0	0	0	0
30	5224	0.500	0.866	1	0.866
60	8097	0.866	0.866	0	−0.866
90	7850	1.000	0	−1	0
120	5499	0.866	−0.866	0	0.866
150	2626	0.500	−0.866	1	−0.866

$$\therefore \quad b_1 = \frac{2}{6}\Sigma y \sin\theta = \frac{1}{3}\{(5224 + 2626)\,0.5 + (8097 + 5499)\,0.866 + 7850\} = 7850$$

$$b_2 = \frac{2}{6}\Sigma y \sin 2\theta = \frac{1}{3}\{(5224 + 8097)\,0.866 + (5499 + 2626)(-0.866)\} = 1500$$

$$b_3 = \frac{2}{6}\Sigma y \sin 3\theta = \frac{1}{3}\{5224 - 7850 + 2626\} = 0.$$

$$b_4 = \frac{2}{6}\Sigma y \sin 4\theta = \frac{1}{3}\{(5224 + 5499)(0.866) + (8097 + 2626)(-0.866)\} = 0$$

Hence $\quad T = 785° \sin\theta + 150° \sin 2\theta$

For $\theta = 75°$, $T = 7850 \sin 75° + 1500 \sin 150°$

$= 7850° (0.9659) + 1500 (0.5) = 8332.$

PROBLEMS 10.9

1. The following values of y give the displacement in inches of a certain machine part for the rotation x of the flywheel. Expand y in terms of a Fourier series :

x :	0	$\pi/6$	$2\pi/6$	$3\pi/6$	$4\pi/6$	$5\pi/6$
y :	0	9.2	14.4	17.8	17.3	11.7

2. Compute the first two harmonics of the Fourier series of $f(x)$ given in the following table :

x :	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
$f(x)$:	1.0	1.4	1.9	1.7	1.5	1.2	1.0

(*Anna, 2009*)

3. Obtain the constant term and the coefficients of the first sine and cosine terms in the Fourier expansion of y as given in the following table :

x :	0	1	2	3	4	5
y :	9	18	24	28	26	20

(*V.T.U., 2011 ; Anna, 2005 S*)

4. In a machine the displacement y of a given point is given for a certain angle θ as follows :

$\theta°$:	0	30	60	90	120	150	180	210	240	270	300	330
y :	7.9	8.0	7.2	5.6	3.6	1.7	0.5	0.2	0.9	2.5	4.7	6.8

Find the coefficient of sin 2θ in the Fourier series representing the above variation.

5. Determine the first two harmonics of the Fourier series for the following values :

$x°$:	30	60	90	120	150	180	210	240	270	300	330	360
y :	2.34	3.01	3.68	4.15	3.69	2.20	0.83	0.51	0.88	1.09	1.19	1.64

(*Madras, 2006 ; Cochin, 2005*)

6. The turning moment T on the crankshaft of a steam engine for the crank angle θ degrees is given as follows :

θ :	0	15	30	45	60	75	90	105	120	135	150	165	180
T :	0	2.7	5.2	7.0	8.1	8.3	7.9	6.8	5.5	4.1	2.6	1.2	0

Expand T in a series of sines upto the fourth harmonics.

10.12 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 10.10

Fill up the blanks or choose the correct answer in each of the following problems :

1. The period of $\cos 3x$ is $x =$ ......
2. If $x = c$ is a point of discontinuity then the Fourier series of $f(x)$ at $x = c$ gives $f(x) =$ ......
3. A function $f(x)$ defined for $0 < x < 1$ can be extended to an odd periodic function in ......
4. The mathematical function representing the following graph is ......
5. Fourier expansion of an odd function has only ...... terms.
6. Formulae for evaluation of Fourier coefficients for a given set of points $(x_i, y_i) : i = 0, 1, 2, \ldots\ldots n$ are......
7. If $f(x) = x^4$ in $(-1, 1)$, then the Fourier coefficient $b_n =$ ......
8. The period of a constant function is ......
9. If $f(t) = \begin{cases} -1, & -1 < t < 0 \\ 1, & 0 < t < 1 \end{cases}$, then $f(t)$ is an ......
10. Fourier expansion of an even function $f(x)$ in $(-\pi, \pi)$ has only ...... terms.
11. If $f(x) = \begin{cases} -x, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$, then $f(x)$ is an ...... function in $(-\pi, \pi)$.
12. The smallest period of the function $\sin\left(\dfrac{2n\pi x}{k}\right)$ is ......
13. In the Fourier series expansion of $f(x) = |\sin x|$ in $(-\pi, \pi)$, the value of $b_n =$ ......
14. In the Fourier series for $f(x) = x$ in $(-\pi \le x \le \pi)$, the ...... terms are absent.
15. If $f(x)$ is an even function in $(-l, l)$, then the value of $b_n =$ ......
16. If $f(x) = x^2$ in $-2 < x < 2$, $f(x + 4) = f(x)$, then a_n is ......
17. If $f(x)$ is a periodic function with period $2T$, then the value of the Fourier coefficient $b_n =$ ......
18. Dirichlet conditions for the expansion of a function as a Fourier series in the interval $c_1 \le x \le c_2$ are ......
19. If $f(x) = x \sin x$ in $(-\pi, \pi)$, then the value of $b_n =$ ......
20. The formulae for finding the half range cosine series for the function $f(x)$ in $(0, l)$ are ......
21. The half-range sine series for 1 in $(0, \pi)$, is ......
22. Period of $|\sin t|$ is ......
23. The value of bn in the Fourier series of $f(x) = |x|$ in $(-\pi, \pi) =$ ......
24. If $f(x)$ is defined in $(0, l)$ then the period of $f(x)$ to expand it as a half range sine series is ......
25. The complex form of Fourier series for e^{-x} in $(-1, 1)$ is ......
26. $f(x)$ is an odd function in $(-\pi, \pi)$, then the graph of $f(x)$ is symmetric about the x-axis. (True or False)
27. $f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi, \end{cases}$ then $f(0) =$ ......
28. If $f(x) = \begin{cases} \pi x & \text{in } 0 \le x \le 1 \\ \pi(2 - x) & \text{in } 1 \le x \le 2, \end{cases}$ then it is ...... function. (odd or even)
29. If $f(x)$ is an odd function in $(-l, l)$, then the values of a_0 and a_n are......
30. The root mean square value of $f(t) = 3 \sin 2t + 4 \cos 2t$ over the range $0 \le t \le \pi$ is...... (*Nagpur, 2009*)
31. In the Fourier series expansion of the function
$$f(x) = \begin{cases} -(\pi + x), & -\pi < x < 0 \\ -(\pi - x), & 0 < x < \pi, \end{cases}$$
the value of b_n is ...... (*P.T.U., 2010*)
32. Let $f(x)$ be defined in $(0, 2\pi)$ by
$$f(t) = \begin{cases} \dfrac{1 + \cos x}{\pi - x}, & 0 < x < \pi \\ \cos x, & \pi < x < 2\pi, \end{cases}$$
$f(x) + 2\pi = f(x)$. The value of $f(\pi)$ is ...... (*Anna, 2009*)

33. The mean value of $f(x) \cos nx$ in $(0, 2\pi) =$ ......

34. Using sine series for $f(x) = 1$ in $0 < x < \pi$, show that $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots \infty =$ .....

35. Fourier series representing $f(x) = |x|$ in $-\pi < x < \pi$, is ......

36. Fourier series of $f(x) = \cos^4 x$ in $(0, 2\pi)$ is ......

37. If $f(x) = x^2 + x$ in $(0, l)$, then the even extension of $f(x)$ in $(-l, 0)$ is ......

38. If $f(x) = x(l - x)$ in $(0, l)$, then the extension of $f(x)$ in $(l, 2l)$ so as to get sine series is ......

39. A function $f(x)$ defined in $(-\pi, \pi)$ can be expanded into Fourier series containing both sine and cosine terms. (True or False)

40. The function $f(x) = \begin{cases} 1 - x \text{ in } & -\pi < x < 0 \\ 1 + x \text{ in } & 0 < x < \pi \end{cases}$, is an odd function. (True or False)

41. If $f(x) = x^2$ in $(-\pi, \pi)$, then the Fourier series of $f(x)$ contains only sine terms. (True or False)

CHAPTER 11

Differential Equations of First Order

11.1 DEFINITIONS

(1) *A* **differential equation** *is an equation which involves differential coefficients or differentials.*

Thus (*i*) $e^x\,dx + e^y\,dy = 0$ (*ii*) $\dfrac{d^2x}{dt^2} + n^2x = 0$

(*iii*) $y = x\dfrac{dy}{dx} + \dfrac{x}{dy/dx}$ (*iv*) $\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2} \Big/ \dfrac{d^2y}{dx^2} = c$

(*v*) $\dfrac{dx}{dt} - wy = a\cos pt,\ \dfrac{dy}{dt} + wx = a\sin pt$ (*vi*) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2u$

(*vii*) $\dfrac{\partial^2 y}{\partial t^2} = c^2\dfrac{\partial^2 y}{\partial x^2}$ are all examples of differential equations.

(2) *An* **ordinary differential equation** *is that in which all the differential coefficients have reference to a single independent variable.* Thus the equations (*i*) to (*v*) are all ordinary differential equations.

A **partial differential equation** *is that in which there are two or more independent variables and partial differential coefficients with respect to any of them.* Thus the equations (*vi*) and (*vii*) are partial differential equations.

(3) *The* **order** *of a differential equation is the order of the highest derivative appearing in it.*

The **degree** *of a differential equation is the degree of the highest derivative occurring in it, after the equation has been expressed in a form free from radicals and fractions as far as the derivatives are concerned.*

Thus, from the examples above,

(*i*) is of the first order and first degree ; (*ii*) is of the second order and first degree ;

(*iii*) written as $y\dfrac{dy}{dx} = x\left(\dfrac{dy}{dx}\right)^2 + x$ is clearly of the first order but of second degree ;

and (*iv*) written as $\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^3 = c^2\left(\dfrac{d^2y}{dx^2}\right)^2$ is of the second order and second degree.

11.2 PRACTICAL APPROACH TO DIFFERENTIAL EQUATIONS

Differential equations arise from many problems in oscillations of mechanical and electrical systems, bending of beams, conduction of heat, velocity of chemical reactions etc., and as such play a very important role in all modern scientific and engineering studies.

The approach of an engineering student to the study of differential equations has got to be practical unlike that of a student of mathematics, who is only interested in solving the differential equations without knowing as to how the differential equations are formed and how their solutions are physically interpreted.

Thus for an applied mathematician, the study of a differential equation consists of three phases :

(*i*) *formulation of differential equation from the given physical situation, called* **modelling.**

(*ii*) *solutions of this differential equation, evaluating the arbitrary constants from the given conditions,* and

(*iii*) *physical interpretation of the solution.*

11.3 FORMATION OF A DIFFERENTIAL EQUATION

An ordinary differential equation is formed in an attempt to eliminate certain arbitrary constant from a relation in the variables and constants. It will, however, be seen later that the partial differential equations may be formed by the elimination of either arbitrary constants or arbitrary functions. In applied mathematics, every geometrical or physical problem when translated into mathematical symbols gives rise to a differential equation.

Example 11.1. *Form the differential equation of simple harmonic motion given by $x = A \cos (nt + \alpha)$.*

Solution. To eliminate the constants A and α differentiating it twice, we have

$$\frac{dx}{dt} = -nA \sin (nt + \alpha) \quad \text{and} \quad \frac{d^2x}{dt^2} = -n^2A \cos (nt + \alpha) = -n^2x$$

Thus $$\frac{d^2x}{dt^2} + n^2x = 0$$

is the desired differential equation which states that the acceleration varies as the distance from the origin.

Example 11.2. *Obtain the differential equation of all circles of radius a and centre (h, k).* (*Andhra, 1999*)

Solution. Such a circle is $(x - h)^2 + (y - k)^2 = a^2$ ...(*i*)

where h and k, the coordinates of the centre, and a are the constants.

Differential it twice, we have

$$x - h + (y - k)\frac{dy}{dx} = 0 \quad \text{and} \quad 1 + (y - k)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

Then $$y - k = -\frac{1 + (dy/dx)^2}{d^2y/dx^2}$$

and $$x - h = -(y - k)\, dy/dx = \frac{\dfrac{dy}{dx}\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]}{d^2y/dx^2}$$

Substituting these in (*i*) and simplifying, we get $[1 + (dy/dx)^2]^3 = a^2 (d^2y/dx^2)^2$ ...(*ii*)

as the required differential equation

Writing (*ii*) in the form $\dfrac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} = a$,

it states that the radius of curvature of a circle at any point is constant.

Example 11.3. *Obtain the differential equation of the coaxial circles of the system $x^2 + y^2 + 2ax + c^2 = 0$ where c is a constant and a is a variable.* (*J.N.T.U., 2003*)

Solution. We have $x^2 + y^2 + 2ax + c^2 = 0$ ...(*i*)

Differentiating w.r.t. x, $2x + 2y dy/dx + 2a = 0$

or
$$2a = -2\left(x + y\frac{dy}{dx}\right)$$

Substituting in (*i*), $x^2 + y^2 - 2(x + y\, dy/dx)x + c^2 = 0$

or
$$2xy\, dy/dx = y^2 - x^2 + c^2$$

which is the required differential equation.

11.4 (1) SOLUTION OF A DIFFERENTIAL EQUATION

A solution (or integral) of a differential equation is a relation between the variables which satisfies the given differential equation.

For example, $$x = A \cos (nt + \alpha) \quad ...(1)$$

is a solution of $$\frac{d^2x}{dt^2} + n^2x = 0 \text{ [Example 11.1]} \quad ...(2)$$

The **general** (*or* **complete**) **solution** *of a differential equation is that in which the number of arbitrary constants is equal to the order of the differential equation.* Thus (1) is a general solution (2) as the number of arbitrary constants (A, α) is the same as the order of (2).

A **particular solution** *is that which can be obtained from the general solution by giving particular values to the arbitrary constants.*

For example, $$x = A \cos (nt + \pi/4)$$

is the particular solution of the equation (2) as it can be derived from the general solution (1) by putting $\alpha = \pi/4$.

A differential equation may sometimes have an additional solution which cannot be obtained from the general solution by assigning a particular value to the arbitrary constant. Such a solution is called a **singular solution** and is not of much engineering interest.

Linearly independent solution. *Two solutions $y_1(x)$ and $y_2(x)$ of the differential equation*

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)\, y = 0 \quad ...(3)$$

are said to be linearly independent if $c_1y_1 + c_2y_2 = 0$ such that $c_1 = 0$ and $c_2 = 0$

If c_1 and c_2 are not both zero, then the two solutions y_1 and y_2 are said to be *linearly dependent.*

If $y_1(x)$ and $y_2(x)$ any two solutions of (3), then their linear combination $c_1y_1 + c_2y_2$ where c_1 and c_2 are constants, is also a solution of (3).

Example 11.4. *Find the differential equation whose set of independent solutions is $[e^x, xe^x]$.*

Solution. Let the general solution of the required differential equation be $y = c_1e^x + c_2xe^x$ ...(*i*)

Differentiating (*i*) w.r.t. x, we get

$$y_1 = c_1e^x + c_2(e^x + xe^x)$$

$$\therefore \quad y - y_1 = c_2e^x \quad ...(ii)$$

Again differentiating (*ii*) w.r.t. x, we obtain

$$y_1 - y_2 = c_2e^x \quad ...(iii)$$

Subtracting (*iii*) from (*ii*), we get

$$y - y_1 - (y_1 - y_2) = 0 \quad \text{or} \quad y - 2y_1 + y_2 = 0$$

which is the desired differential equation.

(2) Geometrical meaning of a differential equation. Consider any differential equation of the first order and first degree

$$\frac{dy}{dx} = f(x, y) \quad ...(1)$$

If $P(x, y)$ be any point, then (1) can be regarded as an equation giving the value of dy/dx (= m) when the values of x and y are known (Fig. 11.1). Let the value of m at the point $A_0(x_0, y_0)$ derived from (1) be m_0. Take a neighbouring

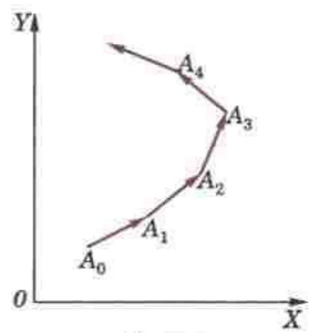


Fig. 11.1

point $A_1(x_1, y_1)$ such that the slope of A_0A_1 is m_0. Let the corresponding value of m at A_1 be m_1. Similarly take a neighbouring point $A_2(x_2, y_2)$ such that the slope of A_1A_2 is m_1 and so on.

If the successive points A_0, A_1, A_2, A_3 ... are chosen very near one another, the broken curve $A_0A_1A_2A_3$ ... approximates to a smooth curve $C[y = \phi(x)]$ which is a solution of (1) associated with the initial point $A_0(x_0, y_0)$. Clearly the slope of the tangent to C at any point and the coordinates of that point satisfy (1).

A different choice of the initial point will, in general, give a different curve with the same property. *The equation of each such curve is thus a* **particular solution** *of the differential equation (1). The equation of the whole family of such curves is the general solution of* (1). The slope of the tangent at any point of each member of this family and the co-ordinates of that point satisfy (1).

Such a simple geometric interpretation of the solutions of a second (or higher) order differential equation is not available.

PROBLEMS 11.1

Form the differential equations from the following equations :

1. $y = ax^3 + bx^2$.
2. $y = C_1 \cos 2x + C_2 \sin 2x$ *(Bhopal, 2008)*
3. $xy = Ae^x + Be^{-x} + x^2$. *(U.P.T.U., 2005)*
4. $y = e^x (A \cos x + B \sin x)$. *(P.T.U., 2003)*
5. $y = ae^{2x} + be^{-3x} + ce^x$.

Find the differential equations of :

6. A family of circles passing through the origin and having centres on the x-axis. *(J.N.T.U., 2006)*
7. All circles of radius 5, with their centres on the y-axis.
8. All parabolas with x-axis as the axis and $(a, 0)$ as focus.
9. If $y_1(x) = \sin 2x$ and $y_2(x) = \cos 2x$ are two solutions of $y'' + 4y = 0$, show that $y_1(x)$ and $y_2(x)$ are linearly independent solutions.
10. Determine the differential equation whose set of independent solutions is $[e^x, xe^x, x^2 e^x]$ *(U.P.T.U., 2002)*
11. Obtain the differential equation of the family of parabolas $y = x^2 + c$ and sketch those members of the family which pass through (0, 0), (1, 1), (0, 1) and (1, – 1) respectively.

11.5 EQUATIONS OF THE FIRST ORDER AND FIRST DEGREE

It is not possible to solve such equations in general. We shall, however, discuss some special methods of solution which are applied to the following types of equations :

(*i*) Equations where variables are separable, (*ii*) Homogeneous equations,
(*iii*) Linear equations, (*iv*) Exact equations.

In other cases, the particular solution may be determined numerically (Chapter 31).

11.6 VARIABLES SEPARABLE

If in an equation it is possible to collect all functions of x and dx on one side and all the functions of y and dy on the other side, then the *variables are said to be separable.* Thus the general form of such an equation is $f(y)\, dy = \phi(x)\, dx$

Integrating both sides, we get $\int f(y)\, dy = \int \phi(x)\, dx + c$ as its solution.

Example 11.5. *Solve* $dy/dx = \dfrac{x(2 \log x + 1)}{\sin y + y \cos y}$. *(V.T.U., 2008)*

Solution. Given equation is $x(2 \log x + 1)\, dx = (\sin y + y \cos y)\, dy$

Integrating both sides, $2\int (\log x \,.\, x + x)\, dx = \int \sin y\, dy + \int y \cos y\, dy + c$

or $$2\left[\left(\log x \,.\, \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2}\, dx\right) + \frac{x^2}{2}\right] = -\cos y + \left[y \sin y - \int \sin y \,.\, 1\, dy + c\right]$$

or $$2x^2 \log x - \frac{x^2}{2} + \frac{x^2}{2} = -\cos y + y \sin y + \cos y + c$$

Hence the solution is $2x^2 \log x - y \sin y = c$.

Example 11.6. *Solve* $\dfrac{dy}{dx} = e^{3x-2y} + x^2 e^{-2y}$.

Solution. Given equation is $\dfrac{dy}{dx} = e^{-2y}(e^{3x} + x^2)$ or $e^{2y}\,dy = (e^{3x} + x^2)\,dx$

Integrating both sides, $\int e^{2y}\,dy = \int (e^{3x} + x^2)\,dx + c$

or $$\frac{e^{2y}}{2} = \frac{e^{3x}}{3} + \frac{x^3}{3} + c \quad \text{or} \quad 3e^{2y} = 2(e^{3x} + x^3) + 6c.$$

Example 11.7. *Solve* $\dfrac{dy}{dx} = \sin(x + y) + \cos(x + y)$. *(V.T.U., 2005)*

Solution. Putting $x + y = t$ so that $dy/dx = dt/dx - 1$

The given equation becomes $\dfrac{dt}{dx} - 1 = \sin t + \cos t$

or $$dt/dx = 1 + \sin t + \cos t$$

Integrating both sides, we get $dx = \int \dfrac{dt}{1 + \sin t + \cos t} + c.$

or $$x = \int \frac{2d\theta}{1 + \sin 2\theta + \cos 2\theta} + c \qquad \text{[Putting } t = 2\theta]$$

$$= \int \frac{2d\theta}{2\cos^2\theta + 2\sin\theta\cos\theta} + c = \int \frac{\sec^2\theta}{1 + \tan\theta}\,d\theta + c$$

$$= \log(1 + \tan\theta) + c$$

Hence the solution is $x = \log\left[1 + \tan \frac{1}{2}(x + y)\right] + c$.

Example 11.8. *Solve* $dy/dx = (4x + y + 1)^2$, *if* $y(0) = 1$.

Solution. Putting $4x + y + 1 = t$, we get $\dfrac{dy}{dx} = \dfrac{dt}{dx} - 4$.

$\therefore$ the given equation becomes $\dfrac{dt}{dx} - 4 = t^2$ or $\dfrac{dt}{dx} = 4 + t^2$

Integrating both sides, we get $\int \dfrac{dt}{4 + t^2} = \int dx + c$

or $$\frac{1}{2}\tan^{-1}\frac{t}{2} = x + c \quad \text{or} \quad \frac{1}{2}\tan^{-1}\left[\frac{1}{2}(4x + y + 1)\right] = x + c.$$

or $$4x + y + 1 = 2\tan 2(x + c)$$

When $x = 0, y = 1$ $\therefore$ $\dfrac{1}{2}\tan^{-1}(1) = c$ *i.e.* $c = \pi/8$.

Hence the solution is $4x + y + 1 = 2\tan(2x + \pi/4)$.

Example 11.9. *Solve* $\dfrac{y}{x}\dfrac{dy}{dx} + \dfrac{x^2 + y^2 - 1}{2(x^2 + y^2) + 1} = 0$. *(V.T.U., 2003)*

Solution. Putting $x^2 + y^2 = t$, we get $2x + 2y\dfrac{dy}{dx} = \dfrac{dt}{dx}$ or $\dfrac{y}{x}\dfrac{dy}{dx} = \dfrac{1}{2x}\dfrac{dt}{dx} - 1$.

Therefore the given equation becomes $\frac{1}{2x}\frac{dt}{dx} - 1 + \frac{t-1}{2t+1} = 0$

or $$\frac{1}{2x}\frac{dt}{dx} = 1 - \frac{t-1}{2t+1} = \frac{t+2}{2t+1} \quad \text{or} \quad 2x\,dx = \frac{2t+1}{t+2}dt$$

or $$2x\,dx = \left(2 - \frac{3}{t+2}\right)dt$$

Integrating, we get $\quad x^2 = 2t - 3\log(t+2) + c$

or $$x^2 + 2y^2 - 3\log(x^2 + y^2 + 2) + c = 0 \qquad [\because \; t = x^2 + y^2]$$

which is the required solution.

PROBLEMS 11.2

Solve the following differential equations :

1. $y\sqrt{(1-x^2)}\,dy + x\sqrt{(1-y^2)}\,dx = 0.$
2. $(x^2 - yx^2)\frac{dy}{dx} + y^2 + xy^2 = 0.$
3. $\sec^2 x \tan y\,dx + \sec^2 y \tan x\,dy = 0.$ *(P.T.U., 2003)*
4. $\frac{y}{x}\frac{dy}{dx} = \sqrt{(1 + x^2 + y^2 + x^2y^2)}.$ *(V.T.U., 2011)*
5. $e^x \tan y\,dx + (1 - e^x)\sec^2 y\,dy = 0.$ *(V.T.U., 2009)*
6. $\frac{dy}{dx} = xe^{y-x^2}$, if $y = 0$ when $x = 0$. *(J.N.T.U., 2006)*
7. $x\frac{dy}{dx} + \cot y = 0$ if $y = \pi/4$ when $x = \sqrt{2}$.
8. $(xy^2 + x)\,dx + (yx^2 + y)\,dy = 0.$
9. $\frac{dy}{dx} = e^{2x-3y} + 4x^2e^{-3y}.$
10. $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right).$
11. $(x+1)\frac{dy}{dx} + 1 = 2e^{-y}.$ *(Madras, 2000 S)*
12. $(x-y)^2\frac{dy}{dx} = a^2.$
13. $(x+y+1)^2\frac{dy}{dx} = 1.$ *(Kurukshetra, 2005)*
14. $\sin^{-1}(dy/dx) = x + y$ *(V.T.U., 2010)*
15. $\frac{dy}{dx} = \cos(x+y+1)$ *(V.T.U., 2003)*
16. $\frac{dy}{dx} - x\tan(y-x) = 1.$
17. $x^4\frac{dy}{dx} + x^3y + \operatorname{cosec}(xy) = 0.$

11.7 HOMOGENEOUS EQUATIONS

are of the form $\frac{dy}{dx} = \frac{f(x, y)}{\phi(x, y)}$

where $f(x, y)$ and $\phi(x, y)$ are homogeneous functions of the same degree in x and y (see page 205).

To solve a homogeneous equation (*i*) Put $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$,

(*ii*) Separate the variables v and x, and integrate.

Example 11.10. *Solve* $(x^2 - y^2)\,dx - xy\,dy = 0.$

Solution. Given equation is $\frac{dy}{dx} = \frac{x^2 - y^2}{xy}$ which is homogeneous in x and y. ...(*i*)

Put $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$. $\therefore$ (*i*) becomes $v + x\frac{dv}{dx} = \frac{1 - v^2}{v}$

or $$x\frac{dv}{dx} = \frac{1-v^2}{v} - v = \frac{1 - 2v^2}{v}.$$

Separating the variables, $\dfrac{v}{1-2v^2}\,dv = \dfrac{dx}{x}$

Integrating both sides, $\displaystyle\int \frac{v\,dv}{1-2v^2} = \int \frac{dx}{x} + c$

or $$-\frac{1}{4}\int \frac{-4v}{1-2v^2}\,dv = \int \frac{dx}{x} + c \quad \text{or} \quad -\frac{1}{4}\log(1-2v^2) = \log x + c$$

or $$4\log x + \log(1-2v^2) = -4c \quad \text{or} \quad \log x^4(1-2v^2) = -4c \qquad [\text{Put } v = y/x]$$

or $$x^4(1-2y^2/x^2) = e^{-3c} = c'$$

Hence the required solution is $x^2(x^2-2y^2) = c'$.

Example 11.11. *Solve* $(x \tan y/x - y \sec^2 y/x)\,dx - x \sec^2 y/x\,dy = 0.$ (V.T.U., 2006)

Solution. The given equation may be rewritten as

$$\frac{dy}{dx} = \left(\frac{y}{x}\sec^2\frac{y}{x} - \tan\frac{y}{x}\right)\cos^2 y/x \qquad \text{...(i)}$$

vy

which is a homogeneous equation. Putting $y = vx$, (*i*) becomes $v + x\dfrac{dv}{dx} = (v\sec^2 v - \tan v)\cos^2 v$

or $$x\frac{dv}{dx} = v - \tan v\cos^2 v - v$$

Separating the variables $\dfrac{\sec^2 v}{\tan v}\,dv = -\dfrac{dx}{x}$

Integrating both sides $\log\tan v = -\log x + \log c$

or $$x\tan v = c \quad \text{or} \quad x\tan y/x = c.$$

Example 11.12. *Solve* $(1+e^{x/y})\,dx + e^{x/y}(1-x/y)\,dy = 0.$

(P.T.U., 2006 ; Rajasthan, 2005 ; V.T.U., 2003)

Solution. The given equation may be rewritten as

$$\frac{dx}{dy} = -\frac{e^{x/y}(1-x/y)}{1+e^{x/y}} \qquad \text{...(i)}$$

which is a homogeneous equation. Putting $x = vy$ so that (*i*) becomes

$$v + y\frac{dv}{dy} = -\frac{e^v(1-v)}{1+e^v} \quad \text{or} \quad y\frac{dv}{dy} = -\frac{e^v(1-v)}{1+e^v} - v = -\frac{v+e^v}{1+e^v}$$

Separating the variables, we get

$$-\frac{dy}{y} = \frac{1+e^v}{v+e^v}\,dv = \frac{d(v+e^v)}{v+e^v}$$

Integrating both sides, $-\log y = \log(v+e^v) + c$

or $$y(v+e^v) = e^{-c} \quad \text{or} \quad x + ye^{x/y} = c' \quad \text{(say)}$$

which is the required solution.

PROBLEMS 11.3

Solve the following differential equations :

1. $(x^2-y^2)\,dx = 2xy\,dy$
2. $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0.$ (Bhopal, 2008)
3. $x^2y\,dx - (x^3+y^3)\,dy = 0.$ (V.T.U., 2010)
4. $y\,dx - x\,dy = \sqrt{x^2+y^2}\,dx.$ (Raipur, 2005)
5. $y^2 + x^2\dfrac{dy}{dx} = xy\dfrac{dy}{dx}.$
6. $(3xy - 2ay^2)\,dx + (x^2 - 2axy)\,dy = 0.$ (S.V.T.U., 2009)

[Equations solvable like homogeneous equations : When a differential equation contains y/x a number of times, solve it like a homogeneous equation by putting $y/x = v$].

7. $\frac{dy}{dx} = \frac{y}{x} + \sin \frac{y}{x}$. (V.T.U., 2000 S)
8. $ye^{x/y}\, dx = (xe^{x/y} + y^2)\, dy$. (V.T.U., 2006)
9. $xy (\log x/y)\, dx + [y^2 - x^2 \log (x/y)]\, dy = 0$.
10. $x\, dx + \sin^2 (y/x) (y dx - x dy) = 0$.
11. $x \cos \frac{y}{x} (y dx + x dy) = y \sin \frac{y}{x} (x dy - y dx)$.

11.8 EQUATIONS REDUCIBLE TO HOMOGENEOUS FORM

The equations of the form $\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'}$ ...(1)

can be reduced to the homogeneous form as follows :

Case I. When $\frac{a}{a'} \neq \frac{b}{b'}$

Putting $x = X + h,\ y = Y + k,$ (h, k being constants)

so that $dx = dX,\ dy = dY$, (1) becomes

$$\frac{dY}{dX} = \frac{aX + bY + (ah + bk + c)}{a'X + b'Y + (a'h + b'k + c')} \qquad ...(2)$$

Choose h, k so that (2) may become homogeneous.

Put $ah + bk + c = 0$, and $a'h + b'k + c' = 0$

so that $$\frac{h}{bc' - b'c} = \frac{k}{ca' - c'a} = \frac{1}{ab' - ba'}$$

or $$h = \frac{bc' - b'c}{ab' - b'a},\ k = \frac{ca' - c'a}{ab' - ba'} \qquad ...(3)$$

Thus when $ab' - ba' \neq 0$, (2) becomes $\frac{dY}{dX} = \frac{aX + bY}{a'X + b'Y}$ which is homogeneous in X, Y and can be solved by putting $Y = vX$.

Case II. When $\frac{a}{a'} = \frac{b}{b'}$.

i.e., $ab' - b'a = 0$, the above method fails as h and k become infinite or indeterminate.

Now $\frac{a}{a'} = \frac{b}{b'} = \frac{1}{m}$ (say)

$\therefore$ $a' = am,\ b' = bm$ and (1) becomes

$$\frac{dy}{dx} = \frac{(ax + by) + c}{m(ax + by) + c'} \qquad ...(4)$$

Put $ax + by = t$, so that $a + b\frac{dy}{dx} = \frac{dt}{dx}$

or $\frac{dy}{dx} = \frac{1}{b}\left(\frac{dt}{dx} - a\right)$ $\therefore$ (4) becomes $\frac{1}{b}\left(\frac{dt}{dx} - a\right) = \frac{t + c}{mt + c'}$

or $$\frac{dt}{dx} = a + \frac{bt + bc}{mt + c'} = \frac{(am + b)t + ac' + bc}{mt + c'}$$

so that the variables are separable. In this solution, putting $t = ax + by$, we get the required solution of (1).

Example 11.13. *Solve* $\frac{dy}{dx} = \frac{y + x - 2}{y - x - 4}$. (Raipur, 2005)

Solution. Given equation is $\frac{dy}{dx} = \frac{y + x - 2}{y - x - 4}$ $\left[\text{Case } \frac{a}{a'} \neq \frac{b}{b'}\right]$ ...(i)

Putting $x = X + h$, $y = Y + k$, (h, k being constants) so that $dx = dX$, $dy = dY$, (i) becomes

$$\frac{dY}{dX} = \frac{Y + X + (k + h - 2)}{Y - X + (k - h - 4)} \qquad ...(ii)$$

Put $k + h - 2 = 0$ and $k - h - 4 = 0$ so that $h = -1, k = 3$.

$\therefore$ (ii) becomes $\dfrac{dY}{dX} = \dfrac{Y + X}{Y - X}$ which is homogeneous in X and Y. ...(iii)

$\therefore$ put $Y = vX$, then $\dfrac{dY}{dX} = v + X\dfrac{dv}{dX}$

$\therefore$ (iii) becomes $\quad v + X\dfrac{dv}{dX} = \dfrac{v+1}{v-1}$ or $X\dfrac{dv}{dX} = \dfrac{v+1}{v-1} - v = \dfrac{1 + 2v - v^2}{v - 1}$

or $$\frac{v-1}{1 + 2v - v^2}\,dv = \frac{dX}{X}.$$

Integrating both sides, $-\dfrac{1}{2}\displaystyle\int \dfrac{2 - 2v}{1 + 2v - v^2}\,dv = \int \dfrac{dX}{X} + c.$

or $$-\frac{1}{2}\log(1 + 2v - v^2) = \log X + c$$

or $$\log\left(1 + \frac{2Y}{X} - \frac{Y^2}{X^2}\right) + \log X^2 = -2c$$

or $$\log(X^2 + 2XY - Y^2) = -2c \quad \text{or} \quad X^2 + 2XY - Y^2 = e^{-2c} = c' \qquad ...(iv)$$

Putting $X = x - h = x + 1$, $Y = y - k = y - 3$, (iv) becomes

$$(x + 1)^2 + 2(x + 1)(y - 3) - (y - 3)^2 = c'$$

or $x^2 + 2xy - y^2 - 4x + 8y - 14 = c'$ which is the required solution.

Example 11.14. *Solve* $(3y + 2x + 4)\,dx - (4x + 6y + 5)\,dy = 0.$ (*Madras, 2000 S*)

Solution. Given equation is $\dfrac{dy}{dx} = \dfrac{(2x + 3y) + 4}{2(2x + 3y) + 5}$ ...(i)

Putting $2x + 3y = t$ so that $2 + 3\dfrac{dy}{dx} = \dfrac{dt}{dx}$ $\therefore$ (i) becomes $\dfrac{1}{3}\left(\dfrac{dt}{dx} - 2\right) = \dfrac{t + 4}{2t + 5}$

or $$\frac{dt}{dx} = 2 + \frac{3t + 12}{2t + 5} = \frac{7t + 22}{2t + 5} \quad \text{or} \quad \frac{2t + 5}{7t + 22}\,dt = dx$$

Integrating both sides, $\displaystyle\int \frac{2t + 5}{7t + 22}\,dt = \int dx + c$

or $$\int\left(\frac{2}{7} - \frac{9}{7}\cdot\frac{1}{7t + 22}\right)dt = x + c \quad \text{or} \quad \frac{2}{7}t - \frac{9}{49}\log(7t + 22) = x + c$$

Putting $t = 2x + 3y$, we have $14(2x + 3y) - 9\log(14x + 21y + 22) = 49x + 49c$

or $\quad 21x - 42y + 9\log(14x + 21y + 22) = c'$ which is the required solution.

PROBLEMS 11.4

Solve the following differential equations :

1. $(x - y - 2)\,dx + (x - 2y - 3)\,dy = 0.$ (*Rajasthan, 2006*)
2. $(2x + y - 3)\,dy = (x + 2y - 3)\,dx.$ (*V.T.U., 2009 S ; Madras, 2000*)
3. $(2x + 5y + 1)\,dx - (5x + 2y - 1)\,dy = 0.$ (*J.N.T.U., 2000*)
4. $\dfrac{dy}{dx} + \dfrac{ax + hy + g}{hx + by + f} = 0.$
5. $\dfrac{dy}{dx} = \dfrac{x + y + 1}{2x + 2y + 3}.$
6. $(4x - 6y - 1)\,dx + (3y - 2x - 2)\,dy = 0.$
7. $(x + 2y)(dx - dy) = dx + dy.$ (*Bhopal, 2002 S ; V.T.U., 2001*)

11.9 LINEAR EQUATIONS

A differential equation is said to be linear if the dependent variable and its differential coefficients occur only in the first degree and not multiplied together.

Thus **the standard form of a linear equation of the first order,** *commonly known as* **Leibnitz's linear equation,*** *is*

$$\frac{dy}{dx} + Py = Q \quad \text{where, } P, Q \text{ are the functions of } x. \qquad ...(1)$$

To solve the equation, multiply both sides by $e^{\int P dx}$ so that we get

$$\frac{dy}{dx} . e^{\int Pdx} + y\,(e^{\int Pdx} P) = Qe^{\int Pdx} \quad i.e., \quad \frac{d}{dx}(ye^{\int Pdx}) = Qe^{\int Pdx}$$

Integrating both sides, we get $ye^{\int Pdx} = \int Qe^{\int Pdx}\, dx + c$ as the required solution.

Obs. The factor $e^{\int Pdx}$ on multiplying by which the left-hand side of (1) becomes the differential coefficient of a single function, is called the **integrating factor (I.F.)** of the linear equation (1).

It is important to remember that **I.F.** $= \mathbf{e^{\int Pdx}}$

and the solution is $\mathbf{y}$ **(I.F.)** $= \int \mathbf{Q(I.F.)\,dx + c}$.

Example 11.15. *Solve* $(x + 1)\dfrac{dy}{dx} - y\, e^{3x}(x + 1)^2$.

Solution. Dividing throughout by $(x + 1)$, given equation becomes

$$\frac{dy}{dx} - \frac{y}{x+1} = e^{3x}(x + 1) \text{ which is Leibnitz's equation.} \qquad ...(i)$$

Here $\qquad P = -\dfrac{1}{x+1}$ and $\int Pdx = -\int \dfrac{dx}{x+1} = -\log(x + 1) = \log(x + 1)^{-1}$

$\therefore \qquad$ I.F. $= e^{\int Pdx} = e^{\log(x+1)^{-1}} = \dfrac{1}{x+1}$

Thus the solution of (1) is $y(\text{I.F.}) = \int [e^{3x}(x + 1)](\text{I.F.})\, dx + c$

or $\qquad \dfrac{y}{x+1} = \int e^{3x}\, dx + c = \dfrac{1}{3}e^{3x} + c \quad$ or $\quad y = \left(\frac{1}{3}e^{3x} + c\right)(x + 1).$

Example 11.16. *Solve* $\left(\dfrac{e^{-2\sqrt{x}}}{\sqrt{x}} - \dfrac{y}{\sqrt{x}}\right)\dfrac{dx}{dy} = 1.$

Solution. Given equation can be written as $\dfrac{dy}{dx} + \dfrac{y}{\sqrt{x}} = \dfrac{e^{-2\sqrt{x}}}{\sqrt{x}} \qquad ...(i)$

$\therefore \qquad$ I.F. $= e^{\int x^{1/2} dx} = e^{2\sqrt{x}}$

Thus solution of (*i*) is $y\,(\text{I.F.}) = \int \dfrac{e^{-2\sqrt{x}}}{\sqrt{x}}(\text{I.F.})\, dx + c$

or $\qquad ye^{2\sqrt{x}} = \int \dfrac{e^{-2\sqrt{x}}}{\sqrt{x}} . e^{2\sqrt{x}}\, dx + c$

or $\qquad ye^{2\sqrt{x}} = \int x^{-1/2}\, dx + c \quad$ or $\quad ye^{2\sqrt{x}} = 2\sqrt{x} + c.$

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* See footnote p. 139.

Example 11.17. *Solve* $3x(1-x^2)\,y^2\,\dfrac{dy}{dx}+(2x^2-1)\,y^3=ax^3$ *(Rajasthan, 2006)*

Solution. Putting $y^3 = z$ and $3y^2\,\dfrac{dy}{dx}=\dfrac{dz}{dx}$, the given equation becomes

$$x(1-x^2)\,\frac{dz}{dx}+(2x^2-1)\,z=ax^3,\quad\text{or}\quad \frac{dz}{dx}+\frac{2x^2-1}{x-x^3}\,z=\frac{ax^3}{x-x^3} \qquad ...(i)$$

which is Leibnitz's equation in z

$$\therefore \qquad \text{I.F.} = \exp\left(\int \frac{2x^2-1}{x-x^3}\,dx\right)$$

Now $\displaystyle\int \frac{2x^2-1}{x-x^3}\,dx=\int\left(-\frac{1}{x}-\frac{1}{2}\,\frac{1}{1+x}+\frac{1}{2}\cdot\frac{1}{1-x}\right)dx=-\log x-\frac{1}{2}\log(1+x)-\frac{1}{2}\log(1-x)$

$$=-\log\,[x\sqrt{(1-x^2)}]$$

$$\therefore \qquad \text{I.F.} = e^{-\log[x\sqrt{(1-x^2)}]}=[x\sqrt{(1-x^2)}]^{-1}$$

Thus the solution of (i) is

$$z(\text{I.F.})=\int\frac{ax^3}{x-x^3}\,(\text{I.F.})\,dx+c$$

or
$$\frac{z}{[x\sqrt{(1-x^2)}]}=a\int\frac{x^3}{x(1-x^2)}\cdot\frac{1}{x\sqrt{(1-x^2)}}\,dx+c=a\int x(1-x^2)^{-3/2}\,dx$$

$$=-\frac{a}{2}\int(-2x)(1-x^2)^{-3/2}\,dx+c=a\,(1-x^2)^{-1/2}+c$$

Hence the solution of the given equation is

$$y^3=ax+cx\sqrt{(1-x^2)}. \qquad [\because\ z=y^3]$$

Example 11.18. *Solve* $y\,(\log y)\,dx+(x-\log y)\,dy=0.$ *(U.P.T.U., 2000)*

Solution. We have $\dfrac{dx}{dy}+\dfrac{x}{y\log y}=\dfrac{1}{y}$ ... (i)

which is a Leibnitz's equation in x

$$\therefore \qquad \text{I.F.} = e^{\int\frac{1}{y\log y}dy}=e^{\log(\log y)}=\log y$$

Thus the solution of (i) is $x\,(\text{I.F.})=\displaystyle\int\frac{1}{y}(\text{I.F.})\,dy+c$

$$x\log y=\int\frac{1}{y}\log y\,dy+c=\frac{1}{2}(\log y)^2+c$$

i.e.,
$$x=\frac{1}{2}\log y+c\,(\log y)^{-1}.$$

Example 11.19. *Solve* $(1+y^2)\,dx=(\tan^{-1}y-x)\,dy.$ *(Bhopal, 2008 ; V.T.U., 2008 ; U.P.T.U., 2005)*

Solution. This equation contains y^2 and $\tan^{-1}y$ and is, therefore, not a linear in y; but since only x occurs, it can be written as

$$(1+y^2)\,\frac{dx}{dy}=\tan^{-1}y-x\quad\text{or}\quad\frac{dx}{dy}+\frac{x}{1+y^2}=\frac{\tan^{-1}y}{1+y^2}$$

which is a Leibnitz's equation in x.

$$\therefore \qquad \text{I.F.} = e^{\int P\,dy}=e^{\int\frac{1}{1+y^2}dy}=e^{\tan^{-1}y}$$

Thus the solution is $x\,(\text{I.F.})=\displaystyle\int\frac{\tan^{-1}y}{1+y^2}\,(\text{I.F.})\,dy+c$

or $$xe^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2} . e^{\tan^{-1} y} dy + c$$ $\left[\begin{array}{l}\text{Put } \tan^{-1} y = t \\ \therefore \ \dfrac{dy}{1+y^2} = dt\end{array}\right]$

$$= \int te^t \, dt + c = t \, . e^t - \int 1 . e^t \, dt + c$$ (Integrating by parts)

$$= t \, . e^t - e^t + c = (\tan^{-1} y - 1) \, e^{\tan^{-1} y} + c$$

or $$x = \tan^{-1} y - 1 + ce^{-\tan^{-1} y}.$$

Example 11.20. *Solve* $r \sin \theta \, d\theta + (r^3 - 2r^2 \cos \theta + \cos \theta) \, dr = 0.$

Solution. Given equation can be rewritten as

$$\sin \theta \frac{d\theta}{dr} + \frac{1}{r} (1 - 2r^2) \cos \theta = - r^2 \quad ...(i)$$

Put $\cos \theta = y$ so that $- \sin \theta \, d\theta/dr = dy/dr$

Then (i) becomes $-\dfrac{dy}{dr} + \left(\dfrac{1}{r} - 2r\right) y = - r^2$ or $\dfrac{dy}{dr} + \left(2r - \dfrac{1}{r}\right) y = r^2$

which is a Leibnitz's equation $\therefore$ I.F. $= e^{\int (2r - 1/r) dr} = e^{r^2 - \log r} = \dfrac{1}{r} e^{r^2}$

Thus its solution is $y \left(\dfrac{1}{r} e^{r^2}\right) = \displaystyle\int r^2 . e^{r^2} . \frac{1}{r} \, dr + c$

or $$ye^{r^2}/r = \frac{1}{2} \int e^{r^2} 2r \, dr + c = \frac{1}{2} e^{r^2} + c$$

or $$2e^{r^2} \cos \theta = re^{r^2} + 2cr \quad \text{or} \quad r(1 + 2ce^{-r^2}) = 2 \cos \theta.$$

PROBLEMS 11.5

Solve the following differential equations :

1. $\cos^2 x \dfrac{dy}{dx} + y = \tan x.$
2. $x \log x \dfrac{dy}{dx} + y = \log x^2.$ (V.T.U., 2011)
3. $2y' \cos x + 4y \sin x = \sin 2x$, given $y = 0$ when $x = \pi/3$. (V.T.U., 2003)
4. $\cosh x \dfrac{dy}{dx} + y \sinh x = 2 \cosh^2 x \sinh x$. (J.N.T.U., 2003)
5. $(1 - x^2) \dfrac{dy}{dx} - xy = 1$ (V.T.U., 2010)
6. $(1 - x^2) \dfrac{dy}{dx} + 2xy = x \sqrt{(1 - x^2)}$ (Nagpur, 2009)
7. $\dfrac{dy}{dx} = -\dfrac{x + y \cos x}{1 + \sin x}$.
8. $dr + (2r \cot \theta + \sin 2\theta) \, d\theta = 0.$ (J.N.T.U., 2003)
9. $\dfrac{dy}{dx} + 2xy = 2e^{-x^2}$ (P.T.U., 2005)
10. $(x + 2y^3) \dfrac{dy}{dx} = y.$ (Marathwada, 2008)
11. $\sqrt{(1 - y^2)} \, dx = (\sin^{-1} y - x) \, dy.$
12. $ye^y \, dx = (y^3 + 2xe^y) \, dy.$
13. $(1 + y^2) \, dx + (x - e^{-\tan^{-1} y}) \, dy = 0.$ (V.T.U., 2006)
14. $e^{-y} \sec^2 y \, dy = dx + x \, dy.$

11.10 BERNOULLI'S EQUATION

The equation $\dfrac{dy}{dx} + Py = Qy^n$ ...(1)

where P, Q are functions of x, is reducible to the Leibnitz's linear equation and is usually called the Bernoulli's equation.*

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*Named after the Swiss mathematician *Jacob Bernoulli* (1654–1705) who is known for his basic work in probability and elasticity theory. He was professor at Basel and had amongst his students his youngest brother *Johann Bernoulli* (1667–1748) and his nephew *Niklaus Bernoulli* (1687–1759). Johann is known for his basic contributions to Calculus while Niklaus had profound influence on the development of Infinite series and probability. His son *Daniel Bernoulli* (1700–1782) is known for his contributions to kinetic theory of gases and fluid flow.

To solve (1), divide both sides by y^n, so that $y^{-n}\dfrac{dy}{dx} + Py^{1-n} = Q$ ...(2)

Put $y^{1-n} = z$ so that $(1-n)\,y^{-n}\dfrac{dy}{dx} = \dfrac{dz}{dx}$.

$\therefore$ (2) becomes $\dfrac{1}{1-n}\dfrac{dz}{dx} + Pz = Q$ or $\dfrac{dz}{dx} + P(1-n)\,z = Q(1-n)$,

which is Leibnitz's linear in z and can be solved easily.

Example 11.21. *Solve* $x\dfrac{dy}{dx} + y = x^3y^6$.

Solution. Dividing throughout by xy^6, $y^{-6}\dfrac{dy}{dx} + \dfrac{y^{-5}}{x} = x^2$ ...(*i*)

Put $y^{-5} = z$, so that $-5y^{-6}\dfrac{dy}{dx} = \dfrac{dz}{dx}$ $\therefore$ (*i*) becomes $-\dfrac{1}{5}\dfrac{dz}{dx} + \dfrac{z}{x} = x^2$

or $$\frac{dz}{dx} - \frac{5}{x}z = -5x^2 \text{ which is Leibnitz's linear in } z. \quad ...(ii)$$

$$\text{I.F.} = e^{-\int (5/x)dx} = e^{-5\log x} = e^{\log x^{-5}} = x^{-5}$$

$\therefore$ the solution of (*ii*) is $z\,(\text{I.F.}) = \int(-5x^2)(\text{I.F.})dx + c$ or $zx^{-5} = \int(-5x^2)\,x^{-5}\,dx + c$

or $$y^{-5}x^{-5} = -5.\frac{x^{-2}}{-2} + c \qquad [\because\ z = y^{-5}]$$

Dividing throughout by $y^{-5}x^{-5}$, $1 = (2.5 + cx^2)\,x^3y^5$ which is the required solution.

Example 11.22. *Solve* $xy\,(1 + xy^2)\dfrac{dy}{dx} = 1$. (*Nagpur, 2009*)

Solution. Rewriting the given equation as

$$\frac{dx}{dy} - yx = y^3x^2$$

and dividing by x^2, we have

$$x^{-2}\frac{dx}{dy} - yx^{-1} = y^3 \quad ...(i)$$

Putting $x^{-1} = z$ so that $-x^{-2}\dfrac{dx}{dy} = \dfrac{dz}{dy}$ (*i*) becomes

$$\frac{dz}{dy} + yz = -y^3 \text{ which is Leibnitz's linear in } z.$$

Here $\text{I.F.} = e^{\int y\,dy} = e^{y^2/2}$

$\therefore$ the solution is $z\,(\text{I.F.}) = \int(-y^3)(\text{I.F.})\,dy + c$

or $$ze^{y^2/2} = -\int y^2 . e^{\frac{1}{2}y^2} . y\,dy + c \qquad \left|\ \begin{array}{l}\text{Put } \tfrac{1}{2}y^2 = t \\ \text{so that } y\,dy = dt\end{array}\right.$$

$$= -2\int t . e^t\,dt + c \qquad \text{[Integrate by parts]}$$

$$= -2\,[t . e^t - \int 1 . e^t\,dt] + c = -2\,[te^t - e^t] + c = (2 - y^2)\,e^{y^2/2} + c$$

or $$z = (2 - y^2) + ce^{-\frac{1}{2}y^2} \quad \text{or} \quad 1/x = (2 - y^2) + ce^{-\frac{1}{2}y^2}.$$

Note. *General equation reducible to Leibnitz's linear is* $f'(y)\dfrac{dy}{dx} + Pf(y) = Q$ ...(*A*)

where P, Q are functions of x. To solve it, put $f(y) = z$.

Example 11.23. *Solve* $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$. *(V.T.U., 2011 ; Marathwada, 2008 ; J.N.T.U., 2005)*

Solution. Dividing throughout by $\cos^2 y$, $\sec^2 y \frac{dy}{dx} + 2x \frac{\sin y \cos y}{\cos^2 y} = x^3$

or $$\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3 \text{ which is of the form } (A) \text{ above.} \quad ...(i)$$

$\therefore$ put $\tan y = z$ so that $\sec^2 y \frac{dy}{dx} = \frac{dz}{dx}$ $\quad\therefore$ (*i*) becomes $\frac{dz}{dx} + 2xz = x^3$.

This is Leibnitz's linear equation in z. $\therefore$ I.F. $= e^{\int 2x\,dx} = e^{x^2}$

$\therefore$ the solution is $ze^{x^2} = \int e^{x^2} x^3 dx + c = \frac{1}{2}(x^2 - 1)e^{x^2} + c$.

Replacing z by $\tan y$, we get $\tan y = \frac{1}{2}(x^2 - 1) + ce^{-x^2}$ which is the required solution.

Example 11.24. *Solve* $\frac{dz}{dx} + \left(\frac{z}{x}\right) \log z = \frac{z}{x}(\log z)^2$.

Solution. Dividing by z, the given equation becomes

$$\frac{1}{z}\frac{dz}{dx} + \frac{1}{x}\log z = \frac{1}{x}(\log z)^2 \quad ...(i)$$

Put $\log z = t$ so that $\frac{1}{z}\frac{dz}{dx} = \frac{dt}{dx}$. $\quad\therefore$ (*i*) becomes

$$\frac{dt}{dx} + \frac{t}{x} = \frac{t^2}{x} \quad \text{or} \quad \frac{1}{t^2}\frac{dt}{dx} + \frac{1}{x}\cdot\frac{1}{t} = \frac{1}{x} \quad ...(ii)$$

This being Bernoulli's equation, put $1/t = v$ so that (*ii*) reduces to

$$-\frac{dv}{dx} + \frac{v}{x} = \frac{1}{x} \quad \text{or} \quad \frac{dv}{dx} - \frac{1}{x}v = -\frac{1}{x}$$

This is Leibnitz's linear in v. $\therefore$ I.F. $= e^{-\int 1/x\,dx} = 1/x$

$\therefore$ the solution is $$v \cdot \frac{1}{x} = -\int \frac{1}{x}\cdot\frac{1}{x}dx + c = \frac{1}{x} + c$$

Replacing v by $1/\log z$, we get $(x \log z)^{-1} = x^{-1} + c$ or $(\log z)^{-1} = 1 + cx$ which is the required solution.

PROBLEMS 11.6

Solve the following equations :

1. $\frac{dy}{dx} = y \tan x - y^2 \sec x$. *(P.T.U., 2005)*
2. $r \sin\theta - \cos\theta \frac{dr}{d\theta} = r^2$. *(V.T.U., 2005)*
3. $2xy' = 10x^3y^5 + y$.
4. $(x^3y^2 + xy)\,dx = dy$. *(B.P.T.U., 2005)*
5. $\frac{dy}{dx} = \frac{x^2 + y^2 + 1}{2xy}$. *(Bhillai, 2005)*
6. $x(x - y)\,dy + y^2\,dx = 0$. *(I.S.M., 2001)*
7. $\frac{dy}{dx} - \frac{\tan y}{1 + x} = (1 + x)e^x \sec y$. *(Bhopal, 2009)*
8. $e^y\left(\frac{dy}{dx} + 1\right) = e^x$. *(V.T.U., 2009)*
9. $\sec^2 y \frac{dy}{dx} + x \tan y = x^3$.
10. $\tan y \frac{dy}{dx} + \tan x = \cos y \cos^2 x$. *(Sambalpur, 2002)*
11. $\frac{dy}{dx} = \frac{y}{x - \sqrt{(xy)}}$. *(V.T.U., 2011)*
12. $(y \log x - 2)\,y dx - x dy = 0$. *(V.T.U., 2006)*

11.11 EXACT DIFFERENTIAL EQUATIONS

(1) Def. *A differential equation of the form* $\mathbf{M}(x, y)\, \mathbf{dx} + \mathbf{N}(x, y)\, \mathbf{dy} = 0$ *is said to be* ***exact*** *if its left hand member is the exact differential of some function* $u(x, y)$ *i.e.,* $du = Mdx + Ndy = 0$. Its solution, therefore, is $u(x, y) = c$.

(2) Theorem. The necessary and sufficient condition for the differential equation $Mdx + Ndy = 0$ to be exact is

$$\frac{\partial \mathbf{M}}{\partial \mathbf{y}} = \frac{\partial \mathbf{N}}{\partial \mathbf{x}}$$

Condition is necessary :

The equation $Mdx + Ndy = 0$ will be exact, if

$$Mdx + Ndy \equiv du \quad \text{...(1)}$$

where u is some function of x any y.

But $$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \quad \text{...(2)}$$

$\therefore$ equating coefficients of dx and dy in (1) and (2), we get $M = \dfrac{\partial u}{\partial x}$ and $N = \dfrac{\partial u}{\partial y}$

$\therefore$ $$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \text{ and } \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}.$$

But $$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} \quad \text{(Assumption)}$$

$\therefore$ $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ which is the necessary condition for exactness.

Condition is sufficient : i.e., if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, then $Mdx + Ndy = 0$ is exact.

Let $\int Mdx = u$, where y is supposed constant while performing integration.

Then $$\frac{\partial}{\partial x}\left(\int Mdx\right) = \frac{\partial u}{\partial x}, \quad i.e., \ M = \frac{\partial u}{\partial x}$$

$\therefore$ $$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad \text{or} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) \quad \left\{\begin{array}{l} \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \text{ (given)} \\ \text{and } \dfrac{\partial^2 u}{\partial y \partial x} = \dfrac{\partial^2 u}{\partial x \partial y} \end{array}\right. \quad \text{...(3)}$$

Interating both sides w.r.t. x (taking y as constant).

$$N = \frac{\partial u}{\partial y} + f(y), \text{ where } f(y) \text{ is a function of } y \text{ alone.} \quad \text{...(4)}$$

$\therefore$ $$Mdx + Ndy = \frac{\partial u}{\partial x} dx + \left\{\frac{\partial u}{\partial y} + f(y)\right\} dy \quad \text{[By (3) and (4)]}$$

$$= \left\{\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy\right\} + f(y)\, dy = du + f(y)\, dy = d\left[u + \int f(y) dy\right] \quad \text{...(5)}$$

which shows that $Mdx + Ndy = 0$ is exact.

(3) Method of solution. By (5), the equation $Mdx + Ndy = 0$ becomes $d[u + \int f(y)dy] = 0$

Integrating $\quad u + \int f(y) dy = 0.$

But $\quad u = \int_{y\text{ constant}} Mdx$ and $f(y)$ = terms of N not containing x.

$\therefore$ *The solution of* $Mdx + Ndy = 0$ *is*

$$\int_{(y \text{ cons.})} \mathbf{M\, dx} + \int (\textbf{terms of N not containing x})\, \mathbf{dy} = \mathbf{c}$$

provided $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Example 11.25. *Solve* $(y^2 e^{xy^2} + 4x^3)\, dx + (2xy\, e^{xy^2} - 3y^2)\, dy = 0.$ *(V.T.U., 2006)*

Solution. Here $M = y^2 e^{xy^2} + 4x^3$ and $N = 2xy\, e^{xy^2} - 3y^2$

$$\therefore \quad \frac{\partial M}{\partial y} = 2y\, e^{xy^2} + y^2\, e^{xy^2} \cdot 2xy = \frac{\partial N}{\partial x}$$

Thus the equation is exact and its solution is

$$\int_{(y \text{ const.})} M dx + \int (\text{terms of } N \text{ not containing } x)\, dy = c$$

i.e.,
$$\int_{(y \text{ const.})} (y^2 e^{y^2 x} + 4x^3) dx + \int (-3y^2)\, dy = c \quad \text{or} \quad e^{xy^2} + x^4 - y^3 = c.$$

Example 11.26. *Solve* $\left\{y\left(1 + \frac{1}{x}\right) + \cos y\right\} dx + (x + \log x - x \sin y)\, dy = 0.$

(Marathwada, 2008 S ; V.T.U., 2006)

Solution. Here $M = y\,(1 + 1/x) + \cos y$ and $N = x + \log x - x \sin y$

$$\therefore \quad \frac{\partial M}{\partial y} = 1 + 1/x - \sin y = \frac{\partial N}{\partial x}$$

Then the equation is exact and its solution is

$$\int_{(y \text{ const})} M dx + \int (\text{terms of } N \text{ not containing } x)\, dy = c$$

$$\int_{(y \text{ const})} \left\{\left(1 + \frac{1}{x}\right) y + \cos y\right\} dx = c \quad \text{or} \quad (x + \log x)\, y + x \cos y = c.$$

Example 11.27. *Solve* $(1 + 2xy \cos x^2 - 2xy)\, dx + (\sin x^2 - x^2)\, dy = 0.$

Solution. Here $M = 1 + 2xy \cos x^2 - 2xy$ and $N = \sin x^2 - x^2$

$$\therefore \quad \frac{\partial M}{\partial y} = 2x \cos x^2 - 2x = \frac{\partial N}{\partial x}$$

Thus the equation is exact and its solution is

$$\int_{(y \text{ const})} M dx + \int (\text{terms of } N \text{ not containing } x) = c$$

i.e.,
$$\int_{(y \text{ const})} (1 + 2xy \cos x^2 - 2xy)\, dx = c \quad \text{or} \quad x + y\left[\int \cos x^2 \cdot 2x dx - \int 2x\, dx\right] = c$$

or
$$x + y \sin x^2 - yx^2 = c.$$

Example 11.28. *Solve* $\frac{dy}{dx} + \frac{y \cos x + \sin y + y}{\sin x + x \cos y + x} = 0.$ *(Kurukshetra, 2005)*

Solution. Given equation can be written as

$$(y \cos x + \sin y + y)\, dx + (\sin x + x \cos y + x)\, dy = 0.$$

Here $M = y \cos x + \sin y + y$ and $N = \sin x + x \cos y + x$.

$$\therefore \quad \frac{\partial M}{\partial y} = \cos x + \cos y + 1 = \frac{\partial N}{\partial x}.$$

Thus the equation is exact and its solution is

$$\int_{(y \text{ const.})} M dx + \int (\text{terms of } N \text{ not containing } x)\, dy = c$$

i.e.,
$$\int_{(y \text{ const.})} (y \cos x + \sin y + y)\, dx + \int (0)\, dx = c \quad \text{or} \quad y \sin x + (\sin y + y)\, x = c.$$

Example 11.29. *Solve* $(2x^2 + 3y^2 - 7)\,xdx - (3x^2 + 2y^2 - 8)\,ydy = 0.$ (*U.P.T.U., 2005*)

Solution. Given equation can be written as

$$\frac{ydy}{xdx} = \frac{2x^2 + 3y^2 - 7}{3x^2 + 2y^2 - 8}$$

or $$\frac{ydy + xdx}{ydy - xdx} = \frac{5(x^2 + y^2 - 3)}{-x^2 + y^2 + 1}$$ [By componendo & dividendo]

or $$\frac{xdx + ydy}{x^2 + y^2 - 3} = 5 \cdot \frac{xdx - ydy}{x^2 - y^2 - 1}$$

Integrating both sides, we get

$$\int \frac{2xdx + 2ydy}{x^2 + y^2 - 3} = 5 \int \frac{2xdx - 2ydy}{x^2 - y^2 - 1} + c$$

or $$\log (x^2 + y^2 - 3) = 5 \log (x^2 - y^2 - 1) + \log c'$$ [Writing $c = \log c'$]

or $$x^2 + y^2 - 3 = c'\,(x^2 - y^2 - 1)^5$$

which is the required solution.

PROBLEMS 11.7

Solve the following equations :

1. $(x^2 - ay)\,dx = (ax - y^2)\,dy.$
2. $(x^2 + y^2 - a^2)\,xdx + (x^2 - y^2 - b^2)\,ydy = 0$ (*Kurukshetra, 2005*)
3. $(x^2 - 4xy - 2y^2)\,dx + (y^2 - 4xy - 2x^2)\,dy = 0.$
4. $(x^4 - 2xy^2 + y^4)\,dx - (2x^2y - 4xy^3 + \sin y)\,dy = 0$
5. $ye^{xy}dx + (xe^{xy} + 2y)\,dy = 0$
6. $(5x^4 + 3x^2y^2 - 2xy^3)\,dx + (2x^3y - 3x^2y^2 - 5y^4)\,dy = 0$ (*V.T.U., 2008*)
7. $(3x^2 + 6xy^2)\,dx + (6x^2y + 4y^3)\,dy = 0$
8. $\dfrac{2x}{y^3}\,dx + \dfrac{y^2 - 3x^2}{y^4}\,dy = 0$
9. $y \sin 2x\,dx - (1 + y^2 + \cos^2 x)\,dy = 0$
10. $(\sec x \tan x \tan y - e^x)\,dx + \sec x \sec^2 y\,dy = 0$ (*Marathwada, 2008*)
11. $(2xy + y - \tan y)\,dx + x^2 - x \tan^2 y + \sec^2 y)\,dy = 0.$ (*Nagpur, 2009*)

11.12 EQUATIONS REDUCIBLE TO EXACT EQUATIONS

Sometimes a differential equation which is not exact, can be made so on multiplication by a suitable factor called an *integrating factor*. The rules for finding integrating factors of the equation $Mdx + Ndy = 0$ are as follows :

(1) I.F. found by inspection. In a number of cases, the integrating factor can be found after regrouping the terms of the equation and recognizing each group as being a part of an exact differential. In this connection the following integrable combinations prove quite useful :

$$xdy + ydx = d\,(xy)$$

$$\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right); \frac{xdy - ydx}{xy} = d\left[\log\left(\frac{y}{x}\right)\right]$$

$$\frac{xdy - ydx}{y^2} = -d\left(\frac{x}{y}\right); \frac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$$

$$\frac{xdy - ydx}{x^2 - y^2} = d\left(\frac{1}{2}\log\frac{x + y}{x - y}\right).$$

Example 11.30. *Solve* $y\,(2xy + e^x)\,dx = e^x\,dy.$ (*Kurukshetra, 2005*)

Solution. It is easy to note that the terms ye^xdx and e^xdy should be put together.

$\therefore$ $$(ye^xdx - e^xdy) + 2xy^2\,dx = 0$$

Now we observe that the term $2xy^2\,dx$ should not involve y^2. This suggests that $1/y^2$ may be I.F. Multiplying throughout by $1/y^2$, it follows

$$\frac{ye^x\,dx - e^x\,dy}{y^2} + 2xdx = 0 \quad \text{or} \quad d\left(\frac{e^x}{y}\right) + 2xdx = 0$$

Integrating, we get $\dfrac{e^x}{y} + x^2 = c$ which is the required solution.

(2) I.F. of a homogeneous equation. *If $Mdx + Ndy = 0$ be a homogeneous equation in x and y, then $1/(Mx + Ny)$ is an integrating factor ($Mx + Ny \neq 0$).*

Example 11.31. *Solve $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$.* (*Osmania, 2003 S*)

Solution. This equation is homogeneous in x and y.

$$\therefore \qquad \text{I.F.} = \frac{1}{Mx + Ny} = \frac{1}{(x^2y - 2xy^2)\,x - (x^3 - 3x^2y)y} = \frac{1}{x^2y^2}$$

Multiplying throughout by $1/x^2y^2$, the equation becomes

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0 \text{ which is exact.}$$

$\therefore$ the solution is $\displaystyle\int_{(y\text{ const})} Mdx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$ or $\dfrac{x}{y} - 2\log x + 3\log y = c$.

(3) I.F. for an equation of the type $f_1(xy)ydx + f_2(xy)xdy = 0$.

If the equation $Mdx + Ndy = 0$ be of this form, then $1/(Mx - Ny)$ is an integrating factor ($Mx - Ny \neq 0$).

Example 11.32. *Solve $(1 + xy)\,ydx + (1 - xy)\,xdy = 0$.* (*S.V.T.U., 2008*)

Solution. The given equation is of the form $f_1(xy)\,ydx + f_2(xy)\,xdy = 0$

Here $\qquad M = (1 + xy)\,y, N = (1 - xy)\,x.$

$$\therefore \qquad \text{I.F.} = \frac{1}{Mx - Ny} = \frac{1}{(1+xy)yx - (1 - xy)xy} = \frac{1}{2x^2y^2}$$

Multiplying throughout by $1/2x^2y^2$, it becomes

$$\left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \left(\frac{1}{2xy^2} - \frac{1}{2y}\right)dy = 0, \text{ which is an exact equation.}$$

$\therefore$ the solution is $\displaystyle\int_{(y\text{ const})} Mdx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$

or
$$\frac{1}{2y}\left(-\frac{1}{x}\right) + \frac{1}{2}\log x - \frac{1}{2}\log y = c \quad \text{or} \quad \log\frac{x}{y} - \frac{1}{xy} = c'.$$

(4) In the equation $Mdx + Ndy = 0$,

(*a*) *if* $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N}$ *be a function of x only = f(x) say, then* $e^{\int f(x)dx}$ *is an integrating factor.*

(*b*) *if* $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M}$ *be a function of y only = F(y) say, then* $e^{\int F(y)dy}$ *is an integrating factor.*

Example 11.33. *Solve $(xy^2 - e^{1/x^3})\,dx - x^2y\,dy = 0$.* (*S.V.T.U., 2009 ; Mumbai, 2007*)

Solution. Here $M = xy^2 - e^{1/x^3}$ and $N = -x^2y$

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \frac{2xy - (-2xy)}{-x^2y} = -\frac{4}{x} \text{ which is a function of } x \text{ only.}$$

$\therefore$ $$\text{I.F.} = e^{\int \frac{-4}{x} dx} = e^{-4 \log x} = x^{-4}$$

Multiplying throughout by x^{-4}, we get $\left(\dfrac{y^2}{x^3} - \dfrac{1}{4^4} e^{1/x^3}\right) dx - \dfrac{y}{x^2}\, dy = 0$

which is an exact equation.

$\therefore$ the solution is $\int_{(y\text{ const})} (M dx) + \int (\text{terms of } N \text{ not containing } x)\, dy = c.$

or $$\int \left(\frac{y^2}{x^3} - \frac{1}{x^4} e^{1/x^3} \right) dx + 0 = c$$

or $$-\frac{y^2 x^{-2}}{2} + \frac{1}{3} \int e^{x^{-3}} (-3x^{-4})\, dx = c \text{ or } \frac{1}{3} e^{x^{-3}} - \frac{1}{2} \frac{y^2}{x^2} = c.$$

Otherwise it can be solved as a Bernoulli's equation (§ 11.10)

Example 11.34. *Solve* $(xy^3 + y)\, dx + 2\,(x^2y^2 + x + y^4)\, dy = 0.$

Solution. Here $M = xy^3 + y$, $N = 2(x^2y^2 + x + y^4)$

$\therefore$ $$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{y(xy^2 + 1)}(4xy^2 + 2 - 3xy^2 - 1) = \frac{1}{y}, \text{ which is a function of } y \text{ alone.}$$

$\therefore$ $$\text{I.F.} = e^{\int 1/y\, dy} = e^{\log y} = y$$

Multiplying throughout by y, it becomes $(xy^4 + y^2)\, dx + (2x^2y^3 + 2xy + 2y^5)\, dy = 0$, which is an exact equation.

$\therefore$ its solution is $\int_{(y\text{ const})} (M dx) + \int (\text{terms of } N \text{ not containing } x)\, dy = 0$

or $$\int_{(y\text{ const})} (xy^4 + y^2) dx + \int 2y^5\, dy = c \quad \text{or} \quad \frac{1}{2} x^2 y^4 + xy^2 + \frac{1}{3} y^6 = c.$$

Example 11.35. *Solve* $(y \log y)\, dx + (x - \log y)\, dy = 0$ *(U.P.T.U., 2004)*

Solution. Here $M = y \log y$ and $N = x - \log y$

$\therefore$ $$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{y \log y}(1 - \log y - 1) = -\frac{1}{y}, \text{ which is a function of } y \text{ alone.}$$

$\therefore$ $$\text{I.F.} = e^{-\int \frac{1}{y} dy} = e^{-\log y} = \frac{1}{y}$$

Multiplying the given equation throughout by $1/y$, it becomes

$$\log y\, dx + \frac{1}{y}(x - \log y)\, dy = 0$$

which is an exact equation $\left[\because \dfrac{\partial}{\partial y}(\log y) = \dfrac{\partial}{\partial x}\left(\dfrac{x - \log y}{y}\right)\right.$

$\therefore$ its solution is $\int_{(y\text{ const})} (M dx) + \int (\text{terms of } N \text{ not containing } x)\, dy = c$

or $$\log y \int dx + \int \left(\frac{-\log y}{y} \right) dy = c \qquad \text{or} \quad x \log y - \frac{1}{2} (\log y)^2 = c.$$

(5) For the equation of the type

$$x^a y^b\, (my dx + nx dy) + x^{a'} y^{b'}\, (m'y dx + n'x dy) = 0,$$

an integrating factor is $x^h y^k$

where $$\frac{a + h + 1}{m} = \frac{b + k + 1}{n}, \quad \frac{a' + h + 1}{m'} = \frac{b' + k + 1}{n'}.$$

Example 11.36. *Solve* $y(xy + 2x^2y^3)\,dx + x\,(xy - x^2y^2)\,dy = 0.$ *(Hissar, 2005 ; Kurukshetra, 2005)*

Solution. Rewriting the equation as $xy\,(ydx + xdy) + x^2y^2\,(2ydx - xdy) = 0$ and comparing with

$$x^ay^b\,(mydx + nxdy) + x^{a'}y^{b'}\,(m'ydx + n'xdy) = 0,$$

we have $a = b = 1$, $m = n = 1$; $a' = b' = 2$, $m' = 2$, $n' = -1$.

$$\therefore \qquad \text{I.F.} = x^hy^k.$$

where $$\frac{a+h+1}{m} = \frac{b+k+1}{n}, \frac{a'+h+1}{m'} = \frac{b'+k+1}{n'}$$

i.e. $$\frac{1+h+1}{1} = \frac{1+k+1}{1}, \frac{2+h+1}{2} = \frac{2+k+1}{-1}$$

or $$h - k = 0,\ h + 2k + 9 = 0$$

Solving these, we get $h = k = -3$. $\therefore$ I.F. $= 1/x^3y^3$.

Multiplying throughout by $1/x^3y^3$, it becomes

$$\left(\frac{1}{x^2y} + \frac{2}{x}\right)dx + \left(\frac{1}{xy^2} - \frac{1}{y}\right)dx = 0, \text{ which is an exact equation.}$$

$\therefore$ The solution is $\int_{(y\ const)} Mdx + \int(\text{terms of } N \text{ not containing } x)\,dy = c$

or $$\frac{1}{y}\left(-\frac{1}{x}\right) + 2\log x - \log y = c \quad \text{or} \quad 2\log x - \log y - 1/xy = c.$$

PROBLEMS 11.8

Solve the following equations :

1. $xdy - ydx + a(x^2 + y^2)\,dx = 0.$
2. $xdx + ydy = \dfrac{a^2(xdy - ydx)}{x^2 + y^2}.$ *(U.P.T.U., 2005)*
3. $ydx - xdy + \log x\,dx = 0.$
4. $\dfrac{dy}{dx} = \dfrac{x^3 + y^3}{xy^2}.$
5. $(x^3y^2 + x)\,dy + (x^2y^3 - y)\,dx = 0.$
6. $(x^2y^2 + xy + 1)ydx + (x^2y^2 - xy + 1)\,xdy = 0.$
7. $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0.$
8. $(4xy + 3y^2 - x)\,dx + x\,(x + 2y)dy = 0$ *(Mumbai, 2006)*
9. $x^4\dfrac{dy}{dx} + x^3y + \text{cosec}\,(xy) = 0.$
10. $(y - xy^2)\,dx - (x + x^2y)\,dy = 0$ *(Mumbai, 2006)*
11. $ydx - xdy + 3x^2y^2e^{x^3}\,dx = 0.$ *(Kurukshetra, 2006)*
12. $(y^2 + 2x^2y)\,dx + (2x^3 - xy)dy = 0.$ *(Rajasthan, 2005)*
13. $2ydx + x(2\log x - y)\,dy = 0.$ *(P.T.U., 2005)*

11.13 EQUATIONS OF THE FIRST ORDER AND HIGHER DEGREE

As dy/dx will occur in higher degrees, it is convenient to denote dy/dx by p. Such equations are of the form $f(x, y, p) = 0$. Three cases arise for discussion :

Case. I. Equation solvable for p. A differential equation of the first order but of the nth degree is of the form

$$p^n + P_1p^{n-1} + P_2p^{n-2} + \ldots + P_n = 0 \qquad \ldots(1)$$

where $P_1, P_2, \ldots, P_n$ are functions of x and y.

Splitting up the left hand side of (1) into n linear factors, we have

$$[p - f_1(x, y)]\,[p - f_2(x, y)] \ldots [p - f_n(x, y)] = 0.$$

Equating each of the factors to zero,

$$p = f_1(x, y), p = f_2(x, y), \ldots, p = f_n(x, y)$$

Solving each of these equations of the first order and first degree, we get the solutions

$$F_1(x, y, c) = 0, F_2(x, y, c) = 0, \ldots, F_n\,(x, y, c) = 0.$$

These n solutions constitute the general solution of (1).

Otherwise, the general solution of (1) may be written as

$$F_1(x, y, c) \,.\, F_2(x, y, c) \,.......\, F_n(x, y, c) = 0.$$

Example 11.37. *Solve* $\dfrac{dy}{dx} - \dfrac{dx}{dy} = \dfrac{x}{y} - \dfrac{y}{x}$.

Solution. Given equation is $p - \dfrac{1}{p} = \dfrac{x}{y} - \dfrac{y}{x}$ where $p = \dfrac{dy}{dx}$ or $p^2 + p\left(\dfrac{y}{x} - \dfrac{x}{y}\right) - 1 = 0.$

Factorising $(p + y/x)(p - x/y) = 0.$

Thus we have $\quad p + y/x = 0 \quad ...(i) \quad$ and $\quad p - x/y = 0 \quad ...(ii)$

From (*i*), $\quad \dfrac{dy}{dx} + \dfrac{y}{x} = 0$ or $xdy + ydx = 0$

i.e., $\quad d(xy) = 0$. Integrating, $xy = c$.

From (*ii*), $\quad \dfrac{dy}{dx} - \dfrac{x}{y} = 0$ or $xdx - ydx = 0$

Integrating, $x^2 - y^2 = c$. Thus $xy = c$ or $x^2 - y^2 = c$, constitute the required solution.

Otherwise, combining these into one, the required solution can be written as

$$(xy - c)(x^2 - y^2 - c) = 0.$$

Example 11.38. *Solve* $p^2 + 2py \cot x = y^2$. (*Bhopal, 2008 ; Kerala, 2005*)

Solution. We have $p^2 + 2py \cot x + (y \cot x)^2 = y^2 + y^2 \cot^2 x$

or $\quad p + y \cot x = \pm\, y \operatorname{cosec} x$

i.e., $\quad p = y(-\cot x + \operatorname{cosec} x) \quad ...(i)$

or $\quad p = y(-\cot x - \operatorname{cosec} x) \quad ...(ii)$

From (*i*), $\dfrac{dy}{dx} = y(-\cot x + \operatorname{cosec} x)$ or $\dfrac{dy}{y} = (\operatorname{cosec} x - \cot x)\, dx$

Integrating, $\log y = \log \tan \dfrac{x}{2} - \log \sin x + \log c = \log \dfrac{c \tan x/2}{\sin x}$

or $\quad y = \dfrac{c}{2 \cos x^2/2}$ or $y(1 + \cos x) = c \quad ...(iii)$

From (*ii*), $\quad \dfrac{dy}{dx} = -y(\cot x + \operatorname{cosec} x)$ or $\dfrac{dy}{y} = -(\cot x + \operatorname{cosec} x)\, dx$

Integrating, $\log y = -\log \sin x - \log \tan \dfrac{x}{2} + \log c = \log \dfrac{c}{\sin x \tan \dfrac{x}{2}}$

or $\quad y = \dfrac{c}{2 \sin^2 \dfrac{x}{2}}$ or $y(1 - \cos x) = c \quad ...(iv)$

Thus combining (*iii*) and (*iv*), the required general solution is

$$y(1 \pm \cos x) = c.$$

PROBLEMS 11.9

Solve the following equations :

1. $y\left(\dfrac{dy}{dx}\right)^2 + (x - y)\dfrac{dy}{dx} - x = 0.$
2. $p(p + y) = x(x + y)$. (*V.T.U., 2011*)
3. $y = x[p + \sqrt{(1 + p^2)}]$.
4. $xy\left(\dfrac{dy}{dx}\right)^2 - (x^2 + y^2)\dfrac{dy}{dx} + xy = 0.$
5. $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0.$ (*Madras, 2003*)

Case II. Equations solvable for y. If the given equation, on solving for y, takes the form

$$y = f(x, p). \qquad ...(1)$$

then differentiation with respect to x gives an equation of the form

$$p = \frac{dy}{dx} = \phi\left(x, p, \frac{dp}{dx}\right).$$

Now it may be possible to solve this new differential equation in x and p.

Let its solution be $F(x, p, c) = 0$. ...(2)

The ellimination of p from (1) and (2) gives the required solution.

In case elimination of p is not possible, then we may solve (1) and (2) for x and y and obtain

$$x = F_1(p, c), y = F_2(p, c)$$

as the required solution, where p is the parameter.

Obs. This method is especially useful for equations which do not contain x.

Example 11.39. *Solve* $y - 2px = \tan^{-1}(xp^2)$.

Solution. Given equation is $y = 2px + \tan^{-1}(xp^2)$ ...(*i*)

Differentiating both sides with respect to x, $\dfrac{dy}{dx} = p = 2\left(p + x\dfrac{dp}{dx}\right) + \dfrac{p^2 + 2xp\dfrac{dp}{dx}}{1 + x^2p^4}$

or $$p + 2x\frac{dp}{dx} + \left(p + 2x\frac{dp}{dx}\right) \cdot \frac{p}{1 + x^2p^4} = 0 \text{ or } \left(p + 2x\frac{dp}{dx}\right)\left(1 + \frac{p}{1 + x^2p^4}\right) = 0$$

This gives $p + 2x\, dp/dx = 0$.

Separating the variables and integrating, we have $\int \dfrac{dx}{x} + 2\int \dfrac{dp}{p}$ = a constant

or $$\log x + 2\log p = \log c \quad \text{or} \quad \log xp^2 = \log c$$

whence $$xp^2 = c \quad \text{or} \quad p = \sqrt{(c/x)} \qquad ...(ii)$$

Eliminating p from (*i*) and (*ii*), we get $y = 2\sqrt{(c/x)}\,x + \tan^{-1} c$

or $y = 2\sqrt{(cx)} + \tan^{-1} c$ which is the general solution of (*i*).

Obs. The significance of the factor $1 + p/(1 + x^2p^4) = 0$ which we didn't consider, will not be considered here as it concerns 'singular solution' of (*i*) whereas *we are interested only in finding general solution.*

Caution. Sometimes one is tempted to write (*ii*) as

$$\frac{dy}{dx} = \sqrt{\left(\frac{c}{x}\right)}$$

and integrating it to say that the required solution is $y = 2\sqrt{(cx)} + c'$. Such a reasoning is *incorrect*.

Example 11.40. *Solve* $y = 2px + p^n$. *(Bhopal, 2009)*

Solution. Given equation is $y = 2px + p^n$ ...(*i*)

Differentiating it with respect to x, we get

$$\frac{dy}{dx} = p = 2p + 2x\frac{dp}{dx} + np^{n-1}\frac{dp}{dx} \quad \text{or} \quad p\frac{dx}{dp} + 2x = -np^{n-1}$$

or $$\frac{dx}{dp} + \frac{2x}{p} = -np^{n-2} \qquad ...(ii)$$

This is Leibnitz's linear equation in x and p. Here I.F. $= e^{\int \frac{2}{p}dp} = e^{\log p^2} = p^2$

$\therefore$ the solution of (*ii*) is

$$x(\text{I.F.}) = \int (-np^{n-2}) \, . \, (\text{I.F.}) \, dp + c \quad \text{or} \quad xp^2 = -n \int p^n \, dp + c = -\frac{np^{n+1}}{n+1} + c$$

or $$x = cp^{-2} - \frac{np^{n-1}}{n+1} \qquad ...(iii)$$

Substituting this value of x in (*i*), we get $y = \dfrac{2c}{p} + \dfrac{1-n}{1+n} p^n$ ...(*iv*)

The equations (*iii*) and (*iv*) taken together, with parameter p, constitute the general solution (*i*).

Obs. In general, the equations of the form $y = xf(p) + \phi(p)$, known as *Lagrange's* equation, are solvable for y and lead to Leibnitz's equation in dx/dp.

PROBLEMS 11.10

Solve the following equations :

1. $y = x + a \tan^{-1} p$.
2. $y + px = x^4p^2$. (*S.V.T.U., 2007*)
3. $x^2 \left(\dfrac{dy}{dx}\right)^4 + 2x \dfrac{dy}{dx} - y = 0$.
4. $xp^2 + x = 2yp$.
5. $y = xp^2 + p$.
6. $y = p \sin p + \cos p$.

Case III. **Equations solvable for x.** If the given equation on solving for x, takes the form

$$x = f(y, p) \qquad ...(1)$$

then differentiation with respect to y gives an equation of the form

$$\frac{1}{p} = \frac{dx}{dy} = \phi\left(y, p, \frac{dp}{dy}\right)$$

Now it may be possible to solve the new differential equation in y and p. Let its solution be $F(y, p, c) = 0$.

The elimination of p from (1) and (2) gives the required solution. In case the elimination is not feasible, (1) and (2) may be expressed in terms of p and p may be regarded as a parameter.

Obs. This method is especially useful for equations which do not contain y.

Example 11.41. *Solve* $y = 2px + y^2p^3$. (*Bhopal, 2008*)

Solution. Given equation, on solving for x, takes the form $x = \dfrac{y - y^2p^3}{2p}$

Differentiating with respect to y, $\dfrac{dx}{dy}\left(= \dfrac{1}{p}\right) = \dfrac{1}{2} \cdot \dfrac{p\left(1 - 2y \, . \, p^3 \, \_ \, y^2 3p^2 \dfrac{dp}{dy}\right) - (y - y^2p^3)\dfrac{dp}{dy}}{p^2}$

or $$2p = p - 2yp^4 - 3y^2p^3 \frac{dp}{dy} - y\frac{dp}{dy} + y^2p^3 \frac{dp}{dy}$$

or $$p + 2yp^4 + 2y^2p^3 \frac{dp}{dy} + y\frac{dp}{dy} = 0 \text{ or } p\,(1 + 2yp^3) + y\frac{dp}{dy}(1 + 2yp^3) = 0.$$

or $$\left(p + y\frac{dp}{dy}\right)(1 + 2py^3) = 0 \text{ This gives } p + y\frac{dp}{dy} = 0. \text{ or } \frac{d}{dy}(py) = 0.$$

Integrating $$py = c. \qquad ...(i)$$

Thus eliminating from the given equation and (*i*), we get $y = 2\,\dfrac{c}{y}x + \dfrac{c^3}{y^3}y^2$ or $y^2 = 2cx + c^3$

which is the required solution.

PROBLEMS 11.11

Solve the following equations :

1. $p^3 - 4xyp + 8y^2 = 0.$ (*Kanpur, 1996*)
2. $p^2y + 2px = y.$
3. $x - yp = ap^2.$ (*Andhra, 2000*)
4. $p = \tan\left(x - \frac{p}{1+p^2}\right).$ (*S.V.T.U., 2008*)

11.14 CLAIRAUT'S EQUATION*

An equation of the form $y = px + f(p)$ is known as Clairaut's equation ...(1)

Differentiating with respect to x, we have $p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$

or $$[x + f'(p)]\frac{dp}{dx} = 0 \quad \therefore \frac{dp}{dx} = 0, \text{ or } x + f'(p) = 0$$

$$\frac{dp}{dx} = 0, \text{ gives } p = c \qquad ...(2)$$

Thus eliminating p from (1) and (2), we get $y = cx + f(c)$ ...(3)

as the general solution of (1).

Hence the solution of the Clairaut's equation is obtained on replacing p by c.

Obs. If we eliminate p from $x + f'(p) = 0$ and (1), we get an equation involving no constant. *This is the* **singular solution** of (1) *which gives the envelope of the family of straight lines* (3).

To obtain the singular solution, we proceed as follows :

(*i*) Find the general solution by replacing p by c *i.e.*, (3)

(*ii*) Differentiate this w.r.t. c giving $x + f'(c) = 0.$ ...(4)

(*iii*) Eliminate c from (3) and (4) which will be the *singular solution*.

Example 11.42. *Solve $p = \sin(y - xp)$. Also find its singular solutions.*

Solution. Given equation can be written as

$\sin^{-1} p = y - xp$ or $y = px + \sin^{-1} p$ which is the *Clairaut's equation.*

$\therefore$ its solution is $y = cx + \sin^{-1} c.$

To find the singular solution, differentiate (*i*) w.r.t. c giving

$$0 = x + \frac{1}{\sqrt{1-c^2}} \qquad ...(ii)$$

To eliminate c from (*i*) and (*ii*), we rewrite (*ii*) as

$$c = N(x^2 - 1)/x$$

Now substituting this value of c in (*i*), we get

$$y = N(x^2 - 1) + \sin^{-1}\{N(x^2 - 1)/x\}$$

which is the desired singular solution.

Obs. Equations reducible to Clairaut's form. Many equations of the first order but of higher degree can be easily reduced to the Clairaut's form by making suitable substitutions.

Example 11.43. *Solve $(px - y)(py + x) = a^2p.$* (*V.T.U., 2011 ; J.N.T.U., 2006*)

Solution. Put $x^2 = u$ and $y^2 = v$ so that $2xdx = du$ and $2ydy = dv$

$$\therefore \quad p = \frac{dy}{dx} = \frac{dv}{y}\bigg/\frac{du}{x} = \frac{x}{y}P, \text{ where } P = \frac{dv}{du}$$

*After the name of a youthful prodigy *Alexis Claude Clairaut* (1713–65) who first solved this equation. A French mathematician who is also known for his work in astronomy and geodesy.

Then the given equation becomes $\left(\frac{xP}{y}.x-y\right)\left(\frac{xP}{y}.y+x\right)=a^2\frac{xP}{y}$

or $\qquad (uP-v)(P+1)=a^2P$ or $uP-v=\frac{a^2P}{P+1}$

or $\qquad v=uP-a^2P/(P+1)$, which is Clairaut's form.

$\therefore$ its solution is $\qquad v=uc-a^2c/(c+1)$, *i.e.*, $y^2=cx^2-a^2c/(c+1)$.

PROBLEMS 11.12

1. Find the general and singular solution of the equations :
   (*i*) $xp^2-yp+a=0$. (*J.N.T.U., 2006*) (*ii*) $p=\log(px-y)$.
   (*iii*) $y=px+\sqrt{(a^2p^2+b^2)}$ (*W.B.T.U., 2005*) (*iv*) $\sin px\cos y=\cos px\sin y+p$ (*P.T.U., 2006*)

Solve the following equations :

2. $y+2\left(\frac{dy}{dx}\right)^2=(x+1)\frac{dy}{dx}$.
3. $(y-px)(p-1)=p$.
4. $(x-a)\left(\frac{dy}{dx}\right)^2+(x-y)\frac{dy}{dx}-y=0$.
5. $x^2(y-px)=yp^2$.
6. $(px+y)^2=py^2$.
7. $(px-y)(x+py)=2p$.

11.15 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 11.13

Fill up the blanks or choose the correct answer in the following problems :

1. $y=cx-c^2$, is the general solution of the differential equation
   (*i*) $(y')^2-xy'+y=0$ (*ii*) $y''=0$ (*iii*) $y'=c$ (*iv*) $(y')^2+xy'+y=0$.
2. The differential equation having a basis for its solution as sinh $6x$ and cosh $6x$ is
   (*i*) $y''+36y=0$ (*ii*) $y''-36y=0$ (*iii*) $y''+6y=0$ (*iv*) none of these.
3. The differential equation $(dx/dy)^2+5y^{1/3}=x$ is
   (*i*) linear of degree 3 (*ii*) non-linear of order 1 and degree 6
   (*iii*) non-linear of order 1 and degree 2.
4. The differential equation $y\,dx/dy+1=y$, $y(0)=1$, has
   (*i*) a unique solution (*ii*) two solutions
   (*iii*) infinite number of solutions (*iv*) no solution
5. Solution of $(x^2+y^2)\,dy=xy\,dx$ is ....
6. Solution of $(3x-2y)\,dx=x\,dy$ is ....
7. Solution of $dy/dx-y=2xy^2e^{-x}$ is ....
8. The differential equation $(y^2e^{xy^2}+6x)\,dx+(2xye^{xy^2}-4y)\,dy=0$ is
   (*i*) linear, homogeneous and exact (*ii*) non-linear, homogeneous and exact
   (*iv*) non-linear, non-homogeneous and exact (*iv*) non-linear, non-homogeneous and inexact.
9. Solution of $x\,dx+y\,dy+\frac{x\,dy-y\,dx}{x^2+y^2}$ is ....
10. Solution of $dy/dx=\frac{x^3+y^3}{xy^2}$ is ......
11. The differential equation $(x+x^8+ay^2)\,dx+(y^8-y+bxy)\,dy=0$ is exact if
    (*i*) $b=2a$ (*ii*) $a=b$ (*iii*) $a\neq 2b$ (*iv*) $a=1, b=3$.
12. Solution of $xy(1+xy^2)\,dy=dx$ is ....
13. Solution of $xp^2-yp+a=0$ is ....
14. The differential equation $p=\log(px-y)$ has the solution ....
15. Solution of $\frac{dy}{dx}-\frac{dx}{dy}=\frac{x}{y}-\frac{y}{x}$ is ....

16. The order of the differential equation $(1 + y_1^2)^{3/2}/y_2 = c$ is ......

17. The general solution of $\frac{1}{x^2y^2}(xdy + ydx) = 0$ is ....

18. Integrating factor of the differential equation $\frac{dx}{dy} + \frac{3x}{y} = \frac{1}{y^2}$ is

(*a*) e^{y^3} (*b*) y^3 (*c*) x^3 (*d*) $-y^3$. (*V.T.U., 2009*)

19. Solution of the equation $\frac{dy}{dx} = \frac{y}{x} - \text{cosec}\frac{y}{x}$ is

(*a*) $\cos(y/x) - \log x = c$ (*b*) $\cos(y/x) + \log x = c$
(*c*) $\cos^2(y/x) + \log x = c$ (*d*) $\cos^2(y/x) - \log x = c$. (*V.T.U., 2010*)

20. Solution of $x\sqrt{(1+x^2)} + y\sqrt{(1+y^2)}\ dy/dx = 0$ is ....

21. Solution of $dy/dx + y = 0$ given $y(0) = 5$ is ....

22. The substitution that transforms the equation $\frac{dy}{dx} = \frac{x+y+1}{2x+2y+3}$ to homogeneous form is ....

23. Integrating factor of $xy' + y = x^3y^6$ is ....

24. Solution of the exact differential equation $Mdx + Ndy = 0$ is ....

25. Solution of $(2x^3y^2 + x^4)\,dx + (x^4y + y^4)\,dy = 0$ is ....

26. The general solution of the differential equation $\frac{dy}{dx} + \frac{y}{x} = \tan 2x$ is ....

27. Degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right)^5 x^2y = 0$ is

(*a*) 2 (*b*) 0 (*c*) 3 (*d*) 5 (*Bhopal, 2008*)

28. Integrating factor of the differential equation $\frac{dy}{dx} + y\cos x = \frac{\sin 2x}{2}$ is

(*a*) $e^{\sin^2 x}$ (*b*) $e^{\sin^3 x}$ (*c*) $e^{\sin x}$ (*d*) $\sin x$

29. The differential equation of the family of circles with centre as origin is ...... (*Nagarjuna, 2008*)

30. Solution of $x e^{-x^2}\,dx + \sin y\,dy = 0$ is .... (*Nagarjuna, 2008*)

31. Solution of $p = \sin(y - xp)$ is

(*a*) $y = \frac{c}{x} + \sin^{-1} c$ (*b*) $y = cx + \sin c$ (*c*) $y = cx + \sin^{-1} c$ (*d*) $y = x + \sin^{-1} c$ (*V.T.U., 2011*)

32. Differential equation obtained by eliminating A and B form $y = A\cos x + B\sin x$ is $d^2y/dx^2 - y = 0$ (True or False)

33. $(x^3 - 3xy^2)\,dx + (y^3 - 2x^2y)\,dy = 0$ is an exact differential equation. (True or False)

CHAPTER 12

Applications of Differential Equations of First Order

12.1 INTRODUCTION

In this chapter, we shall consider only such practical problems which give rise to differential equations of the first order. The fundamental principles required for the formation of such differential equations are given in each case and are followed by illustrative examples.

12.2 GEOMETRIC APPLICATIONS

(*a*) *Cartesian coordinates.* Let $P(x, y)$ be any point on the curve $f(x, y) = 0$ (Fig. 12.1), then [as per 4.6 §(1) & 4.11(1) & (4)], we have

(*i*) slope of the tangent at $P(= \tan \psi) = dy/dx$

(*ii*) equation of the tangent at P is

$$Y - y = \frac{dy}{dx}(X - x)$$

so that its x-intercept $(= OT)$

$= x - y \, . \, dx/dy$

and y-intercept $(= OT') = y - x \, . \, dy/dx$

(*iii*) equation of the normal at P is $Y - y = -\dfrac{dx}{dy}(X - x)$

(*iv*) length of the tangent $(= PT) = y\sqrt{[1 + (dx/dy)^2]}$

(*v*) length of the normal $(= PN) = y\sqrt{[1 + (dy/dx)^2]}$

(*vi*) length of the sub-tangent $(= TM) = y \, . \, dx/dy$

(*vii*) length of the sub-normal $(= MN) = y \, . \, dy/dx$

(*viii*) $\dfrac{ds}{dx} = [1 + (dy/dx)^2]$; $\dfrac{ds}{dy} = \sqrt{[1 + (dx/dy)^2]}$

(*ix*) differential of the area $= ydx$ or xdy

(*x*) ρ, radius of curvature at $P = \dfrac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2}$

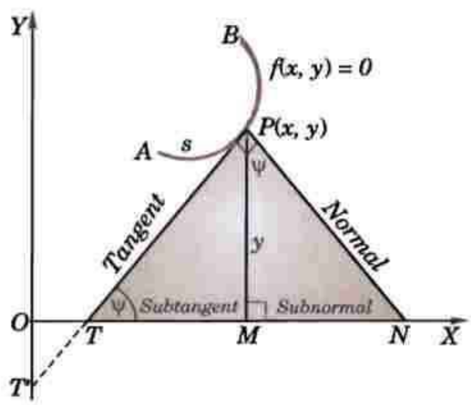


Fig. 12.1

(*b*) *Polar coordinates*. Let $P(r, \theta)$ be any point on the curve $r = f(\theta)$ (Fig. 12.2), then [as per § 4.7, 4.9 (2) & 4.11 (4)], we have

(*i*) $\psi = \theta + \phi$

(*ii*) $\tan\phi = r d\theta/dr$, $p = r \sin\phi$

(*iii*) $\dfrac{1}{p^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$

(*iv*) polar sub-tangent $(= OT) = r^2 d\theta/dr$

(*v*) polar sub-normal $(ON) = dr/d\theta$

(*vi*) $\dfrac{ds}{dr} = \sqrt{\left[1 + \left(r\dfrac{d\theta}{dr}\right)^2\right]}$; $\dfrac{ds}{d\theta} = \sqrt{\left[r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right]}$

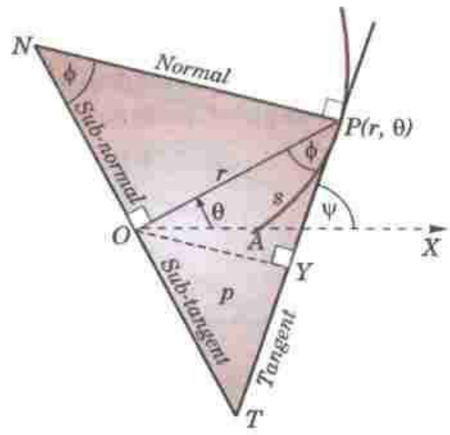


Fig. 12.2

Example 12.1. *Show that the curve in which the portion of the tangent included between the co-ordinates axes is bisected at the point of contact is a rectangular hyperbola.*

Solution. Let the tangent at any point $P(x, y)$ of a curve cut the axes at T and T' (Fig. 12.3).

We know that its x-intercept $(= OT) = x - y \,.\, dx/dy$

and $\qquad y$-intercept $(= OT') = y - x \,.\, dy/dx$

$\therefore$ the co-ordinates of T and T' are

$$(x - y \,.\, dx/dy, 0), (0, y - x\, dy/dx)$$

Since P is the mid-point of TT'

$$\therefore \quad \frac{[x - y.dx/dy] + 0}{2} = x$$

or $\qquad x - y \,.\, dx/dy = 2x$ or $x\,dy + y\,dx = 0$

or $\qquad d(xy) = 0$ Integrating, $xy = c$

which is the equation of a rectangular hyperbola, having x and y axes as its asymptotes.

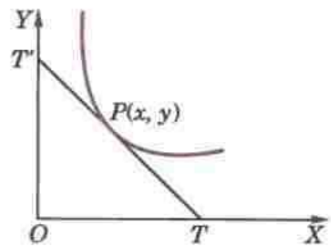


Fig. 12.3

Example 12.2. *Find the curve for which the normal makes equal angles with the radius vector and the initial line.*

Solution. Let PT and PN be the tangent and normal at $P(r, \theta)$ of the curve so that

$$\tan\phi = r\, d\theta/dr$$

By the condition of the problem,

$$\angle OPN = 90° - \phi = \angle ONP \text{ (Fig. 12.4).}$$

$$\therefore \quad \theta = \angle PON = 180° - (180° - 2\phi) = 2\phi$$

or $\qquad \theta/2 = \phi \quad \therefore \quad \tan\dfrac{\theta}{2} = \tan\phi = r\dfrac{d\theta}{dr}.$

Here the variables are separable.

$$\therefore \quad \frac{dr}{r} = \frac{\cos\theta/2}{\sin\theta/2}\, d\theta$$

Integrating both sides $\log r = 2 \log \sin\theta/2 + \log c$

or $\qquad r = c\sin^2\theta/2 = \dfrac{1}{2}\, c(1 - \cos\theta)$

Thus the curve is the cardioid $r = a(1 - \cos\theta)$.

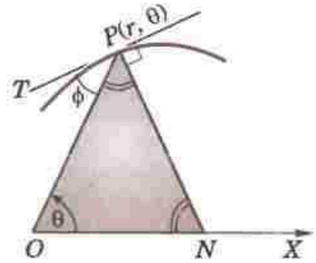


Fig. 12.4

Example 12.3. *Find the shape of a reflector such that light coming from a fixed source is reflected in parallel rays.*

Solution. Taking the fixed source of light as the origin and the X-axis parallel to the reflected rays; the reflector will be a surface generated by the revolution of a curve $f(x, y) = 0$ about X-axis (Fig. 12.5).

In the XY-plane, let PP' be the reflected ray, where P is the point (x, y) on the curve $f(x, y) = 0$.

If TPT' be the tangent at P, then

$\because$ angle of incidence = angle of reflection,

$\therefore$ $$\phi = \angle OPT = \angle P'PT' = \angle OTP = \psi$$

i.e., $$p = \frac{dy}{dx} = \tan \angle XOP = \tan 2\phi$$

$$= \frac{2 \tan \phi}{1 - \tan^2 \phi} = \frac{2p}{1 - p^2}$$

or $$2x = \frac{y}{p} - yp \text{ which is solvable for } x \qquad ...(i)$$

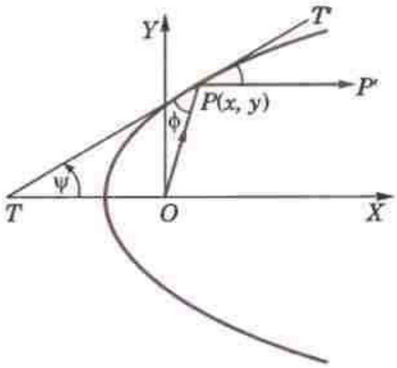


Fig. 12.5

$\therefore$ differentiating (i) w.r.t. y, $\dfrac{2}{p} = \dfrac{1}{p} - \dfrac{y}{p^2}\dfrac{dp}{dy} - p - y\dfrac{dp}{dy}$

i.e., $$\left(\frac{1}{p} + p\right) + \left(\frac{1}{p^2} + 1\right) y \frac{dp}{dy} = 0 \quad \text{or} \quad \left(\frac{1}{p} + p\right)\left(1 + \frac{y}{p}\frac{dp}{dy}\right) = 0$$

This gives $\qquad dp/p = -dy/y$

Integrating, $\qquad \log p = \log c - \log y, \quad$ *i.e.,* $p = c/y \qquad ...(ii)$

Thus eliminating p from (i) and (ii), we have family of curves $y^2 = 2cx + c^2$.

Hence the reflector is a member of the family of paraboloids of revolution $y^2 + z^2 = 2cx + c^2$.

PROBLEMS 12.1

1. Find the equation of the curve which passes through
   (*i*) the point $(3, -4)$ and has the slope $2y/x$ at the point (x, y) on it.
   (*ii*) the origin and has the slope $x + 3y - 1$.
2. At every point on a curve the slope is the sum of the abscissa and the product of the ordinate and the abscissa, and the curve passes through (0, 1). Find the equation of the curve.
3. A curve is such that the length of the perpendicular from origin on the tangent at any point P of the curve is equal to the abscissa of P. Prove that the differential equation of the curve is
$$y^2 - 2xy\frac{dy}{dx} - x^2 = 0, \text{ and hence find the curve.}$$
4. A plane curve has the property that the tangents from any point on the y-axis to the curve are of constant length a. Find the differential equation of the family to which the curve belongs and hence obtain the curve.
5. Determine the curve whose sub-tangent is twice the abscissa of the point of contact and passes through the point (1, 2). *(Sambalpur, 1998)*
6. Determine the curve in which the length of the sub-normal is proportional to the square of the ordinate.
7. The tangent at any point of a certain curve forms with the coordinate axes a triangle of constant area A. Find the equation to the curve.
8. Find the curve which passes through the origin and is such that the area included between the curve, the ordinate and the x-axis is twice the cube of that ordinate.
9. Find the curve whose (*i*) polar sub-tangent is constant.
   (*ii*) polar sub-normal is proportional to the sine of the vectorial angle.
10. Determine the curve for which the angle between the tangent and the radius vector is twice the vectorial angle. *(Kanpur, 1996)*
11. Find the curve for which the tangent at any point P on it bisects the angle between the ordinate at P and the line joining P to the origin.
12. Find the curve for which the tangent, the radius vector r and the perpendicular from the origin on the tangent form a triangle of area kr^2.

12.3 (1) ORTHOGONAL TRAJECTORIES

Two families of curves such that every member of either family cuts each member of the other family at right angles are called **orthogonal trajectories** *of each other (Fig. 12.6).*

The concept of the orthogonal trajectories is of wide use in applied mathematics especially in field problems. For instance, in an electric field, the paths along which the current flows are the orthogonal trajectories of the equipotential curves and *vice versa*. In fluid flow, the stream lines and the equipotential lines (lines of constant velocity potential) are orthogonal trajectories. Likewise, the lines of heat flow for a body are perpendicular to the isothermal curves. The problem of finding the orthogonal trajectories of a given family of curves depends on the solution of the first order differential equations.

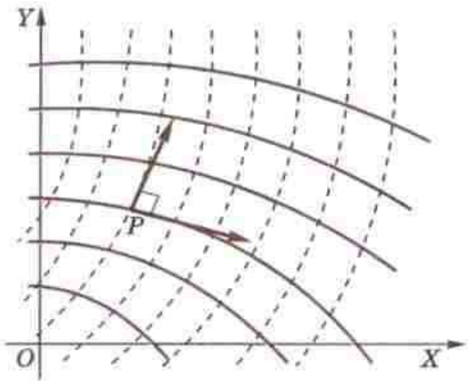


Fig. 12.6

(2) To find the orthogonal trajectories of the family of curves F(x, y, c) = 0.

(*i*) *Form its differential equation in the form* $f(x, y, dy/dx) = 0$ *by eliminating c.*

(*ii*) *Replace, in this differential equation,* dy/dx *by* $-dx/dy$, (so that the product of their slopes at each point of intersection is -1).

(*iii*) *Solve the differential equation of the orthogonal trajectories i.e.,* $f(x, y, -dx/dy) = 0$.

Example 12.4. *If the stream lines (paths of fluid particles) of a flow around a corner are xy = constant find their orthogonal trajectories (called equipotential lines-§ 20.6)* (*Marathwada, 2008*)

Solution. Taking the axes as the walls, the stream lines of the flow around the corner of the walls is

$$xy = c \qquad ...(i)$$

Defferentiating, we get, $$x\frac{dy}{dx} + y = 0 \qquad ...(ii)$$

as the differential equation of the given family (*i*).

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (*ii*), we obtain $x\left(-\frac{dx}{dy}\right) + y = 0$

or $$xdx - ydy = 0 \qquad ...(iii)$$

as the differential equation of the orthogonal trajectories.

Integrating (*iii*), we get $x^2 - y^2 = c'$ as the required orthogonal trajectories of (*i*) *i.e.*, the *equipotential lines*, shown dotted in Fig. 12.7.

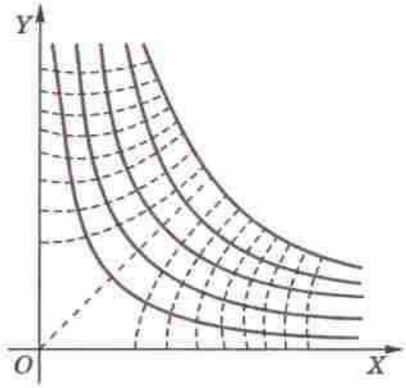


Fig. 12.7

Example 12.5. *Find the orthogonal trajectories of the family of confocal conics* $\frac{x^2}{a^2} + \frac{y^2}{a^2 + \lambda} = 1$, *where* λ *is the parameter.* (*V.T.U., 2009 S*)

Solution. Differentiating the given equation, we get $\frac{2x}{a^2} + \frac{2y}{a^2+\lambda}\frac{dy}{dx} = 0$

or $$\frac{y}{a^2+\lambda} = -\frac{x}{a^2\,(dy/dx)} \quad \text{or} \quad \frac{y^2}{a^2+\lambda} = \frac{-xy}{a^2\,(dy/dx)}$$

Substituting this in the given equation, we get

$$\frac{x^2}{a^2} - \frac{xy}{a^2\,(dy/dx)} = 1 \quad \text{or} \quad (x^2 - a^2)\frac{dy}{dx} = xy \qquad ...(i)$$

which is the differential equation of the given family.

Changing dy/dx to $-dx/dy$ in (*i*), we get $(a^2 - x^2)\,dx/dy = xy$ as the differential equation of the orthogonal trajectories.

Separating the variables and integrating, we obtain

$$\int y\,dy = \int \frac{a^2 - x^2}{x}\,dx + c \quad \text{or} \quad \frac{1}{2}y^2 = a^2 \log x - \frac{1}{2}x^2 + c$$

or $$x^2 + y^2 = 2a^2 \log x + c' \qquad [c' = 2c]$$

which is the equation of the required orthogonal trajectories.

Example 12.6. *Find the orthogonal trajectories of a system of confocal and coaxial parabolas.*

Solution. The equation of the family of confocal parabolas having x-axis as their axis, is of the form

$$y^2 = 4a(x + a) \qquad ...(i)$$

Differentiating, $$y\frac{dy}{dx} = 2a \qquad ...(ii)$$

Substituting the value of a from (ii) in (i), we get $y^2 = 2y\dfrac{dy}{dx}\left(x + \dfrac{1}{2}y\dfrac{dy}{dx}\right)$

i.e., $$y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0 \text{ as the differential equation of the family.} \qquad ...(iii)$$

Replacing $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$ in (iii), we obtain $y\left(\dfrac{dx}{dy}\right)^2 - 2x\dfrac{dx}{dy} - y = 0$

or $$y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0 \text{ which is the same as } (iii).$$

Thus we see that a system of confocal and coaxial parabolas is *self-orthogonal, i.e.,* each member of the family (i) cuts every other member of the same family orthogonally.

(3) To find the orthogonal trajectories of the curves F(r, θ, c) = 0.

(i) Form its differential equation in the form $f(r, \theta, dr/d\theta) = 0$ by eliminating c.

(ii) Replace in this differential equation,

$$\frac{dr}{d\theta} \text{ by } -r^2\frac{d\theta}{dr}$$

[∵ for the given curve through $P(r, \theta)$ $\tan\phi = r d\theta/dr$ and for the orthogonal trajectory through P

$$\tan\phi' = \tan(90° + \phi) = -\cot\phi = -\frac{1}{r}\frac{dr}{d\theta}$$

Thus for getting the differential equation of the orthogonal trajectory

$$r\frac{d\theta}{dr} \text{ is to be replaced by } -\frac{1}{r}\frac{dr}{d\theta}$$

or $$\frac{dr}{d\theta} \text{ is to be replaced by } -r^2\frac{d\theta}{dr}.]$$

(iii) Solve the differential equation of the orthogonal trajectories

i.e., $$f(r, \theta, -r^2 d\theta/dr) = 0.$$

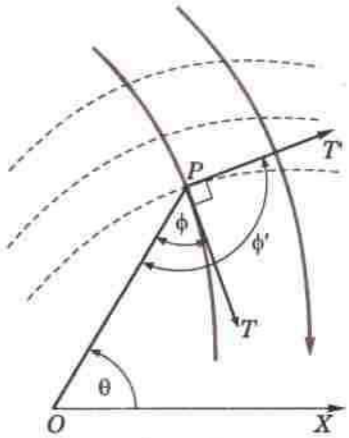


Fig. 12.8

Example 12.7. *Find the orthogonal trajectory of the cardioids $r = a(1 - \cos\theta)$.* (Kurukshetra, 2005)

Solution. Differentiating $$r = a(1 - \cos\theta). \qquad ...(i)$$

with respect to θ, we get $$\frac{dr}{d\theta} = a\sin\theta \qquad ...(ii)$$

Eliminating a from (i) and (ii), we obtain

$$\frac{dr}{d\theta}\cdot\frac{1}{r} = \frac{\sin\theta}{1 - \cos\theta} = \cot\frac{\theta}{2} \text{ which is the differential equation of the given family.}$$

Replacing $dr/d\theta$ by $-r^2\, d\theta/dr$, we obtain

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \cot\frac{\theta}{2} \quad \text{or} \quad \frac{dr}{r} + \tan\frac{\theta}{2}\, d\theta = 0$$

as the differential equation of orthogonal trajectories. It can be rewritten as

$$\frac{dr}{r} = -\frac{(\sin\theta/2)d\theta}{\cos\theta/2}$$

Integrating, $\log r = 2 \log \cos \theta/2 + \log c$

or $$r = c \cos^2 \theta/2 = \frac{1}{2} c(1 + \cos \theta) \quad \text{or} \quad r = a'(1 + \cos \theta)$$

which is the required orthogonal trajectory.

Example 12.8. *Find the orthogonal trajectory of the family of curves $r^n = a \sin n\theta$.* (V.T.U., 2006)

Solution. We have $n \log r = \log a + \log \sin n\theta$.

Differentiating w.r.t. θ, we have

$$\frac{n}{r}\frac{dr}{d\theta} = \frac{n \cos n\theta}{\sin n\theta} \quad \text{or} \quad \frac{1}{r}\frac{dr}{d\theta} = \cot n\theta$$

Replacing $dr/d\theta$ by $-r^2\, d\theta/dr$, we obtain

$$\frac{1}{r}\left(-r^2 \frac{d\theta}{dr}\right) = \cot n\theta \quad \text{or} \quad \tan n\theta.\ d\theta - \frac{dr}{r} = 0$$

Integrating, $$\int \frac{dr}{r} + \int \frac{\sin n\theta}{\cos n\theta}\, d\theta = c,$$

i.e., $$\log r - \frac{1}{n} \log \cos n\theta = c \quad \text{or} \quad \log (r^n/\cos n\,\theta) = nc = \log b. \text{ (say)}$$

or $r^n = b \cos n\theta$, which is the required orthogonal trajectory.

PROBLEMS 12.2

Find the orthogonal trajectories of the family of :

1. Parabolas $y^2 = 4ax$. (*Marathwada, 2009*)
2. Parabolas $y = ax^2$. (*J.N.T.U., 2006*)
3. Semi-cubical parabolas $ay^2 = x^3$. (*J.N.T.U., 2005*)
4. Coaxial circles $x^2 + y^2 + 2\lambda x + c = 2$, λ being the parameter. (*J.N.T.U., 2006*)
5. Confocal conics $\dfrac{x^2}{a^2+\lambda} + \dfrac{y^2}{b^2+\lambda} = 1$, λ being the parameter. (*Kurukshetra, 2006*)
6. Cardioids $r = a\,(1 + \cos \theta)$. (*J.N.T.U., 2003*)
7. $r = 2a\,(\cos \theta + \sin \theta)$ (*V.T.U., 2010 S*)
8. Confocal and coaxial parabolas $r = 2a/(1 + \cos \theta)$. (*Nagpur, 2008*)
9. Curves $r^2 = a^2 \cos 2\theta$. (*V.T.U., 2009 S*)
10. $r^n \cos n\theta = a^n$. (*V.T.U. 2011*)
11. Show that the family of parabolas $x^2 = 4a\,(y + a)$ is self orthogonal. (*Kerala, 2005*)
12. Show that the family of curves $r^n = a \sec n\theta$ and $r^n = b \operatorname{cosec} n\theta$ are orthogonal. (*Mumbai, 2005*)
13. The *electric lines of force* of two opposite charges of the same strength at $(\pm 1, 0)$ are circles (through these points) of the form $x^2 + y^2 - ay = 1$. Find their *equipotential lines* (orthogonal trajectories).

    [**Isogonal trajectories.** *Two families of curves such that every member of either family cuts each member of the other family at a constant angle α (Say), are called* ***isogonal trajectories*** *of each other.* The slopes m, m' of the tangents to the corresponding curves at each point, are connected by the relation $\dfrac{m - m'}{1 + mm'} = \tan \alpha = \text{const.}$]
14. Find the isogonal trajectories of the family of circles $x^2 + y^2 = a^2$ which intersect at 45°.

12.4 PHYSICAL APPLICATIONS

(1) Let a body of mass m start moving from O along the straight line OX under the action of a force F. After any time t, let it be moving at P where $OP = x$, then

(*i*) its velocity $(v) = \dfrac{dx}{dt}$

(*ii*) its acceleration $(a) = \dfrac{dv}{dt}$ or $\dfrac{v\,dv}{dx}$ or $\dfrac{d^2x}{dt^2}$

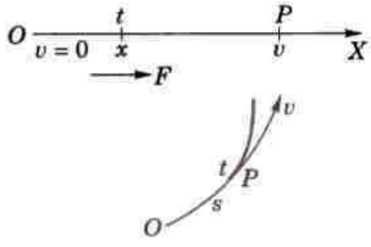


Fig. 12.9

If, however, the body be moving along a curve, then

(*i*) its velocity $(v) = ds/dt$ and

(*ii*) its acceleration $(a) = \dfrac{dv}{dt}, v\dfrac{dv}{ds}$ or $\dfrac{d^2s}{dt^2}$.

The quantity mv is called the *momentum*.

(2) Newton's second law *states that* $F = \dfrac{d}{dt}(mv)$.

If m is constant, then $F = m\dfrac{dv}{dt} = ma$, *i.e., net force = mass × acceleration.*

(3) Hooke's law* *states that tension of an elastic string (or a spring) is proportional to extension of the string (or the spring) beyond its natural length.*

Thus $\mathbf{T} = \lambda e/\mathbf{1}$,

where e is the extension beyond the natural length l and λ is the *modulus of elasticity.*

Sometimes for a spring, we write $\mathbf{T = ke}$,

where e is the extension beyond the natural length and k is the *stiffness of the spring.*

(4) Systems of units

I. **F.P.S.** [foot (*ft.*) pound (*lb.*), second (*sec.*)] **system**. If mass m is in *pounds* and acceleration (a) is in *ft/sec*^{2}, then the force $F(= ma)$ is in *poundals.*

II. **C.G.S.** [centimetre (*cm.*), gram (*g*), second (*sec*)] **system**. If mass m is in *grams* and acceleration a is in *cm/sec*^{2} then the force $F(= ma)$ is *dynes.*

III. **M.K.S.** [metre (*m*), kilogram (*kg.*), second (*sec*)] **system**. If mass m is in kilograms and acceleration a in *m/sec*^{2}, then the force $F(= ma)$ is in *newtons* (*nt*).

These are called *absolute units*. If g is the acceleration due to gravity and w is the weight of the body, then w/g is the mass of the body in *gravitational units.*

$$g = 32 \text{ ft/sec}^2 = 980 \text{ cm/sec}^2 = 9.8 \text{ m/sec}^2 \text{ approx.}$$

Example 12.9. Motion of a boat across a stream. *A boat is rowed with a velocity u directly across a stream of width a. If the velocity of the current is directly proportional to the product of the distances from the two banks, find the path of the boat and the distance down stream to the point where it lands.*

Solution. Taking the origin at the point from where the boat starts, let the axes be chosen as in Fig. 12.10.

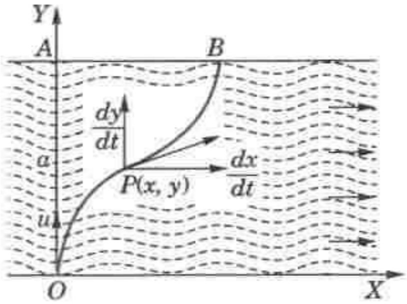


Fig. 12.10

At any time t after its start from O, let the boat be at $P(x, y)$, so that

dx/dt = velocity of the current = $ky(a - y)$

dy/dt = velocity with which the boat is being rowed = u.

$$\therefore \qquad \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{u}{ky(a-y)} \qquad \text{...}(i)$$

This gives the direction of the resultant velocity of the boat which is also the direction of the tangent to the path of the boat.

Now (*i*) is of variables separable form and we can write it as

$$y(a - y)dy = \frac{u}{k}\,dx$$

Integrating, we get $\quad \dfrac{ay^2}{2} - \dfrac{y^3}{3} = \dfrac{u}{k}x + c$

Since $y = 0$ when $x = 0$, $\therefore \; c = 0$.

Hence the equation to the path of the boat is $x = \dfrac{k}{6u}y^2(3a - 2y)$

Putting $y = a$, we get the distance AB, down stream where the boat lands $= ka^3/6u$.

*Named after an English physicist *Robert Hooke* (1635–1703) who had discovered the law of gravitation earlier than Newton.

Example 12.10. Resisted motion. *A moving body is opposed by a force per unit mass of value cx and resistance per unit of mass of value bv² where x and v are the displacement and velocity of the particle at that instant. Find the velocity of the particle in terms of x, if it starts from rest.* (*Marathwada, 2008*)

Solution. By Newton's second law, the equation of motion of the body is $v\dfrac{dv}{dx} = -cx - bv^2$

or $$v\frac{dv}{dx} + bv^2 = -cx \quad \text{...}(i)$$

This is Bernoulli's equation. ∴ Put $v^2 = z$ and $2v\, dv/dx = dz/dx$, so that (*i*) becomes

$$\frac{dz}{dx} + 2bz = -2cx \quad \text{...}(ii)$$

This is Leibnitz's linear equation and I.F. $= e^{2bx}$.

∴ the solution of (*ii*) is $ze^{2bx} = -\int 2cxe^{2bx}\,dx + c'$ [Integrate by parts]

$$= -2c\left[x\cdot\frac{e^{2bx}}{2b} - \int 1\cdot\frac{e^{2bx}}{2b}\,dx\right] + c' = -\frac{cx}{b}e^{2bx} + \frac{c}{2b^2}e^{2bx} + c'$$

or $$v^2 = \frac{c}{2b^2} + c'e^{-2bx} - \frac{cx}{b} \quad \text{...}(iii)$$

Initially $v = 0$ when $x = 0$ ∴ $0 = c/2b^2 + c'$.

Thus, substituting $c' = -c/2b^2$ in (*iii*), we get $v^2 = \dfrac{c}{2b^2}(1 - e^{-2bx}) - \dfrac{cx}{b}$.

Example 12.11. Resisted vertical motion. *A particle falls under gravity in a resisting medium whose resistance varies with velocity. Find the relation between distance and velocity if initially the particle starts from rest.* (*U.P.T.U., 2003*)

Solution. After falling a distance s in time t from rest, let v be velocity of the particle. The forces acting on the particle are its weight mg downwards and resistance $m\lambda v$ upwards.

∴ equating of motion is $m\dfrac{dv}{dt} = mg - m\lambda v$

or $$\frac{dv}{dt} = g - \lambda v \quad \text{or} \quad \frac{dv}{g - \lambda v} = dt$$

Integrating, $\displaystyle\int \frac{dv}{g - \lambda v} = \int dt + c$ or $-\dfrac{1}{\lambda}\log(g - \lambda v) = t + c$

Since $v = 0$ when $t = 0$, ∴ $c = -\dfrac{1}{\lambda}\log g$

Thus $$\frac{1}{\lambda}\log\left[\frac{g}{g - \lambda v}\right] = t \quad \text{or} \quad \frac{g - \lambda v}{g} = e^{-\lambda t}$$

or $$\frac{ds}{dt} = v = \frac{g}{\lambda}(1 - e^{-\lambda t}) \quad \text{...}(i)$$

Integrating, $$s = \frac{g}{\lambda}\int (1 - e^{-\lambda t})dt + c' \quad \text{or} \quad s = \frac{g}{\lambda}\left(t + \frac{1}{\lambda}e^{-\lambda t}\right) + c'$$

Since $s = 0$ when $t = 0$, ∴ $c' = -g/\lambda^2$

Thus $$s = \frac{g}{\lambda}t + \frac{g}{\lambda^2}(e^{-\lambda t} - 1) \quad \text{...}(ii)$$

Eliminating t from (*i*) and (*ii*), we get

$$s = \frac{g}{\lambda^2}\log\left(\frac{g}{g - \lambda v}\right) - \frac{v}{\lambda}$$

which is the desired relation between s and v.

Example 12.12. *A body of mass m, falling from rest is subject to the force of gravity and an air resistance proportional to the square of the velocity (i.e., kv^2). If it falls through a distance x and possesses a velocity v at that instant, prove that*

$$\frac{2kx}{m} = \log \frac{a^2}{a^2 - v^2}, \text{ where } mg = ka^2.$$

Solution. If the body be moving with the velocity v after having fallen through a distance x, then its equation of motion is

$$mv\frac{dv}{dx} = mg - kv^2 \quad \text{or} \quad mv\frac{dv}{dx} = k(a^2 - v^2). \qquad [\because mg = ka^2] \quad ...(i)$$

$\therefore$ separating the variables and integrating, we get $\int \frac{vdv}{a^2 - v^2} = \int \frac{k}{m} dx + c$

or
$$-\frac{1}{2}\log(a^2 - v^2) = \frac{kx}{m} + c \qquad ...(ii)$$

Initially, when $x = 0, v = 0$. $\therefore -\frac{1}{2}\log a^2 = c$ ...(iii)

Subtracting (*iii*) from (*ii*), we have $\frac{1}{2}[\log a^2 - \log(a^2 - v^2)] = kx/m$

or
$$\frac{2kx}{m} = \log\left(\frac{a^2}{a^2 - v^2}\right)$$

Obs. *When the resistance becomes equal to the weight, the acceleration becomes zero and particle continues to fall with a constant velocity, called the* **limiting** *or* **terminal** *velocity.* From (*i*), it follows that the acceleration will become zero when $v = a$. *Thus, the limiting velocity, i.e., the maximum velocity which the particle can attain is a.*

Example 12.13. Velocity of escape from the earth. *Find the initial velocity of a particle which is fired in radial direction from the earth's centre and is supposed to escape from the earth. Assume that it is acted upon by the gravitational attraction of the earth only.*

Solution. According to Newton's law of gravitation, the acceleration α of the particle is proportional to $1/r^2$ where r is the variable distance of the particle from the earth's centre. Thus

$$\alpha = v\frac{dv}{dr} = -\frac{\mu}{r^2}$$

where v is the velocity when at a distance r from the earth's centre. The acceleration is negative because v is decreasing. When $r = R$, the earth's radius then $\alpha = -g$, the acceleration of gravity at the surface.

i.e.,
$$-g = -\mu/R^2, \text{ i.e., } \mu = gR^2 \quad \therefore \quad v\frac{dv}{dr} = -\frac{gR^2}{r^2}$$

Separating the variables and integrating, we obtain $\int v dv = -gR^2 \int \frac{dr}{r^2} + c$

i.e.,
$$v^2 = \frac{2gR^2}{r} + 2c \qquad ...(i)$$

On the earth's surface $r = R$ and $v = v_0$ (say), the initial velocity. Then

$$v_0^2 = 2gR + 2c, \quad \text{i.e.,} \quad 2c = v_0^2 - 2gR$$

Inserting this value of c in (*i*), we get $v^2 = \frac{2gR^2}{r} + v_0^2 - 2gR$

When v vanishes, the particle stops and the velocity will change from positive to negative and the particle will return to the earth. Thus the velocity will remain positive, if and only if $v_0^2 \geq 2gR$ and then the particle projected fron the earth with this velocity will escape from the earth. Hence the minimum such velocity of projection $v_0 = \sqrt{(2gR)}$ *is called the velocity of escape* from the earth [See Problem 9, page 454].

Example 12.14. Rotating cylinder containing liquid. *A cylindrical tank of radius r is filled with water to a depth h. When the tank is rotated with angular velocity ω about its axis, centrifugal force tends to drive the water outwards from the centre of the tank. Under steady conditions of uniform rotation, show that the section of the free surface of the water by a plane through the axis, is the curve*

$$y = \frac{\omega^2}{2g}\left(x^2 - \frac{r^2}{2}\right) + h.$$

Solution. Let the figure represent an axial section of the cylindrical tank. Forces acting on a particle of mass m at $P(x, y)$ on the curve, cut out from the free surface of water, are :

(*i*) the weight mg acting vertically downwards,

(*ii*) the centrifugal force $m\omega^2 x$ acting horizontally outwards.

As the motion is steady, P moves just on the surface of the water and, therefore, there is no force along the tangent to the curve. Thus the resultant R of mg and $m\omega^2 x$ is along the outward normal to the curve.

$\therefore \quad R \cos \psi = mg$ and $R \sin \psi = m\omega^2 x$

whence $$\frac{dy}{dx} = \tan \psi = \frac{m\omega^2 x}{mg} = \frac{\omega^2 x}{g} \qquad ...(i)$$

This is the differential equation of the surface of the rotating liquid.

Integrating (*i*), we get

$$\int dy = \frac{\omega^2}{g} \int x\,dx + c$$

i.e., $$y = \frac{\omega^2 x^2}{2g} + c \qquad ...(ii)$$

To find c, we note that the volume of the liquid remains the same in both cases (Fig. 12.11).

When $x = 0$ in (*ii*), $OA\ (= y) = c$. When $x = r$ in (*ii*), $h'\ (= y) = \dfrac{\omega^2 r^2}{2g} + c \qquad ...(iii)$

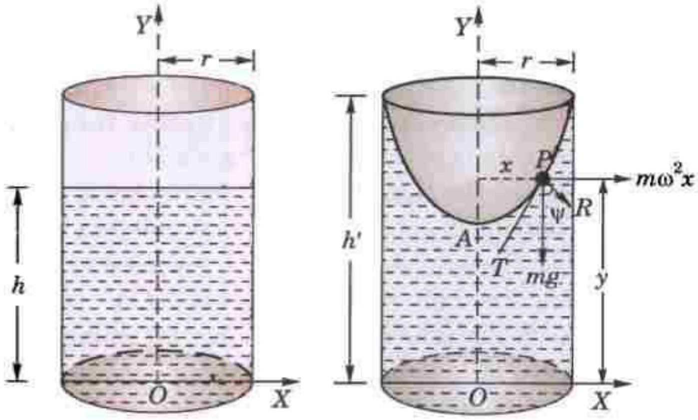


Fig 12.11

Now the volume of the liquid in the non-rotational case $= \pi r^2 h$, and the volume of the liquid in the rotational case

$$= \pi r^2 h' - \int_{OA}^{h'} \pi x^2 dy = \pi r^2 h' - \frac{2\pi g}{\omega^2} \int_c^{h'} (y - c)\, dy \qquad \text{[From } (ii)]$$

$$= \pi r^2 h' - \frac{\pi g}{\omega^2} (h' - c)^2 = \pi r^2 \left(\frac{\omega^2 r^2}{4g} + c \right) \qquad \text{[By } (iii)]$$

Thus $$\pi r^2 h = \pi r^2 \left(\frac{\omega^2 r^2}{4g} + c \right) \text{ whence } c = h - \frac{\omega^2 r^2}{4g}$$

$\therefore$ (*ii*) becomes, $$y = \frac{\omega^2 x^2}{2g} + h - \frac{\omega^2 x^2}{4g} \quad \text{or} \quad y = \frac{\omega^2}{2g}\left(x^2 - \frac{r^2}{2}\right) + h$$

which is the desired equation of the curve.

Example 12.15. Discharge of water through a small hole. *If the velocity of flow of water through a small hole is* $0.6\sqrt{2gy}$ *where g is the gravitational acceleration and y is the height of water level above the hole, find the time required to empty a tank having the shape of a right circular cone of base radius a and height h filled completely with water and having a hole of area* A_0 *in the base.*

Solution. At any time t, let the height of the water level be y and radius of its surface be r (Fig. 12.12) so that

$$\frac{h - y}{r} = \frac{h}{a} \quad \text{or} \quad r = a(h - y)/h$$

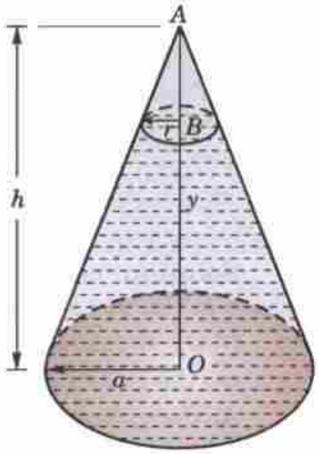


Fig. 12.12

$\therefore$ surface area of the liquid $= \pi r^2 = \pi a^2 (1 - y/h)^2$

Volume of water drained through the hole per unit time

$$= 0.6\sqrt{(2gy)}\,A_0 = 4.8\sqrt{y}\,A_0 \qquad [\because \; g = 32$$

$\therefore$ rate of fall of liquid level $= 4.8\,A_0\sqrt{y} \div \pi a^2(1 - y/h)^2$

i.e., $$\frac{dy}{dt} = -\frac{4.8\,A_0\sqrt{y}}{\pi a^2(1 - y/h)^2}$$ (– ve is taken since the water level decreases)

Hence time to empty the tank (= t)

$$= -\int_h^0 \frac{\pi a^2 (1 - y/h)^2}{4.8\,A_0\sqrt{y}}\,dy = \frac{\pi a^2}{4.8\,A_0}\int_0^h (y^{-1/2} - 2y^{1/2}/h + y^{3/2}/h^2)dy$$

$$= \frac{\pi a^2}{4.8\,A_0}\left|2y^{1/2} - \frac{4}{3h}y^{3/2} + \frac{2}{5h^2}y^{5/2}\right|_0^h = 0.2\,\pi a^2\sqrt{h}/A_0.$$

Example 12.16. Atmospheric pressure. *Find the atmospheric pressure p lb. per ft. at a height z ft. above the sea-level, both when the temperature is constant or variable.*

Solution. Take a vertical column of air of unit cross-section.

Let p be the pressure at a height z above the sea-level and $p + \delta p$ at height $z + \delta z$. Let ρ be the density at a height z. (Fig. 12.13)

Now since the thin column δz of air is being pressured upwards with pressure p and downwards with $p + \delta p$, we get by considering its equilibrium;

$$p = p + \delta p + g\rho\delta z.$$

Taking the limit, we get $\qquad dp/dz = -g\rho \qquad$ ...(*i*)

which is the differential equation giving the atmospheric pressure at height z.

(*i*) *When the temperature is constant,* we have by Boyle's law*, $p = k\rho \qquad$ ...(*ii*)

$\therefore$ Substituting the value of ρ from (*ii*) in (*i*), we get

$$\frac{dp}{dz} = -gp/k \quad \text{or} \quad \int\frac{dp}{p} = -\frac{g}{k}\int dz + c \text{ or } \log p = -\frac{g}{k}z + c$$

At the sea-level, where $z = 0$, $p = p_0$ (say) then $c = \log p_0$

$$\therefore \qquad \log p - \log p_0 = -\frac{g}{k}z \text{ i.e., } \log p/p_0 = -gz/k$$

Hence p is given by $p = p_0 e^{-gz/k}$.

(*ii*) *When the temperature varies*, we have $p = k\rho^n$.

Proceeding as above, we shall find that p is given by $\dfrac{n}{n-1}(p_0^{1-1/n} - p^{1-1/n}) = gk^{-1/n}z.$

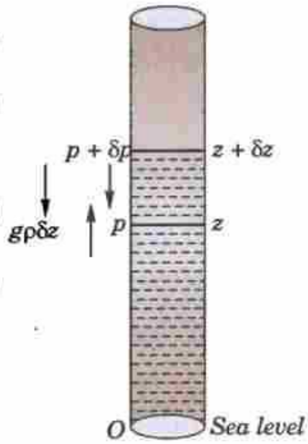


Fig. 12.13

PROBLEMS 12.3

1. A particle of mass m moves under gravity in a medium whose resistance is k times its velocity, where k is a constant. If the particle is projected vertically upwards with a velocity v, show that the time to reach the highest point is $\dfrac{m}{k}\log_e\left(1 + \dfrac{kv}{mg}\right)$.
2. A body of mass m falls from rest under gravity and air resistance proportional to square of velocity. Find velocity as function of time. (*Marathwada, 2008*)
3. A body of mass m falls from rest under gravity in a field whose resistance is mk times the velocity of the body. Find the terminal velocity of the body and also the time taken to acquire one half of its limiting speed.
4. A particle is projected with velocity v along a smooth horizontal plane in the medium whose resistance per unit mass is μ times the cube of the velocity. Show that the distance it has described in time t is $\dfrac{1}{\mu v}(\sqrt{1 + 2\mu v^2 t} - 1)$.

*Named after the English physicist *Robert Boyle* (1627–1691) who was one of the founders of the Royal Society.

5. When a bullet is fired into a sand tank, its retardation is proportional to the square root of its velocity. How long will it take to come to rest if it enters the sand bank with velocity v_0?
6. A particle of mass m is attached to the lower end of a light spring (whose upper end is fixed) and is released. Express the velocity v as a function of the stretch x feet.
7. A chain coiled up near the edge of a smooth table just starts to fall over the edge. The velocity v when a length x has fallen is given by $xv\dfrac{dv}{dx} + v^2 = gx$.

   Show that $v = 8\sqrt{(x/3)}$ ft/sec.
8. A toboggan weighing 200 lb., descends from rest on a uniform slope of 5 in 13 which is 15 yards long. If the coefficient of friction is 1/10 and the air resistance varies as the square of the velocity and is 3 lb. weight when the velocity is 10 ft/sec.; prove that its velocity at the bottom is 38.6 ft/sec and show that however long, the slope is the velocity cannot exceed 44 ft per sec.
   [**Hint.** Fig. 12.14. Equation of motion is
   $$\frac{W}{g}\cdot v\frac{dv}{dx} = -\mu R - kv^2 + W\sin\alpha]$$

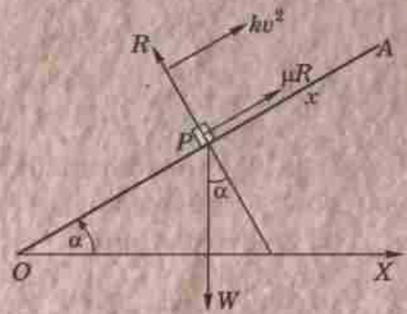


Fig. 12.14

9. Show that a particle projected from the earth's surface with a velocity of 7 miles/sec. will not return to the earth. [Take earth's radius = 3960 miles and $g = 32.17$ ft/sec^2].
10. A cylindrical tank 1.5 m. high stands on its circular base of diameter 1 m. and is initially filled with water. At the bottom of the tank there is a hole of diameter 1 cm., which is opened at some instant, so that the water starts draining under gravity. Find the height of water in the tank at any time t sec. Find the times at which the tank is one-half full, one quarter full, and empty.

    [**Hint.** Take $g = 980$ cm/sec^2 in $v = 0.6\sqrt{(2gy)}$]
11. The rate at which water flows from a small hole at the bottom of a tank is proportional to the square root of the depth of the water. If half the water flows from a cylindrical tank (with vertical axis) in 5 minutes, find the time required to empty the tank.
12. A conical cistern of height h and semi-vertical angle α is filled with water and is held in vertical position with vertex downwards. Water leaks out from the bottom at the rate of kx^2 cubic cms per second, k is a constant and x is the height of water level from the vertex. Prove that the cistern will be empty in $(\pi h \tan^2\alpha)/k$ seconds.
13. Upto a certain height in the atmosphere, it is found that the pressure p and the density ρ are connected by the relation $p = k\rho^n$ $(n > 1)$. If this relation continued to hold upto any height, show that the density would vanish at a finite height.

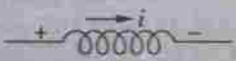
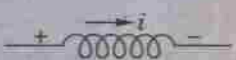
12.5 SIMPLE ELECTRIC CIRCUITS

We shall consider circuits made up of

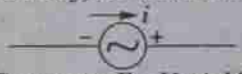
(*i*) three passive elements—resistance, inductance, capacitance and

(*ii*) an active element—voltage source which may be a battery or a generator.

(1) Symbols

	Element	*Symbol*	*Unit**
1.	*Quantity of electricity*	q	coulomb
2.	*Current* (= time rate flow of electricity)	i	ampere (A)
3.	*Resistance, R*	 R	ohm (Ω)
4.	*Inductance, L*	 L	henry (H)
5.	*Capacitance, C*	 C	farad (F)

*These units are respectively named after the French engineer and physicist *Charles Augustin de Coulomb* (1736–1806); French physicist *Andre Marie Ampere* (1775–1836); German physicist *George Simon Ohm* (1789–1854); Italian physicist *Joseph Henry* (1797–1878); American physicist *Michael Faraday* (1791–1867) and the Italian physicist *Alessandro Volta* (1745–1827).

Element	Symbol	Unit
6. *Electromotive force (e.m.f.) or voltage, E*	 *Battery, E = Constant*  *Generator, E = Variable*	volt (V)

7. *Loop* is any closed path formed by passing through two or more elements in series.
8. *Nodes* are the terminals of any of these elements.

(2) Basic relations

(*i*) $i = \dfrac{dq}{dt}$ or $q = \int i dt$ [$\because$ current is the rate of flow of electricity]

(*ii*) Voltage drop across resistance $R = Ri$ [Ohm's Law]

(*iii*) Voltage drop across inductance $L = L\dfrac{di}{dt}$

(*iv*) Voltage drop across capacitance $C = \dfrac{q}{C}$.

(3) Kirchhoff's laws*. The formulation of differential equations for an electrical circuit depends on the following two Kirchhoff's laws which are of cardinal importance :

I. *The algebraic sum of the voltage drops around any closed circuit is equal to the resultant electromotive force in the circuit.*

II. *The algebraic sum of the currents flowing into (or from) any node is zero.*

(4) Differential equations

(*i*) *R, L series circuit.* Consider a circuit containing resistance R and inductance L in series with a voltage source (battery) E. (Fig. 12.15).

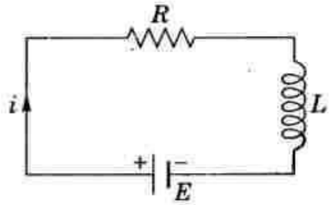


Fig. 12.15

Let i be the current flowing in the circuit at any time t. Then by Kirchhoff's first law, we have sum of voltage drops across R and $L = E$

i.e., $$Ri + L\frac{di}{dt} = E \quad \text{or} \quad \frac{di}{dt} + \frac{R}{L}i = \frac{E}{L} \qquad ...(1)$$

This is a Leibnitz's linear equation.

$$\text{I.F.} = e^{\int \frac{R}{L}dt} = e^{Rt/L} \text{ and therefore, its solution is } i(\text{I.F.}) = \int \frac{E}{L}(\text{I.F.})dt + c$$

or $$i \cdot e^{Rt/L} = \int \frac{E}{L} e^{Rt/L}\, dt + c = \frac{E}{L}\cdot\frac{L}{R}\cdot e^{Rt/L} + c \text{ whence } i = \frac{E}{R} + ce^{-Rt/L} \qquad ...(2)$$

If initially there is no current in the circuit, *i.e.*, $i = 0$, when $t = 0$, we have $c = -E/R$.

Thus (2) becomes $i = \dfrac{E}{R}\left(1 - e^{-Rt/L}\right)$ which shows that i increases with t and attains the maximum value E/R.

(*ii*) *R, L, C series circuit.* Now consider a circuit containing resistance R, inductance L and capacitance C all in series with a constant e.m.f. E (Fig. 12.16)

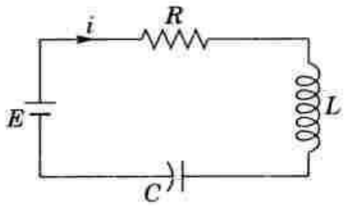


Fig. 12.16

If i be the current in the circuit at time t, then the charge q on the condenser $= \int i\, dt$, *i.e.*, $i = \dfrac{dq}{dt}$.

Applying Kirchhoff's law, we have, sum of the voltage drops across R, L and $C = E$.

i.e., $$Ri + L\frac{di}{dt} + \frac{q}{C} = E$$

*Named after the German physicist *Gustav Robert Kirchhoff* (1824–1887).

or
$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C}=E.$$

This is the desired differential equation of the circuit and will be solved in § 14.5.

Example 12.17. *Show that the differential equation for the current i in an electrical circuit containing an inductance L and a resistance R in series and acted on by an electromotive force E sin ωt satisfies the equation L di/dt + Ri = E sin ωt.*

Find the value of the current at any time t, if initially there is no current in the circuit.

(Kurukshetra, 2005)

Solution. By Kirchhoff's first law, we have sum of voltage drops across R and $L = E \sin \omega t$

i.e.,
$$Ri + L\frac{di}{dt} = E\sin\omega t.$$

This is the required differential equation which can be written as $\frac{di}{dt}+\frac{R}{L}i=\frac{E}{L}\sin\omega t$

This is a *Leibnitz's equation*. Its I.F. $= e^{\int\frac{R}{L}dt}=e^{\frac{Rt}{L}}$

$\therefore$ the solution is $i(\text{I.F.}) = \int\frac{E}{L}\sin\omega t\cdot(\text{I.F.})dt + c$

or
$$ie^{Rt/L}=\frac{E}{L}\int e^{Rt/L}\sin\omega t\,dt + c = \frac{E}{L}\frac{e^{Rt/L}}{\sqrt{[(R/L)^2+\omega^2]}}\sin\left(\omega t-\tan^{-1}\frac{L\omega}{R}\right)+c$$

or
$$i=\frac{E}{\sqrt{(R^2+\omega^2L^2)}}\sin(\omega t-\phi)+ce^{-Rt/L} \quad \text{where } \tan\phi = L\omega/R \qquad ...(i)$$

Initially when $t = 0$; $i = 0$. $\therefore$ $0=\frac{E\sin(-\phi)}{\sqrt{(R^2+\omega^2L^2)}}+c$, *i.e.*, $c=\frac{E\sin\phi}{\sqrt{(R^2+\omega^2L^2)}}$

Thus (i) takes the form $i=\frac{E\sin(\omega t-\phi)}{\sqrt{(R^2+\omega^2L^2)}}+\frac{E\sin\phi}{\sqrt{(R^2+\omega^2L^2)}}\cdot e^{-Rt/L}$

or
$$i=\frac{E}{\sqrt{(R^2+\omega^2L^2)}}[\sin(\omega t-\phi)+\sin\phi\,.\,e^{-Rt/L}]$$ which gives the current at any time t.

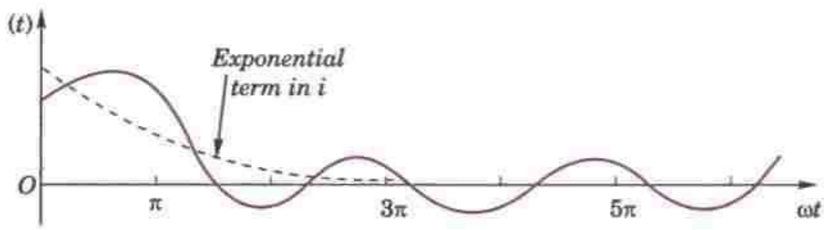


Fig. 12.17

Obs. As t increases indefinitely, the exponential term will approach zero. This implies that after sometime the current $i(t)$ will execute nearly harmonic oscillations only (Fig. 12.17).

PROBLEMS 12.4

1. When a switch is closed in a circuit containing a battery E, a resistance R and an inductance L, the current i builds up at a rate given by $L\,di/dt + Ri = E$.
   Find i as a function of t. How long will it be, before the current has reached one-half its final value if $E = 6$ volts, $R = 100$ ohms and $L = 0.1$ henry ?
2. When a resistance R ohms is connected in series with an inductance L henries with an e.m.f. of E volts, the current i amperes at time t is given by $L\,di/dt + Ri = E$.
   If $E = 10 \sin t$ volts and $i = 0$ when $t = 0$, find i as a function of t.

3. A resistance of 100 Ω, an inductance of 0.5 henry are connected in series with a battery of 20 volts. Find the current in the circuit at $t = 0.5$ sec, if $i = 0$ at $t = 0$. *(Marathwada, 2008)*
4. The equation of electromotive force in terms of current i for an electrical circuit having resistance R and condenser of capacity C in series, is
$$E = Ri + \int \frac{idt}{C}$$
Find the current i at any time t when $E = E_m \sin \omega t$. *(S.V.T.U., 2008, P.T.U., 2006)*
5. A resistance R in series with inductance L is shunted by an equal resistance R with capacity C. An alternating e.m.f. $E \sin pt$ produces currents i_1 and i_2 in two branches. If initially there is no current, determine i_1 and i_2 from the equations
$$L\frac{di_1}{dt} + Ri_1 = E \sin pt \quad \text{and} \quad \frac{i_2}{C} + R\frac{di_2}{dt} = pE \cos pt.$$
Verify that if $R^2C = L$, the total current $i_1 + i_2$ will be $(E \sin pt)/R$.

12.6 NEWTON'S LAW OF COOLING*

According to this law, *the temperature of a body changes at a rate which is proportional to the difference in temperature between that of the surrounding medium and that of the body itself.*

If θ_0 is the temperature of the surroundings and θ that of the body at any time t, then
$$\frac{d\theta}{dt} = -k(\theta - \theta_0), \text{ where } k \text{ is a constant.}$$

Example 12.18. *A body originally at 80°C cools down to 60°C in 20 minutes, the temperature of the air being 40°C. What will be the temperature of the body after 40 minutes from the original ?*

Solution. If θ be the temperature of the body at any time t, then
$$\frac{d\theta}{dt} = -k(\theta - 40), \qquad \text{where } k \text{ is a constant.}$$
Integrating, $$\int \frac{d\theta}{\theta - 40} = -k \int dt + \log c, \qquad \text{where } c \text{ is a constant.}$$
or $$\log(\theta - 40) = -kt + \log c \quad i.e., \quad \theta - 40 = ce^{-kt} \qquad ...(i)$$
When $t = 0$, $\theta = 80°$ and when $t = 20$, $\theta = 60°$. $\therefore$ $40 = c$, and $20 = ce^{-20k}$; $k = \frac{1}{20}\log 2$.

Thus (i) becomes $\theta - 40 = 40e^{-\left(\frac{1}{20}\log 2\right)t}$

When $t = 40$ min., $\theta = 40 + 40e^{-2\log 2} = 40 + 40e^{\log(1/4)} = 40 + 40 \times \frac{1}{4} = 50°C.$

12.7 HEAT FLOW

The fundamental principles involved in the problems of heat conduction are :

(*i*) *Heat flows from a higher temperature to the lower temperature.*

(*ii*) *The quantity of heat in a body is proportional to its mass and temperature.*

(*iii*) *The rate of heat-flow across an area is proportional to the area and to the rate of change of temperature with respect to its distance normal to the area.*

If q (cal./sec.) be the quantity of heat that flows across a slab of area α (cm^2) and thickness δx in one second, where the difference of temperature at the faces is δT, then by (*iii*) above
$$q = -k\alpha dT/dx \qquad ...(A)$$
where k is a constant depending upon the material of the body and is called the *thermal conductivity*.

*Named after the great English mathematician and physicist *Sir Issac Newton* (1642–1727) whose contributions are of utmost importance. He discovered many physical laws, invented Calculus alongwith *Leibnitz* (see footnote p. 139) and created analytical methods of investigating physical problems. He became professor at Cambridge in1699, but his *'Mathematical Principles of Natural Philosophy'* containing development of classical mechanics had been completed in 1687.

Example 12.19. *A pipe 20 cm in diameter contains steam at 150°C and is protected with a covering 5 cm thick for which $k = 0.0025$. If the temperature of the outer surface of the covering is 40°C, find the temperature half-way through the covering under steady state conditions.*

Solution. Let q cal./sec. be the constant quantity of heat flowing out radially through a surface of the pipe having radius x cm. and length 1 cm (Fig. 12.18). Then the area of the lateral surface (belt) = $2\pi x$.

$\therefore$ the equation (A) above gives

$$q = -k \cdot 2\pi x \cdot \frac{dT}{dx} \quad \text{or} \quad dT = -\frac{q}{2\pi k} \cdot \frac{dx}{x}$$

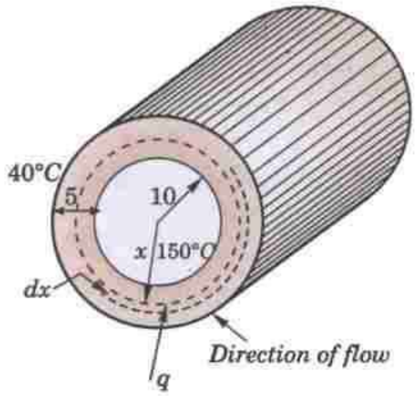


Fig. 12.18

Integrating, we have

$$T = -\frac{q}{2\pi k} \log_e x + c$$

Since $T = 150$, when $x = 10$. $\therefore$ $150 = -\dfrac{q}{2\pi k} \log_e 10 + c$ ...(*i*)

Again since $T = 40$, when $x = 15$, $40 = -\dfrac{q}{2\pi k} \log_e 15 + c$ ...(*ii*)

Subtracting (*ii*) from (*i*), $110 = \dfrac{q}{2\pi k} \log_e 1.5$ ...(*iii*)

Let $T = t$, when $x = 12.5$ $\therefore$ $t = -\dfrac{q}{2\pi k} \log_e 12.5 + c$ ...(*iv*)

Subtracting (*i*) from (*iv*), $t - 150 = -\dfrac{q}{2\pi k} \log_e 1.25$ ...(*v*)

Dividing (*v*) by (*iii*), $\dfrac{t-150}{110} = -\dfrac{\log_e 1.25}{\log_e 1.5}$, whence $t = 89.5$°C.

PROBLEMS 12.5

1. If the temperature of the air is 30°C and the substance cools from 100°C to 70°C in 15 minutes, find when the temperature will 40°C.
2. If the air is maintained at 30°C and the temperature of the body cools from 80°C to 60°C in 12 minutes, find the temperature of the body after 24 minutes.
3. Two friends *A* and *B* order coffee and receive cups of equal temperature at the same time. *A* adds a small amount of cool cream immediately but does not drink his coffee until 10 minutes later, *B* waits for 10 minutes and adds the same amount of cool cream and begins to drink. Assuming the Newton's law of cooling, decide who drinks the hotter coffee ?
4. A pipe 20 cm. in diameter contains steam at 200°C. It is covered by a layer of insulations 6 cm thick and thermal conductivity 0.0003. If the temperature of the outer surface is 30°C, find the heat loss per hour from two metre length of the pipe.
5. A steam pipe 20 cm. in diameter contains steam at 150°C and is covered with asbestos 5 cm thick. The outside temperature is kept at 60°C. By how much should the thickness of the covering be increased in order that the rate of heat loss should be decreased by 25% ?

12.8 RATE OF DECAY OF RADIO-ACTIVE MATERIALS

This law states that disintegration at any instant is proportional to the amount of material present.

of material at any time t, then $\dfrac{du}{dt} = -ku$, where k is a constant.

Example 12.20. *Uranium disintegrates at a rate proportional to the amount then present at any instant. If M_1 and M_2 grams of uranium are present at times T_1 and T_2 respectively, find the half-life of uranium.*

Solution. Let the mass of uranium at any time t be m grams.

Then the equation of disintegration of uranium is $\frac{dm}{dt} = -\mu m$, where μ is a constant.

Integrating, we get $\int \frac{dm}{dt} = -\mu \int dt + c$ or $\log m = c - \mu t$ ...(i)

Initially, when $t = 0$, $m = M$ (say) so that $c = \log M$ $\therefore$ (i) becomes, $\mu t = \log M - \log m$ ...(ii)

Also when $t = T_1$, $m = M_1$ and when $t = T_2$, $m = M_2$

$\therefore$ From (ii), we get $\mu T_1 = \log M - \log M_1$ ...(iii)

$\mu T_2 = \log M - \log M_2$ ...(iv)

Subtracting (iii) from (iv), we get

$$\mu(T_2 - T_1) = \log M_1 - \log M_2 = \log (M_1/M_2) \text{ whence } \mu = \frac{\log (M_1/M_2)}{T_2 - T_1}$$

Let the mass reduce to half its initial value in time T. *i.e.*, when $t = T$, $m = \frac{1}{2}M$.

$\therefore$ from (ii), we get $\mu T = \log M - \log (M/2) = \log 2.$

Thus $$T = \frac{1}{\mu} \log 2 = \frac{(T_2 - T_1)\log 2}{\log (M_1/M_2)}.$$

12.9 CHEMICAL REACTIONS AND SOLUTIONS

A type of problems which are especially important to chemical engineers are those concerning either chemical reactions or chemical solutions. These can be best explained through the following example :

Example 12.21. *A tank initially contains 50 gallons of fresh water. Brine, containing 2 pounds per gallon of salt, flows into the tank at the rate of 2 gallons per minute and the mixture kept uniform by stirring, runs out at the same rate. How long will it take for the quantity of salt in the tank to increase from 40 to 80 pounds ?* (Andhra, 1997)

Solution. Let the salt content at time t be u lb. so that its rate of change is du/dt

$= 2$ gal. $\times$ 2 lb. $= 4$ lb./min.

If C be the concentration of the brine at time t, the rate at which the salt content decreases due to the out-flow

$= 2$ gal. $\times$ C lb. $= 2C$ lb./min.

$$\therefore \quad \frac{du}{dt} = 4 - 2C \quad \text{...(i)}$$

Also since there is no increase in the volume of the liquid, the concentration $C = u/50$.

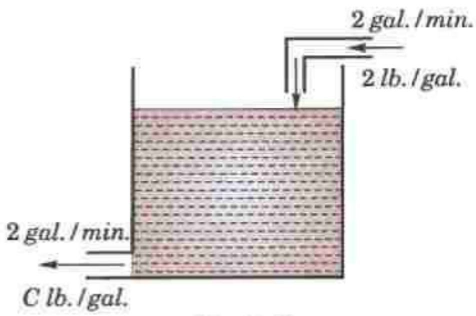


Fig. 12.19

$\therefore$ (i) becomes $\frac{du}{dt} = 4 - 2\frac{u}{50}$

Separating the variables and integrating, we have

$$\int dt = 25 \int \frac{du}{100 - u} + k \quad \text{or} \quad t = -25 \log_e (100 - u) + k \quad \text{...(ii)}$$

Initially when $t = 0$, $u = 0$ $\therefore$ $0 = -25 \log_e 100 + k$ ...(iii)

Eliminating k from (ii) and (iii), we get $t = 25 \log_e \frac{100}{100 - u}$.

Taking $t = t_1$ when $u = 40$ and $t = t_2$ when $u = 80$, we have

$$t_1 = 25 \log_e \frac{100}{60} \text{ and } t_2 = 25 \log_e \frac{100}{20}$$

$\therefore$ The required time $(t_2 - t_1) = 25 \log_e 5 - 25 \log_e 5/3$

$= 25 \log_e 3 = 25 \times 1.0986 = 27$ min. 28 sec.

PROBLEMS 12.6

1. The number N of bacteria in a culture grew at a rate proportional to N. The value of N was initially 100 and increased to 332 in one hour. What would be the value of N after $1\frac{1}{2}$ hours ? *(Nagarjuna, 2008 ; J.N.T.U., 2003)*
2. The rate at which bacteria multiply is proportional to the instantaneous number present. If the original number doubles in 2 hours, in how many hours will it triple ? *(Andhra, 2000)*
3. Radium decomposes at a rate proportional to the amount present. If a fraction p of the original amount disappears in 1 year, how much will remain at the end of 21 years ?
4. If 30% of radio active substance disappeared in 10 days, low long will it take for 90% of it to disappear ? *(Madras, 2000 S)*
5. Under certain conditions cane-sugar in water is converted into dextrose at a rate which is proportional to the amount unconverted at any time. If of 75 gm. at time $t = 0$, 8 gm. are converted during the first 30 minutes, find the amount converted in $1\frac{1}{2}$ hours.
6. In a chemical reaction in which two substances A and B initially of amounts a and b respectively are concerned, the velocity of transformation dx/dt at any time t is known to be equal to the product $(a - x)(b - x)$ of the amounts of the two substances then remaining untransformed. Find t in terms of x if $a = 0.7$, $b = 0.6$ and $x = 0.3$ when $t = 300$ seconds.
7. A tank contains 1000 gallons of brine in which 500 lt. of salt are dissolved. Fresh water runs into the tank at the rate of 10 gallons /minute and the mixture kept uniform by stirring, runs out at the same rate. How long will it be before only 50 lt. of salt is left in the tank ?
   [**Hint.** If u be the amount of salt after t minutes, then $du/dt = -10u/1000$.]
8. A tank is initially filled with 100 gallons of salt solution containing 1 lb. of salt per gallon. Fresh brine containing 2 lb. of salt per gallon runs into the tank at the rate of 5 gallons per minute and the mixture assumed to be kept uniform by stirring, runs out at the same rate. Find the amount of salt in the tank at any time, and determine how long it will take for this amount to reach 150 lb.

12.10 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 12.7

Fill up the blanks or choose the correct answer in the following problems :

1. If a coil having a resistance of 15 ohms and an inductance of 10 henries is connected to 90 volts supply then the current after 2 secs is .....
2. A tennis ball dropped from a height of 6 m, rebounds infinitely often. If it rebounds 80% of the distance that it falls, then the total distance for these bounces is .....
3. Radium decomposes at a rate proportional to the amount present. If 5% of the original amount disappears in 50 years then ......% will remain after 100 years.
4. The curve whose polar subtangent is constant is ......
5. The curve in which the length of the subnormal is proportional to the square of the ordinate, is ......
6. The curve in which the portion of the tangent between the axes is bisected at the point of contact, is ......
7. If the stream lines of a flow around a corner are $xy = c$, then the equipotential lines are ......
8. The orthogonal trajectories of a system of confocal and coaxial parabolas is ......
9. When a bullet is fired into a sand tank, its retardation is proportional to $\sqrt{(\text{velocity})}$. If it enters the sand tank with velocity v_0, it will come to rest after ...... seconds.
10. The rate at which bacteria multiply is proportional to the instantaneous number present. If the original number doubles in two hours, then it will triple after ..... hours.
11. Ram and Sunil order coffee and receive cups simultaneously at equal temperature. Ram adds a spoon of cold cream but doesn't drink for 10 minutes, Sunil waits for 10 minutes and adds a spoon of cold cream and begins to drink. Who drinks the hotter coffee ?
12. The equation $y - 2x = c$ represents the orthogonal trajectories of the family
    (*i*) $y = ae^{-2x}$ (*ii*) $x^2 + 2y^2 = a$ (*iii*) $xy = a$ (*iv*) $x + 2y = a$.

13. In order to keep a body in air above the earth for 12 seconds, the body should be thrown vertically up with a velocity of

(a) $\sqrt{6}$ g m/sec (b) $\sqrt{12}$ g m/sec (c) 6 g m/sec (d) 12g m/sec.

14. The orthogonal trajectory of the family $x^2 + y^2 = c^2$ is

(a) $x + y = c$ (b) $xy = c$ (c) $x^2 + y^2 = x + y$ (d) $y = cx$. *(V.T.U., 2010)*

15. If a thermometer is taken outdoors where the temperature is 0°C, from a room having temperature 21°C and the reading drops to 10°C in 1 minute then its reading will be 5°C after ......minutes.

16. The equation of the curve for which the angle between the tangent and the radius vector is twice the vectorial angle is $r^2 = 2a \sin 2\theta$. This satisfies the differential equation

(a) $r\dfrac{dr}{d\theta} = \tan 2\theta$ (b) $r\dfrac{dr}{d\theta} = \cos 2\theta$ (c) $r\dfrac{d\theta}{dr} = \tan 2\theta$ (d) $r\dfrac{d\theta}{dr} = \cos 2\theta$.

17. Two balls of m_1 and m_2 grams are projected vertically upwards such that the velocity of projection of m_1 is double that of m_2. If the maximum height to which m_1 and m_2 rise be h_1 and h_2 respectively then

(a) $h_1 = 2h_2$ (b) $2h_1 = h_2$ (c) $h_1 = 4h_2$ (d) $4h_1 = h_2$.

18. Two balls are projected simultaneously with same velocity from the top of a tower, one vertically upwards and the other vertically downwards. If they reach the ground in times t_1 and t_2, then the height of the tower is

(a) $\frac{1}{2} g t_1 t_2$ (b) $\frac{1}{2} g(t_1^2 + t_2^2)$ (c) $\frac{1}{2} g(t_1^2 - t_2^2)$ (d) $\frac{1}{2} g(t_1 + t_2)^2$.

19. A particle projected from the earth's surface with a velocity of 7 miles/sec will return to the earth.
(Taking $g = 32.17$ and earth's radius = 3960 miles) (True/False)

20. If a particle falls under gravity with air resistance k times its velocity, then its velocity cannot exceed g/k. (True/False)

CHAPTER 13

Linear Differential Equations

13.1 DEFINITIONS

Linear differential equations *are those in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together.* Thus the general linear differential equation of the nth order is of the form

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + p_2 \frac{d^{n-2} y}{dx^{n-2}} + ... + p_n y = X,$$

where $p_1, p_2, ..., p_n$ and X are functions of x only.

Linear differential equations with constant co-efficients are of the form

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + ... + k_n y = X$$

where $k_1, k_2, ..., k_n$ are constants. Such equations are most important in the study of electro-mechanical vibrations and other engineering problems.

13.2 (1) THEOREM

If y_1, y_2 are only two solutions of the equation

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + ... + k_n y = 0 \quad ...(1)$$

then $c_1 y_1 + c_2 y_2$ (= u) is also its solution.

Since $y = y_1$ and $y = y_2$ are solutions of (1).

$$\therefore \quad \frac{d^n y_1}{dx^n} + k_1 \frac{d^{n-1} y_1}{dx^{n-1}} + k_2 \frac{d^{n-2} y_1}{dx^{n-2}} + ... + k_n y_1 = 0 \quad ...(2)$$

and

$$\frac{d^n y_2}{dx^n} + k_1 \frac{d^{n-1} y_2}{dx^{n-1}} + k_2 \frac{d^{n-2} y_2}{dx^{n-2}} + ... + k_n y_2 = 0 \quad ...(3)$$

If c_1, c_2 be two arbitrary constants, then

$$\frac{d^n (c_1 y_1 + c_2 y_2)}{dx^n} + k_1 \frac{d^{n-1} (c_1 y_1 + c_2 y_2)}{dx^{n-1}} + ... + k_n (c_1 y_1 + c_2 y_2)$$

$$= c_1\left(\frac{d^n y_1}{dx^n} + k_1 \frac{d^{n-1}y_1}{dx^{n-1}} + \dots + k_n y_1\right) + c_2\left(\frac{d^n y_2}{dx^n} + k_1 \frac{d^{n-1}y_2}{dx^{n-1}} + \dots + k_n y_2\right)$$

$$= c_1(0) + c_2(0) = 0 \qquad \text{[By (2) and (3)]}$$

i.e.,
$$\frac{d^n u}{dx^n} + k_1 \frac{d^{n-1}u}{dx^{n-1}} + \dots + k_n y = 0 \qquad \dots(4)$$

This proves the theorem.

(2) Since the general solution of a differential equation of the nth order contains n arbitrary constants, it follows, from above, that if $y_1, y_2, y_3, \dots, y_n$, are n independent solutions of (1), then $c_1y_1 + c_2y_2 + \dots + c_ny_n$ $(= u)$ is its complete solution.

(3) If $y = v$ be any particular solution of

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1}y}{dx^{n-1}} + \dots + k_n y = X \qquad \dots(5)$$

then
$$\frac{d^n v}{dx^n} + k_1 \frac{d^{n-1}v}{dx^{n-1}} + \dots + k_n v = X \qquad \dots(6)$$

Adding (4) and (6), we have $\dfrac{d^n (u+v)}{dx^n} + k_1 \dfrac{d^{n-1}(u+v)}{dx^{n-1}} + \dots + k_n(u+v) = X$

This shows that $y = u + v$ is the complete solution of (5).

The part u is called the **complementary function (C.F.)** *and the part v is called the* **particular integral (P.I.)** *of (5).*

$\therefore$ the complete solution (C.S.) of (5) is **y = C.F. + P.I.**

Thus in order to solve the question (5), we have to first find the C.F., i.e., the complete solution of (1), and then the P.I., i.e. a particular solution of (5).

13.3 OPERATOR D

Denoting $\dfrac{d}{dx}, \dfrac{d^2}{dx^2}, \dfrac{d^3}{dx^3}$ etc. by D, D^2, D^3 etc., so that

$\dfrac{dy}{dx} = Dy$, $\dfrac{d^2y}{dx^2} = D^2y$, $\dfrac{d^3x}{dx^3} = D^3y$ etc., the equation (5) above can be written in the *symbolic form* $(D^n + k_1D^{n-1} + \dots + k_n)y = X$, *i.e.,* $f(D)y = X$,

where $f(D) = D^n + k_1D^{n-1} + \dots + k_n$, *i.e.*, a polynomial in D.

Thus *the symbol D stands for the operation of differentiation and can be treated much the same as an algebraic quantity i.e., $f(D)$ can be factorised by ordinary rules of algebra and the factors may be taken in any order.* For instance

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = (D^2 + 2D - 3)\,y = (D+3)(D-1)y \text{ or } (D-1)(D+3)\,y.$$

13.4 RULES FOR FINDING THE COMPLEMENTARY FUNCTION

To solve the equation $\dfrac{d^n y}{dx^n} + k_1 \dfrac{d^{n-1}y}{dx^{n-1}} + k_2 \dfrac{d^{n-2}y}{dx^{n-2}} + \dots + k_n y = 0 \qquad \dots(1)$

where k's are constants.

The equation (1) in *symbolic form is*

$$(D^n + k_1D^{n-1} + k_2D^{n-2} + \dots + k_n)y = 0 \qquad \dots(2)$$

Its symbolic co-efficient equated to zero *i.e.*

$$D^n + k_1D^{n-1} + k_2D^{n-2} + \dots + k_n = 0$$

is called the *auxiliary equation* (A.E.). Let $m_1, m_2, \dots, m_n$ be its roots.

Case I. *If all the roots be real and different*, then (2) is equivalent to

$$(D - m_1)(D - m_2) \dots (D - m_n)y = 0 \qquad \dots(3)$$

Now (3) will be satisfied by the solution of $(D - m_n)y = 0$, *i.e.*, by $\frac{dy}{dx} - m_n y = 0$.

This is a Leibnitz's linear and I.F. $= e^{-m_n x}$

$\therefore$ its solution is $y e^{-m_n x} = c_n$, *i.e.*, $y = c_n e^{m_n x}$

Similarly, since the factors in (3) can be taken in any order, it will be satisfied by the solutions of $(D - m_1)y = 0$, $(D - m_2)y = 0$ etc. *i.e.*, by $y = c_1 e^{m_1 x}$, $y = c_2 e^{m_2 x}$ etc.

Thus the complete solution of (1) is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$ ...(4)

Case II. *If two roots are equal (i.e.,* $m_1 = m_2$*),* then (4) becomes

$$y = (c_1 + c_2) e^{m_1 x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$$

$$y = C e^{m_1 x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x} \qquad [\because \ c_1 + c_2 = \text{one arbitrary constant } C]$$

It has only $n - 1$ arbitrary constants and is, therefore, not the complete solution of (1). In this case, we proceed as follows :

The part of the complete solution corresponding to the repeated root is the complete solution of $(D - m_1)(D - m_1)y = 0$

Putting $(D - m_1)y = z$, it becomes $(D - m_1) z = 0$ or $\frac{dz}{dx} - m_1 z = 0$

This is a Leibnitz's linear in z and I.F. $= e^{-m_1 x}$. $\therefore$ its solution is $ze^{-m_1 x} = c_1$ or $z = c_1 e^{m_1 x}$

Thus $\qquad (D - m_1)y = z = c_1 e^{m_1 x}$ or $\frac{dy}{dx} - m_1 y = c_1 e^{m_1 x}$ ...(5)

Its I.F. being $e^{-m_1 x}$, the solution of (5) is

$$ye^{-m_1 x} = \int c_1 e^{m_1 x} \, dx + c_2 = c_1 x + c_2 \ \text{ or } \ y = (c_1 x + c_2) e^{m_1 x}$$

Thus the complete solution of (1) is $y = (c_1 x + c_2) e^{m_1 x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$

If, however, the A.E. has three equal roots (*i.e.*, $m_1 = m_2 = m_3$), then the complete solution is

$$y = (c_1 x^2 + c_2 x + c_3)\, e^{m_1 x} + c_4 e^{m_4 x} + \dots + c_n e^{m_n x}$$

Case III. *If one pair of roots be imaginary, i.e.,* $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$, then the complete solution is

$$y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$$

$$= e^{\alpha x}(c_1 e^{i\beta x} + c_2 e^{-i\beta x}) + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$$

$$= e^{\alpha x}[c_1(\cos \beta x + i \sin \beta x) + c_2 (\cos \beta x - i \sin \beta x)] + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$$

[$\because$ by *Euler's Theorem*, $e^{i\theta} = \cos \theta + i \sin \theta$]

$$= e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x) + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$$

where $C_1 = c_1 + c_2$ and $C_2 = i(c_1 - c_2)$.

Case IV. *If two points of imaginary roots be equal i.e.,* $m_1 = m_2 = \alpha + i\beta$, $m_3 = m_4 = \alpha - i\beta$, then by case II, the complete solution is

$$y = e^{\alpha x}[(c_1 x + c_2) \cos \beta x + (c_3 x + c_4) \sin \beta x] + \dots + c_n e^{m_n x}.$$

Example 13.1. *Solve* $\frac{d^2 x}{dt^2} + 5\frac{dx}{dt} + 6x = 0$, *given* $x(0) = 0$, $\frac{dx}{dt}(0) = 15$. *(V.T.U., 2010)*

Solution. Given equation in *symbolic form is* $(D^2 + 5D + 6)\, x = 0$.

Its A.E. is $D^2 + 5D + 6 = 0$, *i.e.*, $(D + 2)(D + 3) = 0$ whence $D = -2, -3$.

$\therefore \qquad$ C.S. is $x = c_1 e^{-2t} + c_2 e^{-3t}$ and $\frac{dx}{dt} = -2ae^{-2t} - 3c_2 e^{-3t}$

When $t = 0$, $x = 0$. $\quad \therefore \quad 0 = c_1 + c_2$ (i)

When $t = 0$, $dx/dt = 15$ $\quad \therefore \quad 15 = -2c_1 - 3c_2$ ...(ii)

Solving (*i*) and (*ii*), $c_1 = 15, c_2 = -15$.
Hence the required solution is $x = 15\,(e^{-2t} - e^{-3t})$.

Example 13.2. *Solve* $\dfrac{d^2x}{dt^2} + 6\dfrac{dx}{dt} + 9x = 0.$

Solution. Given equation in *symbolic form is* $(D^2 + 6D + 9) = 0$

$\therefore$ A.E. is $D^2 + 6D + 9 = 0$, *i.e.*, $(D + 3)^2 = 0$ whence $D = -3, -3$.

Hence the C.S. is $x = (c_1 + c_2t)\,e^{-3t}$.

Example 13.3. *Solve* $(D^3 + D^2 + 4D + 4) = 0.$

Solution. Here the A.E. is $D^3 + D^2 + 4D + 4 = 0$ *i.e.*, $(D^2 + 4)(D + 1) = 0$ $\therefore$ $D = -1, \pm 2i$.

Hence the C.S. is $y = c_1e^{-x} + e^{0x}(c_2 \cos 2x + c_3 \sin 2x)$

i.e., $y = c_1e^{-x} + c_2 \cos 2x + c_3 \sin 2x$.

Example 13.4. *Solve (i)* $(D^4 - 4D + 4)\,y = 0$ *(Bhopal, 2008)*

(ii) $(D^2 + 1)^3\,y = 0$ *where* $D \equiv d/dx$.

Solution. (*i*) The A.E. equation is $D^4 - 4D^2 + 4 = 0$ or $(D^2 - 2)^2 = 0$

$\therefore$ $D^2 = 2, 2$ *i.e.*, $D = \pm\sqrt{2}, \pm\sqrt{2}$.

Hence the C.S. is $((c_1 + c_2x)\,e^{\sqrt{2}x} + (c_3 + c_4x)\,e^{-\sqrt{2}x})$ [Roots being repeated]

(*ii*) The A.E. equation is $(D^2 + 1)^3 = 0$

$\therefore$ $D = \pm i, \pm i, \pm i$.

Hence the C.S. is $y = e^{0x}\,[(c_1 + c_2x + c_3x^2)\cos x + (c_4 + c_5x + c_6x^2)\sin x]$

i.e., $y = (c_1 + c_2 + c_3x^2)\cos x + (c_4 + c_5\,x + c_6x^2)\sin x$.

Example 13.5. *Solve* $\dfrac{d^4x}{dt^4} + 4x = 0.$

Solution. Given equation in *symbolic form is* $(D^4 + 4)\,x = 0$

$\therefore$ A.E. is $D^4 + 4 = 0$ or $(D^4 + 4D^2 + 4) - 4D^2 = 0$ or $(D^2 + 2)^2 - (2D)^2 = 0$

or $(D^2 + 2D + 2)(D^2 - 2D + 2) = 0$

$\therefore$ either $D^2 + 2D + 2 = 0$ or $D^2 - 2D + 2 = 0$

whence $D = \dfrac{-2 \pm \sqrt{(-4)}}{2}$ and $\dfrac{2 \pm \sqrt{(-4)}}{2}$ *i.e.*, $D = -1 \pm i$ and $1 \pm i$.

Hence the required solution is $x = e^{-t}(c_1 \cos t + c_2 \sin t) + e^t(c_3 \cos t + c_4 \sin t)$.

PROBLEMS 13.1

Solve :

1. $\dfrac{d^2x}{dt^2} - 4\dfrac{dx}{dt} + 13x = 0,\ x(0),\ \dfrac{dx(0)}{dt} = 2.$ (V.T.U., 2008)
2. $y'' - 2y' + 10y = 0,\ y(0) = 4,\ y'(0) = 1.$
3. $4y''' + 4y'' + y' = 0.$
4. $\dfrac{d^3y}{dx^3} + y = 0.$ (V.T.U., 2000 S)
5. $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - y = 0.$
6. $\dfrac{d^4y}{dx^4} + 8\dfrac{d^2y}{dx^2} + 16y = 0.$ (J.N.T.U., 2005)
7. $(4D^4 - 8D^3 - 7D^2 + 11D + 6)y = 0.$ (V.T.U., 2008)
8. $(D^2 + 1)^2(D - 1)y = 0.$
9. If $\dfrac{d^4x}{dt^4} = m^4x$, show that $x = c_1 \cos mt + c_2 \sin mt + c_3 \cosh mt + c_4 \sinh mt$.

13.5 INVERSE OPERATOR

(1) Definition. $\frac{1}{f(D)}X$ is that function of x, not containing arbitrary constants which when operated upon by $f(D)$ gives X.

i.e., $$f(D)\left\{\frac{1}{f(D)}X\right\} = X$$

Thus $\frac{1}{f(D)}X$ *satisfies the equation* $f(D)y = X$ *and is, therefore, its particular integral.*

Obviously, $f(D)$ and $1/f(D)$ are inverse operators.

(2) $$\mathbf{\frac{1}{D}X = \int X dx}$$

Let $$\frac{1}{D}X = y \qquad ...(i)$$

Operating by D, $$D\frac{1}{D}X = Dy \quad i.e., \; X = \frac{dy}{dx}$$

Integrating both sides w.r.t. x, $y = \int X\,dx$, no constant being added as (*i*) does not contain any constant.

Thus $$\frac{1}{D}X = \int X\,dx.$$

(3) $$\mathbf{\frac{1}{D-a}X = e^{ax}\int Xe^{-ax}dx.}$$

Let $$\frac{1}{D-a}X = y \qquad ...(ii)$$

Operating by $D - a$, $(D-a)\,.\,\frac{1}{D-a}X = (D-a)y.$

or $$X = \frac{dy}{dx} - ay, \; i.e., \; \frac{dy}{dx} - ay = X$$ which is a Leibnitz's linear equation.

$\therefore$ I.F. being e^{-ax}, its solution is

$$ye^{-ax} = \int Xe^{-ax}dx,$$ no constant being added as (*ii*) doesn't contain any constant.

Thus $$\frac{1}{D-a}X = y = e^{ax}\int Xe^{-ax}dx.$$

13.6 RULES FOR FINDING THE PARTICULAR INTEGRAL

Consider the equation $\frac{d^ny}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + k_2\frac{d^{n-2}y}{dx^{n-2}} + ... + k_ny = X$

which is *symbolic form of* $(D^n + k_1D^{n-1} + k_2D^{n-2} + ... + k_n)y = X.$

$$\therefore \quad \text{P.I.} = \frac{1}{D^n + k_1D^{n-1} + k_2D^{n-2} + ... + k_n}X.$$

Case I. When $\mathbf{X = e^{ax}}$

Since
$$De^{ax} = ae^{ax}$$
$$D^2e^{ax} = a^2e^{ax}$$
....
....
$$D^ne^{ax} = a^ne^{ax}$$

$$\therefore \quad (D^n + k_1D^{n-1} + ... + k_n)e^{ax} = (a^n + k_1a^{n-1} + ... + k_n)e^{ax}, \quad i.e., \quad f(D)e^{ax} = f(a)e^{ax}$$

Operating on both sides by $\frac{1}{f(D)}, \frac{1}{f(D)} f(D) e^{ax} = \frac{1}{f(D)} f(a) e^{ax}$ or $e^{ax} = f(a) \frac{1}{f(D)} e^{ax}$

$\therefore$ dividing by $f(a)$,

$$\mathbf{\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}} \quad \textit{provided } f(a) \neq 0 \quad \text{...(1)}$$

If f(a) = 0, *the above rule fails and we proceed further.*

Since a is a root of A.E. $f(D) = D^n + k_1 D^{n-1} + ... + k_n = 0$.

$\therefore$ $D - a$ is a factor of $f(D)$. Suppose $f(D) = (D - a)\,\phi(D)$, where $\phi(a) \neq 0$. Then

$$\frac{1}{f(D)} e^{ax} = \frac{1}{D-a} \cdot \frac{1}{\phi(D)} e^{ax} = \frac{1}{D-a} \cdot \frac{1}{\phi(a)} e^{ax} \quad \text{[By (1)]}$$

$$= \frac{1}{\phi(a)} \cdot \frac{1}{D-a} e^{ax} = \frac{1}{\phi(a)} \cdot e^{ax} \int e^{ax} . e^{-ax} dx \quad \text{[By §13.5 (3)]}$$

$$= \frac{1}{\phi(a)} e^{ax} \int dx = x \frac{1}{\phi(a)} e^{ax} \quad i.e., \quad \mathbf{\frac{1}{f(D)} e^{ax} = x \frac{1}{f'(a)} e^{ax}} \quad \text{...(2)}$$

$$\left[\begin{array}{l} \because \ f'(D) = (D-a)\phi'(D) + 1 . \phi(D) \\ \therefore \ f'(a) = 0 \times \phi'(a) + \phi(a) \end{array} \right.$$

If $f'(a) = 0$, then applying (2) again, we get $\mathbf{\frac{1}{f(D)} e^{ax} = x^2 \frac{1}{f''(a)} e^{ax}}$, *provided* $f''(a) \neq 0$ ...(3)

and so on.

Example 13.6. *Find the P.I. of $(D^2 + 5D + 6)y = e^x$.*

Solution. P.I. $= \frac{1}{D^2 + 5D + 6} e^x$ [Put $D = 1$] $= \frac{1}{1^2 + 5.1 + 6} e^x = \frac{e^x}{12}$.

Example 13.7. *Find the P.I. of $(D + 2)(D - 1)^2 y = e^{-2x} + 2 \sinh x$.*

Solution. P.I. $= \frac{1}{(D+2)(D-1)^2} [e^{-2x} + 2 \sinh x] = \frac{1}{(D+2)(D-1)^2} [e^{-2x} + e^x - e^{-x}]$

Let us evaluate each of these terms separately.

$$\frac{1}{(D+2)(D-1)^2} e^{-2x} = \frac{1}{D+2} \cdot \left[\frac{1}{(D-1)^2} e^{-2x} \right]$$

$$= \frac{1}{D+2} \cdot \frac{1}{(-2-1)^2} e^{-2x} = \frac{1}{9} \cdot \frac{1}{D+2} e^{-2x}$$

$$= \frac{1}{9} . x . \frac{1}{1} e^{-2x} = \frac{x}{9} e^{-2x} \quad \left[\because \frac{d}{dD}(D+2) = 1 \right]$$

$$\frac{1}{(D+2)(D-1)^2} e^x = \frac{1}{1+2} \cdot \frac{1}{(D-1)^2} e^x = \frac{1}{3} . x^2 . \frac{1}{2} e^x = \frac{x^2}{6} e^x \quad \left[\because \frac{d^2}{dD^2}(D-1)^2 = 2 \right]$$

and $$\frac{1}{(D+2)(D-1)^2} e^{-x} = \frac{1}{(-1+2)(-1-1)^2} e^{-x} = \frac{e^{-x}}{4}$$

Hence, P.I. $= \frac{x}{9} e^{-2x} + \frac{x^2}{6} e^x + \frac{1}{4} e^{-x}$.

Case II. When X = sin (ax + b) or cos (ax + b).

Since
$$D \sin(ax + b) = a \cos(ax + b)$$
$$D^2 \sin(ax + b) = -a^2 \sin(ax + b)$$
$$D^3 \sin(ax + b) = -a^3 \cos(ax + b)$$

$$D^4 \sin(ax+b) = a^4 \sin(ax+b)$$

i.e., $$D^2 \sin(ax+b) = (-a^2)\sin(ax+b)$$

$$(D^2)^2 \sin(ax+b) = (-a^2)^2 \sin(ax+b)$$

In general $$(D^2)^r \sin(ax+b) = (-a^2)^r \sin(ax+b)$$

$\therefore$ $$f(D^2)\sin(ax+b) = f(-a^2)\sin(ax+b)$$

Operating on both sides $1/f(D^2)$,

$$\frac{1}{f(D^2)}\cdot f(D^2)\sin(ax+b) = \frac{1}{f(D^2)} f(-a^2)\sin(ax+b)$$

or $$\sin(ax+b) = f(-a^2)\frac{1}{f(D^2)}\sin(ax+b)$$

$\therefore$ Dividing by $f(-a^2)$. $\mathbf{\dfrac{1}{f(D^2)}\sin(ax+b) = \dfrac{1}{f(-a^2)}\sin(ax+b)}$ *provided* $f(-a^2)\neq 0$ ...(4)

If $\mathbf{f(-a^2)=0}$, *the above rule fails and we proceed further.*

Since $\cos(ax+b) + i\sin(ax+b) = e^{i(ax+b)}$ [Euler's theorem]

$\therefore$ $$\frac{1}{f(D^2)}\sin(ax+b) = \text{I.P. of } \frac{1}{f(D^2)}e^{i(ax+b)}$$ [Since $f(-a^2)=0$ $\therefore$ by (2)]

$$= \text{I.P. of } x\frac{1}{f'(D^2)}e^{i(ax+b)}$$ where $D^2 = -a^2$

$\therefore$ $$\mathbf{\frac{1}{f(D^2)}\sin(ax+b) = x\frac{1}{f'(-a^2)}\sin(ax+b)} \textit{ provided } f'(-a^2)\neq 0 \quad ...(5)$$

If $f'(-a^2)=0$, $\mathbf{\dfrac{1}{f(D^2)}\cdot\sin(ax+b) = x^2\dfrac{1}{f''(-a^2)}\sin(ax+b)}$, *provided* $f''(-a^2)\neq 0$, and so on.

Similarly, $\mathbf{\dfrac{1}{f(D^2)}\cos(ax+b) = \dfrac{1}{f(-a^2)}\cos(ax+b)}$, *provided* $f(-a^2)\neq 0$

If $f(-a^2)=0$, $\mathbf{\dfrac{1}{f(D^2)}\cos(ax+b) = x\cdot\dfrac{1}{f'(-a^2)}\cos(ax+b)}$, *provided* $f'(-a^2)\neq 0$.

If $f'(-a^2)=0$, $\mathbf{\dfrac{1}{f(D^2)}\cos(ax+b) = x^2\dfrac{1}{f''(-a^2)}\cos(ax+b)}$, *provided* $f''(-a^2)\neq 0$ and so on.

Example 13.8. *Find the P.I. of $(D^3+1)y = \cos(2x-1)$.*

Solution. P.I. $= \dfrac{1}{D^3+1}\cos(2x-1)$ [Put $D^2 = -2^2 = -4$]

$$= \frac{1}{D(-4)+1}\cos(2x-1)$$ [Multiply and divide by $1+4D$]

$$= \frac{(1+4D)}{(1-4D)(1+4D)}\cos(2x-1) = (1+4D)\cdot\frac{1}{1-16D^2}\cos(2x-1)$$ [Put $D^2 = -2^2 = -4$]

$$= (1+4D)\frac{1}{1-16(-4)}\cos(2x-1) = \frac{1}{65}[\cos(2x-1) + 4D\cos(2x-1)]$$

$$= \frac{1}{65}[\cos(2x-1) - 8\sin(2x-1)].$$

Example 13.9. *Find the P.I. of $\dfrac{d^3y}{dx^3} + 4\dfrac{dy}{dx} = \sin 2x$.*

Solution. Given equation in *symbolic form* is $(D^3+4D)y = \sin 2x$

$\therefore \quad \text{P.I.} = \dfrac{1}{D(D^2+4)}\sin 2x$ $\quad$ [$\because$ $D^2 + 4 = 0$ for $D^2 = -2^2$, $\therefore$ Apply (5) 477]

$= x\dfrac{1}{3D^2+4}\sin 2x$ $\quad \left[\because \dfrac{d}{dD}[D^3+4D] = 3D^2+4\right]$

[Put $D^2 = -2^2 = -4$]

$$= x\frac{1}{3(-4)+4}\sin 2x = -\frac{x}{8}\sin 2x.$$

Case III. When $\mathbf{X = x^m}$.

Here $\quad \text{P.I.} = \dfrac{1}{f(D)}x^m = [f(D)]^{-1}x^m.$

Expand $[f(D)]^{-1}$ in ascending powers of D as far as the term in D^m and operate on x^m term by term. Since the $(m+1)$th and higher derivatives of x^m are zero, we need not consider terms beyond D^m.

Example 13.10. *Find the P.I. of* $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = x^2 + 2x + 4.$

Solution. Given equation in *symbolic form* is $(D^2 + D)y = x^2 + 2x + 4$.

$$\therefore \quad \text{P.I.} = \frac{1}{D(D+1)}(x^2+2x+4) = \frac{1}{D}(1+D)^{-1}(x^2+2x+4)$$

$$= \frac{1}{D}(1-D+D^2-\ldots)(x^2+2x+4) = \frac{1}{D}[x^2+2x+4-(2x+2)+2]$$

$$= \int(x^2+4)dx = \frac{x^3}{3}+4x.$$

Case IV. When $\mathbf{X = e^{ax}V}$, *V being a function of x.*

If u is a function of x, then

$$D(e^{ax}u) = e^{ax}Du + ae^{ax}u + e^{ax}(D+a)u$$

$$D^2(e^{ax}u) = a^{ax}D^2u + 2ae^{ax}Du + a^2e^{ax}u = e^{ax}(D+a)^2u$$

and in general, $\quad D^n(e^{ax}u) = e^{ax}(D+a)^n u$

$\therefore \quad f(D)(e^{ax}u) = e^{ax}f(D+a)u$

Operating both sides by $1/f(D)$,

$$\frac{1}{f(D)}\cdot f(D)(e^{ax}u) = \frac{1}{f(D)}[e^{ax}f(D+a)u]$$

$$e^{ax}u = \frac{1}{f(D)}[e^{ax}f(D+a)u]$$

Now put $f(D+a)u = V$, *i.e.*, $u = \dfrac{1}{f(D+a)}V$, so that $e^{ax}\dfrac{1}{f(D+a)}V = \dfrac{1}{f(D)}(e^{ax}V)$

i.e., $$\mathbf{\frac{1}{f(D)}(e^{ax}V) = e^{ax}\frac{1}{f(D+a)}V.} \quad \text{...(6)}$$

Example 13.11. *Find P.I. of* $(D^2 - 2D + 4)y = e^x\cos x$.

Solution. $\quad \text{P.I.} = \dfrac{1}{D^2-2D+4}e^x\cos x$ $\quad$ [Replace D by $D+1$]

$= e^x\dfrac{1}{(D+1)^2-2(D+1)+4}\cos x = e^x\dfrac{1}{D^2+3}\cos x$ $\quad$ [Put $D^2 = -1^2 = -1$]

$$= e^x\frac{1}{-1+3}\cos x = \frac{1}{2}e^x\cos x.$$

Case V. When X is any other function of x.

Here $\qquad \text{P.I.} = \dfrac{1}{f(D)} X.$

If $\qquad f(D) = (D - m_1)(D - m_2) \dots (D - m_n)$, resolving into partial fractions,

$$\frac{1}{f(D)} = \frac{A_1}{D - m_1} + \frac{A_2}{D - m_2} + \dots + \frac{A_n}{D - m_n}$$

$\therefore$
$$\text{P.I.} = \left[\frac{A_1}{D - m_1} + \frac{A_2}{D - m_2} + \dots + \frac{A_n}{D - m_n}\right] X$$

$$= A_1 \frac{1}{D - m_1} X + A_2 \frac{1}{D - m_2} X + \dots + A_n \frac{1}{D - m_n} X$$

$$= A_1 . e^{m_1 x} \int X e^{-m_1 x} dx + A_2 . e^{m_2 x} \int X e^{-m_2 x} dx + \dots + A_n . e^{m_n x} \int X e^{-m_n x} dx \qquad \text{[By §13.5 ...(3)]}$$

Obs. This method is a general one and can, therefore, be employed to obtain a particular integral in any given case.

13.7 WORKING PROCEDURE TO SOLVE THE EQUATION

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + k_{n-1} \frac{dy}{dx} + k_n y = X$$

of which the *symbolic form* is

$$(D^n + k_1 D^{n-1} + \dots + k_{n-1} D + k_n) y = X .$$

Step I. To find the complementary function

(*i*) *Write the A.E.*

i.e., $\qquad D^n + k_1 D^{n-1} + \dots + k_{n-1} D + k_n = 0$ and solve it for D.

(*ii*) Write the *C.F.* as follows :

Roots of A.E.	*C.F.*
1. m_1, m_2, m_3 ... (real and different roots)	$c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \dots$
2. m_1, m_1, m_3 ... (two real and equal roots)	$(c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + \dots$
3. m_1, m_1, m_1, m_4 ... (three real and equal roots)	$(c_1 + c_2 x + c_3 x^2) e^{m_1 x} + c_4 e^{m_4 x} + \dots$
4. $\alpha + i\beta, \alpha - i\beta, m_3$ ... (a pair of imaginary roots)	$e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_3 x} + \dots$
5. $\alpha \pm i\beta, \alpha \pm i\beta, m_5$ ... (2 pairs of equal imaginary roots)	$e^{\alpha x}[(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x] + c_5 e^{m_5 x} + \dots$

Step II. To find the particular integral

From symbolic form P.I. $= \dfrac{1}{D^n + k_1 D^{n-1} + \dots + k_{n-1} D + k_n} X = \dfrac{1}{f(D)}$ or $\dfrac{1}{\phi(D^2)} X$

(*i*) *When* $X = e^{ax}$

$$\text{P.I.} = \frac{1}{f(D)} e^{ax}, \text{ put } D = a, \qquad [f(a) \neq 0]$$

$$= x \frac{1}{f'(D)} e^{ax}, \text{ put } D = a, \qquad [f(a) = 0, f'(a) \neq 0]$$

$$= x^2 \frac{1}{f''(D)} e^{ax}, \text{ put } D = a, \qquad [f'(a) = 0, f''(a) \neq 0]$$

and so on.

where $\qquad f'(D) =$ diff. coeff. of $f(D)$ w.r.t. D

$\qquad f''(D) =$ diff. coeff. of $f'(D)$ w.r.t. D, etc.

(*ii*) *When* $X = \sin(ax+b)$ or $\cos(ax+b)$.

$$\text{P.I.} = \frac{1}{\phi(D^2)} \sin(ax+b) \text{ [or } \cos(ax+b)], \text{ put } D^2 = -a^2 \qquad [\phi(-a^2) \neq 0]$$

$$= x \frac{1}{\phi'(D^2)} \sin(ax+b) \text{ [or } \cos(ax+b)], \text{ put } D^2 = -a^2 \qquad [\phi(-a^2) = 0, \phi'(-a^2) \neq 0]$$

$$= x^2 \frac{1}{\phi''(D^2)} \sin(ax+b) \text{ [or } \cos(ax+b)], \text{ put } D^2 = -a^2 \qquad [\phi'(-a^2) \neq 0, \phi''(-a^2) \neq 0]$$

and so on.

where $\phi'(D^2)$ = diff. coeff. of $\phi(D^2)$ w.r.t. D,

$\phi''(D^2)$ = diff. coeff. of $\phi'(D^2)$ w.r.t. D, etc.

(*iii*) *When* $X = x^m$, m being a positive integer.

$$\text{P.I.} = \frac{1}{f(D)} x^m = [f(D)]^{-1} x^m$$

To evaluate it, expand $[f(D)]^{-1}$ in ascending powers of D by Binomial theorem as far as D^m and operate on x^m term by term.

(*iv*) *When* $X = e^{ax}V$, where V is a function of x.

$$\text{P.I.} = \frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V$$

and then evaluate $\frac{1}{f(D+a)} V$ as in (*i*), (*ii*), and (*iii*).

(*v*) *When X is any function of x.*

$$\text{P.I.} = \frac{1}{f(D)} X$$

Resolve $\frac{1}{f(D)}$ into partial fractions and operate each partial fraction on X remembering that

$$\frac{1}{D-a} X = e^{ax} \int X e^{-ax}\, dx.$$

Step III. To find the complete solution

Then the C.S. is y = C.F. + P.I.

Example 13.12. *Solve* $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = (1-e^x)^2$.

Solution. Given equation in *symbolic form* is $(D^2 + D + 1)y = (1-e^x)^2$

(*i*) *To find C.F.*

Its A.E. is $D^2 + D + 1 = 0$, $\therefore$ $D = \frac{1}{2}(-1+\sqrt{3}i)$

Thus C.F. $= e^{-x/2}\left(c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x\right)$

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{D^2+D+1}(1-2e^x+e^{2x}) = \frac{1}{D^2+D+1}(e^{0x}-2e^x+e^{2x})$$

$$= \frac{1}{0^2+0+1}e^{0x} - 2.\frac{1}{1^2+1+1}e^x + \frac{1}{2^2+2+1}e^{2x} = 1 - \frac{2}{3}e^x + \frac{e^{2x}}{7}$$

(*iii*) Hence the C.S. is $y = e^{-x/2}\left(c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x\right) + 1 - \frac{2}{3}e^x + \frac{e^{2x}}{7}$.

Example 13.13. *Solve $y'' + 4y' + 4y = 3 \sin x + 4 \cos x$, $y(0) = 1$ and $y'(0) = 0$.* (J.N.T.U., 2003)

Solution. Given equation in *symbolic form* is $(D^2 + 4D + 4)y = 3 \sin x + 4 \cos x$

(*i*) *To find C.F.*

Its A.E. is $(D + 2)^2 = 0$ where $D = -2, -2$ $\therefore$ C.F. $= (c_1 + c_2 x)e^{-2x}$.

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{D^2 + 4D + 4}(3 \sin x + 4 \cos x) = \frac{1}{-1 + 4D + 4}(3 \sin x + 4 \cos x)$$
$$= \frac{4D - 3}{16D^2 - 9}(3 \sin x + 4 \cos x) = \frac{(4D - 3)}{-16 - 9}(3 \sin x + 4 \cos x)$$
$$= \frac{-1}{25}\{3(4 \cos x - 3 \sin x) + 4(-4 \sin x - 3 \cos x)\} = \sin x$$

(*iii*) C.S. is $y = (c_1 + c_2)\, e^{-2x} + \sin x$

When $x = 0, y = 1,$ $\therefore$ $1 = c_1$

Also $y' = c_2 e^{-2x} + (c_1 + c_2 x)(-2)e^{-2x} + \cos x$.

When $x = 0, y' = 0,$ $\therefore$ $0 = c_2 - 2c_1 + 1$, *i.e.*, $c_2 = 1$.

Hence the required solution is $y = (1 + x)\, e^{-2x} + \sin x$.

Example 13.14. *Solve $(D - 2)^2 = 8(e^{2x} + \sin 2x + x^2)$.*

Solution. (*i*) *To find C.F.*

Its A.E. is $(D - 2)^2 = 0$, $\therefore$ $D = 2, 2$.

Thus C.F. $= (c_1 + c_2 x)\, e^{2x}$.

(*ii*) *To find P.I.*

$$\text{P.I.} = 8\left[\frac{1}{(D-2)^2} e^{2x} + \frac{1}{(D-2)^2} \sin 2x + \frac{1}{(D-2)^2} x^2\right]$$

Now $\dfrac{1}{(D-2)^2} e^{2x} = x^2 \dfrac{1}{2(1)} e^{2x}$ [$\because$ by putting $D = 2$, $(D - 2)^2 = 0$, $2(D - 2) = 0$]

$$= \frac{x^2 e^{2x}}{2}$$

$$\frac{1}{(D-2)^2} \sin 2x = \frac{1}{D^2 - 4D + 4} \sin 2x = \frac{1}{(-2^2) - 4D + 4} \sin 2x$$
$$= -\frac{1}{4}\int \sin 2x\, dx = -\frac{1}{4}\left(\frac{-\cos 2x}{2}\right) = \frac{1}{8} \cos 2x$$

and
$$\frac{1}{(D-2)^2} x^2 = \frac{1}{4}\left(1 - \frac{D}{2}\right)^{-2} . x^2 = \frac{1}{4}\left[1 + (-2)\left(\frac{D}{2}\right) + \frac{(-2)(-3)}{2!}\left(-\frac{D}{2}\right)^2 + \ldots\right] x^2$$
$$= \frac{1}{4}\left(1 + D + \frac{3D^2}{4} + \ldots\right) x^2 = \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right)$$

Thus P.I. $= 4x^2 e^{2x} + \cos 2x + 2x^2 + 4x + 3$.

(*iii*) Hence the C.S. is $y = (c_1 + c_2 x)\, e^{2x} + 4x^2 e^{2x} + \cos 2x + 2x^2 + 4x + 3$.

Example 13.15. *Find the complete solution of $y'' - 2y' + 2y = x + e^x \cos x$.* (U.P.T.U., 2002)

Solution. Given equation in *symbolic form* is $(D^2 - 2D + 2)y = x + e^x \cos x$

(*i*) *To find C.F.*

Its A.E. is $D^2 - 2D + 2 = 0$ $\therefore$ $D = \dfrac{2 \pm \sqrt{(4 - 8)}}{2} = 1 \pm i$.

Thus C.F. $= e^x\, (c_1 \cos x + c_2 \sin x)$

(ii) To find P.I.

$$\text{P.I.} = \frac{1}{D^2 - 2D + 2}(x) + \frac{1}{D^2 - 2D + 2}(e^x \cos x)$$

$$= \frac{1}{2}\left[1 - \left(D - \frac{D^2}{2}\right)\right]^{-1}(x) + e^x \frac{1}{(D+1)^2 - 2(D+1) + 2}(\cos x)$$

$$= \frac{1}{2}\left(1 + D - \frac{D^2}{2}\right)x + e^x \frac{1}{D^2 + 1}\cos x \qquad \text{[Case of failure]}$$

$$= \frac{1}{2}(x + 1 - 0) + e^x \cdot x \frac{1}{2D}\cos x = \frac{1}{2}(x+1) + \frac{xe^x}{2}\int \cos x \, dx = \frac{1}{2}(x+1) + \frac{xe^x}{2}\sin x$$

(iii) Hence the C.S. is $y = e^x(c_1 \cos x + c_2 \sin x) + \frac{1}{2}(x+1) + \frac{xe^x}{2}\sin x$.

Example 13.16. *Solve* $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = xe^{3x} + \sin 2x$.

(V.T.U., 2008 ; Kottayam, 2005 ; U.P.T.U., 2003)

Solution. Given equation in *symbolic form* is $(D^2 - 3D + 2)y = xe^{3x} + \sin 2x$

(i) To find C.F.

Its A.E. is $D^2 - 3D + 2 = 0$ or $(D-2)(D-1) = 0$ whence $D = 1, 2$.

Thus C.F. $= c_1e^x + c_2e^{2x}$

(ii) To find P.I.

$$\text{P.I.} = \frac{1}{D^2 - 3D + 2}(xe^{3x} + \sin 2x) = \frac{1}{D^2 - 3D + 2}(e^{3x} . x) + \frac{1}{D^2 - 3D + 2}(\sin 2x)$$

$$= e^{3x} . \frac{1}{(D+3)^2 - 3(D+3) + 2}(x) + \frac{1}{-4 - 3D + 2}(\sin 2x)$$

$$= e^{3x} . \frac{1}{D^2 + 3D + 2}(x) - \frac{3D - 2}{9D^2 - 4}(\sin 2x) = \frac{e^{3x}}{2} . \left[1 + \left\{\frac{3D + D^2}{2}\right\}\right]^{-1} x - \frac{(3D-2)}{9(-4) - 4}(\sin 2x)$$

$$= \frac{e^{3x}}{2}\left(1 - \frac{3D}{2} ...\right)x + \frac{1}{40}(6\cos 2x - 2\sin 2x) = \frac{e^{3x}}{2}\left(x - \frac{3}{2}\right) + \frac{1}{20}(3\cos 2x - \sin 2x)$$

(iii) Hence the C.S. is $y = c_1e^x + c_2e^{2x} + e^{3x}\left(\frac{x}{2} - \frac{3}{4}\right) + \frac{1}{20}(3\cos 2x - \sin 2x)$.

Example 13.17. *Solve* $\frac{d^2y}{dx^2} - 4y = x \sinh x$. *(Madras, 2000 S)*

Solution. Given equation in *symbolic form* is $(D^2 - 4)y = x \sinh x$.

(i) To find C.F.

Its A.E. is $D^2 - 4 = 0$, whence $D = \pm 2$.

Thus C.F. $= c_1e^{2x} + c_2e^{-2x}$

(ii) To find P.I.

$$\text{P.I.} = \frac{1}{D^2 - 4}x \sinh x = \frac{1}{D^2 - 4}x\left(\frac{e^x - e^{-x}}{2}\right) = \frac{1}{2}\left[\frac{1}{D^2 - 4}e^x \cdot x - \frac{1}{D^2 - 4}e^{-x} \cdot x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{(D+1)^2 - 4}x - e^{-x}\frac{1}{(D-1)^2 - 4}x\right] = \frac{1}{2}\left[e^x \frac{1}{D^2 + 2D - 3}x - e^{-x}\frac{1}{D^2 - 2D - 3}x\right]$$

$$= \frac{1}{2}\left[\frac{e^x}{-3}\left\{1-\left(\frac{2D}{3}+\frac{D^2}{3}\right)\right\}^{-1}.x - \frac{e^{-x}}{-3}\left\{1+\left(\frac{2D}{3}-\frac{D^2}{3}\right)\right\}^{-1}\cdot x\right]$$

$$= -\frac{1}{6}\left[e^x\left(1+\frac{2D}{3}+...\right)x - e^{-x}\left(1-\frac{2D}{3}+...\right)x\right] = -\frac{1}{6}\left[e^x\left(x+\frac{2}{3}\right) - e^{-x}\left(x-\frac{2}{3}\right)\right]$$

$$= -\frac{x}{3}\left(\frac{e^x - e^{-x}}{2}\right) - \frac{2}{9}\left(\frac{e^x + e^{-x}}{2}\right) = -\frac{x}{3}\sinh x - \frac{2}{9}\cosh x.$$

(*iii*) Hence the C.S. is $y = c_1e^{2x} + c_2e^{-2x} - \frac{x}{3}\sinh x - \frac{2}{9}\cosh x$.

Example 13.18. *Solve* $(D^2 - 1)y = x \sin 3x + \cos x$.

Solution. (*i*) *To find C.F.*

Its A.E. is $D^2 - 1 = 0$, whence $D = \pm 1$. $\therefore$ C.F. $= c_1e^x + c_2e^{-x}$

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{D^2 - 1}(x \sin 3x + \cos x) = \frac{1}{D^2-1}x\,(\text{I.P. of } e^{3ix}) + \frac{1}{D^2-1}\cos x$$

$$= \text{I.P. of } \frac{1}{D^2-1}e^{3ix}.x + \frac{1}{(-1)^2-1}\cos x = \text{I.P. of}\left[e^{3ix}\frac{1}{(D+3i)^2-1}x\right] - \frac{\cos x}{2}$$

[Replacing D by $D + 3i$]

$$= \text{I.P. of}\left[e^{3ix}\frac{1}{D^2+6iD-10}x\right] - \frac{\cos x}{2}$$

$$= \text{I.P. of}\left[e^{3ix}.\frac{1}{-10}\left(1-\frac{3iD}{5}-\frac{D^2}{10}\right)^{-1}x\right] - \frac{\cos x}{2}$$ [Expand by Binomial theorem]

$$= \text{I.P. of}\left[e^{3ix}.\frac{1}{-10}\left(1+\frac{3iD}{5}+......\right)x\right] - \frac{\cos x}{2} = \text{I.P. of}\left[-\frac{e^{3ix}}{10}\left(x+\frac{3i}{5}\right)\right] - \frac{\cos x}{2}$$

$$= \text{I.P. of}\left[\frac{-1}{10}(\cos 3x + i \sin 3x)\left(x+\frac{3i}{5}\right)\right] - \frac{\cos x}{2}$$

$$= -\frac{1}{10}\text{ I.P. of}\left[\left(x\cos 3x - \frac{3\sin 3x}{5}\right) + i\left(x\sin 3x + \frac{3}{5}\cos 3x\right)\right] - \frac{\cos x}{2}$$

$$= -\frac{1}{10}\left(x\sin 3x + \frac{3}{5}\cos 3x\right) - \frac{\cos x}{2}.$$

(*iii*) Hence the C.S. is $y = c_1e^x + c_2e^{-x} - \frac{1}{50}(5x \sin 3x + 3\cos 3x + 25\cos x)$.

Example 13.19. *Solve* $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x$. (*S.V.T.U., 2007 ; J.N.T.U., 2006 ; U.P.T.U., 2005*)

Solution. Given equation in *symbolic form* is $(D^2 - 2D + 1)\,y = xe^x \sin x$

(*i*) *To find C.F.*

Its A.E. is $D^2 - 2D + 1 = 0$, *i.e.*, $(D-1)^2 = 0$

$\therefore$ $D = 1, 1$. Thus C.F. $= (c_1 + c_2x)e^x$

(ii) To find P.I.

$$\text{P.I.} = \frac{1}{(D-1)^2} e^x \cdot x \sin x = e^x \cdot \frac{1}{(D+1-1)^2} x \sin x$$

$$= e^x \frac{1}{D^2} x \sin x = e^x \frac{1}{D} \int x \sin x \, dx \qquad \text{[Integrate by parts]}$$

$$= e^x \frac{1}{D}\left[x(-\cos x) - \int 1 . (-\cos x)\, dx\right] = e^x \int [-x \cos x + \sin x]\, dx$$

$$= e^x\left[-\left\{x \sin x - \int 1 . \sin x\, dx\right\} - \cos x\right] = e^x[-x \sin x - \cos x - \cos x]$$

$$= -e^x(x \sin x + 2 \cos x).$$

(iii) Hence the C.S. is $y = (c_1 + c_2x)\, e^x - e^x(x \sin x + 2 \cos x)$.

Example 13.20. *Solve* $(D^4 + 2D^2 + 1)\, y = x^2 \cos x$. *(Nagarjuna, 2008 ; Rajasthan, 2005)*

Solution. *(i) To find C.F.*

Its A.E. is $(D^2 + 1)^2 = 0$ whose roots are $D = \pm i, \pm i$

$\therefore$ C.F. $= (c_1 + c_2x) \cos x + (c_3 + c_4x) \sin x$

(ii) To find P.I.

$$\text{P.I.} = \frac{1}{(D^2+1)^2} x^2 \cos x = \frac{1}{(D^2+1)^2} x^2 (\text{Re.P. of } e^{ix})$$

$$= \text{Re.P. of } \left\{\frac{1}{(D^2+1)^2} e^{ix} . x^2\right\} = \text{Re.P. of } \left\{e^{ix} \frac{1}{[(D+i)^2+1]^2} x^2\right\}$$

$$= \text{Re.P. of } \left\{e^{ix} \frac{1}{(D^2+2iD)^2} x^2\right\} = \text{Re.P. of } \left[e^{ix}\left\{-\frac{1}{4D^2}\left(1 - \frac{i}{2}D\right)^{-2} x^2\right\}\right]$$

$$= \text{Re.P. of } \left[-\frac{1}{4} e^{ix} . \frac{1}{D^2}\left\{1 + 2\frac{iD}{2} + 3\left(\frac{iD}{2}\right)^2 + \ldots\right\} x^2\right]$$

$$= \text{Re.P. of } \left\{-\frac{1}{4} e^{ix} . \frac{1}{D^2}\left(x^2 + 2ix - \frac{3}{2}\right)\right\} = \text{Re.P. of } \left\{-\frac{1}{4} e^{ix} . \frac{1}{D}\left(\frac{x^3}{3} + ix^2 - \frac{3}{2}x\right)\right\}$$

$$= -\frac{1}{4} \text{Re.P. of } \left\{e^{ix}\left(\frac{x^4}{12} + \frac{ix^3}{3} - \frac{3}{4}x^2\right)\right\} = \frac{-1}{48} \text{ Re.P. of } \{(\cos x + i \sin x)(x^4 + 4ix^3 - 9x^2)\}$$

$$= -\frac{1}{48}[(x^4 - 9x^2) \cos x - 4x^3 \sin x]$$

(iii) Hence the C.S. is $y = (c_1 + c_2x) \cos x + (c_3 + c_4x) \sin x + \dfrac{1}{48}\,[4x^3 \sin x - x^2\,(x^2 - 9) \cos x]$.

Example 13.21. *Solve* $(D^2 - 4D + 4)y = 8x^2e^{2x} \sin 2x$. *(J.N.T.U., 2006 ; U.P.T.U., 2004)*

Solution. *(i) To find C.F.*

Its A.E. is $D^2 - 4D + 4 = 0$ *i.e.*, $(D-2)^2 = 0$. $\therefore$ $D = 2, 2$

$\therefore$ C.F. $= (c_1 + c_2x)\, e^{2x}$

(ii) To find P.I.

$$\text{P.I.} = \frac{1}{(D-2)^2}(8x^2e^{2x} \sin 2x) = 8e^{2x} \frac{1}{(D+2-2)^2}(x^2 \sin 2x)$$

$$= 8e^{2x} \frac{1}{D^2}(x^2 \sin 2x) = 8e^{2x} . \frac{1}{D} \int x^2 \sin 2x\, dx$$

$$= 8e^{2x} \cdot \frac{1}{D}\left\{x^2\left(\frac{-\cos 2x}{2}\right) - \int 2x\left(\frac{-\cos 2x}{2}\right)dx\right\}$$

$$= 8e^{2x}\frac{1}{D}\left\{-\frac{x^2}{2}\cos 2x + x\frac{\sin 2x}{2} - \int 1 . \frac{\sin 2x}{2}dx\right\}$$

$$= 8e^{2x}\int\left\{-\frac{x^2}{2}\cos 2x + \frac{x}{2}\sin 2x + \frac{\cos 2x}{4}\right\}dx$$

$$= 8e^{2x}\left[\left\{\frac{-x^2}{2}\frac{\sin 2x}{2} - \int(-x)\frac{\sin 2x}{2}dx\right\} + \left\{\int\frac{x}{2}\sin 2x\,dx\right\} + \frac{\sin 2x}{8}\right]$$

$$= 8e^{2x}\left[\left(\frac{-x^2}{4} + \frac{1}{8}\right)\sin 2x + \int x\sin 2x\,dx\right]$$

$$= 8e^{2x}\left[\left(\frac{1}{8} - \frac{x^2}{4}\right)\sin 2x + x\left(\frac{-\cos 2x}{2}\right) - \int 1 . \left(\frac{-\cos 2x}{2}\right)dx\right]$$

$$= 8e^{2x}\left[\left(\frac{1}{8} - \frac{x^2}{4}\right)\sin 2x - \frac{x\cos 2x}{2} + \frac{\sin 2x}{4}\right]$$

$$= e^{2x}[(3 - 2x^2)\sin 2x - 4x\cos 2x]$$

(*iii*) Hence the C.S. is $y = e^{2x}[c_1 + c_2x + (3 - 2x^2)\sin 2x - 4x\cos 2x]$.

Example 13.22. *Solve* $\dfrac{d^2y}{dx^2} + a^2y = \sec ax$.

Solution. Given equation in *symbolic form* is $(D^2 + a^2)y = \sec ax$.

(*i*) *To find C.F.*

Its A.E. is $D^2 + a^2 = 0 \quad \therefore \quad D = \pm ia$.

Thus C.F. $= c_1 \cos ax + c_2 \sin ax$.

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{D^2 + a^2}\sec ax = \frac{1}{(D + ia)(D - ia)}\sec ax \qquad \text{[Resolving into partial fractions]}$$

$$= \frac{1}{2ia}\left[\frac{1}{D - ia} - \frac{1}{D + ia}\right]\sec ax = \frac{1}{2ia}\left[\frac{1}{D - ia}\sec ax - \frac{1}{D + ia}\sec ax\right]$$

Now $\quad \dfrac{1}{D - ia}\sec ax = e^{iax}\displaystyle\int \sec ax . e^{-iax}\,dx \qquad \left[\because \dfrac{1}{D - a}X = e^{ax}\displaystyle\int Xe^{-ax}\,dx\right]$

$$= e^{iax}\int\frac{\cos ax - i\sin ax}{\cos ax}dx = e^{iax}\int(1 - i\tan ax)dx = e^{iax}\left(x + \frac{i}{a}\log\cos ax\right)$$

Changing i to $-i$, we have

$$\frac{1}{D + ia}\sec ax = e^{-iax}\left\{x - \frac{i}{a}\log\cos ax\right\}$$

Thus $\quad \text{P.I.} = \dfrac{1}{2ia}\left[e^{iax}\left\{x + \dfrac{i}{a}\log\cos ax\right\} - e^{-iax}\left\{x - \dfrac{i}{a}\log\cos ax\right\}\right]$

$$= \frac{x}{a}\frac{e^{iax} - e^{-iax}}{2i} + \frac{1}{a^2}\log\cos ax . \frac{e^{iax} + e^{-iax}}{2} = \frac{x}{a}\sin ax + \frac{1}{a^2}\log\cos ax . \cos ax .$$

(*iii*) Hence the C.S. is

$$y = c_1\cos ax + c_2\sin ax + (1/a)x\sin ax + (1/a^2)\cos ax\log\cos ax.$$

PROBLEMS 13.2

Solve :

1. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 6e^{3x} + 7e^{-2x} - \log 2$ (V.T.U., 2005)

2. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = -2\cosh x$. Also find y when $y = 0$, $\frac{dy}{dx} = 1$ at $x = 0$.

3. $\frac{d^2x}{dt^2} + n^2x = k\cos(nt + \alpha)$.

4. $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 3x = \sin t$.

5. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 4\cos^2 x$. (Bhopal, 2002 S)

6. $(D^2 - 4D + 3)y = \sin 3x \cos 2x$. (Madras, 2000)

7. $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{-x} + \sin 2x$. (V.T.U., 2004)

8. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^{2x} - \cos^2 x$. (Delhi, 2002)

9. $(D^3 - 5D^2 + 7D - 3)y = e^{2x}\cosh x$. (Nagarjuna, 2008)

10. $\frac{d^2y}{dx^2} - y = e^x + x^2e^x$. (Nagpur, 2009)

11. $(D^3 - D)y = 2x + 1 + 4\cos x + 2e^x$. (Mumbai, 2006)

12. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = e^{2x} + \sin x + x$. (V.T.U., 2006)

13. $(D^2 + 1)^2 y = x^4 + 2\sin x \cos 3x$. (Madras, 2006)

14. $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{-2x}\sin 2x$. (Bhopal, 2008)

15. $(D^4 + D^2 + 1)y = e^{-x/2}\cos\frac{\sqrt{3}}{2}x$. (Rajasthan, 2006)

16. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 3y = e^x \cos x$. (V.T.U., 2010)

17. $(D^2 + 4D + 3)y = e^{-x}\sin x + xe^{3x}$. (Raipur, 2005 ; Anna, 2002 S)

18. $\frac{d^2y}{dx^2} + 2y = x^2e^{3x} + e^x\cos 2x$.

19. $\frac{d^4y}{dx^4} - y = \cos x \cosh x$.

20. $(D^3 + 2D^2 + D)y = x^2e^{2x} + \sin^2 x$. (P.T.U., 2003)

21. $\frac{d^2y}{dx^2} + 16y = x\sin 3x$. (V.T.U., 2010 S)

22. $(D^2 + 2D + 1)y = x\cos x$. (Rajasthan, 2006)

23. $(D^2 - 1)y = x\sin x + (1 + x^2)e^x$.

24. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$. (S.V.T.U., 2009)

25. $(D^2 + a^2)y = \tan ax$. (V.T.U., 2005)

13.8 TWO OTHER METHODS OF FINDING P.I.

I. Method of variation of parameters. *This method is quite general and applies to equations of the form*

$$y'' + py' + qy = X \quad \text{...(1)}$$

where p, q, and X are functions of x. It gives P.I. $= -y_1\int\frac{y_2X}{W}dx + y_2\int\frac{y_1X}{W}dx$ ...(2)

where y_1 and y_2 are the solutions of $y'' + py' + qy = 0$ ...(3)

and $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ *is called the Wronskian* of y_1, y_2.*

Proof. Let the C.F. of (1) be $y = c_1y_1 + c_2y_2$

Replacing c_1, c_2 (regarded as *parameters*) by unknown functions $u(x)$ and $v(x)$, let the P.I. be

$$y = uy_1 + vy_2 \quad \text{...(4)}$$

Differentiating (4) w.r.t. x, we get $y' = uy'_1 + vy'_2 + u'y_1 + v'y_2$

*Named after the Polish mathematician and philosopher *Hoene Wronsky* (1778–1853).

$$= uy_1' + vy_2' \qquad ...(5)$$

on assuming that $u'y_1 + v'y_2 = 0$ ...(6)

Differentiate (4) and substitute in (1). Then noting that y_1 and y_2, satisfy (3), we obtain

$$u'y_1' + v'y_2' = X \qquad ...(7)$$

Solving (6) and (7), we get

$$u' = -\frac{y_2X}{W}, v' = \frac{y_1X}{W} \qquad \text{where } W = y_1y_2' - y_2y_1'$$

Integrating $u = -\int \frac{y_2X}{W}\,dx, v = \int \frac{y_1X}{W}\,dx$. Substituting these in (4), we get (2).

Example 13.23. *Using the method of variation of parameters, solve*

$$\frac{d^2y}{dx^2} + 4y = \tan 2x. \qquad \text{(V.T.U., 2008 ; Bhopal, 2007 ; S.V.T.U., 2006 S)}$$

Solution. Given equation in *symbolic form* is $(D^2 + 4)y = \tan 2x$.

(*i*) *To find C.F.*

Its A.E. is $D^2 + 4 = 0, \quad \therefore \quad D = \pm 2i$

Thus C.F. is $\quad y = c_1 \cos 2x + c_2 \sin 2x$.

(*ii*) *To find P.I.*

Here $y_1 = \cos 2x, y_2 = \sin 2x$ and $X = \tan 2x$

$$\therefore \qquad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

Thus,

$$\text{P.I.} = -y_1 \int \frac{y_2X}{W}\,dx + y_2 \int \frac{y_1X}{W}\,dx$$

$$= -\cos 2x \int \frac{\sin 2x \tan 2x}{2}\,dx + \sin 2x \int \frac{\cos 2x \tan 2x}{2}\,dx$$

$$= -\frac{1}{2}\cos 2x \int (\sec 2x - \cos 2x)dx + \frac{1}{2}\sin 2x \int \sin 2x\,dx$$

$$= -\frac{1}{4}\cos 2x[\log(\sec 2x + \tan 2x) - \sin 2x] - \frac{1}{4}\sin 2x \cos 2x$$

$$= -\frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)$$

Hence the C.S. is $y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)$.

Example 13.24. *Solve, by the method of variation of parameters,* $d^2y/dx^2 - y = 2/(1 + e^x)$.

(V.T.U., 2005 ; Hissar, 2005)

Solution. Given equation is $D^2 - 1 = 2/(1 + e^x)$

A.E. is $D^2 - 1 = 0, D = \pm 1, \quad \therefore \quad$ C.F. $= c_1e^x + c_2e^{-x}$

Here $y_1 = e^x, y_2 = e^{-x}$ and $X = 2/(1 + e^x)$

$$\therefore \qquad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -e^xe^{-x} - e^xe^{-x} = -2.$$

Thus

$$\text{P.I.} = -y_1 \int \frac{y_2X}{W}\,dx + y_2 \int \frac{y_1X}{W}\,dx = -e^x \int \frac{e^{-x}}{-2}\cdot\frac{2}{1+e^x}dx + e^{-x}\int \frac{e^x}{-2}\cdot\frac{2}{1+e^x}dx$$

$$= e^x \int \left(\frac{1}{e^x} - \frac{1}{1+e^x}\right)dx - e^{-x}\log(1+e^x) = e^x\left[e^{-x} - \int \frac{e^{-x}}{e^{-x}+1}dx\right] - e^{-x}\log(1+e^x)$$

$$= e^x[-e^{-x} + \log(e^{-x}+1)] - e^{-x}\log(1+e^x) = -1 + e^x\log(e^{-x}+1) - e^{-x}\log(e^x+1)$$

Hence C.S. is $\quad y = c_1e^x + c_2e^{-x} - 1 + e^x\log(e^{-x}+1) - e^{-x}\log(e^x+1)$.

Example 13.25. *Solve by the method of variation of parameters $y'' - 6y' + 9y = e^{3x}/x^2$.*

(Nagpur, 2009 ; S.V.T.U., 2009)

Solution. Given equation is $(D^2 - 6D + 9)y = e^{3x}/x^2$

A.E. is $D^2 - 6D + 9 = 0$ *i.e.* $(D-3)^2 = 0$ $\therefore$ C.F. $= (c_1 + c_2 x)e^{3x}$

Here $y_1 = e^{3x}, y_2 = xe^{3x}$ and $X = e^{3x}/x^2$

$$\therefore \qquad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x} + 3xe^{3x} \end{vmatrix} = e^{6x}.$$

Thus
$$\text{P.I.} = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx = -e^{3x} \int \frac{xe^{3x}}{e^{6x}} \frac{e^{3x}}{x^2} dx + xe^{3x} \int \frac{e^{3x}}{e^{6x}} \cdot \frac{e^{3x}}{x^2} dx$$

$$= -e^{3x} \int \frac{dx}{x} + xe^{3x} \int x^{-2}\, dx = -e^{3x}(\log x + 1)$$

Hence C.S. is $\quad y = (c_1 + c_2 x)e^{3x} - e^{3x}(\log x + 1)$.

Example 13.26. *Solve, by the method of variation of parameters, $y'' - 2y' + y = e^x \log x$.*

(V.T.U., 2006 ; Kurukshetra, 2005 ; Madras, 2003)

Solution. Given equation in symbolic form is $(D^2 - 2D + 1)\, y = e^x \log x$

(i) To find C.F.

Its A.E. is $(D-1)^2 = 0$, $\therefore$ $D = 1, 1$

Thus C.F. is $y = (c_1 + c_2 x)e^x$

(ii) To find P.I.

Here $y_1 = e^x$, $y_2 = xe^x$ and $X = e^x \log x$

$$\therefore \qquad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & (1+x)\, e^x \end{vmatrix} = e^{2x}$$

Thus
$$\text{P.I.} = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$$

$$= -e^x \int \frac{xe^x \cdot e^x \log x}{e^{2x}} dx + xe^x \int \frac{e^x \cdot e^x \log x}{e^{2x}} dx = -e^x \int x \log x\, dx + xe^x \int \log x\, dx$$

$$= -e^x \left(\frac{x^2}{2} \log x - \int \frac{1}{x} \cdot \frac{x^2}{2} dx \right) + x \cdot e^x \left(x \log x - \int \frac{1}{x} \cdot x\, dx \right)$$

$$= -e^x \left(\frac{x^2}{2} \log x - \frac{x^2}{4} \right) + x\; e^x (x \log x - x) = \frac{1}{4}\, x^2 e^x (2 \log x - 3)$$

Hence C.S. is $y = (c_1 + c_2 x)\, e^x + \dfrac{1}{4}\, x^2 e^x (2 \log x - 3)$.

II. Method of undetermined coefficients

To find the P.I. of $f(D)\, y = X$, we assume a trial solution containing unknown constants which are determined by substitution in the given equation. The trial solution to be assumed in each case, depends on the form of X. Thus when *(i)* $X = 2e^{3x}$, trial solution $= ae^{3x}$.

(ii) $X = 3 \sin 2x$, trial solution $= a_1 \sin 2x + a_2 \cos 2x$

(iii) $X = 2x^3$, trial solution $= a_1 x^3 + a_2 x^2 + a_3 x + a_4$

However when $X = \tan x$ or $\sec x$, this method fails, since the number of terms obtained by differentiating $X = \tan x$ or $\sec x$ is infinite.

The above method holds so long as no term in the trial solution appears in the C.F. If any term of the trial solution appears in the C.F., we multiply this trial solution by the lowest positive integral power of x which is large enough so that none of the terms which are then present, appear in the C.F.

Example 13.27. *Solve* $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 2x^2 + 3e^{-x}$. *(V.T.U., 2008)*

Solution. Here C.F. $= e^{-x}(c_1 \cos \sqrt{3}\,x + c_2 \sin \sqrt{3}\,x)$

Assume P.I. as $y = a_1x^2 + a_2x + a_3 + a_4e^{-x}$

$\therefore \quad Dy = 2a_1x + a_2 - a_4e^{-x}$ and $D^2y = 2a_1 + a_4e^{-x}$

Substituting these in the given equation, we get

$$4a_1x^2 + (4a_1 + 4a_2)x + (2a_1 + 2a_2 + 4a_3) + 3a_4e^{-x} = 2x^2 + 3e^{-x}$$

Equating corresponding coefficients on both sides, we get

$$4a_1 = 2,\ 4a_1 + 4a_2 = 0,\ 2a_1 + 2a_2 + 4a_3 = 0,\ 3a_4 = 3$$

Then $a_1 = \frac{1}{2}, a_2 = -\frac{1}{2}, a_3 = 0, a_4 = 1$. Thus P.I. $= \frac{1}{2}x^2 - \frac{1}{2}x + e^{-x}$

$\therefore$ C.S. is $y = e^{-x}(c_1 \cos \sqrt{3x} + c_2 \sin \sqrt{3x}) + \frac{1}{2}x^2 - \frac{1}{2}x + e^{-x}$.

Example 13.28. *Solve* $(D^2 + 1)y = \sin x$.

Solution. Here C.F. $= c_1 \cos x + c_2 \sin x$

We would normally assume a trial solution as $a_1 \cos x + a_2 \sin x$.

However, since these terms appear in the C.F., we multiply by x and assume the trial P.I. as

$$y = x(a_1 \cos x + a_2 \sin x)$$

$\therefore \quad Dy = (a_1 + a_2x)\cos x + (a_2 - a_1x)\sin x$ and $D^2y = (2a_2 - a_1x)\cos x - (2a_1 + a_2x)\sin x$

Substituting these in the given equation, we get $2a_1 \cos x - 2a_2 \sin x = \sin x$

Equating corresponding coefficients,

$$2a_1 = 0, \quad -2a_2 = 1 \quad \text{so that } a_1 = 0, a_2 = -\frac{1}{2}. \quad \text{Thus P.I.} = -\frac{1}{2}x \sin x$$

$\therefore$ C.S. is $y = c_1 \cos x + c_2 \sin x - \frac{1}{2}x \sin x$.

Example 13.29. *Solve by the method of undetermined coefficients,*

$$\frac{d^2y}{dx^2} - y = e^{3x}\cos 2x - e^{2x}\sin 3x.$$

Solution. Its A.E. is $D^2 - 1 = 0$, $\therefore \quad D = \pm 1$.

Thus C.F. $= c_1e^x + c_2e^{-x}$

Assume P.I. as $y = e^{3x}(c_1 \cos 2x + c_2 \sin 2x) - e^{2x}(c_3 \cos 3x + c_4 \sin 3x)$

$$\therefore \quad \frac{dy}{dx} = e^{3x}\{(3c_1 + 2c_2)\cos 2x + (3c_2 - 2c_1)\sin 2x\} - e^{2x}\{(2c_3 + 3c_4)\cos 3x + (2c_4 - 3c_3)\sin 3x\}$$

and $$\frac{d^2y}{dx^2} = e^{3x}\{(5c_1 + 12c_2)\cos 2x + (5c_2 - 12c_1)\sin 2x\} - e^{2x}\{(12c_4 - 5c_3)\cos 3x - (5c_4 + 12c_3)\sin 3x\}$$

Substituting these in the given equation, we get

$$e^{3x}\{(4c_1 + 12c_2)\cos 2x + (4c_2 - 12c_1)\sin 2x\} - e^{2x}\{(12c_4 - 6c_3)\cos 3x - (6c_4 + 12c_3)\sin 3x\}$$
$$= e^{3x}\cos 2x - e^{2x}\sin 3x$$

Equating corresponding coefficients,

$$4c_1 + 12c_2 = 1,\ 4c_2 - 12c_1 = 0\ ;\ 12c_4 - 6c_3 = 0,\ 6c_4 + 12c_3 = -1$$

whence $c_1 = 1/40, c_2 = 3/40, c_3 = -1/15, c_4 = -1/30$

Thus P.I. $= \frac{1}{40}e^{3x}(\cos 2x + 3 \sin 2x) + \frac{1}{30}e^{2x}(2 \cos 3x + \sin 3x)$

Hence C.S. is $y = c_1e^x + c_2e^{-x} + \frac{1}{30}e^{2x}(2 \cos 3x + \sin 3x) + \frac{1}{40}e^{3x}(\cos 2x + 3 \sin 2x)$.

PROBLEMS 13.3

Solve by the method of variation of parameters :

1. $\frac{d^2y}{dx^2} + a^2y = \text{cosec } ax.$
2. $\frac{d^2y}{dx^2} + y = \sec x.$ (*Bhopal, 2007*)
3. $\frac{d^2y}{dx^2} + y = \tan x.$ (*P.T.U., 2005 ; Raipur, 2004*)
4. $\frac{d^2y}{dx^2} + y = x \sin x.$ (*S.V.T.U., 2007 ; J.N.T.U., 2005*)
5. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x/]x.$ (*V.T.U., 2006*)
6. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \frac{1}{1+e^{-x}}.$ (*V.T.U., 2010 S ; U.P.T.U., 2005*)
7. $y'' - 2y' + 2y = e^x \tan x.$ (*V.T.U., 2010*)
8. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^x \sin x.$ (*U.P.T.U., 2003*)
9. $\frac{d^2y}{dx^2} + y = \frac{1}{1+\sin x}.$ (*V.T.U., 2004*)

Solve by the method of undetermined coefficients :

10. $(D^2 - 3D + 2)\, y = x^2 + e^x.$ (*V.T.U., 2003 S*)
11. $\frac{d^2y}{dx^2} + y = 2\cos x.$ (*V.T.U., 2000 S*)
12. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{3x} + \sin x.$ (*V.T.U., 2008*)
13. $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = x + \sin x.$ (*V.T.U., 2010*)
14. $(D^2 - 2D + 3)\, y = x^3 + \cos x.$
15. $(D^2 - 2D)\, y = e^x \sin x.$ (*V.T.U., 2006*)

13.9 EQUATIONS REDUCIBLE TO LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

Now we shall study two such forms of linear differential equations with variable coefficients which can be reduced to linear differential equations with constant coefficients by suitable substitutions.

I. Cauchy's homogeneous linear equation*. An equation of the form

$$x^n \frac{d^n y}{dx^n} + k_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \dots + k_{n-1} x \frac{dy}{dx} + k_n y = X \qquad \text{...(1)}$$

where X is a function of x, is called *Cauchy's homogeneous linear equation.*

Such equations can be reduced to linear differential equations with constant coefficients, by putting

$$x = e^t \quad \text{or} \quad t = \log x. \text{ Then if } D = \frac{d}{dt}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{x}, \quad i.e., \quad x\frac{dy}{dx} = Dy.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx} = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d^2y}{dt^2}\frac{dt}{dx} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

i.e., $$x^2\frac{d^2y}{dx^2} = D(D-1)\,y. \text{ Similarly, } x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)\,y \text{ and so on.}$$

After making these substitutions in (1), there results a linear equation with constant coefficients, which can be solved as before.

Example 13.30. *Solve* $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log x.$ (*V.T.U., 2010*)

Solution. This is a Cauchy's homogeneous linear.

*See footnote p. 144.

Put $x = e^t$, *i.e.*, $t = \log x$, so that $x \dfrac{dy}{dx} = Dy, x^2 \dfrac{d^2y}{dx^2} = D(D-1)y$ where $D = \dfrac{d}{dt}$

Then the given equation becomes $[D(D-1) - D + 1]\, y = t$ or $(D-1)^2 y = t$ ...(*i*)

which is a linear equation with constant coefficients.

Its A.E. is $(D-1)^2 = 0$ whence $D = 1, 1$.

$\therefore$ C.F. $= (c_1 + c_2 t)e^t$ and P.I. $= \dfrac{1}{(D-1)^2}\, t = (1-D)^{-2}\, t = (1 + 2D + 3D^2 + \ldots)\, t = t + 2.$

Hence the solution of (*i*) is $y = (c_1 + c_2 t)e^t + t + 2$ or, putting $t = \log x$ and $e^t = x$, we get

$y = (c_1 + c_2 \log x)\, x + \log x + 2$ as the required solution of (*i*).

Example 13.31. *Solve* $x^2 \dfrac{d^2y}{dx^2} + 3x \dfrac{dy}{dx} + y = \dfrac{1}{(1-x)^2}$. (*P.T.U., 2003*)

Solution. Put $x = e^t$ *i.e.*, $t = \log x$ so that $x\, dy/dx = Dy$, $x^2 \dfrac{d^2y}{dx^2} = D(D-1)\,y$

Then the given equation becomes

$$[D(D-1) + 3D + 1]\, y = \frac{1}{(1-e^t)^2} \quad \text{or} \quad (D^2 + 2D + 1)\, y = \frac{1}{(1-e^t)^2}$$

Its A.E. is $D^2 + 2D + 1 = 0$ or $(D+1)^2 = 0$ *i.e.*, $D = -1, -1$.

$$\therefore \quad \text{C.F.} = (c_1 + c_2 x)\, e^{-t} = (c_1 + c_2 \log x)\, \frac{1}{x}$$

$$\text{P.I.} = \frac{1}{(D+1)^2} \frac{1}{(1-e^t)^2} = \frac{1}{D+1}\, u, \text{ where } u = \frac{1}{D+1} \cdot \frac{1}{(1-e^t)^2} \text{ i.e. } \frac{du}{dt} + u = (1-e^t)^{-2}$$

which is Leibnitz's linear equation having I.F. $= e^t$

$$\therefore \text{ its solution is} \quad ue^t = \int \frac{e^t}{(1-e^t)^2}\, dt = \frac{1}{1-e^t} \quad \text{or} \quad u = \frac{e^{-t}}{1-e^t}$$

$$\therefore \quad \text{P.I.} = \frac{1}{D+1}\left(\frac{e^{-t}}{1-e^t}\right) = e^{-t} \int \frac{1}{1-e^t}\, dt = \frac{1}{x} \int \frac{dx}{x(1-x)}$$

$$= \frac{1}{x} \int \left(\frac{1}{x} + \frac{1}{1-x}\right) dx = \frac{1}{x}\,[\log x - \log(1-x)] = \frac{1}{x} \log \frac{x}{x-1}$$

Hence the solution is $\quad y = \left\{c_1 + c_2 \log x + \log \dfrac{x}{x-1}\right\} \dfrac{1}{x}.$

Example 13.32. *Solve* $x^2 \dfrac{d^2y}{dx^2} + x \dfrac{dy}{dx} + y = \log x \sin(\log x)$.

(*Kurukshetra, 2006 ; Madras, 2006 ; Kerala, 2005*)

Solution. Putting $x = e^t$ *i.e.* $t = \log x$, the given equation becomes

$[D(D-1) + D + 1]\, y = t \sin t$ *i.e.* $(D^2 + 1)\, y = t \sin t$ ...(*i*)

Its A.E. is $D^2 + 1 = 0$ *i.e.* $D = \pm i$.

$\therefore$ C.F. $= c_1 \cos t + c_2 \sin t$

and $\quad$ *P.I.* $= \dfrac{1}{D^2+1}\, t \sin t = \dfrac{1}{D^2+1}\, t\,(\text{I.P. of } e^{it})$

$$= \text{I.P. of } e^{it}\, \frac{1}{(D+i)^2 + 1}\, t = \text{I.P. of } e^{it} \cdot \frac{1}{D^2 + 2iD}\, t$$

$$= \text{I.P. of } e^{it} \frac{1}{2iD\,(1 + D/2i)}\, t = \text{I.P. of } \frac{1}{2i} e^{it} \frac{1}{D}\left(1 - \frac{iD}{2}\right)^{-1} t$$

$$= \text{I.P. of } \frac{1}{2i} e^{it} \frac{1}{D}\left(1 + \frac{iD}{2} + \ldots\right) t = \text{I.P. of } \frac{1}{2i} e^{it} \frac{1}{D}\left(t + \frac{i}{2}\right)$$

$$= \text{I.P. of } \frac{e^{it}}{2i} \int \left(t + \frac{i}{2}\right) dt = \text{I.P. of } \frac{e^{it}}{2i}\left(\frac{t^2}{2} + \frac{it}{2}\right)$$

$$= \text{I.P. of } e^{it}\left(-\frac{i}{4}t^2 + \frac{t}{4}\right) = \text{I.P. of } (\cos t + i \sin t)\left(-\frac{it^2}{4} + \frac{t}{4}\right) = -\frac{t^2}{4} \cos t + \frac{t}{4} \sin t$$

Hence the C.S. of (*i*) is $y = c_1 \cos t + c_2 \sin t - \dfrac{t^2}{4} \cos t + \dfrac{t}{4} \sin t$

or $\quad y = c_1 \cos (\log x) + c_2 \sin (\log x) - \dfrac{1}{4} (\log x)^2 \cos (\log x) + \dfrac{1}{4} \log (\log x) \sin (\log x)$

which is the required solution.

Example 13.33. *Solve* $x^2 \dfrac{d^2 y}{dx^2} - 3x \dfrac{dy}{dx} + y = \log x \dfrac{\sin (\log x) + 1}{x}$. *(I.S.M., 2001)*

Solution. Put $x = e^t$, *i.e.*, $t = \log x$ so that $x \dfrac{dy}{dx} = Dy,\ x^2 \dfrac{d^2y}{dx^2} = D\,(D - 1)\, y$

Then the given equation becomes

$$\{D\,(D-1) - 3D + 1\}\, y = t\, \frac{\sin t + 1}{e^t} \qquad \text{or} \qquad (D^2 - 4D + 1)\, y = e^{-t}\, t\, (\sin t + 1)$$

which is a linear equation with constant coefficients.

Its A.E. is $D^2 - 4D + 1 = 0$ whence $D = 2 \pm \sqrt{3}$

$\therefore \qquad$ C.F. $= c_1 e^{(2+\sqrt{3})t} + c_2 e^{(2-\sqrt{3})t} = e^{2t}(c_1 e^{\sqrt{3}t} + c_2 e^{-\sqrt{3}t})$

and $\qquad$ P.I. $= \dfrac{1}{D^2 - 4D + 1}\, e^{-t}\, t\, (\sin t + 1) = e^{-t} \dfrac{1}{(D-1)^2 - 4\,(D-1) + 1}\, t\, (\sin t + 1)$

$$= e^{-t}\left\{\frac{1}{D^2 - 6D + 6}\, t + \frac{1}{D^2 - 6D + 6}\, t \sin t\right\}$$

$$\frac{1}{D^2 - 6D + 6}\, t = \frac{1}{6}\left(1 - \frac{6D - D^2}{6}\right)^{-1} t = \frac{1}{6}\,(1 + D)\, t = \frac{1}{6}\,(t + 1)$$

$$\frac{1}{D^2 - 6D + 6}\, t \sin t = \text{I.P. of } \frac{1}{D^2 - 6D + 6}\, e^{it} \cdot t$$

$$= \text{I.P. of } e^{it} \frac{1}{(D+i)^2 - 6\,(D+i) + 6}\, t = \text{I.P. of } e^{it} \frac{1}{D^2 + (2i - 6)\, D + (5 - 6i)}\, t$$

$$= \text{I.P. of } \frac{e^{it}}{5 - 6i}\left\{1 + \frac{(2i-6)\, D + D^2}{5 - 6i}\right\}^{-1} t = \text{I.P. } \frac{e^{it}}{5 - 6i}\left(1 - \frac{2i - 6}{5 - 6i}\, D\right) t$$

$$= \text{I.P. of } \frac{(5 + 6i)}{61}\, (\cos t + i \sin t)\left(t - \frac{2i - 6}{5 - 6i}\right)$$

$$= \text{I.P. of } \frac{1}{61}\, \{(5 \cos t - 6 \sin t) + i\,(5 \sin t + 6 \cos t)\}\left(t + \frac{42 + 26i}{61}\right)$$

$$= \frac{26}{3721}\, (5 \cos t - 6 \sin t) + \frac{1}{61}\, (5 \sin t + \cos t)\left(t + \frac{42}{61}\right)$$

$$= \frac{t}{61}(5 \sin t + \cos t) + \frac{2}{3721}(27 \sin t + 191 \cos t)$$

$\therefore$ $$\text{P.I.} = e^{-t}\left[\frac{1}{6}(t+1) + \frac{1}{61}(5 \sin t + 6 \cos t) + \frac{2}{3721}(27 \sin t + 191 \cos t)\right]$$

Hence $$y = e^{2t}(c_1 e^{\sqrt{3}t} + c_2 e^{-\sqrt{3}t}) + e^{-t}\left[\frac{1}{6}(t+1) + \frac{t}{61}(5 \sin t + 6 \cos t) + \frac{2}{3721}(27 \sin t + 191 \cos t)\right]$$

or $$y = x^2(c_1 x^{\sqrt{3}} + c_2 x^{-\sqrt{3}}) + \frac{1}{x}\left[\frac{1}{6}(\log x + 1) + \frac{\log x}{61}\{5 \sin(\log x) + 6 \cos(\log x)\} + \frac{2}{3721}\{27 \sin(\log x) + 191 \cos(\log x)\}\right].$$

Example 13.34. *Solve* $x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y = e^{x}$. *(Kurukshetra, 2005 ; U.P.T.U., 2005)*

Solution. Putting $x = e^t$, *i.e.*, $t = \log x$, the given equation becomes

$[D(D-1) + 4D + 2]\, y = e^{e^t}$ *i.e.*, $(D^2 + 3D + 2)\, y = e^{e^t}$

Its A.E. is $D^2 + 3D + 2 = 0$ whence $D = -1, -2$.

$\therefore$ C.F. $= c_1 e^{-t} + c_2 e^{-2t} = c_1 x^{-1} + c_2 x^{-2}$

and $$\text{P.I.} = \frac{1}{(D^2 + 3D + 2)} e^{e^t} = \frac{1}{(D+1)(D+2)} e^{e^t} = \left(\frac{1}{D+1} - \frac{1}{D+2}\right) e^{e^t}.$$

Now $$\frac{1}{D+1} e^{e^t} = \frac{1}{D+1} e^{-t} \cdot e^t e^{e^t} = e^{-t} \frac{1}{(D-1)+1} e^t e^{e^t}$$

$$= e^{-t} \frac{1}{D} e^t e^{e^t} = e^{-t} \int e^{e^t} d(e^t) = x^{-1} \int e^x dx = x^{-1} e^x$$

$$\frac{1}{D+2} e^{e^t} = \frac{1}{D+2} e^{-2t} \cdot e^{2t} e^{e^t} = e^{-2t} \frac{1}{(D-2)+2} e^{2t} e^{e^t}$$

$$= e^{-2t} \frac{1}{D} e^{e^t} e^{2t} = e^{-2t} \int e^{e^t} e^t \, d(e^t)$$

$$= x^{-2} \int e^x \, x \, dx \qquad [\because \; e^t = x]$$

$$= x^{-2}(xe^x - e^x) \qquad \text{[Integrating by parts]}$$

$\therefore$ $$\text{P.I.} = x^{-1} e^x - x^{-2}(xe^x - e^x) = x^{-2} e^x$$

Hence the required solution is $y = c_1 x^{-1} + x^{-2}(c_2 + e^x)$.

II. Legendre's linear equation*. An equation of the form

$$(ax+b)^n \frac{d^n y}{dx^n} + k_1 (ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + k_n y = X \qquad \ldots(2)$$

where k's are constants and X is a function of x, is called *Legendre's linear equation.*

Such equations can be reduced to linear equations with constant coefficients by the substitution $ax + b = e^t$, *i.e.*, $t = \log(ax + b)$.

Then, if $$D = \frac{d}{dt}, \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{a}{ax+b} \cdot \frac{dy}{dt} \text{ i.e. } (ax+b)\frac{dy}{dx} = a\,Dy$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{a}{ax+b} \frac{dy}{dt}\right) = \frac{-a^2}{(ax+b)^2} \frac{dy}{dt} + \frac{a}{ax+b} \frac{d}{dt}\left(\frac{dy}{dt}\right) \frac{dt}{dx} = \frac{a^2}{(ax+b)^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

* A French mathematician *Adrien Marie Legender* (1752 – 1833) who made important contributions to number theory, special functions, calculus of variations and elliptic integrals.

i.e., $(ax+b)^2 \dfrac{d^2y}{dx^2} = a^2 D(D-1)\,y$. Similarly, $(ax+b)^3 \dfrac{d^3y}{dx^3}\ a^3D(D-1)(D-2)\,y$ and so on.

After making these replacements in (2), there results a linear equation with constant coefficients.

Example 13.35. *Solve* $(1+x)^2 \dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = 2\sin[\log(1+x)].$ ...(*i*)

(*V.T.U., 2009 ; J.N.T.U., 2005 ; Kerala, 2005*)

Solution. This is a Legendre's linear equation.

$\therefore$ put $1+x=e^t$, *i.e.*, $t=\log(1+x)$, so that $(1+x)\dfrac{dy}{dx} = Dy$

and $$(1+x)^2 \frac{d^2y}{dx^2} = D(D-1)\,y, \text{ where } D = \frac{d}{dt}$$

Then (*i*) becomes $D(D-1)\,y + Dy + y = 2\sin t$

or $$(D^2+1)\,y = 2\sin t \qquad ...(ii)$$

This is a linear equation with constant co-efficients

Its A.E. is $D^2+1=0$, whence $D=\pm i$ $\therefore$ C.F. $= c_1\cos t + c_2 \sin t$

and $$\text{P.I.} = 2\frac{1}{D^2+1}\sin t = 2t\,.\,\frac{1}{2D}\sin t$$

$$= t\int \sin t\,dt = -t\cos t \qquad [\because \text{ on replacing } D^2 \text{ by } -1^2,\ D^2+1=0]$$

Hence the solution of (*ii*) is $y = c_1\cos t + c_2\sin t - t\cos t$ and on replacing t by $\log(1+x)$, we get $y = c_1\cos[\log(1+x)] + c_2\sin[\log(1+x)] - \log(1+x)\cos[\log(1+x)]$ as the required solution.

Example 13.36. *Solve* $(2x-1)^2 \dfrac{d^2y}{dx^2} + (2x-1)\dfrac{dy}{dx} - 2y = 8x^2 - 2x + 3.$ (*V.T.U., 2006*)

Solution. This is a Legendre's linear equation.

$\therefore$ put $2x-1=e^t$ *i.e.*, $t=\log(2x-1)$ so that $(2x-1)\dfrac{dy}{dx} = 2Dy$

and $$(2x-1)^2\frac{d^2y}{dx^2} = 4D\,(D-1)\,y, \text{ where } D = \frac{d}{dt}.$$

Then the given equation becomes

$$4D\,(D-1)\,y + 2Dy - 2y = 8\left(\frac{1+e^t}{2}\right)^2 - 2\left(\frac{1+e^t}{2}\right) + 3$$

or $$2D^2y - Dy - y = e^{2t} + \frac{3}{2}e^t + 2 \qquad ...(i)$$

This is a linear equation with constant coefficients.

Its A.E. is $2D^2 - D - 1 = 0$ whence $D = 1, -1/2$.

$\therefore$ C.F. $= c_1e^t + c_2e^{-t/2}$

and $$\text{P.I.} = \frac{1}{2D^2-D-1}\left(e^{2t} + \frac{3}{2}e^t + 2\right) = \frac{1}{2.4-2-1}e^{2t} + \frac{3}{2}\frac{t}{4D-1}e^t + 2\,.\,\frac{1}{2.0^2-0-1}e^{0t}$$

$[\because$ on putting $t=1$, $2D^2-D-1=0$

$$= \frac{1}{5}e^{2t} + \frac{3t}{2}\,.\,\frac{1}{4-1}e^t - 2 = \frac{1}{5}e^{2t} + \frac{t}{2}e^t - 2$$

Hence the solution of (*i*) is

$$y = c_1e^t + c_2e^{-t/2} + \frac{1}{5}e^{2t} + \frac{1}{2}te^t - 2 \text{ and on replacing } t \text{ by } \log(2x-1),$$

$$y = c_1 (2x - 1) + c_2 (2x - 1)^{-1/2} + \frac{1}{5} (2x - 1)^2 + \frac{1}{2} (2x - 1) \log (2x - 1) - 2.$$

which is the required solution.

PROBLEMS 13.4

Solve :

1. $x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + 6y = x^2.$

2. $x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} - 4y = x^4.$

3. $x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 4y = (1 + x^2).$ (*S.V.T.U., 2007*)

4. $x \frac{d^2y}{dx^2} - \frac{2y}{x} = x + \frac{1}{x^2}.$ (*V.T.U., 2005 S*)

5. The radial displacement u in a rotating disc at a distance r from the axis is given by $r^2 \frac{d^2u}{dr^2} + r \frac{du}{dr} - u + kr^3 = 0,$ where k is a constant. Solve the equation under the conditions $u = 0$ when $r = 0$, $u = 0$ when $r = a$.

Solve :

6. $x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y = \log x.$ (*Bhopal, 2009*)

7. $x^3 \frac{d^3y}{dx^3} + 3x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = x + \log x$ (*Bhopal, 2008*)

8. $x^2y'' + xy' + y = 2\cos^2 (\log x).$ (*V.T.U., 2011*)

9. $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10 \left(x + \frac{1}{x}\right)$ (*S.V.T.U., 2006 ; P.T.U., 2003*)

10. $x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} + y = \frac{1}{(1 - x)^2}.$ (*P.T.U., 2003*)

11. $x^2 \frac{d^2y}{dx^2} + 5x \frac{dy}{dx} + 4y = x \log x.$ (*U.P.T.U., 2004*)

12. $x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - 12y = x^3 \log x.$ (*Bhopal, 2008*)

13. $(2x + 3)^2 \frac{d^2y}{dx^2} - (2x + 3) \frac{dy}{dx} - 12y = 6x.$ (*V.T.U., 2007 ; Kerala, 2005 ; Anna, 2002 S*)

14. $(x - 1)^3 \frac{d^3y}{dx^3} + 2(x - 1)^2 \frac{d^2y}{dx^2} - 4 (x - 1) \frac{dy}{dx} + 4y = 4 \log (x - 1).$ (*Nagpur, 2009*)

15. $(1 + x)^2 \frac{d^2y}{dx^2} + (1 + x) \frac{dy}{dx} + y = \sin [2 \log (1 + x)]$ (*P.T.U., 2006 ; V.T.U., 2004*)

16. $(3x + 2)^2 \frac{d^2y}{dx^2} + 5 (3x + 2) \frac{dy}{dx} - 3y = x^2 + x + 1.$ (*Mumbai, 2006*)

13.10 (1) LINEAR DEPENDENCE OF SOLUTIONS

Consider the initial value problem consisting of the homogeneous linear equation

$$y'' + py' + qy = 0 \quad ...(1)$$

with variable coefficients $p(x)$ and $q(x)$ and two initial conditions $y(x_0) = k_0$, $y'(x_0) = k_1$ ...(2)

Let its general solution be $y = c_1y_1 + c_2y_2$ ...(3)

which is made up of two linearly dependent solutions y_1 and y_2.*

If $p(x)$ and $q(x)$ are continuous functions on some open interval I and x_0 is any fixed point on I, then the above initial value problem has a **unique solution** *$y(x)$ on the interval I.*

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* As in §2.12, y_1, y_2 are said to be *linearly dependent* in an interval I, if and only if there exist numbers λ_1, λ_2 not both zero such that $\lambda_1y_1 + \lambda_2y_2 = 0$ for all x in I.
If no such numbers other than zero exist, then y_1, y_2 are said to be *linearly independent*.

(2) Theorem. *If $p(x)$ and $q(x)$ are continuous on an open interval I, then the solutions y_1 and y_2 of (1) are linearly dependent in I if and only if the Wronskian*† $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 0$ *for some x_0 on I. If there is an $x = x_1$ in I at which $W(y_1, y_2) \neq 0$, then y_1, y_2 are linearly independent on I.*

Proof. If y_1, y_2 are linearly dependent solutions of (1) then there exist two constants c_1, c_2 not both zero, such that

$$c_1y_1 + c_2y_2 = 0 \qquad ...(4)$$

Differentiating w.r.t. x, $c_1y_1' + c_2y_2' = 0$ ...(5)

Eliminating c_1, c_2 from (4) and (5), we get

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 0$$

Conversely, suppose $W(y_1, y_2) = 0$ for some $x = x_0$ on I and show that y_1, y_2 are linearly dependent.

Consider the equation

$$\left.\begin{aligned} c_1y_1(x_0) + c_2y_2(x_0) = 0 \\ c_1y_1'(x_0) + c_2y_2'(x_0) = 0 \end{aligned}\right\} \qquad ...(6)$$

which, on eliminating c_1, c_2, give $W(y_1, y_2) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y'_1(x_0) & y'_2(x_0) \end{vmatrix} = 0$

Hence the system has a solution in which c_1, c_2 are not both zero.

Now introduce the function $\bar{y}(x) = c_1y_1(x) + c_2y_2(x)$

Then $y(x)$ is a solution of (1) on I. By (6), this solution satisfies the initial conditions $y(x_0) = 0$ and $y'(x_0) = 0$. Also since $p(x)$ and $q(x)$ are continuous on I, this solution must be unique. But $\mathbf{y = 0}$ is obviously another solution of (1) satisfying the given initial conditions. Hence $\bar{\mathbf{y}} = y$ *i.e.*, $c_1y_1 + c_2y_2 = 0$ in I. Now since c_1, c_2 are not both zero, it implies that y_1 and y_2 are linearly dependent on I.

Example 13.37. *Show that the two functions sin 2x, cos 2x are independent solutions of $y'' + 4y = 0$.*

Solution. Substituting $y_1 = \sin 2x$ (or $y_2 = \cos 2x$) in the given equation we find that y_1, y_2 are its solutions.

Also $$W(y_1, y_2) = \begin{vmatrix} \sin 2x & \cos 2x \\ 2\cos 2x & -2\sin 2x \end{vmatrix} = -2 \neq 0$$

for any value of x. Hence the solutions y_1, y_2 are linearly independent.

PROBLEMS 13.5

Solve :

1. Show that e^{-x}, xe^{-x} are independent solutions of $y'' + 2y' + y = 0$ in any interval.
2. Show that $e^x \cos x, e^x \sin x$ are independent solutions of the equation $xy'' - 2y' = 0$.
3. If y_1, y_2 be two solutions of $y'' + p(x)y' + q(x)y = 0$, show that the *Wronskian* can be expressed as $W(y_1, y_2) = ce^{-\int p dx}$

13.11 SIMULTANEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

Quite often we come across linear differential equations in which there are two or more dependent variables and a single independent variable. Such equations are known as *simultaneous linear equations*. Here we shall deal with systems of linear equations with constant coefficients only. Such a system of equations is solved by eliminating all but one of the dependent variables and then solving the resulting equations as before. Each of the dependent variables is obtained in a similar manner.

Example 13.38. *Solve the simultaneous equations :*

$$\frac{dx}{dt} + 5x - 2y = t, \quad \frac{dy}{dt} + 2x + y = 0$$

being given $x = y = 0$ when $t = 0$. (S.V.T.U., 2009 ; Kurukshetra, 2005)

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† See footnote on p. 486.

Solution. Taking $d/dt = D$, the given equations become

$$(D+5)x - 2y = t \quad ...(i)$$

$$2x + (D+1)y = 0 \quad ...(ii)$$

Eliminate x as if D were an ordinary algebraic multiplier. Multiplying (i) by 2 and operating on (ii) by $D+5$ and then subtracting, we get

$$[-4-(D+5)(D+1)]\,y = 2t \text{ or } (D^2+6D+9)\,y = -2t$$

Its auxiliary equation is $D^2 + 6D + 9 = 0$, *i.e.*, $(D+3)^2 = 0$

whence $D = -3, -3 \quad \therefore \quad \text{C.F.} = (c_1 + c_2 t)\,e^{-3t}$

and $$\text{P.I.} = \frac{1}{(D+3)^2}(-2t) = -\frac{2}{9}\left(1+\frac{D}{3}\right)^{-2} t = -\frac{2}{9}\left(1-\frac{2D}{3}+...\right) t = -\frac{2t}{9}+\frac{4}{27}$$

Hence $$y = (c_1 + c_2 t)\,e^{-3t} - \frac{2t}{9} + \frac{4}{27} \quad ...(iii)$$

Now to find x, either eliminate y from (i) and (ii) and solve the resulting equation or substitute the value of y in (ii). Here, it is more convenient to adopt the latter method.

From (iii), $$Dy = c_2 e^{-3t} + (c_1 + c_2 t)(-3)\,e^{-3t} - \frac{2}{9}$$

$\therefore$ Substituting for y and Dy in (ii), we get

$$x = -\frac{1}{2}[Dy + y] = \left[\left(c_1 - \frac{1}{2}c_2\right) + c_2 t\right] e^{-3t} + \frac{t}{9} + \frac{1}{27} \quad ...(iv)$$

Hence (iii) and (iv) constitute the solutions of the given equations.

Since $x = y = 0$ when $t = 0$, the equations (iii) and (iv) give

$$0 = c_1 + \frac{4}{27} \text{ and } c_1 - \frac{1}{2}c_2 + \frac{1}{27} = 0 \text{ whence } c_1 = -\frac{4}{27},\ c_2 = -\frac{2}{9}.$$

Hence the desired solutions are

$$x = -\frac{1}{27}(1+6t)\,e^{-3t} + \frac{1}{27}(1+3t),\ y = -\frac{2}{27}(2+3t)\,e^{-3t} + \frac{2}{27}(2-3t).$$

Example 13.39. *Solve the simultaneous equations* $\dfrac{dx}{dt} + 2y + \sin t = 0,\ \dfrac{dy}{dt} - 2x - \cos t = 0$ *given that* $x = 0$ *and* $y = 1$ *when* $t = 0$.

Solution. Given equations are

$$Dx + 2y = -\sin t \quad ...(i); \qquad -2x + Dy = \cos t \quad ...(ii)$$

Eliminating x by multiplying (i) by 2 and (ii) by D and then adding, we get

$$4y + D^2y = -2\sin t - \sin t \text{ or } (D^2+4)y = -3\sin t$$

Its A.E. is $D = \pm 2i \quad \therefore \quad \text{C.F.} = c_1 \cos 2t + c_2 \sin 2t$

$$\text{P.I.} = -3\,\frac{1}{D^2+4}\sin t = -3\,\frac{1}{-1+4}\sin t = -\sin t$$

$\therefore$ $$y = c_1 \cos 2t + c_2 \sin 2t - \sin t \quad ...(iii)$$

and $$dy/dt = -2\sin 2t + 2c_2 \cos 2t - \cos t \quad ...(iv)$$

Substituting (iii) in (ii), we get

$$2x = Dy - \cos t = -2c_1 \sin 2t + 2c_2 \cos 2t - 2\cos t$$

or $$x = -c_1 \sin 2t + c_2 \cos 2t + -\cos t \quad ...(v)$$

When $t = 0, x = 0, y = 1$, (iii) and (v) give $1 = c_1,\ 0 = c_2 - 1$

Hence $x = \cos 2t - \sin 2t - \cos t,\ y = \cos 2t + \sin 2t - \sin t$.

Example 13.40. *Solve the simultaneous equations*

$$\frac{dx}{dt} + \frac{dy}{dt} - 2y = 2\cos t - 7\sin t,\ \frac{dx}{dt} - \frac{dy}{dt} + 2x = 4\cos t - 3\sin t. \quad \text{(U.P.T.U., 2001)}$$

Solution. Given equations are

$$Dx + (D-2)y = 2\cos t - 7\sin t \qquad ...(i)$$

$$(D+2)\,x - Dy = 4\cos t - 3\sin t \qquad ...(ii)$$

Eliminate y by operating on (*i*) by D and (*ii*) by $(D-2)$ and then adding, we get

$$D^2x + (D-2)(D+2)x = -2\sin t - 7\cos t + 4(-\sin t - 2\cos t) - 3(\cos t - 2\sin t)$$

or
$$2(D^2-2)\,x = -18\cos t \text{ or } (D^2-2)\,x = -9\cos t$$

Its A.E. is $\quad D^2 - 2 = 0$ or $D = \pm\sqrt{2}$, $\quad \therefore$ C.F. $= c_1 e^{\sqrt{st}} + c_2 e^{-\sqrt{2t}}$

$$\text{P.I.} = (-9)\frac{1}{D^2-2}\cos t = \frac{-9\cos t}{-1-2} = 3\cos t.$$

Hence $x = c_1 e^{\sqrt{2t}} + c_2 e^{-\sqrt{2t}}\ 3\cos t$.

Now substituting this value of x in (*ii*), we get

$$Dy = (D+2)\,(c_1 e^{\sqrt{2t}} + c_2 e^{-\sqrt{2t}} + 3\cos t) - 4\cos t + 3\sin t$$

$$= c_1\sqrt{2}\,e^{\sqrt{2t}} + 2c_1 e^{\sqrt{2t}} + c_2(-\sqrt{2}\,e^{-2t}) + 2c_2\,e^{-\sqrt{2t}} - 3\sin t + 6\cos t - 4\cos t + 3\sin t$$

$$= (2+\sqrt{2})\,c_1 e^{\sqrt{2t}} + (2-\sqrt{2})\,c_2 e^{-\sqrt{2t}} + 2\cos t$$

Hence $\quad y = (\sqrt{2}+1)\,c_1 e^{\sqrt{2t}} - (\sqrt{2}-1)\,c_2 e^{-\sqrt{2t}} + 2\sin t + c_3$.

Example 13.41. *The small oscillations of a certain system with two degrees of freedom are given by the equations*

$$D^2x + 3x - 2y = 0$$

$$D^2x + D^2y - 3x + 5y = 0$$

where $D = d/dt$. If $x = 0$, $y = 0$, $Dx = 3$, $Dy = 2$ when $t = 0$, find x and y when $t = 1/2$.

Solution. Given equations are $(D^2+3)\,x - 2y = 0 \qquad ...(i)$

$$(D^2-3)x + (D^2+5)y = 0 \qquad ...(ii)$$

To eliminate x, operate these equations by $D^2 - 3$ and $D^2 + 3$ respectively and subtract (*i*) from (*ii*). Then

$$[(D^2+3)(D^2+5) + 2(D^2-3)]\,y = 0 \quad \text{or} \quad (D^4 + 10D^2 + 9)y = 0$$

Its auxiliary equation is $D^4 + 10D^2 + 9 = 0$ whence $D = \pm i, \pm 3i$

Thus $\quad y = c_1\cos t + c_2\sin t + c_3\cos 3t + c_4\sin 3t \qquad ...(iii)$

To find x, we eliminate y from (*i*) and (*ii*).

$\therefore$ operating (*i*) by $D^2 + 5$ and multiplying (*ii*) by 2 and adding, we get

$(D^4 + 10D^2 + 9)\,x = 0$. Thus $x = k_1\cos t + k_2\sin t + k_3\cos 3t + k_4\sin 3t \qquad ...(iv)$

To find the relations between the constants in (*iii*) and (*iv*), substitute these values of x and y either of the given equations, say (*i*). This gives

$$2(k_1 - c_1)\cos t + 2\,(k_2 - c_2)\sin t - 2\,(3k_3 + c_3)\cos 3t - 2\,(3k_4 + c_4)\sin 3t = 0$$

which must hold for all values of t.

$\therefore$ Equating to zero the coefficients of $\cos t$, $\sin t$, $\cos 3t$ and $\sin 3t$, we get

$$k_1 = c_1,\ k_2 = c_2,\ k_3 = -c_3/3,\ k_4 = -c_4/3$$

Thus $\quad x = c_1\cos t + c_2\sin t - \frac{1}{3}(c_3\cos 3t + c_4\sin 3t) \qquad ...(v)$

Hence (*iii*) and (*iv*) constitute the solutions of (*i*) and (*ii*).

Since $\quad x = y = 0$, when $t = 0$; $\therefore$ (*iii*) and (*v*) give

$$0 = c_1 + c_3 \text{ and } c_1 - \frac{1}{3}c_3 = 0 \text{ i.e. } c_1 = c_3 = 0$$

Thus (*iii*) and (*v*) reduce to

$$\left.\begin{aligned} y &= c_2\sin t + c_4\sin 3t \\ \text{and} \quad x &= c_2\sin t - \frac{c_4}{3}\sin 3t \end{aligned}\right\} \qquad ...(vi)$$

$\therefore$ $Dx = c_2 \cos t - c_4 \cos 3t$ and $Dy = c_2 \cos t + 3c_4 \cos 3t$.

Since $Dx = 3$ and $Dy = 2$ when $t = 0$

$\therefore$ $3 = c_2 - c_4$ and $2 = c_2 + 3c_4$, whence $c_2 = 11/4$, $c_4 = -\dfrac{1}{4}$.

Hence equation (*vi*) becomes $x = \dfrac{1}{4}\left(11 \sin t + \dfrac{1}{3} \sin 3t\right), y = \dfrac{1}{4}(11 \sin t - \sin 3t)$ ...(*vii*)

$\therefore$ when $t = 1/2$, $x = \dfrac{1}{4}\left[11 \sin (0.5) + \dfrac{1}{3} \sin (1.5)\right] = \dfrac{1}{4}\left[[11 (0.4794 + \dfrac{1}{3}(0.9975)\right] = 1.4015$

and $y = \dfrac{1}{4}[11 \sin (0.5) - \sin (1.5)] = 1.069.$

Example 13.42. *Solve the simultaneous equations:* $\dfrac{dx}{dt} = 2y, \dfrac{dy}{dt} = 2z, \dfrac{dz}{dt} = 2x$.

(*S.V.T.U., 2006 S ; U.P.T.U., 2004*)

Solution. Differentiating first equation w.r.t. t, $\dfrac{d^2x}{dt^2} = 2\dfrac{dy}{dt} = 2(2z)$

Again differentiating w.r.t. t, $\dfrac{d^3x}{dt^2} = 4\dfrac{dz}{dt} = 4(2x)$ ...(*i*)

or $(D^3 - 8)x = 0$

Its A.E. is $D^3 - 8 = 0$ or $(D-2)(D^2 + 2D + 4) = 0$

or $D = 2, -1 \pm i\sqrt{3}$

$\therefore$ the solution of (*i*) is $x = c_1e^{2t} + e^{-t}(c_2 \cos \sqrt{3t} + c_3 \sin \sqrt{3t})$ ...(*ii*)

From the first equation, we have $y = \dfrac{1}{2}\dfrac{dx}{dt}$

$$\therefore \quad y = \frac{1}{2}[2c_1e^{2t} + (-1)e^{-t}(c_2 \cos \sqrt{3t} + c_3 \sin \sqrt{3t}) + e^{t}(-\sqrt{3}\,c_2 \sin \sqrt{3t} + \sqrt{3}\,c_3 \cos \sqrt{3t})]$$

or $$y = c_1e^{2t} + \frac{1}{2}e^{-t}\left\{(\sqrt{3}c_3 - c_2)\cos \sqrt{3t} - (c_3 + \sqrt{3}c_2)\sin \sqrt{3t}\right\} \quad ...(iii)$$

From the second equation, we have $z = \dfrac{1}{2}\dfrac{dy}{dt}$

$$\therefore \quad z = \frac{1}{2}2c_1e^{2t} + \frac{1}{4}\Big[(-1)e^{-t}\left\{(\sqrt{3}c_3 - c_2)\cos \sqrt{3t} - (c_3 + \sqrt{3}c_2)\sin \sqrt{3t}\right\}$$
$$+ e^{-t}\left\{\sqrt{3}(c_2 - \sqrt{3}c_3)\sin \sqrt{3t} - \sqrt{3}(c_3 + \sqrt{3}c_2)\cos \sqrt{3t}\right\}\Big]$$
$$= c_1e^{2t} + \frac{1}{4}e^{-t}\left\{(-2c_2 - 2\sqrt{3}c_3)\cos \sqrt{3t} - (2\sqrt{3}\,c_2 - 2c_3)\sin \sqrt{3t}\right\}$$

or $$z = c_1e^{2t} - \frac{1}{2}e^{-t}\left\{(\sqrt{3}\,c_2 - c_3)\sin \sqrt{3t} + (c_2 + \sqrt{3}c_3)\cos \sqrt{3t}\right\} \quad ...(iv)$$

Hence the equations (*ii*), (*iii*) and (*iv*) taken together give the required solution.

PROBLEMS 13.6

Solve the following simultaneous equations :

1. $\dfrac{dx}{dt} = 5x + y, \dfrac{dy}{dt} = y - 4x.$
2. $\dfrac{dx}{dt} + y = \sin t, \dfrac{dy}{dt} + x = \cos t$; given that $x = 2$ and $y = 0$ when $t = 0$.

(*Bhopal, 2009 ; J.N.T.U., 2006 ; Kerala, 2005*)

3. $\frac{dx}{dt} + 2x + 3y = 0,\ 3x + \frac{dy}{dt} + 2y = 2e^{2t}$. (*Delhi, 2002*)
4. $\frac{dx}{dt} - 7x + y = 0,\ \frac{dy}{dt} - 2x - 5y = 0$.
5. $\frac{dx}{dt} + 2y = e^t,\ \frac{dy}{dt} - 2x = e^{-t}$. (*Bhopal, 2002 S*)
6. $\frac{dx}{dt} + 2x - 3y = t;\ \frac{dy}{dt} - 3x + 2y = e^{2t}$. (*Nagpur, 2009*)
7. $(D-1)x + Dy = 2t + 1,\ (2D+1)x + 2Dy = t$.
8. $(D+1)x + (2D+1)y = e^t,\ (D-1)x + (D+1)y = 1$.
9. $Dx + Dy + 3x = \sin t,\ Dx + y - x = \cos t$. (*U.P.T.U., 2003*)
10. $t\frac{dx}{dt} + y = 0,\ t\frac{dy}{dt} + x = 0$ given $x(1) = 1,\ y(-1) = 0$.
11. $\frac{dx}{dt} + \frac{dy}{dt} + 3x = \sin t,\ \frac{dx}{dt} + y - x = \cos t$.
12. $\frac{d^2x}{dt^2} - 3x - 4y = 0,\ \frac{d^2y}{dt^2} + x + y = 0$. (*U.P.T.U., 2005*)
13. $\frac{d^2x}{dt^2} + y = \sin t,\ \frac{d^2y}{dt^2} + x = \cos t$. (*U.P.T.U., 2004*)
14. A mechanical system with two degrees of freedom satisfies the equations

$$2\frac{d^2x}{dt^2} + 3\frac{dy}{dt} = 4;\ 2\frac{d^2y}{dt^2} - 3\frac{dx}{dt} = 0.$$

Obtain expression for x and y in terms of t, given $x, y, dx/dt, dy/dt$ all vanish at $t = 0$.

13.12 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 13.7

Fill up the blanks or choose the correct answer in the following problems :

1. The complementary function of $(D^4 - a^4)y = 0$ is ....
2. P.I. of the differential equation $(D^2 + D + 1)y = \sin 2x$ is ....
3. P.I. of $y'' - 3y' + 2y = 12$ is ....
4. The Wronskian of x and e^x is ....
5. The C.F. of $y'' - 2y' + y = xe^x \sin x$ is
   (*a*) $C_1e^x + C_2e^{-x}$ (*b*) $(C_1x + C_2)e^x$ (*c*) $(C_1 + C_2x)e^{-x}$ (*d*) None of these. (*V.T.U., 2010*)
6. The general solution of the differential equation $(D^4 - 6D^3 + 12D^2 - 8D)y = 0$ is ....
7. The particular integral of $(D^2 + a^2)y = \sin ax$ is
   (*a*) $-\frac{x}{2a}\cos ax$ (*b*) $\frac{x}{2a}\cos ax$ (*c*) $-\frac{ax}{2}\cos ax$ (*d*) $\frac{ax}{2}\cos ax$.
8. The solution of the differential equation $(D^2 - 2D + 5)^2 y = 0$, is ....
9. The solution of the differential equation $y'' + y = 0$ satisfying the conditions $y(0) = 1$ and $y(\pi/2) = 2$, is ....
10. $e^{-x}(c_1 \cos\sqrt{3x} + c_2 \sin\sqrt{3x}) + c_3e^{2x}$ is the general solution of
    (*a*) $d^3y/dx^3 + 4y = 0$ (*b*) $d^3y/dx^3 - 8y = 0$
    (*c*) $d^3y/dx^3 + 8y = 0$ (*d*) $d^3y/dx^3 - 2d^2y/dx^2 + dy/dx - 2 = 0$.
11. The solution of the differential equation $(D^2 + 1)^2 y = 0$ is ....
12. The particular integral of $d^2y/dx^2 + y = \cosh 3x$ is ....
13. The solution of $x^2y'' + xy' = 0$ is ....
14. The general solution of $(D^2 - 2)^2, y = 0$ is.......
15. P.I. of $(D+1)^2 y = xe^{-x}$ is ....
16. If $f(D) = D^2 - 2,\ \frac{1}{f(D)}e^{2x} =$ ....
17. If $f(D) = D^2 + 5,\ \frac{1}{f(D)}\sin 2x =$ ....
18. The particular integral of $(D+1)^2 y = e^{-x}$ is ....
19. The general solution of $(4D^3 + 4D^2 + D)y = 0$ is ....

20. P.I. of $(D^2+4)y = \cos 2x$ is

(a) $\frac{1}{2}\sin 2x$ (b) $\frac{1}{2}x\sin 2x$ (c) $\frac{1}{4}x\sin 2x$ (d) $\frac{1}{2}x\cos 2x$. *(Bhopal, 2008)*

21. By the method of undetermined coefficients y_p of $y''+3y'+2y=12x^2$ is of the form

(a) $a+bx+cx^2$ (b) $a+bx$ (c) $ax+bx^2+cx^3$ (d) None of these. *(V.T.U., 2010)*

22. In the equation $\frac{dx}{dt}+y=\sin t+1$, $\frac{dy}{dt}+x=\cos t$ if $y=\sin t+1+e^{-t}$, then $x=$ ....

23. $(x^2D^2+xD+7)y=2/x$ converted to a linear differential equation with constant coefficients is ....

24. P.I. of $\frac{d^2y}{dx^2}+\frac{dy}{dx}=x^2+2x+4$ is

(a) $\frac{x^2}{3}+4x$ (b) $\frac{x^3}{3}+4$ (c) $\frac{x^3}{3}+4x$ (d) $\frac{x^3}{3}+4x^2$.

25. The solution of the differential equation $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=e^{3x}$ is given by

(a) $y=C_1e^x+C_2e^{2x}+\frac{1}{2}e^{3x}$ (b) $y=C_1e^{-x}+C_2e^{-2x}+\frac{1}{2}e^{3x}$

(c) $y=C_1e^{-x}+C_2e^{2x}+\frac{1}{2}e^{3x}$ (d) $y=C_1e^{-x}+C_2e^{2x}+\frac{1}{2}e^{-3x}$.

26. The particular integral of the differential equation $(D^3-D)y=e^x+e^{-x}$, $D=\frac{d}{dx}$ is

(a) $\frac{1}{2}(e^x+e^{-x})$ (b) $\frac{1}{2}x(e^x+e^{-x})$ (c) $\frac{1}{2}x^2(e^x+e^{-x})$ (d) $\frac{1}{2}x^2(e^x-e^{-x})$.

27. The complementary function of the differential equation $x^2y''-xy'+y=\log x$ is ....

28. The homogeneous linear differential equation whose auxiliary equation has roots $1, -1$ is ......

29. The particular integral of $(D^2-6D+9)y=\log 2$ is .... *(V.T.U., 2011)*

30. To transform $x\frac{d^2y}{dx^2}+\frac{dy}{dx}=\frac{1}{x}$ into a linear differential equation with constant coefficients, put $x=$ ....

31. The particular integral of $(D^2-4)y=\sin 3x$ is

(a) 1/4 (b) $-1/13$ (c) 1/5 (d) None of these. *(V.T.U., 2010)*

32. The solution of $\frac{d^3y}{dx^3}-3\frac{d^2y}{dx^2}+4y=0$ is ....

33. The differential equation whose auxiliary equation has the roots $0, -1, -1$ is ....

34. Complementary function of $x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}-y=2x\log x$ is

(a) $(C_1+C_2x)e^x$ (b) $(C_1+C_2\log x)x$ (d) $(C_1+C_2x)\log x$ (d) $(C_1+C_2\log x)e^x$. *(Bhopal, 2008)*

35. The general solution of $(D^2-D-2)x=0$ is $x=c_1e^t+c_2e^{-2t}$ (True or False)

36. $\frac{1}{f(D)}(x^2e^{ax})=\frac{1}{f(D+a)}(e^{ax}x^2)$. (True or False)

CHAPTER 14

Applications of Linear Differential Equations

1. Introduction. 2. Simple harmonic motion. 3. Simple Pendulum, Gain and Loss of Oscillations. 4. Oscillations of a spring. 5. Oscillatory electrical circuits. 6. Electro-mechanical analogy. 7. Deflection of Beams. 8. Whirling of Shafts. 9. Applications of simultaneous linear equations. 10. Objective Type of Questions.

14.1 INTRODUCTION

The linear differential equations with constant coefficients find their most important applications in the study of electrical, mechanical and other linear systems. In fact such equations play a dominant role in unifying the theory of electrical and mechanical oscillatory systems.

We shall begin by explaining the types of oscillations of the mechanical systems and the equivalent electrical circuits. Then we shall study at some length the slightly less striking applications such as deflection of beams and whirling of shafts. At the end, we'll take up some of the applications of simultaneous linear differential equations.

14.2 SIMPLE HARMONIC MOTION

When the acceleration of a particle is proportional to its displacement from a fixed point and is always directed towards it, then the motion is said to be *simple harmonic*.

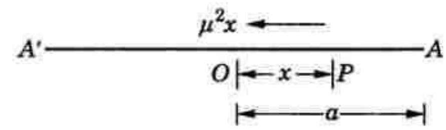

Fig. 14.1

If the displacement of the particle at any time t, from fixed point O is x (Fig. 14.1), then

$$\frac{d^2x}{dt^2} = -\mu^2 x \qquad \text{or} \qquad (D^2 + \mu^2)x = 0, \qquad ...(1)$$

$\therefore$ its solution is $\quad x = c_1 \cos \mu t + c_2 \sin \mu t \qquad ...(2)$

$\therefore$ its velocity at $\quad P = \dfrac{dx}{dt} = \mu(-c_1 \sin \mu t + c_2 \cos \mu t) \qquad ...(3)$

If the particle starts from rest at A, where $OA = a$,

then from (2), $\quad$ (at $t = 0, x = a$) $\quad a = c_1$

and from (3), $\quad$ (at $t = 0, dx/dt = 0$) $\quad 0 = c_2$.

Thus $\quad x = a \cos \mu t \qquad ...(4)$

and $\quad \dfrac{dx}{dt} = -a\mu \sin \mu t = -\sqrt{(a^2 - x^2)} \qquad ...(5)$

which give the displacement and the velocity of the particle at any time t.

Nature of motion. The particle starts from A towards O under acceleration directed towards O which gradually decreases until it vanishes at O, when the particle has acquired the maximum velocity. On passing through O, retardation begins and the particle comes to an instantaneous rest at A', where $OA' = OA$. It then retraces its path and goes on oscillating between A and A'.

The **amplitude** *or maximum displacement from the centre is a.*

The **periodic time,** *i.e., the time of complete oscillation is* $2\pi/\mu$, for when t is increased by $2\pi/\mu$, the values of x and dx/dt remain unaltered.

The **frequency** *or the number of oscillations per second is*

$$1/\text{periodic time, i.e., } \mu/2\pi.$$

Example 14.1. *In the case of a stretched elastic horizontal string which has one end fixed and a particle of mass m attached to the other, find the equation of motion of the particle given that l is the natural length of the string and e is its elongation due to weight mg. Also find the displacement s of particle when initially s = 0, v = 0.*

Solution. Let $OA(= l)$ be the elastic horizonta string with the end O fixed and having a particle of mass m attached to the end A. (Fig. 14.2)

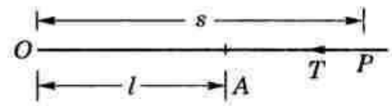

Fig. 14.2

At any time t, let the particle be at P where $OP = s$; so that the elongation $AP = s - l$.

Since for the elongation e, tension $= mg$

$\therefore$ for the elongation $s - l$, tension $T = \dfrac{mg(s-l)}{e}$

Tension being the only horizontal force, the equation of motion is

$$m\frac{d^2s}{dt^2} = -T \quad \text{or} \quad \frac{d^2s}{dt^2} = -\frac{T}{m} = -\frac{g(s-l)}{e} \qquad ...(i)$$

which is the required equation of motion.

Now (i) can be written as $(D^2 + g/e)s = gl/e$, where $D = d/dt$ ...(ii)

$\therefore$ the auxiliary equation is $D^2 + g/e = 0$ or $D = \pm i\sqrt{(g/e)}$

$$\therefore \qquad \text{C.F.} = c_1 \cos\sqrt{(g/e)}t + c_2 \sin\sqrt{(g/e)}t$$

and
$$\text{P.I.} = \frac{1}{D^2 + g/e}\cdot\frac{gl}{e} = \frac{gl}{e}\cdot\frac{l}{D^2 + g/e}e^{0t} = l$$

Thus the solution of (ii) is

$$s = c_1 \cos\sqrt{(g/e)}t + c_2 \sin\sqrt{(g/e)}t + l \qquad ...(iii)$$

When $t = 0$, $s = s_0$, $\quad\therefore\quad s_0 = c_1 + 0 + l$ i.e., $c_1 = s_0 - l$

Again from (iii), $\quad \dfrac{ds}{dt} = -c_1\sqrt{(g/e)}\sin\sqrt{(g/e)}t + c_2\sqrt{(g/e)}\cos\sqrt{(g/e)}t$

When $t = 0$, $ds/dt = 0$; $\quad\therefore\quad 0 = c_2$.

Substituting the values of c_1 and c_2 in (iii), we have

$$s = (s_0 - l)\cos\sqrt{(g/e)}t + l \text{ which is the required result.}$$

Example 14.2. *Two particles of masses m_1 and m_2 are tied to the ends of an elastic string of natural length a and modulus λ. They are placed on a smooth table so that the string is just taut and m_2 is projected with any velocity directly away from m_1. Show that the string will become slack after the lapse of time* $\pi\sqrt{[am_1m_2/\lambda(m_1 + m_2)]}$.

Solution. Taking O as fixed point of reference, let particle m_1 be at O and m_2 at a distance a from m_1 at time $t = 0$ Fig. 14.3. At any time t, let m_1 be of a distance x from O and m_2 be at a distance y from O. Then the equation of motion of m_1 is

Fig. 14.3

$$m_1 d^2x/dt^2 = T \qquad ...(i)$$

and equation of motion of m_2 is $m_2 d^2y/dt^2 = -T$ ...(ii)

where $T = \lambda(y - x)/a$

From (*i*) and (*ii*) $d^2y/dt^2 - d^2x/dt^2 = -\dfrac{T}{m_2} - \dfrac{T}{m_1}$

or $$\frac{d^2(y-x)}{dt^2} = -\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\frac{\lambda(y-x)}{a} \quad \text{or} \quad \frac{d^2u}{dt^2} = -\frac{\lambda(m_1+m_2)u}{m_1\, m_2\, a} \text{ where } u = y - x$$

This is S.H.M. with periodic time $\tau = 2\pi\sqrt{\left\{\dfrac{a m_1 m_2}{\lambda(m_1+m_2)}\right\}}$

The string will acquire its original length (*i.e.* become slack) after time τ_1 of m_2 moving towards m_1 such that

$$\tau_1 = \frac{\tau}{4}+\frac{\tau}{4} = \frac{\tau}{2} = \pi\sqrt{\left\{\frac{a m_1 m_2}{\lambda(m_1+m_2)}\right\}}.$$

Example 14.3. *A particle of mass m executes S.H.M. in the line joining the points A and B, on a smooth table and is connected with these points by elastic strings whose tensions in equilibrium are each T. If l, l′ be the extensions of the strings beyond their natural lengths, find the time of an oscillation.*

Solution. In the equilibrium position, let the particle be at C so that $AC = a + l$ and $BC = a' + l'$, where a, a' are natural lengths of the strings (Fig. 14.4). Then the tensions (at this time) are given by

$$T = \lambda l/a = \lambda' l'/a' \quad \ldots(i)$$

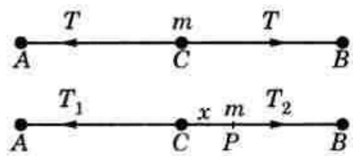

Fig. 14.4

At any time t, let the particle be at P, so that $CP = x$. Then

$$T_1 = \lambda\frac{l+x}{a} \text{ and } T_2 = \lambda\frac{l'-x}{a'}$$

$\therefore$ the equation of motion is $m\dfrac{d^2x}{dt^2} = T_2 - T_1 = \lambda'\dfrac{l'-x}{a'} - \lambda\dfrac{l+x}{a}$

$$= \left(\frac{\lambda' l'}{a'} - \frac{\lambda l}{a}\right) - \left(\frac{\lambda'}{a'}+\frac{\lambda}{a}\right)x = -\left(\frac{T}{l'}+\frac{T}{l}\right)x \quad \text{[By } (i)]$$

or $$\frac{d^2x}{dt^2} = -\mu x \quad \text{where } \mu = \frac{l+l'}{ll'}\cdot\frac{T}{m}$$

Hence the periodic time $= \dfrac{2\pi}{\sqrt{\mu}} = 2\pi\sqrt{\left\{\dfrac{mll'}{(l+l')T}\right\}}.$

14.3 (1) SIMPLE PENDULUM

A heavy particle attached by a light string to a fixed point and oscillating under gravity constitutes a *simple pendulum.*

Let O be the fixed point, l be the length of the string and A be the position of the bob initially (Fig. 14.5). If P be the position of the bob at any time t, such that arc $AP = s$ and $\angle AOP = \theta$, then $s = l\theta$.

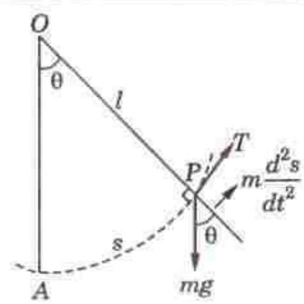

Fig. 14.5

$\therefore$ the equation of motion along PT is $m\dfrac{d^2s}{dt^2} = -mg\sin\theta$

i.e., $$\frac{d^2(l\theta)}{dt^2} = -g\sin\theta$$

or $$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta = -\frac{g}{l}\left(\theta - \frac{\theta^3}{3!}+\ldots\right) = -\frac{g\theta}{l} \text{ to a first approx.}$$

Here the auxiliary equation being $D^2 + g/l = 0$, we have $D = \pm\sqrt{(g/l)}\,i$

$\therefore$ its solution is $\theta = c_1\cos\sqrt{(g/l)}\,t + c_2\sin t.$

Thus the motion of the bob is simple harmonic and the time of an oscillation is $2\pi\sqrt{(l/g)}$.

Obs. The movement of the bob from one end to the other constitutes half an oscillation and is called a *beat or a swing.* Thus the time of one beats = $\pi\sqrt{(l/g)}$.

A seconds pendulum beats 86400 times a day for there are 86,400 seconds in 24 hours.

(2) Gain or loss of oscillations. Let a pendulum of length l make n beats in time T, so that

$$T = \text{time of } n \text{ beats} = n\pi\sqrt{(l/g)} \quad \text{or} \quad n = \frac{T}{\pi}(g/l)^{1/2}$$

Taking logs, $\log n = \log (T/\pi) + \frac{1}{2}(\log g - \log l)$.

Taking differentials of both sides, we get $\dfrac{dn}{n} = \dfrac{1}{2}\left(\dfrac{dg}{g} - \dfrac{dl}{l}\right)$.

If only g changes, l remaining constant, $\dfrac{\mathbf{dn}}{\mathbf{n}} = \dfrac{\mathbf{dg}}{\mathbf{2g}}$ ...(1)

If only l changes, g remaining constant, $\dfrac{\mathbf{dn}}{\mathbf{n}} = -\dfrac{\mathbf{dl}}{\mathbf{2l}}$. ...(2)

Example 14.4. *Find how many seconds a clock would lose per day if the length of its pendulum were increased in the ratio 900 : 901.*

Solution. If the original length l of the string be increased to $l + dl$, then

$$\frac{l+dl}{dl} = \frac{901}{900}. \quad \therefore \quad \frac{dl}{l} = \frac{901}{900} - 1 = \frac{1}{900}.$$

$\therefore$ using (2) above, we have $\dfrac{dn}{n} = -\dfrac{dl}{2l} = -\dfrac{1}{1800}$

i.e.,
$$dn = -\frac{n}{1800} = -\frac{86400}{1800} = -48.$$

Since dn is negative, the clock will lose 4 seconds per day.

Example 14.5. *A simple pendulum of length l is oscillating through a small angle θ in a medium in which the resistance is proportional to the velocity. Find the differential equation of its motion. Discuss the motion and find the period of oscillation.*

Solution. Let the position of the bob (of mass m), at any time t be P and O be the point of suspension such that $OP = l$, $\angle AOP = \theta$ and therefore, arc $AP = s = l\theta$. (Fig. 14.6)

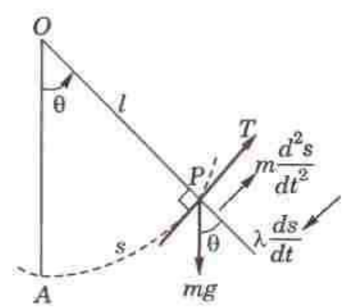

Fig. 14.6

$\therefore$ the equation of motion along the tangent PT is

$$m\frac{d^2s}{dt^2} = -mg\sin\theta - \lambda\frac{ds}{dt} \text{ where } \lambda \text{ is a constant.}$$

or
$$\frac{d^2(l\theta)}{dt^2} + \frac{\lambda}{m}\frac{d(l\theta)}{dt} + g\sin\theta = 0$$

Replacing $\sin\theta$ by θ since it is small and writing $\lambda/m = 2k$, we get

$$\frac{d^2\theta}{dt^2} + 2k\frac{d\theta}{dt} + \frac{g\theta}{l} = 0 \qquad ...(i)$$

which is the required differential equation.

Its auxiliary equation has roots $D = k \pm \sqrt{(k^2 - w^2)}$ where $w = g/l$.

The oscillatory motion of the bob is only possible when $k < w$.

Then the roots of the auxiliary equation are $-k \pm i\sqrt{(w^2 - k^2)}$.

$\therefore$ the solution of (i) is

$$\theta = e^{-kt}$$

which gives a vibratory motion of period $2\pi/\sqrt{(w^2 - k^2)}$.

Example 14.6. *A pendulum of length l has one end of the string fastened to a peg on a smooth plane inclined to the horizon at an angle α. With the string and the weight on the plane, its time of oscillation is t sec. If the pendulum of length l' oscillates in one sec. when suspended vertically, prove that $\alpha = \sin^{-1}\left(\frac{l}{l't^2}\right)$.*

(*Kurukshetra, 2006*)

Solution. At any time t, let the bob of mass m be at P and O be the point of suspension so that $OP = l$ and $\angle AOP = \theta$ (Fig. 14.7).

The component of weight along the plane being $mg \sin \alpha$, the equation of motion of the bob along the tangent at P is

$$m\frac{d^2s}{dt^2} = -mg \sin\alpha \sin\theta$$

or $$\frac{d^2(l\theta)}{dt^2} = -g\sin\alpha\sin\theta \qquad [\because\ s = l\theta]$$

or $$\frac{d^2\theta}{dt^2} = -g\sin\alpha\left(\theta - \frac{\theta^3}{3!} + \dots\right)$$

or $$\frac{d^2\theta}{dt^2} = -\mu\theta \quad \text{where } \mu = \frac{g\sin\alpha}{l}, \text{ to a first approximation.}$$

$\therefore$ the motion being simple harmonic, the time of oscillation t.

$$= \frac{2\pi}{\mu} = 2\pi\sqrt{\frac{l}{g\sin\alpha}} \qquad ...(i)$$

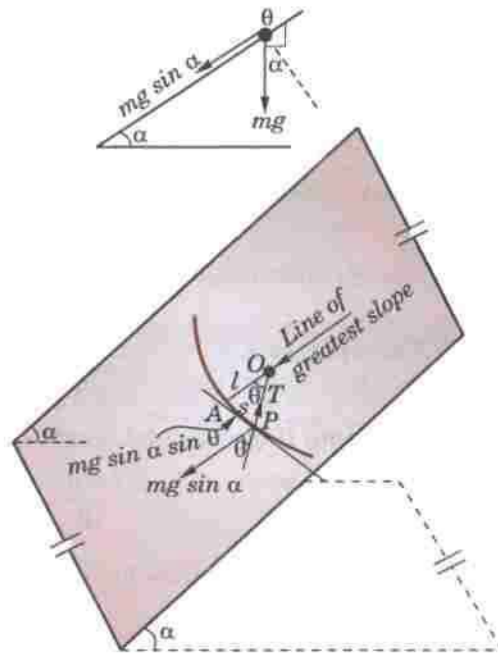

Fig. 14.7

We know that for a pendulum of length l' when suspended vertically, the time of oscillation

$$1 = 2\pi\sqrt{l'/g} \qquad ...(ii)$$

$\therefore$ dividing (*i*) by (*ii*), we have $t = \sqrt{\left(\frac{l}{l'\sin\alpha}\right)}$

or $$t^2 = l/l'\sin\alpha \quad \text{or} \quad \alpha = \sin^{-1}(l/l't^2).$$

PROBLEMS 14.1

1. A particle is executive simple harmonic motion with amplitude 20 cm and time 4 seconds. Find the time required by the particle in passing between points which are at distances 15 cm and 5 cm from the centre of force and are on the same side of it.
2. At the ends of three successive seconds, the distances of a point moving with S.H.M. from its mean position are x_1, x_2, x_3. Show that the time of a complete oscillation is
$$2\pi/\cos^{-1}\left(\frac{x_1+x_3}{2x_2}\right).$$
3. An elastic string of natural length $2a$ and modulus λ is stretched between two points A and B distant $4a$ apart on a smooth horizontal table. A particle of mass m is attached to the middle of the string. Show that it can vibrate in line AB with period $2\pi/\omega$, where $\omega^2 = 2\lambda/am$.
4. A particle of mass m moves in a straight line under the action of force $mn^2(OP)$, which is always directed towards fixed point O in the line. If the resistance to the motion is $2\lambda\ mnv$, where v is the speed and $0 < \lambda < 1$, find the displacement x in terms of the time t given that when $t = 0$, $x = 0$ and $dx/dt = u$ where $OP = x$.
5. A point moves in a straight line towards the centre of force μ/(distance^3) starting from rest at a distance a from the centre of force, show that the time of reaching a point b from the centre of force is $a\sqrt{(a^2-b^2)}/\sqrt{\mu}$ and that its velocity then is $\frac{\sqrt{\mu}}{ab}\sqrt{(a^2-b^2)}$. (*U.P.T.U., 2001*)

6. A clock loses five seconds a day, find the alteration required in the length of its pendulum in order that it may keep correct time.
7. A clock with a seconds pendulum loses 10 seconds per day at a place where $g = 32$ ft/sec^2. What change in the gravity is necessary to make it accurate ?
8. A seconds pendulum which gains 10 seconds per day at one place loses 10 seconds per day at another; compare the acceleration due to gravity at the two places. *(Kurukshetra, 2005)*
9. Show that the free oscillations of a galvanometer needle, as affected by the viscosity of the surrounding air which varies directly as the angular velocity of the needle, are determined by the equation $\frac{d^2\theta}{dt^2} + K\frac{d\theta}{dt} + \mu\theta = 0$, where k is the co-efficient of viscosity and θ is the angular deflection of the needle at time t. Obtain θ in terms of t and discuss the different cases that can arise.
10. If $I = \frac{d^2\theta}{dt^2} = -\,mgl \sin\theta$, where I, m, g, l are constant, given that at $t = 0$, $\theta = 0$ and $d\theta/dt = \omega_0 = m\sqrt{(mgl)}\,/\,I$, then show that $t = \frac{2}{\omega_0}\log\frac{\pi+\theta}{4}$. *(Nagpur, 2009)*

14.4 OSCILLATIONS OF A SPRING

(*i*) **Free oscillations.** Suppose a mass m is suspended from the end A of a light spring, the other end of which is fixed at O. (Fig. 14.8)

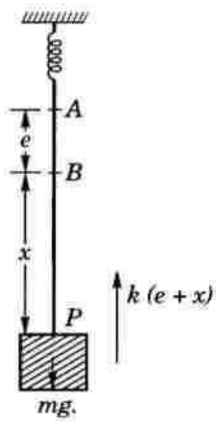

Fig. 14.8

Let e (= AB) be the elongation produced by the mass m hanging in equilibrium. If k be the restoring force per unit stretch of the spring due to elasticity, then for the equilibrium at B,

$$mg = T = ke \qquad \text{...(1)}$$

At any time t, after the motion ensues, let the mass be at P, where $BP = x$. Then the equation of motion of m is

$$m\frac{d^2x}{dt^2} = mg - k(e + x) = -\,kx \qquad \text{[By (1)]}$$

Or writing $k/m = \mu^2$, it becomes

$$\frac{d^2x}{dt^2} + \mu^2 x = 0 \qquad \text{...(2)}$$

This equation represents the free vibrations of the spring which are of the simple harmonic form having centre of oscillation at B—its equilibrium position and the *period of oscillation*

$$= \frac{2\pi}{\mu} = 2\pi\sqrt{\left(\frac{e}{g}\right)}. \qquad \left[\because\ \frac{1}{\mu} = \sqrt{\left(\frac{m}{k}\right)} = \sqrt{\left(\frac{e}{g}\right)}, \text{[By (1)]}\right]$$

(*ii*) **Damped oscillations.** If the mass m be subjected to do damping force proportional to velocity (say : $r\ dx/dt$) (Fig. 14.9), then the equation of motion becomes

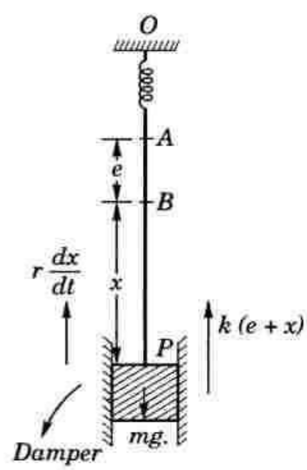

Fig. 14.9

$$m\frac{d^2x}{dt^2} = mg - k(e + x) - r\frac{dx}{dt}$$

$$= -\,kx - r\frac{dx}{dt} \qquad \text{[By (1)]}$$

Or writing $r/m = 2\lambda$ and $k/m = \mu^2$, it becomes

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \mu^2 x = 0 \qquad \text{...(3)}$$

$\therefore$ its auxiliary equation is

$$D^2 + 2\lambda D + \mu^2 = 0 \quad \text{whence} \quad D = -\,\lambda \pm .$$

Case I. *When* $\lambda > \mu$, the roots of the auxiliary equation are real and distinct (say γ_1, γ_2).

$\therefore$ the solution of (3) is $x = c_1 e^{\gamma_1 t} + c_2 e^{\gamma_2 t}$ ...(4)

To determine c_1, c_2 let the spring be stretched to a length $x = l$ and then released so that

$$x = l \text{ and } dx/dt = 0 \text{ at } t = 0.$$

$\therefore$ from (4), $l = c_1 + c_2$

Also from $\dfrac{dx}{dt} = c_1\gamma_1 e^{\gamma_1 t} + c_2\gamma_2 e^{\gamma_2 t}$, we get

$$0 = c_1\gamma_1 + c_2\gamma_2$$

whence $c_1 = \dfrac{-l\gamma_2}{\gamma_1 - \gamma_2}$ and $c_2 = \dfrac{l\gamma_1}{\gamma_1 - \gamma_2}$

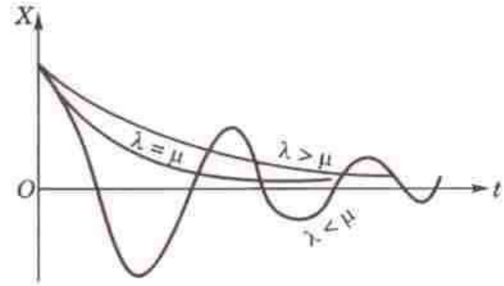

Fig. 14.10

Hence the solution of (3) is

$$x = \frac{l}{\gamma_1 - \gamma_2}(\gamma_1 e^{\gamma_2 t} - \gamma_2 e^{\gamma_1 t}) \qquad ...(5)$$

which shows that x is always positive and decreases to zero as $t \to \infty$ (Fig. 14.10).

The restoring force, in this case, is so great that the motion is non-oscillatory and is, therefore, referred to as *over-damped* or *dead-beat* motion.

Case II. *When* $\lambda = \mu$, the roots of the auxiliary equation are real and equal, (each being $= -\lambda$).

$\therefore$ The general solution of (3) becomes $x = (c_1 + c_2 t)e^{-\lambda t}$.

As in case I, if $x = l$ and $dx/dt = 0$ at $t = 0$, then $c_1 = l$ and $c_2 = \lambda l$.

Hence the solution of (3) is $x = l\,(1 + \lambda t)e^{-\lambda t}$ which also shows that x is always positive and decreases to zero as $t \to \infty$ (Fig. 14.10).

The nature of motion is similar to that of the previous case and is called the *critically damped motion* for it separates the non-oscillatory motion of case I from the most interesting oscillatory motion of case III.

Case III. *When* $\lambda < \mu$, the roots of the auxiliary equation are imaginary, *i.e.* $D = -\lambda \pm i\alpha$, where $\alpha^2 = \mu^2 - \lambda^2$.

$\therefore$ the solution of (3) is $x = e^{-\lambda t}(c_1 \cos \alpha t + c_2 \sin \alpha t)$

As in case I, $x = l$, $dx/dt = 0$ at $t = 0$, then $c_1 = l$ and $c_2 = \lambda l/\alpha$

Thus the solution of (3) becomes $x = le^{-\lambda t}\left(\cos \alpha t + \dfrac{\lambda}{\alpha}\sin \alpha t\right)$.

which can be put in the form $x = l\sqrt{\left\{1 + \left(\dfrac{\lambda}{\alpha}\right)^2\right\}}e^{-\lambda t}\cos\left\{\alpha - \tan^{-1}\dfrac{\lambda}{\alpha}\right\}$ ...(7)

Here the presence of the trigonometric factor in (7) shows that the *motion is oscillatory*, having

(*a*) the variable amplitude $= l\sqrt{[1 + (\lambda/\alpha)^2]}e^{-\lambda t}$ which decreases with time,

(*b*) the periodic time $T = 2\pi/\alpha$.

But the periodic time of free oscillations is $T' = 2\pi/\mu$.

As $\alpha = \sqrt{(\mu^2 - \lambda^2)} < \mu$

$\therefore$ $\dfrac{2\pi}{\alpha} > \dfrac{2\pi}{\mu}$, *i.e.* $T > T'$.

This shows that the *effect of damping is to increase the period of oscillation and the motion ultimately dies away*. Such a motion is termed as *damped oscillatory motion*.

(*iii*) **Forced oscillations** (*without damping*). If the point of the support of the spring is also vibrating with some external periodic force, then the resulting motion is called the *forced oscillatory motion*.

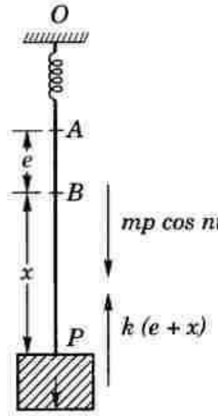

Fig. 14.11

Taking the external periodic force to be $mp \cos nt$, the equation of motion is

$$m\frac{d^2x}{dt^2} = mg - k(e + x) + mp \cos nt$$

$$= -kx + mp \cos nt \qquad [\because \; mg = ke] \qquad ...(8)$$

Or writing $k/m = \mu^2$, (8) takes the form

$$\frac{d^2x}{dt^2} + \mu^2 x = p \cos nt \qquad ...(9)$$

Its C.F. $= c_1 \cos \mu t + c_2 \sin \mu t$ and P.I. $= p \dfrac{1}{D^2 + \mu^2} \cos nt$.

New two cases arise :

Case I. *When* $\mu \neq n$.

$$\text{P.I.} = \frac{p}{\mu^2 - n^2} \cos nt.$$

$\therefore$ the complete solution of (9) is $x = c_1 \cos \mu t + c_2 \sin \mu t + \dfrac{p}{\mu^2 - n^2} \cos nt$.

On writing $c_1 \cos \mu t + c_2 \sin \mu t$ as $r \cos (\mu t + \phi)$, we have

$$x = r \cos (\mu t + \phi) + \frac{p}{\mu^2 - n^2} \cos nt \qquad ...(10)$$

This shows that the motion is compounded of two oscillatory motions : the first (due to the C.F.) *gives free oscillations of period $2\pi/\mu$, and the second* (due to the P.I.) *gives forced oscillations of period $2\pi/n$.*

Also we observe that if the frequency of free oscillations is very high (*i.e.*, μ is large), then the amplitude of forced oscillations is small.

Case II. *When* $\mu = n$.

$$\text{P.I.} = pt \cdot \frac{1}{2D} \cos \mu t = \frac{pt}{2} \int \cos \mu t \, dt = \frac{pt}{2\mu} \sin \mu t$$

$\therefore$ the complete solution of (9) is $x = c_1 \cos \mu t + c_2 \sin \mu t + \dfrac{pt}{2\mu} \sin \mu t$

$$= \left(c_2 + \frac{pt}{2\mu}\right) \sin \mu t + c_1 \cos \mu t.$$

Putting $c_2 + pt/2\mu = \rho \cos \psi$ and $c_1 = \rho \sin \psi$, we get

$$x = \rho \sin (\mu t + \psi) \qquad ...(11)$$

This shows that the oscillations are of period $2\pi/\mu$ and amplitude $\rho = \sqrt{[(c_2 + pt/2\mu)^2 + c_1^2]}$, which clearly increases with time (Fig. 14.12).

Thus the amplitude of the oscillations may become abnormally large causing over-strain and consequently breakdown of the system. In practice, however, collapse rarely occurs, though the amplitudes may become dangerously large since there is always some resistance present in the system.

This phenomenon of the impressed frequency becoming equal to the natural frequency of the system, is refered to as **resonance.**

Thus, while designing a machine or a structure, the occurrence of resonance should always be avoided to check the rupture of the system at any stage. That is why, the soldiers break step while marching over a bridge for the fear that their steps may not be in rhym with the natural frequency of the bridge causing its collapse due to 'resonance'.

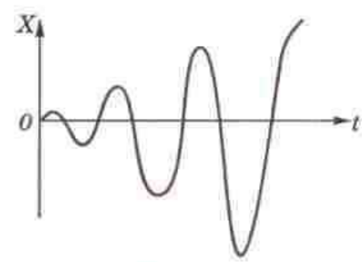

Fig. 14.12

(*iv*) **Forced oscillations** (*with damping*). If, in addition, there is a damping force proportional to velocity (say : $r\, dx/dt$) (Fig. 14.13), then the equation (8) becomes

$$m \frac{d^2x}{dt^2} = mg - k(e + x) + mp \cos nt - r \frac{dx}{dt}$$

$$= -kx + mp \cos nt - r \frac{dx}{dt} \qquad | \because \; mg = ke$$

On writing $r/m = 2\lambda$ and $k/m = \mu^2$, it takes the form

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \mu^2 x = p\cos nt \qquad ...(12)$$

Its auxiliary equation is $D^2 + 2\lambda D + \mu^2 = 0$ whence $D = -\lambda \pm \sqrt{\lambda^2 - \mu^2}$.

$$\therefore \qquad \text{C.F.} = e^{-\lambda t}[c_1 e^{t\sqrt{(\lambda^2-\mu^2)}} + c_2 e^{-t\sqrt{(\lambda^2-\mu^2)}}].$$

It represents the free oscillations of the system which die out as $t \to \infty$.

Also the P.I.

$$= p\frac{1}{D^2 + 2\lambda D + \mu^2}\cos nt = p\frac{1}{-n^2 + 2\lambda D + \mu^2}\cos nt$$

$$= p\frac{(\mu^2 - n^2) - 2\lambda D}{(\mu^2 - n^2)^2 - 4\lambda^2 D^2}\cos nt = p\frac{(\mu^2 - n^2)^2\cos nt + 2\lambda n\sin nt}{(\mu^2 - n^2)^2 + 4\lambda^2 n^2}$$

Putting $\mu^2 - n^2 = R\cos\theta$ and $2\lambda n = R\sin\theta$, we get

$$\text{P.I.} = \frac{p}{\sqrt{[(\mu^2 - n^2)^2 + 4\lambda^2 n^2]}}\cos(nt - \theta)$$

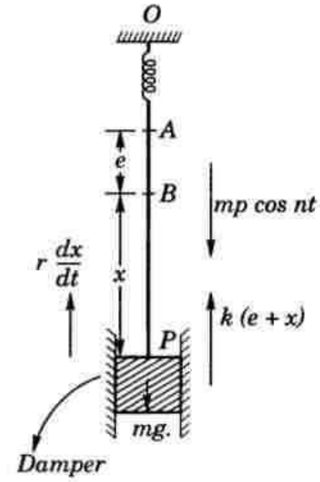

Fig. 14.13

which represents the forced oscillations of the system having

(*a*) a constant amplitude

$$= p/\sqrt{[(\mu^2 - n^2)^2 + 4\lambda^2 n^2]}$$

and (*b*) the period $= 2\pi/n$ which is the same as that of the impressed force.

Thus *with the increase of time, the free oscillations die away while the forced oscillations continue giving the steady state motion.*

Example 14.7. *A body weighing 10 kg is hung from a spring. A pull of 20 kg. wt. will stretch the spring to 10 cm. The body is pulled down to 20 cm below the static equilibrium position and then released. Find the displacement of the body from its equilibrium position at time t sec., the maximum velocity and the period of oscillation.*

Solution. Let O be the fixed end and A, the lower end of the spring (Fig. 14.14).

Since a pull of 20 kg wt. at A stretches the spring by 0.1 m.

$$\therefore \qquad 20 = T_0 = k \times 0.1, \text{ i.e. } k = 200 \text{ kg/m}.$$

Let B be the equilibrium position when a body weighing $W = 10$ kg is hung from A ; then

$$10 = T_B = k \times AB$$

i.e.,

$$AB = \frac{10}{200} = 0.05 \text{ m}$$

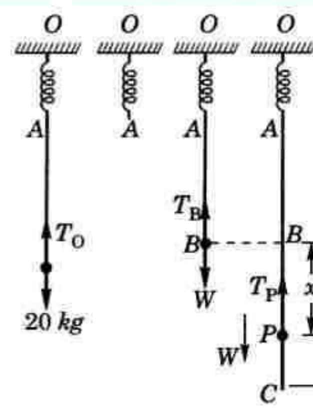

Fig. 14.14

Now the weight is pulled down to C, where $BC = 0.2$ m. After any time t sec. of its release from C, let the weight be at P where $BP = x$.

Then the tension $T_P = k \times AP = 200(0.05 + x) = 10 + 200x$.

$\therefore$ The equation of motion of the body is

$$\frac{W}{g}\frac{d^2x}{dt^2} = W - T_P, \text{ where } g = 9.8 \text{ m/sec}^2.$$

i.e.,

$$\frac{10}{9.8}\frac{d^2x}{dt^2} = 10 - (10 + 200x) \quad \text{or} \quad \frac{d^2x}{dt^2} = -\mu^2 x, \quad \text{where } \mu = 14.$$

This shows that the motion of the body is simple harmonic about B as centre and the period of oscillation $= 2\pi/\mu = 0.45$ sec.

Also the amplitude of motion being $BC = 0.2$ m., the displacement of the body from B at time t is given by $x = 0.2\cos\mu t = 0.2\cos 14t$ m

and the maximum velocity $= \mu$ (amplitude) $= 14 \times 0.2 = 2.8$ m/sec.

Example 14.8. *A spring fixed at the upper end supports a weight of 980 gm at its lower end. The spring stretches $\frac{1}{2}$ cm under a load of 10 gm and the resistance (in gm wt.) to the motion of the weigth is numerically equal to $\frac{1}{10}$ of the speed of the weight in cm/sec. The weight is pulled down $\frac{1}{4}$ cm. below its equilibrium position and then released. Find the expression for the distance of weight from its equilibrium position at time t during its first upward motion.*

Also find the time it takes the damping factor to drop to $\frac{1}{10}$ of its initial value.

Solution. Let O be the fixed end and A the other end of the spring (Fig. 14.15).

Since load of 10 gm attached to A stretches the spring by $\frac{1}{2}$ cm.

$$\therefore \qquad 10 = T_0 = k \cdot \frac{1}{2} \quad i.e., \; k = 20 \text{ gm/cm}.$$

Let B be the equilibrium position when 980 gm. weight is attached to A, then

$$980 = T_B = k \times AB, \quad i.e., \quad AB = \frac{980}{20} = 49 \text{ cm}.$$

Now the 980 gm weight is pulled down to C, where $BC = \frac{1}{4}$ cm.

After any time t of its release from C, let the weight be at P, where $BP = x$.

Then the tension

$$T = k \times AP = 20\,(49 + x) = 980 + 20x \text{ and the resistance to motion} = \frac{1}{10}\frac{dx}{dt}.$$

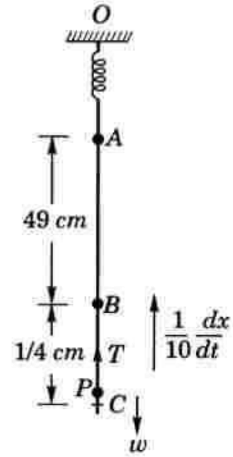

Fig. 14.15

$\therefore$ the equation of motion is

$$\frac{980}{g}\frac{d^2x}{dt^2} = w - T - \frac{1}{10}\frac{dx}{dt} \qquad [\because\ g = 980 \text{ cm/sec}^2 \text{ (p. 449)}$$

$$= 980 - (980 + 20x) - \frac{1}{10}\frac{dx}{dt} \quad i.e. \quad 10\frac{d^2x}{dt^2} + \frac{dx}{dt} + 200x = 0 \qquad ...(i)$$

Its auxiliary equation is $10D^2 + D + 200 = 0$,

whence
$$D = \frac{-1 + \sqrt{[1 - 4 \times 10 \times 200]}}{20} = \frac{-1 + i(89.4)}{20} = -0.05 \pm i\,(4.5)$$

$\therefore$ the solution of (i) is $x = e^{-0.05t}[c_1 \cos(4.5)t + c_2 \sin(4.5)t]$ ...(ii)

Also
$$\frac{dx}{dt} = e^{-0.05t}(-0.05)\,[c_1 \cos(4.5)t + c_2 \sin(4.5)t]$$
$$+ e^{-0.05t}\,[-c_1 \sin(4.5)t + c_2 \cos(4.5)t](4.5) \qquad ...(iii)$$

Initially when the mass is at C, $t = 0$, $x = \frac{1}{4}$ cm. and $dx/dt = 0$.

From (ii), $c_1 = \frac{1}{4}$, and from (iii) $0 = (-0.05)c_1 + c_2(4.5)$, $i.e.$, $c_2 = -0.003$.

Thus, substituting these value in (ii), we get

$$x = e^{-0.05t}[0.25 \cos(4.5)t + 0.003 \sin(4.5)t]$$

which gives the displacement of the weight from the equilibrium position at any time t.

Here damping factor $= re^{-0.05t}$, where r is a constant of proportionality.

Its initial value $= re^0 = r$.

Suppose after time t, the damping factor $= r/10$. $\therefore$ $r/10 = re^{-0.05t}$ or $e^{t/20} = 10$.

Thus
$$t = 20 \log_e 10 = 20 \times 2.3 = 46 \text{ sec}.$$

Example 14.9. *A spring which stretches by an amount e under a force $m\lambda^2 e$ is suspended from a support O and has a mass m at the lower end. Initially the mass is at rest in its equilibrium position at a point A below O. A vertical oscillation is now given to the support O such that at any time (t > 0) its displacement below its initial position is a sin nt. Show that the displacement x of the mass below A is given by*

$$d^2x/dt^2 + \lambda^2 x = \lambda^2 a \sin nt.$$

Hence show that if $n \neq \lambda$, the displacement is given by $x = \lambda a\,(\lambda \sin nt - n \sin \lambda t)/(\lambda^2 - n^2)$. What happens when $n = \lambda$?

Solution. If k be the stiffness of the spring then $m\lambda^2 e = ke$ *i.e.*, $k = m\lambda^2$.

Also in equilibrium $\quad mg = ke \quad$ ...(*i*)

Initially the mass is in equilibrium at A (Fig. 14.7). At time t, the support P is given a downward displacement $a \sin nt$. If the mass is displaced through a further distance x from A, then the equation of motion of the mass is given by

$$m\frac{d^2x}{dt^2} = mg - k(x+e) + ka \sin nt$$

$$= -kx + ka \sin nt \qquad \text{[By (i)]}$$

or
$$\frac{d^2x}{dt^2} + \lambda^2 x = \lambda^2 a \sin nt \qquad [\because \; k = m\lambda^2]$$

or
$$(D^2 + \lambda^2)x = \lambda^2 a \sin nt \qquad \text{...(ii)}$$

Its A.E. $= c_1 \cos \lambda t + c_2 \sin \lambda t$

$$\text{P.I.} = \frac{1}{D^2+\lambda^2}\lambda^2 a \sin nt.$$

Now two cases arise :

Case I. *When* $n \neq \lambda$

$$\text{P.I.} = \lambda^2 a \frac{1}{n^2+\lambda^2} \sin nt$$

$\therefore$ the complete solution of (*ii*) is $x = c_1 \cos \lambda t + c_2 \sin \lambda t + \dfrac{\lambda^2 a}{\lambda^2 - n^2} \sin nt$ ...(*iii*)

$$\therefore \quad \frac{dx}{dt} = -c_1\lambda \sin \lambda t + c_2 \lambda \cos \lambda t + \frac{\lambda^2 an}{\lambda^2 - n^2}\cos nt$$

Initially when $t = 0$, $x = 0$ and $dx/dt = 0$.

$\therefore \quad c_1 = 0$ and $0 = c_2\lambda + \lambda^2 an/(\lambda^2 - n^2)$ *i.e.*, $c_2 = \lambda an/(\lambda^2 - n^2)$

Thus, substituting the values of c_1 and c_2 in (*iii*), we have

$$x = -\frac{\lambda an}{\lambda^2 - n^2}\sin \lambda t + \frac{\lambda^2 a}{\lambda^2 - n^2}\sin nt = \frac{\lambda a}{\lambda^2 - n^2}(\lambda \sin nt - n \sin \lambda t)$$

Case II. *When* $n = \lambda$

$$\text{P.I.} = \lambda^2 a \frac{1}{D^2+\lambda^2}\sin nt = \lambda^2 at\,.\,\frac{1}{2D}\sin \lambda t = \frac{\lambda^2 at}{2}\int \sin \lambda t\, dt = -\frac{\lambda at}{2}\cos \lambda t$$

$\therefore$ the complete solution is

$$x = c_1 \cos \lambda t + c_2 \sin \lambda t - \frac{\lambda at}{2}\cos \lambda t \qquad \text{...(iv)}$$

$$\therefore \quad \frac{dx}{dt} = -c_1\lambda \sin \lambda t + c_2\lambda \cos \lambda t + \frac{\lambda^2 at}{2}\sin \lambda t - \frac{\lambda a}{2}\cos \lambda t$$

When $t = 0$, $x = 0$ and $dx/dt = 0$

$\therefore \quad 0 = c_1$ and $0 = c_2\lambda - \lambda a/2$ *i.e.*, $c_2 = a/2$.

Thus, substituting the values of c_1 and c_2 in (*iv*), we get

$$x = \frac{a}{2}\sin \lambda t - \frac{\lambda at}{2}\cos \lambda t$$

$$= \frac{a}{2}(\sin \lambda t - \lambda t \cos \lambda t) \qquad [\text{Put } 1 = r\cos\phi \text{ and } \lambda t = r \sin \phi]$$

$$= \frac{ar}{2}\sin(\lambda t - \phi)$$

Its amplitude $\left(\frac{ar}{2}\right) = \frac{a}{2}\sqrt{(1+\lambda^2 t^2)}$, which increases with time. Hence the phenomenon of *resonance* occurs.

Example 14.10. *A spring of negligible weight which stretches 1 inch under tension of 2 lb is fixed at one end and is attached to a weight of w lb at the other. It is found that resonance occurs when an axial periodic force 2 cos 2t lb acts on the weight. Show that when the free vibrations have died out, the forced vibrations are given by x = ct sin 2t, and find the values of w and c.*

Solution. As a weight of 2 lb attached to the lower end A of the spring stretched it by $\frac{1}{12}$ ft.

$\therefore \quad 2 = T = k \cdot \frac{1}{12}$, *i.e.*, $k = 24$ lb/ft.

Let B be the equilibrium position of the weight w attached to A (Fig. 14.16), then

$$w = T_B = k \times AB = 24 \times AB$$

$\therefore \quad AB = w/24$ ft.

At any time t, let the weight be at P, where $BP = x$.

Then the tension T at $P = k \times AP = 24\left(\frac{w}{24} + x\right) = w + 24x$

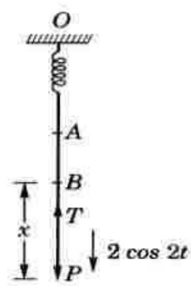

Fig. 14.16

$\therefore$ its equation of motion is

$$\frac{w}{g}\frac{d^2x}{dt^2} = -T + w + 2\cos 2t = -w - 24x + w + 2\cos 2t$$

or $$w\frac{d^2x}{dt^2} + 24gx = 2g\cos 2t \qquad \text{...}(i)$$

The phenomenon of **resonance** *occurs when the period of free oscillations is equal to the period of forced oscillations.*

Writing (i) as $\frac{d^2x}{dt^2} + \mu^2 x = \frac{2g}{w}\cos 2t$, where $\mu^2 = 24g/w$, the period of free oscillations is found to be $2\pi/\mu$ and the period of the force $(2g/w)\cos 2t$ is π.

$\therefore \quad 2\pi/\mu = \pi$ or $24g/w = \mu^2 = 4$. Thus the weight, $w = 6g$.

Taking this value of w, (i) takes the form

$$\frac{d^2x}{dt^2} + 4x = \frac{1}{3}\cos 2t \qquad \text{...}(ii)$$

We know that the free oscillations are given by the C.F. and the forced oscillations by the P.I.

Thus, when the free oscillations have died out, the forced oscillations are given by the P.I. of (ii).

Now P.I. of $(ii) = \frac{1}{3}\cdot\frac{1}{D^2+4}\cos 2t = \frac{1}{3}t\cdot\frac{1}{2D}\cos 2t = \frac{1}{12}t\sin 2t.$

Hence $\quad c = \frac{1}{12}.$

PROBLEMS 14.2

1. An elastic string of natural length a is fixed at one end and a particle of mass m hangs freely from the other end. The modulus of elasticity is mg. The particle is pulled down a further distance l below its equilibrium position and released from rest. Show that the motion of the particle is simple harmonic and find the pariodicity.
2. A mass of 4 lb suspended from a light elastic string of natural length 3 feet extends it to a distance 2 feet. One end of the string is fixed and a mass of 2 lb is attached to other. The mass is held so that the string is just unstretched and is then let go. Find the amplitude, the period and the maximum velocity of the ensuing simple harmonic motion.

3. A light elastic string of natural length l has one extremity fixed at a point A and the other end attached to a stone, the weight of which in equilibrium would extend the string to a depth l_1. Show that if the stone be dropped from rest at A, it will come to instantaneous rest at a depth $\sqrt{(l_1^2 - l^2)}$ below the equilibrium position.
4. A 4 lb weight on a string stretches it 6 in. Assuming that a damping force in lb wt. equal to λ times the instantaneous velocity in ft/sec. acts on the weight, show that the motion is over damped, critically damped or oscillatory according as $\lambda > = < 2$. Find the period of oscillation when $\lambda = 1.5$.
5. A mass of 200 gm is tied at the end of a spring which extends to 4 cm under a force 196,000 dynes. The spring is pulled 5 cm and released. Find the displacement t seconds after release if there be a damping force of 2000 dynes per cm per second.
6. A body weighing 16 lb is suspended by a spring in a fluid whose resistance in lb wt. is twice the speed of the body in ft/sec. A pull of 25 lb wt. would stretch the spring 3 inches. The body is drawn 3 inches below the equilibrium position in the fluid and then released. Find the period of oscillations and the time required for the damping factor to be reduced to one-tenth of its initial value. *(Sambhalpur, 1998)*
7. A mass M suspended from the end of a helical spring is subjected to a periodic force $f = F \sin \omega t$ in the direction of its length. The force f is measured positive vertically downwards and at zero time M is at rest. If the spring stiffness is S, prove that the displacement of M at time t from the commencement of motion is given by
$$x = \frac{F}{M(p^2 - \omega^2)}\left[\sin \omega t - \frac{\omega}{p}\sin pt\right]$$
where $p^2 = S/M$ and damping effects are neglected. *(U.P.T.U., 2002)*
8. A vertical spring having 4.5 lb/ft. has 16 lb wt. suspended from it. An external force of 24 sin $9t$ $(t \geq 0)$ lb wt. is applied. A damping force given numerically in lb. wt. by four times its velocity in ft/sec, is assumed to act. Initially the weight is at rest at its equilibrium position. Determine the position of the weight at any time. Also find the amplitude, period and the frequency of the steady-state solution.
9. A body weighing 4 lb hangs at rest on a spring producing in the spring an extension of 1ft. The upper end of the spring in now made to execute a vertical simple harmonic oscillation $x = \sin 4t$, x being measured vertically downwards in feet. If the body is subject to a frictional resistance whose magnitude in lb wt. is one-quarter of its velocity in feet per second, obtain the differential equation for the motion of the body and find the expression for its displacement at time t, when t is large.
10. A body executes damped forced vibrations given by the equation
$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + b^2x = e^{-kt}\sin nt.$$
Solve the equation for both the cases, when $n^2 \neq b^2 - k^2$ and $n^2 = b^2 - k^2$. *(U.P.T.U., 2004)*

14.5 OSCILLATORY ELECTRICAL CIRCUIT

(i) L-C circuit

Consider an electrical circuit containing an inductance L and capacitance C (Fig. 14.17).

Let i be the current and q the charge in the condenser plate at any time t, so that the voltage drop across

$$L = L\frac{di}{dt} = L\frac{d^2q}{dt^2}$$

and the voltage drop across $C = q/C$.

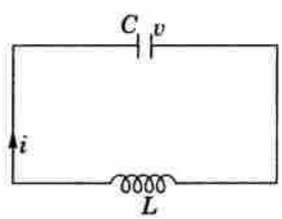

Fig. 14.17

As there is no applied e.m.f. in the circuit, therefore, by Kirchhoff's first law, we have

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = 0.$$

Or dividing by L and writing $1/LC = \mu^2$, we get $\dfrac{d^2q}{dt^2} + \mu^2 q = 0$ ...(1)

This equation is precisely same as (2) on page 507 and, therefore, *it represents free electrical oscillations of the current having period* $2\pi/\mu = 2\pi\sqrt{LC}$.

Thus the discharging of a condenser through an inductance L is same as the motion of the mass m at the end of a spring.

(*ii*) **L–C–R circuit**

Now consider the discharge of a condenser C through an inductanc L and the resistance R (Fig. 14.18). Since the voltage drop across L, C and R are respectively

$$L\frac{d^2q}{dt^2}, \frac{q}{C} \text{ and } R\frac{dq}{dt}.$$

$\therefore$ by Kirchhoff's law, we have $L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{q}{C} = 0$ ...(2)

Or writing $R/L = 2\lambda$ and $1/LC = \mu^2$, we have $\dfrac{d^2q}{dt^2} + 2\lambda\dfrac{dq}{dt} + \mu^2 q = 0$

This equation is same as (3) on page 507 and, therefore has the same solution as for the mass m on a spring with a damper.

Thus *the charging or discharging of a condenser through the resistance R and an inductance L is an electrical analogue of the damped oscillations of mass m on a spring.*

(*iii*) **L–C circuit with e.m.f.** = *p* cos *nt*.

The equation (1) for an L–C circuit (Fig. 14.19), now becomes $L\dfrac{d^2q}{dt^2} + \dfrac{q}{C} = p \cos nt.$

Or writing $1/LC = \mu^2$, we have $\dfrac{d^2q}{dt^2} + \mu^2 q = \dfrac{p}{L}\cos nt$ ...(3)

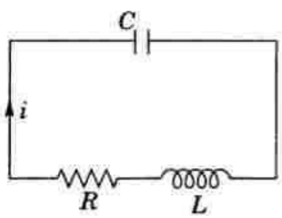

Fig. 14.18

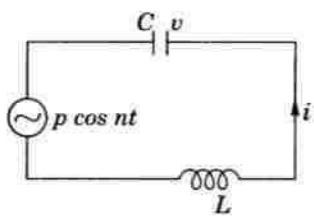

Fig. 14.19

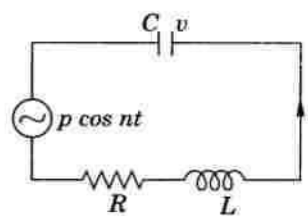

Fig. 14.20

This equation is of the same form as (9) on page 509 and, therefore, has the solution as for the motion of a mass m on a spring with external periodic force $p \cos nt$ acting on it.

Thus the condenser placed in series with source of e.m.f. (= p cos nt) and discharging through a coil containing inductance L is an electrical analogue of the forced oscillations of the mass m on a spring.

An electrical instance of *resonance* phenomena occurs while tuning a radio-station, for the natural frequency of the tuning of L–C circuit is made equal to the frequency of the desired radio-station, giving the maximum output of the receiver at the said receiving station.

(*iv*) **L–C–R circuit with e.m.f.** = *p cos nt*.

The equation of (2) above, now becomes $L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{q}{C} = p \cos nt.$ (Fig. 14.20)

Or writing $R/L = 2\lambda$ and $1/LC = \mu^2$ as before, we have

$$\frac{d^2q}{dt^2} + 2\lambda\frac{dq}{dt} + \mu^2 q = \frac{p}{L}\cos nt \qquad \text{...(4)}$$

This equation is exactly same as (12) on page 510 and, therefore, its C.F. represents the free oscillations of the circuit whereas the P.I. represents the forced oscillations.

Here also as t increases, the free oscillations die out while the forced oscillations persist giving steady motion.

Thus the L–C–R circuit with a source of alternating e.m.f. is an electrical equivalent of the mechanical phenomena of forced oscillations with resistance.

14.6 ELECTRO-MECHANICAL ANALOGY

We have just seen, how merely by renaming the variables, the differential equation representing the oscillation of a weight on a spring represents an analogous electrical circuit. As electrical circuits are easy to assemble and the currents and voltages are accurately measured with ease, this affords a practical method of studying the oscillations of complicated mechanical systems which are expensive to make and unwieldy to handle by considering an equivalent electrical circuit. While making an electrical equivalent of a mechanical system, the following correspondences between the elements should be kept in mind, noting that the circuit may be in series or in parallels :

Mech. System	*Series circuit*	*Parallel circuit*
Displacement	Current i	Voltage E
Force or couple	Voltage E	Current i
Mass m or $M.I.$	Inductance L	Capacitance C
Damping force	Resistance R	Conductance $1/R$
Spring modulus	Elastance $1/C$	Susceptance $1/L$

Example 14.11. *An uncharged condenser of capacity C is charged by applying an e.m.f. E sin $t/\sqrt{(LC)}$, through leads of self-inductance L and negligible resistance. Prove that at any time t, the charge on one of the plates is* $\frac{EC}{2}\left\{\sin\frac{t}{\sqrt{(LC)}}-\frac{t}{\sqrt{(LC)}}\cos\frac{t}{\sqrt{(LC)}}\right\}$ (U.P.T.U., 2003)

Solution. If q be the change on the condenser, the differential equation of the circuit is

$$L\frac{d^2q}{dt^2}+\frac{q}{C}=E\sin\frac{t}{\sqrt{(LC)}} \quad \text{...(i)}$$

Its A.E. is $\quad LD^2+1/C=0 \quad$ or $\quad D=\pm 1/\sqrt{(LC)}$

$\therefore \quad$ C.F. $= c_1\cos t/\sqrt{LC}+c_2\sin t/\sqrt{(LC)}$

and

$$\text{P.I.}=\frac{1}{LD^2+\frac{1}{C}}E\sin\frac{t}{\sqrt{(LC)}} \qquad \left[\text{Putting } D^2=-\frac{1}{LC}, \text{denom.}=0\right]$$

$$=Et\frac{1}{2LD}\sin\frac{t}{\sqrt{(LC)}}=\frac{Et}{2L}\int\sin\frac{t}{\sqrt{(LC)}}dt=-\frac{Et}{2L}\sqrt{(LC)}\cos\frac{t}{\sqrt{(LC)}}=-\frac{Et}{2}\sqrt{\frac{C}{L}}\cos\frac{t}{\sqrt{LC}}$$

Thus the C.S. of (*i*) is $q=c_1\cos\frac{t}{\sqrt{LC}}+c_2\sin\frac{t}{\sqrt{LC}}-\frac{Et}{2}\sqrt{\frac{C}{L}}\cos\frac{t}{\sqrt{LC}}$

When $t=0, q=0, c_1=0$

$$\therefore \quad q=c_2\sin\frac{t}{\sqrt{LC}}-\frac{Et}{2}\sqrt{\frac{C}{L}}\cos\frac{t}{\sqrt{LC}} \quad \text{...(ii)}$$

Differentiating (*ii*) w.r.t. t, we get

$$\frac{dq}{dt}=\frac{c_2}{\sqrt{LC}}\cos\frac{t}{\sqrt{LC}}-\frac{E}{2}\sqrt{\frac{C}{L}}\left\{\cos\frac{t}{\sqrt{LC}}-\frac{t}{\sqrt{LC}}\sin\frac{t}{\sqrt{LC}}\right\}$$

Also when $t=0$, $dq/dt=i=0$,

$$\therefore \quad \frac{c_2}{\sqrt{LC}}-\frac{E}{2}\sqrt{\frac{C}{L}}=0 \quad \text{or} \quad c_2=\frac{EC}{2}.$$

Substituting the value of c_2 in (*ii*), q at any time t is given by

$$q=\frac{EC}{2}\left\{\sin\frac{t}{\sqrt{LC}}-\frac{t}{\sqrt{LC}}\cos\frac{t}{\sqrt{LC}}\right\}.$$

Example 14.12. *In an L—C—R circuit, the charge q on a plate of a condenser is given by*

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C}=E\sin pt.$$

The circuit is tuned to resonance so that $p^2=1/LC$. If initially the current i and the charge q be zero, show that, for small values of R/L, the current in the circuit at time t is given by

$(Et/2L)\sin pt.$ (U.P.T.U., 2004)

Solution. Given differential equation is $(LD^2 + RD + 1/C)q = E \sin pt$ ...(*i*)

Its auxiliary equation is $LD^2 + RD + 1/C = 0$,

which gives $$D = \frac{1}{2L}\left[-R \pm \sqrt{\left(R^2 - \frac{4L}{C}\right)}\right] = -\frac{R}{2L} + \sqrt{\left(\frac{R^2}{4L^2} - \frac{1}{LC}\right)}$$

As R/L is small, therefore, to the first order in R/L,

$$D = -\frac{R}{2L} \pm i\frac{1}{\sqrt{(LC)}} = -\frac{R}{2L} \pm ip \qquad \left[\because\ p^2 = \frac{1}{LC}\right]$$

$\therefore$ C.F. $= e^{-(Rt/2L)}(c_1 \cos pt + c_2 \sin pt)$

$= (1 - Rt/2L)(c_1 \cos pt + c_2 \sin pt)$ rejecting terms in $(R/L)^2$ etc.

and $$\text{P.I.} = \frac{1}{LD^2 + RD + 1/C} E \sin pt = E\frac{1}{-Lp^2 + RD + 1/C}\sin pt$$

$$= \frac{E}{R}\int \sin pt\, dt = -\frac{E}{Rp}\cos pt \qquad \left[\because p^2 = \frac{1}{LC}\right]$$

Thus the complete solution of (*i*) is $q = \left(1 - \frac{Rt}{2L}\right)(c_1 \cos pt + c_2 \sin pt) - \frac{E}{Rp}\cos pt$ ...(*ii*)

$$\therefore \quad i = \frac{dq}{dt} = \left(1 - \frac{Rt}{2L}\right)(-c_1 \sin pt + c_2 \cos pt)\,p - \frac{R}{2L}(c_1 \cos pt + c_2 \sin pt) + \frac{E}{R}\sin pt \qquad ...(iii)$$

Initially, when $t = 0, q = 0, i = 0$ $\therefore$ from (*ii*), $0 = c_1 - E/Rp$ $\therefore c_1 = E/Rp$ and from (*iii*),

$0 = c_2 p - Rc_1/2L$ $\therefore c_2 = Rc_1/2Lp = E/2Lp^2$

Thus, substituting these values of c_1 and c_2 in (*iii*), we get

$$i = \left(1 - \frac{Rt}{2L}\right)\left(-\frac{E}{Rp}\sin pt + \frac{E}{2Lp^2}\cos pt\right)p - \frac{R}{2L}\left(\frac{E}{Rp}\cos pt + \frac{E}{2Lp^2}\sin pt\right) + \frac{E}{R}\sin pt$$

$$= \frac{Et}{2L}\sin pt. \qquad [\because\ R/L \text{ is small}]$$

PROBLEMS 14.3

1. Show that the frequency of free vibrations in a closed electrical circuit with inductance L and capacity C in series is $\frac{30}{\pi\sqrt{(LC)}}$ per minute.
2. The differential equation for a circuit in which self-inductance and capacitance neutralize each other is $L\frac{d^2i}{dt^2} + \frac{i}{C} = 0$. Find the current i as a function of t given that I is the maximum current, and $i = 0$ when $t = 0$.
3. A constant e.m.f. E at $t = 0$ is applied to a circuit consisting of inductance L, resistance R and capacitance C in series. The initial values of the current and the charge being zero, find the current at any time t, if $CR^2 < 4L$. Show that the amplitudes of the successive vibrations are in geometrical progression.
4. The damped LCR circuit is governed by the equation $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$ where, L, R, C are positive constants. Find the conditions under which the circuit is over damped, under damped and critically damped. Find also the critical resistance. (*U.P.T.U., 2005*)
5. A condenser of capacity C discharged through an inductance L and resistance R in series and the charge q at time t satisfies the equation $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$. Given that $L = 0.25$ henries, $R = 250$ ohms, $C = 2 \times 10^{-6}$ farads, and that when $t = 0$, charge q is 0.002 coulombs and the current $dq/dt = 0$, obtain the value of q in terms of t.
6. An e.m.f. $E \sin pt$ is applied at $t = 0$ to a circuit containing a capacitance C and inductance L. The current i satisfies the equation $L\frac{di}{dt} + \frac{1}{C}\int i\, dt = E \sin pt$. If $p^2 = 1/LC$ and initially the current i and the charge q are zero, show that the current at time t is $(Et/2L) \sin pt$, where $i = dq/dt$.

7. For an L—R—C circuit, the charge q on a plate of the condenser is given by $L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C}=E\sin\omega t$, where $i=\frac{dq}{dt}$. The circuit is tuned to resonance so that $\omega^2 = 1/LC$.

If $CR^2 < 4L$ and initially $q = 0, i = 0$, show that $q=\frac{E}{R\omega}\left[e^{-Rt/2C}\left(\cos pt+\frac{R}{2Lp}\sin pt\right)-\cos\omega t\right]$

where $$p^2=\frac{1}{LC}-\frac{R^2}{4L^2}.$$ (U.P.T.U., 2003)

8. An alternating E.M.F. $E \sin pt$ is applied to a circuit at $t = 0$. Given the equation for the current i as $L\frac{d^2i}{dt^2}+R\frac{di}{dt}+\frac{i}{C}=pE\cos pt$, find the current i when (i) $CR^2 > 4L$, (ii) $CR^2 < 4L$.

14.7 DEFLECTION OF BEAMS

Consider a uniform beam as made up of fibres running lengthwise. We have to find its deflection under given loadings.

In the bent form, the fibres of the lower half are stretched and those of upper half are compressed. In between these two, there is a layer of unstrained fibres called the *neutral surface*. The fibre which was initially along the x-axis (the central horizontal axis of the beam) now lies in the neutral surface, in the form of a curve called the *deflection curve* or the *elastic curve*. We shall encounter differential equations while finding the equation of this curve.

Consider a cross-section of the beam cutting the elastic curve in P and the neutral surface in the line AA'—called the neutral axis of this section (Fig. 14.21).

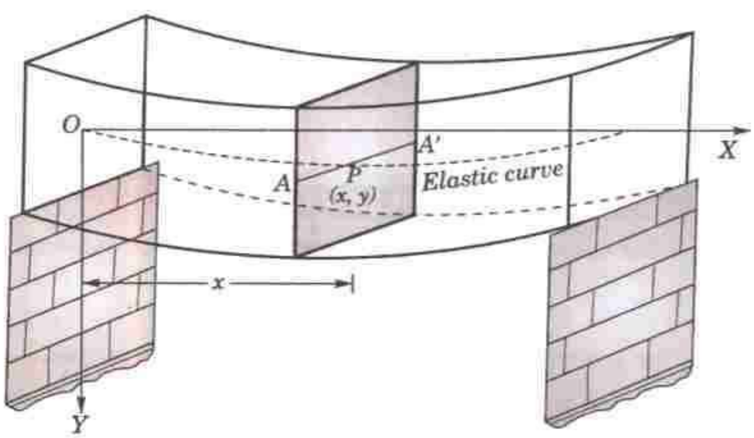

Fig. 14.21

It is well-known from mechanics that the bending moment M about AA', of all forces acting on either side of the two portions of the beam separated by this cross-section, is given by the *Bernoulli-Euler law*

$$M = EI/R$$

where E = modulus of elasticity of the beam,

I = moment of inertia of the cross-section about AA',

and R = radius of curvature of the elastic curve at $P(x, y)$.

If the deflection of the beam is small, the slope of the elastic curve is also small so that we may neglect $(dy/dx)^2$ in the formula,

$$R=\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}\Big/\frac{d^2y}{dx^2}.$$ Thus for small deflections, $R = 1/(d^2y/dx^2)$.

Hence (1) *Bending moment* $\mathbf{M}=\mathbf{EI}\dfrac{\mathbf{d^2y}}{\mathbf{dx^2}}$

(2) *Shear force* $\left(= \dfrac{dM}{dx}\right) = EI\,\dfrac{d^3y}{dx^3}$;

(3) *Intensity of loading* $\left(= \dfrac{d^2M}{dx^2}\right) = EI\,\dfrac{d^4y}{dx^4}$

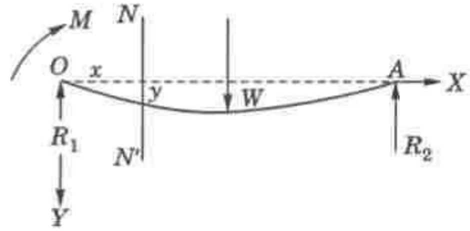

Fig. 14.22

(4) *Convention of signs.* The sum of the moments about a section NN' due to external forces on the left of the section, if anti-clockwise is taken as positive and if clockwise (as in Fig. 14.22) is taken as negative.

The deflection y downwards and length x to the right are taken as positive. The slope dy/dx will be positive if downwards in the direction of x-positive.

(5) *End conditions.* The arbitrary constants appearing in the solution of the differential equation (1) for a given problem are found from the following end conditions :

(*i*) *At a freely supported end* (Fig. 14.23), there being no deflection and no bending moment, we have $y = 0$ and $d^2y/dx^2 = 0$.

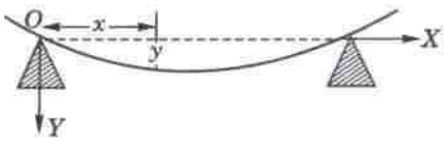

Fig. 14.23

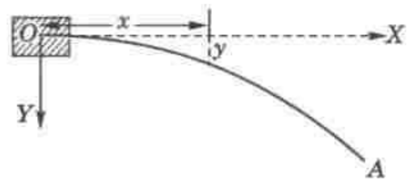

Fig. 14.24

(*ii*) *At a (horizontal) fixed end* (Fig. 14.24), the deflection and the slope of the beam being both zero, we have

$$y = 0 \text{ and } dy/dx = 0.$$

(*ii*) *At a perfectly free end* (*A* in Fig. 14.24), there being no bending moment or shear force, we have

$$\frac{d^2y}{dx^2} = 0 \quad \text{and} \quad \frac{d^3y}{dx^3} = 0$$

(6) *A member of a structure or a machine when subjected to end thrusts only is called a* **strut** *and a vertical strut is called a* **column**.

There are four possible ways of the end fixation of a strut:

(*i*) Both ends fixed, called a *built-in* or *encastre* strut.

(*ii*) One end fixed and the other freely supported, hinged or pin-jointed.

(*iii*) One end fixed and the other end free, called a *cantilever.*

(*iv*) Both ends freely supported or pin-jointed.

Example 14.13. *The deflection of a strut of length l with one end ($x = 0$) built-in and the other supported and subjected to end thrust P, satisfies the equation*

$$\frac{d^2y}{dx^2} + a^2y = \frac{a^2R}{P}(l - x).$$

Prove that the deflection curve is $y = \dfrac{R}{P}\left(\dfrac{\sin ax}{a} - l\cos ax + l - x\right)$, *where* $al = \tan al$.

(U.P.T.U., 2001)

Solution. Given differential equation is $(D^2 + a^2)y = \dfrac{a^2R}{P}(l - x)$ ...(*i*)

Its auxiliary equation is $D^2 + a^2 = 0$, whence $D = \pm\, ai$.

$$\therefore \qquad C.F. = \frac{1}{D^2 + a^2}\,\frac{a^2R}{P}(l - x) = \frac{R}{P}\left(1 + \frac{D^2}{a^2}\right)^{-1}(l - x)$$

$$= \frac{R}{P}\left(1 - \frac{D^2}{a^2} + \ldots\right)(l - x) = \frac{R}{P}(l - x)$$

Thus the complete solution of (*i*) is $y = c_1 \cos ax + c_2 \sin ax + \dfrac{R}{P}(l - x)$ ...(*ii*)

Also $$\frac{dy}{dx} = -c_1 a \sin ax + c_2 a \cos ax - \frac{R}{P}$$ ...(*iii*)

Now as the end O is built in (Fig. 14.25). $\therefore\ y = dy/dx = 0$ at $x = 0$.

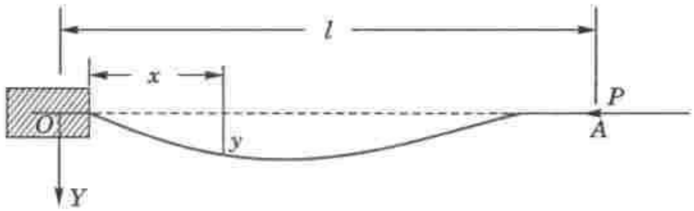

Fig. 14.25

$\therefore$ from (*ii*) and (*iii*), we have

$$0 = c_1 + Rl/P \text{ and } 0 = c_2 a - R/P$$

whence $$c_1 = -Rl/P \quad \text{and} \quad c_2 = R/aP$$

Thus (*ii*) becomes $$y = \frac{R}{P}\left(\frac{\sin ax}{a} - l \cos ax + l - x\right)$$ ...(*iv*)

which is the desired equation of the deflection curve.

The end A being freely supported $y = 0$ when $x = l$ (We don't need the other condition $d^2y/dx^2 = 0$].

$\therefore$ (*iv*) gives $$0 = \frac{R}{P}\left(\frac{\sin al}{a} - l \cos al\right) \text{ whence } al = \tan al.$$

Example 14.14. *A horizontal tie-rod is freely pinned at each end. It carries a uniform load w lb per unit length and has a horizontal pull P. Find the central deflection and the maximum bending moment, taking the origin at one of its ends.*

Solution. Let OA be the given beam of length l (Fig.14.26).

At each end there is a vertical reaction $R = wl/2$.

The external forces acting to the left of the section NN' are : (*i*) the horizontal pull P, (*ii*) the reaction $R = wl/2$ and (*iii*) the weight of the portion $ON = wx$ acting mid-way.

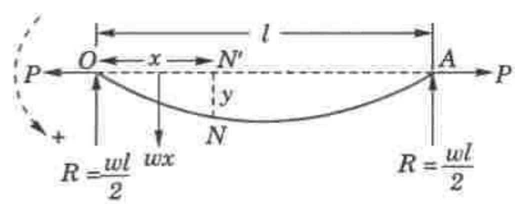

Fig. 14.26

Taking moments about, N, we have

$$EI\frac{d^2y}{dx^2} = Py - \frac{wl}{2}x + wx \cdot \frac{x}{2}$$

or $$EI\frac{d^2y}{dx^2} - Py = \frac{w}{2}(x^2 - lx) \quad \text{or} \quad \frac{d^2y}{dx^2} - a^2y = \frac{w}{2EI}(x^2 - lx), \text{ where } a^2 = \frac{P}{EI}$$ ...(*i*)

This is the differential equation of the elastic curve. Its auxiliary equation is $D^2 - a^2 = 0$, whence $D = \pm a$.

$\therefore$ $$\text{C.F.} = c_1 \cosh ax + c_2 \sinh ax$$

and $$\text{P.I.} = \frac{1}{D^2 - a^2}\frac{w}{2EI}(x^2 - lx) = \frac{-w}{2EIa^2}\left(1 - \frac{D^2}{a^2}\right)^{-1}(x^2 - lx)$$

$$= -\frac{w}{2P}\left(1 + \frac{D^2}{a^2}\ldots\right)(x^2 - lx) = -\frac{w}{2P}\left(x^2 - lx + \frac{2}{a^2}\right).$$

Thus the complete solution of (*i*) is $y = c_1 \cosh ax + c_2 \sinh ax - \dfrac{W}{2P}\left(x^2 - lx + \dfrac{2}{a^2}\right)$ ...(*ii*)

At the end O, $y = 0$ when $x = 0$, [We don't need the other condition $d^2y/dx^2 = 0$]

$\therefore$ (*ii*) gives $0 = c_1 - w/Pa^2$, or $c_1 = w/Pa^2$ ...(*iii*)

At the end A, $y = 0$ when $x = l$, [We don't need the other condition $d^2y/dx^2 = 0$]

$\therefore$ (*ii*) gives $0 = c_1 \cosh al + c_2 \sinh al - w/Pa^2$ or $c_2 \sinh al = \dfrac{W}{Pa^2}(1 - \cosh al)$

whence
$$c_2 = -\frac{w}{Pa^2}\tanh\frac{al}{2} \quad \text{...(iv)}$$

Substituting these values of c_1 and c_2 in (*ii*), we get

$$y = \frac{w}{Pa^2}\left(\cosh ax - \tanh\frac{al}{2}\sinh ax\right) - \frac{w}{2P}\left(x^2 - lx + \frac{2}{a^2}\right)$$

which gives the deflection of the beam at N.

Thus the central deflection $= y$ (at $x = l/2$)

$$= \frac{w}{Pa^2}\left(\cosh\frac{al}{2} - \tanh\frac{al}{2}\sinh\frac{al}{2} - 1\right) + \frac{wl^2}{8P} = \frac{w}{Pa^2}\left(\text{sech}\frac{al}{2} - 1\right) + \frac{wl^2}{8P}$$

Also the bending moment is maximum at the point of maximum deflection ($x = l/2$).

$\therefore$ *The maximum bending moment*

$$= EI\frac{d^2y}{dx^2}(\text{at } x = l/2) = Py + \frac{w}{2}(x^2 - lx)\ (\text{at } x = l/2) = \frac{w}{a}\left(\text{sech}\frac{al}{2} - 1\right)$$

Example 14.15. *A cantilever beam of length l and weighing w lb/unit is subjected to a horizontal compressive force P applied at the free end. Taking the origin at the free end and y-axis upwards, establish the differential equation of the beam and hence find the maximum deflection.*

Solution. Let $N(x, y)$ be any point of the beam referred to axes through the free end as shown (Fig. 14.27).

The external forces acting to the left of the section NN', are

(*i*) the compressive force P,

(*ii*) the weight of the portion $ON = wx$ acting midway.

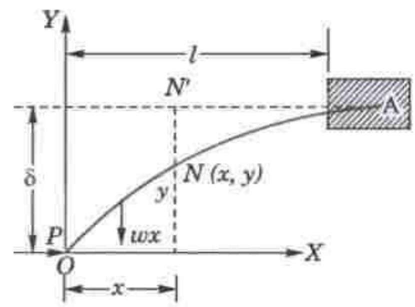

Fig. 14.27

$\therefore$ Taking moments about N, we get $EI\dfrac{d^2y}{dx^2} = -Py - wx \cdot \dfrac{x}{2}$

or
$$EI\frac{d^2y}{dx^2} + Py = -\frac{wx^2}{2} \quad \text{...(i)}$$

which is the desired differential equation.

Dividing by EI and taking $P/EI = n^2$, we get

$$\frac{d^2y}{dx^2} + n^2y = -\frac{wn^2}{2P}\cdot x^2$$

Its auxiliary equation is $D^2 + n^2 = 0$, whence $D = \pm ni$.

$$\text{C.F.} = c_1 \cos nx + c_2 \sin nx$$

$$\therefore \quad \text{P.I.} = \frac{1}{D^2 + n^2}\left(-\frac{wn^2}{2P}x^2\right) = -\frac{w}{2P}\left(1 + \frac{D^2}{n^2}\right)^{-1} x^2 = -\frac{w}{2P}\left(1 - \frac{D^2}{n^2} + \ldots\right)x^2 = \frac{w}{2P}\left(\frac{2}{n^2} - x^2\right)$$

Thus the complete solution of (*i*) is $y = c_1 \cos nx + c_2 \sin nx + \dfrac{w}{2P}\left(\dfrac{2}{n^2} - x^2\right)$ ...(*ii*)

The boundary conditions at the fixed end are

$x = l$, $y = \delta$, the maximum deflection and $dy/dx = 0$.

Using the first condition (*i.e.* $y = \delta$, when $x = l$), (*ii*) gives

$$\delta = c_1 \cos nl + c_2 \sin nl + \frac{w}{2P}\left(\frac{2}{n^2} - l^2\right) \quad \text{...(iii)}$$

Differentiating (*ii*), we get $\dfrac{dy}{dx} = n\,(-c_1 \sin nx + c_2 \cos nx) - \dfrac{wx}{P}$.

Applying the second condition, it gives $0 = n(-c_1 \sin nl + c_2 \cos nl) - wl/P$ ...(*iv*)

Also imposing the boundary condition for the free end (*i.e.* $x = 0$, $d^2y/dx^2 = 0$) on

$$\frac{d^2y}{dx^2} = -n^2(c_1 \cos nx + c_2 \sin nx) - \frac{w}{P},$$

we get
$$0 = -n^2c_1 - w/P, \text{ i.e., } c_1 = -w/Pn^2.$$

Substituting this value of c_1 in (*iv*), we get $c_2 = \dfrac{wl}{Pn} \sec nl - \dfrac{w}{Pn^2} \tan nl$

Thus, substituting the values of c_1 and c_2 in (*iii*), we get

the *maximum deflection* $\delta = \dfrac{w}{Pn^2}\left(1 - \dfrac{l^2n^2}{2} - \sec nl + nl \tan nl\right)$.

14.8 WHIRLING OF SHAFTS

(1) Critical or whirling speeds. A shaft seldom rotates about its geometrical axis for there is always some non-symmetrical crookedness in the shaft. In fact, the dead weight of the shaft causes some deflection which tends to become large at certain speeds. Such speeds at which the deflection of the shaft reaches a stage, where the shaft will fracture unless the speed is lowered are called the *critical or whirling speeds* of the shaft.

(2) Differential equation of the rotating shaft. Consider a shaft of weight W per unit length which is rotating with angular velocity ω.

Take its original horizontal position and the vertical downwards through the end O as the axes of x and y (Fig. 14.28). We know that for a uniformly loaded beam, the intensity of loading at $P(x, y) = EI\, d^4y/dx^4$.

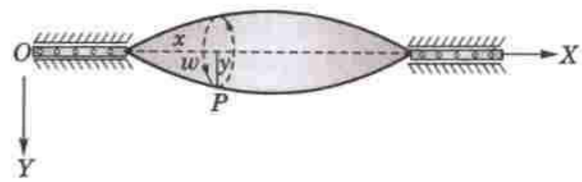

Fig. 14.28

$\therefore$ the restoring force (*i.e.* the internal action to oppose bending at $P(x, y) = EI\, d^4y/dx^4$.

Also the centrifugal force per unit length at $P = mr\omega^2$, *i.e.* $\dfrac{Wy}{g}\,\omega^2$.

As the restoring force arising out of the rigidity or stiffness of the shaft balances the centrifugal force which causes further deflection.

$$\therefore \qquad EI\,\frac{d^4y}{dx^4} = \frac{W}{g}\,y\omega^2 \quad \text{or} \quad \frac{d^4y}{dx^4} - a^4y = 0, \text{ where } a^4 = \frac{W\omega^2}{gEI}$$

which is the desired differential equation.

Its auxiliary equation being $D^4 - a^4 = 0$, we have

$$D = \pm a, \pm ai.$$

Hence its solution is $y = c_1e^{ax} + c_2e^{-ax} + c_3 \cos ax + c_4 \sin ax$ which may be put in the form

$$y = A \cosh ax + B \sinh ax + C \cos ax + D \sin ax.$$

(3) End conditions. To determine the arbitrary constants A, B, C, D we use the following end conditions :

(*i*) *At an end in a short or flexible bearings* (Fig. 14.29), there being no deflection and also no bending moment, we have

$$y = 0 \text{ and } \frac{d^2y}{dx^2} = 0.$$

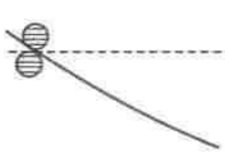
Fig. 14.29

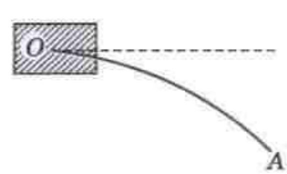

Fig. 14.30

(*ii*) *At an end in long or fixed bearings* (Fig 14.30), the deflection and the slope of the shaft being both zero, we have

$$y = 0 \text{ and } \frac{dy}{dx} = 0.$$

(*ii*) *At a perfectly free end* (such as *A* in Fig. 14.30), there being no bending moment and no shear force, we have

$$\frac{d^2y}{dx^2} = 0 \text{ and } \frac{d^3y}{dx^3} = 0.$$

Example 14.16. *The differential equation for the displacement y of a whirling shaft when the weight of the shaft is taken into account is*

$$EI\frac{d^4y}{dx^4} - \frac{W\omega^2}{g}y = W.$$

Taking the shaft of length 2l with the origin at the centre and short bearings at both ends, show that the maximum deflection of the shaft is

$$\frac{g}{2\omega^2}(\text{sech } al + \sec al - 2).$$

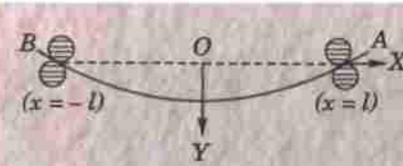

Fig. 14.31

Solution. Given differential equation can be written as

$$\frac{d^4y}{dx^4} - a^4y = \frac{W}{EI}, \text{ where } a^4 = \frac{W\omega^2}{EIg} \quad ...(i)$$

Its C.F. = $A \cosh ax + B \sinh ax + C \cos ax + D \sin ax$

and $$\text{P.I.} = \frac{1}{D^4 - a^4} \cdot \frac{W}{EI} = \frac{W}{EI} \cdot \frac{1}{D^4 - a^4} e^{0.x} = -\frac{W}{EIa^4} = -\frac{g}{\omega^2}$$

Thus the complete solution of (*i*) is

$$y = A \cosh ax + B \sinh ax + C \cos ax + D \sin ax - \frac{g}{\omega^2} \quad ...(ii)$$

Differentiating it twice, we get

$$\frac{1}{a}\frac{dy}{dx} = A \sinh ax + B \cosh ax - C \sin ax + D \cos ax$$

$$\frac{1}{a^2}\frac{d^2y}{dx^2} = A \cosh ax + B \sinh ax - C \cos ax - D \sin ax \quad ...(iii)$$

As the end A of the shaft is in short bearings (Fig. 14.31)

∴ when $x = l$; $y = 0$, $d^2y/dx^2 = 0$

∴ from (*ii*) and (*iii*), we have

$$0 = A \cosh al + B \sinh al + C \cos al + D \sin al - \frac{g}{\omega^2} \quad ...(iv)$$

$$0 = A \cosh al + B \sinh al - C \cos al - D \sin al \quad ...(v)$$

Similarly *at the end B*, $x = -l$, $y = 0$, $d^2y/dx^2 = 0$.

∴ from (*ii*) and (*iii*), we get

$$0 = A \cosh al - B \sinh al + C \cos al - D \sin al - \frac{g}{\omega^2} \quad ...(vi)$$

$$0 = A \cosh al - B \sinh al - C \cos al + D \sin al \quad ...(vii)$$

Adding (*iv*) and (*vi*), and (*v*) and (*vii*), we get

$$A \cosh al + C \cos al = \frac{g}{\omega^2} \quad \text{and} \quad A \cosh al - C \cos al = 0.$$

whence $$A = \frac{g}{2\omega^2 \cosh al} \text{ and } C = \frac{g}{2\omega^2 \cos al}$$

Again subtracting (*vi*) from (*iv*) and (*vii*) from (*v*), we get

$D \sinh al + D \sin al = 0$ and $B \sinh al - D \sin al = 0$, whence $B = 0$ and $D = 0$.

Substituting the values of A, B, C and D in (*ii*), we get

$$y = \frac{g}{2\omega^2}\left[\frac{\cosh ax}{\cosh al} + \frac{\cos ax}{\cos al} - 2\right]$$

Thus the maximum deflection = value of y at the centre (x = 0)

$$= \frac{g}{2\omega^2}(\text{sech } al + \sec al - 2).$$

Example 14.17. *The whirling speed of a shaft of length l is given by*

$$\frac{d^4y}{dx^4} - m^4y = 0 \text{ where } m^4 = \frac{W\omega^2}{gEI},$$

and y is the displacement at distance x from one end. If the ends of the shaft are constrained in long bearings, show that the shaft will whirl when $\cos ml \cosh ml = 1$.

Solution. The solution of the given differential equation is

$$y = A \cosh mx + B \sinh mx + C \cos mx + D \sin mx \quad ...(i)$$

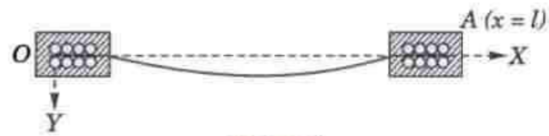

Fig. 14.32

which on differentiation gives,

$$\frac{1}{m}\frac{dy}{dx} = A \sinh mx + B \cosh mx - C \sin mx + D \cos mx \quad ...(ii)$$

As the end O of the shaft is fixed in long bearings (Fig. 14.32).

$\therefore$ when $x = 0$, $y = 0$, $dy/dx = 0$,

$\therefore$ from (*i*) and (*ii*), we have

$$0 = A + C \quad \text{or} \quad C = -A \quad ...(iii)$$

and

$$0 = B + D \quad \text{or} \quad D = -B \quad ...(iv)$$

Similarly, at the end A, $x = l$, $y = 0$, $dy/dx = 0$.

$\therefore$ From (*i*) and (*ii*), we have

$$0 = A \cosh ml + B \sinh ml + C \cos ml + D \sin ml \quad ...(v)$$

$$0 = A \sinh ml + B \cosh ml - C \sin ml + D \cos ml \quad ...(vi)$$

Substituting the values of C and D in (*v*) and (*vi*), we get

$$A(\cosh ml - \cos ml) + B(\sinh ml - \sin ml) = 0$$

and

$$A(\sinh ml + \sin ml) + B(\cosh ml - \cos ml) = 0$$

Eliminating A and B from these equations, we get

$$\frac{\cosh ml - \cos ml}{\sinh ml - \sin ml} = -\frac{B}{A} = \frac{\sinh ml + \sin ml}{\cosh ml - \cos ml}$$

or $\quad \cosh^2 ml - 2 \cosh ml \cos ml + \cos^2 ml = \sinh^2 ml - \sin^2 ml$

or $\quad -2 \cosh ml \cos ml + 2 = 0$ or $\cos ml \cosh ml = 1$

which must be satisfied when the shaft whirls.

The solution of this equation gives $ml = 4.73 = 3\pi/2$ radians approximately.

$$\therefore \quad \omega\sqrt{\left(\frac{W}{gEI}\right)}\, l^2 = m^2l^2 = \frac{9\pi^2}{4}$$

Thus the *whirling speed of a shaft with ends in long bearings.*

$$= \omega = \frac{9\pi^2}{4l^2}\sqrt{\left(\frac{gEI}{W}\right)} \text{ approximately.}$$

Obs. 1. *When the shaft has one long bearing and the other short bearing*, the condition to be satisfied is $\tan ml = \tanh ml$, of which the solution is $ml = 3.927$

or $$\omega\sqrt{\left(\frac{W}{gEI}\right)}\cdot l^2 = m^2l^2 = (3.927)^2 = 15.4 \text{ nearly.}$$

Thus *the whirling speed* $= \omega = \dfrac{15.4}{l^2}\sqrt{\left(\dfrac{gEI}{W}\right)}$

Obs. 2. *When the shaft has both short bearings,* the condition to be satisfied is $\sin ml = 0$ *i.e.* $ml = \pi$ (least non-zero value).

$\therefore$ $$\omega\sqrt{\left(\frac{W}{gEI}\right)}\cdot l^2 = m^2 l^2 = \pi^2. \text{ Thus } \textit{the whirling speed} = \omega = \frac{\pi^2}{l^2}\sqrt{\left(\frac{gEI}{W}\right)}.$$

Obs. 3. *When the shaft has one long bearing,* the condition to the satisfied is $\cos ml \cosh ml = -1$.

Its solution gives $ml = 1.865$ [See Example 1.25]

$\therefore$ $$\omega\sqrt{\left(\frac{W}{gEI}\right)}\cdot l^2 = m^2l^2 = (1.865)^2 = 3.5 \text{ nearly. Thus } \textit{the whirling speed } \omega = \frac{3.5}{l^2}\sqrt{\left(\frac{gEI}{W}\right)}.$$

PROBLEMS 14.4

1. A horizontal tie-rod of length $2l$ with concentrated load W at the centre and ends freely hinged, satisfies the differential equation $EI\dfrac{d^2y}{dx^2} = Py - \dfrac{W}{2}x$. With conditions $x = 0, y = 0$ and $x = l$, $dy/dx = 0$, prove that the deflection δ and the bending moment M at the centre $(x = l)$ are given by $\delta = \dfrac{W}{2Pn}(nl - \tanh nl)$ and $M = -\dfrac{W}{2n}\tanh nl$, where $n^2EI = P$.
2. A light horizontal strut AB is freely pinned at A and B. It is under the action of equal and opposite compressive forces P at its ends and it carries a load W at its centre. Then for $0 < x < l/2$, $EI\dfrac{d^2y}{dx^2} + Py + \dfrac{1}{2}Wx = 0$. Also $y = 0$ at $x = 0$ and $dy/dx = 0$ at $x = l/2$.

   Prove that $y = \dfrac{W}{2P}\left(\dfrac{\sin nx}{n\cos nl/2} - x\right)$ where $n^2 = \dfrac{P}{EI}$.
3. A uniform horizontal strut of length l freely supported at both ends, carries a uniformly distributed load W per unit length. If the thrust at each end is, P, prove that the maximum deflection is $\dfrac{W}{Pa^2}\left(\sec\dfrac{al}{2} - 1\right) - \dfrac{Wl^2}{8P}$, where $\dfrac{P}{EI} = a^2$.

   Prove also that the maximum bending moment is of the magnitude $\dfrac{W}{a^2}\left(\sec\dfrac{al}{2} - 1\right)$.
4. The shape of a strut of length l subjected to an end thrust P and lateral load w per unit length, when the ends are built in, is given by $EI\dfrac{d^2y}{dx^2} + Py = \dfrac{wx^2}{2} - \dfrac{wlx}{2} + M$, where M is the moment at a fixed end. Find y in terms of x, given that $y = 0$, $dy/dx = 0$ at $x = 0$ and $dy/dx = 0$ at $x = l/2$.
5. A light horizontal strut of length l is clamped at one end carries a vertical load W at the free end. If the horizontally thrust at the free end is P, show that the strut satisfies the differential equation

   $El\dfrac{d^2y}{dx^2} = (\delta - y)P + W(l - x)$, where y is the displacement of a point at a distance x from the fixed end and δ, the deflection at the free end.

   Prove that the deflection at the free end is given by $\dfrac{W}{nP}(\tan nl - nl)$, where $n^2EI = P$.
6. A long column fixed at one end $(x = 0)$ and hinged at the other $(x = l)$ is under the action of axial load P. If a force F is applied laterally at the hinge to prevent lateral movement, show that it satisfies the equation $\dfrac{d^2y}{dx^2} + n^2y = \dfrac{En^2}{P}(l - x)$, where $EIn^2 = P$. Hence determine the equation of the deflection curve.
7. A long column of length l is fixed at one end and is completely free at the other end. If y is the lateral deflection at a point distance x from the fixed end, when load P is axially applied, find the differential equation satisfied by x and y. Show that the deflection curve is given by $y = a\{1 - \cos\sqrt{P/EI}\,x\}$ and find the least value of the critical load (a is the lateral deflection of the free end).

8. The differential equation for the displacement y of a heavy whirling shaft is $\frac{d^4y}{dx^4} = a^4\left(y + \frac{g}{\omega^2}\right)$, where $a^4 = \frac{W\omega^2}{gEI}$. If both ends are in short bearings, the ends being $x = 0$ and $x = l$, find the bending moment of the centre of the shaft.

14.9 APPLICATIONS OF SIMULTANEOUS LINEAR EQUATIONS

So far we have considered engineering systems having only one degree of freedom. The analysis of a system having more than one degree of freedom depends on the solution of simultaneous linear equations. In fact such equations form the basis of the theory of projectiles and the coupled circuits having self and mutual inductance. The details of such applications are best explained through the following examples :

Example 14.18. Projectile with resistance. *Find the path of a particle projected with a velocity v at an angle α to the horizon in a medium whose resistance, apart from gravity, varies as velocity. Also find the greatest height attained.*

Solution. Let the axes of x and y be respectively horizontal and vertical with origin at the point of projection (Fig. 14.33).

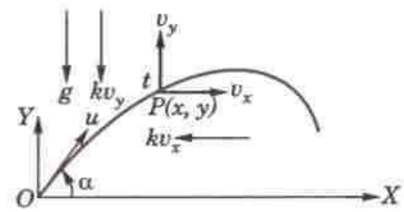

Fig. 14.33

Let $P(x, y)$ be the position of the projectile at the time t, where the velocity components parallel to the axes are

$$v_x = \frac{dx}{dt}, v_y = \frac{dy}{dt}$$

$\therefore$ the equations of motion are:

Parallel to x-axis	*Parallel to y-axis*
$m\frac{dv_x}{dt} = -mkv_x$	$m\frac{dv_y}{dt} = -mg - mkv_y$
or $\frac{dv_x}{dt} = -kv_x$	$\frac{dv_y}{dt} = -(g + kv_y)$

Separating the variables and integrating, we have

$\int \frac{dv_x}{v_x} = -k\int dt + c_1$	$\frac{dv_y}{g + kv_y} = -\int dt + c_2$
or $\log v_x = -kt + c_1$	$\frac{1}{k}\log(g + kv_y) = -t + c_2.$

Initially when $t = 0$, $v_x = u\cos\alpha$, $v_y = u\sin\alpha$.

$\log u\cos\alpha = c_1$	$\frac{1}{k}\log(g + ku\sin\alpha) = c_2$

Subtracting,

$\log\left(\frac{v_x}{u\cos\alpha}\right) = -kt$	$\frac{1}{k}\log\left(\frac{g + kv_y}{g + ku\sin\alpha}\right) = -t$
or $\frac{dx}{dt} = v_x = u\cos\alpha e^{-kt}$ ...(*i*)	$\frac{dy}{dt} = v_y = \frac{1}{k}[(g + ku\sin\alpha)e^{-kt} - g]$ ...(*ii*)

Again integrating, we get

$$x = \frac{u\cos\alpha}{-k}e^{-kt} + c_3, y = -\frac{1}{k}\left(\frac{g}{k} + u\sin\alpha\right)e^{-kt} - \frac{g}{k}t + c_4$$

Initially when $t = 0, x = 0, y = 0,$

$$\therefore \quad 0 = \frac{u\cos\alpha}{k} + c_3, 0 = -\frac{1}{k}\left(\frac{g}{k} + u\sin\alpha\right) + c_4$$

Subtracting, we get $\quad x = \frac{u\cos\alpha}{k}(1 - e^{-kt})$ ...(*iii*)

$$y = \frac{1}{k}\left(\frac{g}{k} + u \sin \alpha\right)\left(1 - e^{-kt}\right) - \frac{gt}{k} \quad ...(iv)$$

Eliminating t from (iii) and (iv), we obtain $y = \left(\frac{g}{k} + u \sin \alpha\right)\frac{x}{u \cos \alpha} + \frac{g}{k^2}\log\left(1 - \frac{kx}{u \cos \alpha}\right)$

which is the required equation of the trajectory.

The projectile will attain the greatest height when $dy/dt = 0$.

i.e., when $e^{-kt} = g/(g + ku \sin \alpha)$, *i.e.*, at time $t = \frac{1}{k}\log\left(1 + \frac{ku \sin \alpha}{g}\right)$. [From (ii)]

Subtituting the value of t in (iv), we get the greatest height attained

$$(= y) = \frac{u \sin \alpha}{k} - \frac{g}{k^2}\log\left(1 + \frac{ku \sin \alpha}{g}\right).$$

Example 14.19. *Two particles each of mass m gm are suspended from two springs of same stiffness k as in Fig. 14.34. After the system comes to rest, the lower mass is pulled l cm downwards and released. Discuss their motion.*

Solution. Let x and y denote the displacement of the upper and lower masses at time t from their respective positions of equilibrium.

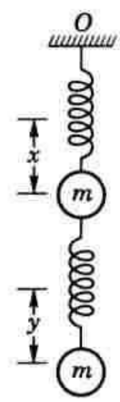

Fig. 14.34

Then the stretch of the upper spring is x and that of the lower spring is $y - x$.

$\therefore$ the restoring force acting on the upper mass

$$= -kx + k(y - x) = k(y - 2x)$$

and that on the lower mass $= -k(y - x)$.

Thus their equations of motion are

$$m\,\frac{d^2x}{dt^2} = k(y - 2x) \text{ and } m\,\frac{d^2y}{dt^2} = -k(y - x)$$

or $\quad (mD^2 + 2k)x - ky = 0 \quad ...(i)$

and $\quad (mD^2 + k)y - kx = 0 \quad ...(ii)$

Operating (i) by $(mD^2 + k)$ and adding to k times (ii), we get

$$[(mD^2 + k)(mD^2 + 2k) - k^2]\,x = 0 \text{ or } (D^4 + 3\lambda D^2 + \lambda^2)x = 0, \text{ where } \lambda^2 = k/m.$$

Its auxiliary equation is $D^4 + 3\lambda D^2 + \lambda^2 = 0$

which gives $\quad D^2 = \dfrac{-3\lambda \pm \sqrt{(9\lambda^2 - 4\lambda^2)}}{2} = -2.62\lambda \text{ or } -0.38\lambda = -\alpha^2, -\beta^2$ (say)

so that $D = \pm i\alpha, \pm i\beta$.

Thus $\quad x = c_1 \cos \alpha t + c_2 \sin \alpha t + c_3 \cos \beta t + c_4 \sin \beta t \quad ...(iii)$

Also from (i), $y = \left(\dfrac{D^2}{\lambda} + 2\right)x = (2 - \alpha^2/\lambda)(c_1 \cos \alpha t + c_2 \sin \alpha t) + (2 - \beta^2/\lambda)(c_3 \cos \beta t + c_4 \sin \beta t) \quad ...(iv)$

Initially when $\quad t = 0, x = y = l, dx/dt = dy/dt = 0$.

$\therefore$ from (iii), $\quad l = c_1 + c_3;\ 0 = \alpha c_2 + \beta c_4$

and from (iv) $\quad l = (2 - \alpha^2/\lambda)c_1 + (2 - \beta^2/\lambda)c_3$ and $0 = (2 - \alpha^2/\lambda)\alpha c_2 + (2 - \beta^2/\lambda)\beta c_4$

whence $\quad c_1 = \dfrac{l(\lambda - \beta^2)}{\alpha^2 - \beta^2},\ c_3 = \dfrac{l(\lambda - \alpha^2)}{\beta^2 - \alpha^2},\ c_3 = c_4 = 0.$

Substituting these values of constants in (iii) and (iv), we get x and y which show that the motion of the spring is a combination of two simple harmonic motions of periods $2\pi/\alpha$ and $2\pi/\beta$.

Example 14.20. *Two coils of a transformer are identical with resistance R, inductance L, mutual inductance M and a voltage E is impressed on the primary. Determine the currents in the coils at any instant, assuming that there is no current in either initially.*

Solution. Let i_1, i_2 ampere be the currents flowing through the primary and secondary coils at time t sec (Fig. 14.35). Then by Kirchhoff's law, we know that sum of the voltage drops across R, L and M = applied voltage.

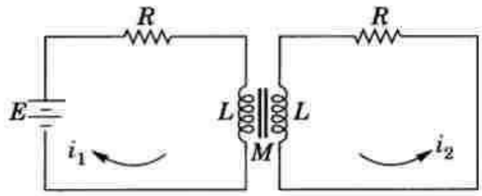

Fig. 14.35

$\therefore$ for the primary circuit,

$$Ri_1 + L\frac{di_1}{dt} + M\frac{di_2}{dt} = E$$

and for the secondary circuit, $Ri_2 + L\dfrac{di_2}{dt} + M\dfrac{di_1}{dt} = 0.$

Replacing d/dt by D and rearranging the terms,

$$(LD + R)i_1 + MDi_2 = E \qquad ...(i)$$

$$MDi_1 + (LD + R)i_2 = 0 \qquad ...(ii)$$

Eliminating i_2, we get $[(LD + R)^2 - M^2D^2]i_1 = (LD + R)E$

i.e., $$[(L^2 - M^2)D^2 + 2LRD + R^2]i_1 = RE \qquad ...(iii)$$

Its auxiliary equation is $(L^2 - M^2)D^2 + 2LRD + R^2 = 0$ whence $D = \dfrac{-R}{L+M}, \dfrac{-R}{L-M}$.

As L is usually $> M$, therefore, both values of D are negative and real.

$\therefore$ $$\text{C.F.} = c_1 e^{-\frac{Rt}{L+M}} + c_2 e^{-\frac{Rt}{L-M}} \text{ and P.I.} = RE \cdot \frac{1}{(L^2 - M^2)D^2 + 2LRD + R^2} e^{0.t} = E/R.$$

Thus the complete solution of (iii) is $i_1 = c_1 e^{-Rt/(L+M)} + c_2 e^{-Rt/(L-M)} + E/R \qquad ...(iv)$

and from (ii), we have $i_2 = -\dfrac{MD}{LD+R} i_1$

$$= -\frac{MD}{LD+R}\left(c_1 e^{-Rt/(L+M)} + c_2 e^{-Rt/(L-M)}\right) - \frac{MD}{LD+R}\left(\frac{E}{R}\right)$$

$$= -\frac{Mc_1}{L\left(\dfrac{-R}{L+M}\right) + R} \cdot De^{-Rt/(L+M)} - \frac{Mc_2}{L\left(\dfrac{-R}{L-M}\right) + R} \cdot De^{-Rt/(L-M)}$$

$$= c_1 e^{-Rt/(L+M)} - c_2 e^{-Rt/(L-M)}$$

Initially, when $t = 0, i_1 = i_2 = 0$.

$\therefore$ $c_1 + c_2 = -E/R, c_1 - c_2 = 0 \qquad \therefore c_1 = c_2 = -E/2R.$

Substituting the values of c_1, c_2 in (iv) and (v), we get

$$i_1 = \frac{E}{2R}\left[2 - e^{-Rt/(L+M)} - e^{-Rt/(L-M)}\right] \qquad ...(vi)$$

and $$i_2 = \frac{E}{2R}\left[e^{-Rt/(L-M)} - e^{-Rt/(L+M)}\right] \qquad ...(vii)$$

Thus (vi) and (vii) give the currents at any instant.

PROBLEMS 14.5

1. A particle is projected with velocity u, at an elevation α. Neglecting air resistance, show that the equation to its path is the parabola $y = x \tan\alpha - \dfrac{gx^2}{2u^2\cos^2\alpha}$. Also find the time of flight and range on the horizontal plane.
2. An inclined plane makes angle α with the horizontal. A projectile is launched from the bottom of the inclined plane with speed V in a direction making angle β with the horizontal. Set up the differential equations and find (i) the range on the incline, (ii) the maximum range up the incline.
3. A particle of unit mass is projected with velocity u at an inclination α above the horizon in a medium whose resistance is k times the velocity. Show that its direction will again make an angle α with the horizon after a time $\dfrac{1}{k}\log\left\{1 + \dfrac{2ku}{g}\sin\alpha\right\}$.

4. A particle moving in a plane is subjected to a force directed towards a fixed point O and proportional to the distance of the particle from O. Show that the differential equations of motion are of the form $\frac{d^2x}{dt^2} = -k^2x$, $\frac{d^2y}{dt^2} = -k^2y$. Find the cartesian equation of the path of the particle if $x = 1$, $y = 0$, $\frac{dx}{dt} = 0$ and $dy/dt = 2$, when $t = 0$.
5. The currents i_1 and i_2 in mesh are given by the differential equations
$\frac{di_1}{dt} - \omega i_2 = a \cos pt$, $\frac{di_2}{dt} + \omega i_1 = a \sin pt$. Find the currents i_1 and i_2 if $i_1 = i_2 = 0$ at $t = 0$.
6. The currents i_1 and i_2 in two coupled circuits are given by
$$L\frac{di_1}{dt} + Ri_1 + R(i_1 - i_2) = E;\ L\frac{di_2}{dt} + Ri_2 - R(i_1 - i_2) = 0,$$
where L, R, E are constants. Find i_1 and i_2 in terms of t given that $i_1 = i_2 = 0$ at $t = 0$.
7. The motion of a particle is governed by the equations
$\frac{d^2x}{dt^2} - n\frac{dy}{dt} = 0$, $\frac{d^2y}{dt^2} + n\frac{dx}{dt} = n^2a$, when $x = y = \frac{dx}{dt} = \frac{dy}{dt} = 0$ at $t = 0$. Find x and y in terms of t.
8. Under certain conditions, the motion of an electron is given by the equations $m\frac{d^2x}{dt^2} + eH\frac{dy}{dt} = eE$ and $m\frac{d^2y}{dt^2} - eH\frac{dx}{dt} = 0$. Find the path of the electron, if it started from rest at the origin.
9. The voltage V and the current i at a distance x from the source satisfy the equations $-dV/dt = Ri$, $-di/dx = GV$, where R, G are constants. If $V = V_0$ at $x = 0$ and $V = 0$ at the receiving end $x = l$, show that $V = V_0 \sinh n(l - x)/\sinh nl$, $i = V_0\sqrt{(G/R)}.\cosh n(l - x)/\sinh nl$, where $n^2 = RG$.

14.10 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 14.6

Fill up the blanks or choose the correct answer in the following problems:

1. A particle executing simple harmonic motion of amplitude 5 cm has a speed of 8 cm/sec when at a distance of 3 cm from the centre of the path. The period of the motion of the particle will be
(*a*) $\pi/2$ sec (*b*) π sec (*c*) 2π sec (*c*) 4π sec.
2. A ball of mass m is suspended from a fixed point O by a light string of natural length l and modulus of elasticity λ. If the ball is displaced vertically, its motion will be S.H.M. of period
(*a*) $2\pi\sqrt{(m/\lambda l)}$ (*b*) $2\pi\sqrt{(ml/\lambda)}$ (*c*) $2\pi\sqrt{(l/m\lambda)}$ (*d*) $2\pi\sqrt{(\lambda m/l)}$.
3. The periodic time of the motion described by the differential equation $\frac{d^2x}{dt^2} + 4x = 0$ is
(*a*) $\pi/2$ (*b*) π (*c*) 2π.
4. A particle is projected with a velocity u at an angle of 60° to the horizontal. The time of flight of the projectile is equal to
(*a*) $\sqrt{3u/2g}$ (*b*) $\sqrt{3u/g}$ (*c*) u/g (*d*) $u/2g$.
5. A body of 6.5 kg is suspended by two strings of lengths 5 and 12 metres attached to two points in the same horizontal line whose distance apart is 13 meters. The tension of the strings are
(*a*) 2 kg & 6.5 kg (*b*) 2.5 kg & 6 kg (*c*) 2.25 kg & 6.25 kg (*d*) 3 kg & 5.5 kg.
6. A particle is projected at an angle of 30° to the horizontal with a velocity of 1962 cm/sec then the time of flight is
(*a*) 1 sec (*b*) 2 sec (*c*) 2.5 sec (*d*) 3 sec.
7. A point moves with S.H.M. whose period is 4 seconds. If it starts from rest at a distance of 4 meters from the centre of its path, then the time it takes, before it has described 2 metres is
(*a*) $\frac{1}{3}$ second (*b*) $\frac{2}{3}$ second (*c*) $\frac{3}{4}$ second (*d*) $\frac{4}{5}$ second.

8. If the length of the pendulum of a clock be increased in the ratio 720 : 721, it would loose .... seconds per day.
9. The frequency of free vibrations in a closed circuit with inductance L and capacity C in series is .... per minute.
10. If a clock with a seconds pendulum loses 10 seconds per day at a place having $g = 32$ ft/sec^2, g should be increased by .... ft/sec^2, to keep correct time.
11. The soldiers break step while marching over a bridge for the fear that their steps may not be in rhym with the natural frequency of the bridge causing its collapse due to ....
12. A horizontal tie-rod is freely pinned at each end. If it carries a uniform load w lb per unit length and has a horizontal pull P, then the differential equation of the elastic curve is ....
13. The conditions for an end of a whirling shaft to be in fixed bearings are .... and ....

CHAPTER 15

Differential Equations of Other Types

1. Introduction. 2. Equations of the form $d^2y/dx^2 = f(x)$. 3. Equations of the form $d^2y/dx^2 = f(y)$. 4. Equations which do not contain y. 5. Equations which do not contain x. 6. Equations whose one solution is known. 7. Equations which can be solved by changing the independent variable. 8. Total differential equation : $Pdx + Qdy + Rdz = 0$. 9. Simultaneous total differential equations. 10. Equations of the form $dx/P = dy/Q = dz/R$.

15.1 INTRODUCTION

In this chapter, we propose to study some other important types of ordinary differential equations which require special methods for their solution and have varied applications as illustrated side by side.

15.2 EQUATIONS OF THE FORM $d^2y/dx^2 = f(x)$

Integrating with respect to x, we have $\dfrac{dy}{dx} = \int f(x)dx + c = F(x)$. (say)

Again integrating, we get $y = \int F(x)dx + c'$ as the required solution.

In general, the solution of the equations of the form $\dfrac{d^n y}{dx^n} = f(x)$ is obtained by integrating it n times successively.

Example 15.1. *Solve* $\dfrac{d^2 y}{dx^2} = xe^x$.

Solution. Integrating, we get $\dfrac{dy}{dx} = xe^x - \int e^x\, dx + c_1 = (x-1)\, e^x + c_1$

Again integrating, we get

$$y = (x-1)e^x - \int e^x dx + c_1 x + c_2 = (x-2)e^x + c_1 x + c_2.$$

PROBLEMS 15.1

Solve :

1. $\dfrac{d^2 y}{dx^2} = x^2 \sin x$.

2. $\dfrac{d^3 y}{dx^3} = x + \log x$.

3. A beam of length $2l$ with uniform load w per unit length is freely supported at both ends. Prove that the maximum deflection of the beam is $\dfrac{5wl^4}{24EI}$.

[**Hint.** Taking the origin at the left end, we have $EI\, d^4y/dx^4 = w$. At each end, $y = 0$ and $d^2y/dx^2 = 0$.]

4. For a cantilever beam of length l with a uniform load of w per unit length, show that the maximum deflection at the free end is wl^4/EI, where the symbols have the usual meaning.

15.3 EQUATIONS OF THE FORM $d^2y/dx^2 = f(y)$

Multiplying both sides by $2dy/dx$, we have $2\dfrac{dy}{dx}\cdot\dfrac{d^2y}{dx^2} = 2f(y)\dfrac{dy}{dx}$

Integrating with respect to x, $\left(\dfrac{dy}{dx}\right)^2 = 2\int f(y)\, dy + c = F(y)$ (say)

or
$$\frac{dy}{dx} = \sqrt{[F(y)]}$$

Separating the variables and integrating, we get $\displaystyle\int \frac{dy}{\sqrt{[F(y)]}} = x + c$, whence follows the desired solution.

Such equations occur quite frequently in Dynamics.

Example 15.2. *Solve $d^2y/dx^2 = 2(y^3 + y)$ under the conditions $y = 0$, $dy/dx = 1$, when $x = 0$.* *(U.P.T.U., 2003)*

Solution. Multiplying by $2\, dy/dx$, the given equation becomes

$$2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} = 4(y^3 + y)\,\frac{dy}{dx}$$

Integrating w.r.t. x, $\left(\dfrac{dy}{dx}\right)^2 = 4\left(\dfrac{y^4}{4} + \dfrac{y^2}{2}\right) + c = y^4 + 2y^2 + c$ ...(*i*)

As $dy/dx = 1$ for $y = 0$, $\therefore\; c = 1$

$\therefore$ (*i*) takes the form $(dy/dx)^2 = y^4 + 2y^2 + 1 = (y^2 + 1)^2$ or $dy/dx = y^2 + 1$

Separating the variables and integrating, we have $\displaystyle\int \frac{dy}{1 + y^2} = \int dx + c'$

or
$$\tan^{-1} y = x + c' \qquad ...(ii)$$

Thus (*ii*) becomes $\tan^{-1} y = x$ or $y = \tan x$ which is the required solution.

Example 15.3. *A point moves in a straight line towards a centre of force $\mu/(\text{distance})^3$, starting from rest at a distance 'a' from the centre of force, show that the time of reaching a point distant 'b' from the centre of force is $\dfrac{a}{\sqrt{\mu}}\sqrt{(a^2 - b^2)}$ and that its velocity is $\dfrac{\sqrt{\mu}}{ab}\sqrt{(a^2 - b^2)}$.* *(U.P.T.U., 2001)*

Solution. Let O be the centre of force and A the point of start so that $OA = a$. At any time t, let the point be at P where $OP = x$ so that

$$\frac{d^2x}{dt^2} = \frac{-\mu}{x^3} \qquad ...(i)$$

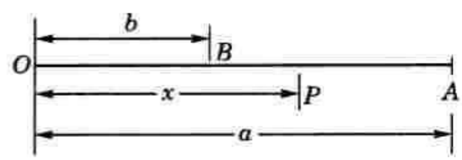

Fig. 15.1

Multiplying both sides by $2\, dx/dt$, we get

$$\frac{2dx}{dt}\cdot\frac{d^2x}{dt^2} = -\frac{\mu}{x^3}\cdot\frac{2dx}{dt}$$

Integrating both sides, we obtain

$$\left(\frac{dx}{dt}\right)^2 = -\mu \int \frac{2}{x^3}\frac{dx}{dt}\,.\,dt + c = +\frac{\mu}{x^2} + c$$

When $x = a$, velocity $dx/dt = 0$. $\therefore$ $c = -\mu/a^2$.

$$\therefore \qquad \left(\frac{dx}{dt}\right)^2 = \mu\left(\frac{1}{x^2} - \frac{1}{a^2}\right) = \frac{\mu(a^2 - x^2)}{a^2x^2} \qquad ...(ii)$$

At B $(x = b)$, velocity towards $O = \dfrac{\sqrt{\mu(a^2 - b^2)}}{ab}$

Again (ii) can be rewritten as $\dfrac{-ax\,dx}{\sqrt{(a^2 - x^2)}} = \sqrt{\mu}\,dt$ [– ve is taken since point is moving towards O]

Integrating both sides, we get

$$\sqrt{\mu}\int dt = -\int \frac{ax\,dx}{\sqrt{(a^2 - x^2)}} + c' \quad \text{or} \quad \sqrt{\mu}t = a\sqrt{(a^2 - x^2)} + c' \qquad ...(iii)$$

Since $t = 0$ at $x = a$, $\therefore$ $c' = 0$

Thus (iii) gives $t = \dfrac{a}{\sqrt{\mu}}\sqrt{(a^2 - x^2)}$

Hence at B $(x = b)$ $t = \dfrac{a}{\sqrt{\mu}}\sqrt{(a^2 - b^2)}$.

PROBLEMS 15.2

Solve :

1. $d^2y/dx^2 = 3\sqrt{y}$ given that $y = 1$, $dy/dx = 2$ when $x = 0$.
2. $\dfrac{d^2y}{dx^2} = \dfrac{36}{y^2}$, given that when $x = 0$, $\dfrac{dy}{dx} = 0$, $y = 8$.
3. If $d^2r/dt^2 = \omega^2 r$, find the value of r in terms of t, if $r = a$ and $dr/dt = v$, when $t = 0$.
4. The motion of a particle let fall from a point outside the earth is given by $d^2x/dt^2 = -ga^2/x^2$. Given that $x = h$ and $dx/dt = 0$, when $t = 0$, find t in terms of x.
5. A particle is acted upon by a force $\mu\,(x + a^4/x^3)$ per unit mass towards the origin, where x is the distance from the origin at time t. If it starts from rest at a distance a, show that it will arrive at the origin in time $\pi/(4\sqrt{\mu})$.

15.4 EQUATIONS WHICH DO NOT CONTAIN y

A second order equation of this form is

$$f(d^2y/dx^2, dy/dx, x) = 0$$

On putting $dy/dx = p$ and $d^2y/dx^2 = dp/dx$, it becomes

$$f(dp/dx, p, x) = 0.$$

This is an equation of the first order in x and p and can, therefore, be solved easily.

If its solution is $(p =)\ dy/dx = \phi(x)$, then $y = \int \phi(x)dx + c$ is the required solution.

Obs. This method may be used to reduce any such equation of the nth order to one of the $(n-1)$th order. If, however, the lowest derivative in such an equation is d^ry/dx^r

(i) put $d^ry/dx^r = p$; (ii) find p and therefrom find y, (*See* Ex. 15.5).

Example 15.4. *Solve* $x\dfrac{d^2y}{dx^2} = \sqrt{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]}$.

Solution. Putting $dy/dx = p$ and $d^2y/dx^2 = dp/dx$, the given equation becomes

$$x\,dp/dx = \sqrt{(1+p^2)}\,.$$

Separating the variables and integrating, we get

$$\int \frac{dp}{\sqrt{(1+p^2)}} = \int \frac{dx}{x} + \text{constant}$$

or $$\log\left[p+\sqrt{(1+p^2)}\right] = \log x + \log c = \log cx.$$

$\therefore$ $$p+\sqrt{(1+p^2)} = cx \quad \text{or} \quad 1+p^2 = (cx-p)^2$$

or $$(p=)\ \frac{dy}{dx} = \frac{1}{2}\left(cx - \frac{1}{cx}\right).$$

$\therefore$ integrating again, we have $y = \dfrac{1}{2}\left(c\dfrac{x^2}{2} - \dfrac{1}{c}\log x\right) + c'$ as the required solution.

Example 15.5. *Solve* $\dfrac{d^4y}{dx^4}\cdot\dfrac{d^3y}{dx^3} = 1.$

Solution. Putting $d^3y/dx^3 = p$ and $d^4y/dx^4 = dp/dx$, the given equation becomes $\dfrac{dp}{dx}\,p = 1.$

Integrating w.r.t. x, $\int p\,dp = x + c_1$, *i.e.* $p^2/2 = x + c_1$ or $(p=)\ d^3y/dx^3 = \sqrt{2}(x+c_1)^{1/2}$.

Integrating thrice successively, we get

$$\frac{d^2y}{dx^2} = \sqrt{2}\,\frac{(x+c_1)^{3/2}}{3/2} + c_2,\quad \frac{dy}{dx} = \frac{2\sqrt{2}}{3}\cdot\frac{(x+c_1)^{5/2}}{5/2} + c_2x + c_3$$

$$y = \frac{4\sqrt{2}}{15}\,\frac{(x+c_1)^{7/2}}{7/2} + c_2\frac{x^2}{2} + c_3x + c_4$$

Hence $y = \dfrac{8\sqrt{2}}{105}(x+c_1)^{7/2} + \dfrac{1}{2}c_2x^2 + c_3x + c_4$ is the desired solution.

PROBLEMS 15.3

Solve the following equations :

1. $x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 6x = 0.$
2. $(1+x^2)\dfrac{d^2y}{dx^2} + 1 + \left(\dfrac{dy}{dx}\right)^2 = 0.$
3. $2x\dfrac{d^3y}{dx^3}\cdot\dfrac{d^2y}{dx^2} = \left(\dfrac{d^2y}{dx^2}\right)^2 - a^2.$
4. $\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2} = a\dfrac{d^2y}{dx^2}.$
5. A particle of mass m grammes is constrained to move in a horizontal circular path of radius a cm and is subjected to a resistance proportional to the square of the speed at any instant. Show that the differential equation of motion is of the form $m\dfrac{d^2\theta}{dt^2} + \mu a\left(\dfrac{d\theta}{dt}\right)^2 = 0$. If the particle starts with an angular velocity ω, find its angular displacement θ at time t sec.
6. When the inner of two concentric spheres of radii r_1 and $r_2 (r_1 < r_2)$ carries an electric charge, the differential equation for the potential v at any point between two spheres at a distance r from their common centre is $\dfrac{d^2v}{dt^2} + \dfrac{2}{r}\dfrac{dv}{dr} = 0$. Solve for v given $v = v_1$ when $r = r_1$ and $v = v_2$ when $r = r_2$.

15.5 EQUATIONS WHICH DO NOT CONTAIN x

A second order equation of this form is

$$f(d^2y/dx^2, dy/dx, y) = 0.$$

On putting $\dfrac{dy}{dx} = p$ and $\dfrac{d^2y}{dx^2} = \dfrac{dp}{dx} = \dfrac{dp}{dy} \cdot \dfrac{dy}{dx} = p\dfrac{dp}{dy}$, it becomes

$$f(p\, dp/dy, p, y) = 0.$$

This is an equation of the first order in y and p and can, therefore, be solved easily.

Example 15.6. *Solve* $y\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx}\left(\dfrac{dy}{dx} - 2y\right) = 0.$

Solution. On putting $dy/dx = p$ and $d^2y/dx^2 = p\, dp/dy$, the given equation becomes

$$yp\frac{dp}{dy} + p(p - 2y) = 0.$$

This gives *either* $p = 0$, of which the solution is $y = c$;

or $$\left(y\frac{dp}{dy} + p\right) - 2y = 0 \quad i.e., \quad (ydp + pdy) = 2ydy \ i.e., \ d(py) = 2ydy.$$

Integrating, $\quad py = 2\int ydy + c_1 = y^2 + c_1.$

Separating the variables and integrating, we get

$$\int \frac{ydy}{y^2 + c_1} = \int dx + c_2 \quad \text{or} \quad \frac{1}{2}\log(y^2 + c_1) = x + c_2 \text{ whence } y^2 + c_1 = c_3e^{2x}$$

Hence the required solutions are $y = c$ and $y^2 + c_1 = c_3\, e^{2x}$.

Example 15.7. *Find the curve in which the radius of curvature is twice the normal and in the opposite direction.*

Solution. At any point $P(x, y)$ of a curve, the radius of curvature

$$\rho = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} \bigg/ \frac{d^2y}{dx^2}$$

and the length of the normal (PN)

$$= y\sqrt{[1 + (dy/dx)^2]}.$$

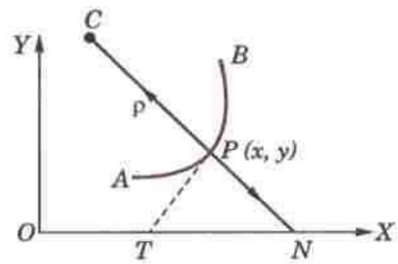

Fig. 15.2

Also we know that ρ is measured inwards and the normal is measured outwards, *i.e.*, both of them are positive when measured in opposite directions. So the sign will be positive (or negative) according as ρ and the normal run in the opposite (or same) directions.

Thus for the given curve $\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2} \div \dfrac{d^2y}{dx^2} = 2y\sqrt{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]}$

or $$1 + \left(\frac{dy}{dx}\right)^2 = 2y\frac{d^2y}{dx^2}.$$

On putting $dy/dx = p$ and $d^2y/dx^2 = p\, dp/dy$, the given equation becomes

$$1 + p^2 = 2y \,.\, p\, dp/dy.$$

$\therefore$ separating variables and integrating, we have

$$\int \frac{2pdp}{1 + p^2} = \int \frac{dy}{y} + \text{constant}$$

or $$\log(1 + p^2) = \log y + \log a = \log ay$$

$\therefore$ $$1 + p^2 = ay \text{ or } (p =)\ dy/dx = \sqrt{(ay - 1)}$$

$\therefore$ separating the variables and integrating, we get

$$\int dx + b = \int (ay-1)^{-1/2} dy$$

or
$$x + b = \frac{2}{a}(ay-1)^{1/2} \quad \text{or} \quad a^2(x+b)^2 = 4(ay-1)$$

which is required equation of the curve and represents a system of parabolas having axes parallel to y-axis.

PROBLEMS 15.4

Solve the following equations :

1. $2\dfrac{d^2y}{dx^2} - \left(\dfrac{dy}{dx}\right)^2 + 4 = 0.$
2. $y\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = 1.$
3. $y\dfrac{d^2y}{dx^2} - \left(\dfrac{dy}{dx}\right)^2 = y^2 \log y.$
4. $y(1-\log y)\dfrac{d^2y}{dx^2} + (1+\log y)\left(\dfrac{dy}{dx}\right)^2 = 0.$
5. Find the curve in which the radius of curvature is equal to the normal and is in the same direction.

15.6 EQUATIONS WHOSE ONE SOLUTION IS KNOWN

Consider the equation $d^2y/dx^2 + P\,dy/dx + Q = R$, where P, Q and R are functions of x only. If $y = u(x)$ is a known solution of this equation, then put $y = uv$ in it. It *reduces the differential equation to one of first order in dv/dx which can be completely solved.*

One integral belonging to the C.F. can be found by inspection as follows ;

(*i*) If $1 + P + Q = 0$, then $y = e^x$ is a solution,

(*ii*) If $I - P + Q = 0$, then $y = e^{-x}$ is a solution,

(*iii*) If $P + Qx = 0$, then $y = x$ is a solution.

Example 15.8. *Solve* $x\dfrac{d^2y}{dx^2} - (2x-1)\dfrac{dy}{dx} + (x-1)y = 0.$ *(Bhopal, 2008 S)*

Solution. The given equation is $\dfrac{d^2y}{dx^2} - \left(2 - \dfrac{1}{x}\right)\dfrac{dy}{dx} + \left(1 - \dfrac{1}{x}\right)y = 0$ ...(*i*)

Here $1 + P + Q = 1 - (2 - 1/x) + (1 - 1/x) = 0$

$\therefore$ $y = e^x$ is a part of C.F. of (*i*)

Now let $y = e^x v$ ...(*ii*)

so that $\dfrac{dy}{dx} = e^x v + e^x \dfrac{dv}{dx}$ ...(*iii*) and $\dfrac{d^2y}{dx^2} = e^x v + 2e^x \dfrac{dv}{dx} + e^x \dfrac{d^2v}{dx^2}$ ...(*iv*)

Substituting (*iv*), (*iii*) and (*ii*) in (*i*), we get

$$x\left(e^x v + 2e^x \frac{dv}{dx} + e^x \frac{d^2v}{dx^2}\right) - (2x-1)\left(e^x v + e^x \frac{dv}{dx}\right) + (x-1)e^x v = 0$$

or cancelling e^x, it becomes $x\dfrac{d^2v}{dx^2} + \dfrac{dv}{dx} = 0$ or $x\dfrac{dp}{dx} + p = 0$, where $p = \dfrac{dv}{dx}$.

Integrating, we get $\displaystyle\int \frac{dp}{p} = -\int \frac{dx}{x} + c$ or $\log p = -\log x + \log c_1$

i.e.,
$$p = \frac{c_1}{x} \quad \text{or} \quad \frac{dv}{dx} = \frac{c_1}{x}.$$

Again integrating, we obtain $v = c_1 \log x + c_2$

Hence the complete solution of (*i*) is $y = e^x(c_1 \log x + c_2)$.

Example 15.9. *Solve* $(1 - x^2)y'' - 2xy' + 2y = 0$ *given that* $y = x$ *is a solution.* (B.P.T.U., 2005 S)

Solution. Let $y = xv$ so that $y' = v + x\dfrac{dv}{dx}$

and $$y'' = x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}$$

Substituting these in the given equation, we get

$$(1 - x^2)\left(x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}\right) - 2x\left(v + x\frac{dv}{dx}\right) + 2xv = 0$$

or $$(x - x^3)\frac{d^2v}{dx^2} + (2 - 4x^2)\frac{dv}{dx} = 0$$

or $$(x - x^3)\frac{dp}{dx} + (2 - 4x^2)p = 0 \text{ where } p = \frac{dv}{dx}$$

Integrating, we get $\displaystyle\int \frac{dp}{p} + \int \frac{2 - 4x^2}{x - x^3}\,dx = c$

or $$\log p + \int \frac{2}{x}dx - \int \frac{dx}{1 - x} - \int \frac{dx}{1 + x} = c$$

or $$\log p + 2\log x + \log(1 - x) - \log(1 + x) = \log c_1$$

$$px^2(1 - x)/(1 + x) = c_1 \text{ or } \frac{dv}{dx} = \frac{c_1(1 + x)}{x^2(1 - x)}$$

Again integrating, $v = c_1\displaystyle\int\left(\frac{2}{x} + \frac{1}{x^2} + \frac{2}{1 - x}\right)dx + c_2$

or $$v = c_1\{2\log(x/1 - x) - 1/x\} + c_2$$

Hence the required complete solution is $y = x\,[c_1\{\log(x/1 - x)^2 - 1/x\} + c_2]$

Obs. Here $P + Qx = 0$. That is why $y = x$ is a solution of the given equation.

PROBLEMS 15.5

1. If $y = e^{x^2}$ is a solution of $y'' - 4xy' + (4x^2 - 2)y = 0$, find a second independent solution. (U.P.T.U., 2004)
2. Solve $x^2y'' - (x^2 + 2x)y' + (x + 2)y = x^3e^x$.
3. Solve $x\dfrac{d^2y}{dx^2} - (2x - 1)\dfrac{dy}{dx} + (x - 1)y = e^x$ given that $y = e^x$ is one integral. (Bhopal, 2007 S)
4. Solve $\sin^2 x\dfrac{d^2y}{dx^2} = 2y$, given that $y = \cot x$ is a solution. (Bhopal, 2007)
5. Solve $\dfrac{d^2y}{dx^2} - \cot x\dfrac{dy}{dx} - (1 - \cot x)y = e^x \sin x$.

15.7 EQUATIONS WHICH CAN BE SOLVED BY CHANGING THE INDEPENDENT VARIABLE

Consider the equation $$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad ...(1)$$

To change the independent variable x to z, let $z = f(x)$

Then $$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} \quad ...(2)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dz}\cdot\frac{dz}{dx}\right) = \frac{dy}{dz}\cdot\frac{d^2z}{dx^2} + \left(\frac{dz}{dx}\right)^2\frac{d^2y}{dx^2} \quad ...(3)$$

Substituting (2) and (3) in (1), we get $\dfrac{d^2y}{dz^2} + P_1 \dfrac{dy}{dz} + Q_1 y = R_1$ ...(4)

where $$P_1 = \left(\frac{d^2z}{dx^2} + P\frac{dz}{dx}\right)\Bigg/\left(\frac{dz}{dx}\right)^2, Q_1 = Q\Bigg/\left(\frac{dz}{dx}\right)^2, R_1 = R\Bigg/\left(\frac{dz}{dx}\right)^2$$

Now equation (4) can be solved by taking Q_1 = a constant.

Example 15.10. *Solve, by changing the independent variable,* $x\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + 4x^3y = x^5$ *(U.P.T.U., 2003)*

Solution. Given equation is $\dfrac{d^2y}{dx^2} - \dfrac{1}{x}\dfrac{dy}{dx} + 4x^2y = x^4$ ...(i)

Here $P = -1/x$, $Q = 4x^2$ and $R = x^4$.

Choose z so that $Q/(dz/dx)^2$ = const. or $(dz/dx)^2 = 4x^2$ (say)

or $$\frac{dz}{dx} = 2x \quad \text{or} \quad z = x^2$$

Changing the independent variable x to z by $z = x^2$, we get

$$\frac{d^2y}{dz^2} + P \cdot \frac{dy}{dz} + Q_1 y = R_1 \qquad ...(ii)$$

where $P_1 = \left(\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}\right)\Bigg/\left(\dfrac{dz}{dx}\right)^2 = [2 + (-x^{-1})\, 2x]/4x^2 = 0$

$$Q_1 = \frac{Q}{(dz/dx)^2} = \frac{4x^2}{4x^2} = 1, R_1 = \frac{R}{(dz/dx)^2} = \frac{x^4}{4x^2} = \frac{x^2}{4} = \frac{z}{4}$$

$\therefore$ (ii) takes the form $\dfrac{d^2y}{dz^2} + y = \dfrac{z}{4}$ or $(D^2 + 1)\, y = \dfrac{z}{4}$

Its A.E. is $D^2 + 1 = 0$, *i.e.*, $D = \pm i$

$$\text{C.F.} = c_1 \cos z + c_2 \sin z$$

$$\text{P.I.} = \frac{1}{D^2+1}\frac{z}{4} = \frac{1}{4}(1 + D^2)^{-1} z = \frac{1}{4}(1 - D^2 \,\ldots\ldots\ldots)\, z = \frac{z}{4}.$$

Hence the complete solution of (*i*) is

$$y = c_1 \cos z + c_2 \sin z + \frac{z}{4} \quad \text{or} \quad y = c_1 \cos x^2 + c_2 \sin x^2 + \frac{x^2}{4}.$$

Example 15.11. *Solve* $\dfrac{d^2y}{dx^2} + \cot x\dfrac{dy}{dx} + 4y \operatorname{cosec}^2 x = 0.$ ...(i)

Solution. Here $P = \cot x, Q = 4 \operatorname{cosec}^2 x$

Choosing z so that $Q\Bigg/\left(\dfrac{dz}{dx}\right)^2$ = const. or $\left(\dfrac{dz}{dx}\right)^2 = \operatorname{cosec}^2 x$ (say)

$$dz/dx = \operatorname{cosec} x \text{ or } z = \int \operatorname{cosec} x\, dx = \log \tan x/2$$

Changing the independent variable x to z, we get

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1 \qquad ...(ii)$$

where $P_1 = \left(\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}\right)\Bigg/\left(\dfrac{dz}{dx}\right)^2 = (-\operatorname{cosec} x \cot x + \cot x \operatorname{cosec} x)/\operatorname{cosec}^2 x = 0$

$$Q_1 = Q\Bigg/\left(\frac{dz}{dx}\right)^2 = \frac{4 \operatorname{cosec}^2 x}{\operatorname{cosec}^2 x} = 4, R_1 = 0$$

$\therefore$ (*ii*) takes the form $\dfrac{d^2y}{dz^2} + 4y = 0$

Its solution is $y = c_1 \cos(2z) + c_2 \sin(2z)$

i.e., $$y = c_1 \cos(2 \log \tan x/2) + c_2 \sin(2 \log \tan x/2)$$

This is the required complete solution of (*i*).

PROBLEMS 15.6

Solve the following equations (by changing the independent variable) :

1. $\dfrac{d^2y}{dx^2} + \tan x \dfrac{dy}{dx} + y\cos^2 x = 0.$ (*Bhopal, 2005*)
2. $\dfrac{d^2y}{dx^2} + \dfrac{2}{x}\dfrac{dy}{dx} + \dfrac{y}{x^4} = 0.$
3. $\dfrac{d^2y}{dx^2} - \cot x \dfrac{dy}{dx} - \sin^2 xy = 0.$
4. $x\dfrac{d^2y}{dx^2} + (4x^2 - 1)\dfrac{dy}{dx} + 4x^3 y = 2x^3.$ (*U.P.T.U., 2006*)
5. $\cos x \dfrac{d^2y}{dx^2} + \sin x \dfrac{dy}{dx} - 2y\cos^3 x = 2\cos^5 x.$ (*Bhopal, 2006 S*)

15.8 TOTAL DIFFERENTIAL EQUATIONS

(1) An ordinary differential equation of the first order and first degree involving three variables is of the form

$$P + Q\frac{dy}{dx} + R\frac{dz}{dx} = 0 \qquad ...(1)$$

where P, Q, R are functions of x, y, z and x is the independent variable.

In terms of differentials, (1) can be written as

$$Pdx + Qdy + Rdz = 0 \qquad ...(2)$$

which is integrable only if

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0 \qquad ...(3)$$

(2) Rule *to solve* $Pdx + Qdy + Rdz = 0$

If the condition of integrability is satisfied, consider one of the variables say : z, as constant so that dz = 0. Then integrate the equation Pdx + Qdy = 0. Replace the arbitrary constant appearing in its integral by $\phi(z)$. *Now differentiate the integral just obtained with respect to x, y, z. Finally, compare this result with the given differential equation to determine* $\phi(z)$.

Example 15.12. *Solve* $(y^2 + yz)dx + (z^2 + zx)dy + (y^2 - xy)dz = 0.$

Solution. Here $P = y^2 + yz,\ Q = z^2 + zx,\ R = y^2 - xy.$

$$\therefore \quad P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$= (y^2 + yz)\,[2z + x - (2y - x)] + (z^2 + zx)\,[-y - y] + (y^2 - xy)\,[(2y + z) - z] = 0$$

Hence the condition of integrability is satisfied.

Considering z as constant, the given equation becomes

$$(y^2 + yz)dx + (z^2 + zx)dy = 0, \quad \text{or} \quad \frac{dx}{z(z+x)} + \frac{dy}{y(y+z)} = 0$$

Integrating and noting that z is a constant, we get

$$\frac{1}{z}\int \frac{dx}{z+x} + \frac{1}{z}\int\left(\frac{1}{y} - \frac{1}{y+z}\right)dy = \text{constant}$$

i.e., $$\log(z + x) + \log y - \log(y + z) = \text{constant.}$$

i.e., $$\frac{y(z+x)}{y+z} = \text{constant} = \phi(z), \text{ say} \qquad ...(i)$$

or $$y(z + x) - (y + z)\,\phi(z) = 0$$

Differentiating w.r.t. x, y, z, we obtain
$$y(dz + dx) + (z + x)dy - [(y + z)\phi'(z)dz + (dy + dz)\phi(z)] = 0$$
or $$ydx + [z + x - \phi(z)]dy + [y - (y + z)\,\phi'(z) - \phi(z)]\,dz = 0 \qquad ...(ii)$$

Comparing (*ii*) with the given differential equation, we get
$$\frac{y^2 + yz}{y} = \frac{z^2 + zx}{z + x - \phi(z)} = \frac{y^2 - xy}{y - (y + z)\phi'(z) - \phi(z)}.$$

The relation $\dfrac{y^2 + yz}{y} = \dfrac{z^2 + zx}{z + x - \phi(z)}$ reduces to (*i*). $\therefore$ it gives no information about $\phi(z)$.

Taking $\dfrac{y^2 + yz}{y} = \dfrac{y^2 - xy}{y - (y + z)\phi'(z) - \phi(z)}$, we get
$$y^2 - xy = (y + z)\,[y - (y + z)\phi'(z) - \phi(z)] = y^2 + yz - (y + z)^2\,\phi'(z) - (y + z)\,\phi(z)$$
$$= y^2 + yz - (y + z)^2\,\phi'(z) - y(z + x) \qquad \text{[From (i)]}$$
$$= y^2 - xy - (y + z)^2\,\phi'(z)$$

i.e., $(y + z)^2\,\phi'(z) = 0$, *i.e.*, $\phi'(z) = 0$ so that $\phi(z) = c$

Hence the required solution is $y(z + x) = (y + z)\,c$. [From (*i*)]

Obs. *Sometimes the integral is readily obtained by simply regrouping the terms in the given equation* as is illustrated below.

Example 15.13. *Solve* $xdx + zdy + (y + 2z)\,dz = 0$.

Solution. Regrouping the terms, we can write the given equation as
$$xdx + (ydz + zdy) + 2z\,dz = 0$$
of which the integral is $\dfrac{x^2}{2} + yz + z^2 = c$.

PROBLEMS 15.7

Solve :

1. $(mz - ny)dx + (nx - lz)dy + (ly - mx)dz = 0.$
2. $(y^2 + z^2 - x^2)dx - 2xydy - 2xzdz = 0.$
3. $yzdx - 2zxdy - 3xydz = 0.$
4. $(2xz - yz)dx + (2yz - zx)dy - (x^2 - xy + y^2)dz = 0.$
5. $(x + z)^2dy + y^2(dx + dz) = 0.$
6. $(yz + xyz)dx + (zx + xyz)\,dy + (xy + xyz)\,dz = 0.$

15.9 SIMULTANEOUS TOTAL DIFFERENTIAL EQUATIONS

These equations in three variables are given by
$$\left.\begin{aligned} Pdx + Qdy + Rdz = 0 \\ P'dx + Q'dy + R'dz = 0 \end{aligned}\right\} \qquad ...(1)$$
where P, Q, R and P', Q', R' are any functions of x, y, z.

(*a*) If each of these equations is integrable and have solutions f$(x, y, z) = c$ and Y$(x, y, z) = c¢$ respectively, then these taken together constitute the solution of the simultaneous equations (1).

(*b*) If one or both the equations (1) is not integrable, then we write these as follows :
$$\frac{dx}{QR' - Q'R} = \frac{dy}{RP' - R'P} = \frac{dz}{PQ' - P'Q}$$
and solve these by the methods explained below.

15.10 EQUATIONS OF THE FORM dx/P = dy/Q = dz/R

(1) Method of grouping

See if it is possible to take two fractions $dx/P = dz/R$ from which y can be cancelled or is absent, leaving equations in x and z only.

If so, integrate it by giving $\phi(x, z) = c$. ...(1)

Again see if one variable say : x is absent or can be removed may be with the help of (1), from the equation $dy/Q = dz/R$.

Then integrate it by giving $\psi(y, z) = c'$ ...(2)

These two independent solutions (1) and (2) taken together constitute the complete solution required.

Example 15.14. *Solve* $\dfrac{dx}{z^2y} = \dfrac{dy}{z^2x} = \dfrac{dz}{y^2x}$.

Solution. Taking the first two fractions and cancelling z^2, we get

$$\frac{dx}{y} = \frac{dy}{x} \quad \text{or} \quad xdx - ydy = 0$$

which on integration gives $x^2 - y^2 = c$. ...(*i*)

Again taking the second and third fractions and cancelling x, we have

$$\frac{dy}{z^2} = \frac{dz}{y^2}, \text{ i.e., } y^2dy - z^2dz = 0.$$

Its integral is $y^3 - z^3 = c'$. ...(*ii*)

Thus (*i*) and (*ii*) taken together constitute the required solution of the given equations.

(2) Method of multipliers

By a proper choice of the multipliers l, m, n which are not necessarily constants, we write

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{ldx + mdy + ndz}{lP + mQ + nR} \text{ such that } lP + mQ + nR = 0.$$

Then $ldx + mdy + ndz = 0$ can be solved giving the integral $\phi(x, y, z) = c$ ...(1)

Again search for another set of multipliers λ, μ, γ

so that $\lambda P + \mu Q + \gamma R = 0$

giving $\lambda dx + \mu dy + \gamma dz = 0$,

which on integration gives the solution $\psi(x, y, z) = c'$ ...(2)

These two solutions (1) and (2) taken together constitute the required solution.

Example 15.15. *Solve* $\dfrac{dx}{x(y^2 - z^2)} = \dfrac{dy}{-y(z^2 + x^2)} = \dfrac{dz}{z(x^2 + y^2)}$.

Solution. Using the multipliers x, y, z

each fraction $= \dfrac{xdx + ydy + zdz}{x^2(y^2 - z^2) - y^2(z^2 + x^2) + z^2(x^2 + y^2)} = \dfrac{xdx + ydy + zdz}{0}$

$\therefore$ $xdx + ydy + zdz = 0$, which on integration gives the solution $x^2 + y^2 + z^2 = c$ ...(*i*)

Again using the multipliers $1/x, -1/y, -1/z$

each fraction $= \dfrac{\frac{1}{x}dx - \frac{1}{y}dy - \frac{1}{z}dz}{(y^2 - z^2) + (z^2 + x^2) - (x^2 + y^2)} = \dfrac{\frac{1}{x}dx - \frac{1}{y}dy - \frac{1}{z}dz}{0}$ so that $\dfrac{dx}{x} - \dfrac{dy}{y} - \dfrac{dz}{z} = 0$

which on integration gives $\log x - \log y - \log z$ = constant or $yz = c'x$. ...(*ii*)

Hence the solution of the given equation is $x^2 + y^2 + z^2 = c$; $yz = c'x$.

PROBLEMS 15.8

Solve :

1. $\dfrac{xdx}{y^2z} = \dfrac{dy}{xz} = \dfrac{dz}{y^2}$.
2. $\dfrac{dx}{mz - ny} = \dfrac{dy}{nx - lz} = \dfrac{dx}{ly - mx}$.
3. $\dfrac{dx}{x^2 - yz} = \dfrac{dy}{y^2 - zx} = \dfrac{dz}{z^2 - xy}$.
4. $\dfrac{dx}{y - zx} = \dfrac{dy}{yz + x} = \dfrac{dz}{x^2 + y^2}$.
5. $\dfrac{dx}{x(y^2 - z^2)} = \dfrac{dy}{y(z^2 - x^2)} = \dfrac{dz}{z(x^2 - y^2)}$.
6. $\dfrac{dx}{x^2 - y^2 - z^2} = \dfrac{dy}{2xy} = \dfrac{dz}{2xz}$.

CHAPTER 16

Series Solution of Differential Equations and Special Functions

16.1 INTRODUCTION

Many differential equations arising from physical problems are linear but have variable coefficients and do not permit a general solution in terms of known functions. Such equations can be solved by numerical methods (Chapter 28), but in many cases it is easier to find a solution in the form of an infinite convergent series.

The series solution of certain differential equations give rise to special functions such as Bessel's function, Legendre's polynomial, Lagurre's polynomial, Hermite's polynomial, Chebyshev polynomials. Strum-Lioville problem based on the orthogonality of functions is also included which shows that Bessel's, Legendre's and other equations can be considered from a common point of view. These special functions have many applications in engineering.

16.2 VALIDITY OF SERIES SOLUTION OF THE EQUATION

$$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0 \qquad ...(i)$$

can be determined with the help of the following theorems :

Def. 1. If $P_0(a) \neq 0$, then $x = a$ is called and **ordinary point** of (i), otherwise a **singular point.**

2. *A singular point $x = a$ of (1) is called* **regular** *if, when (i) is put in the form*

$$\frac{d^2y}{dx^2} + \frac{Q_1(x)}{x-a}\frac{dy}{dx} + \frac{Q_2(x)}{(x-a)^2}y = 0,$$

$Q_1(x)$ and $Q_2(x)$ possess derivatives of all orders in the neighbourhood of a.

3. *A singular point which is not regular is called an* **irregular singular point.**

Theorem I. *When $x = a$ is an ordinary point of (i), its every solution can be expressed in the form*

$$y = a_0 + a_1(x-a) + a_2(x-a)^2 + ... \qquad ...(ii)$$

Theorem II. *When $x = a$ is a regular singularity of (i), at least one of the solutions can be expressed as*

$$y = (x-a)^m [a_0 + a_1(x-a) + a_2(x-a)^2 + ...] \qquad ...(iii)$$

Theorem III. *The series (ii) and (iii) are convergent at every point within the circle of convergence at a.* A solution in series will be valid only if the series is convergent.

16.3 SERIES SOLUTION WHEN X = 0 IS AN ORDINARY POINT OF THE EQUATION

$$P_0 \frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = 0 \qquad ...(1)$$

where P's are polynomials in x and $P_0 \neq 0$ at $x = 0$.

(*i*) Assume its solution to be of the form $y = a_0 + a_1x + a_2x^2 + ... + a_nx^n + ...$ ...(2)

(*ii*) Calculate dy/dx, d^2y/dx^2 from (2) and substitute the values of y, dy/dx, d^2y/dx^2 in (1).

(*iii*) Equate to zero the coefficients of the various powers of x and determine a_2, a_3, a_4 ... in terms of a_0, a_1. (The result obtained by equating to zero is the coefficient of x^n that is called the *recurrence relation*).

(*iv*) Substituting the values of a_2, a_3, a_4 ... in (2), we get the desired series solution having a_0, a_1 as its arbitrary constants.

Example 16.1. *Solve in series the equation* $\frac{d^2y}{dx^2} + xy = 0.$ *(V.T.U., 2010)*

Solution. Here $x = 0$ is an ordinary point since coefficient of $y'' \neq 0$ at $x = 0$.

Assume its solution is $y = a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_nx^n + ...$ ...(*i*)

Then $$\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + ... + na_nx^{n-1} + ...$$

and $$\frac{d^2y}{dx^2} = 2 \,.\, 1a_2 + 3 \,.\, 2a_3x + 4 \,.\, 3a_4x^2 + ... + n(n-1)a_nx^{n-2} + ...$$

Substituting in the given differential equation

$$1 \,.\, 1a_2 + 3 \,.\, 2a_3x + 4 \,.\, 3a_4x^2 + ... + n(n-1)a_nx^{n-2} + ... + x(a_0 + a_1x + a_2x^2 + ... + a_nx^n + ...) = 0$$

or $$2 \,.\, 1a_2 + (3 \,.\, 2a_3 + a_0)x + (4 \,.\, 3a_4 + a_1)x^2 + (5 \,.\, 4a_5 + a_2)x^3 + ... + [(n+2)(n+1)a_{n+2} + a_{n-1}]x^n + ... = 0.$$

Equating to zero the co-efficients of the various powers of x,

$a_2 = 0,$ [Coeff. of $x^0 = 0$]

$3 \,.\, 2a_3 + a_0 = 0$, *i.e.*, $a_3 = -\frac{a_0}{3!}$ [Coeff. of $x = 0$]

$4 \,.\, 3a_4 + a_1 = 0$, *i.e.*, $a_4 = -\frac{2a_1}{4!}$ [Coeff. of $x^2 = 0$]

$5 \,.\, 4a_5 + a_2 = 0$, *i.e.*, $a_5 = -\frac{a_2}{5 \,.\, 4} = 0$ and so on. [Coeff. of $x^3 = 0$]

In general, $(n+2)(n+1)a_{n+2} + a_{n-1} = 0$ [Coeff. of $x^n = 0$]

i.e., $$a_{n+2} = \frac{-a_{n-1}}{(n+2)(n+1)} \qquad ..(ii)$$

which is the *recurrence relation.*

Putting $n = 4, 5, 6, ...$ in (*ii*) successively, $a_6 = -\frac{a_3}{6 \,.\, 5} = \frac{4a_0}{6!}$; $a_7 = -\frac{a_4}{7 \,.\, 6} = \frac{5 \,.\, 2a_1}{7!}$

$$a_8 = -\frac{a_5}{8 \,.\, 7} = 0; a_9 = -\frac{a_6}{9 \,.\, 8} = -\frac{7 \,.\, 4a_0}{9!} \text{ and so on.}$$

Substituting these values in (*i*), we get

$$y = a_0\left(1 - \frac{x^3}{3!} + \frac{1 \,.\, 4x^6}{6!} - \frac{1 \,.\, 4 \,.\, 7x^9}{9!} + ...\right) + a_1\left(x - \frac{2x^4}{4!} + \frac{2 \,.\, 5x^7}{7!} - ...\right)$$

which is the required solution.

Example 16.2. *Solve in series* $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = 0.$ *(Bhopal, 2008 ; U.P.T.U., 2006)*

Solution. Here $x = 0$ is an ordinary point since coefficient of $y'' \neq 0$ at $x = 0$.

Assume the solution of the given equation to be

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n + \ldots \qquad \ldots(i)$$

Then $$\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + \ldots + na_nx^{n-1} + \ldots$$

and $$\frac{d^2y}{dx^2} = 2a_2 + 3.2a_3x + 4.3a_4x^2 + \ldots + n(n-1)a_nx^{n-2} + \ldots$$

Substituting in the given equation, we get

$$(1-x^2)[2a_2 + 3.2a_3x + 4.3a_4x^2 + \ldots + n(n-1)a_nx^{n-2} + \ldots]$$
$$- x[a_1 + 2a_2x + 3a_3x^2 + \ldots + na_nx^{n-1} + \ldots] + 4[a_0 + a_1x + a_2x^2 + \ldots. + a_nx^n + \ldots] = 0$$

Equating to zero the coefficients of the various powers of x,

$$2a_2 + 4a_0 = 0 \quad i.e., \quad a_2 = -2a_0 \qquad [\text{coeff. of } x^0 = 0]$$

$$3.2a_3 - a_1 + 4a_1 = 0 \quad i.e., \quad a_3 = -\frac{1}{2}a_1 \qquad [\text{coeff. of } x^1 = 0]$$

$$4.3a_4 - 2a_2 - 2a_2 + 4a_2 = 0 \quad i.e., \quad a_4 = 0 \qquad [\text{coeff. of } x^2 = 0]$$

$$5.4a_5 - 3.2a_3 - 3a_3 + 4a_3 = 0 \qquad [\text{coeff. of } x^3 = 0]$$

i.e., $$20a_5 - 5a_3 = 0 \quad i.e., \quad a_5 = -\frac{a_1}{8} \text{ and so on.}$$

In general, $(n+2)(n+1)a_{n+2} - n(n-1)a_n - na_n + 4a_n = 0$

or $$a_{n+2} = \frac{n-2}{n+1}a_n \qquad \ldots(ii)$$

which is the *recurrence relation*

Putting $n = 4, 5, 6, 7\ldots$ in (ii) successively,

$$a_6 = 0; \quad a_7 = \frac{3}{6}a_5 = -\frac{3}{6}\frac{a_1}{8}; a_8 = 0; a_9 = -\frac{5.3}{8.6}\cdot\frac{a_1}{8}\ldots$$

Substituting these values in (i), we get

$$y = a_0(1-2x^2) + a_1x\left(1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{3}{6}\cdot\frac{x^6}{8} - \frac{5.3}{8.6}\cdot\frac{x^8}{8} - \ldots\right).$$

PROBLEMS 16.1

Solve the following equations in series :

1. $\frac{d^2y}{dx^2} + y = 0$, given $y(0) = 0$. *(B.P.T.U., 2005 S)*
2. $\frac{d^2y}{dx^2} + x^2y = 0$.
3. $y'' + xy' + y = 0$. *(V.T.U., 2008)*
4. $(1-x^2)y'' + 2y = 0$, given $y(0) = 4$, $y'(0) = 5$. *(P.T.U., 2006)*
5. $(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$. *(S.V.T.U., 2008)*
6. $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 2y = 0$. *(U.P.T.U., 2004)*

16.4 FROBENIUS* METHOD : Series solution when x = 0 is a regular sigularity of the equation

$$\mathbf{P_0\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2y = 0} \qquad \ldots(1)$$

*A German mathematician *F.G. Frobenius* (1849–1917) who is known for his contributions to the theory of matrices and groups.

(*i*) Assume the solution to be $y = x^m(a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n + \dots)$ ...(2)

(*ii*) Substitute from (2) for y, dy/dx, d^2y/dx^2 in (1) as before.

(*iii*) Equate to zero the coefficient of the lowest degree term in x. It gives a quadratic equation known as the *indicial equation*.

(*iv*) Equating to zero the coefficients of the other powers of x, find the values of $a_1, a_2, a_3, \dots$ in terms of a_0.

The complete solution depends on the nature of roots of the indicial equation.

Case I. When roots of the indicial equation are distinct and do not differ by an integer, *the complete solution is*

$$y = c_1(y)_{m_1} + c_2(y)_{m_2}$$

where m_1, m_2 are the roots.

Example 16.3. *Solve in series the equation* $9x(1-x)\dfrac{d^2y}{dx^2} - 12\dfrac{dy}{dx} + 4y = 0.$

(*Madras, 2006 ; Roorkee, 2000*)

Solution. Here $x = 0$ is a singular point since coefficient of $y'' = 0$ at $x = 0$.

Substituting $y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \dots$

$$\therefore \quad \frac{dy}{dx} = ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + \dots$$

and
$$\frac{d^2y}{dx^2} = m(m-1)a_0x^{m-2} + (m+1)ma_1x^{m-1} + (m+2)(m+1)a_2x^m + \dots$$

in the given equation, we obtain

$$9x(1-x)\,[m(m-1)a_0x^{m-2} + (m+1)ma_1x^{m-1} + (m+2)(m+1)a_2x^m + \dots]$$
$$- 12[ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + \dots] + 4[a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \dots] = 0.$$

The lowest power of x is x^{m-1}. Its coefficient equated to zero gives

$$a_0(9m(m-1) - 12m] = 0, \text{ i.e., } m(3m-7) = 0 \quad \text{as} \quad a_0 \neq 0.$$

Thus the roots of the *indicial equation* are $m = 0, 7/3$. *i.e., Roots are distinct and do not differ by an integer.*

The coefficient of x^m equated to zero gives $a_1\{9(m+1)m - 12(m+1)\} + a_0\{4 - 9m(m-1)\} = 0$

i.e., $\quad 3a_1(3m-4)(m+1) - a_0(3m-4)(3m+1) = 0$

i.e., $\quad 3a_1(m+1) = a_0(3m+1).$

Similarly $3a_2(m+2) = a_1(3m+4)$, $3a_3(m+3) = a_2(5m+7)$ and so on.

$$\therefore \quad a_1 = \frac{3m+1}{3(m+1)}a_0,\ a_2 = \frac{(3m+4)a_1}{3(m+2)} = \frac{(3m+4)(3m+1)}{3^2(m+2)(m+1)}a_0,\ a_3 = \frac{(3m+7)(3m+4)(3m+1)}{3^3(m+3)(m+2)(m+1)}a_0 \text{ etc.}$$

When $m = 0$, $a_1 = \dfrac{1}{3}a_0,\ a_2 = \dfrac{1.4}{3.6}a_0,\ a_3 = \dfrac{1.4.7}{3.6.9}a_0$ etc. giving the particular solution

$$y_1 = a_0\left[1 + \frac{1}{3}x + \frac{1.4}{3.6}x^2 + \frac{1.4.7}{3.6.9}x^3 + \dots\right]$$

When $m = 7/3$, the particular solution is

$$y_2 = a_0x^{7/3}\left[1 + \frac{8}{10}x + \frac{8.11}{10.13}x^2 + \frac{8.11.14}{10.13.16}x^3 + \dots\right]$$

Thus the complete solution is $y = c_1y_1 + c_2y_2$

i.e.,
$$y = C_1\left[1 + \frac{1}{3}x + \frac{1.4}{3.6}x^2 + \frac{1.4.7}{3.6.9}x^3 + \dots\right]$$
$$+ C_2x^{7/3}\left[1 + \frac{8}{10}x + \frac{8.11}{10.13}x^2 + \frac{8.11.14}{10.13.16}x^3 + \dots\right], \qquad \text{where } C_1 = c_1a_0,\ C_2 = c_2a_0.$$

Case II. When roots of the indicial equation are equal *the complete solution is*

$$y = c_1(y)_{m_1} + c_2\left(\frac{\partial y}{\partial m}\right)_{m_1}$$

where m_1, m_1 are the roots.

Example 16.4. *Solve in series the equation* $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + xy = 0.$ *(V.T.U., 2010 ; S.V.T.U., 2007)*

Solution. Substituting $y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \ldots$ ...(*i*)

$$\frac{dy}{dx} = ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + \ldots$$

and $$\frac{d^2y}{dx^2} = m(m-1)a_0x^{m-2} + (m+1)ma_1x^{m-1} + (m+2)(m+1)a_2x^m + \ldots$$

in the given equation, we obtain

$$x[m(m-1)a_0x^{m-2} + (m+1)ma_1x^{m-1} + (m+2)(m+1)a_2x^m + \ldots]$$
$$+ [ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + \ldots]$$
$$+ x[a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \ldots] = 0.$$

The lowest power of x is x^{m-1}. Its coefficient equated to zero gives $a_0[m(m-1) + m] = 0$.

i.e., $m^2 = 0$ as $a_0 \neq 0$. $\therefore$ $m = 0, 0$.

The coefficients of $x^m, x^{m+1}, \ldots$ equated to zero give

$$a_1[(m+1)m + m + 1] = 0, \quad i.e., \quad a_1 = 0$$
$$a_2(m+2)^2 + a_0 = 0, a_3(m+3)^2 + a_1 = 0, a_4(m+4)^2 + a_2 = 0 \text{ and so on.}$$

Clearly $a_3 = a_5 = a_7 \ldots\ldots = 0$.

Also $$a_2 = -\frac{a_0}{(m+2)^2}, a_4 = -\frac{a_2}{(m+4)^2} = \frac{a_0}{(m+2)^2(m+4)^2} \text{ etc.}$$

$$\therefore \quad y = a_0x^m\left[1 - \frac{x^2}{(m+2)^2} + \frac{x^4}{(m+2)^2(m+4)^2} - \frac{x^6}{(m+2)^2(m+4)^2(m+6)^2} + \ldots\ldots\right] \quad \ldots(ii)$$

Putting $m = 0$, the first solution is

$$y_1 = a_0\left[1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 . 4^2} - \frac{x^6}{2^2 . 4^2 . 6^2} + \ldots\ldots\right] \quad \ldots(iii)$$

This gives only one solution instead of two. To get the second solution, differentiate (*ii*) partially w.r.t. m.

$$\frac{\partial y}{\partial m} = y \log x + a_0x^m\left\{\frac{x^2}{(m+2)^2}\frac{2}{m+2} - \frac{x^4}{(m+2)^2(m+4)^2}\left[\frac{2}{m+2} + \frac{2}{m+4}\right] + \ldots\right\}$$

$\therefore$ the second solution is $y_2 = \left(\dfrac{\partial y}{\partial m}\right)_{m=0}$

$$= y_1 \log x + a_0\left\{\frac{1}{2^2}x^2 - \frac{1}{2^2 . 4^2}\left(1 + \frac{1}{2}\right)x^4 + \frac{1}{2^2 . 4^2 . 6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right)x^6 - \ldots\right\} \quad \ldots(iv)$$

Hence the complete solution is $y = c_1y_1 + c_2y_2$. [From (*iii*) & (*iv*)]

i.e., $$y = (C_1 + C_2 \log x)\left[1 - \frac{1}{2^2}x^2 + \frac{1}{2^2 . 4^2}x^4 - \frac{1}{2^2 . 4^2 . 6^2}x^6 + \ldots\right]$$
$$+ C_2\left\{\frac{1}{2^2}x^2 - \frac{1}{2^2 . 4^2}\left(1 + \frac{1}{2}\right)x^4 + \frac{1}{2^2 . 4^2 . 6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right)x^6 - \ldots\right\}$$

where $C_1 = a_0c_1, C_2 = a_0c_2$.

Obs. The above differential equation is called *Bessel's equation of order zero*, y_1 is called *Bessel function of the first kind of order zero* and is denoted by $J_0(x)$. It is absolutely convergent for all values of x whether real or complex.

y_2 is called the *Bessel function of the second kind of order zero or the Neumann function* and is denoted by $Y_0(x)$.

Thus the complete solution of the *Bessel's equation of order zero is* $y = AJ_0(x) + BY_0(x)$.

Case III. When roots of indicial equation are distinct and differ by an integer, making a coefficient of y infinite.

Let m_1 and m_2 be the roots such that $m_1 < m_2$. If some of the coefficients of y series become infinite when $m = m_1$, we modify the form of y by replacing a_0 by $b_0\,(m - m_1)$. Then the complete solution is

$$y = C_1 (y)_{m_2} + C_2 \left(\frac{\partial y}{\partial m}\right)_{m_1}$$

Obs. 1. Two independent solution can also be obtained by putting $m = m_1$ (lesser of the two roots) in the modified form of y and $\partial y/\partial m$.

Obs. 2. *If one of the coefficients* (say : a_1) *becomes* **indeterminate** *when* $m = m_2$, *the complete solution is given by putting* $m = m_2$ *in y which contains two arbitrary constants.*

Example 16.5. *Obtain the series solution of the equation*

$$x(1-x)\frac{d^2y}{dx^2} - (1+3x)\frac{dy}{dx} - y = 0.$$

Solution. Here $x = 0$ is a singular point, since coefficient of y'' is zero at $x = 0$.

$\therefore$ substituting $\qquad y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \ldots \qquad \ldots(i)$

$$\frac{dy}{dx} = ma_0x^{m-1} + (m+1)a_1x^m + (m+2)\,a_2\,x^{m+1} + \ldots$$

and
$$\frac{d^2y}{dx^2} = m(m-1)\,a_0x^{m-2} + (m+1)ma_1\,x^{m-1} + (m+2)\,(m+1)\,a_2x^m + \ldots$$

in the given equation, we obtain

$$x(1-x)[m(m-1)\,a_0x^{m-2} + (m+1)ma_1\,x^{m-1} + (m+2)\,(m+1)a_2x^m + \ldots]$$
$$-(1+3x)[ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + \ldots] - [a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \ldots] = 0$$

Equating to zero the coefficients of the lowest power of x, we get $a_0[m(m-1) - m] = 0$, $(a_0 \neq 0)$,

i.e., $\quad m(m-2) = 0$, *i.e.* $m = 0, 2$ *i.e.*, the two roots are distinct and differ by an integer.

Equating to zero the coefficients of successive powers of x, we get

$$(m-1)a_1 = (m+1)a_0,\ ma_2 = (m+2)a_1,\ (m+1)a_3 = (m+3)\,a_2 \text{ and so on.}$$

i.e.,
$$a_1 = \frac{m+1}{m-1}a_0,\ a_2 = \frac{(m+1)(m+2)}{(m-1)m}a_0,\ a_3 = \frac{(m+1)(m+2)(m+3)}{(m-1)m(m+1)}a_0 \text{ etc.}$$

Thus (i) becomes

$$y = a_0x^m\left[1 + \frac{m+1}{m-1}x + \frac{(m+1)(m+2)}{(m-1)m}x^2 + \frac{(m+1)(m+2)(m+3)}{(m-1)m(m+1)}x^3 + \ldots\right] \qquad \ldots(ii)$$

Putting $m = 2$ (greater of the two roots) in (ii), the first solution is

$$y_1 = a_0x^2\left[1 + 3x + \frac{3.4}{2}x^2 + \frac{4.5}{2}x^3 + \ldots\right]$$

If we put $m = 0$ in (ii), the coefficients become infinite.

To obviate this difficulty, put $a_0 = b_0\,(m - 0)$ so that

$$y = b_0x^m\left[m + \frac{m\,(m+1)}{m-1}x + \frac{(m+1)\,(m+2)}{m-1}x^2 + \frac{(m+1)\,(m+2)\,(m+3)}{(m-1)\,(m+1)}x^3 + \ldots\right]$$

$$\therefore \quad \frac{\partial y}{\partial m} = b_0x^m \log x\left[m + \frac{m\,(m+1)\,x}{m-1} + \frac{(m+1)\,(m+2)}{m-1}x^2 + \frac{(m+1)\,(m+2)\,(m+3)}{(m-1)\,(m+1)}x^3 + \ldots\right]$$
$$+ b_0x^m\left[1 + \frac{m^2-2m-1}{(m-1)^2}x + \frac{m^2-m-5}{(m-1)^2}x^2 + \frac{m^2-2m-11}{(m-1)^2}x^3 + \ldots\right]$$

$\therefore$ the second solution is $y_2 = \left(\dfrac{\partial y}{\partial m}\right)_{m=0}$

$$= b_0 \log x\,[-1.2x^2 - 2.3x^3 - 3.4x^4 - \ldots] + b_0[1 - x - 5x^2 - 11x^3 - \ldots]$$

Hence the complete solution is $y = c_1y_1 + c_2y_2$

i.e., $$y = \frac{1}{2}\,c_1a_0[1.2x^2 + 2.3x^3 + 3.4x^4 + \ldots] - b_0c_2 \log x\,[1.2x^2 + 2.3x^3 + 3.4x^4 + \ldots] - b_0c_2\,[-1 + x + 5x^2 + 11x^3 + \ldots]$$

i.e., $$y = (C_1 + C_2 \log x)\,(1.2x^2 + 2.3x^3 + 3.4x^4 + \ldots) + C_2\,(-1 + x + 5x^2 + 11x^3 + \ldots)$$

where $C_1 = \frac{1}{2}\,c_1a_0,\ C_2 = -b_0c_2$

Example 16.6. *Solve in series* $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + (x^2 - 4)\,y = 0.$ *(Bhopal, 2008 S; Rajasthan, 2003)*

Solution. $x = 0$ is a singular point, since coeff. of y'' is zero at $x = 0$.

Substituting $$y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \ldots \qquad \ldots(i)$$

$$\frac{dy}{dx} = ma_0x^{m-1} + (m+1)\,a_1x^m + (m+2)\,a_2x^{m+1} + \ldots$$

and $$\frac{d^2y}{dx^2} = m\,(m-1)\,a_0x^{m-2} + (m+1)\,ma_1x^{m-1} + (m+2)\,(m+1)\,a_2x^m + \ldots$$

in the given equation, we get

$$x^2\,[m\,(m-1)\,a_0x^{m-2} + (m+1)\,ma_1x^{m-1} + (m+2)\,(m+1)\,a_2x^m + \ldots] + x\,[ma_0x^{m-1} + (m+1)\,a_1x^m + (m+2)\,a_2x^{m+1} + \ldots] + (x^2 - 4)\,[a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \ldots] = 0$$

Equating to zero the coefficients of the lowest power of x.

$$a_0\,[m\,(m-1) + m - 4] = 0 \text{ so that } m = \pm 2.$$

i.e., the two roots are distinct and differ by an integer.

Now equating to zero the coefficiens of successive powers of x, we get

$$m(m+4)\,a_2 = -a_0,\ i.e.,\ a_2 = \frac{-1}{m\,(m+4)}\,a_0,\ a_3 = 0$$

$$a_4 = \frac{1}{(m+2)\,(m+6)}\cdot\frac{1}{m\,(m+4)}\,a_0,\ a_5 = a_7 = \ldots = 0.$$

$$a_6 = \frac{-a_0}{m\,(m+2)\,(m+4)^2\,(m+6)\,(m+8)} \text{ etc.}$$

Substituting these values in (i), we get

$$y = a_0x^m\left[1 - \frac{x^2}{m\,(m+4)} + \frac{x^4}{m\,(m+2)\,(m+4)\,(m+6)} - \frac{x^6}{m\,(m+2)\,(m+4)^2\,(m+6)\,(m+8)} + \ldots\right] \qquad \ldots(ii)$$

Putting $m = 2$ (greater of the two roots) in (ii), the first solution is

$$y_1 = a_0x^2\left\{1 - \frac{x^2}{2\cdot 6} + \frac{x^4}{2\cdot 4\cdot 6\cdot 8} - \frac{x^6}{2\cdot 4\cdot 6^2\cdot 8\cdot 10} + \ldots\right\}$$

If we put $m = -2$ in (ii), the coefficients become infinite. To obviate this difficulty, let $a_0 = b_0\,(m+2)$, so that

$$y = b_0x^m\left[(m+2)\left\{1 - \frac{x^2}{m\,(m+4)}\right\} + \frac{x^4}{m\,(m+4)\,(m+6)} - \frac{x^6}{m\,(m+4)^2\,(m+6)\,(m+8)} + \ldots\right]$$

$$\therefore \quad \frac{\partial y}{\partial m} = b_0 x^m \log x \left[(m+2)\left\{1 - \frac{x^2}{m(m+4)}\right\} + \frac{x^4}{m(m+4)(m+6)} - \dots \right]$$
$$+ b_0 x^m \left[1 - \frac{(m+2)}{m(m+4)} \left\{ \frac{1}{m+2} - \frac{1}{m} - \frac{1}{m+4} \right\} x^2 \right.$$
$$\left. + \frac{1}{m(m+4)(m+6)} \left\{ -\frac{1}{m} - \frac{1}{m+4} - \frac{1}{m+6} \right\} x^4 + \dots \right]$$

The second solution is $y_2 = \left(\frac{\partial y}{\partial m} \right)_{m=-2}$

$$= b_0 x^{-2} \log x \left[-\frac{x^4}{2^2 \cdot 4} + \frac{x^6}{2^3 \cdot 4 \cdot 6} \cdots \right] + b_0 x^{-2} \left[1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \dots \right]$$

Hence the complete solution $y = c_1 y_1 + c_2 y_2$

i.e.,
$$y = C_1 x^2 \left\{ 1 - \frac{x^2}{2 \cdot 6} + \frac{x^4}{2 \cdot 4 \cdot 6 \cdot 8} - \frac{x^6}{2 \cdot 4 \cdot 6^2 \cdot 8 \cdot 10} + \dots \right\}$$
$$+ C_2 \left[x^2 \log x \left\{ -\frac{1}{2^2 \cdot 4} + \frac{x^4}{2^3 \cdot 4 \cdot 6} \cdots \right\} + x^{-2} \left\{ 1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \cdots \right\} \right]$$

where $C_1 = c_1 a_0$, $C_2 = c_2 b_0$.

Example 16.7. *Solve in series $xy'' + 2y' + xy = 0$.* *(U.P.T.U., 2003)*

Solution. Here $x = 0$ is a singular point since coefficient of $y'' = 0$ at $x = 0$.

$\therefore$ Substituting $y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} + \dots$ ...(*i*)

$$\frac{dy}{dx} = m a_0 x^{m-1} + (m+1) a_1 x^m + (m+2) a_2 x^{m+1} + (m+3) a_3 x^{m+2} + \dots$$

and
$$\frac{d^2 y}{dx^2} = m(m-1) a_0 x^{m-2} + (m+1) m a_1 x^{m-1} + (m+2)(m+1) a_2 x^m + \dots.$$

in the given equation, we get

$$x [m(m-1) a_0 x^{m-2} + (m+1) m a_1 x^{m-1} + (m+2)(m+1) a_2 x^m + \dots]$$
$$+ 2 [m a_0 x^{m-1} + (m+1) a_1 x^m + (m+2) a_2 x^{m+1} + (m+3) a_3 x^{m+2} + \dots]$$
$$+ x [a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} + \dots]$$

Equating to zero, the coefficients of the lowest power of x,

$$m(m-1) a_0 + 2m a_0 = 0 \text{ so that } m = 0, -1.$$

i.e., the roots are distinct of and differ by an integer.

Equating to zero, the coefficient of x^m, we get

$$(m+1) m a_1 + 2(m+1) a_1 = 0 \textit{ i.e. } (m+1)(m+2) a_1 = 0$$

or
$$(m+1) a_1 = 0 \qquad [\because \; m + 2 \neq 0]$$

When $m = -1$, $a_1 = 0/0$ *i.e., indeterminate.*

Hence the complete solution will be given by putting $m = -1$ in y itself (containing two arbitrary constants a_0 and a_1).

Now equating to zero, the coefficients of successive powers of x, we get

$$(m+2)(m+3) a_2 + a_0 = 0 \qquad [\text{Coeff. of } x^{m+1} = 0]$$
$$(m+3)(m+4) a_3 + a_1 = 0 \qquad [\text{Coeff. of } x^{m+2} = 0]$$
$$(m+4)(m+5) a_4 + a_2 = 0 \qquad [\text{Coeff. of } x^{m+3} = 0]$$
$$(m+5)(m+6) a_5 + a_3 = 0 \text{ etc.} \qquad [\text{Coeff. of } x^{m+4} = 0]$$

i.e.,
$$a_2 = -\frac{a_0}{(m+2)(m+3)}, a_3 = \frac{-a_1}{(m+3)(m+4)}, a_4 = \frac{a_0}{(m+2)(m+3)(m+4)(m+5)},$$

$$a_5 = \frac{a_1}{(m+3)(m+4)(m+5)(m+6)} \text{ and so on.}$$

Substituting the values in (i), we get

$$y = x^m\left[a_0 + a_1x - \frac{a_0}{(m+2)(m+3)}x^2 - \frac{a_1}{(m+3)(m+4)}x^3 + \frac{a_0}{(m+2)(m+3)(m+4)(m+5)}x^4 + \frac{a_1}{(m+3)(m+4)(m+5)(m+6)}x^5 - \cdots\right.$$

Putting $m = -1$, the complete solution is

$$y = x^{-1}\left\{a_0\left(1 - \frac{x^2}{1\cdot2} + \frac{x^4}{1\cdot2\cdot3\cdot4} - \cdots\right) + a_1\left(x - \frac{x^3}{1\cdot2\cdot3} + \frac{x^5}{1\cdot2\cdot3\cdot4\cdot5}\cdots\right)\right\}$$

$$= x^{-1}(a_0 \cos x + a_1 \sin x).$$

PROBLEMS 16.2

Solve the following equations in power series :

1. $4x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 0.$ (*P.T.U., 2005*)
2. $y'' + xy' + (x^2 + 2)y = 0.$ (*P.T.U., 2007*)
3. $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - y = 0.$
4. $3x\dfrac{d^2y}{dx^2} + (1-x)\dfrac{dy}{dx} - y = 0.$ (*S.V.T.U., 2008*)
5. $x\dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + 2y = 0.$ (*J.N.T.U., 2006*)
6. $2x^2y'' + xy' - (x+1)y = 0.$ (*U.P.T.U., 2005*)
7. $8x^2\dfrac{d^2y}{dx^2} + 10x\dfrac{dy}{dx} - (1+x)y = 0.$ (*P.T.U., 2009*)
8. $2x(1-x)\dfrac{d^2y}{dx^2} + (1-x)\dfrac{dy}{dx} + 3y = 0.$ (*U.P.T.U., 2004*)
9. $x(1-x)\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 0.$
10. $(2x + x^3)\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 6xy = 0.$ (*Bhopal, 2008*)

16.5 BESSEL'S EQUATION*

One of the most important differential equations in applied mathematics is

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \qquad \text{...(1)}$$

which is known as *Bessel's equation of order n*. Its particular solutions are called *Bessel functions of order n*. Many physical problems involving vibrations or heat conduction in cylindrical regions give rise to this equation.

Substituting $y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \ldots$

(1) takes the form

$$a_0(m^2 - n^2)x^m + a_1\{(m+1)^2 - n^2\}x^{m+1} + \{a_2[(m+2)^2 - n^2] + a_0\}x^{m+2} + \ldots = 0.$$

Equating to zero the coefficient of x^m, we obtain the indicial equation $m^2 - n^2 = 0$ (as $a_0 \neq 0$) where $m = n$ or $-n$.

$$a_1 = a_3 = a_5 = a_7 = \ldots = 0$$

and
$$a_2 = -\frac{a_0}{(m+2)^2 - n^2},\ a_4 = -\frac{a_2}{(m+4)^2 - n^2} \text{ etc.}$$

These give
$$y = a_0x^m\left(1 - \frac{1}{(m+2)^2 - n^2}x^2 + \frac{1}{[(m+2)^2 - n^2][(m+4)^2 - n^2]}x^4 - \cdots\right)$$

---

* Named after the German mathematician and astronomer *Friederich Wilhelm Bessel* (1784 – 1846) whose paper on Bessel functions appeared in 1826. He studied Astronomy of his own and became director of Königsberg observatory.

For $m = n$, we get

$$y_1 = a_0 x^n \left\{1 - \frac{1}{4(n+1)} x^2 + \frac{1}{4^2 \cdot 2!(n+1)(n+2)} x^4 - \frac{1}{4^3 \cdot 3!(n+1)(n+2)(n+3)} x^6 + \cdots\right\} \qquad ...(2)$$

and for $m = -n$, we have

$$y_2 = a_0 x^{-n} \left\{1 - \frac{1}{4(-n+1)} x^2 + \frac{1}{4^2 \cdot 2!(-n+1)(-n+2)} x^4 - \frac{1}{4^3 \cdot 3!(-n+1)(-n+2)(-n+3)} x^6 + \cdots\right\} \qquad ...(3)$$

Case I. *When n is not integral or zero,* the complete solution of (1) is $y = c_1 y_1 + c_2 y_2$.

If we take $a_0 = 1/2^n\, \Gamma(n+1)$, then the solution given by (2) is called the *Bessel function of the first kind of order n* and is denoted by $J_n(x)$. Thus

$$J_n(x) = \left(\frac{x}{2}\right)^n \left\{\frac{1}{\Gamma(n+1)} - \frac{1}{1!\,\Gamma(n+2)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\Gamma(n+3)}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\,\Gamma(n+4)}\left(\frac{x}{2}\right)^6 + \cdots\right\} \quad (n > 0)$$

i.e.

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{r!\,\Gamma(n+r+1)} \qquad ...(4)$$

and corresponding to (3), we have $J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{-n+2r} \frac{1}{r!\,\Gamma(-n+r+1)}$ ...(5)

which is called the *Bessel function of the first kind of order – n.*

Hence complete solution of the Bessel's equation (1) may be expressed in the form.

$$y = AJ_n(x) + BJ_{-n}(x). \qquad ...(6)$$

Case II. *When n is zero,* $y_1 = y_2$ and the complete solution of (1), which reduces to the *Bessel's equation of order zero,* is obtained as in Example 16.4.

Case III. *When n is integral,* y_2 fails to give a solution for positive values of n and y_1 fails to give a solution for negative values. Thus another independent integral of the Bessel's equation (1) is needed to form its general solution. We now proceed to find an independent solution of (1), when n is an integer.

Let $y = u(x)J_n(x)$ be a solution of (1). Substituting the values of y, $y' = u'J_n + uJ_n'$ and $y'' = u''J_n + 2u'J_n' + uJ_n''$ in (1), we obtain

$$x^2(u''J_n + 2u'J_n' + uJ_n'') + x(u'J_n + uJ_n') + (x^2 - n^2)uJ_n = 0$$

or

$$u\{x^2J_n'' + xJ_n' + (x^2 - n^2)J_n\} + x^2u''J_n + 2x^2u'J_n' + xu'J_n = 0. \qquad ...(7)$$

Now since J_n is a solution of (1), therefore, $x^2J_n'' + xJ_n' + (x^2 - n^2)J_n = 0$

$\therefore$ (7) reduces to $x^2u''J_n + 2x^2u'J_n' + xu'J_n = 0$.

Dividing throughout by $x^2u'J_n$, it becomes $\dfrac{u''}{u'} + 2\dfrac{J_n'}{J_n} + \dfrac{1}{x} = 0$

i.e.,

$$\frac{d}{dx}(\log u') + 2\frac{d}{dx}(\log J_n) + \frac{d}{dx}(\log x) = 0 \text{ or } \frac{d}{dx}\{\log(u'J_n^2x)\} = 0.$$

Integrating, $\log(u'J_n^2x) = \log B$, whence $xu'J_n^2 = B$.

$$\therefore \qquad u' = \frac{B}{xJ_n^2} \text{ or } u = B\int \frac{dx}{xJ_n^2} + A.$$

Thus

$$y = AJ_n(x) + BJ_n(x)\int \frac{dx}{x[J_n(x)]^2}.$$

Hence the complete solution of the Bessel's equation (1) is

$$y = AJ_n(x) + BY_n(x) \qquad ...(8) \quad (V.T.U., 2006)$$

where

$$Y_n(x) = J_n(x)\int \frac{dx}{x[J_n(x)]^2} \qquad ...(9)$$

$Y_n(x)$ is called the *Bessel function of the second kind of order n or Neumann function**.

* Named after the German mathematician and physicist *Carl Neumann* (1832–1925) whose work on potential theory gave impetus for development of integral equations by *Volterra* of Rome, *Fredholm* of Stockholm and *Hilbert* of Gottingen.

Obs. Putting $k = -n + r$, *i.e.* $r = k + n$, and noting that $\Gamma(k + 1) = k\,!$ where k is an integer, (5) may be written as

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+n}\,(x/2)^{2k+n}}{(k+n)!\,k!} = (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k\,(x/2)^{n+2k}}{k\,!\,\Gamma(k+n+1)}$$

Hence $J_{-n}(x) = (-1)^n J_n(x)$. ...(10) (*Bhopal, 2008 ; S.V.T.U., 2008 ; V.T.U., 2006*)

16.6 RECURRENCE FORMULAE FOR $J_n(x)$

The following recurrence formulae can easily be derived from the series expression for $J_n(x)$:

(1) $\dfrac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$.

(2) $\dfrac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$.

(3) $J_n(x) = \dfrac{x}{2n}[J_{n-1}(x) + J_{n+1}(x)]$.

(4) $J_n'(x) = \dfrac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$.

(5) $J_n'(x) = \dfrac{n}{x} J_n(x) - J_{n+1}(x)$.

(6) $J_{n+1}(x) = \dfrac{2n}{x} J_n(x) - J_{n-1}(x)$.

These formulae are very useful in the solution of boundary value problems and in establishing the various properties of Bessel functions.

Proofs. (1) Multiplying (4) of page 551 by x^n, we have

$$x^n J_n(x) = x^n \sum_{r=0}^{\infty} \frac{(-1)^r\,(x/2)^{n+2r}}{r\,!\,\Gamma(n+r+1)} = \sum_{r=0}^{\infty} \frac{(-1)^r\,x^{2(n+r)}}{2^{n+2r}\,r\,!\,\Gamma(n+r+1)}$$

$$\therefore \quad \frac{d}{dx}[x^n J_n(x)] = \sum_{r=0}^{\infty} \frac{(-1)^r\,2(n+r)x^{2(n+r)-1}}{2^{n+2r}\,r\,!\,\Gamma(n+r+1)} = x^n \sum_{r=0}^{\infty} \frac{(-1)^r\,(x/2)^{n-1+2r}}{r\,!\,\Gamma(n-1+r+1)} = x^n J_{n-1}(x).$$

(*Bhopal, 2008 ; V.T.U., 2005 ; U.P.T.U., 2005*)

(2) Multiplying (4) of page 551 by x^{-n}, we have

$$x^{-n} J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r\,x^{2r}}{2^{n+2r}\,r!\,\Gamma(n+r+1)}$$

$$\therefore \quad \frac{d}{dx}[x^{-n} J_n(x)] = \sum_{r=0}^{\infty} \frac{(-1)^r\,2r\,x^{2r-1}}{2^{n+2r}\,r\,!\,\Gamma(n+r+1)} = -x^{-n} \sum_{r=1}^{\infty} \frac{(-1)^{r-1}\,x^{n+1+2(r-1)}}{2^{n+1+2(r-1)}(r-1)\,!\,\Gamma(n+r+1)}$$

$$= -x^{-n} \sum_{k=0}^{\infty} \frac{(-1)^k\,(x/2)^{n+1+2k}}{k\,!\,\Gamma(n+1+k+1)} = -x^{-n} J_{n+1}(x), \text{ where } k = r-1.$$

(*P.T.U., 2006 ; B.P.T.U., 2005*)

(3) From (1), we have $x^n J_n'(x) + nx^{n-1} J_n(x) = x^n J_{n-1}(x)$

or dividing by x^n, $\quad J_n'(x) + (n/x)J_n(x) = J_{n-1}(x)$ ...(*i*)

Similarly from (2), we get $x^{-n} J_n'(x) - nx^{-n-1} J_n(x) = -x^{-n} J_{n+1}(x)$

or $\quad -J_n'(x) + \dfrac{n}{x} J_n(x) = J_{n+1}(x)$ ...(*ii*)

Adding (*i*) and (*ii*), we obtain $\dfrac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$

i.e., $\quad J_n(x) = \dfrac{x}{2n}[J_{n-1}(x) + J_{n+1}(x)]$ (*S.V.T.U., 2008 ; Anna, 2005 S*)

(4) Subtracting (*ii*) from (*i*), we get $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$

i.e., $\quad J_n'(x) = \dfrac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$. (*S.V.T.U., 2007 ; P.T.U., 2005*)

(5) is another way of writing (*ii*). (*J.N.T.U., 2006 ; Anna, 2005*)

(6) is another way of writing (3). (*Madras, 2006 ; V.T.U., 2005*)

16.7 (1) EXPANSIONS FOR J_0 AND J_1

We have from (4) of page 551,

$$J_0(x) = 1 - \frac{1}{(1!)}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \cdots \quad ...(1)$$

and
$$J_1(x) = \frac{x}{2}\left[1 - \frac{1}{1!\,2!}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,3!}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\,4!}\left(\frac{x}{2}\right)^6 + \cdots\right] \quad \text{(B.P.T.U., 2005) ...(2)}$$

Because of their special importance, the values of $J_0(x)$ and $J_1(x)$ are given in Appendix 2 : Table II to four decimal places at intervals of 0.1. With the help of these values, the graphs of $J_0(x)$ and $J_1(x)$ can be drawn as shown in Fig. 16.1, for $x > 0$. Their close resemblance to graphs of $\cos x$ and $\sin x$ is interesting.

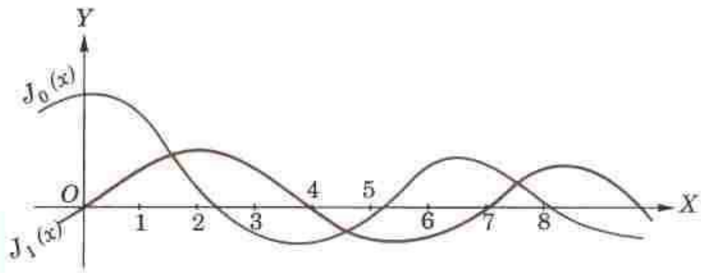

Fig. 16.1

Obs. The roots of the equation $J_0(x) = 0$ are useful in some physical problems. This equation has no complex roots but an infinite number of real roots. Its first four roots are $x = 2.4, 5.52, 8.65, 11.79$ approximately.

16.8 VALUE OF $J_{1/2}$

We may think that $J_0(x)$ is the simplest of the J's but actually $J_{1/2}(x)$ is simpler, for it can be expressed in a finite form. Taking $n = \frac{1}{2}$ in (4) of page 551, we have

$$J_{1/2}(x) = \left(\frac{x}{2}\right)^{1/2}\left\{\frac{1}{\Gamma\left(\frac{3}{2}\right)} - \frac{1}{1!\,\Gamma\left(\frac{5}{2}\right)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\Gamma\left(\frac{7}{2}\right)}\left(\frac{x}{2}\right)^4 - \cdots\right\}$$

$$= \left(\frac{x}{2}\right)^{1/2}\left\{\frac{1}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} - \frac{1}{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^2 + \frac{1}{2\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^4 - \cdots\right\}$$

$$= \frac{\sqrt{x}}{\sqrt{2}\,\Gamma\left(\frac{1}{2}\right)}\left\{\frac{2}{1!} - \frac{2x^2}{3!} + \frac{2x^4}{5!} - \cdots\right\}$$

Now multiplying the series by $x/2$ and outside by $2/x$, we get

$$J_{1/2}(x) = \frac{\sqrt{2}}{\sqrt{x}\sqrt{\pi}}\left\{\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right\} = \sqrt{\left(\frac{2}{\pi x}\right)}\sin x. \quad \text{...(3) (V.T.U., 2009 ; J.N.T.U., 2003)}$$

Similarly taking $n = \frac{1}{2}$ in (5) of page 551, it can be shown that

$$J_{-\frac{1}{2}}(x) = \sqrt{\left(\frac{2}{\pi x}\right)}\cos x. \quad \text{...(4) (Anna, 2005 ; W.B.T.U., 2005 ; V.T.U., 2003)}$$

Example 16.8. *Express $J_5(x)$ in terms of $J_0(x)$ and $J_1(x)$.* *(Bhopal, 2008 S ; V.T.U., 2001)*

Solution. We know that

$$J_n(x) = \frac{x}{2n}[J_{n-1}(x) + J_{n+1}(x)] \text{ i.e. } J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x)$$

Putting $n = 1, 2, 3, 4$ successively, $J_2(x) = \frac{2}{x}J_1(x) - J_0(x)$ ...(*i*) $\qquad J_3(x) = \frac{4}{x}J_2(x) - J_1(x)$ ...(*ii*)

$J_4(x) = \frac{6}{x}J_3(x) - J_2(x)$ ...(*iii*) $\qquad J_5(x) = \frac{8}{x}J_4(x) - J_3(x)$ ...(*iv*)

Substituting the value of $J_2(x)$ in *(ii)*, we have

$$J_3(x) = \frac{4}{x}\left\{\frac{2}{x}J_1(x) - J_0(x)\right\} - J_1(x) = \left(\frac{8}{x^2} - 1\right)J_1(x) - \frac{4}{x}J_0(x) \quad ...(v)$$

(W.B.T.U., 2005 ; Madras, 2003)

Now substituting the values of $J_3(x)$ from *(v)* and $J_2(x)$ from *(i)* in *(iii)*, we get

$$J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right)J_1(x) + \left(1 - \frac{24}{x^2}\right)J_0(x) \quad ...(vi)$$ *(V.T.U., 2003 S)*

Finally putting the values of $J_4(x)$ from *(vi)* and $J_3(x)$ from *(v)* in *(iv)*, we obtain

$$J_5(x) = \left(\frac{384}{x^4} - \frac{72}{x^2} - 1\right)J_1(x) + \left(\frac{12}{x} - \frac{192}{x^3}\right)J_0(x).$$

Example 16.9. *Prove that* $J_{5/2}(x) = \sqrt{\left(\frac{2}{\pi x}\right)}\left\{\frac{3-x^2}{x^2}\sin x - \frac{3}{x}\cos x\right\}.$ *(J.N.T.U., 2006)*

Solution. We know that $J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x)$ ...*(i)*

Putting $n = \frac{1}{2}$, we get $J_{3/2}(x) = \frac{1}{x}J_{1/2}(x) - J_{-1/2}(x) = \sqrt{\left(\frac{2}{\pi x}\right)}\left(\frac{\sin x}{x} - \cos x\right)$

(Bhopal, 2007 ; V.T.U., 2006)

Again putting $n = \frac{3}{2}$ in *(i)*, we get $J_{5/2}(x) = \frac{3}{x}J_{3/2}(x) - J_{1/2}(x)$

$$= \frac{3}{x}\left[\sqrt{\left(\frac{2}{\pi x}\right)}\left(\frac{\sin x}{x} - \cos x\right)\right] - \sqrt{\left(\frac{2}{\pi x}\right)}\sin x = \sqrt{\left(\frac{2}{\pi x}\right)}\left[\frac{3-x^2}{x^2}\sin x - \frac{3}{x}\cos x\right]$$

which is the required result.

Example 16.10. *Prove that*

(a) $J_n''(x) = \frac{1}{4}[J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)].$ *(J.N.T.U., 2006)*

(b) $\frac{d}{dx}[xJ_n(x)J_{n+1}(x)] = x[J_n^2(x) - J_{n+1}^2(x)].$ *(V.T.U., 2006)*

Solution. *(a)* We know that $J_n'(x) = \frac{1}{2}\{J_{n-1}(x) - J_{n+1}(x)\}$ ...*(i)*

Differentiating both sides, we get $J_n''(x) = \frac{1}{2}\{J_{n-1}'(x) - J_{n+1}'(x)\}$ ...*(ii)*

Changing n to $n - 1$ in *(i)*, we obtain $J_{n-1}'(x) = \frac{1}{2}\{J_{n-2}(x) - J_n(x)\}$ ...*(iii)*

Changing n to $n + 1$ in *(i)*, we have $J_{n+1}'(x) = \frac{1}{2}\{J_n(x) - J_{n+2}(x)\}$ ...*(iv)*

Substituting the values of $J_{n-1}'(x)$ and $J_{n+1}'(x)$ from *(iii)* and *(iv)* in *(ii)*, we get

$$J_n'' = \frac{1}{4}[J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)]$$

(b) $\frac{d}{dx}[xJ_n(x)J_{n+1}(x)] = J_n(x)J_{n+1}(x) + x[J_n(x)J'_{n+1}(x) + J_n'(x)J_{n+1}(x)]$ ...*(i)*

From (5) of § 16.6, we have $J_n'(x) = \frac{n}{x}J_n(x) - J_{n+1}(x)$ ...*(ii)*

Changing n to $n + 1$ in *(i)* of page 499, we get $J'_{n+1}(x) = J_n(x) - \frac{n+1}{x}J_{n+1}(x)$ ...*(iii)*

Now substituting from (*iii*) and (*ii*) in (*i*), we get

$$\frac{d}{dx}[xJ_n(x)\,J_{n+1}(x)] = J_n(x)\,J_{n+1}(x) + x\left[J_n(x)\left\{J_n(x) - \frac{n+1}{x}J_{n+1}(x)\right\} + \left\{\frac{n}{x}J_n(x) - J_{n+1}(x)\right\}J_{n+1}(x)\right]$$

$$= x\,\{J_n^2(x) - J_{n+1}^2(x)\}.$$

Example 16.11. *Prove that :*

(*a*) $\int J_3(x)dx = c - J_2(x) - \frac{2}{x}J_1(x).$

(*b*) $\int xJ_0^2(x)dx = \frac{1}{2}x^2\{J_0^2(x) + J_1^2(x)\}.$ (*U.P.T.U., 2004 ; Osmania, 2002*)

Solution. (*a*) We know that $\frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$ [§ 16.6 (2)] ...(*i*)

or $$\int x^{-n}J_{n+1}(x)dx = -x^{-n}J_n(x) \quad ...(ii)$$

$\therefore$ $$\int J_3(x)\,dx = \int x^2\cdot x^{-2}J_3(x)dx + c \quad \text{[Integrate by parts]}$$

$$= x^2\cdot\int x^{-2}J_3(x)dx - \int 2x\left[\int x^{-2}J_3(x)dx\right]dx + c$$

$$= x^2[-x^{-2}J_2(x)] - \int 2x[-x^{-2}J_2(x)]\,dx + c \quad \text{[By (ii) when } n = 2]$$

$$= c - J_2(x) + \int\frac{2}{x}J_2(x)\,dx = c - J_2(x) - \frac{2}{x}J_1(x) \quad \text{[By (ii) when } n = 1]$$

(*b*) $$\int xJ_0^2(x)dx = \int J_0^2(x)\cdot xdx \quad \text{[Integrate by parts]}$$

$$= J_0^2(x)\cdot\frac{1}{2}x^2 - \int 2J_0(x)J_0'(x)\,.\,\frac{1}{2}x^2\,dx$$

$$= \frac{1}{2}x^2J_0^2(x) + \int x^2J_0(x)J_1(x)\,dx \quad \text{[By (i) when } n = 0]$$

$$= \frac{1}{2}x^2J_0^2(x) + \int xJ_1(x)\cdot\frac{d}{dx}[xJ_1(x)]\,dx \quad \left[\because \frac{d}{dx}[xJ_1(x)] = xJ_0(x) \text{ by § 16.6 (1)}\right]$$

$$= \frac{1}{2}x^2J_0^2(x) + \frac{1}{2}[xJ_1(x)]^2 = \frac{1}{2}x^2[J_0^2(x) + J_1^2(x)].$$

16.9 GENERATING FUNCTION FOR $J_n(x)$

To prove that $$e^{\frac{1}{2}x(t-1/t)} = \sum_{n=-\infty}^{\infty}t^nJ_n(x).$$

We have $$e^{\frac{1}{2}x(t-t^{-1})} = e^{xt/2}\times e^{-x/2t}$$

$$= \left[1 + \left(\frac{xt}{2}\right) + \frac{1}{2!}\left(\frac{xt}{2}\right)^2 + \frac{1}{3!}\left(\frac{xt}{2}\right)^3 + \cdots\right]\times\left[1 - \left(\frac{x}{2t}\right) + \frac{1}{2!}\left(\frac{x}{2t}\right)^2 - \frac{1}{3!}\left(\frac{x}{2t}\right)^3 + \cdots\right]$$

The coefficient of t^n in this product

$$= \frac{1}{n!}\left(\frac{x}{2}\right)^n - \frac{1}{(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \frac{1}{2!(n+2)!}\left(\frac{x}{2}\right)^{n+4} - \ldots = J_n(x).$$

As all the integral powers of t, both positive and negative occur, we have

$$e^{\frac{1}{2}x(t-t^{-1})} = J_0(x) + tJ_1(x) + t^2J_2(x) + t^3J_3(x) + \ldots + t^{-1}J_{-1}(x) + t^{-2}J_{-2}(x) + t^{-3}J_{-3}(x) + \ldots$$

$$= \sum_{n=-\infty}^{\infty}t^nJ_n(x) \quad (V.T.U., 2007)$$

This shows that Bessel functions of various orders can be derived as coefficients of different powers of t in the expansion of $e^{\frac{1}{2}x(t-1/t)}$. For this reason, it is known as the *generating function of Bessel functions*.

Example 16.12. *Show that*

(*a*) $J_n(x) = \dfrac{1}{\pi}\displaystyle\int_0^\pi \cos(n\theta - x\sin\theta)\, d\theta$, *n being an integer.* (*V.T.U., 2006*)

(*b*) $J_0(x) = \dfrac{1}{\pi}\displaystyle\int_0^\pi \cos(x\cos\phi)\, d\phi$. (*Madras, 2006*)

(*c*) $J_0^2 + 2J_1^2 + 2J_2^2 + 2J_3^2 + \ldots = 1$. (*Kerala M. Tech, 2005 ; U.P.T.U., 2003; V.T.U., 2003 S*)

Solution. (*a*) We know that

$$e^{\frac{1}{2}x(t-1/t)} = J_0(x) + tJ_1(x) + t^2J_2(x) + t^3J_3(x) + \ldots + t^{-1}J_{-1}(x) + t^{-2}J_{-2}(x) + t^{-3}J_{-3}(x) + \ldots$$

Since $J_{-n}(x) = (-1)^n J_n(x)$

$$\therefore \quad e^{\frac{1}{2}x(t-1/t)} = J_0 + J_1(t - 1/t) + J_2(t^2 + 1/t^2) + J_3(t^3 - 1/t^3) + \ldots \qquad \ldots(i)$$

Now put $t = \cos\theta + i\sin\theta$

so that $t^p = \cos p\theta + i\sin p\theta$ and $1/t^p = \cos p\theta - i\sin p\theta$

giving $t^p + 1/t^p = 2\cos p\theta$ and $t^p - 1/t^p = 2i\sin p\theta$.

Substituting these in (*i*), we get

$$e^{ix\sin\theta} = J_0 + 2[J_2\cos 2\theta + J_4\cos 4\theta + \ldots] + 2i\,[J_1\sin\theta + J_3\sin 3\theta + \ldots] \qquad \ldots(ii)$$

Since $e^{ix\sin\theta} = \cos(x\sin\theta) + i\sin(x\sin\theta)$.

$\therefore$ equating the real and imaginary parts in (*ii*), we get

$$\cos(x\sin\theta) = J_0 + 2\,[J_2\cos 2\theta + J_4\cos 4\theta + \ldots] \qquad \ldots(iii)$$

$$\sin(x\sin\theta) = 2[J_1\sin\theta + J_3\sin 3\theta + \ldots] \qquad \ldots(iv)$$

which are known as *Jacobi series**. (*V.T.U., 2006*)

Now multiplying both sides of (*iii*) by $\cos n\theta$ and both sides of (*iv*) by $\sin n\theta$ and integrating each of the resulting expressions between 0 and π, we obtain

$$\frac{1}{\pi}\int_0^\pi \cos(x\sin\theta)\cos n\theta\, d\theta = \begin{cases} J_n(x), & n \text{ even or zero} \\ 0, & n \text{ odd} \end{cases}$$

and

$$\frac{1}{\pi}\int_0^\pi \sin(x\sin\theta)\sin n\theta\, d\theta = \begin{cases} 0 & n \text{ even} \\ J_n(x), & n \text{ odd} \end{cases}$$

Hence generally, if n is a positive integer,

$$J_n(x) = \frac{1}{\pi}\int_0^\pi [\cos(x\sin\theta)\cos n\theta + \sin(x\sin\theta)\sin n\theta]\, d\theta = \frac{1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta)\, d\theta.$$

[This is Bessel's original definition of $J_n(x)$ given in 1824 while investigating Planetary motion.]

(*b*) Changing θ to $\frac{1}{2}\pi - \phi$ in (*iii*), we get

$$\cos(x\cos\phi) = J_0 + 2J_2\cos(\pi - 2\phi) + 2J_4\cos(2\pi - 4\phi) + \ldots$$
$$= J_0 - 2J_2\cos 2\phi + 2J_4\cos 4\phi - \ldots$$

Integrating both sides w.r.t. ϕ from 0 to π, we get

$$\int_0^\pi \cos(x\cos\phi)\, d\phi = \int_0^\pi [J_0(x) - 2J_2(x)\cos 2\phi + 2J_4(x)\cos 4\phi - \cdots]\, d\phi$$

$$= \left| J_0(x)\cdot\phi - 2J_2(x)\cdot\frac{1}{2}\sin 2\phi + 2J_4(x)\cdot\frac{1}{4}\sin 4\phi - \cdots \right|_0^\pi = J_0(x)\,.\,\pi \text{ whence follows the result.}$$

* See footnot p. 215.

(c) Squaring (iii) and (iv) and integrating w.r.t. ϕ from 0 to π and noting that (m, n being integers),

$$\int_0^\pi \cos m\theta \cos n\theta \, d\theta = \int_0^\pi \sin m\theta \sin n\theta \, d\theta = 0, \qquad (m \neq n)$$

and
$$\int_0^\pi \cos^2 n\theta \, d\theta = \int_0^\pi \sin^2 n\theta \, d\theta = \pi/2, \text{ we obtain}$$

$$[J_0(x)]^2 \pi + 4 [J_2(x)]^2 \frac{\pi}{2} + 4 [J_4(x)]^2 \frac{\pi}{2} + \ldots = \int_0^\pi \cos^2 (x \sin\theta) \, d\theta$$

$$4 [J_1(x)]^2 \frac{\pi}{2} + 4 [J_3(x)]^2 \frac{\pi}{2} + \ldots = \int_0^\pi \sin^2 (x \sin\theta) \, d\theta$$

Adding, $$\pi [J_0^2 + 2J_1^2 + 2J_2^2 + 2J_3^2 + \cdots] = \int_0^\pi d\theta = \pi$$

Hence $$J_0^2 + 2J_1^2 + 2J_2^2 + 2J_3^2 + \cdots = 1.$$

PROBLEMS 16.3

1. Compute $J_0(2)$, $J_1(1)$ correct to three decimal places.
2. Show that (i) $J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) + \left(1 - \frac{24}{x^4}\right) J_0(x)$. (ii) $J_1(x) + J_3(x) = \frac{4}{x} J_2(x)$ (P.T.U., 2003)
3. Show that
   (i) $J_{-1/2}(x) = J_{1/2}(x) \cot x$. (S.V.T.U., 2008) (ii) $J'_{1/2}(x) J_{-1/2}(x) - J'_{-1/2}(x) J_{1/2}(x) = 2/\pi x$ (Delhi, 2002)
   (iii) $J_{-3/2}(x) = -\sqrt{\left(\frac{2}{\pi x}\right)}\left(\sin x + \frac{\cos x}{x}\right)$ (iv) $J_{-5/2}(x) = \sqrt{\left(\frac{2}{\pi x}\right)}\left(\frac{3}{x}\sin x + \frac{3 - x^2}{x^2}\cos x\right)$
   (V.T.U., 2000)
4. Prove that (i) $\frac{d}{dx} J_0(x) = -J_1(x)$. (ii) $\frac{d}{dx} [xJ_1(x)] = xJ_0(x)$.
   (iii) $\frac{d}{dx} [x^n J_n(ax)] = ax^n J_{n-1}(ax)$. (Madras, 2000 S) (iv) $J'_n(x) = -\frac{n}{2} J_n(x) + J_{n-1}(x)$ (P.T.U., 2009 S)
5. Show by the use of recurrence formula, that
   (i) $J_0''(x) = \frac{1}{2} [J_2(x) - J_0(x)]$ (ii) $J_1''(x) = J_1(x) - \frac{1}{x} J_2(x)$.
   (iii) $4J_0'''(x) + 3J_0'(x) + J_3(x) = 0$. (Osmania, 2003)
6. Prove that
   (i) $\frac{d}{dx} \{J_n^2(x)\} = \frac{x}{2n} [J_{n-1}^2(x) - J_{n+1}^2(x)]$ (S.V.T.U., 2008 ; Kerala M.E., 2005)
   (ii) $\frac{d}{dx} \{J_n^2(x) + J_{n+1}^2(x)\} = 2\left\{\frac{n}{2} J_n^2(x) - \frac{n+1}{x} J_{n+1}^2(x)\right\}$. (U.P.T.U., 2005 ; V.T.U., 2000 S)
7. Prove that (i) $\int_0^{\pi/2} \sqrt{(\pi x)} \, J_{1/2}(2x) \, dx = 1$. (P.T.U., 2005) (ii) $\int_0^r x J_0(ax) dx = \frac{r}{a} J_1(ar)$.
   (iii) $\int x^2 J_1(x) dx = x^2 J_2(x)$. (P.T.U., 2007)
8. Prove that (i) $\int J_0(x) J_1(x) \, dx = -\frac{1}{2} [J_0(x)]^2$. (ii) $\int_0^\infty e^{-ax} J_0(bx) dx = \frac{1}{\sqrt{a^2 + b^2}}$.
9. Starting with the series of § 16.9, prove that
   $$2nJ_n(x) = x[J_{n-1}(x) + J_{n+1}(x)] \text{ and } xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x).$$
10. Establish the *Jacobi series*
    $$\cos(x \cos\theta) = J_0 - 2J_2 \cos 2\theta + 2J_4 \cos 4\theta - \ldots$$
    $$\sin(x \cos\theta) = 2[J_1 \cos\theta - J_3 \cos 3\theta + J_5 \cos 5\theta - \ldots]$$ (Madras, 2003 S)
11. Prove that (i) $\sin x = 2[J_1 - J_3 + J_5 - \ldots]$ (Anna, 2005 S)
    (ii) $\cos x = J_0 - 2J_2 + 2J_4 - 2J_6 + \ldots$ (Kerala M. Tech., 2005)
    (iii) $1 = J_0 + 2J_2 + 2J_4 + 2J_6 + \ldots$

16.10 EQUATIONS REDUCIBLE TO BESSEL'S EQUATION

In many problems, we come across such differential equations which can easily be reduced to Bessel's equation and, therefore, can be solved by means of Bessel functions.

(1) *To reduce the differential equation* $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + (k^2x^2 - n^2)y = 0$ *to Bessel form.*

Put $t = kx$, so that $\dfrac{dy}{dx} = k\dfrac{dy}{dt}$ and $\dfrac{d^2y}{dx^2} = k^2\dfrac{d^2y}{dt^2}$.

Then (**1**) becomes $t^2\dfrac{d^2y}{dt^2} + t\dfrac{dy}{dt} + (t^2 - n^2)y = 0$

$\therefore$ its solution is $y = c_1J_n(t) + c_2J_{-n}(t)$, n is non-integral,

or $\quad y = c_1J_n(t) + c_2Y_n(t)$, n is integral.

Hence the solution of (1) is

$y = c_1J_n(kx) + c_2J_{-n}(kx)$, n is non-integral

or $\quad y = c_1J_n(kx) + c_2Y_n(kx)$, n is integral.

(2) *To reduce the differential equation* $x\dfrac{d^2y}{dx^2} + a\dfrac{dy}{dx} + k^2xy = 0$ *to Bessel's equation,* *(Madras, 2006)*

put $\quad y = x^nz,$

so that $\quad \dfrac{dy}{dx} = x^n\dfrac{dz}{dx} + nx^{n-1}z$ and $\dfrac{d^2y}{dx^2} = x^n\dfrac{d^2z}{dx^2} + 2nx^{n-1}\dfrac{dz}{dx} + n(n-1)x^{n-2}z$

Then (**2**) takes the form $x^{n+1}\dfrac{d^2z}{dx^2} + (2n+a)x^n\dfrac{dz}{dx} + [k^2x^2 + n^2 + (a-1)n]x^{n-1}z = 0.$

Dividing throughout by x^{n-1} and putting $2n + a = 1$,

$$x^2\frac{d^2z}{dx^2} + x\frac{dz}{dx} + (k^2x^2 - n^2)z = 0.$$

Its solution by (1) is $z = c_1J_n(kx) + c_2J_{-n}(kx)$, n is non-integral

or $\quad z = c_1J_n(kx) + c_2Y_n(kx)$, n is integral

Hence the solution of (2) is $y = x^n[c_1J_n(kx) + c_2J_{-n}(kx)]$, n is non-integral

or $\quad y = x^n[c_1J_n(kx) + c_2Y_n(kx)]$, n is integral, where $n = (1-a)/2$.

(3) *To reduce the differential equation* $x\dfrac{d^2y}{dx^2} + c\dfrac{dy}{dx} + k^2x^ry = 0$ *to Bessel form,* put $x = t^m$, *i.e.* $t = x^{1/m}$,

so that $\quad \dfrac{dy}{dx} = \dfrac{dy}{dt}\cdot\dfrac{dt}{dx} = \dfrac{1}{m}t^{1-m}\dfrac{dy}{dt}$

and $\quad \dfrac{d^2y}{dx^2} = \dfrac{d}{dt}\left(\dfrac{1}{m}t^{1-m}\dfrac{dy}{dt}\right)\cdot\dfrac{1}{m}t^{1-m}\ \dfrac{1}{m^2}t^{2-2m}\dfrac{d^2y}{dt^2} + \dfrac{1-m}{m^2}t^{1-2m}\dfrac{dy}{dt}$

Then (**3**) takes the form $\dfrac{1}{m^2}t^{2-m}\dfrac{d^2y}{dt^2} + \dfrac{1-m+cm}{m^2}t^{1-m}\dfrac{dy}{dt} + k^2t^{mr}y = 0$

or multiplying throughout by m^2/t^{1-m}, $t\dfrac{d^2y}{dt^2} + (1-m+cm)\dfrac{dy}{dt} + (km)^2t^{mr+m-1}y = 0.$

In order to reduce it to (2), we set $mr + m - 1 = 1$, *i.e.* $m = 2/(r+1)$

and $\quad a = 1 - m + cm = (r + 2c - 1)/(r+1).$

Thus it reduces to $t\dfrac{d^2y}{dt^2} + a\dfrac{dy}{dt} + (km)^2ty = 0$ which is similar to (2).

Hence the solution of (3) is $y = x^{n/m}\{c_1J_n(k_mx^{1/m}) + c_2J_{-n}(kmx^{1/m})\}$, n is a fraction

or $\quad y = x^{n/m}\{c_1J_n(kmx^{1/m}) + c_2Y_n(kmx^{1/m})\}$, n is an integer

where $\quad n = \dfrac{1-a}{2} = \dfrac{1-c}{1+r}$ and $m = \dfrac{2}{1+r}$.

Example 16.13. *Solve the differential equations :*

(i) $y'' + \dfrac{y'}{x} + \left(8 - \dfrac{1}{x^2}\right) y = 0.$ *(ii)* $4y'' + 9xy = 0.$ *(iii)* $xy'' + y' + \dfrac{1}{4}y = 0.$ (Anna, 2005)

Solution. (*i*) Rewriting the given equation as $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + (8x^2 - 1)y = 0,$

and comparing with (**1**) above, we see that $n = 1$ and $k = 2\sqrt{2}$.

$\therefore$ The solution of the given equation is $y = c_1J_n(kx) + c_2Y_n(kx)$

i.e.,
$$y = c_1J_1(2\sqrt{2}x) + c_2Y_1(2\sqrt{2}x).$$

(*ii*) Rewriting the given equation as $x\dfrac{d^2y}{dx^2} + \dfrac{9}{4}x^2y = 0$ ...(α)

and comparing with (**3**) above, we find that $c = 0$, $k = 3/2$ and $r = 2$.

$$\therefore \quad n = \frac{1-c}{1+r} = \frac{1}{3}, \quad m = \frac{2}{1+r} = \frac{2}{3} \quad \text{and} \quad \frac{n}{m} = \frac{1}{2}.$$

Hence the solution of (α) is $y = x^{n/m}\{c_1J_n(kmx^{1/m}) + c_2Y_{-n}(kmx^{1/m})\}$

$$y = \sqrt{x}\,[c_1J_{1/3}(x^{3/2}) + c_2J_{-1/3}(x^{3/2})].$$

(*iii*) Multiplying by x, the given equation becomes

$$x^2y'' + xy' + \frac{1}{4}xy = 0 \qquad ...(\alpha)$$

Comparing with (**3**) above, we get $c = 1$, $k = 1/2$ & $r = 0$. $\therefore\ m = \dfrac{2}{1+r} = 2,\ n = \dfrac{1-c}{1+r} = 0\ \&\ \dfrac{n}{m} = 0$

Hence the solution of (α)

$$y = x^{n/m}\{c_1J_n(kmx^{1/m}) + c_2Y_n(kmx^{1/m})\} = x^0\left\{c_1J_0\left(\frac{1}{2}2x^{1/2}\right) + c_2Y_0\left(\frac{1}{2}.2x^{1/2}\right)\right\}$$

i.e.,
$$y = c_1J_0(\sqrt{x}) + c_2Y_0(\sqrt{x})$$

16.11 (1) ORTHOGONALITY OF BESSEL FUNCTIONS

We shall prove that

$$\int_0^1 xJ_n(\alpha x)J_n(\beta x)\,dx = \begin{cases} 0, & \alpha \neq \beta \\ \dfrac{1}{2}[J_{n+1}(\alpha)]^2, & \alpha = \beta \end{cases}, \textit{ where } \alpha, \beta \textit{ are the roots of } J_n(x) = 0.$$

We know that the solution of the equation

$$x^2u'' + xu' + (\alpha^2x^2 - n^2)u = 0 \qquad ...(1)$$

and
$$x^2v'' + xv' + (\beta^2x^2 - n^2)v = 0 \qquad ...(2)$$

are
$$u = J_n(\alpha x) \text{ and } v = J_n(\beta x) \text{ respectively.}$$

Multiplying (1) by v/x and (2) by u/x and subtracting, we get

$$x(u''v - uv'') + (u'v - uv') + (\alpha^2 - \beta^2)xuv = 0$$

or
$$\frac{d}{dx}\{x(u'v - uv')\} = (\beta^2 - \alpha^2)\,xuv.$$

Now integrating both sides from 0 to 1,

$$(\beta^2 - \alpha^2)\int_0^1 xuv\,dx = \left[x(u'v - uv')\right]_0^1 = (u'v - uv')_{x=1} \qquad ...(3)$$

Since
$$u = J_n(\alpha x),$$

$$\therefore \quad u' = \frac{d}{dx}[J_n(\alpha x)] = \frac{d}{d(\alpha x)}[J_n(\alpha x)].\frac{d(\alpha x)}{dx} = \alpha J_n'(\alpha x)$$

Similarly, $v = J_n(\beta x)$ and $v' = \beta J_n'(\beta x)$. Substituting these values in (3), we get

$$\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = \frac{\alpha J_n'(\alpha)J_n(\beta) - \beta J_n(\alpha)J_n'(\beta)}{\beta^2 - \alpha^2} \quad ...(4)$$

If α and β are distinct roots of $J_n(x) = 0$, then $J_n(\alpha) = J_n(\beta) = 0$, and (4) reduces to

$$\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = 0 \quad ...(5)$$

This is known as the *orthogonality relation of Bessel functions.*

When $\beta = \alpha$, the right side of (4) is of 0/0 form. Its value can be found by considering α as a root of $J_n(x) = 0$ and β as a variable approaching α. Then (4) gives

$$\underset{\beta\to\alpha}{\text{Lt}} \int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = \underset{\beta\to\alpha}{\text{Lt}} \frac{\alpha J_n'(\alpha)J_n(\beta)}{\beta^2 - \alpha^2}$$

or by L'Hospital's rule, $\displaystyle\int_0^1 xJ_n^2(\alpha x)dx = \underset{\beta\to\alpha}{\text{Lt}} \frac{\alpha J_n'(\alpha)J_n'(\beta)}{2\beta} = \frac{1}{2}[J_n'(\alpha)]^2$

$$= \frac{1}{2}[J_{n+1}(\alpha)]^2 \quad ...(6) \quad \text{[By (5) of p. 552]}$$

Obs. If however, the interval be from 0 to 1, it can be shown that

$$\int_0^1 xJ_n^2(\alpha x)\,dx = \frac{1}{2}[J_n'(\alpha)]^2 \quad \text{where } \alpha \text{ is the root of } J_n(x) = 0. \quad ...(7) \quad \text{(V.T.U., 2006)}$$

(2) Fourier-Bessel expansion. If $f(x)$ is a continuous function having finite number of oscillations in the interval $(0, a)$, then we can write

$$f(x) = c_1J_n(\alpha_1 x) + c_2J_n(\alpha_2 x) + ... + c_nJ_n(\alpha_n x) + ... \quad ...(8)$$

where α_1, α_2 ... are the positive roots of $J_n(x) = 0$.

To determine the coefficients c_n, multiply both sides of (8) by $xJ_n(\alpha_n x)$ and integrate from 0 to a. Then all integrals on the right of (1) vanish by (5), except the term in c_n. This gives

$$\int_0^a xf(x)J_n(\alpha_n x)dx = c_n\int_0^a xJ_n^2(\alpha_n x)dx = c_n\frac{a^2}{2}J_{n+1}^2(a\alpha_n) \quad \text{[By (7)]}$$

$$\therefore \quad c_n = \frac{2}{a^2J_{n+1}^2(a\alpha_n)}\int_0^a xf(x)J_n(\alpha_n x)\,dx$$

Equation (8) is known as the *Fourier-Bessel expansion of f (x).*

Example 16.14. *If $\alpha_1, \alpha_2, ... \alpha_n$ are the positive roots of $J_0(x) = 0$, show that*

$$\frac{1}{2} = \sum_{n=1}^{\infty} [J_0(\alpha_n x)/\alpha_n J_1(\alpha_n)].$$

Solution. If $\quad f(x) = c_1J_n(\alpha_1 x) + c_2J_n(\alpha_2 x) + ... + c_rJ_n(\alpha_n x) + ... \quad ...(i)$

then
$$c_r = \frac{2}{a^2J_{n+1}^2(a\alpha_r)}\int_0^a x\,f(x)J_n(\alpha_n x)dx$$

Taking $f(x) = 1$, $a = 1$ and $n = 0$, we get

$$c_r = \frac{2}{J_1^2(\alpha_r)}\int_0^1 xJ_0(\alpha_r x)dx = \frac{2}{J_1^2(\alpha_r)}\left|\frac{xJ_1(\alpha_r x)}{\alpha_r}\right|_0^1 = \frac{2}{\alpha_rJ_1(\alpha_r)}$$

From (i),
$$1 = \sum_{r=1}^{\infty}\frac{2}{\alpha_rJ_1(\alpha_r)}J_0(\alpha_r x) \quad \text{or} \quad \frac{1}{2} = \sum_{n=1}^{\infty}\frac{J_0(\alpha_n x)}{\alpha_nJ_1(\alpha_n)}.$$

Example 16.15. *Expand $f(x) = x^2$ in the interval $0 < x < 2$ in terms of $J_2(\alpha_n x)$, where α_n are determined by $J_2(2\alpha_n) = 0$.*

Solution. Let the Fourier-Bessel expansion of $f(x)$ be $x^2 = \sum_{n=1}^{\infty} c_nJ_2(\alpha_n x)$.

Multiplying both sides by $xJ_2(\alpha_n x)$ and integrating w.r.t. x from 0 to 2, we get

$$\int_0^2 x^3 J_2(\alpha_n x)dx = c_n \int_0^2 xJ_2^2(\alpha_n x)dx = c_n \frac{(2)^2}{2} J_3^2(2\alpha_n) \quad \text{[By (7)]}$$

or
$$\left| \frac{x^3 J_3(\alpha_n x)}{\alpha_n} \right|_0^2 = 2c_n J_3^2(2\alpha_n)$$

$\therefore$
$$c_n = \frac{4}{\alpha_n J_3(2\alpha_n)}$$

Hence
$$x^2 = 4\sum_{n=1}^{\infty} \frac{J_2(\alpha_n x)}{\alpha_n J_3(2\alpha_n)}.$$

16.12 BER AND BEI FUNCTIONS

Consider the differential equation

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} - ixy = 0 \quad ...(1)$$

which occurs in certain problems of electrical engineering. This is equation (1) of §16.10 with $n = 0$ and $k^2 = -i$, so that its particular solution is

$$y = J_0(kx) = J_0[(-i)^{1/2}x] = J_0(i^{3/2}x)$$

Replacing $i^{3/2}x$ in the series for $J_0(x)$ [§16.8], we get

$$y = 1 - \frac{i^3x^2}{2^2} + \frac{i^6x^4}{(2\,!)^2\,2^4} - \frac{i^9x^6}{(3\,!)^2 2^6} + \frac{i^{12}x^8}{(4!)^2\,2^8} - ...$$

$$= \left[1 - \frac{x^4}{2^2\cdot 4^2} + \frac{x^8}{2^2\cdot 4^2\cdot 6^2\cdot 8^2} - ...\right] + i\left[\frac{x^2}{2^2} - \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \frac{x^{10}}{2^2\cdot 4^2\cdot 6^2\cdot 8^2\cdot 10^2} - ...\right] \quad ...(2)$$

which is complex for x real. The series in the above brackets are taken to define *Bessel-real* (*or ber*) and *Bessel-imaginary* (*or bei*) *functions.*

Thus
$$ber\, x = 1 + \sum_{m=1}^{\infty} (-1)^m \frac{x^{4m}}{2^2\cdot 4^2\cdot 6^2 ...(4m)^2} \quad ...(3)$$

and
$$bei\, x = -\sum_{m=1}^{\infty} (-1)^m \frac{x^{4m-2}}{2^2\cdot 4^2\cdot 6^2 ...(4m-2)^2} \quad ...(4)$$

so that $y = ber\, x + i\, bei\, x$ is a solution of (1).

Tables giving numerical values of $ber\, x$ and $bei\, x$ are also available.

Example 16.16. *Prove that* (*i*) $\frac{d}{dx}(x\, ber'\, x) = -x\, bei\, x$ (*ii*) $\frac{d}{dx}(x\, bei'\, x) = x\, ber\, x.$

Solution. We have $x\, ber' x = x\sum_{m=1}^{\infty} (-1)^m \frac{4mx^{4m-1}}{2^2\cdot 4^2\cdot 6^2 ...(4m)^2}$

$$= \sum_{m=1}^{\infty} (-1)^m \cdot \frac{x^{4m}}{2^2\cdot 4^2\cdot 6^2 ...(4m-2)^2\, 4m} = -\int_0^{\infty} x\, bei\, x\, dx$$

or
$$\frac{d}{dx}(x\, ber'\, x) = -x\, bei\, x$$

Again
$$\int_0^x x\, ber\, x\, dx = \frac{x^2}{2} + \sum_{m=1}^{\infty} (-1)^m \frac{x^{4m+2}}{2^2\cdot 4^2\cdot 6^2 ...(4m)^2(4m+2)}$$

$$= -\sum_{m=1}^{\infty} (-1)^m \frac{x^{4m-2}}{2^2\cdot 4^2\cdot 6^2 ...(4m-4)^2(4m-2)} = x\, bei'\, x \quad \text{or} \quad \frac{d}{dx}(x\, bei'\, x) = x\, ber\, x.$$

PROBLEMS 16.4

Obtain the solutions of the following differential equations in terms of Bessel functions :

1. $y'' + \frac{y'}{x} + \left(1 - \frac{1}{9x^2}\right) y = 0.$
2. $y'' + \frac{y'}{2} + \left(1 - \frac{1}{6.25x^2}\right) y = 0.$
3. $xy'' + \alpha y' + k^2 xy = 0.$ (V.T.U., 2010)
4. $x^2y'' - xy' + 4x^2y = 0.$
5. $xy'' + y = 0.$
6. Show that (i) $x^n J_n(x)$ is a solution of the equation $xy'' + (1 - 2n) y' + xy = 0.$ (V.T.U., 2001)
   (ii) $x^{-n} J_n(x)$ is a solution of the equation $xy'' + (1 + 2n)y' + xy = 0.$
7. Show that under the transformation $y = u/\sqrt{x}$, Bessel equation becomes

   $u'' + \left(1 + \frac{1 - 4n^2}{4x^2}\right) u = 0.$ Hence find the solution of this equation.
8. By the use of substitution $y = u/\sqrt{x}$, show that the solution of the equation $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - \frac{1}{4}\right) y = 0$ can be written in the form $y = c_1 \frac{\sin x}{\sqrt{x}} + c_2 \frac{\cos x}{\sqrt{x}}.$
9. Show that $\int_0^p x(ber^2 x + bei^2 x)\, dx = p(ber\, p\, bei'\, p - bei\, p\, ber'p).$
10. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the positive roots of $J_0(x) = 0$, prove that

$$x^2 = 2\sum_{n=1}^{\infty} \frac{\alpha_n^2 - 4}{\alpha_n^3 J_1(\alpha_n)} J_0(\alpha_n x).$$

11. Expand $f(x) = x^3$ in the interval $0 < x < 3$ in terms of functions $J_1(\alpha_n x)$ where α_n are determined by $J_1(3\alpha) = 0.$

16.13 LEGENDRE'S EQUATION*

Another differential equation of importance in Applied Mathematics, particularly in boundary value problems for spheres, is *Legendre's equation,*

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \qquad \ldots(1)$$

Here n is a real number. But in most applications only integral values of n are required.

Substituting $\quad y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \ldots\ (a_0 \neq 0),$

(1) takes the form

$$a_0(m)(m-1)x^{m-2} + a_1(m+1)mx^{m-1} + \ldots + [a_{r+2}(m+r+2)(m+r+1)$$
$$-\{(m+r)(m+r+1) - n(n+1)\}a_r]x^{m+r} + \ldots = 0$$

Equating to zero the coefficient of the lowest power of x, *i.e.*, of x^{m-2}, we get

$$a_0 m(m-1) = 0,\ m = 0, 1 \qquad [\because\ a_0 \neq 0]$$

Equating to zero the coefficients of x^{m-1} and x^{m+r}, we get $a_1(m+1)m = 0 \qquad \ldots(2)$

$$a_{r+2}(m+r+2)(m+r+1) - \{(m+r)(m+r+1) - n(n+1)\}a_r = 0 \qquad \ldots(3)$$

When $m = 0$, (2) is satisfied and therefore, $a_1 \neq 0$. Then (3) gives, taking $r = 0, 1, 2, \ldots$ in turn,

$$a_2 = -\frac{n(n+1)}{2!}a_0, \quad a_3 = -\frac{(n-1)(n+2)}{3!}a_1$$

$$a_4 = \frac{-(n-2)(n+3)}{4.3}a_2 = \frac{n(n-2)(n+1)(n+3)}{4!}a_0$$

$$a_5 = -\frac{(n-3)(n+4)}{5.4}a_3 = \frac{(n-1)(n-3)(n+2)(n+4)}{5!}a_1, \text{ etc.}$$

Hence for $m = 0$, there are two independent solutions of (1) :

$$y_1 = a_0\left\{1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - \ldots\ldots\right\} \qquad \ldots(4)$$

*See footnote p. 493.

$$y_2 = a_1 \left\{ x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} x^5 - \ldots\ldots \right\} \quad \ldots(5)$$

When $m = 1$, (2) shows that $a_1 = 0$. Therefore, (3) gives

$$a_3 = a_5 = a_7 = \ldots = 0$$

and

$$a_2 = -\frac{(n-1)(n+2)}{3!} a_0$$

$$a_4 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!} a_0, \text{ etc.}$$

Thus for $m = 1$, we get the solution (5) again. Hence $y = y_1 + y_2$ is the general solution of (1).

If n is a positive even integer, the series (4) terminates at the term in x^n and y_1 becomes a polynomial. Similarly if n is an odd integer, (5) becomes a polynomial of degree n. Thus, whenever n is a positive integer, the general solution of (1) consists of a polynomial solution and an infinite series solution.

These polynomial solutions, with a_0 or a_1 so chosen that the value of the polynomial is 1 for $x = 1$, are called *Legendre polynomials* of order n and are denoted by $P_n(x)$. The infinite series solution with (a_0 or a_1 properly chosen) is called *Legendre function of the second kind* and is denoted by $Q_n(x)$. (*V.T.U., 2006*)

16.14 (1) RODRIGUE'S FORMULA*

We shall prove that $P_n(x) = \dfrac{1}{n!\,2^n} \dfrac{d^n}{dx^n}(x^2-1)^n$ ...(1)

Let $v = (x^2-1)^n$. Then $v_1 = \dfrac{dv}{dx} = 2nx(x^2-1)^{n-1}$

i.e., $(1-x^2)v_1 + 2nxv = 0$ ...(2)

Differentiating (2), $(n+1)$ times by Leibnitz's theorem

$$(1-x^2)v_{n+2} + (n+1)(-2x)v_{n+1} + \frac{1}{2!}(n+1)n(-2)v_n + 2n[xv_{n+1} + (n+1)v_n = 0$$

or

$$(1-x^2)\frac{d^2(v_n)}{dx^2} - 2x\frac{d(v_n)}{dx} + n(n+1)v_n = 0$$

which is Legendre's equation and cv_n is its solution. Also its finite series solution is $P_n(x)$.

$\therefore$ $P_n(x) = cv_n = c\dfrac{d^n}{dx^n}(x^2-1)^n$ ...(3)

To determine the constant c, put $x = 1$ in (3). Then

$$1 = c\left[\frac{d^n}{dx^n}\{(x-1)^n(x+1)^n\}\right]_{x=1}$$

$$= c[n!\,(x+1)^n]$$

+ terms containing $(x-1)$ and its powers$]_{x=1}$

$$= c \,.\, n!\,2^n, \text{ i.e. } c = 1/n!\,2^n.$$

Substituting this value of c in (3), we get (1), which is known as the *Rodrigue's formula*.

(*V.T.U., 2008 ; Bhopal, 2007 ; U.P.T.U., 2004*)

Obs. All roots of $P_n(x) = 0$ are real and lie between -1 and $+1$. (*Madras, 2003 S*)

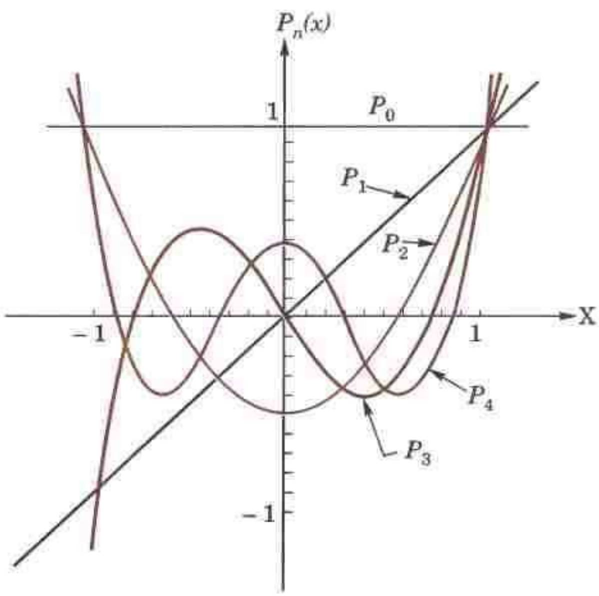

Fig. 16.2. Legendre polynomials.

(2) Legendre polynomials. Using (1), we get

$P_0(x) = 1,$ $\quad P_1(x) = x,$

$P_2(x) = \dfrac{1}{2}(3x^2-1),$ $\quad P_3(x) = \dfrac{1}{2}(5x^3-3x),$

---

* Named after the French mathematician and economist *Olinde Rodrigue* (1794–1851).

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x), \text{ etc.} \qquad (V.T.U., 2009)$$

In general, we have $P_n(x) = \sum_{r=0}^{N} \frac{(-1)^r (2n-2r)!}{2^n r!(n-r)!(n-2r)!} x^{n-2r}$ ...(4)

where $N = \frac{1}{2}n$ or $\frac{1}{2}(n-1)$ according as n is even or odd.

Let us derive (4) from (1).

By Binomial theorem, $(x^2-1)^n = \sum_{r=0}^{n} {}^nC_r (x^2)^{n-r}(-1)^r = \sum_{r=0}^{n} (-1)^r \frac{n!}{r!(n-r)!} x^{2n-2r}$

$$\therefore \text{ by (1), } P_n = \frac{1}{n!2^n} \sum_{n=0}^{n} \frac{(-1)^r n!}{r!(n-r)!} \frac{d^n (x^{2n-2r})}{dx^n} = \sum_{r=0}^{N} \frac{(-1)^r (2n-2r)!}{2^n r!(n-r)!(n-2r)!} x^{n-2r}$$

This is same as (4), and the last term ($r = N$) is such that the power of x (*i.e.*, $n - 2r$) for this term is either 0 or 1.

Example 16.17. *Express $f(x) = x^4 + 3x^3 - x^2 + 5x - 2$ in terms of Legendre polynomials.* *(V.T.U., 2010 ; S.V.T.U., 2007)*

Solution. Since $P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$. $\therefore x^4 = \frac{8}{35}P_4(x) + \frac{6}{7}x^2 - \frac{3}{35}$

$$\therefore \quad f(x) = \left[\frac{8}{35}P_4(x) + \frac{6}{7}x^2 - \frac{3}{35}\right] + 3x^3 - x^2 + 5x - 2$$

$$= \frac{8}{35}P_4(x) + 3x^3 - \frac{1}{7}x^2 + 5x - \frac{73}{35} \qquad \left[\because x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}x ; x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}\right]$$

$$= \frac{8}{35}P_4(x) + 3\left[\frac{2}{5}P_3(x) + \frac{3}{5}x\right] - \frac{1}{7}\left[\frac{2}{3}P_2(x) + \frac{1}{3}\right] + 5x - \frac{73}{35}$$

$$= \frac{8}{35}P_4(x) + \frac{6}{5}P_3(x) - \frac{2}{21}P_2(x) + \frac{34}{5}x - \frac{224}{105} \qquad [\because x = P_1(x), 1 = P_0(x)]$$

$$= \frac{8}{35}P_4(x) + \frac{6}{5}P_3(x) - \frac{2}{21}P_2(x) + \frac{34}{5}P_1x - \frac{224}{105}P_0(x).$$

Example 16.18. *Show that for any function $f(x)$, for which the nth derivative is continuous,*

$$\int_{-1}^{1} f(x)\, P_n(x) dx = \frac{1}{2^n\, n!} \int_{-1}^{1} (1-x^2)^n f^n(x)\, dx.$$

Solution. Using Rodrigue's formula : $P_n(x) = \frac{1}{2^n\, n!} \frac{d^n}{dx^n}(x^2-1)^n$,

$$\int_{-1}^{1} f(x)\, P_n(x)\, dx = \frac{1}{2^n\, n!} \int_{-1}^{1} f(x) \frac{d^n (x^2-1)^n}{dx^n} dx \qquad \text{[Integrate by parts]}$$

$$= \frac{1}{2^n\, n!}\left[\left| f(x) \cdot \frac{d^{n-1}(x^2-1)^n}{dx^{n-1}} \right|_{-1}^{1} - \int_{-1}^{1} f'(x) \cdot \frac{d^{n-1}(x^2-1)^n}{dx^{n-1}} dx\right]$$

$$= \frac{(-1)}{2^n\, n!} \int_{-1}^{1} f'(x) \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n\, dx$$

$$= \frac{(-1)^2}{2^n\, n!} \int_{-1}^{1} f''(x) \frac{d^{n-2}}{dx^{n-2}} (x^2-1)^n\, dx \qquad \text{[Again integrating by parts]}$$

$$= \frac{(-1)^n}{2^n\, n!} \int_{-1}^{1} f^n(x)\ (x^2-1)^n\, dx \qquad \text{[Integrating by parts } (n-2) \text{ times]}$$

$$= \frac{(-1)^{2n}}{2^n\, n!} \int_{-1}^{1} f^n(x)\,(1-x^2)^n\, dx = \frac{1}{2^n n!} \int_{-1}^{1} f^n\,(x)\,(1-x^2)^n\, dx$$

16.15 GENERATING FUNCTION FOR $P_n(x)$

To show that $(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(x)$.

Since
$$(1-z)^{-\frac{1}{2}} = 1 + \frac{1}{2} z + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!} z^2 + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!} z^3 + \ldots$$

$$= 1 + \frac{2!}{(1!)^2\, 2^2} z + \frac{4!}{(2!)^2\, 2^4} z^2 + \frac{6!}{(3!)^2\, 2^6} z^3 + \ldots$$

$\therefore$
$$[1-t(2x-t)]^{-\frac{1}{2}} = 1 + \frac{2!}{(1!)^2\, 2^2} t(2x-t) + \frac{4!}{(2!)^2\, 2^4} t^2(2x-t)^2 + \ldots$$

$$+ \frac{(2n-2r)!}{[(n-r)!]^2\, 2^{2n-2r}} t^{n-r}(2x-t)^{n-r} + \ldots + \frac{(2n)!}{(n!)^2\, 2^{2n}} t^n\,(2x-t)^n + \ldots \qquad \ldots(1)$$

The term in t^n from the term containing $t^{n-r}\,(2x-t)^{n-r}$

$$= \frac{(2n-2r)!}{[(n-r)!]^2\, 2^{2n-2r}}\ t^{n-r}\,.\,{}^{n-r}C_r(-t)^r\,(2x)^{n-2r}$$

$$= \frac{(2n-2r)!}{[(n-r)!]^2\, 2^{2n-2r}} \times \frac{(n-r)!}{r!\,(n-2r)!}\ (-1)^r t^n\,.\,(2x)^{n-2r} = \frac{(-1)^r\ (2n-2r)!}{2^n\, r!\,(n-r)!\,(n-2r)!}\ x^{n-2r}\,.\,t^n.$$

Collecting all terms in t^n which will occur in the term containing $t^n\,(2x-t)^n$ and the preceding terms, we see that terms in t^n.

$$= \sum_{r=0}^{N} \frac{(-1)^r\ (2n-2r)!}{2^n\, r!\,(n-r)!\,(n-2r)!}\ x^{n-2r}\,.\,t^n = P_n(x) t^n$$

where $N = \frac{1}{2}\ n$ or $\frac{1}{2}\ (n-1)$ according as n is even or odd.

Hence (1) may be written as $[1-t(2x-t)]^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n$ ...(2)

(Kerala M.E., 2005 ; U.P.T.U., 2005)

This shows that $P_n(x)$ is the coefficient of t^n in the expansion of $(1-2xt+t^2)^{-1/2}$. That is why, it is known as the *generating function of Legendre polynomials.*

Cor. 1. $P_n(1) = 1$. *(V.T.U., 2003 S ; Delhi, 2002)*

Taking $x = 1$ in (2), we have $(1-2t+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(1)\, t^n$

i.e.,
$$\sum_{n=0}^{\infty} P_n(1)\, t^n = (1-t)^{-1} = 1 + t + t^2 + \ldots + t^n + \ldots$$

Equating coefficients of t^n, we get $P_n(1) = 1$.

Cor. 2. $P_n(-1) = (-1)^n$. *(B.P.T.U., 2005 S ; V.T.U., 2003)*

Taking $x = -1$ in (2), we have

$$\sum_{n=0}^{\infty} P_n(-1)\, t^n = (1+t)^{-1} = 1 - t + t^2 - \ldots + (-1)^n\, t^n + \ldots$$

Equating coefficients of t^n, we get the desired result.

Cor. 3. $P_n(0) = \begin{cases} (-1)^{n/2} \dfrac{1 \times 3 \times 5 \ldots (n-1)}{2 \times 4 \times 6 \times \ldots n}, & \text{when } n \text{ is even} \\ 0, & \text{when } n \text{ is odd} \end{cases}$ *(V.T.U., 2005)*

Putting $x = 0$ in (2), we get $\sum_{n=0}^{\infty} P_n(0)\ t^n = (1 + t^2)^{-1/2}$

$$= 1 - \frac{1}{2}t^2 + \frac{1.3}{2.4}\ t^4 \ldots + (-1)^r \frac{1 \cdot 3 \cdot 5 \cdots (2r+1)}{2 \cdot 4 \cdot 6 \cdots 2r} t^{2r} + \ldots$$

Equating coefficient of t^{2m}, we get $P_{2m}(0) = (-1)^m\ \dfrac{1 \cdot 3 \cdot 5 \cdots (2m-1)}{2 \cdot 4 \cdot 6 \cdots 2m}$

Similarly equating coefficients of t^{2m+1}, we have $P_{2m+1}(0) = 0$.

Cor. 4. $P_n'(1) = \dfrac{1}{2} n (n+1)$ *(U.P.T.U. 2003)*

Since $P_n(x)$ is a solution of Legendre's eduation, $(1 - x^2) P_n''(x) - 2xP_n'(x) + n(n+1) P_n(x) = 0$

Putting $x = 1$, $-2P_n'(1) + n(n+1) P_n(1) = 0$ or $P_n'(1) = \dfrac{1}{2}\ n(n+1)$ $[\because\ P_n(1) = 1]$

16.16 RECURRENCE FORMULAE FOR $P_n(x)$

The following recurrence formulae can be easily derived from the generating function for $P_n(x)$:

(1) $(n+1) P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$

(2) $nP_n(x) = xP_n'(x) - P_{n-1}'(x)$

(3) $(2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)$

(4) $P_n'(x) = xP_{n-1}'(x) + nP_{n-1}(x)$.

(5) $(1 - x^2)P_n'(x) = n\,[P_{n-1}(x) - xP_n(x)]$.

Proofs. **(1)** We know that $(1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n$ ...(*i*)

Differentiating partially w.r.t. t, we get

$$-\frac{1}{2}\ (1 - 2xt + t^2)^{-3/2}\ (-2x + 2t) = \Sigma nP_n(x)\ t^{n-1}$$

or $$(x - t)\ (1 - 2xt + t^2)^{-1/2} = (1 - 2xt + t^2)\ \Sigma nP_n(x)\ t^{n-1}$$

or $$(x - t)\ \Sigma P_n(x)\ t^n = (1 - 2xt + t^2)\ \Sigma nP_n(x)\ t^{n-1}$$

Equating coefficients of t^n from both sides, we get

$$xP_n(x) - P_{n-1}(x) = (n+1)\ P_{n+1}(x) - 2nxP_n(x) + (n-1)\ P_{n-1}(x)$$

whence follows the required result. *(S.V.T.U., 2007 ; V.T.U., 2003)*

(2) Differentiating (*i*) partially w.r.t. x,

$$-\frac{1}{2}\ (1 - 2xt + t^2)^{-3/2}\ .\ (-2t) = \Sigma P_n'(x)t^n$$

i.e., $$t(1 - 2tx + t^2)^{-3/2} = \Sigma P_n'(x)\ t^n \quad \text{...(ii)}$$

Again differentiating (*i*) partially w.r.t. t, we have

$$(x - t)\ (1 - 2tx + t^2)^{-3/2} = \Sigma nP_n(x)t^{n-1} \quad \text{...(iii)}$$

Dividing (*iii*) by (*ii*), we get $\dfrac{x - t}{t} = \dfrac{\Sigma nP_n(x)t^{n-1}}{\Sigma P_n'(x)t^n}$

i.e., $$\Sigma nP_n(x)\ t^n = (x - t)\ \Sigma P_n'(x)\ t^n$$

Equating coefficients of t^n from both sides, we get (2). *(J.N.T.U., 2006 ; U.P.T.U., 2006)*

(3) Differentiating **(1)** w.r.t. x, we get

$$(n+1)P_{n+1}'(x) = (2n+1)P_n(x) + (2n+1)xP_n'(x) - nP_{n-1}'(x) \quad \text{...(iv)}$$

Substituting for $xP_n'(x)$ from (2) in (*iv*), we obtain

$$(n+1)\ P_{n+1}'(x) = (2n+1)\ P_n(x) + (2n+1)\ [nP_n(x) + P_{n-1}'(x)] - nP_{n-1}'(x)$$

or $$(2n+1)\ P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)$$ *(Madras, 2006)*

(4) Rewriting (*iv*) as

$$(n+1)\,P'_{n+1}(x) = (2n+1)\,P_n(x) + (n+1)\,xP_n'(x) + n[xP_n'(x) - P'_{n-1}(x)]$$
$$= (2n+1)\,P_n(x) + (n+1)\,xP_n'(x) + n^2P_n(x) \qquad \text{[by (2)]}$$
$$= (n+1)xP_n'(x) + (n^2+2n+1)P_n(x)$$

or $$P'_{n+1}(x) = xP_n'(x) + (n+1)\,P_n(x)$$

Replacing n by $(n-1)$, we get (4).

(5) Rewriting (2) and (4) as

$$xP_n'(x) - P'_{n-1}(x) = nP_n(x) \qquad ...(v)$$

and $$P_n'(x) - xP'_{n-1}(x) = nP_{n-1}(x) \qquad ...(vi)$$

Multiplying (*v*) by x and substracting from (*vi*), we get

$$(1-x^2)P_n'(x) = n[P_{n-1}(x) - xP_n(x)].$$

Example 16.19. *Prove that* $(2n+1)\,(1-x^2)\,P_n'(x) = n\,(n+1)\,[P_{n-1}(x) - P_{n+1}(x)]$.

Solution. We have the recurrence formula

$$(n+1)\,P_{n+1}(x) = (2n+1)\,xP_n(x) - nP_{n-1}(x)$$

or $$(\overline{n+1} + n)xP_n(x) = (n+1)\,P_{n+1}(x) + nP_{n-1}(x)$$

or $$(n+1)\,[xP_n(x) - P_{n+1}(x)] = n\,[P_{n-1}(x) - xP_n(x)]$$
$$= (1-x^2)\,P_n'(x) \quad [\because\ (1-x^2)\,P_n'(x) = n\,[P_{n-1}(x) - xP_n(x)] \qquad ...(i)$$

or $$xP_n(x) = P_{n+1}(x) + \frac{(1-x^2)\,P_n'(x)}{n+1} \qquad ...(ii)$$

Also from (*i*) $$xP_n(x) = P_{n-1}(x) - \frac{(1-x^2)\,P_n'(x)}{n} \qquad ...(iii)$$

From (*ii*) and (*iii*), $$P_{n-1}(x) - \frac{(1-x^2)\,P_n'(x)}{n} = P_{n+1}(x) + \frac{(1-x^2)\,P_n'(x)}{n+1}$$

or $$n\,(n+1)\,P_{n-1}(x) - (n+1)\,(1-x^2)\,P_n'(x) = n\,(n+1)\,P_{n+1}(x) + n\,(1-x^2)\,P_n'(x)$$

or $$(2n+1)\,(1-x^2)\,P_n'(x) = n\,(n+1)\,[P_{n-1}(x) - P_{n+1}(x)]$$

16.17 (1) ORTHOGONALITY OF LEGENDRE POLYNOMIALS

We shall prove that, $$\int_{-1}^{1} P_m(x)\,P_n(x)\,dx = \begin{cases} 0, & m \neq n \\ \dfrac{2}{2n+1}, & m = n \end{cases}$$

We know that the solutions of

$$(1-x)^2\,u'' - 2xu' + m(m+1)\,u = 0 \qquad ...(1)$$

and $$(1-x^2)v'' - 2xv' + n(n+1)\,v = 0 \qquad ...(2)$$

are $P_m(x)$ and $P_n(x)$ respectively.

Multiplying (1) by v and (2) by u and subtracting, we get

$$(1-x^2)\,(u''v - uv'') - 2x(u'v - uv') + [m(m+1) - n(n+1)]uv = 0$$

or $$\frac{d}{dx}\,\{(1-x^2)\,(u'v - uv')\} + (m-n)\,(m+n+1)\,uv = 0.$$

Now integrating from -1 to 1, we get

$$(m-n)\,(m+n+1)\int_{-1}^{1} uv\,dx = \left|\,(1-x^2)\,(uv' - u'v)\right|_{-1}^{1} = 0.$$

Hence $$\int_{-1}^{1} P_m(x)\,P_n(x)\,dx = 0. \qquad (m \neq n) \qquad ...(3)$$

This is known as the *orthogonality property of Legendre polynomials.*

(S.V.T.U., 2008 ; Madras, 2006 ; V.T.U., 2006)

When $m = n$, we have from Rodrigue's formula,

$$(n\,!\,2^n)^2 \int_{-1}^{1} P_n^2(x)\,dx = \int_{-1}^{1} D^n(x^2-1)^n \cdot D^n(x^2-1)^n\,dx \qquad \text{[Integrate by parts]}$$

$$= \left| D^n(x^2-1)^n \cdot D^{n-1}(x^2-1)^n \right|_{-1}^{1} - \int_{-1}^{1} D^{n+1}(x^2-1)^n \,.\, D^{n-1}(x^2-1)^n\,dx$$

Since $D^{n-1}(x^2-1)^n$ has x^2-1 as a factor, the first term on the right vanishes for $x = \pm 1$. Thus

$$(n\,!\,2^n)^2 \int_{-1}^{1} P_n^2(x)\,dx = -\int_{-1}^{1} D^{n+1}(x^2-1)^n \,.\, D^{n-1}(x^2-1)^n\,dx$$

[Integrate by parts $(n-1)$ times]

$$= (-1)^n \int_{-1}^{1} D^{2n}(x^2-1)^n \,.\, (x^2-1)^n\,dx = (-1)^n \int_{-1}^{1} (2n)\,!\,(x^2-1)^n\,dx$$

$$= 2(2n)\,! \int_0^1 (1-x^2)^n\,dx \qquad \text{[Put } x = \sin\theta\text{]}$$

$$= 2(2n)\,! \int_0^{\pi/2} \cos^{2n+1}\theta\,d\theta = 2(2n)\,!\,\frac{2n(2n-2)\cdots 4\cdot 2}{(2n+1)(2n-1)\cdots 2\,.\,1}$$

$$= 2(2n)\,!\,[2n(2n-2)\ldots 4\,.\,2]^2/(2n+1)\,! = \frac{2}{2n+1}\,(2^n n\,!)^2$$

Hence $\displaystyle\int_{-1}^{1} P_n^2(x)dx = 2/(2n+1).$ ...(4) *(Bhopal, 2008 ; V.T.U., 2007 ; J.N.T.U., 2006)*

(2) Fourier-Legendre expansion of f(x). If $f(x)$ be a function defined from $x = -1$ to $x = 1$, we can write

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x) \qquad ...(5)$$

To determine the coefficient c_n, multiply both sides by $P_n(x)$ and integrate from -1 to 1. Then (3) and (4) give

$$\int_{-1}^{1} f(x)\,P_n(x)dx = c_n \int_{-1}^{1} P_n^2(x)\,dx = \frac{2c_n}{2n+1} \quad \text{or} \quad c_n = \left(n + \frac{1}{2}\right)\int_{-1}^{1} f(x)\,P_n(x)\,dx$$

Equation (5) is known as *Fourier-Legendre expansion of* $f(x)$.

Example 16.20. *Show that* $\displaystyle\int_{-1}^{1} xP_n(x)\,P_{n-1}(x)\,dx = \frac{2n}{4n^2-1}$.

Solution. The recurrence formula (1) can be written as

$$(2n+1)xP_n = (n+1)\,P_{n+1} + nP_{n-1}$$

or $$(2n-1)\,xP_{n-1} = nP_n + (n-1)\,P_{n-2} \qquad \text{[Changing } n \text{ to } n-1\text{]}$$

Multiplying by P_n, we get $\displaystyle xP_nP_{n-1} = \frac{1}{2n-1}\,[nP_n^2 + (n-1)\,P_nP_{n-2}]$

Integrating both sides w.r.t. x from $x = -1$ to $x = 1$, we get

$$\int_{-1}^{1} xP_nP_{n-1}dx = \frac{n}{2n-1}\int_{-1}^{1} P_n^2 dx + \frac{n-1}{2n-1}\int_{-1}^{1} P_nP_{n-2}\,dx$$

$$= \frac{n}{2n-1}\left(\frac{2}{2n+1}\right) + \frac{n-1}{2n-1}(0), \text{ by Orthogonal property}$$

Hence $\displaystyle\int_{-1}^{1} xP_n(x)P_{n-1}(x)\,dx = \frac{2n}{4n^2-1}.$

Example 16.21. *Show that* $\displaystyle\int_{-1}^{1} (1-x^2)\,P_m'(x)\,dx, = \begin{cases} 0, & \text{when } m \neq n \\ \dfrac{2n(n+1)}{2n+1}, & \text{when } m = n \end{cases}$

(S.V.T.U., 2008 ; U.P.T.U., 2006)

Solution. Integrating by parts,

$$\int_{-1}^{1}(1-x^2)\ P_m'(x)\,P_n'(x)dx = \left|(1-x^2)P_m'(x)\cdot P_n(x)\right|_{-1}^{1} - \int_{-1}^{1}\cdot\frac{d}{dx}\left\{(1-x^2)\,P_m'(x)\right\}P_n\,dx$$

$$= -\int_{-1}^{1} P_n\left\{(1-x^2)\,P_m''(x) - 2x\,P_m'(x)\right\}dx \qquad \text{...(i)}$$

Now $P_m(x)$ being a solution of Legendre's equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + m(m+1)\,y = 0, \text{ we have}$$

$$(1-x^2)\,P_m''(x) - 2x\,P_m'(x) = -m(m+1)\,P_m(x)$$

Substituting this in (i), we get

$$\int_{-1}^{1}(1-x^2)\ P_m'(x)\,P_n'(x)\,dx = -\int_{-1}^{1} P_n\ \{-m(m+1)\,P_m(x)\}\,dx$$

$$= m(m+1)\int_{-1}^{1} P_m(x)\,P_n(x)dx \qquad \text{...(ii)}$$

When $m \neq n$, $\int_{-1}^{1} P_m(x)\,P_n(x)\,dx = 0$, by orthogonality property.

$$\therefore \quad \int_{-1}^{1}(1-x^2)\ P_m'(x)\,P_n'(x)\,dx = m(m+1)\,.\,0 = 0 \qquad \text{[from (ii)]}$$

When $m = n$, $\int_{-1}^{1} P_m(x)\,P_n(x)\,dx = \frac{2}{2n+1}$, by orthogonality property.

$$\therefore \quad \int_{-1}^{1}(1-x^2)\ P_m'(x)\,P_n'(x)\,dx = n(n+1)\,.\,\frac{2}{2n+1} = \frac{2n(n+1)}{(2n+1)}.$$

Example 16.22. *Show that* $\int_{-1}^{1} x^2\ P_{n-1}P_{n+1}\,dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$.

(J.N.T.U., 2006 ; Kerala M. Tech., 2005)

Solution. We have from the recurrence relation (1),

$$(2n+1)\,x\,P_n = (n+1)\,P_{n+1} + n\,P_{n-1}$$

$$\therefore \quad xP_{n-1} = \frac{1}{2n-1}\ \{nP_n + (n-1)\,P_{n-2}\}$$

and

$$xP_{n+1} = \frac{1}{2n+3}\ \{(n+2)\,P_{n+2} + (n+1)\,P_n\}P$$

$$\therefore \quad x^2P_{n-1}P_{n+1} = \frac{1}{(2n-1)(2n+3)}\{n(n+2)\,P_nP_{n+2} + n(n+1)P_n^{\,2}$$

$$+ (n-1)(n+2)P_{n-2}P_{n+2} + (n^2-1)P_n\,P_{n-2}\}$$

Integrating both sides from -1 to 1 and using orthogonality of Legendre polynomials, we get

$$\int_{-1}^{1} x^2P_{n-1}P_{n+1}\,dx = \frac{n(n+1)}{(2n-1)(2n+3)}\int_{-1}^{1} P_n^{\,2}\,dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}.$$

Example 16.23. *If* $f(x) = 0, \quad -1 < x \le 0$

$\qquad = x, \qquad 0 < x < 1,$

show that $f(x) = \frac{1}{4}P_0(x) + \frac{1}{2}P_1(x) + \frac{5}{16}\,P_2(x) - \frac{3}{32}\,P_4(x) + \ldots\ldots$ *(U.P.T.U., 2003)*

Solution. Let
$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x)$$

Then c_n is given by $c_n = \left(n+\frac{1}{2}\right)\int_{-1}^{1} f(x)\, P_n(x)\, dx$

$$= \left(n+\frac{1}{2}\right)\left[\int_{-1}^{0} 0\,.\,P_n(x)\,dx + \int_0^1 xP_n(x)\,dx\right] = \left(n+\frac{1}{2}\right)\int_0^1 x\,P_n(x)\,dx$$

$\therefore$
$$c_0 = \frac{1}{2}\int_0^1 xP_0(x)\,dx = \frac{1}{2}\int_0^1 x\,dx = \frac{1}{4}$$

$$c_1 = \frac{3}{2}\int_0^1 xP_1(x)\,dx = \frac{3}{2}\int_0^1 x^2\,dx = \frac{1}{2}$$

$$c_2 = \frac{5}{2}\int_0^1 xP_2(x)\,dx = \frac{5}{2}\int_0^1 x\,.\,\frac{3x^2-1}{2}\,dx = \frac{5}{4}\left|\frac{3x^4}{4}-\frac{x^2}{2}\right|_0^1 = \frac{5}{16}$$

$$c_3 = \frac{7}{2}\int_0^1 xP_3(x)\,dx = \frac{7}{2}\int_0^1 x\,.\,\frac{5x^3-3x}{2}\,dx = \frac{7}{4}\left|5\frac{x^5}{5}-3\frac{x^3}{3}\right|_0^1 = 0$$

$$c_4 = \frac{9}{2}\int_0^1 xP_4(x)\,dx = \frac{9}{2}\int_0^1 x\,.\,\frac{35x^4-30x^2+3}{8}\,dx$$

$$= \frac{9}{16}\left|35\frac{x^6}{6}-35\frac{x^4}{4}+3\frac{x^2}{2}\right|_0^1 = -\frac{3}{32} \text{ and so on.}$$

Hence
$$f(x) = \frac{1}{4}P_0(x) + \frac{1}{2}P_1(x) + \frac{5}{16}P_2(x) - \frac{3}{32}P_4(x) + \ldots\,.$$

PROBLEMS 16.5

1. Show that $P_n(-x) = (-1)^n P_n(x)$. *(Bhopal, 2008 ; V.T.U., 2003 S)*

2. Prove that (*i*) $P_{2n}'(0) = 0$ (*ii*) $P_{2n+1}'(0) = \frac{(-1)^n\,(2n+1)!}{2^{2n}(n!)^2}$. (*iii*) $P_n'(-1) = (-1)^n\,\frac{n(n+1)}{2}$ *(S.V.T.U., 2008)*

3. Express the following in terms of Legendre polynomials : (*i*) $5x^3 + x$
(*ii*) $x^3 + 2x^2 - x - 3$, *(Osmania, 2003)* (*iii*) $4x^3 + 6x^2 + 7x + 2$. *(S.V.T.U., 2008)*
(*iv*) $x^4 + 3x^3 - x^2 + 5x - 2$ *(Bhopal, 2008 ; Madras, 2006)*

4. Prove that (*i*) $(1-x^2)\,P_n'(x) = (n+1)\,[xP_n(x) - P_{n+1}(x)]$,
(*ii*) $P_n(x) = P'_{n+1}(x) - 2xP_n'(x) + P'_{n-1}(x)$. (*iii*) $P_n(x)\,P_{n+1/2}(x) = \frac{\sqrt{\pi}}{2^{2n+1}}\,P_{2m}(x)$ *(Anna, 2005 S)*

5. Prove that (*i*) $\int_{-1}^{1}[P_2(x)]^2\,dx = \frac{2}{5}$. *(P.T.U., 2002)* (*ii*) $\int_0^1 P_{2n}(x)\,dx = 0$.

6. Prove that $\int_{-1}^{1} P_n(x)(1-2hx+h^2)^{-1/2}\,dx = \frac{2h^n}{2n+1}$.

7. Show that $\int_{-1}^{1}(1-x^2)[P_n'(x)]^2\,dx = \frac{2n(n+1)}{2n+1}$. *(U.P.T.U., 2006 ; Kerala M.E., 2005)*

8. Using Rodrigue's formula, show that $P_n(x)$ satisfies the differential equation
$$\frac{d}{dx}\left[(1+x^2)\frac{d}{dx}[P_n(x)]\right] + n\,(n+1)\,P_n(x) = 0.$$

9. Expand the following functions in terms of Legendre polynomials in the interval $-1 < x < 1$:
(*i*) $f(x) = x^3 + 2x^2 - x - 3$ *(V.T.U., 2008)* (*ii*) $f(x) = x^4 + x^3 + 2x^2 - x - 3$.

10. If $f(x) = 0, \quad -1 < x < 0$
$= 1, \quad 0 < x < 1$, show that $f(x) = \frac{1}{2}P_0(x) + \frac{3}{4}P_1(x) - \frac{7}{16}P_3(x) + \ldots\ldots$

16.18 OTHER SPECIAL FUNCTIONS

The following special functions occur in numerous engineering problems. We state below their important properties which can be verified by similar methods :

(1) **Laguerre's polynomials***. These are the solutions of *Laguerre's differential equation*

$$xy'' + (1-x)y' + ny = 0 \qquad ...(1)$$

These polynomials $L_n(x)$, are given by the corresponding Rodrigue's formula

$$L_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x}) \qquad ...(2)$$

In particular, $L_0(x) = 1$; $L_1(x) = 1 - x$, $L_2(x) = 2 - 4x + x^2$; $L_3(x) = 6 - 18x + 9x^2 - x^3$. *(Madras, 2006)*

Their *generating function* is given by

$$\frac{e^{-xt/(1-t)}}{1-t} = \sum_{n=0}^{\infty} \frac{L_n(x)}{n!} t^n \qquad ...(3)$$

The *orthogonal property* for these polynomials is

$$\int_0^{\infty} e^{-x} L_m(x) L_n(x)\, dx = \begin{cases} 0, & m \neq n \\ (n!)^2, & m = n \end{cases} \qquad ...(4)$$

(2) **Hermite's polynomials†.** These are the solutions of Hermite's differential equation

$$y'' - 2xy' + 2ny = 0 \qquad ...(5)$$

These polynomials $H_n(x)$, are given by the Rodrigue's formula

$$H_n(x) = (-1)^n e^{x^2} \frac{d^{2n}}{dx^n}(e^{-x^2}) \qquad ...(6)$$

In particular, $H_0(x) = 1$; $H_1(x) = 2x$; $H_2(x) = 4x^2 - 2$; $H_3(x) = 8x^3 - 12x$. *(Madras, 2006)*

Their *generating function is* given by

$$e^{2tx - t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n \qquad ...(7) \quad \textit{(Madras, 2002 S)}$$

The *orthogonal property* of these polynomials is

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x)\, dx = \begin{cases} 0, & m \neq n \\ 2^n\, n!\sqrt{\pi}, & m = n \end{cases} \qquad ...(8)$$

(3) **Chebyshev polynomials**.** These polynomials denoted by $T_n(x)$, are the solutions of the differential equation

$$(1-x^2)y'' - xy' + n^2 y = 0 \qquad ...(9)$$

Their *generating function* is

$$\frac{1-xt}{1-2xt+t^2} = \sum_{n=0}^{\infty} T_n(x)\, t^n \qquad ...(10)$$

and $$T_n(x) = \frac{n}{2} \sum_{r=0}^{N} (-1)^r \frac{(n-r-1)!}{r!\,(n-2r)!} (2x)^{n-2r} \qquad ...(11)$$

(J.N.T.U., 2006)

where $N = \frac{n}{2}$, if n is even and $N = \frac{1}{2}(n-1)$, if n is odd.

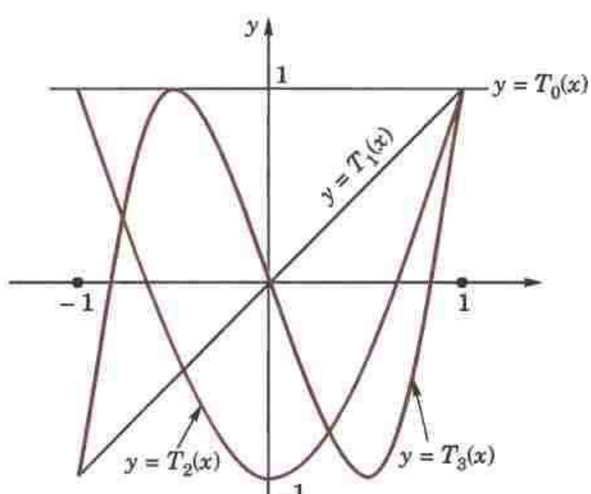

Fig. 16.3. Graphs of $T_0(x)$, $T_1(x)$, $T_2(x)$, $T_3(x)$.

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* Named after the French mathematician *Edmond Laguerre* (1834–86) who is known for his work in infinite series and geometry.

† See footnote p. 68.

** Named after the Russian mathematician *Pafnuti Chebyshev* (1821–1894) who is known for his work in the theory of numbers and approximation theory.

In particular, $T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x$. Also, we have the recurrence relation

$$T_{n+1}(x) = 2x\, T_n(x) - T_{n-1}(x) \qquad ...(12) \quad (Bhopal,\ 2002)$$

which defines T_{n+1} in terms of T_n and T_{n-1}.

Their *orthogonal property* is

$$\int_{-1}^{1} (1-x^2)^{-1/2}\, T_m(x) T_n(x)\, dx = \begin{cases} 0, & m \neq n \\ \dfrac{\pi}{2}, & m = n \\ \pi, & m = n = 0 \end{cases} \qquad ...(13)$$

Example 16.24. *Prove that* $\int_0^\infty e^{-x} L_m(x) L_n(x)\, dx = 0, m \neq n.$ *(Anna, 2006)*

Solution. Since $L_m(x)$ and $L_n(x)$ are the solutions of the Laguerre's differential equation (1).

$$\therefore \qquad xL_m'' + (1-x)L_m' + mL_m = 0 \qquad ...(i)$$

$$xL_n'' + (1-x)L_n' + nL_n = 0 \qquad ...(ii)$$

Multiplying (*i*) by L_n and (*ii*) by L_m and subtracting, we get

$$x(L_nL_m'' - L_mL_n'') + (1-x)(L_nL_m' - L_mL_n') = (n-m) L_mL_n$$

or

$$\frac{d}{dx}(L_nL_m' - L_mL_n') + \frac{1-x}{x}(L_nL_m' - L_mL_n') = \frac{(n-m) L_mL_n}{x}$$

This is Leibnitz's linear equation and its

$$\text{I.F.} = e^{\int \left(\frac{1}{x} - 1\right) dx} = e^{\log x - x} = xe^{-x}.$$

$$\therefore \quad \text{Its solution is } \left| (L_nL_m' - L_mL_n')\, xe^{-x} \right|_0^\infty = \int_0^\infty \frac{(n-m)L_mL_n}{x}\, xe^{-x}\, dx$$

or

$$\int_0^\infty e^{-x} L_mL_n\, dx = \left| \frac{(L_nL_m' - L_mL_n')xe^{-x}}{n-m} \right|_0^\infty = 0 \text{ which proves the result.}$$

Example 16.25. *Prove that* $H_n(x) = (-1)^n e^{x^2} \dfrac{d^{2n}}{dx^n}(e^{-x^2})$.

Solution. The generating function for $H_n(x)$ is $e^{2tx - t^2} = e^{x^2} . e^{-(t-x)^2} = \sum_{n=0}^{\infty} H_n(x) . \dfrac{t^n}{n!}$

Then

$$\left[\frac{\partial^n}{\partial t^n}(e^{2tx - t^2})\right]_{t=0} = H_n(x) \qquad ...(i)$$

Also

$$\left[\frac{\partial^n}{\partial t^n}(e^{2tx - x^2})\right]_{t=0} = e^{x^2}\left[\frac{\partial^n}{\partial t^n}\{e^{-(t-x)^2}\}\right]_{t=0}$$

$$= e^{x^2}\left[\frac{\partial^n}{\partial(-x)^n}\{e^{-(t-x)^2}\}\right]_{t=0} = (-1)^n \frac{d^n}{dx^n}(e^{-x^2}) \qquad ...(ii)$$

Equating (*i*) and (*ii*), we get the desired result.

PROBLEMS 16.6

1. Using the generating function (3) page 571, obtain the recurrence formula
$$L_{n+1}(x) = (2n + 1 - x)\, L_n(x) - n^2 L_{n-1}(x).$$
2. Show that (*i*) $nL_{n-1}(x) = nL'_{n-1}(x) - L'_n(x)$. (*ii*) $L'_n(x) = L'_{n-1}(x) - L_{n-1}(x)$. *(Anna, 2005)*
3. Show that (*i*) $H_{2n}(0) = (-1)^n \dfrac{2n!}{n!}$ (*ii*) $H_{2n+1}(0) = 0$ *(Anna, 2005)*

4. Prove that (i) $H_n'(x) = 2n\, H_{n-1}(x)$ (ii) $\dfrac{d^m}{dx^m}[H_n(x)] = \dfrac{2^m \cdot n!}{(n-m)!} H_{n-m}(x),\ m < n.$
5. Using the generating function (7) page 515, obtain the recurrence formula $2xH_n(x) = 2nH_{n-1}(x) + H_{n+1}(x)$.
6. Prove that (i) $\int_{-\infty}^{\infty} e^{-x^2} H_2(x)\, H_3(x) = 0$, (ii) $\int_{-\infty}^{\infty} e^{-x^2} \{H_2(x)\}^2 = 8\sqrt{\pi}$. *(Madras, 2003)*
7. Express x^3 in terms of Chebyshev polynomials T_1 and T_3. *(U.P.T.U., 2009)*
8. Show that (i) $T_5 = 16x^5 - 20x^3 + 5x$. *(Bhopal, 2002)*
   (ii) $(1 - x^2)T_n' = nT_{n-1}(x) - nxT_n(x)$. *(Osmania, 2003)*
9. Prove that $\dfrac{1 - t^2}{1 - 2xt + t^2} = T_0(x) + 2\sum_{n=1}^{\infty} T_n(x)\, t^n$. *(J.N.T.U., 2006)*

16.19 (1) STRUM*-LIOUVILLE† PROBLEM

Legendre's equation $(1 - x^2)y'' - 2xy' + n(n+1)y = 0$ ...(i)

can be written as, $[(1 - x^2)y']' + \lambda y = 0$ $[\lambda = n(n+1)]$

Bessel's equation $X^2 \dfrac{d^2y}{dx^2} + X\dfrac{dy}{dx} + (X^2 - n^2)y = 0$ can be transformed by putting $X = kx$ (so that $\dfrac{dy}{dx} = \dfrac{dy}{dx}\cdot\dfrac{dx}{dX} = \dfrac{y'}{k}, \dfrac{d^2y}{dX^2} = \dfrac{y''}{k^2}$ to the form

$$x^2y'' + xy' + (k^2x^2 - n^2)\, y = 0$$

or $$(xy'' + y') + (\lambda x - n^2/x)\, y = 0 \qquad [\lambda = k^2]$$

or $$(xy')' + (\lambda x - n^2/x)\, y = 0 \qquad ...(ii)$$

Both the equations (i) and (ii) are of the form

$$[r(x)\, y']' + [\lambda p(x) + q(x)]y = 0 \qquad ...(1)$$

which is known as the *Strum-Liouville equation*. Similarly Laguerre's, Hermite's equations etc. can also be reduced to (1). Thus all the above equations of engineering utility can be considered with a common approach by means of Strum-Liouville's equation.

Eqution (1) considered on some interval $a \le x \le b$, satisfying the conditions

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0,\ \beta_1 y(b) + \beta_2 y'(b) = 0 \qquad ...(2)$$

with the real constants : α_1, α_2 not both zero and β_1, β_2 not both zero. The conditions (2) at the end points are called *boundary condtions.*

A differential equation together with the boundary conditions, is called a **boundary value problem.** *Equation (1) together with boundary conditions (2) is called a* **Strum-Liouville problem.**

Obviously $y = 0$ is a solution of the problem for any value of the parameter λ which is a trivial solution and as such is of no practical utility. Any other solution of (1) satisfying (2) is called an *eigen function* of the problem and the corresponding value of λ is called an eign value of the problem.

A special case. Taking $r = p = 1$ and $q = 0$ in (1), we get

$$y'' + \lambda y = 0 \qquad ...(3)$$

Also if $\alpha_1 = \beta_1 = 1$ and $\alpha_2 = \beta_2 = 0$, then the boundary conditions (2) become

$$y(a) = 0, \quad y(b) = 0 \qquad ...(4)$$

Thus (3) and (4) constitute the *simplest form of Strum-Liouville problem.*

(2) **Orthogonality.** Of the various properties of eigen functions of Strum-Liouville problem the orthogonality is of special importance.

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* Named after the Swiss mathematician *J.C.F. Strum* (1803–1855) who later became Poisson's successor at Sorbonne university, Paris.

† Named after the French professor *Joseph Liouville* (1809–1882) who is known for his important contributions to complex analysis, special functions, number theory and differential geometry.

Def. *Two functions $y_m(x)$ and $y_n(x)$ defined on some interval $a \le x \le b$, are said to be orthogonal on this interval w.r.t. the weight function $p(x) > 0$, if*

$$\int_a^b p(x)\, y_m(x) y_n(x)\, dx = 0 \text{ for } m \ne n.$$

The **norm** *of y_m, denoted by $\| y_m \|$, is defined to be the non-negative square root of $\int_a^b p(x)\,[y_m(x)]^2\, dx$. Thus*

$$\| y_m \| = \sqrt{\left\{\int_a^b p(x)\,[y_m(x)]^2\, dx\right\}}$$

The functions which are orthogonal on $a \le x \le b$ and have norm equal to 1, are called **orthonormal** on this interval.

(3) Orthogonality of eigen functions.

Theorem. *If (i) the functions p, q, r and r′ in the Strum-Liouville equation (1) be continuous in $a \le x \le b$;*

(ii) $y_m(x), y_n(x)$ be two eigen functions of the Strum-Liouville problem corresponding to eigen values λ_m and λ_n respectively ;

then $y_m(x)$ and $y_n(x)$ $(m \ne n)$ are orthogonal on that interval w.r.t. the weight function $p(x)$.

Proof. Since y_m and y_n satisfy (1) above

$$(ry'_m)' + (\lambda_m p + q)\, y_m = 0$$
$$(ry'_n)' + (\lambda_n p + q)\, y_n = 0$$

Multiplying the first equation by y_n and the second by $-y_m$ and adding, we get

$$(\lambda_m - \lambda_n)\, p y_m y_n = y_m(ry'_n) - y_n(ry'_m)$$
$$= \frac{d}{dx}[(ry'_n)\, y_m - (ry'_m) y_n], \text{ after differentiation.}$$

Now integrating both sides w.r.t. x from a to b, we obtain

$$(\lambda_m - \lambda_n) \int_a^b p y_m y_n\, dx = \left[(ry'_n) y_m - (ry'_m) y_n\right]_a^b$$
$$= r(b)\,[y'_n(b)\, y_m(b) - y'_m(b)\, y_n(b)] - r(a)\,[y'_n(a)\, y_m(a) - y'_m(a)\, y_n(a)] \qquad \text{...(A)}$$

The R.H.S. will vanish if the boundary conditions are of one of the following forms :

I. $y(a) = y(b) = 0$; II. $y'(a) = y'(b) = 0$; III. $\alpha_1 y(a) + \alpha_2 y'(a) = 0$, $\beta_1 y(b) + \beta_2 y'(b) = 0$ where either α_1 and α_2 is not zero and either β_1 or β_2 is not zero.

Thus in each case (A) reduces to $\int_a^b p y_m y_n\, dx = 0 \quad (m \ne n)$

which shows that the eigen functions y_m and y_n are orthogonal on $a \le x \le b$ w.r.t. the weight function $p(x) = 0$.

Obs. The third form of the boundary conditions in fact contains the first two forms as special cases.

Cor. 1. Orthogonality of Legendre polynomials has already been established directly in § 16.17. But if follows at once from the above theorem.

We have already seen in para (1) that Legendre's equation is Strum-Liouville equation

$$[(1 - x^2)\, y']' + \lambda y = 0 \qquad [\lambda = n(n+1)]$$

with $r(x) = 1 - x^2$, $p(x) = 1$ and $q(x) = 0$.

Since $y(-1) = y(1) = 0$ and for $n = 0, 1, 2, ..., \lambda = 0, 1.2, 2.3, ...$, the Legendre polynomials are the solutions of the problem *i.e.*, these are the eigen functions. Thus it follows by the above theorem, that they are orthogonal on $-1 \le x \le 1$.

Cor. 2. Orthogonality of Bessel functions has also been established directly in § 16.11. But it can easily be seen to follow from the above theorem.

In para (1), we transformed the Bessel's equation

$$X^2 \frac{d^2 J_n}{dx^2} + X \frac{dJ_n}{dx} + (X^2 - n^2)\, J_n(x) = 0$$

into $[xJ'_n(kx)]' + (k^2 x - n^2/x)\, J_n(kx) = 0$ which is Strum-Liouville equation with $r(x) = x$, $p(x) = x$, $q(x) = -n^2/x$ and $\lambda = k^2$. Since $r(0) = 0$, it follows from the above theorem that those solutions of $J_n(kx)$ which are zero at $x = 0$ form an orthogonal set on $0 \le x \le R$ with weight function $p(x) = x$.

Example 16.26. *For the Strum-Liouville problem $y'' + \lambda y = 0$, $y(0) = 0$, $y(l) = 0$, find the eigen functions and show that they are orthogonal.*

Solution. For $\lambda = -\gamma^2$, the general solution of the equation is $y(x) = c_1 e^{\gamma x} + c_2 e^{-\gamma x}$

The above boundary conditions give $c_1 = c_2 = 0$ and $y = 0$ which is not an eigen function.

For $\lambda = \gamma^2$, the general solution is $y(x) = A \cos \gamma x + B \sin \gamma x$

The first boundary condition gives $y(0) = A = 0$ and the second boundary condition gives $y(l) = B \sin \gamma l = 0$, $\gamma = 0, \pm \pi/l, \pm 2\pi/l, \ldots$ Thus the eigen values are $\lambda = 0, \pi^2/l^2, 4\pi^2/l^2, \ldots$ and taking $B = 1$, the corresponding eigen functions are

$$y_n(x) = \sin (n\pi x/l) \qquad n = 0, 1, 2, \ldots$$

From the above theorem, it follows that the said eigen functions are orthogonal on the interval $0 \le x \le l$.

Obs. This problem concerns an elastic string stretched between fixed points $x = 0$ and $x = l$ and allowed to vibrate. Here $y(x)$ is the space function of the deflection $u(x, t)$ of the string where t is the time. (See § 18.4).

PROBLEMS 16.7

Find the eigen functions of each of the following *Strum-Liouville problems* and verify their orthogonality :

1. $y'' + \lambda y = 0, y(0) = 0, y(\pi) = 0.$
2. $y'' + \lambda y = 0, y(0) = 0, y'(l) = 0.$
3. $y'' + \lambda y = 0, y'(0) = 0, y'(\pi) = 0.$
4. $y'' + \lambda y = 0, y(\pi) = y(-\pi), y'(\pi) = y'(-\pi).$
5. $(xy')' + \lambda x^{-1} y = 0, y(1) = 0, y'(e) = 0.$

Transform each of the following equations to the *Strum-Liouville equations* indicating the weight function :

6. *Laguerre's equation* : $xy'' + (1 - x)y' + ny = 0.$
7. Hermite's equation : $y'' - 2xy' + 2ny = 0.$

16.20 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 16.8

Fill up the blanks or choose the correct answer in the following problems :

1. In terms of Legendre polynomials $2 - 3x + 4x^2$ is ......
2. $J_{-1/2} =$ ....
3. $\int_{-1}^{1} P_n^2(x)\, dx =$ ....
4. $P_{2n+1}(0) =$ ....
5. $\int_{-1}^{1} x^m P_n(x)\, dx =$ ......(m being an integer $< n$)
6. The recurrence relation connecting $J_n(x)$ to $J_{n-1}(x)$ and $J_{n+1}(x)$ is ....
7. Orthogonality relation for Bessel functions is ....
8. Bessel's equation of order zero is ....
9. $J_{1/2} =$ ....
10. $\frac{d}{dx}[x^n J_n(x)] =$ ....
11. Value of $P_2(x)$ is ....
12. $\int_{-1}^{1} P_3(x) P_4(x)\, dx =$ ....
13. $P_n(-1) = (-1)^n$ (True or False)
14. Rodrigue's formula for $P_n(x)$ is ....
15. $\int_{0}^{1} x J_n(\alpha x) J_n(\beta x)\, dx = 0$, if ....
16. Expansion of $5x^3 + x$ interms of Legendre polynomials is ....
17. Generating function of $P_n(x)$ is ....
18. $\frac{d}{dx}[J_0(x)] =$ ....
19. Bessel equation of order 4 is $x^2y'' + xy' + (x^2 - 4)y = 0.$ (True or False)
20. $\frac{d}{dx}[x^2 J_2(x)] = x^2 J_1(x).$ (True or False)
21. Legendre's polynomial of first degree $= x$. (True or False)

22. If α is a root of $P_n(x) = 0$, then $P_{n+1}(\alpha)$ and $P_{n-1}(\alpha)$ are of opposite signs. (True or False)
23. $x = 0$ is a regular singular point of $2x^2y'' + 3xy' + (x^2 - 4)y = 0$. (True or False)
24. $\cos x = 2J_1 - 2J_3 + 2J_5 - \ldots\ldots$ (True or False)
25. If J_0 and J_1 are Bessel functions, then $J_1'(x)$ is given by

(a) $-J_0$ (b) $J_0(x) - 1/x\, J_1(x)$ (c) $J_0(x) + \frac{1}{x} J_1(x)$.

26. If $J_n(x)$ is the Bessel function of first kind, then $\int_0^{\pi} [J_{-2}(x) - J_2(x)]\, dx =$

(a) 2 (b) -2 (c) 0 (d) 1.

27. If $J_{n+1}(x) = \frac{2}{x} J_n(x) - J_0(x)$, then n is

(a) 0 (b) 2 (c) -1 (d) none of these.

28. The series $x - \frac{x^3}{2^2(1!)^2} + \frac{x^5}{2^4(2!)^2} - \frac{x^7}{2^6(3!)^2} + \ldots\ldots \infty$ equals

(a) $J_{1/2}(x)$ (b) $J_0(x)$ (c) $xJ_0(x)$ (d) $xJ_{1/2}(x)$.

29. If $\int_{-1}^{1} P_n(x)\, dx = 2$, then n is

(a) 0 (b) 1 (c) -1 (d) none of these.

30. The value of $\int_{-1}^{1} (2x+1)P_3(x)\, dx$ where $P_3(x)$ is the third degree Legendre polynomial, is

(a) 1 (b) -1 (c) 2 (d) 0.

31. The value of the integral $\int_{-1}^{1} x^3 P_3(x)\, dx$, where $P_3(x)$ is a Legendre polynomial of degree 3, is

(a) 0 (b) $\frac{2}{35}$ (c) $\frac{4}{35}$ (d) $\frac{11}{35}$.

32. The polynomial $2x^2 + x + 3$ in terms of Legendre polynomials is

(a) $\frac{1}{3}(4P_2 - 3P_1 + 11P_0)$ (b) $\frac{1}{3}(4P_2 + 3P_1 - 11P_0)$

(c) $\frac{1}{3}(4P_2 + 3P_1 + 11P_0)$ (d) $\frac{1}{3}(4P_2 - 3P_1 - 11P_0)$.

33. If $P_n(x)$ be the Legendre polynomial, then $P_n'(-x)$ is equal to

(a) $(-1)^n P_n(x)$ (b) $(-1)^n P_n'(x)$ (c) $(-1)^{n+1} P_n'(x)$ (d) $P_n''(x)$.

34. Legendre polynomial $P_5(x) = \lambda(63x^5 - 70x^3 + 15x)$ where λ is equal to

(a) 1/2 (b) 1/5 (c) 1/8 (d) 1/10.

35. $\int_{-1}^{1} (1+x)\, P_n(x)\, dx$, $(n > 1)$, is equal to

(a) $\frac{1}{2n+1}$ (b) $\frac{2}{2n+1}$ (c) $\frac{n}{2n+1}$ (d) 0.

36. The singular points of the differential equation $x^3(x-1)y'' + 2(x-1)y' + y - 0$ are .... (P.T.U., 2009)

CHAPTER 17

Partial Differential Equations

17.1 INTRODUCTION

The reader has, already been introduced to the notion of partial differential equations. Here, we shall begin by studying the ways in which partial differential equations are formed. Then we shall investigate the solutions of special types of partial differential equations of the first and higher orders.

In what follows x and y will, usually be taken as the independent variables and z, the dependent variable so that $z = f(x, y)$ and we shall employ the following notation :

$$\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q, \frac{\partial^2 z}{\partial x^2} = r, \frac{\partial^2 z}{\partial x \partial y} = s, \frac{\partial^2 z}{\partial y^2} = t.$$

17.2 FORMATION OF PARTIAL DIFFERENTIAL EQUATIONS

Unlike the case of ordinary differential equations which arise from the elimination of arbitrary constants; *the partial differential equations can be formed either by the elimination of arbitrary constants or by the elimination of arbitrary functions from a relation involving three or more variables.* The method is best illustrated by the following examples :

Example 17.1. *Derive a partial differential equation (by eliminating the constants) from the equation*

$$2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}. \quad \text{...(i)}$$

Solution. Differentiating (*i*) partially with respect to x and y, we get

$$2\frac{\partial z}{\partial x} = \frac{2x}{a^2} \quad \text{or} \quad \frac{1}{a^2} = \frac{1}{x}\frac{\partial z}{\partial x} = \frac{p}{x}$$

and

$$\frac{2\partial z}{\partial y} = \frac{2y}{b^2} \quad \text{or} \quad \frac{1}{b^2} = \frac{1}{y}\frac{\partial z}{\partial y} = \frac{q}{y}$$

Substituting these values of $1/a^2$ and $1/b^2$ in (*i*), we get

$$2z = xp + yq$$

as the desired partial differential equation of the first order.

Example 17.2. *Form the partial differential equations (by eliminating the arbitrary functions) from*
(*a*) $z = (x + y)\,\phi\,(x^2 - y^2)$ (*P.T.U., 2009*)
(*b*) $z = f(x + at) + g(x - at)$ (*V.T.U., 2009*) (*c*) $f(x^2 + y^2, z - xy) = 0$ (*S.V.T.U., 2007*)

Solution. (*a*) We have $z = (x + y)\,\phi\,(x^2 - y^2)$

Differentiating z partially with respect to x and y,

$$p = \frac{\partial z}{\partial x} = (x + y)\,\phi'\,(x^2 - y^2)\,.\,2x + \phi\,(x^2 - y^2), \quad ...(i)$$

$$q = \frac{\partial z}{\partial y} = (x + y)\,\phi'\,(x^2 - y^2)\,.\,(-2y) + \phi\,(x^2 - y^2) \quad ...(ii)$$

From (*i*), $$p - \frac{z}{x + y} = 2x\,(x + y)\,\phi'\,(x^2 - y^2)$$

From (*ii*), $$q - \frac{z}{x + y} = -2y\,(x + y)\,\phi'\,(x^2 - y^2)$$

Division gives $$\frac{p - z/(x + y)}{q - z/(x + y)} = -\frac{x}{y}$$

i.e., $[p(x + y) - z]y + [q\,(x + y) - z]x$

i.e., $(x + y)\,(py + qx) - z(x + y) = 0$

Hence $py + qz = z$ is required equation.

(*b*) We have $z = f\,(x + at) + g(x - at)$ ...(*i*)

Differentiating z partially with respect to x and t,

$$\frac{\partial z}{\partial x} = f'\,(x + at) + g'(x - at),\ \frac{\partial^2 z}{\partial x^2} = f''\,(x + at) + g''(x - at) \quad ...(ii)$$

$$\frac{\partial z}{\partial t} = af'\,(x + at) - ag'(x - at),\ \frac{\partial^2 z}{\partial t^2} = a^2 f''\,(x + at) + a^2 g''(x - at) = a^2\frac{\partial^2 z}{\partial x^2} \quad \text{[By (ii)]}$$

Thus the desired partial differential equation is $\dfrac{\partial^2 z}{\partial t^2} = a^2\,\dfrac{\partial^2 z}{\partial x^2}$

which is an equation of the second order and (*i*) is its solution.

(*c*) Let $x^2 + y^2 = u$ and $z - xy = v$ so that $f\,(u, v) = 0$.

Differentiating partially w.r.t. x and y, we have

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p\right) = 0$$

or $$\frac{\partial f}{\partial u}(2x) + \frac{\partial f}{\partial v}(-y + p) = 0 \quad ...(i)$$

and $$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q\right) = 0 \quad \text{or} \quad \frac{\partial f}{\partial u}(2y) + \frac{\partial f}{\partial v}(-x + q) = 0 \quad ...(ii)$$

Eliminating $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$ from (*i*) and (*ii*), we get

$$\begin{vmatrix} 2x & -y + p \\ 2y & -x + q \end{vmatrix} = 0 \quad \text{or} \quad xq - yp = x^2 - y^2.$$

Example 17.3. *Find the differential equation of all planes which are at a constant distance a from the origin.* (*V.T.U., 2009 S ; Kurukshetra, 2006*)

Solution. The equation of the plane in 'normal form' is

$$lx + my + nz = a \quad ...(i)$$

where l, m, n are the d.c.s of the normal from the origin to the plane.

Then $l^2 + m^2 + n^2 = 1$ or $n = \sqrt{(1 - l^2 - m^2)}$

$\therefore$ (*i*) becomes $lx + my + \sqrt{(1 - l^2 - m^2)}\ z = a$ ...(*ii*)

Differentiating partially w.r.t. x, we get

$$l + \sqrt{(1 - l^2 - m^2)}\ .\ p = 0 \qquad ...(iii)$$

Differentiating partially w.r.t. y, we get

$$m + \sqrt{(1 - l^2 - m^2)}\ .\ q = 0 \qquad ...(iv)$$

Now we have to eliminate l, m from (*ii*), (*iii*) and (*iv*).

From (*iii*), $l = -\sqrt{(1 - l^2 - m^2)}\ .\ p$ and $m = -\sqrt{(1 - l^2 - m^2)}\ .\ q$

Squaring and adding, $l^2 + m^2 = (1 - l^2 - m^2)(p^2 + q^2)$

or $$(l^2 + m^2)(1 + p^2 + q^2) = p^2 + q^2 \text{ or } 1 - l^2 - m^2 = 1 - \frac{p^2 + q^2}{1 + p^2 + q^2} = \frac{1}{1 + p^2 + q^2}$$

Also $$l = -\frac{p}{\sqrt{(1 + p^2 + q^2)}} \text{ and } m = -\frac{q}{\sqrt{(1 + p^2 + q^2)}}$$

Substituting the values of l, m and $1 - l^2 - m^2$ in (*ii*), we obtain

$$\frac{-px}{\sqrt{(1 + p^2 + q^2)}} - \frac{qy}{\sqrt{(1 + p^2 + q^2)}} + \frac{1}{\sqrt{(1 + p^2 + q^2)}}\ z = a$$

or $z = px + qy + a\sqrt{(1 + p^2 + q^2)}$ which is the required partial differential equation.

PROBLEMS 17.1

From the partial differential equation (by eliminating the arbitrary constants from :

1. $z = ax + by + a^2 + b^2$.
2. $(x - a)^2 + (y - b)^2 + z^2 = c^2$. (*Kottayam, 2005*)
3. $(x - a)^2 + (y - b)^2 = z^2 \cot^2 \alpha$ (*Anna, 2009*)
4. $z = a \log \left\{\frac{b(y-1)}{1-x}\right\}$ (*J.N.T.U., 2002 S*)
5. Find the differential equation of all spheres of fixed radius having their centres in the xy-plane. (*Madras 2000 S*)
6. Find the differential equation of all spheres whose centres lie on the z-axis. (*Kerala, 2005*)

Form the partial differential equations (by eliminating the arbitrary functions) from :

7. $z = f(x^2 - y^2)$ (*S.V.T.U., 2008*)
8. $z = f(x^2 + y^2) + x + y$ (*Anna, 2009*)
9. $z = yf(x) + xg(y)$. (*V.T.U., 2004*)
10. $z = x^2 f(y) + y^2 g(x)$. (*Anna, 2003*)
11. $z = f(x) + e^y g(x)$.
12. $xyz = \phi(x + y + z)$.
13. $z = f_1(x) f_2(y)$.
14. $z = e^{my} \phi(x - y)$. (*P.T.U., 2002*)
15. $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$. (*V.T.U., 2010 ; J.N.T.U., 2010 ; Madras, 2000*)
16. $z = f_1(y + 2x) + f_2(y - 3x)$. (*Kurukshetra, 2005*)
17. $v = \frac{1}{r}[f(r - at) + F(r + at)]$.
18. $z = xf_1(x + t) + f_2(x + t)$.
19. $F(xy + z^2, x + y + z) = 0$. (*V.T.U., 2006*)
20. $F(x + y + z, x^2 + y^2 + z^2) = 0$. (*S.V.T.U., 2007*)
21. If $u = f(x^2 + 2yz, y^2 + 2zx)$, prove that $(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0$.

17.3 SOLUTIONS OF A PARTIAL DIFFERENTIAL EQUATION

It is clear from the above examples that a partial differential equation can result both from elimination of arbitrary constants and from the elimination of arbitrary functions.

The solution $f(x, y, z, a, b) = 0$ ...(1)

of a first order partial differential equation which contains two arbitrary constants is called a *complete integral*.

A solution obtained from the complete integral by assigning particular values to the arbitrary constants is called a particular integral.

If we put $b = \phi(a)$ in (1) and find the envelope of the family of surfaces $f[x, y, z, \phi(a)] = 0$, then we get a solution containing an arbitrary function ϕ, which is called the *general integral*.

The envelope of the family of surfaces (1), with parameters a and b, if it exists, is called a *singular integral*. The singular integral differs from the particular integral in that it is not obtained from the complete integral by giving particular values to the constants.

17.4 EQUATIONS SOLVABLE BY DIRECT INTEGRATION

We now consider such partial differential equations which can be solved by direct integration. In place of the usual constants of integration, we must, however, use arbitrary functions of the variable held fixed.

Example 17.4. *Solve* $\dfrac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x - y) = 0.$ (V.T.U., 2010)

Solution. Integrating twice with respect to x (keeping y fixed),

$$\frac{\partial^2 z}{\partial x \partial y} + 9x^2y^2 - \frac{1}{2}\cos(2x - y) = f(y)$$

$$\frac{\partial z}{\partial y} + 3x^3y^2 - \frac{1}{4}\sin(2x - y) = xf(y) + g(y).$$

Now integrating with respect to y (keeping x fixed)

$$z + x^3y^3 - \frac{1}{4}\cos(2x - y) = x\int f(y)dy + \int g(y)dy + w(x)$$

The result may be simplified by writing

$$\int f(y)dy = u(y) \text{ and } \int g(y)dy = v(y).$$

Thus $z = \dfrac{1}{4}\cos(2x - y) - x^3y^3 + xu(y) + v(y) + w(x)$ where u, v, w are arbitrary functions.

Example 17.5. *Solve* $\dfrac{\partial^2 z}{\partial x^2} + z = 0$, *given that when* $x = 0$, $z = e^y$ *and* $\dfrac{\partial z}{\partial x} = 1$.

Solution. If z were function of x alone, the solution would have been $z = A \sin x + B \cos x$, where A and B are constants. Since z is a function of x and y, A and B can be arbitrary functions of y. Hence the solution of the given equation is $z = f(y) \sin x + \phi(y) \cos x$

$$\therefore \quad \frac{\partial z}{\partial x} = f(y)\cos x - \phi(y)\sin x$$

When $x = 0$; $z = e^y$, $\therefore e^y = \phi(y)$. When $x = 0$, $\dfrac{\partial z}{\partial x} = 1$, $\therefore 1 = f(y)$.

Hence the desired solution is $z = \sin x + e^y \cos x$.

Example 17.6. *Solve* $\dfrac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$, *for which* $\dfrac{\partial z}{\partial y} = -2\sin y$ *when* $x = 0$ *and* $z = 0$ *when* y *is an odd multiple of* $\pi/2$. (V.T.U., 2010 S)

Solution. Given equation is $\dfrac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$

Integrating w.r.t. x, keeping y constant, we get

$$\frac{\partial z}{\partial y} = -\cos x \sin y + f(y) \qquad \ldots(i)$$

When $x = 0$, $\dfrac{\partial z}{\partial y} = -2 \sin y$, $\quad \therefore \quad -2 \sin y = -\sin y + f(y)$ or $f(y) = -\sin y$

$\therefore$ (*i*) becomes $\dfrac{\partial z}{\partial y} = -\cos x \sin y - \sin y$

Now integrating w.r.t. y, keeping x constant, we get

$$z = \cos x \cos y + \cos y + g(x) \qquad ...(ii)$$

When y is an odd multiple of $\pi/2$, $z = 0$.

$$\therefore \qquad 0 = 0 + 0 + g(x) \quad \text{or} \quad g(x) = 0 \qquad [\because \cos (2n+1)\pi/2 = 0]$$

Hence from (*ii*), the complete solution is $z = (1 + \cos x) \cos y$.

PROBLEMS 17.2

Solve the following equations :

1. $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{x}{y} + a.$
2. $\dfrac{\partial^2 z}{\partial x^2} = xy.$
3. $\dfrac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x.$
4. $\dfrac{\partial^3 z}{\partial x^2 \partial y} = \cos (2x + 3y).$
5. $\dfrac{\partial^2 z}{\partial y^2} = z$, gives that when $y = 0$, $z = e^x$ and $\dfrac{\partial z}{\partial y} = e^{-x}$
6. $\dfrac{\partial^2 z}{\partial x^2} = a^2 z$ given that when $x = 0$, $\dfrac{\partial z}{\partial x} = a \sin y$ and $\dfrac{\partial z}{\partial y} = 0.$

17.5 LINEAR EQUATIONS OF THE FIRST ORDER

A linear partial differential equation of the first order, commonly known as *Lagrange's Linear equation**, is of the form

$$Pp + Qq = R \qquad ...(1)$$

where P, Q and R are functions of x, y, z. This equation is called a *quasi-linear equation*. When P, Q and R are independent of z it is known as *linear equation*.

Such an equation is obtained by eliminating an arbitrary function ϕ from $\phi(u, v) = 0$ ...(2)

where u, v are some functions of x, y, z.

Differentiating (2) partially with respect to x and y.

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} p\right) = 0 \text{ and } \frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} q\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} q\right) = 0.$$

Eliminating $\dfrac{\partial \phi}{\partial u}$ and $\dfrac{\partial \phi}{\partial v}$, we get $\begin{vmatrix} \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial z} p & \dfrac{\partial v}{\partial x} + \dfrac{\partial v}{\partial z} p \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} q & \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial z} q \end{vmatrix} = 0$

which simplifies to $\left(\dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial z} - \dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial y}\right)p + \left(\dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial z}\right)q = \left(\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial y} - \dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial x}\right)$ ...(3)

This is of the same form as (1).

Now suppose $u = a$ and $v = b$, where a, b are constants, so that

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = du = 0$$

$$\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz = dv = 0.$$

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*See footnote p. 142.

By cross-multiplication, we have

$$\frac{dx}{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}}=\frac{dy}{\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}}=\frac{dz}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}}.$$

or $$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$ ...(4) [By virtue of (1) and (3)]

The solutions of these equations are $u = a$ and $v = b$.

$\therefore$ $\phi(u, v) = 0$ is the required solution of (1).

Thus to solve the equation $Pp + Qq = R$.

(*i*) *form the subsidiary equations* $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$.

(*ii*) *solve these simultaneous equations by the method of § 16.10 giving $u = a$ and $v = b$ as its solutions.*

(*iii*) *write the complete solution as $\phi(u, v) = 0$* or $u = f(v)$.

Example 17.7. *Solve* $\frac{y^2z}{x}p + xzq = y^2$. *(Kottayam, 2005)*

Solution. Rewriting the given equation as

$$y^2zp + x^2zq = y^2x,$$

The subsidiary equations are $\frac{dx}{y^2z}=\frac{dy}{x^2z}=\frac{dz}{y^2x}$

The first two fractions give $x^2dx = y^2dy$.

Integrating, we get $x^3 - y^3 = a$ ...(*i*)

Again the first and third fractions give $xdx = zdz$

Integrating, we get $x^2 - z^2 = b$ ...(*ii*)

Hence from (*i*) and (*ii*), the complete solution is

$$x^3 - y^3 = f(x^2 - z^2).$$

Example 17.8. *Solve* $(mz - ny)\frac{\partial z}{\partial x} + (nx - lz)\frac{\partial z}{\partial y} = ly - mx.$ *(V.T.U., 2010 ; S.V.T.U., 2009)*

Solution. Here the subsidiary equations are $\frac{dx}{mz-ny}=\frac{dy}{mx-lz}=\frac{dz}{ly-mx}$

Using multipliers x, y, and z, we get each fraction $= \frac{xdx + ydy + zdz}{0}$

$\therefore$ $xdx + ydy + zdz = 0$ which on integration gives $x^2 + y^2 + z^2 = a$ ...(*i*)

Again using multipliers l, m and n, we get each fraction $= \frac{ldx + mdy + ndz}{0}$

$\therefore$ $ldx + mdy + ndx = 0$ which on integration gives $lx + my + nz = b$ ...(*ii*)

Hence from (*i*) and (*ii*), the required solution is $x^2 + y^2 + z^2 = f(lx + my + nz)$.

Example 17.9. *Solve* $(x^2 - y^2 - z^2)p + 2xyq = 2xz.$ *(V.T.U., 2010 ; Anna, 2009 ; S.V.T.U., 2008)*

Solution. Here the subsidiary equations are $\frac{dx}{x^2-y^2-z^2}=\frac{dy}{2xy}=\frac{dz}{2xz}$

From the last two fractions, we have $\frac{dy}{y}=\frac{dz}{z}$

which on integration gives $\log y = \log z + \log a$ or $y/z = a$ ...(*i*)

Using multipliers x, y and z, we have

$$\text{each fraction} = \frac{xdx + ydy + zdz}{x(x^2+y^2+z^2)} \quad \therefore \quad \frac{2xdx + 2ydy + 2zdz}{x^2+y^2+z^2}=\frac{dz}{z}$$

which on integration gives $\log(x^2 + y^2 + z^2) = \log z + \log b$

or $$\frac{x^2 + y^2 + z^2}{z} = b \qquad \text{...(ii)}$$

Hence from (*i*) and (*ii*), the required solution is $x^2 + y^2 + z^2 = zf(y/z)$.

Example 17.10. *Solve* $x^2(y - z)p + y^2(z - x)q = z^2(x - y)$. *(P.T.U., 2009 ; Bhopal, 2008 ; S.V.T.U. 2007)*

Solution. Here the subsidiary equations are

$$\frac{dx}{x^2(y - z)} = \frac{dy}{y^2(z - x)} = \frac{dz}{z^2(x - y)}$$

Using the multipliers $1/x$, $1/y$ and $1/z$, we have

$$\text{each fraction} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$\therefore$ $\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$ which on integration gives

$$\log x + \log y + \log z = \log a \quad \text{or} \quad xyz = a \qquad \text{...(i)}$$

Using the multipliers $\frac{1}{x^2}, \frac{1}{y^2}$ and $\frac{1}{z^2}$, we get

$$\text{each fraction} = \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$$

$\therefore$ $\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2} = 0$, which on integrating gives

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0 \qquad \text{...(ii)}$$

Hence from (*i*) and (*ii*), the complete solution is

$$xyz = f\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right).$$

Example 17.11. *Solve* $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$. *(Bhopal, 2008 ; V.T.U., 2006 ; Madras, 2000)*

Solution. Here the subsidiary equations are

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy} \qquad \text{...(i)}$$

$$\text{Each of these equations} = \frac{dx - dy}{x^2 - y^2 - (y - x)z} = \frac{dy - dz}{y^2 - z^2 - x(z - y)}$$

i.e., $$\frac{d(x - y)}{(x - y)(x + y + z)} = \frac{d(y - z)}{(y - z)(x + y + z)} \quad \text{or} \quad \frac{d(x - y)}{x - y} = \frac{d(y - z)}{y - z}$$

Integrating, $$\log(x - y) = \log(y - z) + \log c \quad \text{or} \quad \frac{x - y}{y - z} = c \qquad \text{...(ii)}$$

$$\text{Each of the subsidiary equations (i)} = \frac{xdx + ydy + zdz}{x^3 + y^3 + z^3 - 3xyz}$$

$$= \frac{xdx + ydy + zdz}{(x + y + z)(x^2 + y^2 + z^2 - yz - zx - xy)} \qquad \text{...(iii)}$$

$$\text{Also each of the subsidiary equations} = \frac{dx + dy + dz}{x^2 + y^2 + z^2 - yz - zx - xy} \qquad \text{...(iv)}$$

Equating (*iii*) and (*iv*) and cancelling the common factor, we get

$$\frac{xdx + ydy + zdz}{x + y + z} = dx + dy + dz$$

or $$\int (xdx + ydy + zdz) = \int (x + y + z) d\,(x + y + z) + c'$$

or $$x^2 + y^2 + z^2 = (x + y + z)^2 + 2c' \quad \text{or} \quad xy + yz + zx + c' = 0 \qquad ...(v)$$

Combining (*ii*) and (*v*), the general solution is

$$\frac{x - y}{y - z} = f\,(xy + yz + zx).$$

PROBLEMS 17.3

Solve the following equations :

1. $xp + yq = 3z$.
2. $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$.
3. $(z - y)p + (x - z)q = y - x$.
4. $p \cos (x + y) + q \sin (x + y) = z$.
5. $pyz + qzx = xy$.
6. $p \tan x + q \tan y = \tan z$.
7. $p - q = \log (x + y)$.
8. $xp - yq = y^2 - x^2$ (*J.N.T.U., 2002 S*)
9. $(y + z)p - (z + x)\,q = x - y$.
10. $x(y - z)p + y(z - x)q = z(x - y)$. (*Bhopal, 2007*)
11. $x(y^2 - z^2)p + y(z^2 - x^2)q - z(x^2 - y^2) = 0$. (*V.T.U., 2010 ; Anna, 2008*)
12. $y^2p - xyq = x(z - 2y)$. (*S.V.T.U., 2008*)
13. $(y^2 + z^2)p - xyq + zx = 0$. (*P.T.U., 2009 ; V.T.U., 2009*)
14. $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$. (*Kerala, 2005*)
15. $px(z - 2y^2) = (z - qy)(z - y^2 - 2x^3)$.

17.6 NON-LINEAR EQUATIONS OF THE FIRST ORDER

Those equations in which p and q occur other than in the first degree are called *non-linear partial differential equations of the first order*. The *complete solution* of such an equation contains only two arbitrary constants (*i.e.*, equal to the number of independent variables involved) and the particular integral is obtained by giving particular values to the constants.]

Here we shall discuss four standard forms of these equations.

Form I. f(p, q) = 0, *i.e., equations containing p and q only.*

Its complete solution is $z = ax + by + c$ ...(1)

where a and b are connected by the relation $f\,(a, b) = 0$ ...(2)

[Since from (1), $p = \frac{\partial z}{\partial x} = a$ and $q = \frac{\partial z}{\partial y} = b$, which when substituted in (2) give $f\,(p, q) = 0$].

Expressing (2) as $b = \phi(a)$ and substituting this value of b in (1), we get the required solution as $z = ax + \phi(a)y + c$ in which a and c are arbitrary constants.

Example 17.12. *Solve $p - q = 1$.* (*Anna, 2009*)

Solution. The complete solution is $z = ax + by + c$ where $a - b = 1$

Hence $z = ax + a - 1y + c$ is the desired solution.

Example 17.13. *Solve $x^2p^2 + y^2q^2 = z^2$.* (*Anna, 2008 ; Bhopal, 2008 ; Kerala, 2005 ; Kurukshetra, 2005*)

Solution. Given equation can be reduced to the above form by writing it as

$$\left(\frac{x}{z} \cdot \frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z} \cdot \frac{\partial z}{\partial y}\right)^2 = 1 \qquad ...(i)$$

and setting $\frac{dx}{x} = du, \frac{dy}{y} = dv, \frac{dz}{z} = dw$ so that $u = \log x, v = \log y, w = \log z$.

Then (*i*) becomes $$\left(\frac{\partial w}{\partial u}\right)^2 + \left(\frac{\partial w}{\partial v}\right)^2 = 1$$

i.e., $$P^2 + Q^2 = 1 \quad \text{where} \quad P = \frac{\partial w}{\partial u} \quad \text{and} \quad Q = \frac{\partial w}{\partial v}.$$

Its complete solution is $w = au + bv + c$ ...(*ii*)

where $a^2 + b^2 = 1$ or $b = \sqrt{(1-a^2)}$.

$\therefore$ (*ii*) becomes $w = au + \sqrt{(1-a^2)}\, v + c$

or $$\log z = a \log x + \sqrt{(1-a^2)} \log y + c \text{ which is the required solution.}$$

Form II. f(z, p, q) = 0, *i.e., equations not containing x and y.*

As a trial solution, assume that z is a function of $u = x + ay$, where a is an arbitrary constant.

$\therefore$ $$p = \frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = \frac{dz}{du} \qquad q = \frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = a\frac{dz}{du}$$

Substituting the values of p and q in $f(z, p, q) = 0$, we get

$$f\left(z, \frac{\partial z}{\partial u}, a\frac{dz}{du}\right) = 0 \text{ which is an ordinary differential equation of the first order.}$$

Rewiting it as $\dfrac{dz}{du} = \phi(z, a)$ it can be easily integrated giving

$F(z, a) = u + b$, or $x + ay + b = F(z, a)$ which is the desired complete solution.

Thus to solve $f(z, p, q) = 0$,

(*i*) *assume* $u = x + ay$ *and substitute* $p = dz/du$, $q = a\, dz/du$ *in the given equation;*

(*ii*) *solve the resulting ordinary differential equation in z and u;*

(*iii*) *replace u by* $x + ay$.

Example 17.14. *Solve* $p(1 + q) = qz$. (*Madras, 2000 S*)

Solution. Let $u = x + ay$, so that $p = dz/du$ and $q = a\, dz/du$.

Substituting these values of p and q in the given equation, we have

$$\frac{dz}{du}\left(1 + a\frac{dz}{du}\right) = az\frac{dz}{du} \quad \text{or} \quad a\frac{dz}{du} = az - 1 \quad \text{or} \quad \int \frac{a\,dz}{az-1} = \int du + b$$

or $$\log(az - 1) = u + b \text{ or } \log(az - 1) = x + ay + b$$

which is the required complete solution.

Example 17.15. *Solve* $q^2 = z^2p^2(1 - p^2)$. (*J.N.T.U., 2005 ; Kerala, 2005*)

Solution. Setting $u = y + ax$ and $z = f(u)$, we get

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = a\frac{dz}{du} \text{ and } q = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = \frac{dz}{du}$$

$\therefore$ The given equation becomes $\left(\dfrac{dz}{du}\right)^2 = a^2z^2\left(\dfrac{dz}{du}\right)^2 \left\{1 - a^2\left(\dfrac{dz}{du}\right)^2\right\}$ ...(*i*)

or $$a^4z^2\left(\frac{dz}{du}\right)^2 = a^2z^2 - 1 \quad \text{or} \quad \frac{dz}{du} = \frac{\sqrt{(a^2z^2 - 1)}}{a^2 z}$$

Integrating, $$\int \frac{a^2 z}{\sqrt{(a^2z^2 - 1)}}\, dz = \int du + c \quad \text{or} \quad (a^2z^2 - 1)^{1/2} = u + c$$

i.e., $$a^2z^2 = (y + ax + c)^2 + 1 \qquad [\because u = y + ax]$$

The second factor in (*i*) is $dz/du = 0$. Its solution is $z = c'$.

Example 17.16. *Solve* $z^2(p^2x^2 + q^2) = 1$. (*Bhopal, 2008 S*)

Solution. Given equation can be reduced to the above form by writing it as

$$z^2\left[\left(x\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]=1 \quad ...(i)$$

Putting $X = \log x$, so that $x\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial X}$, (i) takes the standard form

$$z^2\left[\left(\frac{\partial z}{\partial X}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]=1 \quad ...(ii)$$

Let $u = X + ay$ and put $\dfrac{\partial z}{\partial X}=\dfrac{dz}{du}$ and $\dfrac{\partial z}{\partial y}=a\dfrac{dz}{du}$ in (ii), so that

$$z^2\left[\left(\frac{dz}{du}\right)^2+a^2\left(\frac{dz}{du}\right)^2\right]=1 \quad \text{or} \quad \sqrt{(1+a^2)}\,zdz=\pm du$$

Integrating, $\qquad \sqrt{(1+a^2)}\,z^2=\pm 2u+b=\pm 2(X+ay)+b$

or $\qquad z^2\sqrt{(1+a^2)}=\pm 2(\log x+ay)+b$

which is the complete solution required.

Form III. f(x, p) = F(y, q), *i.e., equations in which z is absent and the terms containing x and p can be separated from those containing y and q.*

As a trial solution assume that $f(x, p) = F(y, q) = a$, say

Then solving for p, we get $\qquad p = \phi(x)$

and solving for q, we get $\qquad q = \psi(y)$

Since $\qquad dz=\dfrac{\partial z}{\partial x}dx+\dfrac{\partial z}{\partial y}dy=pdx+qdy$

$\therefore \qquad dz = \phi(x)dx + \psi(y)dy$

Integrating, $\qquad z=\int\phi(x)dx+\int\psi(y)dy+b$

which is the desired complete solution containing two constants a and b.

Example 17.17. *Solve* $p^2 + q^2 = x + y$. *(Bhopal, 2006 ; Madras, 2003)*

Solution. Given equation is $p^2 - x = y - q^2 = a$, say

$\therefore \qquad p^2 - x = a$ gives $p=\sqrt{(a+x)}$

and $\qquad y - q^2 = a$ gives $q=\sqrt{(y-a)}$

Substituting these values of p and q in $dz = pdx + qdy$, we get

$$dz=\sqrt{(a+x)}\,dx+\sqrt{(y-a)}\,dy$$

$\therefore$ integrating gives, $\qquad z=\dfrac{2}{3}(a+x)^{3/2}+\dfrac{2}{3}(y-a)^{3/2}+b$

which is the required complete solution.

Example 17.18. *Solve* $z^2(p^2 + q^2) = x^2 + y^2$. *(Bhopal, 2008)*

Solution. The equation can be reduced to the above form by writing it as

$$\left(z\frac{\partial z}{\partial x}\right)^2+\left(z\frac{\partial z}{\partial y}\right)^2=x^2+y^2 \quad ...(i)$$

and putting $\qquad zdz = dZ$, *i.e.*, $Z=\dfrac{1}{2}z^2$

$$\therefore \qquad \frac{\partial Z}{\partial x}=\frac{\partial Z}{\partial z}\cdot\frac{\partial z}{\partial x}=z\frac{\partial z}{\partial x}=P$$

$$\text{and} \qquad \frac{\partial Z}{\partial y}=\frac{\partial Z}{\partial z}\cdot\frac{\partial z}{\partial y}=z\frac{\partial z}{\partial y}=Q$$

$\therefore$ (i) becomes $$P^2 + Q^2 = x^2 + y^2$$
or $$P^2 - x^2 = y^2 - Q^2 = a, \text{ say.}$$
$\therefore$ $$P = \sqrt{(x^2 + a)} \text{ and } Q = \sqrt{(y^2 - a)}.$$
$\therefore$ $dZ = Pdx + Qdy$ gives
$$dZ = \sqrt{(x^2 + a)}\, dx + \sqrt{(y^2 - a)}\, dy$$
Integrating, we have
$$Z = \frac{1}{2} x \sqrt{(x^2 + a)} + \frac{1}{2} a \log [x + \sqrt{(x^2 + a)}]$$
$$+ \frac{1}{2} y \sqrt{(y^2 - a)} - \frac{1}{2} a \log [y + \sqrt{(y^2 - a)}] + b$$
or
$$z^2 = x \sqrt{(x^2 + a)} + y \sqrt{(y^2 - a} + a \log \frac{x + \sqrt{(x^2 + a)}}{y + \sqrt{(y^2 - a)}} + 2b$$
which is the required complete solution.

Example 17.19. *Solve* $(x + y)(p + q)^2 + (x - y)(p - q)^2 = 1.$ *(Bhopal, 2006 ; Rajasthan, 2006 ; V.T.U., 2003)*

Solution. This equation can be reduced to the form $f(x, q) = F(y, q)$ by putting $u = x + y$, $v = x - y$ and taking $z = z(u, v)$.

Then $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = P + Q$$
and $$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = P - Q, \text{ where } P = \frac{\partial z}{\partial u}, Q = \frac{\partial z}{\partial v}$$

Substituting these, the given equation reduces to
$$u(2P)^2 + v(2Q)^2 = 1 \quad \text{or} \quad 4P^2 u = 1 - 4Q^2 v = a \text{ (say)}$$
$$P = \pm \frac{1}{2} \sqrt{\frac{a}{u}}, Q = \pm \frac{1}{2} \sqrt{\frac{1 - a}{v}}$$
$\therefore$ $$dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv = Pdu + Qdv$$
$$= \pm \frac{\sqrt{a}}{2} \frac{du}{\sqrt{u}} \pm \frac{\sqrt{1 - a}}{2} \frac{dv}{\sqrt{v}}$$
Integrating, we have $$z = \pm \sqrt{a} \sqrt{u} \pm \sqrt{1 - a} \sqrt{v} + b$$
or $$z = \pm \sqrt{a (x + y)} \pm \sqrt{(1 - a)(x - y)} + b$$
which is the required complete solution.

Form IV. z = px + qy + f(p, q): *an equation analogous to the Clairaut's equation (§ 11.14).*

Its complete solution is $z = ax + by + f(a, b)$ which is obtained by writing a for p and b for q in the given equation.

Example 17.20. *Solve* $z = px + qy + \sqrt{(1 + p^2 + q^2)}$. *(Anna, 2009)*

Solution. Given equation is of the form $z = px + qy + f(p, q)$ where $f(p, q) = \sqrt{(1 + p^2 + q^2)}$

$\therefore$ Its complete solution is $z = ax + by + \sqrt{(1 + a^2 + b^2)}$.

PROBLEMS 17.4

Obtain the complete solution of the following equations :

1. $pq + p + q = 0$.
2. $p^2 + q^2 = 1$. *(Osmania, 2000)*
3. $z = p^2 + q^2$. *(Anna, 2005 S ; J.N.T.U., 2002 S)*
4. $p(1 - q^2) = q(1 - z)$ *(Anna, 2006)*
5. $yp + xq + pq = 0$.
6. $p + q = \sin x + \sin y$.

7. $p^2 - q^2 = x - y$.
8. $\sqrt{p} + \sqrt{q} = x + y$.
9. $p^2 + q^2 = x^2 + y^2$. *(Osmania, 2003)*
10. $z = px + qy + \sin(x + y)$.
11. $\sqrt{p} + \sqrt{q} = 2x$. *(J.N.T.U., 2006)*
12. $z = px + qy - 2\sqrt{(pq)}$.
13. $(x - y)(px - qy) = (p - q)^2$. **[Hint.** Use $x + y = u, xy = v$]

17.7 CHARPIT'S METHOD*

We now explain a general method for finding the complete integral of a non-linear partial differential equation which is due to Charpit.

Consider the equation

$$f(x, y\ z, p, q) = 0 \qquad \text{...(1)}$$

Since z depends on x and y, we have

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = pdx + qdy \qquad \text{...(2)}$$

Now if we can find another relation involving x, y, z, p, q such as $\phi(x, y, z, p, q) = 0$ ...(3)

then we can solve (1) and (3) for p and q and substitute in (2). This will give the solution provided (2) is integrable.

To determine ϕ, we differentiate (1) and (3) with respect to x and y giving

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} p + \frac{\partial f}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial x} = 0 \qquad \text{...(4)}$$

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial z} p + \frac{\partial \phi}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial \phi}{\partial q}\frac{\partial q}{\partial x} = 0 \qquad \text{...(5)}$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} q + \frac{\partial f}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial y} = 0 \qquad \text{...(6)}$$

$$\frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z} q + \frac{\partial \phi}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial \phi}{\partial q}\frac{\partial q}{\partial y} = 0 \qquad \text{...(7)}$$

Eliminating $\frac{\partial p}{\partial x}$ between the equations (4) and (5), we get

$$\left(\frac{\partial f}{\partial x}\frac{\partial \phi}{\partial p} - \frac{\partial \phi}{\partial x}\frac{\partial f}{\partial p}\right) + \left(\frac{\partial f}{\partial z}\frac{\partial \phi}{\partial p} - \frac{\partial \phi}{\partial z}\frac{\partial f}{\partial p}\right) p + \left(\frac{\partial f}{\partial q}\frac{\partial \phi}{\partial p} - \frac{\partial \phi}{\partial q}\frac{\partial f}{\partial p}\right)\frac{\partial q}{\partial x} = 0 \qquad \text{...(8)}$$

Also eliminating $\frac{\partial q}{\partial y}$ between the equations (6) and (7), we obtain

$$\left(\frac{\partial f}{\partial y}\frac{\partial \phi}{\partial q} - \frac{\partial \phi}{\partial y}\frac{\partial f}{\partial q}\right) + \left(\frac{\partial f}{\partial z}\frac{\partial \phi}{\partial q} - \frac{\partial \phi}{\partial z}\frac{\partial f}{\partial q}\right) q + \left(\frac{\partial f}{\partial p}\frac{\partial \phi}{\partial q} - \frac{\partial \phi}{\partial p}\frac{\partial f}{\partial q}\right)\frac{\partial p}{\partial y} = 0 \qquad \text{...(9)}$$

Adding (8) and (9) and using $\frac{\partial q}{\partial x} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial p}{\partial y}$,

we find that the last terms in both cancel and the other terms, on rearrangement, give

$$\left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial p} + \left(\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial q} + \left(-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial z} + \left(-\frac{\partial f}{\partial p}\right)\frac{\partial \phi}{\partial x} + \left(-\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial y} = 0 \qquad \text{...(10)}$$

i.e.,

$$\left(-\frac{\partial f}{\partial p}\right)\frac{\partial \phi}{\partial x} + \left(-\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial y} + \left(-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial z} + \left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial p} + \left(\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial q} = 0 \qquad \text{...(11)}$$

This is Lagrange's linear equation (§ 17.5) with x, y, z, p, q as independent variables and ϕ as the dependent variable. Its solution will depend on the solution of the subsidiary equations

*Charpit's memoir containing this method was presented to the Paris Academy of Sciences in 1784.

$$\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{\partial\phi}{0}$$

An integral of these equations involving p or q or both, can be taken as the required relation (3), which alongwith (1) will give the values of p and q to make (2) integrable. Of course, we should take the simplest of the integrals so that it may be easier to solve for p and q.

Example 17.21. *Solve $(p^2+q^2)y = qz$.* *(V.T.U., 2007 ; Hissar, 2005)*

Solution. Let $f(x, y, z, p, q) = (p^2+q^2)y - qz = 0$ ...(*i*)

Charpit's subsidiary equations are

$$\frac{dx}{-2py}=\frac{dy}{z-2qy}=\frac{dz}{-qz}=\frac{dp}{-pq}=\frac{dq}{p^2}$$

The last two of these give $pdp + qdq = 0$

Integrating, $\quad p^2+q^2=c^2$ ...(*ii*)

Now to solve (*i*) and (*ii*), put $p^2+q^2=c^2$ in (*i*), so that $q = c^2y/z$

Substituting this value of q in (*ii*), we get $p = c\sqrt{(z^2-c^2y^2)/z}$

Hence $\quad dz = pdx + qdy = \frac{c}{z}\sqrt{(z^2-c^2y^2)}\,dx + \frac{c^2y}{z}dy$

or $\quad zdz - c^2y\,dy = c\sqrt{(z^2-c^2y^2)}\,dx \quad$ or $\quad \dfrac{\frac{1}{2}d(z^2-c^2y^2)}{\sqrt{(z^2-c^2y^2)}} = c\,dx$

Integrating, we get $\sqrt{(z^2-c^2y^2)} = cx + a$ or $z^2 = (a+cx)^2 + c^2y^2$ which is the required complete integral.

Example 17.22. *Solve $2xz - px^2 - 2qxy + pq = 0$.* *(Rajasthan, 2006)*

Solution. Let $f(x, y, z, p, q) = 2xz - px^2 - 2qxy + pq = 0$ ...(*i*)

Charpit's subsidiary equations are

$$\frac{dx}{x^2-q}=\frac{dy}{2xy-p}=\frac{dz}{px^2-2pq+2qxy}=\frac{dp}{2z-2qy}=\frac{dq}{0}$$

$\therefore \quad dq = 0 \quad$ or $\quad q = a.$

Putting $q = a$ in (*i*), we get $p = \dfrac{2x(z-ay)}{x^2-a}$

$\therefore \quad dz = pdx + qdy = \dfrac{2x(z-ay)}{x^2-a}dx + ady \quad$ or $\quad \dfrac{dz-ady}{z-ay} = \dfrac{2x}{x^2-a}dx$

Integrating, $\quad \log(z-ay) = \log(x^2-a) + \log b$

or $\quad z - ay = b(x^2-a) \quad$ or $\quad z = ay + b(x^2-a)$

which is the required complete solution.

Example 17.23. *Solve $2z + p^2 + qy + 2y^2 = 0$.* *(J.N.T.U., 2005 ; Kurukshetra, 2005)*

Solution. Let $f(x, y, z, p, q) = 2z + p^2 + qy + 2y^2$

Charpit's subsidiary equations are

$$\frac{dx}{-2p}=\frac{dy}{-y}=\frac{dz}{-(2p^2+qy)}=\frac{dp}{2p}=\frac{dq}{4y+3q}$$

From first and fourth ratios,

$$dp = -dx \quad \text{or} \quad p = -x + a$$

Substituting $p = a - x$ in the given equation, we get

$$q = \frac{1}{y}[-2z - 2y^2 - (a-x)^2]$$

$\therefore \qquad dz = pdx + qdy = (a - x)dx - \dfrac{1}{y}[2z + 2y^2 + (a - x)^2]dy$

Multiplying both sides by $2y^2$,

$$2y^2dz + 4yz\, dy = 2y^2 (a - x)dx - 4y^3dy - 2y(a - x)^2dy$$

Integrating $\qquad 2zy^2 = -[y^2(a - x)^2 + y^4] + b$

or $\qquad y^2[(x - a)^2 + 2z + y^2] = b$, which is the desired solution.

PROBLEMS 17.5

Solve the following equations :

1. $z = p^2x + q^2x$.
2. $z^2 = pq\,xy$. (*Anna, 2009 ; V.T.U., 2004*)
3. $1 + p^2 = qz$.
4. $pxy + pq + qy = yz$. (*J.N.T.U., 2006 ; Kurukshetra, 2006*)
5. $p(p^2 + 1) + (b - z)q = 0$.
6. $q + xp = p^2$. (*Osmania, 2003*)

17.8 HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

An equation of the form

$$\frac{\partial^n z}{\partial x^n} + k_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + \dots + k_n \frac{\partial^n z}{\partial y^n} = F(x, y) \qquad \text{...(1)}$$

in which k's are constants, is called a *homogeneous linear partial differential equation of the nth order with constant coefficients.* It is called homogeneous because all terms contain derivatives of the same order.

On writing, $\dfrac{\partial^r}{\partial x^r} = D^r$ and $\dfrac{\partial^r}{\partial y^r} = D'^r$. (1) becomes $(D^n + k_1D^{n-1}D'^r + D' + \dots + k_nD'^{\,n})\, z = F(x, y)$

or briefly $\qquad f(D, D')z = F(x, y) \qquad$ ...(2)

As in the case of ordinary linear equations with constant coefficients the complete solution of (1) consists of two parts, namely : the *complementary function* and *the particular integral.*

The complementary function is the complete solution of the equation $f(D, D')z = 0$, which must contain n arbitrary functions. The particular integral is the particular solution of equation (2).

17.9 RULES FOR FINDING THE COMPLEMENTARY FUNCTION

Consider the equation $\dfrac{\partial^2 z}{\partial x^2} + k_1 \dfrac{\partial^2 z}{\partial x \partial y} + k_2 \dfrac{\partial^2 z}{\partial y^2} = 0 \qquad$ ...(1)

which in *symbolic form is* $(D^2 + k_1DD' + k_2D'^2)z = 0 \qquad$ ...(2)

Its symbolic operator equated to zero, *i.e.*, $D^2 + k_1DD' + k_2D'^2 = 0$ is called the *auxiliary equation* (A.E.)

Let its root be $\qquad D/D' = m_1, m_2$.

Case I. *If the roots be real and distinct* then (2) is equivalent to

$$(D - m_1D')(D - m_2D')z = 0 \qquad \text{...(3)}$$

It will be satisfied by the solution of

$$(D - m_2D')z = 0, \text{ i.e., } p - m_2q = 0.$$

This is a Lagrange's linear and the subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m_2} = \frac{dz}{0}, \text{ whence } y + m_2x = a \text{ and } z = b.$$

$\therefore$ its solution is $z = \phi(y + m_2x)$.

Similarly (3) will also be satisfied by the solution of

$$(D - m_1D')z = 0, \quad i.e., \quad \text{by } z = f(y + m_1x)$$

Hence the complete solution of (1) is $z = f(y + m_1x) + \phi(y + m_2x)$.

Case II. *If the roots be equal* (*i.e.*, $m_1 = m_2$), then (2) is equivalent to

$$(D - m_1D')^2z = 0 \qquad ...(4)$$

Putting $(D - m_1D')z = u$, it becomes $(D - m_1D')u = 0$ which gives

$$u = \phi(y + m_1x)$$

$\therefore$ (4) takes the form $(D - m_1'D)z = \phi(y + m_1x)$ or $p - m_1q) = \phi(y + m_1x)$

This is again Lagrange's linear and the subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m_1} = \frac{dz}{\phi(y + m_1x)}$$

giving $\quad y + m_1x = a$ and $dz = \phi(a)\,dx$, *i.e.*, $z = \phi(a)x + b$

Thus the complete solution of (1) is

$$z - x\phi(y + m_1x) = f(y + m_1x). \quad i.e., \quad z = f(y + m_1x) + x\phi(y + m_1x).$$

Example 17.24. *Solve* $2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0.$

Solution. Given equation in *symbolic form* is $(2D^2 + 5DD' + 2D'^2)z = 0$.

Its *auxiliary equation* is $2m^2 + 5m + 2 = 0$, where $m = D/D'$.

which gives $\quad m = -2, -1/2.$

Here the complete solution is $z = f_1(y - 2x) + f_2(y - \frac{1}{2}x)$

which may be written as $z = f_1(y - 2x) + f_2(2y - x)$.

Example 17.25. *Solve* $4r + 12s + 9t = 0.$ *(P.T.U., 2010)*

Solution. Given equation in *symbolic form* is $(4D^2 + 12DD' + 9D'^2)z = 0$

for $\quad r = \frac{\partial^2 z}{\partial x^2} = D^2z,\ s = \frac{\partial^2 z}{\partial x \partial y} = DD'z$ and $t = \frac{\partial^2 z}{\partial y^2} = D'^2z.$

$\therefore$ Its *auxiliary equation* is $4m^2 + 12m + 9 = 0$, whence $m = -3/2, -3/2$

Hence the complete solution is $z = f_1(y - 1.5x) + xf_2(y - 1.5x)$.

17.10 RULES FOR FINDING THE PARTICULAR INTEGRAL

Consider the equation $(D^2 + k_1DD' + k_2D'^2)z = F(x, y)$ *i.e.*, $f(D, D')z = F(x, y)$.

$$\therefore \qquad \text{P.I.} = \frac{1}{f(D, D')}F(x, y)$$

Case I. *When* $\quad F(x, y) = e^{ax + by}$

Since $\quad De^{ax+by} = ae^{ax+by}$; $D'e^{ax+by} = be^{ax+by}$

$\therefore \quad D^2e^{ax+by} = a^2e^{ax+by}$; $DD'e^{ax+by} = abe^{ax+by}$

and $\quad D'^2e^{ax+by} = b^2e^{ax+by}$

$\therefore \quad (D^2 + k_1DD' + k_2D'^2)e^{ax+by} = (a^2 + k_1ab + k_2b^2)\,e^{ax+by}$

i.e., $\quad f(D, D')e^{ax+by} = f(a, b)\,e^{ax+by}$

Operating both sides by $1/f(D, D')$, we get

$$\text{P.I.} = \frac{1}{f(D, D')}e^{ax+by} = \frac{1}{f(a, b)}e^{ax+by}$$

Case II. *When* $F(x, y) = \sin(mx + ny)$ or $\cos(mx + ny)$

Since $\quad D^2 \sin(mx + ny) = -m^2 \sin(mx + ny)$

$\quad DD' \sin(mx + ny) = -mn \sin(mx + ny)$

and $\quad D'^2 \sin(mx + ny) = -n^2 \sin(mx + ny).$

$\therefore \quad f(D^2, DD', D'^2) \sin(mx + ny) = f(-m^2, -mn, -n^2) \sin(mx + ny)$

Operating both sides by $1/f(D^2, DD', D'^2)$, we get

$$\text{P.I.} = \frac{1}{f(D^2, DD', D'^2)} \sin(mx + ny) = \frac{1}{f(-m^2 - mn, -n^2)} \sin(mx + ny)$$

Similarly about the P.I. for $\cos(mx + ny)$.

Case III. *When* $F(x, y) = x^m y^n$, m and n being constants.

$$\therefore \quad \text{P.I.} = \frac{1}{f(D, D')} x^m y^n = [f(D, D')]^{-1} x^m y^n.$$

To evaluate it, we expand $[f(D, D')]^{-1}$ in ascending powers of D or D' by Binomial theorem and then operate on $x^m y^n$ term by term.

Case IV. *When $F(x, y)$ is any function of x and y.*

$$\therefore \quad \text{P.I.} = \frac{1}{f(D, D')} F(x, y)$$

To evaluate it, we resolve $1/f(D, D')$ into partial fractions treating $f(D, D')$ as a function of D alone and operate each partial fraction on $F(x, y)$ remembering that

$$\frac{1}{D - mD'} F(x, y) = \int F(x, c - mx)\, dx$$

where c is replaced by $y + mx$ after integration.

17.11 WORKING PROCEDURE TO SOLVE THE EQUATION

$$\frac{\partial^n z}{\partial x^n} + k_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + \dots + k^n \frac{\partial^n z}{\partial y^n} = F(x, y).$$

Its *symbolic form* is $\quad (D^n + k_1 D^{n-1} D' + \dots + k_n D'^n) z = F(x, y)$

or briefly $\quad f(D, D') z = F(x, y)$

Step I. To find the C.F.

(*i*) *Write the A.E.*

i.e., $\quad m^n + k_1 m^{n-1} + \dots + k_n = 0$ *and solve it for* m.

(*ii*) *Write the C.F. as follows*

Roots of A.E.	*C.F.*
1. m_1, m_2, m_3 ... (distinct roots)	$f_1(y + m_1 x) + f_2(y + m_2 x) + f_3(y + m_3 x) + \dots$
2. m_1, m_1, m_3 ... (two equal roots)	$f_1(y + m_1 x) + x f_2(y + m_1 x) + f_3(y + m_3 x) + \dots$
3. m_1, m_1, m_1 ... (three equal roots)	$f_1(y + m_1 x) + x f_2(y + m_1 x) + x^2 f_3(y + m_1 x) + \dots$

Step II. To find the P.I.

From the *symbolic form*, $\text{P.I.} = \dfrac{1}{f(D, D')} F(x, y).$

(*i*) *When* $F(x, y) = e^{ax + by}$ $\text{P.I.} = \dfrac{1}{f(D, D')} e^{ax + by}$ [Put $D = a$ and $D' = b$]

(*ii*) *When* $F(x, y) = \sin(mx + ny)$ or $\cos(mx + ny)$

$$\text{P.I.} = \frac{1}{f(D^2, DD', D'^2)} \sin \text{ or } \cos(mx + ny) \qquad [\text{Put } D^2 = -m^2, DD' = -mn, D'^2 = -n^2]$$

(*iii*) *When* $F(x, y) = x^m y^n$, $\text{P.I.} = \dfrac{1}{f(D, D')} x^m y^n = [f(D, D')]^{-1} x^m y^n.$

Expand $[f(D, D')]^{-1}$ in ascending powers of D or D' and operate on $x^m y^n$ term by term.

(*iv*) *When $F(x, y)$ is any function of x and y* $\text{P.I.} = \dfrac{1}{f(D, D')} F(x, y).$

Resolve $1/f(D, D')$ into partial fractions considering $f(D, D')$ as a function of D alone and operate each partial fraction on $F(x, y)$ remembering that

$$\frac{1}{D - mD'} F(x, y) = \int F(x, c - mx)\,dx \text{ where } c \text{ is replaced by } y + mx \text{ after integration.}$$

Example 17.26. *Solve $(D^2 + 4DD' - 5D'^2)z = \sin(2x + 3y)$.* *(Madras, 2006)*

Solution. A.E. of the given equation is $m^2 + 4m - 5 = 0$ *i.e.*, $m = 1, -5$

$\therefore$ C.F. $= f_1(y + x) + f_2(y - 5x)$

$$\text{P.I.} = \frac{1}{D^2 + 4DD' - 5D'^2} \sin(2x + 3y) \quad [\text{Put } D^2 = -2^2, DD' = -2 \times 3, D'^2 = -3^2$$

$$= \frac{1}{-4 + 4(-6) - 5(-9)} \sin(2x + 3y) = \frac{1}{17} \sin(2x + 3y).$$

Hence the C.S. is $z = f_1(y + x) + f_2(y - 5x) + \frac{1}{17} \sin(2x + 3y)$.

Example 17.27. *Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \cos x \cos 2y$.* *(Bhopal, 2008 S)*

Solution. Given equation in *symbolic form* is $(D^2 - DD')z = \cos x \cos 2y$.
Its A.E. is $m^2 - m = 0$, whence $m = 0, 1$.

$\therefore$ C.F. $= f_1(y) + f_2(y + x)$

$$\text{P.I.} = \frac{1}{D^2 - DD'} \cos x \cos 2y = \frac{1}{2} \frac{1}{D^2 - DD'} [\cos(x + 2y) + \cos(x - 2y)]$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - DD'} \cos(x + 2y)\right. \qquad [\text{Put } D^2 = -1, DD' = -2]$$

$$\left. + \frac{1}{D^2 - DD'} \cos(x - 2y)\right] \qquad [\text{Put } D^2 = -1, DD' = 2]$$

$$= \frac{1}{2}\left[\frac{1}{-1+2} \cos(x + 2y) + \frac{1}{-1-2} \cos(x - 2y)\right] = \frac{1}{2} \cos(x + 2y) - \frac{1}{6} \cos(x - 2y)$$

Hence the C.S. is $z = f_1(y) + f_2(y + x) + \frac{1}{2} \cos(x + 2y) - \frac{1}{6} \cos(x - 2y)$.

Example 17.28. *Solve $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} = 2e^{2x} + 3x^2y$.* *(S.V.T.U., 2007)*

Solution. Given equation in *symbolic form* is

$$(D^3 - 2D^2D')z = 2e^{2x} + 3x^2y$$

Its A.E. is $m^3 - 2m^2 = 0$, whence $m = 0, 0, 2$.

$\therefore$ C.F. $= f_1(y) + xf_2(y) + f_3(y + 2x)$

$$\text{P.I.} = \frac{1}{D^3 - 2D^2D'}(2e^{2x} + 3x^2y) = 2\frac{1}{D^3 - 2D^2D'} e^{2x} + 3\frac{1}{D^3(1 - 2D'/D)} x^2y$$

$$= 2\frac{1}{2^3 - 2 \cdot 2^2(0)} e^{2x} + \frac{3}{D^3}(1 - 2D'/D)^{-1} x^2y = \frac{1}{4} e^{2x} + \frac{3}{D^3}\left(1 + \frac{2D'}{D} + \frac{4D'^2}{D^2} + \ldots\right) x^2y$$

$$= \frac{1}{4} e^{2x} + \frac{3}{D^3}\left(x^2y + \frac{2}{D} x^2 \cdot 1\right) = \frac{1}{4} e^{2x} + \frac{3}{D^3}\left(x^2y + \frac{2}{3}x^3\right) \qquad \left[\because \frac{1}{D} f(x) = \int f(x)\,dx\right]$$

$$= \frac{1}{4} e^{2x} + 3y \frac{x^5}{3 \cdot 4 \cdot 5} + 2 \cdot \frac{x^6}{4 \cdot 5 \cdot 6} \qquad \left[\because \frac{1}{D^3} f(x) = \int\left[\int\left(\int f(x)\,dx\right)dx\right]dx\right]$$

$$= \frac{e^{2x}}{4} + \frac{x^5 y}{20} + \frac{x^6}{60}$$

Hence the C.S. is $z = f_1(y) + xf_2(y) + f_3(y + 2x) + \frac{1}{60}(15e^{2x} + 3x^5y + x^6)$.

Example 17.29. *Solve* $r - 4s + 4t = e^{2x+y}$.

Solution. Given equation is $\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = e^{2x+y}$.

i.e., in *symbolic form* $(D^2 - 4DD' + 4D'^2)z = e^{2x+y}$.

Its A.E. is $(m - 2)^2 = 0$, whence $m = 2, 2$.

$\therefore$ C.F. $= f_1(y + 2x) + xf_2(y + 2x)$

$$\text{P.I.} = \frac{1}{(D - 2D')^2} e^{2x+y}$$

The usual rule fails because $(D - 2D')^2 = 0$ for $D = 2$ and $D' = 1$.

$\therefore$ to obtain the P.I., we find from $(D - 2D')u = e^{2x+y}$, the solution

$$u = \int F(x, c - mx)\, dx = \int e^{2x+(c-2x)}\, dx = xe^c = xe^{2x+y} \qquad [\because\ y = c - mx = c - 2x]$$

and from $(D - 2D')z = u = xe^{2x+y}$, the solution

$$z = \int xe^{2x+(c-2x)} dy = \frac{1}{2}x^2 e^c = \frac{1}{2}x^2 e^{2x+y} \qquad [\because\ y = c - mx = c - 2x]$$

Hence the C.S. is $z = f_1(y + 2x) + xf_2(y + 2x) + \frac{1}{2}x^2 e^{2x+y}$.

Example 17.30. *Solve* $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = \cos(2x + y)$. *(P.T.U., 2010 ; S.V.T.U., 2009)*

Solution. Given equation in *symbolic form* is $(D^2 + DD' - 6D'^2)z = \cos(2x + y)$

Its A.E. is $m^2 + m - 6 = 0$ whence $m = -3, 2$.

$\therefore$ C.F. $= f_1(y - 3x) + f_2(y + 2x)$.

Since $D^2 + DD' - 6D'^2 = -2^2 - (2)(1) - 6(-1)^2 = 0$

$\therefore$ It is a case of failure and we have to apply the general method.

$$\text{P.I.} = \frac{1}{D^2 + DD' - 6D'^2}\cos(2x + y) = \frac{1}{(D + 3D')(D - 2D')}\cos(2x + y)$$

$$= \frac{1}{D + 3D'}\left[\int \cos(2x + \overline{c - 2x})dx\right]_{c \to y+2x} = \frac{1}{D + 3D'}\left[\int \cos c\, dx\right]_{c \to y+2x}$$

$$[\because\ y = c - mx = c - 2x$$

$$= \frac{1}{D + 3D'}x\cos(y + 2x) = \left[\int x\cos(\overline{c + 3x} + 2x)dx\right]_{c \to y-3x} = \left[\int x\cos(5x + c)dx\right]_{c \to y-3x}$$

$$= \left[\frac{x\sin(5x + c)}{5} + \frac{\cos(5x + c)}{25}\right]_{c \to y-3x} \qquad \text{[Integrating by parts]}$$

$$= \frac{x}{5}\sin(5x + \overline{y - 3x}) + \frac{1}{25}\cos(5x + \overline{y - 3x}) = \frac{x}{5}\sin(2x + y) + \frac{1}{25}\cos(2x + y)$$

Hence the C.S. is

$$z = f_1(y - 3x) + f_2(y + 2x) + \frac{x}{5}\sin(2x + y) + \frac{1}{25}\cos(2x + y)$$

$$z = f_1(y - 3x) + f_2(y + 2x) + \frac{x}{5}\sin(2x + y) + \frac{1}{25}\cos(2x + y).$$

Example 17.31. *Solve* $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y \cos x.$ (Anna, 2005 S ; U.P.T.U., 2003)

or $r + s - 6t = y \cos x.$ (Bhopal, 2008 ; S.V.T.U., 2008)

Solution. Its *symbolic form* is $(D^2 + DD' - 6D'^2)z = y \cos x$
and the A.E. is $m^2 + m - 6 = 0$, whence $m = -3, 2$.

$$\therefore \quad \text{C.F.} = f_1(y - 3x) + f_2(y + 2x)$$

$$\text{P.I.} = \frac{1}{(D - 2D')(D + 3D')} y \cos x = \frac{1}{D - 2D'}\left[\int (c + 3x) \cos x \, dx\right]_{c \to y - 3x}$$

$$[\because \; y = c - mx = c + 3x]$$

$$= \frac{1}{D - 2D'}[(c + 3x) \sin x + 3 \cos x]_{c \to y - 3x} \qquad \text{[Integrating by parts]}$$

$$= \frac{1}{D - 2D'}(y \sin x + 3 \cos x) = \left[\int \{(c - 2x) \sin x \; 3 \cos x\} dx\right]_{c \to y - 2x}$$

$$= [(c - 2x)(-\cos x) - (-2)(-\sin x) + 3 \sin x]_{c \to y + 2x}$$

$$= -y \cos x + \sin x$$

Hence the C.S. is $z = f_1(y - 3x) + f_2(y + 2x) + \sin x - y \cos x$.

Example 17.32. *Solve* $4\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 16 \log (x + 2y).$

Solution. Its *symbolic form* is $4D^2 - 4DD' + D'^2 = 16 \log (x + 2y)$
and the A.E. is $4m^2 - 4m + 1 = 0$, $m = 1/2, 1/2$.

$$\therefore \quad \text{C.F.} = f_1\left(y + \frac{1}{2}x\right) + x f_2\left(y + \frac{1}{2}x\right)$$

$$\text{P.I.} = \frac{1}{(2D - D')^2} 16 \log (x + 2y) = 4\frac{1}{\left(D - \frac{1}{2}D'\right)}\left\{\frac{1}{D - \frac{1}{2}D'} \log (x + 2y)\right\}$$

$$= 4\frac{1}{D - \frac{1}{2}D'}\left[\int \log \left\{x + 2\left(c - \frac{x}{2}\right)\right\} dx\right]_{c \to y + x/2} \qquad [\because \; y = c - mx = c - x/2]$$

$$= 4\frac{1}{D - \frac{1}{2}D'}\left[\int \log (2c) dx\right]_{c \to y + x/2} = 4\frac{1}{D - \frac{1}{2}D'}[x \log (x + 2y)]$$

$$= 4\left[\int \left\{x \log \left[x + 2\left(c - \frac{x}{2}\right)\right]\right\} dx\right]_{c \to y + x/2} = 4\left[\log 2c \int x \, dx\right]_{c \to y + x/2} = 2x^2 \log (x + 2y)$$

Hence the C.S. is $z = f_1\left(y + \frac{x}{2}\right) + x f_2\left(y + \frac{x}{2}\right) + 2x^2 \log (x + 2y)$.

PROBLEMS 17.6

Solve the following equations :

1. $\frac{\partial^3 z}{\partial x^3} - 4\frac{\partial^3 x}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial x \partial y^2} = 0.$

2. $\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial y^3} = e^{x + 2y}.$ (Burdwan, 2003)

3. $(D^2 - 2DD' + D'^2)z = e^{x + y}.$ (Bhopal, 2007)

4. $\frac{\partial^3 z}{\partial x^3} - 4\frac{\partial^3 z}{\partial x^2 \partial y} + 5\frac{\partial^3 z}{\partial x \partial y^2} - 2\frac{\partial^3 z}{\partial y^3} = e^{2x + y}.$ (Bhopal, 2008)

5. $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} = \sin x.$ (*P.T.U., 2009 S*)

6. $\dfrac{\partial^2 y}{\partial t^2} - a^2 \dfrac{\partial^2 y}{\partial x^2} = E \sin pt.$

7. $\dfrac{\partial^3 z}{\partial x^3} - \dfrac{4\partial^3 z}{\partial z^2 \partial y} + 4\dfrac{\partial^3 z}{\partial x \partial y^2} = 2 \sin (3x + 2y).$ (*S.V.T.U., 2007*)

8. $(D^3 - 7DD'^2 - 6D'^3)z = \cos (x + 2y) + 4.$ (*Anna, 2008*)

9. $\dfrac{\partial^2 z}{\partial x^2} - 3\dfrac{\partial^2 z}{\partial x \partial y} + 2\dfrac{\partial^2 z}{\partial y^2} = e^{2x-y} + e^{x+y} + \cos (x + 2y).$ (*U.P.T.U., 2006*)

10. $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial x \partial y} = \sin x \cos 2y.$ (*U.P.T.U., 2003*)

11. $(D^2 - DD')z = \cos 2y (\sin x + \cos x).$

12. $(D^2 - D'^2)z = e^{x-y} \sin (x + 2y).$ (*Anna, 2009*)

13. $(D^2 + 3DD' + 2D'^2)z = 24xy.$

14. $\dfrac{\partial^2 z}{\partial x^2} + 2\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} = x^2 + xy + y^2.$

15. $(D^2 - DD' - 2D'^2)z = (y - 1)\, e^x.$ (*Bhopal, 2006*)

16. $(D^3 + D^2D' - DD'^2 - D'^3)z = e^x \cos 2y.$

17. $(D^2 + 2DD' + D'^2)z = 2 \cos y - x \sin y.$ (*P.T.U., 2005*)

17.12 NON-HOMOGENEOUS LINEAR EQUATIONS

If in the equation $\qquad f(D, D')z = F(x, y) \qquad$ ...(1)

the polynomial expression $f(D, D')$ is not homogeneous, then (1) is a non-homogeneous linear partial differential equation. As in the case of homogeneous linear partial differential equations, its complete solution = C.F. + P.I.

The methods to find P.I. are the same as those for homogeneous linear equations.

To find the C.F., we factorize $f(D, D')$ into factors of the form $D - mD' - c$. To find the solution of $(D - mD' - c)z = 0$, we write it as $p - mq = cz$ ...(2)

The subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{cz}$$

Its integrals are $y + mx = a$ and $z = be^{cx}$.

Taking $b = \phi(a)$, we get $z = e^{cx}\, \phi(y + mx)$

as the solution of (2). The solution corresponding to various factors added up, give the C.F. of (1).

Example 17.32. *Solve* $(D^2 + 2DD' + D'^2 - 2D - 2D')z = \sin (x + 2y).$ (*U.P.T.U., 2004*)

Solution. Here $\qquad f(D, D') = (D + D')(D + D' - 2)$

Since the solution corresponding to the factor $D - mD' - c$ is known to be

$$z = e^{cx}\, \phi(y + mx)$$

$$\therefore \qquad \text{C.F.} = \phi_1(y - x) + e^{2x}\, f_2(y - x)$$

$$\therefore \qquad \text{P.I.} = \frac{1}{D^2 + 2DD' + D'^2 - 2D - 2D'} \sin (x + 2y)$$

$$= \frac{1}{-1 + 2(-2) + (-4) - 2D - 2D'} \sin (x + 2y)$$

$$= -\frac{1}{2(D + D') + 9} \sin (x + 2y) = -\frac{2(D + D') - 9}{4(D^2 + 2DD' + D'^2) - 81} \sin (x + 2y)$$

$$= \frac{1}{39} [2 \cos (x + 2y) - 3 \sin (x + 2y)]$$

Hence the complete solution is

$$z = \phi_1(y - x) + e^{2x}\, \phi_2(y - x) + \frac{1}{39} [2 \cos (x + 2y) - 3 \sin (x + 2y)].$$

PROBLEMS 17.7

Solve the following equations :

1. $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial z}{\partial y}-z=e^{-x}$.
2. $(D-D'-1)(D-D'-2)z=e^{2x-y}$.
3. $(D+D'-1)(D+2D'-3)z=4+3x+6y$.
4. $\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial z}{\partial y}=x^2+y^2$. *(Madras, 2000 S)*
5. $(D^2+DD'+D'-1)z=\sin(x+2y)$. *(S.V.T.U., 2009)*
6. $(2DD'+D'^2-3D')z=3\cos(3x-2y)$.

17.13 NON-LINEAR EQUATIONS OF THE SECOND ORDER

We now give a method due to *Monge**, for integrating the equation $Rr+Ss+Tt=V$ ...(1)
in which R, S, T, V are functions of x, y, z, p and q.

Since $dp=\frac{\partial p}{\partial x}dx+\frac{\partial p}{\partial y}dy=rdx+tdy$, and $dq=sdx+tdy$,

we have $r=(dp-tdy)/dx$ and $t=(dq-sdx)/dy$.

Substituting these values of r and t in (1), and rearranging the terms, we get

$$(Rdpdy+Tdqdx-Vdxdy)-s(Rdy^2-Sdydx+Tdx^2)=0 \quad ...(2)$$

Let us consider the equations

$$Rdy^2-Sdydx+Tdx^2=0 \quad ...(3)$$

$$Rdpdy+Tdqdx-Vdxdy=0 \quad ...(4)$$

which are known as *Monge's equations.*

Since (3) can be factorised, we obtain its integral first. In case the factors are different, we may get two distinct integrals of (3). Either of these together with (4) will give an integral of (4). If need be, we may also use the relation $dz=pdx+qdy$ while solving (3) and (4).

Let $u(x,y,z,p,q)=a$ and $v(x,y,z,p,q)=b$ be the integrals of (3) and (4) respectively. Then $u=a$, $v=b$ evidently constitute a solution of (2) and therefore, of (1) also. Taking $b=\phi(a)$, we find a general solution of (1) to be $v=\phi(u)$, which should be further integrated by methods of first order equations.

Example 17.34. *Solve $(x-y)(xr-xs-ys+yt)=(x+y)(p-q)$.* *(S.V.T.U., 2007)*

Solution. Monge's equations are

$$xdy^2+(x+y)dy\,dx+ydx^2=0 \quad ...(i)$$

$$xdpdy+ydqdx-\frac{x+y}{x-y}(p-q)\,dydx=0 \quad ...(ii)$$

(*i*) may be factorised as $(xdy+ydx)(dx+dy)=0$ whose integrals are $xy=c$ and $x+y=c$.

Taking $xy=c$ and dividing each term of (*ii*) by xdy or its equivalent $-ydx$, we get

$$dp-dq-\frac{dx-dy}{x-y}(p-q)=0 \quad \text{or} \quad \frac{d(p-q)}{p-q}-\frac{d(x-y)}{x-y}=0$$

This gives on integration $(p-q)/(x-y)=c$.

Hence a first integral of the given equation is $p-q=(x-y)\,\phi(xy)$ which is a Lagrange's linear equation. Its subsidiary equations are

$$\frac{dx}{1}=\frac{dy}{-1}=\frac{dz}{(x-y)\phi(xy)}$$

From the first two equations, we have $x+y=a$

Using this, we have

$$dz=-\phi(ax-x^2)\,.\,(a-2x)\,dx \quad \text{which gives } z=\phi_1(ax-x^2)+b$$

Writing $b=\phi_2(a)$ and $a=x+y$, we get

$$z=\phi_1(xy)+\phi_2(x+y).$$

* Named after *Gaspard Monge* (1746–1818), Professor at Paris.

Obs. Had we started with the integral $x + y = c$ and divided each term of (*ii*) by dx or $-dy$, we would have arrived at the same solution.

Example 17.35. *Solve* $y^2r - 2ys + t = p + 6y$. *(Osmania, 2002)*

Solution. Monge's equations are $y^2dy^2 + 2ydydx + dx^2 = 0$ ...(*i*)

and $$y^2dpdy + dqdx - (p + 6y)dydx = 0 \quad ...(ii)$$

(*i*) gives $$(ydy + dx)^2 = 0 \text{ i.e. } y^2 + 2x = c \quad ...(iii)$$

Putting $ydy = -dx$ in (*ii*), we get

$$ydp - dq + (p + 6y)dy = 0 \quad \text{or} \quad (ydp + pdy) - dq + 6ydy = 0$$

whose integral is $$py - q + 3y^2 = a$$

Combining this with (*iii*), we get the integral $py - q + 3y^2 = \phi(y^2 + 2x)$

The subsidiary equations for this Lagrange's linear equation are

$$\frac{dx}{y} = \frac{dy}{-1} = \frac{dz}{\phi(y^2 + 2x) - 3y^2}$$

From the first two equations, we have $y^2 + 2x = c$

Using this, we have $dz + [\phi(c) - 3y^2]\, dy = 0$

whose solution is $$z + y\phi(c) - y^3 = b.$$

Hence the required solution is $z = y^3 - y\phi(y^2 + 2x) + \psi(y^2 + 2x)$.

PROBLEMS 17.8

Solve :

1. $(q + 1)s = (p + 1)t$.
2. $r - t\cos^2 x + p\tan x = 0$.
3. $2x^2r - 5xys + 2y^2t + 2(px + qy) = 0$. *(J.N.T.U., 2006)*
4. $xy(t - r) + (x^2 - y^2)(s - 2) = py - qx$.
5. $q^2r - 2pqs + p^2t = pq^2$.
6. $(1 + q)^2r - 2(1 + p + q + pq)s + (1 + p)^2t = 0$.

17.14 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 17.9

Fill up the blanks or choose the correct answer in each of the following problems :

1. The equation $\frac{\partial^2 z}{\partial x^2} + 2xy\left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial z}{\partial y} = 5$ is of order .... and degree ....
2. The complementary function of $(D^2 - 4DD' + 4D'^2)z = x + y$ is ......
3. The solution of $\frac{\partial^2 z}{\partial y^2} = \sin(xy)$ is ......
4. A solution of $(y - z)p + (z - x)q = x - y$ is ....
5. The particular integral of $(D^2 + DD')z = \sin(x + y)$ is ....
6. The partial differential equation obtained from $z = ax + by + ab$ by eliminating a and b is ......
7. Solution of $\sqrt{p} + \sqrt{q} = 1$ is ......
8. Solution of $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$ is ......
9. Solution of $p - q = \log(x + y)$.
10. The order of the partial differential equation obtained by eliminating f from $z = f(x^2 + y^2)$, is ......
11. The solution of $x\, \partial z/\partial x = 2x + y$ is
12. By eliminating a and b from $z = a(x + y) + b$, the p.d.e. formed is ......
13. The solution of $[D^3 - 3D^2D' + 2DD'^2]z = 0$ is ......
14. By eliminating the arbitrary constants from $z = a^2x + ay^2 + b$, the partial differential equation formed is ...... *(Anna, 2008)*
15. A solution of $u_{xy} = 0$ is of the form ....
16. If $u = x^2 + t^2$ is a solution of $c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, then $c =$ ......

17. The general solution of $u_{xx} = xy$ is ....
18. The complementary function of $r - 7s + 6t = e^{x+y}$ is ....
19. The solution of $xp + yq = z$ is

(i) $f(x^2, y^2) = 0$ (ii) $f(xy, yz)$ (iii) $f(x, y) = 0$ (iv) $f\left(\frac{x}{y}, \frac{y}{z}\right) = 0$.

20. The solution of $(y - z)p + (z - x)q = x - y$, is

(i) $f(x^2 + y^2 + z^2) = xyz$ (ii) $f(x + y + z) = xyz$
(iii) $f(x + y + z) = x^2 + y^2 + z^2$ (iv) $f(x^2 + y^2 + z^2, xyz) = 0$.

21. The partial differential equation from $z = (c + x)^2 + y$ is

(i) $z = \left(\frac{\partial z}{\partial x}\right)^2 + y$ (ii) $z = \left(\frac{\partial z}{\partial y}\right)^2 + y$ (iii) $z = \frac{1}{4}\left(\frac{\partial z}{\partial x}\right)^2 + y$ (iv) $z = \frac{1}{4}\left(\frac{\partial z}{\partial y}\right)^2 + y$.

22. The solution of $p + q = z$ is

(i) $f(xy, y \log z) = 0$ (ii) $f(x + y, y + \log z) = 0$
(iii) $f(x - y, y - \log z) = 0$ (iv) None of these.

23. Particular integral of $(2D^2 - 3DD' + D'^2)z = e^{x+2y}$ is

(i) $\frac{1}{2}e^{x+2y}$ (ii) $-\frac{x}{2}e^{x+2y}$ (iii) xe^{x+2y} (iv) x^2e^{x+2y}.

24. The solution of $\frac{\partial^3 z}{\partial x^3} = 0$ is

(i) $z = (1 + x + x^2) f(y)$ (ii) $z = (1 + y + y^2) f(x)$
(iii) $z = f_1(x) + y f_2(x) + y^2 f_3(x)$ (iv) $z = f_1(y) + x f_2(y) + x^2 f_3(y)$.

25. Particular integral of $(D^2 - D'^2)z = \cos(x + y)$ is

(i) $x \cos(x + y)$ (ii) $\frac{x}{2} \cos(x + y)$ (iii) $x \sin(x + y)$ (iv) $\frac{x}{2} \sin(x + y)$

26. The solution of $\partial^2 z/\partial x^2 = \partial^2 z/\partial y^2$ is

(i) $z = f_1(y + x) + f_2(y - x)$ (ii) $z = f_1(y + x) + f_1(y - x)$
(iii) $z = f(x^2 - y^2)$ (iv) $z = f(x^2 + y^2)$.

27. $xu_x + yu_y = u^2$ is a non-linear partial differential equation. (True or False)
28. $xu_x + u_{xx} = 0$ is a non-linear partial differential equation. (True or False)
29. $u = x^2 - y^2$ is a solution of $u_{xx} + u_{yy} = 0$. (True or False)
30. $u = e^{-t} \sin x$ is a solution of $\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = 0$. (True or False)
31. $x\frac{\partial u}{\partial x} + t\frac{\partial u}{\partial t} = 2u$ is an ordinary differential equation. (True or False)

CHAPTER 18

Applications of Partial Differential Equations

1. Introduction. 2. Method of separation of variables. 3. Partial differential equations of engineering. 4. Vibrations of a stretched string—Wave equation. 5. One dimensional heat flow. 6. Two dimensional heat flow. 7. Solution of Laplace's equation. 8. Laplace's equation in polar coordinates. 9. Vibrating membrane—Two dimensional wave equation. 10. Transmission line. 11. Laplace's equation in three dimensions. 12. Solution of three-dimensional Laplace's equation. 13. Objective Type of Questions.

18.1 INTRODUCTION

In physical problems, we always seek a solution of the differential equation which satisfies some specified conditions known as the boundary conditions. The differential equation together with these boundary conditions, constitute a *boundary value problem.*

In problems involving ordinary differential equations, we may first find the general solution and then determine the arbitrary constants from the initial values. But the same process is not applicable to problems involving partial differential equations for the general solution of a partial differential equation contains arbitrary functions which are difficult to adjust so as to satisfy the given boundary conditions. Most of the boundary value problems involving linear partial differential equations can be solved by the following method.

18.2 METHOD OF SEPARATION OF VARIABLES

It involves a solution which breaks up into a product of functions each of which contains only one of the variables. The following example explains this method :

Example 18.1. *Solve (by the method of separation of variables) :*

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0. \qquad \text{(P.T.U., 2009 S ; Bhopal 2008 ; U.P.T.U., 2005)}$$

Solution. Assume the trial solution $z = X(x)Y(y)$ ...(*i*)
where X is a function of x alone and Y that of y alone.

Substituting this value of z in the given equation, we have

$$X''Y - 2X'Y + XY' = 0 \qquad \text{where } X' = \frac{dX}{dx},\ Y' = \frac{dY}{dy} \text{ etc.}$$

Separating the variables, we get $\dfrac{X'' - 2X'}{X} = -\dfrac{Y'}{Y}$ ...(*ii*)

Since x and y are independent variables, therefore, (*ii*) can only be true if each side is equal to the same constant, a (say).

$\therefore \quad \dfrac{X'' - 2X'}{X} = a$, i.e. $X'' - 2X' - aX = 0$ ...(iii)

and $\quad -Y'/Y = a$, i.e., $Y' + aY = 0$ ...(iv)

To solve the ordinary linear equation (*iii*), the auxiliary equation is

$$m^2 - 2m - a = 0, \quad \text{whence } m = 1 \sqrt{(1 + a)}.$$

$\therefore$ the solution of (*iii*) is $X = c_1 e^{[1 + \sqrt{(1+a)}]x} + c_2 e^{[1 - \sqrt{(1+a)}]x}$

and the solution of (*iv*) is $\quad Y = c_3 e^{-ay}$.

Substituting these values of X and Y in (*i*), we get

$$z = \{c_1 e^{[1 + \sqrt{(1+a)}]x} + c_2 e^{[1 - \sqrt{(1+a)}]x}\} \,.\, c_3 e^{-ay}$$

i.e., $$z = \{k_1 e^{[1 + \sqrt{(1+a)}]x} + k_2 e^{[1 - \sqrt{(1+a)}]x}\}\, e^{-ay}$$

which is the required complete solution.

Obs. In practical problems, the unknown constants a, k_1, k_2 are determined from the given boundary conditions.

Example 18.2. *Using the method of separation of variables, solve* $\dfrac{\partial u}{\partial x} = 2\dfrac{\partial u}{\partial t} + u$ *where* $u(x, 0) = 6e^{-3x}$.

(*V.T.U., 2009 ; Kurukshetra, 2006 ; Kerala, 2005*)

Solution. Assume the solution $u(x, t) = X(x)T(t)$

Substituting in the given equation, we have

$$X'T = 2XT' + XT \quad \text{or} \quad (X' - X)\, T = 2XT'$$

or $$\frac{X' - X}{2X} = \frac{T'}{T} = k \text{ (say)}$$

$\therefore \quad X' - X - 2kX = 0$ or $\dfrac{X'}{X} = 1 + 2k$ ...(i) and $\dfrac{T'}{T} = k$ ...(ii)

Solving (*i*), $\quad \log X = (1 + 2k)x + \log c$ or $X = ce^{(1 + 2k)x}$

From (*ii*), $\quad \log T = kt + \log c'$ or $T = c'e^{kt}$

Thus $\quad u(x, t) = XT = cc'\, e^{(1 + 2k)x}\, e^{kt}$ ...(iii)

Now $\quad 6e^{-3x} = u(x, 0) = cc'\, e^{(1 + 2k)x}$

$\therefore \quad cc' = 6$ and $1 + 2k = -3$ or $k = -2$

Substituting these values in (*iii*), we get

$$u = 6e^{-3x}\, e^{-2t} \quad \text{i.e.,} \quad u = 6e^{-(3x + 2t)}$$ which is the required solution.

PROBLEMS 18.1

Solve the following equations by the method of *separation of variables* :

1. $py^3 + qx^2 = 0$. (*V.T.U., 2011 ; S.V.T.U., 2008*)
2. $x^2 \dfrac{\partial u}{\partial x} + y^2 \dfrac{\partial u}{\partial y} = 0$. (*V.T.U., 2008*)
3. $\dfrac{\partial u}{\partial x} = 4\dfrac{\partial u}{\partial y}$, given that $u(0, y) = 8e^{-3y}$. (*J.N.T.U., 2006*)
4. $4\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 3u$, given $u = 3e^{-y} - e^{-5y}$ when $x = 0$. (*S.V.T.U., 2008*)
5. $3\dfrac{\partial u}{\partial x} + 2\dfrac{\partial u}{\partial y} = 0$, $u(x, 0) = 4e^{-x}$. (*V.T.U., 2008 S*)
6. Find a solution of the equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial y} + 2u$ in the form $u = f(x)\, g(y)$. Solve the equation subject to the conditions $u = 0$ and $\partial u/\partial x = 1 + e^{-3y}$, when $x = 0$ for all values of y. (*Andhra, 2000*)

18.3 PARTIAL DIFFERENTIAL EQUATIONS OF ENGINEERING

A number of problems in engineering give rise to the following well-known partial differential equations :

(i) *Wave equation* : $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$.

(ii) *One dimensional heat flow equation* : $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$.

(iii) *Two dimensional heat flow equation* which in steady state becomes the two dimensional *Laplace's equation* : $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

(iv) *Transmission line equations.*

(v) *Vibrating membrane.* Two dimensional wave equation.

(vi) *Laplace's equation* in three dimensions.

Besides these, the partial differential equations frequently occur in the theory of Elasticity and Hydraulics.

Starting with the method of separation of variables, we find their solutions subject to specific boundary conditions and the combination of such solution gives the desired solution. Quite often a certain condition is not applicable. In such cases, the most general solution is written as the sum of the particular solutions already found and the constants are determined using Fourier series so as to satisfy the remaining conditions.

18.4 VIBRATIONS OF A STRETCHED STRING—WAVE EQUATION

Consider a tightly stretched elastic string of length l and fixed ends A and B and subjected to constant tension T (Fig. 18.1). The tension T will be considered to be large as compared to the weight of the string so that the effects of gravity are negligible.

Let the string be released from rest and allowed to vibrate. We shall study the subsequent motion of the string, with no external forces acting on it, assuming that each point of the string makes small vibrations at right angles to the equilibrium position AB, of the string entirely in one plane.

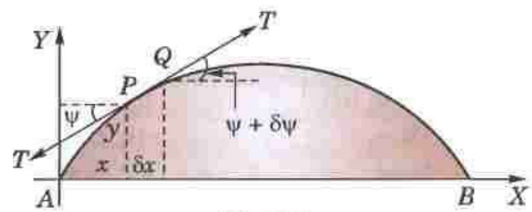


Fig. 18.1

Taking the end A as the origin, AB as the x-axis and AY perpendicular to it as the y-axis ; so that the motion takes place entirely in the xy-plane. Figure 18.1 shows the string in the position APB at time t. Consider the motion of the element PQ of the string between its points $P(x, y)$ and $Q(x + \delta x, y + \delta y)$, where the tangents make angles ψ and $\psi + \delta\psi$ with the x-axis. Clearly the element is moving upwards with the acceleration $\partial^2 y/\partial t^2$. Also the vertical component of the force acting on this element.

$$= T \sin(\psi + \delta\psi) - T \sin\psi = T[\sin(\psi + \delta\psi) - \sin\psi]$$

$$= T\,[\tan(\psi + \delta\psi) - \tan\psi], \text{ since } \psi \text{ is small} = T\left[\left\{\frac{\partial y}{\partial x}\right\}_{x+\delta x} - \left\{\frac{\partial y}{\partial x}\right\}_x\right]$$

If m be the mass per unit length of the string, then by Newton's second law of motion, we have

$$m\delta x \,.\, \frac{\partial^2 y}{\partial t^2} = T\left[\left\{\frac{\partial y}{\partial x}\right\}_{x+\delta x} - \left\{\frac{\partial y}{\partial x}\right\}_x\right] \quad i.e., \quad \frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\left[\frac{\left\{\frac{\partial y}{\partial x}\right\}_{x+\delta x} - \left\{\frac{\partial y}{\partial x}\right\}_x}{\delta x}\right]$$

Taking limits as $Q \to P$ *i.e.*, $dx \to 0$, we have $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$, where $c^2 = \dfrac{T}{m}$ ...(1)

This is the partial differential equation giving the transverse vibrations of the string. It is also called the one dimensional *wave equation*.

(2) Solution of the wave equation. Assume that a solution of (1) is of the form

$$z = X(x)T(t) \text{ where } X \text{ is a function of } x \text{ and } T \text{ is a function of } t \text{ only.}$$

Then $\dfrac{\partial^2 y}{\partial t^2} = X \,.\, T''$ and $\dfrac{\partial^2 y}{\partial x^2} = X'' \,.\, T$

Substituting these in (1), we get $XT'' = c^2X''T$ *i.e.*, $\dfrac{X''}{X} = \dfrac{1}{c^2}\dfrac{T''}{T}$ ...(2)

Clearly the left side of (2) is a function of x only and the right side is a function of t only. Since x and t are independent variables, (2) can hold good if each side is equal to a constant k (say). Then (2) leads to the ordinary differential equations :

$$\frac{d^2X}{dx^2} - kX = 0 \quad ...(3) \quad \text{and} \quad \frac{d^2T}{dt^2} - kc^2T = 0 \quad ...(4)$$

Solving (3) and (4), we get

(*i*) *When k is positive* and $= p^2$, say $X = c_1e^{px} + c_2e^{-px}$; $T = c_3e^{cpt} + c_4e^{-cpt}$.

(*ii*) *When k is negative* and $= -p^2$ say $X = c_5 \cos px + c_6 \sin px$; $T = c_7 \cos cpt + c_8 \sin cpt$.

(*iii*) *When k is zero*. $X = c_9x + c_{10}$; $T = c_{11}t + c_{12}$.

Thus the various possible solutions of wave-equation (1) are

$$y = (c_1e^{px} + c_2e^{-px})(c_3e^{cpt} + c_4e^{-cpt}) \quad ...(5)$$

$$y = (c_5 \cos px + c_6 \sin px)(c_7 \cos cpt + c_8 \sin cpt) \quad ...(6)$$

$$y = (c_9x + c_{10})(c_{11}t + c_{12}) \quad ...(7)$$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. As we will be dealing with problems on vibrations, y must be a periodic function of x and t. Hence their solution must involve trigonometric terms. Accordingly the solution given by (6), *i.e.*, of the form

$$y = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt) \quad ...(8)$$

is the only suitable solution of the *wave equation*. (*Bhopal, 2008*)

Example 18.3. *A string is stretched and fastened to two points l apart. Motion is started by displacing the string in the form $y = a \sin (\pi x/l)$ from which it is released at time $t = 0$. Show that the displacement of any point at a distance x from one end at time t is given by*

$y(x, t) = a \sin (\pi x/l) \cos (\pi ct/l)$. (*V.T.U., 2010 ; S.V.T.U., 2008 ; Kerala, 2005 ; U.P.T.U., 2004*)

Solution. The vibration of the string is given by $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ ...(*i*)

As the end points of the string are fixed, for all time,

$$y(0, t) = 0 \quad ...(ii) \quad \text{and} \quad y(l, t) = 0 \quad ...(iii)$$

Since the initial transverse velocity of any point of the string is zero,

therefore, $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = 0$ ...(*iv*)

Also $y(x, 0) = a \sin (\pi x/l)$ ...(*v*)

Now we have to solve (*i*) subject to the *boundary conditions* (*ii*) and (*iii*) and *initial conditions* (*iv*) and (*v*). Since the vibration of the string is periodic, therefore, the solution of (*i*) is of the form

$$y(x, t) = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt) \quad ...(vi)$$

By (*ii*), $y(0, t) = C_1(C_3 \cos cpt + C_4 \sin cpt) = 0$

For this to be true for all time, $C_1 = 0$.

Hence $y(x, t) = C_2 \sin px (C_3 \cos cpt + C_4 \sin cpt)$ ...(*vii*)

and $\dfrac{\partial y}{\partial t} = C_2 \sin px \,\{C_3(-cp \,.\, \sin cpt) + C_4(cp \,.\, \cos cpt)\}$

$\therefore$ By (*iv*), $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = C_2 \sin px \,.\, (C_4\, cp) = 0$, whence $C_2C_4cp = 0$.

If $C_2 = 0$, (*vii*) will lead to the trivial solution $y(x, t) = 0$,

$\therefore$ the only possibility is that $C_4 = 0$.

Thus (*vii*) becomes $y(x, t) = C_2C_3 \sin px \cos cpt$ ...(*viii*)

$\therefore$ By (*iii*), $y(l, t) = C_2C_3 \sin pl \cos cpt = 0$ for all t.

Since C_2 and $C_3 \neq 0$, we have $\sin pl = 0$. $\therefore$ $pl = n\pi$, *i.e.*, $p = n\pi/l$, where n is an integer.

Hence (*i*) reduces to $y(x, t) = C_2C_3 \sin \dfrac{n\pi x}{l} \cos \dfrac{n\pi ct}{l}$.

[These are the solutions of (*i*) satisfying the boundary conditions. These functions are called the **eigen functions** corresponding to the **eigen values** $\lambda_n = cn\pi/l$ of the vibrating string. The set of values $\lambda_1, \lambda_2, \lambda_3, \ldots$ is called its **spectrum.**]

Finally, imposing the last condition (*v*), we have $y(x, 0) = C_2C_3 \sin \dfrac{n\pi x}{l} = a \sin \dfrac{\pi x}{l}$

which will be satisfied by taking $C_2C_3 = a$ and $n = 1$.

Hence the required solution is $y(x, t) = a \sin \dfrac{\pi x}{l} \cos \dfrac{\pi ct}{l}$ ...(*ix*)

Obs. We have from (*ix*) $\dfrac{\partial^2 y}{\partial t^2} = -a\left(\dfrac{\pi c}{l}\right)^2 \sin \dfrac{\pi x}{l} \cos \dfrac{\pi ct}{l} = -\left(\dfrac{\pi c}{l}\right)^2 y$.

This shows that the motion of each point $y(x, t)$ of the string is simple harmonic with period $= 2\pi/(\pi c/l)$, *i.e.*, $2l/c$.

Thus we can look upon (*ix*) as a sine wave $y = y_0 \sin (\pi x/l)$ of wave length l, wave-velocity c and amplitude $y_0 = a \cos (\pi ct/l)$ which varies harmonically with time t. Whatever t may be, $y = 0$ when $x = 0, l, 2l, 3l$ etc. and these points called *nodes*, remain undisturbed during wave motion. Thus (*ix*) represents a *stationary sine wave* of varying amplitudes whose frequency is $c/2l$. Such waves often occur in electrical and mechanical vibratory systems.

Example 18.4. *A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $y = y_0 \sin^3 (\pi x/l)$. If it is released from rest from this position, find the displacement $y(x, t)$.*

(*Rajasthan, 2006 ; V.T.U., 2003 ; J.N.T.U., 2002*)

Solution. The equation of the vibrating string is $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ ...(*i*)

The boundary conditions are $y(0, t) = 0, y(l, t) = 0$ ...(*ii*)

Also the initial conditions are $y(x, 0) = y_0 \sin^3 \left(\dfrac{\pi x}{l}\right)$ ...(*iii*)

and $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \qquad \text{...(iv)}$$

Since the vibration of the string is periodic, therefore, the solution of (*i*) is of the form

$$y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

By (*ii*), $y(0, t) = c_1(c_3 \cos cpt + c_4 \sin cpt) = 0$

For this to be true for all time, $c_1 = 0$.

$\therefore$ $y(x, t) = c_2 \sin px\,(c_3 \cos cpt + c_4 \sin cpt)$

Also by (*ii*), $y(l, t) = c_2 \sin pl\,(c_3 \cos cpt + c_4 \sin cpt) = 0$ for all t.

This gives $pl = n\pi$ or $p = n\pi/l$, n being an integer.

Thus $$y(x, t) = c_2 \sin \frac{n\pi x}{l}\left(c_3 \cos \frac{cn\pi t}{l} + c_4 \sin \frac{cn\pi t}{l}\right) \qquad \text{...(v)}$$

$$\frac{\partial y}{\partial t} = \left(c_2 \sin \frac{n\pi x}{l}\right)\frac{cn\pi}{l}\left(-c_3 \sin \frac{cn\pi t}{l} + c_4 \cos \frac{cn\pi t}{l}\right)$$

By (*iv*), $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = \left(c_2 \sin \dfrac{n\pi x}{l}\right)\dfrac{cn\pi}{l} \cdot c_4 = 0$, *i.e.* $c_4 = 0$.

Thus (*v*) becomes $y(x, t) = c_2c_3 \sin \dfrac{n\pi x}{l} \cos \dfrac{n\pi ct}{l} = b_n \sin \dfrac{n\pi x}{l} \cos \dfrac{n\pi ct}{l}$

Adding all such solutions the general solution of (*i*) is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \qquad \text{...(vi)}$$

$\therefore$ from (*iii*), $$y_0 \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

or $$y_0 \left\{ \frac{3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l}}{4} \right\} = b_1 \sin \frac{\pi x}{l} + b_2 \sin \frac{2\pi x}{l} + b_3 \sin \frac{3\pi x}{l} + \ldots$$

Comparing both sides, we have

$$b_1 = 3\, y_0/4,\ b_2 = 0,\ b_3 = -\, y_0/4,\ b_4 = b_5 = \ldots = 0.$$

Hence from (*vi*), the desired solution is

$$y(x, t) = \frac{3y_0}{4} \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l} - \frac{y_0}{4} \sin \frac{3\pi x}{l} \cos \frac{3\pi ct}{l}.$$

Example 18.5. *A tightly stretched flexible string has its ends fixed at $x = 0$ and $x = l$. At time $t = 0$, the string is given a shape defined by $F(x) = \mu x(l - x)$, where μ is a constant, and then released. Find the displacement of any point x of the string at any time $t > 0$.*

(Bhopal, 2008 ; Madras, 2006 ; J.N.T.U., 2005 ; P.T.U., 2005)

Solution. The equation of the string is $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ ...(*i*)

The boundary conditions are $y(0, t) = 0, y(l, t) = 0$ ...(*ii*)

Also the initial conditions are $y(x, 0) = \mu x(l - x)$ ...(*iii*)

and $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \qquad \text{...}(iv)$$

The solution of (*i*) is of the form

$$y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

By (*ii*), $y(0, t) = c_1(c_3 \cos cpt + c_4 \sin cpt) = 0$

For this to be true for all time, $c_1 = 0$.

$\therefore$ $y(x, t) = c_2 \sin px\, (c_3 \cos cpt + c_4 \sin cpt)$

Also by (*ii*) $y(l, t) = c_2 \sin pl(c_3 \cos cpt + c_4 \sin cpt) = 0$ for all t.

This gives $pl = n\pi$ or $p = n\pi/l$, n being an integer.

Thus $$y(x, t) = c_2 \sin \frac{n\pi x}{l} \left(c_3 \cos \frac{n\pi ct}{l} + c_4 \sin \frac{n\pi ct}{l} \right) \qquad \text{...}(v)$$

$$\frac{\partial y}{\partial t} = \left(c_2 \sin \frac{n\pi x}{l} \right) \frac{n\pi c}{l} \left(-\, c_3 \sin \frac{n\pi ct}{l} + c_4 \cos \frac{n\pi ct}{l} \right)$$

$\therefore$ by (*iv*) $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \left(c_2 \sin \frac{n\pi x}{l} \right) \frac{n\pi c}{l} \,.\, c_4 = 0$$

Thus (*v*) becomes $$y(x, t) = c_2 c_3 \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} = b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}$$

Adding all such solutions, the general solution of (*i*) is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \qquad \text{...}(vi)$$

From (*iii*), $$\mu(lx - x^2) = y(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

where $$b_n = \frac{2}{l} \int_0^l \mu(lx - x^2) \sin \frac{n\pi x}{l}\, dx, \text{ by Fourier half-range sine series}$$

$$= \frac{2\mu}{l} \left\{ \left| (lx - x^2)\left(-\frac{\cos n\pi x/l}{n\pi/l} \right) \right|_0^l - \int_0^l (l - 2x)\left(-\frac{\cos n\pi x/l}{n\pi/l} \right) dx \right\}$$

$$= \frac{2\mu}{l} \cdot \frac{1}{n\pi}\left\{\int_0^l (l-2x)\frac{\cos n\pi x}{l}\,dx\right\} = \frac{2\mu}{n\pi}\left\{\left|(l-2x)\frac{\sin n\pi x/l}{n\pi/l}\right|_0^l - \int_0^l (-2)\frac{\sin n\pi x/l}{n\pi/l}\,dx\right\}$$

$$= \frac{2\mu}{n\pi}\cdot\frac{2l}{n\pi}\int_0^l \sin\frac{n\pi x}{l}\,dx = \frac{4\mu l}{n^2\pi^2}\left|\frac{-\cos n\pi x/l}{n\pi/l}\right|_0^l = \frac{4\mu l^2}{n^3\pi^3}\{1-(-1)^n\}$$

Hence from (*vi*), the desired solution is

$$y(x,t) = \frac{4\mu l^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^3}\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}$$

$$= \frac{8\mu l^2}{\pi^3}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^3}\sin\frac{(2m-1)\pi}{l}x\cos\frac{(2m-1)\pi ct}{l}.$$

Example 18.6. *A tightly stretched string of length l with fixed ends is initially in equilibrium position. It is set vibrating by giving each point a velocity $v_0 \sin^3 \pi x/l$. Find the displacement $y(x, t)$.*

(S.V.T.U., 2008 ; V.T.U., 2008 ; U.P.T.U., 2006)

Solution. The equation of the vibrating string is $\dfrac{\partial^2 y}{\partial t^2} = c^2\dfrac{\partial^2 y}{\partial x^2}$ ...(*i*)

The boundary conditions are $y(0, t) = 0$, $y(l, t) = 0$ ...(*ii*)

Also the initial conditions are $y(x, 0) = 0$ ...(*iii*)

and $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = v_0 \sin^3\frac{\pi x}{l} \quad ...(iv)$$

Since the vibration of the string is periodic, therefore, the solution of (*i*) is of the form

$$y(x,t) = (c_1\cos px + c_2\sin px)(c_3\cos cpt + c_4\sin cpt)$$

By (*ii*), $y(0,t) = c_1(c_3\cos cpt + c_4\sin cpt) = 0$

For this to be true for all time $c_1 = 0$.

$\therefore$ $y(x,t) = c_2\sin px\,(c_3\cos cpt + c_4\sin cpt)$

Also $y(l,t) = c_2\sin pl\,(c_3\cos cpt + c_4\sin cpt) = 0$ for all t.

This gives $pl = n\pi$ or $p = \dfrac{n\pi}{l}$, n being an integer.

Thus $$y(x,t) = c_2\sin\frac{n\pi x}{l}\left(c_3\cos\frac{cn\pi}{l}t + c_4\sin\frac{cn\pi}{l}t\right)$$

By (*iii*), $0 = c_2c_3\sin\dfrac{n\pi x}{l}$ for all x *i.e.*, $c_2c_3 = 0$

$\therefore$ $y(x,t) = b_n\sin\dfrac{n\pi x}{l}\sin\dfrac{cn\pi t}{l}$ where $b_n = c_2c_4$

Adding all such solutions, the general solution of (*i*) is

$$y(x,t) = \sum b_n\sin\frac{n\pi x}{l}\sin\frac{cn\pi t}{l} \quad ...(v)$$

Now $$\frac{\partial y}{\partial t} = \sum b_n\sin\frac{n\pi x}{l}\cdot\frac{cn\pi}{l}\cos\frac{cn\pi t}{l}$$

By (*iv*), $$v_0\sin^3\frac{\pi x}{l} = \left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum\frac{cn\pi}{l}b_n\sin\frac{n\pi x}{l}$$

or $$\frac{v_0}{4}\left(3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right) = \sum\frac{cn\pi}{l}b_n\sin\frac{n\pi x}{l} \quad [\because \sin 3\theta = 3\sin\theta - 4\sin^3\theta]$$

$$= \frac{c\pi}{l}b_1\sin\frac{\pi x}{l} + \frac{2c\pi}{l}b_2\sin\frac{2\pi x}{l} + \frac{3c\pi}{l}b_3\sin\frac{3\pi x}{l} + \dots$$

Equating coefficients from both sides, we get

$$\frac{3v_0}{4} = \frac{c\pi}{l} b_1, \quad 0 = \frac{2c\pi}{l} b_2, \quad -\frac{v_0}{4} = \frac{3c\pi}{l} b_3, \ldots$$

$$\therefore \qquad b_1 = \frac{3lv_0}{4c\pi}, \quad b_3 = -\frac{lv_0}{12c\pi}, \quad b_2 = b_4 = b_5 = \ldots = 0$$

Substituting in (v), the desired solution is

$$y = \frac{lv_0}{12c\pi}\left(9 \sin \frac{\pi x}{l} \sin \frac{c\pi t}{l} - \sin \frac{3\pi x}{l} \sin \frac{3c\pi t}{l}\right).$$

Example 18.7. *A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If it is vibrating by giving to each of its points a velocity $\lambda x(l - x)$, find the displacement of the string at any distance x from one end at any time t.* (Anna, 2009 ; U.P.T.U., 2002)

Solution. The equation of the vibrating string is $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ ...(i)

The boundary conditions are $y(0, t) = 0, y(l, t) = 0$ ...(ii)

Also the initial conditions are $y(x, 0) = 0$ ...(iii)

and $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \lambda x(l - x) \qquad \text{...(iv)}$$

As in example 18.6, the general solution of (i) satisfying the conditions (ii) and (iii) is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} . \sin \frac{n\pi ct}{l} \qquad \text{...(v)}$$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} . \cos \frac{n\pi ct}{l} . \left(\frac{n\pi c}{l}\right)$$

By (iv), $$\lambda x(l - x) = \left(\frac{\partial y}{\partial t}\right)_{t=0} = \frac{\pi c}{l} \sum_{n=1}^{\infty} nb_n \sin \frac{n\pi x}{l}$$

$$\therefore \qquad \frac{\pi c}{l} nb_n = \frac{2}{l}\int_0^l \lambda x(l - x) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2\lambda}{l}\left| (lx - x^2)\left(-\frac{l}{n\pi} \cos \frac{n\pi x}{l}\right) - (l - 2x)\left(-\frac{l^2}{n^2\pi^2} \sin \frac{n\pi x}{l}\right) + (-2)\left(\frac{l^3}{n^3\pi^3} \cos \frac{n\pi x}{l}\right)\right|_0^l$$

$$= \frac{4\lambda l^2}{n^3\pi^3}(1 - \cos n\pi) = \frac{4\lambda l^2}{n^3\pi^3}\, [1 - (-1)^n]$$

or $$b_n = \frac{4\lambda l^3}{c\pi^4 n^4}[1 - (-1)^n] = \frac{8\lambda l^3}{c\pi^4 (2m - 1)^4} \text{ taking } n = 2m - 1.$$

Hence, from (v), the desired solution is

$$y = \frac{8\lambda l^3}{c\pi^4} \sum_{m=1}^{\infty} \frac{1}{(2m - 1)^4} \sin \frac{(2m - 1)\pi x}{l} \sin \frac{(2m - 1)\pi ct}{l}.$$

Example 18.8. *The points of trisection of a string are pulled aside through the same distance on opposite sides of the position of equilibrium and the string is released from rest. Derive an expression for the displacement of the string at subsequent time and show that the mid-point of the string always remains at rest.* (Kerala, 2005)

Solution. Let B and C be the points of the trisection of the string $OA(= l)$ (Fig. 18.2). Initially the string is held in the form $OB'C'A$, where $BB' = CC' = a$(say).

The displacement $y(x, t)$ of any point of the string is given by

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad ...(i)$$

and the boundary conditions are

$$y(0, t) = 0 \quad ...(ii)$$

$$y(l, t) = 0 \quad ...(iii)$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \quad ...(iv)$$

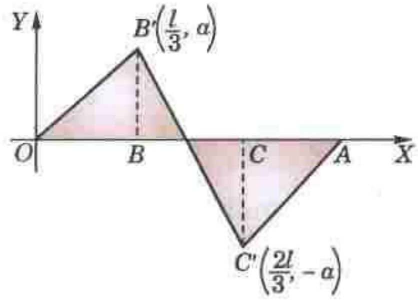


Fig. 18.2

The remaining condition is that at $t = 0$, the string rests in the form of the broken line $OB'C'A$. The equation of OB' is $y = (3a/l)\,x$;

the equation of $B'C'$ is $\quad y - a = \frac{-2a}{(l/3)}\left(x - \frac{l}{3}\right)$, i.e.. $y = \frac{3a}{l}(l - 2x)$

and the equation of $C'A$ is $\quad y = \frac{3a}{l}(x - l)$

Hence the fourth boundary condition is

$$\left.\begin{aligned} y(x, 0) &= \frac{3a}{l}x, \ 0 \le x \le \frac{l}{3} \\ &= \frac{3a}{l}(l - 2x), \ \frac{l}{3} \le x \le \frac{2l}{3} \\ &= \frac{3a}{l}(x - l), \ \frac{2l}{3} \le x \le l \end{aligned}\right\} \quad ...(v)$$

As in example 18.6, the solution of (i) satisfying the boundary conditions (ii), (iii) and (iv), is

$$y(x, t) = b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \quad [\text{Where } b_n = C_2C_3]$$

Adding all such solutions, the most general solution of (i) is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \quad ...(vi)$$

Putting $t = 0$, we have $y(x, 0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$ ...(vii)

In order that the condition (v) may be satisfied, (v) and (vii) must be same. This requires the expansion of $y(x, 0)$ into a Fourier half-range sine series in the interval $(0, l)$.

$\therefore$ by (1) of § 10.7,

$$b_n = \frac{2}{l}\left[\int_0^{l/3}\frac{3ax}{l}\sin\frac{n\pi x}{l}dx + \int_{l/3}^{2l/3}\frac{3a}{l}(l - 2x)\sin\frac{n\pi x}{l}dx + \int_{2l/3}^{l}\frac{3a}{l}(x - l)\sin\frac{n\pi x}{l}dx\right]$$

$$= \frac{6a}{l^2}\left[\left| x\left\{-\frac{\cos(n\pi x/l)}{(n\pi/l)}\right\} - 1\left\{-\frac{\sin(n\pi x/l)}{(n\pi/l)^2}\right\}\right|_0^{l/3}\right.$$

$$+ \left|(l - 2x)\left\{-\frac{\cos(n\pi x/l)}{(n\pi/l)}\right\} - (-2)\left\{\frac{\sin(n\pi x/l)}{(n\pi/l)^2}\right\}\right|_{l/3}^{2l/3}$$

$$\left.+ \left|(x - l)\left\{-\frac{\cos(n\pi x/l)}{(n\pi/l)}\right\} - (1).\left\{-\frac{\sin(n\pi x/l)}{(n\pi/l)^2}\right\}\right|_{2l/3}^{l}\right]$$

$$= \frac{6a}{l^2}\left[\left(-\frac{l^2}{3n\pi}\cos\frac{n\pi}{3} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{3}\right) + \frac{l^2}{3n\pi}\cos\frac{2n\pi}{3} - \frac{2l^2}{n^2\pi^2}\sin\frac{2n\pi}{3} + \frac{l^2}{3n\pi}\cos\frac{n\pi}{3}\right.$$

$$\left. + \frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{3} - \left(\frac{l^2}{3n\pi}\cos\frac{2n\pi}{3} + \frac{l^2}{n^2\pi^2}\sin\frac{2n\pi}{3}\right)\right]$$

$$= \frac{6a}{l^2} \cdot \frac{3l^2}{n^2\pi^2}\left(\sin\frac{n\pi}{3} - \sin\frac{2n\pi}{3}\right)$$

$$= \frac{18a}{n^2\pi^2} \sin\frac{n\pi}{3}\,[1 + (-1)^n] \qquad \left[\because \ \sin\frac{2n\pi}{3} = \sin\left(n\pi - \frac{n\pi}{3}\right) = -(-1)^n \sin\frac{n\pi}{3}\right]$$

Thus $b_n = 0$, when n is odd.

$$= \frac{36a}{n^2\pi^2}\sin\frac{n\pi}{3}, \text{ when } n \text{ is even.}$$

Hence (*vi*) gives

$$y(x, t) = \sum_{n=2,4,\ldots}^{\infty} \frac{36a}{n^2\pi^2}\sin\frac{n\pi}{3}\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \qquad [\text{Take } n = 2m]$$

$$= \frac{9a}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\sin\frac{2m\pi}{3}\sin\frac{2m\pi x}{l}\cos\frac{2m\pi ct}{l} \qquad \ldots(vii)$$

Putting $x = l/2$ in (*vii*), we find that the displacement of the mid-point of the string, *i.e.* $y(l/2, t) = 0$, because $\sin m\pi = 0$ for all integral values of m.

This shows that the mid-point of the string is always at rest.

(3) D'Alembert's solution of the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \qquad \ldots(1)$$

Let us introduce the new independent variables $u = x + ct$, $v = x - ct$ so that y becomes a function of u and v.

Then $\dfrac{\partial y}{\partial x} = \dfrac{\partial y}{\partial u} + \dfrac{\partial y}{\partial v}$

and
$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right) = \frac{\partial}{\partial u}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right) = \frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u\partial v} + \frac{\partial^2 y}{\partial v^2}$$

Similarly,
$$\frac{\partial^2 y}{\partial t^2} = c^2\left(\frac{\partial^2 y}{\partial u^2} - 2\frac{\partial^2 y}{\partial u\partial v} + \frac{\partial^2 y}{\partial v^2}\right)$$

Substituting in (1), we get $\dfrac{\partial^2 y}{\partial u\partial v} = 0$ ...(2)

Integrating (2) w.r.t. v, we get $\dfrac{\partial y}{\partial u} = f(u)$ ...(3)

where $f(u)$ is an arbitrary function of u. Now integrating (3) w.r.t. u, we obtain

$$y = \int f(u)\,du + \psi(v)$$

where $\psi(v)$ is an arbitrary function of v. Since the integral is a function of u alone, we may denote it by $\phi(u)$. Thus

$$y = \phi(u) + \psi(v)$$

i.e.
$$y(x, t) = \phi(x + ct) + \psi(x - ct) \qquad \ldots(4)$$

This is the *general solution of the wave equation* (1).

Now to determine ϕ and ψ, suppose initially $u(x, 0) = f(x)$ and $\partial y(x, 0)/\partial t = 0$.

Differentiating (4) w.r.t. t, we get $\dfrac{\partial y}{\partial t} = c\phi'(x + ct) - c\psi'(x - ct)$

At $t = 0$, $\qquad \phi'(x) = \psi'(x)$ ...(5)

and $\qquad y(x, 0) = \phi(x) + \psi(x) = f(x)$ ...(6)

(5) gives, $\qquad \phi(x) = \psi(x) + k$

$\therefore$ (6) becomes $2\psi(x) + k = f(x)$

or
$$\psi(x) = \frac{1}{2}[f(x) - k] \text{ and } \phi(x) = \frac{1}{2}[f(x) + k]$$

Hence the solution of (4) takes the form

$$y(x, t) = \frac{1}{2}[f(x + ct) + k] + \frac{1}{2}[f(x - ct) - k] = f(x + ct) + f(x - ct) \quad ...(7)$$

which is the *d'Alembert's solution* of the wave equation* (1) *(V.T.U., 2011 S)*

Obs. The above solution gives a very useful method of solving partial differential equations by change of variables.

Example 18.9. *Find the deflection of a vibrating string of unit length having fixed ends with initial velocity zero and initial deflection $f(x) = k(\sin x - \sin 2x)$.* *(V.T.U., 2011)*

Solution. By d'Alembert's method, the solution is

$$y(x, t) = \frac{1}{2}[f(x + ct) + f(x - ct)]$$

$$= \frac{1}{2}[k\{\sin (x + ct) - \sin 2(x + ct)\} + k\{\sin (x - ct) - \sin 2(x - ct)\}]$$

$$= k[\sin x \cos ct - \sin 2x \cos 2ct]$$

Also $\quad y(x, 0) = k(\sin x - \sin 2x) = f(x)$

and $\quad \partial y(x, 0)/\partial t = k(-c \sin x \sin ct + 2c \sin 2x \sin 2ct)_{t=0} = 0$

i.e., the given boundary conditions are satisfied.

PROBLEMS 18.2

1. Solve completely the equation $\partial^2 y/\partial t^2 = c^2 \partial^2 y/\partial x^2$, representing the vibrations of a string of length l, fixed at both ends, given that $y(0, t) = 0$; $y(l, t) = 0$; $y(x, 0) = f(x)$ and $\partial y(x, 0)/\partial t = 0$, $0 < x < l$. *(Bhopal, 2007 S ; U.P.T.U., 2005)*
2. Solve the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ under the conditions $u(0, t) = 0$, $u(l, t) = 0$ for all t ; $u(x, 0) = f(x)$ and $\left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x)$, $0 < x < l$.
3. Find the solution of the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, corresponding to the triangular initial deflection
$$f(x) = \frac{2k}{l} x \text{ when } 0 < x < \frac{l}{2} . = \frac{2k}{l}(l - x) \text{ when } \frac{l}{2} < x < l,$$
and initial velocity zero. *(Bhopal, 2006 ; Kerala, M.E., 2005)*
4. A tightly stretched string of length l has its ends fastened at $x = 0$, $x = l$. The mid-point of the string is then taken to height h and then released from rest in that position. Find the lateral displacement of a point of the string at time t from the instant of release. *(Anna, 2005)*
5. A tightly stretched string with fixed end points at $x = 0$ and $x = 1$, is initially in a position given by
$$f(x) = \begin{cases} x, & 0 \le x \le \frac{1}{2} \\ 1 - x, & \frac{1}{2} \le x \le 1 \end{cases}$$
If it is released from this position with velocity a, perpendicular to the x-axis, show that the displacement $u(x, t)$ at any point x of the string at any time $t > 0$, is given by
$$u(x, t) = \frac{4\sqrt{2}}{\pi^2}\left[\sum_{n=1}^{\infty}\left\{\frac{\sin[(4pi - 3)\pi x]\cos[(4pi - 3)\pi at - \pi/4]}{(4n - 3)^2} - \frac{\sin[(4pi - 1)\pi x]\cos[(4pi - 1)\pi at - \pi/4]}{(4n - 1)^2}\right\}\right]$$
6. If a string of length l is initially at rest in equilibrium position and each of its points is given a velocity v such that $v = cx$ for $< x < l/2$
$c(l \,.\, x)$ for $l/2 < x < l$, determine the displacement $y(x, t)$ at anytime t. *(Anna, 2008)*
7. Using d'Alembert's method, find the deflection of a vibrating string of unit length having fixed ends, with initial velocity zero and initial deflection :
(*i*) $f(x) = a(x - x^2)$ *(Kerala, M. Tech., 2005)* (*ii*) $f(x) = a \sin^2 \pi x$.

*See footnote of p. 373.

18.5 (1) ONE-DIMENSIONAL HEAT FLOW

Consider a homogeneous bar of uniform cross-section $\alpha(\text{cm}^2)$. Suppose that the sides are covered with a material impervious to heat so that the stream lines of heat-flow are all parallel and perpendicular to the area α. Take one end of the bar as the origin and the direction of flow as the positive x-axis (Fig. 18.3). Let ρ be the density (gr/cm^3), s the specific heat (cal./gr. deg.) and k the thermal conductivity (cal./cm. deg. sec.).

Let $u(x, -t)$ be the temperature at a distance x from O. If δu be the temperature change in a slab of thickness δx of the bar, then by § 12.7 (*ii*) p. 466, the quantity of heat in this slab $= s\rho\alpha\,\delta x\delta u$. Hence the rate of increase of heat in this slab, *i.e.*, $s\rho\alpha\delta x \dfrac{\partial u}{\partial t} = R_1 - R_2$, where R_1 and R_2 are respectively the rate (cal./sec.) of inflow and outflow of heat.

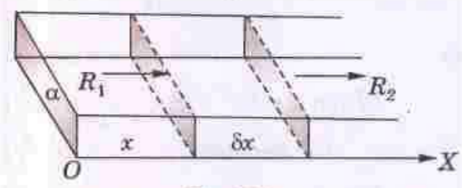


Fig. 18.3

Now by (*A*) of p. 466, $R_1 = -k\alpha\left(\dfrac{\partial u}{\partial x}\right)_x$ and $R_2 = -k\alpha\left(\dfrac{\partial u}{\partial x}\right)_{x+\delta x}$

the negative sign appearing as a result of (*i*) on p. 466.

Hence $s\rho\alpha\delta x \dfrac{\partial u}{\partial t} = -k\alpha\left(\dfrac{\partial u}{\partial x}\right)_x + k\alpha\left(\dfrac{\partial u}{\partial x}\right)_{x+\delta x}$ *i.e.*, $\dfrac{\partial u}{\partial t} = \dfrac{k}{s\rho}\left\{\dfrac{(\partial u/\partial x)_{x+\delta x} - (\partial u/\partial x)_x}{\delta x}\right\}$

Writing $k/s\rho = c^2$, called the *diffusivity* of the substance (cm^2/sec.), and taking the limit as $\delta x \to 0$, we get

$$\frac{\partial \mathbf{u}}{\partial \mathbf{t}} = \mathbf{c}^2 \frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} \qquad \text{...(1)}$$

This is the *one-dimensional heat-flow equation.* (*V.T.U., 2011*)

(2) Solution of the heat equation. Assume that a solution of (1) is of the form

$$u(x, t) = X(x) \,.\, T(t)$$

where X is a function of x alone and T is a function of t only.

Substituting this in (1), we get

$$XT' = c^2X''T, \text{ i.e., } X''/X = T'/c^2T \qquad \text{...(2)}$$

Clearly the left side of (2) is a function of x only and the right side is a function of t alone. Since x and t are independent variables, (2) can hold good if each side is equal to a constant k (say). Then (2) leads to the ordinary differential equations

$$\frac{d^2X}{dx^2} - kX = 0 \quad \text{...(3)} \qquad \text{and} \qquad \frac{dT}{dt} - kc^2T = 0 \qquad \text{...(4)}$$

Solving (3) and (4), we get

(*i*) *When k is positive* and $= p^2$, say :

$$X = c_1e^{px} + c_2e^{-px},\ T = c_3\, e^{c^2p^2t}\ ;$$

(*ii*) *When k is negative* and $= -p^2$, say :

$$X = c_4 \cos px + c_5 \sin px,\ T = c_6\, e^{-c^2p^2t}\ ;$$

(*iii*) *When k is zero :*

$$X = c_7x + c_8,\ T = c_9.$$

Thus the various possible solutions of the heat-equation (1) are

$$u = (c_1e^{px} + c_2e^{-px})\,c_3e^{c^2p^2t} \qquad \text{...(5)}$$

$$u = (c_4 \cos px + c_5 \sin px)\,c_6e^{-c^2p^2t} \qquad \text{...(6)}$$

$$u = (c_7x + c_8)c_9 \qquad \text{...(7)}$$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. As we are dealing with problems on heat conduction, it must be a transient solution, *i.e.*, u is to decrease with the increase of time t. Accordingly, the solution given by (6), *i.e.*, of the form

$$u = (C_1 \cos px + C_2 \sin px)\,e^{-c^2p^2t} \qquad \text{...(8)}$$

is the only suitable solution of the heat equation.

Example 18.10. *Solve the equation* $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$ *with boundary conditions* $u(x, 0) = 3 \sin n\pi x$, $u(0, t) = 0$ *and* $u(1, t) = 0$, *where* $0 < x < 1$, $t > 0$.

Solution. The solution of the equation $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$ ...(*i*)

is $$u(x, t) = (c_1 \cos px + c_2 \sin px)\ e^{-p^2 t} \quad ...(ii)$$

When $x = 0$, $u(0, t) = c_1 e^{-p^2 t} = 0$ *i.e.*, $c_1 = 0$.

$\therefore$ (*ii*) becomes $$u(x, t) = c_2 \sin px e^{-p^2 t} \quad ...(iii)$$

When $x = 1$, $u(1, t) = c_2 \sin p \,.\, e^{-p^2 t} = 0$ or $\sin p = 0$

i.e., $p = n\pi$.

$\therefore$ (*iii*) reduces to $u(x, t) = b_n e^{-(n\pi)^2 t} \sin n\pi x$ where $b_n = c_2$

Thus the general solution of (*i*) is $u(x, t) = \sum b_n e^{-n^2\pi^2 t} \sin n\pi x$ ...(*iv*)

When $t = 0$, $3 \sin n\pi x = u(0, t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin n\pi x$

Comparing both sides, $b_n = 3$

Hence from (*iv*), the desired solution is

$$u(x, t) = 3 \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} \sin n\pi x.$$

Example 18.11. *Solve the differential equation* $\frac{\partial u}{\partial t}=\alpha^2\frac{\partial^2 u}{\partial x^2}$ *for the conduction of heat along a rod without radiation, subject to the following conditions :*

(i) u is not infinite for $t \to \infty$, *(ii)* $\frac{\partial u}{\partial x} = 0$ *for* $x = 0$ *and* $x = l$,

(ii) $u = lx - x^2$ *for* $t = 0$, *between* $x = 0$ *and* $x = l$. (*P.T.U., 2007*)

Solution. Substituting $u = X(x)T(t)$ in the given equation, we get

$$XT' = \alpha^2 X''T \quad i.e., \quad X''/X = \frac{T'}{\alpha^2 T} = -k^2 \text{ (say)}$$

$$\therefore \quad \frac{d^2X}{dx^2} + k^2X = 0 \text{ and } \frac{dT}{dt} + k^2\alpha^2 T = 0 \quad ...(1)$$

Their solutions are $$X = c_1 \cos kx + c_2 \sin kx,\ T = c_3 e^{-k^2\alpha^2 t} \quad ...(2)$$

If k^2 is changed to $-k^2$, the solutions are

$$X = c_4 e^{kx} + c_5 e^{-kx},\ T = c_6 e^{k^2\alpha^2 t} \quad ...(3)$$

If $k^2 = 0$, the solutions are $$X = c_7 x + c_8,\ T = c_9 \quad ...(4)$$

In (3), $T \to \infty$ for $t \to \infty$ therefore, u also $\to \infty$ *i.e.*, the given condition (*i*) is not satisfied. So we reject the solutions (3) while (2) and (4), satisfy this condition.

Applying the condition (*ii*) to (4), we get $c_7 = 0$.

$$\therefore \quad u = XT = c_8 c_9 = a_0 \quad \text{(say)} \quad ...(5)$$

From (2), $$\frac{\partial u}{\partial x} = (-c_1 \sin kx + c_2 \cos kx)\, kc_3 e^{-k^2\alpha^2 t}$$

Applying the condition (*ii*), we get $c_2 = 0$ and $-c_1 \sin kl + c_2 \cos kl = 0$

i.e., $c_2 = 0$ and $kl = n\pi$ (n an integer)

$$\therefore \quad u = c_1 \cos kx \,.\, c_3 e^{-k^2\alpha^2 t} = a_n \cos\left(\frac{n\pi x}{l}\right)\frac{e^{-n^2\pi^2\alpha^2 t}}{l^2} \quad ...(6)$$

Thus the general solution being the sum of (5) and (6), is

$$u = a_0 + \Sigma a_n \cos (n\pi x/l)\, e^{-n^2\pi^2\alpha^2 t/l^2} \quad ...(7)$$

Now using the condition (*iii*), we get

$$lx - x^2 = a_0 + \Sigma a_n \cos (n\pi x/l)$$

This being the expansion of $lx - x^2$ as a half-range cosine series in $(0, l)$, we get

$$a_0 = \frac{1}{l}\int_0^l (lx - x^2)\, dx = \frac{1}{l}\left|\frac{lx^2}{2} - \frac{x^3}{3}\right|_0^l = \frac{l^2}{6}$$

and

$$a_n = \frac{2}{l}\int_0^l (lx - x^2)\cos\frac{n\pi x}{l}\, dx = \frac{2}{l}\left|(lx - x^2)\left(\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right)\right.$$

$$\left. - (l - 2x)\left(-\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right) + (-2)\left(-\frac{l^3}{n^3\pi^3}\sin\frac{n\pi x}{l}\right)\right|_0^l$$

$$= \frac{2}{l}\left\{0 - \frac{l^3}{n^2\pi^2}(\cos n\pi + 1) + 0\right\} = -\frac{4l^2}{n^2\pi^2} \text{ when } n \text{ is even, otherwise } 0.$$

Hence taking $n = 2m$, the required solution is

$$u = \frac{l^2}{6} - \frac{l^2}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\cos\left(\frac{2m\pi x}{l}\right)e^{-4m^2\pi^2\alpha^2 t/l^2}.$$

Example 18.12. (*a*) *An insulated rod of length l has its ends A and B maintained at 0°C and 100°C respectively until steady state conditions prevail. If B is suddenly reduced to 0°C and maintained at 0°C, find the temperature at a distance x from A at time t.* *(U.P.T.U., 2005)*

(*b*) *Solve the above problem if the change consists of raising the temperature of A to 20°C and reducing that of B to 80°C.* *(Madras, 2000 S)*

Solution. (*a*) Let the equation for the conduction of heat be

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \quad ...(i)$$

Prior to the temperature change at the end B, when $t = 0$, the heat flow was independent of time (*steady state condition*). When u depends only on x, (*i*) reduces to $\partial^2 u/\partial x^2 = 0$.

Its general solution is $u = ax + b$ ...(*ii*)

Since $u = 0$ for $x = 0$ and $u = 100$ for $x = l$, therefore, (*ii*) gives $b = 0$ and $a = 100/l$.

Thus the *initial condition* is expressed by $u(x, 0) = \dfrac{100}{l}x$ ...(*iii*)

Also the *boundary conditions* for the subsequent flow are

$$u(0, t) = 0 \text{ for all values of } t \quad ...(iv)$$

and

$$u(l, t) = 0 \text{ for all values of } t \quad ...(v)$$

Thus we have to find a temperature function $u(x, t)$ satisfying the differential equation (*i*) subject to the initial condition (*iii*) and the boundary conditions (*iv*) and (*v*).

Now the solution of (*i*) is of the form

$$u(x, t) = (C_1\cos px + C_2\sin px)\, e^{-c^2p^2t} \quad ...(vi)$$

By (*iv*), $u(0, t) = C_1 e^{-c^2p^2t} = 0$, for all values of t.

Hence $C_1 = 0$ and (*vi*) reduces to $u(x, t) = C_2\sin px \,.\, e^{-c^2p^2t}$ ...(*vii*)

Applying (*v*), (*vii*) gives $u(l, t) = C_2\sin pl \,.\, e^{-c^2p^2t} = 0$, for all values of t.

This requires $\sin pl = 0$ *i.e.*, $pl = n\pi$ as $C_2 \neq 0$. $\therefore\ p = n\pi/l$, where n is any integer.

Hence (*vii*) reduces to $u(x, t) = b_n\sin\dfrac{n\pi x}{l}\cdot e^{-c^2n^2\pi^2t/l^2}$, where $b_n = C_2$.

[These are the solutions of (*i*) satisfying the boundary conditions (*iv*) and (*v*). These are the **eigen functions** corresponding to the **eigen values** $\lambda_n = cn\pi/l$, of the problem.]

Adding all such solutions, the most general solution of (*i*) satisfying the boundary conditions (*iv*) and (*v*) is

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot e^{-c^2 n^2 \pi^2 t / l^2} \qquad ...(viii)$$

Putting $t = 0$,
$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \qquad ...(ix)$$

In order that the condition (*iii*) may be satisfied, (*iii*) and (*ix*) must be same. This requires the expansion of $100x/l$ as a half-range Fourier sine series in $(0, l)$. Thus

$$\frac{100x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \text{ where } b_n = \frac{2}{l}\int_0^l \frac{100x}{l} \cdot \sin \frac{n\pi x}{l} dx$$

$$= \frac{200}{l^2}\left[x\left\{-\frac{\cos(n\pi x/l)}{(n\pi/l)}\right\} - (1)\left\{-\frac{\sin(n\pi x/l)}{(n\pi/l)^2}\right\}\right]_0^l = \frac{200}{l^2}\left(-\frac{l^2}{n\pi}\cos n\pi\right) = \frac{200}{n\pi}(-1)^{n+1}$$

Hence (*viii*) gives $u(x, t) = \dfrac{200}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{n}\sin\dfrac{n\pi x}{l} \cdot e^{-(cn\pi/l)^2 t}$

(*b*) Here the initial condition remains the same as (*iii*) above, and the boundary conditions are

$$u(0, t) = 20 \text{ for all values of } t \qquad ...(x)$$
$$u(l, t) = 80 \text{ for all values of } t \qquad ...(xi)$$

In part (*a*), the boundary values (*i.e.*, the temperature at the ends) being zero, we were able to find the desired solution easily. Now the boundary values being non-zero, we have to modify the procedure.

We split up the temperature function $u(x, t)$ into two parts as

$$u(x, t) = u_s(x) + u_t(x, t) \qquad ...(xii)$$

where $u_s(x)$ is a solution of (*i*) involving x only and satisfying the boundary conditions (*x*) and (*xi*); $u_t(x, t)$ is then a function defined by (*xii*). Thus $u_s(x)$ is a steady state solution of the form (*ii*) and $u_t(x, t)$ may be regarded as a transient part of the solution which decreases with increase of t.

Since $u_s(0) = 20$ and $u_s(l) = 80$, therefore, using (*ii*) we get

$$u_s(x) = 20 + (60/l)x \qquad ...(xiii)$$

Putting $x = 0$ in (*xii*), we have by (*x*),

$$u_t(0, t) = u(0, t) - u_s(0) = 20 - 20 = 0 \qquad ...(xiv)$$

Putting $x = l$ in (*xii*), we have by (*xi*),

$$u_t(l, t) = u(l, t) - u_s(l) = 80 - 80 = 0 \qquad ...(xv)$$

Also
$$u_t(x, 0) = u(x, 0) - u_s(x) = \frac{100x}{l} - \left(\frac{60x}{l} + 20\right) \qquad [\text{by } (iii) \text{ and } (xiii)$$

$$= \frac{40x}{l} - 20 \qquad ...(xvi)$$

Hence (*xiv*) and (*xv*) give the boundary conditions and (*xvi*) gives the initial condition relative to the transient solution. Since the boundary values given by (*xiv*) and (*xv*) are both zero, therefore, as in part (*a*), we have $u_t(x, t) = (C_1 \cos px + C_2 \sin px)\, e^{-c^2 p^2 t}$

By (*xiv*), $\quad u_t(0, t) = C_1 e^{-c^2 p^2 t} = 0$, for all values of t.

Hence $C_1 = 0$ and $\quad u_t(x, t) = C_2 \sin px \,.\, e^{-c^2 p^2 t} \qquad ...(xvii)$

Applying (*xv*), it gives $\quad u_t(l, t) = C_2 \sin pl e^{-c^2 p^2 t} = 0$ for all values of t.

This requires $\quad \sin pl = 0$, *i.e.* $pl = n\pi$ as $C_2 \neq 0$. $p = n\pi/l$, when n is any integer.

Hence (*xvii*) reduces to $u_t(x, t) = b_n \sin \dfrac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t / l^2}$ where $b_n = C_2$.

Adding all such solutions, the most general solution of (*xvii*) satisfying the boundary conditions (*xiv*) and (*xv*) is

$$u_t(x, t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} e^{-c^2n^2\pi^2t/l^2} \quad ...(xviii)$$

Putting $t = 0$, we have $u_t(x, 0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$ ...(*xix*)

In order that the condition (*xvi*) may be satisfied, (*xvi*) and (*xix*) must be same. This requires the expansion of $(40/l)\,x - 20$ as a half-range Fourier sine series in $(0, l)$. Thus

$$\frac{40x}{l} - 20 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad \text{where} \quad b_n = \frac{2}{l}\int_0^l \left(\frac{40x}{l} - 20\right) \sin\frac{n\pi x}{l}\, dx = -\frac{40}{nx}(1 + \cos n\pi)$$

i.e., $b_n = 0$, when n is odd ; $= -80/n\pi$, when n is even

Hence (*xviii*) becomes $u_t(x, t) = \sum_{n=2,4...}^{\infty} \left(\frac{-80}{n\pi}\right) \sin\frac{n\pi x}{l} \cdot e^{-c^2n^2\pi^2t/l^2}$ [Take $n = 2m$]

$$= -\frac{40}{\pi}\sum_{m=1}^{\infty} \frac{1}{m} \sin\frac{2m\pi x}{l} \cdot e^{-4c^2m^2\pi^2t/l^2} \quad ...(xx)$$

Finally combining (*xiii*) and (*xx*), the required solution is

$$u(x, t) = \frac{40x}{l} + 20 - \frac{40}{\pi}\sum_{m=1}^{\infty} \frac{1}{m} \sin\frac{2m\pi x}{l} \cdot e^{-4c^2m^2\pi^2t/l^2}.$$

Example 18.13. *The ends A and B of a rod 20 cm long have the temperature at 30°C and 80°C until steady-state prevails. The temperature of the ends are changed to 40°C and 60°C respectively. Find the temperature distribution in the rod at time t.*

Solution. Let the heat equation be $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ ...(*i*)

In steady state condition, u is independent of time and depends on x only, (*i*) reduces to

$$\partial^2 u/\partial x^2 = 0. \quad ...(ii)$$

Its solution is $u = a + bx$

Since $u = 30$ for $x = 0$ and $u = 80$ for $x = 20$, therefore $a = 30$, $b = (80 - 30)/20 = 5/2$

Thus the initial conditions are expressed by

$$u(x, 0) = 30 + \frac{5}{2}x \quad ...(iii)$$

The boundary conditions are $u(0, t) = 40$, $u(20, t) = 60$

Using (*ii*), the steady state temperature is

$$u(x, 0) = 40 + \frac{60 - 40}{20}x = 40 + x \quad ...(iv)$$

To find the temperature u in the intermediate period,

$$u(x, t) = u_s(x) + u_t(x, t)$$

where $u_s(x)$ is the steady state temperature distribution of the form (*iv*) and $u_t(x, t)$ is the transient temperature distribution which decreases to zero as t increases.

Since $u_t(x, t)$ satisfies one dimensional heat equation

$$\therefore \quad u(x, t) = 40 + x + \sum_{n=1}^{\infty}(a_n \cos px + b_n \sin px)\, e^{-p^2t} \quad ...(v)$$

$$u(0, t) = 40 = 40 + \sum_{n=1}^{\infty} a_n e^{-p^2t} \quad \text{whence } a_n = 0.$$

$\therefore$ (v) reduces to $\qquad u(x, t) = 40 + x + \sum_{n=1}^{\infty} b_n \sin pxe^{-p^2 t}$ ...(vi)

Also $\qquad u(20, t) = 60 = 40 + 20 + \sum_{n=1}^{\infty} b_n \sin 20\, pe^{-p^2 t}$

or $\qquad \sum_{n=1}^{\infty} b_n \sin 20\, pe^{-p^2 t} = 0$ *i.e.*, $\sin 20p = 0$ *i.e.*, $p = n\pi/20$

Thus (vi) becomes $\qquad u(x, t) = 40 + x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20} e^{-n\pi t/20}$ ...(vii)

Using (iii), $\qquad 30 + \frac{5}{2}x = u(0, t) = 40 + x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20}$

or $\qquad \frac{3x}{2} - 10 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20}$

where $b_n = \frac{2}{20}\int_0^{20}\left(\frac{3x}{2} - 10\right)\sin \frac{n\pi x}{20}\,dx = -\frac{20}{n\pi}(1 + 2\cos n\pi)$

Hence from (vii), the desired solution is

$$u = 40 + x - \frac{20}{\pi}\sum_{n=1}^{\infty}\frac{1 + 2\cos n\pi}{n}\sin \frac{n\pi x}{20} e^{-(n\pi/20)^2 t}.$$

Example 18.14. Bar with insulated ends. *A bar 100 cm long, with insulated sides, has its ends kept at 0°C and 100°C until steady state conditions prevail. The two ends are then suddenly insulated and kept so. Find the temperature distribution.*

Solution. The temperature $u(x, t)$ along the bar satisfies the equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad ...(i)$$

By law of heat conduction, the rate of heat flow is proportional to the gradient of the temperature. Thus, if the ends $x = 0$ and $x = l$ (= 100 cm) of the bar are insulated (Fig. 18.4) so that no heat can flow through the ends, the boundary conditions are

$$\frac{\partial u(0, t)}{\partial x} = 0, \frac{\partial u(l, t)}{\partial x} = 0 \text{ for all } t \qquad ...(ii)$$

Initially, under steady state conditions, $\frac{\partial^2 u}{\partial x^2} = 0$. Its solution is $u = ax + b$.

Since $u = 0$ for $x = 0$ and $u = 100$ for $x = l$ $\therefore$ $b = 0$ and $a = 1$.

Thus the initial condition is $u(x, 0) = x \quad 0 < x < l.$ ...(iii)

Now the solution of (i) is of the form $u(x, t) = (c_1 \cos px + c_2 \sin px)\, e^{-c^2 p^2 t}$ ...(iv)

Differentiating partially w.r.t. x, we get

$$\frac{\partial u}{\partial x} = (-c_1 p \sin px + c_2 p \cos px)\, e^{-c^2 p^2 t} \qquad ...(v)$$

Putting $x = 0$, $\qquad \left(\frac{\partial u}{\partial x}\right)_0 = c_2 pe^{-c^2 p^2 t} = 0$ for all t. [By (ii)]

$\therefore \qquad c_2 = 0$

Putting $x = l$ in (v), $\qquad \left(\frac{\partial u}{\partial x}\right)_l = -c_1 p \sin ple^{-c^2 p^2 t}$ for all t. [By (ii)]

$\therefore \qquad c_1 p \sin pl = 0$ *i.e.*, p being $\neq 0$, either $c_1 = 0$ or $\sin pl = 0$.

When $c_1 = 0$, (iv) gives $u(x, t) = 0$ which is a trivial solution, therefore $\sin pl = 0$.

or $\qquad pl = n\pi \quad$ or $\quad p = n\pi/l, \quad n = 0, 1, 2, ...$

Hence (*iv*) becomes $u(x, t) = c_1 \cos \dfrac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t/l^2}$.

$\therefore$ the most general solution of (*i*) satisfying the boundary conditions (*ii*) is

$$u(x, t) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t/l^2} = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t/l^2} \quad \text{(where } A_n = c_1) \quad ...(vi)$$

Putting $t = 0$, $u(x, 0) = A_0 + \sum_{n=1}^{\infty} A_n \cos \dfrac{n\pi x}{l} = x$ [by (*iii*)]

This requires the expansion of x into a half range cosine series in $(0, l)$.

Thus $$x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x/l \qquad \text{where } a_0 = \frac{2}{l}\int_0^l x\,dx = l$$

and $$a_n = \frac{2}{l}\int_0^l x \cos \frac{n\pi x}{l} dx = \frac{2l}{n^2\pi^2}(\cos n\pi - 1)$$
$$= 0, \text{ where } n \text{ is even} ; = -4l/n^2\pi^2, \text{ when } n \text{ is odd.}$$

$\therefore$ $$A_0 = \frac{a_0}{2} = l/2, \text{ and } A_n = a_n = 0 \text{ for } n \text{ even} ; = -4l/n^2\pi^2 \text{ for } n \text{ odd.}$$

Hence (*vi*) takes the form

$$u(x, t) = \frac{l}{2} + \sum_{n=1,3,\ldots}^{\infty} \frac{4l}{n^2\pi^2} \cos \frac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t/l^2}$$
$$= \frac{l}{2} - \frac{4l}{\pi^2}\sum_{1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{l} e^{-c^2(2n-1)^2\pi^2 t/l^2} \quad ...(vii)$$

This is the required temperature at a point P_1 distant x from end A at any time t.

Obs. *The sum of the temperatures at any two points equidistant from the centre is always 100°C, a constant.*

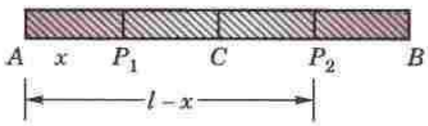


Fig. 18.4

Let P_1, P_2 be two points equidistant from the centre C of the bar so that $CP_1 = CP_2$ (Fig. 18.4).

If $AP_1 = BP_2 = x$ (say), then $AP_2 = l - x$.

$\therefore$ Replacing x by $l - x$ in (*vii*), we get the temperature at P_2 as

$$u(l - x, t) = \frac{l}{2} - \frac{4l}{\pi^2}\sum_{1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi(l-x)}{l} e^{\frac{-c^2(2n-1)^2\pi^2 t}{l^2}}$$
$$= \frac{l}{2} + \frac{4l}{\pi^2}\sum_{1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{l} e^{\frac{-c^2(2n-1)^2\pi^2 t}{l^2}} \quad ...(viii)$$

$$\left\{\because \cos \frac{(2n-1)\pi(l-x)}{l} = \cos\left[2n\pi - \pi - \frac{(2n-1)\pi x}{l}\right] = -\cos \frac{(2n-1)\pi x}{l}\right.$$

Adding (*vii*) and (*viii*), we get $u(x, t) + u(l - x, t) = l = 100°C$.

PROBLEMS 18.3

1. A homogeneous rod of conducting material of length 100 cm has its ends kept at zero temperature and the temperature initially is

$$u(x, 0) = x, \quad 0 \le x \le 50$$
$$= 100 - x, \quad 50 \le x \le 100.$$

Find the temperature $u(x, t)$ at any time. (*Bhopal, 2007 ; S.V.T.U., 2007 ; Kurukshetra, 2006*)

2. Find the temperature $u(x, t)$ in a homogeneous bar of heat conducting material of length l, whose ends are kept at temperature 0°C and whose initial temperature in (°C) is given by $ax(l-x)/l^2$. *(P.T.U., 2009)*
3. A rod 30 cm. long, has its ends A and B kept at 20° and 80°C respectively until steady state conditions prevail. The temperature at each end is then suddenly reduced to 0°C and kept so. Find the resulting temperature function $u(x, t)$ taking $x = 0$ at A. *(Anna, 2008)*
4. A bar of 10 cm long, with insulated sides has its ends A and B maintained at temperatures 50°C and 100°C respectively, until steady-state conditions prevail. The temperature A is suddenly raised to 90°C and at the same time that at B is lowered to 60°C. Find the temperature distribution in the bar at time t. *(P.T.U., 2010)*

   Show that the temperature at the middle point of the bar remains unaltered for all time, regardless of the material of the bar.
5. Solve the following boundary value problem :
$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2},\quad 0 < x < l,\qquad \frac{\partial u(0, t)}{\partial x} = 0, \frac{\partial u(l, t)}{\partial x} = 0,\ u(x, 0) = x. \qquad \textit{(S.V.T.U., 2008)}$$
6. The temperatures at one end of a bar, 50 cm long with insulated sides, is kept at 0°C and that the other end is kept at 100°C until steady-state conditions prevail. The two ends are then suddenly insulated, so that the temperature gradient is zero at each end thereafter. Find the temperature distribution.
7. Find the solution of $\dfrac{\partial \theta}{\partial t} = k\dfrac{\partial^2 \theta}{\partial x^2}$, such that

   (*i*) θ is not infinite when $t \to +\infty$; (*ii*) $\left.\begin{aligned} \frac{\partial \theta}{\partial x} &= 0 \quad \text{when } x = 0 \\ \theta &= 0, \quad \text{when } x = l \end{aligned}\right\}$ for all values of t ;

   (*iii*) $\theta = \theta_0$, when $t = 0$, for all values of x between 0 and l. *(S.V.T.U., 2008)*
8. Find the solution of $\dfrac{\partial V}{\partial t} = k\dfrac{\partial^2 V}{\partial x^2}$ having given that $V = V_0 \sin nt$ when $x = 0$ for all values of t and $V = 0$ when x is very large.

18.6 TWO-DIMENSIONAL HEAT FLOW

Consider the flow of heat in a metal plate of uniform thickness α (cm), density ρ (gr/cm^3), specific heat s (cal/gr deg) and thermal conductivity k (cal/cm sec deg). Let XOY plane be taken in one face of the plate (Fig. 18.5). If the temperature at any point is independent of the z-coordinate and depends only on x, y and time t, then the flow is said to be two-dimensional. In this case, the heat flow is in the XY-plane only and is zero along the normal to the XY-plane.

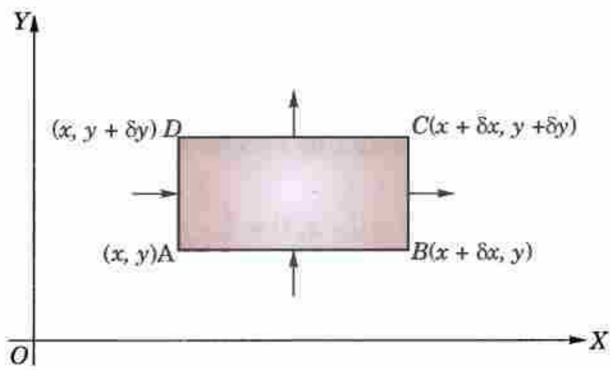


Fig. 18.5

Consider a rectangular element $ABCD$ of the plane with sides δx and δy. By (A) on p. 466, the amount of heat entering the element in 1 sec. from the side AB
$$= -k\alpha\delta x\left(\frac{\partial u}{\partial y}\right)_y$$
and the amount of heat entering the element in 1 second from the side $AD = -k\alpha\delta y\left(\dfrac{\partial u}{\partial x}\right)_x$

The quantity of heat flowing out through the side CD per sec. $= -k\alpha\delta x\left(\dfrac{\partial u}{\partial y}\right)_{y+\delta y}$

and the quantity of heat flowing out through the side BC per second $= -k\alpha\delta y\left(\dfrac{\partial u}{\partial x}\right)_{x+\delta x}$

Hence the total gain of heat by the rectangular element $ABCD$ per second
$$= -k\alpha\delta x\left(\frac{\partial u}{\partial y}\right)_y - k\alpha\delta y\left(\frac{\partial u}{\partial x}\right)_x + k\alpha\delta x\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} + k\alpha\delta y\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$$

$$= k\alpha\delta x\left[\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_y\right] + k\alpha\delta y\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]$$

$$= k\alpha\delta x\delta y\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\partial x} + \frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_y}{\partial y}\right] \quad ...(1)$$

Also the rate of gain of heat by the element

$$= \rho\delta x\delta y\alpha s\,\frac{\partial u}{\partial t} \quad ...(2)$$

Thus equating (1) and (2),

$$k\alpha\delta x\delta y\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x} + \frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_y}{\delta y}\right] = \rho\delta x\delta y\alpha s\,\frac{\partial u}{\partial t}.$$

Dividing both sides by $\alpha\delta x\delta y$ and taking limits as $\delta x \to 0$, $\delta y \to 0$, we get

$$k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \rho s\,\frac{\partial u}{\partial t}$$

i.e.,

$$\frac{\partial \mathbf{u}}{\partial \mathbf{t}} = \mathbf{c}^2\left(\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{u}}{\partial \mathbf{y}^2}\right) \text{ where } c^2 = k/\rho s \text{ is the } \textit{diffusivity}. \quad ...(3)$$

Hence the equation (3) gives the temperature distribution of the plane in the *transient state*.

Cor. In the *steady state*, u is independent of t, so that $\partial u/\partial t = 0$ and the above equation reduces to,

$$\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{u}}{\partial \mathbf{y}^2} = \mathbf{0}$$

which is the well known **Laplace's equation** *in two dimensions.*

Obs. When the stream lines are curves in space, *i.e.*, the heat flow is three dimensional, we shall similarly arrive at the equation

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

In a *steady state*, it reduces to $\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{u}}{\partial \mathbf{y}^2} + \frac{\partial^2 \mathbf{u}}{\partial \mathbf{z}^2} = \mathbf{0}$

which is the *three dimensional* **Laplace's equation.**

18.7 SOLUTION OF LAPLACE'S EQUATION

$$\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{u}}{\partial \mathbf{y}^2} = \mathbf{0} \quad ...(1)$$

Let $u = X(x)Y(y)$ be a solution of (1).

Substituting it in (1), we get $\frac{d^2X}{dx^2}Y + X\frac{\partial^2 Y}{\partial y^2} = 0$

or separating the variables, $\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2}$ ...(2)

Since x and y are independent variables, (2) can hold good only if each side of (2) is equal to a constant k (say). Then (2) leads to the ordinary differential equations

$$\frac{d^2X}{dx^2} - kX = 0 \text{ and } \frac{d^2Y}{dy^2} + kY = 0.$$

Solving these equations, we get

(*i*) *When k is positive* and is equal to p^2, say

$$X = c_1e^{px} + c_2e^{-px},\ Y = c_3 \cos py + c_4 \sin py$$

(*ii*) *When k is negative*, and is equal to $-p^2$, say

$$X = c_5 \cos px + c_6 \sin px,\ Y = c_7e^{py} + c_8e^{-py}$$

(*iii*) *When k is zero* ; $X = c_9x + c_{10}$, $Y = c_{11}y + c_{12}$.

Thus the various possible solutions of (1) are

$$u = (c_1e^{px} + c_2e^{-px})(c_3 \cos py + c_4 \sin py) \qquad ...(3)$$

$$u = (c_5 \cos px + c_6 \sin px)(c_7e^{py} + c_8e^{-py}) \qquad ...(4)$$

$$u = (c_9x + c_{10})(c_{11}y + c_{12}) \qquad ...(5)$$

Of these we take that solution which is consistent with the given boundary conditions.

(*V.T.U., 2011 S ; Kerala, 2005*)

Temperature distribution in long plates

Example 18.15. *An infinitely long plane uniform plate is bounded by two parallel edges and an end at right angles to them. The breadth is π ; this end is maintained at a temperature u_0 at all points and other edges are at zero temperature. Determine the temperature at any point of the plate in the steady-state.*

(*P.T.U., 2005 ; J.N.T.U., 2002 S*)

Solution. In the steady state (Fig. 18.6), the temperature $u(x, y)$ at any point $P(x, y)$ satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad ...(i)$$

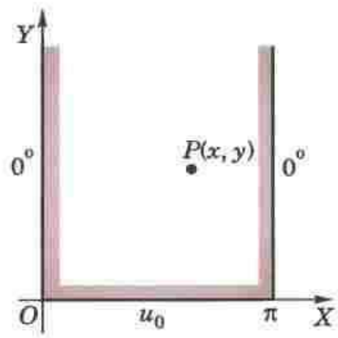


Fig. 18.6

The boundary conditions are $u(0, y) = 0$ for all values of y ...(*ii*)

$u(\pi, y) = 0$ for all values of y ...(*iii*)

$u(x, \infty) = 0$ in $0 < x < \pi$ ...(*iv*)

$u(x, 0) = u_0$ in $0 < x < \pi$ ...(*v*)

Now the three possible solutions of (*i*) are

$$u = (c_1e^{px} + c_2e^{-px})(c_3 \cos py + c_4 \sin py) \qquad ...(vi)$$

$$u = (c_5 \cos px + c_6 \sin px)(c_7\, e^{py} + c_8e^{-py}) \qquad ...(vii)$$

$$u = (c_9x + c_{10})(c_{11}y + c_{12}) \qquad ...(viii)$$

Of these, we have to choose that solution which is consistent with the physical nature of the problem. The solution (*vi*) cannot satisfy the condition (*ii*) for $u \neq 0$ for $x = 0$, for all values of y. The solution (*viii*) cannot satisfy the condition (*iv*). Thus the only possible solution is (*vii*), *i.e.* of the form

$$u(x, y) = (C_1 \cos px + C_2 \sin px)(C_3\, e^{py} + C_4\, e^{-py}) \qquad ...(ix)$$

By (*ii*), $\quad u(0, y) = C_1(C_3e^{py} + C_4e^{-py}) = 0$ for all y.

Hence $C_1 = 0$ and (*ix*) reduces to

$$u(x, y) = C_2 \sin px\,(C_3e^{py} + C_4e^{-py}) \qquad ...(x)$$

By (*iii*), $\quad u(\pi, y) = C_2 \sin p\pi(C_3e^{py} + C_4e^{-py}) = 0$, for all y.

This requires $\sin p\pi = 0$, *i.e.* $p\pi = n\pi$ as $C_2 \neq 0$. $\quad \therefore \quad p = n$, an integer.

Also to satisfy the condition (*iv*), *i.e.*, $u = 0$ as $y \to \infty$, $C_3 = 0$.

Hence (*x*) takes the form $u(x, y) = b_n \sin nx \,.\, e^{-ny}$, where $b_n = C_2C_4$.

$\therefore$ the most general solution satisfying (*ii*), (*iii*) and (*iv*) is of the form

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin nx \,.\, e^{-ny} \qquad ...(xi)$$

Putting $y = 0$, $$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin nx \qquad ...(xii)$$

In order that the condition (*v*) may be satisfied, (*v*) and (*xii*) must be same. This requires the expansion of u as a half-range Fourier sine series in $(0, \pi)$. Thus

$$u = \sum_{n=1}^{\infty} b_n \sin nx \text{ where } b_n = \frac{2}{\pi}\int_0^{\pi} u_0 \sin nx\, dx = \frac{2u_0}{n\pi}[1-(-1)^n]$$

i.e., $b_n = 0$, if n is even ; $= 4u_0/n\pi$, if n is odd.

Hence (xi) becomes $u(x, y) = \dfrac{4u_0}{\pi}\left[e^{-y}\sin x + \dfrac{1}{3}e^{-3y}\sin 3x + \dfrac{1}{5}e^{-5y}\sin 5x + \ldots\right]$.

Temperature distribution in finite plates

Example 18.16. *Solve the Laplace equation* $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ *subject to the conditions* $u(0, y) = u(l, y) = u(x, 0) = 0$ *and* $u(x, a) = \sin n\pi x/l$. *(V.T.U., 2011 ; J.N.T.U., 2006 ; Kerala M. Tech., 2005, U.P.T.U., 2004)*

Solution. The three possible solutions of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \ldots(i)$$

are
$$u = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py) \quad \ldots(ii)$$
$$u = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py}) \quad \ldots(iii)$$
$$u = (c_9 x + c_{10})(c_{11} y + c_{12}) \quad \ldots(iv)$$

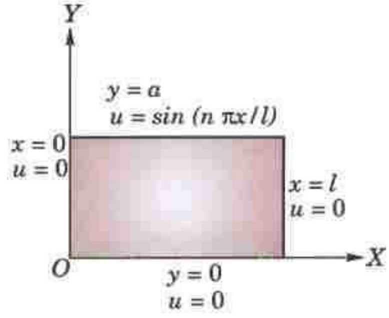


Fig. 18.7

We have to solve (i) satisfying the following boundary conditions

$$u(0, y) = 0 \quad \ldots(v) \qquad u(l, y) = 0 \quad \ldots(vi)$$
$$u(x, 0) = 0 \quad \ldots(vii) \qquad u(x, a) = \sin n\pi x/l \quad \ldots(viii)$$

Using (v) and (vi) in (ii), we get

$$c_1 + c_2 = 0, \quad \text{and} \quad c_1 e^{pl} + c_2 e^{-pl} = 0$$

Solving these equations, we get $c_1 = c_2 = 0$ which lead to trivial solution. Similarly, we get a trivial solution by using (v) and (vi) in (iv). Hence the suitable solution for the present problem is solution (iii). Using (v) in (iii), we have $c_5(c_7 e^{py} + c_8 e^{-py}) = 0$ *i.e.,* $c_5 = 0$

$\therefore$ (iii) becomes $\qquad u = c_6 \sin px(c_7 e^{py} + c_8 e^{-py}) \quad \ldots(ix)$

Using (vi), we have $c_6 \sin pl\,(c_7 e^{py} + c_8 e^{-py}) = 0$

$\therefore$ either $c_6 = 0$ or $\sin pl = 0$

If we take $c_6 = 0$, we get a trivial solution.

Thus $\sin pl = 0$ whence $pl = n\pi$ or $p = n\pi/l$ where $n = 0, 1, 2, \ldots\ldots$

$\therefore$ (ix) becomes $u = c_6 \sin(n\pi x/l)(c_1 e^{n\pi y/l} + c_8 e^{-n\pi y/l}) \quad \ldots(x)$

Using (vii), we have $0 = c_6 \sin n\pi x/l \,.\, (c_7 + c_8)$ *i.e.,* $c_8 = -c_7$.

Thus the solution suitable for this problem is

$$u(x, y) = b_n \sin\frac{n\pi x}{l}(e^{n\pi y/l} - e^{-n\pi y/l}) \quad \text{where} \quad b_n = c_6 c_7$$

Now using the condition $(viii)$, we have

$$u(x, a) = \sin\frac{n\pi x}{l} = b_n \sin\frac{n\pi x}{l}(e^{n\pi a/l} - e^{-n\pi a/l}),$$

we get
$$b_n = \frac{1}{(e^{n\pi a/l} - e^{-n\pi a/l})}$$

Hence the required solution is

$$u(x, y) = \frac{e^{n\pi y/l} - e^{-n\pi y/l}}{e^{n\pi a/l} - e^{-n\pi a/l}}\sin\frac{n\pi x}{l} = \frac{\sinh(n\pi y/l)}{\sinh(n\pi a/l)}\sin\frac{n\pi x}{l}.$$

Example 18.17. *The function* $v(x, y)$ *satisfies the Laplace's equation in rectangular coordinates* (x, y) *and for points within the rectangle* $x = 0, x = a, y = 0, y = b$, *it satisfies the conditions* $v(0, y) = v(a, y) = v(x, b) = 0$ *and* $v(x, 0) = x(a - x)$, $0 < x < a$. *Show that* $v(x, y)$ *is given by*

$$v(x, y) = \frac{8a^2}{\pi^3}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi x/a}{(2n+1)^3}\,\frac{\sinh(2n+1)\pi(b-y)/a}{\sinh(2n+1)\pi b/a} \qquad \textit{(Madras, 2003)}$$

Solution. The only possible solution of

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad ...(i)$$

is of the form $\quad v(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \quad ...(ii)$

The boundary conditions are

$$v(0, y) = 0\,; \quad v(a, y) = 0 \quad ...(iii)$$

$$v(x, b) = 0 \quad ...(iv)$$

$$v(x, 0) = x(a - x), \; 0 < x < a. \quad ...(v)$$

Using (iii) $\quad v(0, y) = c_1(c_3 e^{py} + c_4 e^{-py}) = 0 \quad i.e., \quad c_1 = 0.$

$\therefore$ (ii) becomes $\quad v(x, y) = c_2 \sin px\,(c_3 e^{py} + c_4 e^{-py}) \quad ...(vi)$

Again using (iii), $\quad v(a, y) = c_2 \sin pa\,(c_3 e^{py} + c_4 e^{-py}) = 0.$

i.e., $\quad \sin pa = 0$, *i.e.* $pa = n\pi$ or $p = n\pi/a$

$\therefore$ (vi) becomes $\quad v(x, y) = c_2 \sin \frac{n\pi x}{a}\left(c_3 e^{\frac{n\pi y}{a}} + c_4 e^{-\frac{n\pi y}{a}}\right)$

or $\quad v(x, y) = \sin \frac{n\pi x}{a}\,(Ae^{n\pi y/a} + Be^{-n\pi y/a})$ where $A = c_2 c_3, B = c_2 c_4 \quad ...(vii)$

Now using (iv), $\quad v(x, b) = \sin \frac{n\pi x}{a}\left(Ae^{\frac{n\pi b}{a}} + Be^{-\frac{n\pi b}{a}}\right) = 0$

i.e., $\quad Ae^{n\pi b/a} + Be^{-n\pi b/a} = 0$ or $Ae^{n\pi b/a} - Be^{-n\pi b/a} = -\frac{1}{2} b_n$ (say)

Thus (vii) becomes $\quad v(x, y) = \sin \frac{n\pi x}{a} \cdot \frac{1}{2} b_n \left\{e^{n\pi(b-y)/a} - e^{-n\pi(b-y)/a}\right\}$

$$= b_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi(b - y)}{a}$$

$\therefore$ the most general solution of (i) is

$$v(x, y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi(b-y)}{a} \quad ...(viii)$$

Using the condition (v), we have

$$x(a - x) = v(x, 0) = \sum_{n=1}^{\infty} b_n \sinh \frac{n\pi b}{a} \sin \frac{n\pi x}{a}$$

where $\quad b_n \sinh \frac{n\pi b}{a} = \frac{2}{a}\int_0^a x(a-x) \sin \frac{n\pi x}{a}\,dx$

$$= \frac{2}{a}\left| (ax - x^2)\left(\frac{-\cos n\pi x/a}{n\pi/a}\right) - (a - 2x)\left(-\frac{\sin n\pi x/a}{(n\pi/a)^2}\right) + (-2)\left\{\frac{\cos n\pi x/a}{(n\pi/a)^3}\right\}\right|_0^a$$

$$= 0 - 0 + \frac{4a^2}{n^3\pi^3}(1 - \cos n\pi)$$

$$= \frac{8a^2}{n^3\pi^3} \text{ when } n \text{ is odd, otherwise zero when } n \text{ is even.}$$

Hence from $(viii)$, the required solution is

$$v(x, y) = \frac{8a^2}{\pi^3} \sum_{n=1, 3, 5, \ldots}^{\infty} \frac{\sinh n\pi(b-y)/a}{n^3 \sinh n\pi b/a} \sin \frac{n\pi x}{a}$$

or $$v(x, y) = \frac{8a^2}{\pi^3} \sum_{n=0}^{\infty} \frac{\sinh (2n+1)\pi(b-y)/a}{(2n+1)^3 \sinh (2n+1)\pi b/a} \sin \frac{(2n+1)\pi x}{a}.$$

PROBLEMS 18.4

1. A long rectangular plate of width a cm. with insulated surface has its temperature v equal to zero on both the long sides and one of the short sides so that $v(0, y) = 0$, $v(a, y) = 0$, $v(x, \infty) = 0$, $v(x, 0) = kx$. Show that the steady-state temperature within the plate is

$$v(x, y) = \frac{2ak}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot e^{-n\pi y/a} \sin \frac{n\pi x}{a}. \qquad \textit{(J.N.T.U., 2005)}$$

2. A rectangular plate with insulated surface is 8 cm. wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along one short edge $y = 0$ is given by

$$u(x, 0) = 100 \sin (\pi x/8), \quad 0 < x < 8\,;$$

while the two long edges $x = 0$ and $x = 8$ as well as the other short edge are kept at 0°C, show that the steady-state temperature at any point of the plane is given by

$$u(x, y) = 100 e^{-\pi y/8} \sin (\pi x/8).$$

3. A rectangular plate with insulated surface is 10 cm. wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature of the short edge $y = 0$ is given by

$$u = 20\,x \qquad \text{for } 0 \le x \le 5$$

and

$$u = 20(10 - x) \qquad \text{for } 5 \le x \le 10$$

and the two long edges $x = 0$, $x = 10$ as well as the other short edge are kept at 0°C, prove that the temperature u at any point (x, y) is given by

$$u = \frac{40}{\pi^2} \sum_{1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin \frac{(2n-1)\pi x}{10} \cdot e^{-(2n-1)\pi y/10} \qquad \textit{(Anna, 2009)}$$

4. Solve $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ for $0 < x < \pi$, $0 < y < \pi$, with conditions given : $u(0, y) = u(\pi, y) = u(x, \pi) = 0$, $u(x, 0) = \sin^2 x$.

5. A square plate is bounded by the lines $x = 0$, $y = 0$, $x = 20$ and $y = 20$. Its faces are insulated. The temperature along the upper horizontal edge is given by

$$u(x, 20) = x(20 - x), \text{ when } 0 < x < 20,$$

while other three edges are kept at 0°C. Find the steady state temperature in the plate. *(Madras, 2003)*

6. The temperature u is maintained at 0° along three edges of a square plate of length 100 cm. and the fourth edge is maintained at 100° until steady-state conditions prevail. Find an expression for the temperature u at any point (x, y). Hence show that the temperature at the centre of the plate

$$= \frac{200}{\pi}\left[\frac{1}{\cosh \pi/2} - \frac{1}{3\cosh 3\pi/2} + \frac{1}{5\cosh 5\pi/2} - \ldots\right].$$

7. A square thin metal plate of side a is bounded by the lines $x = 0$, $x = a$, $y = 0$, $y = a$. The edges $x = 0$, $y = a$ are kept at zero temperature, the edge $y = 0$ is insulated and the edge $x = a$ is kept at constant temperature T_0. Show that in the steady state conditions, the temperature $u(x, y)$ at the point (x, y) is given by

$$u(x, y) = \frac{4T_0}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sinh \frac{(2n-1)\pi x}{2a} \cos \frac{(2n-1)\pi y}{2a}}{(2n-1) \sinh \frac{(2n-1)\pi}{2}}.$$

8. A rectangular plate has sides a and b. Taking the side of length a as OX and that of length b as OY and other sides to be $x = a$ and $y = b$, the sides $x = 0$, $x = a$, $y = b$ are insulated and the edge $y = 0$ is kept at temperature $u_0 \cos \dfrac{\pi x}{a}$. Find the temperature $u(x, y)$ in the steady-state.

18.8 (1) LAPLACE'S EQUATION IN POLAR COORDINATES

In the study of steady-state temperature distribution in a rectangular plate, it is usually convenient to employ Cartesian coordinates as hitherto done. Sometimes Polar coordinates (r, θ) are found to be more useful and the Cartesian form of Laplace's equation is replaced by its polar form :

$$r^2 \frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0$$

(See Ex. 5.24, p. 213-214)

(2) Solution of Laplace's equation

$$r^2\frac{\partial^2 u}{\partial r^2}+r\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial \theta^2}=0 \quad ...(1)$$

Assume that a solution of (1) is of the form $u = R(r) \,.\, \phi(\theta)$ where R is a function of r alone and ϕ is a function of θ only.

Substituting it in (1), we get $r^2R''\phi + rR'\phi + R\phi'' = 0$ or $\phi(r^2R'' + rR') + R\phi'' = 0$.

Separating the variables $\dfrac{r^2R''+rR'}{R}=-\dfrac{\phi''}{\phi}$ ...(2)

Clearly the left side of (2) is a function of r only and the right side is a function of θ alone. Since r and θ are independent variables, (2) can hold good only if each side is equal to a constant k (say). Then (2) leads to the ordinary differential equations

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}-kR=0 \quad ...(3) \qquad \text{and} \qquad \frac{d^2\phi}{d\theta^2}+k\phi=0 \quad ...(4)$$

Putting $r = e^z$, (3) reduces to $\dfrac{d^2R}{dz^2}-kR=0$ ...(5)

Solving (5) and (4), we get

(*i*) *When k is positive* and $= p^2$, say :

$$R = c_1e^{pz} + c_2e^{-pz} = c_1r^p + c_2r^{-p},\ \phi = c_3\cos p\theta + c_4 \sin p\theta$$

(*ii*) *When k is negative* and $= -p^2$, say

$$R = c_5\cos pz + c_6 \sin pz = c_5\cos(p\log r) + c_6\sin(p\log r),\ \phi = c_7e^{p\theta}+c_8e^{-p\theta}$$

(*iii*) *When k is zero :*

$$R = c_9z + c_{10} = c_9\log r + c_{10},\ \phi = c_{11}\theta + c_{12}$$

Thus the three possible solutions of (1) are

$$u = (c_1r^p + c_2r^{-p})(c_3\cos p\theta + c_4\sin p\theta) \quad ...(6)$$

$$u = [c_5\cos(p\log r) + c_6\sin(p\log r)]\,(c_7e^{p\theta}+c_8e^{-p\theta}) \quad ...(7)$$

$$u = (c_9\log r + c_{10})(c_{11}\theta + c_{12}) \quad ...(8)$$

Of these solutions, we have to take that solution which is consistent with the physical nature of the problem. The general solution will consist of a sum of terms of type (6), (7) or (8). (*S.V.T.U., 2008*)

Example 18.18. *The diameter of a semi-circular plate of radius a is kept at 0°C and the temperature at the semi-circular boundary is T°C. Show that the steady state temperature in the plate is given by*

$$u(r,\theta)=\frac{4T}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\left(\frac{r}{a}\right)^{2n-1}\sin(2n-1)\theta. \qquad (\textit{Kerala M. Tech., 2005})$$

Solution. Take the centre of the circle as the pole and bounding diameter as the initial line as in Fig. 18.8. Let the steady state temperature at any point $P(r, \theta)$ be $u(r, \theta)$, so that u satisfies the equation

$$r^2\frac{\partial^2 u}{\partial r^2}+r\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial \theta^2}=0 \quad ...(i)$$

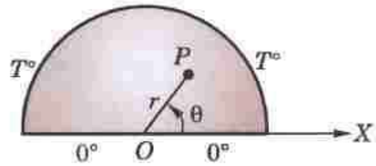


Fig. 18.8

The boundary conditions are :

$$u(r, 0) = 0 \quad \text{in } 0 \le r \le a \quad ...(ii)$$

$$u(r, \pi) = 0 \quad \text{in } 0 \le r \le a \quad ...(iii)$$

and

$$u(a, \theta) = T \quad ...(iv)$$

The three possible solutions of (*i*) are

$$u = (c_1r^p + c_2r^{-p})(c_3\cos p\theta + c_4\sin p\theta) \quad ...(v)$$

$$u = [c_5\cos(p\log r) + c_6\sin(p\log r)]\,(c_7e^{p\theta}+c_8e^{-p\theta}) \quad ...(vi)$$

$$u = (c_9\log r + c_{10})(c_{11}\theta + c_{12}) \quad ...(vii)$$

From (*ii*) and (*iii*), $u = 0$ when $r = 0$ *i.e.*, u must be finite at the origin. Thus the solutions (*vi*) and (*vii*) are to be rejected. Hence the only suitable solution is (*v*).

By (*ii*), $u(r, \theta) = (c_1 r^p + c_2 r^{-p})\, c_3 = 0$

Hence $c_3 = 0$ and (*v*) becomes $u(r, \theta) = (c_1 r^p + c_2 r^{-p})\, c_4 \sin p\theta$ ...(*viii*)

By (*iii*), $u(r, \pi) = (c_1 r^p + c_2 r^{-p})\, c_4 \sin p\pi = 0.$

As $c_4 \neq 0$, $\sin p\pi = 0$, *i.e.*, $p = n$, where n is any integer.

Hence (*viii*) reduces to $u(r, \theta) = (c_1 r^n + c_2 r^{-n})\, c_4 \sin n\theta$ ...(*ix*)

Since $u = 0$, when $r = 0$, $\therefore$ $c_2 = 0$ and (*ix*) becomes

$u(r, \theta) = b_n r^n \sin n\theta$, where $b_n = c_1 c_4$.

$\therefore$ the most general solution of (*i*) is of the form

$$u(r, \theta) = \sum_{n=1}^{\infty} b_n r^n \sin n\theta \qquad ...(x)$$

Putting $r = a$,

$$u(a, \theta) = \sum_{n=1}^{\infty} b_n a^n \sin n\theta. \qquad ...(xi)$$

In order that (*iv*) may be satisfied, (*iv*) and (*xi*) must be same. This requires the expansion of T as a half-range Fourier sine series in $(0, \pi)$. Thus

$$T = \sum_{n=1}^{\infty} B_n \sin n\theta \quad \text{where} \quad B_n = \frac{2}{\pi}\int_0^{\pi} T \sin n\theta\, d\theta = \frac{2T}{n\pi}(1 - \cos n\pi) \quad \text{and } B_n = b_n a^n$$

$$\therefore \qquad b_n = \frac{B_n}{a^n} = \frac{2T}{n\pi a^n}(1 - \cos n\pi)$$

i.e., $b_n = 0$, if n is even

$= \dfrac{4T}{n\pi a^n}$, if n is odd.

Hence (*x*) gives $u(r, \theta) = \dfrac{4T}{\pi}\left\{\dfrac{(r/a)}{1}\sin\theta + \dfrac{(r/a)^3}{3}\sin 3\theta + \dfrac{(r/a)^5}{5}\sin 5\theta + \ldots\right\}$

Example 18.19. *The bounding diameter of a semi-circular plate of radius a cm is kept at 0°C and the temperature along the semi-circular boundary is given by*

$$u(a, \theta) = \begin{cases} 50\theta, \text{ when } 0 < \theta \leq \pi/2 \\ 50(\pi - \theta) \text{ when } \pi/2 < \theta < \pi \end{cases}$$

Find the steady-state temperature function u (r, θ). (*Madras, 2003*)

Solution. We know that $u(r, \theta)$ satisfies the equation

$$r^2 \frac{\partial^2 u}{\partial \theta^2} + r\frac{\partial u}{\partial \theta} + \frac{\partial^2 u}{\partial \theta^2} = 0 \qquad ...(i)$$

The boundary conditions are $u(r, \theta) = 0$, $u(r, \pi) = 0$ ...(*ii*)

and $u(a, \theta) = 50\theta$ for $0 \leq \theta \leq \pi/2$; $u(a, \theta) = 50(\pi - \theta)$ for $\pi/2 \leq \theta \leq \pi$ ...(*iii*)

As in example 18.18, the most general solution of (*i*) satisfying the boundary conditions (*ii*) is of the form

$$u(r, \theta) = \sum_{n=1}^{\infty} b_n r^n \sin n\theta \qquad ...(iv)$$

Putting $r = a$,

$$r(a, \theta) = \sum_{n=1}^{\infty} b_n a^n \sin n\theta$$

In order that the boundary condition (*iii*) is satisfied, we have $r(a, \theta) = \sum_{n=1}^{\infty} B_n \sin n\theta$

where $b_n a^n = B_n = \dfrac{2}{\pi}\left\{\int_0^{\pi/2} 50\theta \sin n\theta\, d\theta + \int_{\pi/2}^{\pi} 50(\pi - \theta)\sin n\theta\, d\theta\right\}$ ...(*v*)

$$= \frac{100}{\pi}\left\{\left| \theta\left(\frac{-\cos n\theta}{\theta}\right) - (1)\left(\frac{-\sin n\theta}{n^2}\right)\right|_0^{\pi/2} + \left|(\pi-\theta)\left(\frac{-\cos n\theta}{n}\right) - (-1)\left(\frac{-\sin n\theta}{n^2}\right)\right|_{\pi/2}^{\pi}\right\}$$

$$= \frac{100}{\pi}\left\{-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{\sin n\pi/2}{n^2} + \frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{\sin n\pi/2}{n^2}\right\} = \frac{200}{\pi n^2}\sin n\pi/2.$$

When n is even $B_n = 0$, so taking $n = 1, 3, 5$ etc, (iv) gives

$$u(r, \theta) = \sum_{n=1,3,5,\ldots}^{\infty} \left(\frac{200}{\pi n^2}\sin\frac{n\pi}{2}\right)\frac{1}{a^n}\cdot r^n \sin n\theta$$

$$= \frac{200}{\pi}\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{(2m-1)^2}\left(\frac{r}{a}\right)^{2m-1} \sin(2m-1)\theta.$$

[Taking $n = 2m - 1$, $n = 1, 3, 5, \ldots$; gives $m = 1, 2, 3$, $\sin n\pi/2 = \sin(2m-1)\pi/2 = (-1)^{m-1}$. This gives the required temperature function.

PROBLEMS 18.5

1. A semi-circular plate of radius a has its circumference kept at temperature $u(a, \theta)$, $= k\theta(\pi - \theta)$ while the boundary diameter is kept at zero temperature. Find the steady state temperature distribution $u(r, \theta)$ of the plate assuming the lateral surfaces of the plate to be insulated.
2. A semi-circular plate of radius 10 cm has insulated faces and heat flows in plane curves. The bounding diameter is kept at 0°C and on the circumference the temperature distribution maintained is $u(10, \theta) = (400/\pi)(\pi\theta - \theta^2)$, $0 \le \theta \le \pi$. Determine the temperature distribution $u(r, \theta)$ at any point on the plate.
3. A plate in the shape of truncated quadrant of a circle, is bounded by $r = a$, $r = b$ and $\theta = 0$, $\theta = \pi/2$. It has its faces insulated and heat flows in plane curves. It is kept at temperature 0°C along three of the edges while along the edge $r = a$, it is kept at temperature $\theta(\pi/2 - \theta)$. Determine the temperature distribution.
4. Determine the steady state temperature at the points on the sector $0 \le \theta \le \pi/4$, $0 \le r \le a$ of a circular plate, if the temperature is maintained at 0°C along the side edges and at a constant temperature k°C along the curved edges.
5. Find the steady-state temperature in a circular plate of radius a which has one-half of its circumference at 0°C and the other half at 60°C.
6. If the radii of the inner and outer boundaries of a circular annulus area 10 cm and 20 cm and
$$u(10, \theta) = 15\cos\theta,\ u(20, \theta) = 30\sin\theta,$$
find the value of $u(r, \theta)$ in the annulus. [$u(r, \theta)$ satisfies Laplace equation in the interior of the annulus.]
7. A plate in the form of a ring is bounded by the lines $r = 2$ and $r = 4$. Its surfaces are insulated and the temperature along the boundaries are
$$u(2, \theta) = 10\sin\theta + 6\cos\theta,\ u(4, \theta) = 17\sin\theta + 15\cos\theta$$
Find the steady-state temperature $u(r, \theta)$ in the ring.

18.9 (1) VIBRATING MEMBRANE—TWO DIMENSIONAL WAVE EQUATION

We shall now derive the equation for the vibrations of a tightly stretched membrane, such as the membrane of a drum. We shall assume that the membrane is uniform and the tension T in it per unit length is the same in all directions at every point.

Consider the forces acting on an element $\delta x \delta y$ of the membrane (Fig. 18.9). Forces $T\delta x$ and $T\delta y$ act on the edges along the tangent to the membrane. Let u be its small displacement perpendicular to the xy-plane, so that the forces $T\delta y$ on its opposite edges of length δy make angles α and β to the horizontal. So their vertical component

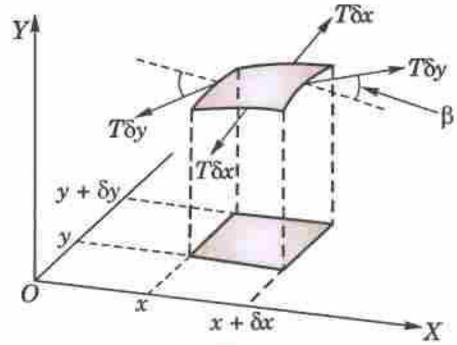


Fig. 18.9

$= T\delta y \sin\beta - T\delta y \sin\alpha$

$= T\delta y(\tan\beta - \tan\alpha)$ approximately, since α and β are small

$$= T\delta y\left\{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right\} = T\delta y \frac{\partial^2 u}{\partial x^2}\delta x, \text{ up to a first order of approximation.}$$

Similarly, the vertical component of the force $T\delta x$ acting on the edges of length δx

$$= T\delta x \frac{\partial^2 u}{\partial y^2}\delta y$$

If m be the mass per unit area of the membrane, then the equation of motion of the element is

$$m\delta x\delta y \frac{\partial^2 u}{\partial t^2} = T\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]\delta x\delta y \quad \text{or} \quad \frac{\partial^2 \mathbf{u}}{\partial \mathbf{t}^2} = \mathbf{c}^2\left[\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{u}}{\partial \mathbf{y}^2}\right] \quad \text{where} \quad c^2 = T/m \qquad ...(1)$$

This is the wave equation in two dimensions.

(2) Solution of the two-dimensional wave equation - Rectangular membrane. Assume that a solution of (1) is of the form $u = X(x)Y(y)T(t)$

Substituting this in (1) and dividing by XYT, we get

$$\frac{1}{c^2T}\frac{d^2T}{dt^2} = \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2}$$

This can hold good if each member is a constant. Choosing the constants suitably, we have

$$\frac{d^2X}{dx^2} + k^2X = 0, \quad \frac{d^2Y}{dy^2} + l^2Y = 0 \quad \text{and} \quad \frac{d^2T}{dt^2} + (k^2 + l^2)c^2T = 0$$

Hence a solution of (1) is

$$u = (c_1 \cos kx + c_2 \sin kx)(c_3 \cos ly + c_4 \sin ly) \times [c_5 \cos\sqrt{(k^2+l^2)}ct + c_6 \sin\sqrt{(k^2+l^2)}ct] \qquad ...(2)$$

Now suppose the membrane is rectangular and is stretched between the lines $x = 0$, $x = a$, $y = 0$, $y = b$. Then the condition $u = 0$ when $x = 0$ gives

$$0 = c_1(c_3 \cos ly + c_4 \sin ly)[c_5 \cos\sqrt{(k^2+l^2)}ct + c_6 \sin\sqrt{(k^2+l^2)}ct] \quad i.e., \quad c_1 = 0.$$

Then putting $c_1 = 0$ in (2) and applying the condition $u = 0$ when $x = a$, we get $\sin ka = 0$ or $k = m\pi/a$. (m being an integer)

Similarly, applying the conditions $u = 0$, when $y = 0$ and $y = b$, we obtain

$$c_3 = 0 \quad \text{and} \quad l = n\pi/b \qquad (n \text{ being an integer})$$

Thus the solution (2) becomes

$$u(x, y, t) = c_2c_4 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}(c_5 \cos p_{mn}t + c_6 \sin p_{mn}t)$$

where $p_{mn} = \pi c\sqrt{[(m/a)^2 + (n/b)^2]}$ ...(3)

[These are the solutions of the wave equation (1) which are zero on the boundary of the rectangular membrane. These functions are called **eigen functions** and the numbers p_{mn} are the **eigen values** of the vibrating membrane.]

Choosing the constants c_2 and c_4 so that $c_2c_4 = 1$, we can write the general solution of the equation (1) as

$$u = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}(A_{mn}\cos pt + B_{mn}\sin pt) \qquad ...(4)$$

If the membrane starts from rest from the initial position $u = f(x, y)$, *i.e.*, $\frac{\partial u}{\partial t} = 0$ when $t = 0$, then (3) gives $B_{mn} = 0$.

Also using the condition $u = f(x, y)$ when $t = 0$, we get

$$f(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

This is *double Fourier series.* Multiplying both sides by $\sin(m\pi x/a)\sin(n\pi y/b)$ and integrating from $x = 0$ to $x = a$ and $y = 0$ to $y = b$, every term on the right except one, becomes zero. Hence we obtain

$$\int_0^a \int_0^b f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dydx = \frac{ab}{4} A_{mn} \qquad ...(5)$$

which gives the coefficients in the solution and is called the **generalised Euler's formula.**

Rectangular Membranes

Example 18.20. *Find the deflection $u(x, y, t)$ of the square membrane with $a = b = 1$ and $c = 1$, if the initial velocity is zero and the initial deflection is $f(x, y) = A \sin \pi x \sin 2\pi y$.*

Solution. Taking $a = b = 1$ and $f(x, y) = A \sin \pi x \sin 2\pi y$, in (5), we get

$$A_{mn} = 4 \int_0^1 \int_0^1 A \sin \pi x \sin 2\pi y \sin m\pi x \sin n\pi y\, dydx$$

$$= 4A \int_0^1 \sin \pi x \sin m\pi x\, dx \left(\int_0^1 \sin 2\pi y \sin n\pi y\, dy \right) = 0, \quad \text{for } m \neq 1$$

$$= 4A \left(\frac{1}{2}\right) \int_0^1 \sin 2\pi y \sin n\pi y\, dy, \text{ for } m = 1 \qquad \left[\because \int_0^1 \sin \pi x \sin \pi x\, dx = \tfrac{1}{2}\right]$$

i.e., $$A_{mn} = 2A \int_0^1 \sin 2\pi y \sin n\pi y\, dy = 0, \text{ for } n \neq 2$$

$$= 2A \left(\frac{1}{2}\right), \text{ for } n = 2.$$

$\therefore$ $A_{12} = A$. Also from (3), $p_{mn} = \pi\sqrt{(m^2 + n^2)}$ $\qquad [\because a = b = 1 = c]$

$\therefore$ $p_{12} = \pi\sqrt{(1^2 + 2^2)} = \sqrt{5}\,\pi$.

Hence from (4), the required solution is $u(x, y, t) = A \sin \pi x \sin 2\pi y \cos (\sqrt{5}\,\pi t)$.

Example 18.21. *Find the vibration $u(x, y, t)$ of a rectangular membrane $(0 < x < a, 0 < y < b)$ whose boundary is fixed given that it starts from rest and $u(x, y, 0) = hxy(a - x)(b - y)$.*

Solution. Proceeding as in § 18.9 (**2**), we have from (4),

$$u = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} (A_{mn} \cos pt + B_{mn} \sin pt) \text{ where } p = \pi c\sqrt{[(m/a)^2 + (n/b)^2}$$

Since the membrane starts from rest $\partial u/\partial t = 0$ when $t = 0$,

$$\therefore \quad \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} (-A_{mn}\, p \sin pt + pB_{mn} \cos pt) = 0 \text{ when } t = 0$$

This gives $B_{mn} = 0$

$$\therefore \quad u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos pt \qquad ...(i)$$

Then $$hxy(a - x)(b - y) = u(x, y, 0) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

where $$A_{mn} = \frac{2}{a} \cdot \frac{2}{b} \int_0^a \int_0^b hxy(a-x)(b-y) \cdot \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dydx$$

$$= \frac{4h}{ab} \left\{ \int_0^a x(a-x) \sin \frac{m\pi x}{a} dx \right\} \left\{ \int_0^b y(b-y) \sin \frac{n\pi y}{b} dy \right\}$$

$$= \frac{4h}{ab} \left| (ax - x^2)\left(\frac{-\cos m\pi x/a}{m\pi/a} \right) - (a - 2x) \left\{ \frac{-\sin \frac{m\pi x}{a}}{(m\pi/a)^2} \right\} + (-2) \frac{\cos m\pi x/a}{(m\pi/a)^3} \right|_0^a$$

$$\times \left| (by - y^2)\left(\frac{-\cos n\pi y/b}{n\pi/b} \right) - (b - 2y) \left\{ \frac{-\sin n\pi/b}{(n\pi/b)^2} \right\} + (-2) \frac{\cos n\pi y/b}{(n\pi/b)^3} \right|_0^b$$

$$= \frac{4h}{ab} \frac{2a^3}{m^3\pi^3} \cdot \frac{2b^3}{n^3\pi^3} (1 - \cos m\pi)(1 - \cos n\pi)$$

Hence from (i), we get

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos pt$$

where $A_{mn} = \dfrac{16\,ha^2b^2}{m^3n^3\pi^6}(1 - \cos m\pi)(1 - \cos n\pi)$ and $p = \pi c\sqrt{[(m/a)^2 + (n/b)^2]}$

Circular Membranes*

Example 18.22. *A circular membrane of unit radius fixed along its boundary starts vibrating from rest and has initial deflection $u(r, 0) = f(r)$. Show that the deflection $u(r, t)$ of the membrane at any instant is given by*

$$u(r, t) = \sum_{m=1}^{\infty} A_m \cos(c\alpha_m t) \,.\, J_o(\alpha_m r) \text{ where } A_m = \frac{2}{J_1^2(\alpha_m)} \int_0^1 r f(r) J_0(\alpha_m r)\, dr,$$

and $\alpha_m (m = 1, 2, \ldots)$ are the positive roots of the Bessel function $J_0(k) = 0$.

Solution. The vibrations of a plane circular membrane are governed by 2-dimensional wave equation in polar coordinates *i.e.*,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \right)$$

For a radially symmetric membrane (in which u does not depend on θ) the above equation reduces to

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \right) \quad \ldots(i)$$

For the given membrane fixed along its boundary, the boundary condition is

$$u(1, t) = 0 \quad \text{for all } t \geq 0 \quad \ldots(ii)$$

For solutions not depending on θ,

initial deflection $u(r, 0) = f(r)$ ...(iii)

and initial velocity $\left(\dfrac{\partial u}{\partial t}\right)_{t=0} = 0$ ...(iv)

which are the initial conditions. We find the solutions $u(r, t) = R(r)T(t)$ ...(v)

satisfying the boundary condition (ii).

Differentiating and substituting (v) in (i), we get

$$\frac{\partial^2 T/\partial t^2}{c^2 T} = \frac{1}{R}\left(\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r}\right) = -k^2 \text{ (say)}$$

This leads to $\dfrac{\partial^2 T}{\partial t^2} + p^2 T = 0$ where $p = ck$ ...(vi)

and $\dfrac{\partial^2 R}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial R}{\partial r} + k^2 R = 0$ ...(vii)

Now putting $s = kr$, (vii) transforms to $\dfrac{d^2R}{ds^2} + \dfrac{1}{s}\dfrac{dR}{ds} + R = 0$ which is Bessel's equation. Its general solution $R = aJ_0(s) + bY_0(s)$ where J_0 and Y_0 are Bessel's functions of the first and second kind of order zero.

Since the deflection of the membrane is always finite, we must have $b = 0$. Then taking $a = 1$, we get

$$R(r) = J_0(s) = J_0(kr)$$

On the boundary of the circular membrane, we must have $J_0(k) = 0$, which is satisfied for

$$k = \alpha_m,\ m = 1, 2, \ldots$$

*Drums, telephones and microphones provide examples of circular membrane and as such are quite useful in engineering.

Thus the solutions of (*vii*) are $R(r) = J_0(\alpha_m r)$, $m = 1, 2, \ldots$ and the corresponding solutions of (*vi*) are $T(t) = A_m \cos p_m t + B_m \sin p_m t$, where $p_m = ck_m = c\alpha_m$.

Hence the general solution of (*i*) satisfying (*ii*) are

$$u(r, t) = (A_m \cos p_m t + B_m \sin p_m t)\, J_0(\alpha_m r)$$

which are the *eigen functions* of the problem and the corresponding *eigen values* are p_m.

To find that solution which also satisfies the initial conditions (*iii*) and (*iv*), consider the series

$$u(r, t) = \sum_{m=1}^{\infty} (A_m \cos p_m t + B_m \sin p_m t)\, J_0(\alpha_m r)$$

Putting $t = 0$ and using (*iii*), we get $u(r, 0) = \sum_{m=1}^{\infty} A_m J_0(\alpha_m r) = f(r)$

Here, the constants A_m must be the coefficients of Fourier-Bessel series [(8) page 560] with $m = 0$, *i.e.*,

$$A_m = \frac{2}{J_1^2(\alpha_m)} \int_0^1 r\, f(r)\, J_0(\alpha_m r)\, dr$$

Using (*iv*), we get $B_m = 0$. Hence the result.

PROBLEMS 18.6

1. A tightly stretched unit square membrane starts vibrating from rest and its initial displacement is $k \sin 2\pi x \sin \pi y$. Show that the deflection at any instant is $k \sin 2\pi x \sin \pi y \cos(\sqrt{5}\, \pi ct)$.
2. Find the deflection $u(r, t)$ of the circular membrane of unit radius if $c = 1$, the initial velocity is zero and the initial deflection is $0.25\,(1 - r^2)$.

18.10 TRANSMISSION LINE

Consider a cable l km in length, carrying an electric current with resistance R ohms/km, inductance L henries/km ; capacitance C farads/km and leakance G mhos/km (Fig. 18.10).

Let the instantaneous voltage and current at any point P, distant x km from the sending end O, and at time t sec be $v(x, t)$ and $i(x, t)$ respectively. Consider a small length $PQ (= \delta x)$ of the cable.

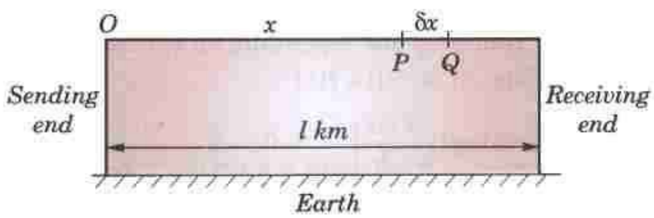


Fig. 18.10

Now since the voltage drop across the segment δx

= voltage drop due to resistance + voltage drop due to inductance

$$\therefore \qquad -\delta v = iR\delta x + L\delta x \,.\, \frac{\partial i}{\partial t}$$

and dividing by δx and taking limits as $\delta x \to 0$, we get

$$-\frac{\partial v}{\partial x} = Ri + L\frac{\partial i}{\partial t} \qquad \ldots(1)$$

Similarly the current loss between P and Q

= current lost due to capacitance and leakance

$$\therefore \qquad -\delta i = C\frac{\partial v}{\partial t}\delta x + Gv\delta x \text{ from which as before, we get}$$

$$-\frac{\partial i}{\partial x} = C\frac{\partial v}{\partial t} + Gv \qquad \ldots(2)$$

Rewriting the simultaneous partial differential equations (1) and (2) as

$$\left(R + L\frac{\partial}{\partial t}\right) i + \frac{\partial v}{\partial x} = 0 \qquad \ldots(3)$$

and $$\frac{\partial i}{\partial x}+\left(C\frac{\partial}{\partial t}+G\right)v=0, \qquad ...(4)$$

we shall eliminate i and v in turn.

$\therefore$ operating (3) by $\frac{\partial}{\partial x}$ and (4) by $\left(R+L\frac{\partial}{\partial t}\right)$ and subtracting

$$\frac{\partial^2 v}{\partial x^2}-\left(R+L\frac{\partial}{\partial t}\right)\left(C\frac{\partial}{\partial t}+G\right)v=0$$

or $$\frac{\partial^2 v}{\partial x^2}=LC\frac{\partial^2 v}{\partial t^2}+(LG+RC)\frac{\partial v}{\partial t}+RGv \qquad ...(5)$$

Again operating (3) by $\left(C\frac{\partial}{\partial t}+G\right)$ and (4) by $\frac{\partial}{\partial x}$ and subtracting

$$\left(C\frac{\partial}{\partial t}+G\right)\left(R+L\frac{\partial}{\partial t}\right)i-\frac{\partial^2 i}{\partial x^2}=0$$

or $$\frac{\partial^2 i}{\partial x^2}=LC\frac{\partial^2 i}{\partial t^2}+(LG+RC)\frac{\partial i}{\partial t}+RGi \qquad ...(6)$$

which is (5) with v replaced by i. The equations (5) and (6) are called the *telephone equations.*

Cor. 1. *If* $L = G = 0$, the equations (5) and (6) become

$$\frac{\partial^2 v}{\partial x^2}=RC\frac{\partial v}{\partial t} \quad ...(7) \qquad \frac{\partial^2 i}{\partial x^2}=RC\frac{\partial i}{\partial t} \quad ...(8)$$

which are known as the *telegraph equations.*

Rewriting (7) as $\frac{\partial v}{\partial t}=\frac{1}{RC}\frac{\partial^2 v}{\partial x^2}$, we observe that it is similar to the heat equation [(1) p. 611].

Cor. 2. *If* $R = G = 0$, the equations (5) and (6) become

$$\frac{\partial^2 v}{\partial x^2}=LC\frac{\partial^2 v}{\partial t^2} \quad ...(9) \qquad \frac{\partial^2 i}{\partial x^2}=LC\frac{\partial^2 i}{\partial t^2} \quad ...(10)$$

which are called the *radio equations.*

Rewriting (9) as $\frac{\partial^2 v}{\partial t^2}=k^2\frac{\partial^2 v}{\partial x^2}$ where $k^2=\frac{1}{LC}$,

its general solution is $v(x, t) = f_1(x + kt) + f_2(x - kt)$. [See (4) p. 609]

Similarly from (10), $i(x, t) = F_1(x + kt) + F_2(x - kt)$.

Thus the voltage $v(x, t)$ for the current $i(x, t)$ at any point along the lossless transmission line can be obtained by the superposition of a progressive wave and a receding wave travelling with equal velocities (k). This is the case of oscillations of $v(x, t)$ and $i(x, t)$ at high frequencies.

Cor. 3. *If* $L = C = 0$, *e.g.*, in the case of a submarine cable, then (5) gives

$$\frac{\partial^2 v}{\partial x^2}=GRv, \text{ i.e. } (D^2-GR)v=0$$

$\therefore$ $$v(x)=A\cosh(\sqrt{GR}\,.\,x)+B\sinh(\sqrt{GR}\,.\,x) \qquad ...(11)$$

Since by (1), $$Ri=-\frac{\partial v}{\partial x}=-\sqrt{GR}\,[A\sinh(\sqrt{GR}\,.\,x)+B\cosh(\sqrt{GR}\,.\,x)]$$

$\therefore$ $$i(x)=-\sqrt{G/R}\,[A\sinh(\sqrt{GR}\,.\,x)+B\cosh(\sqrt{GR}\,.\,x)] \qquad ...(12)$$

If $v(0) = v_0$ and $i(0) = i_0$, then $v_0 = A$ and $i_0 = -\sqrt{(G/R)}B$.

Hence writing $\sqrt{(GR)} = \gamma$ and $\sqrt{(R/G)} = z_0$, (11) and (12) give

$$v(x)=v_0\cosh\gamma x-i_0 z_0\sinh\gamma x \qquad ...(13)$$

and $$i(x)=i_0\cosh\gamma x-\frac{v_0}{z_0}\sinh\gamma x. \qquad ...(14)$$

Obs. Steady-state solutions. We have so far considered the transient state solutions only. The steady-state solutions of transmission lines are however, obtained by assuming $v = Ve^{i\omega t}$ and $i = Ie^{i\omega t}$, where V and I are complex functions of x only. Substituting these in (5) and (6), we get two ordinary linear differential equations of the second order which can be solved at once.

Example 18.23. *Neglecting R and G, find the e.m.f. v(x, t) in a line of length l, t seconds after the ends were suddenly grounded, given that* $i(x, 0) = i_0$ *and* $v(x, 0) = e_1 \sin \frac{\pi x}{l} + e_5 \sin \frac{5\pi x}{l}$. (S.V.T.U., 2008)

Solution. Since R and G are negligible, we use the *Radio equation* $\frac{\partial^2 v}{\partial x^2} = LC \frac{\partial^2 v}{\partial t^2}$ ...(*i*)

Since the ends are suddenly grounded, we have the boundary conditions

$$v(0, t) = 0, \; v(l, t) = 0 \quad ...(ii)$$

Also the initial conditions are $i(x, 0) = i_0$

and
$$v(x, 0) = e_1 \sin \pi x/l + e_5 \sin 5\pi x/l \quad ...(iii)$$

$$\therefore \quad \frac{\partial i}{\partial x} = -c \frac{\partial v}{\partial t} \text{ gives } \frac{\partial v}{\partial t}(x, 0) = 0 \quad ...(iv)$$

Let $v = X(x)T(t)$ be the solution of (*i*).

$\therefore$ (*i*) gives
$$X''T = LCXT''$$

$$\frac{X''}{X} = LC \frac{T''}{T} = -k^2 \text{ (say)}$$

$$\therefore \quad X'' + k^2 X = 0 \quad \text{and} \quad T'' + (k^2/LC)T = 0$$

Solving these equations, we get

$$v = (c_1 \cos kx + c_2 \sin kx)\left(c_3 \cos \frac{k}{\sqrt{LC}} . t + c_4 \sin \frac{k}{\sqrt{LC}} t \right)$$

Using the boundary conditions (*ii*), we get

$$c_1 = 0 \quad \text{and} \quad k = n\pi/l.$$

$$\therefore \quad v = \sin \frac{n\pi x}{l} \left(a_n \cos \frac{n\pi}{l\sqrt{LC}} t + b_n \sin \frac{n\pi}{l\sqrt{LC}} t \right)$$

Using the initial condition (*iv*), we get $b_n = 0$

$$\therefore \quad v = a_n \sin \frac{n\pi x}{l} \cos \frac{n\pi}{l\sqrt{LC}} t$$

Thus the most general solution of (*i*) is

$$v = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \cos \frac{n\pi t}{l\sqrt{LC}}$$

Finally by the initial condition (*iii*), we have

$$e_1 \sin \frac{\pi x}{l} + e_5 \sin \frac{5\pi x}{l} = \sum a_n \sin \frac{n\pi x}{l}$$

$$\therefore \quad a_1 = e_1 \quad \text{and} \quad a_5 = e_5$$ while all other a's are zero.

Hence
$$v = e_1 \sin \frac{\pi x}{l} \cos \frac{\pi t}{l\sqrt{LC}} + e_5 \sin \frac{5\pi x}{l} \cos \frac{5\pi t}{l\sqrt{LC}}$$

which is the required solution.

Example 18.24. *A telephone line 3000 km. long has a resistance of 4 ohms/km. and a capacitance of* 5×10^{-7} *farad/km. Initially both the ends are grounded so that the line is uncharged. At time* $t = 0$, *a constant e.m.f. E is applied to one end, while the other end is left grounded. Assuming the inductance and leakance to be negligible, show that the steady state current of the grounded end at the end of 1 sec. is 5.3%.*

Solution. Since $L = 0$, $G = 0$, we use the telegraph equation

$$\frac{\partial^2 v}{\partial x^2} = RC\frac{\partial v}{\partial t}$$

Let $v = X(x)T(t)$ be its solution so that

$$TX'' = RCXT' \quad \text{or} \quad \frac{X''}{X} = RC\frac{T'}{T} = -k^2 \text{ (say)}$$

$\therefore$ $$X'' + k^2X = 0 \text{ and } T' + (k^2/RC)T = 0$$

Solving these equations, we get

$$X = c_1 \cos kx + c_2 \sin kx,\ T = c_3 e^{-k^2t/RC}$$

giving $$v = (c_1 \cos kx + c_2 \sin kx)\, c_3 e^{-k^2t/RC} \quad ...(i)$$

When $t = 0$; $v = 0$ at $x = 0$ and $v = 0$ at $x = l$

$\therefore$ $$0 = c_1c_3 \ ; \ 0 = (c_1 \cos kl + c_2 \sin kl)c_3$$

i.e., $$c_1c_3 = 0 \text{ and } kl = n\pi \ (n \text{ an integer})$$

Putting these in (i) and adding a linear term, we have

$$v = a_0x + b_0 + \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\, e^{-n^2\pi^2t/RCl^2} \quad ...(ii)$$

The end conditions of the problem are

$$v = 0 \text{ at } x = 0 \text{ and } v = E \text{ at } x = l$$

Using these, (ii) gives $b_0 = 0$ and $a_0 = E/l$

Then (ii) becomes $$v = \frac{E}{l}x + \sum b_n \sin\frac{n\pi x}{l}\, e^{-n^2\pi^2t/RCl^2}$$

Also $v = 0$ when $t = 0$, we get $-Ex/l = \sum b_n \sin n\pi x/l$

This requires the expansion of $(-Ex/l)$ as a half-range sine series in $(0, l)$.

$\therefore$ $$b_n = \frac{2}{l}\int_0^l \left(\frac{-Ex}{l}\right)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left[\left(\frac{-Ex}{l}\right)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - \left(\frac{-E}{l}\right)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_0^l = \frac{2}{l}\left(\frac{El}{n\pi}\cos n\pi\right) = \frac{2E}{n\pi}(-1)^n.$$

Thus $$v = \frac{Ex}{l} + \frac{2E}{\pi}\sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\, e^{-n^2\pi^2t/RCl^2} \quad ...(iii)$$

Also when $L = 0$, $\dfrac{-\partial v}{\partial x} = Ri$

i.e., $$i = -\frac{1}{R}\frac{\partial v}{\partial x} = -\frac{E}{lR} - \frac{2E}{lR}\sum_{n=1}^{\infty}(-1)^n \cos\frac{n\pi x}{l}\, e^{-n^2\pi^2t/RCl^2}$$

At the grounded end $(x = 0)$, the current is

$$i = -\frac{E}{lR} - \frac{2E}{lR}\sum_{n=1}^{\infty}(-1)^n e^{-n^2\pi^2t/RCl^2}$$

When $t = 1$ sec, $$i = -\frac{E}{lR}\left(1 - 2e^{\pi^2/RCl^2} + 2e^{-4\pi^2/RCl^2} - ...\right) \quad ...(iv)$$

Since $$\frac{\pi^2}{RCl^2} = \frac{(3.14)^2}{4(5 \times 10^{-7})(3000)^2} = 0.548$$

$\therefore$ $$e^{-\pi^2/RCl^2} = e^{-0.548} = 0.578$$

When $t \to \infty$, $i \to -E/lR$

Hence from (*iv*), we have

$$i = -\frac{E}{lR}\{1 - 2(0.578) + 2(0.578)^4 - 2(0.578)^9 + ...\}$$
$$= -\frac{E}{lR}\{1 - 1.156 + 0.223 - 0.014 + ...\}$$
$$= i_\infty(0.053) = 5.3\% \text{ of } i_\infty.$$

Example 18.25. *A transmission line 1000 kilometers long is initially under steady-state conditions with potential 1300 volts at the sending end (x = 0) and 1200 volts at the receiving end (x = 1000). The terminal end of the line is suddenly grounded, but the potential at the source is kept at 1300 volts. Assuming the inductance and leakance to be negligible, find the potential v(x, t).* *(Andhra, 2000)*

Solution. The equation of the telegraph line is

$$\frac{\partial^2 v}{\partial x^2} = RC\frac{\partial v}{\partial t} \quad \text{or} \quad \frac{\partial v}{\partial t} = \frac{1}{RC}\frac{\partial^2 v}{\partial x^2} \qquad ...(i)$$

v_s = initial steady voltage satisfying $\frac{\partial^2 v}{\partial x^2} = 0 = 1300 - x/10 = v(x, 0)$ ...(*ii*)

v_s' = steady voltage (after grounding the terminal end) when steady conditions are ultimately reached = $1300 - 1.3x$

$\therefore$ $v(x, t) = v_s' + v_t(x, t)$ where $v_t(x, t)$ is the transient part

$$= 1300 - 1.3x + \sum_{n=1}^{\infty} b_n e^{-(n^2\pi^2 t)/(l^2 RC)} \sin\frac{n\pi x}{l} \qquad \text{[By (viii), p. 614]} \qquad ...(iii)$$

where l = 1000 kilometers.

Putting $t = 0$, we have from (*ii*) and (*iii*)

$$1300 - 0.1x = v(x, 0) = 1300 - 1.3x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

i.e.
$$1.2x = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \text{ where } b_n = \frac{2}{l}\int_0^1 1.2 \sin\frac{n\pi x}{l}\,dx = \frac{2400}{\pi}\cdot\frac{(-1)^{n+1}}{n}$$

Hence
$$v(x, t) = 1300 - 1.3x + \frac{2400}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} e^{-(n^2\pi^2 t)/(l^2 RC)} \sin\frac{n\pi x}{1000}.$$

PROBLEMS 18.7

1. Find the current i and voltage e in a line of length l, t seconds after the ends are suddenly grounded, given that $i(x, 0) = i_0$, $e(x, 0) = e_0 \sin(\pi x/l)$.
   Also R and G are negligible.
2. Show that a transmission line with negligible resistance and leakage propagates waves of current and potential with a velocity equal to $l/\sqrt{(LC)}$, where L is the self-inductance and C is the capacitance.
3. A steady voltage distribution of 20 volts at the sending end and 12 volts at the receiving end is maintained in a telephone wire of length l. At time $t = 0$, the receiving end is grounded. Find the voltage and current t sec later. Neglect leakance and inductance.
4. Obtain the solution of the *radio equation*
$$\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2}$$
   appropriate to the case when a periodic e.m.f. $V_0 \cos pt$ is applied at the end $x = 0$ of the line.

18.11 LAPLACE'S EQUATION IN THREE DIMENSIONS

We have seen that the three dimensional heat flow equation in steady state reduces to

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad ...(1)$$

which is the *Laplace's equation in three dimensions.*

Laplace's equation also arises in the study of gravitational potential at (x, y, z) of a particle of mass m situated at (ξ, η, ζ) given by

$$\frac{Gm}{r} \text{ where } r = \sqrt{[(x-\zeta)^2 + (y-\eta)^2 + (z-\zeta)^2]}$$

This function is called the *potential of the gravitational field* and satisfies the Laplace's equation.

If a mass of density ρ at (ξ, η, ζ) is distributed throughout a region R, then the gravitational potential u at an external point (x, y, z) is given by

$$u(x, y, z) = G \iiint_R \frac{\rho}{r}\, d\xi\, d\eta\, d\zeta \qquad ...(2)$$

Since $\nabla^2(1/r) = 0$ and ρ is independent of x, y and z, we get

$$\nabla^2 u = \iiint_R \rho\, \nabla^2(1/r)\, d\xi\, d\eta\, d\zeta = 0.$$

This shows that the gravitational potential defined by (2) also obeys Laplace's equation.

Thus Laplace's equation (1) is one of the most important equations arising in connection with numerous problems of physics and engineering. *The theory of its solutions is called the* **potential theory** *and its solutions are called the* **harmonic functions.**

In most of the problems leading to Laplace's equation, it is required to solve the equation subject to certain boundary conditions. A proper choice of coordinate system makes the solution of the problem simpler. Now we proceed to take up the solutions of (1) in its other forms.

18.12 SOLUTIONS OF THREE DIMENSIONAL LAPLACE'S EQUATION

(1) **Cartesian form of ∇^2u = 0** is $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$ ...(1)

Let $u = X(x)Y(y)Z(z)$ ...(2)

be a solution of (1). Substituting (2) in (1) and dividing by XYZ, we obtain

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\cdot\frac{d^2Z}{dz^2} = 0 \qquad ...(3)$$

which is of the form $F_1(x) + F_2(y) + F_3(z) = 0$.

As x, y, z are independent, this will hold good only if F_1, F_2, F_3 are constants. Assuming these constants to be $k^2, l^2, -(k^2 + l^2)$ respectively, (3) leads to the equations

$$\frac{d^2X}{dx^2} - k^2X = 0,\ \frac{d^2Y}{dy^2} - l^2Y = 0,\ \frac{d^2Z}{dz^2} + (k^2 + l^2)Z = 0$$

Their solutions are $X = c_1e^{kx} + c_2e^{-kx},\ Y = c_3e^{ly} + c_4e^{-ly}$

$$Z = c_5 \cos \sqrt{(k^2 + l^2)}z + c_6 \sin \sqrt{(k^2 + l^2)}z$$

Thus a possible solution of (1) is

$$u = (c_1e^{kx} + c_2e^{-kx})(c_3e^{ly} + c_4e^{-ly})\{c_5 \cos \sqrt{(k^2 + l^2)}z + c_6 \sin \sqrt{(k^2 + l^2)}z\}.$$

Since the three constants could have been taken as $-k^2, -l^2$ and $k^2 + l^2$, an alternative solution of (1) will be

$$u = (c_1 \cos kx + c_2 \sin kx)(c_3 \cos ly + c_4 \sin ly)\{c_5e^{\sqrt{(k^2+l^2)}z} + c_6e^{-\sqrt{(k^2+l^2)}z}\}.$$

(2) **Cylindrical form of ∇^2u = 0** is $\dfrac{\partial^2 u}{\partial \rho^2} + \dfrac{1}{\rho}\dfrac{\partial u}{\partial \rho} + \dfrac{1}{\rho^2}\dfrac{\partial^2 u}{\partial \phi^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$ ...(1)

Let $$u = R(\rho)\, H(\phi)\, Z(z)$$ [(iv) p. 359]

be a solution of (1). Substituting it in (1), and dividing by RHZ, we get

$$\frac{1}{R}\left(\frac{d^2R}{d\rho^2}+\frac{1}{\rho}\frac{dR}{d\rho}\right)+\frac{1}{\rho^2 H}\frac{d^2H}{d\phi^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}=0 \qquad ...(2)$$

Assuming that $\dfrac{d^2H}{d\phi^2} = -n^2H$ and $\dfrac{d^2Z}{dz^2} = k^2Z$, ..(3)

(2) reduces to $$\frac{1}{R}\left(\frac{d^2R}{d\rho^2}+\frac{1}{\rho}\frac{dR}{d\rho}\right)-\frac{n^2}{\rho^2}+k^2=0$$

or $$\rho^2\frac{d^2R}{d\rho^2}+\rho\frac{dR}{d\rho}+(k^2\rho^2-n^2)R=0.$$

This is Bessel's equation [§ 16.10 (1)] and its solution is $R = c_1J_n(k\rho) + c_2Y_n(k\rho)$.

Also the solutions of equations (3) are

$$H = c_3 \cos n\phi + c_4 \sin n\phi,\ Z = c_5e^{kz} + c_6e^{-kz}$$

Thus a solution of (1) is

$$u = [c_1J_n(k\rho) + c_2Y_n(k\rho)]\,[c_3 \cos n\phi + c_4 \sin n\phi]\,[c_5 e^{kz} + c_6e^{-kz}]$$

which is known as a *cylindrical harmonic.* (*Assam, 1999*)

(3) Spherical form of $\nabla^2 u = 0$ is

$$\frac{\partial^2u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2u}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2u}{\partial\phi^2}=0 \qquad ...(1) \quad [(iv)\ p.\ 361]$$

Let $u = R(r)\, G(\theta)\, H(\phi)$ be a solution of (1).

Then $$\frac{1}{R}\left(r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}\right)+\frac{1}{G}\left(\frac{d^2G}{d\theta^2}+\cot\theta\frac{dG}{d\theta}\right)+\frac{1}{H\sin^2\theta}\frac{d^2H}{d\phi^2}=0$$

Putting $$\frac{1}{R}\left(r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}\right)=n(n+1) \quad ...(2) \qquad \text{and} \quad \frac{1}{H}\frac{d^2H}{d\phi^2}=-m^2, \quad ...(3)$$

the above equation takes the form

$$\frac{d^2G}{d\theta^2}+\cot\theta\frac{dG}{d\theta}+[n(n+1)-m^2\operatorname{cosec}^2\theta]G=0 \qquad ...(4)$$

Now differentiating the *Legendre's equation* (§ 16.13)

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0,$$

m times with respect to x and writing $u = d^my/dx^m$, we get

$$(1-x^2)u'' - 2(m+1)xu' + (n-m)(n+m+1)u = 0 \qquad ...(5)$$

Now putting $G = (1-x^2)^{m/2}\, u$ in (5), we get

$$(1-x^2)\frac{d^2G}{dx^2}-2x\frac{dG}{dx}+\left[n(n+1)-\frac{m^2}{1-x^2}\right]G=0 \qquad ...(6)$$

Now putting $x = \cos\theta$ in (6), it reduces to (4) and its solution is

$$G = c_1P_n^m(\cos\theta) + c_2Q_n^m(\cos\theta)$$

The solution of (3) is $H = c_3 \cos m\phi + c_4 \sin m\phi$

To solve (2), write $R = r^k$, so that $k(k-1) + 2k = n(n+1)$ which gives $k = n$ or $-(n+1)$

Thus $$R = c_5r^n + c_6r^{-n-1}$$

Hence the general solution of (1) is

$$u = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\{c_1P_n^m(\cos\theta) + c_2Q_n^m(\cos\theta)\}(c_3\cos m\phi + c_4\sin m\phi)\times(c_5r^n + c_6r^{-n-1})$$

Any solution of (1) is known as a *spherical harmonic.*

Example 18.26. *Find the potential in the interior of a sphere of unit radius when the potential on the surface is $f(\theta) = \cos^2 \theta$.*

Solution. Take the origin at the centre of the given sphere S. Since the potential is independent of ϕ on S, so also is the potential at any point. Therefore, the Laplace's equation in spherical co-ordinates reduces to

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} = 0 \qquad ...(i)$$

Putting $u(r, \theta) = R(r)\, G(\theta)$ in (i) and proceeding as in § 18.12 (3), we obtain the equations

$$\frac{\partial^2 G}{d\theta^2} + \cot\theta \frac{dG}{d\theta} + n(n+1)G = 0 \qquad ...(ii)$$

and

$$\frac{1}{R}\left(r^2 \frac{d^2 R}{dr^2} + 2r\frac{dR}{dr}\right) = n(n+1) \qquad ...(iii)$$

Putting $\cot\theta = v$, (ii) takes the form

$$(1 - v^2)\frac{d^2 G}{dv^2} - 2v\frac{dG}{dv} + n(n+1)G = 0$$

which is Legendre's equation. Its solutions are

$$G = P_n(v) = P_n(\cos\theta) \text{ for } n = 0, 1, 2, ...$$

The solutions of (iii) are $R_n(r) = r^n$, $\overline{R_n}\,(r) = 1/r^{n+1}$.

Hence the equation (i) has the following two sets of solutions

$$u_n(r, \theta) = c_n r^n P_n\,(\cos\theta) \text{ and } \bar{u}_n\,(r, \theta) = c_n P_n(\cos\theta)/r^{n+1}, \text{ where } n = 0, 1, 2, ...$$

For points inside S, we have the general equation $u(r, \theta) = \sum_{n=0}^{\infty} c_n r^n P_n\,(\cos\theta)$ ...(iv)

On the boundary of S, $u(1, \theta) = f(\theta)$ $\therefore$ $f(\theta) = \sum_{n=0}^{\infty} c_n P_n\,(\cos\theta)$

which is Fourier-Legendre expansion of $f(\theta)$. Hence by (5) p. 560, we have

$$c_n = \left(n + \frac{1}{2}\right)\int_{-1}^{1} f(\theta)\, P_n(x)\, dx \text{ where } x = \cos\theta.$$

$$= \left(n + \frac{1}{2}\right)\int_{-1}^{1} x^2 P_n(x) dx \qquad [\because\ f(\theta) = \cos^2\theta]$$

$$= \left(n + \frac{1}{2}\right)\int_{-1}^{1}\left[\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)\right]P_n(x)\, dx \qquad [\because\ P_2(x) = \tfrac{1}{2}(3x^2 - 1)]$$

Using the orthogonality of Legendre polynomials, we get

$c_n = 0$, except for $n = 0, 2$. Hence

$$c_0 = \frac{1}{2}\cdot\frac{1}{3}\int_{-1}^{1} P_0^2(x)\, dx = \frac{1}{3},\ c_2 = \frac{5}{2}\cdot\frac{2}{3}\int_{-1}^{0} P_2^2(x)\, dx = \frac{2}{3}.$$

Substituting in (iv), we get $u(r, \theta) = \frac{1}{3} + \frac{2}{3}r^2 P_2\,(\cos\theta)$ or $u(r, \theta) = \frac{1}{3} + r^2(\cos^2\theta - \frac{1}{3})$.

PROBLEMS 18.8

1. Show that a solution of Laplace's equation in cylindrical co-ordinates, which remains finite at $r = 0$, may be expressed in the form

$$u = \sum_{n=0}^{\infty} J_n(kr)\,\{e^{kz}(A_n \cos n\theta + B_n \sin n\theta) + e^{-kz}(C_n \cos n\theta + D_n \sin n\theta)\}.$$

2. The potential on the surface of a unit sphere is $f(\theta) = \cos 2\theta$. Show that the potential at all points of space is given by

$$u(r, \theta) = 2r^2(\cos^2\theta - 1/3) - \frac{1}{3} \text{ for } r < 1,$$

and $$u(r, \theta) = 2r^{-3}(\cos^2\theta - 1/3) - r^{-1}/3 \text{ for } r > 1.$$

3. Show that in spherical polar coordinates (r, θ, ϕ), Laplace's equation possesses solutions of the form

$$(Ar^n + B/r^{n+1})P_n(\mu)e^{\pm im\phi},$$

where $\mu = \cos\theta$, A, B, m, n are constants and $P_n(\mu)$ satisfies Legendre's equation

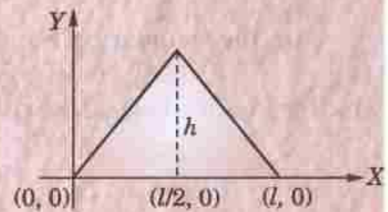
$$(1-\mu^2)\frac{d^2P_n}{d\mu^2} - 2\mu\frac{dP_n}{d\mu} + \left\{n(n+1) - \frac{m^2}{1-\mu^2}\right\}P_n = 0.$$

18.13 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 18.9

Fill up the blanks in each of the following questions :

1. The radio equations for the potential and current are ....
2. The partial differential equation representing variable heat flow in three dimensions, is ....
3. Temperature gradient is defined as ....
4. The differential equation $f_{xx} + 2f_{xy} + 4f_{yy} = 0$ is classified as ....
5. The partial differential equation of the transverse vibrations of a string is ....
6. The steady state temperature of a rod of length l whose ends are kept at 30° and 40° is ....
7. The equation $u_t = c^2u_{xx}$ is classified as ....
8. The two dimensional steady state heat flow equation in polar coordinates is ....
9. The mathematical function of the initial form of the string given by the following graph is ....



10. When a vibrating string fastened to two points l apart, has an initial velocity u_0, its initial conditions are ....
11. In two dimensional heat flow, the temperature along the normal to the xy-plane is ....
12. If a square plate has its faces and the edge $y = 0$ insulated, its edges $x = 0$ and $x = a$ are kept at zero temperature and the fourth edge is kept at temperature u, then the boundary conditions for this problem are ....
13. If the ends $x = 0$ and $x = l$ are insulated in one dimensional heat flow problems, then the boundary conditions are ....
14. D'Alembert's solution of the wave equation is ....
15. The partial differential equation of 2-dimensional heat flow in ....
16. A rod 50 cm long with insulated sides has its end A and B kept at 20° and 70°C respectively. The steady state temperature distribution of the rod is .... *(Anna, 2008)*
17. The three possible solutions of Laplace equation in polar coordinates are ....
18. Solution of $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$, given $u(0, y) = 8e^{-3y}$, is ....
19. Solution of $\frac{\partial z}{\partial x} + 4z = \frac{\partial z}{\partial t}$, given $z(x, 0) = 4e^{-3x}$, is ....
20. In the equation $\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2}, \alpha^2$ represents ....
21. The telegraph equations for potential and current are ......
22. The general solution of one-dimensional heat flow equation when both ends of the bar are kept at zero temperature, is of the form ....
23. The three possible solutions of Laplace equation $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ are ....