CHAPTER 19

Complex Numbers and Functions

1. Complex Numbers. **2.** Argand's diagram. **3.** Geometric representation of $z_1 \pm z_2$; $z_1 z_2$ and z_1/z_2. **4.** De Moivre's theorem. **5.** Roots of a complex number. **6.** To expand sin $n\theta$, cos $n\theta$ and tan $n\theta$ in powers of sin θ, cos θ and tan θ respectively; Addition formulae for any number of angles; To expand $\sin^m \theta$, $\cos^n \theta$ and $\sin^m \theta \cos^n \theta$ in a series of sines or cosines of multiples of θ. **7.** Complex function: Definition. **8.** Exponential function of a complex variable. **9.** Circular functions of a complex variable. **10.** Hyperbolic functions. **11.** Inverse hyperbolic functions. **12.** Real and imaginary parts of circular and hyperbolic functions. **13.** Logarithmic functions of a complex variable. **14.** Summation of series – '$C + iS$' method. **15.** Approximations and Limits. **16.** Objective Type of Questions.

19.1 COMPLEX NUMBERS

Definition. *A number of the form $x + iy$, where x and y are real numbers and $i = \sqrt{(-1)}$, is called a* **complex number.**

x is called the *real part* of $x + iy$ and is written as $R(x + iy)$ and y is called the *imaginary part* and is written as $I(x + iy)$.

A pair of complex numbers $x + iy$ and $x - iy$ are said to be conjugate of each other.

Properties : (1) *If $x_1 + iy_1 = x_2 + iy_2$, then $x_1 - iy_1 = x_2 - iy_2$.*

(2) *Two complex numbers $x_1 + iy_1$ = and $x_2 + iy_2$ are said to be equal when*

$$R(x_1 + iy_1) = R(x_2 + iy_2), \text{ i.e., } x_1 = x_2$$

and
$$I(x_1 + iy_1) = I(x_2 + iy_2), \text{ i.e., } y_1 = y_2.$$

(3) *Sum, difference, product and quotient of any two complex numbers is itself a complex number.*

If $x_1 + iy_1$ and $x_2 + iy_2$ be two given complex numbers, then

(*i*) their sum $= (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$

(*ii*) their difference $= (x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2)$

(*i*) their product $= (x_1 + iy_1) + (x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)$

and (*iv*) their quotient
$$= \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\,\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}$$

(4) *Every complex number $x + iy$ can always be expressed in the form r (cos θ + i sin θ).*

Put
$$R(x + iy), \text{ i.e., } x = r\cos\theta \qquad ...(i)$$

and
$$I(x + iy), \text{ i.e., } y = r\sin\theta \qquad ...(ii)$$

Squaring and adding, we get $x^2 + y^2 = r^2$ *i.e.* $r = \sqrt{(x^2 + y^2)}$ (taking positive square root only)

Dividing (*ii*) by (*i*), we get $y/x = \tan\theta$ *i.e.* $\theta = \tan^{-1}(y/x)$.

Thus $x + iy = r(\cos\theta + i\sin\theta)$ where $r = \sqrt{(x^2 + y^2)}$ and $\theta = \tan^{-1}(y/x)$.

Definitions. *The number $r = +\sqrt{(x^2 + y^2)}$ is called the* **modulus** *of $x + iy$ and is written as mod $(x + iy)$ or $|x + iy|$.*

The angle θ is called the amplitude or argument of $x + iy$ and is written as amp $(x + iy)$ or arg $(x + iy)$.

Evidently the amplitude θ has an infinite number of values. The value of θ which lies between $-\pi$ and π is called the *principal value of the amplitude.* Unless otherwise specified, we shall take amp (z) to mean the principal value.

Note. *$\cos\theta + i\sin\theta$ is briefly written as cis θ (pronounced as 'sis θ')*

(5) *If the conjugate of $z = x + iy$ be $\bar{z}$, then*

(*i*) $R(z) = \dfrac{1}{2}(z + \bar{z}),\ I(z) = \dfrac{1}{2i}(z - \bar{z})$ (*ii*) $|z| = \sqrt{R^2(z) + I^2(z)} = |\bar{z}|$

(*iii*) $z\bar{z} = |z|^2$ (*iv*) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(*v*) $\overline{z_1 z_2} = \bar{z}_1 . \bar{z}_2$ (*vi*) $\overline{(z_1 / z_2)} = \bar{z}_1 / \bar{z}_2$, where $\bar{z}_2 \neq 0$.

Example 19.1. *Reduce $1 - \cos\alpha + i\sin\alpha$ to the modulus amplitude form.*

Solution. Put $1 - \cos\alpha = r\cos\theta$ and $\sin\alpha = r\sin\theta$

$\therefore$ $$r = (1 - \cos\alpha)^2 + \sin^2\alpha = 2 - 2\cos\alpha = 4\sin^2\alpha/2$$

i.e., $$r = 2\sin\alpha/2$$

and $$\tan\theta = \frac{\sin\alpha}{1-\cos\alpha} = \frac{2\sin\alpha/2\cos\alpha/2}{2\sin^2\alpha/2} = \cot\alpha/2$$

$$= \tan\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) \quad \therefore\ \theta = \frac{\pi - \alpha}{2}.$$

Thus $$1 - \cos\alpha + i\sin\alpha = 2\sin\frac{\alpha}{2}\left[\cos\frac{\pi-\alpha}{2} + i\sin\frac{\pi-\alpha}{2}\right].$$

Example 19.2. *Find the complex number z if arg $(z + 1) = \pi/6$ and arg $(z - 1) = 2\pi/3$.* *(Mumbai, 2009)*

Solution. Let $z = x + iy$ so that $z + 1 = (x + 1) + iy$ and $(z - 1) = (x - 1) + iy$

Since $\arg(z + 1) = \pi/6$, $\therefore\ \tan^{-1}\left(\dfrac{y}{x+1}\right) = 30°$

i.e., $$\frac{y}{x+1} = \tan 30° = 1/\sqrt{3}, \text{ or } \sqrt{3}y = x + 1 \quad ...(i)$$

Also since $\arg(z - 1) = 2\pi/3$, $\therefore\ \tan^{-1}\left(\dfrac{y}{x+1}\right) = 120°$

i.e., $$\frac{y}{x-1} = \tan 120° = -\sqrt{3}, \text{ or } y = -\sqrt{3}x + \sqrt{3} \text{ or } \sqrt{3}y = -3x + 3 \quad ...(ii)$$

Subtracting (*ii*) from (*i*), we get $4x - 2 = 0$ *i.e.,* $x = 1/2$

From (*i*), $\sqrt{3}\,y = 1/2 + 1$, *i.e.,* $y = \sqrt{3}/2$

Hence $z = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$.

Example 19.3. *Find the real values of x, y so that $-3 + ix^2y$ and $x^2 + y + 4i$ may represent complex conjugate numbers.*

Solution. If $z = -3 + ix^2y$, then $\bar{z} = x^2 + y + 4i$

so that $$z = (x^2 + y) - 4i$$

$\therefore$ $$-3 + ix^2y = x^2 + y - 4i$$

Equating real and imaginary parts from both sides, we get

$$-3 = x^2 + y, x^2y = -4$$

Eliminating x, $(y + 3)y = -4$

or $y^2 + 3y - 4 = 0$ *i.e.*, $y = 1$ or -4

When $y = 1$, $x^2 = -3 - 1$ or $x = +2i$ which is not feasible

When $y = -4$, $x^2 = 1$ or $x = \pm 1$

Hence $x = 1$, $y - 4$ or $x = -1, y = -4$.

19.2 (1) GEOMETRIC REPRESENTATION OF IMAGINARY NUMBERS

Let all the real numbers be represented along $X'OX$, the positive real numbers being along OX and negative ones along OX'. Let OA be equal to one unit of measurement (Fig. 19.1).

Take a point L on OX such that $OL = x\,(OA)$.

Then L on OX represents the positive real number x and $i.ix = i^2x = -x$ is represented by a point L' on OX' distant OL from O.

From this we infer that the multiplication of the real number x by i twice amounts to the rotation of OL through two right angles to the position OL'.

Thus it naturally follows that the multiplication of a real number by i is equivalent to the rotation of OL through one right angle to the position OL''.

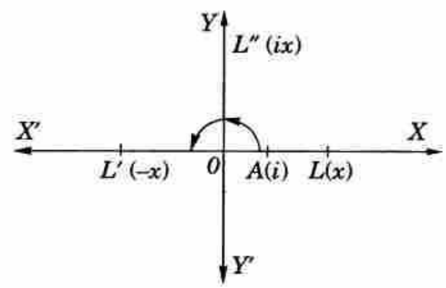

Fig. 19.1

Hence, if Y′OY be a line perpendicular to the real axis X′OX, then all imaginary numbers are represented by points on Y′OY, called the **imaginary axis**, *the positive ones along OY and negative ones along OY′.**

Obs. Geometric interpretation of i*. From the above, it is clear that i is an operation which when multiplied to any real number makes it imaginary and rotates its direction through a right angle on the complex plane.

(2) Geometric representation of complex numbers†

Consider two lines $X'OX$, $Y'OY$ at right angles to each other.

Let all the real numbers be represented by points on the line $X'OX$ (called the *real axis*), positive real numbers being along OX and negative ones along OX'. Let the point L on OX represent the real number x (Fig. 19.2).

Since the multiplication of a real number by i is equivalent to the rotation of its direction through a right angle. Therefore, let all the imaginary numbers be represented by points on the line $Y'OY$ (called the *imaginary axis*), the positive ones along OY and negative ones along OY'. Let the point M on OY represent the imaginary number iy.

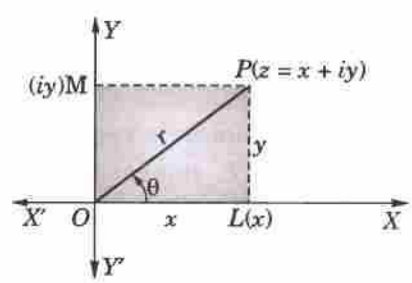

Fig. 19.2

Complete the rectangle $OLPM$. *Then the point whose cartesian coordinates are (x, y) uniquely represents the complex number z = x + iy on the complex plane z.* The diagram in which this representation is carried out is called the **Argand's diagram**.

If (r, θ) be the polar coordinates of P, then r is the modulus of z and θ is its amplitude.

Obs. Since a complex number has magnitude and direction, therefore, it can be represented like a vector. *Hereafter we shall often refer to the complex number $z = x + iy$ as*

(i) the point z whose co-ordinates are (x, y) or (ii) the vector z from O to $P(x, y)$.

Example 19.4. *The centre of a regular hexagon is at the origin and one vertex is given by $\sqrt{3} + i$ on the Argand diagram. Determine the other vertices.*

* The first mathematician to propose a geometric representation of imaginary number i was *Kuhn of Denzig* (1750–51).

† The geometric representation of complex numbers came into mathematics through the memoire of *Jean Robert Argand*, Paris 1806.

Solution. Let $\vec{OA} = \sqrt{3} + i$ so that

$OA = 2$ and $\angle XOA = \tan^{-1} 1/\sqrt{3} = 30°$. (Fig. 19.3)

Being a regular hexagon, $OB = OC = 2$

$\angle XOB = 30° + 60° = 90°$

and $\angle XOC = 30° + 120° = 150°$

$\therefore$ $\vec{OB} = 2(\cos 90° + i \sin 90°) = 2i$

$\vec{OC} = 2(\cos 150° + i \sin 150°) = -\sqrt{3} + i$

Since $\vec{AD}, \vec{BE}, \vec{CF}$ are bisected at O,

$\therefore$ $\vec{OD} = -\vec{OA} = -\sqrt{3} - i$

$\vec{OE} = -\vec{OB} = -2i$ and $\vec{OF} = -\vec{OC} = \sqrt{3} - i$.

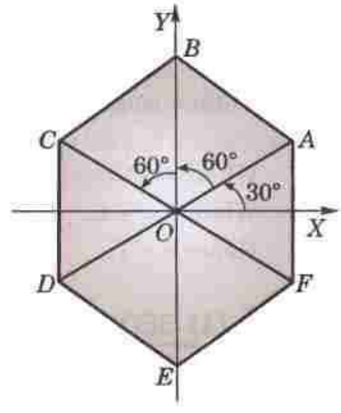

Fig. 19.3

19.3 (1) GEOMETRIC REPRESENTATION OF $z_1 + z_2$

Let P_1, P_2 represent the complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. (Fig. 19.4)

Complete the parallelogram OP_1PP_2. Draw P_1L, P_2M and PN $\perp$ s to OX. Also draw $P_1K \perp PN$.

Since $ON = OL + LN = OL + OM = x_1 + x_2$ $[\because LN = P_1K = OM]$

and $NP = NK + KP = LP_1 + MP_2 = y_1 + y_2$.

The coordinates of P are $(x_1 + x_2, y_1 + y_2)$ and it represents the complex number

$$z = x_1 + x_2 + i(y_1 + y_2) = (x_1 + iy_1) + (x_2 + iy_2) = z_1 + z_2.$$

Thus the point P which is the extremity of the diagonal of the parallelogram having OP_1 and OP_2 as adjacent sides, represents the sum of the complex numbers $P_1(z_1)$ and $P_2(z_2)$ such that

$$|z_1 + z_2| = OP \text{ and amp } (z_1 + z_2) = \angle XOP.$$

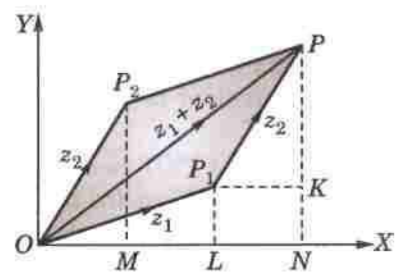

Fig. 19.4

Obs. Vectorially, we have $\vec{OP_1} + \vec{P_1P} = \vec{OP}$.

(2) Geometric representation of $z_1 - z_2$

Let P_1, P_2 represent the complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ (Fig. 19.5). Then the subtraction of z_2 from z_1 may be taken as addition of z_1 to $-z_2$.

Produce P_2O backwards to R such that $OR = OP_2$. Then the coordinates of R are evidently $(-x_2, -y_2)$ and so it corresponds to the complex number $-x_2 - iy_2 = -z_2$.

Complete the parallelogram $ORQP_1$, then the sum of z_1 and $(-z_2)$ is represented by OQ *i.e.*, $z_1 - z_2 = \vec{OQ} = \vec{P_2P_1}$.

Hence the complex number $z_1 - z_2$ is represented by the vector P_2P_1.

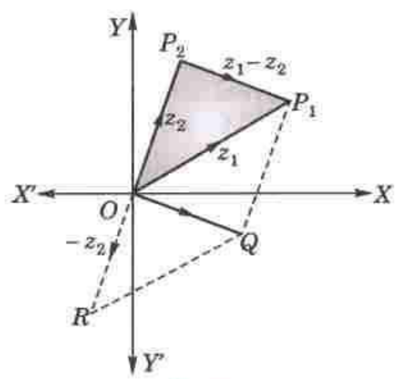

Fig. 19.5

Obs. By means of the relation $\vec{P_2P_1} = \vec{OP_1} - \vec{OP_2}$, any vector $\vec{P_2P_1}$ may be referred to the origin.

Example 19.5. *Find the locus of $P(z)$ when*
(i) $|z - a| = k$;
(ii) *amp* $(z - a) = \alpha$, where k and α are constants. *(Gorakhpur, 1999)*

Solution. Let a, z be represented by A and P in the complex plane, O being the origin (Fig. 19.6).

Then $z - a = \vec{OP} - \vec{OA} = \vec{AP}$

(i) $|z - a| = k$ means that $AP = k$.

Thus the locus of $P(z)$ is a circle whose centre is $A(a)$ and radius k.

(*ii*) amp $(z - a)$, *i.e.*, amp $(\vec{AP}) = \alpha$, means that AP always makes a constant angle with the X-axis.

Thus the locus of $P(z)$ is a straight line through $A(a)$ making an $\angle\alpha$ with OX.

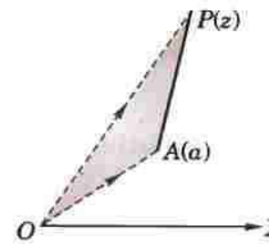

Fig. 19.6

Example 19.6. *Determine the region in the z-plane represented by*
(*i*) $1 < |z + 2i| \le 3$ (*ii*) $R(z) > 3$ (*iii*) $\pi/6 \le \text{amp}\,(z) \le \pi/3$.

Solution. (*i*) $|z + 2i| = 1$ is a circle with centre $(-2i)$ and radius 1 and $|z + 2i| = 3$ is a circle with the same centre and radius 3.

Hence $1 < |z + 2i| \le 3$ represents the region outside the circle $|z + 2i| = 1$ and inside (including circumference of) the circle $|z + 2i| = 3$ [Fig. 19.7].

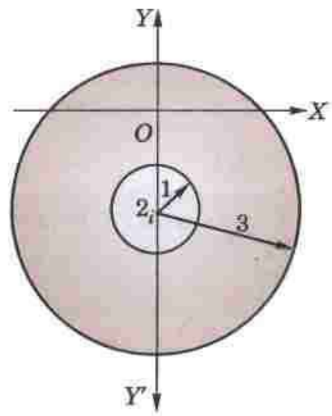

Fig. 19.7

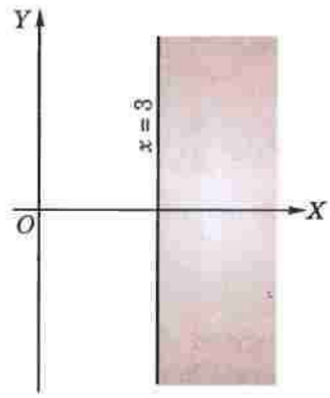

Fig. 19.8

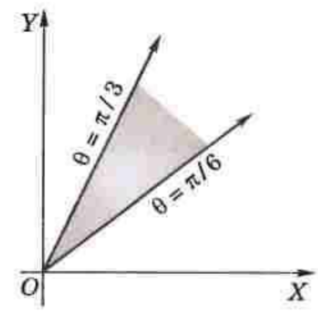

Fig. 19.9

(*ii*) $R(z) > 3$, defines all points (z) whose real part is greater than 3. Hence it represents the region of the complex plane to the right of the line $x = 3$ [Fig. 19.8].

(*iii*) If $z = r(\cos\theta + i\sin\theta)$, then amp $(z) = \theta$.

$\therefore$ $\pi/6 \le \text{amp}\,(z) \le \pi/3$ defines the region bounded by and including the lines $\theta = \pi/6$ and $\theta = \pi/3$. [Fig. 19.9].

Example 19.7. *If z_1, z_2 be any two complex numbers, prove that*
(*i*) $|z_1 + z_2| \le |z_1| + |z_2|$ [*i.e., the modulus of the sum of two complex numbers is less than or at the most equal to the sum of their moduli*].
(*ii*) $|z_1 - z_2| \ge |z_1| - |z_2|$ [*i.e., the modulus of the difference of two complex numbers is greater than or at the most equal to the difference of their moduli*].

Solution. Let P_1, P_2 represent the complex numbers z_1, z_2 (Fig. 19.10). Complete the parallelogram OP_1PP_2, so that

$$|z_1| = OP_1,\ |z_2| = OP_2 = P_1P,$$

and
$$|z_1 + z_2| = OP.$$

Now from ΔOP_1P, $OP \le OP_1 + P_1P$, the sign of equality corresponding to the case when O, P_1, P are collinear.

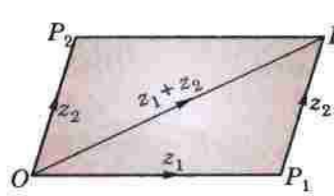

Fig. 19.10

Hence $\qquad |z_1 + z_2| \le |z_1| + |z_2| \qquad$ ...(*i*)

Again $\qquad |z_1| = |(z_1 - z_2) + z_2| \le |z_1 - z_2| + |z_2| \qquad$ [By (*i*)]

Thus $\qquad |z_1 - z_2| \ge |z_1| - |z_2| \qquad$ ...(*ii*)

Obs. $|z_1 + z_2 + z_3| \le |z_1| + |z_2| + |z_3|$.
In general, $|z_1 + z_2 + \ldots + z_n| \le |z_1| + |z_2| + \ldots + |z_n|$.

Example 19.8. *If $|z_1 + z_2| = |z_1 - z_2|$, prove that the difference of amplitudes of z_1 and z_2 is $\pi/2$.* (*Mumbai, 2007*)

Solution. Let $z_1 + z_2 = r(\cos\theta + i\sin\theta)$ and $z_1 - z_2 = r(\cos\phi + i\sin\phi)$

Then $$2z_1 = r[(\cos\theta + \cos\phi) + i(\sin\theta + \sin\phi)]$$
$$= r\left\{2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2} + 2i\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}\right\}$$

or $$z_1 = r\cos\frac{\theta-\phi}{2}\left(\cos\frac{\theta+\phi}{2} + i\sin\frac{\theta+\phi}{2}\right) \text{ i.e., amp }(z_1) = \frac{\theta+\phi}{2} \quad ...(i)$$

Also $$2z_2 = r(\cos\theta - \cos\phi) + i(\sin\theta - \sin\phi)$$
$$= 2r\sin\frac{\theta-\phi}{2}\left(-\sin\frac{\theta+\phi}{2} + i\cos\frac{\theta+\phi}{2}\right)$$

or $$z_2 = r\sin\frac{\theta-\phi}{2}\left\{\cos\left(\frac{\pi}{2} + \frac{\theta+\phi}{2}\right) + i\sin\left(\frac{\pi}{2} + \frac{\theta+\phi}{2}\right)\right\}$$

i.e., $$\text{amp }(z_2) = \frac{\pi}{2} + \frac{\theta+\phi}{2} \quad ...(ii)$$

Hence [(*ii*) – (*i*)], gives amp (z_2) – amp $(z_1) = \dfrac{\pi}{2}$.

Example 19.9. *Show that the equation of the ellipse having foci at z_1, z_2 and major axis $2a$, is $|z - z_1| + |z - z_2| = 2a$.*
Also find its eccentricity.

Solution. Let $P(z)$ be any point on the given ellipse (Fig. 19.11) having foci at $S(z_1)$ and $S'(z_2)$ so that $SP = |z - z_1|$ and $S'P = |z - z_2|$.

We know that $SP + S'P = AA' (= 2a)$

i.e., $$|z - z_1| + |z - z_2| = 2a$$

which is the desired equation of the ellipse.

Also we know that $SS' = 2ae$, e being the eccentricity.

or $$\left|\vec{OS'} - \vec{OS}\right| = 2ae \quad \text{or} \quad |z_2 - z_1| = 2ae$$

or $$|z_1 - z_2| = 2ae \text{ whence } e = |z_1 - z_2|/2a.$$

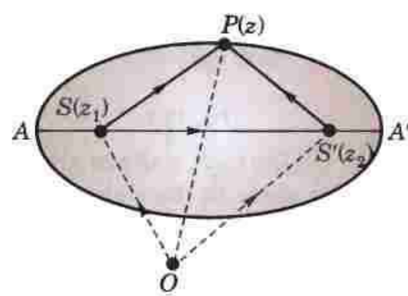

Fig. 19.11

(3) Geometric Representation of z_1z_2. Let P_1, P_2 represent the complex numbers

$$z_1 = x_1 + iy_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$
and $$z_2 = x_2 + iy_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

Measure off $OA = 1$ along OX (Fig. 19.12). Construct ΔOP_2P on OP_2 directly similar to ΔOAP_1,

so that $$OP/OP_1 = OP_2/OA \text{ i.e., } OP = OP_1 \,.\, OP_2 = r_1r_2$$
and $$\angle AOP = \angle AOP_2 + \angle P_2OP = \angle AOP_2 + \angle AOP_1 = \theta_2 + \theta_1$$

$\therefore$ P represents the number
$$r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)].$$

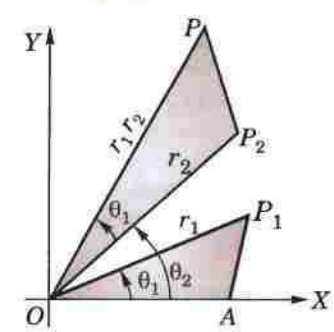

Fig. 19.12

Hence *the product of two complex numbers z_1, z_2 is represented by the point P, such that* (*i*) $|z_1z_2| = |z_1| \cdot |z_2|$.

(*ii*) **amp (z_1z_2) = amp (z_1) + amp (z_2).**

Cor. The effect of multiplication of any complex number z by $\cos\theta + i\sin\theta$ is to rotate its direction through an angle θ, for the modulus of $\cos\theta + i\sin\theta$ is unity.

(4) Geometric representation of z_1/z_2.

Let P_1, P_2 represent the complex numbers

$$z_1 = x_1 + iy_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$

and $$z_2 = x_2 + iy_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

Measure off $OA = 1$, construct triangle OAP on OA directly similar to the triangle OP_2P_1 (Fig. 19.13), so that

$$\frac{OP}{OA} = \frac{OP_1}{OP_2} \quad i.e., \quad OP = \frac{OP_1}{OP_2} = \frac{r_1}{r_2}$$

and $$\angle XOP = \angle P_2OP_1 = \angle AOP_1 - \angle AOP_2 = \theta_1 - \theta_2.$$

$\therefore$ P represents the number

$$(r_1/r_2)\,[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)].$$

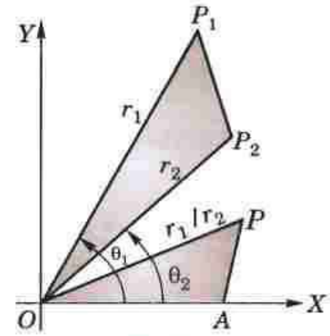

Fig. 19.13

Hence the complex number z_1/z_2 is represented by the point P, such that

(*i*) $|\mathbf{z_1/z_2}| = |\mathbf{z_1}|/|\mathbf{z_2}|$

(*ii*) **amp** $(\mathbf{z_1/z_2})$ **= amp** $(\mathbf{z_1})$ **– amp** $(\mathbf{z_2})$.

Note. *If $P_1(z_1)$, $P_2(z_2)$ and $P_3(z_3)$ be any three points, then*

$$\text{amp}\left(\frac{z_3 - z_2}{z_1 - z_2}\right) = \angle P_1P_2P_3.$$

Join O, the origin, to P_1, P_2, and P_3. Then from the figure 19.14, we have

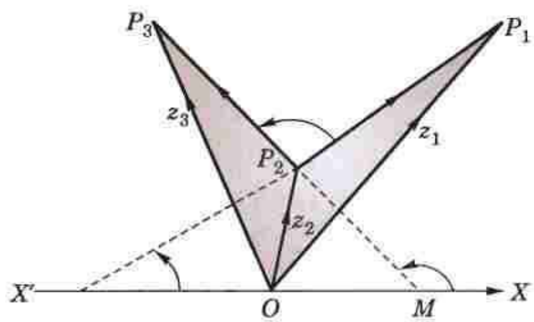

Fig. 19.14

$$\vec{P_2P_1} = z_1 - z_2 \quad \text{and} \quad \vec{P_2P_3} = z_3 - z_2$$

$$\therefore \quad \text{amp}\left(\frac{z_3 - z_2}{z_1 - z_2}\right) = \text{amp}\left[\frac{\vec{P_2P_3}}{\vec{P_2P_1}}\right]$$

$$= \text{amp}(\vec{P_2P_3}) - \text{amp}(\vec{P_2P_1}) = \beta - \alpha = \angle P_1P_2P_3.$$

Example 19.10. *Find the locus of the point z, when*

(*i*) $\left|\dfrac{z-a}{z-b}\right| = k$ (*ii*) *amp* $\left(\dfrac{z-a}{z-b}\right) = \alpha$ *where k and* α *are constants.*

Solution. Let $A(a)$ and $B(b)$ be any two fixed points on the complex plane and let $P(z)$ be any variable point (Fig. 19.15).

(*i*) Since $|z - a| = AP$ and $|z - b| = BP$.

$\therefore$ The point P moves so that $\left|\dfrac{z-a}{z-b}\right| = \left|\dfrac{z-a}{z-b}\right| = \dfrac{AP}{BP} = k$

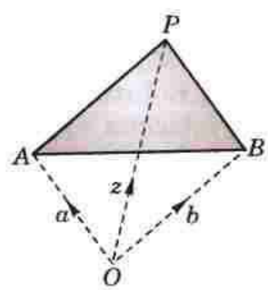

Fig. 19.15

i.e., P moves so that its distances from two fixed points are in a constant ratio, which is obviously the *Appollonius* circle.

When $k = 1$, $BP = AP$ *i.e.*, P moves so that its distance from two fixed points are always equal and thus the locus of P is the right bisector of AB.

Hence the locus of P(z) is a circle (unless k = 1, when the locus is the right bisector of AB).

Obs. For different values of k, the equation represents family of non-intersecting coaxial circles having A and B as its limiting points.

(*ii*) From the figure 19.16, we have amp $\left(\dfrac{z-a}{z-b}\right) = \angle APB = \alpha.$

Hence the locus of P(z) is the arc APB of the circle which passes through the fixed points A and B.

If, however, $P'(z')$ be a point on the lower arc AB of this circle, then

amp $\left(\dfrac{z'-a}{z'-b}\right) = \angle BP'A = \alpha - \pi$, which shows that the locus of P' is the arc $AP'B$ of the same circle.

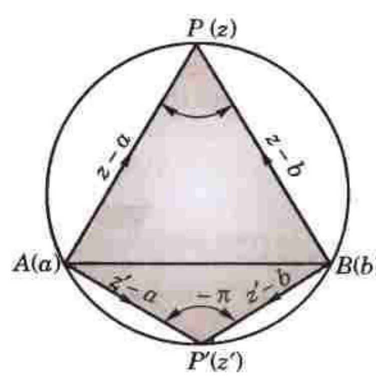

Fig. 19.16

Obs. For different values of α from $-\pi$ to π, the equation represents a family of intersecting coaxial circles having AB as their common radical axis.

Example 19.11. *If z_1, z_2 be two complex numbers, show that*

$$(z_1 + z_2)^2 + (z_1 - z_2)^2 = 2[|z_1|^2 + |z_2|^2].$$

Solution. Let $z_1 = r_1 \text{ cis } \theta_1$ and $z_2 = r_2 \text{ cis } \theta_2$ so that

$$|z_1 + z_2|^2 = (r_1 \cos\theta_1 + r_2 \cos\theta_2)^2 + (r_1 \sin\theta_1 + r_2 \sin\theta_2)^2$$
$$= r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_2 - \theta_1)$$

and
$$|z_1 - z_2|^2 = (r_1 \cos\theta_1 - r_2 \cos\theta_2)^2 + (r_1 \sin\theta_1 - r_2 \sin\theta_2)^2$$
$$= r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1)$$

$$\therefore \quad |z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(r_1^2 + r_2^2) = 2\left\{|z_1|^2 + |z_2|^2\right\}.$$

Example 19.12. *If z_1, z_2, z_3 be the vertices of an isosceles triangle, right angled at z_2, prove that*

$$z_1^2 + z_3^2 + 2z_2^2 = 2z_2(z_1 + z_3).$$

Solution. Let $A(z_1), B(z_2), C(z_3)$ be the vertices of ΔABC such that

$AB = BC$ and $\angle ABC = \pi/2$. (Fig. 19.17)

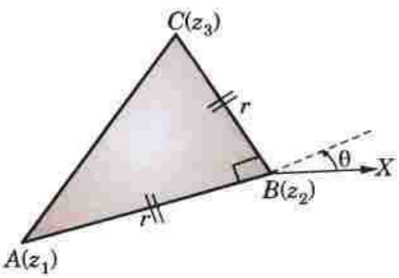

Fig. 19.17

Then $\quad |z_1 - z_2| = |z_3 - z_2| = r$ (say).

If amp $(z_1 - z_2) = \theta$ then amp $(z_3 - z_2) = \pi/2 + \theta$

$$\therefore \quad z_1 - z_2 = r(\cos\theta + i \sin\theta),$$

and
$$z_3 - z_2 = r\left[\cos\left(\frac{\pi}{2} + \theta\right) + i \sin\left(\frac{\pi}{2} + \theta\right)\right] = r(-\sin\theta + i\cos\theta)$$

i.e.,
$$z_3 - z_2 = ir(\cos\theta + i\sin\theta) = i(z_1 - z_2)$$

or
$$(z_3 - z_2)^2 = -(z_1 - z_2)^2 \text{ or } z_1^2 + z_3^2 + 2z_2^2 = 2z_3(z_1 + z_2).$$

Example 19.13. *If z_1, z_2, z_3 be the vertices of an equilateral triangle, prove that*

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1.$$

Solution. Since ΔABC is equilateral, therefore, BC when rotated through 60° coincides with BA (Fig. 19.18). But to turn the direction of a complex number through an $\angle\theta$, we multiply it by $\cos\theta + i\sin\theta$.

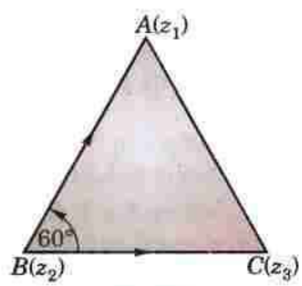

Fig. 19.18

$$\therefore \quad \vec{BC}\,(\cos\pi/3 + i\sin\pi/3) = \vec{BA}$$

i.e.,
$$(z_3 - z_2)\left(\frac{1 + i\sqrt{3}}{2}\right) = z_1 - z_2$$

or
$$i\sqrt{3}\,(z_3 - z_2) = 2z_1 - z_2 - z_3$$

Squaring,
$$-3(z_3 - z_2)^2 = (2z_1 - z_2 - z_3)^2$$

or
$$4(z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1) = 0$$

whence follows the required condition.

PROBLEMS 19.1

1. Express the following in the modulus-amplitude form:

   (*i*) $1 + \sin\alpha + i\cos\alpha$ $\qquad$ (*ii*) $\dfrac{1}{(2+1)^2} - \dfrac{1}{(2-1)^2}$. $\qquad$ (*V.T.U., 2011 S*)

2. If $\dfrac{1}{x + iy} + \dfrac{1}{u + iv} = 1$; x, y, u, v being real quantities, express v in terms of x and y.

3. If x and y are real, solve the equation $\dfrac{iy}{ix+1} - \dfrac{3y+4i}{3x+y} = 0$.
4. If $\alpha - i\beta = \dfrac{1}{a-ib}$, prove that $(\alpha^2+\beta^2)(a^2+b^2) = 1$. *(Mumbai, 2008 S)*
5. Find what curve $z\bar{z} + (1+i)z + (1-i)\bar{z} = 0$ represents?
6. In an Argand diagram, show that $9+i$, $4+13i$, $-8+8i$ and $-3-4i$ form a square.
7. If $|z_1| = |z_2|$ and amp (z_1) + amp $(z_2) = 0$, then show that z_1 and z_2 are conjugate complex numbers.
8. A rectangle is constructed in the complex plane and its sides parallel to the axes and its centre is situated at the origin. If one of the vertices of the rectangle is $1 + i\sqrt{3}$, find the complex numbers representing the other three vertices of the rectangle. Find also the area of the rectangle.
9. An equilateral triangle constructed in the complex plane has its one vertex at the point $1 + i\sqrt{3}$. Find the complex numbers representing the other two vertices, O the origin being its circumcentre.
10. The centre of a regular hexagon is at the origin and one vertex is given by $1+i$ on the Argand diagram. Find the remaining vertices.
11. What domain of the z-plane is represented by
    (*i*) $2 \le |z+3| < 4$ (*ii*) $I(z) > 2$
    (*iii*) $\pi/3 <$ amp $(z) < \pi/2$ (*iv*) $|z+2| + |z-2| < 4$.
12. If $|z^2 - 1| = |z|^2 + 1$, prove that z lies on the imaginary axis. *(Mumbai, 2007)*
13. What are the loci given by (*i*) $|z-1| + |z+1| = 3$ (*ii*) $|z-3| = k\,|z+1|$ for $k = 1$ and 2.
14. Find the locus of z given by :
    (*i*) $|z| = |z-2|$. (*ii*) $|3z-1| = |z-3|$.
15. Find the locus of z :
    (*i*) when $\dfrac{z+i}{z+2}$ is real, (*ii*) when $\dfrac{z-i}{z-2}$ is purely imaginary. *(Osmania, 2003 S)*

19.4 DE MOIVRE'S THEOREM*

Statement : *If n be (i) an integer, positive or negative* $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$; *(ii) a fraction, positive or negative, one of the values of* $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$.

Proof. Case I. *When n is a positive integer.*

By actual multiplication

$$\text{cis}\,\theta_1\,\text{cis}\,\theta_2 = (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)$$
$$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2), \text{ i.e., cis}\,(\theta_1+\theta_2)$$

Similarly $\text{cis}\,\theta_1\,\text{cis}\,\theta_2\,\text{cis}\,\theta_3 = \text{cis}\,(\theta_1+\theta_2)\,\text{cis}\,\theta_3 = \text{cis}\,(\theta_1+\theta_2+\theta_3)$

Proceeding in this way,

$$\text{cis}\,\theta_1\,\text{cis}\,\theta_2\,\text{cis}\,\theta_3 \ldots \text{cis}\,\theta_n = \text{cis}\,(\theta_1+\theta_2+\theta_3+\ldots+\theta_n)$$

Now putting $\theta_1 = \theta_2 = \theta_3 = \ldots = \theta_n = \theta$, we obtain $(\text{cis}\,\theta)^n = \text{cis}\,n\theta$.

Case II. *When n is a negative integer.*

Let $n = -m$, where m is a + ve integer.

$$\therefore \quad (\text{cis}\,\theta)^n = (\text{cis}\,\theta)^{-m} = \frac{1}{(\text{cis}\,\theta)^m} = \frac{1}{\text{cis}\,m\theta} \qquad \text{(By case I)}$$

$$= \frac{\cos m\theta - i\sin m\theta}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)}$$

[Multiplying the num. and denom. by $(\cos m\theta - i\sin m\theta)$]

*One of the remarkable theorems in mathematics; called after the name of its discoverer *Abraham De Moivre* (1667–1754), a French Mathematician.

$$= \frac{\cos m\theta - i \sin m\theta}{\cos^2 m\theta + \sin^2 m\theta} = \cos m\theta - i \sin m\theta$$

$$= \cos(-m\theta) + i \sin(-m\theta) = \text{cis}(-m\theta) = \text{cis}\, n\theta \qquad [\because \ -m = n]$$

Case III. *When n is a fraction, positive or negative.*

Let $n = p/q$, where q is a + ve integer and p is any integer + ve or – ve

Now $(\text{cis}\, \theta/q)^q = \text{cis}\,(q \cdot \theta/q) = \text{cis}\, \theta$

$\therefore$ Taking qth root of both sides cis (θ/q) is one of the q values of $(\text{cis}\, \theta)^{1/q}$,

i.e., one of the values of $(\text{cis}\, \theta)^{1/q} = \text{cis}\, \theta/p$

Raise both sides to power p, then one of the values of $(\text{cis}\, \theta)^{p/q} = (\text{cis}\, \theta/q)^p = \text{cis}\,(p/q)\,\theta$ *i.e.*, one of the values of $(\text{cis}\, \theta)^n = \text{cis}\, n\theta$. (By case I and II)

Thus *the theorem is completely established for all rational values of n.*

Cor. **1.** $\textit{cis}\, \theta_1 \cdot \textit{cis}\, \theta_2 \dots \textit{cis}\, \theta_n = \textit{cis}\,(\theta_1 + \theta_2 + \dots + \theta_n)$

2. $(\cos\theta - i \sin\theta)^n = \cos n\theta - i \sin n\theta = (\cos\theta + i \sin\theta)^{-n}$

3. $(\textit{cis}\, m\theta)^n = \text{cis}\, mn\theta = (\textit{cis}\, n\theta)^m$.

Example 19.14. *Simplify* $\dfrac{(\cos 3\theta + i \sin 3\theta)^4 (\cos 4\theta - i \sin 4\theta)^5}{(\cos 4\theta + i \sin 4\theta)^3 (\cos 5\theta + i \sin 5\theta)^{-4}}$.

Solution. We have, $(\cos 3\theta + i \sin 3\theta)^4 = \cos 12\theta + i \sin 12\theta = (\cos\theta + i \sin\theta)^{12}$

$(\cos 4\theta - i \sin 4\theta)^5 = \cos 20\theta - i \sin 20\theta = (\cos\theta + i \sin\theta)^{-20}$

$(\cos 4\theta + i \sin 4\theta)^3 = \cos 12\theta + i \sin 12\theta = (\cos\theta + i \sin\theta)^{12}$

$(\cos 5\theta + i \sin 5\theta)^{-4} = \cos 20\theta - i \sin 20\theta = (\cos\theta + i \sin\theta)^{-20}$

$$\therefore \quad \text{The given expression} = \frac{(\cos\theta + i \sin\theta)^{12} (\cos\theta + i \sin\theta)^{-20}}{(\cos\theta + i \sin\theta)^{12} (\cos\theta + i \sin\theta)^{-20}} = 1.$$

Example 19.15. *Prove that*

$$(1 + \cos\theta + i \sin\theta)^n + (1 + \cos\theta - i \sin\theta)^n = 2^{n+1} \cos^n (\theta/2) \cdot (\cos n\theta/2).$$

Solution. Put $1 + \cos\theta = r \cos\alpha$, $\sin\theta = r \sin\alpha$.

$$\therefore \qquad r^2 = (1 + \cos\theta)^2 + \sin^2\theta = 2 + 2\cos\theta = 4\cos^2\theta/2 \quad \textit{i.e.,} \quad r = 2\cos\theta/2$$

and
$$\tan\alpha = \frac{\sin\theta}{1 + \cos\theta} = \frac{2\sin\theta/2 \cdot \cos\theta/2}{2\cos^2\theta/2} = \tan\theta/2 \quad \textit{i.e.,} \quad \alpha = \theta/2.$$

$$\begin{aligned}\therefore \qquad \text{L.H.S.} &= [r(\cos\alpha + i \sin\alpha)]^n + [r(\cos\alpha - i \sin\alpha)]^n \\ &= r^n[(\cos\alpha + i \sin\alpha)^n + (\cos\alpha - i \sin\alpha)^n] = r^n(\cos n\alpha + i \sin n\alpha + \cos n\alpha - i \sin n\alpha) \\ &= r^n \cdot 2\cos n\alpha \qquad \text{[Substituting the values of } r \text{ and } \alpha] \\ &= 2^{n+1} \cos^n (\theta/2) \cos (n\theta/2).\end{aligned}$$

Example 19.16. *If* $2\cos\theta = x + \dfrac{1}{x}$, *prove that*

(*i*) $2\cos r\theta = x^r + 1/x^r$, (*ii*) $\dfrac{x^{2n} + 1}{x^{2n-1} + x} = \dfrac{\cos n\theta}{\cos (n-1)\theta}$ (*Madras, 2000 S*)

Solution. Since $x + 1/x = 2\cos\theta \qquad \therefore \qquad x^2 - 2x\cos\theta + 1 = 0$

whence
$$x = \frac{2\cos\theta \pm \sqrt{(4\cos^2\theta - 4)}}{2} = \cos\theta \pm i \sin\theta.$$

(*i*) Taking the + ve sign, $x^r = (\cos\theta + i \sin\theta)^r = \cos r\theta + i \sin r\theta$ (*S.V.T.U., 2009*)

and $\qquad x^{-r} = (\cos\theta + i \sin\theta)^{-r} = \cos r\theta - i \sin r\theta$

Adding $x^r + 1/x^r = 2\cos r\theta$. Similarly with the – ve sign, the same result follows.

(ii) $$\frac{x^{2n}+1}{x^{2n-1}+x} = \frac{(\cos\theta + i\sin\theta)^{2n}+1}{(\cos\theta+i\sin\theta)^{2n-1}+\cos\theta+i\sin\theta}$$

$$= \frac{\cos 2n\theta + i\sin 2n\theta + 1}{\cos(2n-1)\theta + i\sin(2n-1)\theta + \cos\theta + i\sin\theta}$$

$$= \frac{(1+\cos 2n\theta)+i\sin 2n\theta}{(\cos\overline{2n-1}\,\theta+\cos\theta)+i(\sin\overline{2n-1}\,\theta+\sin\theta)}$$

$$= \frac{2\cos^2 n\theta + 2i\sin n\theta\cos\theta}{2\cos n\theta\cos\overline{n-1}\,\theta + 2i\sin n\theta\cos\overline{n-1}\,\theta}$$

$$= \frac{\cos n\theta(2\cos n\theta + 2i\sin n\theta)}{\cos\overline{n-1}\,\theta(2\cos n\theta+2i\sin n\theta)} = \frac{\cos n\theta}{\cos\overline{n-1}\,\theta}.$$

Example 19.17. *If $\sin\alpha+\sin\beta+\sin\gamma=\cos\alpha+\cos\beta+\cos\gamma=0$,*
prove that (i) $\sin 2\alpha+\sin 2\beta+\sin 2\gamma=0$
(ii) $\sin 3\alpha+\sin 3\beta+\sin 3\gamma=3\sin(\alpha+\beta+\gamma)$
(iii) $\sin 4\alpha+\sin 4\beta+\sin 4\gamma=2\Sigma\sin 2(\alpha+\beta)$
(iv) $\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)=0.$ *(Mumbai, 2009)*

Solution. Let $a=\text{cis}\,\alpha,\ b=\text{cis}\,\beta$ and $c=\text{cis}\,\gamma$.

Then $a+b+c=(\cos\alpha+\cos\beta+\cos\gamma)+i(\sin\alpha+\sin\beta+\sin\gamma)=0$ ...(1)

(i) $$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=(\cos\alpha+i\sin\alpha)^{-1}+(\cos\beta+i\sin\beta)^{-1}+(\cos\gamma+i\sin\gamma)^{-1}$$

$$=\sum\frac{\cos\alpha-i\sin\alpha}{\cos\alpha-i\sin\alpha}\cdot\frac{1}{\cos\alpha+i\sin\alpha}=\sum(\cos\alpha-i\sin\alpha)$$

$$=(\cos\alpha+\cos\beta+\cos\gamma)-i(\sin\alpha+\sin\beta+\sin\gamma)=0 \quad \text{(Given)}$$

or $bc+ca+ab=0$ ...(2)

$\therefore\quad a^2+b^2+c^2=(a+b+c)^2-2(bc+ca+ab)=0$ [By (1) & (2) ...(3)]

or $(cis\,\alpha)^2+(cis\,\beta)^2+(cis\,\gamma)^2=cis\,2\alpha+cis\,2\beta+cis\,2\gamma=0$

Equating imaginary parts from both sides, we get

$$\sin 2\alpha+\sin 2\beta+\sin 2\gamma=0$$

(ii) Since $a+b+c=0$, $\quad\therefore\quad a^3+b^3+c^3=3abc$

or $(cis\,\alpha)^3+(cis\,\beta)^3+(cis\,\gamma)^3=3\,cis\,\alpha\,cis\,\beta\,cis\,\gamma$

or $cis\,3\alpha+cis\,3\beta+cis\,3\gamma=3\,cis\,(\alpha+\beta+\gamma)$

Equating imaginary parts from both sides, we get

$$\sin 3\alpha+\sin 3\beta+\sin 3\gamma=3\sin(\alpha+\beta+\gamma)$$

(iii) From (1), $\quad a+b=-c$ or $(a+b)^2=c^2$ or $a^2+b^2-c^2=-2ab$

Again squaring, $\quad a^4+b^4+c^4+2a^2b^2-2b^2c^2-2c^2a^2=4a^2b^2$

i.e., $\quad a^4+b^4+c^4=2(a^2b^2+b^2c^2+c^2a^2)$

or $(cis\,\alpha)^4+(cis\,\beta)^4+(cis\,\gamma)^4=2\sum(cos\,\alpha)^2\,(cis\,\beta)^2$

or $cis\,4\alpha+cis\,4\beta+cis\,4\gamma=2\sum cis\,2\alpha\,cis\,2\beta=2\sum cis\,2(\alpha+\beta)$

Equating imaginary parts from both sides, we get

$$\sin 4\alpha+\sin 4\beta+\sin 4\gamma=2\sum\sin 2(\alpha+\beta)$$

(iv) From (2), $ab+bc+ca=0$

or $cis\,\alpha\,cis\,\beta+cis\,\beta\,cis\,\gamma+cis\,\gamma\,cis\,\alpha=0$

or $cis\,(\alpha+\beta)+cis\,(\beta+\gamma)+cis\,(\gamma+\alpha)=0$

Equating imaginary parts from both sides, we get

$$\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)=0$$

PROBLEMS 19.2

1. Prove that (*i*) $\dfrac{(\cos 5\theta - i \sin 5\theta)^2 (\cos 7\theta + i \sin 7\theta)^{-3}}{(\cos 4\theta - i \sin 4\theta)^9 (\cos \theta + i \sin \theta)^5} = 1$

   (*ii*) $\dfrac{(\cos \alpha + i \sin \alpha)^4}{(\sin \beta + i \cos \beta)^5} = \sin (4\alpha + 5\beta) - i \cos (4\alpha + 5\beta)$.   (*iii*) $\left(\dfrac{\cos \theta + i \sin \theta)^4}{\sin \theta + i \cos \theta}\right) = \cos 8\theta + i \sin 8\theta$.

2. If $p = \text{cis } \theta$ and $q = \text{cis } \phi$, show that

   (*i*) $\dfrac{p-q}{p+q} = i \tan \dfrac{\theta - \phi}{2}$ (*Mumbai, 2008*)   (*ii*) $\dfrac{(p+q)(pq-1)}{(p-q)(pq+1)} = \dfrac{\sin \theta + \sin \phi}{\sin \theta - \sin \phi}$. (*Kurukshetra, 2005*)

3. If $a = \text{cis } 2\alpha$, $b = \text{cis } 2\beta$, $c = \text{cis } 2\gamma$ and $d = \text{cis } 2\delta$, prove that

   (*i*) $\sqrt{\dfrac{ab}{c}} + \sqrt{\dfrac{c}{ab}} = 2 \cos (\alpha + \beta - \gamma)$ (*ii*) $\sqrt{\dfrac{ab}{cd}} + \sqrt{\dfrac{cd}{ab}} = 2 \cos (\alpha + \beta - \gamma - \delta)$.

4. If $x_r = \text{cis } (\pi/2^r)$, show that $\underset{n \to \infty}{\text{Lt}}\ x_1 x_2 x_3 \ldots x_n = -1$. (*S.V.T.U., 2009 ; Mumbai, 2007*)

5. Find the general value of θ which satisfies the equation
$$(\cos \theta + i \sin \theta)(\cos 2\theta + i \sin 2\theta) \ldots (\cos n\theta + i \sin n\theta) = 1.$$

6. Prove that (*i*) $(a + ib)^{m/n} + (a - ib)^{m/n} = 2(a^2 + b^2)^{m/2n} \cos \left(\dfrac{m}{n} \tan^{-1} \dfrac{b}{a}\right)$.

   (*ii*) $(1 + i)^n + (1 - i)^n = 2^{n/2 + 1} \cos n\pi/4$.

7. Simplify $[\cos \alpha - \cos \beta + i (\sin \alpha - \sin \beta)]^n + [\cos \alpha - \cos \beta - i (\sin \alpha - \sin \beta)]^n$

8. Prove that (*i*) $(1 + \sin \theta + i \cos \theta)^n + (1 + \sin \theta - i \cos \theta)^n = 2^{n+1} \cos^n \left(\dfrac{\pi}{4} - \dfrac{\theta}{2}\right) \cos \left(\dfrac{n\pi}{4} - \dfrac{n\theta}{2}\right)$.

   (*ii*) $\left[\dfrac{1 + \sin \alpha + i \cos \alpha}{1 + \sin \alpha - i \cos \alpha}\right]^n = \cos \left(\dfrac{n\pi}{2} - n\alpha\right) + i \sin \left(\dfrac{n\pi}{2} - n\alpha\right)$. (*S.V.T.U., 2006*)

9. If $2 \cos \theta = x + 1/x$ and $2 \cos \phi = y + 1/y$, show that one of the values of

   (*i*) $x^m y^n + \dfrac{1}{x^m y^n}$ is $2 \cos (m\theta + n\phi)$. (*S.V.T.U., 2007*)

   (*ii*) $\dfrac{x^m}{y^n} + \dfrac{y^n}{x^m}$ is $2 \cos (m\theta - n\phi)$. (*Nagpur, 2009*)

10. If α, β be the roots of $x^2 - 2x + 4 = 0$, prove that $\alpha^n + \beta^n = 2^{n+1} \cos n\pi/3$. (*Delhi, 2002*)

11. If α, β are the roots of the equation $z^2 \sin^2 \theta - z \sin \theta + 1 = 0$, then prove that

    (*i*) $\alpha^n + \beta^n = 2 \cos n\theta \operatorname{cosec}^n \theta$ (*ii*) $\alpha^n \beta^n = \operatorname{cosec}^{2n} \theta$. (*Mumbai, 2009*)

12. If $x^2 - 2x \cos \theta + 1 = 0$, show that $x^{2n} - 2x^n \cos n\theta + 1 = 0$.

13. If $x = \cos \alpha + i \sin \alpha$, $y = \cos \beta + i \sin \beta$, $z = \cos \gamma + i \sin \gamma$ and $x + y + z = 0$, then prove that
$$x^{-1} + y^{-1} + z^{-1} = 0.$$

14. If $\sin \theta + \sin \phi + \sin \psi = 0 = \cos \theta + \cos \phi + \cos \psi$, prove that

    (*i*) $\cos 2\theta + \cos 2\phi + \cos 2\psi = 0$ (*Mumbai, 2009*)

    (*ii*) $\cos 3\theta + \cos 3\phi + \cos 3\psi = 3 \cos (\theta + \phi + \psi)$

    (*iii*) $\cos 4\theta + \cos 4\phi + \cos 4\psi = 2 \sum \cos 2(\phi + \psi)$.

15. If $\cos \alpha + \cos \beta + \cos \gamma = 0$ and $\sin \alpha + \sin \beta + \sin \gamma = 0$, prove that

    (*i*) $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 3/2$

    (*ii*) $\cos (\alpha + \beta) + \cos (\beta + \gamma) + \cos (\gamma + \alpha) = 0$ (*Mumbai, 2009 ; S.V.T.U., 2008*)

16. If $\sin \alpha + 2 \sin \beta + 3 \sin \gamma = 0$, $\cos \alpha + 2 \cos \beta + 3 \cos \gamma = 0$, prove that $\sin 3\alpha + 8 \sin 3\beta + 27 \sin 3\gamma = 18 \sin (\alpha + \beta + \gamma)$ and $\cos 3\alpha + 8 \cos 3\beta + 27 \sin 3\gamma = 18 \cos (\alpha + \beta + \gamma)$.

19.5 ROOTS OF A COMPLEX NUMBER

There are q and only q distinct values of $(\cos \theta + i \sin \theta)^{1/q}$, *q being an integer.*

Since $\cos \theta = \cos (2n\pi + \theta)$ and $\sin \theta = \sin (2n\pi + \theta)$, where n is any integer.

$$\therefore \qquad \text{cis } \theta = \text{cis } (2n\pi + \theta).$$

By De Moivre's theorem one of the values of

$$(\text{cis } \theta)^{1/q} = [\text{cis } (2n\pi + \theta)]^{1/q} = \text{cis } (2n\pi + \theta)/q \quad ...(1)$$

Giving n the values 0, 1, 2, 3, ..., $(q - 1)$ successively, we get the following q values of $(\text{cis } \theta)^{1/q}$;

$$\left.\begin{array}{ll} \text{cis } \theta/q & (\text{for } n = 0) \\ \text{cis } (2\pi + \theta)/q & (\text{for } n = 1) \\ \text{cis } (4\pi + \theta)/q & (\text{for } n = 2) \\ \dots\dots\dots & \dots\dots\dots \\ \text{cis } [2(q-1)\pi + \theta]/q & (\text{for } n = q - 1) \end{array}\right\} \quad ...(2)$$

Putting $n = q$ in (1), we get a value of $(\text{cis } \theta)^{1/q} = \text{cis } (2\pi + \theta/q) = \text{cis } \theta/q$, which is the same as the value of $n = 0$.

Similarly for $n = q + 1$, we get a value of $(\text{cis } \theta)^{1/q}$ to be cis $(2\pi + \theta)/q$, which is the same as the value for $n = 1$ and so on.

Thus, the values of $(\text{cis } \theta)^{1/q}$ for $n = q, q + 1, q + 2$ etc. are the mere repetition of the q values obtained in (2).

Moreover, the q values given by (2) are clearly distinct from each other, for no two of the angles involved therein are equal or differ by a multiple of 2π.

Hence $(\text{cis } \theta)^{1/q}$ has q and only q distinct values given by (2).

Obs. $(\text{cis } \theta)^{p/q}$ *where p/q is a rational fraction in its lowest terms, has also q and only q distinct values*; which are obtained by putting $n = 0, 1, 2, ..., q - 1$ successively in cis $p\,(2n\pi + \theta)/q$.

Note that $(\text{cis } \theta)^{6/15}$ has only 5 distinct values and not 15; because 6/15 in its lowest terms = 2/5

$\therefore$ *In order to find the distinct values of* $(\text{cis } \theta)^{p/q}$ *always see that* p/q *is in its lowest terms.*

Note. The above discussion can usefully be employed for extracting any assigned root of a given quantity. We have only to express it in the form $r(\cos\theta + i \sin\theta)$ and proceed as above.

Example 19.18. *Find the cube roots of unity and show that they form an equilateral triangle in the Argand diagram.*

Solution. If x be a cube root of unity, then

$$x = (1)^{1/3} = (\cos 0 + i \sin 0)^{1/3} = (\text{cis } 0)^{1/3} = (\text{cis } 2n\pi)^{1/3} = \text{cis } 2n\pi/3$$

where $n = 0, 1, 2$.

$\therefore$ the three vlaues of x are cis $0 = 1$,

$$\text{cis } 2\pi/3 = \cos 120° + i \sin 120° = -\frac{1}{2} + i\frac{\sqrt{3}}{2},$$

and
$$\text{cis } 4\pi/3 = \cos 240° + i \sin 240° = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

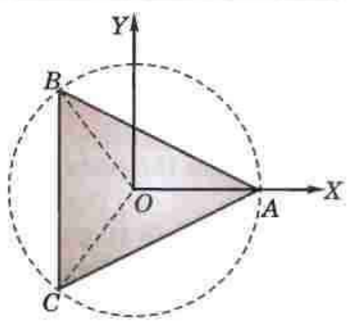

Fig. 19.19

These three cube roots are represented by the points A, B, C on the Argand diagram such that $OA = OB = OC$ and $\angle AOB = 120°$, $\angle AOC = 240°$ (Fig. 19.19).

$\therefore$ these points lie on a circle with centre O and unit radius such that $\angle AOB = \angle BOC = \angle COA = 120°$ *i.e.*, $AB = BC = CA$.

Hence A, B, C form an equilateral triangle.

Example 19.19. *Find all the values of* $\left(\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)^{3/4}$.

Also show that the continued product of these values is 1. (*Nagpur, 2009*)

Solution. Put $1/2 = r\cos\theta$ and $\sqrt{3}/2 = r \sin\theta$ so that $r = 1$ and $\theta = \pi/3$

$$\therefore \quad (1/2 + \sqrt{3}i/2)^{3/4} = [(\cos \pi/3 + i \sin \pi/3)^3]^{1/4} = (\text{cis } \pi)^{1/4}$$
$$= [\text{cis } (2n+1)\pi]^{1/4} = \text{cis } (2n+1)\,\pi/4 \text{ where } n = 0, 1, 2, 3.$$

Hence the required values are cis $\pi/4$, cis $3\pi/4$, cis $5\pi/4$ and cis $7\pi/4$.

$$\therefore \text{ their continued product} = \text{cis}\left(\frac{\pi}{4} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4}\right) = \text{cis } 4\pi = 1.$$

Example 19.20. *Use De Moivre's theorem to solve the equation.* *(P.T.U., 2005)*

$$x^4 - x^3 + x^2 - x + 1 = 0.$$

Solution. '$x^4 - x^3 + x^2 - x + 1$' is a G.P. with common ratio $(-x)$, therefore

$$\frac{1-(-x)^5}{1-(-x)} = 0, \quad x \neq -1 \quad \text{or} \quad x^5 + 1 = 0$$

i.e., $$x^5 = -1 = \text{cis } \pi = \text{cis } (2n+1)\pi$$

$\therefore$ $$x = [\text{cis } (2n+1)\pi]^{1/5} = \text{cis } (2n+1)\pi/5, \text{ where } n = 0, 1, 2, 3, 4$$

Hence the values are cis $\pi/5$, cis $3\pi/5$, cis π, cis $7\pi/5$, cis $9\pi/5$

or $$\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}, \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}, -1, \cos\frac{3\pi}{5} - i\sin\frac{3\pi}{5}, \cos\frac{\pi}{5} - i\sin\frac{\pi}{5}$$

Rejecting the value -1 which corresponds to the factor $x + 1$, the required roots are :

$$\cos \pi/5 \pm i \sin \pi/5, \cos 3\pi/5 \pm i \sin 3\pi/5.$$

Example 19.21. *Show that the roots of the equation $(x-1)^n = x^n$, n being a positive integer are $\frac{1}{2}(1 + i \cot r\pi/n)$, where r has the values 1, 2, 3, ..., n – 1.*

Solution. Given equation is $\left(\frac{x-1}{x}\right)^n = 1$ or $1 - \frac{1}{x} = (1)^{1/n}$

or $$\frac{1}{x} = 1 - (1)^{1/n} = 1 - \text{cis } \frac{2r\pi}{n}, r = 0, 1, 2, \ldots (n-1). \qquad [\because \ 1 = \text{cis } 2\pi r]$$

or $$= \left(1 - \cos\frac{2r\pi}{n}\right) - i\sin\frac{2r\pi}{n} = 2\sin^2\frac{r\pi}{n} - 2i\sin\frac{r\pi}{n}\cos\frac{r\pi}{n}$$

$\therefore$ $$x = \frac{1}{2\sin\frac{r\pi}{n}} \cdot \frac{1}{\left(\sin\frac{r\pi}{n} - i\cos\frac{r\pi}{n}\right)} = \frac{\sin\frac{r\pi}{n} + i\cos\frac{r\pi}{n}}{2\sin\frac{r\pi}{n}}$$

$$= \frac{1}{2}\left(1 + i\cot\frac{r\pi}{n}\right), r = 1, 2, \ldots (n-1). \qquad [\because \ \cot 0 \to \infty]$$

Hence the roots of the given equation are $\frac{1}{2}(1 + i \cot r\pi/n)$ where $r = 1, 2, 3, \ldots (n-1)$.

Example 19.22. *Find the 7th roots of unity and prove that the sum of their nth powers always vanishes unless n be a multiple number of 7, n being an integer, and then the sum is 7.*

(Mumbai, 2008 ; Kurukshetra, 2005)

Solution. We have $(1)^{1/7} = (\cos 2r\pi + i \sin 2r\pi)^{1/7} = \text{cis } \frac{2r\pi}{7} = \left(\text{cis } \frac{2\pi}{7}\right)^r$

Putting $r = 0, 1, 2, 3, 4, 5, 6$, we find that 7th roots of unity are $1, \rho, \rho^2, \rho^3, \rho^4, \rho^5, \rho^6$ where $\rho = \cos 2\pi/7$.

$\therefore$ sum S of the nth powers of these roots $= 1 + \rho^n + \rho^{2n} + \ldots + \rho^{6n}$ ...(*i*)

$$= \frac{1-\rho^{7n}}{1-\rho^n}, \text{ being a G.P. with common ratio } \rho$$

When n is not a multiple of 7, $\rho^{7n} = (\rho^7)^n = (\text{cis } 2\pi)^n = 1$.

i.e., $$1 - \rho^{7n} = 0 \text{ and } 1 - \rho^n \neq 0, \text{ as } n \text{ is not a multiple of 7.}$$

Thus $S = 0$.

When n is a multiple of $7 = 7p$ (say)

From (*i*), $$S = 1 + (\rho^7)^p + (\rho^7)^{2p} + \ldots + (\rho^7)^{6p} = 1 + 1 + 1 + 1 + 1 + 1 + 1 = 7.$$

Example 19.23. *Find the equation whose roots are 2 cos $\pi/7$, 2 cos $3\pi/7$, 2 cos $5\pi/7$.*

Solution. Let $y = \cos\theta + i\sin\theta$, where $\theta = \pi/7, 3\pi/7, \ldots, 13\pi/7$.

Then $\quad y^7 = (\cos\theta + i\sin\theta)^7 = \cos 7\theta + i\sin 7\theta = -1 \quad$ or $\quad y^7 + 1 = 0$

or $\quad (y+1)(y^6 - y^5 + y^4 - y^3 + y^2 - y + 1) = 0$

Leaving the factor $y + 1$ which corresponds to $\theta = \pi$,

We get $\quad y^6 - y^5 + y^4 - y^3 + y^2 - y + 1 = 0 \quad$ ...(*i*)

Its roots are $\quad y = \text{cis}\,\theta$ where $\theta = \pi/7, 3\pi/7, 5\pi/7, 9\pi/7, 11\pi/7, 13\pi/7$.

Dividing (*i*) by y^3, $(y^3 + 1/y^3) - (y^2 + 1/y^2) + (y + 1/y) - 1 = 0$

or $\quad \{(y + 1/y)^3 - 3(y + 1/y)\} - \{(y + 1/y)^2 - 2\} - (y + 1/y) - 1 = 0$

or $\quad x^3 - x^2 - 2x + 1 = 0 \quad$ ...(*ii*)

where $\quad x = y + 1/y = 2\cos\theta$.

Now since $\quad \cos 13\pi/7 = \cos\pi/7,\ \cos 11\pi/7 = \cos 3\pi/7,\ \cos 9\pi/7 = \cos 5\pi/7$

Hence the roots of (*ii*) are $2\cos\dfrac{\pi}{7}, 2\cos\dfrac{3\pi}{7}, 2\cos\dfrac{5\pi}{7}$.

PROBLEMS 19.3

1. Find all the values of
   (*i*) $(1+i)^{1/4}$ (*ii*) $(-1+i)^{2/5}$
   (*iii*) $(-1+i\sqrt{3})^{3/2}$ (*iv*) $(1+i\sqrt{3})^{1/3} + (1-i\sqrt{3})^{1/3}$.
2. If w is a complex cube root of unity, prove that $1 + w + w^2 = 0$.
3. Find all the values of $(-1)^{1/6}$.
4. Mark by points on the Argand diagram, all the values of $(1+i\sqrt{3})^{1/5}$ and verify that they form a pentagon.
5. Use De Moivre's theorem to solve the following equations :
   (*i*) $x^5 + 1 = 0$ (*ii*) $x^7 + x^4 + x^3 + 1 = 0$
   (*iii*) $x^9 + x^5 - x^4 - 1 = 0$ (*Madras, 2000*) (*iv*) $(x-1)^5 + x^5 = 0$.
6. Find the roots common to the equations $x^4 + 1 = 0$ and $x^6 - i = 0$.
7. Solve the equation $x^{12} - 1 = 0$ and find which of its roots satisfy the equation $x^4 + x^2 + 1 = 0$.
8. Show that the roots of $(x+1)^7 = (x-1)^7$ are given by $\pm i\cot r\pi/7$, $r = 1, 2, 3$. (*Mumbai, 2008*)
9. Prove that the nth roots of unity form a geometric progression. (*Mumbai, 2007*)
   Also show that the sum of these n roots is zero and their product is $(-1)^{n-1}$.
10. Find the equation whose roots are $2\cos 2\pi/7$, $2\cos 4\pi/7$, $2\cos 6\pi/7$.

19.6 (1) TO EXPAND sin nθ, cos nθ AND tan nθ IN POWERS OF sin θ, cos θ AND tan θ RESPECTIVELY (n BEING A POSITIVE INTEGER)

We have $\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$ (By De Moivre's theorem)

$= \cos^n\theta + {}^nC_1\cos^{n-1}\theta\,(i\sin\theta) + {}^nC_2\cos^{n-2}\theta\,(i\sin\theta)^2 + {}^nC_3\cos^{n-3}\theta\,(i\sin\theta)^3 + \ldots$ (By Binomial theorem)

$= (\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + \ldots) + i({}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + \ldots)$

Equating real and imaginary parts from both sides, we get

$$\cos n\theta = \cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta - \ldots \quad \text{...(1)}$$

$$\sin n\theta = {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + {}^nC_5\cos^{n-5}\theta\sin^5\theta - \ldots \quad \text{...(2)}$$

Replacing every $\sin^2\theta$ by $1 - \cos^2\theta$ in (1) and every $\cos^2\theta$ by $1 - \sin^2\theta$ in (2), we get the desired expansions of $\cos n\theta$ and $\sin n\theta$.

Dividing (2) by (1),

$$\tan n\theta = \frac{{}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + {}^nC_5\cos^{n-5}\theta\sin^5\theta - \ldots}{\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta - \ldots}$$

and dividing numerator and denominator by $\cos^n\theta$, we get

$$\tan n\theta = \frac{{}^nC_1\tan\theta - {}^nC_3\tan^3\theta + {}^nC_5\tan^5\theta - \ldots}{1 - {}^nC_2\tan^2\theta + {}^nC_4\tan^4\theta - \ldots}.$$

Example 19.24. *Express cos 6θ in terms of cos θ.* (*Madras, 2002*)

Solution. We know that $\cos n\theta = \cos^n \theta - {}^nC_2 \cos^{n-2} \theta \sin^2 \theta + {}^nC_4 \cos^{n-4} \theta \sin^4 \theta - \ldots$

Put $n = 6$, then $\cos 6\theta = \cos^6 \theta - {}^6C_2 \cos^4 \theta \sin^2 \theta + {}^6C_4 \cos^2 \theta \sin^4 \theta - {}^6C_6 \sin^6 \theta$

$$= \cos^6 \theta - 15 \cos^4 \theta (1 - \cos^2 \theta) + 15 \cos^2 \theta (1 - \cos^2 \theta)^2 - (1 - \cos^2 \theta)^3$$
$$= 32 \cos^6 \theta - 48 \cos^4 \theta + 18 \cos^2 \theta - 1.$$

(2) Addition formulae for any number of angles

We have, $\cos (\theta_1 + \theta_2 + \ldots + \theta_n) + i \sin (\theta_1 + \theta_2 + \ldots + \theta_n)$
$$= (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) \ldots (\cos \theta_n + i \sin \theta_n)$$

Now $\cos \theta_1 + i \sin \theta_1 = \cos \theta_1 (1 + i \tan \theta_1)$, $\cos \theta_2 + i \sin \theta_2 = \cos \theta_2 (1 + i \tan \theta_2)$ and so on.

$\therefore$ $\cos (\theta_1 + \theta_2 + \ldots + \theta_n) + i \sin (\theta_1 + \theta_2 + \ldots + \theta_n)$
$$= \cos \theta_1 \cos \theta_2 \ldots \cos \theta_n (1 + i \tan \theta_1)(1 + i \tan \theta_2)\ldots(1 + i \tan \theta_n)$$
$$= \cos \theta_1 \cos \theta_2 \ldots \cos \theta_n [1 + i (\tan \theta_1 + \tan \theta_2 + \ldots + \tan \theta_n)$$
$$+ i^2(\tan \theta_1 \tan \theta_2 + \tan \theta_2 \tan \theta_3 + \ldots) + i^3(\tan \theta_1 \tan \theta_2 \tan \theta_3 + \ldots) + \ldots + \ldots]$$
$$= \cos \theta_1 \cos \theta_2 \ldots \cos \theta_n (1 + is_1 - s_2 - is_3 + s_4 + \ldots)$$

where $s_1 = \tan \theta_1 + \tan \theta_2 + \ldots + \tan \theta_n$, $s_2 = \Sigma \tan \theta_1 \tan \theta_2$, $s_3 = \Sigma \tan \theta_1 \tan \theta_2 \tan \theta_3$ etc.

Equating real and imaginary parts, we have
$$\cos (\theta_1 + \theta_2 + \ldots + \theta_n) = \cos \theta_1 \cos \theta_2 \ldots \cos \theta_n (1 - s_2 + s_4 - \ldots)$$
$$\sin (\theta_1 + \theta_2 + \ldots + \theta_n) = \cos \theta_1 \cos \theta_2 \ldots \cos \theta_n (s_1 - s_3 + s_5 - \ldots)$$

and by division, we get $\mathbf{\tan (\theta_1 + \theta_2 + \ldots + \theta_n) = \dfrac{s_1 - s_3 + s_5 - \ldots}{1 - s_2 + s_4 - s_6 + \ldots}}$.

Example 19.25. *If* $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \pi/2$, *show that* $xy + yz + zx = 1$. (*P.T.U., 2003*)

Solution. Let $\tan^{-1} x = \alpha$, $\tan^{-1} y = \beta$, $\tan^{-1} z = \gamma$ so that $x = \tan \alpha$, $y \tan \beta$, $z = \tan \gamma$

We know that
$$\tan (\alpha + \beta + \gamma) = \frac{\tan \alpha + \tan \beta + \tan \gamma - \tan \alpha \tan \beta \tan \gamma}{1 - \tan \alpha \tan \beta - \tan \beta \tan \gamma - \tan \gamma \tan \alpha}$$

$\therefore$
$$\tan \pi/2 = \frac{x + y + z - xyz}{1 - xy - yz - zx} \quad \text{or} \quad 1 - xy - yz - zx = 0$$

Hence $xy + yz + zx = 1$.

Example 19.26. *If* $\theta_1, \theta_2, \theta_3$ *be three values of* θ *which satisfy the equation* $\tan 2\theta = \lambda \tan (\theta + \alpha)$ *and such that no two of them differ by a multiple of* π, *show that* $\theta_1 + \theta_2 + \theta_3 + \alpha$ *is a multiple of* π.

Solution. Given equation can be written as $\dfrac{2t}{1 - t^2} = \lambda \dfrac{t + \tan \alpha}{1 - t \,.\, \tan \alpha}$ where $t = \tan \theta$

or
$$\lambda t^3 + (\lambda - 2) \tan \alpha \,.\, t^2 + (2 - \lambda) t - \lambda \tan \alpha = 0$$

$\therefore$ $\tan \theta_1, \tan \theta_2, \tan \theta_3$, being its roots, we have

$$s_1 = \Sigma \tan \theta_1 = -\frac{\lambda - 2}{\lambda} \tan \alpha \qquad \text{[By § 1.3]}$$

$$s_2 = \Sigma \tan \theta_1 \tan \theta_2 = \frac{2 - \lambda}{\lambda} \quad \text{and} \quad s_3 = \tan \alpha$$

$\therefore$
$$\tan (\theta_1 + \theta_2 + \theta_3) = \frac{s_1 - s_3}{1 - s_2} = \frac{(-1 + 2/\lambda) \tan \alpha - \tan \alpha}{1 - (2/\lambda - 1)}$$
$$= -\tan \alpha = \tan (n\pi - \alpha)$$

Thus $\theta_1 + \theta_2 + \theta_3 = n\pi - \alpha$, whence follows the result.

(3) To expand $\sin^m \theta$, $\cos^n \theta$ or $\sin^m \theta \cos^n \theta$ in a series of sines or cosines of multiples of θ

If $z = \cos \theta + i \sin \theta$ then $1/z = \cos \theta - i \sin \theta$.

By De Moivre's theorem, $z^p = \cos p\theta + i \sin p\theta$ and $1/z^p = \cos p\theta - i \sin p\theta$

$\therefore$ $z + 1/z = 2 \cos \theta$, $z - 1/z = 2i \sin \theta$; $z^p + 1/z^p = 2 \cos p\theta$, $z^p - 1/z^p = 2i \sin p\theta$

These results are used to expand the powers of sin θ or cos θ or their products in a series of sines or cosines of multiples of θ.

Example 19.27. *Expand $\cos^8\theta$ in a series of cosines of multiples of θ.*

Solution. Let $z = \cos\theta + i\sin\theta$, so that $z + 1/z = 2\cos\theta$ and $z^p + 1/z^p = 2\cos p\theta$.

$$\therefore \quad (2\cos\theta)^8 = (z + 1/z)^8$$

$$= z^8 + {}^8C_1 z^7 \cdot \frac{1}{z} + {}^8C_2 z^6 \cdot \frac{1}{z^2} + {}^8C_3 z^5 \cdot \frac{1}{z^3} + {}^8C_4 z^4 \cdot \frac{1}{z^4} + {}^8C_5 z^3 \cdot \frac{1}{z^5} + {}^8C_6 z^2 \cdot \frac{1}{z^6} + {}^8C_7 z \cdot \frac{1}{z^7} + \frac{1}{z^8}$$

$$= (z^8 + 1/z^8) + {}^8C_1(z^6 + 1/z^6) + {}^8C_2(z^4 + 1/z^4) + {}^8C_3(z^2 + 1/z^2) + {}^8C_4$$

$$= (2\cos 8\theta) + 8(2\cos 6\theta) + 28(2\cos 4\theta) + 56(2\cos 2\theta) + 70.$$

Hence $$\cos^8\theta = \frac{1}{128}[\cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35].$$

Example 19.28. *Expand $\sin^7\theta\cos^3\theta$ in a series of sines of multiples of θ.*

Solution. Let $z = \cos\theta + i\sin\theta$

so that $z + 1/z = 2\cos\theta$, $z - 1/z = 2i\sin\theta$ and $z^p - 1/z^p = 2i\sin p\theta$.

$$\therefore \quad (2i\sin\theta)^7(2\cos\theta)^3 = (z - 1/z)^7(z + 1/z)^3$$

$$= (z - 1/z)^4[(z - 1/z)(z + 1/z)]^3 = (z - 1/z)^4(z^2 - 1/z^2)^3$$

$$= \left(z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4}\right)\left(z^6 - 3z^2 + \frac{3}{z^2} - \frac{1}{z^6}\right)$$

$$= \left(z^{10} - \frac{1}{z^{10}}\right) - 4\left(z^8 - \frac{1}{z^8}\right) + 3\left(z^6 - \frac{1}{z^6}\right) + 8\left(z^4 - \frac{1}{z^4}\right) - 14\left(z^2 - \frac{1}{z^2}\right)$$

$$= 2i\sin 10\theta - 4(2i\sin 8\theta) + 3(2i\sin 6\theta) + 8(2i\sin 4\theta) - 14(2i\sin 2\theta)$$

Since $i^7 = -i$,

$$\therefore \quad \sin^7\theta\cos^3\theta = -\frac{1}{2^9}[\sin 10\theta - 4\sin 8\theta + 3\sin 6\theta + 8\sin 4\theta - 14\sin 2\theta].$$

Obs. The expansion of $\sin^m\theta\cos^n\theta$ is a series of sines or cosines of multiples of θ according as m is odd or even.

PROBLEMS 19.4

1. Express $\sin 6\theta/\sin\theta$ as a polynomial in $\cos\theta$?

Prove that (2–5) :

2. $\sin 7\theta/\sin\theta = 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta$.
3. $\dfrac{1+\cos 7\theta}{1+\cos\theta} = (x^3 - x^2 - 2x + 1)^2$, where $x = 2\cos\theta$. (*Madras, 2002*)
4. $2(1 + \cos 8\theta) = (x^4 - 4x^2 + 2)^2$ where $x = 2\cos\theta$.
5. $\tan 5\theta = \dfrac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$ where $t = \tan\theta$.
6. If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$, show that $x + y + z = xyz$.
7. If α, β, γ be the roots of the equation $x^3 + px^2 + qx + p = 0$, prove that $\tan^{-1}\alpha + \tan^{-1}\beta + \tan^{-1}\gamma = n\pi$ radians except in one particular case.

Prove that (8-12) :

8. $\cos^7\theta = \dfrac{1}{16}(\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta)$. (*Madras, 2003 S*)
9. $\cos^6\theta - \sin^6\theta = \dfrac{1}{16}(\cos 6\theta + 15\cos 2\theta)$. (*Mumbai, 2007*)
10. $\sin^8\theta = 2^{-7}(\cos 8\theta - 8\cos 6\theta + 28\cos 4\theta - 56\cos 2\theta + 35)$.
11. $32\sin^4\theta\cos^2\theta = \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$.
12. $\sin^5\theta\cos^2\theta = \dfrac{1}{64}(\sin 7\theta - 3\sin 5\theta + \sin 3\theta + 5\sin\theta)$. (*Madras, 2003*)

13. Expand $\cos^5 \theta \sin^7 \theta$ in a series of sines of multiples of θ ? (*Madras, 2002*)
14. If $\cos^5 \theta = A \cos \theta + B \cos 3\theta + C \cos 5\theta$, find $\sin^5 \theta$ in terms of A, B, C.
15. If $\sin^4 \theta \cos^3 \theta = A_1 \cos \theta + A_3 \cos 3\theta + A_5 \cos 5\theta + A_7 \cos 7\theta$, prove that
$$A_1 + 9A_3 + 25A_5 + 49A_7 = 0.$$

19.7 COMPLEX FUNCTION

Definition. *If for each value of the complex variable z (= $x + iy$) in a given region R, we have one or more values of w (= $u + iv$), then w is said to be a* **complex function** *of z* and we write $w = u(x, y) + iv(x, y) = f(z)$ where u, v are real functions of x and y.

If to each value of z, there corresponds one and only one value of w, then w is said to be a *single-valued function of z* otherwise a *multi-valued function*. For example, $w = 1/z$ is a single-valued function and $w = \sqrt{z}$ is a multi-valued function of z. The former is defined at all points of the z-plane except at $z = 0$ and the latter assumes two values for each value of z except at $z = 0$.

19.8 EXPONENTIAL FUNCTION OF A COMPLEX VARIABLE

(1) Definition. When x is real, we are already familiar with the exponential function

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots \infty .$$

Similarly, *we define the exponential function of the complex variable $z = x + iy$, as*

$$\mathbf{e^z \text{ or } \exp(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \dots + \frac{z^n}{n!} + \dots \infty} \quad \dots(i)$$

(2) Properties :

I. Exponential form of $z = re^{i\theta}$

Putting $x = 0$ in (i), we get

$$e^{iy} = 1 + \frac{iy}{1!} + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \dots \infty$$

$$= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots\right) = \cos y + i \sin y$$

Thus $\quad e^z = e^x \cdot e^{iy} = e^x (\cos y + i \sin y)$

Also $\quad x + iy = r(\cos \theta + i \sin \theta) = re^{i\theta}$. Thus, $z = re^{i\theta}$

II. e^z is periodic function having imaginary period $2\pi i$, $[\because \ e^{z + 2n\pi i} = e^z \cdot e^{2n\pi i} = e^z]$.

III. e^z is not zero for any value of z.

Since $\quad e^z = e^{x + iy} = re^{i\theta}$ or $e^x \cdot e^{iy} = re^{i\theta}$

$\therefore \quad r = e^x > 0, y = \theta, |e^{iy}| = 1,$

Thus $\quad |e^z| = |e^x| \cdot |e^{iy}| = e^x \neq 0.$

IV. $\quad e^{\bar{z}} = \overline{e^z}$

Since $\quad e^{\bar{z}} = e^{x - iy} = e^x \cdot e^{-iy} = e^x (\cos y - i \sin y)$

$$= \overline{e^x (\cos y + i \sin y)} = \overline{e^z}$$

19.9 CIRCULAR FUNCTIONS OF A COMPLEX VARIABLE

(1) Definitions:

Since $\quad e^{iy} = \cos y + i \sin y$ and $e^{-iy} = \cos y - i \sin y$.

$\therefore$ the circular functions of real angles can be written as

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i}, \cos y = \frac{e^{iy} + e^{-iy}}{2} \text{ and so on.}$$

It is, therefore, natural to *define the circular functions of the complex variable z by the equations :*

$$\mathbf{\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \cos z = \frac{e^{iz} + e^{-iz}}{2}, \tan z = \frac{\sin z}{\cos z}}$$

with cosec z, sec z and cot z as their respective reciprocals.

(2) Properties :

I. Circular functions are periodic : $\sin z$, $\cos z$ are periodic functions having real period 2π while $\tan z$, $\cot z$ have period π. [a $\sin (z + 2n\pi) = \sin z$, $\tan (z + n\pi) = \tan z$ etc.]

II. Even and odd functions : cos z, sec z are even functions while sin z, cosec z are odd functions. [$\because$ $\cos z$ $= \dfrac{e^{-iz} + e^{iz}}{2} = \cos z$, and $\sin (-z) = \dfrac{e^{-iz} - e^{iz}}{2i} = \dfrac{e^{iz} - e^{-iz}}{2i} = -\sin z$]

III. Zeros of sin z are given by $z = \pm 2n\pi$ *and zeros of cos z are given by* $z = \pm \frac{1}{2}(2n + 1)\pi$, $n = 0, 1, 2, ...$

IV. All the formulae for real circular functions are valid for complex circular functions

e.g., $\sin^2 z + \cos^2 z = 1$, $\sin (z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$.

(3) Euler's theorem $e^{iz} = \cos z + i \sin z$.

By definition $\cos z + i \sin z = \dfrac{e^{iz} + e^{-iz}}{2} + i\,\dfrac{e^{iz} - e^{-iz}}{2i} = e^{iz}$ where $z = x + iy$.

Also we have shown that $e^{iy} = \cos y + i \sin y$, where y is real.

Thus $e^{i\theta} = \mathbf{\cos}\ \theta + \boldsymbol{i}\ \mathbf{sin}\ \theta$, *where θ is real or complex. This is called the Euler's theorem.**

Cor. *De Moivre's theorem for complex numbers*

Whether θ is real or complex, we have

$$(\cos \theta + i \sin \theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos n\theta + i \sin n\theta$$

Thus De Moivre's theorem is true for all θ (real or complex).

Example 19.29. *Prove that (i)* $[\sin (\alpha + \theta) - e^{i\alpha} \sin \theta]^n = \sin^n \alpha\, e^{-in\theta}$

(ii) $\sin (\alpha - n\theta) + e^{-i\alpha} \sin n\theta = e^{-in\theta} \sin \alpha$.

Solution. (*i*) L.H.S. $= [\sin \alpha \cos \theta + \cos \alpha \sin \theta - (\cos \alpha + i \sin \alpha) \sin \theta]^n$

$= (\sin \alpha \cos \theta - i \sin \alpha \sin \theta)^n$

$= \sin^n \alpha (\cos \theta - i \sin \theta)^n = \sin^n \alpha (e^{-i\theta})^n = \sin^n \alpha e^{-in\theta}$

(*ii*) L.H.S. $= \sin \alpha \cos n\theta - \cos \alpha \sin n\theta + (\cos \alpha - i \sin \alpha) \sin n\theta$

$= \sin \alpha \cos n\theta - i \sin \alpha \sin n\theta$

$= \sin \alpha (\cos n\theta - i \sin n\theta) = \sin \alpha \,.\, e^{-in\theta}$.

Example 19.30. *Given* $\dfrac{1}{\rho} = \dfrac{1}{L\rho i} + C\rho i + \dfrac{1}{R}$, *where L, ρ, R are real, express ρ in the form* $Ae^{i\theta}$ *giving the values of A and θ.*

Solution.
$$\frac{1}{\rho} = \frac{R + L\rho^2 CR(-1) + L\rho i}{L\rho Ri} = \frac{(R - L\rho^2 CR) + iLR}{L\rho Ri}$$

or
$$\rho = L\rho \frac{Ri}{(R - L\rho^2 CR) + iL\rho} \times \frac{(R - L\rho^2 CR) - iL\rho}{(R - L\rho^2 CR) - iL\rho}$$

$$= \frac{L^2\rho^2 R + iL\rho R\,(R - L\rho^2 CR)}{(R - L\rho^2 CR)^2 + (L\rho)^2} = A(\cos \theta + i \sin \theta), \text{ say}$$

*See footnote p. 205.

Equating real and imaginary parts, we have

$$A\cos\theta = \frac{L^2\rho^2 R}{(R - L\rho^2 CR)^2 + (L\rho)^2} \qquad ...(i)$$

$$A\sin\theta = \frac{L\rho R(R - L\rho^2 CR)}{(R - L\rho^2 CR)^2 + (L\rho)^2} \qquad ...(ii)$$

Squaring and adding (*i*) and (*ii*),

$$A^2 = \frac{(L^2\rho^2 R)^2 + (L\rho R)^2 (R - L\rho^2 CR)^2}{[(R - L\rho^2 CR)^2 + (L\rho)^2]^2} \quad \text{or} \quad A = \frac{L\rho R}{\sqrt{[(R - L\rho^2 CR)^2 + (L\rho)]^2}} \qquad ...(iii)$$

Dividing (*ii*) by (*i*),

$$\tan\theta = \frac{R - L\rho^2 CR}{L\rho} \quad \text{or} \quad \theta = \tan^{-1}\left\{\frac{R(1 - LC\rho^2)}{L\rho}\right\} \qquad ...(iv)$$

Hence $\qquad P = A(\cos\theta + i\sin\theta) = Ae^{i\theta}$

where A and θ are given by (*iii*) and (*iv*).

19.10 HYPERBOLIC FUNCTIONS

(1) Definitions: *If x be real or complex,*

(*i*) $\dfrac{e^x - e^{-x}}{2}$ *is defined as* **hyperbolic sine of x** and is written as **sinh x.**

(*ii*) $\dfrac{e^x + e^{-x}}{2}$ *is defined as* **hyperbolic cosine of x** *and is written as* **cosh x.**

Thus $\sinh x = \dfrac{e^x - e^{-x}}{2}$ and $\cosh x = \dfrac{e^x + e^{-x}}{2}$

Also we define,

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}; \ \coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}; \ \operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

(2) Properties

I. Periodic functions : sinh z and cosh z are periodic functions having imaginary period $2\pi i$.

[$\because$ $\sinh(z + 2\pi i) = \sinh z$; $\cosh(z + 2\pi i) = \cosh z$]

II. Even and odd functions : cosh z is an even function while sinh z is an odd function

III. $\sinh 0 = 0$, $\cosh 0 = 1$, $\tanh 0 = 0$.

IV. **Relations between hyperbolic and circular functions.**

Since for all values of θ, $\qquad \sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$ and $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

$\therefore$ Putting $\theta = ix$, we have $\qquad \sin ix = \dfrac{e^{-x} - e^{x}}{2i} = -\dfrac{e^x - e^{-x}}{2i}$ $\qquad [\because \ e^{i\theta} = e^{i\cdot ix} = e^{-x}]$

$$= i^2\,\frac{e^x - e^{-x}}{2i} = i\,.\,\frac{e^x - e^{-x}}{2} = i\sinh x$$

and, therefore, $\qquad \cos ix = \dfrac{e^{-x} + e^{x}}{2} = \cosh x$

Thus $\qquad$ **sin ix = i sinh x** $\qquad ...(i)$

cos ix = cosh x $\qquad ...(ii)$

and $\therefore$ $\qquad$ **tan ix = i tanh x** $\qquad ...(iii)$

Cor. $\qquad$ **sinh ix = i sin x** $\qquad ...(iv)$

cosh ix = cos x $\qquad ...(v)$

tanh ix = i tan x $\qquad ...(vi)$

V. Formulae of hyperbolic functions

(a) Fundamental formulae

(1) $\cosh^2 x - \sinh^2 x = 1$ (2) $\text{sech}^2 x + \tanh^2 x = 1$ (3) $\coth^2 x - \text{cosech}^2 x = 1$.

(b) Addition formulae

(4) $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$ (5) $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

(6) $\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$

(c) Functions of 2x.

(7) $\sinh 2x = 2 \sinh x \cosh x$

(8) $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$

(9) $\tanh 2x = \dfrac{2 \tanh x}{1 + \tanh^2 x}$

(d) Functions of 3x

(10) $\sinh 3x = 3 \sinh x + 4 \sinh^3 x$

(11) $\cosh 3x = 4\cosh^3 x - 3\cosh x$ (12) $\tanh 3x = \dfrac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}$

(e) (13) $\sinh x + \sinh y = 2 \sinh \dfrac{x+y}{2} \cosh \dfrac{x-y}{2}$ (14) $\sinh x - \sinh y = 2\cosh \dfrac{x+y}{2} \sinh \dfrac{x-y}{2}$

(15) $\cosh x + \cosh y = 2\cosh \dfrac{x+y}{2} \cosh \dfrac{x-y}{2}$ (16) $\cosh x - \cosh y = 2 \sinh \dfrac{x+y}{2} \sinh \dfrac{x-y}{2}$.

Proofs. (1) Since, for all values of θ, we have $\cos^2\theta + \sin^2\theta = 1$.

∴ putting $\theta = ix$, we get $\cos^2 ix + \sin^2 ix = 1$ or $\cosh^2 x - \sinh^2 x = 1$

Otherwise : $\cosh^2 x - \sinh^2 x = \left(\dfrac{e^x + e^{-x}}{2}\right)^2 - \left(\dfrac{e^x - e^{-x}}{2}\right)^2 = \dfrac{1}{4}[e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2] = 1$.

Similarly we can establish the formulae (2) and (3).

(4) $\sinh(x+y) = (1/i)\sin i(x+y) = -i[\sin ix \cos iy + \cos ix \sin iy]$

$$= -i[i \sinh x \,.\, \cosh y + \cosh x \,.\, i \sinh y] = \sinh x \cosh y + \cosh x \sinh y.$$

Otherwise : $\sinh x \cosh y + \cosh x \sinh y$

$$= \frac{e^x - e^{-x}}{2} \cdot \frac{e^y + e^{-y}}{2} + \frac{e^x + e^{-x}}{2} \cdot \frac{e^y - e^{-y}}{2} = \frac{e^{x+y} - e^{-(x+y)}}{2} = \sinh(x+y)$$

Similarly we can establish the formulae (5) and (6).

(12) $$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

Putting $A = ix$, $\tan 3ix = \dfrac{3\tan ix - \tan^3 ix}{1 - 3\tan^2 ix}$ or $i \tanh 3x = \dfrac{3(i\tanh x) - (i \tanh x)^3}{1 - 3(i\tanh x)^2}$

$$\tanh 3x = \frac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}$$

Similarly, we can establish the formulae (7) to (11).

(16) $\cos C - \cos D = -2 \sin \dfrac{C+D}{2} \sin \dfrac{C-D}{2}$

Putting $C = ix$, and $D = iy$, $\cos ix - \cos iy = -2 \sin i\dfrac{x+y}{2} \sin i \dfrac{x-y}{2}$

or $$\cosh x - \cosh y = -2\left(i \sinh \frac{x+y}{2}\right)\left(i \sinh\frac{x-y}{2}\right) = 2\sinh\frac{x+y}{2} \sinh \frac{x-y}{2}$$

Similarly, we can establish the formulae (13) to (15).

19.11 INVERSE HYPERBOLIC FUNCTIONS

(1) Definitions: If *sinh* $u = z$, then u is called the hyperbolic sine inverse of z and is written as $u = \sinh^{-1} z$. Similarly we define $\cosh^{-1} z$, $\tanh^{-1} z$, etc.

The inverse hyperbolic functions like other inverse functions are many-valued, but we shall consider only their principal values.

(2) *To show that (i)* $\sinh^{-1} z = \log [z + \sqrt{(z^2+1)}]$ *(Mumbai, 2009)*

(ii) $\cosh^{-1} z = \log [z + \sqrt{(z^2-1)}]$, *(iii)* $\tanh^{-1} z = \dfrac{1}{2} \log \dfrac{1+z}{1-z}$.

(*i*) Let $\sinh^{-1} z = u$, then $z = \sinh u = \dfrac{1}{2}(e^u - e^{-u})$

or $$2z = e^u - 1/e^u \quad \text{or} \quad e^{2u} - 2ze^u - 1 = 0$$

This being a quadratic in e^u, we have

$$e^u = \frac{2z \pm \sqrt{(4z^2+4)}}{2} = z \pm \sqrt{(z^2+1)}$$

$\therefore$ Taking the positive sign only, we have

$$e^u = z + \sqrt{(z^2+1)} \quad \text{or} \quad u = \log [z + \sqrt{(z^2+1)}]$$

Similarly we can establish (*ii*)

(*iii*) Let $\tanh^{-1} z = u$, then $z = \tanh u$

i.e., $$z = \frac{e^u - e^{-u}}{e^u + e^{-u}}.$$

Applying componendo and dividendo, we get $\dfrac{1+z}{1-z} = e^u/e^{-u} = e^{2u}$

or $$2u = \log\left(\frac{1+z}{1-z}\right) \text{ whence follows the result.}$$ *(P.T.U., 2005)*

Example 19.31. *If* $u = \log \tan (\pi/4 + \theta/2)$, *prove that*

(i) $\tanh u/2 = \tan \theta/2$ *(Mumbai, 2008 ; P.T.U., 2006 ; Madras, 2003)*

(ii) $\theta = -i \log \tan\left(\dfrac{\pi}{4} + \dfrac{iu}{2}\right)$. *(Kurukshetra, 2006)*

Solution. We have $$e^u = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) \quad \text{or} \quad \frac{e^{u/2}}{e^{-u/2}} = \frac{1+\tan \theta/2}{1-\tan \theta/2}$$

By componendo and dividendo, we get

$$\frac{e^{u/2} - e^{-u/2}}{e^{u/2} + e^{-u/2}} = \tan \theta/2 \quad \text{i.e.,} \quad \tanh \frac{u}{2} = \tan \frac{\theta}{2} \qquad \text{...(i)}$$

or $$\frac{1}{i} \tan \frac{iu}{2} = \frac{1}{i} \tanh \frac{i\theta}{2} \quad \text{or} \quad \frac{i\theta}{2} = \tanh^{-1}\left(\tan \frac{iu}{2}\right) = \frac{1}{2} \log \frac{1+\tan iu/2}{1-\tan iu/2}$$

or $$\theta = \frac{1}{i} \log \tan\left(\frac{\pi}{4} + \frac{iu}{2}\right) = -i \log \tan\left(\frac{\pi}{4} + \frac{iu}{2}\right). \qquad \text{...(ii)}$$

Example 19.32. *Show that* $\tanh^{-1} (\cos \theta) = \cosh^{-1} (\text{cosec } \theta)$. *(Kurukshetra, 2005)*

Solution. Let $\tanh^{-1} (\cos \theta) = \phi$ so that $\cos \theta = \tanh \phi$

or $\tanh^2 \phi = \cos^2 \theta$ or $1 - \text{sech}^2 \phi = \cos^2 \theta$

or $\text{sech}^2 \phi = 1 - \cos^2 \theta = \sin^2 \theta$ or $\text{sech } \phi = \sin \theta$

or $\cosh \phi = \text{cosec } \theta$ or $\phi = \cosh^{-1} (\text{cosec } \theta)$.

Example 19.33. *Find* $\tanh x$, *if* $5 \sinh x - \cosh x = 5$. (*Mumbai, 2004*)

Solution. We have $5(\sinh x - 1) = \cosh x$

or $25 (\sinh x - 1)^2 = \cosh^2 x = 1 + \sinh^2 x$

or $24 \sinh^2 x - 50 \sinh x + 24 = 0$ or $12 \sinh^2 x - 25 \sinh x + 12 = 0$

or $(3 \sinh x - 4)(4 \sinh x - 3) = 0$ whence $\sinh x = 4/3$ or $3/4$.

$\therefore$ $\cosh x = \sqrt{(1 + \sinh^2 x)} = 5/3$ or $-5/4$ [$\because$ $\cosh x = 5/4$ doesn't satisfy (*i*)]

Hence $\tanh x = \dfrac{4}{5}$ or $-\dfrac{3}{5}$.

PROBLEMS 19.5

1. Separate into real and imaginary parts
   (*i*) $\exp(z^2)$ where $z = x + iy$ (*ii*) $\exp(5 + i\pi/2)$ (*iii*) $\exp(5 + 3i)^2$.
2. From the definitions of $\sin z$ and $\cos z$, prove that
   (*i*) $\cos 2z = 2\cos^2 z - 1$ (*ii*) $\dfrac{\sin 2z}{1 - \cos 2z} = \cot z$ (*iii*) $\sin 3z = 3 \sin z - 4 \sin^3 z$.
3. Prove that $[\sin(\alpha - \theta) + e^{-i\alpha} \sin\theta]^n = \sin^{n-1}\alpha \{\sin(\alpha - n\theta) + e^{-i\alpha}\sin n\theta\}$
4. If $z = e^{i\theta}$, show that $\dfrac{z^2 - 1}{z^2 + 1} = i \tan\theta$.
5. Eliminate z from p cosech $z + q$ sech $z + r = 0$, p' cosech $z + q'$ sech $z + r' = 0$.
6. If $y = \log\tan x$, show that $\sinh ny = \dfrac{1}{2}(\tan^n x - \cot^n x)$.
7. If $\tan y = \tan\alpha \tanh\beta$ and $\tan z = \cot\alpha \tanh\beta$, prove that $\tan(y + z) = \sinh 2\beta \operatorname{cosec} 2\alpha$.
8. Prove that
   (*i*) $\cosh(\alpha + \beta) - \cosh(\alpha - \beta) = 2 \sinh\alpha \sinh\beta$
   (*ii*) $\sinh(\alpha + \beta)\cosh(\alpha - \beta) = \dfrac{1}{2}(\sinh 2\alpha + \sinh 2\beta)$.
9. Prove that (*i*) $(\cosh\theta \pm \sinh\theta)^n = \cosh n\theta + \sinh n\theta$; (*ii*) $\left(\dfrac{1 + \tanh\theta}{1 - \tanh\theta}\right)^3 = \cosh 6\theta + \sinh 6\theta$.
10. Express $\cosh^7\theta$ in terms of hyperbolic cosines of multiples of θ.
11. If $\sin\theta = \tanh x$, prove that $\tan\theta = \sinh x$.
12. If $\tan x/2 = \tanh u/2$, prove that
    (*i*) $\tan x = \sinh u$ and $\cos x \cosh u = 1$; (*ii*) $u = \log_e \tan(\pi/4 + x/2)$.
13. If $\cosh x = \sec\theta$, prove that
    (*i*) $\tanh^2 x/2 = \tan^2 \theta/2$ (*ii*) $x = \log_e \tan(\pi/4 + \theta/2)$.
14. Show that $\tan^{-1} z = \dfrac{i}{2} \log \dfrac{i + z}{i - z}$.
15. Prove that
    (*i*) $\sinh^{-1} x = \cosh^{-1}\sqrt{(1 + x^2)} = \tanh^{-1} \dfrac{x}{\sqrt{(1 - x^2)}} = \dfrac{1}{2} \operatorname{cosech}^{-1} \dfrac{1}{2x\sqrt{(1 + x^2)}}$
    (*ii*) $\tanh^{-1} x = \sinh^{-1} \dfrac{x}{\sqrt{(1 - x^2)}}$.
16. Show that
    (*i*) $\sinh^{-1}(\tan\theta) = \log\tan(\pi/4 + \theta/2)$ (*ii*) $\operatorname{sech}^{-1}(\sin\theta) = \log\cot\theta/2$.
17. Solve the equation $7\cosh x + 8\sinh x = 1$ for real values of x. (*Mumbai, 2008*)
18. Find $\tanh x$ if $\sinh x - \cosh x = 5$.

19.12 REAL AND IMAGINARY PARTS OF CIRCULAR AND HYPERBOLIC FUNCTIONS

(1) *To separate the real and imaginary parts of*

(i) sin $(x + iy)$; (ii) cos $(x + iy)$; (iii) tan $(x + iy)$; (iv) cot $(x + iy)$; (v) sec $(x + iy)$; (vi) cosec $(x + iy)$.

Proofs. (*i*) $\sin(x + iy) = \sin x \cos iy + \cos x \sin iy = \sin x \cosh y + i \cos x \sinh y$.

Similarly, $\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$

(*iii*) Let $\alpha + i\beta = \tan(x + iy)$ then $\alpha - i\beta = \tan(x - iy)$

Adding, $2\alpha = \tan(x + iy) + \tan(x - iy)$

i.e.,
$$\alpha = \frac{\sin(x + iy + x - iy)}{2\cos(x + iy)\cos(x - iy)} = \frac{\sin 2x}{\cos 2x + \cos 2iy} = \frac{\sin 2x}{\cos 2x + \cosh 2y}$$

Subtracting, $2i\beta = \tan(x + iy) - \tan(x - iy)$

i.e.,
$$i\beta = \frac{\sin 2iy}{2\cos(x + iy)\cos(x - iy)} = \frac{i \sinh 2y}{\cos 2x + \cosh 2y}$$

$\therefore$
$$\beta = \frac{\sinh 2y}{\cos 2x + \cosh 2y}$$

Similarly, $\cot(x + iy) = \dfrac{\sin 2x - i \sinh 2y}{\cosh 2y - \cos 2x}$.

(*v*) Let $\alpha + i\beta = \sec(x + iy)$ then $\alpha - i\beta = \sec(x - iy)$

Adding, $2\alpha = \sec(x + iy) + \sec(x - iy)$

i.e.,
$$\alpha = \frac{\cos(x - iy) + \cos(x + iy)}{2\cos(x + iy)\cos(x - iy)} = \frac{2\cos x \cos iy}{\cos 2x + \cos 2iy} = \frac{2\cos x \cosh y}{\cos 2x + \cosh 2y}$$

Subtracting, $2i\beta = \sec(x + iy) - \sec(x - iy)$

i.e.,
$$i\beta = \frac{\cos(x - iy) - \cos(x + iy)}{2\cos(x + iy)\cos(x - iy)} = \frac{2\sin x \sin iy}{\cos 2x + \cos 2iy} = \frac{2i \sin x \sinh y}{\cos 2x + \cosh 2y}$$

$\therefore$
$$\beta = \frac{2\sin x \sinh y}{\cos 2x + \cosh 2y}$$

Similarly, $\operatorname{cosec}(x + iy) = 2\,\dfrac{\sin x \cosh y - i \cos x \sinh y}{\cosh 2y - \cos 2x}$.

(2) *To separate the real and imaginary parts of*

(i) sinh $(x + iy)$; (ii) cosh $(x + iy)$; (iii) tanh $(x + iy)$.

Proofs. (*i*) $\sinh(x + iy) = (1/i)\sin i(x + iy) = (1/i)\sin(ix - y)$

$= (1/i)[\sin ix \cos y - \cos ix \sin y] = (1/i)[i \sinh x \cos y - \cosh x \sin y]$

$= \sinh x \cos y + i \cosh x \sin y$

Similarly, $\cosh(x + iy) = \cosh x \cos y + i \sinh x \sin y$.

(*iii*) If $\alpha + i\beta = \tanh(x + iy) = (1/i)\tan(ix - y)$

then $\alpha - i\beta = \tanh(x - iy) = (1/i)\tan(ix + y)$

Adding, $2\alpha = (1/i)[\tan(ix - y) + \tan(ix + y)]$

$$\alpha = \frac{\sin(ix - y + ix + y)}{i\,.\,2\cos(ix - y)\cos(ix + y)} = \frac{(1/i)\sin 2ix}{\cos 2ix + \cos 2y} = \frac{\sinh 2x}{\cosh 2x + \cos 2y}.$$

Subtracting, $2i\beta = (1/i)[\tan(ix - y) - \tan(ix + y)]$

i.e.,
$$i\beta = -\frac{\sin[(ix + y) - (ix - y)]}{i\,.\,2\cos(ix + y)\cos(ix - y)}$$

$\therefore$
$$\beta = \frac{\sin 2y}{\cos 2ix + \cos 2y} = \frac{\sin 2y}{\cosh 2x + \cos 2y}.$$

Example 19.34. *If* $\cosh(u + iv) = x + iy$, *prove that*

$$\frac{x^2}{\cosh^2 u} + \frac{y^2}{\sinh^2 u} = 1 \quad \text{(P.T.U., 2009 S)} \qquad \frac{x^2}{\cos^2 v} - \frac{y^2}{\sin^2 v} = 1. \quad \text{(Madras, 2000)}$$

Solution. Since $x + iy = \cosh(u + iv) = \cos(iu - v)$

$$= \cos iu \cos v + \sin iu \sin v = \cosh u \cos v + i \sinh u \sin v.$$

$\therefore$ equating real and imaginary parts, we get $x = \cosh u \cos v$; $y = \sinh u \sin v$

i.e., $$\frac{x}{\cosh u} = \cos v \quad \text{and} \quad \frac{y}{\sinh u} = \sin v$$

Squaring and adding, we get the first result.

Again $$\frac{x}{\cos v} = \cosh u \quad \text{and} \quad \frac{v}{\sin v} = \sinh u.$$

$\therefore$ squaring and subtracting, we get the second result.

Example 19.35. *If* $\tan(\theta + i\phi) = e^{i\alpha}$, *show that*

$$\theta = (n + 1/2)\pi/2 \quad \text{and} \quad \phi = \frac{1}{2}\log\tan(\pi/4 + \alpha/2).$$ *(S.V.T.U., 2007 ; Rohtak, 2005)*

Solution. Since $\tan(\theta + i\phi) = \cos\alpha + i\sin\alpha \quad \therefore \quad \tan(\theta - i\phi) = \cos\alpha - i\sin\alpha$

$\therefore$ $$\tan 2\theta = \tan[(\theta + i\phi) + (\theta - i\phi)]$$

$$= \frac{\tan(\theta + i\phi) + \tan(\theta - i\phi)}{1 - \tan(\theta + i\phi)\tan(\theta - i\phi)} = \frac{2\cos\alpha}{1 - (\cos^2\alpha + \sin^2\alpha)} = \frac{2\cos\alpha}{0} \to \infty$$

i.e., $$2\theta = n\pi + \pi/2 \quad \text{or} \quad \theta = (n + 1/2)\pi/2$$

Also $$\tan 2i\phi = \tan[(\theta + i\phi) - (\theta - i\phi)] = \frac{\tan(\theta + i\phi) - \tan(\theta - i\phi)}{1 + \tan(\theta + i\phi)\tan(\theta - i\phi)}$$

or $$i\tanh 2\phi = \frac{2i\sin\alpha}{1 + (\cos^2\alpha + \sin^2\alpha)} = i\sin\alpha \quad \text{or} \quad \frac{e^{2\phi} - e^{-2\phi}}{e^{2\phi} + e^{-2\phi}} = \frac{\sin\alpha}{1}$$

By componendo and dividendo, we get

$$\frac{e^{2\phi}}{e^{-2\phi}} = \frac{1 + \sin\alpha}{1 - \sin\alpha} = \frac{\cos^2\alpha/2 + \sin^2\alpha/2 + 2\sin\alpha/2\cos\alpha/2}{\cos^2\alpha/2 + \sin^2\alpha/2 - 2\sin\alpha/2\cos\alpha/2}$$

or $$e^{4\phi} = \frac{(\cos\alpha/2 + \sin\alpha/2)^2}{(\cos\alpha/2 - \sin\alpha/2)^2} = \left(\frac{1 + \tan\alpha/2}{1 - \tan\alpha/2}\right)^2$$

or $$e^{2\phi} = \frac{1 + \tan\alpha/2}{1 - \tan\alpha/2} = \tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right). \quad \text{Hence } \phi = \frac{1}{2}\log\tan(\pi/4 + \alpha/2).$$

Example 19.36. *Separate* $\tan^{-1}(x + iy)$ *into real and imaginary parts.* *(S.V.T.U., 2009)*

Solution. Let $\alpha + i\beta = \tan^{-1}(x + iy)$. Then $\alpha - i\beta = \tan^{-1}(x - iy)$

Adding, $$2\alpha = \tan^{-1}(x + iy) + \tan^{-1}(x - iy)^* = \tan^{-1}\frac{(x + iy) + (x - iy)}{1 - (x + iy)(x - iy)}$$

$\therefore$ $$\alpha = \frac{1}{2}\tan^{-1}\frac{2x}{1 - x^2 - y^2}$$

Subtracting, $$2i\beta = \tan^{-1}(x + iy) - \tan^{-1}(x - iy) = \tan^{-1}\frac{(x + iy) - (x - iy)}{1 + (x + iy)(x - iy)}$$

$$= \tan^{-1} i\frac{2y}{1 + x^2 + y^2} = i\tanh^{-1}\frac{2y}{1 + x^2 + y^2} \qquad [\because \tan^{-1} iz = i\tanh^{-1} z]$$

$\therefore$ $$\beta = \frac{1}{2}\tanh^{-1}\frac{2y}{1 + x^2 + y^2}.$$

Example 19.37. *Separate* $\sin^{-1}(\cos\theta + i\sin\theta)$ *into real and imaginary parts, where* θ *is a positive acute angle.*

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* $\tan^{-1} A \pm \tan^{-1} B = \tan^{-1}\dfrac{A \pm B}{1 \mp AB}$

Solution. Let $\sin^{-1}(\cos\theta + i\sin\theta) = x + iy$

Then $$\cos\theta + i\sin\theta = \sin(x+iy) = \sin x\cosh y + i\cos x\sinh y$$

$\therefore$ $\cos\theta = \sin x\cosh y$ ...(i) and $\sin\theta = \cos x\sinh y$ ...(ii)

Squaring and adding, we have

$$1 = \sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y = \sin^2 x(1+\sinh^2 y) + \cos^2 x\sinh^2 y$$
$$= \sin^2 x + \sinh^2 y(\sin^2 x + \cos^2 x)$$

or $$1 - \sin^2 x = \sinh^2 y, \quad i.e. \quad \cos^2 x = \sinh^2 y.$$

Hence from (ii), we have $\sin^2\theta = \cos^4 x$, *i.e.*, $\cos^2 x = \sin\theta$ because θ being a positive acute angle, $\sin\theta$ is positive.

As x is to be between $-\pi/2$ and $\pi/2$, therefore, we have

$$\cos x = +\sqrt{(\sin\theta)} \quad \text{or} \quad x = \cos^{-1}\sqrt{(\sin\theta)}$$

The relation (ii), then, gives $\sinh y = \sqrt{(\sin\theta)}$ so that $y = \log[\sqrt{(\sin\theta)} + \sqrt{(1+\sin\theta)}]$.

PROBLEMS 19.6

1. If $\sin(A+iB) = x+iy$, prove that

   (i) $\dfrac{x^2}{\cosh^2 B} + \dfrac{y^2}{\sinh^2 B} = 1$ (ii) $\dfrac{x^2}{\sin^2 A} - \dfrac{y^2}{\cos^2 A} = 1.$ *(P.T.U., 2010)*

2. If $\cos(\alpha+i\beta) = r(\cos\theta + i\sin\theta)$, prove that (i) $e^{2\beta} = \dfrac{\sin(\alpha-\theta)}{\sin(\alpha+\theta)}$. *(Kurukshetra, 2005 ; Madras, 2003)*

   (ii) $\beta = \dfrac{1}{2}\log\dfrac{\sin(\alpha-\theta)}{\sin(\alpha+\theta)}$. *(V.T.U., 2006)*

3. If $\cos(\theta+i\phi) = \cos\alpha + i\sin\alpha$, prove that

   (i) $\sin^2\theta = \pm\sin\alpha$ *(Madras, 2003)* (ii) $\cos 2\theta + \cosh 2\phi = 2$.

4. If $\tan(A+iB) = x+iy$, prove that

   (i) $x^2+y^2+2x\cot 2A = 1$. (ii) $x^2+y^2-2y\coth 2B + 1 = 0$. (iii) $x\sinh 2B = y\sin 2A$.

5. If $\tan(\theta+i\phi) = \tan\alpha + i\sec\alpha$, prove that $e^{2\phi} = \pm\cot\alpha/2$ and $2\theta = \left(n+\dfrac{1}{2}\right)\pi + \alpha$. *(Nagpur, 2009 ; S.V.T.U., 2008)*

6. If $\tan(x+iy) = \sin(u+iv)$, prove that $\dfrac{\sin 2x}{\sinh 2y} = \dfrac{\tan u}{\tan v}$. *(S.V.T.U., 2006)*

7. If $\operatorname{cosec}(\pi/4 + ix) = u+iv$, prove that $(u^2+v^2) = 2(u^2-v^2)$. *(Mumbai, 2009)*

8. If $x = 2\cos\alpha\cosh\beta$, $y = 2\sin\alpha\sinh\beta$, prove that $\sec(\alpha+i\beta) + \sec(\alpha-i\beta) = \dfrac{4x}{x^2+y^2}$.

9. If $a+ib = \tanh(v + i\pi/4)$, prove that $a^2+b^2 = 1$.

10. Reduce $\tan^{-1}(\cos\theta + i\sin\theta)$ to the form $a+ib$. *(Mumbai, 2009)*

    Hence show that $\tan^{-1}(e^{i\theta}) = \dfrac{n\pi}{2} + \dfrac{\pi}{4} - \dfrac{i}{2}\log\tan\left(\dfrac{\pi}{4} - \dfrac{\theta}{2}\right)$.

11. Separate $\cos^{-1}(\cos\theta + i\sin\theta)$ into real and imaginary parts, where θ is a positive acute angle.

12. If $\sin^{-1}(u+iv) = \alpha + i\beta$, prove that $\sin^2\alpha$ and $\cosh^2\beta$ are the roots of the equation

    $$x^2 - x(1+u^2+v^2) + u^2 = 0.$$

13. If $\cos^{-1}(x+iy) = \alpha + i\beta$, show that

    (i) $x^2\sec^2\alpha - y^2\operatorname{cosec}^2\alpha = 1$, (ii) $x^2\operatorname{sech}^2\beta + y^2\operatorname{cosech}^2\beta = 1$.

14. Prove that (i) $\sin^{-1}(ix) = 2n\pi + i\log(\sqrt{1+x^2}+x)$ (ii) $\sin^{-1}(\operatorname{cosec}\theta) = \pi/2 + i\log\cot\theta/2$.

19.13 LOGARITHMIC FUNCTION OF A COMPLEX VARIABLE

(1) *Definition. If $z(=x+iy)$ and $w(=u+iv)$ be so related that $e^w = z$, then w is said to be a logarithm of z to the base e and is written as $w = \log_e z$.* ...(i)

Also $$e^{w+2in\pi} = e^w \cdot e^{2in\pi} = z \qquad [\because e^{2in\pi} = 1]$$

$\therefore$ $$\log z = w + 2in\pi \qquad ...(ii)$$

i.e., the logarithm of a complex number has an infinite number of values and is, therefore, a multi-valued function.

The general value of the logarithm of z is written as $Log\ z$ (beginning with capital L) so as to distinguish it from its principal value which is written as $\log z$. This principal value is obtained by taking $n = 0$ in $Log\ z$.

Thus from (*i*) and (*ii*), $Log\ (x + iy) = 2in\pi + \log (x + iy)$.

Obs. 1. If $y = 0$, then $Log\ x = 2in\pi + \log x$.

This shows that the logarithm of a real quantity is also multi-valued. Its principal value is real while all other values are imaginary.

2. *We know that the logarithm of a negative quantity has no real value. But we can now evaluate this.*

e.g. $\log_e (-2) = \log_e 2(-1) = \log_e 2 + \log_e (-1) = \log_e 2 + i\pi$ $\quad [\because\ -1 = \cos \pi + i \sin \pi = e^{i\pi}]$

$= 0.6931 + i\ (3.1416)$.

(2) *Real and imaginary parts of Log* $(x + iy)$.

$$Log\ (x + iy) = 2in\pi + \log (x + iy)$$

$$= 2in\pi + \log [r (\cos \theta + i \sin \theta)] \quad \left\{\begin{array}{l}\text{Put } x = r \cos \theta,\ y = r \sin \theta \text{ so that} \\ r = \sqrt{(x^2 + y^2)} \text{ and } \theta = \tan^{-1} (y/x)\end{array}\right.$$

$$= 2in\pi + \log (re^{i\theta})$$

$$= 2in\pi + \log r + i\theta = \log \sqrt{(x^2 + y^2)} + i\ \{2n\pi + \tan^{-1} (y/x)\}$$

(3) *Real and imaginary parts of* $(\alpha + i\beta)^{x + iy}$

$$(\alpha + i\beta)^{x + iy} = e^{(x + iy)\ Log\ (\alpha + i\beta)} = e^{(x + iy)\ [2in\pi + \log (\alpha + i\beta)]} \quad \left\{\begin{array}{l}\text{Put } \alpha = r \cos \theta,\ \beta = r \sin \theta \text{ so that} \\ r = \sqrt{(\alpha^2 + \beta^2)} \text{ and } \theta = \tan^{-1} \beta/\alpha\end{array}\right.$$

$$= e^{(x+iy)\,[2in\pi + \log r e^{i\theta}]} = e^{(x + iy)\ [\log r + i\ (2n\pi + \theta)]}$$

$$= e^{A + iB} = e^A (\cos B + i \sin B).$$

where $A = x \log r - y\ (2n\pi + \theta)$ and $B = y \log r + x\ (2n\pi + \theta)$.

$\therefore$ the required real part $= e^A \cos B$ and the imaginary part $= e^A \sin B$.

Example 19.38. *Find the general value of* $\log (-i)$.

Solution. $Log\ (-i) = 2in\pi + \log [0 + i(-1)]$ $\quad \left\{\begin{array}{l}\text{Put } 0 = r \cos \theta,\ -1 = r \sin \theta 2 \\ \text{so that } r = 1 \text{ and } \theta = -\pi/2\end{array}\right.$

$$= 2in\pi + \log [r (\cos \theta + i \sin \theta)] = 2in\pi + \log (re^{i\theta})$$

$$= 2in\pi + \log r + i\theta = 2in\pi + \log 1 + i\ (-\pi/2) = i\left(2n - \frac{1}{2}\right)\pi.$$

Example 19.39. *Prove that* (*i*) $i^i = e^{-(4n + 1)\pi/2}$ *and* $Log\ i^i = -\left(2n + \frac{1}{2}\right)\pi$.

(*ii*) $(\sqrt{i})^{\sqrt{i}} = e^{-\alpha}$ *cis* α *where* $\alpha = \pi/4\sqrt{2}$. (*Mumbai, 2008*)

Solution. (*i*) By definition, we have

$$i^i = e^{i\ Log\ i} = e^{i\ (2in\pi + \log i)} = e^{-2n\pi + i \log [\exp (i\pi/2)]} \quad [\because\ i = \text{cis}\ \pi/2 = \exp (i\pi/2)]$$

$$= e^{-2n\pi + i(i\pi/2)} = e^{-(2n + 1/2)\pi}$$

Taking logarithms, we get (*ii*)

(*ii*) $$(\sqrt{i})^{\sqrt{i}} = e^{\sqrt{i} \log \sqrt{i}}$$

Now $$\sqrt{i}\ \log \sqrt{i} = \frac{1}{2}\left(\cos \frac{\pi}{2} + i \sin \pi/2\right)^{1/2} \log \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)$$

$$= \frac{1}{2}\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right) \log (e^{i\pi/2}) = \frac{1}{2}\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right) \frac{i\pi}{2}$$

$$= \frac{i\pi}{4}\left(\frac{i}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = -\frac{\pi}{4\sqrt{2}} + i\,\frac{\pi}{4\sqrt{2}}$$

Hence $\quad (\sqrt{i})^{\sqrt{i}} = e^{-\alpha + i\alpha}$ where $\alpha = \pi/4\sqrt{2}$

$$= e^{-\alpha} \cdot e^{i\alpha} = e^{-\alpha}(\cos \alpha + i \sin \alpha).$$

Example 19.40. *If $(a + ib)^p = m^{x+iy}$, prove that one of the values of y/x is $2 \tan^{-1}(b/a) \div \log(a^2 + b^2)$.*

Solution. Taking logarithms, $(a + ib)^p = m^{x+iy}$ gives $p \log(a + ib) = (x + iy) \log m$

or
$$p\left(\frac{1}{2}\log(a^2 + b^2) + i \tan^{-1}\frac{b}{a}\right) = x \log m + iy \log m$$

Equating real and imaginary parts from both sides, we get

$$\frac{p}{2}\log(a^2 + b^2) = x \log m \quad ...(i), \qquad p \tan^{-1}\frac{b}{a} = y \log m \quad ...(ii)$$

Division of (*ii*) by (*i*) gives

$$y/x = 2 \tan^{-1}(b/a)/\log(a^2 + b^2).$$

Example 19.41. *If $i^{i^{\cdots\infty}} = A + iB$, prove that $\tan \pi A/2 = B/A$ and $A^2 + B^2 = e^{-\pi/B}$.* (S.V.T.U., 2006 S)

Solution. $\quad i^{i^{\cdots\infty}} = A + iB \quad$ *i.e.* $\quad i^{A+iB} = A + iB$

or
$$A + iB = e^{(A+iB)\log i} = e^{(A+iB)\log(\cos \pi/2 + i \sin \pi/2)}$$
$$= \exp[(A + iB)\log(e^{i\pi/2})] = e^{(A+iB)(i\pi/2)}$$
$$= e^{-B\pi/2} \cdot e^{i\pi A/2} = e^{-B\pi/2}\left(\cos\frac{\pi A}{2} + i \sin\frac{\pi A}{2}\right)$$

Equating real and imaginary parts, we get

$$A = e^{-B\pi/2}\cos\frac{\pi A}{2} \quad ...(i) \qquad B = e^{-B\pi/2}\sin\frac{\pi A}{2} \quad ...(ii)$$

Division of (*ii*) by (*i*) gives $B/A = \tan \pi A/2$

Squaring and adding (*i*) and (*ii*), $A^2 + B^2 = e^{-B\pi}$.

Example 19.42. *Prove that $\log\left(\frac{a+ib}{a-ib}\right) = 2i \tan^{-1}\left(\frac{b}{a}\right)$. Hence evaluate $\cos\left[i \log\left(\frac{a+ib}{a-ib}\right)\right]$.* (P.T.U., 2006)

Solution. Putting $a = r\cos\theta$, $b = r\sin\theta$ so that $\theta = \tan^{-1} b/a$, we have

$$\log\left(\frac{a+ib}{a-ib}\right) = \log\frac{r(\cos\theta + i\sin\theta)}{r(\cos\theta - i\sin\theta)} = \log(e^{i\theta} \div e^{-i\theta})$$
$$= \log e^{2i\theta} = 2i\theta = 2i \tan^{-1} b/a.$$

Thus
$$\cos\left[i\log\left(\frac{a+ib}{a-ib}\right)\right] = \cos[i(2i\theta)] = \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-(b/a)^2}{1+(b/a)^2} = \frac{a^2-b^2}{a^2+b^2}.$$

Example 19.43. *Separate into real and imaginary parts $\log \sin(x + iy)$.*

Solution.
$$\log\sin(x + iy) = \log(\sin x \cos iy + \cos x \sin iy)$$
$$= \log(\sin x \cosh y + i \cos x \sinh y) = \log r(\cos\theta + i\sin\theta),$$

where $\quad r\cos\theta = \sin x \cosh y$ and $r\sin\theta = \cos x \sinh y$,

so that
$$r = \sqrt{(\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y)}$$
$$= \sqrt{\frac{1-\cos 2x}{2}\cdot\frac{1+\cosh 2y}{2} + \frac{1+\cos 2x}{2}\cdot\frac{\cosh 2y - 1}{2}} = \sqrt{\left[\frac{1}{2}(\cosh 2y - \cos 2x)\right]}$$

and $\quad \theta = \tan^{-1}(\cot x \tanh y)$.

Thus $\log\sin(x + iy) = \log(re^{i\theta}) = \log r + i\theta$

$$= \frac{1}{2}\log\left[\frac{1}{2}(\cosh 2y - \cos 2x)\right] + i \tan^{-1}(\cot x \tanh y).$$

Example 19.44. *Find all the roots of the equation*
(i) sin z = cosh 4 *(ii) sinh z = i.*

Solution. (*i*) $\sin z = \cosh 4 = \cos 4i = \sin (\pi/2 - 4i)$

$\therefore$ $$z = n\pi + (-1)^n (\pi/2 - 4i)$$ $\left\{\begin{array}{l}\because \text{ If } \sin\theta = \sin\alpha \\ \text{then } \theta = n\pi + (-1)^n \alpha\end{array}\right.$

(*ii*) $$i = \sinh z = \frac{e^z - e^{-z}}{2}$$

or $$e^{2z} - 2ie^z - 1 = 0, \quad i.e. \quad (e^z - i)^2 = 0 \quad i.e., \quad e^z = i$$

or $$z = Log\, i = 2in\pi + \log i = 2in\pi + \log e^{i\pi/2} = 2in\pi + i\pi/2 = i\left(2n + \frac{1}{2}\right)\pi.$$

PROBLEMS 19.7

1. Find the general value of
(*i*) $\log (6 + 8i)$ (*Rohtak, 2006*) (*ii*) $\log (-1)$. (*J.N.T.U., 2003*)
2. Show that (*i*) $\log (1 + i \tan \alpha) = \log (\sec \alpha) + i\alpha$, where α is an acute angle.
(*ii*) $\text{Log}_e \dfrac{3-i}{3+i} = 2i\left(n\pi - \tan^{-1}\dfrac{1}{3}\right)$.
3. If $(a_1 + ib_1)(a_2 + ib_2)......(a_n + ib_n) = A + iB$, prove that
(*i*) $(a_1^2 + b_1^2)(a_2^2 + b_2^2)......(a_n^2 + b_n^2) = A^2 + B^2$
(*ii*) $\tan^{-1}\dfrac{b_1}{a_1} + \tan^{-1}\dfrac{b_2}{a_2} + ... + \tan^{-1}\dfrac{b_n}{a_n} = \tan^{-1}\dfrac{B}{A}$.
4. Find the modulus and argument of (*i*) $(1 - i)^{1+i}$. (*P.T.U., 2010*) (*ii*) $i^{\log (1+i)}$
5. If $i^{\alpha + i\beta} = \alpha + i\beta$, prove that $\alpha^2 + \beta^2 = e^{-(4n+1)\pi\beta}$. (*Kurukshetra, 2005*)
6. Prove that $\log \left\{\dfrac{\sin (x + iy)}{\sin (x - iy)}\right\} = 2i \tan^{-1} (\cot x \tanh y)$. (*Mumbai, 2007*)
7. Prove that $\tan\left[i \log\left(\dfrac{a - ib}{a + ib}\right)\right] = \dfrac{2ab}{a^2 - b^2}$.
8. If $\tan \log (x + iy) = a + ib$ where $a^2 + b^2 \neq 1$, show that $\tan \log (x^2 + y^2) = \dfrac{2a}{1 - a^2 - b^2}$.
9. If $\sin^{-1} (x + iy) = \log (A + iB)$, show that $\dfrac{x^2}{\sin^2 u} - \dfrac{y^2}{\cos^2 u} = 1$, where $A^2 + B^2 = e^{2u}$.
10. Separate into real and imaginary parts $\log \cos (x + iy)$.
11. Find all the roots of the equation, (*i*) $\cos z = 2$, (*ii*) $\tanh z + 2 = 0$.

19.14 SUMMATION OF SERIES – 'C + iS' METHOD

This is the most general method and is applied to find the sum of a series of the form
$$a_0 \sin \alpha + a_1 \sin (\alpha + \beta) + a_2 \sin (\alpha + 2\beta) + ...$$
or $$a_0 \cos \alpha + a_1 \cos (\alpha + \beta) + a_2 \cos (\alpha + 2\beta) + ...$$

Procedure. (*i*) Put the given series = S (or C) according as it is a series of sines (or cosines). Then write C (or S) = a similar series of cosines (or sines).

e.g., If $$S = a_0 \sin \alpha + a_1 \sin (\alpha + \beta) + a_2 \sin (\alpha + 2\beta) + ...$$
then $$C = a_0 \cos \alpha + a_1 \cos (\alpha + \beta) + a_2 \cos (\alpha + 2\beta)...$$

(*ii*) Multiply the series of sines by i and add to the series of cosines, so that
$$C + iS = a_0 [\cos \alpha + i \sin \alpha] + a_1 [\cos (\alpha + \beta) + i \sin (\alpha + \beta)] + ...$$
$$= a_0 e^{i\alpha} + a_1 e^{i(\alpha + \beta)} + a_2 e^{i(\alpha + 2\beta)} + ...$$

(*iii*) Sum up this last series using any of the following standard series :

(1) Exponential series *i.e.*, $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \infty = e^x$

(2) Sine, cosine, sinh or cosh series

i.e., $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \infty = \sin x, \quad 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \infty = \cos x$

$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \infty = \sinh x, \quad 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots \infty = \cosh x$

(3) Logarithmic series

i.e., $x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots \infty = \log(1 + x), \quad -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots \infty\right) = \log(1 - x)$

(4) Gregory's series

i.e., $x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots \infty = \tan^{-1} x, \quad x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots \infty = \tanh^{-1} x = \frac{1}{2} \log \frac{1+x}{1-x}$

(5) Binomial series

i.e., $1 + nx + \frac{n(n-1)}{1.2} x^2 + \frac{n(n-1)(n-2)}{1.2.3} x^3 + \ldots \infty = (1 + x)^n$

$1 - nx + \frac{n(n+1)}{2!} x^2 - \frac{n(n+1)(n+2)}{3!} x^3 + \ldots \infty = (1 + x)^{-n}$

$1 + nx + \frac{n(n+1)}{2!} x^2 + \frac{n(n+1)(n+2)}{3!} x^3 + \ldots \infty = (1 - x)^{-n}$

(6) Geometric series

i.e., $a + ar + ar^2 + \ldots$ to n terms $= a\,\frac{1 - r^n}{1 - r}$, $a + ar + ar^2 + \ldots\infty = \frac{a}{1-r}, |r| < 1.$

(*iv*) Finally express the sum thus obtained in the form $A + iB$ so that by equating the real and imaginary parts, we get $C = A$ and $S = B$.

Series depending on exponential series

Example 19.45. *Sum the series* $\sin \alpha + x \sin(\alpha + \beta) + \frac{x^2}{2!} \sin(\alpha + 2\beta) + \ldots \infty$.

Solution. Let $S = \sin \alpha + x \sin(\alpha + \beta) + \frac{x^2}{2!} \sin(\alpha + 2\beta) + \ldots \infty$

and $C = \cos \alpha + x \cos(\alpha + \beta) + \frac{x^2}{2!} \cos(\alpha + 2\beta) + \ldots \infty$

$$\therefore \quad C + iS = [\cos \alpha + i \sin \alpha] + x[\cos(\alpha + \beta) + i \sin(\alpha + \beta)] + \frac{x^2}{2!}[\cos(\alpha + 2\beta) + i \sin(\alpha + 2\beta)] + \ldots\ldots \infty$$

$$= e^{i\alpha} + xe^{i(\alpha+\beta)} + \frac{x^2}{2!} \cdot e^{i(\alpha+2\beta)} + \ldots \infty = e^{i\alpha}\left[1 + \frac{xe^{i\beta}}{1!} + \frac{x^2 e^{2i\beta}}{2!} + \ldots \infty\right]$$

$$= e^{i\alpha} \cdot e^{xe^{i\beta}} = e^{i\alpha} e^{x(\cos\beta + i \sin\beta)} = e^{x\cos\beta + i(\alpha + x\sin\beta)} = e^{x\cos\beta} e^{i(\alpha + x\sin\beta)}$$

$$= e^{x\cos\beta}[\cos(\alpha + x\sin\beta) + i \sin(\alpha + x\sin\beta)]$$

Equating imaginary parts from both sides, we have $S = e^{x\cos\beta} \sin(\alpha + x\sin\beta)$.

Series depending on logarithmic series

Example 19.46. *Sum the series*

$$\sin^2\theta - \frac{1}{2}\sin 2\theta \sin^2\theta + \frac{1}{3}\sin 3\theta \sin^3\theta - \frac{1}{4}\sin 4\theta \sin^4\theta + \ldots \infty.$$

(*P.T.U., 2010 ; V.T.U., 2006 S*)

Solution. Let $S = \sin\theta . \sin\theta - \frac{1}{2}\sin 2\theta . \sin^2\theta + \frac{1}{3}\sin 3\theta . \sin^3\theta - \dots \infty$

and $C = \cos\theta . \sin\theta - \frac{1}{2}\cos 2\theta . \sin^2\theta + \frac{1}{3}\cos 3\theta . \sin^3\theta - \dots \infty$

$\therefore$ $C + iS = e^{i\theta}\sin\theta - \frac{e^{2i\theta}\sin^2\theta}{2} + \frac{e^{3i\theta}\sin^3\theta}{3} - \dots \infty$

$= \log(1 + e^{i\theta}\sin\theta) = \log[1 + (\cos\theta + i\sin\theta)\sin\theta]$

$= \log[1 + \cos\theta\sin\theta + i\sin^2\theta]$ [Put $1 + \cos\theta\sin\theta = r\cos\alpha$; $\sin^2\theta = r\sin\alpha$] ...(*i*)

$= \log r(\cos\alpha + i\sin\alpha) = \log re^{i\alpha} = \log r + i\alpha$

Equating imaginary parts, we have $S = \alpha = \tan^{-1}\left(\frac{\sin^2\theta}{1 + \cos\theta\sin\theta}\right)$. [from (*i*)]

Series depending on binomial series

Example 19.47. *Find the sum to infinity of the series*

$$1 - \frac{1}{2}\cos\theta + \frac{1.3}{2.4}\cos 2\theta - \frac{1.3.5}{2.4.6}\cos 3\theta + \dots (-\pi < \theta < \pi).$$ (*S.V.T.U., 2009*)

Solution. Let $C = 1 - \frac{1}{2}\cos\theta + \frac{1.3}{2.4}\cos 2\theta - \frac{1.3.5}{2.4.6}\cos 3\theta + \dots \infty$

and $S = 0 - \frac{1}{2}\sin\theta + \frac{1.3}{2.4}\sin 2\theta - \frac{1.3.5}{2.4.6}\sin 3\theta + \dots \infty$

$\therefore$ $C + iS = 1 - \frac{1}{2}e^{i\theta} + \frac{1.3}{2.4}e^{2i\theta} - \frac{1.3.5}{2.4.6}e^{3i\theta} - \dots$

$$= 1 + \left(-\frac{1}{2}\right)e^{i\theta} + \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)}{1.2}e^{2i\theta} + \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{1.2.3}e^{3i\theta} + \dots$$

$$= (1 + e^{i\theta})^{-1/2} = (1 + \cos\theta + i\sin\theta)^{-1/2} = \left(2\cos^2\frac{\theta}{2} + i . 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{-1/2}$$

$$= \left(2\cos\frac{\theta}{2}\right)^{-1/2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)^{-1/2} = \left(2\cos\frac{\theta}{2}\right)^{-1/2}\left(\cos\frac{\theta}{4} - i\sin\frac{\theta}{4}\right).$$

Equating real parts, we have $C = (2\cos\theta/2)^{-1/2}\cos\theta/4$.

PROBLEMS 19.8

Sum the following series :

1. $\cos\theta + \sin\theta\cos 2\theta + \frac{\sin^2\theta}{1.2}\cos 3\theta + \dots \infty$. (*P.T.U., 2005*)
2. $\sin\alpha - \frac{\sin(\alpha + 2\beta)}{2!} + \frac{\sin(\alpha + 4\beta)}{4!} - \dots \infty$.
3. $x\sin\theta - \frac{1}{2}x^2\sin 2\theta + \frac{1}{3}x^3\sin 3\theta - \dots \infty$. (*Kurukshetra, 2005*)
4. $\cos\theta - \frac{1}{2}\cos 2\theta + \frac{1}{3}\cos 3\theta \dots \infty$. (*S.V.T.U., 2006*)
5. $e^{\alpha}\cos\beta - \frac{e^{3\alpha}}{3}\cos 3\beta + \frac{e^{5\alpha}}{5}\cos 5\beta - \dots \infty$.
6. $c\sin\alpha + \frac{c^3}{3}\sin 3\alpha + \frac{c^5}{5}\sin 5\alpha + \dots \infty$.
7. $1 - \frac{1}{2}\cos 2\theta + \frac{1.3}{2.4}\cos 4\theta - \frac{1.3.5}{2.4.6}\cos 6\theta + \dots \infty$. (*Kurukshetra, 2006*)
8. $n\sin\alpha + \frac{n(n+1)}{1.2}\sin 2\alpha + \frac{n(n+1)(n+2)}{1.2.3}\sin 3\alpha + \dots \infty$.
9. $\sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \dots \sin(\alpha + \overline{n-1}\,\beta)$ (*P.T.U., 2009 S*)
10. $\cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \dots$ to n terms. (*Kurukshetra, 2006*)
11. $\sin\alpha\cos\alpha + \sin^2\alpha\cos 2\alpha + \sin^3\alpha\cos 3\alpha + \dots\infty$.
12. $1 + x\cos\theta + x^2\cos 2\theta + \dots + x^{n-1}\cos(n-1)\theta$.

19.15 APPROXIMATIONS AND LIMITS

Example 19.48. *If* $\frac{\sin\theta}{\theta} = \frac{599}{600}$, *find an approximate value of* θ *in radians.*

Solution. Since $\frac{\sin\theta}{\theta} = 1 - \frac{1}{600}$ which is nearly equal to 1. $\therefore$ θ must be very small.

We know that $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots$

$$\therefore \quad \frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{6} + \frac{\theta^4}{5!}$$

Omitting θ^4 and higher powers, we have

$$\frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{6} = 1 - \frac{1}{600} \quad \text{or} \quad \theta^2 = \frac{1}{100}. \text{ Hence } \theta = 0.1 \text{ radians.}$$

Example 19.49. *Solve approximately* $\sin\left(\frac{\pi}{6} + \theta\right) = 0.51$.

Solution. Since 0.51 is nearly equal to 1/2, which is the value of $\sin \pi/6$, so θ must be very small.

$$\therefore \quad \sin\left(\frac{\pi}{6} + \theta\right) = \sin\frac{\pi}{6}\cos\theta + \cos\frac{\pi}{6}\sin\theta = \frac{1}{2}\left(1 - \frac{\theta^2}{2!} + \ldots\right) + \frac{\sqrt{3}}{2}\left(\theta - \frac{\theta^3}{3!} + \ldots\right)$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2}\theta, \text{ omitting } \theta^2 \text{ and higher powers of } \theta.$$

Hence the given equation becomes,

$$\frac{1}{2} + \frac{\sqrt{3}}{2}\theta = 0.51 \quad \text{or} \quad \theta = \frac{1}{50\sqrt{3}}$$

or
$$\theta = \frac{1}{50\sqrt{3}} \text{ radian} = \frac{\sqrt{3}}{150} \times 57.29 \text{ degrees nearly} = 39.7'.$$

PROBLEMS 19.9

1. Given $\frac{\sin\theta}{\theta} = \frac{5045}{5046}$, show that θ is $1°58'$ nearly.
2. If $\frac{\sin\theta}{\theta} = \frac{2165}{2166}$, find an approximate value of θ in radians. *(Madras, 2003)*
3. If $\cos\theta = \frac{1681}{1682}$, find θ approximately.
4. Solve approximately the equation $\cos\left(\frac{\pi}{3} + \theta\right) = 0.49$.

19.16 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 19.10

Choose the correct answer or fill up the blanks in each of the following problems :

1. If $x + iy = \sqrt{2} + 3i$, then $x^2 + y$ is
   (*a*) 7 (*b*) 5 (*c*) 13 (*d*) $\sqrt{2} + 3$
2. The real part of $(\sin x + i\cos x)^5$ is
   (*a*) $-\cos 5x$ (*b*) $-\sin 5x$ (*c*) $\sin 5x$ (*d*) $\cos 5x$.

3. The number $(i)^i$ is
   (a) a purely imaginary number (b) an irrational number
   (c) a rational number (d) an integer.
4. The relation $|3 - z| + |3 + z| = 5$ represents
   (a) a circle (b) a parabola (c) an ellipse (d) a hyperbola.
5. z is a complex number with $|z| = 1$ and arg $(z) = 3\pi/4$. The value of z is
   (a) $(1 + i)/\sqrt{2}$ (b) $(-1 + i)/\sqrt{2}$ (c) $(1 - i)/\sqrt{2}$ (d) $(-1 - i)/\sqrt{2}$.
6. If $f(z) = e^{2z}$, then the imaginary part of $f(z)$ is
   (a) $e^y \sin x$ (b) $e^x \cos y$ (c) $e^{2x} \cos 2y$ (d) $e^{2x} \sin 2y$.
7. Expansion of $\sin^m \theta \cos^n \theta$ is a series of sines of multiples of θ when m is ......
8. Expansion of $\cos 6\theta$ in terms of $\cos \theta$ is ......
9. If $f(z) = 3\bar{z}$, then the value of $f(z)$ at $z = 2 + 4i$ is ......
10. If $x = \cos\theta + i \sin\theta$, then $x^n - 1/x^n$ = ...
11. Imaginary part of $(2 + i3)/(3 - i4)$ is ......
12. Real part of $\cosh(x + iy)$ is ......
13. If $\dfrac{\sin\theta}{\theta} = \dfrac{2165}{2166}$, then θ = ... approximately.
14. If $\tan x/2 = \tanh y/2$, then $\cos x \cosh y$ = ...
15. Imaginary part of $\sin \bar{z}$ is ......
16. Modulus of $(\sqrt{i})^{\sqrt{i}}$ = ...
17. If $\sin\alpha + \sin\beta + \sin\gamma = 0 = \cos\alpha + \cos\beta + \cos\gamma$, then $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3 \cos$ (.............)
18. $\log(-1)$ = ...
19. $(i)^i$ is purely real or imaginary
20. If $\sin\theta = \tanh\phi$, then $\tan\theta$ = ...
21. Imaginary part of $\tan(\theta + i\phi)$ = ...
22. $\cos 5\alpha$ = (.........) $\cos^5\alpha$ + (.....) $\cos^3\alpha$ + (.....) $\cos\alpha$.
23. Cube roots of unity form .... triangle.
24. If $|z_1 + z_2| = |z_1 - z_2|$ then amp (z_1) – amp (z_2) is ....
25. If $-3 + ix^2y$ and $x^2 + y + 4i$ represent conjugate complex number then x = ... and y = ...
26. If $\left|\dfrac{z-a}{z-b}\right| = k\ (\neq 1)$, then the locus of z is ....
27. $(-i)^{-i}$ is purely real. (True or False)
28. The statements Re $z > 0$ and $|z - 1| < |z + 1|$ are equivalent. (*Mumbai, 2007*) (True or False)
29. Hyperbolic functions are periodic. (True or False)
30. nth roots of unity form a G.P. (True or False)
31. $\sin ix = -i \sinh x$. (*Mumbai, 2008*) (True or False)
32. If the sum and product of two complex numbers are real, then the two numbers must be either real or conjugate. (*Mumbai, 2008*) (True or False)
33. The modulus of the sum of two complex numbers $\geq$ to the sum of their moduli. (True or False)

CHAPTER 20

Calculus of Complex Functions

20.1 INTRODUCTION

In the previous chapter, we have dealt with some elementary complex functions—the exponential, logarithmic, circular and hyperbolic functions, evaluated at specific complex values. These functions are useful in the study of fluid mechanics, thermodynamics and electric fields. It, therefore, seems desirable to study the calculus of such functions.

20.2 (1) LIMIT OF A COMPLEX FUNCTION

A *function $w = f(z)$ is said to tend to* **limit** *l as z approaches a point z_0, if for every real ϵ, we can find a positive real δ such that*

$$| f(z) - l | < \epsilon \quad \text{for} \quad | z - z_0 | < \delta$$

i.e., for every $z \neq z_0$ in the δ–disc (dotted) of z-plane, $f(z)$ has a value lying in the ϵ-disc of w-plane (Fig. 20.1). In symbols, we write $\underset{z \to z_0}{\text{Lt}} f(z) = l$.

This definition of limit though similar to that in ordinary calculus, is quite different for in real calculus x approaches x_0 only along the line whereas here *z approaches z_0 from any direction in the z-plane.*

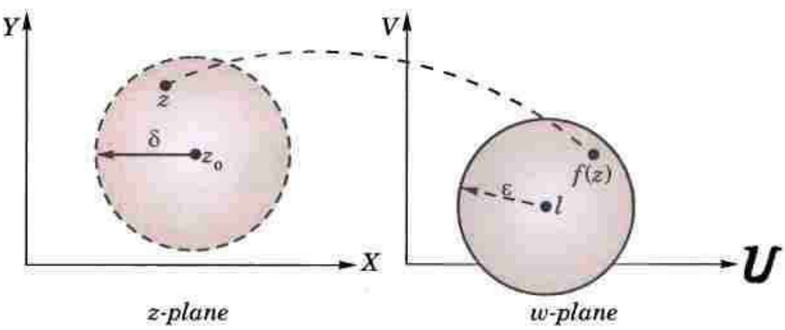

Fig. 20.1

(2) Continuity of f(z). *A function $w = f(z)$ is said to be* **continuous** *at $z = z_0$, if*

$$\underset{z \to z_0}{Lt}\ f(z) = f(z_0).$$

Further $f(z)$ is said to be continuous in any region R of the z-plane, if it is continuous at every point of that region.

Also if $w = f(z) = u(x, y) + iv(x, y)$ is continuous at $z = z_0$, then $u(x, y)$ and $v(x, y)$ are also continuous at $z = z_0$, *i.e.*, at $x = x_0$ and $y = y_0$. Conversely if $u(x, y)$ and $v(x, y)$ are continuous at (x_0, y_0), then $f(z)$ will be continuous at $z = z_0$. [cf. § 5.1 (3)].

20.3 (1) DERIVATIVE OF f(z)

Let $w = f(z)$ be a single-valued function of the variable $z = x + iy$. Then *the derivative of $w = f(z)$ is defined to be*

$$\frac{dw}{dz} = f'(z) = \underset{\delta z \to 0}{Lt}\ \frac{f(z + \delta z) - f(z)}{\delta z},$$

provided the limit exists and has the same value for all the different ways in which δz approaches zero.

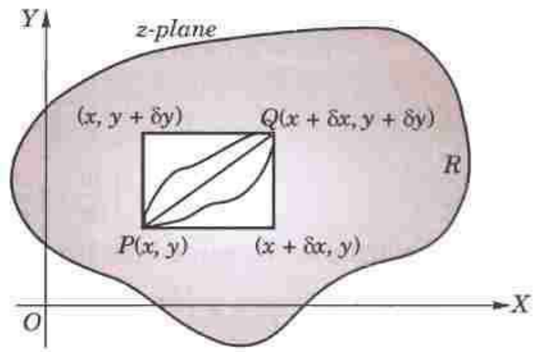

Fig. 20.2

Suppose $P(z)$ is fixed and $Q(z + \delta z)$ is a neighbouring point (Fig. 20.2). The point Q may approach P along any straight or curved path in the given region, *i.e.*, δz may tend to zero in any manner and dw/dz may not exist. It, therefore, becomes a fundamental problem to determine the necessary and sufficient conditions for dw/dz to exist. The fact is settled by the following theorem.

(2) Theorem. *The necessary and sufficient conditions for the derivative of the function $w = u(x, y) + iv(x, y) = f(z)$ to exist for all values of z in a region R, are*

(*i*) $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ *are continuous functions of x and y in R;*

(*ii*) $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$.

The relations (ii) are known as **Cauchy-Riemann* equations** or briefly *C-R* equations.

(*a*) *Condition is necessary.*

If $f(z)$ possesses a unique derivative at $P(z)$, then

$$f'(z) = \underset{\delta z \to 0}{\text{Lt}}\ \frac{f(z + \delta z) - f(z)}{\delta z}$$

$$= \underset{\substack{\delta x \to 0 \\ \delta y \to 0}}{\text{Lt}}\ \frac{\{u(x + \delta x, y + \delta y) + iv(x + \delta x, y + \delta y)\} - \{u(x, y) + iv(x, y)\}}{\delta x + i\delta y}$$

Since δz can approach zero in any manner, we can first assume δz to be wholly real and then wholly imaginary. When δz is wholly real, then $\delta y = 0$ and $\delta z = \delta x$.

$$\therefore \quad f'(z) = \underset{\delta x \to 0}{\text{Lt}} \left(\frac{u(x + \delta x, y) - u(x, y)}{\delta x} + i\, \frac{v(x + \delta x, y) - v(x, y)}{\delta x} \right) = \frac{\partial u}{\partial x} + i\, \frac{\partial v}{\partial x} \quad ...(1)$$

When δz is wholly imaginary, then $\delta x = 0$ and $\delta z = i\delta y$.

$$\therefore \quad f'(z) = \underset{\delta y \to 0}{\text{Lt}} \left(\frac{u(x, y + \delta y) - u(x, y)}{i\delta y} + i\, \frac{v(x, y + \delta y) - v(x, y)}{i\delta y} \right) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\delta y} - i\, \frac{\partial u}{\partial y} \quad ...(2)$$

Now the existence of $f'(z)$ requires the equality of (1) and (2).

* Named after *Cauchy* (p. 144) and the German mathematician *Bernhard Riemann* (1826—1866) who along with *Weierstrass* (p. 390) laid the foundations of complex analysis. Riemann introduced the concept of integration and made basic contributions to number theory and mathematical analysis. He developed the Riemannian geometry which formed the mathematical base for Einstein's relativity theory.

$$\therefore \qquad \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

On equating the real and imaginary parts from both sides, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad ...(3)$$

Thus the necessary conditions for the existence of the derivative of $f(z)$ is that the C-R equations should be satisfied. (*V.T.U., 2011 S*)

(*b*) *Condition is sufficient.* Suppose $f(z)$ is a single-valued function possessing partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ at each point of the region and the *C-R* equations (3) are satisfied.

Then by Taylor's theorem for functions of two variables (p. 220)

$$f(z + \delta z) = u(x + \delta x, y + \delta y) + iv(x + \delta x, y + \delta y)$$

$$= u(x, y) + \left(\frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y\right) + ... + i\left[v(x, y) + \left(\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y\right) + ...\right]$$

$$= f(z) + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\partial x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$$

[Omitting terms beyond the first powers of δx and δy]

or $$f(z + \delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\partial x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y.$$

Now using the *C-R* equation (3), replace $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ by $-\frac{\partial v}{\partial x}$ and $\frac{\partial u}{\partial x}$ respectively.

Then $$f(z + \delta z) - f(z) = \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\delta x + \left[-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right]\delta y = \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\delta x + \left[i\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\right]i\delta y$$

$$= \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right](\delta x + i\delta y) = \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\delta z$$

$$\therefore \qquad f'(z) = \underset{\delta z \to 0}{\text{Lt}} \frac{f(z + \delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{or} \quad \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

which by (1) or (2) proves the sufficiency of conditions.

20.4 ANALYTIC FUNCTIONS

A function $f(z)$ which is single-valued and possesses a unique derivative with respect to z at all points of a region R, is called an **analytic function** *of z in that region. An analytic function is also called a regular function* or an *holomorphic function.*

A function which is analytic everywhere in the complex plane, is known as an **entire function.** As derivative of a polynomial exists at every point, a polynomial of any degree is an *entire function.*

A point at which an analytic function ceases to possess a derivative is called a **singular point** *of the function.*

Thus if u and v are real single-valued functions of x and y such that $\delta u/\delta x$, $\delta u/\delta y$, $\delta v/\delta x$, $\delta v/\delta y$ are continuous throughout a region R, then the Cauchy-Riemann equations

$$\frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \frac{\partial \mathbf{v}}{\partial \mathbf{y}} \text{ and } \frac{\partial \mathbf{u}}{\partial \mathbf{y}} = -\frac{\partial \mathbf{v}}{\partial \mathbf{x}} \qquad ...(1)$$

are both necessary and sufficient conditions for the function $f(z) = u + iv$ to be analytic in R. The derivative of $f(z)$ is then, given by (1) of p. 664 or (2) of p. 665.

The real and imaginary parts of an analytic function are called *conjugate functions*. The relation between two conjugate functions is given by *C-R* equation (1).

Example 20.1. *If $w = \log z$, find dw/dz and determine where w is non-analytic.*
(*U.P.T.U., 2005 ; J.N.T.U., 2005*)

Solution. We have $w = u + iv = \log(x + iy) = \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1} y/x$ [By **(2)**, p. 665]

so that $u = \frac{1}{2}\log(x^2 + y^2),\ v = \tan^{-1} y/x.$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2} = \frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2} = -\frac{\partial v}{\partial x}.$$

Since the Cauchy-Riemann equations are satisfied and the partial derivatives are continuous except at (0, 0). Hence w is analytic everywhere except at $z = 0$.

$$\therefore \quad \frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2} = \frac{x - iy}{(x + iy)(x - iy)} = \frac{1}{x + iy} = \frac{1}{z}\ (z \neq 0).$$

Obs. The definition of the derivative of a function of complex variable is identical in form to that of the derivative of a function of real variable. Hence the rules of differentiation for complex functions are the same as those of real calculus. *Thus* **if, a complex function is once known to be analytic, it can be differentiated just in the ordinary way.**

Example 20.2. *If $f(z)$ is an analytic function with constant modulus, show that $f(z)$ is constant.* *(U.P.T.U., 2008 ; Mumbai, 2005 S ; Madras 2003 ; Bhopal, 2002 S)*

Solution. If $f(z) = u + iv$ is an analytic function, then

$$|f(z)| = \sqrt{(u^2 + v^2)} \text{ is constant} = c \text{ (say) or } u^2 + v^2 = c^2 \quad ...(i)$$

Differentiating (*i*) partially w.r.t. x and y, we get

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0; \qquad 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$$

or
$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \quad ...(ii) \qquad u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0 \quad ...(iii)$$

Since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ by C-R equations,

$$\therefore \quad (iii) \text{ becomes } -u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} = 0 \quad ...(iv)$$

Squaring and adding (*ii*) and (*iv*), we obtain

$$u^2\left(\frac{\partial u}{\partial x}\right)^2 + v^2\left(\frac{\partial v}{\partial x}\right)^2 + u^2\left(\frac{\partial v}{\partial x}\right)^2 + v^2\left(\frac{\partial u}{\partial x}\right)^2 = 0$$

or
$$(u^2 + v^2)\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] = 0 \text{ or } \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = 0 \quad [\because\ u^2 + v^2 = c^2 \neq 0] \quad ...(v)$$

Now
$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\therefore \quad |f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = 0 \quad \text{[By } (v)]$$

or $f'(z) = 0.$ or $f(z) = \text{constant}.$

Example 20.3. *Show that the function $f(z) = \sqrt{|xy|}$ is not analytic at the origin even though C.R. equations are satisfied thereof.* *(A.M.I.E.T.E., 2005 S ; Osmania, 2003)*

Solution. If $f(z) = \sqrt{|xy|} = u(x, y) + iv(x, y)$, then $u(x, y) = \sqrt{|xy|},\ v(x, y) = 0$

At the origin, we have

$$\frac{\partial u}{\partial x} = \underset{x \to 0}{\text{Lt}} \frac{u(x, 0) - u(0, 0)}{x} = \underset{x \to 0}{\text{Lt}} \frac{0 - 0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \underset{y \to 0}{\text{Lt}} \frac{u(0, y) - u(0, 0)}{y} = \underset{x \to 0}{\text{Lt}} \frac{0 - 0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \underset{x \to 0}{\text{Lt}} \frac{v(x, 0) - v(0, 0)}{x} = \underset{x \to 0}{\text{Lt}} \frac{0 - 0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \underset{y \to 0}{\text{Lt}} \frac{v(0, y) - v(0, 0)}{y} = \underset{y \to 0}{\text{Lt}} \frac{0 - 0}{y} = 0$$

$\therefore$ $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

i.e., C.R. equations are satisfied at the origin.

However $$f'(0) = \underset{z \to 0}{\text{Lt}} \frac{f(z) - f(0)}{z - 0} = \underset{z \to 0}{\text{Lt}} \frac{\sqrt{|xy|} - 0}{x + iy}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\sqrt{|mx^2|} - 0}{x(1 + im)}, \text{ when } z \to 0 \text{ along the line } y = mx$$

$$= \frac{\sqrt{|m|}}{1 + im} \text{ which is not unique.}$$

$\therefore$ $f'(0)$ does not exist. Hence $f(z)$ is not analytic at the origin.

Example 20.4. *Prove that the function $f(z)$ defined by*

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} \; (z \neq 0), f(0) = 0$$

is continuous and the Cauchy-Riemann equations are satisfied at the origin, yet $f'(0)$ does not exist.

(S.V.T.U., 2009 ; V.T.U., 2001)

Solution. $$\underset{z \to 0}{\text{Lt}} f(z) = \underset{\substack{x \to 0 \\ y \to 0}}{\text{Lt}} \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} = \underset{y \to 0}{\text{Lt}} \frac{-y^3(1-i)}{y^2} = \underset{y \to 0}{\text{Lt}} [-y(1-i)] = 0$$

$$\underset{z \to 0}{\text{Lt}} f(z) = \underset{\substack{y \to 0 \\ x \to 0}}{\text{Lt}} \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} = \underset{x \to 0}{\text{Lt}} \frac{x^3(1+i)}{x^2} = \underset{x \to 0}{\text{Lt}} [x(1+i)] = 0$$

Also $f(0) = 0$ (given).

Thus $\underset{z \to 0}{\text{Lt}} f(z) = f(0)$ when $x \to 0$ first and then $y \to 0$ and also vice-versa. Now let both x and y tend to zero simultaneously along the path $y = mx$. Then

$$\underset{z \to 0}{\text{Lt}} f(z) = \underset{\substack{y \to mx \\ x \to 0}}{\text{Lt}} \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{x^3(1+i) - m^3x^3(1-i)}{(1+m^2)x^2} = \underset{x \to 0}{\text{Lt}} \frac{x[1 + i - m^3(1-i)]}{1 + m^2} = 0$$

Hence $\underset{z \to 0}{\text{Lt}} f(z) = f(0)$, in whatever manner $z \to 0$. $\therefore$ $f(z)$ is continuous at the origin.

Now $$f(z) = \frac{x^3 - y^3}{x^2 + y^2} + i\frac{x^3 + y^3}{x^2 + y^2} = u(x, y) + iv(x, y).$$

Also $u(0, 0) = 0$, and $v(0, 0) = 0$ $\qquad [\because \; f(0) = 0]$

$\therefore$ $$\left(\frac{\partial u}{\partial x}\right)_{0,0} = \underset{x \to 0}{\text{Lt}} \frac{u(x, 0) - u(0, 0)}{x} = \underset{x \to 0}{\text{Lt}} \frac{x}{x} = 1$$

$$\left(\frac{\partial u}{\partial y}\right)_{0,0} = \underset{y \to 0}{\text{Lt}} \frac{u(0, y) - u(0, 0)}{y} = \underset{y \to 0}{\text{Lt}} \frac{-y}{y} = -1$$

$$\left(\frac{\partial v}{\partial x}\right)_{0,0} = \underset{x \to 0}{\text{Lt}} \frac{v(x, 0) - v(0, 0)}{x} = \underset{x \to 0}{\text{Lt}} \frac{x}{x} = 1$$

and $$\left(\frac{\partial v}{\partial y}\right)_{0,0} = \underset{y \to 0}{\text{Lt}} \frac{v(0, y) - v(0, 0)}{y} = \underset{y \to 0}{\text{Lt}} \frac{y}{y} = 1.$$

Hence at (0, 0), $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

Thus the *C-R* equations are satisfied at the origin.

But $$f'(0)=\underset{z\to 0}{\text{Lt}}\frac{f(z)-f(0)}{z}=\underset{z\to 0}{\text{Lt}}\frac{f(z)}{z}=\underset{z\to 0}{\text{Lt}}\frac{x^3-y^3+i(x^3+y^3)}{(x^2+y^2)(x+iy)}.$$

If $z\to 0$ along the path $y=mx$, then $f'(0)=\frac{1-m^3+i(1+m^3)}{(1+m^2)(1+im)}$

which assumes different values as m varies. So $f'(z)$ is not unique at (0, 0) *i.e.*, $f'(0)$ does not exist. Thus $f(z)$ is not analytic at the origin even though it is continuous and satisfies the *C-R* equations thereat.

Example 20.5. *Show that polar form of Cauchy-Riemann equations are*

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta},\ \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}.$$ (U.P.T.U., 2008 ; V.T.U., 2006)

Deduce that $\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0.$ (Bhopal, 2009 ; Kurukshetra, 2005)

Solution. If (r, θ) be the coordinates of a point whose cartesian coordinates are (x, y), then $z=x+iy=re^{i\theta}$.

$$\therefore \qquad u+iv=f(z)=f(re^{i\theta})$$

where u and v are now expressed in terms of r and θ.

Differentiating it partially w.r.t. r and θ, we have

$$\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}=f'(re^{i\theta})\,.\,e^{i\theta}$$

and $$\frac{\partial u}{\partial \theta}+i\frac{\partial v}{\partial \theta}=f'(re^{i\theta})\,.\,ire^{i\theta}=ir\left(\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right)$$

Equating real and imaginary parts, we get

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta} \qquad ...(i) \qquad \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta} \qquad ...(ii)$$

Differentiating (i) partially w.r.t. r, we get

$$\frac{\partial^2 u}{\partial r^2}=-\frac{1}{r^2}\frac{\partial v}{\partial \theta}+\frac{1}{r}\frac{\partial^2 v}{\partial \theta \partial r} \qquad ...(iii)$$

Differentiating (ii) partially w.r.t. θ, we have

$$\frac{\partial^2 u}{\partial \theta^2}=-r\frac{\partial^2 v}{\partial r \partial \theta} \qquad ...(iv)$$

Thus using (i), (ii) and (iv)

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=-\frac{1}{r^2}\frac{\partial v}{\partial \theta}+\frac{1}{r}\frac{\partial^2 v}{\partial \theta \partial r}+\frac{1}{r}\left(\frac{1}{r}\frac{\partial v}{\partial \theta}\right)+\frac{1}{r^2}\left(-r\frac{\partial^2 v}{\partial r \partial \theta}\right)=0 \qquad \left[\because \frac{\partial^2 v}{\partial \theta \partial r}=\frac{\partial^2 v}{\partial r \partial \theta}\right]$$

20.5 (1) HARMONIC FUNCTIONS

If $f(z)=u+iv$ be an analytic function in some region of the z-plane, then the Cauchy-Riemann equations are satisfied.

i.e., $$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \qquad ...(1) \qquad \text{and} \qquad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}. \qquad ...(2)$$

Differentiating (1) with respect to x and (2) with respect to y, we obtain

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial x \partial y} \qquad ...(3) \qquad \text{and} \qquad \frac{\partial^2 u}{\partial y^2}=-\frac{\partial^2 v}{\partial y \partial x}. \qquad ...(4)$$

Adding (3) and (4) and assuming that $\frac{\partial^2 v}{\partial x \partial y}=\frac{\partial^2 v}{\partial y \partial x}$, we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \quad ...(5)$$

Similarly, by differentiating (1) with respect to y and (2) with respect to x and subtracting, we obtain

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \quad ...(6)$$

Thus both the functions u and v satisfy the Laplace's equation in two variables. For this reason, they are known as **harmonic functions** and their theory is called **potential theory.** *(Rohtak, 2005)*

(2) Orthogonal system. Consider the two families of curves

$$u(x, y) = c_1 \quad ...(7) \qquad \text{and} \qquad v(x, y) = c_2 \quad ...(8)$$

Differentiating (7), we get $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} \cdot \dfrac{dy}{dx} = 0$

or $$\frac{dy}{dx} = -\frac{\partial u / \partial x}{\partial u / \partial y} = \frac{\partial v / \partial y}{\partial v / \partial x} = m_1 \text{ (say)} \quad \text{[By (1) and (2)]}$$

Similarly (8) gives $$\frac{dy}{dx} = -\frac{\partial v / \partial x}{\partial v / \partial y} = m_2 \text{ (say)}$$

$\therefore \quad m_1 m_2 = -1$, *i.e.*, (7) and (8) form an orthogonal system.

Hence *every analytic function $f(z) = u + iv$ defines two families of curves $u(x, y) = c_1$ and $v(x, y) = c_2$, which form an orthogonal system.* (U.P.T.U., 2009)

20.6 APPLICATIONS TO FLOW PROBLEMS

As the real and imaginary parts of an analytic function are the solutions of the Laplace's equation in two variables, the conjugate functions provide solutions to a number of field and flow problems.

As an illustration, consider the irrotational motion of an incompressible fluid in two dimensions. Assuming the flow to be in planes parallel to the xy-plane, the velocity $\mathbf{V}$ of a fluid particle can be expressed as

$$\mathbf{V} = v_x \mathbf{I} + v_y \mathbf{J} \quad ...(1)$$

Since the motion is irrotational, therefore, by § 6.18 (1), there exist a scalar function $\phi(x, y)$ such that

$$\mathbf{V} = \nabla\phi(x, y) = \frac{\partial \phi}{\partial x}\mathbf{I} + \frac{\partial \phi}{\partial y}\mathbf{J} \quad ...(2)$$

[The function $\phi(x, y)$ is called the *velocity potential* and the curves $\phi(x, y) = c$ are known as *equipotential lines.*]

Thus from (1) and (2), $v_x = \dfrac{\partial \phi}{\partial x}$ and $v_y = \dfrac{\partial \phi}{\partial y}$ ...(3)

Also the fluid being incompressible div $\mathbf{V} = 0$ [by § 8.7 (1)] *i.e.*, $\dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} = 0.$

Substituting the values of v_x and v_y from (3), we get $\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = 0$

which shows that the velocity potential ϕ is *harmonic*. It follows that there must exist a conjugate harmonic function $\psi(x, y)$ such that $w(z) = \phi(x, y) + i\psi(x, y)$ ...(4)
is analytic.

Also the slope at any point of the curve $\psi(x, y) = c'$ is given by

$$\frac{dy}{dx} = -\frac{\partial\psi/\partial x}{\partial\psi/\partial y} = \frac{\partial\phi/\partial y}{\partial\phi/\partial x} \quad \text{[By } C\text{-}R \text{ equations]}$$

$$= v_y / v_x \quad \text{[By (3)]}$$

This shows that the velocity of the fluid particle is along the tangent to the curve $\psi(x, y) = c'$, *i.e.* the particle moves along this curve. Such curves are known as *stream lines* and $\psi(x, y)$ is called the *stream function.* Also the equipotential lines $\phi(x, y) = c$ and the stream lines $\psi(x, y) = c'$ cut orthogonally.

From (4), $\dfrac{\partial w}{dz} = \dfrac{\partial\phi}{\partial x} + i\dfrac{\partial\psi}{\partial x} = \dfrac{\partial\phi}{\partial x} - i\dfrac{\partial\phi}{\partial y}$ [By C-R equations]

$= v_x - iv_y$ [By (3)]

$\therefore$ The magnitude of the fluid velocity $= \sqrt{(v_x^2 + v_y^2)} = |\,dw/dz\,|$.

Thus the flow pattern is fully represented by the function $w(z)$ which is known as the **complex potential.**

Similarly the complex potential $w(z)$ can be taken to represent any other type of 2-dimensional steady flow. In electrostatics and gravitational fields, the curves $\phi(x, y) = c$ and $\psi(x, y) = c'$ are *equipotential lines* and *lines of force*. In heat flow problems, the curves $\phi(x, y) = c$ and $\psi(x, y) = c'$ are known as *isothermals* and *heat flow lines* respectively.

Given $\phi(x, y)$, we can find $\psi(x, y)$ and *vice-versa*.

Example 20.6. *If $w = \phi + i\psi$ respresents the complex potential for an electric field and $\psi = x^2 - y^2 + \dfrac{x}{x^2+y^2}$, determine the function ϕ.* (V.T.U., 2011 ; Mumbai, 2008 ; Bhopal, 2002 S)

Solution. It is readily verified that ψ satisfies the Laplace's equation.

$\therefore$ ϕ and ψ must satisfy the Cauchy-Riemann equations :

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \quad ...(i) \qquad \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} \quad ...(ii)$$

$\therefore$ by (*i*), $$\frac{\partial\phi}{\partial x} = \frac{\partial}{\partial y}\left[x^2 - y^2 + \frac{x}{x^2+y^2}\right] = -2y - \frac{2xy}{(x^2+y^2)^2}$$

Integrating w.r.t. x, we get $\phi = -2xy + \dfrac{y}{x^2+y^2} + \eta(y)$ where $\eta(y)$ is an arbitrary function of y.

$\therefore$ (*ii*) gives $-2x + \dfrac{x^2-y^2}{(x^2+y^2)^2} + \eta'(y) = -2x + \dfrac{x^2-y^2}{(x^2+y^2)^2}$

whence $\eta'(y) = 0$, *i.e.*, $\eta(y) = c$, an arbitrary constant.

Thus $$\phi = -2xy + \frac{y}{x^2+y^2} + c$$

Otherwise (Milne-Thomson's method*) :

We have $$\frac{dw}{dz} = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial y} + i\frac{\partial\psi}{\partial x} = \left[-2y - \frac{2xy}{(x^2+y^2)^2}\right] + i\left[2x + \frac{y^2-x^2}{(x^2+y^2)^2}\right]$$

By Milne-Thomson's method, we express dw/dz in terms of z, on replacing x by z and y by 0.

$$\therefore \quad \frac{dw}{dz} = i\left(2z - \frac{1}{z^2}\right)$$

Integrating w.r.t. z, we get $w = i\,(z^2 + 1/z) + A$ where A is a complex constant.

---

* Since $z = x + iy$ and $\bar{z} = x - iy$, we have

$$x = \frac{1}{2}(z+\bar{z}), \qquad y = \frac{1}{2i}(z+\bar{z})$$

$$\therefore \quad f(z) = \phi(x, y) + i\psi(x, y) \quad ...(1)$$

$$= \phi\left[\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right] + i\psi\left[\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right]$$

Now considering this as a formal identity in the two independent variables z, $\bar{z}$ and putting $\bar{z} = z$, we get

$$f(z) = \phi(z, 0) + i\psi(z, 0) \quad ...(2)$$

$\therefore$ (2) is the same as (1), if we replace x by z and y by 0.

Thus **to express any function in terms of z, replace x by z and y by 0.** *This provides an elegant method of finding $f(z)$ when its real part or the imaginary part is given. It is due to Milne-Thomson.*

Hence $$\phi = R\left[i\left(z^2 + \frac{1}{z}\right) + A\right] = -2xy + \frac{y}{x^2 + y^2} + c.$$

Example 20.7. *Find the analytic function, whose real part is $\sin 2x/(\cosh 2y - \cos 2x)$.*

(J.N.T.U., 2005 ; Anna, 2003)

Solution. Let $f(z) = u + iv$, where $u = \sin 2x/(\cosh 2y - \cos 2x)$

$$\therefore \quad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \text{[By } C\text{-}R \text{ equations]}$$

$$= \frac{(\cosh 2y - \cos 2x)\, 2\cos 2x - \sin 2x\,(2\sin 2x)}{(\cosh 2y - \cos 2x)^2} - i\frac{\sin 2x(-2\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2\cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2x)^2} + i\frac{2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

By Milne-Thomson's method, we express $f'(z)$ in terms of z by putting $x = z$ and $y = 0$.

$$\therefore \quad f'(z) = \frac{2\cos 2z - 2}{(1 - \cos 2z)^2} + i(0) = \frac{-2}{1 - \cos 2z} = \frac{-2}{2\sin^2 z} = -\operatorname{cosec}^2 z$$

Integrating w.r.t. z, we get $f(z) = \cot z + ic$, taking the constant of integration as imaginary since u does not contain any constant.

Example 20.8. *Determine the analytic function $f(z) = u + iv$, if $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$ and $f(\pi/2) = 0$.*

(A.M.I.E.T.E., 2005 ; Osmania, 2003)

Solution. We have $$u - v = \frac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$$

$$\therefore \quad \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = \frac{(\sin x - \cos x)\cosh y + 1 - e^{-y}\sin x}{2(\cos x - \cosh y)^2} \qquad ...(i)$$

and $$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = \frac{(\cos x - \cosh y)\, e^{-y} + (\cos x + \sin x - e^{-y})\sinh y}{2(\cos x - \cosh y)^2}$$

or $$-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} = \frac{(\sin x + \cos x)\sinh y + e^{-y}(\cos x - \cosh y - \sinh y)}{2(\cos x - \cosh y)^2} \qquad ...(ii)$$

Subtracting (ii) from (i), we get

$$2\frac{\partial u}{\partial x} = \frac{(\sin x - \cos x)\cosh y - (\sin x + \cos x)\sinh y + 1 - e^{-y}(\sin x + \cos x - \cosh y - \sinh y)}{2(\cos x - \cosh y)^2}$$

Adding (i) and (ii), we have

$$-2\frac{\partial v}{\partial x} = \frac{(\sin x - \cos x)\cosh y + (\sin x + \cos x)\sinh y + 1 + e^{-y}(-\sin x + \cos x - \cosh y - \sinh y)}{2(\cos x - \cosh y)^2}$$

Thus $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{1 - \cos z}{2(1 - \cos z)^2} \qquad \text{[Putting } x = z \text{ and } y = 0]$$

$$= \frac{1}{2(1 - \cos z)} = \frac{1}{4\sin^2 z/2} = \frac{1}{4}\operatorname{cosec}^2\frac{z}{2} \quad \text{or} \quad f(z) = -\frac{1}{2}\cot\frac{z}{2} + c$$

Since $f(\pi/2) = 0$, $\quad 0 = -\dfrac{1}{2}\cot \pi/4 + c$, whence $c = \dfrac{1}{2}$

Hence $$f(z) = \frac{1}{2}\left(1 - \cot\frac{z}{2}\right).$$

Example 20.9. *Find the conjugate harmonic of $v(r, \theta) = r^2 \cos 2\theta - r \cos \theta + 2$. Show that v is harmonic.* (*Marathwada, 2008*)

Solution. Let $f(z) = u + v$. Using C-R equations in polar coordinates (Ex. 20.5),

$$r \frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} = -2r^2 \sin 2\theta + r \sin \theta \quad ...(i)$$

$$-\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial r} = 2r \cos 2\theta - \cos \theta \quad ...(ii)$$

$\therefore$ (*i*) gives, $\dfrac{\partial u}{\partial r} = -2r \sin 2\theta + \sin \theta$

Integrating w.r.t., r

$$u = -r^2 \sin 2\theta + r \sin \theta + \phi(\theta)$$ where $\phi(\theta)$ is an arbitrary function.

$$\therefore \quad \frac{\partial u}{\partial \theta} = -2r^2 \cos 2\theta + r \cos \theta + \phi'(\theta) \quad ...(iii)$$

From (*ii*) and (*iii*), we get

$$-2r^2 \cos 2\theta + r \cos \theta = \frac{\partial u}{\partial \theta} = -2r^2 \cos 2\theta + r \cos \theta + \phi'(\theta)$$

$\therefore \quad \phi'(\theta) = 0 \quad$ or $\quad \phi(\theta) = c$

Thus $\quad u = -r^2 \sin 2\theta + r \sin \theta + c$ is the conjugate harmonic of v.

Now v will be harmonic if it satisfies the Laplace equation $\dfrac{\partial^2 v}{\partial r^2} + \dfrac{1}{r} + \dfrac{\partial v}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 v}{\partial \theta^2} = 0$

From (*i*), $\dfrac{\partial^2 v}{\partial \theta^2} = -4r^2 \cos 2\theta + r \cos \theta$. From (*ii*), $\dfrac{\partial^2 v}{\partial r^2} = 2 \cos 2\theta$

$$\therefore \quad \frac{\partial t^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 2 \cos 2\theta + \frac{1}{r}(2r \cos 2\theta - \cos \theta) + \frac{1}{r^2}(-4r^2 \cos 2\theta + r \cos \theta)$$

$$= 4 \cos 2\theta - \frac{1}{r}\cos \theta - 4 \cos 2\theta + \frac{1}{r}\cos \theta = 0$$

Hence v is harmonic.

Example 20.10. (*a*) *Find the orthogonal trajectories of the family of curves*

$$x^4 + y^4 - 6x^2y^2 = \text{constant}$$

(*b*) *Show that the curves $r^n = \alpha \sec n\theta$ and $r^n = \beta \operatorname{cosec} n\theta$ cut orthogonally.*

(*Mumbai, 2005 ; J.N.T.U., 2003*)

Solution. (*a*) Take $u(x, y) = x^4 + y^4 - 6x^2y^2$. Then the family of curves $v(x, y) =$ constant will be the required trajectories if $f(z) = u + iv$ is analytic.

Now $\quad \dfrac{\partial u}{\partial x} = 4x^3 - 12xy^2, \; \dfrac{\partial u}{\partial y} = 4y^3 - 12x^2y$

$$\therefore \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 4x^3 - 12xy^2$$

Integrating, $\quad v = 4x^3y - 4xy^3 + c(x)$

Differentiating partially w.r.t. x

$$12x^2y - 4y^3 + \frac{dc(x)}{dx} = \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -4y^3 + 12x^2y$$

$$\therefore \quad \frac{dc(x)}{dx} = 0 \quad \text{or} \quad c = \text{constant}$$

Thus the required orthogonal trajectories are v = constant or $x^3y - xy^3$ = constant.

(*b*) Writing $u(r, \theta) = r^n \cos n\theta = \alpha \quad$ and $\quad v(r, \theta) = r^n \sin n\theta = \beta$,

we have $u(r, \theta) + iv(r, \theta) = \alpha + i\beta = r^n(\cos n\theta + i \sin n\theta) = r^n \, . \, e^{in\theta} = (re^{i\theta})^n = z^n$

This is an analytic function.

Thus $f(z) = u + iv$, gives the curves $u = \alpha$ and $v = \beta$ which cut orthogonally.

Example 20.11. *Two concentric circular cylinders of radii r_1, r_2 $(r_1 < r_2)$ are kept at potentials ϕ_1 and ϕ_2 respectively. Using complex function $w = a \log z + c$, prove that the capacitance per unit length of the capacitor formed by them is $2\pi\lambda / \log(r_2/r_1)$ where λ is the dielectric constant of the medium.*

Solution. We have $\quad \phi + i\psi = a \log(re^{i\theta}) + c \quad$ where $z = x + iy = re^{i\theta}$

$\therefore \qquad \phi = a \log r + c, \quad$ and $\quad \psi = a\theta$

so that $\qquad \phi_1 = a \log r_1 + c, \quad \phi_2 = a \log r_2 + c$

Thus the potential difference $= \phi_2 - \phi_1 = a(\log r_2 - \log r_1)$

Also the total charge (or flux) $= \int_0^{2\pi} d\psi = \int_0^{2\pi} a\, d\theta = 2\pi a.$

The capacitance being the charge required to maintain a unit potential difference ; the capacitance without dielectric

$$= \frac{\text{charge}}{\text{potential difference}} = \frac{2\pi a}{a(\log r_2 - \log r_1)} = \frac{2\pi}{\log(r_2/r_1)}.$$

A medium of dielectric constant λ increases the potential difference to λ times that in vacuum for the same charge. Thus the capacitance with dielectric $= 2\pi\lambda/\log(r_2/r_1)$.

Example 20.12. *If $f(z)$ is a regular function of z, prove that*

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\,|f'(z)|^2. \qquad \text{(J.N.T.U., 2006 ; Kottayam, 2005)}$$

or

$$\nabla^2\,|f(z)|^2 = 4\,|f'(z)|^2 \qquad \text{(Madras, 2006)}$$

Solution. Let $f(z) = u(x, y) + iv(x, y)$ so that $|f(z)|^2 = u^2 + v^2 = \phi(x, y)$, (say).

$$\therefore \quad \frac{\partial\phi}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial^2\phi}{\partial x^2} = 2\left\{u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + v\frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2\right\}$$

Similarly,
$$\frac{\partial^2\phi}{\partial y^2} = 2\left\{u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2 + v\frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial v}{\partial y}\right)^2\right\}$$

Adding, we have

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 2\left\{u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)\right\} + 2\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right\} \qquad ...(i)$$

Since u, v have to satisfy Cauchy-Riemann equations and the Laplace's equation.

$$\therefore \quad \left(\frac{\partial u}{\partial x}\right)^2 = \left(\frac{\partial v}{\partial y}\right)^2 ; \left(\frac{\partial u}{\partial y}\right)^2 = \left(-\frac{\partial v}{\partial x}\right)^2 \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}.$$

Thus (*i*) takes the form $\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = 4\left\{\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial v}{\partial x}\right)^2\right\}$

Hence
$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\,|f'(z)|^2 \quad \text{or} \quad \nabla^2\,|f(z)|^2 = 4\,|f'(z)|^2.$$

PROBLEMS 20.1

1. If $f(z) = \begin{cases} x^3 y(y - ix)/(x^6 + y^2), & z \neq 0 \\ 0, & z = 0 \end{cases}$ prove that $|f(z) - f(0)|/z \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ along the curve $y = ax^3$.

2. Show that (*a*) $f(z) = xy + iy$ is everywhere continuous but is not analytic. (*Osmania, 2003 S*)

(*b*) $f(z) = z + 2\bar{z}$ is not analytic anywhere in the complex plane. (*J.N.T.U., 2003*)

3. If $f(z) = u + iv$ is analytic, then show that $|f'(z)|^2 = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$. (*Mumbai, 2007*)

4. Find the constants a, b, c, d and e if $f(z) = (ax^4 + bx^2y^2 + cy^4 + dx^2 - 2y^2) + i(4x^3y - exy^3 + 4xy)$ is analytic. (*Mumbai, 2008*)

5. Show that z^n is analytic. Hence find its derivative. (*V.T.U., 2010 S*)

6. Determine which of the following functions are analytic :

(*i*) $2xy + i(x^2 - y^2)$ (*ii*) $(x - iy)/(x^2 + y^2)$ (*iii*) $\cosh z$.

7. (*a*) Determine p such that the function $f(z) = \frac{1}{2}\log_e(x^2 + y^2) + i\tan^{-1}(px/y)$ be an analytic function. (*Mumbai, 2007 ; J.N.T.U., 2003*)

(*b*) Show that $u = \frac{1}{2}\log(x^2 + y^2)$ is harmonic and find its harmonic conjugate function. (*U.P.T.U., 2010*)

8. Show that each of the following functions is not analytic at any point :

(*i*) $\bar{z}$ (*J.N.T.U., 2003*) (*ii*) $|z|^2$.

9. Show that $u + iv = (x - iy)/(x - iy + a)$ where $a \neq 0$, is not an analytic function of $z = x + iy$ whereas $u - iv$ is such a function.

10. Show that $f(z) = \begin{cases} xy^2(x + iy) \div (x^2 + y^4), & z \neq 0 \\ 0 & , \quad z = 0 \end{cases}$

is not analytic at $z = 0$, although *C-R* equations are satisfied at the origin. (*J.N.T.U., 2003*)

11. Verify if $f(z) = \dfrac{xy^2(x + iy)}{x^2 + y^4}, z \neq 0; f(0) = 0$ is analytic or not. (*U.P.T.U., 2008*)

12. Examine the nature of the function $f(z) = \dfrac{x^2y^5(x + iy)}{x^4 + y^{10}}, z \neq 0 ; f(0) = 0$. (*Rohtak, 2004*)

13. For the function $f(z)$ defined by $f(z)^2 = (\bar{z})^2/z, z \neq 0; f(0) = 0$, show that the C-R equations are satisfied at (0, 0), but $f(z)$ is not differentiable at (0, 0). (*P.T.U., 2010*)

14. Determine the analytic function whose real part is

(*i*) $x^3 - 3xy^2 + 3x^2 - 3y^2$ (*Bhopal, 2009*) (*ii*) $\cos x \cosh y$ (*Rohtak, 2004*)

(*iii*) $y/(x^2 + y^2)$ (*iv*) $y + e^x \cos y$ (*S.V.T.U., 2008 ; V.T.U., 2006*)

(*v*) $e^{-x}(x \sin y - y \cos y)$ (*U.P.T.U., 2008*)

(*vi*) $e^{2x}(x \cos 2y - y \sin 2y)$ (*V.T.U., 2008 S ; Mumbai, 2005 ; Kottayam, 2005*)

(*vii*) $x \sin x \cosh y - y \cos x \sinh y$ (*V.T.U., 2006*)

(*viii*) $e^x[(x^2 - y^2)\cos y - 2xy \sin y]$. (*V.T.U., 2010 S ; Rohtak, 2005*)

15. Find the regular function whose imaginary part is

(*i*) $(x - y)/(x^2 + y^2)$ (*ii*) $-\sin x \sinh y$ (*iii*) $e^x \sin y$

(*iv*) $e^{-x}(x \sin y - y \cos y)$ (*v*) $e^{-x}(x \cos y + y \sin y)$ (*U.P.T.U., 2009*) (*vi*) $\dfrac{2 \sin x \sin y}{\cos 2x + \cosh 2y}$. (*Mumbai, 2006*)

16. Find the analytic function $z = u + iv$, if

(*i*) $u - v = (x - y)(x^2 + 4xy + y^2)$ (*Mumbai, 2008 ; V.T.U., 2007 ; W.B.T.U., 2005*)

(*ii*) $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - e^y - e^{-y}}$ when $f\left(\frac{\pi}{2}\right) = 0$ (*Mumbai, 2007*)

(*iii*) $u + v = \dfrac{2 \sin 2x}{e^{2y} - e^{-2y} - 2\cos 2x}$. (*P.T.U., 2002*)

17. An electrostatic field in the xy-plane is given by the potential function $\phi = 3x^2y - y^3$, find the stream function.

18. If the potential function is $\log(x^2 + y^2)$, find the flux function and the complex potential function.

19. Prove that $u = x^2 - y^2$ and $v = \dfrac{y}{x^2 + y^2}$ are harmonic functions of (x, y) but are not harmonic conjugates. (*U.P.T.U., 2004 S*)

20. Show that the function $u = e^{-2xy} \sin(x^2 - y^2)$ is harmonic. Find the conjugate function v and express $u + iv$ as an analytic function of z. *(Bhopal, 2007)*

21. For $w = \exp(z^2)$, find u and v, and prove that the curves $u(x, y) = c_1$ and $v(x, y) = c_2$ where c_1 and c_2 are constants, cut orthogonally. *(J.N.T.U., 2003)*

22. Find the orthogonal trajectories of the family of curves

 (*i*) $x^3y - xy^3 = c$ *(Mumbai, 2007)* (*ii*) $e^x \cos y - xy = c$ *(Mumbai, 2008)* (*iii*) $r^2 \cos 2\theta = c$.

23. In a two dimensional fluid flow, the stream function ψ is given, find the velocity potential ϕ :

 (*i*) $\psi = -y/(x^2 + y^2)$ (*ii*) $\psi = \tan^{-1}(y/x)$.

24. Find the analytic function $f(z) = u + iv$, given

 (*i*) $u = a(1 + \cos\theta)$ (*ii*) $v = (r - 1/r)\sin\theta$, $r \neq 0$.

25. If $f(z)$ is an analytic function of z, show that

$$\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = |f'(z)|^2.$$ *(U.P.T.U., 2009 ; V.T.U., 2008 S ; P.T.U., 2005)*

26. If $f(z)$ is an analytic function of z, prove that

 (*i*) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log|f(z)| = 0$ *(Madras, 2000 S)* (*ii*) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|R\,f(z)|^2 = 2\,|f'(z)|^2$

 (*iii*) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^p = p^2\,|f'(z)|^2\,|f(z)|^{p-2}$. *(Kerala, 2005)*

27. Prove that $\psi = \log[(x-1)^2 + (y-2)^2]$ is harmonic in every region which does not include the point (1, 2). Find a function ϕ such that $\phi + i\psi$ is an analytic function of the complex variable $z = x + iy$. Express $\phi + i\psi$ as a function of z.

20.7 GEOMETRICAL REPRESENTATION OF w = f(z)

To find the geometrical representation of a function of a complex variable, it requires a departure from the usual practice of cartesian plotting, where we associate a curve to a real function $y = f(x)$.

In the complex domain, the function $w = f(z)$

i.e., $$u + iv = f(x + iy) \qquad ...(1)$$

involves four real variables x, y, u, v. Hence a four dimensional region is required to plot (1) in the cartesian fashion. As it is not possible to have 4-dimensional graph papers, we make use of two complex planes, one for the variable $z = x + iy$, and the other for the variable $w = u + iv$. If the point z describes some curve C in the z-plane, the point w will move along a corresponding curve C' in the w-plane, since to each point (x, y), there corresponds a point (u, v) (Fig. 20.3). We then, say that *a curve C in the z-plane is mapped into the corresponding curve C′ in the w-plane by the function w = f (z) which defines a* **mapping** *or* **transformation** *of the z-plane into the w-plane.*

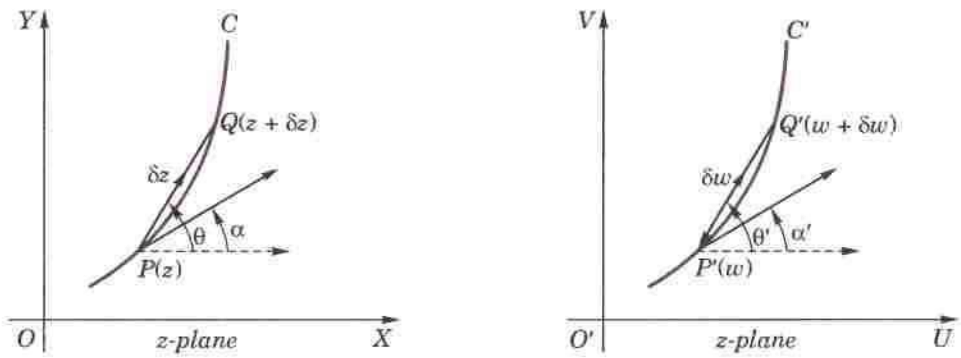

Fig. 20.3

20.8 SOME STANDARD TRANSFORMATIONS

(1) Translation. $w = z + c$, where c is a complex constant.

If $z = x + iy$, $c = c_1 + ic_2$ and $w = u + iv$, then the transformation becomes $u + iv = x + iy + c_1 + ic_2$ whence $u = x + c_1$ and $v = y + c_2$, *i.e.* the point $P(x, y)$ in the z-plane is mapped onto the point $P'(x + c_1, y + c_2)$ in the

w-plane. Every point in the z-plane is mapped onto w-plane in the same way. Thus if the w-plane is superposed on the z-plane, figure is shifted through a distance given by the vector c. Accordingly, this transformation maps a figure in the z-plane into a figure in the w-plane of the same shape and size.

In particular, this transformation changes circles into circles.

(2) Magnification and rotation. w = cz, where c is a complex constant.

If $\quad c = \rho e^{i\alpha}$, $z = re^{i\theta}$ and $w = Re^{i\phi}$, then

$$Re^{i\phi} = \rho e^{i\alpha} \,.\, re^{i\theta} = \rho r e^{i(\theta+\alpha)}$$

whence $R = \rho r$ and $\phi = \theta + \alpha$, *i.e.* the point $P(r, \theta)$ in the z-plane is mapped onto the point $P'(\rho r, \theta + \alpha)$ in the w-plane. Hence the transformation consists of magnification (or contraction) of the radius vector of P by $\rho = |\, c \,|$ and its rotation through an $\angle\alpha = \text{amp}\,(c)$. Accordingly any figure in the z-plane is transformed into a geometrically similar figure in the w-plane. In particular, this transformation maps circles into circles.

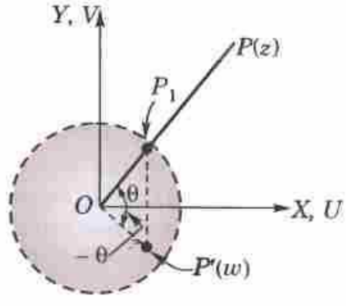

Fig. 20.4

(3) Inversion and reflection. w = 1/z.

Here it is convenient to think the w-plane as superposed on z-plane (Fig. 20.4).

If $z = re^{i\theta}$ and $w = Re^{i\phi}$, then $Re^{i\phi} = \dfrac{1}{r} e^{-i\theta}$

whence $R = 1/r$ and $\phi = -\theta$. Thus, if P be (r, θ) and P_1 be $(1/r, \theta)$, *i.e.* P_1 is the inverse* of P w.r.t. the unit circle with centre O, then the reflection P' of P_1 in the real axis represents $w = 1/z$.

Hence this transformation is an inversion of z w.r.t. the unit circle $|\, z \,| = 1$ followed by reflection of the inverse into the real axis.

Obs. 1. Clearly the function $w = 1/z$ maps the interior of the unit circle $|\, z \,| = 1$ onto the exterior of the unit circle $|\, w \,| = 1$ and the exterior of $|\, z \,| = 1$ onto the interior of $|\, w \,| = 1$. In particular, the origin $z = 0$ corresponds to the improper point $w = \infty$, called the *point at infinity* and the image of the improper point $z = \infty$ is the origin $w = 0$.

2. *This transformation maps a circle onto a circle or to a straight line if the former goes through the origin.*

To prove this, we write $z = 1/w$ as $x + iy = \dfrac{1}{u + iv} = \dfrac{u - iv}{u^2 + v^2}$

so that $$x = \frac{u}{u^2 + v^2} \text{ and } y = \frac{-v}{u^2 + v^2}. \qquad ...(1)$$

Now the general equation of any circle in the z-plane is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad ...(2)$$

which on substituting from (1), becomes $\dfrac{u^2}{(u^2+v^2)^2} + \dfrac{v^2}{(u^2+v^2)^2} + 2g\dfrac{u}{u^2+v^2} + 2f\dfrac{-v}{u^2+v^2} + c = 0$

or $$c(u^2 + v^2) + 2gu - 2fv + 1 = 0 \qquad ...(3)$$

This is the equation of a circle in the w-plane. If $c = 0$, the circle (2) passes through the origin and its image, *i.e.*, (3) reduces to a straight line. Hence the result.

Regarding a straight line as the limiting form of a circle with infinite radius, we conclude that the *transformation $w = 1/z$ always maps a circle into a circle.*

(4) Bilinear transformation. The transformation

$$w = \frac{az + b}{cz + d} \qquad ...(1)$$

where a, b, c and d are complex constants and $ad - bc \neq 0$ is known as the **bilinear transformation.**** The condition $ad - bc \neq 0$ ensures that $dw/dz \neq 0$, *i.e.*, the transformation is conformal. If $ad - bc = 0$ every point of the z-plane is a *critical point.*

The inverse mapping of (1) is

$$z = \frac{-dw + b}{cw - a} \qquad ...(2)$$

which is also a bilinear transformation.

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* The inverse of a point A w.r.t. a circle with centre O and radius k is defined as the point B on the line OA such that $OA.OB = k^2$.

** First studied by *Möbius* (p. 337). Hence, sometimes called *Möbius transformation*

Obs. 1. From (1), we see that each point in the z-plane except $z = -d/c$, corresponds a unique point in the w-plane. Similarly, (2) shows that each point in the w-plane except $w = a/c$, maps into a unique point in the z-plane. Including the images of the two exceptional points as the infinite points in the two planes, it follows that *there is one to one correspondence between all points in the two planes.*

Obs. 2. Invariant points of bilinear transformation. If z maps into itself in the w-plane (*i.e.*, $w = z$), then (1) gives

$$z = \frac{az+b}{cz+d} \quad \text{or} \quad cz^2 + (d-a)z - b = 0$$

The roots of this equation (say : z_1, z_2) *are defined as the invariant or fixed points of the bilinear transformation (1).*

If however, the two roots are equal, the bilinear transformation is said to be *parabolic*.

Obs. 3. Dividing the numerator and denominator of the right side of (1) by one of the four constants, it is clear that (1) has only three essential arbitrary constants. Hence *three conditions are required to determine a bilinear transformation.* For instance, three distinct points z_1, z_2, z_3 can be mapped into any three specified points w_1, w_2, w_3.

Two important properties :

I. A bilinear transformation maps circles into circles.

By actual division, (1) can be written as $w = \dfrac{a}{c} + \dfrac{bc-ad}{c^2} \cdot \dfrac{1}{z + d/c}$

which is a combination of the transformations

$$w_1 = z + d/c,\ w_2 = 1/w_1,\ w_3 = \frac{bc-ad}{c^2} w_2,\ w = \frac{a}{c} + w_3 .$$

By these transformations, we successively pass from z-plane to w_1-plane, from w_1-plane to w_2-plane, from w_2-plane to w_3-plane and finally from w_3-plane to w-plane. Now each of these transformations is one or other of the standard transformations $w = z + c$, $w = cz$, $w = 1/z$ and under each of these a circle always maps onto a circle. Hence the bilinear transformation maps circles into circles.

II. A bilinear transformation preserves cross-ratio† *of four points.*

Let the points z_1, z_2, z_3, z_4 of the z-plane map onto the points w_1, w_2, w_3, w_4 of the w-plane respectively under the bilinear transformation (1). If these points are finite, then from (1), we have

$$w_j - w_k = \frac{az_j + b}{cz_j + d} - \frac{az_k + b}{cz_k + d} = \frac{ad - bc}{(cz_j + d)(cz_k + d)} (z_j - z_k).$$

Using this relation for $j, k = 1, 2, 3, 4$, we get $\dfrac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_4)(w_3 - w_2)} = \dfrac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$

Thus the cross-ratio of four points is invariant under bilinear transformation.

This property is very useful in finding a bilinear transformation. If one of the points, say : $z_1 \to \infty$, the quotient of those two differences which contain z_1, is replaced by 1 *i.e.*,

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)} = \frac{z_3 - z_4}{z_3 - z_2}.$$

Example 20.13. *Find the bilinear transformation which maps the points $z = 1, i, -1$ onto the points $w = i, 0, -i$.*

Hence find (a) the image of $|z| < 1$, *(Mumbai, 2006 ; Delhi, 2002)*

(b) the invariant points of this transformation. *(U.P.T.U., 2008 ; V.T.U., 2000)*

Solution. Let the points $z_1 = 1, z_2 = i, z_3 = -1$ and $z_4 = z$ map onto the points $w_1 = i, w_2 = 0, w_3 = -i$ and $w_4 = w$. Since the cross-ratio remains unchanged under a bilinear transformation.

$$\therefore \quad \frac{(1-i)(-1-z)}{(1-z)(-1-i)} = \frac{(i-0)(-i-w)}{(i-w)(-i-0)} \quad \text{or} \quad \frac{w+i}{w-i} = \frac{(z+1)(1-i)}{(z-1)(1+i)}$$

By componendo dividendo, we get $\dfrac{2w}{2i} = \dfrac{(z+1)(1-i) + (z-1)(1+i)}{(z+1)(1-i) - (z-1)(1+i)}$

† **Def.** *If t_1, t_2, t_3, t_4 be any four numbers, then* $\dfrac{(t_1 - t_2)(t_3 - t_4)}{(t_1 - t_4)(t_3 - t_2)}$ *is said to be their cross-ratio and is denoted* (t_1, t_2, t_3, t_4).

$$\therefore \qquad w = \frac{1 + iz}{1 - iz} \qquad ...(i)$$

which is the required bilinear transformation.

(*a*) Rewriting (*i*) as $\quad z = i\,\dfrac{1 - w}{1 + w}$

$$\therefore \qquad \left|\frac{i(1 - w)}{1 + w}\right| = |\,z\,| < 1 \quad \text{or} \quad |\,i\,|\,|\,1 - w\,| < |\,1 + w\,|$$

or $\qquad |\,1 - u - iv\,| < |\,1 + u + iv\,| \qquad [\because \ |\,i\,| = 1]$

or $\qquad (1 - u)^2 + v^2 < (1 + u)^2 + v^2$ which reduces to $u > 0$.

Hence the interior of the circle $x^2 + y^2 = 1$ in the z-plane is mapped onto the entire half of the w-plane to the right of the imaginary axis.

(*b*) To find the invariant points of the transformation, we put $w = z$ in (*i*).

$$\therefore \qquad z = \frac{1 + iz}{1 - iz} \quad \text{or} \quad iz^2 + (i - 1)\,z + 1 = 0$$

or
$$z = \frac{1 - i \pm \sqrt{[(i - 1)^2 - 4i]}}{2i} = -\tfrac{1}{2}\{1 + i \pm \sqrt{(6i)}\}$$

which are the required invariant points.

Example 20.14. *Show that* $w = \dfrac{i - z}{i + z}$ *maps the real axis of z-plane into the circle* $|\,w\,| = 1$ *and the half plane* $y > 0$ *into the interior of the unit circle* $|\,w\,| = 1$ *in the w-plane.* (*Mumbai, 2007*)

Solution. Since $\qquad w = (i - z)/(i + z)$,

$\therefore \qquad |\,w\,| = 1$ becomes $(i - z)/(i + z)\,| = 1 \quad$ or $\quad |\,i - z\,| = |\,i + z\,|$

i.e., $\qquad |\,i - x - iy\,| = |\,i + x + iy\,| \quad$ or $\quad |\,-x + i(1 - y)\,| = |\,x + i\,(1 + y)\,|$

$\therefore \qquad \sqrt{[x^2 + (1 - y)^2} = \sqrt{(x^2 + (1 + y)^2]}$ or $(1 - y)^2 = (1 + y)^2$

$\therefore \qquad 4y = 0 \quad$ or $\quad y = 0$ which is the real axis.

Hence the real axis of the z-plane is mapped to the circle $|\,w\,| = 1$

Now for the interior of the circle $|\,w\,| = 1$

$\qquad |\,w\,| < 1 \quad$ *i.e.*, $\quad |\,i - z\,| < |\,i + z\,| \quad$ or $\quad (1 - y)^2 < (1 + y)^2$

$\therefore \qquad -4y < 0 \quad$ *i.e.*, $\quad y > 0$

Hence the half plane $y > 0$ is mapped into the interior of the circle $|\,w\,| = 1$.

PROBLEMS 20.2

1. Find the invariant points of the transformation $w = (z - 1)/(z + 1)$. (*Madras, 2003*)
2. Find the transformation which maps the points $-1, i, 1$ of the z-plane onto $1, i, -1$ of the w-plane respectively. Also find its *invariant points*. (*V.T.U., 2011*)
3. Find the bilinear transformation which maps $1, i, -1$ to $2, i, -2$ respectively. Find the fixed and critical points of the transformation. (*S.V.T.U., 2008 ; Mumbai, 2007 ; V.T.U., 2006*)
4. Determine the bilinear transformation that maps the points $1 - 2i$, $2 + i$, $2 + 3i$ respectively into $2 + 2i$, $1 + 3i$, 4. (*J.N.T.U., 2003 ; Coimbatore, 1999*)
5. Find the bilinear transformation which maps
   (*i*) the points $z = 1, i, -1$ into the points $w = 0, 1, \infty$ (*V.T.U., 2008 ; Mumbai, 2007*)
   (*ii*) the points $z = 0, 1, i$ into the points $w = 1 + i, -i, 2 - i$ (*V.T.U., 2010 S*)
   (*iii*) $R(z) > 0$ into interior of unit circle so that $z = \infty, i, 0$ map into $w = -1, -i, 1$.
6. Under the transformation $w = \dfrac{z - 1}{z + 1}$, show that the map of the straight line $x = y$ is a circle and find its centre and radius. (*Marathwada, 2008*)

7. Show that the bilinear transformation $w = (2z + 3)/(z - 4)$ maps the circle $x^2 + y^2 - 4x = 0$ into the line $4u + 3 = 0$. *(Mumbai, 2007 ; J.N.T.U., 2003 ; Bhopal, 2002)*
8. Show that the condition for transformation $w = (az + b)/(cz + d)$ to make the circle $|w| = 1$ correspond to a straight line in the z-plane is $|a| = |c|$.
9. Show that the transformation $w = i(1 - z)/(1 + z)$ maps the circle $|z| = 1$ into the real axis of the w-plane and the interior of the circle $|z| < 1$ into the upper half of the w-plane. *(Osmania, 2003 S ; V.T.U., 2001)*
10. If z_0 is the upper half of the z-plane, show that the bilinear transformation $w = e^{i\alpha}\left(\dfrac{z - z_0}{z - \bar{z}_0}\right)$ maps the upper half of the z-plane into the interior of the unit circle at the origin in the w-plane.

20.9 (1) CONFORMAL TRANSFORMATION

Suppose two curves C, C_1 in the z-plane intersect at the point P and the corresponding curves C' and C_1' in the w-plane intersect at P' (Fig. 20.5). *If the angle of intersection of the curves at P is the same as the angle of intersection of the curves at P′ in magnitude and sense, then the transformation is said to be* **conformal.**

(2) Theorem. *The transformation effected by an analytic function $w = f(z)$ is conformal at every point of the z-plane where $f'(z) \neq 0$.*

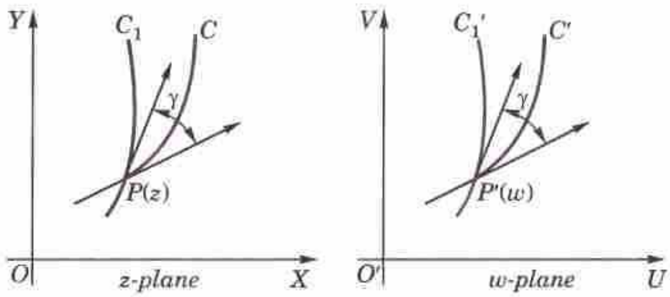

Fig. 20.5

Let $P(z)$ be a point in the region R of the z-plane and $P'(w)$ the corresponding point in the region R' of the w-plane (Fig. 20.3). Suppose z moves on a curve C and w moves on the corresponding curve C'. Let $Q(z + \delta z)$ be a neighbouring point on C and $Q'(w + \delta w)$ be the corresponding point on C' so that $\overrightarrow{PQ} = \delta z$ and $\overrightarrow{P'Q'} = \delta w$. Then δz is a complex number whose modulus r is the length PQ and amplitude θ is the angle which PQ makes with the x-axis.

$$\therefore \qquad \delta z = re^{i\theta}$$

Similarly, if the modulus and amplitude of δw be r' and θ', then $\delta w = r'e^{i\theta'}$.

Hence
$$\frac{\delta w}{\delta z} = \frac{r'}{r} e^{i(\theta' - \theta)}$$

Now if the tangent at P to the curve C makes an $\angle\alpha$ with the x-axis and the tangent at P' to C' makes an $\angle\alpha'$ with the u-axis, then as $\delta z \to 0$, $\theta \to \alpha$ and $\theta' \to \alpha'$.

$$\therefore \qquad f'(z) = \frac{dw}{dz} = \left(\text{Lt}\,\frac{r'}{r}\right) . e^{i(\alpha' - \alpha)} \qquad \text{...(1)}$$

If ρ is the modulus and ϕ the amplitude of the function $f(z)$ which is supposed to be non-zero, then

$$f'(z) = \rho e^{i\phi} \qquad \text{...(2)}$$

$\therefore$ from (1) and (2), we have
$$\rho = \text{Lt}\,\frac{r'}{r} \qquad \text{...(3)}$$

and
$$\phi = \alpha' - \alpha. \qquad \text{...(4)}$$

Now let C_1 be another curve through P in the z-plane and C_1' the corresponding curve through P' in the w-plane. If the tangent at P to C_1 makes an $\angle\beta$ with the x-axis and tangent at P' to C_1' makes an $\angle\beta'$ with the u-axis, then as in (4),

$$\psi = \beta' - \beta \qquad \text{...(5)}$$

Equating (4) and (5), $\qquad \alpha' - \alpha = \beta' - \beta \quad \text{or} \quad \beta - \alpha = \beta' - \alpha' = \gamma$ (Fig. 20.5)

Thus the angle between the curves before and after the mapping is preserved in magnitude and direction. Hence the mapping by the analytic function $w = f(z)$ is conformal at each point where $f'(z) \neq 0$.

Obs. 1. *A point at which $f'(z) = 0$ is called a* **critical point** *of the transformation.*

Obs. 2. The relation (4), *i.e.*, $\alpha' = \alpha + \phi$ shows that the tangent at P to the curve C is rotated through an $\angle\phi$ = amp $\{f'(z)\}$ under the given transformation.

Obs. 3. The relation (3) shows that in the transformation, elements of arc passing through P in any direction are changed in the ratio $\rho : 1$, where $\rho = |f'(z)|$, *i.e.*, *an infinitesimal length in the z-plane is magnified by the factor* $|f'(z)|$. Consequently *the infinitesimal areas are magnified by the factor* $|f'(z)|^2$ *in a conformal transformation.*

If $w = f(z)$ is analytic then u and v must satisfy *C-R* equations.

$$\therefore \quad J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{-\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{vmatrix} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left|\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right|^2 = |f'(z)|^2$$

Hence *in a conformal transformation, infinitesimal areas are magnified by the factor* $J\left(\frac{u,v}{x,y}\right)$.

Also the condition of a conformal mapping is $J\left(\frac{u,v}{x,y}\right) \neq 0$.

Obs. 4. *The angle preserving property of the conformal transformation has many important physical applications.* For instance, consider the flow of an incompressible fluid in a plane with velocity potential $\phi(x, y)$ and stream function $\psi(x, y)$. We know that ϕ and ψ are real and imaginary parts of some analytic function $w = f(z)$. As ϕ = constant and ψ = constant represent a system of orthogonal curves; these are transformed by the function $w = f(z)$ into a set of orthogonal lines in the w-plane and *vice-versa*.

Thus, *the conjugate functions ϕ and ψ when subjected to conformal transformation remain conjugate functions, i.e., the solutions of Laplace's equation remain solutions of the Laplace's equation after the transformation*. This is the main reason for the great importance of the conformal transformation in applications.

20.10 SPECIAL CONFORMAL TRANSFORMATIONS

(1) *Transformation* $w = z^2$.

We have $\quad u + iv = (x + iy)^2 = x^2 - y^2 + 2ixy.$

$$\therefore \quad u = x^2 - y^2 \text{ and } v = 2xy \qquad ...(1)$$

If u is constant (say, a), then $x^2 - y^2 = a$ which is a rectangular hyperbola. Similarly, if v is constant (say, b), then $xy = b/2$ which also represents a rectangular hyperbola.

Hence a pair of lines $u = a$, $v = b$ parallel to the axes in the w-plane, map into pair of orthogonal rectangular hyperbolae in the z-plane as shown in Fig. 12.7 (p. 455).

Again, if x is constant (say, c), then $y = v/2c$ and $y^2 = c^2 - u$. Elimination of y from these equations gives $v^2 = 4c^2(c^2 - u)$, which represents a parabola. Similarly, if y is a constant (say, d), then elimination of x from the equations (1) gives $v^2 = 4d^2(d^2 + u)$ which is also a parabola.

Hence the pair of lines $x = c$ and $y = d$ parallel to the axes in the z-plane map into orthogonal parabolas in the w-plane as shown in Fig. 20.6.

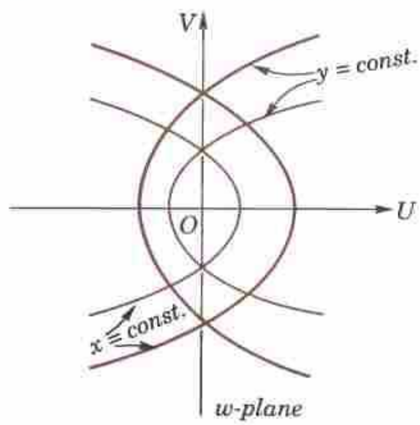

Fig. 20.6

Also since $\frac{dw}{dz} = 2z = 0$ for $z = 0$, therefore, it is a *critical point* of the mapping.

Taking $z = re^{i\theta}$ and $w = Re^{i\phi}$ then in polar form $w = z^2$ becomes $Re^{i\phi} = r^2e^{2i\theta}$.

This shows that upper half of the z-plane $0 < \theta < \pi$ transforms into the entire w-plane $0 \le \phi < 2\pi$. The same is true of the lower half. (P.T.U., 2003)

Obs. 1. Taking the axes to represent two walls, a single quadrant could be used to represent fluid flow at a corner wall. This transformation can also represent the electrostatic field in the vicinity of a corner conductor.

Obs. 2. *For the transformation $w = z^n$, n being a positive integer,* we have $dw/dz = 0$ at $z = 0$.

Also $\quad Re^{i\phi} = (re^{i\theta})^n = r^ne^{in\theta}$

$\therefore \quad R = r^n$ and $\phi = n\theta$, when $0 < \theta < \pi/n$, correspondingly $0 < \phi < \pi$.

Hence $w = z^n$ gives a conformal mapping of the z-plane everywhere except at the origin and that is fans out a sector of z-plane of central angle π/n to cover the upper half of the w-plane.

(2) *Joukowski's transformation** $w = z + 1/z$.

Since $\frac{dw}{dz} = \frac{(z+1)(z-1)}{z^2}$, the mapping is conformal except at the points $z = 1$ and $z = -1$ which correspond to the points $w = 2$ and $w = -2$ of the w-plane.

* Named after the Russian mathematician *Nikolai Jegorovich Joukowshi* (1847-1921).

Changing to polar coordinates,

$$w = u + iv = r(\cos\theta + i\sin\theta) + \frac{1}{r(\cos\theta + i\sin\theta)}$$

$$= r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$

$\therefore$ $$u = (r + 1/r)\cos\theta \text{ and } v = (r - 1/r)\sin\theta$$

Elimination of θ gives $$\frac{u^2}{(r+1/r)^2} + \frac{v^2}{(r-1/r)^2} = 1 \quad ...(1)$$

while the elimination of r gives $$\frac{u^2}{4\cos^2\theta} - \frac{v^2}{4\sin^2\theta} = 1 \quad ...(2)$$

From (1), it follows that the circles r = constant of z-plane transform into a family of ellipses of the w-plane (Fig. 20.7). These ellipses are confocal for $(r + 1/r)^2 - (r - 1/r)^2 = 4$, *i.e.*, a constant.

In particular, the unit circle ($r = 1$) in the z-plane flattens out to become the segment $u = -2$ to $u = 2$ of the real axis in w-plane. Thus the exterior of the unit circle in the z-plane maps into the entire w-plane.

(A.M.I.E.T.E., 2005 S)

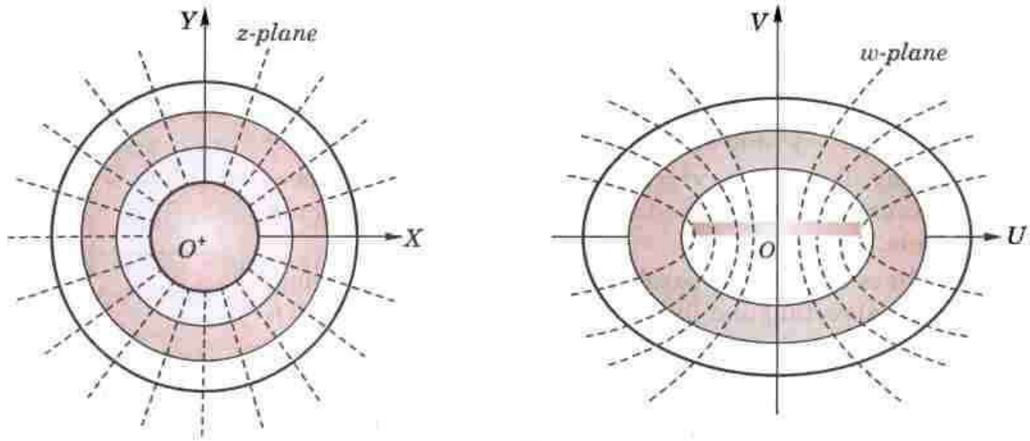

Fig. 20.7

From (2), it is clear that the radial lines θ = constant of the z-plane transform into a family of hyperbolae which are also confocal (Fig. 20.7).

Obs. 1. $v = \left(r - \frac{1}{r}\right)\sin\theta = 0$ gives $r = \pm 1$ or $\theta = 0, \pi$, *i.e.*, this streamline consists of the unit circle $r = 1$ and the x-axis ($\theta = 0$ to $\theta = \pi$). For large z, the flow is nearly uniform and parallel to the x-axis. This can be interpreted as a flow around a circular cylinder of unit radius having two **stagnation points*** at $A(z = 1)$ and $B(z = -1)$. (Fig. 20.8)

[$\because$ $dw/dz = 0$ at $z = \pm 1$]

Obs. 2. This transformation is also used to map the exterior of the profile of an aeroplane wing on the exterior of a nearly circular region. These airfoils are known as *Joukowski airfoils*.

(3) *Transformation* $w = e^z$.

Writing $z = x + iy$ and $w = \rho e^{i\phi}$, we have $\rho e^{i\phi} = e^{x+iy} = e^x \cdot e^{iy}$

whence $\rho = e^x$ ...(1) and $\phi = y$ ...(2)

From (1), it is clear that the lines parallel to y-axis (x = const.) map into circles (ρ = const.) in the w-plane, their radii being less than or greater than 1 according as x is less than or greater than 0 (Fig. 20.9). *(V.T.U., 2011)*

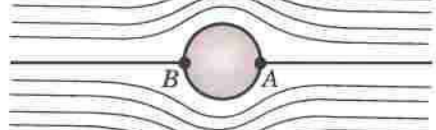

Fig. 20.8

Similarly, it follows from (2) that the lines parallel to the x-axis (y = const.) map into the radial lines (ϕ = const.) of the w-plane. Thus any horizontal strip of height 2π in the z-plane will cover once the entire w-plane.

* *Stagnation points* are those at which the fluid velocity is zero.

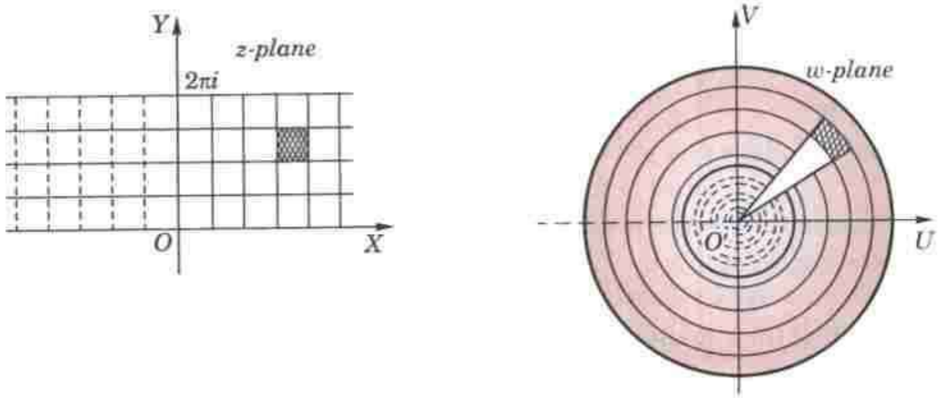

Fig. 20.9

The rectangular region $a_1 \leq x \leq a_2$, $b_1 \leq y \leq b_2$ in the z-plane (shown shaded) transforms into the region $e^{a_1} \leq \rho \leq e^{a_2}$, $b_1 \leq \phi \leq b_2$ in the w-plane bounded by circles and rays (shown shaded).

(P.T.U., 2005 ; Kerala, 2005)

Obs. This transformation can be used to obtain the circulation of a liquid around a cylindrical obstacle, the electrostatic field due to a charged circular cylinder etc.

(4) *Transformation $w = \cosh z$.*

We have $$u + iv = \cosh(x + iy) = \cosh x \cos y + i \sinh x \sin y$$ [By (2) (*ii*), p. 662]

so that $$u = \cosh x \cos y \text{ and } v = \sinh x \sin y.$$

Elimination of x from these equations gives

$$\frac{u^2}{\cos^2 y} - \frac{v^2}{\sin^2 y} = 1 \qquad ...(1)$$

while elimination of y gives $$\frac{u^2}{\cosh^2 x} + \frac{v^2}{\sinh^2 z} = 1 \qquad ...(2)$$

(1) shows that the lines parallel to x-axis (*i.e.*, y = const.) in the z-plane map into hyperbolae in the w-plane.

(2) shows that the lines parallel to the y-axis (*i.e.*, x = const.) in the z-plane map into ellipse in the w-plane (Fig. 20.10). The rectangular region $a_1, \leq x \leq a_2$, $b_1 \leq y \leq b_2$ in the z-plane (shown shaded) transforms into the shaded region in the w-plane bounded by the corresponding hyperbolae and ellipses. *(Kerala M. Tech., 2005)*

Obs. This transformation can be used.

(*i*) to obtain the circulation of liquid around an elliptic cylinder;

(*ii*) to determine the electrostatic field due to a charged cylinder;

(*iii*) to determine the potential between two confocal elliptic (or hyperbolic) cylinders.

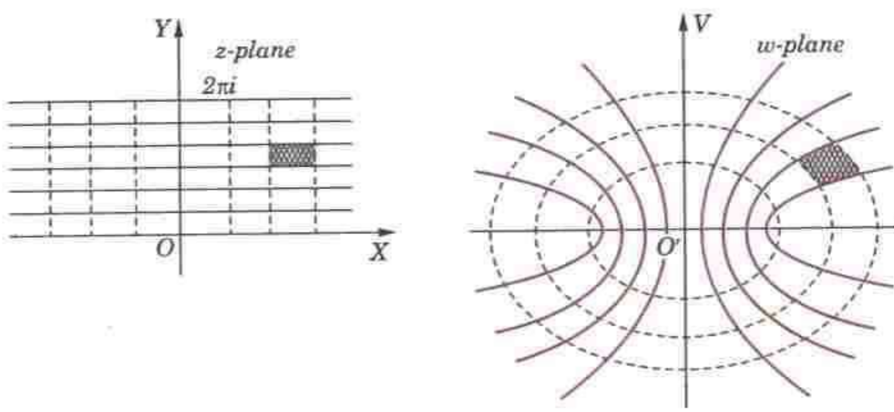

Fig. 20.10

Example 20.15. *Show that under the transformation $w = (z - i)/(z + i)$, real axis in the z-plane is mapped into the circle $|w| = 1$. Which portion of the z-plane corresponds to the interior of the circle?* (J.N.T.U., 2003)

Solution. We have $$|w| = \left|\frac{z-i}{z+i}\right| = \frac{|z-i|}{|z+i|} = \frac{|x+i(y-1|}{|x+i(y+1)|}$$
$$= \sqrt{\{x^2+(y-1)^2\}}/\sqrt{\{x^2+(y+1)^2\}}$$

Now the real axis in z-plane *i.e.*, $y = 0$, transforms into
$$|w| = \sqrt{(x^2+1)}/\sqrt{\{x^2+1)} = 1.$$

Hence the real axis in the z-plane is mapped into the circle $|w| = 1$.

The interior of the circle, *i.e.*, $|w| < 1$, gives
$$\{x^2+(y-1)^2\}/\{x^2+(y+1)^2\} < 1 \text{ i.e., } -4y < 0 \text{ or } y > 0.$$

Thus the upper half of the z-plane corresponds to the interior of the circle $|w| = 1$.

PROBLEMS 20.3

1. Determine the region of the w-plane into which the following regions are mapped by the transformation $w = z^2$:
   (*i*) first quadrant of z-plane (*J.N.T.U., 2000*)
   (*ii*) region bounded by $x = 1,\ y = 1, x + y = 1$ (*Kottayam, 2005 ; V.T.U., 2000 S*)
   (*iii*) the region $1 \le x \le 2$ and $1 \le y \le 2$ (*Osmania, 2003 ; V.T.U., 2000*)
   (*iv*) circle $|z - 1| = 2$.
2. Find the transformation which maps the triangular region $0 \le \arg z \le \pi/3$ into the unit circle $w \le 1$.
3. Discuss the transformation $w = \sqrt{z}$. Is it conformal at the origin? (*Delhi, 2002*)
4. Under the transformation $w = 1/z$, find the image of
   (*a*) the circle $|z - 2i| = 2$ (*Bhopal, 2009 ; Kerala M.Tech., 2005*)
   (*b*) the straight line $y - x + 1 = 0$ (*P.T.U., 2007*)
   (*c*) the hyperbola $x^2 - y^2 = 1$. (*Mumbai, 2005 ; J.N.T.U., 2005*)
5. Show that under the transformation $w = 1/z$, (*a*) circle $x^2 + y^2 - 6x = 0$ is transformed into a straight line in the w-plane.
   (*b*) the circle $(x - 3)^2 + y^2 = 2$ is transformed into a circle with centre $(3/7, 0)$ and radius $\sqrt{2}/17$. (*Mumbai, 2007*)
6. Show that the transformation $w = 1/z$ transforms all circles and straight lines into the circles and straight lines in the w-plane. Which circles in the z-plane become straight lines in the w-plane, and which straight lines are transformed into other straight lines ? (*Anna, 2003*)
7. Show that the transformation $w = z + 1/z$ converts the straight line $\arg z = \alpha$ ($|\alpha| < \alpha/2$) into a branch of hyperbola of eccentricity $\sec \alpha$. (*Mumbai, 2005 S*)
8. Show that the transformation $w = z + (a^2 - b^2)/4z$ transforms the circle of radius $\frac{1}{2}(a + b)$, centre at the origin, in the z-plane into ellipse of semi-axes a, b in the w-plane.
9. Show that the transformation $w = z + a^2/z$ transforms circles with origin at the cent`re in the z-plane into co-axial concentric, confocal ellipses in the w-plane. (*Kurukshetra, 2005 ; J.N.T.U., 2005*)
10. Show that the function $w = A(z + a^2/z)$ may be used to represent the flow of a perfect incompressible fluid past a circular cylinder. Also find the stagnation points.
11. Show that by the relation $u + iv = \cos(x + iy)$, the infinite strip bounded by $x = c$, $x = d$, where c and d lie between 0 and $\pi/2$, is mapped into the region between the two branches of the hyperbola lying in $u > 0$. (*Osmania, 2002*)
12. Prove that the transformation $w = \sin z$, maps the families of lines x = constant and y = constant into two families of confocal central conics. (*J.N.T.U., 2003*)
13. Discuss the transformation $w = e^z$, and show that it transforms the region between the real azis and a line parallel to real axis at $y = \pi$, into the upper half of the w-plane.
14. Discuss fully the transformation $w = c \cosh z$, where c is a real number. What physical problem can we study with the help of this transformation ?

20.11 SCHWARZ-CHRISTOFFEL TRANSFORMATION

This transformation maps the interior of a polygon of the w-plane into the upper half of the z-plane and the boundary of the polygon into the real axis. The formula of this transformation is

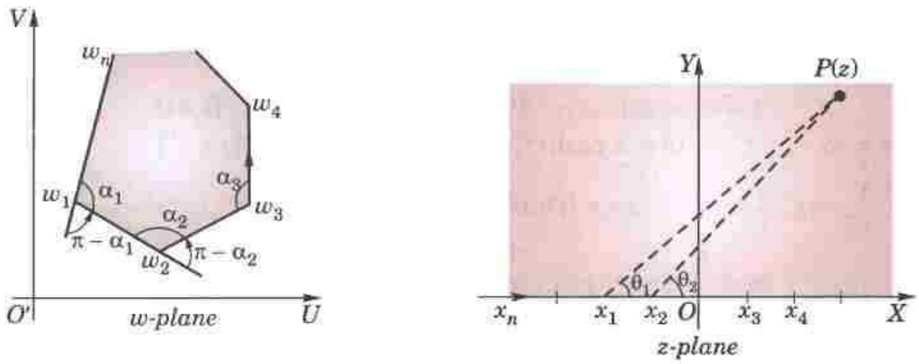

Fig. 20.11

$$\frac{dw}{dz} = A(z - x_1)^{\frac{\alpha_1}{\pi} - 1} (z - x_2)^{\frac{\alpha_2}{\pi} - 1} \dots (z - x_n)^{\frac{\alpha_n}{\pi} - 1} \quad \dots(1)$$

or
$$w = A \int (z - x_1)^{\frac{\alpha_1}{\pi} - 1} (z - x_2)^{\frac{\alpha_2}{\pi} - 1} \dots (z - x_n)^{\frac{\alpha_n}{\pi} - 1} \, dz + B \quad \dots(2)$$

where $\alpha_1, \alpha_2, \dots, \alpha_n$ are the interior angles of the polygon having vertices $w_1, w_2, \dots, w_n$ which map into the points $x_1, x_2, \dots, x_n$ on the real-axis of the z-plane (Fig. 20.11). Also A and B are complex constants which determines the size and position of the polygon.

Proof. We have from (1),

$$\text{amp}\left(\frac{dw}{dz}\right) = \text{amp}\,[(A) + \left(\frac{\alpha_1}{\pi} - 1\right) \text{amp}\,(z - x_1) + \left(\frac{\alpha_2}{\pi} - 1\right) \text{amp}\,(z - x_2)$$

$$\dots. + \left(\frac{\alpha_n}{\pi} - 1\right) \text{amp}\,(z - x_n) \quad \dots(3)$$

As z moves along the real axis from the left towards x_1, suppose that w moves along the side $w_n w_1$ of the polygon towards w_1. As z crosses x_1 from left to right, $\theta_1 = \text{amp}\,(z - x_1)$ changes from π to 0 while all other terms of (3) remain unaffected. Hence only $\left(\frac{\alpha_1}{\pi} - 1\right)$ amp $(z - x_1)$ decreases by $\left(\frac{\alpha_1}{\pi} - 1\right) \pi = \alpha_1 - \pi$, *i.e.* increases by $\pi - \alpha_1$ in the anti-clockwise direction. In other words, amp (dw/dz) increases by $\pi - \alpha_1$. Thus the direction of w_1 turns through the angle $\pi - \alpha_1$ and w now moves along the side $w_1 w_2$ of the polygon.

Similarly when z passes through x_2, $\theta_1 = \text{amp}\,(z - x_1)$ and $\theta_2 = \text{amp}\,(z - x_2)$ change from π to 0 while all other terms remain unchanged. Hence the side $w_1 w_2$ turns through the angle $\pi - \alpha_2$. Proceeding in this way, we see that as z moves along x-axis, w traces the polygon $w_1 w_2 w_3 \dots w_n$ and conversely.

Example 20.16. *Find the transformation which maps the semi-infinite strip in the w-plane (Fig. 20.12) into the upper half of the z-plane* *(V.T.U., M.E. 2006 ; Osmania, 2003)*

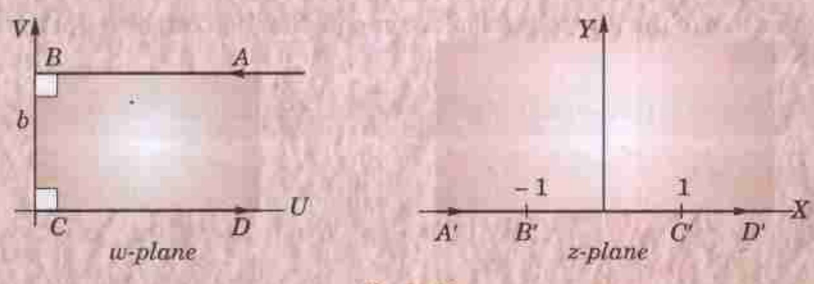

Fig. 20.12

Solution. Consider $ABCD$ as the limiting case of a triangle with two vertices B and C and the third vertex A or D at infinity. Let the vertices B and C map into the points $B'(-1)$and $C'(1)$ of the z-plane. Since the interior angles at B and C are $\pi/2$, we have by the Schwarz-Christoffel transformation,

$$\frac{dw}{dz} = A(z+1)^{\frac{\pi/2}{\pi} - 1} (z - 1)^{\frac{\pi/2}{\pi} - 1} = A / \sqrt{(z^2 - 1)}$$

$\therefore$ $$w = A \int \frac{dz}{\sqrt{(z^2-1)}} + B = A \cosh^{-1} z + B$$

When $z = 1, w = 0$. $\therefore$ $0 = A \cosh^{-1}(1) + B$, *i.e.*, $B = 0$.

When $z = -1, w = ib$. $\therefore$ $ib = A \cosh^{-1}(-1) + 0$, *i.e.*, $\cosh(ib/A) = -1$

or $$\cos \frac{b}{A} = -1 = \cos \pi. \text{ Thus } A = \frac{b}{\pi}.$$

Hence $$w = \frac{b}{\pi} \cosh^{-1} z \text{ or } z = \cosh \frac{\pi w}{b}.$$

PROBLEMS 20.4

1. Find the transformation which maps the semi-infinite strip of width π bounded by the lines $v = 0, v = \pi$ and $u = 0$ into the upper half of the z plane.
2. Show how you will use Schwarz-Christoffel transformation to map the semi-infinite strip enclosed by the real axis and the lines $u = \pm 1$ of the w-plane into the upper half of the z-plane.
3. Find the mapping function which maps semi-infinite strip in the z-plane $-\pi/2 \le x \le \pi/2, y \ge 0$ into half w-plane for which $v \ge 0$, such that the points $(-\pi/2, 0), (\pi/2, 0)$ in the π-plane are mapped into the points $(-1, 0), (1, 0)$ respectively in w-plane.
4. Find the transformation which will map the interior of the infinite strip bounded by the lines $v = 0, v = \pi$ onto the upper half of the z-plane.

20.12 COMPLEX INTEGRATION

We have already discussed the concept of the line integral as applied to vector fields in § 8.11. Now we shall consider the line integral of a complex function.

Fig. 20.13

Consider a continuous function $f(z)$ of the complex variable $z = x + iy$ defined at all points of a curve C having end points A and B. Divide C into n parts at the points
$$A = P_0(z_0), P_1(z_1), ..., P_i(z_i), ... P_n(z_n) = B.$$

Let $\delta z_i = z_i - z_{i-1}$ and ζ_i be any point on the arc $P_{i-1}P_i$. The limit of the sum $\sum_{i=1}^{n} f(\zeta_i)\,\delta z_i$ as $n \to \infty$ *in such a way that the length of the chord* δz_i *approaches zero, is called the* **line integral of f (z)** *taken along the path C, i.e.,*
$$\int_C f(z)\, dz.$$

Writing $f(z) = u(x, y) + iv(x, y)$ and noting that $dz = dx + idy$,
$$\int_C f(z)\, dz = \int_C (udx - vdy) + i \int_C (vdx + udy)$$
which shows that the evaluation of the line integral of a complex function can be reduced to the evaluation of two line integrals of real functions.

Obs. *The value of the integral is independent of the path of integration when the integrand is analytic.*

Example 20.17. *Prove that*

(*i*) $\int_C \frac{dz}{z-a} = 2\pi i$. (*ii*) $\int_C (z-a)^n\, dz = 0$ [*n, any integer* $\neq -1$]

where C is the circle $|z - a| = r$. (*U.P.T.U.*, *2003*)

Solution. The parametric equation of C is $z - a = re^{i\theta}$, where θ varies from 0 to 2π as z describes C once in the positive (anti-clockwise) sense. (Fig. 20.14)

(*i*) $$\int_C \frac{dz}{z-a} = \int_0^{2\pi} \frac{1}{re^{i\theta}} \cdot ire^{i\theta}\, d\theta \qquad [\because dz = ire^{i\theta}\, d\theta]$$
$$= i\int_0^{2\pi} d\theta = 2\pi i$$

(ii) $\int_C (z-a)^n\, dz = \int_0^{2\pi} r^n e^{ni\theta} \,.\, ire^{i\theta}\, d\theta$

$$= ir^{n+1} \int_0^{2\pi} e^{(n+1)\theta i}\, d\theta = \frac{r^{n+1}}{n+1} \left| e^{(n+1)\theta i} \right|_0^{2\pi}, \text{ provided } n \neq -1$$

$$= \frac{r^{n+1}}{n+1} [e^{2(n+1)\pi i} - 1] = 0 \qquad [\because \; e^{2(n+1)\pi i} = 1]$$

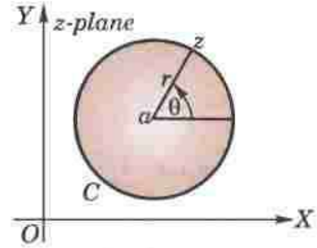

Fig. 20.14

Example 20.18. *Evaluate* $\int_0^{2+i} (\bar{z})^2\, dz$, *along (i) the line* $y = x/2$, (*Bhopal, 2007 ; U.P.T.U., 2002*)
(ii) the real axis to 2 and then vertically to $2 + i$. (*S.V.T.U., 2009 ; P.T.U., 2008 S ; Mumbai, 2006*)

Solution. (i) Along the line OA, $x = 2y$, $z = (2+i)y$, $\bar{z} = (2-i)y$ and $dz = (2+i)\, dy$ (Fig. 20.15)

$$\therefore \qquad I = \int_0^{2+i} (\bar{z})^2\, dz = \int_0^1 (2-i)^2 y^2 \,.\, (2+i)\, dy$$

$$= 5(2-i) \left| \frac{y^3}{3} \right|_0^1 = \frac{5}{3}(2-i)$$

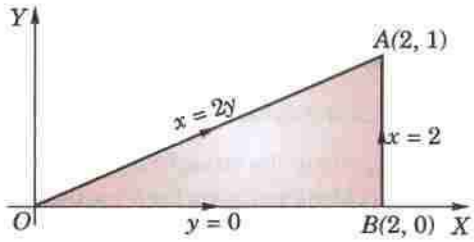

Fig. 20.15

(ii) $$I = \int_{OB} (\bar{z})^2\, dz + \int_{BA} (\bar{z})^2\, dz.$$

Now along OB, $z = x$, $\bar{z} = x$, $dz = dx$;
and along BA, $z = 2 + iy$, $\bar{z} = 2 - iy$, $dz = idy$

$$\therefore \qquad I = \int_0^2 x^2 dx + \int_0^1 (2-iy)^2 \,.\, idy = \left| \frac{x^3}{3} \right|_0^2 + \int_0^1 [\,4y + (4-y^2)i\,]\, dy$$

$$= \frac{8}{3} + 4 \,.\, \frac{1}{2} + \left(4.1 - \frac{1}{3}\right) i = \frac{1}{3}(14 + 11i).$$

Example 20.19. *Evaluate* $\int_C (z^2 + 3z + 2)\, dz$ *where C is the arc of the cycloid* $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ *between the points* $(0, 0)$ *and* $(\pi a, 2a)$. (*Rohtak, 2004*)

Solution. $f(z) = z^2 + 3z + 2$ is analytic in the z-plane being a polynomial. As such, the line integral of $f(z)$ between O and A is independent of the path (Fig. 20.16). We therefore, take the path from O to L and L to A so that

$$\int_C f(z)\, dz = \int_{OL} f(z)\, dz + \int_{LA} f(z)\, dz \qquad ...(i)$$

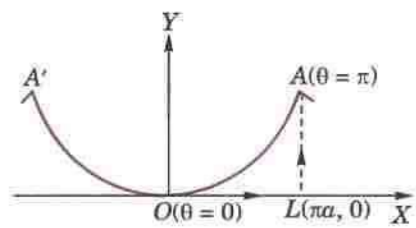

Fig. 20.16

$$\therefore \qquad \int_{OL} f(z)\, dz = \int_0^{\pi a} (x^2 + 3x + 2)\, dx$$

[$\because$ along OL, $y = 0$, $x = 0$ at O, $x = \pi a$ at L

$$= \left| \frac{x^3}{3} + \frac{3x^2}{2} + 2x \right|_0^{\pi a} = \frac{\pi a}{6}(2\pi^2 a^2 + 9\pi a + 12) \qquad ...(ii)$$

and $$\int_{LA} f(z)\, dz = \int_0^{2a} [(\pi a + iy)^2 + 3(\pi a + iy) + 2]\, idy$$

[$\because$ along LA, $x = \pi a$, $z = \pi a + iy$, $dz = idy$ and y varies from 0 (at L) to $2a$ (at A)

$$= L \left| \frac{(\pi a + iy)^3}{3i} + 3\frac{(\pi a + iy)^2}{2i} + 2y \right|_0^{2\pi} = \frac{a^3}{3}(\pi + 2i)^3 + \frac{3a^2}{2}(\pi + 2i)^2 + 4ia \qquad ...(iii)$$

$\therefore$ substituting from (ii) and (iii) in (i), we get

$$\int_C f(z)\, dz = \frac{\pi a}{6}(2\pi^2 a^2 + 9\pi a + 12) + \frac{a^3}{3}(\pi + 2i)^3 + \frac{3a^2}{2}(\pi + 2i)^2 + 4ia$$

PROBLEMS 20.5

1. Evaluate $\int_0^{1+i}(x^2 - iy)\,dz$ along the paths (a) $y = x$ and (b) $y = x^2$. (U.P.T.U., 2010)

2. Evaluate $\int_{1-i}^{2+i}(2x + iy + 1)\,dz$, along the two paths: (U.P.T.U., 2010)
(i) $x = t + 1, y = 2t^2 - 1$ (ii) the straight line joining $1 - i$ and $2 + i$. (U.P.T.U., 2006)

3. Evaluate $\int_{1-i}^{2+3i}(z^2 + z)\,dz$ along the line joining the points (1, – 1) and (2, 3). (V.T.U., 2004)

4. Show that for every path between the limits, $\int_{-2}^{-2+i}(2 + z)^2\,dz = -i/3$. (Delhi, 2002)

5. Show that $\oint (z + 1)\,dz = 0$, where C is the boundary of the square whose vertices are at the points $z = 0$, $z = 1$, $z = 1 + i$ and $z = i$. (Rohtak, 2006)

6. Evaluate $\int_C |z|\,dz$, where C is the contour
(i) straight line from $z = -i$ to $z = i$.
(ii) left half of the unit circle $|z| = 1$ from $z = -i$ to $z = i$.
(iii) circle given by $|z + 1| = 1$ described in the clockwise sense.

7. Find the value of $\int_0^{1+i}(x - y + ix^2)\,dz$
(i) along the straight line from $z = 0$ to $z = 1 + i$
(ii) along real axis from $z = 0$ to $z = 1$ and then along a line parallel to the imaginary axis from $z = 1$ to $z = 1 + i$. (U.P.T.U., 2003)

8. Prove that $\int dz/z = -\pi i$ or πi, according as C is the semi-circular arc $|z| = 1$ above or below the real axis. (Rohtak, 2005)

9. Evaluate $\int_C (z - z^2)\,dz$, where C is the upper half of the circle $|z| = 1$.
What is the value of this integral if C is the lower half of the above circle ?

20.13 CAUCHY'S THEOREM

If $f(z)$ is an analytic function and $f'(z)$ is continuous at each point within and on a closed curve C, then $\oint_C f(z)\,dz = 0$.

Writing $f(z) = u(x, y) + iv(x, y)$ and noting that $dz = dx + idy$

$$\oint_C f(z)dz = \oint_C (udx - vdy) + i\oint_C (vdx + udy) \qquad ...(1)$$

Since $f'(z)$ is continuous, therefore, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous in the region D enclosed by C. Hence the Green's theorem (p. 376) can be applied to (1), giving

$$\oint_C f(z)dz = -\iint_D \left[\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\right]dxdy + i\iint_D \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right]dxdy \qquad ...(2)$$

Now $f(z)$ being analytic, u and v necessarily satisfy the Cauchy-Riemann equations and thus the integrands of the two double integrals in (2) vanish identically.

Hence $\oint_C f(z)dz = 0.$

Obs. 1. The Cauchy-Riemann equations are precisely the conditions for the two real integrals in (1) to be independent of the path. *Hence the line integral of a function f (z) which is analytic in the region D, is independent of the path joining any two points of D.*

Obs. 2. Extension of Cauchy's theorem. *If f (z) is analytic in the region D between two simple closed curves C and* C_1, then $\oint_C f(z)dz = \oint_{C_1} f(z)dz$.

To prove this, we need to introduce the cross-cut AB. Then $\oint f(z)dz = 0$ where the path is as indicated by arrows in Fig. 20.17, *i.e.*, along AB—along C_1 in clockwise sense and along BA—along C in anti-clockwise sense.

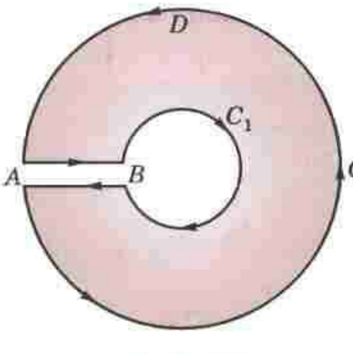

Fig. 20.17(*a*)

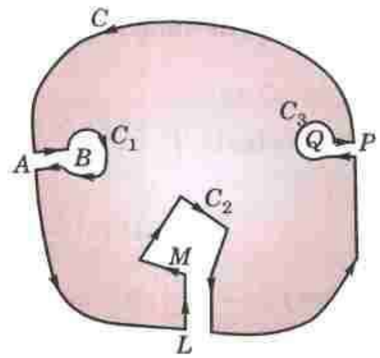

Fig. 20.17(*b*)

i.e.,
$$\int_{AB} f(z)dz + \int_{C_1} f(z)dz + \int_{AB} f(z)dz + \int_{C} f(z)dz = 0$$

But, since the integrals along AB and along BA cancel, it follows that

$$\int_{C} f(z)dz + \int_{C_1} f(z)dz = 0$$

Reversing the direction of the integral around C_1 and transposing, we get

$$\int_{C} f(z)dz + \int_{C_1} f(z)dz \text{ each integration being taken in the anti-clockwise sense.}$$

If $C_1, C_2, C_3, \ldots$ *be any number of closed curves within* C (*Fig. 20.17(b)*), *then*

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \oint_{C_3} f(z)dz + \ldots$$

20.14 CAUCHY'S INTEGRAL FORMULA

If $f(z)$ *is analytic within and on a closed curve and if* a *is any point within* C, *then*

$$f(a) = \frac{1}{2\pi i} = \oint_C \frac{f(z)dz}{z-a}$$

Consider the function $f(z)/(z-a)$ which is analytic at all points within C except at $z = a$. With the point a as centre and radius r, draw a small circle C_1 lying entirely within C.

Now $f(z)/(z-a)$ being analytic in the region enclosed by C and C_1, we have by Cauchy's theorem,

$$\oint_C \frac{f(z)}{z-a}\,dz = \oint_{C_1} \frac{f(z)}{z-a}\,dz \qquad \left\{\begin{array}{l}\text{For any point on } C_1, \\ z-a = re^{i\theta} \text{ and } dz = ire^{i\theta}\,d\theta\end{array}\right.$$

$$= \oint_C \frac{f(a+re^{i\theta})}{re^{i\theta}} . ire^{i\theta}\,d\theta = i\oint_{C_1} f(a+re^{i\theta})d\theta \qquad \ldots(1)$$

In the limiting form, as the circle C_1 shrinks to the point a, *i.e.*, as $r \to 0$, the integral (1) will approach to

$$\oint_C f(a)d\theta = if(a)\int_0^{2\pi} d\theta = 2\pi i f(a)\oint f(z)dz. \text{ Thus } \oint_C \frac{f(z)}{z-a}dz = 2\pi i f(a)$$

i.e.,
$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz \qquad \ldots(2)$$

which is the desired *Cauchy's integral formula.* (*V.T.U., 2011 S*)

Cor. Differentiating both sides of (2) w.r.t. a,

$$f'(a) = \frac{1}{2\pi i}\oint_C \frac{\partial}{\partial a}\left[\frac{f(z)}{z-a}\right]dz = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^2}\,dz \qquad \ldots(3)$$

Similarly,
$$f''(a) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^3}\oint_C \,dz \qquad \ldots(4)$$

and in general, $$f^n(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz. \quad ...(5)$$

Thus it follows from the results (2) to (5) that *if a function f (z) is known to be analytic on the simple closed curve C then the values of the function and all its derivatives can be found at any point of C.* Incidently, we have established a remarkable fact that **an analytic function possesses derivatives of all orders and these are themselves all analytic.**

Example 20.20. *Evaluate* $\int_C \frac{z^2 - z + 1}{z-1}\,dx$, *where C is the circle*

(*i*) $|z| = 1$, (*ii*) $|z| = \frac{1}{2}$. (*S.V.T.U., 2007*)

Solution. (*i*) Here $f(z) = z^2 - z + 1$ and $a = 1$.

Since $f(z)$ is analytic within and on circle $C: |z| = 1$ and $a = 1$ lies on C.

$\therefore$ by Cauchy's integral formula $\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a} = f(a) = 1$ *i.e.*, $\int_C \frac{z^2-z+1}{z-1}\,dz = 2\pi i.$

(*ii*) In this case, $a = 1$ lies outside the circle $C: |z| = 1/2$. So $(z^2 - z + 1)/(z-1)$ is analytic everywhere within C.

$\therefore$ by Cauchy's theorem $\int_C \frac{z^2-z+1}{z-1}\,dz = 0.$

Example 20.21. *Evaluate, using Cauchy's integral formula:*

(*i*) $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz$ *where C is the circle* $|z| = 3$ (*U.P.T.U., 2010*)

(*ii*) $\oint_C \frac{\cos \pi z}{z^2-1}\,dz$ *around a rectangle with vertices* $2 \pm i, -2 \pm 1$

(*iii*) $\oint_C \frac{e^{tz}}{z^2+1}\,dz$ *where C is the circle* $|z| = 3$. (*U.P.T.U., 2009*)

Solution. (*i*) $f(z) = \sin \pi z^2 + \cos \pi z^2$ is analytic within the circle $|z| = 3$ and the two singular points $z = 1$ and $z = 2$ lie inside this circle.

$$\therefore \quad \oint_C \frac{f(z)dz}{(z-1)(z-2)} = \oint_C (\sin \pi z^2 + \cos \pi z^2) = \left(\frac{1}{z-2} - \frac{1}{z-1}\right)$$

$$= \oint_C \frac{(\sin \pi z^2 + \cos \pi z^2)}{z-2}\,dz - \oint_C \frac{(\sin \pi z^2 + \cos \pi z^2)}{z-1}\,dz$$

$$= 2\pi i\,[\sin \pi(2)^2 + \cos \pi(2)^2] - 2\pi i[\sin \pi(1)^2 + \cos \pi(1)^2]$$

[By Cauchy's integral formula]

$$= 2\pi i\,(0 + 1) - 2\pi i\,(0 - 1) = 4\pi i$$

(*ii*) $f(z) = \cos \pi z$ is analytic in the region bounded by the given rectangle and the two singular points $z = 1$ and $z = -1$ lie inside this rectangle. (Fig. 20.18)

$$\therefore \quad \oint_C \frac{\cos \pi z}{z^2-1}\,dz = \frac{1}{2}\oint_C \left(\frac{1}{z-1} - \frac{1}{z+1}\right)\cos \pi z\,dz$$

$$= \frac{1}{2}\oint_C \frac{\cos \pi z}{z-1}\,dz - \oint_C \frac{\cos \pi z}{z+1}\,dz$$

$$= \frac{1}{2}\{2\pi i \cos \pi\,(1)\} - \frac{1}{2}\{2\pi i \cos \pi\,(-1)\} = 0.$$

[By Cauchy's integral formula]

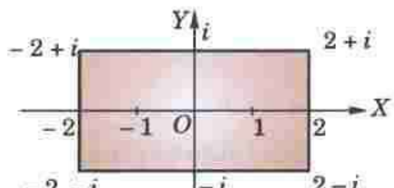

Fig. 20.18

(*iii*) $f(z) = e^{tz}$ is analytic within the circle $|z| = 3$.

The singular points are given by $z^2 + 1 = 0$ *i.e.*, $z = i$ and $z = -i$ which lie within this circle.

$$\therefore \quad \oint_C \frac{e^{tz}}{z^2+1}dz = \oint_C \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right)e^{tz}dz = \frac{1}{2i}\left\{\oint_C \frac{e^{tz}}{z-i}dz - \oint_C \frac{e^{tz}}{z+i}dz\right\}$$

$$= \frac{1}{2i}\{2\pi i e^{t(i)} - 2\pi i e^{t(-i)}\} \quad \text{[By Cauchy's integral formula]}$$

$$= 2\pi i\left(\frac{e^{it} - e^{-it}}{2i}\right) = 2\pi i \sin t.$$

Example 20.22. *Evaluate*

(*i*) $\oint_C \frac{\sin^2 z}{(z-\pi/6)^3}\,dz$, *where C is the circle* $|z| = 1$ *(Rohtak, 2005)*

(*ii*) $\oint_C \frac{e^{2z}}{(z+i)^4}\,dz$, *where C is the circle* $|z| = 3$ *(U.P.T.U., 2008)*

(*iii*) $\oint_C \frac{e^z}{(z^2+\pi^2)^2}\,dz$, *where C is* $|z| = 4$. *(U.P.T.U., 2008; J.N.T.U., 2000)*

Solution. (*i*) $f(z) = \sin^2 z$ is analytic inside the circle C: $|z| = 1$ and the point $a = \pi/6$ (= 0.5 approx.) lies within C.

$\therefore$ by Cauchy's integral formula $f''(a) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^3}dz$,

we get
$$\oint_C \frac{\sin^2 z}{(z-\pi/6)^3}dz = \pi i\left[\frac{d^2}{dz^2}(\sin^2 z)\right]_{z=\pi/6}$$

$$= \pi i(2\cos 2z)_{z=\pi/6} = 2\pi i \cos \pi/3 = \pi i.$$

(*ii*) $f(z) = e^{2z}$ is analytic within the circle C: $|z| = 3$. Also $z = -1$ lies inside C.

$\therefore$ By Cauchy's integral formula: $f'''(a) = \frac{3!}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^4}$

we get
$$\int_C \frac{e^{2z}}{(z+1)^4}dz = \frac{2\pi i}{6}\left|\frac{d^3(e^{2z})}{dz^3}\right|_{z=-1} = \frac{\pi i}{3}[8e^{2z}]_{z=-1} = \frac{8\pi i}{3}e^{-2}$$

(*iii*) $\frac{e^z}{(z^2+\pi^2)^2} = \frac{e^z}{(z+\pi i)^2(z-\pi i)^2}$ is not analytic at $z = \pm \pi i$.

However both $z = \pm \pi i$ lie within the circle $|z| = 4$.

Now
$$\frac{1}{(z+\pi i)^2(z-\pi i)^2} = \frac{A}{z+\pi i} + \frac{B}{(z+\pi i)^2} + \frac{C}{z-\pi i} + \frac{D}{(z-\pi i)^2}$$

where $A = 7/2\pi^3 i$, $C = -7/2\pi^3 i$, $B = D = -1/4\pi^2$

$$\therefore \quad \int_C \frac{e^z}{(z^2+\pi^2)^2}dz = \frac{7}{2\pi^3 i}\left\{\int_C \frac{e^z}{z+\pi i}dz - \int_C \frac{e^z}{z-\pi i}dz\right\} - \frac{1}{4\pi^2}\left\{\int_C \frac{e^z}{(z+\pi i)^2}dz + \int \frac{e^z}{(z-\pi i)^2}dz\right\}$$

$$= \frac{7}{2\pi^3 i}[2\pi i f(-\pi i) - 2\pi i f(\pi i)] - \frac{1}{4\pi^2}[2\pi i f'(-\pi i) + 2\pi i f'(\pi i)] \quad \text{where } f(z) = e^z.$$

$$= \frac{7}{\pi^2}(e^{-\pi i} - e^{\pi i}) - \frac{i}{2\pi}(e^{-\pi i} + e^{\pi i}) = -\frac{14i}{\pi^2}\left(\frac{e^{\pi i} - e^{-\pi i}}{2i}\right) - \frac{i}{\pi}\left(\frac{e^{\pi i} + e^{-\pi i}}{2}\right) \quad \text{[§ 19.9]}$$

$$= -\frac{14i}{\pi^2}\sin \pi - \frac{i}{\pi}\cos \pi = \frac{i}{\pi}.$$

Example 20.23. *Verify Cauchy's theorem by integrating e^{iz} along the boundary of the triangle with the verticles at the points $1 + i, -1 + i$ and $-1 - i$.* (U.P.T.U., 2006)

Solution. The boundary of the given triangle consists of three lines AB, BC, CA. (Fig. 29.19).

$$\oint_{ABC} e^{iz}dz = \int_{AB} e^{iz}dz + \int_{BC} e^{iz}dz + \int_{CA} e^{iz}dz$$

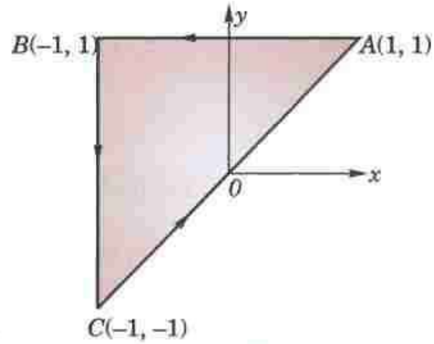

Fig. 20.19

Now $$\int_{AB} e^{iz}dz = \int_1^{-1} e^{i(x+i)}dx \quad \left| \begin{array}{l} \because \text{Along } AB : y = 1 \\ \therefore z = x + i \text{ and } dz = dx \end{array} \right.$$

$$= \int_1^{-1} e^{(ix-1)}dx = \left| \frac{e^{ix-1}}{i} \right|_1^{-1} = \frac{e^{-i-1} - e^{i-1}}{i}$$

$$\int_{BC} e^{iz}dz = \int_1^{-1} e^{i(-1+1iy)}\, idy$$

$$\left| \begin{array}{l} \because \text{Along } BC : x = -1 \\ \therefore z = -1 + iy, dz = idy \end{array} \right.$$

$$= i\int_1^{-1} e^{-i-y}\, dy = i\left| \frac{e^{-i-y}}{-1} \right|_1^{-1} = \frac{e^{-i+1} - e^{-i-1}}{i}$$

$$\int_{CA} e^{iz}dz = \int_{-1}^{1} e^{i(1+i)x}\,(1+i)dx \quad \left| \begin{array}{l} \because \text{Along } CA : y = x \\ \therefore z = (1+i, x, dz = (1+i)dx \end{array} \right.$$

$$= (1+i)\, \frac{e^{(i-1)} - e^{-(i-1)}}{i(1+i)} = \frac{e^{i-1} - e^{-i+1}}{i}$$

Thus from (*i*) $$\oint_{ABC} e^{iz}\,dz = \frac{e^{-i-1} - e^{i-1}}{i} + \frac{e^{-i+1} - e^{-i-1}}{i} + \frac{e^{i-1} - e^{-i+1}}{i} = 0 \qquad ...(ii)$$

Also since $f(x) = e^{iz}$ is analytic everywhere,

$\therefore$ by Cauchy's theorem $\oint f(z) = 0$ ...(*iii*)

Hence from (*ii*) and (*iii*), $\oint_{ABC}$ Cauchy's theorem is verified.

Example 20.24. *If* $F(\zeta) = \oint_C \frac{4z^2 + z + 5}{z - \zeta}\, dz$, *where C is the ellipse* $(x/2)^2 + (y/3)^2 = 1$, *find the value of (a) F(3.5); (b) F(i), F″ (− 1) and F″(− i).* (Bhopal, 2009 ; Marathwada, 2008 ; Mumbai, 2006)

Solution. (*a*) $$F(3.5) = \oint_C \frac{z^3 + z + 1}{z^2 - 7z + 2}dz$$

Since $\zeta = 3.5$ is the only singular point of $(4z^2 + z + 5)/(z - 3.5)$ and it lies outside the ellipse C, therefore, $(4z^2 + z + 5)/(z - 3.5)$ is analytic everywhere within C.

Hence by Cauchy's theorem,

$$\oint_C \frac{4z^2 + z + 5}{z - 3.5}\, dz = 0, \text{ i.e., } F(3.5) = 0.$$

(*b*) Since $f(z) = 4z^2 + z + 5$ is analytic within C and $\zeta = i, -1$ and $-i$ all lie within C, therefore, by Cauchy's integral formula

$$f(\zeta) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - \zeta}\, dz$$

i.e., $$\oint_C \frac{4z^2+z+5}{z-\zeta}dz = 2\pi i(4\zeta^2+\zeta+5)$$

i.e., $$F(\zeta) = 2\pi i(4\zeta^2+\zeta+5)$$

$\therefore$ $$F'(\zeta) = 2\pi i\,(8\zeta+1) \text{ and } F''(\zeta) = 16\pi i$$

Thus $$F(i) = 2\pi i(-4+i+5) = 2\pi(i-1)$$

$$F'(-1) = 2\pi i[8(-1)+1] = -14\pi i \text{ and } F''(-i) = 16\pi i.$$

20.15 (1) CONVERSE OF CAUCHY'S THEOREM: MORERA'S THEOREM*

If $f(z)$ is continuous in a region D and $\oint_C f(z)dz = 0$ around every simple closed curve C in D, then $f(z)$ is analytic in D.

Since $\oint f(z)dz = 0$, then the line integral of $f(z)$ from a fixed point z_0 to a variable point z must be independent of the path and hence must be a function of z only. Thus

$$\int_{z_0}^{z} f(z)dz = \phi(z), \text{ (say)},$$

Let $$\phi(z) = U + iV \text{ and } f(z) = u + iv$$

Then $$U+iV = \int_{(x_0,y_0)}^{(x,y)} (u+iv)(dx+idy) = \int_{(x_0,y_0)}^{(x,y)} (udx - vdy) + i\int_{(x_0,y_0)}^{(x,y)} (vdx+udy)$$

$\therefore$ $$U = \int_{(x_0,y_0)}^{(x,y)} (udx - vdy),\ V = \int_{(x_0,y_0)}^{(x,y)} (vdx+udy)$$

Differentiating under the integral sign,

$$\frac{\partial U}{\partial x} = u, \frac{\partial U}{\partial y} = -v, \frac{\partial V}{\partial x} = v, \frac{\partial V}{\partial y} = u \quad \therefore \quad \frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}, \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

Thus U and V satisfy C-R equations.

Also, since $f(z)$ is given to be continuous, u and v and therefore, $\partial U/\partial x$, $\partial U/\partial y$, $\partial V/\partial x$, $\partial V/\partial y$, are also continuous.

$\therefore$ $\phi(z)$ is an analytic function and

$$\phi'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = u + iv = f(z).$$

Thus, $f(z)$ is the derivative of an analytic function $\phi(z)$. Hence $f(z)$ is analytic by § 20.14 Cor.

(2) Cauchy's inequality†. *If $f(z)$ is analytic within and on the circle C: $|z-a| = r$, then*

$$|f^n(a)| \le \frac{Mn!}{r^n} \qquad ...(I)$$

where M is the maximum value of $|f(z)|$ on C.

From (5) of § 20.14, we get

$$|f^n(a)| = \frac{n!}{2\pi}\left|\oint_C \frac{f(z)dz}{(z-a)^{n+1}}\right|$$

$$\le \frac{n!}{2\pi}\cdot\frac{M}{r^{n+1}}\oint_C |z| \qquad [\because |f(z)| < M]$$

$$= \frac{n!\,M}{2\pi r^{n+1}}\oint_C ds = \frac{Mn!}{2\pi r^{n+1}}2\pi r = \frac{Mn!}{r^n} \qquad \textit{(U.P.T.U., 2005)}$$

(3) Liouville's theorem$. *If $f(z)$ is analytic and bounded for all z in the entire complex plane, then $f(z)$ is a constant.* *(U.P.T.U., 2008)*

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* Named after the Italian mathematician, *Giacinto Morera* (1856–1909) who worked in Turin.

† See footnote p. 144

$ See footnote p. 573.

Taking $n = 1$ and replacing a by z, (1) gives

$$|f'(z)| \le M/r$$

As $r \to \infty$, it gives $f'(z) = 0$ *i.e.*, $f(z)$ is constant for all z.

(4) Poisson's integral formulae. *If $f(z)$ is analytic within and on the circle C: $|z| = \rho$ and $z = re^{i\theta}$ is any point within C, then*

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2 - r^2}{\rho^2 - 2r\rho\cos(\theta - \phi) + r^2} f(re^{i\phi})d\phi$$

By Cauchy's integral formula, $f(z) = \dfrac{1}{2\pi i}\oint_C \dfrac{f(w)}{w - z}\, dw$ ...(1)

As the inverse of the point x w.r.t. C lies outside C and is given by $\rho^2/\bar{z}$. [See footnote p. 685]

$\therefore$ by Cauchy's theorem,

$$0 = \frac{1}{2\pi i}\int \frac{f(w)}{w - \rho^2/z}\, dw \quad ...(2)$$

Subtracting (2) from (1), $f(z) = \dfrac{1}{2\pi i}\displaystyle\int \left(\frac{1}{w - z} - \frac{1}{w - \rho^2/\bar{z}}\right) f(w)\, dw$

$$= \frac{1}{2\pi i}\oint_C \frac{z\bar{z} - \rho^2}{\bar{z}w^2 - (z\bar{z} + \rho^2)\, w + z\rho^2)} f(w)\, dw \quad ...(3)$$

Taking $w = \rho e^{i\phi}$ and noting that $\bar{z} = re^{-i\theta}$, (3) gives

$$f(re^{i\theta}) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{(r^2 - \rho^2)\, f(\rho e^{i\phi})\,.\,\rho i e^{i\phi}\, d\phi}{re^{-i\theta}\,.\,\rho^2 e^{2i\phi} - (r^2 + \rho^2)\,\rho e^{i\phi} + re^{i\theta}\rho^2}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(\rho^2 - r^2)\, f(\rho e^{i\phi})d\phi}{\rho^2 + r^2 - r\rho[e^{i(\theta-\phi)} + e^{-i(\theta-\phi)}]} = \frac{1}{2\pi}\int_0^{2\pi} \frac{(\rho^2 - r^2)\, f(\rho e^{i\phi})d\phi}{\rho^2 - 2r\rho\cos(\theta - \phi) + r^2} \quad ...(4)$$

This is called *Poisson's integral formula** for a circle. It expresses the values of a harmonic function within a circle in terms of its values on the boundary.

Writing $f(re^{i\theta}) = u(r, \theta) + iv(r, \theta)$ and $f(\rho e^{i\phi}) = u(\rho, \phi) + iv(\rho, \phi)$ in (4) and equating real and imaginary parts from both sides, we get the formulae:

$$u(r, \theta) = \int_0^{2\pi} \frac{(\rho^2 - r^2)u(\rho, \phi)d\phi}{\rho^2 - 2r\rho\cos(\theta - \phi) + r^2} \quad ...(5)$$

and

$$v(r, \theta) = \int_0^{2\pi} \frac{(\rho^2 - r^2)v(\rho, \phi)d\phi}{\rho^2 - 2r\rho\cos(\theta - \phi) + r^2} \quad ...(6)$$

PROBLEMS 20.6

1. Evaluate $\oint_C (z - a)^{-1} dz$, where C is a simple closed curve and the point $z = a$ is (*i*) outside C, (*ii*) inside C.
2. Evaluate $\oint_C \dfrac{dz}{(z - a)^n}$, $n = 2, 3, 4, \ldots$, where C is a closed curve containing the point $z = a$.
3. Evaluate (*i*) $\oint_C \dfrac{e^z}{z^2 + 1} dz$, where C is the circle $|z| = 1/2$. (*P.T.U., 2010*)

   (*ii*) $\oint_C \dfrac{e^{3iz}}{(z + \pi)^3} dz$, where C is the circle $|z - \pi| = 3$. (*U.P.T.U., 2007*)

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* Named after the French mathematician *Simeon Denis Poisson* (1781–1840) who was a professor in Paris and made contributions to partial differential equations, potential theory and probability.

4. Use Cauchy's integral formula to calculate:

(i) $\oint_C \frac{3z\ 5}{z^2+2z}dz$, where C is $|z| = 1$. (P.T.U., 2005 S) (ii) $\oint_C \frac{z^2+1}{z(2z+1)}dz$, where C is $|z| = 1$.

(iii) $\oint_C \frac{\sin \pi z + \cos \pi z}{(z-1)(z-2)} dz$ where C is $|z| = 4$. (U.P.T.U., 2008)

5. Evaluate (a) $\oint_C \frac{z^3 - 2z + 1}{(z-i)^2}$ where C is $|z| = 2$.

(b) $\oint_C \frac{e^{-z}}{(z-1)(z-2)^2}dz$ where C is $|z| = 3$. (Rohtak, 2003)

6. Evaluate, using Cauchy's integral formulae:

(i) $\oint_C \frac{z}{z^2-3z+2}dz$, where C is $|z-2| = \frac{1}{2}$. (U.P.T.U., 2009 ; Hissar, 2007 ; Madras, 2000)

(ii) $\oint_C \frac{e^z dz}{(z+1)^2}$, where C is $|z-1| = 3$. (Bhopal, 2009)

(iii) $\oint_C \frac{\log z}{(z-1)^3}dz$ where C is $|z-1| = \frac{1}{2}$. (J.N.T.U., 2003)

7. Evaluate $f(2)$ and $f(3)$ where $f(a) = \oint_C \frac{2z^2 - z - 2}{z-a}dz$ and C is the circle $|z| = 2.5$.

8. If $\phi(\zeta) = \oint_C \frac{3z^2+7z+1}{z-\zeta} dz$, where C is the circle $|z| = 2$ find the values of

(i) $\phi(3)$, (ii) $\phi'(1-i)$ (iii) $\phi''(1-i)$. (Mumbai, 2006)

9. Evaluate $\oint_C \frac{z^3+z+1}{z^2-7z+2} dz$, where C is the ellipse $4x^2 + 9y^2 = 1$. (Rohtak, 2006)

10. Verify Cauchy's theorem for the integral of z^3 taken over the boundary of the (i) rectangle with vertices $-1, 1, 1+i, -1+i$; (ii) triangle with vertices (1, 2), (1, 4), (3, 2). (V.T.U., 2003)

20.16 (1) SERIES OF COMPLEX TERMS

Let
$$(a_1 + ib_1) + (a_2 + ib_2) + \ldots + (a_n + ib_n) + \ldots \qquad \ldots(1)$$
be an infinite series of complex terms ; a's and b's being real numbers. *If the series Σa_n and Σb_n converge to the sums A and B, then series (1) is said to* **converge** *to the sum $A + iB$.* Also if (1) is a convergent series, then $\underset{n\to\infty}{\text{Lt}}\ (a_n + ib_n) = 0$.

The series (1) is said to be **absolutely convergent** *if the series*
$$|a_1 + ib_1| + |a_2 + ib_2| + \ldots + |a_n + ib_n| + \ldots$$
is convergent. Since $|a_n|$ and $|b_n|$ are both $\le |a_n + ib_n|$, it follows *that an absolutely convergent series is convergent.*

Next let the series of functions $u_1(z) + u_2(z) + \ldots + u_n(z) + \ldots\ldots \qquad \ldots(2)$

converge to the sum $S(z)$ and $S_n(z)$ be the sum of its first n terms. Then *the series (2) is said to be* **uniformly convergent** *in a region R, if corresponding to any positive number ε, there exists a positive number N, depending on ε, but not on z, such that for every z in R.*
$$|S(z) - S_n(z)| < \varepsilon \text{ for } n > N. \qquad \text{[cf. Def. p. 389]}$$

As in the case of real series (p. 390) **Weirstrass's M-test** holds for series of complex terms. So *the series (2) is uniformly convergent in a region R if there is a convergent series of positive constants ΣM_n such that $|u_n(z)| \le M_n$ for all z in R.*

Also a *uniformly convergent series of continuous complex functions is itself continuous and can be integrated term by term.*

Obs. *If a power series $\Sigma a_n z^n$ converges for $z = z_1$, then it converges absolutely for $|z| < |z_1|$.*

Since $\Sigma a_n z_1^n$ converges, therefore, $\underset{n\to\infty}{\text{Lt}}\ a_1 z_1^n = 0$ and so we can find a number k such that $|a_n z_1^n| < k$ for all n. Then $\Sigma a_n z^n = \Sigma |a_n z_1^n| \,.\, |z/z_1|^n < \Sigma k t^n$ where $t = |z/z_1|$.

But the series Σt^n converges for $t < 1$. Hence the series $\Sigma a_n z^n$ converges absolutely for $|z| < |z_1|$, *i.e.*, if a circle with centre at the origin and radius $|z|$ be drawn, then the given series converges absolutely at all points inside the circle.

Such a circle $|z| = R$ within which series $\Sigma a_n z^n$ converges, is called the *circle of convergence and R is called the radius of convergence*.

A power series is uniformly convergent in any region which lies entirely within its circle of convergence.

(2) Taylor's series*. *If $f(z)$ is analytic inside a circle C with centre at a, then for z inside C,*

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \dots + \frac{f^n(a)}{n!}(z-a)^n + \dots \qquad \dots(i)$$

Proof. Let z be any point inside C. Draw a circle C_1 with centre at a enclosing z (Fig. 20.20). Let t be a point on C_1. We have

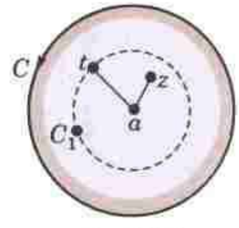


Fig. 20.20

$$\frac{1}{t-z} = \frac{1}{t-a-(z-a)} = \frac{1}{t-a}\left(1 - \frac{z-a}{t-a}\right)^{-1}$$

$$= \frac{1}{t-a}\left[1 + \frac{z-a}{t-a} + \left(\frac{z-a}{t-a}\right) + \dots + \left(\frac{z-a}{t-a}\right)^n + \dots\right] \qquad \dots(ii)$$

As $|z-a| < |t-a|$, *i.e.* $|(z-a)/(t-a)| < 1$, this series converges uniformly. So, multiplying both sides of (*ii*) by $f(t)$, we can integrate over C_1.

$$\therefore \quad \oint_{C_1} \frac{f(t)}{t-z}dz = \oint_{C_1} \frac{f(t)}{t-a}dz + (z-a)\oint_{C_1} \frac{f(t)}{(t-a)^2}dt + \dots + (z-a)^n \,.\, \oint_{C_1} \frac{f(t)}{(t-a)^{n+1}}dt + \dots \qquad \dots(iii)$$

Since $f(t)$ is analytic on and inside C_1, therefore, applying the formulae (2) to (5) of p. 697-698 (*iii*), we get (*i*) which is known as *Taylor's series.*

Obs. Another remarkable fact is that *complex analytic functions can always be represented by power series of the form* (*i*).

(3) Laurent's series†. *If f(z) is analytic in the ring-shaped region R bounded by two concentric circles C and C_1 of radii r and r_1 ($r > r_1$) and with centre at a, then for all z in R*

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \dots$$

where $a_n = \dfrac{1}{2\pi i}\oint \dfrac{f(t)}{(t-a)^{n+1}}\,dt,$

Γ *being any curve in R, encircling C_1* (as in Fig. 20.21).

Proof. Introduce cross-out AB, then $f(z)$ is analytic in the region D bounded by AB, C_1 described clockwise, BA and C described anti-clockwise (see Fig. 20.17). Then if z be any point in D, we have

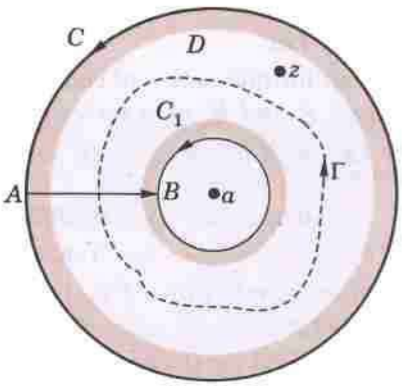


Fig. 20.21

$$f(z) = \frac{1}{2\pi i}\left[\int_{AB}\frac{f(t)}{t-z}dt + \oint_{C_1}\frac{f(t)}{t-z}dt + \int_{BA}\frac{f(t)}{t-z}dt + \oint_{C_1}\frac{f(t)}{t-z}dt\right]$$

$$= \frac{1}{2\pi i}\left[\oint_C \frac{f(t)}{t-z}dt - \oint_{C_1}\frac{f(t)}{t-z}dt\right] \qquad \dots(i)$$

where both C and C_1 are described anti-clockwise in (*i*) and integrals along AB and BA cancel (Fig. 20.21).

For the first integral in (*i*), expanding $1/(t-z)$ as in § 20.16 (2), we get

$$\frac{1}{2\pi i}\oint_C \frac{f(t)}{t-z}dt = \sum_{n=1}^{\infty}\frac{(z-a)^n}{2\pi i}\oint_C \frac{f(t)}{(t-a)^{n+1}}dt$$

* See footnote p. 145.

† Named after the French engineer and mathematician *Pierre Alphonse Laurent* (1813–1854) who published this theorem in 1843.

$$= \Sigma a_n (z-a)^n \text{ where } a_n = \frac{1}{2\pi i} \oint_C \frac{f(t)}{(t-a)^{n+1}} dt \qquad ...(ii)$$

For the second integral in (i), let t lie on C_1. Then we write

$$\frac{1}{t-z} = \frac{1}{(t-a)-(z-a)} = -\frac{1}{z-a}\left(1 - \frac{t-a}{z-a}\right)^{-1}$$

$$= -\frac{1}{z-a}\left[1 + \frac{t-a}{z-a} + \left(\frac{t-a}{z-a}\right)^2 + ... + \left(\frac{t-a}{z-a}\right)^{n-1} + ...\right]$$

As $|t-a| < |z-a|$, *i.e.*, $|(t-a)/(z-a)| < 1$, this series converges uniformly. So multiplying both sides by $f(t)$ and integrating over C_1, we get

$$-\frac{1}{2\pi i} \oint_C \frac{f(t)}{t-z} dt = \sum_{n=1}^{\infty} \frac{1}{(z-a)^n} \cdot \frac{1}{2\pi i} \oint_C (t-a)^{n-1} f(t) dt = \sum_{n=1}^{\infty} a_{-n}(z-a)^{-n} \qquad ...(iii)$$

where $$a_{-n} = \frac{1}{2\pi i} \oint_{C_1} \frac{f(t)}{(t-a)^{-n+1}} dt$$

Substituting from (ii) and (iii) in (i), we get

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} a_{-n}(z-a)^{-n}. \qquad ...(iv)$$

Now $f(t)/(t-a)^{n+1}$ being analytic in the region between C and Γ, we can take the integral giving a_n over Γ. Similarly we can take the integral giving a_{-n} over Γ. Hence (iv) can be written as

$$f(z) = \sum_{-\infty}^{\infty} a_n (z-a)^n \text{ where } a_n = \frac{1}{2\pi i} \oint_\Gamma \frac{f(t)}{(t-a)^{n+1}} dt$$

which is known as *Laurent's series.*

Obs. 1. *As $f(z)$ is not given to be analytic inside Γ,* $a_n = \frac{1}{2\pi i} \oint_\Gamma \frac{f(t)}{(t-a)^{n+1}} dt \neq \frac{f^n(a)}{n!}$

However, *if $f(z)$ is analytic inside Γ*, then $a_{-n} = 0$; $a_n = \frac{1}{2\pi i} \oint_\Gamma \frac{f(t)}{(t-a)^{n+1}} dt = \frac{f^n(a)}{n!}$

and *Laurent's series reduces to Taylor's series.*

Obs. 2. *To obtain Taylor's or Laurent's series, simply expand $f(z)$ by binomial theorem* instead of finding a_n by complex integration which is quite complicated.

Obs. 3. *Laurent series of a given analytic function $f(z)$ in its annulus of convergence is unique.* There may be different Laurent series of $f(z)$ in two annuli with the same centre.

Example 20.25. *Show that the series $z(1-z) + z^2(1-z) + z^3(1-z) + ... \infty$ converges for $|z| < 1$. Determine whether it converges absolutely or not.*

Solution. Let the sum of the first n terms of the series be s_n, so that

$$s_n = z - z^2 + z^2 - z^3 + z^3 - z^4 + ... + z^n - z^{n+1} = z - z^{n+1}$$

For $|z| < 1$, $z^{n+1} \to 0$ as $n \to \infty$.

$\therefore$ $\underset{n\to\infty}{\text{Lt}} (s_n) = z$, *i.e.*, the given series converges for $|z| < 1$.

$$|s_n(z)| = |z(1-z)| + |z^2(1-z)| + ... + |z^n(1-z)|$$
$$= |1-z|\{|z| + |z|^2 + |z|^3 + ... + |z|^n\}$$

For $|z| < 1$, $\underset{n\to\infty}{\text{Lt}} |s_n(z)| = |1-z| \frac{|z|}{1-|z|}$ [G.P.]

Hence the given series converges absolutely.

Example 20.26. *Expand sin z in a Taylor's series about z = 0 and determine the region of convergence.* (P.T.U., 2009 S)

Solution. Given $f(z) = \sin z, f'(z) = \cos\theta, f''(z) = -\sin z, f'''(z) = -\cos z, \ldots$

$\therefore$ $f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = 1$

By Taylor's series about $z = 0$, we have

$$f(z) = f(0) + \frac{(z-0)}{1!}f'(0) + \frac{(z-0)^2}{2!}f''(0) + \frac{(z-0)^3}{3!}f'''(0) + \ldots$$

i.e.,
$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots + (-1)^{n-1}\frac{z^{2n}-1}{(2n-1)!} + \ldots$$

Hence
$$\sin z = \sum_{n=1}^{\infty} a_n (z-0)^{2n-1} \text{ where } a_n = \frac{(-1)^{n-1}}{(2n-1)!}$$

Since
$$\rho = \underset{n\to\infty}{\text{Lt}}\left|\frac{a_{n+1}}{a_n}\right| = \underset{n\to\infty}{\text{Lt}}\left|\frac{(2n-1)!}{(2n+1)!}\right| = 0$$

Thus the radius of convergence of $f(z) = 1/\rho = \infty$

i.e., the region of convergence of $f(z)$ is all reals.

Example 20.27. *Find Taylor's expansion of*

(*i*) $f(z) = \dfrac{1}{(z+1)^2}$ *about the point* $z = -i$. (V.T.U., 2009 S)

(*ii*) $f(z) = \dfrac{2z^3+1}{z^2+z}$ *about the point* $z = i$. (P.T.U., 2003)

Solution. (*i*) To expand $f(z)$ about $z = -i$, *i.e.*, in powers of $z + i$, put $z + i = t$. Then

$$f(z) = \frac{1}{(t-i+1)^2} = (1-i)^{-2}\,[1 + t/(1-i)]^{-2} = \frac{i}{2}\left[1 - \frac{2t}{1-i} + \frac{3t^2}{(1-i)^2} - \frac{4t^3}{(1-i)^3} + \ldots\right]$$

[Expanding by Binomial theorem]

$$= \frac{i}{2}\left[1 + \sum_{n=1}^{\infty}(-1)^n \frac{(n+1)(z+i)^n}{(1-i)^n}\right]$$

(*ii*)
$$f(z) = \frac{2z^3+1}{z(z+1)} = 2z - 2 + \frac{2z+1}{z(z+1)} = (2i-2) + 2(z-i) + \frac{1}{z} + \frac{1}{z+1} \quad \ldots(i)$$

[By partial fractions]

To expand $1/z$ and $1/(z + 1)$ about $z = i$, put $z - i = t$, so that

$$\frac{1}{z} = \frac{1}{(t+i)} = \frac{1}{i}\left(1 + \frac{t}{i}\right)^{-1}$$ [Expanding by Binomial theorem]

$$= \frac{1}{i}\left[1 - \frac{t}{i} + \frac{t^2}{i^2} - \frac{t^3}{i^3} + \frac{t^4}{i^4} - \ldots\infty\right] = \frac{1}{i} + \frac{t}{1} + \frac{t^2}{i^3} - \frac{t^3}{i^4} + \frac{t^4}{i^5} - \ldots\infty$$

$$= -i + (z-i) + \sum_{n=2}^{\infty}(-1)^n \frac{(z-i)^n}{i^{n+1}} \quad \ldots(ii)$$

and
$$\frac{1}{z+1} = \frac{1}{t+i+1} = \frac{1}{1+i}\left(1 + \frac{t}{1+i}\right)^{-1}$$ [Expanding by Binomial theorem]

$$= \frac{1}{1+i}\left[1 - \frac{t}{1+i} + \frac{t^2}{(1+t)^2} - \frac{t^3}{(1+i)^3} + \frac{t^4}{(1+i)^4} - \ldots\infty\right]$$

$$= \frac{1-i}{2} - \frac{t}{2i} + \left[\frac{t^2}{(1+i)^3} - \frac{t^3}{(1+i)^4} + \frac{t^4}{(1+i)^5} - \ldots\infty\right] = \frac{1}{2} - \frac{i}{2} - \frac{z-i}{2i} + \sum_{n=2}^{\infty}(-1)^n\frac{(z-i)^n}{(1+i)^{n+1}} \quad \ldots(iii)$$

Substituting from (*ii*) and (*iii*) in (*i*), we get

$$f(z) = \left(2i - 2 - i + \frac{1}{2} - \frac{i}{2}\right) + \left(2 + 1 - \frac{1}{2i}\right)(z - i) + \sum_{n=2}^{\infty} (-1)^n \left\{\frac{1}{i^{n+1}} + \frac{1}{(1+i)^{n+1}}\right\}(z-i)^n$$

$$= \left(\frac{i}{2} - \frac{3}{2}\right) + \left(3 + \frac{i}{2}\right)(z - i) + \sum_{n=2}^{\infty} (-1)^n \left\{\frac{1}{i^{n+1}} + \frac{1}{(1+i)^{n+1}}\right\}(z-i)^n.$$

Example 20.28. *Expand* $f(z) = 1/[(z-1)(z-2)]$ *in the region:*
(*a*) $|z| < 1$, (*b*) $1 < |z| < 2$, (*c*) $|z| > 2$, (*d*) $0 < |z - 1| < 1$.
(U.P.T.U., 2010 ; V.T.U., 2010 ; Bhopal, 2009)

Solution. (*a*) By partial fractions $\dfrac{1}{(z-1)(z-2)} = \dfrac{1}{z-2} - \dfrac{1}{z-1}$ ...(*i*)

$$= -\frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1} + (1-z)^{-1} \qquad ...(ii)$$

For $|z| < 1$, both $|z/2|$ and $|z|$ are less than 1. Hence (*ii*) gives on expansion

$$f(z) = -\frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \ldots\right) + (1 + z + z^2 + z^3 + \ldots)$$

$$= \frac{1}{2} + \frac{3}{4}z + \frac{7}{8}z^2 + \frac{15}{16}z^3 + \ldots \text{ which is a } \textit{Taylor's series.}$$

(*b*) *For* $1 < |z| < 2$, we write (*i*) as

$$f(z) = -\frac{1}{2}\frac{1}{(1 - z/2)} - \frac{1}{z(1 - z^{-1})} \qquad ...(iii)$$

and notice that both $|z/2|$ and $|z^{-1}|$ are less than 1. Hence (*iii*) gives on expansion

$$f(z) = -\frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \ldots\right) - \frac{1}{z}(1 + z^{-1} + z^{-2} + z^{-3} + \ldots)$$

$$= \ldots - z^{-4} - z^{-3} - z^{-2} - z^{-1} - \frac{1}{2} - \frac{1}{4}z - \frac{1}{8}z^2 - \frac{1}{16}z^3 - \ldots$$

which is a *Laurent's series.*

(*c*) *For* $|z| > 2$, we write (*i*) as

$$f(z) = \frac{1}{z(1 - 2z^{-1})} - \frac{1}{z(1 - z^{-1})}$$

$$= z^{-1}(1 + 2z^{-1} + 4z^{-2} + 8z^{-3} + \ldots) - z^{-1}(1 + z^{-1} + z^{-2} + z^{-3} + \ldots)$$

$$= \ldots + 7z^{-4} + 3z^{-3} + z^{-2} + \ldots$$

(*d*) *For* $0 < |z - 1| < 1$, we write (*i*) as

$$f(z) = \frac{1}{(z-1)-1} - \frac{1}{z-1}$$

$$= -(z-1)^{-1} - [1 - (z-1)]^{-1}$$

$$= -(z-1)^{-1} - [1 + (z-1) + (z-1)^2 + (z-1)^3 + \ldots].$$

Example 20.29. *Find the Laurents' expansion of* $f(z) = \dfrac{7z-2}{(z+1)z(z-2)}$ *in the region* $1 < z + 1 < 3$.
(S.V.T.U., 2009 ; Anna, 2003 ; V.T.U., 2003)

Solution. Writing $z + 1 = u$, we have

$$f(z) = \frac{7(u-1)-2}{u(u-1)(u-1-2)} = \frac{7u-9}{u(u-1)(u-3)}$$

$$= -\frac{3}{u} + \frac{1}{u-1} + \frac{2}{u-3} \quad \text{(splitting into partial fraction)}$$

$$= -\frac{3}{u} + \frac{1}{u(1-1/u)} - \frac{2}{3(1-u/3)} = -\frac{3}{u} + \frac{1}{u}\left(1 - \frac{1}{u}\right)^{-1} - \frac{2}{3}\left(1 - \frac{u}{3}\right)^{-1}$$

Since $1 < u < 3$ or $1/u < 1$ and $u/3 < 1$, expanding by Binomial theorem,

$$f(z) = \frac{-3}{u} + \frac{1}{u}\left(1 + \frac{1}{u} + \frac{1}{u^2} + \frac{1}{u^3} + \dots \infty\right) - \frac{2}{3}\left(1 + \frac{u}{3} + \frac{u^2}{3^2} + \frac{u^3}{3^3} + \dots \infty\right)$$

$$= -\frac{2}{u} + \frac{1}{u^2} + \frac{1}{u^3} + \dots \infty - \frac{2}{3}\left(1 + \frac{u}{3} + \frac{u^2}{3^2} + \frac{u^3}{3^3} + \dots \infty\right)$$

Hence $\quad f(z) = -\dfrac{2}{z+1} + \dfrac{1}{(z+1)^2} + \dfrac{1}{(z+1)^3} + \dots \infty - \dfrac{2}{3}\left[1 + \dfrac{z+1}{3} + \dfrac{(z+1)^2}{3^2} + \dfrac{(z+1)^3}{3^3} + \dots \infty\right]$

which is valid in the region $1 < z + 1 < 3$.

20.17 (1) ZEROS OF AN ANALYTIC FUNCTION

Def. *A zero of an analytic function f(z) is that value of z for which f(z) = 0.*

If $f(z)$ is analytic in the neighbourhood of a point $z = a$, then by Taylor's theorem

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + a_n(z-a)^n + \dots \qquad \text{where } a_n = \frac{f^n(a)}{n!}.$$

If $a_0 = a_1 = a_2 = \dots = a_{m-1} = 0$ but $a_m \neq 0$, then $f(z)$ is said to have a *zero of order m* at $z = a$.

When $m = 1$, the zero is said to be *simple*. In the neighbourhood of zero ($z = a$) of order m,

$$f(z) = a_m(z-a)^m + a_{m+1}(z-a)^{m+1} + \dots \infty$$
$$= (z-a)^m \,\phi(z) \text{ where } \phi(z) = a_m + a_{m+1}(z-a) + \dots$$

Then $\phi(z)$ is analytic and non-zero in the neighbourhood of $z = a$.

(2) Singularities of an analytic function

We have already defined a *singular point of a function as the point at which the function ceases to be analytic.*

(*i*) **Isolated singularity.** *If $z = a$ is a singularity of $f(z)$ such that $f(z)$ is analytic at each point in its neighbourhood* (*i.e.*, there exists a circle with centre a which has no other singularity), *then $z = a$ is called an* **isolated singularity.**

In such a case, $f(z)$ can be expanded in a Laurent's series around $z = a$, giving

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \dots \qquad \dots(1)$$

For example, $f(z) = \cot(\pi/z)$ is not analytic where $\tan(\pi/z) = 0$ *i.e.* at the points $\pi/z = 4\pi$ or $z = 1/n$ ($n = 1, 2, 3, \dots$).

Thus $z = 1, 1/2, 1/3, \dots$ are all *isolated singularities* as there is no other singularity in their neighbourhood.

But when n is large, $z = 0$ is such a singularity that there are infinite number of other singularities in its neighbourhood. Thus $z = 0$ is the *non-isolated singularity* of $f(z)$.

(*ii*) **Removable singularity.** If all the negative powers of $(z - a)$ in (1) are zero, then $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$.

Here the singularity can be removed by defining $f(z)$ at $z = a$ in such a way that it becomes analytic at $z = a$. Such a singularity is called a *removable singularity.*

Thus *if* $\underset{z\to a}{\text{Lt}}\, f(z)$ *exists finitely, then $z = a$ is a removable singularity.*

(*iii*) **Poles.** *If all the negative powers of $(z - a)$ in (i) after the nth are missing, then the singularity at $z = a$ is called a* **pole of order n.**

A pole of first order is called a **simple pole.**

(*iv*) **Essential singularity.** *If the number of negative powers of $(z - a)$ in (1) is infinite, then $z = a$ is called an essential singularity.* In this case, $\underset{z\to a}{\text{Lt}}\, f(z)$ does not exist.

Example 20.30. *Find the nature and location of singularities of the following functions:*

(*i*) $\dfrac{z - \sin z}{z^2}$ $\qquad$ (*ii*) $(z+1)\sin\dfrac{1}{z-2}$ $\qquad$ (*iii*) $\dfrac{1}{\cos z - \sin z}$.

Solution. (*i*) Here $z = 0$ is a singularity.

Also $$\frac{z - \sin z}{z^2} = \frac{1}{z^2}\left\{z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots\right)\right\} = \frac{z}{3!} - \frac{z^3}{5!} + \frac{z^5}{7!} - \dots$$

Since there are no negative powers of z in the expansion, $z = 0$ *is a removable singularity.*

(*ii*) $(z + 1)\sin\frac{1}{z-2} = (t + 2 + 1)\sin\frac{1}{t}$ where $t = z - 2$

$$= (t+3)\left\{\frac{1}{t} - \frac{1}{3!t^3} + \frac{1}{5!t^5} - \dots\right\} = \left(1 - \frac{1}{3!t^2} + \frac{1}{5!t^4} - \dots\right) + \left(\frac{3}{t} - \frac{1}{2t^3} + \frac{3}{5!t^5} - \dots\right)$$

$$= 1 + \frac{3}{t} - \frac{1}{6t^2} - \frac{1}{2t^3} + \frac{1}{120t^4} - \dots = 1 + \frac{3}{z-2} - \frac{1}{6(z-2)^2} - \frac{1}{2(z-2)^3} + \dots$$

Since there are infinite number of terms in the negative powers of $(z - 2)$, $z = 2$ *is an essential singularity.*

(*iii*) Poles of $f(z) = \frac{1}{\cos z - \sin z}$ are given by equating the denominator to zero, *i.e.*, by $\cos z - \sin z = 0$ or $\tan z = 1$ or $z = \pi/4$. Clearly $z = \pi/4$ is a *simple pole of* $f(z)$.

Example 20.31. *What type of singularity have the following functions :*

(*i*) $\frac{1}{1 - e^z}$ (*ii*) $\frac{e^{2z}}{(z-1)^4}$ (*iii*) $\frac{e^{1/z}}{z^2}$. (*U.P.T.U., 2009*)

Solution. (*i*) Poles of $f(z) = 1/(1 - e^z)$ are found by equating to zero $1 - e^z = 0$ or $e^z = 1 = e^{2n\pi i}$

$\therefore$ $z = 2n\pi i$ $(n = 0, \pm 1, \pm 2, \dots)$

Clearly $f(z)$ *has a simple pole at* $z = 2\pi i$.

(*ii*) $\frac{e^{2z}}{(z-1)^4} = \frac{e^{2(t+1)}}{t^4} = \frac{e^2}{t^4}\cdot e^{2t}$ where $t = z - 1$

$$= \frac{e^2}{t^4}\left\{1 + \frac{2t}{1!} + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} + \frac{(2t)^5}{5!} + \dots\right\} = e^2\left\{\frac{1}{t^4} + \frac{2}{t^3} + \frac{2}{t^2} + \frac{4}{3t} + \frac{2}{3} + \frac{4t}{15} + \dots\right\}$$

$$= e^2\left\{\frac{1}{(z-1)^4} + \frac{2}{(z-1)^3} + \frac{2}{(z-1)^2} + \frac{4}{3(z-1)} + \frac{2}{3} + \frac{4}{15}(z-1) + \dots\right\}$$

Since there are finite (4) number of terms containing negative powers of $(z - 1)$,

$\therefore$ $z = 1$ *is a pole of 4th order.*

(*iii*) $$f(z) = \frac{e^{1/z}}{z^2} = \frac{1}{z^2}\left\{1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \dots\right\} = z^{-2} + z^{-3} + \frac{z^{-4}}{2} + \dots \infty$$

Since there are infinite number of terms in the negative powers of z, therefore $f(z)$ has *an essential singularity* at $z = 0$.

PROBLEMS 20.7

1. Obtain the expansion of $(z - 1)/z^2$ in a Taylor's series in powers of $(z - 1)$ and determine the region of convergence.
2. Find the first three times of the Taylor's series expansion of $f(z) = 1/(z^2 + 4)$ about $z = -i$. Also find the region of convergence. (*U.P.T.U., 2006*)
3. Expand in Taylor's series (*i*) $(z - 1)/(z + 1)$ about the point $z = 1$. (*Andhra, 2000*)

   (*ii*) $\cos z$ about the point $z = \pi/2$. (*Marathwada, 2008*) (*iii*) $\frac{1}{z^2 - z - 6}$ about (*a*) $z = -1$ (*b*) $z = 1$ (*P.T.U., 2009*)
4. Expand the following functions in Laurent's series :

   (*i*) $f(z) = \frac{1}{z - z^2}$ for $1 < |z + 1| < 2$. (*Madras, 2006*)

(ii) $f(z) = \dfrac{1}{(z-1)(z+3)}$ for $1 < |z| < 3$. (J.N.T.U., 2006)

(iii) $f(z) = z/[(z-1)(z-3)]$ for $|z-1| < 2$. (V.T.U., 2007)

5. Find the Laurent's expansion of (i) $\dfrac{e^z}{(z-1)^2}$, about $z = 1$. (Rohtak, 2006)

(ii) $e^{2z}/(z-1)^3$ about the singularity $z = 1$.

6. Expand the following functions in Laurent series.

(i) $(z-1)/z^2$ for $|z-1| > 1$ (ii) $\dfrac{1-\cos z}{z^3}$, about $z = 0$. (Rohtak, 2004)

7. Find the Laurent's series expansion of

(i) $\dfrac{z^2-1}{z^2+5z+6}$ about $z = 0$ in the region $2 < |z| < 3$ (V.T.U., 2011 S ; Osmania, 2003)

(ii) $\dfrac{z^2-6z-1}{(z-1)(z-3)(z+2)}$ in the region $3 < |z+2| < 5$

(iii) $\dfrac{7z^2-9z-18}{z^3-9z}$ in the region (a) $|z| > 3$ (b) $0 < |z-3| < 3$. (V.T.U., 2010 S)

8. Find the Laurent's expansion of $1/[(z^2+1)(z^2+2)]$ for (a) $0 < |z| < 1$; (b) $1 < |z| < \sqrt{2}$; (c) $|z| > 2$.

Find the nature and location of the singularities of the following functions : (P.T.U., 2005)

9. $\dfrac{1}{z(2-z)}$. 10. $\sin(1/z)$. (U.P.T.U., 2009) 11. $\tan\left(\dfrac{1}{z}\right)$. (P.T.U., 2006)

12. $\dfrac{z^2-1}{(z-1)^3}$. (Osmania, 2003) 13. $\dfrac{e^z}{(z-1)^4}$. 14. $\dfrac{\cot \pi z}{(z-a)^2}$. (U.P.T.U., 2008)

20.18 (1) RESIDUES

The coefficient of $(z-a)^{-1}$ *in the expansion of* $f(z)$ *around an isolated singularity is called the* **residue** *of* $f(z)$ *at that point.* Thus in the Laurent's series expansion of $f(z)$ around $z = a$ *i.e.*, $f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \dots$, the residue of $f(z)$ at $z = a$ is a_{-1}.

$\therefore$
$$Res\, f(a) = \frac{1}{2\pi i}\oint_C f(z)$$

i.e.,
$$\oint_C f(z)\,dz = 2\pi i \text{ Res } f(a). \qquad \dots(1)$$

(2) Residue Theorem

If $f(z)$ *is analytic in a closed curve C except at a finite number of singular points within C, then* $\oint_C f(z)\,dz = 2\pi i \times$ *(sum of the residues at the singular points within C).*

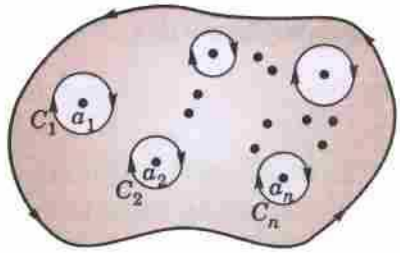


Fig. 20.22

Let us surround each of the singular points $a_1, a_2, \dots, a_n$ by a small circle such that it encloses no other singular point (Fig. 20.22). Then these circles $C_1, C_2, \dots, C_n$ together with C, form a multiply connected region in which $f(z)$ is analytic.

$\therefore$ applying Cauchy's theorem, we have

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \dots + \oint_{C_n} f(z)dz \qquad \text{[by (1)}$$

$$= 2\pi i\, [\text{Res } f(a_1) + \text{Res } f(a_2) + \dots + \text{Res } f(a_n)] \text{ which is the desired result.}$$

20.19 CALCULATION OF RESIDUES

(1) *If $f(z)$ has a simple pole at $z = a$, then*

$$\textbf{Res } \mathbf{f(a)} = \underset{z \to a}{\mathrm{Lt}} \, [(\mathbf{z - a})\mathbf{f(z)}]. \qquad ...(1)$$

Laurent's series in this case is

$$f(z) = c_0 + c_1(z-a) + c_2(z-a)^2 \, ... + c_{-1}(z-a)^{-1}$$

Multiplying throughout by $z - a$, we have

$$(z-a)f(z) = c_0(z-a) + c_1(z-a)^2 + ... + c_{-1}.$$

Taking limits as $z \to a$, we get

$$\underset{z \to a}{\mathrm{Lt}} \, [(z-a)f(z)] = c_{-1} = \text{Res } f(a).$$

(2) *Another formula for Res $f(a)$:*

Let $f(z) = \phi(z)/\psi(z)$, where $\psi(z) = (z-a)\,F(z)$, $F(a) \neq 0$.

Then $$\underset{z \to a}{\mathrm{Lt}} \, [(z-a)\phi(z)/\psi(z)]$$

$$= \underset{z \to a}{\mathrm{Lt}} \frac{(z-a)[\phi(a) + (z-a)\phi'(a) + ...]}{\psi(a) + (z-a)\psi'(a) + ...}$$

$$= \underset{z \to a}{\mathrm{Lt}} \frac{\phi(a) + (z-a)\phi'(a) + ...}{\psi'(a) + (z-a)\psi''(a) + ...}, \quad \text{since } \psi(a) = 0$$

Thus $$\textbf{Res } \mathbf{f(a)} = \frac{\boldsymbol{\phi}\mathbf{(a)}}{\boldsymbol{\psi}'\mathbf{(a)}}.$$

(3) *If $f(z)$ has a pole of order n at $z = a$, then*

$$\textbf{Res } \mathbf{f(a)} = \frac{\mathbf{1}}{\mathbf{(n-1)!}} \left\{ \frac{\mathbf{d^{n-1}}}{\mathbf{dz^{n-1}}} [(\mathbf{z-a})^{\mathbf{n}} \mathbf{f(z)}] \right\}_{\mathbf{z=a}}$$

Here $$f(z) = c_0 + c_1(z-a) + c_2(z-a)^2 + ... + c_{-1}(z-a)^{-1} + ... + c_{-n}(z-a)^{-n}.$$

Multiplying throughout by $(z-a)^n$, we get

$$(z-a)^n f(z) = c_0(z-a)^n + c_1(z-a)^{n+1} + c_2(z-a)^{n+2} + ... + c_{-1}(z-a)^{n-1} + c_{-2}(z-a)^{n-2} + ... + c_{-n}.$$

Differentiating both sides w.r.t. z, $n - 1$ times and putting $z = a$, we get

$$\left\{ \frac{d^{n-1}}{dz^{n-1}} [(z-a)^n f(z)] \right\}_{z=a} = (n-1)!\, c_{-1} \text{ whence follows the result.}$$

Obs. In many cases, the residue of a pole ($z = a$) can be found, by putting $z = a + t$ in $f(z)$ and expanding it in powers of t where $|\,t\,|$ is quite small.

Example 20.32. *Find the sum of the residues of $f(z) = \dfrac{\sin z}{z \cos z}$ at its poles inside the circle $|z| = 2$.* *(Rohtak, 2004)*

Solution. $f(z)$ has simple poles at $z = 0, \pm \pi/2, \pm 3\pi/2, - ...$

Only the poles $z = 0$ and $z = \pm \pi/2$ lies inside $|z| = 2$.

$$\therefore \qquad \text{Res } f(0) = \underset{z \to 0}{\mathrm{Lt}} \, [z \,.\, f(z)] = \underset{z \to 0}{\mathrm{Lt}} \left(\frac{\sin z}{\cos z} \right) = 0.$$

$$\text{Res } f(\pi/2) = \underset{z \to \pi/2}{\mathrm{Lt}} \left[\left(z - \frac{\pi}{2} \right) f(z) \right] = \underset{z \to \pi/2}{\mathrm{Lt}} \left\{ \frac{(z - \pi/2) \sin z}{z \cos z} \right\}$$

$$= \underset{z \to \pi/2}{\mathrm{Lt}} \frac{(z - \pi/2)\cos z + \sin z}{\cos z - z \sin z} \qquad \left[\text{Being } \frac{0}{0} \text{ form} \right]$$

$$= \frac{1}{-\pi/2} = -\frac{2}{\pi}$$

and $$\text{Res } f(-\pi/2) = \underset{z \to -\pi/2}{\text{Lt}} \left\{ \frac{(z+\pi/2)\sin z}{z \cos z} \right\} = \underset{z \to -\pi/2}{\text{Lt}} \frac{(z+\pi/2)\cos z + \sin z}{\cos z - z \sin z} = \frac{-1}{-\pi/2} = \frac{2}{\pi}$$

Hence sum of residues $= 0 - \frac{2}{\pi} + \frac{2}{\pi} = 0.$

Example 20.33. *Determine the poles of the function*

$f(z) = z^2/(z-1)^2(z+2)$ and the residue at each pole. (S.V.T.U., 2008 ; J.N.T.U., 2005)

Hence evaluate $\oint_C f(z)dz$, where C is the circle $|z| = 2.5$.

Solution. Since $\underset{z \to -2}{\text{Lt}} \{(z+2)f(z)\} = \underset{z \to -2}{\text{Lt}} \frac{z^2}{(z-1)^2} = \frac{4}{9}.$

which is finite and non-zero, the function has a simple pole at $z = -2$ and Res $f(-2) = 4/9$.

Also since $\underset{z \to 1}{\text{Lt}} [(z-1)^2 f(z)]$ is finite and non-zero, $f(z)$ has a pole of order two at $z = 1$.

$$\therefore \quad \text{Res } f(1) = \frac{1}{1!}\left[\frac{d}{dz}\{(z-1)^2 f(z)\}\right]_{z=1} = \left[\frac{d}{dz}\left(\frac{z^2}{z+2}\right)\right]_{z=1} = \left[\frac{z^2+4z}{(z+2)^2}\right]_{z=1} = \frac{5}{9}.$$

[*Otherwise writing* $z = 1 + t$,

$$f(z) = \frac{(1+t)^2}{t^2(3+t)} = \frac{1}{3t^2}(1+t)^2(1+t/3)^{-1} = \frac{1}{3t^2}(1+t)^2\left(1 - \frac{t}{3} + \frac{t^2}{9} - \ldots\right)$$

$$= \frac{1}{3t^2}\left(1 + \frac{5}{3}t + \frac{4}{9}t^2 - \ldots\right) = \frac{1}{3t^2} + \frac{5}{9t} + \frac{4}{27} - \ldots \qquad \ldots(i)$$

$$\therefore \quad \text{Res } f(1) = \text{coefficient of } \frac{1}{t} \text{ in } (i) = \frac{5}{9}.]$$

Clearly $f(z)$ is analytic on $|z| = 2.5$ and at all points inside except the poles $z = -2$ and $z = 1$. Hence by residue theorem

$$\oint_C f(z)dz = 2\pi i\,[\text{Res } f(-2) + \text{Res } f(1)] = 2\pi i\left[\frac{4}{9} + \frac{5}{9}\right] = 2\pi i.$$

Example 20.34. *Find the residue of $f(z) = \dfrac{z^3}{(z-1)^4(z-2)(z-3)}$ at its poles and hence evaluate $\oint_C f(z)dz$ where C is the circle $|z| = 2.5$.* (U.P.T.U., 2003)

Solution. The poles of $f(z)$ are given by $(z-1)^4(z-2)(z-3) = 0$.

$\therefore$ $z = 1$ is a pole of order 4, while $z = 2$ and $z = 3$ are simple poles.

$$\text{Res } f(1) = \frac{1}{3!}\frac{d^3}{dz^3}\left\{(z-1)^4 \cdot \frac{z^3}{(z-1)^4(z-2)(z-3)}\right\}_{z=1} = \frac{1}{6}\frac{d^3}{dz^3}\left\{\frac{z^3}{(z-2)(z-3)}\right\}_{z=1}$$

$\therefore$ $z = 1$ is a pole of order 4, while $z = 2$ and $z = 3$ are simple poles.

$$\text{Res } f(1) = \frac{1}{3!}\frac{d^3}{dz^3}\left\{(z-1)^4 \cdot \frac{z^3}{(z-1)^4(z-2)(z-3)}\right\}_{z=1} = \frac{1}{6}\frac{d^3}{dz^3}\left\{\frac{z^3}{(z-2)(z-3)}\right\}_{z=1}$$

$$= \frac{1}{6}\frac{d^3}{dz^3}\left[z + 5 - \frac{8}{z-2} + \frac{27}{z-3}\right] = \frac{1}{6}\left[-8 \cdot \frac{(-1)^3\,3!}{(z-2)^4} + \frac{27 \cdot (-1)^3\,3!}{(z-2)^4}\right]_{z=1}$$

$$= -\left[-8 + \frac{27}{16}\right] = \frac{101}{16}.$$

$$\text{Res } f(2) = \underset{z \to 2}{\text{Lt}} \left\{ (z-2) \cdot \frac{z^3}{(z-1)^4 (z-2)(z-3)} \right\} = \underset{z \to 2}{\text{Lt}} \left\{ \frac{z^3}{(z-1)^4 (z-3)} \right\} = \frac{8}{(1)^4 (-1)} = -8$$

$$\text{Res } f(3) = \underset{z \to 3}{\text{Lt}} \left\{ (z-3) \cdot \frac{z^3}{(z-1)^4 (z-2)(z-3)} \right\} = \frac{27}{(2)^4 \cdot 1} = \frac{27}{16}$$

Now $\oint_C f(z)dz = 2\pi i \, [\text{Res } f(1) + \text{Res } f(2)]$ [$\because$ Pole $z = 3$ is outside C]

$$= 2\pi i \left(\frac{101}{16} - 8 \right) = \frac{-27\pi i}{8}.$$

Example 20.35. *Evaluate*

$$\oint_C \frac{z-3}{z^2+2z+5} dz, \text{ where } C \text{ is the circle}$$

(i) $|z| = 1$, *(ii)* $|z + 1 - i| = 2$, *(iii)* $|z + 1 + i| = 2$. *(J.N.T.U., 2003)*

Solution. The poles of $f(z) = \dfrac{z-3}{z^2+2z+5}$ are given by $z^2 + 2z + 5 = 0$

i.e., by $$z = \frac{-2 \pm \sqrt{(4-20)}}{2} = -1 \pm 2i.$$

(i) Both the poles $z = -1 + 2i$ and $z = -1 - 2i$ lie outside the circle $|z| = 1$. Therefore, $f(z)$ is analytic everywhere within C.

Hence by Cauchy's theorem, $\oint_C \dfrac{z-3}{z^2+2z+5} dz = 0$.

(ii) Here only one pole $z = -1 + 2i$ lies inside the circle $C : |z + 1 - i| = 2$. Therefore, $f(z)$ is analytic within C except at this pole.

$$\therefore \quad \text{Res } f(-1+2i) = \underset{z \to -1+2i}{\text{Lt}} [\{z - (-1+2i)\} f(z)] = \underset{z \to -1+2i}{\text{Lt}} \frac{(z+1-2i)(z-3)}{z^2+2z+5}$$

$$= \underset{z \to -1+2i}{\text{Lt}} \frac{z-3}{z+1+2i} = \frac{-4+2i}{4i} = i + 1/2.$$

Hence by residue theorem $\oint_C f(z)dz = 2\pi i \text{ Res } f(-1+2i) = 2\pi i(i + 1/2) = \pi(i - 2)$.

(iii) Here only the pole $z = -1 - 2i$ lies inside the circle $C : |z + 1 + i| = 2$. Therefore, $f(z)$ is analytic within C except at this pole.

$$\therefore \quad \text{Res } f(-1-2i) = \underset{z \to -1-2i}{\text{Lt}} \frac{(z+1+2i)(z-3)}{z^2+2z+5}$$

$$= \underset{z \to -1-2i}{\text{Lt}} \frac{z-3}{z+1-2i} = \frac{-4-2i}{-4i} = \tfrac{1}{2} - i$$

Hence by residue theorem, $\oint_C f(z)dz = 2\pi i \text{ Res } f(-1-2i) = 2\pi i (\tfrac{1}{2} - i) = \pi(2 + i)$.

Example 20.36. *Evaluate* $\oint_C \dfrac{e^z}{\cos \pi z} dz$, *where C is the unit circle* $|z| = 1$. *(Rohtak, 2006)*

Solution. $f(z) = e^x/\cos \pi z$ has simple poles at $z = \pm 1/2, \pm 3/2, \pm 5/2, \ldots$

Out of these only the poles at $z = 1/2$ and $z = -1/2$ lie inside the given circle $|z| = 1$.

$$\therefore \quad \text{Res } f(1/2) = \underset{z \to 1/2}{\text{Lt}} \left[\left(z - \frac{1}{2} \right) f(z) \right] = \underset{z \to 1/2}{\text{Lt}} \left\{ \frac{\left(z - \frac{1}{2} \right) e^z}{\cos \pi z} \right\} \quad \left[\frac{0}{0} \text{ form} \right]$$

$$= \underset{z \to 1/2}{\text{Lt}} \frac{e^z + \left(z - \frac{1}{2}\right)e^z}{-\pi \sin \pi z} = \frac{e^{1/2}}{-\pi}$$

and $$\text{Res } f(-1/2) = \underset{z \to -1/2}{\text{Lt}} \left\{ \frac{\left(z + \frac{1}{2}\right)e^z}{\cos \pi z} \right\} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \underset{z \to -1/2}{\text{Lt}} \frac{e^z + \left(z + \frac{1}{2}\right)e^z}{-\pi \sin \pi z} = \frac{e^{-1/2}}{\pi}$$

Hence $$\oint_C \frac{e^z}{\cos \pi z} dz = 2\pi i \left(\text{Res } f\left(\frac{1}{2}\right) + \text{Res } f\left(-\frac{1}{2}\right) \right\}$$

$$= 2\pi i \left(-\frac{e^{1/2}}{\pi} + \frac{e^{-1/2}}{\pi} \right) = -4i \left(\frac{e^{1/2} - e^{-1/2}}{2} \right) = -4i \sinh \frac{1}{2}.$$

Example 20.37. *Evaluate* $\oint_C \tan z \, dz$ *where C is the circle* $|z| = 2$. *(V.T.U., 2010 S)*

Solution. The poles of $f(z) = \sin z/\cos z$ are given by $\cos z = 0$ *i.e.* $z = \pm \pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots$

Of these poles, $z = \pi/2$, and $-\pi/2$ only are within the given circle.

$\therefore$ $$\text{Res } f(\pi/2) = \underset{z \to \pi/2}{\text{Lt}} \frac{\sin z}{\frac{d}{dz}(\cos z)} = \underset{z \to \pi/2}{\text{Lt}} \left(\frac{\sin z}{-\sin z} \right) = -1 \qquad \text{[By § 20.19 (2)]}$$

Similarly $$\text{Res } f(-\pi/2) = \underset{z \to -\pi/2}{\text{Lt}} \frac{\sin z}{\frac{d}{dz}(\cos z)} = -1.$$

Hence by residue theorem,

$$\oint_C f(z) dz = 2\pi i \{\text{Res } f(\pi/2) + \text{Res } f(-\pi/2)\} = 2\pi i (-1 - 1) = -4\pi i.$$

Example 20.38. *Evaluate* $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2 (z-2)} dz$, *where C is the circle* $|z| = 3$.

(V.T.U., 2010 ; Anna, 2003 S ; U.P.T.U., 2002)

Solution. $$f(z) = \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2 (z-2)}$$

is analytic within the circle $|z| = 3$ excepting the poles $z = 1$ and $z = 2$.

Since $z = 1$ is a pole of order 2.

$\therefore$ $$\text{Res } f(1) = \frac{1}{1!} \left[\frac{d}{dz} \{(z-1)^2 f(z)\} \right]_{z=1} = \left[\frac{d}{dz} \left(\frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)} \right) \right]_{z=1}$$

$$= \left[\frac{(z-2)(2\pi z \cos \pi z^2 - 2\pi z \sin \pi z^2) - (\sin \pi z^2 + \cos \pi z^2)}{(z-2)^2} \right]_{z=1}$$

$$= (-1)(-2\pi) - (-1) = 2\pi + 1$$

Also $$\text{Res } f(2) = \underset{z \to 2}{\text{Lt}} [(z-2) f(z)] = \underset{z \to 2}{\text{Lt}} \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2} = 1$$

Hence by residue theorem,

$$\oint_C f(z) \, dz = 2\pi i \, [\text{Res } f(1) + \text{Res } f(2)] = 2\pi i \, (2\pi + 1 + 1) = 4\pi(\pi + 1)i.$$

PROBLEMS 20.8

1. Expand $f(z) = 1/[z^2 (z-i)]$ as a Laurent's series about i and hence find the residue thereat.
2. Find the residue of (*i*) $ze^z/(z-1)^3$ at its pole. (*J.N.T.U., 2003*)
   (*ii*) $z^2/(z^2+a^2)$ at $z = ai$. (*P.T.U., 2009 S*)
3. Determine the poles of the following functions and the residue at each pole :
   (*i*) $\dfrac{z^2+1}{z^2-2z}$ (*ii*) $\dfrac{z^2-2z}{(z+1)^2(z^2+1)}$ (*J.N.T.U., 2005*) (*iii*) $\dfrac{2z+4}{(z+1)(z^2+1)}$ (*J.N.T.U., 2006*)
4. Find the residues of the following functions at each pole.
   (*i*) $(1-e^{2z})/z^4$ (*ii*) $ze^{iz}/(z^2+1)$ (*P.T.U., 2010*) (*iii*) $\cot z$.
5. $\oint_C \dfrac{z^2+4}{(z-2)(z+3)}\, dz$, where C is (*i*) $|z+1| = 2$ (*ii*) $|z-2| = 2$. (*Mumbai, 2006*)
6. Evaluate the following integrals :
   (*i*) $\oint_C \dfrac{e^{2z}\, dz}{(z+2)(z+4)(z+7)}$ for C as circle $|z| = 3$. (*V.T.U., 2009*)
   (*ii*) $\oint_C \dfrac{4z^2-4z+1}{(z-2)(4+z^2)}\, dz$, $C : |z| = 1$
   (*iii*) $\oint_C \dfrac{3z^2+z+1}{(z^2-1)(z+3)}\, dz$, $C : |z| = 2$. (*U.P.T.U., 2004*)
7. Evaluate
   (*i*) $\int_C \dfrac{2z+1}{(2z-1)^2}\, dz$, where C is $|z| = 1$ (*ii*) $\oint_C \dfrac{z+4}{z^2+2z+5}\, dz$, where C is $|z+1-i| = 2$
   (*iii*) $\int_C \dfrac{z^2-2z}{(z+1)^2(z^2+4)}\, dz$, where C is the circle $|z| = 10$. (*U.P.T.U., 2009*)
8. Evaluate :
   (*i*) $\oint_C \dfrac{z\, dz}{(z-1)(z-2)^2}$, $C : |z-2| = \frac{1}{2}$. (*Madras, 2006*)
   (*ii*) $\oint_C \dfrac{3z^2+2}{(z-1)(z^2+9)}\, dz$, $C : |z-2| = 2$. (*Rohtak, 2005*)
   (*iii*) $\oint_C \dfrac{dz}{(z^2+4)^2}$, $C : |z-i| = 2$. (*Hissar, 2007 ; Anna, 2003 S ; Osmania, 2003*)
9. Evaluate :
   (*i*) $\oint_C \dfrac{e^{-z}}{z^2}\, dz$, $C : |z| = 1$. (*ii*) $\oint_C z^2 e^{1/z}$, $C : |z| = 1$.
   (*iii*) $\oint_C \dfrac{e^z\, dz}{z^2+4}$, $C : |z-i| = 2$. (*V.T.U., 2006*) (*iv*) $\oint_C \dfrac{e^{2z}\, dz}{(z+1)^4}$, $C : |z| = 2$.
10. Evaluate the following integrals : (*i*) $\oint_C \dfrac{\sin^6 z}{(z-\pi/6)^3}\, dz$, $C : |z| = 1$
    (*ii*) $\oint_C \dfrac{z \sec z}{(1-z)^2}\, dz$, $C : |z| = 3$ (*iii*) $\oint_C \dfrac{z \cos z}{(z-\pi/2)^3}\, dz$, $C : |z-1| = 1$. (*V.T.U., 2007*)
11. Evaluate $\oint_C \dfrac{dz}{\sinh 2z}$ where C is the circle $|z| = 2$. (*Marathwada, 2008*)
12. Obtain Laurent's expansion for the function $f(z) = 1/z^2 \sinh z$ and evaluate
    $\oint_C \dfrac{z}{z^2 \sinh z}$, where C is the circle $|z-1| = 2$. (*J.N.T.U., 2005*)

20.20 EVALUATION OF REAL DEFINITE INTEGRALS

Many important definite integrals can be evaluated by applying the Residue theorem to properly chosen integrals. The contours chosen will consist of straight lines and circular arcs.

(*a*) **Integration around the unit circle.** An integral of the type $\int_0^{2\pi} f(\sin\theta, \cos\theta)\, d\theta$, where the integrand is a rational function of $\sin\theta$ and $\cos\theta$ can be evaluated by writing $e^{i\theta} = z$.

Since $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$ and $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$, then integral takes the form $\int_C f(z)\, dz$, where $f(z)$ is a rational function of z and C is a unit circle $|z| = 1$.

Hence *the integral is equal to* $2\pi i$ *times the sum of the residues at those poles of* $f(z)$ *which are within* C.

Example 20.39. *Show that*

$$\int_0^{2\pi} \frac{\cos 2\theta\, d\theta}{1 - 2a\cos\theta + a^2} = \frac{2\pi a^2}{1 - a^2},\ (a^2 < 1).$$

(Bhopal, 2009 ; Rohtak, 2003)

Solution. Putting $z = e^{i\theta}$, $d\theta = dz/iz$, $\cos\theta = \frac{1}{2}(z + 1/z)$ and $\cos 2\theta = \frac{1}{2}(e^{2i\theta} + e^{-2i\theta}) = \frac{1}{2}(z^2 + 1/z^2)$

$\therefore$ the given integral

$$I = \int_C \frac{\frac{1}{2}(z^2 + 1/z^2)}{1 - a(z + 1/z) + a^2} \cdot \frac{dz}{iz} = \frac{1}{2i}\int_C \frac{(z^4 + 1)\, dz}{z^2(z - az^2 - a + a^2 z)}$$

$$= \frac{1}{2i}\int_C \frac{(z^4 + 1)\, dz}{z^2(z - a)(1 - az)} = \int_C f(z)\, dz \qquad \text{where } C \text{ is the unit circle } |z| = 1.$$

Now $f(z)$ has simple poles at $z = a$, $1/a$ and the second order pole at $z = 0$, of which the poles at $z = 0$ and $z = a$ lie within the unit circle.

$$\therefore \quad \text{Res}\, f(a) = \underset{z \to a}{\text{Lt}}\, [(z - a) f(z)] = \frac{1}{2i}\underset{z \to a}{\text{Lt}}\left[\frac{z^4 + 1}{z^2(1 - az)}\right] = \frac{a^4 + 1}{2ia^2(1 - a^2)}$$

and

$$\text{Res}\, f(0) = \underset{z \to 0}{\text{Lt}}\, \frac{d}{dz}[z^2 f(z)] = \frac{1}{2i}\underset{z \to 0}{\text{Lt}}\, \frac{d}{dz}\left[\frac{z^4 + 1}{z - az^2 - a + a^2 z}\right]$$

$$= \frac{1}{2i}\underset{z \to 0}{\text{Lt}}\, \frac{(z - az^2 - a + a^2 z)(4z^3) - (z^4 + 1)(1 - 2az + a^2)}{(z - az^2 - a + a^2 z)^2} = -\frac{1 + a^2}{2ia^2}$$

Hence

$$I = 2\pi i\,[\text{Res}\, f(a) + \text{Res}\, f(0)] = 2\pi i\left[\frac{a^4 + 1}{2ia^2(1 - a^2)} - \frac{1 + a^2}{2ia^2}\right] = \frac{2\pi a^2}{1 - a^2}.$$

Example 20.40. *By integrating around a unit circle, evaluate* $\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta}\, d\theta$.

(S.V.T.U, 2009 ; U.P.T.U., 2009 ; Madras, 2003)

Solution. Putting $z = e^{i\theta}$, $d\theta = dz/iz$, $\cos\theta = \frac{1}{2}(z + 1/z)$

and

$$\cos 3\theta = \frac{1}{2}(e^{3i\theta} + e^{-3i\theta}) = \frac{1}{2}(z^3 + 1/z^3).$$

$\therefore$ the given integral

$$I = \int_C \frac{\frac{1}{2}(z^3 + 1/z^3)}{5 - 2(z + 1/z)} \cdot \frac{dz}{iz}$$

$$= -\frac{1}{2i}\int_C \frac{z^6 + 1}{z^3(2z^2 - 5z + 2)}\, dz = -\frac{1}{2i}\int_C \frac{(z^6 + 1)\, dz}{z^3(2z - 1)(z - 2)}$$

$$= -\frac{1}{2i}\int f(z)\,dz, \quad \text{where } C \text{ is the unit circle } |z| = 1.$$

Now $f(z)$ has a pole of order 3 at $z = 0$ and simple poles at $z = \frac{1}{2}$ and $z = 2$. Of these only $z = 0$ and $z = 1/2$ lie within the unit circle.

$$\therefore \quad \text{Res}\, f(1/2) = \underset{z\to 1/2}{\text{Lt}} \frac{(z-1/2)(z^6+1)}{(2z-1)(z-2)} = \underset{z\to 1/2}{\text{Lt}} \left\{\frac{z^6+1}{2z^3(z-2)}\right\} = -\frac{65}{24}$$

$$\text{Res}\, f(0) = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-0)^n f(z)]\right\}_{z=0} \quad \text{where } n = 3$$

$$= \frac{1}{2}\left\{\frac{d^2}{dz^2}\left(\frac{z^6+1}{2z^2-5z+2}\right)\right\}_{z=0} = \frac{d}{dz}\left[\frac{(2z^2-5z+2)\,6z^5-(z^6+1)(4z-5)}{2(2z^2-5z+2)^2}\right] \text{ at } z = 0$$

$$= \left\{\frac{d}{dz}\left[\frac{8z^7-25z^6+12z^5-4z+5}{2(2z^2-5z+2)^2}\right]\right\}_{z=0}$$

$$= \left[\frac{(2z^2-5z+2)^2(56z^6-150z^5+60z^4-4)-(8z^7-25z^6+12z^5-4z+5)2(2z^2-5z+2)(4z-5)}{2(2z^2-5z+2)^4}\right]_{z=0}$$

$$= \frac{4(-4)-5(-20)}{2\times 16} = \frac{84}{32} = \frac{21}{8}$$

Hence $$I = \frac{-1}{2i}\{2\pi i\,[\text{Res}\, f(1/2) + \text{Res}\, f(0)]\} = -\pi\left[-\frac{65}{24}+\frac{21}{8}\right] = -\pi\left(-\frac{1}{12}\right) = \frac{\pi}{12}.$$

(*b*) **Integration around a small semi-circle.** To evaluate $\int_{-\infty}^{\infty} f(x)\,dx$, we consider $\int_C f(z)$, where C is the contour consisting of the semi-circle $C_R : |z| = R$, together with the diameter that closes it.

Supposing that $f(z)$ has no singular point on the real axis, we have, by the Residue theorem,

$$\int_{C_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx = 2\pi i \sum \text{Res}\, f(a).$$

Finally making R tend to ∞, we find the value of $\int_{-\infty}^{\infty} f(x)\,dx$, provided $\int_{C_R} f(z)\,dz \to 0$ as $R \to \infty$.

Example 20.41. *Evaluate* $\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)}$. (*U.P.T.U., 2008*)

Solution. Consider $\int_C \frac{z^2 dz}{(z^2+1)(z^2+4)} = \int_C f(z)\,dz$ ...(*i*)

where C is the contour consisting of the semi-circle C_R of radius R together with the part of the real axis from $-R$ to R as shown in Fig. 20.23.

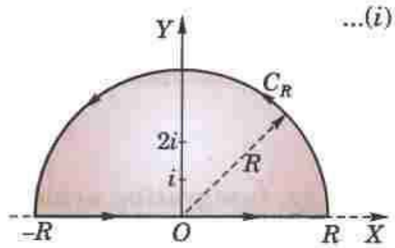


Fig. 20.23

The integrand has simple poles at $z = \pm i$, $z = \pm 2i$ of which $z = i, 2i$ only lie inside C.

$\therefore$ by the Residue theorem,

$$\int_C f(z)\,dz = 2\pi i\,[\text{Res}\, f(i) + \text{Res}\, f(2i)]$$

$$= 2\pi i\,[\underset{z\to i}{\text{Lt}}\,(z-i)\,f(z) + \underset{z\to 2i}{\text{Lt}}\,(z-2i)\,f(z)]$$

$$= 2\pi i\left[\frac{i^2}{2i(i^2+4)} + \frac{4i^2}{(4i^2+1)(4i)}\right] = 2\pi i\left(\frac{i}{6}-\frac{i}{3}\right) = \frac{\pi}{3} \quad ...(ii)$$

Also $$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz \qquad ...(iii)$$

Now let $R \to \infty$, so as to show that the second integral in (*iii*) vanishes. For any point on C_R as $|z| \to \infty$

$$f(z) = \frac{1}{z^2} \cdot \frac{1}{(1+z^{-2})(1+4z^{-2})}$$

decreases as $1/z^2$ and tends to zero whereas the length of C_R increases with z.

Consequently, $\underset{|z|\to\infty}{\text{Lt}} \int_{C_R} f(z)\,dz = 0.$

Hence from (*i*), (*ii*) and (*iii*), we get $\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)} = \frac{\pi}{3}.$

Example 20.42. *Evaluate* $\int_0^{\infty} \frac{\cos ax}{x^2+1}\,dx.$ (*U.P.T.U., 2006 ; Delhi, 2002*)

Solution. Consider $\int_C \frac{e^{iaz}}{z^2+1}\,dz = \int_C f(z)\,dz$

where C is the contour consisting of the semi-circle C_R of radius R together with the part of the real axis from $-R$ to R as shown in Fig. 20.23.

The integrand has simple poles at $z = i$ amd $z = -i$, of which $z = i$ only lies inside C.

$\therefore$ by Residue theorem, $\int_C f(z)\,dz = 2\pi i \operatorname{Res} f(i) = 2\pi i \underset{z\to i}{\text{Lt}} [(z-i)\,f(z)]$

$$= 2\pi i \underset{z\to i}{\text{Lt}} \frac{(z-i)\,e^{iaz}}{z^2+1} = 2\pi i \underset{z\to i}{\text{Lt}} \frac{e^{iaz}}{z+i} = 2\pi i\,\frac{e^{-a}}{2i} = \pi e^{-a} \qquad ...(i)$$

Also $$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz \qquad ...(ii)$$

Now $|z| = R$ on C_R and $|z^2+1| \geq R^2 - 1.$

Also $|e^{iaz}| = |e^{ia(x+iy)}| = |e^{iax} \cdot e^{-ay}| = e^{-ay} < 1$ $[\because\ y > 0]$

$$\therefore \qquad \left|\frac{e^{iaz}}{z^2+1}\right| = |e^{iaz}| \cdot \frac{1}{|z^2+1|} < 1 \cdot \frac{1}{R^2-1}$$

Thus $$\int_{C_R} f(z)\,dz = \left|\int_{C_R} \frac{e^{iaz}}{z^2+1}\,dz\right| < \int_{C_R} \frac{1}{R^2-1}|dz| < \frac{\pi R}{R^2-1} \text{ which} \to \text{to 0 as } R \to \infty. \qquad ...(iii)$$

Hence from (*i*), (*ii*) and (*iii*), we get

$$\pi e^{-a} = \int_{-\infty}^{\infty} f(x)\,dx + 0 \quad \text{or} \quad \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}\,dx = \pi e^{-a}$$

Equating real parts from both sides, we obtain

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}\,dx = \pi e^{-a}$$

Since $\cos ax/(x^2+1)$ is an even function of x, we have

$$2\int_0^{\infty} \frac{\cos ax}{x^2+1}\,dx = \pi e^{-a} \quad \text{or} \quad \int_0^{\infty} \frac{\cos ax}{x^2+1}\,dx = \frac{\pi}{2} e^{-a}.$$

(*c*) **Integration around rectangular contours**

Example 20.43. *Evaluate* $\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}\,dx.$

Solution. Consider $\int_C \frac{e^{az}}{e^z+1}\,dz = \int_C f(z)\,dz$ where C is the rectangle $ABCD$ with vertices at $(R, 0)$, $(R, 2\pi)$, $(-R, 2\pi)$ and $(-R, 0)$, R being positive (Fig. 20.24).

$f(z)$ has finite poles given by

$$e^z = -1 = e^{(2n+1)\pi i}$$

or $\quad z = (2n+1)\pi i$, where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

The only pole inside the rectangle is $z = \pi i$.

$\therefore$ by Residue theorem,

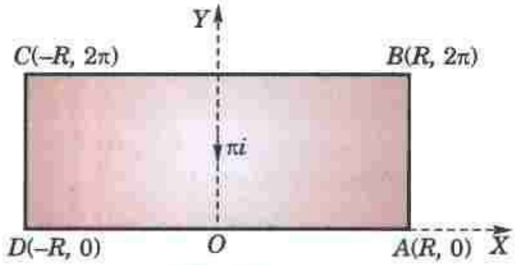


Fig. 20.24

$$\int_C f(z)dz = 2\pi i \text{ Res } f(\pi i)$$

$$= 2\pi i \left[e^{az} \Big/ \frac{d}{dz}(e^z + 1) \right]_{z=\pi i}$$

$$= 2\pi i e^{a\pi i}/e^{\pi i} = -2\pi i e^{a\pi i} \qquad [\because \; e^{\pi i} = -1] \qquad \ldots(i)$$

Also $\quad \int_C f(z)dz = \int_{AB} f(z)dz + \int_{BC} f(z)dz + \int_{CD} f(z)dz + \int_{DA} f(z)dz$

$$= \int_0^{2\pi} f(R+iy)idy + \int_R^{-R} f(x+2\pi i)dx + \int_{2\pi}^0 f(-R+iy)idy + \int_{-R}^R f(x)dx$$

[$\because \; z = R + iy$ along AB, $z = x + 2\pi i$ along BC, $z = -R + iy$ along CD and $z = x$ along DA.]

or $$\int_C f(z)dz = i\int_0^{2\pi} \frac{e^{a(R+iy)}}{e^{R+iy}+1}dy - \int_{-R}^R \frac{e^{a(x+2\pi i)}}{e^{x+2\pi i}+1}dx - i\int_0^{2\pi} \frac{e^{a(-R+iy)}}{e^{-R+iy}+1}dy + \int_{-R}^R \frac{e^{ax}}{e^x+1}dx \qquad \ldots(ii)$$

Now for any two complex numbers z_1, z_2

$$|z_1| \geq |z_2|, \text{ we have } |z_1 + z_2| \geq |z_1| - |z_2|$$

so that $\quad |e^{R+iy} + 1| \geq e^R - 1$. Also $|e^{a(R+iy)}| = e^{aR}$

$\therefore$ for the integrand of first integral in (ii), we have

$$\left| \frac{e^{a(R+iy)}}{e^{R+iy}+1} \right| \leq \frac{e^{aR}}{e^R - 1} \text{ which} \to 0 \text{ as } R \to \infty. \qquad [\because \; a > 1]$$

Similarly, for the integrand of the third integral in (ii), we get

$$\left| \frac{e^{a(-R+iy)}}{e^{-R+iy}+1} \right| \leq \frac{e^{-aR}}{1 - e^{-R}} \text{ which also} \to 0 \text{ as } R \to \infty. \qquad [\because \; a < 0]$$

Hence as $R \to \infty$, since the first and third integrals in (ii) approach zero, we get

$$\int_C f(z)dz = -e^{2a\pi i}\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}dx + \int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}dx = \left(1 - e^{2a\pi i}\right)\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}dx \qquad \ldots(iii)$$

Thus from (i) and (iii), we obtain $(1 - e^{2a\pi i}) \int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}dx = -2\pi i e^{a\pi i}$

$\therefore$ equating real parts, we get $\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}dx = \frac{\pi}{\sin a\pi}$.

Example 20.44. *Show that* $\int_0^{\infty} e^{-x^2} \cos 2mx \, dx = \frac{1}{2}\sqrt{\pi e^{-m^2}}$.

Solution. Integrate $f(z) = e^{-z^2}$ along the rectangle ABCDA having vertices $A(-l)$, $B(l)$, $C(l + im)$, $D(-l + im)$ (Fig. 20.25). $f(z)$ has no poles inside this contour. As such

$$\int_{AB} f(z)dz + \int_{BC} f(z)dz + \int_{CD} f(z)dz + \int_{DA} f(z)dz = 0 \qquad \ldots(i)$$

On $AB : z = x$, on $BC : z = l + iy$, on $CD : z = x + im$ and on $DA : z = -l + iy$.

Therefore, (i) becomes

$$\int_{-l}^{l} e^{-x^2} dx + \int_0^m e^{-(l+iy)^2} idy + \int_l^{-l} e^{-(x+im)^2} dx + \int_m^0 e^{-(-l+iy)^2} dy = 0$$

or $$\int_{-l}^{l} e^{-x^2} dx - \int_{-l}^{l} e^{-x^2 - 2imx + m^2} dx + \int_0^m e^{-l^2 - 2ily + y^2} . idy$$

$$- \int_0^m e^{-l^2 + 2ily + y^2} . idy = 0 \qquad \ldots(ii)$$

Now let $l \to \infty$. Then the last two integrals

$$= ie^{-l^2} \int_0^m e^{y^2} (e^{-2ily} - e^{2ily})\, dy = 2e^{-l^2} \int_0^m e^{y^2} \sin 2ly\, dy \to 0$$

[$\because$ As $l \to \infty$, $e^{-l^2} \to 0$ and $\sin 2ly$ is finite]

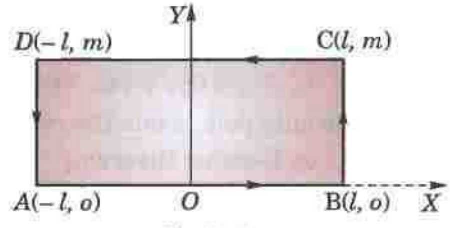


Fig. 20.25

Hence (*ii*) reduces to

$$\int_{-\infty}^{\infty} e^{-x^2} dx - e^{m^2} \int_{-\infty}^{\infty} e^{-x^2} (\cos 2mx - i \sin 2mx) dx = 0$$

Equating real parts, we get

$$e^{m^2} \int_{-\infty}^{\infty} e^{-x^2} \cos 2mx\, dx = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$ [See p. 289]

or $$\int_0^{\infty} e^{-x^2} \cos 2mx\, dx = \frac{1}{2}\sqrt{\pi e^{-m^2}}$$

(*d*) **Indenting the contours having poles on the real axis.** So far we have considered such cases in which there is no pole on the real axis. When the integrand has a simple pole on the real axis, we delete it from the region by indenting the contour (*i.e.*, by drawing a small semi-circle having the pole for the centre). The method will be clear from the following example.

Example 20.45. *Evaluate* $\int_0^{\infty} \frac{\sin mx}{x} dx$, *when* $m > 0$. *(U.P.T.U., 2007)*

Solution. Consider the integral $\int_C \frac{e^{miz}}{z} dz = \int_C f(z) dz$ where C consists of

(*i*) the real axis from r to R,
(*ii*) the upper half of the circle $C_R : |z| = R$,
(*iii*) the real axis $-R$ to $-r$,
(*iv*) the upper half of the circle $C_r : |z| = r$ (Fig. 20.26).

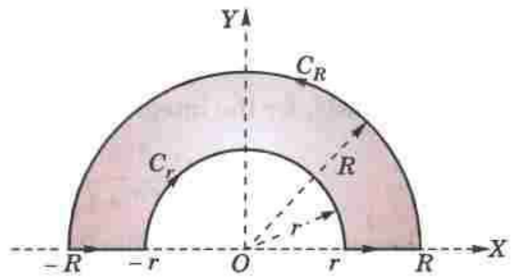


Fig. 20.26

Since $f(z)$ has no singularity inside C (its only singular point being a simple pole at $z = 0$ which has been deleted by drawing C_r), we have by Cauchy's theorem :

$$\int_r^R f(x)dx + \int_{C_R} f(z)dz + \int_{-R}^{-r} f(x)dx + \int_{C_r} f(z)dz = 0 \quad ...(i)$$

Now $$\int_{C_R} f(z)dz = \int_0^{\pi} \frac{e^{imR(\cos\theta + i\sin\theta)}}{Re^{i\theta}} . Rie^{i\theta}\, d\theta$$ [$\because$ $z = Re^{i\theta}$]

$$= i\int_0^{\pi} e^{imR(\cos\theta + i\sin\theta)}\, d\theta$$

Since $|e^{imR(\cos\theta + i\sin\theta)}| = |e^{-mR\sin\theta + imR\cos\theta}| = e^{-mR\sin\theta}$

$\therefore$ $$\left|\int_{C_R} f(z)dz\right| \le \int_0^{\pi} e^{-mR\sin\theta}\, d\theta = 2\int_0^{\pi/2} e^{-mR\sin\theta}\, d\theta$$

$$= 2\int_0^{\pi/2} e^{-2mR\theta/\pi}\, d\theta$$ [$\because$ for $0 \le \theta \le \pi/2$, $\sin\theta/\theta \ge 2/\pi$]

$$= \frac{\pi}{mR}\left(1 - e^{-mR}\right) \text{ which} \to 0 \text{ as } R \to \infty,$$

Also $$\int_{C_r} f(z)dz = i\int_{\pi}^{0} e^{imr(\cos\theta + i\sin\theta)}\, d\theta \to i\int_{\pi}^{0} d\theta \ \ i.e., -i\pi \text{ as } r \to 0.$$

Hence as $r \to 0$ and $R \to \infty$, we get from (*i*) $\int_0^{\infty} f(x)dx + 0 + \int_{-\infty}^{0} f(x)dx - i\pi = 0$

or $$\int_{-\infty}^{\infty} f(x)dx = i\pi \ \ i.e., \int_{-\infty}^{\infty} \frac{e^{imx}}{x} dx = i\pi \quad ...(ii)$$

Equating imaginary parts from both sides,

$$\int_{-\infty}^{\infty} \frac{\sin mx}{x} dx = \pi. \text{ Hence } \int_0^{\infty} \frac{\sin mx}{x} dx = \frac{\pi}{2}.$$

Obs. Equating real parts from both sides of (*ii*), we get

$$\int_{-\infty}^{\infty} \frac{\cos mx}{x} dx = 0.$$

Example 20.46. *Show that* $\int_0^\infty \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin px}$, $0 < p < 1$.

Solution. Integrate $f(z) = \frac{z^{p-1}}{1+z}$ along the contour consisting of the circles α and γ of radii a and R and the lines AB and FG along x-axis (Fig. 20.27). There is a simple pole at $z = -1$ which is within the contour.

$\therefore \quad \text{Res } f(-1) = \underset{z \to -1}{\text{Lt}} (1+z) \cdot \frac{z^{p-1}}{1+z} = \underset{z \to -1}{\text{Lt}} z^{p-1} = (-1)^{p-1} = e^{i\pi(p-1)}$

Thus $\quad \int_{AB} f(z)dz + \int_\gamma f(z)dz + \int_{FG} f(z)dz + \int_\alpha f(z)dz = 2\pi i\, e^{i\pi(p-1)}$ ...(*i*)

On $AB : z = x$ and on $FG : z = xe^{2\pi i}$

$\therefore \quad \int_{AB} f(z)dz + \int_{FG} f(z)dz = \int_a^R \frac{x^{p-1}}{1+x} dx + \int_R^a \frac{(xe^{2\pi i})^{p-1}}{1+xe^{2\pi i}} dx\, e^{2\pi i}$

$$= \int_a^R \frac{x^{p-1}}{1+x} [1 - e^{2\pi i(p-1)}]\, dx$$

On the circle $\gamma : z = Re^{i\theta}$. So

$$\int_\gamma f(z)dz = \int_0^{2\pi} \frac{(Re^{i\theta})^{p-1}}{1+Re^{i\theta}} Re^{i\theta}\, i\, d\theta$$

For large R, the integrand is of the order $\frac{R^{p-1} \cdot R}{1+R}$ i.e.

R^{p-1} which tends to zero as $R \to \infty$. $(\because \; p < 1)$

Hence $\quad \int_\gamma f(z)dz \to 0$ as $R \to \infty$

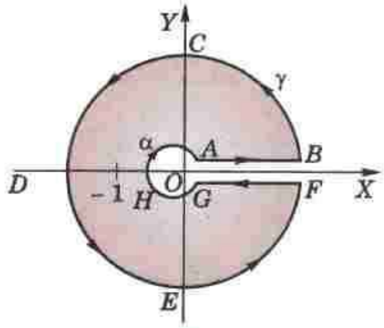


Fig. 20.27

On the circle $\alpha : z = ae^{i\theta}$. So

$$\int_\alpha f(z)dz = \int_{2\pi}^0 \frac{(ae^{i\theta})^{p-1}}{1+ae^{i\theta}} ae^{i\theta}\, i d\theta$$

For small α, the integrand is of the order a^p which tends to zero as $a \to 0$. $(\because \; p > 0)$

Thus on taking limits as $a \to 0$ and $R \to \infty$, (*i*) gives

$$\int_0^\infty \frac{x^{p-1}}{1+x} \{1 - e^{2\pi i(p-1)}\} dx = 2\pi i\, e^{i\pi(p-1)}$$

or $$\int_0^\infty \frac{x^{p-1}}{1+x} dx = \frac{2\pi i\, e^{i\pi(p-1)}}{1 - e^{2\pi i(p-1)}} = \frac{2\pi i\, e^{ip\pi}(-1)}{1 - e^{2ip\pi}\,(1)} = \frac{2i \cdot \pi}{e^{ip\pi} - e^{-ip\pi}} = \frac{\pi}{\sin p\pi}.$$

Example 20.47. *Prove that* $\int_0^\infty \sin x^2\, dx = \int_0^\infty \cos x^2\, dx = \frac{1}{2}\sqrt{\left(\frac{\pi}{2}\right)}$. *(Osmania, 2003)*

Solution. Consider $\int_C e^{-z^2} dz$ where C consists of the real axis from O to A, part of circle AB of radius R and the line $\theta = \frac{\pi}{4}$. (Fig. 20.28).

e^{-z^2} has no singularity within C.

$\therefore \quad \int_{OA} e^{-z^2} dz + \int_{AB} e^{-z^2} dz + \int_{BO} e^{-z^2} dz = 0$ ...(*i*)

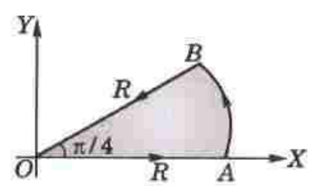


Fig. 20.28

On $OA : z = x$, $\therefore \int_{OA} e^{-z^2} dz = \int_0^R e^{-x^2} dx \to \sqrt{\pi/2}$ as $R \to \infty$ [See p. 289]

On $AB : z = Re^{i\theta}$,

$$\therefore \quad \int_{AB} e^{-z^2} dz = \int_0^{\pi/4} e^{-R^2(\cos 2\theta + i \sin 2\theta)} . Re^{i\theta} . id\theta \to 0 \quad \text{as} \quad R \to \infty.$$

[$\because$ integrand $\to 0$ as $R \to \infty$]

On $BO : z = re^{i\pi/4}$ and $z^2 = r^2 e^{i\pi/2} = ir^2$

$$\therefore \quad \int_{BO} e^{-z^2} dz = \int_R^0 e^{-ir^2} . e^{i\pi/4} dr = -\int_0^R e^{-ix^2} \frac{1+i}{\sqrt{2}} dx$$

$$\to -\int_0^\infty (\cos x^2 - i \sin x^2) \frac{1+i}{\sqrt{2}} dx \qquad \text{when } R \to \infty$$

Substituting these in (i), we get

$$\frac{1}{2}\sqrt{\pi} + 0 - \int_0^\infty (\cos x^2 - i \sin x^2)\left(\frac{1+i}{\sqrt{2}}\right) dx = 0$$

Equating real and imaginary parts, we obtain

$$\int_0^\infty (\cos x^2 + \sin x^2) dx = \frac{1}{2}\sqrt{(2\pi)} \quad \text{and} \quad \int_0^\infty (\cos x^2 - \sin x^2) dx = 0$$

Hence $$\int_0^\infty \sin x^2 \, dx = \int_0^\infty \cos x^2 \, dx = \frac{1}{2}\sqrt{\left(\frac{\pi}{2}\right)}.$$

PROBLEMS 20.9

Apply the calculus of residues, to prove that

1. $\int_0^{2\pi} \frac{d\theta}{1 - 2p \sin\theta + p^2} = \frac{2\pi}{1-p^2} (0 < p < 1)$. *(Hissar, 2007 ; Mumbai, 2006 ; Kerala, 2005)*

2. $\int_0^{2\pi} \frac{d\theta}{1 - 2r\cos\theta + r^2} = \frac{\pi}{1-r^2}$. *(J.N.T.U., 2006 ; Madras, 2006 ; Anna, 2003)*

3. $\int_0^{2\pi} \frac{d\theta}{a + \cos\theta} = \frac{2\pi}{\sqrt{(a^2-1)}} (a > 1)$. *(P.T.U., 2010)*

4. $\int_0^{2\pi} \frac{\cos 2\theta}{5 + 4\cos\theta} d\theta = \frac{\pi}{6}$. *(U.P.T.U., 2010)*

5. $\int_0^{2\pi} \frac{\sin^2\theta \, d\theta}{a + b\cos\theta} = \frac{2\pi}{b^2}[a - \sqrt{(a^2 - b^2)}], (0 < b < a)$. *(J.N.T.U., 2003)*

6. $\int_0^{2\pi} \frac{a d\theta}{a^2 + \sin^2\theta} = \frac{2\pi}{\sqrt{(1+a^2)}}, (a > 0)$. *(S.V.T.U., 2009)*

7. $\int_0^{2\pi} \frac{d\theta}{(5 - 3\cos\theta)^2} = \frac{5\pi}{32}$.

8. $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a+b} (a, b > 0)$. *(P.T.U., 2007 ; Mumbai, 2006 ; Anna, 2003)*

9. $\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx = \frac{5\pi}{12}$. *(A.M.I.E.T.E., 2003 ; Delhi, 2002)*

10. $\int_{-\infty}^{\infty} \frac{dx}{x^4 + 1} = \frac{\pi}{\sqrt{2}}$. *(J.N.T.U., 2006)*

11. $\int_0^\infty \frac{dx}{(1+x^2)^2} = \frac{\pi}{4}$. *(Madras, 2006 ; Kerala, 2005)*

12. $\int_0^\infty \frac{dx}{x^6 + 1} = \frac{\pi}{3}$. *(Kerala, 2005)*

13. $\int_0^\infty \frac{x^2}{x^6+1} dx = \frac{\pi}{6}$. *(Rohtak, 2006)*

14. $\int_{-\infty}^{\infty} \frac{\cos mx}{e^x + e^{-x}} dx = \frac{\pi}{2} \operatorname{sech} \frac{m\pi}{2}$.

15. $\int_0^\infty \frac{1 - \cos x}{x^2} dx = \pi/2$. *(P.T.U., 2005)*

16. $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$. *(Kerala, 2005)*

17. $\int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 4x + 5} dx = \frac{-\pi \sin 2}{e}$.

18. $\int_{-\infty}^{\infty} \frac{\cos x \, dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2 - b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right) (a > b > 0)$.

19. $\int_0^\infty \frac{x \sin x}{x^2 + a^2} dx = \frac{\pi}{2} e^{-a}$.

20.12. OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 20.10

Select the correct answer or fill up the blanks in each of the following questions :

1. The only function that is analytic from the following is
   (i) $f(z) = \sin z$ (ii) $f(z) = \bar{z}$ (iii) $f(z) = \text{Im}(z)$ (iv) $R(iz)$.
2. If $f(z) = u(x, y) + iv(x, y)$ is analytic, then $f'(z) =$
   (i) $\frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x}$ (ii) $\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ (iii) $\frac{\partial v}{\partial y} - i\frac{\partial v}{\partial x}$.
3. If $2x - x^2 + ay^2$ is to be harmonic, then a should be
   (a) 1 (b) 2 (c) 3 (d) 0.
4. The analytic function which maps the angular region $0 \le \theta \le \pi/4$ onto the upper half plane is
   (i) z^2 (ii) $4z$ (iii) z^4 (iv) 2θ.
5. An angular domain in the complex plane is defined by $0 < \text{amp}(z) < \pi/4$. The mapping which maps this region onto the left half plane is
   (i) $w = z^4$ (ii) $w = iz^4$ (iii) $w = -z^4$ (iv) $w = -iz^4$.
6. The mapping $w = z^2 - 2z - 3$ is
   (i) conformal within $|z| = 1$ (ii) not conformal at $z = 1$
   (iii) not conformal at $z = -1$ and $z = 3$ (iv) conformal everywhere.
7. If $z = re^{i\theta}$, then the image of θ = constant under the mapping $w(z) = Re^{i\phi} = iz^3$ is
   (i) $\phi = 3\theta$ (ii) $\phi = 3\theta + \pi/2$ (iii) $\phi = 3\theta - \pi/2$ (iv) $\phi = \theta^3$.
8. The fixed points of the mapping $w = (5z + 4)/(z + 5)$ are
   (i) 2, 2 (ii) 2, – 2 (iii) – 2, – 2 (iv) – 4/5, 5.
9. The value of $\int_C (4x^3\, dx + 3y^2z^2\, dy + 2y^3 z dz)$ where C is any path joining $A(-1, 1, 0)$ to $B(1, 2, 1)$ is
   (i) 0 (ii) 1 (iii) 8 (iv) – 8.
10. The value of $\int_C \frac{3z^2 + 7z + 1}{z + 1} dz$ where C is $|z| = 1/2$ is
    (i) $2\pi i$ (ii) 0 (iii) πi (iv) $\pi i/2$.
11. The value of $\int_C \frac{3z + 4}{z(2z + 1)} dz$ where C is the circle $|z| = 1$ is
    (i) $2\pi i$ (ii) $3\pi i$ (iii) 4 (iv) – 4.
12. The residue of a function can be found if the pole is an isolated singularity :
    (i) True (ii) False (iii) Partially false (iv) none of these.
13. The value of $\int_C \frac{z dz}{\sin z}$ where $C : |z| = 4$ is
    (i) $2\pi i$ (ii) 0 (iii) $-2\pi i$ (iv) $4\pi i$.
14. The value of $\int_C \tanh z\, dz$, where $C : |z| = 3$, is
    (i) 0 (ii) πi (iii) $2\pi i$ (iv) $4\pi i$.
15. The harmonic conjugate of the function $u(x, y) = 2x(1 - y)$ is ...... (U.P.T.U., 2009)
16. Harmonic conjugate of $x^3 - 3xy^2$ is ....
17. The curves $u(x, y) = c$ and $v(x, y) = c'$ are orthogonal if ......
18. The value of $\int_0^{1+i} z^2\, dz$ along the line $x = y$ is ......
19. Residue of $\frac{\cos z}{z}$ at $z = 0$ is ......
20. The critical point of the transformation $w^2 = (z - a)(z - b)$ is ......
21. Image of $|z + 1| = 1$ under the mapping $w = 1/z$ is......
22. The poles of $f(z) = (z^3 - 1)/(z^3 + 1)$ are $z =$ ......
23. $w = \log z$ is analytic everywhere except at $z =$ ......
24. If $f(z) = -\frac{1}{z - 1} - 2[1 + (z - 1) + (z - 1)^2 + ...]$, then the residue of $f(z)$ at $z = 1$ is ......
25. If $|z| < 1$ then Taylor's series expansion of $\log(1 + z)$ about $z = 0$ is ......

26. The value of $\int_C \frac{4z^2+z+5}{z-4}dz$ where C is $9x^2+4y^2=36$, is ......

27. The value of $\int z^4 e^{1/z}\,dz$, where C is $|z|=1$, is
(i) πi (ii) $\pi i/12$ (iii) $\pi i/60$ (iv) $-\pi i/60$.

28. If $f(z)$ has a pole of order three at $z=a$ Res $[f(a)]=$ ......

29. The value of $\int_C \frac{e^z\,dz}{(z-3)^2}$, C being $|z|=2$, is ......

30. The CR equations for $f(z)=u(x,y)+iv(x,y)$ to be analytic are ......

31. If $f(z)$ is analytic in a simply connected domain D and C is any simple closed path then $\int_C f(z)\,dz=$ ......

32. The harmonic conjugate of $e^x\cos y$ is ......

33. The value of $\oint_C \cos z\,dz$ where C is the circle $|z|=1$, is ......

34. The singularity of $f(z)=z/(z-2)^3$ is ......

35. The function $f(z)=\bar{z}$ is analytic at ......

36. C-R equations for a function to be analytic, in polar form, are ......

37. If C is the circle $|z-a|=r$, $\int_C (z-a)^n\,dz$ [n, any integer $\neq -1$] = ...

38. A simply connected region is that ......

39. A holomorphic function is that ......

40. The poles of the function $f(z)=\frac{z^2}{(z-1)^2(z+2)}$ are at $z=$ ......

41. The cross-ratio of four points z_1, z_2, z_3, z_4 is ......

42. The value of $\int_C |z|\,dz$, where C is the contour represented by the straight line from $z=-i$ to $z=i$, is ......

43. Taylor's series expansion of $\left(\frac{1}{z-2}-\frac{1}{z-1}\right)$ in the region $|z|<1$, is ......

44. The invariant points of the transformation $w=(1+z)/(1-z)$ are $z=$ ......

45. The residue at $z=0$ of $\frac{1+e^z}{z\cos z+\sin z}$ is ......

46. The transformation $w=Cz$ consists of ......

47. The residue of $f(z)$ at a pole is ......

48. The value of $\int_C \frac{1}{z-1}dz$, C being $|z|=2$, is ......

49. If C is $|z|=1/2$, $\int_C \frac{z^2-z+1}{z-1}dz=$ ......

50. Singular points of $\frac{\cos \pi z}{(z-1)(z-2)}$ are ......

51. Taylor series expansion of $\frac{1}{z-2}$ in $|z|<1$ is ......

52. $\underset{z\to\infty}{\text{Lt}}\ \frac{iz^3+iz-1}{(2z+3)(z-1)^2}=$ .... (P.T.U., 2007)

53. The poles of $\frac{(z-1)^2}{z(z-2)^2}$ are at $z=$ ......

54. Cauchy's integral theorem states that ......

55. The critical points of the transformation $w=z+1/z$ are ......

56. $\int_c \frac{dz}{2z-3}$, where $|z|=1$, is ......

57. The zeroes and singularities of $\frac{z^2+1}{1-z^2}$ are ......

58. Residue of $\tan z$ at $z=\pi/2$ is ......

59. Singularity of $e^{z^{-1}}$ at $z=0$ is of the type ......

60. Res $(e^{1/z})_{z=0}=$ ......

61. Taylor's series expansion of $\sin z$ about $z=\pi/4$ is ......

62. Image of $|z|=2$ under $w=z+3+2i$ is ....

63. The poles of $\cot z$ are ....

64. If a is simple pole, then Res $[\phi(z)/\psi(z)]_{z=a}=$ ....

65. Bilinear transformation always transforms circles into ....

66. If $f(z)$ and $\overline{f(z)}$ are analytic functions, then $f(z)$ is constant. (True or False) (*Mumbai, 2006*)

67. The function $u(x,y)=2xy+3xy^2-2y^3$ is a harmonic functions. (True or False) (*P.T.U., 2009 S*)

68. The function $e^x\cos y$ is harmonic. (True or False)

69. $\int_C \frac{f(z)}{z-a}dz = 2\pi i\, f(a)$, if $z = a$ is a point within C. (True or False)

70. The transformation affected by an analytic function $w = f(z)$ is conformal at every point of the z-plane where $f'(z) = 0$. (True or False)

71. The function $\bar{z}$ is not analytic at any point. (True or False)

72. Under the transformation $w = 1/z$, circle $x^2 + y^2 - 6x = 0$ transforms into a straight line in the w-plane. (True or False)

73. If $w = f(z)$ is analytic, then $\frac{dw}{dz} = -i\frac{\partial w}{\partial y}$. (True or False)

74. An analytic function with constant imaginary part is constant. (True or False)

75. If $u + iv$ is analytic, then $v - iu$ is also analytic. (True or False)

76. $f(z) = I_m(z)$ is not analytic. (True or False)

77. The cross-ratio of four points is not invariant under bilinear transformation. (True or False)

78. $z = 0$ is not a critical point of the mapping $w = z^2$. (True or False)

79. $f(z) = \text{Re}\,(z^2)$ is analytic. (True or False)

80. An analytic function with constant modulus is constant. (True or False)

81. The function $|\bar{z}|^2$ is not analytic at any point. (True or False)

82. If $f(z) = z^2$, then the family of curves $x^2 - y^2 = C_1$, and $xy = C_2$ are orthogonal. (True or False)

CHAPTER 21

Laplace Transforms

21.1 INTRODUCTION

The knowledge of Laplace transforms has in recent years become an essential part of mathematical background required of engineers and scientists. This is because the transform methods provide an easy and effective means for the solution of many problems arising in engineering.

This subject originated from the operational methods applied by the English engineer Oliver Heaviside (1850–1925), to problems in electrical engineering. Unfortunately, Heaviside's treatment was unsystematic and lacked rigour, which was placed on sound mathematical footing by Bromwich and Carson during 1916–17. It was found that Heaviside's operational calculus is best introduced by means of a particular type of definite integrals called Laplace transforms.*

The method of Laplace transforms has the advantage of directly giving the solution of differential equations with given boundary values without the necessity of first finding the general solution and then evaluating from it the arbitrary constants. Moreover, the ready tables of Laplace transforms reduce the problem of solving differential equations to mere algebraic manipulation.

21.2 (1) DEFINITION

Let f(t) be a function of t defined for all positive values of t. Then the **Laplace transforms** *of f(t), denoted by* $L\{f(t)\}$ *is defined by*

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt \qquad ...(1)$$

provided that the integral exists. s is a parameter which may be a real or complex number.

$L\{f(t)\}$ being clearly a function of s is briefly written as $\bar{f}(s)$ *i.e.*, $L\{f(t)\} = \bar{f}(s)$,

which can also be written as $f(t) = L^{-1}\{\bar{f}(s)\}$.

Then $f(t)$ is called the **inverse Laplace transform** of $\bar{f}(s)$. The symbol L, which transforms $f(t)$ into $\bar{f}(s)$, is called the *Laplace transformation operator*.

**Pierre de Laplace* (1749–1827) (See footnote p. 18) used such transforms, much earlier in 1799, while developing the theory of probability.

(2) Conditions for the existence

The Laplace transform of $f(t)$ i.e., $\int_0^\infty e^{-st} f(t)\,dt$ exists for $s > a$, if

(*i*) *$f(t)$ is continuous* (*iii*) $\underset{t \to \infty}{\text{Lt}} \{e^{-at} f(t)\}$ *is finite.*

It should however, be noted that the above conditions are sufficient and not necessary.

For example, $L(1/\sqrt{t})$ exists, though $1/\sqrt{t}$ is infinite at $t = 0$.

21.3 TRANSFORMS OF ELEMENTARY FUNCTIONS

The direct application of the definition gives the following formulae :

(1) $\mathbf{L\,(1) = \dfrac{1}{s}}$ $(s > 0)$

(2) $\mathbf{L\,(t^n) = \dfrac{n\,!}{s^{n+1}}}$, when $n = 0, 1, 2, 3, \ldots$ $\left[\text{Otherwise } \dfrac{\Gamma(n+1)}{s^{n+1}}\right]$

(3) $\mathbf{L\,(e^{at}) = \dfrac{1}{s-a}}$ $(s > a)$

(4) $\mathbf{L\,(\sin at) = \dfrac{a}{s^2 + a^2}}$ $(s > 0)$

(5) $\mathbf{L\,(\cos at) = \dfrac{s}{s^2 + a^2}}$ $(s > 0)$

(6) $\mathbf{L\,(\sinh at) = \dfrac{a}{s^2 - a^2}}$ $(s > |\,a\,|)$

(7) $\mathbf{L\,(\cosh at) = \dfrac{s}{s^2 - a^2}}$ $(s > |\,a\,|)$

Proofs. **(1)** $$L\,(1) = \int_0^\infty e^{-st}\,.\,1\,dt = \left| -\frac{e^{-st}}{s} \right|_0^\infty = \frac{1}{s} \text{ if } s > 0.$$

(2) $$L\,(t^n) = \int_0^\infty e^{-st}\,.\,t^n\,dt = \int_0^\infty e^{-p}\,.\left(\frac{p}{s}\right)^n \frac{dp}{s}, \text{ on putting } st = p$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-p}\,.\,p^n\,dp = \frac{\Gamma(n+1)}{s^{n+1}}, \text{ if } n > -1 \text{ and } s > 0. \text{ [Page 302]}$$

In particular $L\,(t^{-1/2}) = \dfrac{\Gamma(1/2)}{s^{1/2}} = \sqrt{\dfrac{\pi}{s}}$; $L\,(t^{1/2}) = \dfrac{\Gamma(3/2)}{s^{3/2}} = \dfrac{\sqrt{\pi}}{2\,s^{3/2}}$

In n be a positive integer, $\Gamma(n + 1) = n\,!$ [(*v*) p. 302],

therefore, $L\,(t^n) = n\,!/s^{n+1}$.

(3) $$L\,(e^{at}) = \int_0^\infty e^{-st}\,.\,e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \left| \frac{e^{-(s-a)t}}{-(s-a)} \right|_0^\infty = \frac{1}{s-a}, \text{ if } s > a.$$

(4) $$L\,(\sin at) = \int_0^\infty e^{-st} \sin at\,dt = \left| \frac{e^{-st}}{s^2 + a^2} (-\,s \sin at - a \cos at) \right|_0^\infty = \frac{a}{s^2 + a^2}$$

Similarly, the reader should prove **(5)** himself.

(6) $$L\,(\sinh at) = \int_0^\infty e^{-st} \sinh at\,dt = \int_0^\infty e^{-st} \left(\frac{e^{at} - e^{-at}}{2}\right) dt$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t}\,dt - \int_0^\infty e^{-(s+a)t}\,dt\right] = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{a}{s^2 - a^2} \text{ for } s > |\,a\,|.$$

Similarly, the reader should prove **(7)** himself.

21.4 PROPERTIES OF LAPLACE TRANSFORMS

I. Linearity property. *If a, b, c be any constants and f, g, h any functions of t, then*

$$\mathbf{L\,[af(t) + bg(t) - ch(t)] = aL\,\{f(t)\} + bL\,\{g(t)\} - cL\,\{h(t)\}}$$

For by definition,

$$\text{L.H.S.} = \int_0^\infty e^{-st}\,[af(t) + bg(t) - ch(t)]\,dt$$

$$= a\int_0^\infty e^{-st}\,f(t)\,dt + b\int_0^\infty e^{-st}\,g(t)\,dt - c\int_0^\infty e^{-st}\,h(t)\,dt = aL\{f(t)\} + bL\{g(t)\} - cL\{h(t)\}$$

This result can easily be generalised.

Because of the above property of L, it is called a *linear operator.*

II. First shifting property. *If* $L\{f(t)\} = \bar{f}(s)$, then

$$\mathbf{L\,\{e^{at}f(t)\} = \bar{f}(s-a)}.$$

By definition, $L\{e^{at}f(t)\} = \int_0^\infty e^{-st}\,e^{at}\,f(t)\,dt = \int_0^\infty e^{-(s-a)t}\,f(t)\,dt$

$$= \int_0^\infty e^{-rt}\,f(t)\,dt, \text{ where } r = s - a = \bar{f}(r) = \bar{f}(s-a).$$

Thus, if we know the transform $\bar{f}(s)$ *of* $f(t)$, *we can write the transform of* $e^{at}f(t)$ *simply replacing s by* $s - a$ *to get* $\bar{f}(s-a)$.

Application of this property leads us to the following useful results :

(1) $\mathbf{L\,(e^{at}) = \dfrac{1}{s-a}}$ $\left[\because\ L(1) = \dfrac{1}{s}\right.$

(2) $\mathbf{L\,(e^{at}t^n) = \dfrac{n!}{(s-a)^{n+1}}}$ $\left[\because\ L(t^n) = \dfrac{n!}{s^{n+1}}\right.$

(3) $\mathbf{L\,(e^{at}\sin bt) = \dfrac{b}{(s-a)^2 + b^2}}$ $\left[\because\ L(\sin bt) = \dfrac{b}{s^2+b^2}\right.$

(4) $\mathbf{L\,(e^{at}\cos bt) = \dfrac{s-a}{(s-a)^2 + b^2}}$ $\left[\because\ L(\cos bt) = \dfrac{s}{s^2+b^2}\right.$

(5) $\mathbf{L\,(e^{at}\sinh bt) = \dfrac{b}{(s-a)^2 - b^2}}$ $\left[\because\ L(\sinh bt) = \dfrac{b}{s^2-b^2}\right.$

(6) $\mathbf{L\,(e^{at}\cosh bt) = \dfrac{s-a}{(s-a)^2 - b^2}}$ $\left[\because\ L(\cosh bt) = \dfrac{s}{s^2-b^2}\right.$

where in each case $s > a$.

Example 21.1. *Find the Laplace transforms of*
(*i*) *sin 2t sin 3t* (*ii*) *cos² 2t* (*iii*) *sin³ 2t.*

Solution. (*i*) Since $\sin 2t \sin 3t = \frac{1}{2}[\cos t - \cos 5t]$

$$\therefore \quad L(\sin 2t \sin 3t) = \tfrac{1}{2}[L(\cos t) - L(\cos 5t)] = \frac{1}{2}\left[\frac{s}{s^2+1^2} - \frac{s}{s^2+5^2}\right] = \frac{12s}{(s^2+1)(s^2+25)}$$

(*ii*) Since $\cos^2 2t = \dfrac{1}{2}(1 + \cos 4t)$

$$\therefore \quad L(\cos^2 2t) = \frac{1}{2}[L(1) + L\cos 4t] = \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+16}\right)$$

(*iii*) Since $\sin 6t = 3 \sin 2t - 4 \sin^3 2t$

or $\sin^3 2t = \frac{3}{4} \sin 2t - \frac{1}{4} \sin 6t$

$\therefore$ $L(\sin^3 2t) = \frac{3}{4} L(\sin 2t) - \frac{1}{4} L(\sin 6t)$

$$= \frac{3}{4} \cdot \frac{2}{s^2 + 2^2} - \frac{1}{4} \cdot \frac{2}{s^2 + 6^2} = \frac{48}{(s^2 + 4)(s^2 + 36)}.$$

Example 21.2. *Find the Laplace transform of*

(*i*) $e^{-3t}(2 \cos 5t - 3 \sin 5t)$. (*ii*) $e^{2t} \cos^2 t$ (V.T.U., 2006) (*iii*) $\sqrt{t}e^{3t}$. (P.T.U., 2009)

Solution. (*i*) $L\{e^{-3t}(2 \cos 5t - 3 \sin 5t)\} = 2L(e^{-3t} \cos 5t) - 3L(e^{-3t} \sin 5t)$

$$= 2 \cdot \frac{s+3}{(s+3)^2 + 5^2} - 3 \cdot \frac{5}{(s+3)^2 + 5^2} = \frac{2s - 9}{s^2 + 6s + 34}.$$

(*ii*) Since $L(\cos^2 t) = \frac{1}{2} L(1 + \cos 2t) = \frac{1}{2}\left\{\frac{1}{s} + \frac{s}{s^2 + 4}\right\}$

$\therefore$ by shifting property, we get

$$L(e^{2t} \cos^2 t) = \frac{1}{2}\left\{\frac{1}{s-2} + \frac{s-2}{(s-2)^2 + 4}\right\}.$$

(*iii*) Since $L(\sqrt{t}) = \frac{\Gamma(3/2)}{s^{3/2}} = \frac{(1/2) \cdot \Gamma\pi}{s^{3/2}}$

$\therefore$ by shifting property, we obtain $L(e^{3t}\sqrt{t}) = \frac{\sqrt{\pi}}{2} \frac{1}{(s-3)^{3/2}}$.

Example 21.3. *If* $L f(t) = \bar{f}(s)$, *show that*

$$L[(\sinh at) f(t)] = \frac{1}{2}[\bar{f}(s-a) - \bar{f}(s+a)]$$

$$L[(\cosh at) f(t)] = \frac{1}{2}[\bar{f}(s-a) + \bar{f}(s+a)]$$

Hence evaluate (*i*) $\sinh 2t \sin 3t$ (*ii*) $\cosh 3t \cos 2t$.

Solution. We have $L[(\sinh at) f(t)] = L\left[\frac{1}{2}(e^{at} - e^{-at}) f(t)\right] = \frac{1}{2}[L\{e^{at} f(t) - L\{e^{-at} f(t)\}]$

$$= \frac{1}{2}[\bar{f}(s-a) - \bar{f}(s+a)], \text{ by shifting property.}$$

Similarly, $L[(\cosh at) f(t)] = \frac{1}{2}[L\{e^{at} f(t)\} + \{e^{-at} f(t)\}]$

$$= \frac{1}{2}[\bar{f}(s-a) + \bar{f}(s+a)], \text{ by shifting property.}$$

(*i*) Since $L(\sin 3t) = \frac{3}{s^2 + 3^2}$, the first result gives

$$L(\sinh 2t \sin 3t) = \frac{1}{2}\left\{\frac{3}{(s-2)^2 + 3^2} - \frac{3}{(s+2)^2 + 3^2}\right\} = \frac{12s}{s^4 + 10s^2 + 169}$$

(*ii*) Since $L(\cos 2t) = \frac{s}{s^2 + 2^2}$, the second result gives

$$L(\cosh 3t \cos 2t) = \frac{1}{2}\left\{\frac{s-3}{(s-3)^2 + 2^2} + \frac{s+3}{(s+3)^2 + 2^2}\right\} = \frac{2s(s^2 - 5)}{s^4 - 10s^2 + 169}.$$

Example 21.4. *Show that*

(i) $L(t \sin at) = \dfrac{2as}{(s^2+a^2)^2}$ *(Bhopal, 2001)* (ii) $L(t \cos at) = \dfrac{s^2-a^2}{(s^2+a^2)^2}$.

Solution. Since $L(t) = 1/s^2$. $\therefore$ $L(te^{iat}) = \dfrac{1}{(s-ia)^2} = \dfrac{(s+ia)^2}{[(s-ia)(s+ia)]^2}$

or
$$L[t(\cos at + i \sin at)] = \frac{(s^2-a)^2 + i(2as)}{(s^2+a^2)^2}$$

Equating the real and imaginary parts from both sides, we get the desired results.

Example 21.5. *Find the Laplace transform of $f(t)$ defined as*

(i) $f(t) = t/\tau$, *when* $0 < t < \tau$

$= 1$, *when* $t > \tau$. *(Kerala, 2005)*

(ii) $f(t) = \begin{cases} 1, & 0 < t \le 1 \\ t, & 1 < t \le 2 \\ 0, & t > 2 \end{cases}$ *(J.N.T.U., 2006 ; W.B.T.U., 2005)*

Solution. (i) $$Lf(t) = \int_0^\tau e^{-st} \cdot \frac{t}{\tau} dt + \int_\tau^\infty e^{-st} \cdot 1\, dt = \frac{1}{\tau}\left[\left| t \cdot \frac{e^{-st}}{-s}\right|_0^\tau - \int_0^\tau 1 \cdot \frac{e^{-st}}{-s} dt\right] + \left|\frac{e^{-st}}{-s}\right|_\tau^\infty$$

$$= \frac{1}{\tau}\left[\frac{\tau e^{-s\tau} - 0}{-s} - \left|\frac{e^{-st}}{s^2}\right|_0^\tau\right] + \frac{0 - e^{-s\tau}}{-s} = \frac{-e^{-s\tau}}{s} - \frac{e^{-s\tau}-1}{\tau s^2} + \frac{e^{-s\tau}}{s} = \frac{1-e^{-s\tau}}{\tau s^2}.$$

(ii) $$L\{f(t)\} = \int_0^1 e^{-st} \cdot 1\, dt + \int_1^2 e^{-st} \cdot t\, dt + \int_2^\infty e^{-st} \cdot (0)\, dt$$

$$= \left|\frac{e^{-st}}{-s}\right|_0^1 + \left| t \cdot \frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right|_1^2 = \frac{1-e^{-s}}{s} + \left\{\left(-\frac{2e^{-2s}}{s} - \frac{e^{-2s}}{s^2}\right) - \left(\frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2}\right)\right\}$$

$$= \frac{1}{s} - \frac{2e^{-2s}}{s} + \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2}.$$

Example 21.6. *Find the Laplace transform of* (i) $\left(\sqrt{t} - \dfrac{1}{\sqrt{t}}\right)^3$. *(Kurukshetra, 2005)*

(ii) $\dfrac{\cos\sqrt{t}}{\sqrt{t}}$ *(Mumbai, 2009)*

Solution. (i) Since $(\sqrt{t} - 1/\sqrt{t})^3 = t^{3/2} - 3t^{1/2} + 3t^{-1/2} - t^{-3/2}$

$\therefore$ $$L(\sqrt{t} - 1/\sqrt{t}) = L(t^{3/2}) - 3L(t^{1/2}) + 3L(t^{-1/2}) - L(t^{-3/2})$$

$$= \frac{\Gamma(3/2+1)}{s^{3/2+1}} - 3\frac{\Gamma(1/2+1)}{s^{1/2+1}} + 3\frac{\Gamma(-1/2+1)}{s^{-1/2+1}} - \frac{\Gamma(-3/2+1)}{s^{-3/2+1}}$$

$$= \frac{\frac{3}{2}\Gamma\left(\frac{3}{2}\right)}{s^{5/2}} - 3\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{3/2}} + 3\frac{\Gamma\left(\frac{1}{2}\right)}{s^{1/2}} - \frac{\Gamma\left(-\frac{1}{2}\right)}{s^{-1/2}}$$

$$= \frac{3}{4}\frac{\sqrt{\pi}}{s^{5/2}} - \frac{3}{2}\frac{\sqrt{\pi}}{s^{3/2}} + \frac{3\sqrt{\pi}}{s^{1/2}} + \frac{2\sqrt{\pi}}{s^{-1/2}} \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}\right]$$

$$= \frac{\sqrt{\pi}}{4}\left(\frac{3}{s^{5/2}} - \frac{6}{s^{3/2}} + \frac{12}{s^{1/2}} + \frac{8}{s^{-1/2}}\right).$$

(*ii*) We know that $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots \infty$

$$\therefore \quad \cos\sqrt{t} = 1 - \frac{t}{2!} + \frac{t^2}{4!} - \frac{t^3}{6!} + \dots$$

and
$$\frac{\cos\sqrt{t}}{\sqrt{t}} = t^{-1/2} - \frac{t^{1/2}}{2!} + \frac{t^{3/2}}{4!} - \frac{t^{5/2}}{6!} + \dots$$

and
$$L\left(\frac{\cos\sqrt{t}}{\sqrt{t}}\right) = \frac{\Gamma(1/2)}{s^{1/2}} - \frac{1}{2!}\frac{\Gamma(3/2)}{s^{3/2}} + \frac{1}{4!}\frac{\Gamma(5/2)}{s^{5/2}} - \frac{1}{6!}\frac{\Gamma(7/2)}{s^{7/2}} + \dots$$

$$= \frac{\Gamma 1/2}{\sqrt{s}} - \frac{1}{2}\cdot\frac{1/2\,\Gamma 1/2}{s^{3/2}} + \frac{1}{4!}\frac{3/2\,.\,1/2\,.\,\Gamma(1/2)}{s^{5/2}} - \frac{1}{6!}\frac{5/2\,.\,3/2\,.\,1/2\,.\,\Gamma(1/2)}{s^{7/2}} + \dots$$

$$= \sqrt{\left(\frac{\pi}{2}\right)}\left[1 - \frac{1}{(4s)} + \frac{1}{2!}\frac{1}{(4s)^2} - \frac{1}{3!}\frac{1}{(4s)^3}\dots\right] = \sqrt{\left(\frac{\pi}{s}\right)}\,e^{-1/4s}.$$

Example 21.7. *Find the Laplace transform of the function*

(i) $f(t)\ |\ t-1\ |\ +\ |\ t+1\ |, t \geq 0$ *(S.V.T.U., 2009)*

(ii) $f(t) = [t]$, *where* [ ] *stands for the greatest integer function.* *(P.T.U., 2010)*

Solution. (*i*) Given function is equivalent to

$$f(t) = \begin{cases} 2, & 0 \leq t < 1 \\ 2t, & t \geq 1 \end{cases}$$

$$\therefore \quad L\,f(t) = \int_0^1 e^{-st}(2)\,dt + \int_1^\infty e^{-st}\,(2t)\,dt = 2\left[\left|\frac{e^{-st}}{-s}\right|_0^1 + 2\left|\frac{t.e^{-st}}{-s}\right|_1^\infty - \left|\frac{e^{-st}}{(-s)^2}\right|_1^\infty\right]$$

$$= 2\left(\frac{e^{-s}}{-s} + \frac{1}{s}\right) + 2\left(\frac{0-e^{-s}}{-s} - \frac{0-e^{-s}}{s^2}\right) = \frac{2}{s}\left(1 + \frac{e^{-s}}{s}\right)$$

(*ii*) Given function is equivalent to

$$[t] = 0 \text{ in } (0, 1) + 1 \text{ in } (1, 2) + 2 \text{ in } (2, 3) + 3 \text{ in } (3, 4) + \dots$$

$$\therefore \quad L[f(t)] = \int_0^\infty e^{-st}[f(t)]dt = \int_0^\infty e^{-st}\,[t]dt$$

$$= \int_0^1 e^{-st}(0)dt + \int_1^2 e^{-st}(1)dt + \int_2^3 e^{-st}(2)dt + \int_3^4 e^{-st}(3)dt + \dots \infty$$

$$= 0 + \left|\frac{e^{-st}}{-s}\right|_1^2 + 2\left|\frac{e^{-st}}{-s}\right|_2^3 + 3\left|\frac{e^{-st}}{-s}\right|_3^4 + \dots \infty$$

$$= -\frac{1}{s}[(e^{-2s} - e^{-s}) + 2(e^{-3s} - e^{-2s}) + 3(e^{-4s} - e^{-3s}) + \dots \infty]$$

$$= \frac{1}{s}(e^{-s} + e^{-2s} + e^{-3s} + \dots \infty) = \frac{1}{s}\left(\frac{e^{-s}}{1-e^{-s}}\right) = \frac{1}{s(e^s - 1)}.$$

III. Change of scale property. *If* $L\{f(t)\} = \bar{f}(s)$, *then* $L\{f(at)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$

$$L\{f(at)\} = \int_0^\infty e^{-st} f(at)\,dt = \int_0^\infty e^{-su/a} f(u)\,.\,du/a \qquad \begin{array}{|l} \text{Put } at = u \\ \quad dt = du/a \end{array}$$

$$= \frac{1}{a}\int_0^\infty e^{-su/a} f(u)du = \frac{1}{a}\bar{f}(s/a).$$

Example 21.8. *Find* $L\left\{\frac{\sin at}{t}\right\}$, *given that* $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\left\{\frac{1}{s}\right\}$.

Solution. By the above property,

$$L\left\{\frac{\sin at}{at}\right\} = \frac{1}{a}\tan^{-1}\left\{\frac{1}{(s/a)}\right\} = \frac{1}{a}\tan^{-1}\left(\frac{a}{s}\right) \text{ i.e., } L\left\{\frac{\sin at}{t}\right\} = \tan^{-1}\left\{\frac{a}{s}\right\}.$$

PROBLEMS 21.1

Find the Laplace transforms of

1. $e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t$. (*J.N.T.U., 2003*)
2. $1 + 2\sqrt{t} + 3/\sqrt{t}$.
3. $3\cosh 5t - 4\sinh 5t$. (*Nagarjuna, 2006*)
4. $\cos(at + b)$.
5. $(\sin t - \cos t)^2$.
6. $\sin 2t \cos 3t$. (*Kottayam, 2005*)
7. $\sin\sqrt{t}$.
8. $\sin^5 t$. (*Mumbai, 2007*)
9. $\cos^3 2t$.
10. $e^{-at}\sinh bt$
11. $e^{2t}(3t^5 - \cos 4t)$. (*P.T.U., 2007*)
12. $e^{-3t}\sin 5t \sin 3t$. (*V.T.U., 2006*)
13. $e^{-t}\sin^2 t$. (*Mumbai, 2009*)
14. $e^{2t}\sin^4 t$ (*Mumbai, 2007*)
15. $\cosh at \sin at$. (*Delhi, 2002*)
16. $\sinh 3t \cos^2 t$ (*Madras, 2000*)
17. $t^2 e^{2t}$. (*V.T.U., 2008 S*)
18. $(1 + te^{-t})^3$
19. $t\sqrt{(1+\sin t)}$ (*Mumbai, 2007*)
20. $f(t) = \begin{cases} 4, & 0 \le t <= 1 \\ 3, & t > 1 \end{cases}$ (*U.P.T.U., 2009*)
21. $f(t) = \begin{cases} \sin t, 0 < t < \pi \\ 0, t > \pi. \end{cases}$ (*Madras, 2000 S*)
22. $f(x) = \begin{cases} \sin(x - \pi/3), & x > \pi/3 \\ 0 & , x < \pi/3 \end{cases}$ (*Rajasthan, 2006*)
23. $f(t) = \begin{cases} \cos(t - 2\pi/3), & t > 2\pi/3 \\ 0, & t < 2\pi/3 \end{cases}$
24. $f(t) = \begin{cases} t^2, 0 < t < 2 \\ t - 1, 2 < t < 3 \\ 7, t > 3. \end{cases}$ (*Mumbai, 2007*)
25. If $L[f(t)] = \frac{1}{s(s^2+1)}$, find $L[e^{-t} f(2t)]$.

21.5 TRANSFORMS OF PERIODIC FUNCTIONS

If $f(t)$ is a periodic function with period T, i.e., $f(t + T) = f(t)$, then

$$L\{f(t)\} = \frac{\int_0^T e^{-st} f(t)dt}{1 - e^{-sT}}$$

We have $$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = \int_0^T e^{-st} f(t)dt + \int_T^{2T} e^{-st} f(t)dt + \int_{2T}^{3T} e^{-st} f(t)dt + \ldots$$

In the second integral put $t = u + T$, in the third integral put $t = u + 2T$, and so on. Then

$$L\{f(t)\} = \int_0^T e^{-st} f(t)dt + \int_0^T e^{-s(u+T)} f(u+T)du + \int_0^T e^{-s(u+2T)} f(u+2T)du + \ldots$$

$$= \int_0^T e^{-st} f(t)dt + e^{-sT}\int_0^T e^{-su} f(u)du + e^{-2sT}\int_0^T e^{-su} f(u)du + \ldots$$

$$[\because \quad f(u) = f(u+T) = f(u+2T) \text{ etc.}]$$

$$= \int_0^T e^{-st} f(t)dt + e^{-sT}\int_0^T e^{-st} f(t)dt + e^{-2sT}\int_0^T e^{-st} f(t)dt + \ldots$$

$$= (1 + e^{-sT} + e^{-2sT} + \ldots \infty)\int_0^T e^{-st} f(t)dt$$

$$= \frac{1}{1 - e^{-sT}}\int_0^T e^{-st} f(t)dt. \qquad \text{(V.T.U., 2008 ; Mumbai, 2006)}$$

Example 21.9. *Find the Laplace transform of the function*

$$f(t) = \sin \omega t, \quad 0 < t < \pi/\omega$$
$$= 0, \quad \pi/\omega < t < 2\pi/\omega.$$

(Kurukshetra, 2005 ; Madras, 2003)

Solution. Since $f(t)$ is a periodic function with period $2\pi/\omega$.

$$\therefore \quad L\{f(t)\} = \frac{1}{1-e^{-2\pi s/\omega}} \int_0^{2\pi/\omega} e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left[\int_0^{\pi/\omega} e^{-st} \sin \omega t \, dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st} . 0 \, dt\right]$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left| \frac{e^{-st}(-s \sin \omega t - \omega \cos \omega t)}{s^2+\omega^2} \right|_0^{\pi/\omega} = \frac{\omega e^{-\pi s/\omega} + \omega}{(1-e^{-2\pi s/\omega})(s^2+\omega^2)} = \frac{\omega}{(1-e^{-\pi s/\omega})(s^2+\omega^2)}.$$

Example 21.10. *Draw the graph of the periodic function*

$$f(t) = \begin{cases} t, & 0 < t < \pi \\ \pi - t, & \pi < t < 2\pi. \end{cases}$$

and find its Laplace transform. *(U.P.T.U., 2003)*

Solution. Here the period of $f(t) = 2\pi$ and its graph is as in Fig. 21.1.

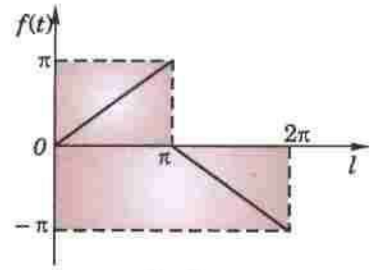


Fig. 21.1

$$\therefore \quad L f(t) = \frac{1}{1-e^{-2\pi s}} \left\{\int_0^{\pi} e^{-st} t \, dt + \int_{\pi}^{2\pi} e^{-st} (\pi - t) dt\right\}$$

$$= \frac{1}{1-e^{-2\pi s}} \left\{\left| t\left(\frac{e^{-st}}{-s}\right) - 1.\left(\frac{e^{-st}}{s^2}\right)\right|_0^{\pi} + \left|(\pi - t)\left(\frac{e^{-st}}{-s}\right) - (-1)\left(\frac{e^{-st}}{s^2}\right)\right|_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{1-e^{-2\pi s}} \left\{\frac{-\pi e^{-\pi s}}{s} - \frac{e^{-\pi s}}{s^2} + \frac{1}{s^2} + \frac{\pi e^{-2\pi s}}{s} + \frac{e^{-2\pi s}}{s^2} - \frac{e^{-\pi s}}{s^2}\right\}$$

$$= \frac{1}{1-e^{-2\pi s}} \left\{\frac{\pi}{s}\left(e^{-2\pi s} - e^{-\pi s}\right) + \frac{1}{s^2}\left(1 + e^{-2\pi s} - 2e^{-\pi s}\right)\right\}.$$

21.6 TRANSFORMS OF SPECIAL FUNCTIONS

(1) Transform of Bessel functions $J_0(x)$ and $J_1(x)$.

We know that $$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 . 4^2} - \frac{x^6}{2^2 . 4^2 . 6^2} + \ldots$$ [§ 16.7 (1), p. 553]

$$\therefore \quad L\{J_0(x)\} = \frac{1}{s} - \frac{1}{2^2} \frac{2!}{s^3} + \frac{1}{2^2 . 4^2} \cdot \frac{4!}{s^5} - \frac{1}{2^2 . 4^2 . 6^2} \cdot \frac{6!}{s^7} + \ldots$$

$$= \frac{1}{s}\left\{1 - \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{1.3}{2.4}\left(\frac{1}{s^4}\right) - \frac{1.3.5}{2.4.6}\left(\frac{1}{s^6}\right) + \ldots\right\}$$

$$= \frac{1}{s}\left(1 + \frac{1}{s^2}\right)^{-1/2} = \frac{1}{\sqrt{(s^2+1)}} \qquad \ldots(1)$$

Also since $$J_0'(x) = -J_1(x).$$ [Problem 4(*i*), p. 557]

$$\therefore \quad L\{J_1(x)\} = -L\{J_0'(x)\} = -[sL\{J_0(x)\} - 1] = 1 - \frac{s}{\sqrt{(s^2+1)}} \qquad \ldots(2)$$

(2) Transform of Error function

We know that $$erf(\sqrt{x}) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{x}} e^{-t^2} dt$$ (§ 7.18, p. 312)

$$= \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{x}} \left(1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \ldots\right) dt = \frac{2}{\sqrt{\pi}} \left(x^{1/2} - \frac{x^{3/2}}{3} + \frac{x^{5/2}}{5.2!} - \frac{x^{7/2}}{7.3!} + \ldots\right)$$

$$\therefore \quad L\{erf(\sqrt{x})\} = \frac{2}{\sqrt{\pi}} \left\{\frac{\Gamma(3/2)}{s^{3/2}} - \frac{\Gamma(5/2)}{3s^{5/2}} + \frac{\Gamma(7/2)}{5.2!\,s^{7/2}} - \frac{\Gamma(9/2)}{7.3!\,s^{9/2}} + \ldots\right\}$$

$$= \frac{1}{s^{3/2}} - \frac{1}{2}\frac{1}{s^{5/2}} + \frac{1.3}{2.4}\cdot\frac{1}{s^{7/2}} - \frac{1.3.5}{2.4.6}\cdot\frac{1}{s^{9/2}} + \ldots$$

$$= \frac{1}{s^{3/2}} \left\{1 - \frac{1}{2}\cdot\frac{1}{s} + \frac{1.3}{2.4}\cdot\frac{1}{s^2} - \frac{1.3.5}{2.4.6}\cdot\frac{1}{s^3} + \ldots\right\}$$

$$= \frac{1}{s^{3/2}} \left[1 + \frac{1}{s}\right]^{-1/2} = \frac{1}{s\sqrt{(s+1)}}. \qquad \textit{(Mumbai, 2009)} \quad ...(3)$$

(3) Transform of Laguerre's polynomials $L_n(x)$

We know that $L_n(x) = e^x \dfrac{d^n}{dx^n}(x^n e^{-x})$ (§ 16.18, p. 571)

$$L[L_n(t)] = \int_0^\infty e^{-st} e^t \frac{d^n}{dt^n}(t^n e^{-t}) dt = \int_0^\infty e^{-(s-1)t} \frac{d^n}{dt^n}(e^{-t} t^n) dt$$

$$= \left| e^{-(s-1)t} \frac{d^{n-1}}{dt^{n-1}}(e^{-t} t^n) \right|_0^\infty + \int_0^\infty e^{-(s-1)t\,(s-1)}\,(s-1) \frac{d^{n-1}}{dt^{n-1}}(e^{-t} t^n) dt$$

$$= (s-1) \int_0^\infty e^{-(s-1)t} \frac{d^{n-1}}{dt^{n-1}}(e^{-t} t^n) dt. \qquad \text{(Integrating by parts)}$$

$$= (s-1)^n \int_0^\infty e^{-(s-1)t} . e^{-t} . t^n\, dt = (s-1)^n \int_0^\infty e^{-st} . t^n\, dt$$

$$= (s-1)^n L(t^n) = (s-1)^n . \frac{n!}{s^{n+1}}$$

Hence $\quad L[L_n(x)] = \dfrac{n!(s-1)^n}{s^n + 1} \ (s > 1).$

Example 21.11. *Evaluate (i) $L\{e^{-at} J_0(at)\}$ (ii) $L(erf\, 2\sqrt{t}\,)$.* *(Mumbai, 2006)*

Solution. (*i*) We know that $L\{J_0(at)\} = \dfrac{1}{\sqrt{(s^2 + a^2)}}$

By shifting property, we get

$$L\{e^{-at} J_0(at)\} = \frac{1}{\sqrt{[(s+a)^2 + a^2]}} = \frac{1}{\sqrt{(s^2 + 2sa + 2a^2)}}$$

(*ii*) We know that $\quad L(erf\sqrt{t}\,) = \dfrac{1}{s(s+1)}$

$$\therefore \quad L(erf\ 2\sqrt{t}\,) = L[erf\sqrt{(4t)}] = \frac{1}{4}\cdot\frac{1}{\frac{s}{4}\sqrt{\left(\frac{s}{4}+1\right)}} = \frac{2}{s\sqrt{(s+4)}}.$$

PROBLEMS 21.2

1. Find the Laplace transform of the *saw-toothed wave* of period T, given $f(t) = t/T$ for $0 < t < T$. *(V.T.U., 2007)*
2. Find the Laplace transform of the *full-wave rectifier*

   $f(t) = E \sin wt,\ 0 < t < \pi/w$, having period π/w.

3. Find the Laplace transform of the *square-wave* (or *meander*) function of period a defined as

$$f(t) = k, \quad \text{when} \quad 0 < t < \alpha$$
$$= -k, \quad \text{when} \quad \alpha < t < 2\alpha.$$

(V.T.U., 2011)

4. Find the Laplace transform of the *triangular wave* of period $2a$ given by

$$f(t) = t, \quad 0 < t < a$$
$$= 2a - t, \quad a < t < 2a.$$

(Nagarjuna, 2008 ; V.T.U., 2008 S ; U.P.T.U., 2002)

Find the Laplace transform of the following functions :

5. $J_0(ax)$. 6. $e^{-at} J_0(bt)$. 7. $e^{2t} \, erf(\sqrt{t})$.

21.7 TRANSFORMS OF DERIVATIVES

(1) *If $f'(t)$ be continuous and $L\{f(t)\} = f(s)$, then $L\{f'(t)\} = s\bar{f}(s) - f(0)$.*

$$L\{f(t)\} = \int_0^\infty e^{-st} f'(t)dt \qquad \text{[Integrate by parts]}$$

$$= \left| e^{-st} f(t) \right|_0^\infty - \int_0^\infty (-s)e^{-st} \cdot f(t)dt.$$

Now assuming $f(t)$ to be such that $\underset{t\to\infty}{\text{Lt}} \; e^{-st} f(t) = 0$. When this condition is satisfied, $f(t)$ is said to be *exponential order s*.

Thus, $$L\{f'(t)\} = f(0) + s \int_0^\infty e^{-st} f(t)dt$$

whence follows the desired result.

(2) *If $f'(t)$ and its first $(n-1)$ derivatives be continuous, then*

$$\mathbf{L\{f^n(t)\} = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{n-1}(0).}$$

Using the general rule of integration by parts (Footnote p. 398).

$$L\{f^n(t)\} = \int_0^\infty e^{-st} f^n(t)dt$$

$$= |e^{-st} f^{n-1}(t) - (-s)e^{-st} f^{n-2}(t) + (-s)^2 e^{-st} f^{n-3}(t) - \ldots$$

$$+ (-1)^{n-1}(-s)^{n-1} e^{-st} \cdot f(t) |_0^\infty + (-1)^n (-s)^n \int_0^\infty e^{-st} f(t)dt$$

$$= -f^{n-1}(0) - sf^{n-2}(0) - s^2 f^{n-3}(0) - \ldots - s^{n-1} f(0) + s^n \int_0^\infty e^{-st} f(t)dt$$

Assuming that $\underset{t\to\infty}{\text{Lt}} \; e^{-st} f^m(t) = 0$ for $m = 0, 1, 2, \ldots, n-1$.

This proves the required result.

21.8 TRANSFORMS OF INTEGRALS

If $L\{f(t)\} = \bar{f}(s)$, then $\mathbf{L\left\{\int_0^t f(u)du\right\} = \frac{1}{s}\bar{f}(s).}$

Let $$\phi(t) = \int_0^t f(u)\,du, \text{ then } \phi'(t) = f(t) \text{ and } \phi(0) = 0$$

$\therefore$ $$L\{\phi'(t)\} = s\bar{\phi}(s) - \phi(0) \qquad \text{[By § 21.7 (1)]}$$

or $$\bar{\phi}(s) = \frac{1}{s} L\{\phi'(t)\} \quad i.e., \quad L\left(\int_0^t f(u)du\right) = \frac{1}{s}\bar{f}(s).$$

21.9 MULTIPLICATION BY t^n

If $L\{f(t)\} = \bar{f}(s)$, then

$$\mathbf{L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[\bar{f}(s)]}, \text{ where } n = 1, 2, 3 \ldots$$

We have $\int_0^\infty e^{-st} f(t)\,dt = \bar{f}(s)$.

Differentiating both sides with respect to s, $\frac{d}{ds}\left\{\int_0^\infty e^{-st} f(t)\,dt\right\} = \frac{d}{ds}\{\bar{f}(s)\}$

or By Leibnitz's rule for differentiation under the integral sign (p. 233).

$$\int_0^\infty \frac{\partial}{\partial s}(e^{-st}) f(t)\,dt = \frac{d}{ds}\{\bar{f}(s)\}$$

or $$\int_0^\infty \{-te^{-st} f(t)\}\,dt = \frac{d}{ds}[\bar{f}(s)] \quad \text{or} \quad \int_0^\infty e^{-st}\,[tf(t)\}\,dt = -\frac{d}{ds}[\bar{f}(s)]$$

which proves the theorem for $n = 1$.

Now assume the theorem to be true for $n = m$ (say), so that

$$\int_0^\infty e^{-st}\,[t^m f(t)]\,dt = (-1)^m \frac{d^m}{ds^m}[\bar{f}(s)]$$

Then $$\frac{d}{ds}\left[\int_0^\infty e^{-st}\,t^m f(t)\,dt\right] = (-1)^m \frac{d^{m+1}}{ds^{m+1}}[\bar{f}(s)]$$

or By Leibnitz's rule, $\int_0^\infty (-te^{-st})\,.\,t^m f(t)\,dt = (-1)^m \frac{d^{m+1}}{ds^{m+1}}[\bar{f}(s)]$

or $$\int_0^\infty e^{-st}\,[t^{m+1} f(t)]\,dt = (-1)^{m+1} \frac{d^{m+1}}{ds^{m+1}}[\bar{f}(s)].$$

This shows that, if the theorem, is true for $n = m$, it is also true for $n = m + 1$. But it is true for $n = 1$. Hence it is true for $n = 1 + 1 = 2$, and $n = 2 + 1 = 3$ and so on.

Thus the theorem is true for all positive integral values of n. *(U.P.T.U., 2005)*

Example 21.12. *Find the Laplace transforms of*

(i) t cos at *(Raipur, 2005)* *(ii) t^2 sin at* *(S.V.T.U., 2007)*

(iii) $t^3 e^{-3t}$ *(Kottayam, 2005)* *(iv) te^{-t} sin 3t.* *(Kurukshetra, 2005)*

Solution. (*i*) Since $L(\cos at) = s/(s^2 + a^2)$

$$\therefore \quad L(t\cos at) = -\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right) = -\frac{s^2+a^2-s\,.\,2s}{(s^2+a^2)^2}$$

$$= \frac{s^2-a^2}{(s^2+a^2)^2}$$ [*cf.* Example 21.4]

(*ii*) Since $$\sin at = \frac{a}{s^2+a^2},$$

$$\therefore \quad L(t^2 \sin at) = (-1)^2 \frac{d^2}{ds^2}\left(\frac{a}{s^2+a^2}\right) = \frac{d}{ds}\left\{\frac{-2as}{(s^2+a^2)^2}\right\} = \frac{2a(3s^2-a^2)}{(s^2+a^2)^3}.$$

(*iii*) Since $$L(e^{-3t}) = 1/(s+3),$$

$$\therefore \quad L(t^3 e^{-3t}) = (-1)^3 \frac{d^3}{ds^3}\left(\frac{1}{s+3}\right) = -\frac{(-1)^3\,.\,3\,!}{(s+3)^{3+1}} = 6/(s+3)^4.$$

(*iv*) Since $$L(\sin 3t) = \frac{3}{s^2+3^2}, \text{ therefore } L(t\sin 3t) = -\frac{d}{ds}\left(\frac{3}{s^2+3^2}\right) = \frac{6s}{(s^2+9)^2}$$

Now using the shifting property (§ 21.4 II), we get

$$L(e^{-t}\,t\sin 3t) = \frac{6(s+1)}{[(s+1)^2+9]^2} = \frac{6(s+1)}{(s^2+2s+10)^2}.$$

Example 21.13. *Evaluate (i) $L\{t\, J_0(at)\}$ (ii) $L\{t\, J_1(t)\}$ (iii) $L\{t\, erf\, 2\sqrt{t}\}$.*

Solution. (*i*) Since $L\{J_0(at)\} = \dfrac{1}{\sqrt{(s^2+a^2)}}$

$$\therefore \quad L\{tJ_0(at)\} = -\frac{d}{ds}\{L\, J_0(at)\} = -\frac{d}{ds}\frac{1}{\sqrt{(s^2+a^2)}} = \frac{s}{(s^2+a^2)^{3/2}}$$

(*ii*) Since $\quad L\{J_1(t)\} = 1 - \dfrac{s}{\sqrt{(s^2+1)}}$

$$\therefore \quad L\{t\, J_1(t)\} = -\frac{d}{ds}\{L\, J_0(t)\} = -\frac{d}{ds}\left\{1 - \frac{s}{(\sqrt{s^2+1})}\right\} = \frac{1}{(s^2+1)^{3/2}}$$

(*iii*) Since $\quad L(erf\sqrt{t}) = \dfrac{1}{s\sqrt{(s+1)}}$

$$\therefore \quad L(erf\, 2\sqrt{t}) = L\, erf\sqrt{4(t)} = \frac{1}{4}\cdot\frac{1}{\frac{s}{4}\sqrt{\left(\frac{s}{4}+1\right)}} = \frac{2}{s\sqrt{(s+4)}}$$

Thus $\quad L(t\, erf\, 2\sqrt{t}) = -\dfrac{d}{ds}\left\{\dfrac{2}{s\sqrt{(s+4)}}\right\} = -\dfrac{d}{ds}\left\{\dfrac{2}{\sqrt{(s^3+4s^2)}}\right\} = \dfrac{3s+8}{s^2(s+4)^{3/2}}$

21.10 DIVISION BY t

If $L\{f(t)\} = \bar{f}(s)$, then $\mathbf{L}\left\{\dfrac{\mathbf{1}}{\mathbf{t}}\mathbf{f(t)}\right\} = \displaystyle\int_s^\infty \bar{\mathbf{f}}(\mathbf{s})\, \mathbf{ds}$ *provided the integral exists.*

We have $\bar{f}(s) = \displaystyle\int_0^\infty e^{-st} f(t)\, dt$

Integrating both sides with respect to s from s to ∞.

$$\int_s^\infty \bar{f}(s)\, ds = \int_s^\infty\left[\int_0^\infty e^{-st} f(t)\, dt\right] ds = \int_0^\infty\int_s^\infty f(t)\, e^{-st}\, ds\, dt$$

[Changing the order of integration]

$$= \int_0^\infty f(t)\left[\int_s^\infty e^{-st}\, ds\right] dt \qquad [\because \; t \text{ is independent of } s]$$

$$= \int_0^\infty f(t)\left|\frac{e^{-st}}{-t}\right|_s^\infty dt = \int_0^\infty e^{-st}\cdot\frac{f(t)}{t}\, dt = L\left\{\frac{1}{t}f(t)\right\}.$$

Example 21.14. *Find the Laplace transform of (i) $(1-e^t)/t$* (*Madras, 2000*)
(ii) $\dfrac{\cos at - \cos bt}{t} + t\sin at$. (*V.T.U., 2010*)

Solution. (*i*) Since $\quad L(1-e^t) = L(1) - L(e^t) = \dfrac{1}{s} - \dfrac{1}{s-1}$

$$\therefore \quad L\left(\frac{1-e^t}{t}\right) = \int_s^\infty\left(\frac{1}{s} - \frac{1}{s-1}\right)ds = |\log s - \log(s-1)|_0^\infty$$

$$= \left|\log\left(\frac{s}{s-1}\right)\right|_s^\infty = -\log\left[\frac{1}{1-1/s}\right] = \log\left(\frac{s-1}{s}\right)$$

(*ii*) Since $\quad L(\cos at - \cos bt) = \dfrac{s}{s^2+a^2} - \dfrac{s}{s^2+b^2}$ and $L(\sin at) = \dfrac{a}{s^2+a^2}$

$$\therefore \quad L\left(\frac{\cos at - \cos bt}{t}\right) + L(t \sin at) = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds - \frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)$$

$$= \left| \frac{1}{2}\log(s^2+a^2) - \frac{1}{2}\log(s^2+b^2)\right|_s^\infty - a\,\frac{-2s}{(s^2+a^2)^2}$$

$$= \frac{1}{2}\underset{s\to\infty}{\text{Lt}}\ \log\frac{s^2+a^2}{s^2+b^2} - \frac{1}{2}\log\frac{s^2+a^2}{s^2+b^2} + \frac{2as}{(s^2+a^2)^2}$$

$$= \frac{1}{2}\log\left(\frac{1+0}{1+0}\right) - \frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right) + \frac{2as}{(s^2+a^2)^2} = \log\left(\frac{s^2+b^2}{s^2+a^2}\right)^{1/2} + \frac{2as}{(s^2+a^2)^2}$$

$$[\because \log 1 = 0]$$

Example 21.15. *Evaluate (i)* $L\left\{e^{-t}\int_0^t \frac{\sin t}{t}\,dt\right\}$ *(Madras, 2006)*

(ii) $L\left\{t\int_0^t \frac{e^{-t}\sin t}{t}\,dt\right\}$ *(P.T.U., 2005)* *(iii)* $L\left\{\int_0^t\int_0^t\int_0^t (t\sin t)\,dt\,dt\,dt\right\}$. *(Mumbai, 2006)*

Solution. (*i*) We know that $L(\sin t) = \dfrac{1}{s^2+1}$

$$L\left(\frac{\sin t}{t}\right) = \int_0^\infty \frac{1}{s^2+1}\,ds = \frac{\pi}{2} - \tan^{-1}s = \cot^{-1}s$$

$$\therefore \quad L\left\{\int_0^t \frac{\sin t}{t}\,dt\right\} = \frac{1}{s}\cot^{-1}s$$

Thus by shifting property, $L\left\{e^{-t}\left(\int_0^t \frac{\sin t}{t}\,dt\right)\right\} = \dfrac{1}{s+1}\cot^{-1}(s+1)$.

(*ii*) Since $\quad L\left(\dfrac{\sin t}{t}\right) = \cot^{-1}s$

$$\therefore \quad L\left(e^{-t}\cdot\frac{\sin t}{t}\right) = \cot^{-1}(s+1)$$

and $\quad L\left\{\displaystyle\int_0^t e^{-t}\frac{\sin t}{t}\,dt\right\} = \dfrac{1}{s}\cot^{-1}(s+1)$

Hence $\quad L\left\{t\cdot\displaystyle\int_0^t e^{-t}\frac{\sin t}{t}\,dt\right\} = -\dfrac{d}{ds}\left\{\dfrac{\cot^{-1}(s+1)}{s}\right\}$

$$= -\frac{s\cdot\left[\dfrac{-1}{1+(s+1)^2}\right] - \cot^{-1}(s+1)}{s^2} = \frac{s + (s^2+2s+2)\cot^{-1}(s+1)}{s^2(s^2+2s+2)}$$

(*iii*) Since $\quad L(\sin t) = \dfrac{1}{s^2+1}$

$$\therefore \quad L(t\sin t) = -\frac{d}{ds}\frac{1}{(s^2+1)} = \frac{2s}{(s^2+1)^2}$$

Thus $\quad L\left\{\displaystyle\int_0^t\int_0^t\int_0^t (t\sin t)\,dt\,.\,dt\,.\,dt\right\} = \dfrac{1}{s^3}L(t\sin t) = \dfrac{1}{s^3}\cdot\dfrac{2s}{(s^2+1)^2} = \dfrac{2}{s^2(s^2+1)^2}$

21.11 EVALUATION OF INTEGRALS BY LAPLACE TRANSFORMS

Example 21.16. *Evaluate* (i) $\int_0^\infty te^{-3t}\sin t\,dt$ *(V.T.U., 2007)*

(ii) $\int_0^\infty \frac{\sin mt}{t}\,dt$ (iii) $\int_0^\infty e^{-t}\left(\frac{\cos at-\cos bt}{t}\right)dt$ *(Mumbai, 2009)*

(iv) $L\left\{\int_0^t \frac{e^{-t}\sin t}{t}\,dt\right\}$.

Solution. (i) $\int_0^\infty te^{-3t}\sin t\,dt = \int_0^\infty e^{-st}(t\sin t)\,dt$ where $s = 3$

$= L(t\sin t)$, by definition.

$$= (-1)\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2} = \frac{2\times 3}{(3^2+1)^2} = \frac{3}{50}.$$

(ii) Since $L(\sin mt) = m/(s^2+m^2) = f(s)$, say.

$\therefore$ Using § 21.10, $L\left(\frac{\sin mt}{t}\right) = \int_s^\infty f(s)\,ds = \int_0^\infty \frac{m\,ds}{s^2+m^2} = \left|\tan^{-1}\frac{s}{m}\right|_s^\infty$

or by Def., $$\int_0^\infty e^{-st}\frac{\sin mt}{t}\,dt = \frac{\pi}{2} - \tan^{-1}\frac{s}{m}$$

Now $\underset{s\to 0}{\text{Lt}}\ \tan^{-1}(s/m) = 0$ if $m > 0$ or π if $m < 0$.

Thus taking limits as $s \to 0$, we get

$$\int_0^\infty \frac{\sin mt}{t}\,dt = \frac{\pi}{2} \text{ if } m > 0 \quad\text{or}\quad -\pi/2 \text{ if } m < 0$$

(iii) We know that $L(\cos at) = \frac{s}{s^2+a^2}$ and $L(\cos bt) = \frac{s}{s^2+b^2}$

$$\therefore \quad L\,\frac{\cos at-\cos bt}{t} = \int_s^\infty\left(\frac{s}{s^2+a^2}-\frac{s}{s^2+b^2}\right)ds$$

$$= \frac{1}{2}\left\{\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\}_s^\infty = \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

This implies that $\int_0^\infty e^{-st}\left(\frac{\cos at-\cos bt}{t}\right)dt = \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$

Taking $s = 1$, we get $\int_0^\infty\left(e^{-t}\,\frac{\cos at-\cos bt}{t}\right)dt = \frac{1}{2}\log\left(\frac{1+b^2}{1+a^2}\right)$

(iv) Since $L\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{ds}{s^2+1} = \tan^{-1} s = \frac{\pi}{2} - \tan^{-1}s = \cot^{-1}s.$

$\therefore$ $L\left\{e^t\left(\frac{\sin t}{t}\right)\right\} = \cot^{-1}(s-1)$, by shifting property (§ 21.4 *II*).

Thus $L\left[\int_0^t\left\{e^t\left(\frac{\sin t}{t}\right)\right\}dt\right] = \frac{1}{s}\cot^{-1}(s-1)$, by § 21.8.

PROBLEMS 21.3

1. Find $L\left(\int_0^t e^{-t}\cos t\,dt\right)$.
2. Given $L\{2\sqrt{(t/\pi)}\} = 1/s^{3/2}$, show that $L\{1/\sqrt{(\pi t)}\} = 1/\sqrt{s}$. (*U.P.T.U., 2005 ; Madras, 2003*)
3. Given $L[\sin(\sqrt{t})] \frac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4\pi}$, prove that $L\left[\frac{\cos(\sqrt{t})}{\sqrt{t}}\right] = \sqrt{\left(\frac{\pi}{s}\right)} e^{-1/4\pi}$. (*Mumbai, 2009*)

Find the Laplace transforms of the following functions :

4. $t \sin^2 t$ (*Nagarjuna, 2008*)
5. $\sin 2t - 2t\cos 2t$. (*Anna, 2003*)
6. $t^2 \cos at$.
7. $t \sinh at$.
8. $te^{2t} \sin 3t$. (*Madras, 2003*)
9. $te^{-2t} \sin 4t$. (*V.T.U., 2008*)
10. $t^2 e^{-3t} \sin 2t$. (*Madras, 2000 S*)
11. $(e^{-at} - e^{-bt})/t$. (*Anna, 2005 S*)
12. $(\sin t)/t$. (*P.T.U., 2010*)
13. $\frac{(\sin t \sin 5t)}{t}$. (*Mumbai, 2008*)
14. $(e^{at} - \cos bt)/t$. (*U.P.T.U., 2003*)
15. $(e^{-t}\sin t)/t$. (*V.T.U., 2009 S*)
16. $(1 - \cos 3t)/t$. (*V.T.U., 2006*)
17. $(1 - \cos t)/t^2$. (*Hazaribag, 2008*)
18. $2^t + \frac{\cos 2t - \cos 3t}{t} + t \sin t$. (*V.T.U., 2004*)
19. Evaluate (*i*) $\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\,dT$ (*Mumbai, 2008 ; P.T.U., 2006*)

    (*ii*) $\int_0^\infty \frac{e^{-\sqrt{2}t}\sinh t \sin t}{t}\,dt$ (*Mumbai, 2005*) (*iii*) $\int_0^\infty te^{-2t} \sin 3t\,dt$ (*V.T.U., 2008*)

    (*iv*) $\int_0^\infty te^{-t}\sin^4 t\,dt$.
20. Prove that (*i*) $\int_0^\infty \frac{e^{-at} - e^{-bt}}{t}\,dt = \log\frac{b}{a}$. (*S.V.T.U., 2009 ; Mumbai, 2007 ; J.N.T.U., 2006*)

    (*ii*) $\int_0^\infty \frac{e^{-2t}\sinh t}{t}\,dt = \frac{1}{2}\log 3$ (*Mumbai, 2008*) (*iii*) $\int_0^\infty \frac{e^{-t}\sin t}{t}\,dt = \frac{\pi}{4}$. (*V.T.U., 2009 S*)

    (*iv*) $\int_0^\infty \frac{e^{-t}\sin^2 t}{t}\,dt = \frac{1}{4}\log 5$. (*Kurukshetra, 2006*)
21. Evaluate (*i*) $L\left(\int_0^t \frac{\sin t}{t}\,dt\right)$ (*J.N.T.U., 2005*)

    (*ii*) $L\left(\int_0^t e^{-t}\cos t\,dt\right)$ (*iii*) $L\int_0^t \frac{e^t \sin t}{t}\,dt$. (*P.T.U., 2009 S ; S.V.T.U., 2009 ; Bhopal, 2008*)
22. Show that (*i*) $L[tJ_0(at)] = \frac{s}{(s^2+a^2)^{3/2}}$ (*ii*) $\int_0^\infty te^{-3t} J_0(4t)\,dt = 3/125$.

21.12 INVERSE TRANSFORMS — METHOD OF PARTIAL FRACTIONS

Having found the Laplace transforms of a few functions, let us now determine the inverse transforms of given functions of s. We have seen that $L\{f(t)\}$ in each case, is a rational algebraic function. Hence *to find the* inverse transforms, we first express the given function of s into partial fractions which will, then, be recognizable as one of the following standard forms :

(1) $L^{-1}\left[\frac{1}{s}\right] = 1$. **(2)** $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$.

(3) $L^{-1}\left[\frac{1}{s^n}\right]=\frac{t^{n-1}}{(n-1)!}$, n = 1, 2, 3, ...

(4) $L^{-1}\left[\frac{1}{(s-a)^n}\right]=\frac{e^{at}t^{n-1}}{(n-1)!}$.

(5) $L^{-1}\left(\frac{1}{s^2+a^2}\right)=\frac{1}{a}\sin at$.

(6) $L^{-1}\left(\frac{s}{s^2+a^2}\right)=\cos at$.

(7) $L^{-1}\left(\frac{1}{s^2-a^2}\right)=\frac{1}{a}\sinh at$.

(8) $L^{-1}\left(\frac{s}{s^2-a^2}\right)=\cosh at$.

(9) $L^{-1}\left[\frac{1}{(s-a)^2+b^2}\right]=\frac{1}{b}e^{at}\sin bt$.

(10) $L^{-1}\left[\frac{s-a}{(s-a)^2+b^2}\right]=e^{at}\cos bt$.

(11) $L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right)=\frac{1}{2a}t\sin at$.

(12) $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]=\frac{1}{2a^3}(\sin at-at\cos at)$.

The reader is strongly advised to commit these results to memory. The results (1) to (10) follow at once from their corresponding results in § 21.3 and 21.4. As illustrations, we shall prove (11) and (12). Example 21.4 gives

$$L(t\sin at)=\frac{2as}{(s^2+a^2)^2}\text{ and }L(t\cos at)=\frac{s^2-a^2}{(s^2+a^2)^2}$$

$\therefore$ $$t\sin at=2a\,L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]\text{, whence follows (11).}$$

Also $$t\cos at=L^{-1}\left[\frac{s^2-a^2}{(s^2+a^2)^2}\right]=L^{-1}\left[\frac{(s^2+a^2)-2a^2}{(s^2+a^2)^2}\right]$$

$$=L^{-1}\left[\frac{1}{s^2+a^2}\right]-2a^2\,L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]$$

$$=\frac{1}{a}\sin at-2a^2\,L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]\text{ whence follows (12).}$$

Obs. *Go through the note on the 'partial fractions' given in para 10 of 'useful information' in Appendix I.*

Example 21.17. *Find the inverse transforms of*

(*i*) $\frac{s^2-3s+4}{s^3}$ (*ii*) $\frac{s+2}{s^2-4s+13}$ (*V.T.U., 2008*)

Solution. (*i*) $L^{-1}\left(\frac{s^2-3s+4}{s^3}\right)=L^{-1}\left(\frac{1}{s}\right)-3L^{-1}\left(\frac{1}{s^2}\right)+4L^{-1}\left(\frac{1}{s^3}\right)=1-3t+4\,.\,t^2/2\,!=1-3t+2t^2.$

(*ii*) $$L^{-1}\left(\frac{s+2}{s^2-4s+13}\right)=L^{-1}\left[\frac{s+2}{(s-2)^2+9}\right]=L^{-1}\left[\frac{s-2+4}{(s-2)^2+3^2}\right]$$

$$=L^{-1}\left[\frac{s-2}{(s-2)^2+3^2}\right]+4L^{-1}\left[\frac{1}{(s-2)^2+3^2}\right]=e^{2t}\cos 3t+\tfrac{4}{3}e^{2t}\sin 3t.$$

Example 21.18. *Find the inverse transforms of*

(*i*) $\frac{2s^2-6s+5}{s^3-6s^2+11s-6}$ (*V.T.U., 2007 ; U.P.T.U., 2004*)

(*ii*) $\frac{4s+5}{(s-1)^2(s+2)}$ (*Kurukshetra, 2005*)

Solution. (*i*) Here the denominator = $(s-1)(s-2)(s-3)$.

So let $$\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)} = \frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{s-3}$$

Then $A = [2 \,.\, 1^2 - 6 \,.\, 1 + 5]/(1-2)(1-3) = \frac{1}{2}$

$B = [2 \,.\, 2^2 - 6 \,.\, 2 + 5]/(2-1)(2-3) = -1$

and $C = [2 \,.\, 3^2 - 6 \,.\, 3 + 5]/(3-1)(3-2) = \frac{5}{2}$.

$$\therefore \quad L^{-1}\left(\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right) = \frac{1}{2}L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left(\frac{1}{s-2}\right) + \frac{5}{2}L^{-1}\left(\frac{1}{s-3}\right)$$

$$= \tfrac{1}{2}e^t - e^{2t} + \tfrac{5}{2}e^{3t}.$$

(*ii*) Let $$\frac{4s+5}{(s-1)^2(s+2)} = \frac{A}{s-1}+\frac{B}{(s-1)^2}+\frac{4(-2)+5}{(-2-1)^2(s+2)}$$

Multiplying both sides by $(s-1)^2(s+2)$, $4s+5 = A(s-1)(s+2) + B(s+2) - \frac{1}{3}(s-1)^2$

Putting $s = 1$, $9 = 3B$, $\therefore \quad B = 3$.

Equating the coefficients of s^2 from both sides,

$$0 = A - \tfrac{1}{3}, \quad \therefore \quad A = \tfrac{1}{3}.$$

$$\therefore \quad L^{-1}\left[\frac{4s+5}{(s-1)^2(s+2)}\right] = \frac{1}{3}L^{-1}\left(\frac{1}{s-1}\right) + 3L^{-1}\left[\frac{1}{(s-1)^2}\right] - \frac{1}{3}L^{-1}\left(\frac{1}{s+2}\right)$$

$$= \tfrac{1}{3}e^t + 3te^t - \tfrac{1}{3}e^{-2t}.$$

Example 21.19. *Find the inverse transforms of*

(*i*) $\dfrac{5s+3}{(s-1)(s^2+2s+5)}$ (*Rohtak, 2009 ; U.P.T.U., 2005*)

(*ii*) $\dfrac{s}{s^4+4a^4}$. (*Mumbai, 2008*)

Solution. (*i*) Let $$\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{5(1)+3}{(s-1)(1^2+2\,.\,1+5)} + \frac{As+B}{s^2+2s+5}$$

Multiplying both sides by $(s-1)(s^2+2s+5)$,

$$5s+3 = 1\,.\,(s^2+2s+5) + (As+B)(s-1).$$

Equating the coefficients of s^2 from both sides,

$$0 = 1 + A, \quad \therefore \quad A = -1.$$

Putting $s = 0$, $3 = 5 - B$, $\therefore \quad B = 2$.

$$\therefore \quad L^{-1}\left[\frac{5s+3}{(s-1)(s^2+2s+5)}\right] = L^{-1}\left(\frac{1}{s-1}\right) + L^{-1}\left(\frac{-s+2}{s^2+2s+5}\right)$$

$$= L^{-1}\left(\frac{1}{s-1}\right) + L^{-1}\left[\frac{-(s+1)+3}{(s+1)^2+4}\right] = L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left[\frac{s+1}{(s+1)^2+2^2}\right] + 3L^{-1}\left[\frac{1}{(s+1)^2+2^2}\right]$$

$$= e^t - e^{-t}\cos 2t + \tfrac{3}{2}e^{-t}\sin 2t.$$

(*ii*) Since $s^4 + 4a^4 = (s^2+2a^2)^2 - (2as)^2 = (s^2+2as+2a^2)(s^2-2as+2a^2)$

$\therefore$ Let $$\frac{s}{s^4+4a^4} = \frac{As+B}{s^2+2as+2a^2} + \frac{Cs+D}{s^2-2as+2a^2}$$

Multiplying both sides by $s^4 + 4a^4$,

$$s = (As+B)(s^2-2as+2a^2) + (Cs+D)(s^2+2as+2a^2)$$

Equating coefficients of s^3, $0 = A + C$ ...(*i*)

Equating coefficients of s^2, $0 = -2aA + B + 2aC + D$ ...(*ii*)

Equating coefficients of s, $1 = 2a^2A - 2aB + 2a^2C + 2aD$ ...(*iii*)

Putting $s = 0$, $\quad 0 = 2a^2B + 2a^2D$ ...(*iv*)

From (*iv*), $B + D = 0$ ...(*v*)

$\therefore$ (*ii*) becomes $-A + C = 0$, and by (*i*), we get $A = C = 0$.

Then (*iii*) reduces to $D - B = 1/2a$ and by (*v*), $B = -1/4a$, $D = 1/4a$.

$$\therefore \quad L^{-1}\left(\frac{s}{s^4 + 4a^4}\right) = -\frac{1}{4a}L^{-1}\left(\frac{1}{s^2 + 2as + 2a^2}\right) + \frac{1}{4a}L^{-1}\left(\frac{1}{s^2 - 2as + 2a^2}\right)$$

$$= -\frac{1}{4a}L^{-1}\left[\frac{1}{(s+a)^2 + a^2}\right] + \frac{1}{4a}L^{-1}\left[\frac{1}{(s-a)^2 + a^2}\right]$$

$$= -\frac{1}{4a}\cdot\frac{1}{a}e^{-at}\sin at + \frac{1}{4a}\cdot\frac{1}{a}e^{at}\sin at = \frac{1}{2a^2}\sin at\left(\frac{e^{at} - e^{-at}}{2}\right) = \frac{1}{2a^2}\sin at \sinh at.$$

PROBLEMS 21.4

Find the inverse Laplace transforms of :

1. $\dfrac{2s-5}{4s^2+25} + \dfrac{4s-18}{9-s^2}$.
2. $\dfrac{1}{s^2-5s+6}$. (*S.V.T.U., 2008*)
3. $\dfrac{s}{(2s-1)(3s-1)}$. (*V.T.U., 2010*)
4. $\dfrac{3s}{s^2+2s-8}$.
5. $\dfrac{3s+2}{s^2-s-2}$. (*V.T.U., 2010 S*)
6. $\dfrac{1}{s(s^2-1)}$. (*Nagarjuna, 2008*)
7. $\dfrac{1-7s}{(s-3)(s-1)(s+2)}$. (*B.P.T.U., 2005 S*)
8. $\dfrac{s^2-10s+13}{(s-7)(s^2-5s+6)}$.
9. $\dfrac{2p^2-6p+5}{p^3-6p^2+11p-6}$. (*U.P.T.U., 2004*)
10. $\dfrac{s}{(s^2-1)^2}$. (*Kurukshetra, 2005*)
11. $\dfrac{1+2s}{(s+2)^2(s-1)^2}$.
12. $\dfrac{s}{(s-3)(s^2+4)}$.
13. $\dfrac{s}{(s+1)^2(s^2+1)}$.
14. $\dfrac{s^3}{s^4-a^4}$. (*Kurukshetra, 2005*)
15. $\dfrac{1}{s^3-a^3}$.
16. $\dfrac{s^2+6}{(s^2+1)(s^2+4)}$.
17. $\dfrac{2s-3}{s^2+4s+13}$.
18. $\dfrac{s^2+s}{(s^2+1)(s^2+2s+2)}$.
19. $\dfrac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$ (*Mumbai, 2008*)
20. $\dfrac{s}{s^4+s^2+1}$. (*Raipur, 2005*)
21. $\dfrac{a(s^2-2a^2)}{s^4+4a^4}$. (*Mumbai, 2009*)

21.13 OTHER METHODS OF FINDING INVERSE TRANSFORMS

We have seen that the most effective method of finding the inverse transforms is by means of partial fractions. However, various other methods are available which depend on the following *important inversion formulae.*

I. Shifting property for inverse Laplace transforms.

If $L^{-1}[\bar{f}(s)] = f(t)$, *then*

$$L^{-1}[\bar{f}'(s-a)] = e^{at} f(t) = e^{at} L^{-1}[\bar{f}(s)].$$

II. *If* $L^{-1}\{\bar{f}(s)\} = f(t)$ *and* $f(0) = 0$, *then*

$$L^{-1}[s\,\bar{f}(s)] = \frac{d}{dt}\{f(t)\}$$

In general, $L^{-1}\{s^n \bar{f}(s)\} = \dfrac{d^n}{dt^n}\{f(t)\}$ provided $f(0) = f'(0) = \ldots = f^{n-1}(0) = 0$.

The above formulae at once follow from the results of § 21.7 (Transforms of derivatives).

III. *If* $L^{-1}\{\bar{f}(s)\} = f(t)$, *then*

$$L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(t)\, dt$$

This result follows from § 21.8 (Transforms of integrals)

Also $$L^{-1}\left\{\frac{\bar{f}(s)}{s^2}\right\} = \int_0^t \left\{\int_0^t f(t)\,dt\right\} dt$$

$$L^{-1}\left\{\frac{\bar{f}(s)}{s^3}\right\} = \int_0^t \left\{\int_0^t \left(\int_0^t f(t)\,dt\right) dt\right\} dt \text{ and so on.}$$

IV. *If* $L^{-1}\{\bar{f}(s)\} = f(t)$, *then*

$$t\,f(t) = L^{-1}\left\{-\frac{d}{ds}[\bar{f}(s)]\right\}$$

This result follows from $L\{t\,f(t)\} = -\dfrac{d}{ds}\{\bar{f}(s)\}$ (§ 21.9)

V. The formula of § 21.10, *i.e.*,

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)\, ds$$

is useful in finding $f(t)$ when $f(s)$ is given, provided the inverse transform of $\int_s^\infty \bar{f}(s)\, ds$ can be conveniently calculated.

Example 21.20. *Find the inverse Laplace transforms of the following :*

(*i*) $\dfrac{s^2}{(s-2)^3}$ (*ii*) $\dfrac{s+3}{s^2-4s+13}$ (*iii*) $\dfrac{(s+2)^2}{(s^2+4s+8)^2}$ (*Mumbai, 2005*)

Solution. (*i*) Since $s^2 = (s-2)^2 + 4(s-2) + 4$

$$\therefore \quad \frac{s^2}{(s-2)^3} = \frac{1}{s-2} + \frac{4}{(s-2)^2} + \frac{4}{(s-2)^3}$$

$$\therefore \quad L^{-1}\left\{\frac{s^2}{(s-2)^3}\right\} = L^{-1}\left\{\frac{1}{s-2}\right\} + 4L^{-1}\left\{\frac{1}{(s-2)^2}\right\} + 4L^{-1}\left\{\frac{1}{(s-2)^3}\right\}$$

$$= e^{2t} + 4e^{2t}\,t + 2e^{2t}\,t^2. \quad \text{[using shifting property]}$$

(*ii*) $$\frac{s+3}{s^2-4s+13} = \frac{s-2}{(s-2)^2+3^2} + \frac{5}{(s-2)^2+3^2}$$

$$\therefore \quad L^{-1}\left\{\frac{s+3}{s^2-4s+13}\right\} = L^{-1}\left\{\frac{s-2}{(s-2)^2+3^2}\right\} + \frac{5}{3}L^{-1}\left\{\frac{3}{(s-2^2)+3^2}\right\}$$

$$= e^{2t}\cos 3t + \frac{5}{3} e^{2t}\sin 3t.$$ [Using shifting property]

(iii) $$L^{-1}\frac{(s+2)^2}{(s^2+4s+8)^2} = L^{-1}\frac{(s+2)^2}{(s^2+4s+4+4)^2} = L^{-1}\frac{(s+2)^2}{[(s+2)^2+4]^2}$$

$$= e^{-2t}L^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\} = e^{-2t}L^{-1}\left\{\frac{s^2+4-4}{(s^2+4)^2}\right\}$$

$$= e^{-2t}L^{-1}\left\{\frac{1}{s^2+4} - \frac{4}{(s^2+4)^2}\right\} = \frac{e^{-2t}\sin 2t}{2} - 4e^{-2t}L^{-1}\left\{\frac{1}{(s^2+4)^2}\right\}$$

$$= \frac{e^{-2t}\sin 2t}{2} - 4e^{-2t}\left\{\frac{1}{4}\left(\frac{\sin 2t}{4} - \frac{t\cos 2t}{2}\right)\right\}$$

$$= e^{-2t}\left\{\frac{\sin 2t}{2} - \frac{\sin 2t}{4} + \frac{t\cos 2t}{2}\right\} = e^{-2t}\left\{\frac{\sin 2t}{4} + \frac{t\cos 2t}{2}\right\}.$$

Example 21.21. *Find the inverse transform of (i) $1/s\,(s^2+a^2)$* *(P.T.U., 2003)*
(ii) $1/s\,(s+a)^3$.

Solution. (*i*) Since $L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at.$

therefore, by formula III above,

$$L^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\} = \int_0^t \frac{1}{a}\sin at\,dt = \frac{1}{a^2}[-\cos at]_0^t = (1-\cos at)/a^2$$

(*ii*) $$L^{-1}\left\{\frac{1}{s(s+a)^3}\right\} = L^{-1}\left\{\frac{1}{[(s+a)-a](s+a)^3}\right\} = e^{-at}L^{-1}\left\{\frac{1}{(s-a)s^3}\right\}$$

Now $$L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at} \quad \therefore \quad L^{-1}\left\{\frac{1}{(s-a)s}\right\} = \int_0^t e^{at}\,dt = \frac{e^{at}}{a} - \frac{1}{a}$$, by III above

$\therefore$ $$L^{-1}\left\{\frac{1}{(s-a)s^2}\right\} = \frac{1}{a}\int_0^t (e^{at}-1)\,dt = \frac{1}{a^2}(e^{at}-at-1)$$

$$L^{-1}\left\{\frac{1}{(s-a)s^3}\right\} = \frac{1}{a^2}\int_0^t (e^{at}-at-1)\,dt = \frac{1}{a^3}\left(e^{at} - \frac{a^2}{2}t^2 - at - 1\right)$$

Hence $$L^{-1}\left\{\frac{1}{s(s+a)^3}\right\} = e^{-at}\cdot\frac{1}{a^3}\left(e^{at} - \frac{a^2t^2}{2} - at - 1\right) = \frac{1}{a^3}\left(1 - e^{-at} - ate^{-at} - \frac{a^2}{2}t^2e^{-at}\right).$$

Example 21.22. *Find the inverse Laplace transforms of :*

(*i*) $\frac{s}{(s^2+a^2)^2}$ *(S.V.T.U., 2009)* (*ii*) $\frac{s^2}{(s^2+a^2)^2}$ *(Hazaribag, 2009)* (*iii*) $\frac{1}{(s^2+a^2)^2}$.

Solution. (*i*) If $f(t) = L^{-1}\frac{s}{(s^2+a^2)^2}$, then by formula V above,

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \frac{s}{(s^2+a^2)^2}\,ds = \frac{1}{2}\int_s^\infty \frac{2s}{(s^2+a^2)^2}\,ds = -\frac{1}{2}\left(\frac{1}{s^2+a^2}\right)_s^\infty = \frac{1}{2}\cdot\frac{1}{s^2+a^2}$$

$\therefore$ $$\frac{f(t)}{t} = \frac{1}{2}L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{\sin at}{2a}$$

Hence, $f(t) = \dfrac{1}{2a}\, t \sin at.$

Otherwise : Let $\quad f(t) = L^{-1}\left(\dfrac{1}{s^2 + a^2}\right) = \dfrac{\sin at}{a}$ so that $\bar{f}(s) = \dfrac{1}{s^2 + a^2}$

Then by (IV) above, $\quad t\, f(t) = L^{-1}\left\{-\dfrac{d}{ds}[\bar{f}(s)]\right\} = L^{-1}\left\{-\dfrac{d}{ds}\left(\dfrac{1}{s^2 + a^2}\right)\right\}$

or $$\frac{t \sin at}{a} = L^{-1}\left\{\frac{2s}{(s^2 + a^2)^2}\right\}. \text{ Hence } L^{-1}\left\{\frac{s}{(s^2 + a^2)^2}\right\} = \frac{1}{2a}\, t \sin at.$$

(*ii*) In (*i*), we have proved that

$$L^{-1}\left\{\frac{s}{(s^2 + a^2)^2}\right\} = \frac{1}{2a}\, t \sin at = f(t), \text{ say}$$

Since $f(0) = 0$, we get from formula II above, that

$$L^{-1}\left\{\frac{s^2}{(s^2 + a^2)^2}\right\} = L^{-1}\left\{s \cdot \frac{s}{(s^2 + a^2)^2}\right\} = \frac{d}{dt}\{f(t)\}$$

$$= \frac{d}{dt}\left(\frac{1}{2a}\, t \sin at\right) = \frac{1}{2a}\,(\sin at + at \cos at)$$

(*iii*) In (*i*), we have shown that

$$L^{-1}\left\{\frac{s}{(s^2 + a^2)^2}\right\} = \frac{1}{2a}\,(t \sin at) = f(t), \text{ say}$$

By formula III above, we have

$$L^{-1}\left\{\frac{1}{(s^2 + a^2)^2}\right\} = L^{-1}\left\{\frac{1}{s} \cdot \frac{s}{(s^2 + a^2)^2}\right\} = \int_0^t f(t)\, dt = \int_0^t \frac{t \sin at}{2a}\, dt$$

$$= \frac{1}{2a}\left\{\left| t \cdot \frac{-\cos at}{a}\right|_0^t - \int_0^t 1 \cdot \left(\frac{-\cos at}{a}\right) dt\right\}$$

$$= \frac{1}{2a}\left\{\frac{-t \cos at}{a} + \frac{\sin at}{a^2}\right\} = \frac{1}{2a^3}\,(\sin at - at \cos at).$$

Example 21.23. *Find the inverse Laplace transforms of*

(*i*) $\dfrac{s+2}{s^2(s+1)(s-2)}$ (*V.T.U., 2003*) (*ii*) $\dfrac{s+2}{(s^2+4s+5)^2}$. (*S.V.T.U., 2009 ; P.T.U., 2005*)

Solution. (*i*) $L^{-1}\left\{\dfrac{s+2}{(s+1)(s-2)}\right\} = \dfrac{4}{3} L^{-1}\left(\dfrac{1}{s-2}\right) - \dfrac{1}{3} L^{-1}\left(\dfrac{1}{s+1}\right) = \dfrac{4}{3} e^{2t} - \dfrac{1}{3} e^{-t}$

By III above, $L^{-1}\left\{\dfrac{s+2}{s(s+1)(s-2)}\right\} = \displaystyle\int_0^t L^{-1}\left(\dfrac{s+2}{(s+1)(s-2)}\right) dt$

$$= \int_0^t \left(\frac{4}{3}e^{2t} - \frac{1}{3}e^{-t}\right) dt = \frac{2}{3}e^{2t} + \frac{1}{3}e^{-t} - 1$$

Again by III above, $L^{-1}\dfrac{s+2}{s^2(s+1)(s-2)} = \displaystyle\int_0^t L^{-1}\left\{\dfrac{s+2}{s(s+1)(s-2)}\right\} dt$

$$= \int_0^t \left(\frac{2}{3}e^{2t} + \frac{1}{3}e^{-t} - 1\right) dt = \frac{1}{3}(e^{2t} - e^{-t} - t).$$

(ii) $$L^{-1}\left(\frac{1}{s^2+4s+5}\right) = L^{-1}\left\{\frac{1}{(s+2)^2+1}\right\} = e^{-2t}\sin t$$

By II above, $$L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+4s+5}\right)\right\} = (-1)^1 t \,.\, e^{-2t}\sin t$$

i.e., $$L^{-1}\left\{\frac{-(2s+4)}{(s^2+4s+5)^2}\right\} = -t\,.\,e^{-2t}\sin t$$

or $$L^{-1}\left\{\frac{s+2}{(s^2+4s+5)^2}\right\} = \frac{1}{2}t\,.\,e^{-2t}\sin t.$$

Example 21.24. *Find the inverse Laplace transforms of the following :*

(i) $\log\frac{s+1}{s-1}$ *(S.V.T.U., 2009 ; Bhopal, 2008)* *(ii)* $\log\frac{s^2+1}{s(s+1)}$ *(S.V.T.U., 2009 ; V.T.U., 2008)*

(iii) $\cot^{-1}\left(\frac{s}{2}\right)$ *(iv)* $\tan^{-1}\left(\frac{2}{s^2}\right)$. *(V.T.U., 2011 ; Mumbai, 2005 S)*

Solution. (i) If $f(t) = L^{-1}\log\frac{s+1}{s-1}$, then by IV above,

$$t\,f(t) = L^{-1}\left\{-\frac{d}{ds}\log\left(\frac{s+1}{s-1}\right)\right\} = -L^{-1}\left\{\frac{d}{ds}\log(s+1)\right\} + L^{-1}\left\{\frac{d}{ds}\log(s-1)\right\}$$

$$= -L^{-1}\left(\frac{1}{s+1}\right) + L^{-1}\left(\frac{1}{s-1}\right) = -e^{-t}+e^{t} = 2\sinh t$$

Thus $f(t) = (2\sinh t)/t.$

(ii) If $f(t) = L^{-1}\log\frac{s^2+1}{s(s+1)}$, then by IV above,

$$t\,f(t) = L^{-1}\left\{-\frac{d}{ds}\log\left(\frac{s^2+1}{s(s+1)}\right)\right\} = -L^{-1}\left\{\frac{d}{ds}\log(s^2+1)\right\} + L^{-1}\left\{\frac{d}{ds}\log s\right\} + L^{-1}\left\{\frac{d}{ds}\log(s+1)\right\}$$

$$= -L^{-1}\left(\frac{2s}{s^2+1}\right) + L^{-1}\left(\frac{1}{s}\right) + L^{-1}\left(\frac{1}{s+1}\right) = -2\cos t + 1 + e^{-t}$$

Thus $f(t) = \frac{1}{t}(1+e^{-t}-2\cos t).$

(iii) If $f(t) = L^{-1}\cot^{-1}\left(\frac{s}{2}\right)$, then by IV above,

$$t\,f(t) = L^{-1}\left\{-\frac{d}{ds}\cot^{-1}\left(\frac{s}{2}\right)\right\} = L^{-1}\left(\frac{2}{s^2+2^2}\right) = \sin 2t$$

Thus $f(t) = (\sin 2t)/t.$

(iv) If $f(t) = L^{-1}\left(\tan^{-1}\frac{2}{s^2}\right)$, then by IV above,

$$t\,f(t) = L^{-1}\left\{-\frac{d}{ds}\tan^{-1}\left(\frac{2}{s^2}\right)\right\} = L^{-1}\left\{\frac{4s}{s^4+4}\right\}$$

$$= L^{-1}\left\{\frac{4s}{(s^2+2)^2-(2s)^2}\right\} = L^{-1}\left\{\frac{4s}{(s^2+2+2s)(s^2+2-2s)}\right\}$$

$$= L^{-1}\left\{\frac{1}{s^2-2s+2} - \frac{1}{s^2+2s+2}\right\} = L^{-1}\left\{\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right\}$$

$$= e^t \sin t - e^{-t}\sin t = 2\sinh t \sin t.$$

21.14 CONVOLUTION THEOREM

If $L^{-1}\{\bar{f}(s)\} = f(t)$, *and* $L^{-1}\{\bar{g}(s)\} = g(t)$,

then $$L^{-1}\{\bar{f}(s)\,\bar{g}(s)\} = \int_0^t f(u)\,g(t-u)\,du = F * G$$

[*F* * *G is called the* **convolution** *or* **falting** *of F and G.*]

Let $$\phi(t) = \int_0^t f(u)\,g(t-u)\,du$$

$$L\{\phi(t)\} = \int_0^\infty e^{-st}\left\{\int_0^t f(u)\,g(t-u)\,du\right\}dt = \int_0^\infty \int_0^t e^{-st} f(u)\,g(t-u)\,du\,dt \qquad ...(1)$$

The domain of integration for this double integral is the entire area lying between the lines $u = 0$ and $u = t$ (Fig. 21.2).

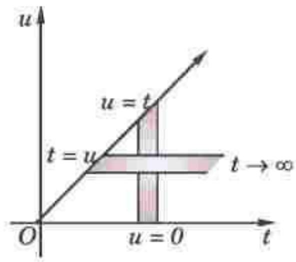


Fig. 21.2

On changing the order of integration, we get

$$L\{\phi(t)\} = \int_0^\infty \int_u^\infty e^{-st} f(u)\,g(t-u)\,dt\,du$$

$$= \int_0^\infty e^{-su} f(u)\left\{\int_0^\infty e^{-s(t-u)}\,g(t-u)\,dt\right\}du$$

$$= \int_0^\infty e^{-su} f(u)\left\{\int_0^\infty e^{-sv}\,g(v)\,dv\right\}du \text{ on putting } t-u=v$$

$$= \int_0^\infty e^{-su} f(u)\,\bar{g}(s)\,du = \int_0^\infty e^{-su} f(u)\,du \,.\, \bar{g}(s)$$

$$= \bar{f}(s)\,.\,\bar{g}(s) \quad \text{whence follows the desired result.}$$

Example 21.25. *Apply Convolution theorem to evaluate*

(i) $L^{-1}\dfrac{s}{(s^2+a^2)^2}$. *(V.T.U., 2010)*

(ii) $L^{-1}\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$. *(V.T.U., 2011 S ; Bhopal, 2008 ; Mumbai, 2007)*

Solution. (i) Since $f(t) = L^{-1}\left(\dfrac{s}{s^2+a^2}\right) = \cos at$ and $g(t) = L^{-1}\left(\dfrac{s}{s^2+a^2}\right) = \dfrac{1}{a}\sin at$

$\therefore$ by Convolution theorem, we get

$$L^{-1}\left[\frac{s}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right] = \int_0^t \cos au\,\frac{\sin a(t-u)}{a}\,du \qquad \left[\because \begin{array}{l} f(u) = \cos au \\ g(t-u) = \frac{1}{a}\sin a(t-u)\end{array}\right.$$

$$= \frac{1}{2a}\int_0^t [\sin at - \sin(2au - at)]dt = \frac{1}{2a}\left| u\sin at + \frac{1}{2a}\cos(2au-at)\right|_0^t = \frac{1}{2a}\,t\sin at$$

Hence $$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{1}{2a}\,t\sin at.$$

(*ii*) Since $f(t) = L^{-1}\left(\dfrac{s}{s^2+a^2}\right) = \cos at$ and $g(t) = L^{-1}\left(\dfrac{s}{s^2+b^2}\right) = \cos bt$,

$\therefore$ by Convolution theorem, we get

$$L^{-1}\left(\frac{s}{s^2+a^2}\cdot\frac{s}{s^2+b^2}\right) = \int_0^t \cos au \cos b(t-u)\,du \qquad [\because f(u) = \cos au, g(t-u) = \cos b(t-u)]$$

$$= \frac{1}{2}\int_0^t \{\cos[(a-b)u + bt] + \cos[(a+b)u - bt]\}\,du$$

$$= \frac{1}{2}\left|\frac{\sin[(a-b)u+bt]}{a-b} + \frac{\sin[(a+b)u-bt]}{a+b}\right|_0^t$$

$$= \frac{1}{2}\left\{\frac{\sin at - \sin bt}{a-b} + \frac{\sin at + \sin bt}{a+b}\right\} = \frac{a\sin at - b\sin bt}{a^2-b^2}.$$

Example 21.26. *Evaluate* (*i*) $L^{-1}\dfrac{1}{(s^2+1)(s^2+9)}$ (*Mumbai, 2005 S*)

(*ii*) $L^{-1}\dfrac{s}{(s^2+1)(s^2+4)(s^2+9)}$. (*Madras, 2006*)

Solution. (*i*) Since $L^{-1}\left(\dfrac{1}{s^2+1}\right) = \sin t$, $L^{-1}\left(\dfrac{1}{s^2+9}\right) = \dfrac{\sin 3t}{3}$

$\therefore$ by Convolution theorem, we get

$$L^{-1}\left(\frac{1}{s^2+1}\cdot\frac{1}{s^2+9}\right) = \int_0^t \sin u \,.\, \frac{\sin 3(t-u)}{3}du$$

$$= \frac{1}{6}\int_0^t [\cos(4u-3t) - \cos(3t-2u)]\,du = \frac{1}{6}\left|\frac{\sin(4u-3t)}{4} - \frac{\sin(3t-2u)}{-2}\right|_0^t$$

$$= \frac{1}{6}\left\{\frac{1}{4}(\sin t + \sin 3t) + \frac{1}{2}(\sin t - \sin 3t)\right\} = \frac{1}{8}\left(\sin t - \frac{1}{3}\sin 3t\right)$$

(*ii*) Since $L^{-1}\left(\dfrac{s}{s^2+4}\right) = \cos 2t$ and $L^{-1}\left\{\dfrac{1}{(s^2+1)(s^2+9)}\right\} = \dfrac{1}{8}\left[\sin t - \dfrac{1}{3}\sin 3t\right]$ [By (*i*)]

$\therefore$ by Convolution theorem, we get

$$L^{-1}\frac{s}{(s^2+1)(s^2+4)(s^2+9)} = L^{-1}\left\{\frac{1}{(s^2+1)(s^2+9)}\cdot\frac{s}{s^2+4}\right\}$$

$$= \int_0^t \frac{1}{8}\left(\sin u - \frac{1}{3}\sin 3u\right).\cos 2(t-u)\,du$$

$$= \frac{1}{8}\int_0^t \left[\sin u\cos 2(t-u) - \frac{1}{3}\sin 3u\cos 2(t-u)\right]du$$

$$= \frac{1}{8}\int_0^t \left[\frac{1}{2}\{\sin(2t-u) - \sin(3u-2t)\} - \frac{1}{6}\{\sin(u+2t) - \sin(5u-2t)\}\right]du$$

$$= \frac{1}{16}\left[\left|\frac{-\cos(2t-u)}{-1} + \frac{\cos(3u-2t)}{3}\right|_0^t\right] - \frac{1}{48}\left[\left|-\cos(u+2t) + \frac{\cos(5u-2t)}{5}\right|_0^t\right]$$

$$= \frac{1}{12}\cos t - \frac{1}{10}\cos 2t + \frac{1}{60}\cos 3t.$$

PROBLEMS 21.5

Find the inverse transforms of :

1. $\dfrac{1}{s^2(s+5)}$. *(Madras, 2003 S)*
2. $\dfrac{1}{s(s+2)^3}$.
3. $\dfrac{s}{a^2s^2+b^2}$. *(Madras, 2000 S)*
4. $\dfrac{1}{s^2(s^2+a^2)}$.
5. $\dfrac{1}{s^3(s^2+1)}$.
6. $\dfrac{s+2}{(s^2+4s+8)^2}$. *(Mumbai, 2006)*
7. $\dfrac{2as}{(s^2+a^2)^2}$.
8. $\dfrac{s^2}{(s+a)^3}$.
9. $\log\left(\dfrac{1+s}{s}\right)$.
10. $\log\left(\dfrac{s+a}{s+b}\right)$. *(Anna, 2003 ; U.P.T.U., 2003)*
11. $\log\left\{\dfrac{s+1}{(s+2)(s+3)}\right\}$.
12. $\dfrac{1}{2}\log\left(\dfrac{s^2+b^2}{s^2+a^2}\right)$. *(Mumbai, 2008 ; V.T.U., 2008)*
13. $\log\left(1-\dfrac{a^2}{s^2}\right)$.
14. $\log\dfrac{s^2+1}{(s-1)^2}$. *(Madras, 2000 S)*
15. $\tan^{-1}\left(\dfrac{2}{s}\right)$ *(Mumbai, 2007 ; P.T.U., 2005)*
16. $\cot^{-1}(s)$. *(V.T.U., 2005)*
17. $s\log\dfrac{s-1}{s+1}$ *(Madras, 1999)*

Using Convolution theorem, evaluate :

18. $L^{-1}\left\{\dfrac{1}{(s+a)(s+b)}\right\}$.
19. $L^{-1}\dfrac{1}{(s^2+a^2)^2}$.
20. $L^{-1}\dfrac{1}{s^2(s^2+a^2)}$.
21. $L^{-1}\left\{\dfrac{1}{s^2(s+1)^2}\right\}$.
22. $L^{-1}\left\{\dfrac{1}{(s-2)(s+2)^2}\right\}$. *(Mumbai, 2009)*
23. $L^{-1}\left\{\dfrac{s}{(s+2)(s^2+9)}\right\}$. *(V.T.U., 2008 S)*
24. $\dfrac{1}{s^3(s^2+1)}$. *(V.T.U., 2007 ; U.P.T.U., 2005)*
25. $\dfrac{1}{(s^2+4s+13)^2}$. *(Mumbai, 2008)*
26. Show that (*i*) $L^{-1}\left(\dfrac{1}{s}\sin\dfrac{1}{s}\right)=t-\dfrac{t^3}{(3!)^2}+\dfrac{t^5}{(5!)^2}-\dfrac{t^7}{(7!)^2}+\dots$

    (*ii*) $L^{-1}\left(\dfrac{1}{s}\cos\dfrac{1}{s}\right)=1-\dfrac{t^2}{(2!)^2}+\dfrac{t^4}{(4!)^2}-\dfrac{t^6}{(6!)^2}+\dots$

21.15 (1) APPLICATION TO DIFFERENTIAL EQUATIONS

The Laplace transform method of solving differential equations yields particular solutions without the necessity of first finding the general solution and then evaluating the arbitrary constants. This method is, in general, shorter than our earlier methods and is specially useful for solving linear differential equations with constant coefficients.

(2) Working procedure *to solve a linear differential equation with constant coefficients by transform method :*

1. Take the Laplace transform of both sides of the differential equation using the formula of § 21.7, and the given initial conditions.

2. Transpose the terms with minus signs to the right.

3. Divide by the coefficient of $\bar{y}$, getting $\bar{y}$ as a known function of s.

4. Resolve this function of s into partial fractions and take the inverse transform of both sides. This gives y as a function of t which is the desired solution satisfying the given conditions.

Example 21.27. *Solve by the method of transforms, the equation*
$$y''' + 2y'' - y' - 2y = 0 \text{ given } y(0) = y'(0) = 0 \text{ and } y''(0) = 6.$$ (V.T.U., 2011 S)

Solution. Taking the Laplace transform of both sides, we get
$$[s^3\bar{y} - s^2y(0) - sy'(0) - y''(0)] + 2[s^2\bar{y} - sy(0) - y'(0)] - [s\bar{y} - y(0)] - 2\bar{y} = 0$$
Using the given conditions, it reduces to
$$(s^3 + 2s^2 - s - 2)\bar{y} = 6$$
$$\therefore \qquad \bar{y} = \frac{6}{(s-1)(s+1)(s+2)} = \frac{6}{(s-1)(6)} + \frac{6}{(-2)(s+1)} + \frac{6}{3(s+2)}$$
On inversion, we get $\quad y = L^{-1}\frac{1}{(s-1)} - 3L^{-1}\frac{1}{(s+2)} + 2L^{-1}\left(\frac{1}{s+2}\right)$

or $\quad y = e^t - 3e^{-t} + 2e^{-2t}$ which is the desired result.

Example 21.28. *Use transform method to solve*
$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t \text{ with } x = 2, \frac{dx}{dt} = -1 \text{ at } t = 0.$$ (Anna, 2005 S)

Solution. Taking the Laplace transforms of both sides, we get
$$[s^2\bar{x} - sx(0) - x'(0)] - 2\{s\bar{x} - x(0)\} + \bar{x} = \frac{1}{s-1}$$
Using the given conditions, it reduces to
$$(s^2 - 2s + 1)\bar{x} = \frac{1}{s-1} + 2s - 5 = \frac{2s^2 - 7s + 6}{s-1}$$
$$\therefore \qquad \bar{x} = \frac{2s^2 - 7s + 6}{(s-1)^3} = \frac{2}{s-1} - \frac{3}{(s-1)^2} + \frac{1}{(s-1)^3}$$ on breaking into partial fractions.

On inversion, we obtain $x = 2L^{-1}\left(\frac{1}{s-1}\right) - 3L^{-1}\frac{1}{(s-1)^2} + L^{-1}\frac{1}{(s-1)^3}$
$$= 2e^t - \frac{3e^t \cdot t}{1!} + \frac{e^t \cdot t^2}{2!} = 2e^t - 3te^t + \tfrac{1}{2}t^2e^t.$$

Example 21.29. *Solve* $(D^2 + n^2)x = a \sin(nt + \alpha)$, $x = Dx = 0$ *at* $t = 0$.

Solution. Taking the Laplace transforms of both sides, we get
$$[s^2\bar{x} - sx(0) - x'(0)] + n^2\bar{x} = aL\{\sin nt \cdot \cos\alpha + \cos nt \cdot \sin\alpha\}$$
On using the given conditions,
$$(s^2 + n^2)\bar{x} = a\cos\alpha \cdot \frac{n}{s^2 + n^2} + a\sin\alpha \cdot \frac{s}{s^2 + n^2}$$
$$\therefore \qquad \bar{x} = an\cos\alpha \cdot \frac{1}{(s^2 + n^2)^2} + a\sin\alpha \cdot \frac{s}{(s^2 + n^2)^2}$$
On inversion, we obtain
$$x = an\cos\alpha \cdot \frac{1}{2n^3}(\sin nt - nt\cos nt) + a\sin\alpha \cdot \frac{t}{2n}\sin nt$$ [By (11) and (12) p. 741]
$$= a\{\sin nt\cos\alpha - nt\cos(nt + \alpha)\}/2n^2.$$

Example 21.30. *Solve* $(D^3 - 3D^2 + 3D - 1)y = t^2e^t$ *given that* $y(0) = 1$, $y'(0) = 0$, $y''(0) = -2$.
(S.V.T.U., 2009)

Solution. Taking the Laplace transforms of both sides, we get

$$[s^3 \bar{y} - s^2 y(0) - sy'(0) - y''(0)] - 3[s^2 \bar{y} - sy(0) - y'(0)] + 3[s\bar{y} - y(0)] - \bar{y} = \frac{2}{(s-1)^3}$$

Using the given conditions, it reduces to

$$\bar{y} = \frac{s^2 - 3s + 1}{(s-1)^3} + \frac{2}{(s-1)^6} = \frac{(s-1)^2 - (s-1) - 1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$= \frac{1}{s-1} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

On inversion, we obtain $y = L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\frac{1}{(s-1)^2} - L^{-1}\frac{1}{(s-1)^3} + 2L^{-1}\frac{1}{(s-1)^6}$

$$= e^t\left(1 - t - \frac{1}{2}t^2 + \frac{1}{60}t^5\right).$$

Example 21.31. *Solve* $\frac{d^2x}{dt^2} + 9x = \cos 2t$, *if* $x(0) = 1$, $x(\pi/2) = -1$. *(Bhopal, 2008 ; U.P.T.U., 2006)*

Solution. Since $x'(0)$ is not given, we assume $x'(0) = a$.

Taking the Laplace transforms of both sides of the equation, we have

$$L(x'') + 9L(x) = L(\cos 2t) \text{ i.e., } [s^2\bar{x} - s\,x(0) - x'(0)] + 9\bar{x} = \frac{s}{s^2+4}$$

$$(s^2+9)\,\bar{x} = s + a + \frac{s}{s^2+4} \quad \text{or} \quad \bar{x} = \frac{s+a}{s^2+9} + \frac{s}{(s^2+4)(s^2+9)}$$

or $$\bar{x} = \frac{a}{s^2+9} + \frac{1}{5}\cdot\frac{s}{s^2+4} + \frac{4}{5}\cdot\frac{s}{s^2+9}.$$

On inversion, we get $\quad x = \frac{a}{3}\sin 3t + \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t$

When $t = \pi/2$, $-1 = -\frac{a}{3} - \frac{1}{5}$ or $\frac{a}{3} = \frac{4}{5}$ $\quad \left[\because\ x\left(\frac{\pi}{2}\right) = -1\right]$

Hence the solution is $x = \frac{1}{5}(\cos 2t + 4\sin 3t + 4\cos 3t)$.

Obs. *Laplace transform method can also be used for solving ordinary differential equations with variable coefficients of the form* $t^m y^{(n)}(t)$ *because* $L[t^m y^{(n)}(t)] = (-1)^m \frac{d^m}{ds^m}[L\,y^{(n)}(t)]$.

Example 21.32. *Solve* $ty'' + 2y' + ty = \cos t$ *given that* $y(0) = 1$. *(S.V.T.U., 2009)*

Solution. Taking Laplace transform of both sides of the equation and noting that

$$L[t\{f(t)\}] = -\frac{d}{ds}[L\{f(t)\}], \text{ we get}$$

$$-\frac{d}{ds}[s^2\bar{y} - sy(0) - y'(0)] + 2[s\bar{y} - y(0)] - \frac{d}{ds}(\bar{y}) = \frac{s}{s^2+1}$$

or $$-\left(s^2\frac{d\bar{y}}{ds} + 2s\bar{y}\right) + y(0) + 0 + 2s\bar{y} - 2y(0) - \frac{d}{ds}(\bar{y}) = \frac{s}{s^2+1}$$

or $$(s^2+1)\frac{d\bar{y}}{ds} + 1 = -\frac{s}{s^2+1} \quad \text{or} \quad \frac{d\bar{y}}{ds} = -\frac{1}{s^2+1} - \frac{s}{(s^2+1)^2}$$

On inversion and noting that $L^{-1}\{\bar{f}'(s)\} = -t\,f(t)$, we get

$$-ty = -\sin t - \tfrac{1}{2} t \sin t \qquad \text{[See § 21.12 (11)]}$$

or $$y = \frac{1}{2}\left(1 + \frac{2}{t}\right) \sin t$$ which is the desired solution.

Example 21.33. *Solve* $x \dfrac{d^2 y}{dx^2} + \dfrac{dy}{dx} + xy = 0,\ y(0) = 2,\ y'(0) = 0.$

Solution. Taking Laplace transform of both sides of the equation, we get

$$L(xy'') + L(y') + L(xy) = 0$$

or $$-\frac{d}{ds}[s^2 \bar{y} - sy(0) - y'(0)] + [s\bar{y} - y(0)] - \frac{d\bar{y}}{ds} = 0 \quad \text{or} \quad (s^2 + 1)\frac{d\bar{y}}{ds} + s\bar{y} = 0$$

Separating the variables, $\displaystyle\int \frac{d\bar{y}}{\bar{y}} + \int \frac{s\,ds}{s^2 + 1} = c$

or $$\log \bar{y} + \tfrac{1}{2} \log(s^2 + 1) = \log c' \quad \text{or} \quad \bar{y} = \frac{c'}{\sqrt{(s^2 + 1)}}$$

Inversion gives $\quad y = c' J_0(x)$

To find c', we have $\quad y(0) = c' J_0(0)$, *i.e.*, $c' = 2$

Hence $\quad y = 2J_0(x)$.

Example 21.34. *An alternating e.m.f. E sin ωt is applied to an inductance L and a capacitance C in series. Show by transform method, that the current in the circuit is* $\dfrac{E\omega}{(p^2 - \omega^2) L}(\cos \omega t - \cos pt)$, *where* $p^2 = 1/LC$.

Solution. If i be a current and q the charge at time t in the circuit, then its differential equation is

$$L\frac{di}{dt} + \frac{q}{c} = E \sin \omega t \qquad [\because \ R = 0]$$

Taking Laplace transform of both sides, we get

$$L[s\bar{i}(s) - i(0)] + \frac{1}{C}L(q) = E \cdot \frac{\omega}{s^2 + \omega^2}$$

Since $i = 0$ and $q = 0$ at $t = 0$

$$\therefore \quad L s\bar{i}(s) + \frac{1}{C}L(q) = \frac{E\omega}{s^2 + \omega^2} \qquad ...(i)$$

Also taking Laplace transform of $i = dq/dt$, we get

$$\bar{i}(s) = L(dq/dt) = s\,L(q) - q(0)$$

i.e. $$L(q) = \bar{i}(s)/s \qquad [\because \ q(0) = 0]$$

$\therefore$ (*i*) becomes $L s\bar{i}(s) + \dfrac{1}{C}[\bar{i}(s)/s] = \dfrac{E\omega}{s^2 + \omega^2}$

or $$\left(Ls + \frac{1}{Cs}\right)\bar{i}(s) = \frac{E\omega}{s + \omega^2} \quad \text{or} \quad \bar{i}(s) = \frac{E\omega s}{L(s^2 + 1/LC)(s^2 + \omega^2)}$$

or $$\bar{i}(s) = \frac{E\omega}{L} \cdot \frac{s}{(s^2 + p^2)(s^2 + \omega^2)} \quad \text{where } p^2 = 1/LC$$

$$\bar{i}(s) = \frac{E\omega}{L(p^2 - \omega^2)}\left\{\frac{s}{s^2 + \omega^2} - \frac{s}{s^2 + p^2}\right\}$$

Now taking inverse Laplace transform of both sides, we get

$$i(t) = \frac{E\omega}{L(p^2 - \omega^2)} L^{-1} \left\{ \frac{s}{s^2 + \omega^2} - \frac{s}{(s^2 + p^2)} \right\}$$

or

$$i(t) = \frac{E\omega}{L(p^2 - \omega^2)} (\cos \omega t - \cos pt).$$

PROBLEMS 21.6

Solve the following equations by the transform method :

1. $y'' + 4y' + 3y = e^{-t}, y(0) = y'(0) = 1.$ *(V.T.U., 2008 S ; Kurukshetra, 2005)*
2. $(D^2 - 1)x = a \cosh t, x(0) = x'(0) = 0.$
3. $y'' + y = t, y(0) = 1, y'(0) = 0.$ *(Mumbai, 2009)*
4. $y'' - 3y' + 2y = e^{3t}$, when $y(0) = 1$ and $y'(0) = 0$. *(V.T.U., 2010)*
5. $(D^2 - 3D + 2)y = 4e^{2t}$ with $y(0) = -3, y(0) = 5.$ *(Mumbai, 2008)*
6. $y'' + 25y = 10 \cos 5t$ given that $y(0) = 2, y'(0) = 0.$ *(S.V.T.U., 2008)*
7. $(D^2 + \omega^2)y = \cos \omega t, t > 0$, given that $y = 0$ and $Dy = 0$ at $t = 0$.
8. $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} - 3y = \sin t, y = \dfrac{dy}{dt} = 0$ when $t = 0$. *(Kurukshetra, 2005 ; Madras, 2003)*
9. $\dfrac{d^4y}{dt^4} - k^4y = 0$, where $y(0) = 1, y'(0) = y''(0) = y'''(0) = 0.$
10. $y''''(t) + 2y''(t) + y(t) = \sin t$, when $y(0) = y'(0) = y''(0) = y'''(0) = 0.$
11. $\dfrac{d^2y}{dt^2} + 5\dfrac{dy}{dt} + 5y = e^{-t} \sin t$, where $y(0) = 0$ and $y'(0) = 1$. *(P.T.U., 2010)*
12. $y'' + 2y' + 5y = 5y = 5(t - 2), y(0) = 0, y'(0) = 0.$ *(P.T.U., 2005 S)*
13. $\dfrac{d^3y}{dt^3} - 3\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} - y = t^2 e^{2t}$, where $y = 1, \dfrac{dy}{dt} = 0, \dfrac{d^2y}{dt^2} = -2$ at $t = 0$.
14. $(D^2 + 1)x = t \cos 2t, x = Dx = 0$ at $t = 0$. *(Raipur, 2005 ; U.P.T.U., 2005)*
15. $ty'' + 2y' + ty = \sin t$, when $y(0) = 1$.
16. $ty'' + (1 - 2t)y' - 2y = 0$, when $y(0) = 1, y'(0) = 2.$ *(P.T.U., 2002)*
17. $y'' + 2ty' - y = t$, when $y(0) = 0, y'(0) = 1.$ *(U.P.T.U., 2003)*
18. $ty'' + y' + 4ty = 0$ when $y(0) = 3, y'(0) = 0.$
19. A voltage Ee^{-at} is applied at $t = 0$ to a circuit of inductance L and resistance R. Show (by the transform method) that the current at time t is $\dfrac{E}{R - aL}(e^{-at} - e^{-Rt/L})$. *(V.T.U., 2000)*
20. Workout example 12.17, p. 465 by the transform method.
21. Obtain the equation for the forced oscillation of a mass m attached to the lower end of an elastic spring whose upper end is fixed and whose stiffness is k, when the driving force is $F_0 \sin at$. Solve this equation (using the Laplace transforms) when $a^2 \neq k/m$, given that initial velocity and displacement (from equilibrium position) are zero.

   $\left[\textbf{Hint :} \text{ The equation of motion is } \dfrac{d^2x}{dt^2} + \dfrac{k}{m}x = \dfrac{F_0}{m} \sin at \text{ and } x = \dfrac{dx}{dt} = 0 \text{ when } t = 0.\right]$

21.16 SIMULTANEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

The Laplace transform method can also be applied with advantage to the solution of simultaneous linear differential equations.

Example 21.35. *Solve the simultaneous equations* $\dfrac{dx}{dt} + 5x - 2y = t, \dfrac{dy}{dt} + 2x + y = 0$ *being given* $x = y = 0$ *when* $t = 0$. [Ex. 13.38]

Solution. Taking the Laplace transforms of the given equations, we get

$$[s\bar{x} - x(0)] + 5\bar{x} - 2\bar{y} = 1/s^2 \quad i.e., \quad (s+5)\bar{x} - 2\bar{y} = 1/s^2 \qquad ...(i) \; [\because \; x(0) = 0]$$

and $$s\bar{y} - y(0) + 2\bar{x} + \bar{y} = 0 \quad i.e., \quad 2\bar{x} + (s+1)\bar{y} = 0 \qquad ...(ii) \; [\because \; y(0) = 0]$$

Solving (*i*) and (*ii*) for $\bar{x}$, we get

$$\bar{x} = \begin{vmatrix} 1/s^2 & -2 \\ 0 & s+1 \end{vmatrix} \div \begin{vmatrix} s+5 & -2 \\ 2 & s+1 \end{vmatrix} = \frac{s+1}{s^2(s+3)^2} = \frac{1}{27s} + \frac{1}{9s^2} - \frac{1}{27(s+3)} - \frac{2}{9(s+3)^2}$$

Substituting the value of $\bar{x}$ in (*ii*), we get

$$\bar{y} = -\frac{2}{s^2(s+3)^2} = \frac{4}{27s} - \frac{2}{9s^2} - \frac{4}{27(s+3)} - \frac{2}{9(s+3)^2}$$

On inversion, we get

$$x = \frac{1}{27} + \frac{t}{9} - \frac{1}{27}e^{-3t} - \frac{2}{9}te^{-3t}, \; y = \frac{4}{27} - \frac{2t}{9} - \frac{4}{27}e^{-3t} - \frac{2}{9}te^{-3t}.$$

Example 21.36. *The coordinates (x, y) of a particle moving along a plane curve at any time t, are given by $dy/dt + 2x = \sin 2t$, $dx/dt - 2y = \cos 2t$, $(t > 0)$. If at $t = 0$, $x = 1$ and $y = 0$, show by transforms, that the particle moves along the curve $4x^2 + 4xy + 5y^2 = 4$.* (U.P.T.U., 2003)

Solution. Taking the Laplace transforms of the given equations and noting that $y(0) = 0$, $x(0) = 1$,

we get $$[s\bar{y} - y(0)] + 2\bar{x} = \frac{2}{s^2+2^2} \quad \text{or} \quad 2\bar{x} + s\bar{y} = \frac{2}{s^2+4} \qquad ...(i)$$

and $$[s\bar{x} - x(0)] - 2\bar{y} = \frac{s}{s^2+2^2} \quad \text{or} \quad s\bar{x} - 2\bar{y} = \frac{s}{s^2+4} + 1 \qquad ...(ii)$$

Multiplying (*i*) by s and (*ii*) by 2 and subtracting, we get

$$(s^2+4)\bar{y} = -2 \quad \text{or} \quad \bar{y} = -2/(s^2+4)$$

On inversion, $$y = -2L^{-1}\left[\frac{1}{s^2+4}\right] = -\sin 2t$$

From the given first equation,

$$2x = \sin 2t - dy/dt = \sin 2t - \frac{d}{dt}(-\sin 2t)$$

or $$2x = \sin 2t + 2\cos 2t \quad \text{or} \quad 4x^2 = (\sin 2t + 2\cos 2t)^2 \qquad ...(iii)$$

Also $$4xy = (\sin 2t + 2\cos 2t)(-2\sin 2t) = -2(\sin^2 2t + 2\sin 2t\cos 2t) \qquad ...(iv)$$

and $$5y^2 = 5\sin^2 2t. \qquad ...(v)$$

Adding (*iii*), (*iv*), and (*v*), we obtain

$$4x^2 + 4xy + 5y^2 = \sin^2 2t + 4\sin 2t\cos 2t + 4\cos^2 2t - 2\sin^2 2t$$
$$- 4\sin 2t\cos 2t + 5\sin^2 2t = 4\sin^2 2t + 4\cos^2 2t = 4.$$

Example 21.37. *The small oscillations of a certain system with two degrees of freedom are given by the equations : $D^2x + 3x - 2y = 0$, $D^2x + D^2y - 3x + 5y = 0$ where $D = d/dt$. If $x = 0$, $y = 0$, $x = 3$, $y = 2$ when $t = 0$, find x and y when $t = 1/2$.* [Example 13.41]

Solution. Taking the Laplace transform of both the equations, we get

$$[s^2\bar{x} - sx(0) - x'(0)] + 3\bar{x} - 2\bar{y} = 0 \quad i.e., \quad (s^2+3)\bar{x} - 2\bar{y} = 3 \qquad ...(i)$$

and $$[s^2\bar{x} - sx(0) - x'(0)] + [s^2\bar{y} - sy(0) - y'(0)] - 3\bar{x} + 5\bar{y} = 0 \; i.e., (s^2-3)\bar{x} + (s^2+5)\bar{y} = 5 \qquad ...(ii)$$

Solving (*i*) and (*ii*) for $\bar{x}$ and $\bar{y}$, we get

$$\bar{x} = \begin{vmatrix} 3 & -2 \\ 5 & s^2+5 \end{vmatrix} \div \begin{vmatrix} s^2+3 & -2 \\ s^2-3 & s^2+5 \end{vmatrix} = \frac{3s^2+25}{(s^2+1)(s^2+9)}$$
$$= \frac{11}{4}\cdot\frac{1}{s^2+1} + \frac{1}{4}\cdot\frac{1}{s^2+9}$$

and $$\bar{y} = \begin{vmatrix} s^2+3 & 3 \\ s^2-3 & 5 \end{vmatrix} \div \begin{vmatrix} s^2+3 & -2 \\ s^2-3 & s^2+5 \end{vmatrix} = \frac{2s^2+24}{(s^2+1)(s^2+9)} = \frac{11}{4} \cdot \frac{1}{s^2+1} + \frac{3}{4} \cdot \frac{1}{s^2+9}.$$

On inversion, we get $x = \frac{11}{4} \sin t + \frac{1}{12} \sin 3t$; $y = \frac{11}{4} \sin t - \frac{1}{4} \sin 3t$

which are the same as the solution in (*vii*) on p. 499.

Obs. The student should compare the earlier solutions of the above examples with those given now and appreciate the superiority of the transform method over others.

PROBLEMS 21.7

Solve the following simultaneous equations (by using Laplace transforms):

1. $\frac{dx}{dt} - y = e^t$, $\frac{dy}{dt} + x = \sin t$, given $x(0) = 1$, $y(0) = 0$. (*U.P.T.U., 2006 ; Delhi, 2002*)
2. $\frac{dx}{dt} + y = \sin t$, $\frac{dy}{dt} + x = \cos t$, given that $x = 2$ and $y = 0$ when $t = 0$. (*Kerala, 2005 ; U.P.T.U., 2004*)
3. $\frac{d^2x}{dt^2} - x = y$, $\frac{d^2y}{dt^2} + y = -x$, given that at $t = 0$; $x = 2$, $y = -1$, $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$. (*P.T.U., 2009 S*)
4. $3\frac{dx}{dt} + \frac{dy}{dt} + 2x = 1$, $\frac{dx}{dt} + 4\frac{dy}{dt} + 3y = 0$; given $x = 0$, $y = 0$ when $t = 0$. (*Madras, 2003 S*)
5. $(D-2)x - (D+1)y = 6e^{3t}$; $(2D-3)x + (D-3)y = 6e^{3t}$ given $x = 3$, $y = 0$ when $t = 0$.
6. The currents i_1 and i_2 in mesh are given by the differential equations ; $di_1/dt - \omega i_2 = a \cos pt$, $di_2/dt + \omega i_1 = a \sin pt$. Find the currents i_1 and i_2 by Laplace transform, if $i_1 = i_2 = 0$ at $t = 0$.

21.17 (1) UNIT STEP FUNCTION

At times, we come across such fractions of which the inverse transform cannot be determined from the formulae so far derived. In order to cover such cases, we introduce the *unit step function* (or *Heaviside's unit function**).

Def. *The unit step function $u(t-a)$ is defined as follows :*

$$u(t-a) = \begin{cases} 0 \text{ for } t < a \\ 1 \text{ for } t \geq a \end{cases}$$

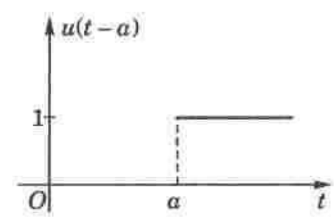


Fig. 21.3

where, a is always positive (Fig. 21.3). It is also denoted as $H(t-a)$.

(2) Transform of unit function.

$$L\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a)\, dt = \int_0^a e^{-st} .\, 0\, dt + \int_a^\infty e^{-st} .\, 1\, dt = 0 + \left| \frac{e^{-st}}{-s} \right|_a^\infty$$

Thus $\quad L\{u(t-a)\} = e^{-as}/s.$

The product $f(t)\, u(t-a) = \begin{cases} 0 & \text{for } t < a \\ f(t) & \text{for } t \geq a. \end{cases}$

The function $f(t-a) \,.\, u(t-a)$ represents the graph of $f(t)$ shifted through a distance a to the right and is of special importance.

Second shifting property. If $L\{f(t)\} = \bar{f}(s)$, *then*

$$\mathbf{L\{f(t-a).\, u(t-a)\} = e^{-as}\, \bar{f}(s)}$$

$$L\{f(t-a) \,.\, u(t-a)\} = \int_0^\infty e^{-st} f(t-a)\, u(t-a)\, dt$$

*Named after the British Electrical Engineer Oliver Heaviside (1850–1925).

$$= \int_0^a e^{-st} f(t-a)(0)\,dt + \int_a^\infty e^{-st} f(t-a)\,dt \qquad \text{[Put } t-a=u\text{]}$$

$$= \int_0^\infty e^{-s(u+a)} f(u)\,du = e^{-sa}\int_0^\infty e^{-su} f(u)\,du = e^{-as}\,\bar{f}(s).$$

Example 21.38. *Express the following function (Fig. 21.4) in terms of unit step function and find its Laplace transform.* *(U.P.T.U., 2002)*

Solution. We have $f(t) = \begin{cases} 0, & 0<t<1 \\ t-1, & 1<t<2 \\ 1, & t>2 \end{cases}$

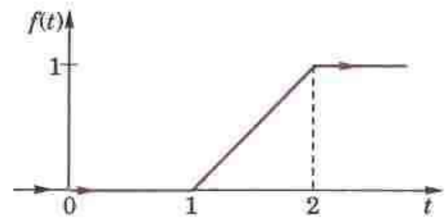


Fig. 21.4

or $$f(t) = (t-1)\,[u(t-1)-u(t-2)] + u(t-2)$$
$$= (t-1)\,u(t-1) - (t-2)\,u(t-2)$$

By second shifting property,
$$L[f(t-a)\,u(t-a)] = e^{-as}\,L[f(t)].$$

Also $$L[f(t)] = L(t) = 1/s^2.$$

$\therefore$ $$L[(t-1)\,u(t-1)]$$
$$= e^{-s}.\,\frac{1}{s^2} \text{ and } L\,[(t-2)\,u(t-2)] = e^{-2s}.\,\frac{1}{s^2}$$

Hence $$L\,[f(t)] = L\,[(t-1)\,u(t-1)-(t-2)\,u(t-2)] = \frac{e^{-s}-e^{-2s}}{s^2}.$$

Example 21.39. *Using unit step function, find the Laplace transform of*

$$f(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ \sin 2t, & \pi \le t < 2\pi. \\ \sin 3t, & t \ge 2\pi \end{cases}$$ *(V.T.U., 2004)*

Solution. $$f(t) = \sin t\,[u(t-0)-u(t-\pi)] + \sin 2t\,[u(t-\pi)-u(t-2\pi)] + \sin 3t\,.\,u(t-2\pi)$$
$$= \sin t + (\sin 2t - \sin t)\,u(t-\pi) + (\sin 3t - \sin 2t)\,u(t-2\pi)$$

Since $L[f(t-a)\,u(t-a)] = e^{-as}\,\bar{f}(s)$ and $L(\sin at) = \dfrac{a}{s^2+a^2}$,

$$L[f(t)] = L(\sin t) + L[(\sin 2t - \sin t)\,.\,u(t-\pi)] + L[(\sin 3t - \sin 2t)\,.\,u(t-2\pi)]$$

$$= \frac{1}{s^2+1} + e^{-\pi s}\left(\frac{2}{s^2+4} - \frac{1}{s^2+1}\right) + e^{-2\pi s}\left(\frac{3}{s^2+9} - \frac{2}{s^2+4}\right).$$

Example 21.40. (*i*) *Express the function (Fig. 21.5) in terms of unit step function and find its Laplace transform.* *(P.T.U., 2005 S)*

(*ii*) *Obtain the Laplace transform of* $e^{-t}[1-u(t-2)]$.

Solution. (*i*) We have $f(t) = \begin{cases} t-1, & 1<t<2 \\ 3-t, & 2<t<3. \end{cases}$

or $$f(t) = (t-1)\,\{u(t-1)-u(t-2)\} + (3-t)\,\{u(t-2)-u(t-3)\}$$
$$= (t-1)\,u(t-1) - 2(t-2)\,u(t-2) + (t-3)\,u(t-3)$$

Since $L\,\{f(t-a)\,u\,(t-a)\} = e^{-as}\,\bar{f}(s)$ ...(λ)

$\therefore$ $$L\,[f(t)] = e^{-s}\,.\,\frac{1}{s^2} - 2e^{-2s}\,.\,\frac{1}{s^2} + e^{-3s}\,.\,\frac{1}{s^2} = \frac{e^{-s}(1-e^{-s})^2}{s^2} \qquad [\because\ f(t)=t]$$

(*ii*) $L\,\{e^{-t}\,[1-u\,(t-2)]\} = L\,(e^{-t}) - L\,\{e^{-t}\,u\,(t-2)\} = \dfrac{1}{s+1} - e^{-2}\,L\,\{e^{-(t-2)}\,u\,(t-2)\}$

Taking $f(t) = e^{-t}$, $\bar{f}(s) = \dfrac{1}{s+1}$ and using (λ) above,

$$L\{e^{-(t-2)} u(t-2)\} = e^{-2s} \cdot \frac{1}{s+1}$$

Hence $Le^{-t}\{1 - u(t-2)\} = \{1 - e^{-2(s+1)}\}/(s+1)$.

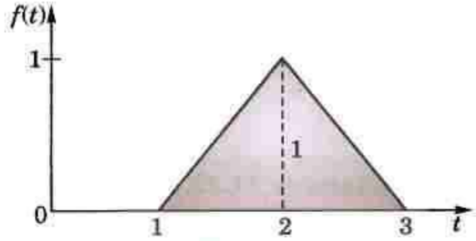


Fig. 21.4

Example 21.41. *Using Laplace transform, evaluate* $\int_0^\infty e^{-t}(1 + 2t - t^2 + t^3) H(t-1)\, dt$.

(Mumbai, 2007)

Solution. We have $L\{(1 + 2t - t^2 + t^3) H(t-1)\}$

$$= e^{-s} L[1 + 2(t+1) - (t+1)^2 + (t+1)^3] = e^{-s} L(3 + 3t + 2t^2 + t^3)$$

$$= e^{-s}\left(3 \cdot \frac{1}{s} + 3 \cdot \frac{1}{s^2} + 2 \cdot \frac{2!}{s^3} + \frac{3!}{s^4}\right) = e^{-s}\left(\frac{3}{s} + \frac{3}{s^2} + \frac{4}{s^3} + \frac{6}{s^4}\right)$$

By definition, this implies that

$$\int_0^\infty e^{-st}\,(1 + 2t - t^2 + t^3) H(t-1)\, dt = e^{-s}\left(\frac{3}{s} + \frac{3}{s^2} + \frac{4}{s^3} + \frac{6}{s^4}\right)$$

Taking $s = 1$, we obtain

$$\int_0^\infty e^{-t}(1 + 2t - t^2 + t^3) H(t-1)\, dt = e^{-1}(3 + 3 + 4 + 6) = 16/e.$$

Example 21.42. *Evaluate* (*i*) $L^{-1}\left\{\dfrac{e^{-s} - 3e^{-3s}}{s^2}\right\}$ *(U.P.T.U., 2002)*

(*ii*) $L^{-1}\left\{\dfrac{se^{-as}}{s^2 - w^2}\right\}, a > 0.$

Solution.

$$L^{-1}\left\{e^{-s} \cdot \frac{1}{s^2}\right\} = \begin{cases} t-1, & t > 1 \\ 0, & t < 1 \end{cases} = (t-1)\, u(t-1)$$

$$L^{-1}\left\{e^{-3s} \cdot \frac{1}{s^2}\right\} = \begin{cases} t-3, & t > 3 \\ 0, & t < 3 \end{cases} = (t-3)\, u(t-3)$$

$$\therefore \quad L^{-1}\left\{\frac{e^{-s} - 3e^{-3s}}{s^2}\right\} = L^{-1}\left(\frac{e^{-s}}{s^2}\right) - 3L^{-1}\left(\frac{e^{-3s}}{s^2}\right) = (t-1)\, u(t-1) - 3(t-3)\, u(t-3)$$

(*ii*) We know that $L^{-1}\left(\dfrac{s}{s^2 - w^2}\right) = \cosh wt$

$$\therefore \quad L^{-1}\left(\frac{se^{-as}}{s^2 - w^2}\right) = \begin{cases} \cosh w(t-a), & t > a \\ 0, & t < a \end{cases}$$

$= \cosh w(t-a)\, u(t-a)$, by second shifting property.

Example 21.43. *Find the inverse Laplace transform of :*

(*i*) $\dfrac{se^{-s/2} + \pi e^{-s}}{s^2 + \pi^2}$ *(V.T.U., 2000)* (*ii*) $\dfrac{e^{-cs}}{s^2(s+a)}$ $(c > 0)$. *(Kurukshetra, 2005)*

Solution. (*i*) Since $L^{-1}\dfrac{s}{s^2 + \pi^2} = \cos \pi t$, $L^{-1}\left(\dfrac{\pi}{s^2 + \pi^2}\right) = \sin \pi t$

and $$L^{-1}[e^{-as}\bar{f}(s)] = f(t-a)\,.\,u(t-a) \qquad ...(\lambda)$$

$$\therefore \quad L^{-1}\left\{\frac{se^{-s/2}+\pi e^{-s}}{s^2+\pi^2}\right\} = L^{-1}\left\{e^{-s/2}\,.\,\frac{s}{s^2+\pi^2}\right\} + L^{-1}\left\{e^{-s}\,.\,\frac{\pi}{s^2+\pi^2}\right\}$$

$$= \cos\pi(t-1/2)\,.\,u(t-1/2) + \sin\pi(t-1)\,.\,u(t-1)$$

$$= \sin\pi t\,.\,u(t-1/2) - \sin\pi t\,.\,u(t-1) = \{u(t-1/2)-u(t-1)\}\sin\pi t$$

(*ii*) $$L^{-1}\left\{\frac{e^{-cs}}{s^2(s+a)}\right\} = L^{-1}\left\{e^{-cs}\left(-\frac{1}{a^2}\,.\,\frac{1}{s}+\frac{1}{a}\,.\,\frac{1}{s^2}+\frac{1}{a^2}\,.\,\frac{1}{s+a}\right)\right\}$$

Using (λ) above, we have

$$L^{-1}\left\{\frac{e^{-cs}}{s^2(s+a)}\right\} = -\frac{1}{a^2}\{1\,.\,u(t-c)\} + \frac{1}{a}\{(t-c)\,.\,u(t-c)\} + \frac{1}{a^2}\{e^{-a(t-c)}\,.\,u(t-c)\}$$

$$= \frac{1}{a^2}\{a(t-c)-1+e^{-a(t-c)}\}\,u(t-c).$$

Example 21.44. *A particle of mass m can oscillate about the position of equilibrium under the effect of a restoring force mk^2 times the displacement. It started from rest by a constant force F which acts for time T and then ceases. Find the amplitude of the subsequent oscillation.*

Solution. The constant force F acting from $t=0$ to $t=T$ can be expressed as

$$F[1-u(t-T)],\ 0<t<T$$

$\therefore$ equation of motion of the particle is

$$m\frac{d^2x}{dt^2} = F[1-u(t-T)] - mk^2x \quad \text{or} \quad \frac{d^2x}{dt^2}+k^2x = \frac{F}{m}[1-u(t-T)]$$

Taking Laplace transform of both sides, we get

$$(s^2+k^2)\,\bar{x} = \frac{F}{ms}(1-e^{-sT}) \qquad [\because\ x=0,\ \dot{x}=0 \text{ at } t=0]$$

or $$\bar{x} = \frac{F}{m}\,.\,\frac{1-e^{-sT}}{s(s^2+k^2)} = \frac{F}{m}(1-e^{-sT})\,.\,\frac{1}{k^2}\left(\frac{1}{s}-\frac{s}{s^2+k^2}\right)$$

$$= \frac{F}{mk^2}\left\{(1-e^{-sT})\frac{1}{s}-(1-e^{-sT})\,.\,\frac{s}{s^2+k^2}\right\}$$

Taking inverse Laplace transform, we obtain

$$x = \frac{F}{mk^2}[(1-\cos kt)-\{1-\cos k(t-T)\}]\,u(t-T)$$

i.e., $$x = \frac{F}{mk^2}(1-\cos kt) \text{ for } 0<t<T$$

and $$x = \frac{F}{mk^2}(1-\cos kt)-\{1-\cos k(t-T)\} \text{ for } t>T$$

$$= \frac{F}{mk^2}\{\cos k(t-T)-\cos kT\} \text{ for } t>T$$

or $$x = \frac{2F}{mk^2}\sin\frac{kT}{2}\,.\,\sin k(t-T/2) \text{ for } t>T$$

Hence the amplitude of subsequent oscillation (*i.e.*, for $t>T$) $= \dfrac{2F}{mk^2}\sin\dfrac{kT}{2}$.

Example 21.45. *In an electrical circuit with e.m.f. E(t), resistance R and inductance L, the current i builds up at the rate given by*

$$L\,di/dt + Ri = E(t). \qquad ...(i)$$

If the switch is connected at t = 0 and disconnected at t = a, find the current i at any instant.

Solution. We have $i = 0$ at $t = 0$ and $E(t) = \begin{cases} E \text{ for } 0 < t < a \\ 0 \text{ for } t > a \end{cases}$

$\therefore$ taking the Laplace transform of both sides, (*i*) becomes

$$(Ls + R)\bar{i} = \int_0^\infty e^{-st} E(t)\, dt = \int_0^a e^{-st} E dt = \frac{E}{s}(1 - e^{-as})$$

or
$$\bar{i} = \frac{E}{s(Ls + R)} - \frac{Ee^{-as}}{s(Ls + R)}$$

On inversion, we get
$$i = L^{-1}\left\{\frac{E}{s(Ls + R)}\right\} - L^{-1}\left\{\frac{Ee^{-as}}{s(Ls + R)}\right\} \qquad ...(ii)$$

Now
$$L^{-1}\left\{\frac{E}{s(Ls + R)}\right\} = \frac{E}{R}\left\{L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s + R/L}\right)\right\} = \frac{E}{R}(1 - e^{-Rt/L})$$

and
$$L^{-1}\left\{\frac{Ee^{-as}}{s(Ls + R)}\right\} = \frac{E}{R}\{1 - e^{-R(t-a)/L}\}\, u(t - a) \qquad \text{[By the second shifting property]}$$

Thus (*ii*) becomes
$$i = \frac{E}{R}\{1 - e^{-Rt/L}\} - \frac{E}{R}\{1 - e^{-R(t-a)/L}\}\, u(t - a)$$

Hence
$$i = \frac{E}{R}\{1 - e^{-Rt/L}\} \text{ for } 0 < t < a$$

and
$$i = \frac{E}{R}[\{1 - e^{-Rt/L}\} - \{1 - e^{-R(t-a)/L}\}] = \frac{E}{R}e^{-Rt/L}(e^{-Ra/L} - 1) \text{ for } t > a.$$

Example 21.46. *Calculate the maximum deflection of an encastre beam 1 ft. long carrying a uniformly distributed load w lb./ft. on its central half length.*

Solution. Taking the origin at the end A, we have

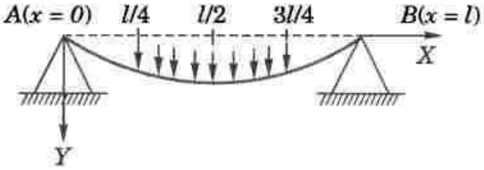


Fig. 21.6

$$EI\frac{d^4y}{dx^4} = w(x)$$

where $w(x) = w\{u(x - l/4) - u(x - 3l/4)\}$

Taking the Laplace transform of both sides, (Fig. 21.6), we get

$$EI[s^4\bar{y} - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)]$$
$$= w\left(\frac{e^{-ls/4}}{s} - \frac{e^{-3ls/4}}{s}\right)$$

Using the conditions $y(0) = y'(0) = 0$ and taking $y''(0) = c_1$ and $y'''(0) = c_2$, we have

$$EI\bar{y} = w\left(\frac{e^{-ls/4}}{s^5} - \frac{e^{-3ls/4}}{s^5}\right) + \frac{c_1}{s^3} + \frac{c_2}{s^4}$$

On inversion, we get $EIy = \dfrac{w}{24}[(x - l/4)^4 u(x - l/4) - (x - 3l/4)^4 u(x - 3l/4)] + \frac{1}{2}c_1x^2 + \frac{1}{6}c_2x^3$ ...(*i*)

For $x > 3l/4$,
$$EIy = \frac{w}{24}[(x - l/4)^2 - (x - 3l/4)^2] + \tfrac{1}{2}c_1x^2 + \tfrac{1}{6}c_2x^3$$

and
$$EIy' = \frac{w}{6}[(x - l/4)^3 - (x - 3l/4)^3] + c_1x + \tfrac{1}{2}c_2x^2$$

Using the conditions $y(l) = 0$ and $y'(l) = 0$, we get $0 = \dfrac{w}{24}\left\{\left(\dfrac{3l}{4}\right)^4 - \left(\dfrac{l}{4}\right)^4\right\} + \frac{1}{2}c_1l^2 + \frac{1}{6}c_2l^3$

and
$$0 = \frac{w}{6}\left\{\left(\frac{3l}{4}\right)^3 - \left(\frac{l}{4}\right)^3\right\} + c_1l + \tfrac{1}{2}c_2l^2$$

whence $c_1 = 11\, wl^2/192$; $c_2 = -\, wl/4$.

Thus for $l/4 < x < 3l/4$, (i) gives $EIy = \frac{w}{24}\left(x + \frac{1}{4}\right)^4 + \frac{11wl^2}{384}x^2 - \frac{wl}{24}x^3$

Hence the maximum deflection $= y(l/2) = \frac{13wl^4}{6144EI}$.

21.18 (1) UNIT IMPULSE FUNCTION

The idea of a very large force acting for a very short time is of frequent occurrence in mechanics. To deal with such and similar ideas, we introduce the *unit impulse function* (also called *Dirac delta function**).

Thus unit impulse function is considered as the limiting form of the function (Fig. 21.7) :

$$\delta_\varepsilon(t-a) = 1/\varepsilon, \quad a \le t \le a+\varepsilon$$
$$= 0, \quad \text{otherwise}$$

as $\varepsilon \to 0$. It is clear from Fig. 21.7 that as $\varepsilon \to 0$, the height of the strip increases indefinitely and the width decreases in such a way that its area is always unity.

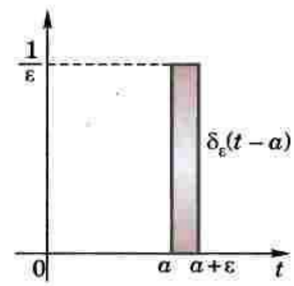


Fig. 21.7

Thus the unit impulse function $\delta(t-a)$ is defined as follows :

$$\delta(t-a) = \infty \text{ for } t = a \; ; = 0 \text{ for } t \ne a,$$

such that $$\int_0^\infty \delta(t-a)\, dt = 1. \qquad (a \ge 0)$$

As an illustration, a load w_0 acting at the point $x = a$ of a beam may be considered as the limiting case of uniform loading w_0/ε per unit length over the portion of the beam between $x = a$ and $x = a + \varepsilon$. Thus

$$w(x) = w_0/\varepsilon \qquad a < x < a+\varepsilon,$$
$$= 0, \qquad \text{otherwise}$$

i.e., $$w(x) = w_0\delta(x+a).$$

(2) Transform of unit impulse function. If $f(t)$ be a function of t continuous at $t = a$, then

$$\int_0^\infty f(t)\,\delta_\varepsilon(t-a)\,.\,dt = \int_0^{a+\varepsilon} f(t)\,.\,\frac{1}{\varepsilon}\,dt$$

$$= (a+\varepsilon-a)f(\eta)\,.\,\frac{1}{\varepsilon} = f(\eta), \qquad \text{where } a < \eta < a+\varepsilon.$$

by Mean value theorem for integrals.

As $\varepsilon \to 0$, we get $\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)$.

In particular, when $f(t) = e^{-st}$, we have $\mathbf{L\{\delta(t-a)\} = e^{-as}}$.

Example 21.47. *Evaluate* (i) $\int_0^\infty \sin 2t\,\delta(t-\pi/4)\,dt$ $\qquad (ii)$ $L\frac{1}{t}\,\delta(t-a)$.

Solution. (i) We know that $\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)$

$$\therefore \quad \int_0^\infty \sin 2t\,\delta(t-\pi/4)\,dt = \sin(2\,.\,\pi/4) = 1$$

(ii) We know that $L\{\delta(t-a)\} = e^{-as}$

$$\therefore \quad L\left[\frac{1}{t}\delta(t-a)\right] = \int_s^\infty L[\delta(t-a)]\,ds = \int_s^\infty e^{-as}\,ds$$

$$= \left|\frac{e^{-as}}{-a}\right|_s^\infty = \frac{1}{a}e^{-as}.$$

* After the English physicist *Paul Dirac* (1902-84) who was awarded the Nobel prize in 1933 for his work in Quantum mechanics.

Example 21.48. *An impulsive voltage $E\delta(t)$ is applied to a circuit consisting of L, R, C in series with zero initial conditions. If i be the current at any subsequent time t, find the limit of i as $t \to 0$?*

Solution. The equation of the circuit governing the current i is

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t i\,dt = E\delta(t) \qquad \text{where } i = 0, \text{ when } t = 0.$$

Taking Laplace transform of both sides, we get

$$L[s\bar{i} - i(0)] + R\bar{i} + \frac{1}{C}\frac{1}{s}\bar{i} = E \qquad \text{[Using § 21.7 and 21.8]}$$

or $$\left(s^2 + \frac{R}{L}s + \frac{1}{CL}\right)\bar{i} = \frac{E}{L}s \quad \text{or} \quad (s^2 + 2as + a^2 + b^2)\bar{i} = (E/L)\,s$$

where $R/L = 2a$ and $1/CL = a^2 + b^2$

or $$\bar{i} = \frac{E}{L}\frac{(s+a)-a}{(s+a)^2+b^2} = \frac{E}{L}\left\{\frac{s+a}{(s+a)^2+b^2} - a\frac{1}{(s+a)^2+b^2}\right\}$$

On inversion, we get

$$i = \frac{E}{L}\left\{e^{-at}\cos bt - \frac{a}{b}e^{-at}\sin bt\right\}$$

Taking limits as $t \to 0$, $i \to E/L$

Although the current $i = 0$ initially, yet a large current will develop instantaneously due to impulsive voltage applied at $t = 0$. In fact, we have determined the limit of this current which is E/L.

Example 21.49. *A beam is simply supported at its end $x = 0$ and is clamped at the other end $x = l$. It carries a load w at $x = l/4$. Find the resulting deflection at any point.*

Solution. The differential equation for deflection is

$$\frac{d^4y}{dx^4} = \frac{w}{EI}\delta(x - l/4)$$

Taking the Laplace transform, we have $s^4\bar{y} - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) = \frac{w}{EI}e^{-ls/4}$

Using the conditions $y(0) = 0$, $y''(0) = 0$ and taking $y'(0) = c_1$ and $y'''(0) = c_2$, we get

$$\bar{y} = \frac{c_1}{s^2} + \frac{c_2}{s^4} + \frac{w}{EI}\frac{e^{-ls/4}}{s^4}.$$

On inversion, it gives $$y = c_1x + c_2\frac{x^3}{3!} + \frac{w}{EI}\frac{(x-l/4)^3}{3!}\,u(x - l/4)$$

$$\left.\begin{aligned} &\text{i.e.,} \quad y = c_1x + \tfrac{1}{6}c_2x^3, && 0 < x < l/4 \\ &\text{and} \quad y = c_1x + \tfrac{1}{6}c_2x^3 + \frac{\omega}{6EI}(x - l/4)^3, && l/4 < x < l \end{aligned}\right\} \qquad ...(i)$$

Using the conditions $y(l) = 0$ and $y'(l) = 0$, we get

$$0 = c_1l + \tfrac{1}{6}c_2l^3 + 9wl^3/128EI \quad \text{and} \quad 0 = c_1 + \tfrac{1}{2}c_2l^2 + 9wl^2/32EI$$

whence $c_1 = 9wl^2/256\,EI$, $\qquad c_2 = -81w/128EI$.

Substituting the values of c_1 and c_2 in (i), we get the deflection at any point.

PROBLEMS 21.8

1. Represent $f(t) = \sin 2t$, $2\pi < t < 4\pi$ and 0 otherwise, in terms of the unit step function and hence find its Laplace transform. *(Mumbai, 2005)*
2. Sketch the graph of the following functions and express them in terms of unit step function. Hence find their Laplace transforms :

(i) $f(t) = 2t$ for $0 < t < \pi$, $f(t) = 1$ for $t > \pi$ *(Assam, 1999)*

(ii) $f(t) = t^2$ for $0 < t \le 2$, $f(t) = 0$ for $t > 2$

(iii) $f(t) = \cos(wt + \phi)$ for $0 < t < T$, $f(t) = 0$ for $t > T$.

3. Express the following functions in terms of unit step function and hence find its Laplace transform.

(i) $f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ 1, & \pi < t < 2\pi \\ \sin t, & t > 2\pi \end{cases}$ *(V.T.U., 2007)* (ii) $f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ \cos 2t, & \pi < t < 2\pi \\ \cos 3t, & t > \pi \end{cases}$

(Mumbai, 2008 ; V.T.U., 2003 S)

(iii) $f(t) = \begin{cases} t^2, & 0 < t < 2 \\ 4t, & 2 < t < 4 \\ 8, & t > 4 \end{cases}$ *(V.T.U., 2011)*

4. Evaluate (i) $L\{e^{t-1}\,u(t-1)\}$ (ii) $L\{(t-1)^2\,u(t-1)\}$

(iii) $L(1 + 2t - 3t^2 + 4t^3)\,H(t-2)$ *(Mumbai, 2007)* (iv) $L\{t^2\,u(t-1) + \delta(t-1)\}$.

5. Evaluate $\int_0^\infty e^{-t}(1 + 3t + t^2)\,u(t-2)\,dt$.

6. Find the inverse Laplace transforms of :

(i) $\dfrac{e^{-\pi s}}{s^2+1}$ (ii) $\dfrac{e^{-2s}}{s^2+8s+25}$ *(Mumbai, 2006)*

(iii) $\dfrac{e^{-s}}{(s+1)^3}$ *(P.T.U., 2010)* (iv) $\dfrac{3}{s} - 4\dfrac{e^{-s}}{s^2} + 4\dfrac{e^{-3s}}{s^2}$. *(P.T.U., 2002 S)*

7. Solve using Laplace transforms $\dfrac{d^2y}{dt^2} + 4y = f(t)$ with conditions

$$y(0) = 0,\ y'(0) = 1 \text{ and } f(t) = \begin{cases} 1 \text{ when } 0 < t < 1 \\ 0 \text{ when } t > 1 \end{cases}$$ *(Mumbai, 2007)*

8. Using Laplace transforms, solve $x''(t) + x(t) = u(t)$, $x(0) = 1$, $x'(0) = 0$

where $u(t) = \begin{cases} 3, & 0 \le t \le 4 \\ 2t - 5, & t > 4. \end{cases}$

9. A beam has its ends clamped at $x = 0$ and $x = l$. A concentrated load W acts vertically downwards at the point $x = l/3$. Find the resulting deflection.

$\left[\textbf{Hint.}\right.$ The differential equation and the boundary conditions are $\dfrac{d^4y}{dx^2} = \dfrac{W}{EI}\,\delta(x - l/3)$ and

$$\left. y(0) = y'(0) = 0,\ y(l) = y'(l) = 0.\right]$$

10. A cantilever beam is clamped at the end $x = 0$ and is free at the end $x = l$. It carries a uniform load w per unit length from $x = 0$ to $x = l/2$. Calculate the deflection y at any point. *(Kurukshetra, 2006)*

[**Hint.** The differential equation and boundary conditions are

$$\frac{d^4y}{dx^4} = \frac{W(x)}{EI}\ (0 < x < l) \text{ where } \quad W(x) = \begin{cases} W_0, & 0 < x < l/2 \\ 0, & x > l/2 \end{cases}$$

and $y(0) = y'(0) = 0,\ y''(0) = y'''(0) = 0$]

11. An impulse I (kg-sec) is applied to a mass m attached to a spring having a spring constant k. The system is damped with damping constant μ. Derive expressions for displacement and velocity of the mass, assuming initial conditions $x(0) = x'(0) = 0$.

$\left[\textbf{Hint.}\right.$ The equation of motion is $m\dfrac{d^2x}{dt^2} = I\,\delta(x) - kx - \mu\,\dfrac{dx}{dt}.\left.\right]$

21.19 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 21.9

Fill up the blanks or choose the correct answer in each of the following problems :

1. Laplace transform of $(t \sin t)$ = ...
2. $L\{\delta(t)\}$ = ...
(a) 0 (b) e^{-as} (c) ∞ (d) 1.
3. If $L\{f(t)\} = f(s)$, then $L\{e^{-at} f(t)\}$ is
(a) $f(s-a)$ (b) $f(s+a)$ (c) $f(s)$ (d) none of these.
4. $L(e^{2t} \sin t)$ = ...
5. Inverse Laplace transform of $(s+2)^{-2}$ is ....
6. Inverse Laplace transform of $1/(s^2+4s+13)$ = ...
7. Laplace transform of $f'(t)$ = ...
8. $L^{-1}\left[\dfrac{s}{(2s+3)^2}\right]$ = ...
9. $L(\cosh^2 2t)$ = ...
10. $L(e^t)$ = ...
11. $L(e^{-t} t^k)$ = ...
12. $\int_0^{\infty} e^{-2t} \cos 3t \, dt$ = ...
13. $L\{u(t-a)\}$ = ...
14. $L^{-1}(\sqrt{t})$ = ...
15. If $L\{F(t)\} = f(s)$, then $L\left\{\dfrac{d^2F(t)}{dt^2}\right\}$ = ...
16. $L(\cos^3 4t)$ = ...
17. $L\left(\dfrac{\sin at}{t}\right)$ = ...
18. $L^{-1}\left\{\dfrac{1}{(s+3)^5}\right\}$ = ...
19. $L \cos(2t+3)$ = ...
20. $L^{-1}(1/s^n)$ is possible only when n is
(a) zero (b) –ve integer (c) +ve integer (d) negative rational.
21. If $L^{-1}\{\phi(s)\} = f(t)$, the $L^{-1}\{e^{-as}\phi(s)\}$ = ...
22. $L\{u(t+2)\}$ = ...
(a) e^{-2s}/s^2 (b) e^{2s} (c) $\dfrac{e^{2s}}{s}$ (d) $\dfrac{e^{-2s}}{s}$ (V.T.U., 2011 S)
23. $L^{-1}\left\{\dfrac{s^2-3s+4}{s^3}\right\}$ = ... (V.T.U., 2010 S)
24. If $L\{f(t)\} = \bar{f}(s)$, then $L^{-1}\left\{\dfrac{\bar{f}(s)}{s}\right\}$ = ...
25. If $f(t)$ is a periodic function with period T, then $L\{f(t)\}$ = ...
26. If y satisfies $y'' + 3y' + 2y = e^{-t}$ with $y(0) = y'(0) = 0$, then $L\{y(t)\}$ = ...
27. $L\{e^{3t}(2\cos 5t + 3\sin 4t)\}$ = ...
28. $L(4^t)$ = ...
29. $L^{-1}\left\{\dfrac{1}{\sqrt{(s+3)}}\right\}$ = ...
30. Laplace transform of $\sin 2t\, \delta(t-2)$ is
(a) $e^{2s} \sin 4$ (b) $e^{-2s} \sin 2$ (c) $e^{-4s} \sin 2$ (d) $e^{-2s} \sin 4$. (V.T.U., 2009 S)
31. If $L^{-1}\left\{\dfrac{s}{(s+1)^2}\right\} = \dfrac{t \sin t}{2}$ then $L^{-1}\left\{\dfrac{8s}{(4s^2+1)^2}\right\}$ (P.T.U., 2009)
32. $L^{-1}\{e^{-as} F(s)\}$ =
(a) $f(t)\,u(t)$ (b) $f(t-a)\,u(t)$ (c) $f(t-a)\,u(t-a)$ (d) None of these. (V.T.U., 2009 S)
33. $L^{-1}\left\{\dfrac{1}{(s+a)^2}\right\}$ =
(a) e^{at} (b) e^{-at} (c) te^{-at}
(d) te^{at} (e) $-t$.

34. Laplace transform of $t^4 e^{-at}$ is

(*i*) $\dfrac{4!}{(s+a)^4}$ (*ii*) $\dfrac{4!}{(s+a)^5}$ (*iii*) $\dfrac{4!}{(s-a)^4}$ (*iv*) $\dfrac{5!}{(s-a)^5}$.

35. Laplace transform of $te^{at}\sin(at)$, $t > 0$, is

(*i*) $\dfrac{s-a}{(s-a)^2+a^2}$ (*ii*) $\dfrac{a(s-a)}{(s-a)^2+a^2}$ (*iii*) $\dfrac{2a(s-a)}{[(s-a)^2+a^2]^2}$ (*iv*) $\dfrac{(s-a)^2}{(s-a)^2+a^2}$.

36. $L^{-1}\dfrac{s^2}{(s^2+4)^2}$ is

(*i*) $\dfrac{1}{4}\sin 2t + t\cos 2t$ (*ii*) $\dfrac{1}{4}\sin 2t + \dfrac{1}{2}\cos 2t$ (*iii*) $\sin 2t + \dfrac{t}{2}\cos 2t$ (*iv*) $\dfrac{1}{2}\sin 2t + \dfrac{t}{4}\cos 2t$.

37. $L^{-1}\dfrac{1}{s(s^2+1)}$ is

(*i*) $1 + \sin t$ (*ii*) $1 - \sin t$ (*iii*) $1 + \cos t$ (*iv*) $1 - \cos t$.

38. $L[u(t-a)]$ where $u(t-a)$ is a unit step function, is

(*i*) $\dfrac{e^{-as}}{s}$ (*ii*) $\dfrac{e^{as}}{s}$ (*iii*) $\dfrac{e^{-as}}{s^2}$ (*iv*) $\dfrac{e^{as}}{s^2}$. (*V.T.U., 2011*)

39. For a periodic function of period 2π, $\displaystyle\int_{a+2\pi}^{b+2\pi} f(x)\,dx =$ .... (*P.T.U., 2009*)

40. $L[\delta(t-a)]$ where $\delta(t-a)$ is a unit impulse function, is

(*i*) e^{as} (*ii*) e^{-as} (*iii*) e^{s} (*iv*) e^{-as}/s (*V.T.U., 2010 S*)

41. Laplace transform of $\sin^2 3t$ is

(*i*) $\dfrac{3}{s^2+36}$ (*ii*) $\dfrac{6}{(s^2+36)}$ (*iii*) $\dfrac{18}{s(s^2+36)}$ (*iv*) $\dfrac{18}{s^2+36}$ (*V.T.U., 2010*)

42. $[L(t^2 e^{-3t})] =$

(*i*) $\dfrac{1}{(s+3)^3}$ (*ii*) $\dfrac{2}{(s+3)^2}$ (*iii*) $\dfrac{3}{(s+3)^3}$ (*iv*) $\dfrac{2}{(s+3)^3}$ (*V.T.U., 2011*)

43. $\dfrac{d^2}{ds^2}[L f(t)] - L(t^2 f(t)) = 0.$ (True or False)

44. Laplace transform of $f(t)$ is defined for +ve and –ve values of t. (True or False)

45. If $L[f(t)] = \phi(s)$, then $L[t f(t)] = \dfrac{d}{ds}[\phi(s)]$. (True or False)

CHAPTER 22

Fourier Transforms

22.1 INTRODUCTION

In the previous chapter, the reader has already been acquainted with the use of Laplace transforms in the solution of ordinary differential equations. In this chapter, the well-known Fourier transforms will be introduced and their properties will be studied which will be used in the solution of partial differential equations. The choice of a particular transform to be employed for the solution of an equation depends on the boundary conditions of the problem and the ease with which the transform can be inverted. A Fourier transform when applied to a partial differential equation reduces the number of its independent variables by one.

The theory of integral transforms afford mathematical devices through which solutions of numerous boundary value problems of engineering can be obtained *e.g.*, conduction of heat, transverse vibrations of a string, transverse oscillations of an elastic beam, free and forced vibrations of a membrane, transmission lines etc. Some of these applications will be illustrated in the last section.

22.2 DEFINITION

The integral transform of a function f(x) denoted by I[f(x)], is defined by

$$\bar{f}(s) = \int_{x_1}^{x_2} f(x) K(s, x)\, dx$$

where $K(s, x)$ is called the *kernel* of the transform and is a known function of s and x. The function $f(x)$ is called the *inverse transform* of $\bar{f}(s)$.

Three simple examples of a kernel are as follows :

(*i*) *When* $K(s, x) = e^{-sx}$, *it leads to the* **Laplace transform** *of* $f(x)$, *i.e.*,

$$\bar{f}(s) = \int_0^\infty f(x) e^{-sx}\, dx. \qquad \text{[Chap. 21]}$$

(*ii*) *When* $K(s, x) = e^{isx}$, *we have the* **Fourier transform** *of* $f(x)$, *i.e.*,

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{isx}\, dx.$$

(*iii*) When $K(s, x) = x^{s-1}$, it gives the *Mellin transform* of $f(x)$ *i.e.*,

$$M(s) = \int_0^\infty f(x) x^{s-1}\, dx.$$

Other special transforms arise when the kernel is a sine or a cosine function or a Bessel's function. These lead to *Fourier sine* or *cosine transforms* and the *Hankel transform* respectively.

In order to introduce the *Fourier transforms*, we shall first derive the Fourier integral theorem.

22.3 (1) FOURIER INTEGRAL THEOREM

Consider a function $f(x)$ which satisfies the Dirichlet's conditions (Art. 10.3) in every interval $(-c, c)$ so that, we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{c} + b_n \sin \frac{n\pi x}{c} \right) \qquad ...(1)$$

where $$a_0 = \frac{1}{c} \int_{-c}^{c} f(t)\, dt,\ a_n = \frac{1}{c} \int_{-c}^{c} f(t) \cos \frac{n\pi t}{c}\, dt, \text{ and } b_n = \frac{1}{c} \int_{-c}^{c} f(t) \sin \frac{n\pi t}{c}\, dt.$$

Substituting the values of a_0, a_n and b_n in (1), it takes the form

$$f(x) = \frac{1}{2c} \int_{-c}^{c} f(t)\, dt + \frac{1}{c} \sum_{n=1}^{\infty} \int_{-c}^{c} f(t) \cos \frac{n\pi(t-x)}{c}\, dt \qquad ...(2)$$

If we assume that $\int_{-\infty}^{\infty} |f(x)|\, dx$ converges, the first term on the right side of (2) approaches 0 as $c \to \infty$, since

$$\left| \frac{1}{2c} \int_{-c}^{c} f(t)\, dt \right| \le \frac{1}{2c} \int_{-\infty}^{\infty} |f(t)|\, dt$$

The second term on the right side of (2) tends to

$$\operatorname*{Lt}_{c \to \infty} \frac{1}{c} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} f(t) \cos \frac{n\pi(t-x)}{c}\, dt$$

$$= \operatorname*{Lt}_{\delta\lambda \to 0} \frac{1}{\pi} \sum_{n=1}^{\infty} \delta\lambda \int_{-\infty}^{\infty} f(t) \cos n\delta\lambda\, (t-x)\, dt, \text{ on writing } \pi/c = \delta\lambda$$

This is of the form $\operatorname*{Lt}_{\delta\lambda \to 0} \sum_{n=1}^{\infty} F(n\delta\lambda)$, *i.e.*, $\int_0^\infty F(\lambda)\, d\lambda$

Thus as $c \to \infty$, (2) becomes $f(x) = \dfrac{1}{\pi} \displaystyle\int_0^\infty \int_{-\infty}^{\infty} f(t) \cos \lambda(t-x)\, dt d\lambda$ ...(3)

which is known as the **Fourier integral** of $f(x)$.

Obs. We have given a heuristic demonstration of the Fourier integral theorem which simply helps in deriving the result (3). It cannot however, be taken as a rigorous proof for that would, involve a proof of the convergence of the Fourier integral which is beyond the scope of this book. When $f(x)$ satisfies the above-mentioned conditions, equation (3) holds good at a point of continuity. If however, x is point of discontinuity, we replace $f(x)$ by $\frac{1}{2}[f(x+0) + f(x-0)]$ as in the case of Fourier series.

(2) Fourier sine and cosine integrals. Expanding $\cos \lambda(t - x)$, (3) may be written as

$$f(x) = \frac{1}{\pi} \int_0^\infty \cos \lambda x \int_{-\infty}^{\infty} f(t) \cos \lambda t\, dt\, d\lambda + \frac{1}{\pi} \int_0^\infty \sin \lambda x \int_{-\infty}^{\infty} f(t) \sin \lambda t\, dt\, d\lambda \qquad ...(4)$$

If $f(x)$ is an odd function, $f(t) \cos \lambda t$ is also an odd function while $f(t) \sin \lambda t$ is even. Then the first term on the right side of (4) vanishes and, we get

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin \lambda x \int_0^\infty f(t) \sin \lambda t\, dt\, d\lambda \qquad ...(5)$$

which is known as the *Fourier sine integral*.

Similarly, if $f(x)$ is an even function, (4) takes the form

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos \lambda x \int_0^\infty f(t) \cos \lambda t \, dt \, d\lambda \qquad ...(6)$$

which is known as the *Fourier cosine integral.*

Obs. A function $f(x)$ defined in the interval $(0, \infty)$ is expressed either as a Fourier sine integral or as a Fourier cosine integral, merely looking upon it as an odd or even function in $(-\infty, \infty)$ on the lines of half-range Fourier series.

(3) **Complex form of Fourier integrals.** Equation (3) can be written as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t) \cos \lambda(t-x) dt d\lambda \qquad ..(7)$$

because $\cos \lambda(t-x)$ is an even function of λ. Also since $\sin \lambda(t-x)$ is an odd function of λ, we have

$$0 = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t) \sin \lambda(t-x) dt d\lambda \qquad ...(8)$$

Now multiply (8) by i and add it to (7), so that

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t)\, e^{i\lambda(t-x)} dt d\lambda \qquad ...(9)$$

which is the *complex* form of the *Fourier integral.*

(4) Fourier integral representation of a function

Using (4), *a function F(x) may be represented by a Fourier integral as*

$$F(x) = \frac{1}{\pi}\int_0^\infty [A(\lambda) \cos \lambda x + B(\lambda) \sin \lambda x] \, d\lambda$$

where $$A(\lambda) = \int_{-\infty}^\infty f(t) \cos \lambda t \, dt \, ; B(\lambda) = \int_{-\infty}^\infty f(t) \sin \lambda t \, dt \qquad ...(10)$$

If f (x) is an odd function, then [By (5)]

$$f(x) = \frac{1}{\pi}\int_0^\infty B(\lambda) \sin \lambda x \, d\lambda \text{ where } B(\lambda) = 2\int_0^\infty f(t) \sin \lambda t \, dt \qquad ...(11)$$

If f (x) is an even function, then [By (6)]

$$f(x) = \frac{1}{\pi}\int_0^\infty A(\lambda) \cos \lambda x \, d\lambda \text{ where } A(\lambda) = 2\int_0^\infty f(t) \cos \lambda t \, dt \qquad ...(12)$$

Example 22.1. *Express* $f(x) = 1$ *for* $0 \le x \le \pi$,
$= 0$ *for* $x > \pi$,
as a Fourier sine integral and hence evaluate

$$\int_0^\infty \frac{1-\cos(\pi\lambda)}{\lambda} \sin(x\lambda) d\lambda.$$ *(Kottayam, 2005 ; J.N.T.U., 2004 S)*

Solution. The Fourier sine integral for $f(x) = \dfrac{2}{\pi}\displaystyle\int_0^\infty \sin(\lambda x) \, d\lambda \int_0^\infty f(t) \sin(\lambda t) \, dt$

$$= \frac{2}{\pi}\int_0^\infty \sin(\lambda x) \, d\lambda \int_0^\infty \sin(\lambda t) \, dt$$

$$= \frac{2}{\pi}\int_0^\infty \sin(\lambda x) \, d\lambda \left| \frac{-\cos(\lambda t)}{\lambda} \right|_0^\pi = \frac{2}{\pi}\int_0^\infty \frac{1-\cos(\lambda\pi)}{\lambda} \sin(\lambda x) d\lambda$$

$$\therefore \quad \int_0^\infty \frac{1-\cos(\lambda\pi)}{\lambda} \sin(\lambda x) d\lambda = \frac{\pi}{2} f(x) = \begin{cases} \pi/2 & \text{for } 0 \le x < \pi \\ 0 & \text{for } x > \pi \end{cases}$$

At $x = \pi$, which is a point of discontinuity of $f(x)$, the value of the above integral

$$= \frac{\pi}{2}\left[\frac{f(\pi-0)+f(\pi+0)}{2}\right] = \frac{\pi}{2} \cdot \frac{1+0}{2} = \frac{\pi}{4}.$$

22.4 (1) FOURIER TRANSFORMS

Rewriting (9) of § 22.3 as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx}\, ds \int_{-\infty}^{\infty} f(t) e^{ist}\, dt,$$

it follows that if $$F(s) = \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt \quad ...(1)$$

then $$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds \quad ...(2)$$

The function F(s), defined by (1), is called the **Fourier transform** of $f(x)$. *Also the function* $f(x)$, *as given by* (2), *is called the* **inverse Fourier transform** *of* $F(s)$. Sometimes, we call (2) as an *inversion formula* corresponding to (1).

(2) Fourier sine and cosine transforms. From (5) of § 22.3, it follows that if

$$F_s(s) = \int_0^{\infty} f(x) \sin sx\, dx \quad ...(3)$$

then $$f(x) = \frac{2}{\pi} \int_0^{\infty} F_s(s) \sin sx\, ds \quad ...(4)$$

The function $F_s(s)$, *as defined by* (3), *is known as the* **Fourier sine transform** *of* $f(x)$ *in* $0 < x < \infty$. *Also the function* $f(x)$, *as given by* (4) *is called the* **inverse Fourier sine transform** *of* $F_s(s)$.

Similarly, if follows from (6) of § 22.3 that if

$$F_c(s) = \int_0^{\infty} f(x) \cos sx\, dx \quad ...(5)$$

then $$f(x) = \frac{2}{\pi} \int_0^{\infty} F_c(s) \cos sx\, ds \quad ...(6)$$

The function $F_c(s)$ *as defined by* (5) *is known as the* **Fourier cosine transform** *of* $f(x)$ *in* $0 < x < \infty$. *Also the function* $f(x)$, *as given by* (6), *is called the* **inverse Fourier cosine transform** *of* $F_c(s)$.

(3) **Finite Fourier sine and cosine transforms.** These transforms are useful for such a boundary-value problem in which at least two of the boundaries are parallel and separated by a finite distance.

The **finite Fourier sine transform** *of* $f(x)$, *in* $0 < x < c$, *is defined as*

$$F_s(n) = \int_0^c f(x) \sin \frac{n\pi x}{c}\, dx \quad ...(7)$$

where n is an integer.

The function $f(x)$ *is then called the* **inverse finite Fourier sine transform** *of* $F_s(n)$ *which is given by*

$$f(x) = \frac{2}{c} \sum_{n=1}^{\infty} F_s(n) \sin \frac{n\pi x}{c} \quad ...(8)$$

The **finite Fourier cosine transform** *of* $f(x)$, *in* $0 < x < c$, *is defined as*

$$F_c(n) = \int_0^c f(x) \cos \frac{n\pi x}{c}\, dx \quad ...(9)$$

where n is an integer.

The function $f(x)$ *is then called the* **inverse finite Fourier cosine transform** *of* $F_c(n)$ *which is given by*

$$f(x) = \frac{1}{c} F_c(0) + \frac{2}{c} \sum_{n=1}^{\infty} F_c(n) \cos \frac{n\pi x}{c} \quad ...(10)$$

Obs. The finite Fourier sine transform is useful for problems involving boundary conditions of heat distribution on two parallel boundaries, while the finite cosine transform is useful for problems in which the velocities normal to two parallel boundaries are among the boundary conditions.

22.5 PROPERTIES OF FOURIER TRANSFORMS

(1) **Linear property.** *If F(s) and G(s) are Fourier transforms of* $f(x)$ *and* $g(x)$ *respectively, then*

$$F[a\, f(x) + bg(x)] = a\, F(s) + bG(s)$$

where a and b are constants.

We have $F(s) = \int_{-\infty}^{\infty} e^{isx} f(x)\,dx$ and $G(s) = \int_{-\infty}^{\infty} e^{isx} g(x)\,dx$

$$\therefore \quad F[af(x) + bg(x)] = \int_{-\infty}^{\infty} e^{isx} [af(x) + bg(x)]\,dx = a\int_{-\infty}^{\infty} e^{isx} f(x)\,dx + b\int_{-\infty}^{\infty} e^{isx} g(x)dx$$
$$= aF(s) + bG(s)$$

(2) Change of scale property. *If F(s) is the complex Fourier transform of f(x), then*

$$F\{f(ax)\} = \frac{1}{a}F\left(\frac{s}{a}\right), a \neq 0$$

We have $$F(s) = \int_{-\infty}^{\infty} e^{isx} f(x)dx \qquad ...(i)$$

$$\therefore \quad F\{f(ax)\} = \int_{-\infty}^{\infty} e^{isx} f(ax)dx \qquad \left| \begin{array}{l} \text{Put } ax = t \\ \text{so that } dx = dt/a \end{array} \right.$$

$$= \int_{-\infty}^{\infty} e^{ist/a} f(t)dt/a = \frac{1}{a}\int_{-\infty}^{\infty} e^{i(s/a)t} f(t)dt = \frac{1}{a}F\left(\frac{s}{a}\right) \qquad \text{[By } (i)]$$

Cor. *If $F_s(s)$ and $F_c(s)$ are the Fourier sine and cosine transforms of f (x) respectively, then*

$$F_s\{f(ax)\} = \frac{1}{a}F_s\left(\frac{s}{a}\right) \quad \text{and} \quad F_c\{f(ax)\} = \frac{1}{a}F_s\left(\frac{s}{a}\right).$$

(3) Shifting property. *If F(s) is the complex Fourier transform of f (x), then*

$$F\{f(x - a)\} = e^{isa} F(s)$$

We have $$F(s) = \int_{-\infty}^{\infty} e^{isx} f(x)dx \qquad ...(i)$$

$$\therefore \quad F\{f(x - a)\} = \int_{-\infty}^{\infty} e^{isx} f(x - a)dx \qquad \left| \begin{array}{l} \text{Put } x - a = t \\ \text{so that } dx = dt \end{array} \right.$$

$$= \int_{-\infty}^{\infty} e^{is(t+a)} f(t)dt = e^{isa}\int_{-\infty}^{\infty} e^{ist} f(t)dt = e^{isa} F(s) \qquad \text{[By } (i)]$$

(4) Modulation theorem. *If F(s) is the complex Fourier transform of f (x), then*

$$F\{f(x)\cos ax\} = \frac{1}{2}[F(s + a) + F(s - a)]$$

We have $$F(s) = \int_{-\infty}^{\infty} e^{isx} f(x)dx \qquad ...(i)$$

$$\therefore \quad F\{f(x)\cos ax\} = \int_{-\infty}^{\infty} e^{isx} f(x)\cos ax\,dx = \int_{-\infty}^{\infty} e^{isx}\,.\,f(x)\,.\,\frac{e^{iax} + e^{-iax}}{2}\,dx$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} e^{i(s+a)x} f(x)dx + \int_{-\infty}^{\infty} e^{i(s-a)x} f(x)dx\right] = \frac{1}{2}[F(s + a) + F(s - a)].$$

Cor. *If $F_s(s)$ and $F_c(s)$ are Fourier sine and cosine transforms of f (x) respectively, then*

(i) $F_s\{f(x)\cos ax\} = \frac{1}{2}[F_s(s + a) + F_s(s - a)]$ *(Anna, 2008)*

(ii) $F_c\{f(x)\sin ax\} = \frac{1}{2}[F_s(s + a) - F_s(s - a)]$

(iii) $F_s\{f(x)\sin ax\} = \frac{1}{2}[F_c(s - a) - F_c(s + a)]$

Obs. This theorem is of great importance in radio and television where the harmonic carrier wave is modulated by an envelope.

Example 22.2. *Find the Fourier transform of*

$$f(x) = \begin{cases} 1 \text{ for } |x| < 1 \\ 0 \text{ for } |x| > 1 \end{cases}.$$

Hence evaluate $\int_0^{\infty} \frac{\sin x}{x}\,dx.$ *(V.T.U., 2010 ; S.V.T.U., 2009 ; U.P.T.U., 2008)*

Solution. The Fourier transform of $f(x)$, *i.e.*,

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{isx}\,dx = \int_{-1}^{1}(1)\,e^{isx}\,dx = \left|\frac{e^{isx}}{is}\right|_{-1}^{1} = \frac{e^{is}-e^{-is}}{is}$$

Thus $\quad F\{f(x)\} = F(s) = 2\,\dfrac{\sin s}{s}$, $s \neq 0$. For $s = 0$, we have $F(s) = 2$.

Now by the inversion formula, we get

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{-isx}ds, \text{ or } \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\sin s}{s}\,e^{-isx}ds = \begin{cases}1 \text{ for } |x|<1\\ 0 \text{ for } |x|>1\end{cases}$$

Putting $x = 0$, we get

$$\int_{-\infty}^{\infty}\frac{\sin s}{s}\,ds = \pi \;\therefore\; \int_{0}^{\infty}\frac{\sin s}{s}\,ds = \frac{\pi}{2}, \text{ since the integrand is even.}$$

Example 22.3. *Find the Fourier transform of :*

$$f(x) = \begin{cases}1-x^2, & |x| \leq 1\\ 0 & , |x|>1\end{cases}$$

Hence evaluate $\displaystyle\int_0^{\infty}\frac{x\cos x - \sin x}{x^3}\cos\frac{x}{2}\,dx.$ *(V.T.U., 2011 S ; Anna, 2005 S ; Mumbai, 2005 S)*

Solution. $F\{f(x)\} = \displaystyle\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx = F(s)$, say

$$= \int_{-\infty}^{-1}(0)\,e^{isx}\,dx + \int_{-1}^{1}(1-x^2)\,e^{isx}\,dx + \int_{1}^{\infty}(0)\,e^{isx}\,dx = \left|(1-x^2)\frac{e^{isx}}{is} - (2x)\frac{e^{isx}}{(is)^2} + (-2)\frac{e^{isx}}{(is)^3}\right|_{-1}^{1}$$

$$= 2\left(\frac{e^{is}+e^{-is}}{-s^2}\right) - 2\left(\frac{e^{is}-e^{-is}}{-is^3}\right) = -\frac{4}{s^3}(s\cos s - \sin s)$$

Now by inversion formula, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}F(s)e^{-isx}\,ds$$

or $$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4}{s^3}(s\cos s - \sin s)\,e^{-isx}\,ds = \begin{cases}1-x^2, & |x| \leq 1\\ 0, & |x|>1\end{cases}$$

Putting $x = 1/2$, we obtain

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4}{s^3}(s\cos s - \sin s)e^{-is/2}ds = \frac{3}{4}$$

or $$\int_{-\infty}^{\infty}\frac{s\cos s - \sin s}{s^3}\left(\cos\frac{s}{2} - i\sin\frac{s}{2}\right)ds = -\frac{3\pi}{8}$$

or $$\int_{-\infty}^{\infty}\frac{s\cos s - \sin s}{s^3}\,.\cos\frac{s}{2}\,ds = -\frac{3\pi}{8}$$

or $$\int_{0}^{\infty}\frac{x\cos x - \sin x}{x^3}\,.\cos\frac{x}{2}\,dx = -\frac{3\pi}{16}, \text{ since the integral is even.}$$

Example 22.4. (*a*) *Find the Fourier transform of* $e^{-a^2x^2}$, $a < 0$. *Hence deduce that* $e^{-x^2/2}$ *is self reciprocal in respect of Fourier transform.* *(Madras, 2006 ; Kottayam, 2005)*

(*b*) *Find Fourier transform of* (*i*) $e^{-2(x-3)^2}$ (*ii*) $e^{-x^2}\cos 3x$.

Solution. (*a*) $\quad F(e^{-a^2x^2}) = \displaystyle\int_{-\infty}^{\infty} e^{-a^2x^2}\,.\,e^{isx}\,dx = \int_{-\infty}^{\infty} e^{-a^2(x^2 - isx/a^2)}dx$

$$= \int_{-\infty}^{\infty} e^{-a^2(x - is/2a^2)^2}\,.\,e^{-s^2/4a^2}\,dx$$

$$= \int_{-\infty}^{\infty} e^{-t^2} \cdot e^{-s^2/4a^2}\, dt/a \qquad \text{[Putting } a(x - is/2a^2) = t,\ dx = dt/a$$

$$= \frac{e^{-s^2/4a^2}}{a} \int_{-\infty}^{\infty} e^{-t^2}\, dt = \frac{e^{-s^2/4a^2}}{a} \sqrt{\pi} \qquad \left[\because \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}\right]$$

Hence $F(e^{-a^2x^2}) = \dfrac{\sqrt{\pi}}{a} e^{-s^2/4a^2}$

Taking $a^2 = 1/2$, we have

$$F(e^{-x^2/2}) = \frac{\sqrt{\pi}}{(1/\sqrt{2})} e^{-s^2/2} = \sqrt{2\pi} e^{-s^2/2}$$

i.e., Fourier transform of $e^{-x^2/2}$ is a constant times $e^{-s^2/2}$. Also the functions $e^{-x^2/2}$ and $e^{-s^2/2}$ are the same. Hence it follows that $e^{-x^2/2}$ *is self-reciprocal under the Fourier transform.*

(*b*) Since $e^{-2x^2} = e^{-(2x)^2/2} = f(2x)$ where $f(x) = e^{-x^2/2}$

$\therefore$ by change of scale property, $F[f(2x)] = \dfrac{1}{2} F(s/2)$

i.e., $$F(e^{-2x^2}) = F[e^{-(2x)^2/2}] = \sqrt{2\pi} e^{-(s/2)^2/2} = \sqrt{2\pi} e^{-s^2/8}$$

By shifting property $F f(x-3) = e^{i3s} F(3)$

$$\therefore \quad F[e^{-2(x-3)^2}] = e^{3is} \sqrt{2\pi} e^{-s^2/8} = \sqrt{2\pi} e^{(3is - s^2/8)} \qquad ...(i)$$

Also by modulation theorem,

$$F[f(x) \cos 2x] = \frac{1}{2} [F(s+a) + F(s-a)]$$

$$F(e^{-x^2} \cos 3x) = \frac{1}{2} \sqrt{2\pi} [e^{-(s+3)^2/2} + e^{-(s-3)^2/2}]. \qquad ...(ii)$$

Example 22.5. *Find the Fourier cosine transform of* e^{-x^2}. (*V.T.U., 2010; Rajasthan, 2006*)

Solution. We have $F_c(e^{-x^2}) = \int_0^{\infty} e^{-x^2} \cos sx\, dx = I$ (say)

Differentiating under the integral sign w.r.t. s,

$$\frac{dI}{ds} = -\int_0^{\infty} x e^{-x^2} \sin sx\, dx = \frac{1}{2} \int_0^{\infty} (\sin sx)(-2xe^{-x^2}) dx$$

$$= \frac{1}{2} \left\{ \left| \sin sx \cdot e^{-x^2} \right|_0^{\infty} - s \int_0^{\infty} \cos sx \cdot e^{-x^2}\, dx \right\}$$

$$= -\frac{s}{2} \int_0^{\infty} e^{-x^2} \cos sx\, dx = -\frac{s}{2} I \quad \text{or} \quad \frac{dI}{I} = -\int \frac{s}{2} ds + \log c$$

or $$\log I = -\frac{s^2}{4} + \log c = \log (ce^{-s^2/4})$$

$$\therefore \quad I = ce^{-s^2/4} \quad \text{or} \quad \int_0^{\infty} e^{-x^2} \cos sx\, dx = ce^{-s^2/4}$$

Putting $s = 0$, $$c = \int_0^{\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}. \ \text{i.e. } I = \frac{\sqrt{\pi}}{2} e^{-s^2/4}.$$

Hence $$F_c(e^{-x^2}) = \frac{\sqrt{\pi}}{2} e^{-s^2/4}.$$

Example 22.6. *Find the Fourier sine transform of* $e^{-|x|}$.

Hence show that $\displaystyle\int_0^{\infty} \frac{x \sin mx}{1+x^2} dx = \frac{\pi e^{-m}}{2}$, $m > 0$. (*V.T.U., 2010 ; S.V.T.U., 2008 ; Kottayam, 2005*)

Solution. x being positive in the interval $(0, \infty)$, $e^{-|x|} = e^{-x}$

$\therefore$ Fourier sine transform of $f(x) = e^{-|x|}$ is given by

$$F_s\{f(x)\} = \int_0^\infty f(x) \sin sx \, dx = \int_0^\infty e^{-x} \sin sx \, dx$$

$$= \left| \frac{e^{-x}}{1+s^2} (-\sin sx - s \cos sx) \right|_0^\infty = \frac{s}{1+s^2}$$

Using Inversion formula for Fourier sine transforms, we get

$$f(x) = \frac{2}{\pi} \int_0^\infty F_s\{f(x)\} \sin sx \, dx \quad \text{or} \quad e^{-x} = \frac{2}{\pi} \int_0^\infty \frac{s}{1+s^2} \sin sx \, ds$$

or changing x to m, $$e^{-m} = \frac{2}{\pi} \int_0^\infty \frac{s \sin ms}{1+s^2} ds = \frac{2}{\pi} \int_0^\infty \frac{x \sin mx}{1+m^2} dx$$

Hence $$\int_0^\infty \frac{x \sin mx}{1+m^2} dx = \frac{\pi e^{-m}}{2}.$$

Example 22.7. *Find the Fourier cosine transform of* $f(x) = \begin{cases} x, \text{ for } 0 < x < 11 \\ 2 - x, \text{ for } 1 < x < 2 \\ 0, \text{ for } x > 2 \end{cases}$ *(J.N.T.U., 2006)*

Solution. Fourier cosine transform of $f(x)$ *i.e.*, $F_c[f(x)]$

$$= \int_0^\infty f_c(x) \cos sx \, dx = \int_0^1 x \cos sx \, dx + \int_1^2 (2-x) \cos sx \, dx + \int_2^\infty 0 \,.\, dx$$

$$= \left| x \frac{\sin sx}{s} - \left(\frac{-\cos sx}{s^2} \right) \right|_0^1 + \left| (2-x) \frac{\sin sx}{s} - (-1) \frac{-\cos sx}{s^2} \right|_1^2$$

$$= \left(\frac{\sin s}{s} + \frac{\cos s}{s^2} - \frac{1}{s^2} \right) + \left(-\frac{\cos 2s}{s^2} - \frac{\sin s}{s} + \frac{\cos s}{s^2} \right)$$

$$= \frac{2 \cos s}{s^2} - \frac{\cos 2s}{s^2} - \frac{1}{s^2}.$$

Example 22.8. *Find the Fourier sine transform of* e^{-ax}/x. *(V.T.U., 2010 S ; P.T.U., 2006 ; Rohtak, 2005)*

Solution. Let $f(x) = e^{-ax}/x$, then its Fourier sine transform

i.e. $$F_s\{f(x)\} = \int_0^\infty f(x) \sin sx \, dx = \int_0^\infty \frac{e^{-ax}}{x} \sin sx \, dx = F(s), \text{ say}$$

Differentiating both sides w.r.t. s, we get

$$\frac{d}{ds}\{F(s)\} = \int_0^\infty \frac{xe^{-ax} \cos sx}{x} dx = \int_0^\infty e^{-ax} \cos sx \, dx = \frac{a}{s^2 + a^2}$$

Integrating w.r.t. s, we obtain $F(s) = \int_0^\infty \frac{a}{s^2+a^2} ds = \tan^{-1} \frac{s}{a} + c$

But $F(s) = 0$, when $s = 0$; $\therefore$ $c = 0$. Hence $F(s) = \tan^{-1}(s/a)$.

Example 22.9. *Find the Fourier cosine transform of* $f(x) = 1/(1+x^2)$. *(V.T.U., 2011 S ; Anna, 2009)*
Hence derive Fourier sine transform of $\phi(x) = x/(1+x^2)$. *(V.T.U., 2009 S)*

Solution. $$F_c[f(x)] = \int_0^\infty \frac{\cos sx}{1+x^2} dx = I, \text{ say} \quad ...(i)$$

$\therefore$ $$\frac{dI}{ds} = \int_0^\infty \frac{-x \sin sx}{1+x^2} dx = -\int_0^\infty \frac{x^2 \sin sx}{x(1+x^2)} dx \quad ...(ii)$$

$$= -\int_0^\infty \frac{[(1+x^2)-1]\sin sx}{x(1+x^2)}\,dx = -\int_0^\infty \frac{\sin sx}{x}\,dx + \int_0^\infty \frac{\sin sx}{x(1+x^2)}\,dx$$

or $$\frac{dI}{ds} = -\frac{\pi}{2} + \int_0^\infty \frac{\sin sx}{x(1+x^2)}\,dx \quad \text{...(iii)}$$

$\therefore$ $$\frac{d^2I}{ds^2} = \int_0^\infty \frac{x\cos sx}{x(1+x^2)}\,dx = I$$

or $$\frac{d^2I}{ds^2} - I = 0 \quad \text{or} \quad (D^2-1)I = 0, \text{ where } D = \frac{dI}{ds}$$

Its solution is $$I = c_1e^s + c_2e^{-s} \quad \text{...(iv)}$$

$\therefore$ $$dI/ds = c_1e^s - c_2e^{-s} \quad \text{...(v)}$$

When $s = 0$, (*i*) and (*iv*) give $c_1 + c_2 = \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$

Also when $s = 0$, (*iii*) and (*v*) give $c_1 - c_2 = -\pi/2$.

Solving these, $c_1 = 0, c_2 = \pi/2$.

Thus from (*i*) and (*iv*), we have $F_c[f(x)] = I = (\pi/2)e^{-s}$

Now $$F_s[\phi(x)] = \int_0^\infty \frac{x\sin sx}{1+x^2}\,dx = -\frac{dI}{ds}, \text{ from (ii)}$$

$$= (\pi/2)e^{-s}, \text{ from (v), with } c_1 = 0, c_2 = \pi/2.$$

Example 22.10. *Find the Fourier sine and cosine transform of x^{n-1}, $n > 0$.* *(Madras, 2006)*

Solution. We know that $$F_s(x^{n-1}) = \int_0^\infty x^{n-1}\sin sx\,dx \quad \text{...(i)}$$

and $$F_c(x^{n-1}) = \int_0^\infty x^{n-1}\cos sx\,dx \quad \text{...(ii)}$$

$\therefore$ $$F_c(x^{n-1}) + i\,F_s(x^{n-1}) = \int_0^\infty (\cos sx + i\sin sx)x^{n-1}\,dx$$

$$= \int_0^\infty e^{isx}x^{n-1}\,dx = \int_0^\infty e^{-t}\left(-\frac{t}{is}\right)^{n-1}\left(-\frac{dt}{is}\right) \quad \text{[Where } isx = -t]$$

$$= \left(-\frac{1}{(i)}\right)^n \int_0^\infty e^{-t}t^{n-1}\,dt = \frac{(i)^{2n}}{(i)^n s^n}\Gamma(n) = \frac{(i)^n}{s^n}\Gamma(n)$$

$$= \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^n \Gamma(n)/s^n = \left(\cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2}\right)\Gamma(n)/s^n$$

Equating real and imaginary parts, we get

$$F_c(x^{n-1}) = \frac{\Gamma(n)}{s^n}\cos\frac{n\pi}{2} \quad \text{and} \quad F_s(x^{n-1}) = \frac{\Gamma(n)}{s^n}\sin\frac{n\pi}{2}.$$

Example 22.11. (*a*) *Show that $F_s[x\,f(x)] = -\frac{d}{ds}\{F_c(s)\}$; $F_c[x\,f(x)] = \frac{d}{ds}\{F_s(s)\}$.*

(*b*) *Find the Fourier sine and cosine transform of xe^{-ax}* *(Madras, 2006)*

Solution. (*a*) $$\frac{d}{ds}\{F_c(s)\} = \frac{d}{ds}\left\{\int_0^\infty f(x)\cos sx\,dx\right\} = \int_0^\infty f(x)(-x\sin sx)dx$$

$$= -\int_0^\infty \{x\,f(x)\}\sin sx\,dx = -F_s[x\,f(x)] \quad \text{...(i)}$$

$$\frac{d}{ds}\{F_s(s)\} = \frac{d}{ds}\left\{\int_0^\infty f(x)\sin sx\,dx\right\} = \int_0^\infty f(x)(x\cos sx)dx$$

$$= \int_0^\infty \{x\,f(x)\}\cos sx\,dx = F_c[x\,f(x)] \quad \text{...(ii)}$$

(b) We have $$F_s(e^{-ax}) = \int_0^\infty e^{-ax} \sin sx\, dx = \frac{e^{-ax}}{a^2+s^2} \left| -a \sin sx - s \cos sx \right|_0^\infty$$

$$= \frac{s}{a^2+s^2} \qquad ...(iii)$$

and $$F_c(e^{-ax}) = \int_0^\infty e^{-ax} \cos sx\, dx = \frac{e^{-ax}}{a^2+s^2} \left| -a \cos sx + s \sin sx \right|_0^\infty$$

$$= \frac{a}{a^2+s^2} \qquad ...(iv)$$

Now $$F_s(xe^{-ax}) = -\frac{d}{ds}\{F_c(e^{-ax})\} \qquad \text{[by (i)]}$$

$$= -\frac{d}{ds}\left(\frac{a}{a^2+s^2}\right) = \frac{2as}{(a^2+s^2)^2} \qquad \text{[by (iv)]}$$

$$F_c(xe^{-ax}) = \frac{d}{ds}\{F_s(e^{-ax})\} \qquad \text{[by (ii)]}$$

$$= \frac{d}{ds}\left(\frac{s}{a^2+s^2}\right) = \frac{(a^2+s^2) - s(2s)}{(a^2+s^2)^2} = \frac{a^2-s^2}{(a^2+s^2)^2}. \qquad \text{[by (iii)]}$$

Example 22.12. *If the Fourier sine transform of $f(x) = \dfrac{1-\cos n\pi}{n^2\pi^2}$ $(0 \le x \le \pi)$, find $f(x)$.* (*Delhi, 2002*)

Solution. We have $f(x)$ = inverse finite Fourier sine transform of $F_s(n)$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty} F_s(n) \sin \frac{n\pi x}{\pi} = \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{1-\cos n\pi}{n^2\pi^2}\right\}\sin nx$$

$$= \frac{2}{\pi^3}\sum_{n=1}^{\infty}\left\{\frac{1-\cos n\pi}{n^2}\right\}\sin nx.$$

Example 22.13. *Solve the integral equation**

$$\int_0^\infty f(\theta)\cos \alpha\theta\, d\theta = \begin{cases} 1-\alpha, & 0 \le a \le 1 \\ 0, & \alpha > 1 \end{cases}$$

Hence evaluate $\int_0^\infty \frac{\sin^2 t}{t^2}\, dt$. (*V.T.U., 2011 S ; Kurukshetra, 2005*)

Solution. We have $$\int_0^\infty f(\theta)\cos \alpha\theta\, d\theta = F_c(\alpha)$$

$$\therefore \quad F_c(\alpha) = \begin{cases} 1-\alpha, & 0 \le a \le 1 \\ 0, & \alpha > 1 \end{cases} \qquad ...(i)$$

By the inversion formula, we have

$$f(\theta) = \frac{2}{\pi}\int_0^\infty F_c(\alpha)\cos\alpha\theta\, d\alpha = \frac{2}{\pi}\int_0^1 (1-\alpha)\cos\alpha\theta\, d\alpha \qquad \text{[Integrating by parts]}$$

$$= \frac{2}{\pi}\left[\left|(1-\alpha)\frac{\sin\alpha\theta}{\theta}\right|_0^1 - \int_0^1 (-1)\frac{\sin\alpha\theta}{\theta}\, d\alpha\right] = \frac{2}{\pi\theta}\left|-\frac{\cos\alpha\theta}{\theta}\right|_0^1 = \frac{2(1-\cos\theta)}{\pi\theta^2}$$

Now $$F_c(\alpha) = \int_0^\infty f(\theta)\cos\alpha\theta\, d\theta = \int_0^\infty \frac{2(1-\cos\theta)}{\pi\theta^2}\cos\alpha\theta\, d\theta \qquad ...(ii)$$

* Refer to Chapter 26.

$\therefore$ From (*i*) and (*ii*), we have

$$\frac{2}{\pi}\int_0^\infty \frac{1-\cos\theta}{\theta^2}\cos\alpha\theta\, d\theta = \begin{cases}1-\alpha, & 0\le\alpha\le 1\\ 0 & ,\alpha>1\end{cases}$$

Now letting $\alpha \to 0$, we get $\frac{2}{\pi}\int_0^\infty \frac{1-\cos\theta}{\theta^2}\, d\theta = 1$ (*V.T.U., 2008*)

or $$\int_0^\infty \frac{2\sin^2\theta/2}{\theta^2}\, d\theta = \pi/2$$ [Put $\theta/2 = t$, so that $d\theta = 2dt$]

$\therefore$ $$\int_0^\infty \frac{\sin^2 t}{t^2} = \pi/2.$$

PROBLEMS 22.1

1. Express the function $f(x) = \begin{cases}1 \text{ for } |x|\le 1\\ 0 \text{ for } |x|>1\end{cases}$ as a Fourier integral.

   Hence evaluate $\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}\, d\lambda$. (*Kottayam, 2005*)
2. Find the Fourier integral representation for

   (*i*) $f(x) = \begin{cases}1-x^2, \text{ for } |x|\le 1\\ 0, \text{ for } |x|>1\end{cases}$ (*Mumbai, 2008*) (*ii*) $f(x) = \begin{cases}e^{ax}, \text{ for } x\le 0, a>0\\ e^{-ax}, \text{ for } x\ge 0\ a<0\end{cases}$
3. Using the Fourier integral representation, show that

   (*i*) $\int_0^\infty \frac{\omega\sin x\omega}{1+\omega^2}\, d\omega = \frac{\pi}{2}e^{-x}\ (x>0)$ (*ii*) $\int_0^\infty \frac{\cos\omega x}{1+\omega^2}\, d\omega = \frac{\pi}{2}e^{-x}\ (x\ge 0)$ (*U.P.T.U., 2008*)

   (*iii*) $\int_0^\infty \frac{\sin\omega\cos x\omega}{\omega}\, d\omega = \frac{\pi}{2}$ when $0\le x<1$. (*iv*) $\int_0^\infty \frac{\sin\pi\alpha\sin\alpha\theta}{1-\alpha^2}\, d\alpha = \begin{cases}\frac{1}{2}\pi\sin\theta, & 0\le\theta\le\pi\\ 0, & \theta>\pi\end{cases}$
4. Find the Fourier transforms of

   (*i*) $f(x) = \begin{cases}1, |x|<a\\ 0, |x|>a\end{cases}$ (*W.B.T.U., 2005 ; Madras, 2003 ; P.T.U., 2003*)

   Hence evaluate $\int_{-\infty}^\infty \frac{\sin ax}{x}\, dx$ (*Mumbai, 2009*)

   (*ii*) $f(x) = \begin{cases}x^2, |x|<a\\ 0, |x|>a\end{cases}$ (*S.V.T.U., 2008*)
5. Find the Fourier transform of $f(x) = \begin{cases}a^2-x^2 \text{ for } |x|\le a\\ 0 \text{ for } |x|>a\end{cases}$ (*V.T.U., 2007*)

   Hence deduce that $\int_0^\infty \frac{\sin t - t\cos t}{t^3}\, dt = \frac{\pi}{4}$. (*Anna, 2009*)
6. Given $F(e^{-x^2}) = \sqrt{\pi}e^{-s^2/4}$, find the Fourier transform of

   (*i*) $e^{-x^2/3}$ (*ii*) $e^{-4(x-3)^2}$
7. Find the Fourier sine and cosine transforms of $f(x) = \begin{cases}1, 0\le x<2\\ 0, x\ge 2\end{cases}$ (*V.T.U., 2008*)
8. Using the Fourier sine transform of e^{-ax} $(a>0)$, show that $\int_0^\infty \frac{x\sin kx}{a^2+x^2}\, dx = \frac{\pi}{2}e^{-ak}\ (k>0)$.

   Hence obtain the Fourier sine transform of $x/(a^2+x^2)$. (*Rohtak, 2006 ; Madras, 2003 S*)
9. Find the Fourier cosine transform of e^{-ax}. (*Anna, 2009*)

   Hence evaluate $\int_0^\infty \frac{\cos\lambda x}{x^2+\alpha^2}\, dx$. (*V.T.U., 2003 S*)
10. If the Fourier sine transform of $f(x)$ is e^{-as}/s, find $f(x)$. Hence obtain the inverse Fourier sine transform of $1/s$. (*Mumbai, 2009*)

11. Find the Fourier cosine transform of e^{-x^2} and hence evaluate Fourier sine transform of xe^{-x^2}.
12. Find the Fourier cosine transform of $e^{-a^2x^2}$ for any $a > 0$ and hence prove that $e^{-x^2/2}$ is self-reciprocal under Fourier cosine transform. *(Anna, 2009)*
13. Find the Fourier sine transform of (*i*) $\dfrac{1}{x(x^2+a^2)}$. *(Rohtak, 2006)*
(*ii*) $[e^{-ax}/x]$, $a > 0$ *(U.P.T.U., 2008)*
14. Obtain Fourier sine transform of
(*i*) $f(x) = \begin{cases} \sin x, & 0 < x < a \\ 0, & x > a \end{cases}$ *(Madras, 2000)* (*ii*) $f(x) = \begin{cases} 4x, & \text{for } 0 < x < 1 \\ 4 - x, & \text{for } 1 < x < 4 \\ 0, & \text{for } x > 4 \end{cases}$ *(V.T.U., 2006)*
15. Find the Fourier cosine transform of $(1 - x/\pi)^2$. *(P.T.U., 2006)*
16. Find the finite Fourier sine and cosine transforms of $f(x) = 2x$, $0 < x < 4$. *(V.T.U., 2011)*
17. Find the finite sine transform of $f(x) = \begin{cases} -x, & x < c \\ \pi - x, & x > c \end{cases}$ where $0 \le c \le \pi$.
18. Show that the inverse finite Fourier sine transform of $F_s(n) = \dfrac{1}{\pi}\left\{1 + \cos n\pi - 2\cos\dfrac{n\pi}{2}\right\}$ is
$$f(x) = \begin{cases} 1, & 0 < x < \pi/2 \\ -1, & \pi/2 < x < \pi \end{cases}$$ *(V.T.U., 2008)*
19. Solve the integral equation $\displaystyle\int_0^\infty f(x)\sin tx\,dx = \begin{cases} 1, & 0 \le t < 1, \\ 2, & 1 \le t < 2, \\ 0, & t \ge 2 \end{cases}$ *(Kottayam, 2005)*
20. Solve the integral equation $\displaystyle\int_0^\infty f(x)\cos \alpha x\,dx = e^{-\alpha}$. *(S.V.T.U., 2009 ; Rohtak, 2004)*

22.6 (1) CONVOLUTION

The convolution of two functions $f(x)$ and $g(x)$ over the interval $(-\infty, \infty)$ is defined as
$$f * g = \int_{-\infty}^{\infty} f(u)\, g(x-u)\,du = h(x).$$

(2) **Convolution theorem for Fourier transforms.** *The Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transforms, i.e.,*
$$F\{f(x) * g(x)\} = F\{f(x)\} \,.\, F\{g(x)\}$$

We have
$$F\{f(x) * g(x)\} = F\left\{\int_{-\infty}^{\infty} f(u)\, g(x-u)\,du\right\}$$
$$= \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(u)\, g(x-u)\,du\right\} e^{isx}\,dx = \int_{-\infty}^{\infty} f(u)\left\{\int_{-\infty}^{\infty} g(x-u)\,.\,e^{isx}\,dx\right\}du$$
[Changing the order of integration]
$$= \int_{-\infty}^{\infty} f(u)\left\{\int_{-\infty}^{\infty} e^{is(x-u)}\,.\,g(x-u)\,d(x-u)\right\} e^{isu}\,du$$
$$= \int_{-\infty}^{\infty} e^{isu} f(u)\left\{\int_{-\infty}^{\infty} e^{ist}\, g(t)\,dt\right\} du \text{ where } x - u = t$$
$$= \int_{-\infty}^{\infty} e^{isu} f(u)\,du \,.\, F\{g(t)\} = \int_{-\infty}^{\infty} e^{isx} f(x)\,dx \,.\, F\{g(x)\} = F\{f(x)\}\,.\,F\{g(x)\}$$

22.7 PARSEVAL'S IDENTITY FOR FOURIER TRANSFORMS

If the Fourier transforms of $f(x)$ and $g(x)$ are $F(s)$ and $G(s)$ respectively, then

(*i*) $\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} F(s)\overline{G}(s)\,ds = \int_{-\infty}^{\infty} f(x)\overline{g}(x)\,dx$ (*ii*) $\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} |F(s)|^2\,ds = \int_{-\infty}^{\infty} |f(x)|^2\,dx$

where bar implies the complex conjugate.

(i) $\int_{-\infty}^{\infty} f(x)\,\bar{g}\,(dx) \int_{-\infty}^{\infty} f(x)\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{G}(s)\,e^{isx}\,ds\right\}dx$ [Using the inversion formula for Fourier transform]

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{G}(s)\left\{\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx\right\}ds$$ [Changing the order of integration]

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{G}(s)F(s)ds,$$ by definition of F-transform.

(ii) Taking $g(x) = f(x)$, we get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\,\bar{F}\,(s)ds = \int_{-\infty}^{\infty} f(x)\,\bar{f}(x)dx \text{ or } \frac{1}{2\pi}\int_{-\infty}^{\infty} |\,F(s)\,|^2\,ds = \int_{-\infty}^{\infty} |\,f(x)\,|^2\,dx$$

Obs. The following Parseval's identities for Fourier cosine and sine transforms can be proved as above :

(i) $\frac{2}{\pi}\int_0^{\infty} F_c(s)\,G_c(s)ds = \int_0^{\infty} f(x)\,g(x)dx$ (ii) $\frac{2}{\pi}\int_0^{\infty} F_s(s)\,G_s(s)ds = \int_0^{\infty} f(x)\,g(x)dx$

(iii) $\frac{2}{\pi}\int_0^{\infty} |F_c(s)|^2\,ds = \int_0^{\infty} |f(x)|^2\,dx$ (iv) $\frac{2}{\pi}\int_0^{\infty} |F_s(s)|^2\,ds = \int_0^{\infty} |f(x)|^2\,dx.$

Example 22.14. *Using Parseval's identities, prove that*

(i) $\int_0^{\infty} \frac{dt}{(a^2+t^2)(b^2+t^2)} = \frac{\pi}{2ab(a+b)}$ *(S.V.T.U., 2009 ; ... 2008)*

(ii) $\int_0^{\infty} \frac{t^2}{(t^2+1)^2}\,dt = \frac{\pi}{4}$ (iii) $\int_0^{\infty} \frac{\sin at}{t(a^2+t^2)}\,dt = \frac{\pi}{2}\cdot\frac{1-e^{-a^2}}{a^2}.$

Solution. (i) Let $f(x) = e^{-ax}$ and $g(x) = e^{-bx}$. Then $F_c(s) = \frac{a}{a^2+s^2}, G_c(s) = \frac{b}{b^2+s^2}$

Now using Parseval's identity for Fourier cosine transforms, *i.e.*,

$$\frac{2}{\pi}\int_0^{\infty} F_c(s)G_c(s)ds = \int_0^{\infty} f(x)\,g(x)dx \quad ...(1)$$

We have
$$\frac{2}{\pi}\int_0^{\infty} \frac{ab}{(a^2+s^2)(b^2+s^2)}\,ds = \int_0^{\infty} e^{-(a+b)x}dx$$

or
$$\frac{2ab}{\pi}\int_0^{\infty} \frac{ds}{(a^2+s^2)(b^2+s^2)} = \left|\frac{e^{-(a+b)x}}{-(a+b)}\right|_0^{\infty} = \frac{1}{a+b}$$

Thus
$$\int_0^{\infty} \frac{dt}{(a^2+t^2)(b^2+t^2)} = \frac{\pi}{2ab(a+b)}$$

(ii) Let $f(x) = \frac{x}{x^2+1}$ so that $F_s[f(x)] = \frac{\pi}{2}e^{-s}$

Now using Parseval's identity for sine transform, *i.e.*,

$$\frac{2}{\pi}\int_0^{\infty} [F_s\{f(x)\}]^2\,ds = \int_0^{\infty} [f(x)]^2\,dx$$

or
$$\int_0^{\infty}\left(\frac{x}{x^2+1}\right)^2 dx = \frac{2}{\pi}\int_0^{\infty}\left(\frac{\pi}{2}e^{-s}\right)^2 ds = \frac{\pi}{2}\left|\,e^{-2s}/-2\right|_0^{\infty} = \frac{\pi}{-4}(0-1) = \frac{\pi}{4}$$

Hence
$$\int_0^{\infty} \frac{t^2}{(t^2+1)^2}\,dt = \frac{\pi}{4}$$

(iii) Let $f(x) = e^{-ax}$ and $g(x) = \begin{cases} 1, & 0 < x < a \\ 0, & x > a \end{cases}$. Then $F_c(s) = \frac{a}{a^2+s^2}, G_c(s) = \frac{\sin as}{s}$

Now using (1) above, we have $\dfrac{2}{\pi}\int_0^{\infty}\dfrac{a\sin as}{s(a^2+s^2)}ds = \int_0^a e^{-ax}.1\,dx = \dfrac{1-e^{-a^2}}{a}$

Thus $$\int_0^{\infty}\frac{\sin at}{t(a^2+t^2)}dt = \frac{\pi}{2a^2}(1-e^{-a^2}).$$

Example 22.15. *Find the Fourier transform of $f(x)$ given by* $f(x) = \begin{cases} a-|x|, & \text{for } |x|<a \\ 0 & \text{, for } |x|>a \end{cases}$

Hence show that $\int_0^{\infty}\left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$ *and* $\int_0^{\infty}\left(\frac{\sin t}{t}\right)^4 dt = \pi/3.$ (Anna, 2008)

Solution. Fourier transform of $f(x)$ *i.e.* $F[f(x)] = \int_{-\infty}^{\infty} f(x)e^{isx}dx = \int_{-a}^{a}[a-|x|]e^{isx}dx$

$$= \int_{-a}^{a}[a-|x|](\cos x + i\sin sx)dx$$

$$= 2\int_0^a (a-x)\cos sx\,dx + 0 \qquad \left\{\because \begin{array}{l}[a-|x|]\cos x \text{ in an even function} \\ [a-|x|]\sin x \text{ is an odd function}\end{array}\right.$$

$$= 2\left|(a-x)\frac{\sin sx}{s} - \frac{\cos sx}{s^2}\right|_0^a = 2\,\frac{1-\cos as}{s^2} = 4\,\frac{\sin^2 as/2}{s^2}$$

(*i*) By inversion formula,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}F(s)e^{-isx}ds = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4\sin^2 as/2}{s^2}e^{-isx}ds$$

To evaluate $\int_0^{\infty}\left(\frac{\sin t}{t}\right)^2 dt$, put $x = 0$ and $a = 2$ so that

$$f(0) = \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\sin^2 s}{s^2}ds = \frac{4}{\pi}\int_0^{\infty}\left(\frac{\sin s}{s}\right)^2 ds \qquad \left[\because \frac{\sin s}{s} \text{ is an even function}\right]$$

$$\therefore \quad \int_0^{\infty}\left(\frac{\sin s}{s}\right)^2 ds = \frac{\pi}{4}f(0) = \frac{\pi}{2}. \qquad [\because f(0) = a = 2$$

(*ii*) Using Parseval's identity

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}[F(s)]^2 ds = \int_{-\infty}^{\infty}|f(x)|^2 dx$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{4\sin^2 as/2}{s^2}\right)^2 dx = \int_{-a}^{a}|[a-|x|]^2 dx$$

$$\frac{16}{\pi}\int_0^{\infty}\left(\frac{\sin as/2}{s}\right)^4 ds = 2\int_0^a (a-x)^2 dx = 2\left|\frac{(a-x)^3}{-3}\right|_0^a = \frac{2}{3}a^3$$

Putting $t = as/2$ and $dt = ads/2$

$$\frac{16}{\pi}\int_0^{\infty}\left(\frac{\sin t}{2t/a}\right)\frac{2}{a}dt = \frac{2}{3}a^3 \quad \text{or} \quad \frac{2a^3}{\pi}\int_0^{\infty}\left(\frac{\sin t}{t}\right)^4 dt = \frac{2}{3}a^3$$

Hence $$\int_0^{\infty}\left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}.$$

PROBLEMS 22.2

1. Verify Convolution theorem for $f(x) = g(x) = e^{-x^2}$. (V.T.U., 2000 S)
2. Use Convolution theorem to find the inverse Fourier transform of $\frac{i}{(1+s^2)^2}$, given that $\frac{2}{(1+s^2)}$ is the Fourier transform of $e^{-|x|}$. (V.T.U., 2010 S)
3. Using Parseval's identity, show that

   (i) $\int_0^\infty \frac{dx}{(t^2+1)^2} = \frac{\pi}{4}$, (Hissar, 2007) (ii) $\int_0^\infty \frac{t^2}{(4+t^2)(9+t^2)}\,dt = \frac{\pi}{10}$. (Rohtak, 2003)
4. Find the Fourier transform of $f(x) = \begin{cases} 1 \text{ if } |x| < 1 \\ 0 \text{ if } |x| > 1 \end{cases}$.

   Hence deduce that $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$. (Anna, 2009)
5. Evaluate $\int_0^\infty \left(\frac{1-\cos x}{x}\right)^2 dx$.

22.8 RELATION BETWEEN FOURIER AND LAPLACE TRANSFORMS

If $$f(t) = \begin{cases} e^{-xt}\, g(t), & t > 0 \\ 0, & t < 0 \end{cases} \qquad ...(i)$$

then $$F\{f(t)\} = L\{g(t)\}.$$

We have $$F\{f(t)\} = \int_{-\infty}^{\infty} e^{ist} f(t)dt = \int_{-\infty}^{0} e^{ist}\,.\,0\,.\,dt + \int_0^\infty e^{ist}\,.\,e^{-xt}\, g(t)dt$$

$$= \int_0^\infty e^{(is-x)t}\, g(t)\, dt = \int_0^\infty e^{-pt}\, g(t)dt \quad \text{where } p = x - is$$

Hence *the Fourier transform of* $f(t)$ [defined by (*i*)] is the *Laplace transform of* $g(t)$.

22.9 FOURIER TRANSFORMS OF THE DERIVATIVES OF A FUNCTION

The Fourier transform of the function $u(x, t)$ is given by

$$F[u(x,t)] = \int_{-\infty}^{\infty} ue^{isx}\, dx$$

Then the Fourier transform of $\partial^2 u/\partial x^2$, *i.e.*

$$F\left[\frac{\partial^2 u}{\partial x^2}\right] = \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{isx}\, dx = \left| e^{isx}\frac{\partial u}{\partial x} - is\, e^{isx}\,.\,u \right|_{-\infty}^{\infty} + (is)^2 \int_{-\infty}^{\infty} ue^{isx}\, dx,$$

on applying the general rule of integration by parts (p. 398). If u and $\frac{\partial u}{\partial x}$ tend to zero as x tends to $\pm\infty$, then

$$\mathbf{F}\left[\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}\right] = -\mathbf{s}^2\,\mathbf{F}[\mathbf{u}] \qquad ...(1)$$

Similarly in the case of Fourier sine and cosine transforms, we have

$$\mathbf{F_s}\left[\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}\right] = \mathbf{s(u)_{x=0}} - \mathbf{s^2 F_s[u]} \qquad ...(2)$$

and $$\mathbf{F_c}\left[\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}\right] = -\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)_{\mathbf{x=0}} - \mathbf{s^2 F_c[u]} \qquad ...(3)$$

In general, the *Fourier transform of the nth derivative of* $f(x)$ is given by

$$\mathbf{F}\left[\frac{\mathbf{d^n f}}{\mathbf{dx^n}}\right] = (-\mathbf{is})^{\mathbf{n}}\ \mathbf{F[f(x)]} \qquad ...(4)$$

provided the first $n - 1$ derivatives vanish as $x \to \pm \infty$.

For $\quad F[f^n(x)] = \int_{-\infty}^{\infty} f^n(x)\, e^{isx}\, dx$

$$= \left| e^{isx} f^{n-1} - ise^{isx} f^{n-2} + (is)^2 e^{isx} f^{n-3} - \ldots \right|_{-\infty}^{\infty} + (-is)^n \int_{-\infty}^{\infty} f \cdot e^{isx}\, dx$$

by the general rule of integration by parts, whence follows (4).

22.10 INVERSE LAPLACE TRANSFORMS BY METHOD OF RESIDUES

Let the Laplace transform of $f(x)$ be $\bar{f}(s)$ so that

$$\bar{f}(s) = \int_0^{\infty} f(t) e^{-st}\, dt \qquad ...(1)$$

Multiply both sides by e^{xs} and integrate w.r.t. s within the limits $a - ir$ and $a + ir$. Then

$$\int_{a-ir}^{a+ix} e^{xs}\, \bar{f}(s) ds = \int_{a-ir}^{a+ir} e^{xs} \int_0^{\infty} f(t) e^{-st}\, dt\, ds \qquad [\text{Put } s = a - iu]$$

$$= \int_r^{-r} e^{x(a-iu)} \int_0^{\infty} f(t) e^{-(a-iu)t}\, dt\,(-idu) = ie^{ax} \int_{-r}^{r} e^{-ixu} \int_0^{\infty} [e^{-at} f(t)] e^{iut}\, dtdu$$

$$= ie^{ax} \int_{-r}^{r} e^{-ixu} \int_{-\infty}^{\infty} \phi(t)\, e^{iut}\, dt\, du$$

where $\phi(t) = \begin{cases} e^{-at} f(t) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$.

Proceeding to limits as $r \to \infty$, we get

$$\int_{a-i\infty}^{a+i\infty} e^{xs}\, \bar{f}(s)\, ds = ie^{ax} \cdot 2\pi\phi(x), \text{ by (2) of § 22.4} = 2\pi i e^{ax} . e^{-ax} f(x) \text{ for } x > 0.$$

Hence $\quad f(x) = \int_{a-i\infty}^{a+i\infty} e^{xs}\, \bar{f}(s) ds \ (x > 0) \qquad ...(2)$

which is called the *complex inversion formula*. It provides a direct means for obtaining the inverse Laplace transform of a given function.

The integration in (2) is performed along a line LM parallel to the imaginary axis in the complex plane $z = x + iy$ such that all the singularities of $\bar{f}(s)$ lie to its left* (Fig. 22.1). Let us take a contour C which is composed of the line LM and the semi-circle C' (*i.e.*, MNL). Then from (2)

$$\frac{1}{2\pi i} \int_{LM} e^{xs}\, \bar{f}(s)\, ds = \frac{1}{2\pi i} \int_C e^{xs}\, \bar{f}(s)\, ds - \frac{1}{2\pi i} \int_{C'} e^{xs}\, \bar{f}(s)\, dx$$

The integral over C' tends to zero as $r \to \infty$ (under certain conditions†). Therefore,

$$\mathbf{f(x)} = \underset{r \to \infty}{\mathbf{Lt}} \frac{\mathbf{1}}{\mathbf{2\pi i}} \int_C \mathbf{e^{xs}\, \bar{f}(s)\, dx}$$

= sum of the residues of $e^{xs}\bar{f}(s)$ at the poles of $f(s)$ ...(3)

[By §20.18]

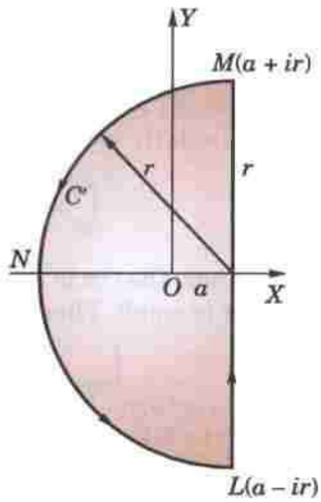


Fig. 22.1

* This has been so assumed simply to ensure the convergence of the integral (1).

† If positive constants A and k can be so found that $|\bar{f}(s)| < Ar^{-k}$ for every point on C', then

$$\underset{r \to \infty}{\text{Lt}} \frac{1}{2\pi i} \int_{C'} e^{xs} \bar{f}(s)\, ds = 0. \qquad (\textit{Jordan's Lemma})$$

Example 22.16. *Evaluate* $L^{-1}\left\{\dfrac{1}{(s-1)(s^2+1)}\right\}$ *by the method of residues.*

Solution. Since $\left|\dfrac{1}{(s-1)(s^2+1)}\right| \sim \left|\dfrac{1}{s^3}\right|$ for $|s| \to \infty$, therefore,

$$L^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right] = \text{sum of Res}\left[\frac{e^{xs}}{(s-1)(s^2+1)}\right] \text{at the poles } s = 1, \pm i$$

Now $$(\text{Res})_{s=1} = \underset{s\to 1}{\text{Lt}}\left[\frac{(s-1)\,.\,e^{xs}}{(s-1)(s^2+1)}\right] = \frac{e^x}{2}$$ [By § 20.19 (1)]

$$(\text{Res})_{s=i} = \underset{s\to i}{\text{Lt}}\left[\frac{(s-i)\,.\,e^{xs}}{(s-1)(s^2+1)}\right] = \frac{e^{ix}}{(i-1)(i-1)} = -\frac{1}{2}\cdot\frac{e^{ix}}{1+i}$$

Changing i to $-i$, we get $(\text{Res})_{s=-i} = -\dfrac{1}{2}\cdot\dfrac{e^{ix}}{1-i}$

$$\therefore \quad L^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right] = \frac{e^x}{2} - \frac{1}{2}\left(\frac{e^{ix}}{1+i} + \frac{e^{-ix}}{1-i}\right) = \frac{1}{2}(e^x - \sin x - \cos x).$$

Example 22.17. *Prove that* $L^{-1}\left(\dfrac{e^{-c\sqrt{s}}}{s}\right) = 1 - erf\left(\dfrac{c}{\sqrt{2x}}\right)$.

Solution. By the complex inversion formula,

$$L^{-1}\left(\frac{e^{-c\sqrt{s}}}{s}\right) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{xs}\,.\,\frac{e^{-c\sqrt{s}}}{s}\,ds.$$

Since $s = 0$ is a branch point of the integrand, we take a contour $LMNPQST$ as shown in Fig. 22.2, so that it doesn't include any singularity. Therefore, by Cauchy's theorem (§ 20.13), we have

$$\left\{\int_{LM} + \int_{MN} + \int_{NP} + \int_{PQS} + \int_{ST} + \int_{TL}\right\} \times e^{xs}\,\frac{e^{-c\sqrt{s}}}{s}\,ds = 0 \qquad ...(i)$$

If $ON = \rho$ and $OP = \varepsilon$, then along NP, $s = Re^{i\pi}$, therefore,

$$\int_{NP} = \int_\rho^\varepsilon e^{-xR}\,\frac{e^{-ic\sqrt{R}}}{R}\,dR$$

Similarly along ST, $s = Re^{-i\pi}$, therefore,

$$\int_{ST} = \int_\epsilon^\rho e^{-xR}\,\frac{e^{c\sqrt{R}}}{R}\,dR$$

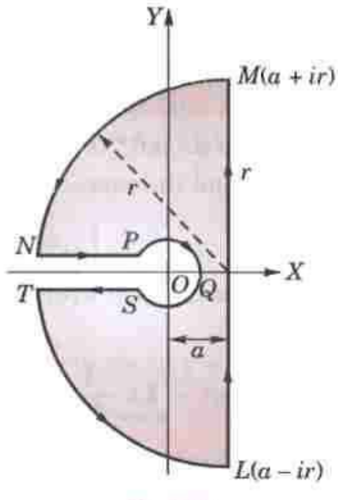


Fig. 22.2

Along the circle PQS, $s = \varepsilon\,e^{i\theta}$. Also e^{xs} and $e^{-c\sqrt{\sigma}}$ are both approximately 1 since ε is small. Therefore,

$$\int_{PQS} = \int_\pi^{-\pi}\frac{1}{\varepsilon e^{i\theta}}\,.\,\varepsilon e^{i\theta}\,id\theta = -2\pi i \text{ approximately.}$$

For $c > 0$, $|\,e^{-c\sqrt{s}}/s\,| < |s|^{-1}$.

But $\int_{MN}$ and $\int_{TL}$ both tend to zero as $r \to \infty$

Thus (i) takes the form

$$\int_{a-ir}^{a+ir}\frac{e^{xs-c\sqrt{s}}}{s}\,ds + \int_\varepsilon^\rho e^{-xR}\,\frac{e^{ic\sqrt{R}} - e^{-ic\sqrt{R}}}{R}\,dR - 2\pi i = 0$$

Taking limits as $\varepsilon \to 0$ and $\rho \to \infty$, we get

$$\int_{a-i\infty}^{a+i\infty}\frac{e^{xs-c\sqrt{s}}}{s}\,ds = 2\pi i - 2i\int_0^\infty e^{-xR}\,\frac{\sin c\sqrt{R}}{R}\,dR$$

or $$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{xs-c\sqrt{s}}}{s}\,ds = 1-\frac{2}{\pi}\int_0^\infty e^{-t^2}\,\frac{\sin(ct/\sqrt{x})}{t}\,dt^*,\text{ where } R = t^2/x$$

$$= 1-\frac{2}{\pi}\cdot\frac{\pi}{2}\,erf\left(\frac{C}{2\sqrt{x}}\right)\text{ whence follows the result.}$$

PROBLEMS 22.3

Using the method of residues, evaluate the inverse Laplace transform of each of the following :

1. $\dfrac{1}{(s+1)(s-2)^2}$.
2. $\dfrac{1}{(s-2)(s^2+1)}$.
3. $\dfrac{1}{s^2(s^2-a^2)}$.
4. $\dfrac{1}{(s-1)^2(s^2+1)}$.
5. $\dfrac{1}{(s^2+1)^2}$. *(V.T.U., 2008 S)*

22.11 APPLICATION OF TRANSFORMS TO BOUNDARY VALUE PROBLEMS

In one dimensional boundary value problems, the partial differential equation can easily be transformed into an ordinary differential equation by applying a suitable transform. The required solution is then obtained by solving this equation and inverting by means of the complex inversion formula or by any other method. In two dimensional problems, it is sometimes required to apply the transforms twice and the desired solution is obtained by double inversion.

(*i*) If in a problem $u(x, t)_{x=0}$ is given then we use infinite sine transform to remove $\partial u^2/\partial x^2$ from the differential equation.

In case $[\partial u(x, t)/\partial x]_{x=0}$ is given then we employ infinite cosine transform to remove $\partial^2 u/\partial x^2$.

(*ii*) If in a problem $u(0, t)$ and $u(l, t)$ are given, then we use finite sine transform to remove $\partial^2 u/\partial x^2$ from the differential equation.

In case $(\partial u/\partial x)_{x=0}$ and $(\partial u/\partial x)_{x=l}$ are given, then we employ finite cosine transform to remove $\partial^2 u/\partial x^2$.

The method of solution is best explained through the following examples.

Heat conduction

Example 22.18. *Determine the distribution of temperature in the semi-infinite medium $x \geq 0$, when the end $x = 0$ is maintained at zero temperature and the initial distribution of temperature is $f(x)$.* *(Osmania, 2003)*

Solution. Let $u(x, t)$ be the temperature at any point x and at any time t. We have to solve the heat-flow equation (§ 18.5)

$$\frac{\partial u}{\partial t} = c^2\,\frac{\partial^2 u}{\partial x^2}\ (x > 0, t > 0) \qquad ...(i)$$

subject to the initial condition $u(x, 0) = f(x)$ ...(*ii*)

and the boundary condition $u(0, t) = 0$ ...(*iii*)

Taking Fourier sine transform of (1) and denoting $F_s[u(x, t)]$ by $\bar{u}_s$, we have

$$\frac{d\bar{u}_s}{dt} = c^2\,[su(0,t) - s^2\bar{u}_s] \qquad \text{[By (2) of § 22.9]}$$

---

* We know that $\int_0^\infty e^{-t^2}\cos 2mt\,dt = \frac{1}{2}\sqrt{\pi}e^{-m^2}$ [Example 20.44

Integrating both sides w.r.t. m from 0 to $c/2\sqrt{x}$.

$$\int_0^\infty e^{-t^2}\left|\frac{\sin 2mt}{2t}\right|_0^{c/2\sqrt{x}} dt = \frac{1}{2}\sqrt{\pi}\int_0^{c/2\sqrt{x}} e^{-m^2}\,dm$$

or $$\int_0^\infty e^{-t^2}\,\frac{\sin(ct/\sqrt{x})}{t}\,dt = \frac{\pi}{2}\,erf\left(\frac{c}{2\sqrt{x}}\right). \qquad \text{[By § 7.18(1)]}$$

or $$\frac{d\bar{u}_s}{dt} + c^2s^2\bar{u}_s = 0$$ [By (iii)] ...(*iv*)

Also the Fourier sine transform of (*ii*) is $\bar{u}_s = \bar{f}(s)$ at $t = 0$. ...(*v*)

Solving (*iv*) and using (*v*), we get $\bar{u}_s = \bar{f}_s(s)e^{-c^2s^2t}$

Hence taking its inverse Fourier sine transform, we obtain

$$u(x, t) = \frac{2}{\pi}\int_0^\infty \bar{f}_s(s)e^{-c^2s^2t} \sin xs\, ds.$$

Example 22.19. *Solve $\partial u/\partial t = 2\partial^2 u/\partial x^2$, if $u(0, t) = 0$, $u(x, 0) = e^{-x}$ $(x > 0)$, $u(x, t)$ is bounded where $x > 0$, $t > 0$.* (*Rohtak, 2006*)

Solution. Given $\partial u/\partial t = 2\partial^2 u/\partial x^2, x > 0, t > 0$ ...(*i*)

with boundary conditions : $u(0, t) = 0$, $u(x, t)$ is bounded ...(*ii*)

and initial condition $u(x, 0) = e^{-x}, x > 0$ ...(*iii*)

Since $u(0, t)$ is given, we take Fourier sine transform of both sides of (*i*) so that

$$\int_0^\infty \frac{\partial u}{\partial t}\sin px\, dx = 2\int_0^\infty \frac{\partial^2 u}{\partial x^2}\sin px\, dx$$

or $$\frac{d}{dt}\int_0^\infty u(x,t)\sin px\, dx = 2\left[\left|\frac{\partial u}{\partial x}\sin px\right|_0^\infty - \int_0^\infty \frac{\partial u}{\partial x}.\, p\cos px\, dx\right]$$ (Integrating by parts)

or $$\frac{d\bar{u}_s}{dt} = -2p\int_0^\infty \frac{\partial u}{\partial x}\cos px\, dx, \text{ if } \frac{\partial u}{\partial x} \to \frac{\partial u}{\partial x} \to 0 \text{ as } x \to \infty \text{ where } \bar{u}_s(p, t) = \int_0^\infty u(x, t)\sin px\, dx$$

$$= -2p\left[\left| u(x,t)\cos px\right|_0^\infty - \int_0^\infty u(x,t) - (-p\sin px)\, dx\right]$$ [Again integrating by parts]

$$= -2p\left[0 - u(0, t) + p\int_0^\infty u(x, t)\sin px\, dx\right]$$ [$\because$ $u(x, t) \to 0$ as $x \to \infty$ by (*ii*)]

$$= 2pu(0, t) - 2p^2\bar{u}_s$$

or $$\frac{d\bar{u}_s}{dt} = -2p^2\bar{u}_s$$ [By (*ii*)]

Integrating $\int \frac{\partial \bar{u}_s}{\bar{u}_s} - \log c = -2p^2\int dt$ or $\log \bar{u}_s - \log c = -2p^2t$

$\therefore$ $$\bar{u}_s(p, t) = ce^{-2p^2t}$$ ...(*iv*)

Taking Fourier sine transform of both sides of (*iii*), we get

$$\int_0^\infty u(x, 0)\sin px\, dx = \int_0^\infty e^{-x}\sin px\, dx$$

or $$\bar{u}_s(p, 0) = \left|\frac{e^{-x}}{1 + p^2}(-\sin px - p\cos px)\right|_0^\infty = \frac{p}{1 + p^2}$$ ...(*v*)

Putting $t = 0$ in (*iv*) and using (*v*), we obtain $p/(1 + p^2) = c$

Thus (*iv*) becomes $\bar{u}_s(p, t) = \frac{p}{1 + p^2}e^{-2p^2t}$

Now taking inverse Fourier sine transform, we get

$$u(x, t) = \frac{2}{\pi}\int_0^\infty \frac{pe^{-2p^2t}}{1 + p^2}\sin px\, dp.$$

Example 22.20. *Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $(x > 0, t > 0)$ subject to the conditions*

(i) $u = 0$, when $x = 0, t > 0$ (ii) $u = \begin{cases} 1, & 0 < x < 1 \\ 0, & x \le 1, \text{ when } t = 0\end{cases}$ (iii) $u(x, t)$ is bounded. (*U.P.T.U., 2003 S*)

Solution. Since $u(0, t) = 0$, we take Fourier sine transform of both sides of the given equation, we get

$$\int_0^\infty \frac{\partial u}{\partial t} \sin sx\, dx = \int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin sx\, dx$$

$$\frac{\partial}{\partial t}\int_0^\infty u \sin sx\, dx = -s^2\, \bar{u}(s) + s\, u(0) \qquad [\because u = 0, \text{ when } x = 0]$$

or $$\frac{\partial \bar{u}}{\partial t} = -s^2 \bar{u} \quad \text{or} \quad \frac{\partial \bar{u}}{\partial t} + s^2 \bar{u} = 0 \quad \text{or} \quad (D^2 + s^2)\bar{u} = 0 \text{ i.e., } D = \pm is$$

$\therefore$ Its solution is $\bar{u}(s, t) = e^{-s^2 t}$ ...(1)

Since $$\bar{u}(s, t) = \int_0^\infty u(x, t) \sin sx\, dx$$

$\therefore$ $$\bar{u}(s, 0) = \int_0^\infty u(x, 0) \sin sx\, dx = \int_0^1 1 . \sin sx\, dx \qquad [\text{By } (ii)]$$

$$= \frac{1 - \cos s}{s} \qquad ...(2)$$

From (1) and (2), $$c = \bar{u}(s, 0) = \frac{1 - \cos s}{s}$$

Thus (1) gives $$\bar{u}(s, t) = \frac{1 - \cos s}{s} e^{-s^2 t}$$

Now taking inverse Fourier sine transform, we get

$$u(x, t) = \int_0^\infty \frac{1 - \cos s}{s} e^{-s^2 t}\, ds$$

which is the desired solution.

Example 22.21. *Using finite Fourier transform, solve*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

given $u(0, t) = 0$, $u(4, t) = 0$ and $u(x, 0) = 2x$ where $0 < x < 4$, $t > 0$. *(Rajasthan, 2006)*

Solution. Since $u(0, t) = 0$, we take finite Fourier sine transform of both sides of the given equation

$$\int_0^4 \frac{\partial u}{\partial t} \sin \frac{n\pi}{4} x\, dx = \int_0^4 \frac{\partial^2 u}{\partial x^2} \sin \frac{n\pi}{4} x\, dx$$

or $$\frac{d}{dt}(\bar{u}_s) = F_s\left(\frac{\partial^2 u}{\partial x^2}\right)$$

$$= -\frac{n^2\pi^2}{16}\bar{u}_s + \frac{n\pi}{4}[u(0, t) - (-1)^n u(4, t)]$$

$$= -\frac{n^2\pi^2}{16}\bar{u}_s \qquad [\because u(0, t) = 0, u(4, t) = 0.]$$

or $$\frac{d\bar{u}_s}{\bar{u}_s} = -\frac{n^2\pi^2}{16}\, dt$$

Integrating both sides, $$\log \bar{u}_s = -\frac{n^2\pi^2}{16} t + c$$

or $$\bar{u}_s(x, 0) = \alpha e^{\frac{-n^2\pi^2 t}{16}} \qquad ...(i)$$

Putting $t = 0$, $$a = \bar{u}_s(x, 0) = \int_0^4 u(x, 0) \sin \frac{n\pi x}{4}\, dx \qquad [\because \quad u(x, 0) = 2x]$$

$$= \int_0^4 2x \sin \frac{n\pi x}{4}\, dx = -\frac{32}{n\pi} \cos n\pi$$

Thus (*i*) gives, $$\bar{u}_s(x, 0) = -\frac{32}{n\pi}\cos n\pi e^{-n^2\pi^2 t/16} = -\frac{32}{n\pi}(-1)^n e^{-n^2\pi^2 t/16}$$

Now taking inverse Fourier sine transform, we get

$$u(x, 0) = \frac{2}{4}\sum_{n=1}^{\infty}\frac{32}{n\pi}(-1)^{n+1} e^{-n^2\pi^2 t/16}\sin\left(\frac{n\pi x}{4}\right)$$

$$= 16\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\pi} e^{-n^2\pi^2 t/16}\sin\left(\frac{n\pi x}{4}\right).$$

Example 22.22. *If the initial temperature of an infinite bar is given by*

$$\theta(x) = \begin{cases}\theta_0 \text{ for } |x| < a\\ 0 \text{ for } |x| > a,\end{cases}$$

determine the temperature at any point x and at any instant t. (S.V.T.U., 2008 ; Rohtak, 2004)

Solution. To determine the temperature $\theta(x, t)$ at any point at any time, we have to solve the equation

$$\frac{\partial \theta}{\partial t} = c^2\frac{\partial^2\theta}{\partial x^2} \quad (t > 0) \qquad ...(i)$$

subject to the initial condition $\theta(x, 0) = \begin{cases}\theta_0 \text{ for } |x| < a\\ 0 \text{ for } |x| > a\end{cases}$ ...(*ii*)

Taking Fourier transform of (*i*) and denoting $F[\theta(x, t)]$ by $\bar{\theta}$, we find

$$\frac{d\bar{\theta}}{dt} = -c^2 s^2\bar{\theta} \qquad \text{[by (1) of § 22.9] } ...(iii)$$

Also the Fourier transform of (2) is

$$\bar{\theta}(s, 0) = \int_{-\infty}^{\infty}\theta(x, 0)e^{isx}\,dx = \int_{-a}^{a}\theta_0 e^{isx}\,dx = \theta_0\frac{e^{isa} - e^{-isa}}{is} = 2\theta_0\frac{\sin as}{s} \qquad ...(iv)$$

Solving (*iii*) and using (*iv*), we get $\bar{\theta} = \dfrac{2\theta_0 \sin as}{s}e^{-c^2s^2t}$

Hence taking its inverse Fourier transform, we get

$$\theta(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\theta_0\sin as}{s}e^{-c^2s^2t}e^{-isx}\,ds = \frac{\theta_0}{\pi}\int_{-\infty}^{\infty}\frac{\sin as}{s}e^{-c^2s^2t}(\cos xs - i\sin xs)\,ds$$

$$= \frac{2\theta_0}{\pi}\int_0^{\infty}\frac{\sin as}{s}e^{-c^2s^2t}\cos xs\,ds \qquad \left\{\begin{array}{l}\text{The second integral vanishes as}\\ \text{its integrand is an odd function}\end{array}\right.$$

$$= \frac{\theta_0}{\pi}\int_0^{\infty}e^{-c^2s^2t}\frac{\sin(a + x)s + \sin(a - x)s}{s}\,ds$$

$$= \frac{\theta_0}{\pi}\int_0^{\infty}e^{-v^2}\left\{\sin\frac{(a + x)v}{c\sqrt{t}} + \sin\frac{(a - x)v}{c\sqrt{t}}\right\}\frac{dv}{v} \text{ where } v^2 = c^2s^2t$$

$$= \frac{\theta_0}{\pi}\left\{erf\frac{(a + x)}{2c\sqrt{t}} + erf\frac{(a - x)}{2c\sqrt{t}}\right\}.$$ [See footnote on p. 783]

Example 22.23. *A bar of length a is at zero temperature. At t = 0, the end x = a is suddenly raised to temperature u_0 and the end x = 0 is insulated. Find the temperature at any point x of the bar at any time t > 0, assuming that the surface of the bar is insulated.*

Solution. Here we have to solve the differential equation

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \quad (0 < x < a, t > 0) \qquad ...(i)$$

subject to the conditions

$u(x, 0) = 0$ ...(*ii*) ; $u_x(0, t) = 0$ ...(*iii*) and $u(a, t) = u_0$ (*Rohtak, 2005*) ...(*iv*)

The Laplace transform of (*i*), if $L[u(x,t)] = \bar{u}(x,s)$, is

$$s\bar{u} - u(x,0) = c^2 \frac{d^2\bar{u}}{dx^2}$$

Using (*ii*), we get $\qquad \dfrac{d^2\bar{u}}{dx^2} - \dfrac{s}{c^2}\bar{u} = 0 \qquad ...(v)$

Similarly the Laplace transform of (*iii*) and (*iv*) are

$$\bar{u}_x(0,s) = 0 \quad ...(vi); \qquad \bar{u}(a,s) = \frac{u_0}{s} \quad ...(vii)$$

Solving (*v*), we have $\qquad \bar{u} = C_1 e^{x\sqrt{s}/c} + C_2 e^{-x\sqrt{s}/c}$

Using (*vi*), we find $C_1 = C_2$ so that

$$\bar{u} = C_1(e^{\sqrt{s}x/c} + e^{-\sqrt{s}x/c}) = 2C_1 \cosh(\sqrt{s}x/c)$$

Now using (*vii*), we have $\qquad \bar{u} = \dfrac{u_0 \cosh(\sqrt{s}x/c)}{s\cosh(\sqrt{s}a/c)}$

By the inversion formula (3) § 22.10, we get

$$u(x,t) = \text{sum of the residues of } \left(\frac{e^{st}.\,u_0\cosh(\sqrt{s}x/c)}{s\cosh(\sqrt{s}a/c)}\right) \text{ at all the poles which occur at } s = 0$$

and $\qquad \cosh(\sqrt{s}a/c) = 0$ *i.e.*, at $s = 0$, $\sqrt{s}a/c = \left(n - \dfrac{1}{2}\right)\pi i,\ n = 0, \pm 1, \pm 2, ...$

or at $\qquad s = 0,\ s\,(= s_n) = -\dfrac{(2n-1)^2 c^2 \pi^2}{4a^2} = 0, 1, 2, ...$

Now $\qquad (\text{Res})_{s=0} = \underset{s\to 0}{\text{Lt}} \left\{s\,.\,\dfrac{u_0 e^{st}\cosh(\sqrt{s}x/c)}{s\cosh(\sqrt{s}a/c)}\right\} = u_0$

$$(\text{Res})_{s=s_n} = u_0 \underset{s\to s_n}{\text{Lt}} \left\{(s - s_n)\,.\,\frac{u_0 e^{st}\cosh(\sqrt{s}x/c)}{s\cosh(\sqrt{s}a/c)}\right\}$$

$$= u_0 \underset{s\to s_n}{\text{Lt}} \left\{\frac{s - s_n}{\cosh(\sqrt{s}a/c)}\right\} . \underset{s\to s_n}{\text{Lt}} \left\{\frac{e^{st}\cosh(\sqrt{s}x/c)}{s}\right\} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= u_0 \underset{s\to s_n}{\text{Lt}} \frac{1}{\sinh(\sqrt{s}a/c).(a/2\sqrt{s}/c)} . \underset{s\to s_n}{\text{Lt}} \left\{\frac{e^{st}\cosh(\sqrt{s}x/c)}{s}\right\}$$

$$= \frac{4u_0(-1)^n}{(2n-1)\pi} e^{-(2n-1)^2\pi^2c^2t/4a^2} \cos\frac{(2n-1)\pi x}{2a}$$

Thus we get $\qquad u(x,t) = u_0 + \dfrac{4u_0}{\pi}\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} e^{-(2n-1)^2\pi^2c^2t/4a^2} \cos\frac{(2n-1)\pi x}{2a}.$

Vibrations of a string

Example 22.24. *An infinite string is initially at rest and that the initial displacement is $f(x)$, $(-\infty < x < \infty)$. Determine the displacement $y(x, t)$ of the string.* (Rohtak, 2000)

Solution. The equation for the vibration of the string is

$$\partial^2 y/\partial t^2 = c^2 \partial^2 y/\partial x^2 \qquad ...(i)$$

and the initial conditions are

$$(\partial y/\partial t)_{t=0} = 0\ ;\ y(x,0) = f(x) \qquad ...(ii)$$

Multiplying (*i*) by e^{isx} and integrating w.r.t. x from $-\infty$ to ∞, we get

$\partial^2 Y/\partial t^2 = c^2(-s^2 Y)$ provided y and $\partial y/\partial x \to 0$ as $x \to \infty$

$\therefore$ a solution of $d^2Y/dt^2 + c^2s^2Y = 0$ is $Y = A_1 \cos cst + A_2 \sin cst$ ...(*iii*)

Also Fourier transforms of (*ii*) are

$$\partial y/\partial t = 0 \quad \text{and} \quad Y = F(s) \text{ when } t = 0$$

Applying these to (*iii*), we get

$$A_2 = 0 \quad \text{and} \quad A_1 = F(s)$$

Thus $Y = F(s) \cos cst$

Now taking inverse Fourier transforms, we get

$$y(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) \cos cst \,.\, e^{-isx}\, dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) \frac{e^{icst} + e^{-icst}}{2} \,.\, e^{-isx}\, dx$$

$$= \frac{1}{4\pi} \int_{-\infty}^{\infty} [F(s) e^{-is(x-ct)} + F(s) e^{-is(x+ct)}]\, ds$$

$$= \frac{1}{2} [f(x-ct) + f(x+ct)] \qquad \left[\because\ f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds\right.$$

Example 22.25. *An infinitely long string having one end at $x = 0$, is initially at rest along the x-axis. The end $x = 0$ is given a transverse displacement $f(t)$, $t > 0$. Find the displacement of any point of the string at any time.*

Solution. Let $y(x, t)$ be the transverse displacement of any point x of the string at any time t. Then we have to solve the wave equation (§ 18.4)

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad (x > 0, t > 0) \qquad ...(i)$$

subject to the conditions $y(x, 0) = 0$, $y_t(x, 0) = 0$, $y(0, t) = f(t)$ and the displacement $y(x, t)$ is bounded.

The Laplace transform of (*i*), writing $L[y(x, t)] = \bar{y}(x, s)$ is

$$s^2 \bar{y} - sy(x, 0) - \frac{\partial y(x, 0)}{\partial t} = c^2 \frac{\partial^2 \bar{y}}{\partial x^2}$$

Using the first two conditions, we have

$$\frac{\partial^2 \bar{y}}{\partial x^2} = \left[\frac{s}{c}\right]^2 \bar{y} \qquad ...(ii)$$

Similarly the Laplace transforms of the third and fourth conditions are

$$\bar{y}(0, s) = \bar{f}(s) \text{ at } x = 0 \quad ...(iii) \quad \text{and} \quad \bar{y}(x, s) \text{ is bounded.} \qquad ...(iv)$$

Solving (*ii*), we get

$$\bar{y}(x, s) = C_1 e^{sx/c} + C_2 e^{-sx/c}$$

To satisfy condition (*iv*), we must have $C_1 = 0$

Using the condition (*iii*), we get $C_2 = \bar{f}(s)$.

$\therefore$ $\bar{y}(x, s) = \bar{f}(s) e^{-sx/c}$

Using the complex inversion formula, we obtain

$$y = \frac{1}{2\pi i} \int_{a - i\infty}^{a + i\infty} e^{(t - x/c)s}\, \bar{f}(s)\, ds = f(t - x/c).$$

Example 22.26. *A tightly stretched flexible string has its ends fixed at $x = 0$ and $x = l$. At time $t = 0$, the string is given a shape defined by $F(x) = \mu x(l - x)$, where μ is a constant and then released. Find the displacement of any point x of the string at any time $t > 0$.* (V.T.U., M.E., 2006)

Solution. We have to solve the wave equation $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ $\qquad (x > 0, t > 0)$

subject to the conditions $y(0, t) = 0, y(l, t) = 0$
and $y(x, 0) = \mu x(l - x), y_t(x, 0) = 0$

Now taking Laplace transform, writing $L[y(x, t)] = \bar{y}(x, s)$, we get

$$s^2\bar{y} - s\bar{y}(x, 0) - \frac{\partial y(x, 0)}{\partial t} = c^2 \frac{\partial^2 \bar{y}}{\partial x^2} \quad ...(i)$$

where $\bar{y}(0, s) = 0,\ \bar{y}(l, s) = 0$ ...(ii)

$\therefore$ (i) reduces to $\dfrac{\partial^2 \bar{y}}{\partial x^2} - \left(\dfrac{s}{c}\right)^2 \bar{y} = -\dfrac{\mu s x(l - x)}{c^2}$

Its solution is $\bar{y}(x, s) = c_1 \cosh (sx/c) + c_2 \sinh (sx/c) + \dfrac{\mu x(l - x)}{s} - \dfrac{2c^2\mu}{s^3}$

Applying the conditions (ii), we get

$$c_1 = 2c^2\mu/s^2 \quad \text{and} \quad c_2 = \frac{2c^2\mu}{s^3}\left[\frac{1 - \cosh (sl/c)}{\sinh (sl/c)}\right] - \frac{2c^2\mu}{s^3} \tanh (s/2c)$$

Thus $\bar{y}(x, s) = \dfrac{2c^2\mu}{s^3}\left[\dfrac{\cosh \{s(2x - l)/2c\}}{\cosh (sl/2c)}\right] + \dfrac{\mu x(l - x)}{s} - \dfrac{2c^2\mu}{s^3}$

Now using the inversion formula (3) § 22.10, we get

$y(x, t) =$ sum of the residues of

$$2c^2\mu\left[e^{st} \frac{\cosh \{s(2x - l)/2c\}}{s^3 \cosh (sl/2c)}\right] \text{ at all the poles } + \mu x(l - x) - c^2\mu t^2$$

Proceeding exactly as in Example 22.23, we have,

sum of the residues of $2c^2\mu\left[\dfrac{e^{st} \cosh \{s(2x - l)/2c\}}{s^3 \cosh sl/2c}\right]$ at all the poles

$$= c^2\mu\left[t^2 + \left(\frac{2x - l}{2c}\right)^2 - \left(\frac{l}{2c}\right)^2\right]$$

$$- \frac{32c^2\mu}{\pi^3}\left(\frac{l}{2c}\right)^2 \sum_{n=1}^{\infty}\left[\frac{(-1)^n}{(2n - 1)^3} \cos \left\{\frac{(2n - 1)\pi(2x - l)}{2l}\right\} \cos \left\{\frac{(2n - 1)\pi ct}{l}\right\}\right]$$

$$= c^2\mu t^2 - \mu x(l - x) + \frac{8\mu l^2}{\pi^3} \sum_{n=1}^{\infty}\left[\frac{1}{(2n - 1)^3} \sin \frac{(2n - 1)\pi x}{l} \cos \frac{(2n - 1)\pi ct}{l}\right]$$

Hence $y(x, t) = \dfrac{8\mu l^2}{\pi^3} \displaystyle\sum_{n=1}^{\infty}\left[\frac{1}{(2n - 1)^3} \sin \frac{(2n - 1)\pi x}{l} \cos \frac{(2n - 1)\pi ct}{l}\right].$

Transmission lines

Example 22.27. *A semi-infinite transmission line of negligible inductance and leakance per unit length has its voltage and current equal to zero. A constant voltage v_0 is applied at the sending end ($x = 0$) at $t = 0$. Find the voltage and current at any point ($x > 0$) and at any instant.*

Solution. Let $v(x, t)$ and $i(x, t)$ be the voltage and current at any point x and at any time t. If $L = 0$ and $G = 0$, then the transmission line equations [(1) and (2) of § 18.10] become

$$\frac{\partial v}{\partial x} = -Ri, \frac{\partial i}{\partial x} = -C\frac{\partial v}{\partial t} \quad i.e., \quad \frac{\partial^2 v}{\partial x^2} = RC\frac{\partial v}{\partial t} \quad ...(i)$$

The boundary conditions are $v(0, t) = v_0$ and $i(x, t)$ is finite for all x and t.

The initial conditions are $v(x, 0) = 0, i(x, 0) = 0$. ...(ii)

Laplace transforms of (*i*), are

$$\frac{d^2\bar{v}}{dx^2} = RC(s\bar{v} - 0) \quad \text{or} \quad \frac{d^2\bar{v}}{dx^2} - RCs\bar{v} = 0 \qquad ...(iii)$$

Laplace transforms of the conditions in (*ii*), are

$$\bar{v}(0, s) = \frac{v_0}{s} \text{ at } x = 0 \qquad ...(iv)$$

and $$\bar{v}(x, s) \text{ remains finite as } x \to \infty \qquad ...(v)$$

∴ the solution of (*iii*) is

$$\bar{v}(x, s) = C_1 e^{\sqrt{RCs}\,x} + C_2 e^{-\sqrt{RCs}\,x}$$

To satisfy condition (*v*), we must have $C_1 = 0$.

Using the condition (*iv*), we get $C_2 = v_0/s$

Thus $$\bar{v}(x, s) = \frac{v_0}{s} e^{-\sqrt{RCs}\,x}$$

Using the inversion formula, we obtain

$$v(x, t) = v_0 L^{-1}\left\{\frac{e^{-\sqrt{RC}\,x\,.\sqrt{s}}}{s}\right\} = v_0 \; erfc\left(x\frac{\sqrt{RC}}{2\sqrt{t}}\right) \qquad \text{[By Ex. 22.17]}$$

$$= v_0 \frac{x\sqrt{RC}}{2\sqrt{\pi}} \int_0^t u^{-3/2} e^{-(RC\,x^2/4u)}\, du$$

∴ since $i = -\dfrac{1}{R}\dfrac{\partial v}{\partial x}$, we obtain by differentiation,

$$i(x, t) = \frac{v_0 x}{2\sqrt{x}}\sqrt{\frac{C}{R}}\, t^{-3/2} e^{(-RC\,x^2/4t)}.$$

Example 22.28. *A transmission line of length l has negligible inductance and leakance. A constant voltage v_0 is applied at the sending end (x = 0) and is open circuited at the far end. Assuming the initial voltage and current to be zero, determine the voltage and current.*

Solution. For a transmission line with $L = G = 0$, the voltage v and current i are given by the equations

$$\frac{\partial^2 v}{\partial x^2} = RC\frac{\partial v}{\partial t} \text{ and } \frac{\partial v}{\partial x} + Ri = 0 \qquad ...(i)$$

The boundary conditions are (for $t > 0$)

$$v = v_0 \text{ at } x = 0 \text{ and } i = \frac{\partial v}{\partial x} = 0 \quad \text{at } x = l \qquad ...(ii)$$

The initial condition is $v = 0$ at $t = 0$ $(x > 0)$

Laplace transforms of (*i*) and (*ii*) are

$$\frac{\partial^2 \bar{v}}{\partial x^2} = RC(s\bar{v} - 0) \qquad ...(iii)$$

and $$\bar{v} = v_0/s \text{ at } x = 0, \; \frac{\partial \bar{v}}{\partial x} = 0 \text{ at } x = l \qquad ...(iv)$$

∴ the solution of (*iii*) is

$$\bar{v} = c_1 \cosh \sqrt{(RCs)}\, x + c_2 \sinh \sqrt{RCs}\, x$$

Applying conditions (*iv*), it gives

$$v_0/s = c_1,\; 0 = c_1 \sinh \sqrt{(RCs)}\, l + c_2 \cosh \sqrt{(RCs)}\, l$$

∴ $$\bar{v} = \frac{v_0}{s}\left[\cosh \sqrt{(RCs)}x - \frac{\sinh \sqrt{(RCs)}l}{\cosh \sqrt{(RCs)}l} \sinh \sqrt{(RCs)}x\right]$$

$$= \frac{v_0}{s} \frac{\cosh pq\sqrt{s}}{\cosh p\sqrt{s}} \qquad \text{where } p = \sqrt{(RC)}\,l \text{ and } q = (l - x)/l$$

By the inversion formula (3) § 22.10, we get

$$v(x, t) = \text{sum of the residues of } (e^{st}\,\bar{v}) \text{ at all poles of } e^{st}\,\bar{v}. \qquad ...(iv)$$

These poles are at $s = 0$ and $p\sqrt{s} = \pm i(2n-1)\pi/2 = \pm ipk$ (say)

Now $\quad Res\,(e^{st}\,\bar{v})_{s=0} = \underset{s \to 0}{\text{Lt}} \dfrac{se^{st}\,v_0 \cosh pq\sqrt{s}}{s \cosh p\sqrt{s}} = v_0$

and $\quad Res\,(e^{st}\,\bar{v})_{s=-k^2} = \underset{s \to -k^2}{\text{Lt}} \dfrac{(s+k^2)e^{st}\,v_0 \cosh pq\sqrt{s}}{s \cosh p\sqrt{s}} \qquad \left(\dfrac{0}{0} \text{ form}\right)$

$$= \underset{s \to -k^2}{\text{Lt}} \frac{v_0 . e^{st} \cosh pq\sqrt{s} + (s + k^2)(...)}{\cosh p\sqrt{s} + s \sinh p\sqrt{s} \,.\, \frac{1}{2} ps^{-1/2}}$$

$$= \frac{v_0\,e^{-k^2 t} \cosh (ipqk) + 0}{0 + 1/2\,(ipk) \sinh (ipk)} = \frac{2v_0 e^{-k^2 t} \cos (pqk)}{-pk \sin pk}$$

Adding up all the residues, (iv) gives

$$v(x, t) = v_0 + \frac{4v_0}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} e^{-[(2n-1)^2 \pi^2 t/4RCl^2]} \cos [(2n-1)\,\pi(l-x)/2l]$$

$\left[\because \quad pk = (2n-1)\,\pi/2, -\sin pk = (-1)^n, pqk = \frac{1}{2}(2n-1)\,\pi(l-x)/l, \; k^2 = (2n-1)^2\pi^2/4RCl^2\right]$

Also $\quad i = -\dfrac{1}{R}\dfrac{\partial v}{\partial x}. \qquad$ [By (i)]

PROBLEMS 22.4

1. Solve the differential equation using Laplace transform method, $\dfrac{\partial y}{\partial t} = 3\dfrac{\partial^2 y}{\partial x^2}$ where $y(\pi/2, t) = 0$, $(\partial y/\partial x)_{x=0} = 0$ and $y(x, 0) = 30 \cos 5x$. *(U.P.T.U., 2005)*
2. Using suitable transforms, solve the differential equation $\dfrac{\partial^2 V}{\partial x^2} = \dfrac{\partial V}{\partial t}$, $0 \le x \le \pi$, $t \ge 0$. where $V(0, t) = 0 = V(\pi, t)$ and $V(x, 0) = V_0$ constant.
3. The initial temperature along the length of an infinite bar is given by $u(x, 0) = \begin{cases} 2, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$. If the temperature $u(x, t)$ satisfies the equation $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$, $-\infty < x < \infty$, $t > 0$, find the temperature at any point of the bar at any point t. *(Rohtak, 2006)*
4. Use the complex form of the Fourier transform to show that
$$V = \frac{1}{2\sqrt{(\pi t)}} \int_{-\infty}^{\infty} \bar{f}(u) e^{[-(x-u)^2/4t]}\,du$$
is the solution of the boundary value problem
$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2}, -\infty < x < \infty, t > 0 \,;\, V = f(x) \text{ when } t = 0.$$ *(U.P.T.U., 2008)*
5. A semi-infinite solid $(x > 0)$ is initially at temperature zero. At time $t = 0$, a constant temperature $\theta_0 > 0$ is applied and maintained at the face $x = 0$. Show that the temperature at any point x and at any time t, is given by $\theta(x, t) = \theta_0\,erfc(x/2c\sqrt{t})$.

6. A solid is initially at constant temperature θ_0, while the ends $x = 0$ and $x = a$ are maintained at temperature zero. Determine the temperature at any point of the solid at any later time $t > 0$.
7. An infinite string is initially at rest along the x-axis. Its one end which is at $x = 0$, is given a periodic transverse displacement $a_0 \sin \omega t$, $t > 0$. Show that the displacement of any point of the string at any time is given by
$$y(x, t) = \begin{cases} a_0 \sin \omega(t - x/c), & t > x/c \\ 0, & t < x/c, \end{cases}$$
where c is the wave velocity.
8. An infinite string has an initial transverse displacement $y(x, 0) = f(x)$, $-\infty < x < \infty$, and is initially at rest. Show that
$$y(x, t) = \frac{1}{2}[f(x + ct) + f(x - ct)].$$
9. A semi-infinite transmission line has negligible inductance and leakance per unit length. A voltage v is applied at the sending end ($x = 0$) which is given by
$$v(0, t) = \begin{cases} v_0, & 0 < t < \tau \\ 0, & t > \tau \end{cases}$$
Show that the voltage at any point $x > 0$ at any time $t > 0$ is given by
$$v(x, t) = v_0\, erfc\left[\frac{x}{2}\sqrt{(RC/t)}\right].$$

22.12 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 22.5

Fill in the blanks or choose the correct answer in each of the following problems :

1. Fourier cosine transform of $f(t)$ is ....
2. Fourier sine transform of $1/x$ is ....
3. Convolution theorem for Fourier transforms states that ....
4. If Fourier transform of $f(x)$ is $F(s)$, then the inversion formula is ....
5. $F[x^n f(x)] =$ ....
6. If $F\{f(x)\} = F(s)$, then $F\{f(x - a)\} =$ ....
7. Fourier sine integral representation of a function $f(x)$ is given by ....
8. If $F_c[f(ax)] = k\, F_c(s/a)$, then $k =$ ....
9. Fourier transform of second derivative of $u(x, t)$ is ....
10. If $f(x) = \begin{cases} 1, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$, then Fourier sine integral of $f(x)$ is ....
11. Fourier sine transform of $f'(x)$ in the interval $(0, l)$ is ....
12. If $F(\lambda)$ is the Fourier transform of $f(x)$, then the Fourier transform of $f(ax)$ is ....
13. Inverse finite Fourier sine transform of $F_s(p) = \dfrac{1 - \cos p\pi}{(p\pi)^2}$ for $p = 1, 2, 3, \ldots$ and $0 < x < \pi$ is ....
14. If Fourier transform of $f(x) = F(s)$, then Fourier Transform of $f(2x)$ is ....
15. Fourier cosine transform of e^{-x} is ....
16. $f(x) = 1$, $0 < x < \infty$ cannot be represented by a Fourier integral. (True or False)
17. $\int_0^\infty |f(x)|^2\, dx = \int_0^\infty |F_c(s)|^2\, dx.$ (True or False)
18. Fourier transform is a linear operation. (True or False)
19. $F_s[x\, f(x)] = -\dfrac{d}{ds} F_c(s).$ (True or False)
20. Kernel of Fourier transform is e^{isx}. (True or False)
21. Finite Fourier cosine transform of $f(x) = 1$ in $(0, \pi)$ is zero. (True or False)

CHAPTER 23

Z-Transforms

23.1 INTRODUCTION

The development of communication branch is based on discrete analysis. Z-transform plays the same role in discrete analysis as Laplace transform in continuous systems. As such, Z-transform has many properties similar to those of the Laplace transform (§ 21.2). The main difference is that the Z-transform operates not on functions of continuous arguments but on sequences of the discrete integer-valued arguments, *i.e.* $n = 0, \pm 1, \pm 2,$ ...... The analogy of Laplace transform to Z-transform can be carried further. For every operational rule of Laplace transforms, there is a corresponding operational rule of Z-transforms and for every application of the Laplace transform, there is a corresponding application of Z-transform. A discrete system is expressible as a difference equation (§ 30.2) and its solutions are found using Z-transforms.

23.2 DEFINITION

If the function u_n is defined for discrete values ($n = 0, 1, 2, ...$) and $u_n = 0$ for $n < 0$, then its Z-transform is defined to be

$$Z(u_n) = U(z) = \sum_{n=0}^{\infty} u_n z^{-n} \text{ whenever the infinite series converges.} \qquad ...(i)$$

The inverse Z-transform is written as $Z^{-1}[U(z)] = u_n$.

If we insert a particular complex number z into the power series (i), the resulting value of $Z(u_n)$ will be a complex number. Thus the Z-transform $U(z)$ is a complex valued function of a complex variable z.

23.3 SOME STANDARD Z-TRANSFORMS

The direct application of the definition gives the following results :

(1) $Z(a^n) = \dfrac{z}{z-a}$ **(2)** $Z(n^p) = -z\dfrac{d}{dz}Z(n^{p-1})$, *p being a +ve integer.*

Proof. **(1)** By definition, $Z(a^n) = \sum_{n=0}^{\infty} a^n z^{-n}$

$$= 1 + (a/z) + (a/z)^2 + (a/z)^3 + \ldots = \frac{1}{1-(a/z)} = \frac{z}{z-a}$$ (*Kottayam, 2005*)

(2) $$Z(n^p) = \sum_{n=0}^{\infty} n^p z^{-n} = z \sum_{n=0}^{\infty} n^{p-1} . n . z^{-(n+1)} \quad \ldots(i)$$

Changing p to $p-1$, we get $Z(n^{p-1}) = \sum_{n=0}^{\infty} n^{p-1} . z^{-n}$

Differentiating it w.r.t. z,

$$\frac{d}{dz}[Z(n^{p-1})] = \sum_{n=0}^{\infty} n^{p-1} . (-n) z^{-(n+1)} \quad \ldots(ii)$$

Substituting (*ii*) in (*i*), we obtain $Z(n^p) = -z\frac{d}{dz}[Z(n^{p-1})]$

which is the desired *recurrence formula*.

In particular, we have the following formulae :

(3) $Z(1) = \dfrac{z}{z-1}$ [Taking $a = 1$ in (1)] **(4)** $Z(n) = \dfrac{z}{(z-1)^2}$ [Taking $p = 1$ in (2)]

(5) $Z(n^2) = \dfrac{z^2+z}{(z-1)^3}$ (*V.T.U., 2006*) **(6)** $Z(n^3) = \dfrac{z^3+4z^2+z}{(z-1)^4}$

(7) $Z(n^4) = \dfrac{z^4+11z^3+11z^2+z}{(z-1)^5}$.

23.4 LINEARITY PROPERTY

If a, b, c be any constants and u_n, v_n, w_n be any discrete functions, then

$$Z(au_n + bv_n - cw_n) = aZ(u_n) + bZ(v_n) - cZ(w_n)$$

Proof. By definition, $Z(au_n + bv_n - cw_n) = \sum_{n=0}^{\infty} (au_n + bv_n - cw_n)z^{-n}$

$$= a\sum_{n=0}^{\infty} u_n z^{-n} + b\sum_{n=0}^{\infty} v_n z^{-n} - c\sum_{n=0}^{\infty} w_n z^{-n}$$

$$= aZ(u_n) + bZ(v_n) - cZ(w_n).$$

23.5 DAMPING RULE

If $Z(u_n) = U(z)$, then $Z(a^{-n} u_n) = U(az)$

Proof. By definition, $Z(a^{-n} u_n) = \sum_{n=0}^{\infty} a^{-n} u_n . z^{-n} = \sum_{n=0}^{\infty} u_n . (az)^{-n} = U(az)$. (*Madras, 2006*)

Cor. $Z(a^n u_n) = U(z/a)$

Obs. The geometric factor a^{-n} when $|a| < 1$, damps the function u_n, hence the name *damping rule.*

23.6 SOME STANDARD RESULTS

The application of the damping rule leads to the following standard results :

(1) $Z(na^n) = \dfrac{az}{(z-a)^2}$ **(2)** $Z(n^2a^n) = \dfrac{az^2+a^2z}{(z-a)^3}$

(3) $Z(\cos n\theta) = \dfrac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}$ (4) $Z(\sin n\theta) = \dfrac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$

(5) $Z(a^n \cos n\theta) = \dfrac{z(z - a\cos\theta)}{z^2 - 2az\cos\theta + a^2}$ (6) $Z(a^n \sin n\theta) = \dfrac{az\sin\theta}{z^2 - 2az\cos\theta + a^2}$.

Proofs. **(1)** We know that $Z(n) = \dfrac{z}{(z-1)^2}$. Applying damping rule, we have

$$Z(na^n) = U(a^{-1}z) = \frac{a^{-1}z}{(a^{-1}z-1)^2} = \frac{az}{(z-a)^2}.$$ *(Madras, 2000 S)*

(2) We know that $Z(n^2) = \dfrac{z^2+z}{(z-1)^3}$. Applying damping rule, we have

$$Z(n^2a^n) = U(a^{-1}z) = \frac{(a^{-1}z)^2 + a^{-1}z}{(a^{-1}z-1)^3} = \frac{a(z^2+az)}{(z-a)^3}.$$

(3) and **(4)** We know that $Z(1) = \dfrac{z}{z-1}$. Applying damping rule, we have

$$Z(e^{-in\theta}) = Z(e^{-i\theta})^n \,.\, 1 = \frac{ze^{i\theta}}{ze^{i\theta}-1} = \frac{z}{z-e^{-i\theta}} = \frac{z(z-e^{i\theta})}{(z-e^{-i\theta})(z-e^{i\theta})}$$

$$= \frac{z(z-\cos\theta) - iz\sin\theta}{z^2 - z(e^{i\theta}+e^{-i\theta})+1} = \frac{z(z-\cos\theta)-iz\sin\theta}{z^2-2z\cos\theta+1}$$

Equating real and imaginary parts, we get (3) and (4). *(V.T.U., 2010 S; Anna, 2009)*

(5) We know that $Z(\cos n\theta) = \dfrac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$. By damping rule, we have

$$Z(a^n\cos n\theta) = \frac{a^{-1}z(a^{-1}z-\cos\theta)}{(a^{-1}z)^2 - 2(a^{-1}z)\cos\theta+1} = \frac{z(z-a\cos\theta)}{z^2-2az\cos\theta+a^2}$$ *(V.T.U., 2006)*

Similarly using (4) above, we get (6).

Example 23.1. *Find the Z-transform of the following :*

(i) $3n - 4\sin n\pi/4 + 5a$ *(ii)* $(n+1)^2$ *(V.T.U., 2010)*

(iii) $\sin(3n+5)$. *(V.T.U., 2009 S ; Kottayam, 2005)*

Solution. *(i)* $Z\left(3n - 4\sin\dfrac{n\pi}{4} + 5a\right) = 3Z(n) - 4Z\left(\sin\dfrac{n\pi}{4}\right) + 5a\,Z(1)$ [By Linearty property[

$$= 3\,.\,\frac{z}{(z-1)^2} - 4\,.\,\frac{z\sin n\pi/4}{z^2-2z\cos\pi/4+1} + 5a\,.\,\frac{z}{z-1}$$ [Using formulae for $Z(1)$, $Z(n)$, $Z(\sin n\theta)$]

$$= \frac{(3-5a)z+5az^2}{(z-1)^2} - \frac{2\sqrt{2}z}{z^2-\sqrt{2}z+1}$$

(ii) $Z(n+1)^2 = Z(n^2+2n+1) = Z(n^2) + 2Z(n) + Z(1)$

$$= \frac{z^2+z}{(z-1)^3} + 2\frac{z}{(z-1)^2} + \frac{z}{z-1} = \frac{z^2(2z+1)}{(z-1)^3}$$

(iii) $Z[\sin(3n+5)] = Z(\sin 3n\cos 5 + \cos 3n\sin 5)$

$= \cos 5 . Z(\sin 3n) + \sin 5 . Z(\cos 3n)$ (using formulae for $Z(\sin n\theta)$, $Z(\cos n\theta)$

$$= \cos 5\,.\,\frac{z\sin 3}{z^2-2z\cos 3+1} + \sin 5\,.\,\frac{z(z-\cos 3)}{z^2-2z\cos 3+1} = z\,.\,\frac{(z\sin 5-\sin 2)}{z^2-2z\cos 3+1}.$$

Example 23.2. *Find the Z-transforms of the following*
(*i*) e^{an} (*ii*) ne^{an} (*iii*) n^2e^{an}.

Solution. (*i*) Let $u_n = 1$, $e^{an} = (e^{-a})^{-n} = k^{-n}$ where $k = e^{-a}$. By damping rule $Z(k^{-n}\, u_n) = U(kz)$,

$$\therefore \quad Z(e^{an}) = Z(k^{-n}\,.\,1) = U(kz) = \frac{kz}{kz-1} \qquad \left[\because \; U(z) = Z(1) = \frac{z}{z-1}\right]$$

$$= \frac{z}{z-1/k} = \frac{z}{z-e^a}$$

(*ii*) Let $\quad u_n = n$, $e^{an} = (e^{-a})^{-n} = k^{-n}$ where $k = e^{-a}$

By damping rule, $Z\,(e^{an}\,.\,n) = Z(k^{-n}\,.\,n) = U(k\,z)$ where $U(z) = Z(n) = \dfrac{z}{(z-1)^2}$

$$\frac{kz}{(kz-1)^2} = \frac{z}{k(z-1/k)^2} = \frac{e^a z}{(z-e^a)^2}$$

(*iii*) Let $\quad u_n = n^2$, $e^{an} = (e^{-a})^{-n} = k^{-n}$ where $k = e^{-a}$

By damping rule,

$$Z(e^{an}\,.\,n^2) = Z(k^{-n}\,.\,n^2) = U(kz) \quad \text{where} \quad U(z) = Z(n^2) = \frac{z^2+z}{(z-1)^3}$$

$$= \frac{(kz)^2+kz}{(kz-1)^3} = \frac{z(z+1/k)}{(z-1/(k)^3} = \frac{ze^a(z+e^a)}{(z-e^a)^3}.$$

Example 23.3. *Find the Z-transform of* (*i*) $\cosh n\theta$ (V.T.U., 2011) (*ii*) $a^n \cosh n\theta$.

Solution. (*i*)

$$Z\,(\cosh n\theta) = Z\left(\frac{e^{n\theta}+e^{-n\theta}}{2}\right)$$

$$= \frac{1}{2}\left[Z\left\{(e^{-\theta})^{-n}\,.\,1\right\} + Z\left\{(e^{\theta})^{-n}\,.\,1\right\}\right]$$

Apply damping rule to both terms, taking $u_n = 1$.

$$Z(\cosh n\theta) = \frac{1}{2}\left[\frac{ze^{-\theta}}{ze^{-\theta}-1} + \frac{ze^{\theta}}{ze^{\theta}-1}\right] \qquad \left[\because \; z(1) = \frac{z}{z-1}\right]$$

$$= \frac{1}{2}\left\{\frac{2z^2 - z(e^\theta + e^{-\theta})}{z^2 - z(e^\theta+e^{-\theta})+1}\right\} = \frac{z^2 - z\cosh\theta}{z^2 - 2z\cosh\theta + 1}$$

(*ii*)

$$Z(a^n \cosh n\theta) = Z[(a^{-1})^{-n}\,.\,\cosh n\theta] \qquad \text{[Apply damping rule using } (i)]$$

$$= \frac{(a^{-1}z)^2 - (a^{-1}\,z)\cosh\theta}{(a^{-1}\,z)^2 - 2(a^{-1}\,z)\cosh\theta + 1} = \frac{z(z-a\cosh\theta)}{z^2 - 2az\cosh\theta + a^2}.$$

Example 23.4. *Find the Z-transforms of*
(*i*) $e^t \sin 2t$ (Madras, 2003) (*ii*) $c^k \cos k\alpha$, $(k \geq 0)$ (U.P.T.U., 2004 S)

Solution. (*i*) We know that $Z\,(\sin 2t) = \dfrac{z\sin 2}{z^2 - 2z\cos 2 + 1}$ ...(A)

$$\therefore \quad Z\,(e^t \sin 2t) = Z\,[(e^{-1})^{-t}\,.\,\sin 2t] \qquad \text{[Apply damping rule, using (A)]}$$

$$= \frac{(e^{-1}\,z)\sin 2}{(e^{-1}z)^2 - 2(e^{-1}\,z)\cos 2 + 1} = \frac{ez\sin 2}{z^2 - 2ez\cos 2 + e^2}.$$

(*ii*) We know that $\quad Z\,(\cos k\alpha) = \dfrac{z(z-\cos\alpha)}{z^2 - 2z\cos\alpha + 1}$ ...(B)

$$\therefore \quad Z(c^k \cos k\alpha) = Z\,[(c^{-1})^{-k}\,.\,\cos k\alpha] \qquad \text{[Apply damping rule, using (B)]}$$

$$= \frac{(c^{-1}\,z)\,[c^{-1}z - \cos\alpha]}{(c^{-1}\,z)^2 - 2(c^{-1}\,z)\cos\alpha + 1} = \frac{z(z-c\cos\alpha)}{z^2 - 2cz\cos\alpha + c^2}.$$

Example 23.5. *Find the Z-transforms of*

(i) $\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)$ (V.T.U., 2011 S) (ii) $\cosh\left(\frac{n\pi}{2}+\theta\right)$. (U.P.T.U., 2008)

Solution. (i) $Z\left[\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)\right]=Z\left(\cos\frac{n\pi}{2}\cos\frac{\pi}{4}-\sin\frac{n\pi}{2}\sin\frac{\pi}{4}\right)$

$$=\cos\frac{\pi}{4}.Z\left(\cos\frac{n\pi}{2}\right)-\sin\frac{\pi}{2}.Z\left(\sin\frac{n\pi}{2}\right)$$ [Using formulae for $Z(\sin n\alpha)$ and $Z(\cos n\alpha)$]

$$=\frac{1}{\sqrt{2}}\left\{\frac{z(z-\cos\pi/2)}{z^2-2z\cos\pi/2+1}-\frac{z\sin\pi/2}{z^2-2z\cos\pi/2+1}\right\}=\frac{1}{\sqrt{2}}\left(\frac{z^2}{z^2+1}-\frac{z}{z^2+1}\right)=\frac{z(z-1)}{\sqrt{2}(z^2+1)}$$

(ii) $$Z\left[\cosh\left(\frac{n\pi}{2}+\theta\right)\right]=Z\left[\frac{e^{n\pi/2+\theta}+e^{-(n\pi/2+\theta)}}{2}\right]=\frac{1}{2}\left[e^{\theta}\,Z(e^{n\pi/2})+e^{-\theta}\,Z(e^{-n\pi/2})\right]$$

Since, $Z(a^n)=\frac{z}{z-a}$, $\therefore\ Z(e^{n\pi/2})=Z(e^{\pi/2})^n=\frac{z}{z-e^{\pi/2}}$, $Z(e^{-n\pi/2})=\frac{z}{z-e^{-\pi/2}}$

Thus $$Z\left[\cosh\left(\frac{n\pi}{2}+\theta\right)\right]=\frac{1}{2}\left\{e^{\theta}.\frac{z}{z-e^{\pi/2}}+e^{-\theta}.\frac{z}{z-e^{-\pi/2}}\right\}$$

$$=\frac{z}{2}\left\{\frac{z(e^{\theta}+e^{-\theta})-[e^{(\pi/2-\theta)}+e^{-(\pi/2-\theta)}]}{z^2-z(e^{\pi/2}+e^{-\pi/2})+1}\right\}=\frac{z^2\cosh\theta-z\cosh\left(\frac{\pi}{2}-\theta\right)}{z^2-2z\cosh\left(\frac{\pi}{2}\right)+1}.$$

Example 23.6. *Find the Z-transform of*

(i) ${}^nC_p\ (0\le p\le n)$ (ii) ${}^{n+p}C_p$.

Solution. (i) $$Z({}^nC_p)=\sum_{p=0}^{n}\left({}^nC_pz^{-p}\right)=1+{}^nC_1z^{-1}+{}^nC_2z^{-2}+\dots+{}^nC_nz^{-n}=(1+z^{-1})^n$$

(ii) $$Z({}^{n+p}C_n)=\sum_{p=0}^{n}{}^{n+p}C_pz^{-p}$$

$$=1+{}^{n+1}C_1z^{-1}+{}^{n+2}C_2z^{-2}+{}^{n+3}C_3z^{-3}+\dots\infty$$

$$=1+(n+1)z^{-1}+\frac{(n+2)(n+1)}{2!}z^{-2}+\frac{(n+3)(n+2)(n+1)}{3!}z^{-3}+\dots\infty$$

$$=1+(-n-1)(-z^{-1})+\frac{(-n-1)(-n-2)}{2!}(-z^{-1})^2+\frac{(-n-1)(-n-2)(-n-3)}{3!}(-z^{-1})^3+\dots\infty$$

$$=(1-z^{-1})^{-n-1}.$$

Example 23.7. *Find the Z-transform of*

(i) *unit impulse sequence* $\delta(n)=\begin{cases}1 & n=0\\ 0 & n\ne 0\end{cases}$ (ii) *unit step sequence* $u(n)=\begin{cases}0 & n<0\\ 1 & n\ge 0\end{cases}$.

Solution. (i) $$Z[\delta(n)]=\sum_{n=0}^{\infty}\delta(n)z^{-n}=1+0+0+\dots=1$$

(ii) $$Z[u(n)]=\sum_{n=0}^{\infty}u(n)z^{-n}=1+z^{-1}+z^{-2}+z^{-3}+\dots=\frac{1}{1-z^{-1}}=\frac{z}{z-1}.$$

23.7 (1) SHIFTING U_N TO THE RIGHT

If $Z(u_n) = U(z)$, *then* $Z(u_{n-k}) = z^{-k}U(z)$ $(k > 0)$

Proof. By definition,

$$Z(u_{n-k}) = \sum_{n=0}^{\infty} u_{n-k}z^{-n} = u_0 z^{-k} + u_1 z^{-(k+1)} + \dots = z^{-k}\sum_{n=0}^{\infty} u_n z^{-n} = z^{-k}U(z)$$

Obs. This rule will be very useful in applications to difference equations.

(2) **Shifting u_n to the left.** If $Z(u_n) = U(z)$, then

$$Z(u_{n+k}) = z^k[U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} - \dots - u_{k-1}z^{-(k-1)}]$$

Proof. $Z(u_{n+k}) = \sum_{n=0}^{\infty} u_{n+k}z^{-n} = z^k \sum_{n=0}^{\infty} u_{n+k}z^{-(n+k)}$

$$= z^k\left[\sum_{n=0}^{\infty} u_n z^{-n} - \sum_{n=0}^{k-1} u_n z^{-n}\right]$$

Hence $Z(u_{n+k}) = z^k[U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} - \dots - u_{k-1}z^{-(k-1)}]$ (*J.N.T.U., 2002*)

In particular, we have the following standard results :

(1) $Z(u_{n+1}) = z[U(z) - u_0]$; (2) $Z(u_{n+2}) = z^2[U(z) - u_0 - u_1 z^{-1}]$

(3) $Z(u_{n+3}) = z^3[U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2}]$.

Example 23.8. *Show that* $Z\left(\frac{1}{n!}\right) = e^{1/z}$.

Hence evaluate $Z[1/(n+1)!]$ *and* $Z[1/(n+2)!]$. (*Madras, 2006*)

Solution. We have $Z\left(\frac{1}{n!}\right) = \sum_{n=0}^{\infty} \frac{1}{n!}z^{-n} = 1 + \frac{z^{-1}}{1!} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \dots = e^{1/z}$.

Shifting $(1/n!)$ one unit to the left gives

$$Z\left[\frac{1}{(n+1)!}\right] = z\left[Z\left(\frac{1}{n!}\right) - 1\right] = z(e^{-1/z} - 1)$$

Similarly shifting $(1/n!)$ two units to the left gives

$$Z\left[\frac{1}{(n+2)!}\right] = z^2(e^{1/z} - 1 - z^{-1}).$$

23.8 MULTIPLICATION BY n

If $Z(u_n) = u(z)$, then $Z(nu_n) = -z\frac{dU(z)}{dz}$

Proof.

$$Z(nu_n) = \sum_{n=0}^{\infty} n\,.\,u_n z^{-n} = -z\sum_{n=0}^{\infty} u_n(-n)z^{-n-1} = -z\sum_{n=0}^{\infty} u_n \frac{d}{dz}(z^{-n}).$$

$$= -z\sum_{n=0}^{\infty}\frac{d}{dz}(u_n z^{-n}) = -z\frac{d}{dz}\left(\sum_{n=0}^{\infty} u_n z^{-n}\right) = -z\frac{d}{dz}U(z).$$

Obs. We have, $Z(n^2 u_n) = \left(-z\frac{d}{dz}\right)^2 u(z)$ (*Madras, 2006*)

In general, $Z(n^m u_n) = \left(-z\frac{d}{dz}\right)^m u(z)$.

Example 23.9. *Find the Z-transform of (i) $n \sin n\theta$ (ii) $n^2 e^{n\theta}$.*

Solution. (*i*) We know that $Z(nu_n) = -z\dfrac{dU(z)}{dz}$ and $Z(\sin n\theta) = \dfrac{z \sin\theta}{z^2 - 2z\cos\theta + 1}$

$$\therefore \quad Z(n \sin n\theta) = -z\frac{d}{dz}[Z(\sin n\theta)] = -z\frac{d}{dz}\left(\frac{z\sin\theta}{z^2 - 2z\cos\theta+1}\right)$$

$$= -z\frac{\sin\theta - z^2\sin\theta}{(z^2-2z\cos\theta+1)^2} = \frac{z(z^2-1)\sin\theta}{(z^2-2z\cos\theta+1)^2}$$

(*ii*) We know that $\quad Z(e^{n\theta}) = \dfrac{z}{z-e^{\theta}}$

$$\therefore \quad Z(n^2e^{n\theta}) = \left(-z\frac{d}{dz}\right)^2 (Ze^{n\theta}) = \left(-z\frac{d}{dz}\right)\left[-z\frac{d}{dz}\left(\frac{z}{z-e^{\theta}}\right)\right]$$

$$= \left(-z\frac{d}{dz}\right)\left\{-z\frac{(z-e^{\theta})(1)-z(1)}{(z-e^{\theta})^2}\right\} = -z\frac{d}{dz}\left\{\frac{ze^{\theta}}{(z-e^{\theta})^2}\right\}$$

$$= -ze^{\theta}\left\{\frac{(z-e^{\theta})^2(1) - z[2(z-e^{\theta})]}{(z-e^{\theta})^4}\right\} = -ze^{\theta}\frac{z-e^{\theta}-2z}{(z-e^{\theta})^3} = \frac{z(z+e^{\theta})e^{\theta}}{(z-e^{\theta})^3}.$$

23.9 TWO BASIC THEOREMS

In applications, we often need the values of u_n for $n = 0$ or as $n \to \infty$ without requiring complete knowledge of u_n. We can find this as the behaviour of u_n for small values of n is related to the behaviour of $U(z)$ as $z \to \infty$ and *vice-versa*. The precise relationship is given by the following *initial and final value theorems :*

(1) Initial value theorem. *If $Z(u_n) = U(z)$, then $u_0 = \underset{z\to\infty}{Lt}\, U(z)$*

Proof. We know that $U(z) = Z(u_n) = u_0 + u_1z^{-1} + u_2z^{-2} + \ldots$

Taking limits as $z \to \infty$, we get $\underset{z\to\infty}{\text{Lt}}\,[U(z)] = u_0$, as required.

Similarly additional initial values can be found successively, giving :

$$u_1 = \underset{z\to\infty}{\text{Lt}}\left\{z\,[U(z) - u_0]\right\};\ u_2 = \underset{z\to\infty}{\text{Lt}}\left\{z^2\,[U(z) - u_0 - u_1z^{-1}]\right\} \text{ and so on.}$$

(2) Final value theorem. *If $Z(u_n) = U(z)$, then*

$$\underset{n\to\infty}{Lt}\,(u_n) = \underset{z\to 1}{Lt}\,(z-1)\,U(z)$$

Proof. By definition, $Z(u_{n+1} - u_n) = \displaystyle\sum_{n=0}^{\infty}(u_{n+1} - u_n)z^{-n}$

or $$Z(u_{n+1}) - Z(u_n) = \sum_{n=0}^{\infty}(u_{n+1} - u_n)z^{-n}$$

or $$z[U(z) - u_0] - U(z) = \sum_{n=0}^{\infty}(u_{n+1} - u_n)z^{-n}$$

or $$U(z)\,(z-1) - u_0z = \sum_{n=0}^{\infty}(u_{n+1} - u_n)z^{-n}$$

Taking limits of both sides as $z \to 1$, we get

$$\underset{z\to1}{\text{Lt}}\,[(z-1)\,U(z)] - u_0 = \sum_{n=0}^{\infty}(u_{n+1}-u_n) = \underset{n\to\infty}{\text{Lt}}\,[(u_1-u_0)+(u_2-u_1)+\ldots+(u_{n+1}-u_n)]$$

$$= \underset{n\to\infty}{\text{Lt}} [u_{n+1}] - u_0 = u_\infty - u_0$$

Hence $$u_\infty = \underset{z\to 1}{\text{Lt}} [(z-1)\,U(z)].$$ *(Anna, 2005 S)*

Example 23.10. *If* $U(z) = \dfrac{2z^2+5z+14}{(z-1)^4}$, *evaluate* u_2 *and* u_3.

Solution. Writing $$U(z) = \frac{1}{z^2}\cdot\frac{2+5z^{-1}+14z^{-2}}{(1-z^{-1})^4}$$

By initial value theorem, $$u_0 = \underset{z\to\infty}{\text{Lt}}\, U(z) = 0$$

Similarly, $$u_1 = \underset{z\to\infty}{\text{Lt}} \{z\,[U(z)-u_0]\} = 0$$

Now $$u_2 = \underset{z\to\infty}{\text{Lt}} \{z^2\,[U(z)-u_0-u_1z^{-1}]\} = 2-0-0 = 2$$

and $$u_3 = \underset{z\to\infty}{\text{Lt}}\, z^3[U(z)-u_0-u_1z^{-1}-u_2z^{-2}] = \underset{z\to\infty}{\text{Lt}}\, z^3[U(z)-0-0-2z^{-2}]$$

$$= \underset{z\to\infty}{\text{Lt}}\, z^3\left[\frac{2z^2+5z+14}{(z-1)^4}-\frac{2}{z^2}\right] = \underset{z\to\infty}{\text{Lt}}\, z^3\left\{\frac{13z^3+2z^2+8z-2}{z^2(z-1)^4}\right\} = 13.$$

PROBLEMS 23.1

1. Find the Z-transforms of the following sequences :
   (*i*) $\dfrac{a^n}{n!}$ $(n \ge 0)$ *(S.V.T.U., 2009)* (*ii*) $\dfrac{1}{(n+1)!}$ (*iii*) $(\cos\theta + i\sin\theta)^n$.
2. Using the linearity property, find the Z- transforms of the following functions :
   (*i*) $2n + 5\sin n\pi/4 - 3a^4$ (*ii*) $\frac{1}{2}(n-1)(n+2)$ *(S.V.T.U., 2007)*
   (*iii*) $(n+1)(n+2)$ *(Anna, 2008)* (*iv*) $(2n-1)^2$ *(V.T.U., 2011 S)*
3. Show that (*i*) $Z(\sinh n\theta) = \dfrac{z\sinh\theta}{z^2-2z\cosh\theta+1}$ *(V.T.U., 2011)* (*ii*) $Z(a^n\sinh n\theta) = \dfrac{az\sinh\theta}{z^2-2az\cosh\theta+a^2}$.
4. Show that (*i*) $Z(e^{-an}\cos n\theta) = \dfrac{ze^a(ze^a-\cos\theta)}{z^2e^{2a}-2ze^a\cos\theta+1}$; (*ii*) $Z(e^{-an}\sin n\theta) = \dfrac{ze^a\sin\theta}{z^2e^{2a}-2ze^a\cos\theta+1}$
   Also evaluate $Z(e^{3n}\sin 2n)$. *(S.V.T.U., 2007)*
5. Using $Z(n^2) = \dfrac{z^2+z}{(z-1)^3}$, show that $Z(n+1)^2 = \dfrac{z^3+z^2}{(z-1)^3}$.
6. Find the Z-transforms of (*i*) $\sin(n+1)\theta$, (*ii*) $\cos\left(\dfrac{h\pi}{8}+\alpha\right)$. *(Marathwada, 2008)*
7. Find the Z-transform of $\cos n\theta$ and hence find $Z(n\cos n\theta)$. *(Anna, 2009)*
8. Find the Z-transform of $\cos(n\pi/2)$ and $a^n\cos(n\pi/2)$. *(Anna, 2008 S)*
9. Find the Z-transforms of the following
   (*i*) e^{-an} (*ii*) e^{-2n} *(V.T.U., 2010 S)* (*iii*) $e^{-an}n^2$.
10. Show that (*i*) $Z[\delta(n+1)] = 1/z$ (*ii*) $(1/2)^n u(n) = \dfrac{2z}{2z-1}$.
11. Show that $Z({}^{n+p}C_p) = (1-1/z)^{-(p+1)}$. Using the damping rule, deduce that
    $$Z({}^{n+p}C_p\, a^n) = (1+a/z)^{-(p+1)}.$$
12. If $Z(u_n) = \dfrac{z}{z-1}+\dfrac{z}{z^2+1}$, find the Z-transform of u_{n+2}. *(S.V.T.U., 2009)*
13. If $U(z) = \dfrac{2z^2+3z+12}{(z-1)^4}$, find the value of u_2 and u_3.

14. Given that $Z(u_n) = \dfrac{2z^2+3z+4}{(z-3)^3}$, $|z| > 3$, show that $u_1 = 2$, $u_2 = 21$, $u_3 = 139$.

15. Show that (i) $Z\left(\dfrac{1}{n}\right) = z \log \dfrac{z}{z-1}$. (*Madras, 2003 S*) (ii) $Z\left\{\dfrac{1}{n(n+1)}\right\}$. (*Anna, 2005 S*)

16. Using $Z(n) = \dfrac{z}{(z-1)^2}$, show that $Z(n \cos n\theta) = \dfrac{(z^3+z)\cos\theta - 2z^2}{(z^2-2z\cos\theta+1)^2}$.

23.10 SOME USEFUL Z-TRANSFORMS

Sr. No.	*Sequence u_n $(n \geq 0)$*	*Z-transform $U(z) = Z(u_n)$*
1.	k	$kz/(z-1)$
2.	$-k$	$kz/(z+1)$
3.	n	$z/(z-1)^2$
4.	n^2	$(z^2+z)/(z-1)^3$
5.	n^p	$-z\, d/dz\, [Z(n^{p-1})]$, p + ve integer.
6.	$\delta(n) = \begin{cases} 1 & n=0 \\ 0 & n \neq 0 \end{cases}$	1
7.	$u(n) = \begin{Bmatrix} 0 & n<0 \\ 1 & n \geq 0 \end{Bmatrix}$	$z/(z-1)$
8.	a^n	$z/(z-a)$
9.	na^n	$az/(z-a)^2$
10.	n^2a^n	$(az^2+a^2z)/(z-a)^3$
11.	$\sin n\theta$	$\dfrac{z\sin\theta}{z^2-2z\cos\theta+1}$
12.	$\cos n\theta$	$\dfrac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$
13.	$a^n \sin n\theta$	$\dfrac{az\sin\theta}{z^2-2az\cos\theta+a^2}$
14.	$a^n \cos n\theta$	$\dfrac{z(z-a\cos\theta)}{z^2-2az\cos\theta+a^2}$
15.	$\sinh n\theta$	$\dfrac{z\sinh\theta}{z^2-2z\cosh\theta+1}$
16.	$\cosh n\theta$	$\dfrac{z(z-\cosh\theta)}{z^2-2z\cosh\theta+1}$
17.	$a^n \sinh n\theta$	$\dfrac{az\sinh\theta}{z^2-2az\cosh\theta+a^2}$
18.	$a^n \cosh n\theta$	$\dfrac{z(z-a\cosh\theta)}{z^2-2az\cosh\theta+a^2}$
19.	$a^n u_n$	$U(z/a)$
20.	u_{n+1}	$z[U(z)-u_0]$
	u_{n+2}	$z^2[U(z)-u_0-u_1z^{-1}]$
	u_{n+3}	$z^3[U(z)-u_0-u_1z^{-1}-u_2z^{-2}]$
21.	u_{n-k}	$z^{-k}U(z)$
22.	nu_n	$-zd/dz[U(z)]$
23.	u_0	$\underset{z\to\infty}{\text{Lt}}\ U(z)$
24.	$\underset{n\to\infty}{\text{Lt}}\ (u_n)$	$\underset{z\to 1}{\text{Lt}}\ [(z-1)\,U(z)]$

23.11 SOME USEFUL INVERSE Z-TRANSFORMS

Sr. No.	$U(z)$	Inverse Z-transform $u_n = z^{-1}[U(z)]$
1.	$\frac{1}{z-a}$	a^{n-1}
2.	$\frac{1}{z+a}$	$(-a)^{n-1}$
3.	$\frac{1}{(z-a)^2}$	$(n-1)a^{n-2}$
4.	$\frac{1}{(z-a)^3}$	$\frac{1}{2}(n-1)(n-2)a^{n-3}$
5.	$\frac{z}{z-a}$	a^n
6.	$\frac{z}{z+a}$	$(-a)^n$
7.	$\frac{z^2}{(z-a)^2}$	$(n+1)a^n$
8.	$\frac{z^3}{(z-a)^3}$	$\frac{1}{2!}(n+1)(n+2)a^n u(n)$

23.12 CONVOLUTION THEOREM

If $Z^{-1}[U(z)] = u_n$ and $Z^{-1}[V(z)] = v_n$, then

$$Z^{-1}[U(z) \,.\, V(z)] = \sum_{m=0}^{n} u_m \,.\, v_{n-m} = u_n * v_n$$

where the symbol $*$ denotes the convolution operation.

Proof. We have $U(z) = \sum_{n=0}^{\infty} u_n z^{-n}$, $V(z) = \sum_{n=0}^{\infty} v_n z^{-n}$

$\therefore$ $$U(z)\,V(z) = (u_0 + u_1 z^{-1} + u_2 z^{-2} + \ldots + u_n z^{-n} + \ldots \infty) \times (v_0 + v_1 z^{-1} + v_2 z^{-2} + \ldots + v_n z^{-n} + \ldots \infty)$$

$$= \sum_{n=0}^{\infty} (u_0 v_n + u_1 v_{n-1} + u_2 v_{n-2} + \ldots + u_n v_0) z^{-n} = Z(u_0 v_n + u_1 v_{n-1} + \ldots + u_n v_0)$$

whence follows the desired result.

Obs. The convolution theorem plays an important role in the solution of difference equations and in probability problems involving sums of two independent random variables.

Example 23.11. *Use convolution theorem to evaluate* $Z^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\}$.

Solution. We know that $Z^{-1}\left\{\frac{z}{z-a}\right\} = a^n$ and $Z^{-1}\left\{\frac{z}{z-b}\right\} = b^n$

$\therefore$ $$Z^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\} = Z^{-1}\left\{\frac{z}{z-a} \cdot \frac{z}{z-b}\right\} = a^n * b^n$$

$$= \sum_{m=0}^{n} a^m \,.\, b^{n-m} = b^n \sum_{m=0}^{n} \left(\frac{a}{b}\right)^m \text{ which is a G.P.}$$

$$= b^n \,.\, \frac{(a/b)^{n+1} - 1}{a/b - 1} = \frac{a^{n+1} - b^{n+1}}{a - b}.$$

23.13 CONVERGENCE OF Z-TRANSFORMS

Z-transform operation is performed on a sequence u_n which may exist in the range of integers $-\infty < n < \infty$, and we write

$$U(z) = \sum_{n=-\infty}^{\infty} u_n z^{-n} \qquad ...(1)$$

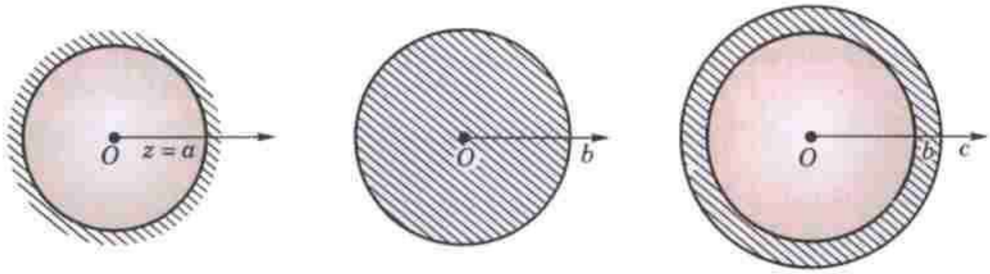
where u_n represents a number in the sequence for n = an integer. *The region of the z-plane in which (1) converges absolutely is known as the region of convergence (ROC) of U(z).*

We have so far discussed *one-sided Z-transform only for which* $n \geq 0$. Here the sequence is always *right-sided* and the *ROC* is always outside a prescribed circle say $|z| > |a|$ Fig. 23.2 (*i*)]. For a left-handed sequence, the ROC is always inside any prescribed contour $|z| < |b|$. [Fig. 23.2 (*ii*)].

23.14 TWO-SIDED Z-TRANSFORM OF u_n IS DEFINED BY

$$U(z) = \sum_{n=-\infty}^{\infty} u_n z^{-n} \qquad ...(2)$$

In this case, the sequence is two-sided and the region of convergence for (2) is the *annular region* $|b| < |z| < |c|$ [Fig. 23.2 (*iii*)]. The inner circle bounds the terms in negative powers of z and the outer circle bounds the terms in positive powers of z. *The shaded annulus of convergence is necessary for the two sided sequence and its Z-transform to exist.*



$|z| > |a|$ (*i*) $\qquad$ $|z| < |b|$ (*ii*) $\qquad$ $|b| < |z| < |c|$ (*iii*)

Fig. 23.1

Example 23.12. *Find the Z-transfrom and region of convergence of*

(*a*) $u(n) = \begin{cases} 4^n & \text{for } n < 0 \\ 2^n & \text{for } n \geq 0 \end{cases}$ $\qquad$ (*b*) $u(n) = {}^nc_k,\ n \geq k.$

Solution. By definition $\quad Z[u(n)] = \sum_{-\infty}^{\infty} u(n) Z^{-n} = \sum_{-\infty}^{-1} 4^n z^{-n} + \sum_{n=0}^{\infty} 2^n z^{-n}$

Putting $-n = m$ in the first series, we get

$$Z[u(n)] = \sum_{1}^{\infty} 4^{-m} z^m + \sum_{n=0}^{\infty} 2^n z^{-n}$$

$$\left\{\frac{z}{4} + \frac{z^2}{4^2} + \frac{z^3}{4^3} + \ldots\right\} + \left\{1 + \frac{2}{z} + \frac{2^2}{z^2} + \ldots\right\}$$

$$= \frac{z}{4}\left\{1 + \left(\frac{z}{4}\right) + \left(\frac{z}{4}\right)^2 + \ldots\right\} + \left\{1 + \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 + \ldots\right\} \qquad ...(i)$$

$$= \frac{z}{4} \cdot \frac{1}{1-(z/4)} + \frac{1}{1-(2/z)} = \frac{z}{4-z} + \frac{z}{z-2} = \frac{2z}{(4-z)(z-2)}$$

Now the two series in (*i*) being G.P. will be convergent if $| z/4 | < 1$ and $| 2/z | < 1$ *i.e.*, if $| z | < 4$ and $2 < | z |$. *i.e.* $2 < z < 4$.

Hence $Z[u(n)]$ is convergent if z lies between the annulus as shown shaded in Fig. 23.3. Hence ROC is $2 < z < 4$.

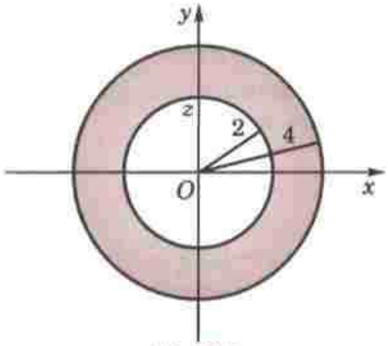


Fig. 23.2

(*b*) By definition, $Z[u(n)] = \sum_{-\infty}^{\infty} {}^nC_k z^{-n} = \sum_{n=k}^{\infty} {}^nC_k 2^n z^{-n}$

To find the sum of this series, we replace n by $k + r$

$$\therefore \quad Z[u(n)] = \sum_{r=0}^{\infty} {}^{k+r}C_k z^{-(k+r)} = z^{-k} \sum_{r=0}^{\infty} {}^{k+r}C_r z^{-r} \qquad [\because \ {}^kC_r = {}^kC_{k-r}]$$

$$= z^{-k}[1 + {}^{k+1}C_1 z^{-1} + {}^{k+1}C_2 z^{-2} + \ldots]$$

$$= z^{-k}(1 - 1/z)^{-(k+1)}$$

This series is convergence for $| 1/z | < 1$ *i.e.*, for $| z | > 1$.

Hence ROC is $| z | > 1$.

Example 23.13. *Find the Z-transform and the radius of convergence of*

(*a*) $f(n) = 2^n, n < 0$ (*b*) $f(n) = 5^n/n!, n \geq 0$. (*Mumbai, 2009*)

Solution. (*a*) Assuming that $f(n) = 0$ for $n \geq 0$ we have

$$Z[f(n)] = \sum_{-\infty}^{\infty} f(n) z^{-n} = \sum_{n=-\infty}^{-1} 2^n z^{-n} = \sum_{m=1}^{\infty} 2^{-m} z^m \quad \text{where} \quad m = -n$$

$$= \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \ldots\infty = \frac{z}{2}\{1 + (z/2) + (z/2)^2 + \ldots \infty\}$$

$$= \frac{z}{2} \cdot \frac{1}{1-(z/2)} = \frac{z}{2-z}$$

This series being a G.P. is convergent if $| z/2 | < 1$ *i.e.*, $| z | < 2$.

Hence ROC is $| z | < 2$.

(*b*) By definition, $Z[u(n)] = \sum_{n=0}^{\infty} \frac{5^n}{n!} \cdot z^{-n} = \sum_{0}^{\infty} \frac{(5/z)^n}{n!} = 1 + \left(\frac{5}{z}\right) + \frac{1}{2!}\left(\frac{5}{z}\right)^2 + \frac{1}{3!}\left(\frac{5}{z}\right)^3 + \ldots \infty$

$$= e^{5/z}$$

The above series is convergent for all values of z.

Hence ROC is the entire z-plane.

PROBLEMS 23.2

Find the Z-transform and its ROC in each of the following sequences :

1. $u(n) = 4^n, n \geq 0$.
2. $u(n) = 2^n, n < 0$.
3. $u(n) = 4^n$, for $n < 0$ and $= 3^n$ for $n \geq 0$.
4. $u(n) = n5^n, n \geq 0$.
5. $u(n) = 2^n/n, n > 1$.
6. $u(n) = 3^n/n!, n \geq 0$.
7. $u(n) = e^{an}, n \geq 0$.

23.15 EVALUATION OF INVERSE Z-TRANSFORMS

We can obtain the inverse Z-transforms using any of the following three methods :

I. **Power series method.** This is the simplest of all the methods of finding the inverse Z-transform. If $U(z)$ is expressed as the ratio of two polynomials which cannot be factorized, we simply divide the numerator by the denominator and take the inverse Z-transform of each term in the quotient.

Example 23.14. *Find the inverse Z-transform of* $\log(z/z+1)$ *by power series method.*

Solution. Putting $z = 1/t$, $U(z) = \log\left(\frac{1/y}{1/y+1}\right) = -\log(1+y) = -y + \frac{1}{2}y^2 - \frac{1}{3}y^3 + \ldots$

$$= -z^{-1} + \frac{1}{2}z^{-2} - \frac{1}{3}z^{-3} + \ldots$$

Thus
$$u_n = \begin{cases} 0 & \text{for } n = 0 \\ (-1)^n/n & \text{otherwise} \end{cases}.$$

Example 23.15. *Find the inverse Z-transform of* $z/(z+1)^2$ *by division method.*

Solution.
$$U(z) = \frac{z}{z^2+2z+1} = z^{-1} - \frac{2+z^{-1}}{z^2+2z+1}, \text{ by actual division}$$

$$= z^{-1} - 2z^{-2} + \frac{3z^{-1}+2z^{-2}}{z^2+2z+1} = z^{-1} - 2z^{-2} + 3z^{-3} - \frac{4z^{-2}+3z^{-3}}{z^2+2z+1}$$

Continuing this process of division, we get an infinite series *i.e.*,

$$U(z) = \sum_{n=0}^{\infty} (-1)^{n-1}\, nz^{-n}$$

Thus $\quad u_n = (-1)^{n-1}\, n.$

II. **Partial fractions method.** This method is similar to that of finding the inverse Laplace transforms using partial fractions. The method consists of decomposing $U(z)/z$ into partial fractions, multiplying the resulting expansion by z and then inverting the same.

Example 23.16. *Find the inverse Z-transforms of*

(*i*) $\dfrac{2z^2+3z}{(z+2)(z-4)}$ (*V.T.U., 2008 S ; S.V.T.U., 2007*) (*ii*) $\dfrac{z^3-20z}{(z-2)^3(z-4)}$. (*V.T.U., 2011*)

Solution. (*i*) We write $U(z) = \dfrac{2z^2+3z}{(z+2)(z-4)}$ as $\dfrac{U(z)}{z} = \dfrac{2z+3}{(z+2)(z-4)} = \dfrac{A}{z+2} + \dfrac{B}{z-4}$

where $A = 1/6$ and $B = 11/6$

$$\therefore \quad U(z) = \frac{1}{6}\frac{z}{z+2} + \frac{11}{6}\frac{z}{z-4}$$

On inversion, we have

$$u_n = \frac{1}{6}(-2)^n + \frac{11}{6}(4)^n \qquad \text{[Using § 23.10 (9)]}$$

(*ii*) We write
$$U(z) = \frac{z^3-20z}{(z-2)^3(z-4)}$$

as
$$\frac{U(z)}{z} = \frac{z^2-20}{(z-2)^3(z-4)} = \frac{A+Bz+Cz^2}{(z-2)^3} + \frac{D}{z-4}$$

Readily we get $D = 1/2$. Multiplying throughout by $(z-2)^3(z-4)$, we get

$$z^2 - 20 = (A + Bz + Cz^2)(z-4) + D(z-2)^3.$$

Putting $z = 0, 1, -1$ successively and solving the resulting simultaneous equations, we get $A = 6$, $B = 0$, $C = 1/2$.

Thus
$$U(z) = \frac{1}{2}\cdot\frac{12z+z^3}{(z-2)^3} - \frac{z}{z-4} = \frac{1}{2}\frac{z(z-2)^2+4z^2+8z}{(z-2)^3} - \frac{z}{z-4}$$

$$= \frac{1}{2}\left\{\frac{z}{z-2} + 2\frac{2z^2+4z}{(z-2)^3}\right\} - \frac{z}{z-4}$$

On inversion, we get $u_n = \frac{1}{2}(2^n + 2 \,.\, n^2 2^n) - 4^n$ [Using § 23.10 (9) & (11)]

$= 2^{n-1} + n^2 2^n - 4^n.$

Example 23.17. *Find the inverse Z-transform of*

$2(z^2 - 5z + 6.5)/[(z-2)(z-3)^2]$, for $2 < |z| < 3$.

Solution. Splitting into partial fractions, we obtain

$$U(z) = \frac{2(z^2 - 5z + 6.5)}{(z-2)(z-3)^2} = \frac{A}{z-2} + \frac{B}{z-3} + \frac{C}{(z-3)^2} \quad \text{where } A = B = C = 1$$

$$\therefore \quad U(z) = \frac{1}{z-2} + \frac{1}{z-3} + \frac{1}{(z-3)^2}$$

$$= \frac{1}{z}\left(1 - \frac{2}{z}\right)^{-1} - \frac{1}{3}\left(1 - \frac{z}{3}\right)^{-1} + \frac{1}{9}\left(1 - \frac{z}{3}\right)^{-2} \quad \text{so that } 2/z < 1 \text{ and } z/3 < 1$$

$$= \frac{1}{z}\left(1 + \frac{2}{z} + \frac{4}{z^2} + \frac{8}{z^3} + \ldots\right) - \frac{1}{3}\left(1 + \frac{z}{3} + \frac{z^2}{9} + \frac{z^3}{27} + \ldots\right) + \frac{1}{9}\left(1 + \frac{2z}{3} + \frac{3z^2}{9} + \frac{4z^3}{27} + \ldots\right)$$

where $2 < |z| < 3$.

$$= \left(\frac{1}{2} + \frac{2}{z^2} + \frac{2^2}{z^3} + \frac{2^3}{z^4} + \ldots\right) - \left(\frac{1}{3} + \frac{z}{3^2} + \frac{z^2}{3^3} + \frac{z^3}{3^4} + \ldots\right) + \left(\frac{1}{3^2} + \frac{2z}{3^3} + \frac{3z^2}{3^4} + \frac{4z^3}{3^5} + \ldots\right)$$

$$= \sum_{n=1}^{\infty} 2^{n-1} z^{-n} - \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{n+1} z^n + \sum_{n=0}^{\infty} (n+1)\left(\frac{1}{3}\right)^{n+2} z^n$$

On inversion, we get $u_n = 2^{n-1}, n \geq 1$ and $u_n = -(n+2)3^{n-2}, n \leq 0$.

III. **Inversion integral method.** *The inverse Z-transform of $U(z)$ is given by the formula*

$$u_n = \frac{1}{2\pi i}\int_c U(z) z^{n-1}\, dz$$

= sum of residues of $U(z)\, z^{n-1}$ at the poles of $U(z)$ which are inside the contour C drawn according to the ROC given.

The following examples will illustrate the application of this formula :

Example 23.18. *Using the inversion integral method, find the inverse Z-transform of*

$$\frac{z}{(z-1)(z-2)}$$

(V.T.U., 2010 S)

Solution. Let $U(z) = \frac{z}{(z-1)(z-2)}$. Its poles are at $z = 1$ and $z = 2$.

Using $U(z)$ in the inversion integral, we have

$$u_n = \frac{1}{2\pi i}\int_C U(z) z^{n-1}\, dz,$$

where C is a circle large enough to enclose both the poles of $U(z)$.

$= \text{sum of residues of } U(z)\, z^{n-1} \text{ at } z = 1 \text{ and } z = 2.$

Now $$Res\,[U(z)\, z^{n-1}]_{z=1} = \underset{z \to 1}{\text{Lt}} \left\{(z-1) \,.\, \frac{z^n}{(z-1)(z-2)}\right\} = -1$$

and $$Res\,[U(z)\, z^{n-1}]_{z=2} = \underset{z \to 2}{\text{Lt}} \left\{(z-2) \,.\, \frac{z^n}{(z-1)(z-2)}\right\} = 2^n$$

Thus the required inverse Z-transform $u_n = 2^n - 1, n = 0, 1, 2, \ldots$

Example 23.19. *Find the inverse Z-transform of $2z/[(z-1)(z^2+1)]$.* (*Madras, 2000 S*)

Solution. Let $U(z) = \dfrac{2z}{(z-1)(z+i)(z-i)}$. It has three poles at $z = 1, z = \pm i$.

Using $U(z)$ in the inversion integral, we have

$$u_n = \frac{1}{2\pi i}\int_C U(z) \,.\, z^{n-1}\,dz \text{, where } C \text{ is a circle large enough to enclose the poles of } U(z).$$

$$= \text{sum of residues of } U(z) \,.\, z^{n-1} \text{ at } z = 1, z = \pm i.$$

Now $$Res\,[U(z)\,z^{n-1}]_{z=1} = \underset{z\to 1}{\text{Lt}}\left\{(z-1)\frac{2z^n}{(z-1)(z^2+1)}\right\} = 1$$

$$Res\,[U(z)\,z^{n-1}]_{z=i} = \underset{z\to i}{\text{Lt}}\left\{(z-i)\frac{2z^n}{(z-1)(z+i)(z-i)}\right\} = \frac{-(i)^n}{1+i}$$

$$Res\,[U(z)\,z^{n-1}]_{z=-i} = \underset{z\to -i}{\text{Lt}}\left\{(z+i)\frac{2z^n}{(z-1)(z+i)(z-i)}\right\} = \frac{(-i)^n}{i-1}$$

Hence $$u_n = 1 - \frac{(i)^n}{1+i} - \frac{(-i)^n}{1-i}.$$

PROBLEMS 23.3

Using convolution theorem, evaluate the inverse Z-transforms of the following :

1. $\dfrac{z^2}{(z-1)(z-3)}$.
2. $\left(\dfrac{z}{z-a}\right)^2$ (*Madras, 2003*)
3. $\left(\dfrac{z}{z-1}\right)^3$.
4. Show that (*a*) $\dfrac{1}{n!} * \dfrac{1}{n!} = \dfrac{2^n}{n!}$ (*b*) $Z^{-1}\left(\dfrac{z^2}{(z+a)(z+b)}\right) = \dfrac{(-1)}{b-a}(b^{n+1} - a^{n+1})$. (*Anna, 2009*)

Find the inverse Z-transforms of the following :

5. $\dfrac{4z}{z-a}$, $|z| > |a|$. (*Kottayam, 2005*)
6. $\dfrac{5z}{(2-z)(3z-1)}$. (*Madras, 1999*)
7. $\dfrac{z}{(z-1)^2}$.
8. $\dfrac{18z^2}{(2z-1)(4z+1)}$. (*S.V.T.U., 2009*)
9. $\dfrac{8z-z^3}{(4-z)^3}$.
10. $\dfrac{3z^2-18z+26}{(z-2)(z-3)(z-4)}$. (*Anna, 2005 S*)
11. $\dfrac{4z^2-2z}{z^3-5z^2+8z-4}$. (*V.T.U., 2011 S*)
12. $\dfrac{z^3+3z}{(z-1)^2(z^2+1)}$. (*Anna, 2009*)
13. $\dfrac{(1-e^{at})z}{(z-1)(z-e^{-at})}$.
14. Obtain $Z^{-1}[1/[(z-2)(z-3)]]$ for (*i*) $|z| < 2$; (*ii*) $2 < |z| < 3$; (*iii*) $|z| > 3$. (*Marathwada, 2008*)
15. Evaluate $Z^{-1}[(z-5)^{-3}]$ for $|z| > 5$. (*Mumbai, 2009*)

Using inversion integral, find the inverse Z-transform of the following functions :

16. $\dfrac{z+3}{(z+1)(z-2)}$.
17. $\dfrac{(2z-1)z}{2(z-1)(z+0.5)}$.
18. $\dfrac{1}{z(z-1)(z+0.5)}$. (*S.V.T.U., 2008*)
19. $\dfrac{z^2+z}{(z-1)(z^2+1)}$. (*Madras, 2003*)
20. $\dfrac{2z(z^2-1)}{(z^2+1)^2}$.

23.16 (1) APPLICATION TO DIFFERENCE EQUATIONS

Just as the Laplace transforms method is quite effective for solving linear differential equations (§ 21.15), the Z-transforms are quite useful for solving linear difference equations.

The performance of discrete systems is expressed by suitable difference equations. Also Z-transform plays an important role in the analysis and representation of discrete-time systems. To determine the frequency response of such systems, the solution of difference equations is required for which Z-transform method proves useful.

(2) **Working procedure** *to solve a linear difference equation with constant coefficients by Z-transforms :*

1. *Take the Z-transform of both sides of the difference equations using the formulae of § 26.16 and the given conditions.*

2. *Transpose all terms without U(z) to the right.*

3. *Divide by the coefficient of U(z), getting U(z) as a function of z.*

4. *Express this function in terms of the Z-transforms of known functions and take the inverse Z-transform of both sides. This gives u_n as a function of n which is the desired solution.*

Example 23.20. *Using the Z-transform, solve*

$$u_{n+2} + 4u_{n+1} + 3u_n = 3^n \text{ with } u_0 = 0, u_1 = 1.$$ *(U.P.T.U., 2003)*

Solution. If $Z(u_n) = U(z)$, then $Z(u_{n+1}) = z[U(z) - u_0]$,

$$Z(u_{n+2}) = z^2[U(z) - u_0 - u_1 z^{-1}]$$

Also $$Z(2^n) = z/(z-2)$$

$\therefore$ taking the Z-transforms of both sides, we get

$$z^2[U(z) - u_0 - u_1 z^{-1}] + 4z\,[U(z) - u_0] + 3U(z) = z/(z-3)$$

Using the given conditions, it reduces to

$$U(z)\,(z^2 + 4z + 3) = z + z/(z-3)$$

$$\therefore \quad \frac{U(z)}{z} = \frac{1}{(z+1)(z+3)} + \frac{1}{(z-3)(z+1)(z+3)} = \frac{3}{8}\frac{1}{z+1} + \frac{1}{24}\frac{1}{z-3} - \frac{5}{12}\frac{1}{z+3},$$

on breaking into partial fractions.

$$U(z) = \frac{3}{8}\frac{z}{z+1} + \frac{1}{24}\frac{z}{z-3} - \frac{5}{12}\frac{z}{z+3}$$

On inversion, we obtain

$$u_n = \frac{3}{8}Z^{-1}\left(\frac{z}{z+1}\right) + \frac{1}{24}Z^{-1}\left(\frac{z}{z-3}\right) - \frac{5}{12}Z^{-1}\left(\frac{z}{z+3}\right) = \frac{3}{8}(-1)^n + \frac{1}{24}3^n - \frac{5}{12}(-3)^n.$$

Example 23.21. *Solve $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$ with $y_0 = y_1 = 0$, using Z-transforms.*

(V.T.U., 2011 ; Anna, 2009 ; S.V.T.U., 2009)

Solution. If $Z(y_n) = Y(z)$, then $Z(y_{n+1}) = z\,[(Y(z) - y_0]$, $Z(y_{n+2}) = z^2[Y(z) - y_0 - y_1 z^{-1}]$

Also $Z(2^n) = z/(z-2)$.

Taking Z-transforms of both sides, we get

$$z^2[Y(z) - y_0 - y_1 z^{-1}] + 6z[Y(z) - y_0] + 9Y(z) = z/(z-2)$$

Since $y_0 = 0$, and $y_1 = 0$, we have $Y(z)\,(z^2 + 6z + 9) = z/(z-2)$

or $$\frac{Y(z)}{z} = \frac{1}{(z-2)(z+3)^2} = \frac{1}{25}\left[\frac{1}{z-2} - \frac{1}{z+3} - \frac{5}{(z+3)^2}\right],$$ on splitting into partial fractions.

or $$Y(z) = \frac{1}{25}\left[\frac{z}{z-2} - \frac{z}{z+3} - 5\frac{z}{(z+3)^2}\right]$$

On taking inverse Z-transform of both sides, we obtain

$$y_n = \frac{1}{25}\left[Z^{-1}\left(\frac{z}{z-2}\right) - Z^{-1}\left(\frac{z}{z+3}\right) + \frac{5}{3}Z^{-1}\left(-\frac{3z}{(z+3)^2}\right)\right]$$

$$= \frac{1}{25}\{2^n - (-3)^n + \frac{5}{3}n(-3)^n\} \qquad \left[\because Z^{-1}\left\{\frac{az}{(z-a)^2}\right\} = na^n\right]$$

Example 23.22. *Find the response of the system $y_{n+2} - 5y_{n+1} + 6y_n = u_n$, with $y_0 = 0, y_1 = 1$ and $u_n = 1$ for $n = 0, 1, 2, 3, \ldots$ by Z-transform method.* (V.T.U., 2010)

Solution. Taking Z-transform of both sides of the given equation, we get

$$z^2(Y(z) - y_0 - y_1 z^{-1}) - 5z(Y(z) - y_0) + 6Y(z) = \frac{z}{z-1}$$

Substituting the values $y_0 = 0, y_1 = 1$, it reduces to

$$(z^2 - 5z + 6)\, Y(z) = \frac{z}{z-1} + z = \frac{z^2}{z-1}$$

or
$$\frac{Y(z)}{z} = \frac{z}{(z-1)(z-2)(z-3)}$$

$$= \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{z-3} \qquad \text{where } A = \frac{1}{2}, B = -2, C = \frac{3}{2}$$

so that
$$Y(z) = \frac{1}{2}\frac{z}{z-1} - 2\frac{z}{z-2} + \frac{3}{2}\frac{z}{z-3}$$

On inversion, we obtain
$$y_n = \frac{1}{2} - 2(2)^n + \frac{3}{2}(3)^n$$

Obs. The initial values given in the problem automatically appear in the generated sequence.

Example 23.23. *Solve the difference equation $y_n + \frac{1}{4}y_{n-1} = u_n + \frac{1}{3}u_{n-1}$ where u_n is a unit step sequence.*

Solution. Taking Z-transform of both sides of the given equation, we get

$$Y(z) + \frac{1}{4}z^{-1}\, Y(z) = 1 + \frac{1}{3}z^{-1}$$

or
$$Y(z) = \left(1 + \frac{1}{3}z^{-1}\right)\Big/\left(1 + \frac{1}{4}z^{-1}\right) = \left(z + \frac{1}{3}\right)\Big/\left(z + \frac{1}{4}\right)$$

There being only one simple pole at $z = -1/4$, consider the contour $|z| > 1/4$.

$$\therefore \quad Res\, [Y(z)\, z^{n-1}]_{z=-1/4} = \underset{z \to -1/4}{\text{Lt}} \left\{\left(z + \frac{1}{4}\right) . \left(z + \frac{1}{3}\right) z^{n-1} \Big/ \left(z + \frac{1}{4}\right)\right\}$$

$$= \underset{z \to -1/4}{\text{Lt}} \left(z + \frac{1}{3}\right) z^{n-1} = \left(-\frac{1}{4} + \frac{1}{3}\right)\left(-\frac{1}{4}\right)^{n-1} = \frac{1}{12} . \left(-\frac{1}{4}\right)^{n-1}$$

Hence by inversion integral method, we have

$$y_n = \frac{1}{12}\left(-\frac{1}{4}\right)^{n-1}.$$

Example 23.24. *Using the Z-transform, solve $u_{n+2} - 2u_{n+1} + u_n = 3n + 5$.* (S.V.T.U., 2007)

Solution. Given equation is $u_{n+2} - 2u_{n+1} + u_n = 3n + 5$.
Taking the Z-transforms of both sides, we get

$$z^2[U(z) - u_0 - u_1 z^{-1}] - 2z[U(z) - u_0] + U(z) = 3 . \frac{z}{(z-1)^2} + 5 . \frac{z}{z-1}$$

or
$$U(z)\,(z^2 - 2z + 1) = \frac{5z^2 - 2z}{(z-1)^2} + u_0(z^2 - 2z) + u_1 z$$

or
$$U(z) = \frac{5z^2 - 2z}{(z-1)^4} + u_0\frac{z^2 - 2z}{(z-1)^2} + u_1\frac{z}{(z-1)^2}$$

On inversion, we obtain

$$u_n = Z^{-1}\left\{\frac{5z^2 - 2z}{(z-1)^4}\right\} + u_0 Z^{-1}\left\{\frac{z^2 - 2z}{(z-1)^2}\right\} + u_1 Z^{-1}\left\{\frac{z}{(z-1)^2}\right\} \qquad ...(i)$$

Noting that $Z(1) = \dfrac{z}{z-1},\ Z(n) = \dfrac{z}{(z-1)^2}$

$$Z(n^2) = \frac{z^2+z}{(z-1)^3},\ Z(n^3) = \frac{z^3+4z^2+z}{(z-1)^4}$$

We write $\dfrac{5z^2 - 2z}{(z-1)^4} \equiv A\,\dfrac{z^3+4z^2+z}{(z-1)^4} + B\,\dfrac{z^2+z}{(z-1)^3} + C\,\dfrac{z}{(z-1)^2} + D\,\dfrac{z}{z-1}$

Equating coefficients of like powers of z, we find

$$A = \frac{1}{2},\ B = 1,\ C = -\frac{3}{2},\ D = 0$$

$\therefore$ $$Z^{-1}\left\{\frac{5z^2 - 2z}{(z-1)^4}\right\} = \frac{1}{2}n^3 + n^2 - \frac{3}{2}n = \frac{1}{2}n(n-1)(n+3)$$

Also $$Z^{-1}\left\{\frac{z^2 - 2z}{(z-1)^2}\right\} = Z^{-1}\left\{\frac{z}{z-1}\right\} - Z^{-1}\left\{\frac{z}{(z-1)^2}\right\} = 1 - n$$

and $$Z^{-1}\left\{\frac{z}{(z-1)^2}\right\} = n.$$

Substituting these values in (i) above, we get

$$u_n = \frac{1}{2}n(n-1)(n+3) + u_0(1-n) + u_1 n$$

$$= \frac{1}{2}n(n-1)(n+3) + c_0 + c_1 n. \qquad \text{where } c_0 = u_0,\ c_1 = u_1 - u_0$$

Example 23.25. *Using residue method, solve* $y_k + \dfrac{1}{9}y_{k-2} = \dfrac{1}{3^k}\cos\dfrac{k\pi}{2},\ k \geq 0.$

Solution. Taking Z-transform of both sides of the given equation, we get

$$Z\left\{y_k + \frac{1}{9}y_{k-2}\right\} = Z\left\{\frac{1}{3^k}\cos\frac{k\pi}{2}\right\}$$

or $$Y(z) + \frac{1}{9}z^{-2}\,Y(z) = \frac{z^2}{z^2 + 1/9} \quad \text{or} \quad \left(1 + \frac{1}{9}z^{-2}\right)Y(z) = \frac{z^2}{z^2 + \frac{1}{9}}$$

or $$Y(z) = \frac{z^2}{\left(1+\frac{1}{9}z^{-2}\right)\left(z^2+\frac{1}{9}\right)} = \frac{z^4}{\left(z^2+\frac{1}{9}\right)^2}$$

There are two poles of second order at $z = i/3$ and $z = -i/3$.

$\therefore$ $$\text{Residue at } (z = i/3) = \left[\frac{d}{dz}\left\{\left(\frac{z-i}{3}\right)^2 \frac{z^{k-1}\,z^4}{(z^2+1/9)^2}\right\}\right]$$

$$= \left[\frac{d}{dz}\left\{\frac{z^{k+3}}{(z+i/3)^2}\right\}\right]_{z=i/3} = \left[\frac{(z+i/3)^2\,(k+3)z^{k+2} - z^{(k+3)}\,.\,2(z+i/3)}{(z+i/3)^4}\right]_{z=i/3}$$

$$= \left[\frac{(z+i/3)\,(k+3)z^{k+2} - 2z^{k+3}}{(z+i/3)^3}\right]_{z=i/3} = \left(\frac{3}{2i}\right)^3\left[(2k+6)\left(\frac{i}{3}\right)^{k+3} - 2\left(\frac{i}{3}\right)^{k+3}\right]$$

$$= \frac{1}{8}(2k+4)\left(\frac{i}{3}\right)^k = \frac{1}{4}(k+2)\left(\frac{1}{3}\right)^k\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^k = \frac{1}{4}(k+2)\left(\frac{1}{3}\right)^k\left(\cos\frac{k\pi}{2} + i\sin\frac{k\pi}{2}\right) \quad ...(i)$$

Changing i to $-i$ in (i), we have

$$\text{Residue at } (z = -i/3) = \frac{1}{4}(k+2)\left(\frac{1}{3}\right)^k\left(\cos\frac{k\pi}{2} - i\sin\frac{k\pi}{2}\right) \quad ...(ii)$$

Adding (i) and (ii), we obtain $y_k = \frac{1}{2}(k+2)\left(\frac{1}{3}\right)^k \cos\frac{k\pi}{2}$.

PROBLEMS 23.4

Solve the following difference equations using Z-transforms (1 – 8) :

1. $6y_{k+2} - y_{k+1} - y_k = 0$, given that $y(0) = y(1) = 1$. *(Kottayam, 2005)*
2. $y(n+2) + 2y(n+1) + y(n) = 0$, given that $y(0) = y(1) = 0$. *(V.T.U., 2008 S)*
3. $y_{n+2} - 4y_n = 0$ given that $y_0 = 0, y_1 = 2$. *(U.P.T.U., 2008)*
4. $f(n) + 3f(n-1) - 4f(n-2) = 0, n \geq 2$, given that $f(0) = 3, f(1) = -2$. *(Madras, 2003 S)*
5. $y_{(n+3)} - 3y(n+1) + 2y(n) = 0$, given that $y(0) = 4$, $y(1) = 0$ and $y(2) = 8$. *(Anna, 2005 S)*
6. $y_{n+2} - 5y_{n+1} + 6y_n = 36$, given that $y(0) = y(1) = 0$. *(Anna, 2009)*
7. $y_{n+2} - 6y_{n+1} + 9y_n = 3^n$.
8. $y_{n+2} - 4y_{n+1} + 3y_n = 5^n$.
9. $y_{n+1} + \frac{1}{4}y_n = \left(\frac{1}{4}\right)^n \quad (n \geq 0), y_0 = 0$.
10. $u_{x+2} + u_x = 5(2^x)$ given that $u_0 = 1, u_1 = 0$. *(Marathwada, 2008)*
11. $y_{n+2} + 4y_{n+1} + 3y_n = 2^n$ with $y_0 = 0, y_1 = 1$. *(Madras, 2006)*
12. $u_{k+2} - 2u_{k+1} + u_k = 2^k$ with $y_0 = 2, y_1 = 1$.
13. $y_{n+2} - 6y_{n+1} + 8y_n = 2^n + 6n$.
14. $y_k + \frac{1}{25}y_{k-2} = \left(\frac{1}{5}\right)^k \cos\frac{k\pi}{2}, \quad (k \geq 0)$.
15. Find the response of the system given by $y_n + 3y_{(n-1)} = u_n$ where u_n is a unit step sequence and $y_{(-1)} = 1$.
16. Find the impulse response of a system described by $y_{(n+1)} + 2y_{(n)} = \delta_n$; $y_0 = 0$.

23.1 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 23.5

Choose the correct answer or fill up the blanks in each of the following problems :

1. $Z(1) =$ ......
2. If u_n is defined for $n = 0, 1, 2,$ ...... only, then $Z(u_n) =$ ....
3. Z-transform of $n =$ .... *(Anna, 2009)*
4. $Z(na^n) =$ ......
5. $Z(\sin n\theta) =$ ......
6. Z-transform of $(1/n!)$ is
7. $Z(n^2) =$ ......
8. Linear property of Z-transform states that...
9. $Z^{-1}\left(\frac{1}{z-2}\right) =$ ....
10. $Z^{-1}\left\{\frac{z}{(z+1)^2}\right\} =$ ......
11. Initial value theorem on Z-transform states that ....
12. Z-transform is linear. (True or False)
13. If $Z(u_n) = u(z)$, then $\underset{n\to\infty}{\text{Lt}}(u_n) = \underset{z\to\infty}{\text{Lt}}(z-1)\,u(z)$. (True or False)
14. Z-transform of the sequence $\{2^k\}, k \geq 0$ is $z/(z-2)$. (True or False)
15. Z-transform of $\{a^k/k!\}, k \geq 0 = e^{a/z}$. (True or False)
16. Z-transform of $\{^nC_r\}, (0 \leq r \leq n)$ is $(1+z)^n$. (True or False)
17. Z-transform of unit impulse sequence $\delta(n) = \begin{cases} 1, & n < 0 \\ 0, & n \geq 0 \end{cases}$, is $z/z - 1$. (True or False)
18. Z-transform of unit step sequence $u(n) = \begin{cases} 0, & n < 0, \\ 1, & n \geq 0, \end{cases}$ is 1. (True or False)

CHAPTER 24

Empirical Laws and Curve-fitting

1. Introduction. 2. Graphical method. 3. Laws reducible to the linear law. 4. Principle of Least squares. 5. Method of Least squares. 6. Fitting of other curves. 7. Method of Group averages. 8. Fitting a parabola. 9. Method of Moments. 10. Objective Type of Questions.

24.1 INTRODUCTION

In many branches of applied mathematics, it is required to express a given data, obtained from observations, in the form of a *Law* connecting the two variables involved. Such a *Law* inferred by some scheme is known as *Empirical Law*. For example, it may be desired to obtain the law connecting the length and the temperature of a metal bar. At various temperatures, the length of the bar is measured. Then, by one of the methods explained below, a law is obtained that represents the relationship existing between temperature and length for the observed values. This relation can then be used to predict the length at an arbitrary temperature.

(2) Scatter diagram. To find a relationship between the set of paired observations x and y (say), we plot their corresponding values on the graph taking one of the variables along the x-axis and other along the y-axis *i.e.* (x_1, y_1), (x_2, y_2), (x_n, y_n). The resulting diagram showing a collection of dots is called a *scatter diagram*. A smooth curve that approximates the above set of points is known as the *approximating curve*.

(3) Curve fitting. Several equations of different types can be obtained to express the given data approximately. But the problem is to find the equation of the curve of '*best fit*' which may be most suitable for predicting the unknown values. The process of finding such an equation of 'best fit' is known as *curve-fitting*.

If there are n pairs of observed values then it is possible to fit the given data to an equation that contains n arbitrary constants for we can solve n simultaneous equations for n unknowns. If it were desired to obtain an equation representing these data but having less than n arbitrary constants, then we can have recourse to any of the four methods : *Graphical method, Method of Least squares, Method of Group averages* and *Method of Moments*. The graphical method fails to give the values of the unknowns uniquely and accurately while the other methods do. *The method of Least squares is, probably, the best to fit a unique curve to a given data.* It is widely used in applications and can be easily implemented on a computer.

24.2 GRAPHICAL METHOD

When the curve representing the given data is a **linear law** $y = mx + c$ *; we proceed as follows :*

(*i*) *Plot the given points on the graph paper* taking a suitable *scale.*

(*ii*) *Draw the straight line of best fit* such that the points are evenly distributed about the line.

(*iii*) *Taking two suitable points* (x_1, y_1) *and* (x_2, y_2) *on the line, calculate m,* the slope of the line and c, its intercept on y-axis.

When the points representing the observed values do not approximate to a straight line, a smooth curve is drawn through them. From the shape of the graph, we try to infer the law of the curve and then reduce it to the form $y = mx + c$.

24.3 LAWS REDUCIBLE TO THE LINEAR LAW

We give below some of the laws in common use, indicating the way these can be reduced to the linear form by suitable substitutions :

(**1**) *When the law is* $y = mx^n + c$.

Taking $x^n = X$ and $y = Y$ the above law becomes $Y = mX + c$

(**2**) *When the law is* $y = ax^n$.

Taking logarithms of both sides, it becomes $\log_{10} y = \log_{10} a + n \log_{10} x$

Putting $\log_{10} x = X$ and $\log_{10} y = Y$, it reduces to the form $Y= nX + c$, where $c = \log_{10} a$.

(**3**) *When the law is* $y = ax^n + b \log x$.

Writing it as $\frac{y}{\log x} = a\frac{x^n}{\log x} + b$ and taking $x^n/\log x = X$ and $y/\log x = Y$,

the given law becomes, $Y = aX + b$.

(**4**) *When the law is* $y = ae^{bx}$

Taking logarithms, it becomes $\log_{10} y = (b \log_{10} e)\, x + \log_{10} a$

Putting $x = X$ and $\log_{10} y = Y$, it takes the form $Y = mX + c$ where $m = b \log_{10} e$ and $c = \log_{10} a$.

(**5**) *When the law is* $xy = ax + by$.

Dividing by x, we have $y = b\,\frac{y}{x} + a$.

Putting $y/x = X$ and $y = Y$, it reduces to the form $Y = bX + a$.

Example 24.1. *R is the resistance to maintain a train at speed V; find a law of the type* $R = a + bV^2$ *connecting R and V, using the following data :*

V (miles / hour) :	*10*	*20*	*30*	*40*	*50*
R (lb / ton) :	*8*	*10*	*15*	*21*	*30*

Solution. Given law is $R = a + bV^2$ ...(*i*)

Taking $V^2 = x$ and $R = y$, (*i*) becomes

$$y = a + bx \quad ...(ii)$$

which is a *linear law*.

Table for the values of x and y is as follows :

x	100	400	900	1600	2500
y	8	10	15	21	30

Plot these points. Draw the straight line of best fit through these points (Fig. 24.1)

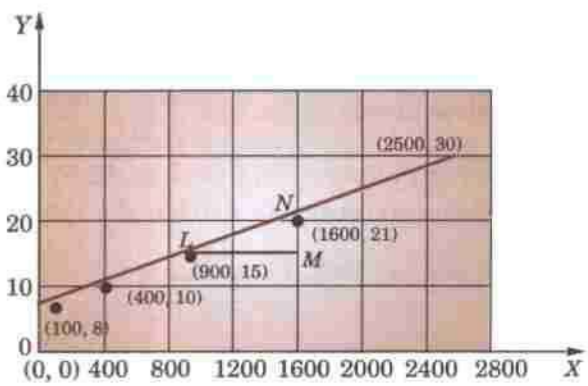


Fig. 24.1

Slope of this line (= b)

$$= \frac{MN}{LM} = \frac{21 - 15}{1600 - 900} = \frac{6}{700} = 0.0085 \text{ nearly.}$$

Since L (900, 15) lies on (*ii*),

$\therefore$ $15 = a + 0.0085 \times 900$,

whence $a = 15 - 7.65 = 7.35$ nearly.

Example 24.2. *The following values of x and y are supposed to follow the law* $y = ax^2 + b \log_{10} x$. *Find graphically the most probable values of the constants a and b.*

x	*2.85*	*3.88*	*4.66*	*5.69*	*6.65*	*7.77*	*8.67*
y	*16.7*	*26.4*	*35.1*	*47.5*	*60.6*	*77.5*	*93.4*

Solution. Given law is $y = ax^2 + b \log_{10} x$

i.e. $$\frac{y}{\log_{10} x} = a\frac{x^2}{\log_{10} x} + b \quad ...(i)$$

Taking $x^2/\log_{10} x = X$ and $y/\log_{10} x = Y$

(i) becomes $Y = aX + b$ ...(ii)

This is a *linear law*. Table for the values of X and Y is as follows :

$X = x^2/\log_{10} x$	17.93	25.56	32.49	42.87	53.75	67.80	80.83
$Y = y/\log_{10} x$	35.59	44.83	52.50	62.90	73.65	87.04	99.56
Points	P_1	P_2	P_3	P_4	P_5	P_6	P_7

Plot these points and draw the straight line of best fit through these points (Fig. 24.2).

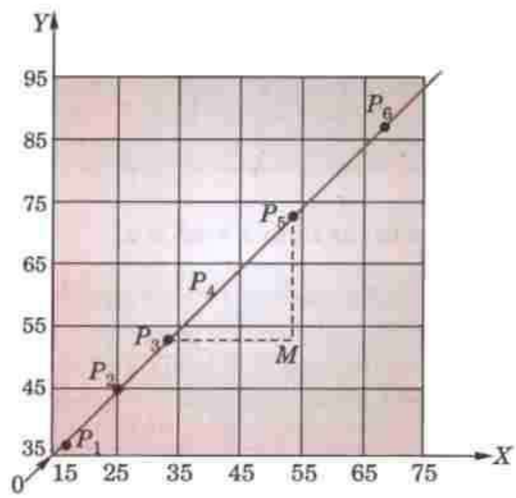


Fig. 24.2

Slope of this line (= a) = $\dfrac{MP_5}{P_3M} = \dfrac{73.65 - 52.50}{53.75 - 32.49} = \dfrac{21.15}{21.26} = 0.99$

Since P_3 lies on (ii), therefore, $52.50 = 0.99 \times 32.49 + b$ whence $b = 20.2$

Hence (i) becomes $y = (0.99)\,x^2 + (20.2)\log_{10} x$.

Example 24.3. *The values of x and y obtained in an experiment are as follows :*

x	2.30	3.10	4.00	4.92	5.91	7.20
y	33.0	39.1	50.3	67.2	85.6	125.0

Th probable law is $y = ae^{bx}$. Test graphically the accuracy of this law and if the law holds good, find the best values of the constants.

Solution. Given law is $y = ae^{bx}$ ...(i)

Taking logarithms to base 10, we have $\log_{10} y = \log_{10} a + (b \log_{10} e)\,x$

Putting $x = X$ and $\log_{10} y = Y$, it becomes $y = (b \log_{10} e)\,X + \log_{10} a$ ...(ii)

Table for the values of X and Y is as under :

$X = x$	2.30	3.10	4.00	4.92	5.91	7.20
$Y = \log_{10} y$	1.52	1.59	1.70	1.83	1.93	2.1
Points	P_1	P_2	P_3	P_4	P_5	P_6

Scale : 1 small division along x–axis = 0.1

10 small divisions along y–axis = 0.1.

Plot these points and draw the line of best fit. As these points are lying almost along a straight line, the given law is nearly accurate (Fig. 24.3).

Now slope of this line (= $b \log_{10} e$)

$$= \frac{MN}{NM} = 0.12$$

whence $$b = \frac{0.12}{\log_{10} e} = 0.12 \times 2.303 = 0.276$$

Since the point L (4, 1.71) lies on (*ii*), therefore, $1.71 = 0.12 \times 4 + \log_{10} a$ whence $a = 17$ nearly.

Hence the curve of best fit is $y = 17\, e^{0.276x}$.

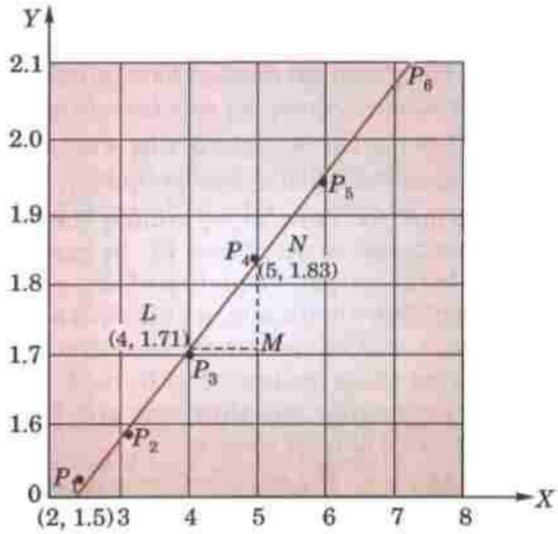


Fig. 24.3

PROBLEMS 24.1

1. If p is the pull required to lift the weight by means of a pulley block, find a linear law of the form $p = a + bw$, connecting p and w, using the following data :

w (lb) :	50	70	100	120
p (lb) :	12	15	21	25

Compute p, when w = 150 lb.

2. The resistance R of a carbon filament lamp was measured at various values of the voltage V and the following observations were made :

Voltage	V...	62	70	78	84	92
Resistance	R...	73	70.7	69.2	67.8	66.3

Assuming a law of the form $R = \frac{a}{V} + b$, find by graphical method the best value of a and b.

3. Verify if the values of x and y, related as shown in the following table, obey the law $y = a + b\sqrt{x}$. If so, find graphically the values of a and b.

x :	500	1,000	2,000	4,000	6,000
y :	0.20	0.33	0.38	0.45	0.51

4. The following values of T and l follow the law $T = al^n$. Test if this is so and find the best values of a and n.

T = 1.0	1.5	2.0	2.5
l = 25	56.2	100	1.56

5. Find the best value of a and b if $y = ax + b \log_{10} x$ is the curve which represents most closely the observed values given below :

x :	2	3	4	5	6
y :	3.74	5.99	7.47	8.92	9.86

6. Fit the curve $y = ae^{bx}$ to the following data :

x :	0	2	4
y :	5.1	10	31.1

(*Coimbatore, 1997*)

7. The following are the results of an experiment on friction of bearings. The speed being constant, corresponding values of the coefficient of friction and the temperature are shown in the table :

t :	120	110	100	90	80	70	60
μ :	0.0051	0.0059	0.0071	0.0085	0.00102	0.00124	0.00148

If μ and t are given by the law $\mu = ae^{bt}$, find the values of a and b by plotting the graph for μ and t.

24.4 PRINCIPLE OF LEAST SQUARES

The graphical method has the obvious drawback of being unable to give a unique curve of fit. *The principle of least squares, however, provides an elegant procedure for fitting a unique curve to a given data.*

Let the curve, $y = a + bx + cx^2 + ... + kx^{m-1}$ ...(1)

be fitted to the set of n data points $(x_1, y_1), (x_2, y_2)...(x_n, y_n)$.

Now we have to determine the constants $a, b, c,...k$ such that it represents the curve of best fit. In case $n = m$, on substituting the values (x_i, y_i) in (1), we get n equations from which a unique set of n constants can be found. But when $n > m$, we obtain n equations which are more than the m constants and hence cannot be solved for these constants. So we try to determine those values of $a, b, c,...k$ which satisfy all the equations as nearly as possible and thus may give the best fit. In such cases, we apply the *principle of least squares*.

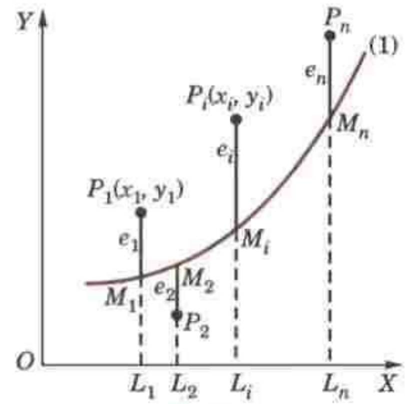


Fig. 24.4

At $x = x_i$, the *observed* (or *experimental*) *value* of the ordinate is $y_i = P_iL_i$ and the corresponding value on the fitting curve (1) is $a + bx_i + cx_i^2 + ... + kx_i^m = M_iL_i$ ($= \eta_i$, say) which is the *expected* (or *calculated*) *value* (Fig. 24.4). The difference of the observed and the expected values *i.e.* $y_i - \eta_i (= e_i)$ is called the *error* (or *residual*) at $x = x_i$. Clearly some of the errors $e_1, e_2, ... e_n$ will be positive and others negative. Thus to give equal weightage to each error, we square each of these and form their sum *i.e.* $E = e_1^2 + e_2^2 + ... e_n^2$.

The curve of best fit is that for which e's are as small as possible i.e., E, the sum of the squares of the errors is a minimum. This is known as the *principle of least squares* and was suggested by *Legendre** in 1806.

Obs. The principle of least squares does not help us to determine the form of the appropriate curve which can fit a given data. It only determines the best possible values of the constants in the equation when the form of the curve is known before hand. The selection of the curve is a matter of experience and practical considerations.

24.5 (1) METHOD OF LEAST SQUARES

For clarity, suppose it is required to fit the curve

$$y = a + bx + cx^2 \qquad ...(1)$$

to a given set of observations $(x_1, y_1), (x_2, y_2), ..., (x_5, y_5)$. For any x_i, the observed value is y_i and the expected value is $\eta_i = a + bx_i + cx_i^2$ so that the error $e_i = y_i - \eta_i$.

$\therefore$ the sum of the squares of these errors is

$$E = e_1^2 + e_2^2 + ... + e_5^2$$
$$= [y_1 - (a + bx_1 + cx_1^2)]^2 + [y_2 - (a + bx_2 + cx_2^2)]^2 + ... + [y_5 - (a + bx_5 + cx_5^2)]^2 \qquad \text{[See § 5.12 (3)]}$$

For E to be minimum, we have

$$\frac{\partial E}{\partial a} = 0 = 2[y_1 - (a + bx_1 + cx_1^2)] - 2[y_2 - (a + bx_2 + cx_2^2)] - ... - 2[y_5 - (a + bx_5 + cx_5^2)] \qquad ...(2)$$

$$\frac{\partial E}{\partial b} = 0 = -2x_1[y_1 - (a + bx_1 + cx_1^2)] - 2x_2[y_2 - (a + bx_2 + cx_2^2)] - ... - 2x_5[y_5 - (a + bx_5 + cx_5^2)] \qquad ...(3)$$

$$\frac{\partial E}{\partial c} = 0 = -2x_1^2[y_1 - (a + bx_1 + cx_1^2)] - 2x_2^2[y_2 - (a + bx_2 + cx_2^2)] - ... - 2x_5^2[y_5 - (a + bx_5 + cx_5^2)] \qquad ...(4)$$

Equation (2) simplifies to

$$y_1 + y_2 + ... + y_5 = 5a + b(x_1 + x_2 + ... + x_5) + c(x_1^2 + x_2^2 + ... + x_5^2)$$

i.e.,

$$\Sigma y_i = 5a + b\Sigma x_i + c\,\Sigma x_i^2 \qquad ...(5)$$

* See footnote on p. 311.

Equation (3) becomes

$$x_1y_1 + x_2y_2 + \ldots + x_5y_5 = a(x_1 + x_2 + \ldots + x_5) + b(x_1^2 + x_2^2 + \ldots + x_5^2) + c(x_1^3 + x_2^3 + \ldots + x_5^3)$$

i.e., $$\Sigma x_i y_i = a\Sigma x_i + b\Sigma x_i^2 + c\Sigma x_i^3 \qquad \ldots(6)$$

Similarly (4) simplifies to $\Sigma x_i^2 y_i = a\Sigma x_i^2 + b\Sigma x_i^3 + c\Sigma x_i^4$ ...(7)

The equations (5), (6) and (7) are known as *Normal equations* and can be solved as simultaneous equations in a, b, c. The values of these constants when substituted in (1) give the desired curve of best fit.

(2) Working procedure

(*a*) **To fit the straight line $y = a + bx$**

(*i*) Substitute the observed set of n values in this equation.

(*ii*) Form normal equations for each constant

i.e., $$\Sigma y = na + b\Sigma x, \ \Sigma xy = a\Sigma x + b\Sigma x^2$$

[The normal equation for the unknown a is obtained by multiplying the equations by the coefficient of a and adding. The normal equation for b is obtained by multiplying the equations by the coefficient of b (*i.e.*, x) and adding.]

(*iii*) Solve these normal equations as simultaneous equations for a and b.

(*iv*) Substitute the values of a and b in $y = a + bx$, which is the required line of best fit.

(*b*) **To fit the parabola : $y = a + bx + cx^2$**

(*i*) Form the normal equations $\Sigma y = na + b\Sigma x + c\Sigma x^2$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$$

and $$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$$

[The normal equation for c has been obtained by multiplying the equations by the coefficient of c (*i.e.*, x^2) and adding.]

(*ii*) Solve these as simultaneous equations for a, b, c.

(*iii*) Substitute the values of a, b, c in $y = a + bx + cx^2$, which is the required parabola of best fit.

(*c*) In general, the curve $y = a + bx + cx^2 + \ldots + kx^{m-1}$ can be fitted to a given data by writing m normal equations.

Example 24.4. *If P is the pull required to lift a load W by means of a pulley block, find a linear law of the form $P = mW + c$ connecting P and W, using the following data :*

$P = 12$	15	21	25
$W = 50$	70	100	120

where P and W are taken in kg-wt. Compute P when W = 150 kg. wt. (U.P.T.U., 2007 ; V.T.U., 2002)

Solution. The corresponding normal equations are

$$\left.\begin{aligned} \Sigma P &= 4c + m\Sigma W \\ \Sigma WP &= c\Sigma W + m\Sigma W^2 \end{aligned}\right\} \qquad \ldots(i)$$

The values of ΣW etc. are calculated by means of the following table :

W	P	W^2	WP
50	12	2500	600
70	15	4900	1050
100	21	10000	2100
120	25	14400	3000
Total = 340	73	31800	6750

$\therefore$ The equations (*i*) becomes $73 = 4c + 340m$ and $6750 = 340c + 31800m$

i.e., $$2c + 170m = 365 \qquad \ldots(ii)$$

and $$34c + 3180m = 675 \qquad \ldots(iii)$$

Multiplying (*ii*) by 17 and substracting from (*iii*), we get

$$m = 0.1879 \qquad \therefore \ \text{from } (ii), \quad c = 2.2785$$

Hence the line of best fit is

$$P = 2.2759 + 0.1879\ W$$

When $W = 150$ kg., $P = 2.2785 + 0.1879 \times 150 = 30.4635$ kg.

Obs. The calculations get simplified when the central values of x is zero. It is therefore, advisable to make the central value zero, if it be not so. This is illustrated by the next example.

Example 24.5. *Fit a second degree parabola to the following data :*

x	0	1	2	3	4
y	1	1.8	1.3	2.5	6.3

(P.T.U., 2006)

Solution. Let $u = x - 2$ and $v = y$ so that the parabola of fit $y = a + bx + cx^2$ becomes

$$v = A + Bu + Cu^2 \qquad ...(i)$$

The normal equations are

$$\Sigma v = 5A + B\Sigma u + C\Sigma u^2 \quad \text{or} \quad 12.9 = 5A + 10C$$
$$\Sigma uv = A\Sigma u + B\Sigma u^2 + C\Sigma u^2 \quad \text{or} \quad 11.3 = 10B$$
$$\Sigma u^2 v = A\Sigma u^2 + B\Sigma u^3 + C\Sigma u^4 \quad \text{or} \quad 33.5 = 10A + 34C$$

Solving these as simultaneous equations, we get

$$A = 1.48, \qquad B = 1.13, \qquad C = 0.55.$$

$\therefore$ (*i*) becomes, $\qquad v = 1.48 + 1.13u + 0.55u^2$

or $\qquad y = 1.48 + 1.13\ (x - 2) + 0.55\ (x - 2)^2$

Hence $y = 1.42 - 1.07x + 0.55x^2$.

Example 24.6. *Fit a second degree parabola to the following data :*

$x = 1.0$	1.5	2.0	2.5	3.0	3.5	4.0
$y = 1.1$	1.3	1.6	2.0	2.7	3.4	4.1

(V.T.U., 2009 ; Bhopal, 2008)

Solution. We shift the origin to (2.5, 0) and take 0.5 as the new unit. This amounts to changing the variable x to X, by the relation $X = 2x - 5$.

Let the parabola of fit be $y = a + bX + cX^2$. The values of ΣX etc., are calculated as below :

x	X	y	Xy	X^2	X^2y	X^3	X^4
1.0	–3	1.1	–3.3	9	9.9	–27	81
1.5	–2	1.3	–2.6	4	5.2	–8	16
2.0	–1	1.6	–1.6	1	1.6	–1	1
2.5	0	2.0	0.0	0	0.0	0	0
3.0	1	2.7	2.7	1	2.7	1	1
3.5	2	3.4	6.8	4	13.6	8	16
4.0	3	4.1	12.3	9	36.9	27	81
Total	0	16.2	14.3	28	69.9	0	196

The normal equations are

$$7a + 28c = 16.2\ ; \qquad 28b = 14.3\ ; \qquad 28a + 196c = 69.9$$

Solving these as simultaneous equations, we get

$$a = 2.07,\ b = 0.511,\ c = 0.061$$

$\therefore \qquad y = 2.07 + 0.511X + 0.061X^2$

Replacing X by $2x - 5$ in the above equation, we get

$$y = 2.07 + 0.511\ (2x - 5) + 0.061\ (2x - 5)^2$$

which simplifies to $y = 1.04 - 0.198x + 0.244x^2$. This is the required parabola of best fit.

Example 24.7. *Fit a second degree parabola to the following data :*

x	1989	1990	1991	1992	1993	1994	1995	1996	1997
y	352	356	357	358	360	361	361	360	359

(U.P.T.U., 2009)

Solution. Taking $u = x - 1993$ and $v = y - 357$, the equation $y = a + bx + cx^2$ becomes

$$v = A + Bu + Cu^2 \quad ...(i)$$

x	$u = x - 1993$	y	$v = y - 357$	uv	u^2	u^2v	u^3	u^4
1989	− 4	352	− 5	20	16	− 80	− 64	256
1990	− 3	360	− 1	3	9	− 9	− 27	81
1991	− 2	357	0	0	4	0	− 8	16
1992	− 1	358	1	− 1	1	1	− 1	1
1993	0	360	3	0	0	0	0	0
1994	1	361	4	4	1	4	1	1
1995	2	361	4	8	4	16	8	16
1996	3	360	3	9	9	27	27	81
1997	4	359	2	8	16	32	64	256
Total	$\Sigma u = 0$		$\Sigma v = 11$	$\Sigma uv = 51$	$\Sigma u^2 = 60$	$\Sigma u^2v = -9$	$\Sigma u^3 = 0$	$\Sigma u^4 = 708$

The normal equations are

$$\Sigma v = 9A + B\Sigma u + C\Sigma u^2 \quad \text{or} \quad 11 = 9A + 60C$$

$$\Sigma uv = A\Sigma u + B\Sigma u^2 + C\Sigma u^3 \quad \text{or} \quad 51 = 60B \quad \text{or} \quad B = \frac{17}{20}$$

$$\Sigma u^2v = A\Sigma u^2 + B\Sigma u^3 + C\Sigma u^4 \quad \text{or} \quad -9 = 60A + 708C$$

On solving these equations, we get $A = \frac{694}{231}, B = \frac{17}{20}, C = -\frac{247}{924}$

$\therefore$ (*i*) becomes $\quad v = \frac{694}{231} + \frac{17}{20}u - \frac{247}{924}u^2$

or $$y - 357 = \frac{694}{231} + \frac{17}{20}(x - 1933) - \frac{247}{924}(x - 1933)^2$$

or $$y = \frac{694}{231} - \frac{32861}{20} - \frac{247}{924}(1933)^2 + \frac{17}{20}x + \frac{247 \times 3866}{924}x - \frac{247}{924}x^2$$

or $$y = 3 - 1643.05 - 998823.36 + 357 + 0.85x + 1033.44x - 0.267x^2$$

Hence $$y = -1000106.41 + 1034.29x - 0.267x^2.$$

PROBLEMS 24.2

1. By the method of least squares, find the straight line that best fits the following data :

x :	1	2	3	4	5
y :	14	27	40	55	68

(U.P.T.U., 2008)

2. Fit a straight line to the following data :

Year x :	1961	1971	1981	1991	2001
Production y :	8	10	12	10	16

(in thousand tons)

and find the expected production in 2006.

3. A simply supported beam carries a concentrated load P (lb) at its mid-point. Corresponding to various values of P, the maximum deflection Y (in) is measured. The data are given below :

P:	100	120	140	160	180	200
Y:	0.45	0.55	0.60	0.70	0.80	0.85

Find a law of the form $Y = a + bP$.

4. The results of measurement of electric resistance R of a copper bar at various temperatures t°C are listed below :

t :	19	25	30	36	40	45	50
R:	76	77	79	80	82	83	85

Find a relation $R = a + bt$ where a and b are constants to be determined by you.

5. Find the best possible curve of the form $y = a + bx$, using method of least squares for the data :

x :	1	3	4	6	8	9	11	14
y:	1	2	4	4	5	7	8	9

(V.T.U., 2011)

6. Fit a straight line to the following data

(a)

x:	1	2	3	4	5	6	7	8	9
y:	9	8	10	12	11	13	14	16	5

(Bhopal, 2008)

(b)

x:	6	7	7	8	8	8	9	9	10
y:	5	5	4	5	4	3	4	3	3

(J.N.T.U., 2008)

7. Find the parabola of the form $y = a + bx + cx^2$ which fits most closely with the observations :

x:	−3	−2	−1	0	1	2	3
y:	4.63	2.11	0.67	0.09	0.63	2.15	4.58

(V.T.U., 2006 ; J.N.T.U., 2000 S)

8. Fit a parabola $y = a + bx + cx^2$ to the following data :

x:	2	4	6	8	10
y:	3.07	12.85	31.47	57.38	91.29

(V.T.U., 2003 S)

9. Fit a second degree parabola to the following data :

x:	1	2	3	4	5	6	7	8	9	10
y:	124	129	140	159	228	289	315	302	263	210

(U.P.T.U., 2009)

10. The following table gives the results of the measurements of train resistances ; V is the velocity in miles per hour. R is the resistance in pounds per ton :

V:	20	40	60	80	100	120
R:	5.5	9.1	14.9	22.8	33.3	46.0

If R is related to V by the relation $R = a + bV + cV^2$, find a, b, and c. (U.P.T.U., 2002)

11. The velocity V of a liquid is known to vary with temperature according to a quadratic law $V = a + bT + cT^2$. Find the best values of a, b and c for the following table :

T:	1	2	3	4	5	6	7
V:	2.31	2.01	3.80	1.66	1.55	1.47	1.41

(U.P.T.U., MCA, 2010)

24.6 FITTING OF OTHER CURVES

(1) $y = ax^b$

Taking logarithms, $\log_{10} y = \log_{10} a + b \log_{10} x$

i.e., $Y = A + bX$ where $X = \log_{10} x$, $Y = \log_{10} y$ and $A = \log_{10} a$. ...(i)

∴ The normal equations for (i) are : $\Sigma Y = nA + b\Sigma X$, $\Sigma XY = A\Sigma X + b\Sigma X^2$

from which A and b can be determined. Then a can be calculated from $A = \log_{10} a$.

(2) $y = ae^{bx}$ *(Exponential curve)*

Taking logarithms, $\log_{10} y = \log_{10} a + bx \log_{10} e$

i.e., $Y = A + Bx$ where $Y = \log_{10} y$, $A = \log_{10} a$ and $B = b \log_{10} e$

Here the normal equations are : $\Sigma Y = nA + B\Sigma x$, $\Sigma xY = A\Sigma x + B\Sigma x^2$

from which A, B can be found and consequently a, b can be calculated.

(3) $xy^a = b$ (or $pv^\gamma = k$) *(Gas equation)*

Taking logarithms, $\log_{10} x + a \log_{10} y = \log_{10} b$ or $\log_{10} y = \dfrac{1}{a} \log_{10} b - \dfrac{1}{a} \log_{10} x$.

This is of the form $Y = A + BX$

where $X = \log_{10} x$, $Y = \log_{10} y$, $A = \dfrac{1}{a} \log_{10} b$, $B = -\dfrac{1}{a}$.

Here also the problem reduces to finding a straight line of best fit through the given data.

Example 24.8. *Find the least squares fit of the form $y = a_0 + a_1x^2$ to the following data :*

x :	-1	0	1	2
y :	2	5	3	0

(U.P.T.U., 2008)

Solution. Putting $x^2 = X$, we have $y = a_0 + a_1X$

$\therefore$ the normal equations are : $\Sigma y = 4a + a_1\Sigma X$; $\Sigma Y = a_0\Sigma X + a_1\Sigma X^2$.

The values of ΣX, ΣX^2 etc. are calculated below :

x	y	X	X^2	XY
-1	2	1	1	2
0	5	0	0	0
1	3	1	1	3
2	0	4	16	0
	$\Sigma y = 10$	$\Sigma X = 10$	$\Sigma X^2 = 18$	$\Sigma XY = 5$

$\therefore$ the normal equations become $\quad 10 = 400 + 6a_1$; $5 = 600 + 18a_1$

Solving these equations we get, $\quad a_0 = 4.167, a_1 = -1.111.$

Hence the curve of best fit is $\quad y = 4.167 - 1.111X$ *i.e.*, $y = 4.167 - 1.111x^2$.

Example 24.9. *An experiment gave the following values :*

v (ft/min) :	350	400	500	600
t (min) :	61	26	7	26

It is known that v and t are connected by the relation $v = at^b$. Find the best possible values of a and b.

Solution. We have $\quad \log_{10} v = \log_b a + b \log_{10} t$

or $\quad y = A + 6X$, where $X = \log_{10} t$, $y = \log_{10} v$, $A = \log_{10} a$

$\therefore$ the normal equations are

$$\Sigma Y = 4A + b\Sigma X \qquad \text{...}(i)$$

$$\Sigma XY = A\Sigma X + b\Sigma X^2 \qquad \text{...}(ii)$$

Now ΣX etc. are calculated as in the following table :

v	t	$X = \log_{10} t$	$y = \log_{10} v$	XY	X^2
350	61	1.7853	2.5441	4.542	3.187
400	26	1.4150	2.6021	3.682	2.002
500	7	0.8451	2.6990	2.281	0.714
600	2.6	0.4150	2.7782	1.153	0.172
Total ·		4.4604	10.6234	11.658	6.075

$\therefore$ Equations (i) and (ii) become

$$4A + 4.46b = 10.623 \text{ ; } 4.46A + 6.075b = 11.658$$

Solving these, $A = 2.845$, $b = -0.1697$

$\therefore$ $a = \text{antilog } A = \text{antilog } 2.845 = 699.8.$

Example 24.10. *Predict the mean radiation dose at an altitude of 3000 feet by fitting an exponential curve to the given data :*

Altitude (x) :	50	450	780	1200	4400	4800	5300
Dose of radiation (y) :	28	30	32	36	51	58	69

(S.V.T.U., 2007 ; J.N.T.U., 2003)

Solution. Let $y = ab^x$ be the exponential curve.

Then $\log_{10} y = \log_{10} a + x \log_{10} b$

or $Y = A + Bx$ where $Y = \log_{10} y$, $A = \log_{10} a$, $B = \log_{10} b$

$\therefore$ the normal equations are

$$\Sigma Y = 7A + B\,\Sigma x \quad ...(i)$$

$$\Sigma x\, Y = A\,\Sigma x + B\,\Sigma x^2 \quad ...(ii)$$

Now Σx etc. are calculated as follows :

x	y	$Y = \log_{10} y$	xY	x^2
50	28	1.447158	72.3579	2500
450	30	1.477121	664.7044	202500
780	32	1.505150	1174.0170	608400
1200	36	1.556303	1867.5636	1440000
4400	51	1.707570	7513.3080	19360000
4800	58	1.763428	8464.4544	23040000
5300	69	1.838849	9745.8997	28090000
$\Sigma = 16980$		11.295579	29502.305	72743400

$\therefore$ equations (i) and (ii) become

$$11.295579 = 7A + 16980B$$

$$29502.305 = 16980A + 72743400B$$

Solving these equations, we get $A = 1.4521015$, $B = 0.0000666289$

$$\therefore \quad \log_{10} y = Y = 1.4521015 + 0.0000666289x$$

Hence y (at $x = 3000$) $= 44.874$ *i.e.* 44.9 approx.

Example 24.11. *The pressure and volume of a gas are related by the equation $pv^\gamma = k$, γ and k being constants. Fit this equation to the following set of observations :*

p (kg/cm²) :	*0.5*	*1.0*	*1.5*	*2.0*	*2.5*	*3.0*
v (litres) :	*1.62*	*1.00*	*0.75*	*0.62*	*0.52*	*0.46*

(V.T.U., 2011)

Solution. We have $\log_{10} p + \gamma \log_{10} v = \log_{10} k$

or $$\log_{10} v = \frac{1}{\gamma} \log_{10} k - \frac{1}{\gamma} \log_{10} p \quad \text{or} \quad Y = A + BX$$

where $X = \log_{10} p$, $Y = \log_{10} v$, $A = \dfrac{1}{\gamma} \log_{10} k$, $B = -\dfrac{1}{\gamma}$.

$\therefore$ the normal equations are

$$\Sigma Y = 6A + B\Sigma X \quad ...(i)$$

$$\Sigma XY = A\Sigma X + B\Sigma X^2 \quad ...(ii)$$

Now ΣX etc. are calculated as follows :

p	v	$X = \log_{10} p$	$Y = \log_{10} v$	XY	X^2
.5	1.62	– 0.3010	0.2095	– 0.0630	0.0906
1.0	1.00	0.0000	0.0000	– 0.0000	0.0000
1.5	0.75	0.1761	– 0.1249	– 0.0220	0.0310
2.0	0.62	0.3010	– 0.2076	– 0.0625	0.0906
2.5	0.52	0.3979	– 0.2840	– 0.1130	0.1583
3.0	0.46	0.4771	– 0.3372	– 0.1609	0.2276
Total		1.0511	– 0.7442	– 0.4214	0.5981

$\therefore$ equations (i) and (ii) become

$$6A + 1.0511B = -0.7442$$

$$1.0511A + 0.5981B = -0.4214$$

Solving these, we get $A = 0.0132$, $B = -0.7836$.

$\therefore \quad \gamma = -1/B = 1.1276$ and $k = \text{antilog}(A\gamma) = \text{antilog}(0.0168) = 1.039$.

Hence the equation of best fit is $pv^{1.276} = 1.039$.

PROBLEMS 24.3

1. If V (km/hr) and R (kg/ton) are related by a relation of the type $R = a + bV^2$, find by the method of least squares a and b with the help of the following table :

V :	10	20	30	40	50
R :	8	10	15	21	30

(*Indore, 2008*)

2. Using the method of least squares fit the curve $y = ax + bx^2$ to following observations :

x :	1	2	3	4	5
y :	1.8	5.1	8.9	14.1	19.8

3. Fit the curve $y = ax + b/x$ to the following data :

x :	1	2	3	4	5	6	7	8
y :	5.4	6.3	8.2	10.3	12.6	14.9	17.3	19.5

(*U.P.T.U., 2010*)

4. Estimate y at $x = 2.25$ by fitting the *indifference curve* of the form $xy = Ax + B$ to the following data :

x :	1	2	3	4
y :	3	1.5	6	7.5

(*J.N.T.U., 2003*)

5. Find the least square curve $y = ax + b/x$ for the following data :

x :	1	2	3	4
y :	-1.5	0.99	3.88	7.66

(*Madras, 2003*)

6. Predict y at $x = 3.75$, by fitting a *power curve* $y = ax^b$ to the given data :

x :	1	2	3	4	5	6
y :	298	4.26	5.21	6.10	6.80	7.50

(*J.N.T.U., 2003*)

7. Fit the curve of the form $y = ae^{bx}$ to the following data :

x :	77	100	185	239	285
y :	2.4	3.4	7.0	11.1	19.6

(*V.T.U., 2011 S ; J.N.T.U., 2006*)

8. Obtain the least squares fit of the form $f(t) = ae^{-3t} + be^{-2t}$ for the data :

x :	0.1	0.2	0.3	0.4
$f(t)$:	0.76	0.58	0.44	0.35

(*U.P.T.U., 2008*)

9. The voltage v across a capacitor at time t seconds is given by the following table :

t :	0	2	4	6	8
v :	150	63	28	12	5.6

Use the method of least squares to fit a curve of the form $v = ae^{kt}$ to this data.

10. Using method of least squares, fit a relation of the form $y = ab^x$ to the following data :

x :	2	3	4	5	6
y :	144	172.8	207.4	248.8	298.5

(*Tiruchirapalli, 2001*)

24.7 METHOD OF GROUP AVERAGES

Let the straight line, $\quad y = a + bx \quad$ ...(1)

fit the set of n observations

$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ quite closely. (Fig. 24.5)

When $x = x_1$, the *observed* (or *experimental*) *value* of $y = y_1 = L_1P_1$ and from (1),

$$y = a + bx_1 = L_1M_1,$$

which is known as the *expected* (or *calculated*) *value* of y at L_1.

Then e_1 = observed value at L_1 – expected value at L_1

$$= y_1 - (a + bx_1) = M_1P_1,$$

which is called the *error* (or *residual*) at x_1. Similarly the errors for the other observations are

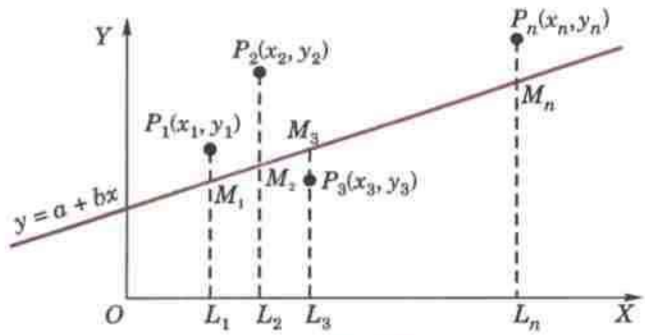


Fig. 24.5

$$e_2 = y_2 - (a + bx_2) = M_2P_2$$

....

$$e_n = y_n - (a + bx_n) = M_nP_n$$

Some of these errors may be positive and others negative.

The method of group averages is based on the assumption that the sum of the residuals is zero. To find the constants a and b is (1), we require two equations. As such we divide the data into two groups : the first containing k observations

$$(x_1, y_1), (x_2, y_2) \dots (x_k, y_k) ;$$

and the second group having the remaining $n - k$ observations

$$(x_{k+1}, y_{k+1}), (x_{k+2}, y_{k+2}), \dots, (x_n, y_n).$$

Assuming that the sum of the errors in each group is zero, we get

$$\{y_1 - (a + bx_1)\} + \{y_2 - (a + bx_2)\} + \dots \{y_k - (a + bx_k)\} = 0$$

$$\{y_{k+1} - (a + bx_{k+1})\} + \{y_{k+2} - (a + bx_{k+2})\} + \dots + \{y_n - (a + bx_n)\} = 0$$

On simplification, we obtain

$$\frac{y_1 + y_2 + \dots + y_k}{k} = a + b\,\frac{x_1 + x_2 + \dots + x_k}{k} \qquad \dots(2)$$

$$\frac{y_{k+1} + y_{k+2} + \dots + y_n}{n - k} = a + b\,\frac{x_{k+1} + x_{k+2} + \dots + x_n}{n - k} \qquad \dots(3)$$

In (2), $\frac{1}{k}(x_1 + x_2 + \dots + x_k)$ and $\frac{1}{k}(y_1 + y_2 + \dots + y_k)$ are simply the average values of x's and y's of the first group. Hence the equations (2) and (3) are obtained from (1) by replacing x and y by their respective averages of the two groups. Solving (2) and (3), we get a and b.

Obs. The main drawback of this method is that a different grouping of the observations will give different values of a and b. In practice, we divide the data in such a way that each group contains almost an equal number of observations.

Example 24.12. *The latent heat of vaporisation of steam r, is given in the following table at different temperatures t :*

t :	*40*	*50*	*60*	*70*	*80*	*90*	*100*	*110*
r :	*1069.1*	*1063.6*	*1058.2*	*1052.7*	*1049.3*	*1041.8*	*1036.3*	*1030.8*

For this range of temperature, a relation of the form r = a + bt is known to fit the data. Find the values of a and b by the method of group averages. (*Madras, 2003*)

Solution. Let us divide the data into two groups each containing four readings. Then we have

t	r	t	r
40	1069.1	80	1049.3
50	1063.6	90	1041.8
60	1058.2	100	1036.3
70	1052.7	110	1030.8
$\Sigma t = 220$	$\Sigma r = 4243.6$	$\Sigma t = 380$	$\Sigma r = 4158.2$

Substituting the averages of t's and r's of the two groups in the given relation, we get

$$\frac{4243.6}{4} = a + b\,\frac{220}{4} \quad i.e., \; 1060.9 = a + 55b \qquad ...(i)$$

$$\frac{4158.2}{4} = a + b\,\frac{380}{4} \quad i.e., \; 1039.55 = a + 95b \qquad ...(ii)$$

Solving (*i*) and (*ii*), we obtain

$$a = 1090.26,\; b = -0.534.$$

24.8 FITTING A PARABOLA

We have applied the method of averages to *linear law* involving two constants only. To fit the parabola

$$y = a + bx + cx^2 \qquad ...(1)$$

which contains three constants, to a set of observations, we proceed as follows :

Let (x_1, y_1) be a point on (1) satisfying the given data so that

$$y_1 = a + bx_1 + cx_1^2$$

Then $$y - y_1 = b\,(x - x_1) + c\,(x^2 - x_1^2)$$

or $$\frac{y - y_1}{x - x_1} = b + c\,(x + x_1)$$

Putting $x + x_1 = X$ and $(y - y_1)/(x - x_1) = Y$, it takes the linear form

$$Y = b + cX.$$

Now b and c can be found as before.

Example 24.13. *The corresponding values of x and y are given by the following table :*

x :	87.5	84.0	77.8	63.7	46.7	36.9
y :	292	283	270	235	197	181

Solution. Taking $x = 84, y = 283$ as a particular point on $y = a + bx + cx^2$,

we get $$283 = a + b\,(84) + c\,(84)^2 \qquad ...(i)$$

$\therefore$ $$y - 283 = b\,(x - 84) + c\,[x^2 - (84)^2]$$

or $$\frac{y - 283}{x - 84} = b + c\,(x + 84)$$

i.e., $$Y = b + cX \qquad ...(ii)$$

where $X = x + 84$, $Y = (y - 283)/(x - 84)$.

Now we have the following table of values :

x	y	$X = x + 84$	$Y = (y - 283)/(x - 84)$
87.5	292	171.5	2.571
84.0	283	—	—
77.8	270	161.8	2.097
		$\Sigma X = 333.3$	$\Sigma Y = 4.668$
63.7	235	147.7	2.364
46.7	197	130.7	2.306
36.9	181	120.9	2.166
		$\Sigma X = 399.3$	$\Sigma Y = 6.836$

Substituting the averages of X and Y in (*ii*), we get

$$\frac{4.668}{2} = b + c\,\frac{333.3}{2} \quad i.e., \; 2.33 = b + 166.65\,c \qquad ...(iii)$$

$$\frac{6.836}{3} = b + c\,\frac{399.3}{3} \quad i.e., \; 2.28 = b + 131.1\,c \qquad ...(iv)$$

(iv)–(iii) gives $c = 0.0014$

and (iii) gives $b = 2.0967$ *i.e.*, 2.1 nearly

From (i), we get $a = 96.9988$ *i.e.*, 97 nearly.

Hence the parabola of fit is

$$y = 97 + 2.1x + .0014x^2.$$

Example 24.14. *The train resistance R (lbs/ton) is measured for the following values of its velocity V (km/hr) :*

V :	20	40	60	80	100
R :	5	9	14	25	36

If R is related to V by the formula $R = a + bV^n$, find a, b, and n.

Solution. To find a, we take the following three values of v which are in G.P. :

$$v_1 = 20, \quad v_2 = 40, \quad v_3 = 80$$

Then $R_1 = 5, \quad R_2 = 9, \quad R_3 = 25$

$\therefore \quad (R_1 - a)(R_3 - a) = (R_2 - a)^2$

whence $$a = \frac{R_1R_3 - R_2^2}{R_1 + R_3 - 2R_2} = 3.67$$

Thus $R - 3.67 = bV^n$ or $\log_{10}(R - 3.67) = \log_{10} b + n \log_{10} V$

i.e., $$Y = k + nX \qquad ...(i)$$

where $X = \log_{10} V$, $Y = \log_{10}(R - 3.67)$, $k = \log_{10} b$.

Now we have the following table of values :

V	R	$X = \log_{10} V$	$Y = \log_{10}(R - 3.67)$
20	5	1.3010	0.1238
40	9	1.6021	0.7267
60	14	1.7782	1.0141
		$\Sigma X = 4.6813$	$\Sigma Y = 1.8646$
80	25	1.9031	1.3290
100	36	2.0000	1.5096
		$\Sigma X = 3.9031$	$\Sigma X = 2.8396$

Substituting the averages of X's and Y's in (i), we obtain

$$\frac{1.8646}{2} = k + n\,\frac{4.6813}{2} \qquad i.e., \ 0.6215 = k + 1.5604\,n \qquad ...(ii)$$

$$\frac{2.8386}{2} = k + n\,\frac{3.9031}{2} \qquad i.e., \ 1.4193 = k + 1.9516\,n \qquad ...(iii)$$

Solving (ii) and (iii), we get $n = 2.04$, $k = -2.56$ approx.

$$b = \text{antilog } k = \text{antilog } (-2.56) = 0.0028.$$

PROBLEMS 24.4

1. Fit a straight line of the form $y = a + bx$ to the following data by the method of group averages :

x :	0	5	10	15	20	25
y :	12	15	17	22	24	30

(*Tiruchirapalli, 2001*)

2. The weights of a calf taken at weekly intervals are given below :

Age :	1	2	3	4	5	6	7	8	9	10
Weight :	52.5	58.7	65.0	70.2	75.4	81.1	87.2	95.5	102.2	108.4

Find a straight line of best fit.

3. Using the method of averages, fit a parabola $y = ax^2 + bx + c$ to the following data :

x :	20	40	60	80	100	120
y :	5.5	9.1	14.9	22.8	33.3	46.0

4. While testing a centrifugal pump, the following data is obtained. It is assumed to fit the equation $y = a + bx + cx^2$, where x is the discharge in litre/sec and y, head in metres of water. Find the values of the constants a, b, c by the method of group averages.

x :	2	2.5	3	3.5	4	4.5	5	5.5	6
y :	18	17.8	17.5	17	15.8	14.8	13.3	11.7	9

5. By the method of averages, fit a curve of the form $y = ae^{bx}$ to the following data :

x :	5	15	20	30	35	40
y :	10	14	25	40	50	62

(*Madras, 2002*)

24.9 METHOD OF MOMENTS

Let $(x_1, y_1), (x_2, y_2), \ldots (x_n, y_n)$ be the set of n observations such that

$$x_2 - x_1 = x_3 - x_2 = \ldots = x_n - x_{n-1} = h \text{ (say)}$$

We define the *moments of the observed values* of y as follows :

m_1, the 1st moment $= h\Sigma y$

m_2, the 2nd moment $= h\Sigma xy$

m_3, the 3rd moment $= h\Sigma x^2 y$ and so on.

Let the curve fitting the given data be $y = f(x)$. Then the *moments of the calculated values* of y are

μ_1, the 1st moment $= \int y\,dx$

μ_2, the 2nd moment $= \int xy\,dx$

μ_3, the 3rd moment $= \int x^2 y\,dx$ and so on.

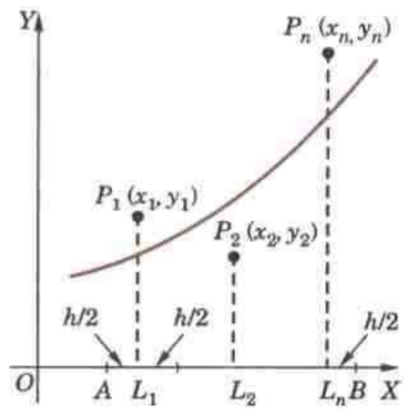


Fig. 24.6

This method is based on the assumption that the moment of the observed values of y are respectively equal to the moments of the calculated values of y i.e., $m_1 = \mu_1, m_2 = \mu_2, m_3 = \mu_3$ *etc. These equations (known as observation equations)* are used to determine the constants in $f(x)$.

m's are calculated from the tabulated values of x and y while μ's are computed as follows :

In Fig. 24.6, y_1 the ordinate of P_1 $(x = x_1)$, can be taken as the value of y at the mid-point of the interval $(x_1 - h/2, x_1 + h/2)$. Similarly, y_n, the ordinate of P_n $(x = x_n)$, can be taken as the value of y at the mid-point of the interval $(x_n - h/2, x_n + h/2)$. If A and B be the points such that

$$OA = x_1 - h/2 \text{ and } OB = x_n + h/2,$$

then
$$\mu_1 = \int y\,dx = \int_{x_1 - h/2}^{x_n + h/2} f(x)\,dx$$

$$\mu_2 = \int_{x_1 - h/2}^{x_n + h/2} xf(x)\,dx$$

and
$$\mu_3 = \int_{x_1 - h/2}^{x_n + h/2} x^2 f(x)\,dx.$$

Example 24.15. *Fit a straight line $y = a + bx$ to the following data by the method of moments :*

x :	1	2	3	4
y :	16	19	23	26

(*Madras, 2001 S*)

Solution. Since only two constants a and b are to be found, it is sufficient to calculate the first two moments in each case. Here $h = 1$.

$$m_1 = h\Sigma y = 1\,(16 + 19 + 23 + 26) = 84$$
$$m_2 = h\Sigma xy = 1\,(1 \times 16 + 2 \times 19 + 3 \times 22 + 4 \times 26) = 227$$

To compute the moments of calculated values of $y = a + bx$, the limits of integration will be $1 - h/2$ and $4 + h/2$ *i.e.*, 0.5 to 4.5

$$\therefore \qquad \mu_1 = 2\int_{0.5}^{4.5}(a+bx)\,dx = \left| ax + b\frac{x^2}{2} \right|_{0.5}^{4.5} = 4a + 10b$$

$$\mu_2 = \int_{0.5}^{4.5} x(a+bx)\,dx = 10a + \frac{91}{3}b.$$

Thus, the observation equations $m_r = \eta_r$ $(r = 1, 2)$ are $4a + 10b = 84$; $10a + \frac{91}{3}b = 227$

Solving these, $a = 13.02$ and $b = 3.19$.

Hence the required equation is $y = 13.02 + 3.19x$.

Example 24.16. *Given the following data :*

x :	0	1	2	3	4
y :	1	5	10	22	38

find the parabola of best fit by the method of moments.

Solution. Let the parabola of best fit be $y = a + bx + cx^2$ ...(*i*)

Since three constants are to be found, we calculate the first three moments in each case. Here $h = 1$.

$$m_1 = h\Sigma y = 1\,(1 + 5 + 10 + 22 + 38) = 76$$
$$m_2 = h\Sigma xy = 1\,(0 + 5 + 20 + 66 + 152) = 243$$
$$m_3 = h\Sigma x^2 y = 1\,(0 + 5 + 40 + 198 + 608) = 851$$

For computing the moments of calculated values of (*i*), the limits of integration will be $0 - h/2$ and $4 + h/2$ *i.e.*, -0.5 and 4.5.

$$\therefore \qquad \mu_1 = \int_{-0.5}^{4.5}(a + bx + cx^2)dx = 5a + 10b + 30.4c$$

$$\mu_2 = \int_{-0.5}^{4.5} x\,(a + bx + cx^2)\,dx = 10a + 30.4b + 102.5c$$

$$\mu_3 = \int_{-0.5}^{4.5} x^2\,(a + bx + cx^2)\,dx = 30.4a + 102.5b + 369.1c$$

Thus the observation equations $m_r = \mu_r$ $(r = 1, 2, 3)$ are

$$5a + 10b + 30.4c = 76\ ;\ 10a + 30.4 + 102.5c = 243\ ;\ 30.4a + 102.5b + 369.1c = 851$$

Solving these equations, we get $a = 0.4$, $b = 3.15$, $c = 1.4$.

Hence the parabola of best fit is $y = 0.4 + 3.15x + 1.4x^2$.

PROBLEMS 24.5

1. Use the method of moments to fit the straight line $y = a + bx$ to the data :

x :	1	2	3	4
y :	0.17	0.18	0.23	0.32

2. Fit a straight line to the following data, using the method of moments :

x :	1	3	5	7	9
y :	1.5	2.8	4.0	4.7	6.0

(*Madras, 2001*)

3. Fit a parabola of the form $y = a + bx + cx^2$ to the data :

x :	1	2	3	4
y :	1.7	1.8	2.3	3.2

by the method of moments.

4. By using the method of moments, fit a parabola to the following data :

x :	1	2	3	4
y :	0.30	0.64	1.32	5.40

(*Madras, 2000 S*)

24.10 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 24.6

Fill up the blanks or choose the correct answer in the following problems :

1. The law $y = ax^2 + bx$ converted to linear form is ....
2. The gas equation $pv^r = k$ can be reduced to $y = a + bx$ where $a =$ .... and $b =$ ....
3. The principle of *'least squares'* states that ....
4. $y = ax^b + c$ in linear form is ....
5. To fit the straight line $y = mx + c$ to n observations, the normal equations are
   (*i*) $\Sigma y = n\, \Sigma x + \Sigma cm$, $\Sigma xy = c\, \Sigma x^2 + c\, \Sigma n$.
   (*ii*) $\Sigma y = m\, \Sigma x + nc$, $\Sigma xy = m\, \Sigma x^2 + c\, \Sigma x$.
   (*iii*) $\Sigma y = c\, \Sigma x + m\, \Sigma n$, $\Sigma xy = c\, \Sigma x^2 + m\, \Sigma x$.
6. To fit $y = ab^x$ by least square method, normal equations are ....
7. The observation equations for fitting a straight line by *method of moments* are ....
8. The *method of group averages* is based on the assumption that the sum of the residuals is ....
9. $y = ax^2 + b \log_{10} x$ reduced to linear law takes the form ....
10. Given $\begin{bmatrix} x: & 0 & 1 & 2 \\ y: & 0 & 1.1 & 2.1 \end{bmatrix}$ then the straight line of best fit is ....
11. The *method of moments* is based on the assumption that ....
12. In $y = a + bx$, $\Sigma x = 50$, $\Sigma y = 80$, $\Sigma xy = 1030$, $\Sigma x^2 = 750$ and $n = 10$, then $a =$ ...., $b =$ ....
13. $y = x/(ax + b)$ in linear form is ....
14. If $y = a + bx + cx^2$ and

x :	0	1	2	3	4
y :	1	1.8	1.3	2.5	7.3

then the first normal equation is :
(α) $15 = 5a + 10b + 29c$, (β) $15 = 5a + 10b + 31c$
(γ) $12.9 = 5a + 10b + 30c$ (δ) $34 = 5a + 10b + 27c$.

15. If $y = 2x + 5$ is the best fit for 8 pairs of values (x, y) by the method of least squares and $\Sigma y = 120$, then $\Sigma X =$
   (*a*) 35 (*b*) 40 (*c*) 45 (*d*) 30.

CHAPTER 25

Statistical Methods

1. Introduction. 2. Collection and classification of data. 3. Graphical representation. 4. Comparison of frequency distributions. 5. Measures of central tendency. 6. Measures of dispersion. 7. Coefficient of variation; Relations between measures of dispersion. 8. Standard deviation of the combination of two groups. 9. Moments. 10. Skewness. 11. Kurtosis. 12. Correlation. 13. Coefficient of correlation. 14. Lines of regression. 15. Standard error of estimate. 16. Rank correlation. 17. Objective Type of Questions.

25.1 INTRODUCTION

Statistics deals with the methods for collection, classification and analysis of numerial data for drawing valid conclusions and making reasonable decisions. It has meaningful applications in production engineering, in the analysis of experimental data, etc. The importance of statistical methods in engineering is on the increase. As such we shall now introduce the student to this interesting field.

25.2 (1) COLLECTION OF DATA

The collection of data constitutes the starting point of any statistical investigation. Data may be collected for each and every unit of the whole lot (*population*), for it would ensure greater accuracy. But complete enumeration is prohibitively expensive and time consuming. As such out of a very large number of items, a few of them (a *sample*) are selected and conclusions drawn on the basis of this sample are taken to hold for the population.

(2) **Classification of data.** The data collected in the course of an inquiry is not in an easily assimilable form. As such, its proper classification is necessary for making intelligent inferences. The classification is done by dividing the raw data into a convenient number of groups according to the values of the variable and finding the frequency of the variable in each group.

Let us, for example, consider the raw data relating to marks obtained in Mechanics by a group of 64 students :

79	88	75	60	93	71	59	85
84	75	82	68	90	62	88	76
65	75	87	74	62	95	78	63
78	82	75	91	77	69	74	68
67	73	81	72	63	76	75	85
80	73	57	88	78	62	76	53
62	67	97	78	85	76	65	71
78	89	61	75	95	60	79	83

This data can conveniently be grouped and shown in a tabular form as follows :

Class	Frequency	Cumulative frequency
50—54	1	1
55—59	2	3
60—64	9	12
65—69	7	19
70—74	8	27
75—79	17	44
80—84	6	50
85—89	8	58
90—94	3	61
95—99	3	64
	Total = 64	

It would be seen from the above table that there is one student getting marks between 50—54, two students getting marks between 55—59, nine students getting marks between 60—64 and so on. Thus the 64 figure have been put into only 10 groups, called the **classes**. The width of the class is called the **class interval** and the number in that interval is called the **frequency.** The mid-point or the mid-value of the class is called the **class mark.** The above table showing the classes and the corresponding frequencies is called a *frequency table. Thus a set of raw data summarised by distributing it into a number of classes alongwith their frequencies is known as a* **frequency distribution**.

While forming a frequency distribution, the number of classes should not ordinarily exceed 20, and should not, in general, be less than 10. As far as possible, the class intervals should be of equal width.

(3) **Cumulative frequency.** In some investigations, we require the number of items less than a certain value. We add up the frequencies of the classes upto that value and call this number as the *cumulative frequency*. In the above table, the third column shows the cumulative frequencies, *i.e.*, the number of students, getting less than 54 marks, less than 59 marks and so on.

25.3 GRAPHICAL REPRESENTATION

A convenient way of representing a sample frequency distribution is by means of graphs. It gives to the eyes the general run of the observations and at the same time makes the raw data readily intelligible. We give below the important types of graphs in use :

(1) **Histogram.** A histogram is drawn by erecting rectangles over the class intervals, such that the areas of the rectangles are proportional to the class frequencies. If the class intervals are of equal size, the height of the rectangles will be proportional to the class frequencies themselves (Fig. 25.1).

(2) **Frequency polygon.** A frequency polygon for an ungrouped data can be obtained by joining points plotted with the variable values as the abscissae and the frequencies as the ordinates. For a grouped distribution, the abscissae of the points will be the mid-values of the class intervals. In case the intervals are equal, the frequency polygon can be obtained by joining the middle points of the upper sides of the rectangles of the histogram by straight lines (shown by dotted lines in Fig. 25.1). If the class intervals become very very small, the frequency polygon takes the form of a smooth curve called the *frequency curve*.

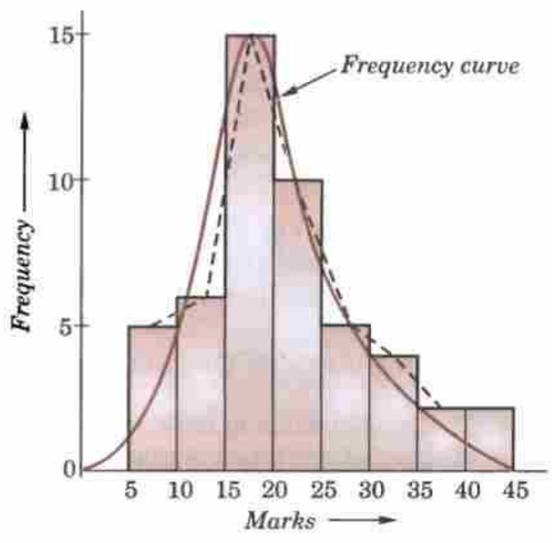


Fig. 25.1

(3) **Cumulative frequency curve-Ogive.** Very often, it is desired to show in a diagrammatic form, not the relative frequencies in the various intervals, but the cumulative frequencies above or below a given value. For example, we may wish to read off from a diagram the number or proportions of people whose income is not less than any given amount, or proportion of people whose height does not exceed any stated value. Diagrams of

this type are known as *cumulative frequency curves* or *ogives*. These are of two kinds '*more than*' or '*less than*' and typically they look somewhat like a long drawn S (Fig. 25.2).

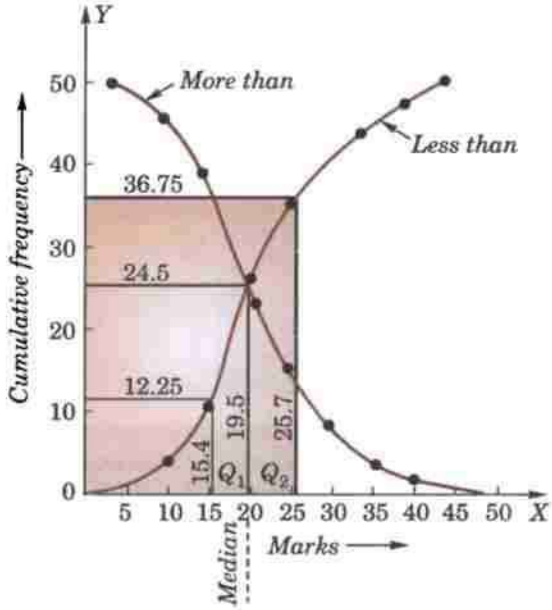


Fig. 25.2

Example 25.1. *Draw the histogram, frequency polygon, frequency curve and the ogive 'less than' and 'more than' from the following distribution of marks obtained by 49 students :*

Class (Marks group)	*Frequency (No. of students)*	*Cumulative frequency (Less than)*	*Cumulative frequency (More than)*
5—10	5	5	49
10—15	6	11	44
15—20	15	26	38
20—25	10	36	23
25—30	5	41	13
30—35	4	45	8
35—40	2	47	4
40—45	2	49	2

Solution. In Fig. 25.1, the rectangles show the *histogram*; the dotted polygon represents the *frequency polygon* and the smooth curve is the *frequency curve*.

The *ogives* 'less than' and 'more than' are shown in Fig. 25.2.

25.4 COMPARISON OF FREQUENCY DISTRIBUTIONS

The condensation of data in the form of a frequency distribution is very useful as far as it brings a long series of observations into a compact form. But in practice, we are generally interested in comparing two or more series. The inherent inability of the human mind to grasp in its entirety even the data in the form of a frequency distribution compels us to seek for certain constants which could concisely give an insight into the important characteristics of the series. The chief constants which summarise the fundamental characteristics of the frequency distributions are (*i*) *Measures of central tendency*, (*ii*) *Measures of dispersion and* (*ii*) *Measures of skewness*.

25.5 MEASURES OF CENTRAL TENDENCY

A frequency distribution in general, shows clustering of the data around some central value. Finding of this central value or the average is of importance, as it gives a most representative value of the whole group.

Different methods give different averages which are known as the *measures of central tendency*. The commonly used measures of central value are *Mean, Median, Mode, Geometric mean and Harmonic mean.*

(1) Mean. If $x_1, x_2, x_3, ..., x_n$ are a set of n values of a variate, then the *arithmetic mean* (or simply *mean*) is given by

$$\bar{\mathbf{x}} = \frac{x_1 + x_2 + x_3 + ... + x_n}{n}, \text{ i.e. } \frac{\Sigma \mathbf{x}_i}{\mathbf{n}} \quad ...(1)$$

In a *frequency distribution*, if $x_1, x_2, ..., x_n$ be the mid-values of the class-intervals having frequencies $f_1, f_2, ..., f_n$ respectively, we have

$$\bar{\mathbf{x}} = \frac{f_1x_1 + f_2x_2 + ... + f_nx_n}{f_1 + f_2 + ... + f_n} = \frac{\Sigma \mathbf{f}_i \mathbf{x}_i}{\Sigma \mathbf{f}_i} \quad ...(2)$$

Calculation of mean. Direct method of computing especially when applied to grouped data involves heavy calculations and in order to avoid these, the following formulae are generally used :

I. Short-cut method $\quad \bar{\mathbf{x}} = \mathbf{A} + \dfrac{\Sigma \mathbf{f}_i d_i}{\Sigma \mathbf{f}_i} \quad ...(3)$

II. Step-deviation method $\quad \bar{\mathbf{x}} = \mathbf{A} + \mathbf{h}\dfrac{\Sigma \mathbf{f}_i \mathbf{u}_i}{\Sigma \mathbf{f}_i} \quad ...(4)$

where $d = x - A$ and $u = (x - A)/h$, A being an arbitrary origin and h the equal class interval.

Proof. If $x_1, x_2, ..., x_n$ are the mid-values of the classes with frequencies $f_1, f_2, ..., f_n$, we have

$$\Sigma f_i x_i = \Sigma f_i (A + d_i) = A\Sigma f_i + \Sigma f_i d_i$$

$$\therefore \quad \bar{x} = \frac{\Sigma f_i x_i}{\Sigma f_i} = A + \frac{\Sigma f_i d_i}{\Sigma f_i}$$

Further $u_i = d_i/h$ or $d_i = hu_i$. Substituting this value in (3), we get (4).

Obs. *The algebraic sum of the deviations of all the variables from their mean is zero, for*

$$\Sigma f_i (x_i - \bar{x}) = \Sigma f_i x_i - \bar{x}\,\Sigma f_i = \Sigma f_i x_i - \frac{\Sigma f_i x_i}{\Sigma f_i} . \Sigma f_i = 0.$$

Cor. *If $\bar{x}_1, \bar{x}_2$ be the means of two samples of size n_1 and n_2, then the mean $\bar{x}$ of the combined sample of size $n_1 + n_2$ is given by*

$$\bar{\mathbf{x}} = \frac{\mathbf{n_1\bar{x}_1 + n_2\bar{x}_2}}{\mathbf{n_1 + n_2}}$$

For $\quad n_1\bar{x}_1$ = sum of all observations of the first sample,

and $\quad n_2\bar{x}_2$ = sum of all observations of the second sample.

$\therefore$ sum of the observations of the combined sample $= n_1\bar{x}_1 + n_2\bar{x}_2$.

Also number of the observations in the combined sample $= n_1 + n_2$.

$\therefore$ mean of the combined sample $= \dfrac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$.

Example 25.2. *The following is the frequency distribution of a random sample of weekly earnings of 509 employees :*

Weekly earnings :	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40
No. of employees :	3	6	10	15	24	42	75	90	79	55	36	26	19	13	9	7

Calculate the average weekly earnings.

Solution. The calculations are arranged in the following table. The arbitrary origin is generally taken as the value corresponding to the maximum frequency.

By direct method, we have

$$\text{Mean } \bar{x} = \frac{\Sigma fx}{\Sigma f} = \frac{13,315}{509} = 26.16$$

By step-deviation method, we have

$$\bar{x} = A + h\frac{\Sigma fu}{\Sigma f} = 25 + 2 \times \frac{295}{509}$$

$$= 25 + 1.16 = 26.16, \text{ which is same as found above.}$$

Weekly earnings	Mid value	No. of employees	Step deviations		
	x	f	$f \times x$	$u = (x - 25)/2$	$f \times u$
10—12	11	3	33	−7	−21
12—14	13	6	78	−6	−36
14—16	15	10	150	−5	−50
16—18	17	15	255	−4	−60
18—20	19	24	456	−3	−72
20—22	21	42	882	−2	−84
22—24	23	75	1725	−1	−75
24—26	25	90	2250	0	−398
26—28	27	79	2133	1	79
28—30	29	55	1595	2	110
30—32	31	36	1116	3	108
32—34	33	26	858	4	104
34—36	35	19	665	5	95
36—38	37	13	481	6	78
38—40	39	9	351	7	63
40—42	41	7	287	8	56
					+693
		$\Sigma f = 509$	$\Sigma fx = 13,315$		$\Sigma fu = 295$

(2) Median. *If the values of a variable are arranged in the ascending order of magnitude, the median is the middle item if the number is odd and is the mean of the two middle items if the number is even.* Thus the median is equal to the mid-value, *i.e.*, the value which divides the total frequency into two equal parts.

For the grouped data,

$$\textbf{Median} = \mathbf{L} + \frac{(\frac{1}{2}\mathbf{N} - \mathbf{C})}{\mathbf{f}} \times \mathbf{h}$$

where L = lower limit of the median class, N = total frequency,
f = frequency of the median class, h = width of the median class,
and C = cumulative frequency upto the class preceding the median class.

Quartiles. Quartiles are those values which divide the frequency into four equal parts, when the values are arranged in the ascending order of magnitude. The **lower quartile** (Q_1) is mid-way between the lower extreme and the median. The **upper quartile** (Q_3) is midway between the median the upper extreme.

For the grouped data, these are calculated by the formulae :

$$\mathbf{Q_1} = \mathbf{L} + \frac{(\frac{1}{4}\mathbf{N} - \mathbf{C})}{\mathbf{f}} \times \mathbf{h}$$

and

$$\mathbf{Q_3} = \mathbf{L} + \frac{(\frac{3}{4}\mathbf{N} - \mathbf{C})}{\mathbf{f}} \times \mathbf{h}$$

where L = lower limit of the class in which Q_1 or Q_3 lies, f = frequency of this class, h = width of the class
and C = cumulative frequency upto the class preceding the class in which Q_1 or Q_3 lies.

The difference between the upper and lower quartiles, i.e., Q_3—Q_1 is called the **inter-quartile range.**

Obs. The ogives give a ready method of marking on the curve the values of the median and the quartiles. The two ogives 'less than' and 'more than' cut each other at the median (Fig. 25.2).

(3) Mode. *The mode is defined as that value of the variable which occurs most frequently, i.e., the value of the maximum frequency.*

For a grouped distribution, it is given by the formula

$$\textbf{Mode} = \mathbf{L} + \frac{\Delta_1}{\Delta_1 + \Delta_2}\mathbf{h}$$

where L = lower limit of the class containing the mode,

Δ_1 = excess of modal frequency over frequency of preceding class,
Δ_2 = excess of modal frequency over following class,
and h = size of modal class.

For a frequency curve (Fig. 25.1), the abscissa of the highest ordinate determines the value of the mode. There may be one or more modes in a frequency curve. Curves having a single mode are termed as *unimodal*, those having two modes as *bi-modal* and those having more than two modes as *multi-modal*.

Obs. In a symmetrical distribution, the mean, median and mode coincide. For other distributions, however, they are different and are known to be connected by the empirical relationship :

Mean – Mode = 3(Mean – Median).

Example 25.3. *Calculate median and the lower and upper quartiles from the distribution of marks obtained by 49 students of example 25.1. Find also the semi-interquartile range and the mode.*

Solution. *Median* (or 49/2) falls in the class (15—20) and is given by

$$15 + \frac{(49/2) - 11}{15} \times 5 = 15 + \frac{13.5}{3} = 19.5 \text{ marks.}$$

Lower quartile Q_1 (or 49/4) = 12.25) also falls in the class 15—20.

$$\therefore \quad Q_1 = 15 + \frac{(49/4) - 11}{15} \times 5 = 15 + \frac{12.5}{3} = 15.4 \text{ marks}$$

Upper quartile (or $\frac{3}{4} \times 49 = 36.75$) falls in the class 25—30.

$$\therefore \quad Q_3 = 25 + \frac{36.75 - 36}{5} \times 5 = 25.75 \text{ marks.}$$

Semi-interquartile range $= \frac{1}{2}(Q_3 - Q_1) = \frac{25.75 - 15.4}{2} = \frac{10.35}{2} = 5.175.$

Mode. It is seen that the mode value falls in the class 15—20. Employing the formula for the grouped distribution, we have

$$\text{Mode} = 15 + \frac{15 - 6}{(15 - 6) + (15 - 10)} \times 5 = 18.2 \text{ marks.}$$

Obs. In Fig. 25.2, the ogives meet at a point whose abscissa is 19.5 which is the *median* of the distribution. The values for the lower and upper quartiles are similarly seen to be 15.4 (for frequency 12.25) and 25.7 (for frequency 36.75).

Example 25.4. *Given below are the marks obtained by a batch of 20 students in a certain class test in Physics and Chemistry.*

Roll No. of students	*Marks in Physics*	*Marks in Chemistry*	*Roll No. of students*	*Marks in Physics*	*Marks in Chemistry*
1	53	58	11	25	10
2	54	55	12	42	42
3	52	25	13	33	15
4	32	32	14	48	46
5	30	26	15	72	50
6	60	85	16	51	64
7	47	44	17	45	39
8	46	80	18	33	38
9	35	33	19	65	30
10	28	72	20	29	36

In which subject is the level of knowledge of the students higher ?

Solution. The subject for which the value of the median is higher will be the subject in which the level of knowledge of the students is higher. To find the median in each case, we arrange the marks in ascending order of magnitude :

Sr. No.	Marks in Physics	Marks in Chemistry	Sr. No.	Marks in Physics	Marks in Chemistry
1	25	10	11	46	42
2	28	15	12	47	44
3	29	25	13	48	46
4	30	26	14	51	50
5	32	30	15	52	55
6	33	32	16	53	58
7	33	33	17	54	64
8	35	36	18	60	72
9	42	38	19	65	80
10	45	39	20	72	85

Median marks in Physics = A.M. of marks of 10th and 11th terms

$$= \frac{45+46}{2} = 45.5$$

Median marks in Chemistry = A.M. of marks of 10th and 11th items.

$$= \frac{39+42}{2} = 40.5$$

Since the median marks in Physics is greater than the median marks in Chemistry; the level of knowledge in Physics is higher.

Example 25.5. *An incomplete frequency distribution is given as below :*

Variable :	*10—20*	*20—30*	*30—40*	*40—50*	*50—60*	*60—70*	*70—80*
Frequency :	*12*	*30*	*?*	*65*	*?*	*25*	*18*

Given that the total frequency is 229 and median is 46, find the missing frequencies.

Solution. Let f_1, f_2 be the missing frequencies of the classes 30—40 and 50—60 respectively.

Since the median lies in the class 40—50,

$$\therefore \qquad 46 = 40 + \frac{229/2 - (12+30+f_1)}{65} \times 10$$

which gives $f_1 = 33.5$ which can be taken as 34.

$$\therefore \qquad f_2 = 229 - (12+30+34+65+25+18) = 45.$$

(4) Geometric mean. If $x_1, x_2, ..., x_n$ are a set of n observations, then the *geometric mean* is given by

$$\text{G.M.} = (x_1\, x_2 \dots x_n)^{1/n}$$

or
$$\log \text{G.M.} = \frac{1}{n}(\log x_1 + \log x_2 + \dots + \log x_n) \qquad \text{...(1)}$$

In a frequency distribution, let $x_1, x_2, ..., x_n$ be the central values with corresponding frequencies $f_1, f_2, ..., f_n$, we have

$$\text{G.M.} = \left[(x_1)^{f_1} . (x_2)^{f_2} \dots (x_n)^{f_n}\right]^{1/n} \quad \text{where } n = \Sigma f_i.$$

or
$$\log \text{G.M.} = \frac{1}{n}[f_1 \log x_1 + f_2 \log x_2 + \dots + f_n \log x_n] \qquad \text{...(2)}$$

Hence (1) and (2) show that logarithm of G.M. = A.M. of logarithms of the values.

(5) Harmonic mean. If $x_1, x_2, \dots x_n$ be a set of n observations, then the *harmonic mean is defined as the reciprocal of the (arithmetic) mean of the reciprocals of the quantities.* Thus

$$\text{H.M.} = \frac{1}{\dfrac{1}{n}\left(\dfrac{1}{x_1} + \dfrac{1}{x_2} + \dots + \dfrac{1}{x_n}\right)}$$

In a frequency distribution, $\text{H.M.} = \dfrac{1}{\dfrac{1}{n}\left(\dfrac{f_1}{x_1} + \dfrac{f_2}{x_2} + \dots + \dfrac{f_n}{x_n}\right)}$ where $n = \Sigma f_i$.

Example 25.6. *Three cities A, B, C are equidistant from each other. A motorist travels from A to B at 30 km / hr, from B to C at 40 km / hr, from C to A at 50 km / hr. Determine the average speed.*

Solution. Let $AB = BC = CA = s$ km

Time taken to travel from A to $B = s/30$

Time taken to travel from B to $C = s/40$

Time taken to travel from C to $A = s/50$

$\therefore$ average time taken $= \dfrac{1}{3}\left(\dfrac{s}{30} + \dfrac{s}{40} + \dfrac{s}{50}\right)$

Thus the average speed $= \dfrac{s}{\dfrac{1}{3}\left(\dfrac{s}{30} + \dfrac{s}{40} + \dfrac{s}{50}\right)}$

In other words, the average speed is the harmonic mean of 30, 40, 50 km / hr.

Hence the average speed $= \dfrac{1}{\dfrac{1}{3}\left(\dfrac{1}{30} + \dfrac{1}{40} + \dfrac{1}{50}\right)} = 38.3$ km/hr.

Obs. Of the various measures of central tendency, *the mean* is the most important for it can be computed easily. *The median*, though more easily calculable, cannot be applied with ease to theoretical analysis. Median is of advantage when there are exceptionally large and small values at the ends of the distribution.

The mode, though most easily calculated, has the least significance. It is particularly misleading in distributions which are small in numbers or highly unsymmetrical.

The geometrical mean though difficult to compute, finds application in cases like populations where we are concerned with a quantity whose changes tend to be directly proportional to the quantity itself.

The harmonic mean is useful in limited situations where time and rate or prices are involved.

PROBLEMS 25.1

1. Draw the histogram and frequency polygon for the following distribution. Also calculate the arithmetic mean :

Class interval	0—99	100—199	200—299	300—399	400—499	500—599	600—699	700—799
Frequency	10	54	184	264	246	40	1	1

2. The following marks were given to a batch of candidates :

66	62	45	79	32	51	56	60	51	49
25	42	54	54	58	70	43	58	50	52
38	67	50	59	48	65	71	30	46	55
82	51	63	45	53	40	35	56	70	52
67	55	57	30	63	42	74	58	44	55

Draw a cumulative frequency curve.

Hence find the proportion of candidates securing more than 50 marks. Also mark off the median, the first and third quartiles.

3. Find the mean, median and mode for the following :

Mid Value :	15	20	25	30	35	40	45	50	55
Frequency :	2	22	19	14	3	4	6	1	1

(*Kerala, 1990*)

4. Calculate mean, median and mode of the following data relating to weight of 120 articles :

Weight (in gm) :	0—10	10—20	20—30	30—40	40—50	50—60
No. of articles :	14	17	22	26	23	18

5. The population of a country was 300 million in 1971. It became 520 million in 1989. Calculate the percentage compound rate of growth per annum.

   [**Hint.** Use $P_n = P_0(1 + r)^n$, r being the growth rate.]

6. The number of divorces per 1000 marriages in the United States increased from 84 in 1970 to 108 in 1990. Find the annual increase of the divorce rate for the period 1970 to 1990.

7. An aeroplane flies along the four sides of a square at speeds of 100, 200, 300 and 400 km/hr. respectively. What is the average speed of the plane in its flight around the square.

8. A man having to drive 90 km. wishes to achieve an average speed of 30 km/hr. For the first half of the journey, he averages only 20 km/hr. What must be his average speed for the second half of the journey if his overall average is to be 30 km/hr.

9. Following table gives the cumulative frequency of the age of a group of 199 teachers. Find the mean and median age of the group.

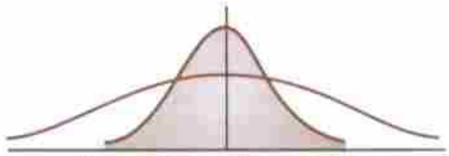
Age in years :	20—25	25—30	30—35	35—40	40—45	45—50	50—55	55—60	60—65	65—70
Cum. frequ. :	21	40	90	130	146	166	176	186	195	199

10. Recast the following cumulative table in the form of an ordinary frequency distribution and determine the median and the mode :

No. of days absent	*No. of students*	*No. of days absent*	*No. of students*
Less than 5	29	Less than 30	644
Less than 10	224	Less than 35	650
Less than 15	465	Less than 40	653
Less than 20	582	Less than 45	655
Less than 25	634		

25.6 MEASURES OF DISPERSION

Although measures of central tendency do exhibit one of the important characteristics of a distribution, yet they fail to give any idea as to how the individual values differ from the central value, *i.e.*, whether they are closely packed around the central value or widely scattered away from it. Two distributions may have the same mean and the same total frequency, yet they may differ in the extent to which the individual values may be spread about the average (*See* Fig. 25.3). The magnitude of such a variation is called *dispersion*. The important measures of dispersion are given below :



Same mean different dispersion

Fig. 25.3

(1) Range. *This is the simplest measure of dispersion and is given by the difference between the greatest and the least values in the distribution.* If the weekly wages of a group of labourers are

₹ 21 23 28 25 35 42 39 48

then *range* = Max. value – Min. value = 48 – 21 = ₹ 27.

(2) Quartile deviation or semi-interquartile range. One half of the interquartile range is called *quartile deviation*, or *semi-interquartile range*. If Q_1 and Q_3 are the first and third quartiles, the semi-interquartile range

$$Q = \tfrac{1}{2}(Q_3 - Q_1).$$

(3) Mean deviation. The mean deviation is the mean of the absolute differences of the values from the mean, median or mode. Thus *mean deviation* (*M.D.*)

$$= \frac{1}{n}\Sigma f_i \, |x_i - A|$$

where A is either the mean or the median or the mode. As the positive and negative differences have equal effects, only the absolute value of differences is taken into account.

(4) Standard deviation. The most important and the most powerful measure of dispersion is the *standard deviation* (*S.D.*) : generally denoted by σ. It is computed as the square root of the mean of the squares of the differences of the *variate* values from their mean.

Thus *standard deviation* (*S.D.*)

$$\sigma = \sqrt{\left[\frac{\Sigma \mathbf{f}_i(\mathbf{x}_i - \bar{\mathbf{x}})^2}{\mathbf{N}}\right]} \qquad ...(1)$$

where N is the total frequency Σf_i.

If however, the deviations are measured from any other value, say A, instead of $\bar{x}$, it is called the *root-mean-square deviation*.

The square of the standard deviation is known as the **variance.**

Calculation of S.D. The change of origin and the change of scale considerably reduces the labour in the calculation of standard deviation. The formulae for the computation of σ are as follows :

I. Short-cut method

$$\boldsymbol{\sigma} = \sqrt{\left[\frac{\Sigma f_i d_i^2}{\Sigma f_i} - \left(\frac{\Sigma f_i d_i}{\Sigma f_i}\right)^2\right]} \quad ...(2)$$

II. Step-deviation method

$$\boldsymbol{\sigma} = \mathbf{h}\sqrt{\left\{\frac{\Sigma f_i d_i'^2}{\Sigma f_i} - \left(\frac{\Sigma f_i d_i'}{\Sigma f_i}\right)^2\right\}} \quad ...(3)$$

where $d_i = x_i - A$ and $d'_i = (x_i - A)/h$, being the assumed mean and h the equal class interval.

Proof. We know that $x_i - \bar{x} = (x_i - A) - (\bar{x} - A)$

$\therefore$ $$\Sigma f_i(x_i - \bar{x})^2 = \Sigma f_i[d_i - (\bar{x} - A)]^2 = \Sigma f_i d_i^2 + (\bar{x} - A)^2 \Sigma f_i - 2(\bar{x} - A)\Sigma f_i d_i$$

$$= \Sigma f_i d_i^2 - \frac{(\Sigma f_i d_i)^2}{\Sigma f_i} \qquad \left[\because \ \bar{x} = A + \frac{\Sigma f_i d_i}{\Sigma f_i}\right]$$

Hence $$\sigma^2 = \frac{\Sigma f_i(x - \bar{x})^2}{\Sigma f_i} = \frac{\Sigma f_i d_i^2}{\Sigma f_i} - \left(\frac{\Sigma f_i d_i}{\Sigma f_i}\right)^2$$

Further $d'_i = (x_i - A)/h = d_i/h$ or $d_i = hd'_i$, then substituting this value in (2), we get (3).

Obs. *The root mean square deviation is least when measured from the mean.*
The root mean square deviation is given by

$$s^2 = \frac{\Sigma f_i d_i^2}{\Sigma f_i} \quad \text{and} \quad \frac{\Sigma f_i d_i}{\Sigma f_i} = \left[A + \frac{\Sigma f_i d_i}{\Sigma f_i}\right] - A = \bar{x} - A$$

$\therefore$ from (2), we have $s^2 = \sigma^2 + (\bar{x} - A)^2$ ...(4)

This shows that s^2 is always $> \sigma^2$ and the least value of $s^2 = \sigma^2$. This occurs when $A = \bar{x}$.

25.7 (1) COEFFICIENT OF VARIATION

The ratio of the standard deviation to the mean, is known as the *coefficient of variation*. As this is a ratio having no dimension, it is used for comparing the variations between the two groups with different means. It is often expressed as a percentage.

$\therefore$ *Coefficient of variation* $= \dfrac{\sigma}{\bar{x}} \times 100$

(2) Relations between measures of dispersion

(*i*) *Quartile deviation* $= 2/3$ *(standard deviation)*

(*ii*) *Mean deviation* $= 4/5$ *(standard deviation)*

25.8 STANDARD DEVIATION OF THE COMBINATION OF TWO GROUPS

If m_1, σ_1 be the mean and S.D. of a sample of size n_1 and m_2, σ_2 be those for a sample of size n_2, then the S.D. σ of the combined sample of size $n_1 + n_2$ is given by

$$\mathbf{(n_1 + n_2)\sigma^2 = n_1\sigma_1^2 + n_2\sigma_2^2 + n_1 D_1^2 + n_2 D_2^2}$$

where $D_i = m_i - m$, m being the mean of combined sample.

From (4), we have $ns^2 = n\sigma^2 + n(\bar{x} - A)^2$ where n is the size of the sample.

i.e. sum of the squares of the deviations from $A = n\sigma^2 + n(\bar{x} - A)^2$.

Now let us apply this result to the first given sample taking A at m. Then, sum of the squares of the deviations of n_1 items from $m = n_1\sigma_1^2 + n_1(m_1 - m)^2$ ...(5)

Similarly for the second given sample taking A at m, sum of the squares of the deviations of n_2 items from $m = n_2\sigma_2^2 + n_2(m_2 - m)^2$ ...(6)

Adding (5) and (6), sum of the squares of the deviations of $n_1 + n_2$ items from m

$$= n_1\sigma_1^2 + n_2\sigma_2^2 + n_1(m_1 - m)^2 + n_2(m_2 - m)^2$$

$\therefore$ $$(n_1 + n_2)\,\sigma^2 = n_1\sigma_1^2 + n_2\sigma_2^2 + n_1 D_1^2 + n_2 D_2^2$$

This result can be extended to the combination of any number of samples, giving a result of the form

$$(\Sigma n_i)\,\sigma^2 = \Sigma(n_i\sigma_i^2) + \Sigma(n_i D_i^2).$$

Example. 25.7. *Calculate the mean and standard deviation for the following :*

Size of item :	6	7	8	9	10	11	12
Frequency :	3	6	9	13	8	5	4

(V.T.U., 2001)

Solution. The calculations are arranged as follows :

Size of item x	*Frequency f*	*Deviation* $d = x - 9$	$f \times d$	$f \times d^2$
6	3	−3	−9	27
7	6	−2	−12	24
8	9	−1	−9	9
9	13	0	0	0
10	8	1	8	8
11	5	2	10	20
12	4	3	12	36
	$\Sigma f = 48$		$\Sigma fd = 0$	$\Sigma fd^2 = 124$

$$\therefore \qquad mean = 9 + \frac{\Sigma fd}{\Sigma f} = 9$$

$$Standard\ deviation = \sqrt{\left[\frac{\Sigma f(x-\bar{x})^2}{\Sigma f}\right]} = \sqrt{\left(\frac{\Sigma fd^2}{\Sigma f}\right)} = \sqrt{\left(\frac{124}{48}\right)} = 1.607.$$

Example 24.8. *Calculate the mean and standard deviation of the following frequency distribution :*

Weekly wages is ₹	*No. of men*
4.5—12.5	4
12.5—20.5	24
20.5—28.5	21
28.5—36.5	18
36.5—44.5	5
44.5—52.5	3
52.5—60.5	5
60.5—68.5	8
68.5—76.5	2

Solution. The calculations are arranged in the table below :

Wages class ₹	*Mid value x*	*No. of men f*	*Step deviation* $d' = \frac{x - 32.5}{8}$	fd'	fd'^2
4.5—12.5	8.5	4	−3	−12	36
12.5—20.5	16.5	24	−2	−48	96
20.5—28.5	24.5	21	−1	−21	21
28.5—36.5	32.5	18	0	0	0
36.5—44.5	40.5	5	1	5	5
44.5—52.5	48.5	3	2	6	12
52.5—60.5	56.5	5	3	15	45
60.5—68.5	64.5	8	4	32	128
68.5—76.5	72.5	2	5	10	50
		$\Sigma f = 90$		$\Sigma fd' = -13$	$\Sigma fd'^2 = 393$

$$\therefore \qquad mean\ wage = 32.5 + 8 \times \frac{\Sigma fd'}{\Sigma f} = 32.5 + 8\left(\frac{-13}{90}\right) = ₹\ 31.35$$

$$Standard\ deviation = 8\sqrt{\frac{\Sigma fd'^2}{\Sigma f} - \left(\frac{\Sigma fd'}{\Sigma f}\right)^2} = 8\sqrt{\frac{393}{90} - \left(\frac{-13}{90}\right)^2} = ₹\ 16.64.$$

Example 25.9. *The following are scores of two batsmen A and B in a series of innings :*

A :	*12*	*115*	*6*	*73*	*7*	*19*	*119*	*36*	*84*	*29*
B :	*47*	*12*	*16*	*42*	*4*	*51*	*37*	*48*	*13*	*0*

Who is the better score getter and who is more consistent ? (V.T.U., 2004)

Solution. Let x denote score of A and y that of B.

Taking 51 as the origin, we prepare the following table :

x	$d(= x - 51)$	d^2	y	$\delta(= y - 51)$	δ^2
12	−39	1521	47	−4	16
115	64	4096	12	−39	1521
6	−45	2025	16	−35	1225
73	22	484	42	−9	81
7	−44	1936	4	−47	2209
19	−32	1024	51	0	0
119	68	4624	37	−14	196
36	−15	225	48	−3	9
84	33	1089	13	−38	1444
29	−22	484	0	−51	2601
Total	−10	17508		−240	9302

For A, A.M. $\bar{x} = 51 + \dfrac{\Sigma d}{n} = 51 - \dfrac{10}{10} = 50$

$$\text{S.D. } \sigma_1 = \sqrt{\left\{\frac{\Sigma d^2}{n} - \left(\frac{\Sigma d}{n}\right)^2\right\}} = \sqrt{\{1750.8 - (-1)^2\}} = 41.8$$

$\therefore$ coefficient of variation $= \dfrac{\sigma_1}{\bar{x}} \times 100 = \dfrac{41.8}{50} \times 100 = 83.6\%$

For B, A.M. $\bar{y} = 51 + \dfrac{\Sigma\delta}{n} = 51 - \dfrac{240}{10} = 27$

$$\text{S.D. } \sigma_2 = \sqrt{\left\{\frac{\Sigma \delta^2}{n} - \left(\frac{\Sigma \delta}{n}\right)^2\right\}} = \sqrt{\{930.2 - (-24)^2\}} = 18.8$$

$\therefore$ coefficient of variation $= \dfrac{\sigma^2}{\bar{y}} \times 100 = \dfrac{18.8}{27} \times 100 = 69.6\%$

Since the A.M. of A > A.M. of B, it follows that A is a better score getter (*i.e.*, more efficient) than B.

Since the coefficient of variation of B < the coefficient of variation of A, it means that B is more consistent than A. Thus even though A is a better player, he is less consistent.

Example 25.10. *The numbers examined, the mean weight and S.D. in each group of examination by three medical examiners are given below. Find the mean weight and S.D. of the entire data when grouped together.*

Med. Exam.	*No. Examined*	*Mean Wt. (lbs.)*	*S.D. (lbs.)*
A	*50*	*113*	*6*
B	*60*	*120*	*7*
C	*90*	*115*	*8*

Solution. We have $n_1 = 50,\ \bar{x}_1 = 113,\ \sigma_1 = 6$

$n_2 = 60,\ \bar{x}_2 = 120,\ \sigma_2 = 7$

$n_3 = 90,\ \bar{x}_3 = 115,\ \sigma_3 = 8.$

If $\bar{x}$ is the mean of the entire data,

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2 + n_3\bar{x}_3}{n_1 + n_2 + n_3} = \frac{50 \times 113 + 60 \times 120 + 90 \times 115}{50 + 60 + 90} = \frac{23200}{200} = 116 \text{ lb.}$$

If σ is the S.D. of the entire data,

$$N\sigma^2 = n_1\sigma_1^2 + n_2\sigma_2^2 + n_3\sigma_3^2 + n_1D_1^2 + n_2D_2^2 + n_3D_3^2$$

where $N = n_1 + n_2 + n_3 = 200$, $D_1 = \bar{x}_1 - \bar{x} = -3$, $D_2 = \bar{x}_2 - \bar{x} = 4$ and $D_3 = \bar{x}_3 - \bar{x} = -1$.

$$\therefore \quad 200\sigma^2 = 50 \times 36 + 60 \times 49 + 90 \times 64 + 50 \times 9 + 60 \times 16 + 90 \times 1$$

$$= 1800 + 2940 + 5760 + 450 + 960 + 90$$

$$\sigma^2 = \frac{12000}{200} = 60. \quad \text{Hence} \quad \sigma = \sqrt{60} = 7.746 \text{ lb.}$$

PROBLEMS 25.2

1. The crushing strength of 8 cement concrete experimental blocks, in metric tonnes per sq. cm., was 4.8, 4.2, 5.1, 3.8, 4.4, 4.7, 4.1 and 4.5. Find the mean crushing strength and the standard deviation.
2. Show that the variance of the first n positive integers is $\frac{1}{12}(n^2 - 1)$. *(V.T.U., 2003)*
3. The mean of five items of an observation is 4 and the variance is 5.2. If three of the items are 1, 2 and 6, then find the other two. *(V.T.U., 2002)*
4. For the distribution

x :	5	6	7	8	9	10	11	12	13	14	15
f :	18	15	34	47	68	90	80	62	35	27	11

find the mean, median and lower and upper quartiles, variance and the standard deviation.

5. The following table shows the marks obtained by 100 candidates in an examination. Calculate the mean, median and standard deviation :

Marks obtained :	1—10	11—20	21—30	31—40	41—50	51—60
No. of candidates :	3	16	26	31	16	8

(Osmania, 2003 S ; V.T.U., 2003 S)

6. Compute the quartile deviation and standard deviation for the following :

x :	100—109	110—119	120—129	130—139	140—149	150—159	160—169	170—179
f :	15	44	133	150	125	82	35	16

7. Calculate (*i*) mean deviation about the mean, (*ii*) mean deviation about the median for the following distribution :

Class :	3—4.9	5—6.9	7—8.9	9—10.9	11—12.9	13—14.9	15—16.9
f :	5	8	30	82	45	24	6

(Madras, 2002)

8. Two observers bring the following two sets of data which represent measurements of the same quantity :

I.	105.1	103.4	104.2	104.7	104.8	105.0	104.9
II.	105.3	105.1	104.8	105.2	106.7	102.9	103.1

Calculate the standard deviation in each case. Which set of data is more reliable ? Can the same conclusion be reached by calculating the mean deviation ?

Obs. The smaller the coefficient of variation, the greater is the *reliability* or *consistency* in the data.

9. The heights and weights of the 10 armymen are given below. In which characteristics are they more variable ?

Height in cm.	170	172	168	177	179	171	173	178	173	179
Weight in kg.	75	74	75	76	77	73	76	75	74	75

10. The index number of prices of two articles A and B for six consecutive weeks are given below :

A :	314	326	336	368	404	412
B :	330	331	320	318	321	330

Find which has a more variable price ?

11. The scores of two golfers A and B in 12 rounds are given below. Who is the better player and who is the more consistent player ?

A :	74	75	78	72	78	77	79	81	79	76	72	71
B :	87	84	80	88	89	85	86	82	82	79	86	80

12. The scores obtained by two batsmen A and B in 10 matches are given below :

A :	30	44	66	62	60	34	80	46	20	38
B :	34	46	70	38	55	48	60	34	45	30

Calculating mean, S.D. and coefficient of variation for each batsman, determine who is more efficient and who is more consistent.

13. Find the mean and standard deviation of the following two samples put together :

Sample No.	*Size*	*Mean*	*S.D.*
1	50	158	5.1
2	60	164	4.6

14. A distribution consists of three components with frequencies 200, 250 and 300 having means 25, 10 and 15 and S.Ds. 3, 4 and 5 respectively. Show that the mean of the combined distribution is 16 and its S.D. is 7.2 approximately.

25.9 (1) MOMENTS

The rth moment about the mean $\bar{x}$ of a distribution is denoted by μ_r and is given by

$$\mu_r = \frac{1}{N}\Sigma f_i (x_i - \bar{x})^r \qquad ...(1)$$

The corresponding moment about any point a is defined as

$$\mu_r' = \frac{1}{N}\Sigma f_i (x_i - a)^r \qquad ...(2)$$

In particular, we have $\mu_0 = \mu_0' = 1$ ...(3)

$$\mu_1 = \frac{1}{N}\Sigma f_i (x_i - \bar{x}) = 0;\ \mu_1' = \frac{1}{N}\Sigma f_i (x_i - a) = \bar{x} - a = d, \text{ say} \qquad ...(4)$$

$$\mu_2 = \frac{1}{N}\Sigma f_i (x_i - \bar{x})^2 = \sigma^2. \qquad ...(5)$$

(2) Moments about the mean in terms of moments about any point.

We have

$$\mu_r = \frac{1}{N}\Sigma f_i (x_i - \bar{x})^r = \frac{1}{N}\Sigma f[(x_i - a) - (\bar{x} - a)]^r$$

$$= \frac{1}{N}\Sigma f_i (X_i - d)^r \quad \text{where} \quad X_i = x_i - a,\ d = \bar{x} - a.$$

$$= \frac{1}{N}[\Sigma f_i X_i^r - {}^rC_1 d\Sigma f_i X_i^{r-1} + {}^rC_2 d^2 \Sigma f_i X_i^{r-2} - ...]$$

$$= \mu_r' - {}^rC_1\, d\mu_{r-1}' + {}^rC_2 d^2 \mu_{r-2}' - ...... \qquad ...(6)$$

In particular,

$$\mu_2 = \mu'_2 - \mu'^2_1 \qquad ...(7)$$

$$\mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu'^3_1 \qquad ...(8)$$

$$\mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2\mu'^2_1 - 3\mu'^4_1 \qquad ...(9)$$

These three results should be committed to memory. It should be noted that in each of these relations, the sum of the coefficients of the various terms on the right side is zero. Also each term on the right side is of the same dimension as the term on the left.

25.10 SKEWNESS

Skewness measures the degree of asymmetry or the departure from symmetry. If the frequency curve has a longer 'tail' to the right, *i.e.*, the mean is to the right of the mode [as in Fig. 25.4 (*a*)], then the distribution is said to have *positive skewness*. If the curve is more elongated to the left, then it is said to have *negative skewness* [Fig. 25.4 (*b*)].

The following three measures of skewness deserve mention :

(*i*) *Pearson's* coefficient of skewness* $= \dfrac{\text{mean} - \text{mode}}{\sigma}$

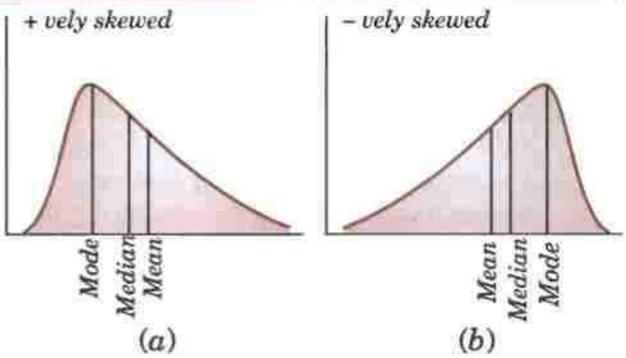


Fig. 25.4

* After the English statistician and biologist *Karl Pearson* (1857–1936) who did pioneering work and found the English school of statistics.

(*ii*) *Quartile coefficient of skewness* $= \dfrac{Q_3 + Q_1 - 2Q_2}{Q_3 - Q_1}$

Its value always lies between – 1 and + 1.

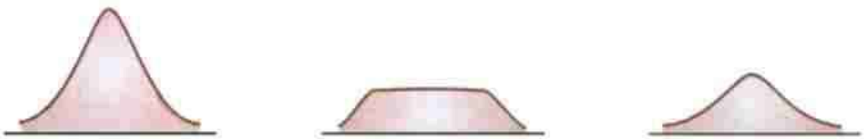
(*iii*) *Coefficient of skewness based on third moment* $\gamma_1 = \sqrt{\beta_1}$.

where $\beta_1 = \mu_3^2/\mu_2^3$

Thus $\gamma_1 = \sqrt{\beta_1}$ gives the simplest measure of *skewness.*

25.11 KURTOSIS

Kurtosis measures the degree of peakedness of a distribution and is given by $\beta_2 = \mu_4/\mu_2^2$.



(*a*) *Leptokurtic* (*b*) *Platykurtic* (*c*) *Mesokurtic*

Fig. 25.5

$\gamma_2 = \beta_2 - 3$ gives the excess of *Kurtosis*. The curves with $\beta_2 > 3$ are called *Leptokurtic* and those with $\beta_2 < 3$ as *Platykurtic*. The normal curve for which $\beta_2 = 3$, is called *Mesokurtic* [Fig. 25.5].

Example 25.11. *The first four moments about the working mean 28.5 of a distribution are 0.294, 7.144, 42.409 and 454.98. Calculate the moments about the mean. Also evaluate β_1, β_2 and comment upon the skewness and kurtosis of the distribution.* (V.T.U., 2... S)

Solution. The first four moments about the arbitrary origin 28.5 are $\mu'_1 = 0.294$, $\mu'_2 = 7.144$, $\mu'_3 = 42.409$, $\mu'_4 = 454.98$.

$$\therefore \quad \mu'_1 = \frac{1}{N}\sum f_i(x_i - 28.5) = \frac{1}{N}\sum f_i x_i - 28.5 = \bar{x} - 28.5 = 0.294 \text{ or } \bar{x} = 28.794$$

$$\mu_2 = \mu'_2 - \mu'^2_1 = 7.144 - (0.294)^2 = 7.058$$

$$\mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu'^3_1 = 42.409 - 3(7.144)(0.294) + 2(0.294)^3 = 36.151.$$

$$\mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2\mu'^2_1 - 3\mu'^4_1$$
$$= 454.98 - 4(42.409) \times (0.294) + 6(7.144)(0.294)^2 - 3(0.294)^4 = 408.738$$

Now $\beta_1 = \mu_3^2/\mu_2^3 = (36.151)^2/(7.058)^3 = 3.717$

$\beta_2 = \mu_4/\mu_2^2 = 408.738/(7.058)^2 = 8.205.$

$\therefore$ $\gamma_1 = \sqrt{\beta_1} = 1.928$, which indicates considerable skewness of the distribution.

$\gamma_2 = \beta_2 - 3 = 5.205$ which shows that the distribution is *leptokurtic.*

Example 25.12. *Calculate the median, quartiles and the quartile coefficient of skewness from the following data :*

Weight (lbs) :	*70–80*	*80–90*	*90–100*	*100–110*	*110–120*	*120–130*	*130–140*	*140–150*
No. of persons :	*12*	*18*	*35*	*42*	*50*	*45*	*20*	*8*

Solution. Here total frequency $N = \Sigma f_i = 230$.

The cummulative frequency table is

Weight (lbs) :	*70–80*	*80–90*	*90–100*	*100–110*	*110–120*	*120–130*	*130–140*	*140–150*
f :	12	18	35	42	50	45	20	8
cum. f. :	12	30	65	107	157	202	222	230

Now $N/2 = 230/2 = 115$th item which lies in 110–120 group.

$$\therefore \quad \text{median or } Q_2 = L + \frac{N/2 - C}{f} \times h = 110 + \frac{115 - 107}{50} \times 10 = 111.6$$

Also $N/4 = 230/4 = 57.5$ *i.e.* Q_1 is 57.5th or 58th item which lies in 90–100 group.

$$\therefore \quad Q_1 = L + \frac{N/4 - C}{f} \times h = 90 + \frac{57.5 - 30}{35} \times 10 = 97.85$$

Similarly, $3N/4 = 172.5$ *i.e.* Q_3 is 173rd item which lies in 120–130 group.

$$\therefore \quad Q_3 = L + \frac{3N/4 - C}{f} \times h = 120 + \frac{172.5 - 157}{45} \times 10 = 123.44$$

Hence quartile coefficient of skewness $= \dfrac{Q_1 + Q_3 - 2Q_2}{Q_3 - Q_1}$

$$= \frac{97.85 + 123.44 - 2 \times 111.6}{123.44 - 97.85} = -0.07 \text{ (approx.)}.$$

PROBLEMS 25.3

1. Calculate the first four moments of the following distribution about the mean :

x :	0	1	2	3	4	5	6	7	8
f :	1	8	28	56	70	56	28	8	1

Also evaluate β_1 and β_2. *(V.T.U., 2004 ; Madras, 2003)*

2. The following table gives the monthly wages of 72 workers in a factory. Compute the standard deviation, quartile deviation, coefficients of variation and skewness. *(V.T.U., 2001)*

Monthly wages (in ₹)	*No. of workers*	*Monthly wages (in ₹)*	*No. of workers*
12.5—17.5	2	37.5—42.5	4
17.5—22.5	22	42.5—47.5	6
22.5—27.5	19	47.5—52.5	1
27.5—32.5	14	52.5—57.5	1
32.5—37.5	3		

3. Find Pearson's coefficient of skewness for the following data :

Class :	10–19	20–29	30–39	40–49	50–59	60–69	70–79	80–89
Frequency :	5	9	14	20	25	15	8	4

(V.T.U., 2000 S)

4. Compute the quartile coefficient of skewness for the following distribution :

x :	3–7	8–12	13–17	18–22	23–27	28–32	33–37	38–42
f :	2	108	580	175	80	32	18	5

(Madras, 2002 ; V.T.U., 2000)

Also compute the measure of skewness based on the third moment.

5. The first three moments of a distribution about the value 2 of the variable are 1, 16 and -40. Show that the mean $= 3$, the variance $= 15$ and $\mu_3 = -86$. *(V.T.U., 2003 S)*
6. Compute skewness and kurtosis, if the first four moments of a frequency distribution $f(x)$ about the value $x = 4$ are respectively 1, 4, 10 and 45. *(Coimbatore, 1999)*
7. In a certain distribution, the first four moments about a point are -1.5, 17, -30 and 108. Calculate the moments about the mean, β_1 and β_2 ; and state whether the distribution is leptokurtic or platykurtic ?

25.12 CORRELATION

So far we have confined our attention to the analysis of observations on a single variable. There are, however, many phenomenae where the changes in one variable are related to the changes in the other variable. For instance, the yield of a crop varies with the amount of rainfall, the price of a commodity increases with the reduction in its supply and so on. Such a simultaneous variation, *i.e.* when the changes in one variable are associated or followed by changes in the other, is called *correlation.* Such a data connecting two variables is called *bivariate population.*

If an increase (or decrease) in the values of one variable corresponds to an increase (or decrease) in the other, the correlation is said to be *positive.* If the increase (or decrease) in one corresponds to the decrease (or increase) in the other, the correlation is said to be *negative.* If there is no relationship indicated between the variables, they are said to be *independent or uncorrelated.*

To obtain a measure of relationship between the two variables, we plot their corresponding values on the graph, taking one of the variables along the x-axis and the other along the y-axis. (Fig. 25.6).

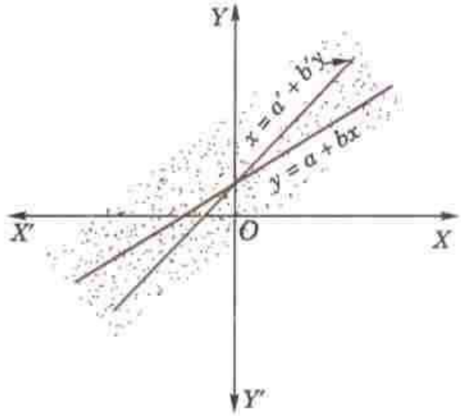


Fig. 25.6

Let the origin be shifted to $(\bar{x}, \bar{y})$, where $\bar{x}, \bar{y}$ are the means of x's and y's that the new co-ordinates are given by

$$X = x - \bar{x}, \qquad Y = y - \bar{y}.$$

Now the points (X, Y) are so distributed over the four quadrants of XY-plane that the product XY is positive in the first and third quadrants but negative in the second and fourth quadrants. The algebraic sum of the products can be taken as describing the trend of the dots in all the quadrants.

$\therefore$ (*i*) If ΣXY is positive, the trend of the dots is through the first and third quadrants,

(*ii*) if ΣXY is negative the trend of the dots is in the second and fourth quadrants, and

(*iii*) if ΣXY is zero, the points indicate no trend *i.e.* the points are evenly distributed over the four quadrants.

The ΣXY or better still $\frac{1}{n}\Sigma XY$, *i.e.*, the average of n products may be taken as a measure of correlation. If we put X and Y in their units, *i.e.*, taking σ_x as the unit for x and σ_y for y, then

$$\frac{1}{n}\Sigma\frac{X}{\sigma_x}\cdot\frac{Y}{\sigma_y}, \text{ i.e., } \frac{\Sigma XY}{n\sigma_x\sigma_y}$$

is the *measure of correlation.*

25.13 COEFFICIENT OF CORRELATION

The numerical measure of correlation is called the *coefficient of correlation* and is defined by the relation

$$r = \frac{\Sigma XY}{n\sigma_x\sigma_y}$$

where X = deviation from the mean $\bar{x} = x - \bar{x}$, Y = deviation from the mean $\bar{y} = y - \bar{y}$,
σ_x = S.D. of x-series, σ_y = S.D. of y-series and n = number of values of the two variables.

Methods of calculation :

(*a*) *Direct method.* Substituting the value of σ_x and σ_y in the above formula, we get

$$r = \frac{\Sigma XY}{\sqrt{(\Sigma X^2 \Sigma Y^2)}} \qquad \text{...(1)}$$

Another form of the formula (1) which is quite handy for calculation is

$$r = \frac{n\Sigma xy - \Sigma x\Sigma y}{\sqrt{[\{n\Sigma x^2 - (\Sigma x)^2\} \times \{n\Sigma y^2 - (\Sigma y)^2\}]}} \qquad \text{...(2)}$$

(*b*) *Step-deviation method.* The direct method becomes very lengthy and tedious if the means of the two series are not integers. In such cases, use is made of assumed means. If d_x and d_y are step-deviations from the assumed means, then

$$r = \frac{n\Sigma d_x d_y - \Sigma d_x \Sigma d_y}{\sqrt{[\{n\Sigma d_x^2 - (\Sigma d_x)^2\} \times \{n\Sigma d_y^2 - (\Sigma d_y)^2\}]}} \qquad \text{...(3)}$$

where $d_x = (x - a)/h$ and $d_y = (y - b)/k$.

Obs. The change of origin and units do not alter the value of the correlation coefficient since r is a pure number.

(*c*) *Co-efficient of correlation for grouped data.* When x and y series are both given as frequency distributions, these can be represented by a two-way table known as the *correlation-table.* It is double-entry table with one series along the horizontal and the other along the vertical as shown on page 848. The co-efficient of correlation for such a *bivariate frequency distribution* is calculated by the formula.

$$r = \frac{n(\Sigma f d_x d_y) - (\Sigma f d_x)(\Sigma f d_y)}{\sqrt{[\{n\Sigma f d_x^2 - (\Sigma f d_x)^2\} \times \{n\Sigma f d_y^2 - (\Sigma f d_y)^2\}]}} \quad ...(4)$$

where d_x = deviation of the central values from the assumed mean of x-series,
d_y = deviation of the central values from the assumed mean of y-series,
f is the frequency corresponding to the pair (x, y)
and $n(= \Sigma f)$ is the total number of frequencies.

Example 25.13. *Psychological tests of intelligence and of engineering ability were applied to 10 students. Here is a record of ungrouped data showing intelligence ratio (I.R.) and engineering ratio (E.R.). Calculate the co-efficient of correlation.*

Student	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*
I.R.	105	104	102	101	100	99	98	96	93	92
E.R.	101	103	100	98	95	96	104	92	97	94

(*Andhra, 2000*)

Solution. We construct the following table :

Student	*Intelligence ratio*		*Engineering ratio*				
	x	$x - \bar{x} = X$	y	$y - \bar{y} = Y$	X^2	Y^2	XY
A	105	6	101	3	36	9	18
B	104	5	103	5	25	25	25
C	102	3	100	2	9	4	6
D	101	2	98	0	4	0	0
E	100	1	95	−3	1	9	−3
F	99	0	96	−2	0	4	0
G	98	−1	104	6	1	36	−6
H	96	−3	92	−6	9	36	18
I	93	−6	97	−1	36	1	6
J	92	−7	94	−4	49	16	28
Total	990	0	980	0	170	140	92

From this table, mean of x, *i.e.*, $\bar{x}$ = 990/10 = 99 and mean of y, *i.e.* $\bar{y}$ = 980/10 = 98.

$$\Sigma X^2 = 170, \Sigma Y^2 = 140 \text{ and } \Sigma XY = 92.$$

Substituting these values in the formula (1) p. 744, we have

$$r = \frac{\Sigma XY}{\sqrt{(\Sigma X^2 \Sigma Y^2)}} = \frac{92}{\sqrt{(170 \times 140)}} = 92/154.3 = 0.59.$$

Example 25.14. *The correlation table given below shows that the ages of husband and wife of 53 married couples living together on the census night of 1991. Calculate the coefficient of correlation between the age of the husband and that of the wife.* (*J.N.T.U., 2003*)

Age of husband	*Age of wife* 15–25	25–35	35–45	45–55	55–65	65–75	*Total*
15–25	1	1	–	–	–	–	2
25–35	2	12	1	–	–	–	15
35–45	–	4	10	1	–	–	15
45–55	–	–	3	6	1	–	10
55–65	–	–	–	2	4	2	8
65–75	–	–	–	–	1	2	3
Total	3	17	14	9	6	4	53

Solution.

Age of husband				*Age of wife x-series*							*Suppose* $d_x = \frac{x-40}{10}$, $d_y = \frac{y-40}{10}$		
				15–25	25–35	35–45	45–55	55–65	65–75				
Years			*Mid pt. x*	20	30	40	50	60	70	*Total f*			
				– 20	– 10	0	10	20	30				
Age group	*Mid pt. y*		d_y \ d_x	–2	–1	0	1	2	3		fd_y	fd_y^2	fd_xd_y
15–25	20	–20	–2	**4** 1	**2** 1					2	–4	8	**6**
25–35	30	–10	–1	**4** 2	**12** 12	**0** 1				15	–15	15	**16**
35–45	40	0	0		**0** 4	**0** 10	**0** 1			15	0	0	**0**
45–55	50	10	1			**0** 3	**6** 6	**2** 1		10	10	10	**8**
55–65	60	20	2				**4** 2	**16** 4	**12** 2	8	16	32	**32**
65–75	70	30	3					**6** 1	**18** 2	3	9	27	**24**
Total f				3	17	14	9	6	4	53 = n	16	92	**86**
fd_x				– 6	–17	0	9	12	12	10	Thick figures in small sqs. stand for fd_xd_y		
fd_x^2				12	17	0	9	24	36	98			
fd_xd_y				**8**	**14**	**0**	**10**	**24**	**30**	**86**	**Check :** $\Sigma fd_xd_y = 86$ from both sides		

With the help of the above correlation table, we have

$$r = \frac{n(\Sigma fd_xd_y) - (\Sigma fd_x)(\Sigma fd_y)}{\sqrt{[\{n\Sigma fd_x^2 - (\Sigma fd_x)^2\} \times [\{n\Sigma fd_y^2 - (\Sigma fd_y)^2\}]}}$$

$$= \frac{53 \times 86 - 10 \times 16}{\sqrt{[(53 \times 98 - 100) \times (53 \times 92 - 256)]}} = \frac{4398}{\sqrt{(5094 \times 4620)}} = \frac{4398}{4850} = 0.91 \text{ (approx.)}.$$

25.14 LINES OF REGRESSION

It frequently happens that the dots of the scatter diagram generally, tend to cluster along a well defined direction which suggests a linear relationship between the variables x and y. Such a line of best-fit for the given distribution of dots is called the *line of regression* (Fig. 25.6). In fact there are two such lines, one giving the best possible mean values of y for each specified value of x and the other giving the best possible mean values of x for given values of y. The former is known as the *line of regression of y on x* and the latter as the *line of regression of x on y*.

Consider first the line of regression of y on x. Let the straight line satisfying the general trend of n dots in a scatter diagram be

$$y = a + bx \qquad ...(1)$$

We have to determine the constants a and b so that (1) gives for each value of x, the best estimate for the average value of y in accordance with the *principle of least squares* (page 816), therefore, the normal equations for a and b are

$$\Sigma y = na + b\Sigma x \qquad ...(2)$$

and
$$\Sigma xy = a\Sigma x + b\Sigma x^2 \qquad ...(3)$$

(2) gives $\quad \frac{1}{n}\Sigma y = a + b \cdot \frac{1}{n}\Sigma x \;$ *i.e.*, $\bar{y} = a + b\bar{x}$.

This shows that $(\bar{x}, \bar{y})$, *i.e.*, the means of x and y, lie on (1).

Shifting the origin to $(\bar{x}, \bar{y})$, (3) takes the form

$$\Sigma(x - \bar{x})(y - \bar{y}) = a\Sigma(x - \bar{x}) + b\Sigma(x - \bar{x})^2, \text{ but } a\Sigma(x - \bar{x}) = 0,$$

$$\therefore \quad b = \frac{\Sigma(x - \bar{x})(y - \bar{y})}{\Sigma(x - \bar{x})^2} = \frac{\Sigma XY}{\Sigma X^2} = \frac{\Sigma XY}{n\sigma_x^2} = r\frac{\sigma_y}{\sigma_x} \qquad \left[\because\ r = \frac{\Sigma XY}{n\sigma_x\sigma_y}\right]$$

Thus the line of best fit becomes $y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})$ $\qquad ...(4)$

which is the equation of *the line of regression of y on x*. Its slope is called the *regression coefficient of y on x*.

Interchanging x and y, we find that the line of regression of x on y is

$$x - \bar{x} = r\frac{\sigma_x}{\sigma_y}(y - \bar{y}) \qquad ...(5)$$

Thus the *regression coefficient of y on x* $= r\sigma_y/\sigma_x$ $\qquad ...(6)$

and the *regression coefficient of x on y* $= r\sigma_x/\sigma_y$ $\qquad ...(7)$

Cor. *The correlation coefficient r is the geometric mean between the two regression co-efficients.*

For $\quad r\frac{\sigma_y}{\sigma_x} \times r\frac{\sigma_x}{\sigma_y} = r^2.$

Example 25.15. *The two regression equations of the variables x and y are $x = 19.13 - 0.87y$ and $y = 11.64 - 0.50x$. Find (i) mean of x's, (ii) mean of y's and (iii) the correlation coefficient between x and y.*

(V.T.U., 2004 ; Anna, 2003 ; Burdwan, 2003)

Solution. Since the mean of x's and the mean of y's lie on the two regression lines, we have

$$\bar{x} = 19.13 - 0.87\bar{y} \qquad ...(i)$$

$$\bar{y} = 11.64 - 0.50\bar{x} \qquad ...(ii)$$

Multiplying (*ii*) by 0.87 and subtracting from (*i*), we have

$$[1 - (0.87)(0.50)]\ \bar{x} = 19.13 - (11.64)(0.87) \text{ or } 0.57\ \bar{x} = 9.00 \text{ or } \bar{x} = 15.79$$

$$\therefore \quad \bar{y} = 11.64 - (0.50)(15.79) = 3.74$$

$\therefore$ regression coefficient of y on x is -0.50 and that of x on y is -0.87.

Now since the coefficient of correlation is the geometric mean between the two regression coefficients.

$$\therefore \quad r = \sqrt{[(-0.50)(-0.87)]} = \sqrt{(0.43)} = -0.66.$$

[– ve sign is taken since both the regression coefficients are – ve]

Example 25.16. *In the following table are recorded data showing the test scores made by salesmen on an intelligence test and their weekly sales :*

Salesmen	1	2	3	4	5	6	7	8	9	10
Test scores	40	70	50	60	80	50	90	40	60	60
Sales (000)	2.5	6.0	4.5	5.0	4.5	2.0	5.5	3.0	4.5	3.0

Calculate the regression line of sales on test scores and estimate the most probable weekly sales volume if a salesman makes a score of 70.

Solution. With the help of the table below, we have

$$\bar{x} = \text{mean of } x \text{ (test scores)} = 60 + 0/10 = 60$$

$$\bar{y} = \text{mean of } y \text{ (sales)} = 4.5 + (-4.5)/10 = 4.05.$$

Regression line of sales (y) on scores (x) is given by

$$y - \bar{y} = r(\sigma_y / \sigma_x)(x - \bar{x})$$

where
$$r\frac{\sigma_y}{\sigma_x} = \frac{\Sigma XY}{\sigma_x \sigma_y} \times \frac{\sigma_y}{\sigma_x} = \frac{\Sigma XY}{(\sigma_x)^2} = \left[\Sigma d_x d_y - \frac{\Sigma d_x \Sigma d_y}{n}\right] \Big/ \left[\Sigma d_x^2 - (\Sigma d_x)^2 / n\right]$$

$$= \frac{140 - \dfrac{0 \times (-4.5)}{10}}{2400 - 0^2/10} = \frac{140}{2400} = 0.06$$

$\therefore$ the required regression line is

$$y - 4.05 = 0.06(x - 60) \quad \text{or} \quad y = 0.06x + 0.45.$$

For $x = 70$, $y = 0.06 \times 70 + 0.45 = 4.65$.

Thus the most probable weekly sales volume for a score of 70 is 4.65.

Test scores x	*Sales* y	*Deviation of x from assumed mean (= 60)* d_x	*Deviation of y from assumed average (= 4.5)* d_y	$d_x \times d_y$	d_x^2	d_y^2
40	2.5	-20	-2	40	400	4
70	6.0	10	1.5	15	100	2.25
50	4.5	-10	0	0	100	0
60	5.0	0	0.5	0	0	2.25
80	4.5	20	0	0	400	0
50	2.0	-10	-2.5	25	100	6.25
90	5.5	30	1	30	900	1.00
40	3.0	-20	-1.5	30	400	2.25
60	4.5	0	0	0	0	0
60	3.0	0	-1.5	0	0	2.25
		$\Sigma d_x = 0$	$\Sigma d_y = -4.5$	$\Sigma d_x d_y = 140$	$\Sigma d_x^2 = 2400$	$\Sigma d_y^2 = 18.25$

Example 25.17. *If θ is the angle between the two regression lines, show that*

$$\tan \theta = \frac{1 - r^2}{r} \cdot \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}$$

Explain the significance when r = 0 and r = ± 1. (U.P.T.U., 2007 ; V.T.U., 2007)

Solution. The equations to the line of regression of y on x and x on y are

$$y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x}) \text{ and } x - \bar{x} = r\frac{\sigma_x}{\sigma_y}(y - \bar{y})$$

$\therefore$ their slopes are $m_1 = r\sigma_y/\sigma_x$ and $m_2 = \sigma_y/r\sigma_x$

Thus
$$\tan \theta = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{\sigma_y / r\sigma_x - r\sigma_y / \sigma_x}{1 + \sigma_y^2 / \sigma_x^2} = \frac{1 - r^2}{r} \cdot \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}$$

When r = 0, tan θ → ∞ or θ = π/2 i.e. when the variables are independent, the two lines of regression are perpendicular to each other.

When r = ± 1, tan θ = 0 i.e., θ = 0 or π. Thus the lines of regression coincide i.e., there is perfect correlation between the two variables.

Example 25.18. *In a partially destroyed laboratory record, only the lines of regression of y on x and x on y are available as $4x - 5y + 33 = 0$ and $20x - 9y = 107$ respectively. Calculate $\bar{x}, \bar{y}$ and the coefficient of co-relation between x and y.* (S.V.T.U., 2009 ; U.P.T.U., 2009 ; V.T.U., 2005)

Solution. Since the regression lines pass through $(\bar{x}, \bar{y})$, therefore,

$$4\bar{x} - 5\bar{y} + 33 = 0, \quad 20\bar{x} - 9\bar{y} = 107.$$

Solving these equations, we get $\bar{x} = 13$, $\bar{y} = 17$.

Rewriting the line of regression of y on x as $y = \frac{4}{5}x + \frac{33}{5}$, we get

$$b_{yx} = r\frac{\sigma_y}{\sigma_x} = \frac{4}{5} \qquad ...(i)$$

Rewriting the line of regression of x on y as $x = \frac{9}{20}y + \frac{107}{9}$, we get

$$b_{xy} = r\frac{\sigma_x}{\sigma_y} = \frac{9}{20} \qquad ...(ii)$$

Multiplying (i) and (ii), we get

$$r^2 = \frac{4}{5} \times \frac{9}{20} = 0.36 \qquad \therefore \qquad r = 0.6$$

Hence $r = 0.6$, the positive sign being taken as b_{yx} and b_{xy} both are positive.

Example 25.19. *Establish the formula* $r = \dfrac{\sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2}{2\sigma_x\sigma_y}$

Hence calculate r from the following data :

x :	21	23	30	54	57	58	72	78	87	90
y :	60	71	72	83	110	84	100	92	113	135

(U.P.T.U., 2002)

Solution. (*a*) Let $z = x - y$ so that $\bar{z} = \bar{x} - \bar{y}$.

$$\therefore \qquad z - \bar{z} = (x - \bar{x}) - (y - \bar{y})$$

or $$(z - \bar{z})^2 = (x - \bar{x})^2 + (y - \bar{y})^2 - 2(x - \bar{x})(y - \bar{y})$$

Summing up for n terms, we have

$$\Sigma(z - \bar{z})^2 = \Sigma(x - \bar{x})^2 + \Sigma(y - \bar{y})^2 - 2\Sigma(x - \bar{x})(y - \bar{y})$$

or $$\frac{\Sigma(z - \bar{z})^2}{n} = \frac{\Sigma(x - \bar{x})^2}{n} + \frac{\Sigma(y - \bar{y})^2}{n} - 2\frac{\Sigma(x - \bar{x})(y - \bar{y})}{n}$$

i.e., $$\sigma_z^2 = \sigma_x^2 + \sigma_y^2 - 2r\sigma_x\sigma_y \qquad \left[\because r = \frac{\Sigma(x - \bar{x})(y - \bar{y})}{n\sigma_x\sigma_y}\right.$$

which is the required result.

(*b*) To fine r, we have to calculate σ_x, σ_y and σ_{x-y}. We make the following table :

x	$X = x - 54$	X^2	y	$Y = y - 100$	Y^2	$y - x$	$(x - y)^2$
21	– 33	1089	60	– 40	1600	39	1521
23	– 31	961	71	– 29	841	48	2304
30	– 24	576	72	– 28	784	42	1764
54	0	0	83	– 17	289	29	841
57	3	9	110	10	100	53	2809
58	4	16	84	– 16	256	26	676
72	18	324	100	0	0	28	784
78	24	576	92	– 8	64	14	196
87	33	1089	113	13	169	26	676
90	36	1296	135	35	1225	45	2025
Total	30	5936		– 80	5328	350	13596

$$\therefore \qquad \sigma_x^2 = \frac{\Sigma X^2}{N} - \left(\frac{\Sigma X}{N}\right)^2 = \frac{5636}{10} - \left(\frac{30}{10}\right)^2 = 593.6 - 9 = 584.6$$

$$\sigma_y^2 = \frac{\Sigma Y^2}{N} - \left(\frac{\Sigma Y}{N}\right)^2 = \frac{5328}{10} - \left(\frac{-80}{10}\right)^2 = 532.8 - 64 = 468.8$$

$$\sigma^2_{x-y} = \frac{\Sigma(x-y)^2}{N} - \left\{\frac{\Sigma(x-y)}{N}\right\}^2 = 1359.6 - 1225 = 134.6$$

From the above formula,

$$r = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2}{2\sigma_x\sigma_y} = \frac{584.6 + 468.8 - 134.6}{2 \times 24.18 \times 23.85} = 0.876.$$

Example 25.20. *While calculating correlation coefficient between two variables x and y from 25 pairs of observations, the following results were obtained : $n = 25$, $\Sigma x = 125$, $\Sigma x^2 = 650$, $\Sigma y = 100$, $\Sigma y^2 = 460$, $\Sigma xy = 508$.*

Later it was discovered at the time of checking that the pairs of values $\begin{array}{c|c} x & y \\ \hline 8 & 12 \\ 6 & 8 \end{array}$ *were copied down as* $\begin{array}{c|c} x & y \\ \hline 6 & 14 \\ 8 & 6 \end{array}$.

Obtain the correct value of correlation coefficient. *(V.T.U., 2011 S ; S.V.T.U., 2009)*

Solution. To get the correct results, we subtract the incorrect values and add the corresponding correct values.

$\therefore$ The correct results would be

$$\Sigma n = 25,\ \Sigma x = 125 - 6 - 8 + 8 + 6 = 125,\ \Sigma x^2 = 650 - 6^2 - 8^2 + 8^2 + 6^2 = 650$$
$$\Sigma y = 100 - 14 - 6 + 12 + 8 = 100,\ \Sigma y^2 = 460 - 14^2 - 6^2 + 12^2 + 8^2 = 436$$
$$\Sigma xy = 508 - 6 \times 14 - 8 \times 6 + 8 \times 12 + 6 \times 8 = 520$$

$$r = \frac{n\Sigma xy - (\Sigma x)(\Sigma y)}{\sqrt{[\{n\Sigma x^2 - (\Sigma x)^2\}\{n\Sigma y^2 - (\Sigma y^2)\}]}} = \frac{25 \times 520 - 125 \times 100}{\sqrt{[\{25 \times 650 - (125)^2\}\{25 \times 436 - (100)^2\}]}}$$

$$= \frac{20}{\sqrt{(25 \times 36)}} = \frac{2}{3}.$$

25.15 STANDARD ERROR OF ESTIMATE

The sum of the squares of the deviations of the points from the line of regression of y on x is

$\Sigma(y - a - bx)^2 = \Sigma(Y - bX)^2$, where $X = x - \bar{x}$, $Y = y - \bar{y}$

$$= \sum\left(Y - r\frac{\sigma_y}{\sigma_x}X\right)^2 = \Sigma Y^2 - 2r(\sigma_y/\sigma_x)\,\Sigma XY + r^2(\sigma_y^2/\sigma_x^2)\ \Sigma X^2$$

$$= n\sigma_y^2 - 2r(\sigma_y/\sigma_x)\,r\,.\,n\sigma_x\sigma_y + r^2(\sigma_y^2/\sigma_x^2)\,.\,n\sigma_x^2 = n\sigma_y^2(1 - r^2).$$

Denoting this sum of squares by nS_y^2, we have $S_y = \sigma_y\sqrt{(1 - r^2)}$ ...(1)

Since S_y is the root mean square deviation of the points from the regression line of y on x, it is called the *standard error of estimate* of y. Similarly the standard error of estimate of x is given by

$$S_x = \sigma_x\sqrt{(1 - r^2)} \qquad ...(2)$$

Since the sum of the squares of deviations cannot be negative, it follows that

$$r^2 \le 1 \quad \text{or} \quad -1 \le r \le 1.$$

i.e., correlation coefficient lies between – 1 and 1. *(J.N.T.U., 2006)*

If $r = 1$ or -1, the sum of the squares of deviations from either line of regression is zero. Consequently each deviation is zero and all the points lie on both the lines of regression. These two lines coincide and we say that the correlation between the variables is *perfect*. The nearer r^2 is to unity the closer are the points to the lines of regression. Thus *the departure of r^2 from unity is a measure of departure from linearity of the relationship between the variables.*

25.16 RANK CORRELATION

A group of n individuals may be arranged in order to merit with respect to some characteristic. The same group would give different orders for different characteristics. Considering the orders corresponding to two characteristics A and B, the correlation between these n pairs of ranks is called the *rank correlation* in the characteristics A and B for that group of individuals.

Let x_i, y_i be the ranks of the ith individuals in A and B respectively. Assuming that no two individuals are bracketed equal in either case, each of the variables taking the values 1, 2, 3, ..., n, we have

$$\bar{x} = \bar{y} = \frac{1+2+3+\ldots+n}{n} = \frac{n(n+1)}{2n} = \frac{n+1}{2}$$

If X, Y be the deviations of x, y from their means, then

$$\Sigma X_i^2 = \Sigma(x_i - \bar{x})^2 = \Sigma x_i^2 + n(\bar{x})^2 - 2\bar{x}\,\Sigma x_i = \Sigma n^2 + \frac{n(n+1)^2}{4} - 2\frac{n+1}{2}\,.\,\Sigma n$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)^2}{4} - \frac{n(n+1)^2}{2} = \frac{1}{12}(n^3 - n)$$

Similarly $\quad \Sigma Y_i^2 = \frac{1}{12}(n^3 - n)$

Now let $\quad d_i = x_i - y_i \quad$ so that $\quad d_i = (x_i - \bar{x}) - (y_i - \bar{y}) = X_i - Y_i$

$\therefore \quad \Sigma d_i^2 = \Sigma X_i^2 + \Sigma Y_i^2 - 2\Sigma X_i Y_i$

or $\quad \Sigma X_i Y_i = \frac{1}{2}(\Sigma X_i^2 + \Sigma Y_i^2 - \Sigma d_i^2) = \frac{1}{12}(n^3 - n) - \frac{1}{2}\Sigma d_i^2.$

Hence the correlation coefficient between these variables is

$$r = \frac{\Sigma X_i Y_i}{\sqrt{(\Sigma X_i^2 \Sigma Y_i^2)}} = \frac{\frac{1}{12}(n^3 - n) - \frac{1}{2}\Sigma d_i^2}{\frac{1}{12}(n^3 - n)} = 1 - \frac{6\,\Sigma d_i^2}{n^3 - n}$$

This is called the *rank correlation coefficient* and is denoted by ρ.

Example 25.21. *Ten participants in a contest are ranked by two judges as follows :*

x :	1	6	5	10	3	2	4	9	7	8
y :	6	4	9	8	1	2	3	10	5	7

Calculate the rank correlation coefficient ρ. (V.T.U., 2002)

Solution. If $\quad d_i = x_i - y_i$, then $d_i = -5, 2, -4, 2, 2, 0, 1, -1, 2, 1$

$\therefore \quad \Sigma d_i^2 = 25 + 4 + 16 + 4 + 4 + 0 + 1 + 1 + 4 + 1 = 60$

Hence $\quad \rho = 1 - \frac{6\Sigma d_i^2}{n^3 - n} = 1 - \frac{6 \times 60}{990} = 0.6$ nearly.

Example 25.22. *Three judges, A, B, C, give the following ranks. Find which pair of judges has common approach*

A :	1	6	5	10	3	2	4	9	7	8
B :	3	5	8	4	7	10	2	1	6	9
C :	6	4	9	8	1	2	3	10	5	7

(J.N.T.U., 2003)

Solution. Here $n = 10$.

	Ranks by		d_1	d_2	d_3	d_1^2	d_2^2	d_3^2
$A\ (= x)$	$B\ (= y)$	$C\ (= z)$	$x - y$	$y - z$	$z - x$			
1	3	6	−2	−3	5	4	9	25
6	5	4	1	1	−2	1	1	4
5	8	9	−3	−1	4	9	1	16
10	4	8	6	−4	−2	36	16	4
3	7	1	−4	6	−2	16	36	4
2	10	2	−8	8	0	64	64	0
4	2	3	2	−1	−1	4	1	1
9	1	10	8	−9	1	64	81	1
7	6	5	1	1	−2	1	1	4
8	9	7	−1	2	−1	1	4	1
Total			0	0	0	200	214	60

$$\therefore \qquad \rho(x, y) = 1 - \frac{6\Sigma d_1^2}{n(n^2 - 1)} = 1 - \frac{6 \times 200}{10 \times 99} = -0.2$$

$$\rho(y, z) = 1 - \frac{6\Sigma d_2^2}{n(n^2 - 1)} = 1 - \frac{6 \times 214}{10 \times 99} = -0.3$$

$$\rho(z, x) = 1 - \frac{6\Sigma d_3^2}{n(n - 1)} = 1 - \frac{6 \times 60}{10 \times 99} = 0.6$$

Since $\rho(z, x)$ is maximum, the pair of judges A and C have the nearest common approach.

PROBLEMS 25.4

1. Find the correlation co-efficient and the regression lines of y and x and x on y for the following data :

x :	1	2	3	4	5
y :	2	5	3	8	7

(*V.T.U., 2010*)

2. Find the correlation coefficient between x and y from the given data :

x :	78	89	97	69	59	79	68	57
y :	125	137	156	112	107	138	123	108

(*J.N.T.U., 2005*)

3. Find the co-efficient of correlation between industrial production and export using the following data and comment on the result.

Production (in crore tons) :	55	56	58	59	60	60	62
Exports (in crore tons) :	35	38	38	39	44	43	45

(*Madras, 2000*)

4. Ten people of various heights as under, were requested to read the letters on a car at 25 yards distance. The number of letters correctly read is given below :

Height (in feet) :	5.1	5.3	5.6	5.7	5.8	5.9	5.10	5.11	6.0	6.1
No. of letters :	11	17	19	14	8	15	20	6	8	12

Is there any correlation between heights and visual power ?

5. Using the formula $r = \dfrac{\sigma_x^{\ 2} + \sigma_y^{\ 2} - \sigma_{x-y}^{\ \ 2}}{2\sigma_x\sigma_y}$, find r from the following data :

x :	92	89	87	86	83	77	71	63	53	50
y :	86	88	91	77	68	85	52	82	37	57

6. Find the correlation between x (marks in Mathematics) and y (marks in Engineering Drawing) given in the following data :

y \ x	10—40	40—70	70—100	Total
0—30	5	20	—	25
30—60	—	28	2	30
60—90	—	32	13	45
Total	5	80	15	100

7. Find two lines of regression and coefficient of correlation for the data given below :

$n = 18$, $\Sigma x = 12$, $\Sigma y = 18$, $\Sigma x^2 = 60$, $\Sigma y^2 = 96$, $\Sigma xy = 48$. *(U.P.T.U., MCA, 2009)*

8. If the coefficient of correlation between two variables x and y is 0.5 and the acute angle between their lines of regression is $\tan^{-1}(3/8)$, show that $\sigma_x = \frac{1}{2}\sigma_y$. *(V.T.U., 2004)*

9. For two random variables x and y with the same mean, the two regression lines are $y = ax + b$ and $x = \alpha y + \beta$. Show that $\frac{b}{\beta} = \frac{1-a}{1-\alpha}$. Find also the common mean. *(U.P.T.U., 2010)*

10. Two random variables have the regression lines with equations $3x + 2y = 26$ and $6x + y = 31$. Find the mean values and the correlation coefficient between x and y. *(Madras, 2002)*

11. The regression equations of two variables x and y are $x = 0.7y + 5.2$, $y = 0.3x + 2.8$. Find the means of the variables and the coefficient of correlation between them. *(Osmania, 2002)*

12. In a partially destroyed laboratory data, only the equations giving the two lines of regression of y on x and x on y are available and are respectively, $7x - 16y + 9 = 0$, $5y - 4x - 3 = 0$. Calculate the co-efficient of correlation, $\bar{x}$ and $\bar{y}$.

13. The following results were obtained from records of age (x) and blood pressure (y) of a group of 10 men :

$$\left.\begin{array}{lcc} & x & y \\ \textit{Mean} & 53 & 142 \\ \textit{Variance} & 130 & 165 \end{array}\right\} \quad \text{and} \quad \Sigma(x-\bar{x})(y-\bar{y}) = 1220.$$

Find the appropriate regression equation and use it to estimate the blood pressure of a man whose age is 45.

14. Compute the standard error of estimate S_x for the respective heights of the following 12 couples :

Height x of husband (inches) :	68	66	68	65	69	66	68	65	71	67	68	70
Height y of wife (inches) :	65	63	67	64	68	62	70	66	68	67	69	71

15. Calculate the rank correlation coefficient from the following data showing ranks of 10 students in two subjects :

Maths :	3	8	9	2	7	10	4	6	1	5
Physics :	5	9	10	1	8	7	3	4	2	6

16. Find the rank correlation for the following data :

x :	56	42	72	36	63	47	55	49	38	42	68	60
y :	147	125	160	118	149	128	150	145	115	140	152	155

(S.V.T.U., 2009 ; J.N.T.U., 2003)

25.17 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 25.5

Select the correct answer or fill up the blanks in each of the following questions :

1. The median of the numbers 11, 10, 12, 13, 9 is

(a) 12.5 (b) 12 (c) 10.5 (d) 11.

2. The mode of the numbers 7, 7, 7, 9, 10, 11, 11, 11, 12 is

(a) 11 (b) 12 (c) 7 (d) 7 and 11.

3. S.D. is defined as

(a) $\sqrt{\dfrac{\Sigma f\,(x-\bar{x})^2}{\Sigma f}}$ (b) $\dfrac{\Sigma f\,(x-\bar{x})}{\Sigma f}$ (c) $\dfrac{\Sigma f\,(x-\bar{x})^2}{\Sigma f}$.

4. Coefficient of variation is

(a) $\dfrac{\sigma}{\bar{x}} \times 100$ (b) $\dfrac{\sigma}{x}$ (c) $\sqrt{\dfrac{\sigma^2}{x}} \times 100$.

5. Average scores of three batsman A, B, C are respectively 40, 45 and 55 and their S.D.s are respectively 9, 11, 16. Which batsman is more consistent ?

(a) A (b) B (c) C.

6. The equations of regression lines are $y = 0.5x + a$ and $x = 0.4y + b$. The correlation coefficient is

(a) $\sqrt{0.2}$ (b) 0.45 (c) $-\sqrt{0.2}$.

7. If the correlation coefficient is 0, the two regression lines are

(a) parallel (b) perpendicular (c) coincident (d) inclined at 45° to each other.

8. If r_1 and r_2 are two regression coefficients, then signs of r_1 and r_2 depend on ....
9. Regression coefficient of y on x is 0.7 and that of x on y is 3.2. Is the correlation coefficient r consistent ?
10. The standard deviation of the numbers 24, 48, 64, 36, 53 is ....
11. If $y = x + 1$ and $x = 3y - 7$ are the two lines of regression then $\bar{x}$ = ..., $\bar{y}$ = ... and r = ...
12. If the two regression lines are perpendicular to each other, then their coefficient of correlation is ....
13. Quartile deviation is defined as ....
14. The minimum value of correlation coefficient is ....
15. Prediction error of Y is defined as ....
16. If X and Y are independent, then the correlation coefficient between X and Y is ....
17. The point of intersection of the two regression lines is ....
18. The smaller the coefficient of variation, the greater is the .... in the data.
19. The moment coefficient of skewness is given by ....
20. Kurtosis measures the .... of a distribution.
21. The equation of the line of regression of y on x is ....
22. Coefficient of variation = ...
23. The angle between two regression lines is given by ....
24. A frequency curve is said to be Mesokurtic when β_2 is ....
25. Correlation coefficient is the geometrical mean between ....
26. When the variables are independent, the two lines of regression are ....
27. Arithmetic mean of the coefficients of regression is .... than the coefficient of correlation.
28. If two regression lines coincide then the coefficient of correlation is ....
29. The rank coefficient is given by ....
30. The ratio of the standard deviation to the mean is known as ....
31. The value of $\Sigma f(x - \bar{x})$ = ...
32. The value of coefficient of correlation lies between .... and ....
33. If the two regression coefficients are -0.4 and -0.9, then the correlation coefficient is ....
34. A distribution with the following constants is positively skew : $Q_1 = 25.8$, median = 49.0, $Q_3 = 64.2$. (True or False)
35. Quartile coefficient of skewness is $\dfrac{Q_3 + Q_1 - 2Q_2}{Q_3 - Q_1}$. (True or False)
36. Skewness indicates peakedness of the frequency distribution. (True or False)

CHAPTER 26

Probability and Distributions

26.1 (1) INTRODUCTION

We often hear such statements : 'It is likely to rain today', 'I have a fair chance of getting admission', and 'There is an even chance that in tossing a coin the head may come up'. In each case, we are not certain of the outcome, but we wish to assess the chances of our predictions coming true. The study of probability provides a mathematical framework for such assertions and is essential in every decision making process. Before defining probability, let us explain a few terms :

(2) Principle of counting. If an event can happen in n_1 ways and thereafter for each of these events a second event can happen in n_2 ways, and for each of these first and second events a third event can happen for n_3 ways and so on, then the number of ways these m event can happen is given by the product $n_1 \cdot n_2 \cdot n_3 \ldots n_m$.

(3) Permutations. A permutation of a number of objects is their arrangement in some definite order. Given three letters a, b, c, we can permute them two at a time as "bc, cb ; ca, ac; ab, ba" yielding 6 permutations. The combinations or groupings are only 3, *i.e.*, bc, ca, ab. Here the order is immetrial.

The number of permutations of n different thing taken r at a time is

$n(n-1)(n-2) \ldots (n-r+1)$, which is denoted by ${}^nP_r$.

Thus $$ {}^nP_r = n(n-1)(n-2) \ldots (n-r+1) = \frac{n!}{(n-r)!}$$

Permutations with repetitions. The number of permutations of n objects of which n_1 are alike, n_2 are alike and n_3 are alike is $\dfrac{n!}{n_1!\,n_2!\,n_3!}$.

(4) Combinations. *The number of combinations of n different objects taken r at a time is denoted by* ${}^nC_r$. If we take any one of the combinations, its r objects can be arranged in $r!$ ways. So the total number of arrangements which can be obtained from all the combinations is ${}^nP_r = {}^nC_r \cdot r!$.

Thus $$ {}^nC_r = \frac{{}^nP_r}{r!} = \frac{n!}{r!\,(n-r)!}$$

Also $$ {}^nC_{n-r} = {}^nC_r$$

e.g., $$ {}^{25}P_4 = 25 \times 24 \times 23 \times 22; \; {}^{25}C_{21} = {}^{25}C_4 = \frac{25 \times 24 \times 23 \times 22}{4 \times 3 \times 2 \times 1}.$$

Example 26.1. *In how many ways can one make a first, second, third and fourth choice among 12 firms leasing construction equipment.* (J.N.T.U., 2003)

Solution. First choice can be made from any of the 12 firms. Thereafter the second choice can be made from among the remaining 11 firms. Then the third choice can be made from the remaining 10 firms and the fourth choice can be made from the 9 firms.

Thus from the *principle of counting*, the number of ways in which first, second, third and fourth choice can be affected $= 12 \times 11 \times 10 \times 9 = 11880$.

Example 26.2. *Find the number of permutations of all the letters of the word (i) Committee (ii) Engineering.*

Solution. (*i*) $n = 9, n_1\,(m, m) = 2, n_2\,(t, t) = 2, n_3\,(e, e) = 2$

$\therefore$ no. of permutations $= \dfrac{n!}{n_1!\,.\,n_2!\,.\,n_3!} = \dfrac{9!}{2!\,.\,2!\,.\,2!} = 45360.$

(*ii*) $n = 11, n_1(e\text{'s}) = 3, n_2\,(g, g) = 2, n_3\,(i, i) = 2, n_4\,(n\text{'s}) = 3$

$\therefore$ no. of permutations $= \dfrac{11!}{3!\,2!\,2!\,3!} = 277200.$

Example 26.3. *From six engineers and five architects a committee is to be formed having three engineers and two architects. How many different committees can be formed if (i) there is no restriction. (ii) two particular engineers must be included. (iii) one particular architect must be excluded.*

Solution. (*i*) Number of committees ${}^6C_3 \times {}^5C_2 = \dfrac{6\,.\,5\,.\,4}{3\,.\,2\,.\,1} \times \dfrac{5\,.\,4}{2\,.\,1} = 200.$

(*ii*) Here we have to choose one engineer from the remaining four engineers.

$\therefore$ no. of committees $= {}^4C_1 \times {}^5C_2 = 4 \times \dfrac{5\,.\,4}{2\,.\,1} = 40$

(*iii*) Here we have to choose two architects from the remaining four architects.

$\therefore$ no. of committees $= {}^6C_3 \times {}^4C_2 = \dfrac{6\,.\,5\,.\,4}{3\,.\,2\,.\,1} \times \dfrac{4\,.\,3}{2\,.\,1} = 120.$

PROBLEMS 26.1

1. If a test consists of 12 true-false questions, in how many different ways can a student make the test paper with one answer to each question. (J.N.T.U., 2003)
2. How many 4-digit numbers can be formed from the six digits 2, 3, 5, 6, 7 and 9, without repetition ? How many of these are less than 500 ?
3. A student has to answer 9 out of 12 questions. How many choices has he (*i*) if he must answer first two questions (*ii*) if he must answer at least four of the first five questions.
4. How many car number plates can be made if each plate contains two different letters followed by three different digits ? Solve the problem (*a*) with repetitions and (*b*) without repetitions.

26.2 (I) BASIC TERMINOLOGY

(*i*) **Exhaustive events.** A set of events is said to be *exhaustive*, if it includes all the possible events. For example, in tossing a coin there are two exhaustive cases either head or tail and there is no third possibility.

(*ii*) **Mutually exclusive events.** If the occurrence of one of the events procludes the occurrence of all other, then such a set of events is said to be *mutually exclusive*. Just as tossing a coin, either head comes up or the tail and both can't happen at the same time, *i.e.*, these are two mutually exclusive cases.

(*iii*) **Equally likely events.** If one of the events cannot be expected to happen in preference to another then such events are said to be *equally likely*. For instance, in tossing a coin, the coming of the head or the tail is equally likely.

Thus when a *die** is thrown, the turning up of the six different faces of the die are exhaustive, mutually exclusive and equally likely.

(*iv*) **Odds in favour of an event.** If the number of ways favourable to an event A is m and the number of ways not favourable to A is n then *odds in favour of* $A = m/n$ and *odds against* $A = n/m$.

(2) Definition of probability. *If there are n exhaustive, mutually exclusive and equally likely cases of which m are favourable to an event A, then probability (p) of the happening of A is*

$$\mathbf{P(A) = m/n.}$$

As there are $n - m$ cases in which A will not happen (denoted by A'), *the chance of A not happening is q or P (A′) so that*

$$q = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - p$$

i.e., $P(A') = 1 - P(A)$ so that $\mathbf{P(A) + P(A') = 1}$,

i.e., if an event is certain to happen then its probability is unity, while if it is certain not to happen, its probability is zero.

> **Obs.** This definitions of probability fails when
> (*i*) number of outcomes is infinite (not exhaustive) and (*ii*) outcomes are not equally likely.

(3) Statistical (or Empirical) definition of probability. *If in n trials, an event A happens m times, then the probability (p) of happening of A is given by*

$$p = P(A) = \underset{n \to \infty}{Lt} \frac{m}{n}$$

Example 26.4. *Find the chance of throwing (a) four, (b) an even number with an ordinary six faced die.*

Solution. (*a*) There are six possible ways in which the die can fall and of these there is only one way of throwing 4. Thus the required chance $= \frac{1}{6}$.

(*b*) There are six possible ways in which the die can fall. Of these there are only 3 ways of getting 2, 4 or 6. Thus the required chance $= 3/6 = \frac{1}{2}$.

Example 26.5. *What is the chance that a leap year selected at random will contain 53 Sundays ?* (*Madras, 2003*)

Solution. A leap year consists of 366 days, so that there are 52 full weeks (and hence 52 Sundays) and two extra days. These two days can be (*i*) Monday, Tuesday (*ii*) Tuesday, Wednesday, (*iii*) Wednesday, Thursday (*iv*) Thursday, Friday (*v*) Friday, Saturday (*vi*) Saturday, Sunday (*vii*) Sunday, Monday.

Of these 7 cases, the last two are favourable and hence the required probability $= \frac{2}{7}$.

Example 26.6. *A five figure number is formed by the digits 0, 1, 2, 3, 4 without repetition. Find the probability that the number formed is divisible by 4.*

Solution. The five digits can be arranged in 5! ways, out of which 4! will begin with zero.

$\therefore$ total number of 5-figure numbers formed $= 5! - 4! = 96$.

Those numbers formed will be divisible by 4 which will have two extreme right digits divisible by 4, *i.e.*, numbers ending in 04, 12, 20, 24, 32, 40.

Now numbers ending in $04 = 3! = 6$, numbers ending in $12 = 3! - 2! = 4$,
numbers ending in $20 = 3! = 6$, numbers ending in $24 = 3! - 2! = 4$,
numbers ending in $32 = 3! - 2! = 4$, and numbers ending in $40 = 3! = 6$.

[The numbers having 12, 24, 32 in the extreme right are $(3! - 2!)$ since the numbers having zero on the extreme left are to excluded.]

---

* *Die* is a small cube. Dots 1, 2, 3, 4, 5, 6 are marked on its six faces. The outcome of throwing a die is the number of dots on its upper face.

$\therefore$ total number of favourable ways = 6 + 4 + 6 + 4 + 4 + 6 = 30.

Hence the required probability $= \frac{30}{96} = \frac{5}{16}$.

Example 26.7. *A bag contains 40 tickets numbered 1, 2, 3, ... 40, of which four are drawn at random and arranged in ascending order ($t_1 < t_2 < t_3 < t_4$). Find the probability of t_3 being 25?*

Solution. Here exhaustive number of cases $= {}^{40}C_4$

If $t_3 = 25$, then the tickets t_1 and t_2 must come out of 24 tickets numbered 1 to 24. This can be done in ${}^{24}C_2$ ways.

Then t_4 must come out of the 15 tickets (numbering 25 to 40) which can be done in ${}^{15}C_1$ ways.

$\therefore$ favourable number of cases $= {}^{24}C_2 \times {}^{15}C_1$

Hence the probability of t_3 being 25 $= \frac{{}^{24}C_2 \times {}^{15}C_1}{{}^{40}C_4} = \frac{414}{9139}$.

Example 26.8. *An urn contains 5 red and 10 black balls. Eight of them are placed in another urn. What is the chance that the latter then contains 2 red and 6 black balls?*

Solution. The number of ways in which 8 balls can be drawn out of 15 is ${}^{15}C_8$.

The number of ways of drawing 2 red balls is ${}^5C_2$ and corresponding to each of these ${}^5C_2$ ways of drawing a red ball, there are ${}^{10}C_6$ ways of drawing 6 black balls.

$\therefore$ the total number of ways in which 2 red and 6 black balls can be drawn is ${}^5C_2 \times {}^{10}C_6$.

$\therefore$ the required probability $= \frac{{}^5C_2 \times {}^{10}C_6}{{}^{15}C_8} = \frac{140}{429}$.

Example 26.9. *A committee consists of 9 students two of which are from 1st year, three from 2nd year and four from 3rd year. Three students are to be removed at random. What is the chance that (i) the three students belong to different classes, (ii) two belong to the same class and third to the different class, (iii) the three belong to the same class?* (V.T.U., 2002 S)

Solution. (*i*) The total number of ways of choosing 3 students out of 9 is ${}^9C_3$, *i.e.*, 84.

A student can be removed from 1st year students in 2 ways, from 2nd year in 3 ways and from 3rd year in 4 ways, so that the total number of ways of removing three students, one from each group is 2 × 3 × 4.

Hence the required chance $= \frac{2 \times 3 \times 4}{{}^9C_3} = \frac{24}{84} = \frac{2}{7}$.

(*ii*) The number of ways of removing two from 1st year students and one from others

$$= {}^2C_2 \times {}^7C_1.$$

The number of ways of removing two from 2nd year students and one from others

$$= {}^3C_2 \times {}^6C_1.$$

The number of ways of removing 2 from 3rd year students and one from others

$$= {}^4C_2 \times {}^5C_1.$$

$\therefore$ the total number of ways in which two students of the same class and third from the others may be removed

$$= {}^2C_2 \times {}^7C_1 + {}^3C_2 \times {}^6C_1 + {}^4C_2 \times {}^5C_1 = 7 + 18 + 30 = 55.$$

Hence, the required chance $= \frac{55}{84}$.

(*iii*) Three students can be removed from 2nd year group in ${}^3C_3$, *i.e.* 1 way and from 3rd year group in ${}^4C_3$, *i.e.*, 4 ways.

$\therefore$ the total number of ways in which three students belong to the same class = 1 + 4 = 5.

Hence the required chance $= \frac{5}{84}$.

Example 26.10. *A has one share in a lottery in which there is 1 prize and 2 blanks ; B has three shares in a lottery in which there are 3 prizes and 6 blanks ; compare the probability of A's success to that of B's success.*

Solution. A can draw a ticket in ${}^3C_1 = 3$ ways.

The number of cases in which A can get a prize is clearly 1.

$\therefore$ the probability of A's success $= \dfrac{1}{3}$.

Again B can draw a ticket in ${}^9C_3 = \dfrac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} = 84$ ways.

The number of ways in which B gets all blanks $= {}^6C_3 = \dfrac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20$

$\therefore$ the number of ways of getting a prize $= 84 - 20 = 64$.

Thus the probability of B's success $= 64/84 = 16/21$.

Hence A's probability of success : B's probability of success $= \dfrac{1}{3} : \dfrac{16}{21} = 7 : 16$.

26.3 PROBABILITY AND SET NOTATIONS

(1) Random experiment. Experiments which are performed essentially under the same conditions and whose results cannot be predicted are known as *random experiments. e.g.,* Tossing a coin or rolling a die are random experiments.

(2) Sample space. The set of all possible outcomes of a random experiment is called *sample space* for that experiment and is denoted by S.

The elements of the sample space S are called the *sample points.*

e.g., On tossing a coin, the possible outcomes are the head (H) and the tail (T). Thus $S = \{H, T\}$.

(3) Event. The outcome of a random experiment is called an *event*. Thus every subset of a sample space S is an *event.*

The null set ϕ is also an event and is called an *impossible event. Probability of an impossible event* is zero *i.e.,* $P(\phi) = 0$.

(4) Axioms

(*i*) *The numerical value of probability lies between 0 and 1.*

i.e., for any event A of S, $0 \le P(A) \le 1$.

(*ii*) *The sum of probabilities of all sample events is unity i.e.,* $P(S) = 1$.

(*iii*) *Probability of an event made of two or more sample events is the sum of their probabilities.*

(5) Notations

(*i*) *Probability of happening of events* **A** *or* **B** is written as $P(A + B)$ or $P(A \cup B)$.

(*ii*) *Probability of happening of* **both the events A and B** is written as $P(AB)$ or $P(A \cap B)$.

(*iii*) '*Event A implies* ($\Rightarrow$) *event B*' is expressed as $A \subset B$.

(*iv*) '*Events A and B are mutually exclusive*' is expressed as $A \cap B = \phi$.

(6) *For any two events A and B,*

$$\mathbf{P(A \cap B') = P(A) - P(A \cap B)}$$

Proof. From Fig. 26.1,

$$(A \cap B') \cup (A \cap B) = A$$

$\therefore$ $P[(A \cap B') \cup (A \cap B)] = P(A)$

or $P(A \cap B') + P(A \cap B) = P(A)$

or $P(A \cap B') = P(A) - P(A \cap B)$

Similarly, $P(A' \cap B) = P(B) - P(A \cap B)$

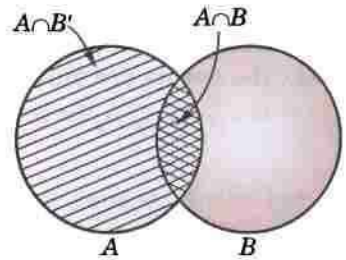


Fig. 26.1

26.4 ADDITION LAW OF PROBABILITY *or* THEOREM OF TOTAL PROBABILITY

(1) *If the probability of an event A happening as a result of a trial is P(A) and the probability of a* **mutually exclusive** *event B happening is P(B), then the probability of* **either** *of the events happening as a result of the trial is* **P(A + B) or P(A ∪ B) = P(A) + P(B).**

Proof. Let n be the total number of equally likely cases and let m_1 be favourable to the event A and m_2 be favourable to the event B. Then the number of cases favourable to A or B is $m_1 + m_2$. Hence the probability of A or B happening as a result of the trial

$$= \frac{m_1 + m_2}{n} = \frac{m_1}{n} + \frac{m_2}{n} = P(A) + P(B).$$

(2) *If A, B, are any two events* **(not mutually exclusive),** *then*

$$\boldsymbol{P(A + B) = P(A) + P(B) - P(AB)}$$

or

$$\boldsymbol{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

If the events A and B are any two events then, there are some outcomes which favour both A and B. If m_3 be their number, then these are included in both m_1 and m_2. Hence the total number of outcomes favouring either A or B or both is

$$m_1 + m_2 - m_3.$$

Thus the probability of occurrence of A or B or both

$$= \frac{m_1 + m_2 - m_3}{n} = \frac{m_1}{n} + \frac{m_2}{n} - \frac{m_3}{n}$$

Hence $\quad P(A + B) = P(A) + P(B) - P(AB)$

or $\quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Obs. *When A and B are mutually exclusive* $P(AB)$ or $P(A \cap B) = 0$ and we get

$$P(A + B) \text{ or } P(A \cup B) = P(A) + P(B).$$

In general, for a number of **mutually exclusive** events $A_1, A_2, \dots A_n$, we have

$$P(A_1 + A_2 + \dots + A_n) \text{ or } P(A_1 \cup A_2 \cup \dots \cup A_n) = P(A_1) + P(A_2) + \dots + P(A_n).$$

(3) *If A, B, C are any three events, then*

$$\boldsymbol{P(A + B + C) = P(A) + P(B) + P(C) - P(AB) - P(BC) - P(CA) + P(ABC).}$$

or

$$\boldsymbol{P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)}$$

Proof. Using the above result for any two events, we have

$$P(A \cup B \cup C) = P[(A \cup B) \cup C]$$
$$= P(A \cup B) + P(C) - P[(A \cup B) \cap C]$$
$$= [P(A) + P(B) - P(A \cap B)] + P(C) - P[(A \cap C) \cup (B \cap C)] \quad \text{(Distributive Law)}$$
$$= P(A) + P(B) + P(C) - P(A \cap B) - \{P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)\}$$
$$[\because (A \cap C) \cap (B \cap C) = A \cap B \cap C]$$
$$= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C) \quad [\because A \cap C = C \cap A.]$$

Example 26.11. *In a race, the odds in favour of the four horses H_1, H_2, H_3, H_4 are 1 : 4, 1 : 5, 1 : 6, 1 : 7 respectively. Assuming that a dead heat is not possible, find the chance that one of them wins the race.*

Solution. Since it is not possible for all the horses to cover the same distance in the same time (a dead heat), the events are mutually exclusive.

If p_1, p_2, p_3, p_4 be the probabilities of winning of the horses H_1, H_2, H_3, H_4 respectively, then

$$p_1 = \frac{1}{1+4} = \frac{1}{5} \quad [\because \text{ Odds in favour of } H_1 \text{ are } 1:4]$$

and

$$p_2 = \frac{1}{6}, p_3 = \frac{1}{7}, p_4 = \frac{1}{8}.$$

Hence the chance that one of them wins $= p_1 + p_2 + p_3 + p_4$

$$= \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} = \frac{533}{840}.$$

Example 26.12. *A bag contains 8 white and 6 red balls. Find the probability of drawing two balls of the same colour.*

Solution. Two balls out of 14 can be drawn in ${}^{14}C_2$ ways which is the total number of outcomes.

Two white balls out of 8 can be drawn in ${}^{8}C_2$ ways. Thus the probability of drawing 2 white balls

$$= \frac{{}^{8}C_2}{{}^{14}C_2} = \frac{28}{91}$$

Similarly 2 red balls out of 6 can be drawn in ${}^{6}C_2$ ways. Thus the probability of drawing 2 red balls

$$= \frac{{}^{6}C_2}{{}^{14}C_2} = \frac{15}{91}.$$

Hence the probability of drawing 2 balls of the same colour (either both white or both red)

$$= \frac{28}{91} + \frac{15}{91} = \frac{43}{91}.$$

Example 26.13. *Find the probability of drawing an ace or a spade or both from a deck of cards* ?*

Solution. The probability of drawing an ace from a deck of 52 cards = 4/52.

Similarly the probability of drawing a card of spades = 13/52, and the probability of drawing an ace of spades = 1/52.

Since the two events (*i.e.*, a card being an ace and a card being of spades) are not mutually exclusive, therefore, the probability of drawing an ace or a spade

$$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{4}{13}.$$

26.5 (1) INDEPENDENT EVENTS

Two events are said to be *independent*, if happening or failure of one does not affect the happening or failure of the other. Otherwise the events are said to be *dependent*.

For two dependent events A and B, the symbol $P(B/A)$ denotes the probability of occurrence of B, when A has already occurred. It is known as the **conditional probability** and is read as a '*probability of B given A*'.

(2) Multiplication law of probability *or* **Theorem of compound probability.** *If the probability of an event A happening as a result of trial is $P(A)$ and after A has happened the probability of an event B happening as a result of another trial* (*i.e.*, **conditional probability of B given A**) *is $P(B/A)$, then the probability of* **both** *the events A and B happening as a result of two trials is* $\mathbf{P(AB)}$ **or** $\mathbf{P(A \cap B) = P(A) \,.\, P(B/A)}$.

Proof. Let n be the total number of outcomes in the first trial and m be favourable to the event A so that $P(A) = m/n$.

Let n_1 be the total number of outcomes in the second trial of which m_1 are favourable to the event B so that $P(B/A) = m_1/n_1$.

Now each of the n outcomes can be associated with each of the n_1 outcomes. So the total number of outcomes in the combined trial is nn_1. Of these mm_1 are favourable to both the events A and B. Hence

$$P(AB) \text{ or } P(A \cap B) = \frac{mm_1}{nn_1} = P(A) \,.\, P(B/A).$$

Similarly, the *conditional probability of A given B is $P(A/B)$.*

$\therefore$ $\quad P(AB)$ or $P(A \cap B) = P(B) \,.\, P(A/B)$

Thus $\quad P(A \cap B) = P(A) \,.\, P(B/A) = P(B) \,.\, P(A/B).$

(3) *If the events A and B are* **independent,** *i.e.*, if the happening of B does not depend on whether A has happened or not, then $P(B/A) = P(B)$ and $P(A/B) = P(A)$.

$\therefore$ $\quad \mathbf{P(AB)}$ **or** $\mathbf{P(A \cap B) = P(A) \,.\, P(B).}$

In general, $\quad P(A_1A_2...A_n)$ or $P(A_1 \cap A_2 \cap ... \cap A_n) = P(A_1) \,.\, P(A_2) ... \,.\, P(A_n).$

* *Cards* : A pack of cards consists of four suits *i.e.*, Hearts, Diamonds, Spades and Clubs. Each suit has 13 cards : an Ace, a King, a Queen, a Jack and nine cards numbered 2, 3, 4, ..., 10. Hearts and Diamonds are *red* while Spades and Clubs are *black*.

Cor. If p_1, p_2 be the probabilities of happening of two independent events, then

(i) the probability that the first event happens and the second fails is $p_1(1 - p_2)$.

(ii) the probability that both events fail to happen is $(1 - p_1)(1 - p_2)$.

(iii) the probability that at least one of the events happens is

$1 - (1 - p_1)(1 - p_2)$. This is commonly known as their **cumulative probability.**

In general, if $p_1, p_2, p_3, \ldots p_n$ be the chances of happening of n independent events, then their cumulative probability (i.e., the chance that at least one of the events will happen) is

$$1 - (1 - p_1)(1 - p_2)(1 - p_3) \ldots (1 - p_n).$$

Example 26.14. *Two cards are drawn in succession from a pack of 52 cards. Find the chance that the first is a king and the second a queen if the first card is (i) replaced, (ii) not replaced.*

Solution. (i) The probability of drawing a king $= \frac{4}{52} = \frac{1}{13}$.

If the card is replaced, the pack will again have 52 cards so that the probability of drawing a queen is 1/13.

The two events being independent, the probability of drawing both cards in succession $= \frac{1}{13} \times \frac{1}{13} = \frac{1}{169}$.

(ii) The probability of drawing a king $= \frac{1}{13}$.

If the card is not replaced, the pack will have 51 cards only so that the chance of drawing a queen is 4/51.

Hence the probability of drawing both cards $= \frac{1}{13} \times \frac{4}{51} = \frac{4}{663}$.

Example 26.15. *A pair of dice is tossed twice. Find the probability of scoring 7 points (a) once, (b) at least once (c) twice.* *(Kurukshetra, 2009 S ; V.T.U., 2004)*

Solution. In a single toss of two dice, the sum 7 can be obtained as (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1) *i.e.*, in 6 ways, so that the probability of getting 7 = 6/36 = 1/6.

Also the probability of not getting 7 = 1 – 1/6 = 5/6.

(a) The probability of getting 7 in the first toss and not getting 7 in the second toss = 1/6 × 5/6 = 5/36.

Similarly, the probability of not getting 7 in the first toss and getting 7 in the second toss = 5/6 × 1/6 = 5/36.

Since these are mutually exclusive events, addition law of probability applies.

$\therefore$ required probability $= \frac{5}{36} + \frac{5}{36} = \frac{5}{18}$.

(b) The probability of not getting 7 in either toss $= \frac{5}{6} \times \frac{5}{6} = \frac{25}{36}$

$\therefore$ the probability of getting 7 at least once $= 1 - \frac{25}{36} = \frac{11}{36}$.

(c) The probability of getting 7 twice $= \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$.

Example 26.16. *There are two groups of objects : one of which consists of 5 science and 3 engineering subjects, and the other consists of 3 science and 5 engineering subjects. An unbiased die is cast. If the number 3 or number 5 turns up, a subject is selected at random from the first group, otherwise the subject is selected at random from the second group. Find the probability that an engineering subject is selected ultimately*

Solution. Prob. of turning up 3 or 5 $= \frac{2}{6} = \frac{1}{3}$.

Prob. of selecting an engg. subject from first group $= \frac{3}{8}$

$\therefore$ Prob of selecting an engg. subject from first group on turning up 3 or 5

$$= \frac{1}{3} \times \frac{3}{8} = \frac{1}{8} \qquad \ldots(i)$$

Now prob. of not turning 3 or 5 $= 1 - \frac{1}{3} = \frac{2}{3}$.

Prob. of selecting an engg. subject from second group $= \frac{5}{8}$

$\therefore$ prob. of selecting an engg. subject from second group on turning up 3 or 5

$$= \frac{2}{3} \times \frac{5}{8} = \frac{5}{12} \qquad ...(ii)$$

Thus the prob. of selecting an engg. subject

$$= \frac{1}{8} + \frac{5}{12} = \frac{13}{24}. \qquad \text{[From } (i) \text{ and } (ii)]$$

Example 26.17. *A box A contains 2 white and 4 black balls. Another box B contains 5 white and 7 black balls. A ball is transferred from the box A to the box B. Then a ball is drawn from the box B. Find the probability that it is white.* (V.T.U., 2004)

Solution. The probability of drawing a white ball from box B will depend on whether the transferred ball is black or white.

If a black ball is transferred, its probability is 4/6. There are now 5 white and 8 black balls in the box B. Then the probability of drawing white ball from box B is $\frac{5}{13}$.

Thus the probability of drawing a white ball from urn B, if the transferred ball is black

$$= \frac{4}{6} \times \frac{5}{13} = \frac{10}{39}.$$

Similarly the probability of drawing a white ball from urn B, if the transferred ball is white

$$= \frac{2}{6} \times \frac{6}{13} = \frac{2}{13}.$$

Hence required probability $= \frac{10}{39} + \frac{2}{13} = \frac{16}{39}$.

Example 26.18. *(a) A biased coin is tossed till a head appears for the first time. What is the probability that the number of required tosses is odd.* (Mumbai, 2006)

(b) Two persons A and B toss an unbiased coin alternately on the understanding that the first who gets the head wins. If A starts the game, find their respective chances of winning. (Madras, 2000 S)

Solution. (*a*) Let p be the probability of getting a head and q the probability of getting a tail in a single toss, so that $p + q = 1$.

Then probability of getting head on an odd toss

= Probability of getting head in the 1st toss
+ Probability of getting head in the 3rd toss
+ Probability of getting head in the 5th toss + ... ∞

$$= p + qqp + qqqqp + \ldots \infty$$

$$= p\,(1 + q^2 + q^4 + \ldots) = p \cdot \frac{1}{1-q^2} \quad (q < 1)$$

$$= p \cdot \frac{1}{(1-q)(1+q)} = p \cdot \frac{1}{p(1+q)} = \frac{1}{1+q}.$$

(*b*) Probability of getting a head = 1/2. Then A can win in 1st, 3rd, 5th, ... throws.

$$\therefore \text{ the chances of } A\text{'s winning } = \frac{1}{2} + \left(\frac{1}{2}\right)^2 \frac{1}{2} + \left(\frac{1}{2}\right)^4 \frac{1}{2} + \left(\frac{1}{2}\right)^6 \frac{1}{2} + \cdots$$

$$= \frac{1/2}{1-(1/2)^2} = \frac{1}{2} \cdot \frac{4}{3} = \frac{2}{3}.$$

Hence the chance of B's winning $= 1 - 2/3 = 1/3$.

Example 26.19. *Two cards are selected at random from 10 cards numbered 1 to 10. Find the probability p that the sum is odd, if*

(i) the two cards are drawn together.

(ii) the two cards are drawn one after the other without replacement.

(iii) the two cards are drawn one after the other with replacement. (J.N.T.U., 2003)

Solution. (*i*) Two cards out of 10 can be selected in ${}^{10}C_2 = 45$ ways. The sum is odd if one number is odd and the other number is even. There being 5 odd numbers (1, 3, 5, , 9) and 5 even numbers (2, 4, 6, 8, 10), an odd and an even number is chosen in $5 \times 5 = 25$ ways.

Thus $$p = \frac{25}{45} = \frac{5}{9}.$$

(*ii*) Two cards out of 10 can be selected one after the other *without replacement* in $10 \times 9 = 90$ ways.

An odd number is selected in $5 \times 5 = 25$ ways and an even number in $5 \times 5 = 25$ ways

Thus $$p = \frac{25 + 25}{90} = \frac{5}{9}.$$

(*iii*) Two cards can be selected one after the other *with replacement* in $10 \times 10 = 100$ ways.

An odd number in selected in $5 \times 5 = 25$ ways and an even number in $5 \times 5 = 25$ ways.

Thus $$p = \frac{25 + 25}{100} = \frac{1}{2}.$$

Example 26.20. *Given $P(A) = 1/4$, $P(B) = 1/3$ and $P(A \cup B) = 1/2$, evaluate $P(A/B)$, $P(B/A)$, $P(A \cap B')$ and $P(A/B')$.*

Solution. (*i*) Since $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\therefore$ $$\frac{1}{2} = \frac{1}{4} + \frac{1}{3} - P(A \cap B) \text{ or } P(A \cap B) = \frac{1}{12}$$

Thus $$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1/12}{1/3} = \frac{1}{4}.$$

(*ii*) $$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{1/12}{1/4} = \frac{1}{3}.$$

(*iii*) $$P(A \cap B') = P(A) - P(A \cap B) = \frac{1}{4} - \frac{1}{12} = \frac{1}{6}.$$

(*iv*) $$P(A/B') = \frac{P(A \cap B')}{P(B')} = \frac{1/6}{1 - P(B)} = \frac{1/6}{1 - 1/3} = \frac{1}{4}.$$

Example 26.21. *The odds that a book will be reviewed favourably by three independent critics are 5 to 2, 4 to 3 and 3 to 4. What is the probability that of the three reviews, a majority will be favourable.* (V.T.U., 2003 S)

Solution. The probability that the book shall be reviewed favourably by first critic is 5/7, by second 4/7 and by third 3/7.

A majority of the three reviews will be favourable when two or three are favourable.

$\therefore$ prob. that the first two are favourable and the third unfavourable

$$= \frac{5}{7} \times \frac{4}{7} \times \left(1 - \frac{3}{7}\right) = \frac{80}{343}$$

Prob. that the first and third are favourable and second unfavourable

$$= \frac{5}{7} \times \frac{3}{7} \times \left(1 - \frac{4}{7}\right) = \frac{45}{343}$$

Prob. that the second and third are favourable and the first unfavourable

$$= \frac{4}{7} \times \frac{3}{7} \times \left(1 - \frac{5}{7}\right) = \frac{24}{343}$$

Finally, prob. that all the three are favourable $= \frac{5}{7} \times \frac{4}{7} \times \frac{3}{7} = \frac{60}{343}$

Since they are mutually exclusive events, the required prob.

$$= \frac{80}{343} + \frac{45}{343} + \frac{24}{343} + \frac{60}{343} = \frac{209}{343}.$$

Example 26.22. *I can hit a target 3 times in 5 shots, B 2 times in 5 shots and C 3 times in 4 shots. They fire a volley. What is the probability that (i) two shots hit, (ii) atleast two shots hit ?*

(A.M.I.E.T.E., 2003 ; Madras, 2000 S)

Solution. Prob. of A hitting the target = 3/5, prob. of B hitting the target = 2/5

Prob. of C hitting the target = 3/4.

(*i*) In order that two shots may hit the target, the following cases must be considered :

$$p_1 = \text{Chance that } A \text{ and } B \text{ hit and } C \text{ fails to hit} = \frac{3}{5} \times \frac{2}{3} \times \left(1 - \frac{3}{4}\right) = \frac{6}{100}$$

$$p_2 = \text{Chance that } B \text{ and } C \text{ hit and } A \text{ fails to hit} = \frac{2}{5} \times \frac{3}{4} \times \left(1 - \frac{3}{5}\right) = \frac{12}{100}$$

$$p_3 = \text{Chance that } C \text{ and } A \text{ hit and } B \text{ fails to hit} = \frac{3}{4} \times \frac{3}{5} \times \left(1 - \frac{2}{5}\right) = \frac{27}{100}$$

Since these are mutually exclusive events, the probability that any 2 shots hit

$$= p_1 + p_2 + p_3 = \frac{6}{100} + \frac{12}{100} + \frac{27}{100} = 0.45.$$

(*ii*) In order that at least two shots may hit the target, we must also consider the case of all A, B, C hitting the target [in addition to the three cases of (*i*)] for which

$$p_4 = \text{chance that } A, B, C \text{ all hit} = \frac{3}{5} \times \frac{2}{5} \times \frac{3}{4} = \frac{18}{100}$$

Since all these are mutually exclusive events, the probability of *atleast* two shots hit

$$= p_1 + p_2 + p_3 + p_4 = \frac{6}{100} + \frac{12}{100} + \frac{27}{100} + \frac{18}{100} = 0.63.$$

Example 26.23. *A problem in mechanics is given to three students A, B, and C whose chances of solving it are $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$ respectively. What is the probability that the problem will be solved.* *(V.T.U., 2004)*

Solution. The probability that A can solve the problem is 1/2.

The probability that A cannot solve the problem is $1 - \frac{1}{2}$.

Similarly the probabilities that B and C cannot solve the problem are $1 - \frac{1}{3}$ and $1 - \frac{1}{4}$.

$\therefore$ the probability that A, B and C cannot solve the problem is $\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right)$.

Hence the probability that the problem will be solved, *i.e.*, at least one student will solve it

$$= 1 - \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right) = \frac{3}{4}.$$

Example 26.24. *The students in a class are selected at random, one after the other, for an examination. Find the probability p that the boys and girls in the class alternate if*

(i) the class consists of 4 boys and 3 girls.

(ii) the class consists of 3 boys and 3 girls. *(J.N.T.U., 2003)*

Solution. (*i*) As there are 7 students in the class, the first examined must be a boy.

$\therefore$ prob. that first is a boy $= \dfrac{4}{7}$

Then the prob. that the second is a girl $= \dfrac{3}{6}\cdot$

$\therefore$ prob. of the next boy $= \dfrac{3}{5}$

Similarly the prob. that the fourth is a girl $= \dfrac{2}{4}$,

the prob. that the fifth is a boy $= \dfrac{2}{3}$,

the prob. that the sixth is a girl $= \dfrac{1}{2}$

and the last is a boy $= \dfrac{1}{1}\cdot$

Thus $$p = \frac{4}{7}\cdot\frac{3}{6}\cdot\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}\cdot\frac{1}{1} = \frac{1}{35}\cdot$$

(*ii*) The first student is a boy and the first student is a girl are two mutually exclusive cases. If the first student is a boy, then the probability p_1 that the students alternate is

$$p_1 = \frac{3}{6}\cdot\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}\cdot\frac{1}{1} = \frac{1}{20}\cdot$$

If the first student is a girl, then the probability p_2 that the students alternate is

$$p_2 = \frac{3}{6}\cdot\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}\cdot\frac{1}{1} = \frac{1}{20}\cdot$$

Thus the required prob. $p = p_1 + p_2 = \dfrac{1}{20} + \dfrac{1}{20} = \dfrac{1}{10}\cdot$

Example 26.25. *(Huyghen's problem) A and B throw alternately with a pair of dice. A wins if he throws 6 before B throws 7 and B wins if he throws 7 before A throws 6. If A begins, find his chance of winning.*

(Madras, 2006 ; J.N.T.U., 2003)

Solution. The sum 6 can be obtained as follows : (1, 5), (2, 4), (3, 3), (4, 2), (5, 1), *i.e.*, in 5 ways.

The probability of A's throwing 6 with 2 dice is $\dfrac{5}{36}\cdot$

$\therefore$ the probability of A's not throwing 6 is 31/36.

Similarly the probability of B's throwing 7 is 6/36, *i.e.*, $\dfrac{1}{6}\cdot$

$\therefore$ the probability of B's not throwing 7 is 5/6.

Now A can win if he throws 6 in the first, third, fifth, seventh etc. throws.

$\therefore$ the chance of A's wining

$$= \frac{5}{36} + \frac{31}{36}\times\frac{5}{6}\times\frac{5}{36} + \frac{31}{36}\times\frac{5}{6}\times\frac{31}{36}\times\frac{5}{6}\times\frac{5}{36} + \cdots$$

$$= \frac{5}{36}\left[1 + \left(\frac{31}{36}\times\frac{5}{6}\right) + \left(\frac{31}{36}\times\frac{5}{6}\right)^2 + \left(\frac{31}{36}\times\frac{5}{6}\right)^3 + \cdots\right]$$

$$= \frac{5}{36}\cdot\frac{1}{1-(31/36)\times(5/6)} = \frac{5}{36}\times\frac{36\times 6}{61} = \frac{30}{61}\cdot$$

PROBLEMS 26.2

1. (*i*) Given $P(A) = 1/2$, $P(B) = 1/3$ and $P(AB) = 1/4$, find the value $P(A + B)$. *(Burdwan, 2003)*

(*ii*) Let A and B be two events with $P(A) = 1/2$, $P(B) = 1/3$ and $P(A \cap B) = 1/4$. Find $P(A/B)$, $P(A \cup B)$, $P(A'/B')$.

(Kurukshetra, 2009 ; V.T.U., 2003 S)

2. In a single throw with two dice, what is the chance of throwing
(*a*) two aces ? (*b*) 7 ? Is this probability the same as that for getting 7 in two throws of a single die ?

3. Compare the chances of throwing 4 with one dice, 8 with two dice and 12 with three dice.

4. Find the probability that a non-leap year should have 53 Saturdays ? (*Madras, 2003*)

5. When a coin is tossed four times, find the probability of getting (*i*) exactly one head, (*ii*) at most three heads and (*iii*) at least two heads ? (*V.T.U., 2000 S*)

6. Ten coins are thrown simultaneously. Find the probability of getting at least seven heads. (*P.T.U., 2003*)

7. If all the letters of word 'ENGINEER' be written at random, what is the probability that all the latters E are found together.

8. A ten digit number is formed using the digits from zero to nine, every digit being used only once. Find the probability that the number is divisible by 4.

9. Four cards are drawn from a pack of 52 cards. What is the chance that
(*i*) no two cards are of equal value ? (*ii*) each belongs to a different suit ?

10. Suppose 5 cards are drawn at random from a pack of 52 cards. If all cards are red what is the probability that all of them are hearts? (*Mumbai, 2005*)

11. Out of 50 rare books, 3 of which are especially valuable, 5 are stolen at random by a thief. What is the probability that
(*a*) none of the 3 is included ? (*b*) 2 of the 3 are included ?

12. Five men in a company of twenty are graduates. If 3 men are picked out of 20 at random, what is the probability that
(*a*) they are all graduates ? (*b*) at least one is graduate ?

13. From 20 tickets marked from 1 to 20, one ticket is drawn at random. Find the probability that it is marked with a multiple of 3 or 5.

14. Five balls are drawn from a bag containing 6 white and 4 black balls. What is the chance that 3 white and 2 black balls are drawn ?

15. The probability of n independent events are $p_1, p_2, p_3, \ldots p_n$. Find the probability that at least one of the events will happen. Use this result to find the chance of getting at least one six in a throw of 4 dice.

16. Find the probability of drawing 4 white balls and 2 black balls without replacement from a bag containing 1 red, 4 black and 6 white balls.

17. A bag contains 10 white and 15 black balls. Two balls are drawn in succession. What is the probability that one of them is black and the other white ?

18. A purse contains 2 silver and 4 copper coins and a second purse contains 4 silver and 4 copper coins. If a coin is selected at random from one of the two purses, what is the probability that it is a silver coin ? (*Osmania, 2002*)

19. A box I contains 5 white balls and 6 black balls. Another box II contains 6 white balls and 4 black balls. A box is selected at random and then a ball is drawn from it : (*i*) what is the probability that the ball drawn will be white ? (*ii*) Given that the ball drawn is white, what is the probability that it came from box I. (*Mumbai, 2006*)

20. A party of n persons take their seats at random at a round table ; find the probability that two specified persons do not sit together.

21. A speaks the truth in 75% cases, and B in 80% of the cases. In what percentage of cases, are they likely to contradict each other in stating the same fact ? (*V.T.U., 2002 S*)

22. The probability that Sushil will solve a problem is 1/4 and the probability that Ram will solve it is 2/3. If Sushil and Ram work independently, what is the probability that the problem will be solved by (*a*) both of them, (*b*) at least one of them ?

23. A student takes his examination in four subjects, P, Q, R, S. He estimates his chances of passing in P as 4/5, in Q as 3/4, in R as 5/6 and in S as 2/3. To qualify, he must pass in P and at least two other subjects. What is the probability that he qualifies ? (*Madras, 2000 S*)

24. The probability that a 50 year old man will be alive at 60 is 0.83 and the probability that a 45 year old women will be alive at 55 is 0.87. What is the probability that a man who is 50 and his wife who is 45 will both be alive 10 years hence ?

25. If on an average one birth in 80 is a case of twins, what is the probability that there will be at least one case of twins in a maternity hospital on a day when 20 births occur ?

26. Two persons A and B fire at a target independently and have a probability 0.6 and 0.7 respectively of hitting the target. Find the probability that the target is destroyed.

27. A and B throw alternately with a pair of dice. The one who throws 9 first wins. Show that the chances of their winning are 9 : 8.

26.6 BAYE'S THEOREM

An event A corresponds to a number of exhaustive events $B_1, B_2, ..., B_n$. If $P(B_i)$ and $P(A/B_i)$ are given, then

$$P(B_i/A) = \frac{P(B_i)\,P(A/B_i)}{\Sigma P(B_i)P(A/B_i)}.$$

Proof. By the multiplication law of probability,

$$P(AB_i) = P(A)\,P(B_i/A) = P(B_i)\,P(A/B_i) \quad ...(1)$$

$$\therefore \quad P(B_i/A) = \frac{P(B_i)\,P(A/B_i)}{P(A)} \quad ...(2)$$

Since the event A corresponds to $B_1, B_2, ..., B_n$, we have by the addition law of probability,

$$P(A) = P(AB_1) + P(AB_2) + ... + P(AB_n) = \Sigma P(AB_i) = \Sigma P(B_i)\,P(A/B_i) \quad \text{[By (1)]}$$

Hence from (2), we have $P(B_i/A) = \dfrac{P(B_i)\,P(A/B_i)}{\Sigma P(B_i)\,P(A/B_i)}$

which is known as the *theorem of inverse probability.*

Obs. The probabilities $P(B_i)$, $i = 1, 2, ..., n$ are called *apriori probabilities* because these exist before we get any information from the experiment.

The probabilities $P(A/B_i)$, $i = 1, 2, ..., n$ are called *posteriori probabilities*, because these are found after the experiment results are known.

Example 26.26. *Three machines M_1, M_2 and M_3 produce identical items. Of their respective output 5%, 4% and 3% of items are faulty. On a certain day, M_1 has produced 25% of the total output, M_2 has produced 30% and M_3 the remainder. An item selected at random is found to be faulty. What are the chances that it was produced by the machine with the highest output?*

Solution. Let the event of drawing a faulty item from any of the machines be A, and the event that an item drawn at random was produced by M_i be B_i. We have to find $P(B_i/A)$ for which we proceed as follows :

	M_1	M_2	M_3	*Remarks*
$P(B_i)$	0.25	0.30	0.45	$\because$ sum = 1
$P(A/B_i)$	0.05	0.04	0.03	
$P(B_i)\,P(A/B_i)$	0.0125	0.012	0.0135	sum = 0.38
$P(B_i/A)$	$\frac{0.0125}{0.038}$	$\frac{0.012}{0.038}$	$\frac{0.0135}{0.038}$	by Baye's theorem

The highest output being from M_3, the required probability = 0.0135/0.038 = 0.355.

Example 26.27. *There are three bags : first containing 1 white, 2 red, 3 green balls ; second 2 white, 3 red, 1 green balls and third 3 white, 1 red, 2 green balls. Two balls are drawn from a bag chosen at random. These are found to be one white and one red. Find the probability that the balls so drawn came from the second bag.* (J.N.T.U., 2003)

Solution. Let B_1, B_2, B_3 pertain to the first, second, third bags chosen and A : the two balls are white and red.

Now $\quad P(B_1) = P(B_2) = P(B_3) = \dfrac{1}{3}$

$P(A/B_1) = P$ (a white and a red ball are drawn from first bag)

$$= ({}^1C_1 \times {}^2C_1)/{}^6C_2 = \frac{2}{15}$$

Similarly $\quad P(A/B_2) = ({}^2C_1 \times {}^3C_1)/{}^6C_2 = \dfrac{2}{5}$, $P(A/B_3) = ({}^3C_1 \times {}^1C_1)/{}^6C_2 = \dfrac{1}{5}$

By Baye's theorem, $\quad P(B_2/A) = \dfrac{P(B_2)\,P(A/B_2)}{P(B_1)\,P(A/B_1) + P(B_2)\,P(A/B_2) + P(B_3)\,P(A/B_3)}$

$$= \frac{\frac{1}{3} \times \frac{2}{5}}{\frac{1}{3} \times \frac{2}{15} + \frac{1}{3} \times \frac{2}{5} + \frac{1}{3} \times \frac{1}{5}} = \frac{6}{11}.$$

PROBLEMS 26.3

1. In a certain college, 4% of the boys and 1% of girls are taller than 1.8 m. Further more 60% of the students are girls. If a student is selected at random and is found to be taller than 1.8 m., what is the probability that the student is a girl ?
2. In a bolt factory, machines *A*, *B* and *C* manufacture 25%, 35% and 40% of the total. Of their output 5%, 4% and 2% are defective bolts. A bolt is drawn at random from the product and is found to be defective. What are the probabilities that it was manufactured by machines *A*, *B* or *C* ? (*V.T.U., 2006 ; Rohtak, 2005 ; Madras, 2000 S*)
3. In a bolt factory, there are four machines *A*, *B*, *C*, *D* manufacturing 20%, 15%, 25% and 40% of the total output respectively. Of their outputs 5%, 4%, 3% and 2% in the same order are defective bolts. A bolt is chosen at random from the factory's production and is found defective. What is the probability that the bolt was manufactured by machine *A* or machine *D* ? (*Hissar, 2007 ; J.N.T.U., 2003*)
4. The contents of three urns are : 1 white, 2 red, 3 green balls ; 2 white, 1 red, 1 green balls and 4 white, 5 red, 3 green balls. Two balls are drawn from an urn chosen at random. These are found to be one white and one green. Find the probability that the balls so drawn came from the third urn. (*Kurukshetra, 2007*)

26.7 RANDOM VARIABLE

If a real variable X be associated with the outcome of a random experiment, then since the values which X takes depend on chance, it is called a *random variable* or a *stochastic variable* or simply a *variate*. For instance, if a random experiment E consists of tossing a pair of dice, the sum X of the two numbers which turn up have the value 2, 3, 4, ..., 12 depending on chance. Then X is the random variable. It is a function whose values are real numbers and depend on chance.

If in a random experiment, the event corresponding to a number a occurs, then the corresponding random variable X is said to assume the value a and the probability of the event is denoted by $P(X = a)$. Similarly the probability of the event X assuming any value in the interval $a < X < b$ is denoted by $P(a < X < b)$. The probability of the event $X \leq c$ is written as $P(X \leq c)$.

If a random variable takes a finite set of values, it is called a *discrete variate*. On the other hand, if it assumes an infinite number of uncountable values, it is called a *continuous variate*.

26.8 (1) DISCRETE PROBABILITY DISTRIBUTION

Suppose a discrete variate X is the outcome of some experiment. If the probability that X takes the values x_i, is p_i, then

$$P(X = x_i) = p_i \text{ or } p(x_i) \text{ for } i = 1, 2, \ldots$$

where (*i*) $p(x_i) \geq 0$ for all values of i, (*ii*) $\Sigma p(x_i) = 1$

The set of values x_i with their probabilities p_i constitute a **discrete probability distribution** *of the discrete variate X.*

For example, the discrete probability distribution for X, the sum of the numbers which turn on tossing a pair of dice is given by the following table :

$X = x_i$	2	3	4	5	6	7	8	9	10	11	12
$p(x_i)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

[$\because$ There are 6 × 6 = 36 equally likely outcomes and therefore, each has the probability 1/36. We have $X = 2$ for one outcome, *i.e.* (1, 1) ; $X = 3$ for two outcomes (1, 2) and (2, 1); $X = 4$ for three outcomes (1, 3), (2, 2) and (3, 1) and so on.]

(2) Distribution function. *The distribution function F (x) of the discrete variate X is defined by*

$$F(x) = P(X \leq x) = \sum_{i=1}^{x} p(x_i)$$ where x is any integer. The graph of $F(x)$ will be stair step form (Fig. 26.2). The distribution function is also sometimes called *cumulative distribution function.*

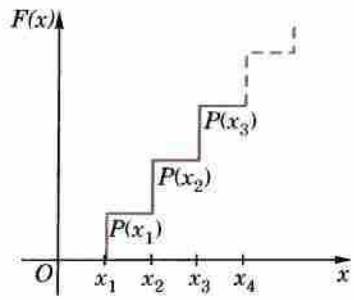


Fig. 26.2

Example 26.28. *A die is tossed thrice. A success is 'getting 1 or 6' on a toss. Find the mean and variance of the number of successes.* *(V.T.U., 2011 S ; Rohtak, 2004)*

Solution. Probability of a success $= \frac{2}{6} = \frac{1}{3}$, Probability of failures $= 1 - \frac{1}{3} = \frac{2}{3}$.

$\therefore$ prob. of no success = Prob. of all 3 failures $= \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27}$

Probability of one successes and 2 failures $= 3c_1 \times \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$

Probability of two successes and one failure $= 3c_2 \times \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} = \frac{2}{9}$

Probability of three successes $= \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27}$

Now				
	$x_i = 0$	1	2	3
	$p_i = 8/27$	4/9	2/9	1/27

$\therefore$ mean $\qquad \mu = \Sigma p_i x_i = 0 + \frac{4}{9} + \frac{4}{9} + \frac{1}{9} = 1.$

Also $\qquad \Sigma p_i x_i^2 = 0 + \frac{4}{9} + \frac{8}{9} + \frac{9}{27} = \frac{5}{3}$

$\therefore$ variance $\qquad \sigma^2 = \Sigma p_i x_i^2 - \mu^2 = \frac{5}{3} - 1 = \frac{2}{3}$.

Example 26.29. *The probability density function of a variate X is*

X :	0	1	2	3	4	5	6
$p(X)$:	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$

(i) Find $P(X < 4)$, $P(X \geq 5)$, $P(3 < X \leq 6)$. *(V.T.U., 2010)*

(ii) What will be the minimum value of k so that $P(X \leq 2) > 3$.

Solution. (*i*) If X is a random variable, then

$$\sum_{i=0}^{6} p(x_i) = 1 \text{ i.e., } k + 3k + 5k + 7k + 9k + 11k + 13k = 1 \text{ or } k = 1/49.$$

$\therefore \qquad P(X < 4) = k + 3k + 5k + 7k = 16k = 16/49.$

$P(X \geq 5) = 11k + 13k = 24k = 24/49.$

$P(3 < X \leq 6) = 9k + 11k + 13k = 33k = 33/49.$

(*ii*) $\qquad P(X \leq 2) = k + 3k + 5k = 9k > 0.3$ or $k > 1/30$

Thus minimum value of $k = 1/30$.

Example 26.30. *A random variable X has the following probability function :*

x :	0	1	2	3	4	5	6	7
$p(x)$:	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

(i) Find the value of the k *(ii) Evaluate $P(X < 6)$, $P(X \geq 6)$*

(iii) $P(0 < X < 5)$. *(W.B.T.U., 2005 ; J.N.T.U., 2003)*

Solution. (*i*) If X is a random variable, then

$$\sum_{i=0}^{7} p(x_i) = 1, \text{ i.e., } 0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$$

i.e., $\qquad 7k^2 + 9k - 1 = 0$ *i.e.* $(10 - k)(k + 1) = 0$ *i.e.,* $k = \frac{1}{10}$

(*ii*) $P(X < 6) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5)$

$$= 0 + k + 2k + 2k + 3k + k^2 = 8k + k^2 = \frac{8}{10} + \frac{1}{100} = \frac{81}{100}$$

$$P(X \geq 6) = P(X = 6) + P(X = 7) = 2k^2 + 7k^2 + k = \frac{9}{100} + \frac{1}{10} = \frac{19}{100}$$

(ii) $P(0 < X < 5) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$

$$= k + 2k + 2k + 3k = 8k = \frac{8}{10} = \frac{4}{5}.$$

26.9 (1) CONTINUOUS PROBABILITY DISTRIBUTION

When a variate X takes every value in an interval, it gives rise to *continuous distribution* of X. The distributions defined by the variates like heights or weights are continuous distributions.

A major conceptual difference, however, exists between discrete and continuous probabilities. When thinking in discrete terms, the probability associated with an event is meaningful. With continuous events, however, where the number of events is infinitely large, the probability that a specific event will occur is practically zero. For this reason, continuous probability statements must be worded somewhat differently from discrete ones. Instead of finding the probability that x equals some value, we find the probability of x falling in a small interval.

Thus the probability distribution of a continuous variate x is defined by a function $f(x)$ such that the probability of the variate x falling in the small interval $x - \frac{1}{2}dx$ to $x + \frac{1}{2}dx$ is $f(x)\,dx$. Symbolically it can be expressed as $P\left(x - \frac{1}{2}dx \leq x \leq x + \frac{1}{2}dx\right) = f(x)\,dx$. Then $f(x)$ is called the *probability density function* and the continuous curve $y = f(x)$ is called the *probability curve.*

The range of the variable may be finite or infinite. But even when the range is finite, it is convenient to consider it as infinite by supposing the density function to be zero outside the given range. Thus if $f(x) = \phi(x)$ be the density function denoted for the variate x in the interval (a, b), then it can be written as

$$\begin{aligned} f(x) &= 0, & x &< a \\ &= \phi(x), & a &\leq x \leq b \\ &= 0, & x &> b. \end{aligned}$$

The density function $f(x)$ is always positive and $\int_{-\infty}^{\infty} f(x)\,dx = 1$ (*i.e.*, the total area under the probability curve and the x-axis is unity which corresponds to the requirements that the total probability of happening of an event is unity).

(2) Distribution function

If $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx,$

then $F(x)$ is defined as the **cumulative distribution function** *or simply the* **distribution function** *of the continuous variate X.* It is the probability that the value of the variate X will be $\leq x$. The graph of $F(x)$ in this case is as shown in Fig. 26.3(*b*).

The distribution function $F(x)$ has the following properties :

(*i*) $F'(x) = f(x) \geq 0$, so that $F(x)$ is a non-decreasing function.

(*i*) $F(-\infty) = 0$; (*iii*) $F(\infty) = 1$

(*iv*) $P(a \leq x \leq b) = \int_a^b f(x)\,dx = \int_{-\infty}^{b} f(x)\,dx - \int_{-\infty}^{a} f(x)\,dx = F(b) - F(a).$

Example 26.31. (*i*) *Is the function defined as follows a density function?*

$$\begin{aligned} f(x) &= e^{-x}, & x &\geq 0 \\ &= 0, & x &< 0, \end{aligned}$$

(*ii*) *If so, determine the probability that the variate having this density will fall in the interval (1, 2) ?*

(*iii*) *Also find the cumulative probability function F(2) ?*

Solution. (*i*) $f(x)$ is clearly ≥ 0 for every x in $(1, 2)$ and

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} 0 \cdot dx + \int_0^{\infty} e^{-x}\,dx = 1$$

Hence the function $f(x)$ satisfies the requirements for a density function.

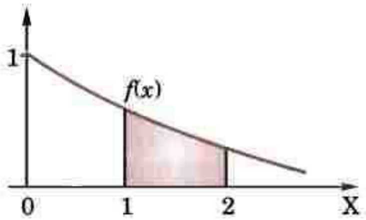
(*ii*) Required probability $= P(1 \le x \le 2) = \int_1^2 e^{-x}\, dx = e^{-1} - e^{-2} = 0.368 - 0.135 = 0.233.$

This probability is equal to the shaded area in Fig. 26.3 (*a*).

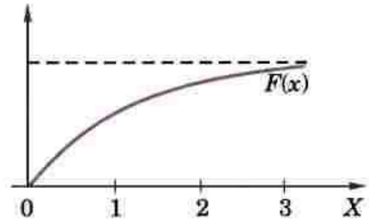
(*iii*) Cumulative probability function $F(2)$

$$\int_{-\infty}^{2} f(x)\, dx = \int_{-\infty}^{0} 0 \cdot dx + \int_0^2 e^{-x}\, dx = 1 - e^{-2} = 1 - 0.135 = 0.865$$

which is shown in Fig. 26.3 (*b*).



(*a*)



(*b*)

Fig. 26.3

26.10 (1) EXPECTATION

The mean value (μ) *of the probability distribution of a variate X is commonly known as its* **expectation** *and is denoted by E(X)*. If $f(x)$ is the probability density function of the variate X, then

$$\sum_i x_i f(x_i) \qquad \text{(discrete distribution)}$$

or
$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx \qquad \text{(continuous distribution)}$$

In general, expectation of any function $\phi(x)$ is given by

$$E[\phi(x)] = \sum_i \phi(x_i) f(x_i) \qquad \text{(discrete distribution)}$$

or
$$E[\phi(x)] = \int_{-\infty}^{\infty} \phi(x) f(x)\, dx \qquad \text{(continuous distribution)}$$

(2) **Variance** *of a distribution is given by*

$$\sigma^2 = \sum_i (x_i - \mu)^2 f(x_i) \qquad \text{(discrete distribution)}$$

or
$$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\, dx \qquad \text{(continuous distribution)}$$

where σ is the *standard deviation* of the distribution.

(3) *The* **rth moment** *about the mean* (denoted by μ_r) *is defined by*

$$\mu_r = \Sigma (x_i - \mu)^r f(x_i) \qquad \text{(discrete distribution)}$$

or
$$\mu_r = \int_{-\infty}^{\infty} (x - \mu)^r f(x)\, dx \qquad \text{(continuous distribution)}$$

(4) **Mean deviation from the mean** *is given by*

$$\Sigma\, |\, x_i - \mu\, |\, f(x_i) \qquad \text{(discrete distribution)}$$

or by
$$\int_{-\infty}^{\infty} |\, x - \mu\, |\, f(x)\, dx \qquad \text{(continuous distribution)}$$

Example 26.32. *In a lottery, m tickets are drawn at a time out of n tickets numbered from 1 to n. Find the expected value of the sum of the numbers on the tickets drawn.*

Solution. Let $x_1, x_2, ..., x_n$ be the variables representing the numbers on the first, second, ..., nth ticket. The probability of drawing a ticket out of n tickets being in each case $1/n$, we have

$$E(x_i) = 1 \cdot \frac{1}{n} + 2 \cdot \frac{1}{n} + 3 \cdot \frac{1}{n} + ... + n \cdot \frac{1}{n} = \frac{1}{2}(n+1)$$

$\therefore$ expected value of the sum of the numbers on the tickets drawn

$$= E(x_1 + x_2 + ... + x_m) = E(x_1) + E(x_2) + ... + E(x_m)$$

$$= mE(x_i) = \frac{1}{2} m (n+1).$$

Example 26.33. *X is a continuous random variable with probability density function given by*

$$f(x) = kx \ (0 \le x < 2)$$
$$= 2k \ (2 \le x < 4)$$
$$= -kx + 6k \ (4 \le x < 6)$$

Find k and mean value of X. (J.N.T.U., 2003)

Solution. Since the total probability is unity

$\therefore$ $$\int_0^6 f(x)\,dx = 1$$

i.e., $$\int_0^2 kx\,dx + \int_2^4 2k dx + \int_4^6 (-kx + 6k)\,dx = 1$$

or $$k \left| x^2/2 \right|_0^2 + 2k \left| x \right|_2^4 + \left(-kx^2/2 + 6kx\right)_4^6 = 1$$

or $$2k + 4k + (-10k + 12k) = 1 \text{ i.e., } k = 1/8.$$

$$\text{Mean of } X = \int_0^6 x\,f(x)\,dx$$

$$= \int_0^2 kx^2\,dx + \int_2^4 2kx\,dx + \int_4^6 x(-kx + 6k)dx$$

$$= k\left| x^3/3 \right|_0^2 + 2k\left| x^2/2 \right|_2^4 + \left(-k\left| x^3/3 \right|_4^6 + 6k\left| x^2/2 \right|_4^6\right)$$

$$= k\,(8/3) + k\,(12) - k\,(152/3) + 3k\,(20) = \frac{1}{8}(24) = 3.$$

Example 26.34. *A variate X has the probability distribution*

x :	-3	6	9
$P(X = x)$:	$1/6$	$1/2$	$1/3$

Find E (X) and E (X²). Hence evaluate E (2X + 1)².

Solution. $$E(X) = -3 \times \frac{1}{6} + 6 \times \frac{1}{2} + 9 \times \frac{1}{3} = 11/2.$$

$$E(X)^2 = 9 \times \frac{1}{6} + 36 \times \frac{1}{2} + 81 \times \frac{1}{3} = 93/2$$

$\therefore$ $$E(2X+1)^2 = E(4X^2 + 4X + 1) = 4E(X^2) + 4E(X) + 1$$
$$= 4\,(93/2) + 4\,(11/2) + 1 = 209.$$

Example 26.35. *The frequency distribution of a measurable characteristic varying between 0 and 2 is as under*

$$f(x) = x^3, \quad 0 \le x \le 1$$
$$= (2-x)^3, \quad 1 \le x \le 2.$$

Calculate the standard deviation and also the mean deviation about the mean.

Solution. Total frequency $N = \int_0^1 x^3\,dx + \int_1^2 (2-x)^3\,dx = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$$\therefore \quad \mu_1' \text{ (about the origin)} = \frac{1}{N}\left[\int_0^1 x \cdot x^3\,dx + \int_1^2 x(2-x)^3\,dx\right]$$

$$= 2\left\{\left|\frac{x^5}{5}\right|_0^1 + \left|-x \cdot \frac{(2-x)^4}{4}\right|_1^2 - \left|\frac{(2-x)^5}{20}\right|_1^2\right\} = 2\left(\frac{1}{5} + \frac{1}{4} + \frac{1}{20}\right) = 1$$

$$\mu_2' \text{ (about the origin)} = \frac{1}{N}\left[\int_0^1 x^2 \cdot x^3 dx + \int_1^2 x^2\,(2-x)^3\,dx\right]$$

$$= 2\left\{\left|\frac{x^6}{6}\right|_0^1 + \left|-x^2\,\frac{(2-x)^4}{4}\right|_1^2 + \frac{1}{2}\int_1^2 x\,(2-x)^4\,dx\right\}$$

$$= 2\left\{\frac{1}{6} + \frac{1}{4} + \frac{1}{2}\left[\frac{1}{5} + \frac{1}{30}\right]\right\} = \frac{16}{15}$$

Hence $\sigma^2 = \mu_2 = \mu_2' - (\mu_1')^2 = \frac{1}{15}$

i.e., standard deviation $\sigma = \frac{1}{\sqrt{15}}$.

Mean deviation about the mean

$$= \frac{1}{N}\left\{\int_0^1 |x-1|\,x^3\,dx + \int_1^2 |x-1|\,(2-x)^3\,dx\right\}$$

$$= 2\left\{\int_0^1 (1-x)x^3\,dx + \int_1^2 (x-1)\,(2-x)^3\,dx\right\}$$

$$= 2\left\{\left(\frac{1}{4} - \frac{1}{5}\right) + \left(0 + \frac{1}{20}\right)\right\} = \frac{1}{5}.$$

26.11 MOMENT GENERATING FUNCTION

(1) *The moment generating function (m.g.f.) of the discrete probability distribution of the variate X about the value $x = a$ is defined as the expected value of* $e^{t(x-a)}$ and is denoted by $M_a(t)$. Thus

$$M_a(t) = \Sigma p_i\, e^{t(x_i - a)} \quad \text{...(1)}$$

which is a function of the parameter t only.

Expanding the exponential in (1), we get

$$M_a(t) = \Sigma p_i + t\Sigma p_i\,(x_i - a) + \frac{t^2}{2!}\,\Sigma p_i(x_i - a)^2 + \ldots + \frac{t^r}{r!}\,\Sigma x_i\,(x_i - a)^r + \ldots$$

$$= 1 + t\mu_1' + \frac{t^2}{2!}\,\mu'_2 + \ldots + \frac{t^r}{r!}\,\mu'_r + \ldots \quad \text{...(2)}$$

where μ'_r is the moment of order r about a. Thus $M_a(t)$ generates moments and that is why it is called the moment generating function. From (2), we find

$$\mu'_r = \text{coefficient of } t^r/r! \text{ in the expansion of } M_a(t).$$

Otherwise differentiating (2) r times with respect to t and then putting $t = 0$, we get

$$\mu'_r = \left[\frac{d^r}{dt^r}\,M_a(t)\right]_{t=0} \quad \text{...(3)}$$

Thus the moment about any point $x = a$ can be found from (2) or more conveniently from the formula (3).

Rewriting (1) as

$$M_a(t) = e^{-at}\,\Sigma p_i e^{tx_i} \quad \text{or} \quad M_a(t) = e^{-at}\,M_0(t) \quad \text{...(4)}$$

Thus the m.g.f. about the point $a = e^{-at}$ *(m.g.f. about the origin).*

Obs. *The m.g.f. of the sum of two independent variables is the product of their m.g.fs.* ...(5)

(2) *If $f(x)$ is the density function of a continuous variate X, then the moment generating function of this continuous probability distribution about $x = a$ is given by*

$$M_a(t) = \int_{-\infty}^{\infty} e^{t(x-a)} f(x)\, dx.$$

Example 26.36. *Find the moment generating function of the exponential distribution $f(x) = \frac{1}{c} e^{-x/c}$, $0 \le x \le \infty$, $c > 0$. Hence find its mean and S.D.* (Kurukshetra, 2009)

Solution. The moment generating function about the origin is

$$M_0(t) = \int_0^{\infty} e^{tx} \cdot \frac{1}{c} e^{-x/c}\, dx = \frac{1}{c}\int_0^{\infty} e^{(t-1/c)x}\, dx \qquad \left[\because\ |t| < \frac{1}{c}\right]$$

$$= \frac{1}{c} \frac{\left| e^{(t-1/c)x} \right|_0^{\infty}}{|(t-1/c)|} = (1-ct)^{-1} = 1 + ct + c^2t^2 + c^3t^3 + \ldots$$

$\therefore$
$$\mu'_1 = \left[\frac{d}{dt} M_0(t)\right]_{t=0} = (c + 2c^2t + 3c^3t^2 + \ldots)_{t=0} = c$$

$$\mu'_2 = \left[\frac{d^2}{dt^2} M_0(t)\right]_{t=0} = 2c^2, \text{ and } \mu_2 = \mu'_2 - (\mu'_1)^2 = 2c^2 - c^2 = c^2.$$

Hence the mean is c and S.D. is also c.

26.12 PROBABILITY GENERATING FUNCTION

The probability generating function (p.g.f.) $P_x(t)$ for a random variable x which takes integral values 0, 1, 2, 3, ... only, is defined by

$$P_x(t) = p_0 + p_1t + p_2t^2 + \ldots = \sum_{n=0}^{\infty} p_n t^n = E(t^x)$$

The coefficient of t^n in the expansion of $P(t)$ in powers of t gives $P(t)_{x=n}$.

$$\frac{\partial P}{\partial t} = \sum_{n=0}^{\infty} np_n t^{n-1} \quad \text{or} \quad \left(\frac{\partial P}{\partial t}\right)_{t=1} = \Sigma\, np_n = \mu_1'$$

$$\frac{\partial^2 P}{\partial t^2} = \sum_{n=0}^{\infty} n(n-1)\, p_n t^{n-2} \quad \text{or} \quad \left(\frac{\partial^2 P}{\partial t^2}\right)_{t=1} = \Sigma\, n(n-1)\, p_n = \mu_2' - \mu_1'$$

$$= \mu_2 + \mu_1'^2 - \mu_1' \text{ and so on}$$

Also $$\left(\frac{\partial^k P}{\partial t^n}\right)_{t=0} = n!\, p_n,\ k = 1, 2, \ldots n.$$

For integral valued variates, we have

$$P_x(e^t) = E(e^{tx}) = m.g.f. \text{ for } x.$$

Obs. The p.g.f. of the sum of two independent random variables is the product of their p.g.f.'s.

Example 26.37. *If x be a random variable with probability generating function $P_x(t)$, find the probability generating function of*

(i) $x + 2$ *(ii)* $2x$.

Solution. We have $P_x(t) = \sum_{k=0}^{\infty} p_k t^k$

(*i*) Probability generating function of $x + 2 = \sum_{k=0}^{\infty} p_k\, t^{k+2} = t^2 \sum_{k=0}^{\infty} p_k\, t^k = t^2 P_x(t)$.

(*ii*) Probability generating function of $2x = \sum_{k=0}^{\infty} p_k\, t^{2k} = \sum_{k=0}^{\infty} p_k (t^2)^k = P(t^2)$.

PROBLEMS 26.4

1. A random variable x has the following probability function :

Values of x :	-2	-1	0	1	2	3
$p(x)$:	0.1	k	0.2	$2k$	0.3	k

Find the value of k and calculate mean and variance. (*S.V.T.U., 2007 ; V.T.U., 2004 ; Madras, 2003*)

2. Find the standard deviation for the following discrete distribution :

x :	8	12	16	20	24
$p(x)$:	1/8	1/6	3/8	1/4	1/12

3. Obtain the distribution function of the total number of heads occurring in three tosses of an unbiased coin.
4. Show that for any discrete distribution $\beta_2 \geq 1$.
5. From an urn containing 3 red and 2 white balls, a man is to draw 2 balls at random without replacement, being promised Rs. 20 for each red ball he draws and Rs. 10 for each white one. Find his expectation.
6. Four coins are tossed. What is the expectation of the number of heads ?
7. The diameter of an electric cable is assumed to be a continuous variate with p.d.f. $f(x) = 6x(1-x)$, $0 \leq x \leq 1$. Verify that the above is a p.d.f. Also find the mean and variance.
8. A random variable gives measurements X between 0 and 1 with a probability function

$$f(x) = 12x^3 - 21x^2 + 10x, \quad 0 \leq x \leq 1$$
$$= 0$$

(*i*) Find $P\left(X \leq \frac{1}{2}\right)$ and $P\left(X > \frac{1}{2}\right)$

(*ii*) Find a number k such that $P(X \leq k) = \frac{1}{2}$ (*J.N.T.U., 2003*)

9. The power reflected by an aircraft that is received by a radar can be described by an exponential random variable X.

The probability density of X is given by $f(x) = \begin{cases} \dfrac{1}{x_0} e^{-x/x_0}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

where x_0 is the average power received by the radar.

(*i*) What is the probability that the radar will receive power larger than the power received on the average ? (*ii*) What is the probability that the radar will receive power less than the power received on the average ?

(*Mumbai, 2006*)

10. A function is defined as follows :

$$f(x) = 0, \qquad x < 2$$
$$= \frac{1}{18}(2x+3), \qquad 2 \leq x \leq 4$$
$$= 0, \qquad x > 4.$$

Show that it is a density function. Find the probability that a variate having this density will fall in the interval $2 \leq x \leq 3$?

11. A continuous distribution of a variable x in the range $(-3, 3)$ is defined as

$$f(x) = \frac{1}{16}(3+x)^2, \qquad -3 \leq x < -1$$
$$= \frac{1}{16}(2-6x^2), \qquad -1 \leq x < 1$$
$$= \frac{1}{16}(3-x)^2, \qquad 1 \leq x \leq 3.$$

Verify that the area under the curve is unity. Show that the mean is zero. *(Kurukshetra, 2005)*

12. The frequency function of a continuous random variable is given by
$$f(x) = y_0 x\,(2 - x),\ 0 \le x \le 2.$$
Find the value of y_0, mean and variance of x. *(Kerala, 2005 ; J.N.T.U., 2003)*

13. The probability density $p(x)$ of a continuous random variable is given by
$$p(x) = y_0 e^{-|x|},\ -\infty < x < \infty.$$
Prove that $y_0 = 1/2$. Find the mean and variance of the distribution. *(S.V.T.U., 2008 ; Kurukshetra, 2007 ; V.T.U., 2004)*

14. If $f(x) = \begin{cases} \frac{1}{2}(x+1), & -1 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$
represents the density of a random variable X, find $E(X)$ and Var (X).

15. A function is defined as under :
$$f(x) = 1/k,\ x_1 \le x \le x_2$$
$$= 0, \text{ elsewhere.}$$
Find the cumulative distribution of the variate x when k satisfies the requirements for $f(x)$ to be a density function.

26.13 REPEATED TRIALS

We know that the probability of getting a head or a tail on tossing a coin is $\frac{1}{2}$. If the coin is tossed thrice, the probability of getting one head and two tails can be combined as $H—T—T$, $T—H—T$, $T—T—H$. The probability of each one of these being $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$, *i.e.*, $\left(\frac{1}{2}\right)^3$, their total probability shall be $3(1/2)^3$.

Similarly if a trial is repeated n times and if p is the probability of a success and q that of a failure, then the probability of r successes and $n - r$ failures is given by $p^r q^{n-r}$.

But these r successes and $n - r$ failures can occur in any of the ${}^nC_r$ ways in each of which the probability is same.

Thus the probability of r successes is ${}^nC_r\, p^r q^{n-r}$.

Cor. *The probabilities of at least r successes in n trials*
$$= \text{the sum of the probabilities of } r, r+1, \ldots, n \text{ successes}$$
$$= {}^nC_r\, p^r q^{n-r} + {}^nC_{r+1}\, p^{r+1} q^{n-r-1} + \ldots + {}^nC_n\, p^n.$$

26.14 (1) BINOMIAL DISTRIBUTION*

It is concerned with trials of a repetitive nature in which only the occurrence or non-occurrence, success or failure, acceptance or rejection, yes or no of a particular event is of interest.

If we perform a series of independent trials such that for each trial p is the probability of a success and q that of a failure, then the probability of r successes in a series of n trials is given by ${}^nC_r\, p^r q^{n-r}$, where r takes any integral value from 0 to n. The probabilities of 0, 1, 2, ... r, ..., n successes are, therefore, given by
$$q^n,\ {}^nC_1 pq^{n-1},\ {}^nC_2 p^2 q^{n-2},\ \ldots,\ {}^nC_r\, p^r q^{n-r},\ \ldots,\ p^n.$$

The probability of the number of successes so obtained is called *the* **binomial distribution** for the simple reason that the probabilities are the successive terms in the expansion of the binomial $(q+p)^n$.

$\therefore$ the sum of the probabilities
$$= q^n + {}^nC_1 pq^{n-1} + {}^nC_2 p^2 q^{n-2} + \ldots + p^n = (q+p)^n = 1.$$

(2) Constants of the binomial distribution. The moment generating function about the origin is
$$M_0(t) = E(e^{tx}) = \Sigma\, {}^nC_x\, p^x q^{n-x} e^{tx} \qquad \text{[By (1) § 26.11]}$$
$$= \Sigma\, {}^nC_x\, (pe^t)^x q^{n-x} = (q + pe^t)^n$$

* It was discovered by a Swiss mathematician *Jacob Bernoulli* and was published posthumously in 1713.

Differentiating with respect to t and putting $t = 0$ and using (3) § 26.11, we get the mean

$$\mu'_1 = np.$$

Since $M_a(t) = e^{-at} M_0(t)$, the m.g.f. of the binomial distribution about its mean $(m) = np$, is given by

$$M_m(t) = e^{-npt}(q + pe^t)^n = (qe^{-pt} + pe^{qt})^n$$

$$= \left(1 + pq\frac{t^2}{2!} + pq(q^2 - p^2)\frac{t^3}{3!} + pq(q^3 + p^3)\frac{t^4}{4!} + \cdots\right)^n$$

or

$$1 + \mu_1 t + \mu_2\frac{t^2}{2!} + \mu_3\frac{t^3}{3!} + \mu_4\frac{t^4}{4!} + \cdots$$

$$= 1 + npq\frac{t^2}{2!} + npq(q - p)\frac{t^3}{3!} + npq\,[1 + 3(n - 2)pq]\frac{t^4}{4!} + \ldots$$

Equating the coefficients of like powers of t on either side, we have

$$\mu_2 = npq,\ \mu_3 = npq(q - p),\ \mu_4 = npq\,[1 + 3(n - 2)pq].$$

Also $$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(q - p)^2}{npq} = \frac{(1 - 2p)^2}{npq} \quad \text{and} \quad \beta_2 = \frac{\mu_4}{\mu_2^2} = 3 + \frac{1 - 6pq}{npq}$$

Thus *mean* $= np$, *standard deviation* $= \sqrt{(npq)}$.

skewness $= (1 - 2p)/\sqrt{(npq)}$, *kurtosis* $= \beta_2$.

Obs. The skewness is positive for $p < \frac{1}{2}$ and negative for $p > \frac{1}{2}$. When $p = \frac{1}{2}$, the skewness is zero, *i.e.*, the probability curve of the binomial distribution will be symmetrical (bell-shaped).

As n the number of trials increase indefinitely, $\beta_1 \to 0$, and $\beta_2 \to 3$.

(3) Binomial frequency distribution. If n independent trials constitute one experiment and this experiment be repeated N times, then the frequency of r successes is $N\,{}^nC_r\,p^r\,q^{n-r}$. The possible number of successes together with these expected frequencies constitute the *binomial frequency distribution.*

(4) Applications of Binomial distribution. This distribution is applied to problems concerning : (*i*) Number of defectives in a sample from production line,

(*ii*) Estimation of reliability of systems,

(*iii*) Number of rounds fired from a gun hitting a target,

(*iii*) Radar detection.

Example 26.38. *The probability that a pen manufactured by a company will be defective is 1/10. If 12 such pens are manufactured, find the probability that*

(*a*) *exactly two will be defective.* (*b*) *at least two will be defective.*

(*c*) *none will be defective.* (*V.T.U., 2004 ; Burdwan, 2003*)

Solution. The probability of a defective pen is 1/10 = 0.1

$\therefore$ The probability of a non-defective pen is $1 - 0.1 = 0.9$

(*a*) *The probability that exactly two will be defective*

$$= {}^{12}C_2\,(0.1)^2\,(0.9)^{10} = 0.2301$$

(*b*) *The probability that at least two will be defective*

$$= 1 - (\text{prob. that either none or one is non-defective})$$

$$= 1 - [{}^{12}C_0(0.9)^{12} + {}^{12}C_1(0.1)\,(0.9)^{11}] = 0.3412$$

(*c*) *The probability that none will be defective*

$$= {}^{12}C_{12}(0.9)^{12} = 0.2833.$$

Example 26.39. *In 256 sets of 12 tosses of a coin, in how many cases one can expect 8 heads and 4 tails.* (*J.N.T.U., 2003*)

Solution. $P(\text{head}) = \frac{1}{2}$ and $P(\text{tail}) = \frac{1}{2}$

By binomial distribution, probability of 8 heads and 4 tails in 12 trials is

$$P(X=8) = {}^{12}C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^4 = \frac{12!}{8!\,4!}\cdot\frac{1}{2^{12}} = \frac{495}{4096}$$

$\therefore$ the expected number of such cases in 256 sets

$$= 256 \times P(X=8) = 256\ \frac{495}{4096} = 30.9 = 31 \text{ (say)}.$$

Example 26.40. *In sampling a large number of parts manufactured by a machine, the mean number of defectives in a sample of 20 is 2. Out of 1000 such samples, how many would be expected to contain at least 3 defective parts.* *(V.T.U., 2004)*

Solution. Mean number of defectives $= 2 = np = 20p$.

$\therefore$ The probability of a defective part is $p = 2/20 = 0.1$.

and the probability of a non-defective part = 0.9

$\therefore$ The probability of at least three defectives in a sample of 20.

$= 1 -$ (prob. that either none, or one, or two are non-defective parts)

$= 1 - [{}^{20}C_0(0.9)^{20} + {}^{20}C_1(0.1)(0.9)^{19} + {}^{20}C_2(0.1)^2(0.9)^{18}]$

$= 1 - (0.9)^{18} \times 4.51 = 0.323.$

Thus *the number of samples having at least three defective parts* out of 1000 samples

$= 1000 \times 0.323 = 323.$

Example 26.41. *The following data are the number of seeds germinating out of 10 on damp filter paper for 80 sets of seeds. Fit a binomial distribution to these data :*

x :	0	1	2	3	4	5	6	7	8	9	10
f :	6	20	28	12	8	6	0	0	0	0	0

Solution. Here $n = 10$ and $N = \Sigma f_i = 80$

$$\therefore \qquad \text{mean} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{20+56+36+32+30}{80} = \frac{174}{80} = 2.175$$

Now the mean of a binomial distribution $= np$

i.e., $np = 10p = 2.175 \quad \therefore \quad p = 0.2175,\ q = 1 - p = 0.7825$

Hence the binomial distribution to be fitted is

$$N(q+p)^n = 80\,(0.7825 + 0.2175)^{10}$$
$$= 80.{}^{10}C_0\,(0.7825)^{10} + 80.{}^{10}C_1\,(0.7825)^9\,(0.2175)^1 + {}^{10}C_2\,(0.7825)^8\,(0.2175)^2 + \ldots + {}^{80}C_9\,(0.7825)^1\,(0.2175)^9 + {}^{80}C_{10}\,(0.2175)^{10}$$
$$= 6.885 + 19.13 + 23.94 + \ldots + 0.0007 + 0.00002$$

$\therefore$ the successive terms in the expansion give the expected or theoretical frequencies which are

x :	0	1	2	3	4	5	6	7	8	9	10
f :	6.9	19.1	24.0	17.8	8.6	2.9	0.7	0.1	0	0	0

PROBLEMS 26.5

1. Determine the binomial distribution for which mean = 2 (variance) and mean + variance = 3. Also find $P(X \leq 3)$. *(Kerala, 2005)*
2. An ordinary six-faced die is thrown four times. What are the probabilities of obtaining 4, 3, 2, 1 and 0 faces ?
3. If the chance that one of the ten telephone lines is busy at an instant is 0.2.
   (*a*) What is the chance that 5 of the lines are busy ?
   (*b*) What is the most probable number of busy lines and what is the probability of this number ?
   (*c*) What is the probability that all the lines are busy ? *(V.T.U., 2002 S)*
4. If the probability that a new-born child is a male is 0.6, find the probability that in a family of 5 children there are exactly 3 boys. *(Kurukshetra, 2005)*

5. If on an average 1 vessel in every 10 is wrecked, find the probability that out of 5 vessels expected to arrive, at least 4 will arrive safely. *(P.T.U., 2005)*
6. The probability that a bomb dropped from a plane will strike the target is 1/5. If six bombs are dropped, find the probability that (*i*) exactly two will strike the target, (*ii*) at least two will strike the target.
7. A sortie of 20 aeroplanes is sent on an operational flight. The chances that an aeroplane fails to return is 5%. Find the probability that (*i*) one plane does not return (*ii*) at the most 5 planes do not return, and (*iii*) what is the most probable number of returns ? *(Hissar, 2007)*
8. The probability that an entering student will graduate is 0.4. Determine the probability that out of 5 students (*a*) none (*b*) one and (*c*) at least one will graduate.
9. Out of 800 families with 5 children each, how many would you expect to have (*a*) 3 boys, (*b*) 5 girls, (*c*) either 2 or 3 boys ? Assume equal probabilities for boys and girls. *(V.T.U., 2004)*
10. If 10 per cent of the rivets produced by a machine are defective, find the probability that out of 5 rivets chosen at random (*i*) none will be defective, (*ii*) one will be defective, and (*iii*) at least two will be defective.
11. In a bombing action there is 50% chance that any bomb will strike the target. Two direct hits are needed to destroy the target completely. How many bombs are required to be dropped to give a 99% chance or better of completely destroying the target. *(V.T.U., 2003 S)*
12. A product is 0.5% defective and is packed in cartons of 100. What percentage contains not more than 3 defectives ?
13. If in a lot of 500 solenoids 25 are defective, find the probability of 0, 1, 2, 3 defective solenoids in a random sample of 20 solenoids.
14. 500 articles were selected at random out of a batch containing 10,000 articles, and 30 were found to be defective. How many defectives articles would you reasonably expect to have in the whole batch ? *(J.N.T.U., 2003)*
15. Fit a binomial distribution for the following data and compare the theoretical frequencies with the actual ones :

x :	0	1	2	3	4	5
f :	2	14	20	34	22	8

(Bhopal, 2006)

16. Fit a binomial distribution to the following frequency distribution :

x :	0	1	2	3	4	5	6
f :	13	25	52	58	32	16	4

(Kurukshetra, 2009 ; S.V.T.U., 2007)

26.15 (1) POISSON DISTRIBUTION*

It is a distribution related to the probabilities of events which are extremely rare, but which have a large number of independent opportunities for occurrence. The number of persons born blind per year in a large city and the number of deaths by horse kick in an army corps are some of the phenomena, in which this law is followed.

This distribution can be derived as a limiting case of the binomial distribution by making n very large and p very small, keeping np fixed (= m, say).

The probability of r successes in a binomial-distribution is

$$P(r) = {}^nC_r\, p^r q^{n-r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}\, p^r q^{n-r}$$

$$= \frac{np(np-p)(np-2p)\cdots(np-\overline{r-1}p)}{r!}\,(1-p)^{n-r}$$

As $n \to \infty$, $p \to 0$ $(np = m)$, we have

$$P(r) = \frac{m^r}{r!}\,\underset{n\to\infty}{\text{Lt}}\,\frac{(1-m/n)^n}{(1-m/n)^r} = \frac{m^r}{r!}\,e^{-m}$$

so that the probabilities of 0, 1, 2..., r,... successes in a Poisson distribution are given by

$$e^{-m},\ me^{-m},\ \frac{m^2e^{-m}}{2!},\ldots,\frac{m^re^{-m}}{r!},\ldots$$

The sum of these probabilities is unity as it should be.

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* It was discovered by a French mathematician S.D. Poisson in 1837.

(2) Constants of the Poisson distribution. These constants can easily be derived from the corresponding constants of the binomial distribution simply by making $n \to \infty$, $p \to 0$, $(q \to 1)$ and noting that $np = m$

$$Mean = \text{Lt}\,(np) = m$$

$$\mu_2 = \text{Lt}\,(npq) = m\ \text{Lt}\,(q) = m$$

$\therefore$ *Standard deviation* $= \sqrt{m}$

Also $\mu_3 = m$, $\mu_4 = m + 3m^2$

$\therefore$ *Skewness* $(= \sqrt{\beta_1}) = 1/m$, *Kurtosis* $(= \beta_2) = 3 + 1/m$.

Since μ_3 is positive, Poisson distribution is positively skewed and since $\beta_2 > 3$, it is *Leptokurtic*.

(3) Applications of Poisson distribution. This distribution is applied to problems concerning : (*i*) Arrival pattern of 'defective vehicles in a workshop', 'patients in a hospital' or 'telephone calls'.

(*ii*) Demand pattern for certain spare parts.

(*iii*) Number of fragments from a shell hitting a target.

(*iv*) Spatial distribution of bomb hits.

Example 26.42. *If the probability of a bad reaction from a certain injection is 0.001, determine the chance that out of 2,000 individuals more than two will get a bad reaction.* (V.T.U., 2008 ; Kottayam, 2005)

Solution. It follows a Poisson distribution as the probability of occurrence is very small.

Mean $m = np = 2000(0.001) = 2$

Probability that more than 2 will get a bad reaction

= 1 – [prob. that no one gets a bad reaction + prob. that one gets a bad reaction + prob. that two get bad reaction]

$$= 1 - \left[e^{-m} + \frac{m^1 e^{-m}}{1!} + \frac{m^2 e^{-m}}{2!}\right] = 1 - \left[\frac{1}{e^2} + \frac{2}{e^2} + \frac{2}{e^2}\right] \qquad [\because \ m = 2]$$

$$= 1 - \frac{5}{e^2} = 0.32. \qquad [\because \ e = 2.718]$$

Example 26.43. *In a certain factory turning out razor blades, there is a small chance of 0.002 for any blade to be defective. The blades are supplied in packets of 10, use Poisson distribution to calculate the approximate number of packets containing no defective, one defective and two defective blades respectively in a consignment of 10,000 packets.* (Kurukshetra, 2009 S; Madras, 2006 ; V.T.U., 2004)

Solution. We know that $m = np = 10 \times 0.002 = 0.02$

$$e^{-0.02} = 1 - 0.02 + \frac{(0.02)^2}{2!} - \ldots = 0.9802 \text{ approximately}$$

Probability of no defective blade is $e^{-m} = e^{-0.02} = 0.9802$

$\therefore$ no. of packets containing no defective blade is

$$10{,}000 \times 0.9802 = 9802$$

Similarly the number of packets containing one defective blade $= 10{,}000 \times me^{-m}$

$$= 10{,}000 \times (0.02) \times 0.9802 = 196$$

Finally the number of packets containing two defective blades

$$= 10{,}000 \times \frac{m^2 e^{-m}}{2!} = 10{,}000 \times \frac{(0.02)^2}{2!} \times 0.9802 = 2 \text{ approximately.}$$

Example 26.44. *Fit a Poisson distribution to the set of observations :*

x :	0	1	2	3	4
f :	122	60	15	2	1

(Bhopal, 2007 S ; V.T.U., 2004 ; U.P.T.U., 2003)

Solution. Mean $= \dfrac{\Sigma f_i x_i}{\Sigma f_i} = \dfrac{60 + 36 + 6 + 4}{200} = 0.5.$

$\therefore$ mean of Poisson distribution *i.e.*, $m = 0.5$.

Hence the theoretical frequency for r successes is

$$\frac{Ne^{-m}(m)^r}{r!} = \frac{200e^{-0.5}(.5)^r}{r!} \quad \text{where } r = 0, 1, 2, 3, 4$$

$\therefore$ the theoretical frequencies are

x:	0	1	2	3	4
f:	121	61	15	2	0

$(\because \; e^{-.5} = 0.61)$

PROBLEMS 26.6

1. If a random variable has a Poisson distribution such that $P(1) = P(2)$, find
   (*i*) mean of the distribution. (*ii*) $P(4)$. (*V.T.U., 2003*)
2. X is a Poisson variable and it is found that the probability that $X = 2$ is two-thirds of the probability that $X = 1$. Find the probability that $X = 0$ and the probability that $X = 3$. What is the probability that X exceeds 3 ?
3. For Poisson distribution, prove that $m\,\mu_2\,\gamma_1\,\gamma_2 = 1$, where symbols have their usual meanings. (*S.V.T.U., 2008*)
4. A certain screw making machine produces on average of 2 defective screws out of 100, and packs them in boxes of 500. Find the probability that a box contains 15 defective screws. (*Kurukshetra, 2006*)
5. A manufacturer knows that the condensers he makes contain on the average 1 % defectives. He packs them in boxes of 100. What is the probability that a box picked at random will contain 3 or more faulty condensers ?
6. A car-hire firm has two cars which it hires out day by day. The number of demands for a car on each day is distributed as a Poisson distribution with mean 1.5. Calculate the proportion of days (*i*) on which there is no demand, (*ii*) on which demand is refused. ($e^{-1.5} = 0.2231$). (*Bhopal, 2008 S ; J.N.T.U., 2003*)
7. The incidence of occupational disease in an industry is such that the workmen have a 10% chance of suffering from it. What is probability that in a group of 7, five or more will suffer from it ?
8. The frequency of accidents per shift in a factory is as shown in the following table :

Accidents per shift :	0	1	2	3	4
Frequency :	180	92	24	3	1

   Calculate the mean number of accidents per shift and the corresponding Poisson distribution and compare with actual observations.
9. A source of liquid is known to contain bacteria with the mean number of bacteria per cubic centimetre equal to 3. Ten 1 c.c., test-tubes are filled with the liquid. Assuming that Poisson distribution is applicable, calculate the probability that all the test-tubes will show growth *i.e.*, contain atleast 1 bacterium each.
10. Find the expectation of the function $\phi(x) = xe^{-x}$ in a Poisson distribution. (*V.T.U., 2003*)
    [**Hint** : If m be the mean of the Poisson distribution, then expectation of

$$\phi(x) = \sum_{x=0}^{\infty} \frac{\phi(x)\cdot m^x\, e^{-m}}{x!} = m \exp. m\,(e^{-1} - m - 1)$$

11. Fit a Poisson distribution to the following :

x:	0	1	2	3	4
f:	46	38	22	9	1

   (*Kurukshetra, 2009 ; Bhopal, 2008 ; V.T.U., 2003 S*)
12. Fit a Poisson distribution to the following data given the number of yeast cells per square for 400 squares :

No. of cells per sq. :	0	1	2	3	4	5	6	7	8	9	10
No. of squares :	103	143	98	42	8	4	2	0	0	0	0

   (*S.V.T.U., 2007*)

26.16 (1) NORMAL DISTRIBUTION*

Now we consider a continuous distribution of fundamental importance, namely the normal distribution. *Any quantity whose variation depends on random causes is distributed according to the normal law.* Its importance lies in the fact that a large number of distributions approximate to the normal distribution.

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* In 1924, *Karl Pearson* found this distribution which *Abraham De Moivre* had discovered as early as 1733. See footnote p. 843 and 647.

Let us define a variate $z = \dfrac{x - np}{\sqrt{(npq)}}$ ...(1)

where x is a binomial variate with mean np and S.D. $\sqrt{(npq)}$ so that z is a variate with mean zero and variance unity. In the limit as n tends to infinity, the distribution of z becomes a continuous distribution extending from $-\infty$ to ∞.

It can be shown that *the limiting form of the binomial distribution (1) for large values of n when neither p nor q is very small, is the normal distribution.* The normal curve is of the form

$$y = \frac{1}{\sigma\sqrt{(2\pi)}} e^{-(x-\mu)^2/2\sigma^2} \qquad \text{...(2)}$$

where μ and σ are the mean and standard deviation respectively.

(2) Properties of the normal distribution

I. The normal curve (2) is bell-shaped and is symmetrical about its mean. It is unimodal with ordinates decreasing rapidly on both sides of the mean (Fig. 26.3). The maximum ordinate is $1/\sigma\sqrt{(2\pi)}$, found by putting $x = \mu$ in (2).

As it is symmetrical, its mean, median and mode are the same. Its points of inflexion (found by putting $d^2y/dx^2 = 0$ and verifying that at these points $d^3y/dx^3 \neq 0$) are given by $x = \mu \pm \sigma$, *i.e.*, these points are equidistant from the mean on either side.

II. Mean deviation from the mean μ

$$= \int_{-\infty}^{\infty} |x-\mu| \cdot \frac{1}{\sigma\sqrt{(2\pi)}} e^{-(x-\mu)^2/2\sigma^2} dx \qquad [\text{Put } z = (x-\mu)/\sigma]$$

$$= \frac{\sigma}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} |z| e^{-z^2/2} dz$$

$$= \frac{\sigma}{\sqrt{(2\pi)}} \left[\int_{-\infty}^{0} -ze^{-z^2/2} dz + \int_{0}^{\infty} ze^{-z^2/2} dz \right] = \frac{2\sigma}{\sqrt{(2\pi)}} \int_0^{\infty} ze^{-z^2/2} dz$$

$$= \frac{2\sigma}{\sqrt{(2\pi)}} \left| -e^{-z^2/2} \right|_0^{\infty} = -\sqrt{\left(\frac{2}{\pi}\right)} \sigma(0-1) = 0.7979\,\sigma \approx (4/5)\,\sigma$$

III. Moments about the mean

$$\mu_{2n+1} = \int_{-\infty}^{\infty} (x-\mu)^{2n+1} \frac{1}{\sigma\sqrt{(2\pi)}} e^{-(x-\mu)^2/2\sigma^2} dx$$

$$= \frac{\sigma^{2n+1}}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} z^{2n+1} e^{-z^2/2} dz \text{ where } z = (x-\mu)/\sigma$$

= 0, since the integral is an odd function.

Thus all odd order moments about the mean vanish.

$$\mu_{2n} = \int_{-\infty}^{\infty} (x-\mu)^{2n} \frac{1}{\sigma\sqrt{(2\pi)}} e^{-(x-\mu)^2/2\sigma^2} dx$$

$$= \frac{\sigma^{2n}}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} z^{2n-1} e^{-z^2/2} \,.\, z dz \qquad [\text{Integrate by parts}]$$

$$= \frac{\sigma^{2n}}{\sqrt{(2\pi)}} \left[\left| -z^{2n-1} e^{-z^2/2} \right|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} (2n-1) z^{2n-2} e^{-z^2/2} dz \right]$$

$$= \frac{\sigma^{2n}}{\sqrt{(2\pi)}} (0-0) + (2n-1)\sigma^2 \mu_{2n-2}$$

Repeated application of this reduction formula, gives

$$\mu_{2n} = (2n-1)(2n-3) \ldots 3 \,.\, 1\sigma^{2n}$$

In particular, $\mu_2 = \sigma^2$, $\mu_4 = 3\sigma^4$.

Hence $\qquad \beta_1 = \dfrac{\mu_3^{\,2}}{\mu_2^{\,3}} = 0$ and $\beta_2 = \dfrac{\mu_4}{\mu_2^{\,2}} = 3$

i.e., the coefficient of skewness is zero (*i.e.* the curve is symmetrical) and the Kurtosis is 3. This is the basis for the choice of the value 3 in the definitions of platykurtic and leptokurtic (page 844).

IV. The probability of x lying between x_1 and x_2 is given by the area under the normal curve from x_1 to x_2, *i.e.*, $P(x_1 \le x \le x_2)$

$$= \frac{1}{\sigma\sqrt{(2\pi)}} \int_{x_1}^{x_2} e^{-(x-\mu)^2/2\sigma^2}\, dx$$

$$= \frac{1}{\sqrt{(2\pi)}} \int_{z_1}^{z_2} e^{-z^2/2}\, dz \text{ where } z = (x-\mu)/\sigma,\ dz = dx/\sigma \text{ and } z_1 = (x_1-\mu)/\sigma,\ z_2 = (x_2-\mu)/\sigma.$$

$$= \frac{1}{\sqrt{(2\pi)}} \left\{ \int_0^{z_2} e^{-z^2/2}\, dz - \int_0^{z_1} e^{-z^2/2}\, dz \right\} = P_2(z) - P_1(z)$$

The values of each of the above integrals can be found from the table III–Appendix 2, which gives the values of

$$P(z) = \frac{1}{\sqrt{(2\pi)}} \int_0^z e^{-z^2/2}\, dz$$

for various values of z. This integral is called the *probability integral* or the *error function* due to its use in the theory of sampling and the theory of errors.

Using this table, we see that the area under the normal curve from $z = 0$ to $z = 1$, *i.e.* from $x = \mu$ to $\mu + \sigma$ is 0.3413.

∴ (*i*) The area under the normal curve between the ordinates $x = \mu - \sigma$ and $x = \mu + \sigma$ is 0.6826, ~ *68% nearly. Thus approximately 2/3 of the values lie within these limits.*

(*ii*) The area under the normal curve between $x = \mu - 2\sigma$ and $x = \mu + 2\sigma$ is 0.9544 ~ 95.5%, which implies that *about $4\frac{1}{2}$ % of the values lie outside these limits.*

(*ii*) *99.73% of the values lie between* $x = \mu - 3\sigma$ and $x = \mu + 3\sigma$ *i.e., only a quarter % of the whole lies outside these limits.*

(*iv*) *95% of the values lie between $x = \mu - 1.96\sigma$ and $x = \mu + 1.96\,\sigma$ i.e., only 5% of the values lie outside these limits.*

(*v*) *99% of the values lie between $x = \mu - 2.58\sigma$ and $x = \mu + 2.58\sigma$ i.e., only 1% of the values lie outside these limits.*

(*vi*) 99.9% of the values lie between $x = \mu - 3.29\sigma$ and $x = \mu + 3.29\sigma$.

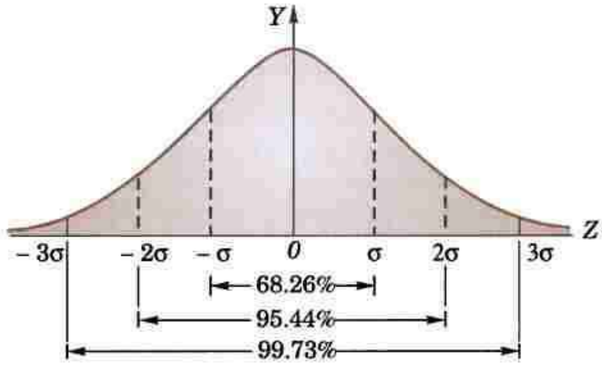

Fig. 26.4

In other words, a value that deviates more than σ from μ occurs about once in 3 trials. A value that deviates more than 2σ or 3σ from μ occurs about once in 20 or 400 trials. Almost all values lie within 3σ of the mean.

The shape of the standardised normal curve is

$$y = \frac{1}{\sqrt{(2\pi)}} e^{-z^2/2} \text{ where } z = (x-\mu)/\sigma \qquad ...(3)$$

and the respective areas are shown in Fig. 26.4. 'z' is called a *normal variate*.

(3) Normal frequency distribution. We can fit a normal curve to any distribution. If N be the total frequency, μ the mean and σ the standard deviation of the given distribution then the curve

$$y = \frac{N}{\sigma\sqrt{(2\pi)}} e^{-(x-\mu)^2/2\sigma^2} \qquad ...(4)$$

will fit the given distribution as best as the data will permit. The frequency of the variate between x_1 and x_2 as given by the fitted curve, will be the area under (1) from x_1 to x_2.

(4) Applications of normal distribution. This distribution is applied to problems concerning :

(*i*) Calculation of errors made by chance in experimental measurements.

(*ii*) Computation of hit probability of a shot.

(*iii*) Statistical inference in almost every branch of science.

26.17 PROBABLE ERROR

Any lot of articles manufactured to certain specifications is subject to small errors. In fact, measurement of any physical quantity shows slight error. In general, these errors of manufacture or experiment are of random nature and therefore, follow a normal distribution. While quoting a specification of an experimental result, we usually mention the *probable error* (λ). *It is such that the probability of an error falling within the limits* $\mu - \lambda$ *and* $\mu + \lambda$ *is exactly equal to the chance of an error falling outside these limits, i.e.* the chance of an error lying within $\mu - \lambda$ and $\mu + \lambda$ is $\frac{1}{2}$.

$$\therefore \quad \frac{1}{\sigma\sqrt{(2\pi)}} \int_{\mu-\lambda}^{\mu+\lambda} e^{-(x-\mu)^2/2\sigma^2}\, dx = \frac{1}{2}$$

or
$$\frac{1}{\sqrt{(2\pi)}} \int_0^{\lambda/\sigma} e^{-z^2/2} dz = \frac{1}{4} \qquad \left[z = \frac{x-\mu}{\sigma}\right]$$

The table V, (Appendix 2) gives $\lambda/\sigma = 0.6745$

Hence the probable error $\lambda = 0.6745\sigma \sim \frac{2}{3}\sigma$.

Obs. Quartile deviation $= \frac{1}{2}(Q_3 - Q_1) \sim \frac{2}{3}\sigma$; Mean deviation $\approx \frac{4}{5}\sigma$ [p. 839]

$\therefore$ *Q.D. : M.D. : S.D.* = 10 : 12 : 15. *(Madras, 2003)*

Example 26.45. *X is a normal variate with mean 30 and S.D. 5, find the probabilities that (i)* $26 \le X \le 40$, *(ii)* $X \ge 45$ *and (iii)* $|X - 30| > 5$. *(J.N.T.U., 2005)*

Solution. We have $\mu = 30$ and $\sigma = 5$

$$\therefore \quad z = \frac{X-\mu}{\sigma} = \frac{X-30}{5}$$

(*i*) When $X = 26, z = -0.8$; when $X = 40, z = 2$

$$\begin{aligned} \therefore \quad P(26 \le X \le 40) &= P(-0.8 \le z \le 2) \\ &= P(-0.8 \le z \le 0) + P(0 \le z \le 2) \qquad \text{[Using Table } III\text{]} \\ &= P(0 \le z \le 0.8) + 0.4772 \\ &= 0.2881 + 0.4772 = 0.7653 \end{aligned}$$

(*ii*) When $X = 45, z = 3$

$$\begin{aligned} \therefore \quad P(X \ge 45) &= P(z \ge 3) = 0.5 - P(0 \le z \le 3) \\ &= 0.5 - 0.4986 = 0.0014 \end{aligned}$$

(*iii*)
$$\begin{aligned} P[|X - 30| \le 5] &= P[25 \le X \le 35] \\ &= P(-1 \le z \le 1) = 2P(0 \le z \le 1) \\ &= 2 \times 0.3413 = 0.6826 \\ \therefore \quad P[|X-30| > 5] &= 1 - P[|X-30| \le 5] \\ &= 1 - 0.6826 = 0.3174. \end{aligned}$$

Example 26.46. *A certain number of articles manufactured in one batch were classified into three categories according to a particular characteristic, being less than 50, between 50 and 60 and greater than 60. If this characteristic is known to be normally distributed, determine the mean and standard deviation for this batch if 60%, 35% and 5% were found in these categories.*

Solution. Let μ be the mean (at $z = 0$) and σ the standard deviation of the normal curve (Fig. 26.5).

Now 60% of the articles have the characteristic below 50, 35% between 50 and 60 and only 5% greater than 60.

Let the area to the left of the ordinate *PQ* be 60% and that between the ordinates *PQ* and *ST* be 35% so that the areas to the left of *PQ* ($z = z_1$) and *ST* ($z = z_2$) are 0.6 and 0.95 respectively, *i.e.*, the area *OPQR* = 0.6 – 0.5 = 0.1 and the area *OSTR* = 0.45.

$\therefore$ area corresponding to $z_1 \left(= \dfrac{50-\mu}{\sigma}\right) = 0.1$

and that corresponding to $z_2 \left(= \dfrac{60-\mu}{\sigma}\right) = 0.45$

From the table III, we have

$(50 - \mu)/\sigma = 0.2533$ and $(60 - \mu)/\sigma = 1.645$

whence $\sigma = 7.543$ and $\mu = 48.092.$

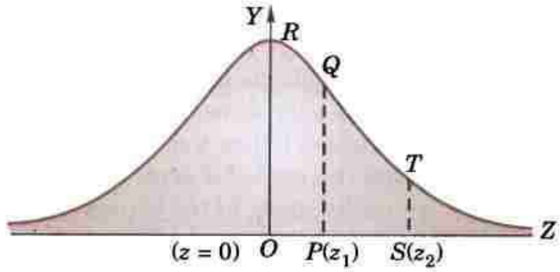

Fig. 26.5

Example 26.47. *In a normal distribution, 31% of the items are under 45 and 8% are over 64. Find the mean and S.D. of the distribution.* (V.T.U., 2009 ; S.V.T.U., 2008 ; Kurukshetra, 2007 S)

Solution. Let $\bar{x}$ be the mean and σ the S.D. 31% of the items are under 45 means area to the left of the ordinate $x = 45$. (Fig. 26.6)

When $x = 45$, let $z = z_1$ so that $z_1 = \dfrac{45-\bar{x}}{\sigma}$ ...(*i*)

$$\therefore \quad \int_{-\infty}^{z_1} \phi(z)\,dz = 0.31 \quad \text{or} \quad \int_{-\infty}^{0} \phi(z)\,dz - \int_{z_1}^{0} \phi(z)\,dz = 0.31$$

$$\text{Hence} \quad \int_{z_1}^{0} \phi(z)\,dz = \int_{-\infty}^{0} \phi(z)\,dz - 0.31 = 0.5 - 0.31 = 0.19$$

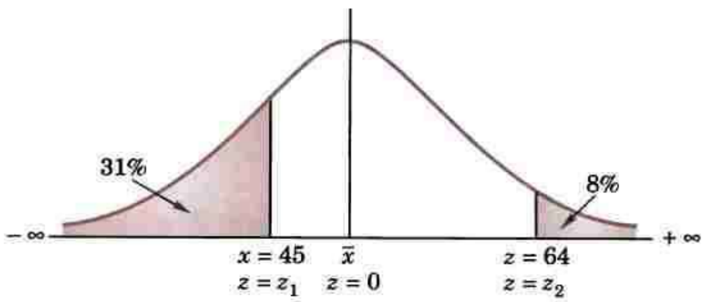

Fig. 26.6

From table III, $z_1 = -0.5$ ...(*ii*)

When $x = 64$, let $z = z_2$ so that $z_2 = (64 - \bar{x})/\sigma$ ...(*iii*)

$$\therefore \quad \int_{z_2}^{\infty} \phi(z)\,dz = 0.08 \quad \text{or} \quad \int_{0}^{\infty} \phi(z)\,dz - \int_{0}^{z_2} \phi(z)\,dz = 0.08$$

$$\text{Hence} \quad \int_{0}^{z_2} \phi(z)\,dz = \int_{0}^{\infty} \phi(z)\,dz - 0.08 = 0.5 - 0.08 = 0.42$$

From table III, $z_2 = 1.4$ ...(*iv*)

From (*i*) and (*ii*), $45 - \bar{x} = -0.5\sigma$

From (*iii*) and (*iv*), $64 - \bar{x} = 1.4\sigma$

Solving these equations, we get $\bar{x} = 50$ and $\sigma = 10$.

Example 26.48. *In a test on 2000 electric bulbs, it was found that the life of a particular make, was normally distributed with an average life of 2040 hours and S.D. of 60 hours. Estimate the number of bulbs likely to burn for*

(a) more than 2150 hours, *(b) less than 1950 hours and*

(c) more than 1920 hours and but less than 2160 hours. (Bhopal, 2008 S ; U.P.T.U., 2008)

Solution. Here $\mu = 2040$ hours and $\sigma = 60$ hours.

(*a*) For $x = 2150$, $\quad z = \dfrac{x-\mu}{\sigma} = 1.833.$

$\therefore$ area against $z = 1.83$ in the table III = 0.4664.

We, however, require the area to the right of the ordinate at $z = 1.83$. This area = 0.5 – 0.4664 = 0.0336.

Thus the number of bulbs expected to burn for more than 2150 hours

$$= 0.0336 \times 2000 = 67 \text{ approximately.}$$

(b) For $x = 1950$, $z = \dfrac{x-\mu}{\sigma} = -1.5$

The area required in this case is to the left of $z = -1.33$

i.e., $$= 0.5 - 0.4082 \text{ (table value for } z = 1.33)$$
$$= 0.0918.$$

$\therefore$ the number of bulbs expected to burn for less than 1950 hours

$$= 0.0918 \times 2000 = 184 \text{ approximately.}$$

(c) When $x = 1920$, $\quad z = \dfrac{1920 - 2040}{60} = -2$

When $x = 2160$, $\quad z = \dfrac{2160 - 2040}{60} = 2.$

The number of bulbs expected to burn for more than 1920 hours but less than 2160 hours will be represented by the area between $z = -2$ and $z = 2$. This is twice the area from the table for $z = 2$, *i.e.*, $= 2 \times 0.4772 = 0.9544$.

Thus the required number of bulbs = 0.9544 × 2000 = 1909 nearly.

Example 26.49. *If the probability of committing an error of magnitude x is given by*

$$y = \frac{h}{\sqrt{\pi}} e^{-h^2x^2};$$

compute the probable error from the following data :

$m_1 = 1.305$; $m_2 = 1.301$; $m_3 = 1.295$; $m_4 = 1.286$;
$m_5 = 1.318$; $m_6 = 1.321$; $m_7 = 1.283$; $m_8 = 1.289$;
$m_9 = 1.300$; $m_{10} = 1.286$. (*Kurukshetra, 2005*)

Solution. From the given data which is normally distributed, we have

$$\text{mean} = \frac{1}{10}\Sigma m_i = \frac{12.984}{10} = 1.2984$$

and
$$\sigma^2 = \frac{1}{10} \Sigma(m_i - \text{mean})^2$$
$$= \frac{1}{10} [(0.007)^2 + (0.003)^2 + (0.003)^2 + (0.012)^2 + (0.02)^2 + (0.023)^2$$
$$+ (0.015)^2 + (0.009)^2 + (0.002)^2 + (0.012)^2]$$
$$= 0.0001594 \text{ whence } \sigma = 0.0126.$$

$$\therefore \quad \textit{probable error} = \frac{2}{3}\sigma = 0.0084 \text{ approx.}$$

Example 26.50. *Fit a normal curve to the following distribution.*

x :	*2*	*4*	*6*	*8*	*10*	
f :	*1*	*4*	*6*	*4*	*1*	(*V.T.U., 2001*)

Solution. $$\text{Mean} = \frac{\Sigma fx}{\Sigma f} = \frac{2 + 16 + 36 + 32 + 10}{16} = 6$$

$$\text{S.D.} = \sqrt{\left[\frac{\Sigma fx^2}{\Sigma f} - \left(\frac{\Sigma fx}{\Sigma f}\right)^2\right]} = \sqrt{(40 - 36)} = 2$$

Taking $\mu = 6$, $\sigma = 2$ and $N = 16$, the equation of the normal curve is

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \text{ or } y = \frac{1}{2\sqrt{2\pi}} e^{-(x-6)^2/8} \quad ...(i)$$

Area under (i) in (x_1, x_2) or (z_1, z_2)

$$= \frac{1}{\sqrt{2\pi}} \int_0^{z_2} e^{-z^2/2} dz - \frac{1}{\sqrt{2\pi}} \int_0^{z_1} e^{-z^2/2} dz \qquad \text{where } z = \frac{x-6}{2}$$

To evaluate these integrals, we refer to table III.

Calculations :

Mid x	(x_1, x_2)	(z_1, z_2)	*Area under (i) in* (z_1, z_2)	*Expected frequency*
2	(1, 3)	(– 2.5, – 1.5)	0.4938 – 0.4332	16 × 0.606 = 0.97
4	(3, 5)	(– 1.5, – 0.5)	0.4332 – 0.1915	16 × 0.2417 = 3.9
6	(5, 7)	(– 0.5, 0.5)	0.1915 + 0.1915	16 × 0.383 = 6.1
8	(7, 9)	(0.5, 1.5)	0.4332 – 0.1915	16 × 0.2417 = 3.9
10	(9, 11)	(1.5, 2.5)	0.4938 – 0.4332	16 × 0.606 = 0.97

Hence the expected (theoretical) frequencies corrected to nearest integer are 1, 4, 6, 4, 1 which agree with the observed frequencies. This shows that the normal curve (i) is a proper fit to the given distribution.

PROBLEMS 26.7

1. Show that the standard deviation for a normal distribution is approximately 25% more than the mean deviation.
2. For a normally distributed variate with mean 1 and S.D. 3, find the probabilities that
   (*i*) $3.43 \le x \le 6.19$ (*ii*) $-1.43 \le x \le 6.19$.
3. If z is normally distributed with mean 0 and variance 1, find
   (*i*) $P_r\,[z \le -1.64]$; (*ii*) z_1 if $P_r[z \ge z_1] = 0.84$.
4. In a certain examination, the percentage of candidates passing and getting distinctions were 45 and 9 respectively. Estimate the average marks obtained by the candidates, the minimum pass and distinction marks being 40 and 75 respectively. (Assume the distribution of marks to be normal). *(Kottayam, 2005)*
5. A manufacturer of air-mail envelopes knows from experience that the weight of the envelopes is normally distributed with mean 1.95 gm and standard deviation 0.05 gm. About how many envelopes weighing (*i*) 2 gm or more ; (*ii*) 2.05 gm or more can be expected in a given packet of 100 envelopes.
6. The mean height of 500 students is 151 cm. and the standard deviation is 15 cm. Assuming that the heights are normally distributed, find how many students' heights lie between 120 and 155 cm. *(Burdwan, 2003)*
7. The mean and standard deviation of the marks obtained by 1000 students in an examination are respectively 34.4 and 16.5. Assuming the normality of the distribution, find the approximate number of students expected to obtain marks between 30 and 60.
8. In an examination taken by 500 candidates, the average and the standard deviation of marks obtained (normally distributed) are 40% and 10%. Find approximately
   (*i*) how many will pass, if 50% is fixed as a minimum ?
   (*ii*) what should be the minimum if 350 candidates are to pass ?
   (*iii*) how many have scored marks above 60% ?
9. The mean inside diameter of a sample of 200 washers produced by a machine is 5.02 mm and the standard deviation is 0.05 mm. The purpose for which these washers are intended allows a maximum tolerance in the diameter of 4.96 to 5.08 mm, otherwise the washers are considered defective. Determine the percentage of defective washers produced by the machine, assuming the diameters are normally distributed.
   [**Hint.** 4.96 in standard units = (4.96 – 5.02)/0.05 = – 1.2
   5.08 in standard units = (5.08 – 5.02)/0.05 = 1.2
   Proportion of non-defective washers = 2 (area between $z = 0$ and $z = 1.2$)
   = 0.7698 or 77% nearly.
   ∴ percentage of defective washers = 100 – 77 = 23%.]
10. Assuming that the diameters of 1000 brass plugs taken consecutively from a machine, form a normal distribution with mean 0.7515 cm. and standard deviation 0.0020 cm., how many of the plugs are likely to be rejected if the approved diameter is 0.752 ± 0.004 cm. ? *(Bhopal, 2002)*

11. It is given that the age of thermostats of a particular make follow the normal law with mean 5 years and S.D. 2 years. 1000 units are sold out every month. How many of them will have to be replaced at the end of the second year.
12. The income of a group of 10,000 persons was found to be normally distributed with mean Rs. 750 p.m., and standard deviation of Rs. 50. Show that, of this group, about 95% had income exceeding Rs. 668 and only 5% had income exceeding Rs. 832. Also find the lowest income among the richest 100. *(U.P.T.U., 2004 S)*
13. Find the equation of the best fitting normal curve to the following distribution :

x :	0	1	2	3	4	5
y :	13	23	34	15	11	4

14. Obtain the equation of the normal probability curve that may be fitted to the following data :

Variable :	4	6	8	10	12	14	16	18	20	22	24
Frequency:	1	7	15	22	35	43	38	20	13	5	1

15. A factory turns out an article by mass production and it is found that 10% of the product is rejected. Find the S.D. of the number of rejects and the equation to the normal curve to represent the number of rejects.

[**Hint.** $p = 0.1, q = 0.9, n = 100$.

$\therefore$ binomial distribution of rejects gives mean $= np = 10$, S.D. $= \sqrt{(npq)} = 3$

If this binomial distribution is approximated by a normal distribution, then the equation to the normal curve is

$$y = \frac{100}{\sigma\sqrt{(2\pi)}} e^{-(x-\mu)^2/2\sigma^2} \quad \text{where } \mu = 10, \sigma = 3.]$$

16. Given that the probability of committing an error of magnitude x is

$$y = \frac{h}{\sqrt{\pi}} e^{-h^2x^2}, \text{ show that the probable error is } 0.4769/h.$$

26.18 NORMAL APPROXIMATION TO BINOMIAL DISTRIBUTION

If the number of successes in a Binomial distribution range from x_1 to x_2, then the probability of getting these successes

$$= \sum_{r=x_1}^{x_2} {}^nC_r\, p^r\, q^{n-r}$$

As the number of trials increases, the Binomial distribution becomes approximated to the Normal distribution. The mean np and the variance npq of the binomial distribution will be quite close to the mean and standard deviation of the approximated normal distribution. Thus *for n sufficiently large (≥ 30), the binomial distribution with probability of success p, is approximated by the normal distribution with* $\mu = np$, $\sigma = \sqrt{npq}$.

We must however, be careful to get the correct values of z. For any success x, real class interval is $(x - 1/2, x + 1/2)$. Hence

$$z_1 = \frac{x_1 - \frac{1}{2} - \mu}{\sigma} = \frac{x_1 - \frac{1}{2} - np}{\sqrt{npq}}; \; z_2 = \frac{x_2 + \frac{1}{2} - np}{\sqrt{npq}}$$

so that $P(x_1 < x < x_2) = P(z_1 < z < z_2) = \int_{z_1}^{z_2} \phi(z)\, dz$ which can be calculated by using table III–Appendix 2.

Example 26.51. *In a referendum 60% of voters voted in favour. A random sample of 200 voters was selected. What is the probability that in the sample*

(a) more than 130 voted in favour ?

(b) between 105 and 130 inclusive voted in favour ?

(c) 120 voted in favour ?

Solution. Here $n = 200, p = 0.6, q = 0.4$

$$\therefore \quad \mu = np = 200 \times 0.6 = 120; \; \sigma = \sqrt{npq} = \sqrt{48} = 6.928$$

$$(a)\; P(x > 130) = P(x > 130.5) = P\left(x > \frac{130.5 - 120}{\sqrt{48}}\right) = P(z > 1.516) = 0.0648$$

(b) $P(105 < x < 130) = P(105.5 < x < 129.5)$

$$= P\left(\frac{105.5 - 120}{\sqrt{48}} < z < \frac{129.5 - 120}{\sqrt{48}}\right) = P(-2.09 < z < 1.37) = 0.8964$$

(c) $P(x = 120) = P(119.5 < x < 120.5)$

$$= P(-0.072 < z < 0.072) = 0.0575.$$

PROBLEMS 26.8

1. A pair of unbiased dice are rolled 180 times and their score recorded. Find
   (a) $P(x \le 20)$, (b) $P(20 \le x \le 40)$, (c) $P(20 < x \le 30)$.
2. A marksman has a probability of 0.9 of hitting a target on a single shot. If the marksman has 40 shots, what is the probability that he hits the target (a) at least 35 times; (b) between 34 and 36 times; (c) 37 times.
3. A certain drug is effective in 72% of cases. Given 2000 people are treated with the drug, what is the probability that it will be effective for (a) at least 1400 patients, (b) less than 1390 patients, (c) 1420 patients.

26.19 SOME OTHER DISTRIBUTIONS

Discrete distributions

(1) Geometric distribution. If p be the probability of success and k be the numbers of failures preceding the first success then this distribution is

$$P(k) = q^k p, \qquad k = 0, 1, 2, \ldots, q = 1 - p.$$

Obviously $\displaystyle\sum_{k=0}^{\infty} P(k) = p \sum_{k=0}^{\infty} q^k = p \cdot \frac{1}{1-q} = 1.$

It can easily be shown that mean $= q/p$, and variance $= q/p^2$.

(2) Negative binomial distribution. This distribution gives the probability that the event occurs for the kth time on the rth trial ($r \ge k$). If p be the probability of occurrence of an event then

$$P(k, r) = {}^{r-1}C_{k-1} p^k q^{r-k}.$$

It contains two parameters p and k. If $k = 1$, the Negative binomial distribution reduces to the geometric distribution.

(3) Hypergeometric distribution. Suppose a bag contains m white and n black balls. If r balls are drawn one at a time (*with replacement*), then the probability that k of them will be white is

$$P(k) = {}^mC_k\, {}^nC_{r-k} / {}^{m+n}C_r,\ k = 0, 1, \ldots r, r \le m, r \le n.$$

This distribution is known as *Hypergeometric distribution*.

For $\displaystyle\sum_{k=0}^{r} P(k) = 1$, since $\displaystyle\sum_{k=0}^{r} {}^mC_k\ {}^nC_{r-k} = {}^{m+n}C_r$

This can be proved by equating the coefficient of t^r in

$$(1 + t)^m (t + 1)^n = (1 + t)^{m+n}$$

Continuous distributions

(4) Uniform (or Rectangular) distribution. A random variable X is said to be uniformly distributed over the interval $-\infty < a < b < \infty$, if its density is given by

$$f(x) = \frac{1}{b-a},\ a < x < b \qquad \ldots(i)$$

The distribution given by (*i*) is called a *uniform distribution*. In this distribution, X takes the values with the same probability.

Its *mean* $\mu = \displaystyle\int_a^b x \cdot f(x)\, dx = \frac{1}{b-a}\left|\frac{x^2}{2}\right|_a^b = \frac{a+b}{2}.$

and *variance* $\sigma^2 = \mu_2' - (\mu)^2 = \displaystyle\int_a^b x^2 \cdot \frac{1}{b-a}\, dx - \left(\frac{a+b}{2}\right)^2 = \frac{1}{12}(b-a)^2.$

(5) Gamma distribution. This continuous distribution is given by $f(x) = \dfrac{\lambda}{\Gamma(r)}(\lambda x)^{r-1} e^{-\lambda x}$ for all $x \geq 0$, where r and λ (both > 0) are called the parameters of the *gamma distribution*. Its mean $= r/\lambda$ and variance $= r/\lambda^2$.

Gamma distribution tends to normal distribution as the parameter r tends to infinity.

(6) Exponential distribution. This distribution is a special case of gamma distribution when $r = 1$ so that $f(x) = \lambda e^{-\lambda x}$ for $x > 0$, where λ is a parameter.

It can be seen that mean $= 1/\lambda$, standard deviation $= 1/\lambda$.

This distribution plays an important role in the reliability and queuing theory.

(7) Weibull distribution*. This distribution is given by

$$f(x) = \frac{\alpha}{c} x^{\alpha-1} e^{-x^{\alpha}/c}, x > 0, c > 0$$

where c is a scale parameter and α a shape parameter.

Initially this distribution was used to describe experimentally observed variation in the fatigue resistance of steel and its elastic limits. But it has also been employed to study the variation of length of service of radio service equipment.

Example 26.52. *A die is cast until 6 appears. What is the probability that it must be cast more than 5 times ?*

Solution. Here probability of getting 6 is $p = \dfrac{1}{6}$. Then $q = \dfrac{5}{6}$.

If X is the number of tosses required for the first success, then

$$P(X = x) = q^{x-1} p \text{ for } x = 1, 2, 3, \ldots$$

$\therefore$ required probability $= P(X > 5) = 1 - P(X \leq 5)$

$$= 1 - \sum_{x=1}^{5}\left(\frac{5}{6}\right)^{x-1} \cdot \left(\frac{1}{6}\right) = 1 - \frac{1}{6}\left\{1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^3 + \left(\frac{5}{6}\right)^4\right\} = \left(\frac{5}{6}\right)^5.$$

Example 26.53. *A random variable X has a uniform distribution over (– 3, 3), find k for which*

$$P(X > k) = \frac{1}{3}.$$

Also evaluate $P(X < 2)$ and $P[|X - 2| < 2]$.

Solution. (*i*) Density of $X = f(x) = \dfrac{1}{b-a} = \dfrac{1}{3-(-3)} = \dfrac{1}{6}$

$\therefore$ $$P(X > k) = 1 - P(X \leq k) = 1 - \int_{-3}^{k} f(x)\, dx$$

$$= 1 - \frac{1}{6}\int_{-3}^{k} dx = 1 - \frac{1}{6}(k+3) = \frac{1}{3} \qquad \text{(given)}$$

This gives $k = 1$.

(*ii*) $$P(X < 2) = \int_{-3}^{2} f(x)\, dx = \frac{1}{6}\int_{-3}^{2} dx = \frac{5}{6}.$$

(*iii*) $$P[|X - 2| < 2] = P[2 - 2 < X < 2 + 2] = P[0 < x < 4] = \int_{0}^{3} f(x)\, dx = \frac{1}{6}\int_{0}^{3} dx = \frac{1}{2}.$$

PROBLEMS 26.9

1. Show that the mode of the *geometric distribution* $P(x) = \left(\dfrac{1}{2}\right)^x, x = 1, 2, 3 \ldots$, is unity.
2. Show that for the *rectangular distribution* $f(x) = 1, 0 \leq x \leq 1$,

$$\text{mean} = \frac{1}{2}, \text{ variance} = \frac{1}{12} \text{ and mean deviation} = \frac{1}{4}.$$

---

* It was first used by Swedish scientist *Weibull* in 1951.

3. Find the mean and variance of the *uniform distribution* given by $f(x) = 1/n$, $x = 1, 2, ..., n$.
4. Show that for the *exponential distribution*
   $dP = y_0 e^{-x/\sigma}$, $0 \le x \le \infty$,
   the mean and S.D. are both equal to σ.
5. Find the mean and variance of the *exponential distribution* $f(x) = \frac{1}{b} e^{-(x-a)/b}$ $x > a$. *(Mumbai, 2005)*
6. Find the moment generating function for the *triangular distribution* given by
   $$f(x) = x, \quad 0 \le x \le 1$$
   $$= 2 - x, \quad 1 \le x \le 2.$$
7. Show that for the *Gamma distribution* $f(x) = \dfrac{e^{-x} x^{l-1}}{\Gamma(l)}$, $0 < x < \infty$, the mean and variance are both equal to l.
8. Find the moment generating function of the *Gamma distribution* $f(x) = \dfrac{1}{\Gamma\left(\frac{1}{4}\right)} e^{-x} x^{-3/4}$, $x \ge 0$, at the origin.
   (J.N.T.U., 2006 ; Madras, 2000 S)

   [**Chebyshev's inequality***. If x is a continuous random variable with mean μ and variance σ^2, then for any positive real parameter t,
   $$P(|x - 2| \ge t) \le \sigma^2/t^2 \text{ or } P(|x - \mu| \le t) \ge 1 - \sigma^2/t^2.$$
   This result is known as *Chebyshev's inequality*. It gives limits to the probability that the value of the variate chosen at random will differ from mean by more than t.]
9. For the points on a symmetrical die, prove that *Chebyshev's inequality* gives
   $$P(|x - \bar{x}| > 2.5) < 0.478,$$
   while the actual probability is zero.
10. For the *Geometrical distribution* $P(x) = 2^{-x}$, $x = 1, 2, 3, ...$, prove that *Chebyshev's inequality* gives
    $$P(|x - 2| < 2) > \frac{1}{2},$$
    while the actual probability is 15/16.

26.20 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 26.10

Select the correct answer or fill up the blanks in each of the following problems :

1. The probability that A happens is 1/3. The odds against happening of A are
   (*a*) 2 : 1 (*b*) 2 : 3 (*c*) 3 : 2 (*d*) 5 : 2.
2. The odds in favour of an event A are 5 to 4. The probability of success of A is
   (*a*) 4/5 (*b*) 5/9 (*c*) 4/9.
3. The probability that A passes a test is 2/3 and the probability that B passes the same test is 3/5. The probability that only one of them passes is
   (*a*) 2/5 (*b*) 4/15 (*c*) 2/15 (*d*) 7/15.
4. A buys a lottery ticket in which the chance of winning is 1/10 ; B has a ticket in which his chance of winning is 1/20. The chance that atleast one of them wins is
   (*a*) 1/200 (*b*) 29/200 (*c*) 30/200 (*d*) 170/200.
5. The probability that a non-leap year should have 53 Tuesdays is ...
6. The probability of getting 2 or 3 or 4 from a throw of single dice is ...
7. The mean of the Binomial distribution with n observations and probability of success p, is
   (*a*) pq (*b*) np (*c*) $\sqrt{np}$ (*d*) $\sqrt{pq}$.
8. If the mean of a Poisson distribution is m, then S.D. of this distribution is
   (*a*) m^2 (*b*) $\sqrt{m}$ (*c*) m (*d*) none of these.

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* See footnote on page 571.

9. The S.D. of the Binomial distribution is

(a) $\sqrt{npq}$ (b) $\sqrt{np}$ (c) npq (d) pq.

10. In a Poisson distribution if $2P(x = 1) = P(x = 2)$, then the variance is

(a) 0 (b) -1 (c) 4 (d) 2.

11. If the probability of hitting a target by one shot be $p = 0.8$, then the probability that out of ten shots, seven will hit the target is ...
12. For a Poisson variate $x : P(x = 1) = P(x = 2)$, then the mean of x is ...
13. If $P(A) = 0.35$, $P(B) = 0.73$ and $P(A \cap B) = 0.14$, then $P(A \cap B') = ...$
14. If A and B are independent, $P(B) = 0.14$ and $P(A/B) = 0.24$, then $P(A) = ...$
15. The probability distribution of the number of heads, when two coins are tossed, is ...
16. The multiplication law of probability states that ...
17. The area under the standard normal curve which lies between $z = 0.90$ and $z = -1.85$ is ...
[Given $P(0 < z < 1.85) = 0.4678$, $P(0 < z < 0.9) = 0.3159$]
18. The mean, median and mode of a normal distribution are ...
19. The mean and variance of a Poisson distribution are ...
20. If A and B are two mutually exclusive events, then $P(A \cup B) = ...$
21. For a normal distribution $\beta_1 = ...$ and $\beta_2 = ...$
22. The number of ways in which five people can be lined up to get on a bus are ...
23. A shipment of 10 television sets contains 3 defective sets. The number of ways in which one can purchase 4 of these sets and receive 2 defective sets are ...
24. The probability of getting a total of 5 when a pair of dice is tossed is ...
25. If $P(B) = 0.81$ and $P(A \cap B) = 0.18$, then $P(A/B) = ...$
26. If two unbiased dice are thrown simultaneously, the probability that the sum of the numbers on them is at least 10, is ....
27. If X is a Poisson variate such that $P(X = 2) = P(X = 3)$, then $P(X = 0) = ...$
28. An unbiased die is tossed twice, then the probability of obtaining the sum 6, is ...
29. The variance of Poisson distribution with parameter $\lambda = 2$ is ...
30. The distribution in which mean, median, mode are equal is ...
31. For the Poisson variate, probability of getting at least one success is ...
32. Total number of events in rolling of an ideal die is ...
33. If X be normal with mean 10 and variance 4, then $P(X < 11) = ...$
34. If X is a binomial variate with parameters n and p, then its m.g.f. about the origin is ...
35. In a normal distribution, mean deviation : standard deviation = ...
36. If A and B are independent and $P(A) = 1/2$, $P(B) = 1/3$ then $P(A \cap B) = ...$
37. If X is the random variable representing the outcome of the roll of an ideal die, then $E(X) = ...$
38. If X is a binomial variate with $p = 1/5$ for the experiment of 50 trials, then the standard deviation is ...
39. The area under the whole normal curve is ...
40. Given $X = B(n, p)$, then the conditions under which X tends to a Poisson distribution, are ...
41. If A and B are mutually exclusive events then $P(A \cup B) = ...$
42. The probability of selecting x white balls from a bag containing y white and z red balls is ...
43. The mean of the binomial distribution is ...
44. If A and B are mutually exclusive events, $P(A) = 0.29$, $P(B) = 0.43$, then $P(A \cup B) = ...$ and $P(A \cap B') = ...$
45. If the mean and variance of a binomial variate are 12 and 4, then the distribution is ...
46. If x is a Poisson variable such that $P(x = 2) = 9\,P(x = 4) + 90\,P(x = 6)$, then the mean = ...
47. μ_r' the rth moment about the origin in terms of the m.g.f. is ...
48. The chance of throwing 7 in a single throw with two dice is ...
49. If A and B are any two events with $P(A) = 1/2$, $P(B) = 1/3$ and $P(A \cap B) = 1/4$, then $P(A/B) = ...$
50. In the roll of an ideal die, the probability of getting a prime number is ...
51. If A and B are mutually exclusive events, $P(A \cup B) = 0.6$, $P(B) = 0.4$, then $P(A) = ...$
52. The probability that a leap year should have 53 Sundays is ...
53. The probability density function of a binomial distribution is ...
54. The probable error is ... times S.D. approximately.
55. To fit a normal distribution, the parameters required are ...

56. A card is drawn from a well-shuffled pack of 52 cards, then the probability of this card being a red coloured ace is ...
57. If $P(1) = P(2)$, then the mean of the Poisson distribution is ...
58. Baye's theorem states that ...
59. If x is a Poisson variate such that $P(x = 1) = 0.3$ and $P(x = 2) = 0.2$, then $P(x = 0) = ...$
60. If A and B are events such that $P(A \cup B) = 3/4$, $P(A \cap B) = 1/4$, and $P(A) = 2/3$, then $P(A) = ...$
61. The chance of throwing 7 in a single throw with two dice is the same as that of getting 7 in two throws of a single die. (True or False)
62. If the mean of a Poisson distribution is 5, then its variance is 10. (True or False)
63. If X is normal with mean 3 and variance 1, then $X - 3$ is a variate with mean 0 and variance 1. (True or False)
64. If X is a binomial variable with parameters $n = 10$, $p = 1/4$, then its standard deviation is 2.25. (True or False)
65. The mean of a binomial distribution is 5 and S.D. is 3. (True or False)
66. The mean and variance of Poisson distribution are equal. (True or False)
67. The graph of the normal distribution is symmetric with respect to the line $y = x$. (True or False)
68. The standard deviation of a binomial distribution is np. (True or False)
69. $f(x) = kx$ in $0 < x < 1$ is a valid probability density function, if $k = ...$
70. If $V(x) = 2$, then $V(2x + 3) = ...$
71. The p.d.f. of an exponential distribution is ...
72. If X is uniformly distributed in $(-2, 3)$, then its variance is ...
73. The variance of Poisson distribution with parameter $\lambda = 2$ is ...
74. In Gamma distribution with parameter l, the variance is ...
75. If $f(x) = kx^3$, $0 < x < 1$ and 0 elsewhere, is a p.d.f., then $k = ...$
76. The m.g.f. of a random variable X is $(1 - 2t)^{-4}$, then $E(x)$ is ...
77. A random variable X has F-distribution with (m, n) degrees of freedom, then $1/X$ has the same distribution with ... degrees of freedom.
78. If X is a continuous random variable having the p.d.f. $f(x)$, then the m.g.f. about the origin is given by ...
79. If $f(X) = X + 2/k$, $X = 1, 2, 3, 4, 5$ is the probability distribution of a discrete random variable, then $k = ...$
80. The p.d.f. of Gamma variate is ...
81. If X is uniformly distributed in $[a, b]$, then $E(X) = ...$
82. The marks obtained by students were found normally distributed with mean 75 and variance 100. The percentage of students who scored more than 75 marks is ...
83. When four unbiased coins are tossed, the probability of getting two heads is ...
84. If $f(x) = \begin{cases} kxe^{-x}, & x > 0 \\ 0, & \text{elsewhere, is the p.d.f. of } x, \text{ then } k = ... \end{cases}$
85. If x is a uniform distribution defined in the interval $(4, 7)$, then its variance is ...
86. The p.d.f. of a continuous random variable is $f(x) = k/x^3$, $5 \le x \le 10$; 0 elsewhere, then the value of k is
    (*a*) 1 (*b*) 50 (*c*) 200/3 (*d*) 200.
87. The relation between probability density function and cumulative density function of a random variable is ...
88. If X has Poisson distribution with parameter λ, then P (X is even) $= ...$
89. Range of t- distribution is ...
90. If the p.d.f. of x is $f(x) = kx(1 - x)$, $0 < x < 1$, then $k = ...$
91. The function $f(x) = \begin{cases} kx^2, & 0 < x < 3 \\ 0, & \text{otherwise, is a probability function, then } k = ... \end{cases}$
92. If the random variable x is uniformly distributed in $[0, 3]$, then its p.d.f. is $f(x) = 3$, $0 < x < 3$; 0, elsewhere. (True or False)
93. Exponential distribution $f(x)$ is defined by $f(x) = ae^{-2x}$, $0 < x < \infty$, then $a = ...$
94. The p.d.f. of Beta distribution with $\alpha = 1$, $\beta = 4$ is $f(x) = ...$
95. For a standard normal variate z, $P(-0.72 \le z \le 0) = ...$

CHAPTER 27

Sampling & Inference

27.1 (1) INTRODUCTION

We know that a small section selected from the population is called a *sample* and the process of drawing a sample is called *sampling*. It is essential that a sample must be a *random* selection so that each member of the population has the same chance of being included in the sample. Thus the fundamental assumption underlying theory of sampling is *Random sampling*.

A special case of random sampling in which each event has the same probability p of success and the chance of success of different events are independent whether previous trials have been made or not, is known as *simple sampling*.

The statistical constants of the population such as mean (μ), standard deviation (σ) etc. are called the *parameters*. Similarly, constants for the *sample* drawn from the given population *i.e.*, mean ($\bar{x}$), standard deviation (S) etc. are called the *statistic*. The population parameters are in general, not known and their estimates given by the corresponding sample statistic are used. We use the Greek letters to denote the population parameters and Roman letters for sample statistic.

(2) Objectives of sampling. Sampling aims at gathering the maximum information about the population with the minimum effort, cost and time. The object of sampling studies is to obtain the best possible values of the parameters under specific conditions. Sampling determines the reliability of these estimates. The logic of the sampling theory is the logic of induction in which we pass from a particular (sample) to general (population). Such a generalisation from sample to population is called **Statistical Inference.**

27.2 SAMPLING DISTRIBUTION

Consider all possible samples of size n which can be drawn from a given population at random. For each sample, we can compute the mean. The means of the samples will not be identical. If we group these different means according to their frequencies, the frequency distribution so formed is known as *sampling distribution of the mean*. Similarly we can have *sampling distribution of the standard deviation* etc.

While drawing each sample, we put back the previous sample so that the parent population remains the same. This is called *sampling with replacement* and all the subsequent formulae will pertain to sampling with replacement.

(2) Standard error. The standard deviation of the sampling distribution is called the *standard error* (*S.E.*). Thus the standard error of the sampling distribution of means is called standard error of means. The standard error is used to assess the difference between the expected and observed values. The reciprocal of the standard error is called *precision*.

If $n \geq 30$, a sample is called *large* otherwise *small*. The sampling distribution of large samples is assumed to be normal.

27.3 (1) TESTING A HYPOTHESIS*

To reach decisions about populations on the basis of sample information, we make certain assumptions about the populations involved. Such assumptions, which may or may not be true, are called *statistical hypothesis*. By testing a hypothesis is meant a process for deciding whether to accept or reject the hypothesis. The method consists in assuming the hypothesis as correct and then computing the probability of getting the observed sample. If this probability is less than a certain preassigned value the hypothesis is rejected.

(2) Errors. If a hypothesis is rejected while it should have been accepted, we say that a *Type I error* has been committed. On the other hand, if a hypothesis is accepted while it should have been rejected, we say that a *Type II error* has been made. The statistical testing of hypothesis aims at limiting the Type I error to a preassigned value (say : 5% or 1%) and to minimize the Type II error. The only way to reduce both types of errors is to increase the sample size, if possible.

(3) Null hypothesis. The hypothesis formulated for the sake of rejecting it, under the assumption that it is true, is called the *null hypothesis* and is denoted by H_0. To test whether one procedure is better than another, we assume that there is no difference between the procedures. Similarly to test whether there is a relationship between two variates, we take H_0 that there is no relationship. By accepting a null hypothesis, we mean that on the basis of the statistic calculated from the sample, we do not reject the hypothesis. It however, does not imply that the hypothesis is proved to be true. Nor its rejection implies that it is disproved.

27.4 (1) LEVEL OF SIGNIFICANCE

The probability level below which we reject the hypothesis is known as the *level of significance*. The region in which a sample value falling is rejected, is known as the *critical region*. We generally take two critical regions which cover 5% and 1% areas of the normal curve. The shaded portion in the figure corresponds to 5% level of significance. Thus the *probability of the value of the variate falling in the critical region is the level of significance*.

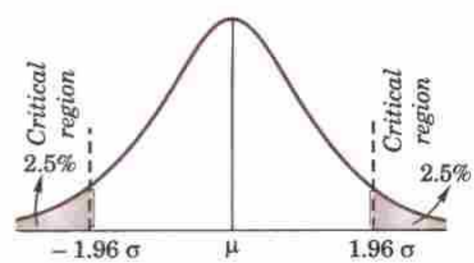

Fig. 27.1

Depending on the nature of the problem, we use a *single-tail test* or *double-tail test* to estimate the significance of a result. In a double-tail test, the areas of both the tails of the curve representing the sampling distribution are taken into account whereas in the single tail test, only the area on the right of an ordinate is taken into consideration. For instance, to test whether a coin is biased or not, double-tail test should be used, since a biased coin gives either more number of heads than tails (which corresponds to right tail), or more number of tails than heads (which corresponds to left tail only).

(2) Tests of significance. The procedure which enables us to decide whether to accept or reject the hypothesis is called the *test of significance*. Here we test whether the differences between the sample values and the population values (or the values given by two samples) are so large that they signify evidence against the hypothesis or these differences are so small as to account for fluctuations of sampling.

27.5 CONFIDENCE LIMITS**

Suppose that the sampling distribution of a statistic S is normal with mean μ and standard deviation σ. As in the Fig. 27.1 the sample statistic S can be expected to lie in the interval $(\mu - 1.96\sigma, \mu + 1.96\sigma)$ for 95% times *i.e.*, we can be confident of finding μ in the interval $(S - 1.96\sigma, S + 1.96\sigma)$ in 95% cases. Because of this, we call

*The American statistician *J. Neyman* (1894—1981) and the English statistician *E.S. Pearson* (1895—1980)–son of *Karl Pearson* (See footnote p. 843), developed a systematic theory of tests around 1930.

***J. Neyman* developed the modern theory and terminology of confidence limits.

$(S - 1.96\sigma, S + 1.96\sigma)$ the 95% confidence interval for estimation of μ. The ends of this interval (*i.e.* $S \pm 1.96\sigma$) are called 95% *confidence limits* (or *fiducial limits*) for S. Similarly $S \pm 2.58\sigma$ are 99% confidence limits. The numbers 1.96, 2.58 etc. are called *confidence coefficients*. The values of confidence coefficients corresponding to various levels of significance can be found from the normal curve area table VI – Appendix 2.

27.6 SIMPLE SAMPLING OF ATTRIBUTES

The sampling of attributes may be regarded as the selection of samples from a population whose members possess the attribute K or not K. The presence of K may be called a success and its absence a failure.

Suppose we draw a simple sample of n items. Clearly it is same as a series of n independent trials with the same probability p of success. The probabilities of 0, 1, 2, ..., n successes are the terms in the binomial expansion of $(q + p)^n$ where $q = 1 - p$.

We know that the mean of this distribution is np and standard deviation is $\sqrt{(npq)}$ *i.e.*, the expected value of success in a sample of size n is np and the standard error is $\sqrt{(npq)}$.

If we consider the proportion of successes, then

(*i*) mean proportion of successes $= np/n = p$.

(*ii*) standard error of the proportion of successes

$$= \sqrt{\left(n \cdot \frac{p}{n} \cdot \frac{q}{n}\right)} = \sqrt{\left(\frac{pq}{n}\right)}$$

and (*iii*) precision of the proportion of successes $= \sqrt{(n/pq)}$, which varies as $\sqrt{n}$, since p and q are constants.

27.7 TEST OF SIGNIFICANCE FOR LARGE SAMPLES

We know that the binomial distribution tends to normal for large n. Suppose we wish to test the hypothesis that the probability of success in such trial is p. Assuming it to be true, the mean μ and the standard deviation σ of the sampling distribution of number of successes are np and $\sqrt{(npq)}$ respectively.

For a normal distribution, only 5% of the members lie outside $\mu \pm 1.96\sigma$ while only 1% of the members lie outside $\mu \pm 2.58\sigma$.

If x be the observed number of successes in the sample and z is the standard normal variate then $z = (x - \mu)/\sigma$.

Thus we have the following test of significance :

(*i*) *If* $|z| < 1.96$, *difference between the observed and expected number of successes is not significant.*

(*ii*) *If* $|z| > 1.96$, *difference is significant at 5% level of significance.*

(*iii*) *If* $|z| > 2.58$, *difference is significant at 1% level of significance.*

Example 27.1. *A coin was tossed 400 times and the head turned up 216 times. Test the hypothesis that the coin is unbiased at 5% level of significance.* (V.T.U., 2007)

Solution. Suppose the coin is unbiased.

Then the probability of getting the head in a toss $= \frac{1}{2}$

$\therefore$ expected number of successes $= \frac{1}{2} \times 400 = 200$

and the observed value of successes = 216

Thus the excess of observed value over expected value = 216 – 200 = 16

Also S.D. of simple sampling $= \sqrt{npq} = \sqrt{\left(400 \times \frac{1}{2} \times \frac{1}{2}\right)} = 10$

Hence $z = \dfrac{x - np}{\sqrt{(npq)}} = \dfrac{16}{10} = 1.6$

As $z < 1.96$, the hypothesis is accepted at 5% level of significance *i.e.*, we conclude that the coin is unbiased at 5% level of significance.

Example 27.2. *A die was thrown 9000 times and a throw of 5 or 6 was obtained 3240 times. On the assumption of random throwing, do the data indicate an unbiased die ?* (V.T.U., 2010)

Solution. Suppose the die is unbiased.

Then the probability of throwing 5 or 6 with one die $= \frac{1}{3}$

The expected number of successes $= \frac{1}{3} \times 9000 = 3000$

and the observed value of successes = 3240

Thus the excess of observed value over expected value $3240 - 3000 = 240$

Also S.D. of simple sampling $= \sqrt{npq} = \sqrt{\left(9000 \times \frac{1}{3} \times \frac{2}{3}\right)} = 44.72$

Hence $z = \frac{x - np}{\sqrt{(npq)}} = \frac{240}{44.72} = 5.4$ nearly.

As $z > 2.58$, the hypothesis has to be rejected at 1% level of significance and we conclude that the die is biased.

Example 27.3. *In a locality containing 18000 families, a sample of 840 families was selected at random. Of these 840 families, 206 families were found to have a monthly income of ₹ 250 or less. It is desired to estimate how many out of 18,000 families have a monthly income of ₹ 250 or less. Within what limits would you place your estimate ?*

Solution. Here $p = \frac{206}{840} = \frac{103}{420}$ and $q = \frac{317}{420}$

$\therefore$ standard error of the population of families having a monthly income of ₹ 250 or less

$$= \sqrt{\left(\frac{pq}{n}\right)} = \sqrt{\left(\frac{103}{420} \times \frac{317}{420} \times \frac{1}{840}\right)} = .015 = 1.5\%$$

Hence taking $\frac{103}{420}$ (or 24.5%) to be the estimate of families having a monthly income of ₹ 250 or less in the locality, the limits are $(24.5 \pm 3 \times 1.5)\%$ *i.e.*, 20% and 29% approximately.

27.8 COMPARISON OF LARGE SAMPLES

Two large samples of sizes n_1, n_2 are taken from two populations giving proportions of attributes A's as p_1, p_2 respectively.

(*a*) On the hypothesis that the populations are similar as regards the attribute A, we combine the two samples to find an estimate of the common value of proportion of A's in the populations which is given by

$$p = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2}$$

If e_1, e_2 be the standard errors in the two samples then

$$e_1^2 = \frac{pq}{n_1} \text{ and } e_2^2 = \frac{pq}{n_2}$$

If e be the standard error of the difference between p_1 and p_2, then

$$e^2 = e_1^2 + e_2^2 = pq\left(\frac{1}{n_1} + \frac{1}{n_2}\right) \quad \therefore \quad z = \frac{p_1 \sim p_2}{e}$$

If $z > 3$, the difference between p_1 and p_2 is real one.

If $z < 2$, the difference may be due to fluctuations of simple sampling.

But if z lies between 2 and 3, then the difference is significant at 5% level of significance.

(*b*) If the proportions of A's are not the same in the two populations from which the samples are drawn, but p_1 and p_2 are the true values of proportions then S.E. e of the difference $p_1 \sim p_2$ is given by

$$e^2 = \frac{p_1q_1}{n_1} + \frac{p_2q_2}{n_2}$$

If $z = \frac{p_1 \sim p_2}{e} < 3$, the difference could have arisen due to fluctuations of simple sampling.

Example 27.4. *In a city A 20% of a random sample of 900 school boys had a certain slight physical defect. In another city B, 18.5% of a random sample of 1600 school boys had the same defect. Is the difference between the proportions significant ?* (V.T.U., 2003 S)

Solution. We have $n_1 = 900, n_2 = 1600$

and $p_1 = \frac{20}{100} = \frac{1}{5}, p_2 = \frac{18.5}{100}$

$\therefore$ $p = \frac{n_1p_1 + n_2p_2}{n_1 + n_2} = \frac{180 + 296}{900 + 1600} = 0.19$

and $q = 1 - 0.19 = 0.81$

Thus $e^2 = pq\left(\frac{1}{n_1} + \frac{1}{n_2}\right) = 0.19 \times 0.81\left(\frac{1}{900} + \frac{1}{1600}\right) = 0.0017$

giving $e = 0.04$ nearly.

Also $p_1 - p_2 = \frac{1.5}{100} = 0.015 \quad \therefore \quad z = \frac{p_1 - p_2}{e} = \frac{.015}{.04} = 0.37$

As $z < 1$, the difference between the proportions is not significant.

Example 27.5. *In two large populations there are 30% and 25% respectively of fair haired people. Is this difference likely to be hidden in samples of 1200 and 900 respectively from the two populations ?* (Coimbatore, 2001)

Solution. Here $p_1 = 0.3, p_2 = 0.25$ so that $p_1 - p_2 = 0.05$.

$\therefore$ $e^2 = \frac{p_1q_1}{n_1} + \frac{p_2q_2}{n_2} = \frac{0.3 \times 0.7}{1200} + \frac{0.25 \times 0.75}{900}$

so that $e = 0.0195$

$\therefore$ $z = \frac{p_1 - p_2}{e} = \frac{0.05}{0.0195} = 2.5$ nearly

Hence it is unlikely that the real difference will be hidden.

PROBLEMS 27.1

1. A die is tossed 960 times and it falls with 5 upwards 184 times. Is the die biased ? (V.T.U., 2006)
2. 12 dice are thrown 3086 times and a throw of 2, 3, 4 is reckoned as a success. Suppose that 19142 throws of 2, 3, 4 have been made out. Do you think that this observed value deviates from the expected value ? If so, can the deviation from the expected value be due to fluctuations of simple sampling ?
3. Balls are drawn from a bag containing equal number of black and white balls, each ball being replaced before drawing another. In 2250 drawings 1018 black and 1232 white balls have been drawn. Do you suspect some bias on the part of the drawer ?
4. A sample of 1000 days is taken from meteorological records of a certain district and 120 of them are found to be foggy. What are the probable limits to the percentage of foggy days in the district ?
5. In a group of 50 first cousines there were found to be 27 males and 23 females. Ascertain if the observed proportions are inconsistent with the hypothesis that the sexes should be in equal proportion.
6. A random sample of 500 apples was taken from a large consignment and 65 were found to be bad. Estimate the proportion of the bad apples in the consignment as well as the standard error of the estimate. Deduce that the percentage of bad apples in the consignment almost certainly lies between 8.5 and 17.5.
7. 400 children are chosen in an industrial town and 150 are found to be under weight. Assuming the conditions of simple sampling, estimate the percentage of children who are under weight in the industrial town and assign limits within which the percentage probably lies ?

8. A machine produces 16 imperfect articles in a sample of 500. After machine is overhauled, it produces 3 imperfect articles in a batch of 100. Has the machine been improved? (*Rohtak, 2005; Madras, 2003*)
9. One type of aircraft is found to develop engine trouble in 5 flights out of a total of 100 and another type in 7 flights out of a total of 200 flights. Is there a significant difference in the two types of aircrafts so far as engine defects are concerned?
10. In a sample of 600 men from a certain city, 450 are found smokers. In another sample of 900 men from another city, 450 are smokers. Do the data indicate that the cities are significantly different with respect to the habit of smoking among men? (*J.N.T.U., 2003*)
11. In a sample of 500 people from a state 280 take tea and rest take coffee. Can we assume that tea and coffee are equally popular in the state at 5% level of significance?

27.9 (1) SAMPLING OF VARIABLES

We now consider sampling of a variable such as weight, height, etc. Each member of the population gives a value of the variable and the population is a frequency distribution of the variable. Thus a random sample of size n from the population is same as selecting n values of the variables from those of the distribution.

(2) Sampling distribution of the mean. *If a population is distributed normally with mean μ and standard deviation σ, then the means of all positive random samples of size n, are also distributed normally with mean μ and standard error $\sigma/\sqrt{n}$.* This result shows how the precision of a sample mean increases as the sample size increases.

27.10 CENTRAL LIMIT THEOREM

This is a very important theorem regarding the distribution of the mean of a sample *if the parent population is non-normal and the sample size is large.*

If the variable X has a non-normal distribution with mean μ and standard deviation σ, then the limiting distribution of

$z = \dfrac{\bar{x}-\mu}{\sigma\sqrt{n}}$ as $n \to \infty$, *is the standard normal distribution* (*i.e.*, with mean 0 and unit S.D.)

There is no restriction upon the distribution of X except that it has a finite mean and variance. This theorem holds well for a sample of 25 or more which is regarded as large.

Thus *if the population is normal, the sampling distribution of the mean is also normal with mean μ and S.E. $\sigma/\sqrt{n}$, while for large samples the same result holds even if the distribution of the population is non-normal.* This property is of universal use and throws light on the importance of normal distribution in statistical theory.

27.11 CONFIDENCE LIMITS FOR UNKNOWN MEAN

Let the population from which a random sample of size n is drawn, have mean μ and S.D. σ. If μ is not known, there will be a range of values of μ for which observed mean $\bar{x}$ of the sample is not significant at any assigned level of probability. The relative deviation of $\bar{x}$ from μ is $(\bar{x}-\mu)/\sqrt{\sigma}$.

If $\bar{x}$ is not significant at 5% level of probability, then

$$|(\bar{x}-\mu)\sqrt{n}/\sigma| < 1.96 \quad i.e. \quad \bar{x} - 1.96\sigma/\sqrt{n} < \mu < \bar{x} + 1.96\sigma/\sqrt{n}$$

Thus 95% confidence or fiducial limits for the mean of the population corresponding to given sample are $\bar{x} \pm 1.96\sigma/\sqrt{n}$.

Similarly *99% confidence limits for μ are $\bar{x} \pm 2.58\sigma/\sqrt{n}$.*

Example 27.6. *A sample of 900 members is found to have a mean of 3.4 cm. Can it be reasonably regarded as a truly random sample from a large population with mean 3.25 cm and S.D. 1.61 cm.*

Solution. Here $\bar{x} = 3.4$ cm, $n = 900$, $\mu = 3.25$ and $\sigma = 1.61$ cm

$$\therefore \qquad z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} = \frac{3.4-3.25}{1.61/30} = 2.8$$

As $z > 1.96$, the deviation of the sample mean from the mean of the population is significant at 5% level of significance. Hence it cannot be regarded as a random sample.

Example 27.7. *The mean of a certain normal population is equal to the standard error of the mean of the samples of 100 from that distribution. Find the probability that the mean of the sample of 25 from the distribution will be negative ?*

Solution. If μ be the mean and σ the S.D. of the distribution, then

$$\mu = \text{S.E. of the sample means} = \frac{\sigma}{\sqrt{100}} = \frac{\sigma}{10}$$

Also for a sample of size 25, we have $z = \dfrac{\bar{x} - \mu}{\sigma/\sqrt{25}} = \dfrac{\bar{x} - \sigma/10}{6/5} = \dfrac{5\bar{x}}{\sigma} - \dfrac{1}{2}$

Since $\bar{x}$ is negative, $z < -\frac{1}{2}$.

$\therefore$ the probability that a normal variate $z < -\frac{1}{2}$

$$= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{1}{2}} e^{-\frac{1}{2}z^2} dz = \frac{1}{\sqrt{2\pi}} \int_{1/2}^{\infty} e^{-\frac{1}{2}z^2} dz$$

$$= 0.5 - 0.915 = 0.3085, \text{ from the tables.}$$

Example 27.8. *An unbiased coin is thrown n times. It is desired that the relative frequency of the appearance of heads should lie between 0.49 and 0.51. Find the smallest value of n that will ensure this result with 90% confidence.*

Solution. S.E. of the proportion of heads $= \sqrt{\dfrac{pq}{n}} = \sqrt{\dfrac{1}{2} \cdot \dfrac{1}{2} \cdot \dfrac{1}{n}} = \dfrac{1}{2\sqrt{n}}$

90% of confidence = 45% or .45 of the total area under the normal curve on each side of the mean.

$\therefore$ the corresponding value of $z = 1.645$, from the tables.

Thus $\qquad p \mp 1.645\sigma = 0.49 \quad \text{or} \quad 0.51.$

i.e., $\qquad 0.5 - 1.645 \cdot \dfrac{1}{2\sqrt{n}} = 0.49 \quad \text{and} \quad 0.5 + 1.645 \cdot \dfrac{1}{2\sqrt{n}} = 0.51$

whence $\qquad \dfrac{1.645}{2\sqrt{n}} = 0.01 \quad \text{or} \quad \sqrt{n} = \dfrac{329}{4} \quad \text{or} \quad n = 6765$ approximately.

27.12 TEST OF SIGNIFICANCE FOR MEANS OF TWO LARGE SAMPLES

(*a*) Suppose two random samples of sizes n_1 and n_2 have been *drawn from the same population* with S.D. σ. We wish to test whether the difference between the sample means $\bar{x}_1$ and $\bar{x}_2$ is significant or is merely due to fluctuations of sampling.

If the samples are independent, then the standard error e of the difference of their means is given by

$$e^2 = e_1^2 + e_2^2$$

where $e_1 = \sigma/\sqrt{n_1}$, $e_2 = \sigma/\sqrt{n_2}$ are the S.E.s of the means of the two samples.

$$\therefore \qquad e = \sigma\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}. \quad \text{Hence} \quad z = \frac{\bar{x}_1 - \bar{x}_2}{\sigma\sqrt{(1/n_1 + 1/n_2)}}$$

is normally distributed with mean zero and S.D. 1.

Test of significance (n_1, n_2 being large) :

If $z > 1.96$, then the difference is significant at 5% level of significance.

If $z > 3$, it is highly probable that either the samples have not been drawn from the same population or the sampling is not simple.

(*b*) *If the samples are known to be drawn from different populations* with means μ_1, μ_2 and standard deviations σ_1 and σ_2. Then the standard error *e* of their means is given by

$$e = \sqrt{\left(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)}$$

Assuming that the two populations have the same mean (*i.e.*, $\mu_1 = \mu_2$), the difference of the means of the samples will be normally distributed with mean zero and S.D. *e*. Now the same procedure of test of significance is applied.

Example 27.9. *The means of simple samples of sizes 1000 and 2000 are 67.5 and 68.0 cm respectively. Can the samples be regarded as drawn from the same population of S.D. 2.5 cm.* *(Madras, 2002)*

Solution. We have $\bar{x}_1 = 67.5,\ \bar{x}_2 = 68.0$
$n_1 = 1000,\ n_2 = 2000.$

On the hypothesis, that the samples are drawn from the same population of S.D. $\sigma = 2.5$, we get

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sigma\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{67.5 - 68.0}{2.5\sqrt{\left(\frac{1}{1000} + \frac{1}{2000}\right)}}$$

$$= \frac{0.5}{2.5 \times 0.0387} = \frac{0.5}{0.09675} = 5.1$$

Hence the difference between the sample means *i.e.*, 5.1 is very much greater than 1.96 and is therefore significant. Thus, the samples cannot be regarded as drawn from the same population.

Example 27.10. *A sample of height of 6400 soldiers has a mean of 67.85 inches and a standard deviation of 2.56 inches while a simple sample of heights of 1600 sailors has a mean of 68.55 inches and a standard deviation of 2.52 inches. Do the data indicate that the sailors are on the average taller than soldiers?*

Solution. Here $\bar{x}_1 = 67.85,\ \sigma_1 = 2.56,\ n_1 = 6400$
$\bar{x}_2 = 68.55,\ \sigma_2 = 2.52,\ n_2 = 1600.$

$\therefore$ S.E. of the difference of the mean heights is

$$e = \sqrt{\left[\frac{{\sigma_1}^2}{n_1} + \frac{{\sigma_2}^2}{n_2}\right]} = \sqrt{\left[\frac{(2.56)^2}{6400} + \frac{(2.52)^2}{1600}\right]}$$

$$= \sqrt{[.001024 + .003969]} = 0.005 \text{ nearly.}$$

Also difference between the means $= \bar{x}_2 - \bar{x}_1 = 0.7$, which > 10*e*. This is highly significant. Hence the data indicates that the sailors are on the average taller than the soldiers.

PROBLEMS 27.2

1. A sample of 400 items is taken from a normal population whose mean is 4 and variance 4. If the sample mean is 4.45, can the samples be regarded as a simple sample ?
2. To know the mean weights of all 10-year old boys in Delhi, a sample of 225 is taken. The mean weight of the sample is found to be 67 pounds with a S.D. of 12 pounds. Can you draw any inference from it about the mean weight of the population ?
3. A normal population has a mean 0.1 and a S.D. of 2.1. Find the probability that the mean of simple sample of 900 members will be negative.
4. If the mean breaking strength of copper wire is 575 lbs. with a standard deviation of 8.3 lbs., how large a sample must be used in order that there be one chance in 100 that the mean breaking strength of the sample is less than 572 lbs. ?

   [**Hint.** $|z| = \left|\frac{\bar{x} - \mu}{\sigma}\sqrt{n}\right| = \frac{3}{8.3}\sqrt{n}$

   Also from table IV, $z = 2.33$. Hence $n = 42$ nearly.]

5. A research worker wishes to estimate mean of a population by using sufficiently large sample. The probability is 95% that sample mean will not differ from the true mean by more than 25% of the S.D. How large a sample should be taken ?
6. The density function of a random variable x is $f(x) = ke^{-2x^2+10x}$. Find the upper 5% point of the distribution of the means of the random sample of size 25 from the above population.
7. The means of two large samples of 1000 and 2000 members are 168.75 cms. and 170 cms. respectively. Can the samples be regarded as drawn from the same population of standard deviation 6.25 cms.
8. If 60 new entrants in a given university are found to have a mean height of 68.60 inches and 50 seniors a mean height of 69.51 inches ; is the evidence conclusive that the mean height of the seniors is greater than that of the new entrants ? Assume the standard deviation of height to be 2.48 inches.
9. A sample of 100 electric bulbs produced by manufacturer A showed a mean life time of 1190 hours and a standard deviation of 90 hours. A sample of 75 bulbs produced by manufacturer B showed a mean life time of 1230 hours, with a standard deviation of 120 hours. Is there a difference between the mean life time of two brands at a significance level of (*i*) 0.05 (*ii*) 0.01.
10. A random sample of 1000 men from North India shows that their mean wage is ₹ 5 per day with a S.D. of ₹ 1.50. A sample of 1500 men from South India gives a mean wage of ₹ 4.50 per day with a standard deviation of ₹ 2. Does the mean rate of wages varies as between the two regions ?

27.13 SAMPLING OF VARIABLES—SMALL SAMPLES

In case of large samples, sampling distribution approaches a normal distribution and values of sample statistic are considered best estimates of the parameters in a population. It will no longer be possible to assume that statistics computed from small samples are normally distributed. As such, *a new technique has been devised for small samples which involves the concept of 'degrees of freedom' which we explain below.*

Number of degrees of freedom *is the number of values in a set which may be assigned arbitrarily.* For instance, if $x_1 + x_2 + x_3 = 15$ and we assign any values of two of the variables (say : x_1, x_2), then the values of x_3 will be known. The two variables are therefore, free and independent choices for finding the third. Hence these are the degrees of freedom. *If there are n observations, the degrees of freedom (d.f.) are (n – 1).* In other words, while finding the mean of a small sample, one degree of freedom is used up and $(n - 1)$ *d.f.* are left to estimate the population variance.

27.14 (1) STUDENT'S t-DISTRIBUTION

Consider a small sample of size n, drawn from a normal population with mean μ and s.d. σ. If $\bar{x}$ and σ_s be the sample mean and s.d., then the statistic, 't' is defined as

$$t = \frac{\bar{x} - \mu}{\sigma}\sqrt{n} \quad \text{or} \quad t = \frac{\bar{x} - \mu}{\sigma_s}\sqrt{(n-1)},$$

where $\nu = n - 1$ denotes the *df.* of t. If we calculate t for each sample, we obtain the sampling distribution for t. This distribution known as *Student's t-distribution**, is given by

$$y = \frac{y_0}{(1 + t^2/\nu)^{(\nu+1)/2}} \qquad ...(1)$$

where y_0 is constant such that the area under the curve is unity.

(2) Properties of t-distribution.

I. This curve is symmetrical about the line $t = 0$, like the normal curve, since only even powers of t appear in (1). But it is more peaked than the normal curve with the same S.D. The t-curve approaches the horizontal axis less rapidly than the normal curve. Also t-curve attains its maximum value at $t = 0$ so that its mode coincides with the mean. (Fig. 27.2)

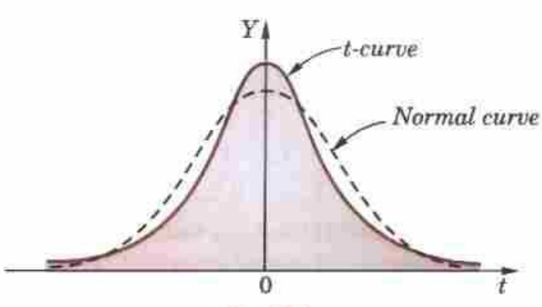

Fig. 27.2

*This distribution was first found by the English statistician *W.S. Gosset* in 1908 who wrote under the pen-name of '*Student*'. *R.A. Fisher* defined t correctly and found its distribution in 1926.

II. The limiting form of t-distribution when $\nu \to \infty$ is given by $y = y_0 e^{-\frac{1}{2}t^2}$ which is a normal curve. This shows that *t is normally distributed for large samples.*

III. The probability P that the value of t will exceed t_0 is given by

$$P = \int_{t_0}^{\infty} y\, dx$$

The values of t_0 have been tabulated for various values of P for various values of ν from 1 to 30 (Table IV – Appendix 2).

IV. Moments about the mean

All the moments of odd order about the mean are zero, due to its symmetry about the line $t = 0$.

Even order moments about the mean are

$$\mu_2 = \frac{\nu}{\nu - 2}, \ \mu_4 = \frac{3\nu^2}{(\nu-2)(\nu-4)}, \ldots$$

The t-distribution is often used in tests of hypothesis about the mean when the population standard deviation σ is unknown.

27.15 SIGNIFICANCE TEST OF A SAMPLE MEAN

Given a random small sample $x_1, x_2, x_3 \ldots x_n$ from a normal population, we have to test the hypothesis that mean of the population is μ. For this, we first calculate $t = (\bar{x} - \mu)\sqrt{n}/\sigma_s$

where $$\bar{x} = \frac{1}{n}\sum_1^n x_i, \sigma_s^2 = \frac{1}{n-1}\sum_1^n (x_i - \bar{x})^2$$

Then find the value of P for the given df from the table.

If the calculated value of $t > t_{0.05}$, the difference between $\bar{x}$ and μ is said to be significant at 5% level of significance.

If $t > t_{0.01}$, the difference is said to be significant at 1% level of significance.

If $t < t_{0.05}$, the data is said to be consistent with the hypothesis that μ is the mean of the population.

Example 27.11. *A certain stimulus administered to each of 12 patients resulted in the following increases of blood pressure : 5, 2, 8, – 1, 3, 0, – 2, 1, 5, 0, 4, 6. Can it be concluded that the stimulus will in general be accompanied by an increase in blood pressure.* (V.T.U., 2007)

Solution. Let us assume that the stimulus administered to all the 12 patients will increase the B.P. Taking the population to be normal with mean $\mu = 0$ and S.D. σ,

$$\bar{d} = \frac{5+2+8-1+3+0-2+1+5+0+4+6}{12} = 2.583$$

$$\sigma^2 = \frac{\Sigma d^2}{n} - \bar{d}^2 = \frac{1}{12}[5^2 + 2^2 + 8^2 + (-1^2) + 3^2 + 0^2 + (-2)^2 + 1^2 + 5^2 + 0^2 + 4^2 + 6^2] - (2.583)^2$$

$$= 8.744. \quad \therefore \quad \sigma = 2.9571$$

Now $$t = \frac{\bar{d} - \mu}{\sigma_s}\sqrt{(n-1)} = \frac{2.583 - 0}{2.9571}\sqrt{(12-1)} = 2.897$$

Here $d.f.$ $\gamma = 12 - 1 = 11$.

For $\gamma = 11$, $t_{0.05} = 2.2$, from table IV.

Since the $|t| > t_{0.05}$, our assumption is rejected *i.e.*, the stimulus does not increase the B.P.

Example 27.12. *The nine items of a sample have the following values : 45, 47, 50, 52, 48, 47, 49, 53, 51. Does the mean of these differ significantly from the assumed mean of 47.5 ?* (V.T.U., 2010)

Solution. We find the mean and standard deviation of the sample as follows :

x	$d = x - 48$	d^2
45	−3	9
47	−1	1
50	2	4
52	2	4
48	0	0
47	−1	1
49	1	1
53	5	25
51	3	9
Total	10	66

$$\therefore \qquad \bar{x} = \text{mean} = 48 + \frac{\Sigma d}{9} = 48 + \frac{10}{9} = 49.1$$

$$\sigma_s^{\ 2} = \frac{\Sigma d^2}{9} - \left(\frac{\Sigma d}{9}\right)^2 = \frac{66}{9} - \frac{100}{81} = \frac{494}{81}$$

$$\therefore \qquad \sigma_s = 2.47$$

Hence
$$t = \frac{\bar{x} - \mu}{\sigma_s}\sqrt{(n-1)} = \frac{49.1 - 47.5}{2047}\sqrt{8} = 1.83$$

Here $\qquad d.f.\ \nu = 9 - 1 = 8$

For $\nu = 8$, we get from table IV, $t_{0.05} = 2.31$.

As calculated value of $t < t_{0.05}$, the value of t is not significant at 5% level of significance which implies that there is no significant difference between $\bar{x}$ and μ. Thus the test provides no evidence against the population mean being 47.5.

Example 27.13. *A machinist is making engine parts with axle diameter of 0.7 inch. A random sample of 10 parts shows mean diameter 0.742 inch with a standard deviation of 0.04 inch. On the basis of this sample, would you say that the work is inferior ?* (V.T.U., 2009)

Solution. Here we have $\mu = 0.700$, $\bar{x} = 0.742$, $\sigma_x = 0.040$, $n = 10$.

Taking the hypothesis that the product is not inferior *i.e.*, there is no significant difference between $\bar{x}$ and μ.

$$\therefore \qquad t = \frac{\bar{x} - \mu}{\sigma_x}\sqrt{(n-1)} = \frac{0.742 - 0.700}{0.040}\sqrt{(10-1)} = \frac{0.126}{0.040} = 3.16$$

Degrees of freedom $\rho = 10 - 1 = 9$.

For $\rho = 9$, we get from table IV, $t_{0.05} = 2.262$.

As the calculated value of $t > t_{0.05}$, the value of t is significant at 5% level of significance. This implies that $\bar{x}$ differs significantly from μ and the hypothesis is rejected. Hence the work is inferior. In fact, the work is inferior even at 2% level of significance.

Example 27.14. *Show that 95% confidence limits for the mean μ of the population are* $\bar{x} \pm \frac{\sigma_s}{\sqrt{n}} t_{0.05}$.

Deduce that for a random sample of 16 values with mean 41.5 inches and the sum of the squares of the deviations from the mean 135 inches^2 and drawn from a normal population, 95% confidence limits for the mean of the population are 39.9 and 43.1 inches.

Solution. (*a*) The probability P that $t \le t_{0.05}$ is 0.95. Hence the 95% confidence limits for μ are given by

$$\left|\frac{\bar{x} - \mu}{\sigma_s}\sqrt{n}\right| \le t_{0.05}$$

or
$$|\bar{x} - \mu| \le \frac{\sigma_s}{\sqrt{n}} t_{0.05} \quad \text{or} \quad \bar{x} - \frac{\sigma_s}{\sqrt{n}} t_{0.05} \le \mu \le \bar{x} + \frac{\sigma_s}{\sqrt{n}} t_{0.05}$$

We can, therefore, say with a confidence coefficient 0.95 that the confidence interval $\bar{x} \pm \frac{\sigma_s}{\sqrt{n}} t_{0.05}$ contains the population mean μ.

(b) Here, $n = 16$, $\nu = n - 1 = 15$, $\sigma_s = \sqrt{\frac{135}{15}} = 3$.

Also from table IV, $t_{0.05}$ (for $\nu = 15$) = 2.13

$$\therefore \qquad \frac{\sigma_s}{\sqrt{n}} t_{0.05} = \frac{3}{4} \times 2.13 = 1.6 \text{ approx.}$$

Hence the required confidence limits are 41.5 ± 1.6 *i.e.*, 39.9 and 43.1 inches.

27.16 SIGNIFICANCE TEST OF DIFFERENCE BETWEEN SAMPLE MEANS

Given two independent samples $x_1, x_2, x_3, \ldots, x_{n_1}$ and $y_1, y_2, \ldots, y_{n_2}$ with means $\bar{x}$ and $\bar{y}$ and standard deviations σ_x and σ_y from a normal population with the same variance, we have to test the hypothesis that the population means μ_1 and μ_2 are the same.

For this, we calculate
$$t = \frac{\bar{x} \sim \bar{y}}{\sigma \sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \qquad \ldots(1)$$

where
$$\bar{x} = \frac{1}{n_1} \sum_1^{n_1} x_i, \; \bar{y} = \frac{1}{n_2} \sum_1^{n_2} y_i$$

and
$$\sigma_s^2 = \frac{1}{n_1 + n_2 - 2}\left[(n_1 - 1)\sigma_x^2 + (n_2 - 1)\sigma_y^2\right] = \frac{1}{n_1 + n_2 - 2}\left\{\sum_1^{n_1} (x_i - \bar{x})^2 + \sum_1^{n_2} (y_j - \bar{y})^2\right\}$$

It can be shown that the variate t defined by (1) follows the t-distribution with $n_1 + n_2 - 2$ degrees of freedom.

If the calculated value of $t > t_{0.05}$, the difference between the sample means is said to be significant at 5% level of significance.

If $t > t_{0.01}$, the difference is said to be significant at 1% level of significance.

If $t < t_{0.05}$, the data is said to be consistent with the hypothesis, that $\mu_1 = \mu_2$.

Cor. If the two samples are of the same size and the data are paired, then t is defined by

$$t = \frac{\bar{d} - 0}{(\sigma/\sqrt{n})} \quad \text{where} \quad \sigma^2 = \frac{1}{n-1} \sum_1^{n} (d_i - \bar{d})^2$$

d_i = difference of the ith members of the samples ;

$\bar{d}$ = mean of the differences = $\Sigma d_i/n$; and the number of *d.f.* = $n - 1$.

Example 27.15. *Eleven students were given a test in statistics. They were given a month's further tuition and a second test of equal difficulty was held at the end of it. Do the marks give evidence that the students have benefitted by extra coaching ?*

Boys	1	2	3	4	5	6	7	8	9	10	11
Marks I test	23	20	19	21	18	20	18	17	23	16	19
Marks II test	24	19	22	18	20	22	20	20	23	20	17

(V.T.U., 2011 S)

Solution. We compute the mean and the S.D. of the difference between the marks of the two tests as under :

$$\bar{d} = \text{mean of } d\text{'s} = \frac{11}{11} = 1 \; ; \; \sigma_s{}^{-2} = \frac{\Sigma(d - \bar{d})^2}{n - 1} = \frac{50}{10} = 5 \quad i.e., \quad \sigma_s = 2.24$$

Assuming that the students have not been benefited by extra coaching, it implies that the mean of the difference between the marks of the two tests is zero *i.e.*, $\mu = 0$.

Then $$t = \frac{\bar{d} - \mu}{\sigma_s}\sqrt{n} = \frac{1-0}{2.24}\sqrt{11} = 1.48 \text{ nearly and } df\ \nu = 11 - 1 = 10.$$

Students	x_1	x_2	$d = x_2 - x_1$	$d - \bar{d}$	$(d - \bar{d})^2$
1	23	24	1	0	0
2	20	19	– 1	– 2	4
3	19	22	3	2	4
4	21	18	– 3	– 4	16
5	18	20	2	1	1
6	20	22	2	1	1
7	18	20	2	1	1
8	17	20	3	2	4
9	23	23	—	– 1	1
10	16	20	4	3	9
11	19	17	– 2	– 3	9
			$\Sigma d = 11$		$\Sigma(d - \bar{d})^2 = 50$

From table IV, we find that $t_{0.05}$ (for $\nu = 10$) = 2.228. As the calculated value of $t < t_{0.05}$, the value of t is not significant at 5% level of significance *i.e.*, the test provides no evidence that the students have benefited by extra coaching.

Example 27.16. *From a random sample of 10 pigs fed on diet A, the increases in weight in a certain period were 10, 6, 16, 17, 13, 12, 8, 14, 15, 9 lbs. For another random sample of 12 pigs fed on diet B, the increases in the same period were 7, 13, 22, 15, 12, 14, 18, 8, 21, 23, 10, 17 lbs. Test whether diets A and B differ significantly as regards their effect on increases in weight ?*

Solution. We calculate the means and standard deviations of the samples as follows :

	Diet A			Diet B	
x_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$	y_i	$y_i - \bar{y}$	$(y_i - \bar{y})^2$
10	– 2	4	7	– 8	64
6	– 6	36	13	– 2	4
16	4	16	22	7	49
17	5	25	15	0	0
13	1	1	12	– 3	9
12	0	0	14	– 1	1
8	– 4	16	18	3	9
14	2	4	8	– 7	49
15	3	9	21	6	36
9	– 3	9	23	8	64
			10	– 5	25
			17	2	4
120	0	120	180	0	314

$$\bar{x} = \frac{120}{10} = 12 \text{ lbs.}, \bar{y} = \frac{180}{12} = 15 \text{ lbs.}$$

$$\sigma_s^2 = [\Sigma(x_i - \bar{x})^2 + \Sigma(y_i - \bar{y})^2]/(n_1 + n_2 - 2)$$

$$= (120 + 314)/(10 + 12 - 2) = (434/20) = 21.1$$

$\therefore$ $$\sigma_s = 4.65$$

Assuming that the samples do not differ in weight so far as the two diets are concerned *i.e.*, $\mu_1 - \mu_2 = 0$.

Hence $$t = \frac{(\bar{y} - \bar{x}) - 0}{\sigma_s\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{15 - 12}{4.65\sqrt{\left(\frac{1}{10} + \frac{1}{12}\right)}} = \frac{3}{4.65}\sqrt{\frac{120}{22}} = 1.6 \text{ nearly}$$

Here $\qquad d.f.\ \nu = n_1 + n_2 - 2 = 10 + 12 - 2 = 20.$

For $\nu = 20$, we find $t_{0.05} = 2.09$ [From table IV

$\therefore$ the calculated value of $t < t_{0.05}$.

Hence the difference between the sample means is not significant *i.e.*, the two diets do not differ significantly as regards their effect on increase in weight.

PROBLEMS 27.3

1. Find the student's t for the following variable values in a sample of eight : $-4, -2, -2, 0, 2, 2, 3, 3$; taking the mean of the universe to be zero.
2. A random sample of 10 boys had the following I.Q. :

   70, 120, 110, 101, 88, 83, 95, 98, 107, 100.

   Do these data support the assumption of a population mean I.Q. of 100 (at 5% level of significance) ? *(V.T.U., 2006 ; Coimbatore, 2001)*
3. A sample of 10 measurements of the diameter of a sphere gave a mean of 12 cm and a standard deviation 0.15 cm. Find 95% confidence limits for the actual diameter.
4. A random sample of size 25 from a normal population has the mean $\bar{x} = 47.5$ and s.d. $S = 8.4$. Does this information refute the claim that the mean of the population is $\mu = 42.1$. *(J.N.T.U., 2003)*
5. A process for making certain bearings is under control if the diameter of the bearings have the mean 0.5 cm. What can we say about this process if a sample of 10 of these bearings has a mean diameter of 0.506 cm. and s.d. of 0.004 cm ?
6. A machine is supposed to produce washers of mean thickness 0.12 cm. A sample of 10 washers was found to have a mean thickness of 0.128 cm and standard deviation 0.008. Test whether the machine is working in proper order at 5% level of significance.
7. Find out the reliability of the sample mean of the following data : *Breaking strength of 10 specimens of 1.04 cms diameter hard-drawn copper wire :*

Specimen :	1	2	3	4	5	6	7	8	9	10
Breaking Strength (kgs) :	578	572	570	568	572	570	570	572	526	584

8. Test runs with 6 models of an experimental engine showed that they operated for 24, 28, 21, 23, 32 and 22 minutes with a gallon of fuel. If the probability of a Type I error is at the most 0.01, is this evidence against a hypothesis that on the average this kind of engine will operate for atleast 29 minutes per gallon of the same fuel. Assume normality. *(J.N.T.U., 2003)*
9. Two horses A and B were tested according to the time (in seconds) to run a particular race with the following results :

Horse A :	28	30	32	33	33	29	and 34
Horse B :	29	30	30	24	27	and 29	

   Test whether you can discriminate between two horses ? *(Rohtak, 2005 ; Coimbatore, 2001)*

10. A group of 10 rats fed on a diet A and another group of 8 rats fed on a different diet B, recorded the following increase in weights :

Diet A :	5	6	8	1	12	4	3	9	6	10	gm
Diet B :	2	3	6	8	10	1	2	8	gm.		

   Does it show that superiority of diet A over that of B ? *(Madras, 2003)*

11. A group of boys and girls were given an intelligence test. The mean score, S.D.s and numbers in each group are as follows :

	Boys	*Girls*
Mean	124	121
S.D.	12	10
n	18	14

   Is the mean score of boys significantly different from that of girls ?

12. The means of two random samples of sizes 9 and 7 are 196.42 and 198.82 respectively. The sum of the squares of the deviations from the mean are 26.94 and 18.73 respectively. Can the sample be considered to have been drawn from the same normal population ? *(Mumbai, 2004)*

27.17 (1) CHI-SQUARE (χ^2) TEST

When a fair coin is tossed 80 times, we expect from theoretical considerations that heads will appear 40 times and tail 40 times. But this never happens in practice *i.e.*, the results obtained in an experiment do not agree exactly with the theoretical results. *The magnitude of discrepancy between observation and theory is given by the quantity* χ^2 (pronounced as *chi-square*). If $\chi^2 = 0$, the observed and theoretical frequencies completely agree. As the value of χ^2 increases, the discrepancy between the observed and theoretical frequencies increases.

(1) Definition. *If* $O_1, O_2, ..., O_n$ *be a set of observed (experimental) frequencies and* $E_1, E_2, ..., E_n$ *be the corresponding set of expected (theoretical) frequencies, then* χ^2 *is defined by the relation*

$$\chi^2 = \frac{(O_1 - E_1)^2}{E_1} + \frac{(O_2 - E_2)^2}{E_2} + ... + \frac{(O_n - E_n)^2}{E_n}$$

$$= \sum \frac{(O_i - E_i)^2}{E_i} \qquad ...(1)$$

with n – 1 degrees of freedom. [$\Sigma O_i = \Sigma E_i = n$ the total frequency]

(2) Chi-square distribution*

If $x_1, x_2, ..., x_n$ be n independent normal variates with mean zero and s.d. unity, then it can be shown that $x_1^2 + x_2^2 + ... + x_n^2$, is a random variate having χ^2-distribution with *ndf*.

The equation of the χ^2-curve is

$$y = y_0 e^{-\chi^2/2} (\chi^2)^{(\nu - 1)/2} \qquad ...(2)$$

where $\nu = n - 1$ (Fig. 27.3).

(3) Properties of χ^2-distribution

I. If $\nu = 1$, the χ^2-curve (2) reduces to $y = y_0 e^{-\chi^2/2}$, which is the exponential distribution.

II. If $\nu > 1$, this curve is tangential to x-axis at the origin and is positively skewed as the mean is at ν and mode at $\nu - 2$.

III. The probability P that the value of χ^2 from a random sample will exceed χ_0^2 is given by

$$P = \int_{\chi_0^2}^{\infty} y\, dx.$$

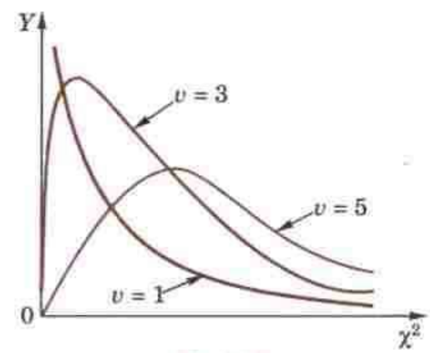

Fig. 27.3

The values of χ_0^2 have been tabulated for various values of P and for values of ν from 1 to 30. (Table-V-Appendix 2)

For $\nu > 30$, the χ^2-curve approximates to the normal curve and we should refer to normal distribution tables for significant values of χ^2.

IV. Since the equation of χ^2-curve does not involve any parameters of the population, *this distribution does not depend on the form of the population* and is therefore, very useful in a large number of problems.

V. Mean = γ and *variance* = 2γ.

27.18 GOODNESS OF FIT

The value of χ^2 is used to test whether the deviations of the observed frequencies from the expected frequencies are significant or not. It is also used to test how will a set of observations fit a given distribution, χ^2 therefore, provides a test of goodness of fit and may be used to examine the validity of some hypothesis about an observed frequency distribution. As a test of goodness of fit, it can be used to study the correspondence between theory and fact.

This is a non-parametric distribution-free test since in this we make no assumption about the distribution of the parent population.

**Hamlet* discovered this distribution in 1875. *Karl Pearson* rediscovered it independently in 1900 and applied it to test 'goodness of fit'.

Procedure to test significance and goodness of fit.

(*i*) Set up a 'null hypothesis' and calculate

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i}$$

(*ii*) Find the *df* and read the corresponding values of χ^2 at a prescribed significance level from Table V.

(*iii*) From χ^2-table, we can also find the probability P corresponding to the calculated values of χ^2 for the given *d.f.*

(*iv*) If $P < 0.05$, the observed value of χ^2 is significant at 5% level of significance.

If $P < 0.01$, the value is significant at 1% level.

If $P > 0.05$, it is a good fit and the value is not significant.

Example 27.17. *In experiments on pea breeding, the following frequencies of seeds were obtained :*

Round and yellow	*Wrinkled and yellow*	*Round and green*	*Wrinkled and green*	*Total*
315	*101*	*108*	*32*	*556*

Theory predicts that the frequencies should be in proportions 9 : 3 : 3 : 1. Examine the correspondence between theory and experiment.

Solution. The corresponding frequencies are

$$\frac{9}{16} \times 556, \frac{3}{16} \times 556, \frac{3}{16} \times 556, \frac{1}{16} \times 556 \text{ i.e., } 313, 104, 104, 35.$$

Hence $$\chi^2 = \frac{(315-313)^2}{313} + \frac{(101-104)^2}{104} + \frac{(108-104)^2}{104} + \frac{(32-35)^2}{35}$$

$$= \frac{4}{313} + \frac{9}{104} + \frac{16}{104} + \frac{9}{35} = 0.51 \quad \text{and} \quad df\ \nu = 4 - 1 = 3.$$

For $\nu = 3$, we have $\chi^2_{0.05} = 7.815$ [From Table V

Since the calculated value of χ^2 is much less than $\chi^2_{0.05}$, there is a very high degree of agreement between theory and experiment.

Example 27.18. *A set of five similar coins is tossed 320 times and the result is*

No. of heads :	*0*	*1*	*2*	*3*	*4*	*5*
Frequency :	*6*	*27*	*72*	*112*	*71*	*32*

Test the hypothesis that the data follow a binomial distribution.

(*Kottayam, 2005 ; P.T.U., 2005 ; V.T.U., 2004*)

Solution. For $\nu = 5$, we have $\chi^2_{0.05} = 11.07$.

p, probability of getting a head $= \frac{1}{2}$; q, probability of getting a tail $= \frac{1}{2}$.

Hence the theoretical frequencies of getting 0, 1, 2, 3, 4, 5 heads are the successive terms of the binomial expansion $320(p+q)^5$

$$= 320\,[p^5 + 5p^4q + 10p^3q^2 + 10p^2q^3 + 5pq^4 + q^5]$$

$$= 320\left[\frac{1}{32} + \frac{5}{32} + \frac{10}{32} + \frac{10}{32} + \frac{5}{32} + \frac{1}{32}\right] = 10 + 50 + 100 + 100 + 50 + 10$$

Thus the theoretical frequencies are 10, 50, 100, 100, 50, 10.

Hence

$$\chi^2 = \frac{(6-10)^2}{10} + \frac{(27-50)^2}{50} + \frac{(72-100)^2}{100} + \frac{(112-100)^2}{100} + \frac{(71-50)^2}{50} + \frac{(32-10)^2}{10}$$

$$= \frac{1}{100}(160 + 1058 + 784 + 144 + 882 + 4840) = \frac{7868}{100} = 78.68$$

and $\quad df\ \nu = 6 - 1 = 5.$

Since the calculated value of χ^2 is much greater than $\chi^2_{0.05}$, the hypothesis that the data follow the binomial law is rejected.

Example 27.19. *Fit a Poisson distribution to the following data and test for its goodness of fit at level of significance 0.05.*

x :	0	1	2	3	4
f :	419	352	154	56	19

(V.T.U., 2008)

Solution. Mean $m = \dfrac{\Sigma f x_i}{\Sigma f_i} = \dfrac{904}{1000} = 0.904$. $\therefore$ $e^{-m} = e^{-0.904} = 0.4049$.

Hence the theoretical frequencies are $\dfrac{1000 \times e^{-m}(m)^r}{r!}$, $r = 0, 1, 2, 3, 4$

i.e.,

x :	0	1	2	3	4	Total
f :	404.9 (406.2)	366	165.4	49.8	11.3 (12.6)	997.4

In order that the total observed and expected frequencies may agree, we take the first and last theoretical frequencies as 406.2 and 12.6 instead of 404.9 and 11.3 as shown in brackets. (In case, the expected frequencies are less than 10, we group together such classes. Here of course, none of the frequencies < 10).

Hence

$$\chi^2 = \frac{(419-406.2)^2}{406.2} + \frac{(352-366)^2}{366} + \frac{(154-165.4)^2}{165.4} + \frac{(56-49.8)^2}{49.8} + \frac{(19-12.6)^2}{12.6}$$
$$= 0.403 + 0.536 + 0.786 + 0.772 + 3.251 = 5.748$$

Since the mean of the theoretical distribution has been estimated from the given data and the totals have been made to agree, there are two constraints so that the number of degrees of freedom $\nu = 5 - 2 = 3$.

For $\nu = 3$, we have $\chi^2_{0.05} = 7.82$. [From Table V]

Since the calculated value of $\chi^2 < \chi^2_{0.05}$, the agreement between the fact and theory is good and hence the Poisson distribution can be fitted to the data.

Example 27.20. *Fit a normal distribution to the following data of weights of 100 students of Delhi University and test the goodness of fit.*

Weights (kg) :	60-62	63-65	66-68	69-71	72-74
Frequency :	5	18	42	27	8

Solution. Here $N = 100$, mean $m = 67.45$ and S.D. $\sigma = 2.92$. The calculations are arranged as follows* :

Class boundary (x)	$z = (x - m)/\sigma$	*Area under normal curve from 0 to z*	*Area for each class (A)**	*Expected frequency* $(f_e = N \times A)$
59.5	− 2.72	0.4967		
			0.0413	4.13
62.5	− 1.70	0.4554		
			0.2068	20.68
65.5	− 0.67	0.2486		
			0.3892	38.92
68.5	0.36	0.1406		
			0.2771	27.71
71.5	1.39	0.4177		
			0.0743	7.43
74.5	2.41	0.4920		

* *A* is obtained by subtracting the successive areas in the third column when the corresponding values of z have the same sign and adding them when the z values have opposite signs (which occurs only once).

$$\therefore \quad \chi^2 = \frac{(5-4.13)^2}{4.13} + \frac{(18-20.68)^2}{20.68} + \frac{(42-38.92)^2}{38.92} + \frac{(27-27.71)^2}{27.71} + \frac{(8-7.43)^2}{7.43}$$
$$= 0.1833 + 0.3473 + 0.2437 + 0.0182 + 0.0437 = 0.8362$$

As regards the number of degrees of freedom (γ), there are three constraints (*i*) discrepancy between total observed and total estimated frequencies (*ii*) and (*iii*) mean (m) and standard deviation (σ) have been estimated from the sample data. $\therefore \quad r = 5 - 3 = 2$.

For $\gamma = 2$, $\chi^2_{0.05} = 0.103$ from table V.

Since $\chi^2 = 0.8362 > 0.103$. Hence the fit is not good.

PROBLEMS 27.4

1. Five dice were thrown 96 times and the number of times 4, 5 or 6 were thrown were :

No. of dice showing 4, 5 or 6 :	5	4	3	2	1	0
Frequency :	8	18	35	24	10	1

Find the probability of getting this result by chance ?

2. Genetic theory states that children having one parent of blood type M and other blood type N will always be one of the three types M, MN, N and that the proportions of these types will on average be 1 : 2 : 1. A report states that out of 300 children having one M parent and one N parent, 30% were found to be of type M, 45% of type MN and remainder of type N. Test the hypothesis by χ^2 test.

3. A die was thrown 60 times and the following frequency distribution was observed :

Faces :	1	2	3	4	5	6
f_0 :	15	6	4	7	11	17

Test whether the die is unbiased ?

4. The following table gives the number of aircraft accidents that occurred during the various days of the week. Find whether the accidents are uniformly distributed over the week ?

Days :	Sun	Mon	Tue	Wed	Thu	Fri	Sat	Total
No. of accidents :	14	16	8	12	11	9	14	84

(Hissar, 2005)

5. Fit a binomial distribution to the data :

x :	0	1	2	3	4	5
f :	38	144	342	287	164	25

and test for goodness of fit, at the level of significance 0.05.

6. In 1000 extensive sets of trials for an event of small probability, the frequencies f_0 of the number x of successes proved to be :

x :	0	1	2	3	4	5	6	7
f_0 :	305	366	210	80	28	9	2	1

Fit a Poisson distribution to the data and test the goodness of fit.

7. The frequencies of localities according to the number of deaths due to cholera during eight years in 1000 localities is as follows :

No. of deaths :	0	1	2	3	4	5	6	7
No. of localities :	314	355	204	86	29	9	3	0

Fit a suitable distribution to the data and test the goodness of fit.

8. Obtain the equation of the normal curve that may be fitted to the data and test the goodness of fit.

x :	4	6	8	10	12	14	16	18	20	22	24	Total
$f(x)$:	1	7	15	22	35	43	38	20	13	5	1	200

27.19 (1) F-DISTRIBUTION*

Let $x_1, x_2, \ldots x_{n_1}$ and $y_1, y_2, \ldots y_{n_2}$ be the values of two independent random samples drawn from the normal populations σ^2 having equal variances.

* This distribution was introduced by the English statistician *Prof. R.A. Fisher* (1890–1962) who had greatly influenced the development of modern statistics.

Let $\bar{x}_1$ and $\bar{x}_2$ be the sample means and $s_1^2 = \frac{1}{n_1 - 1}\sum_1^{n_1}(x_i - \bar{x})^2$, $s_2^2 = \frac{1}{n_2 - 1}\sum_1^{n_2}(y_i - \bar{y})^2$ be the sample variances.

Then we define F by the relation

$$F = \frac{s_1^2}{s_2^2} \qquad (s_1^2 > s_2^2)$$

This gives F-distribution (also known as variance ratio distribution) with $\gamma_1 = n_1 - 1$ and $\gamma_2 = n - 2$ degrees of freedom. *The larger of the variances is placed in the numerator.*

(2) Properties. I. The F-distribution curve lies entirely in the first quadrant and is *unimodal*.

II. The F-distribution is independent of the population variance σ^2 and depends on γ_1 and γ_2 only.

III. $F_\alpha(\gamma_1, \gamma_2)$ is the value of F for γ_1 and γ_2 of such that the area to the right of F_α is α.

IV. It can be shown that the mode of F-distribution is less than unity.

(3) Significance test. Snedecor's F-tables give 5% and 1% points of significance for F. (Table VI – Appendix 2). 5% points of F mean that area under the F-curve to the right of the ordinate at a value of F, is 0.05. Clearly value of F at 5% significance is lower than that at 1%. F-distribution is very useful for testing the equality of population means by comparing sample variances. As such it forms the basis of *analysis of variance*.

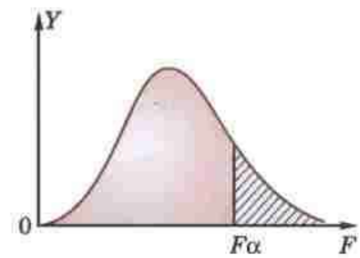

Fig. 27.4

Example 27.21. *Two samples of sizes 9 and 8 give the sum of squares of deviations from their respective means equal to 160 inches² and 91 inches² respectively. Can these be regarded as drawn from the same normal population ?* (V.T.U., 2002)

Solution. We have $\Sigma(x - \bar{x})^2 = 160$ and $\Sigma(y - \bar{y})^2 = 91$

$$\therefore \quad s_1^2 = \frac{160}{8} = 20$$

and $$s_2^2 = \frac{91}{7} = 13.$$

Hence $$F = \frac{s_1^2}{s_2^2} = \frac{20}{13} = 1.54 \text{ nearly.}$$

For $\gamma_1 = 8, \gamma_2 = 7$, we have $F_{0.05} = 3.73$. [From Table VI]

Since the calculated value of $F < F_{0.05}$, the population variances are not significantly different. Thus the two samples can be regarded as drawn from two normal populations with the same variance. If the two populations are to be same, their means should also be the same which can be verified by applying t-test provided the sample means are known.

Example 27.22. *Measurements on the length of a copper wire were taken in 2 experiments A and B as under :*

A's measurements (mm) : 12.29, 12.25, 11.86, 12.13, 12.44, 12.78, 12.77, 11.90, 12.47.

B's measurements (mm) : 12.39, 12.46, 12.34, 12.22, 11.98, 12.46, 12.23, 12.06.

Test whether B's measurements are more accurate than A's. (The readings taken in both cases being unbiased)

Solution. Readings in both cases being unbiased, B's measurements will be taken more accurate if its population variance is less than that of A's measurements.

Under the hypothesis that the two populations have the same variance (*i.e.* $\sigma_1^2 = \sigma_2^2$), we have

$$F = \frac{s_1^2}{s_2^2}$$

with $\gamma_1 = n_1 - 1 = 8$ and $\gamma_2 = n_2 - 1 = 7$.

We calculate the s.d.'s of the two series as follows :

A's measurements			*B's measurements*		
x	$u = 100(x - 12)$	u^2	y	$v = 100(y - 12)$	v^2
12.29	29	841	12.39	39	1521
12.25	25	625	12.46	46	2116
11.86	– 14	196	12.34	34	1156
12.13	13	169	12.22	22	484
12.44	44	1936	11.98	– 2	4
12.78	78	6084	12.46	46	2116
12.77	77	5929	12.23	23	529
11.90	– 10	100	12.06	6	36
12.47	47	2209			
	289	18089		214	7962

$$\therefore \quad s_1^2 = \frac{1}{n_1 - 1}\left[18089 - \frac{(289)^2}{n_1}\right] = \frac{1}{8}(18089 - 9280) = 1101.1$$

$$s_2^2 = \frac{1}{n_2 - 1}\left[7962 - \frac{(214)^2}{n_2}\right] = \frac{1}{7}(7962 - 5724) = 319.7$$

$$\therefore \quad F = \frac{s_1^2}{s_2^2} = \frac{1101.1}{319.7} = 3.44$$

For $\gamma_1 = 8$ and $\gamma_2 = 7$, from table VI, $F_{0.05} = 3.73$ and $F_{0.01} = 6.84$.

Since the calculated value of F is less than both $F_{0.05}$ and $F_{0.01}$, the result is insignificant at both 5% and 1% level.

Hence there is no reason to say that B's measurements are more accurate than those of A's.

27.20 (1) FISHER'S z-DISTRIBUTION

Changing the variable F to z by the substitution $z = \frac{1}{2}\log_e F$ or $F = e^{2z}$ in the F-distribution, we get the *Fisher's z-distribution.*

It is more nearly symmetrical than F-distribution. Table showing the values of z that will be exceeded in simple sampling with probabilities 0.05 and 0.01 have been prepared for various values of v_1 and v_2.

(2) Significance test. As z-table give only critical values corresponding to right hand tail areas, therefore 5% (or 1%) points of z imply that the area to the right of the ordinate at z is 0.05 (or 0.01). In other words, *5% and 1% points of z correspond to 10% and 2% levels of significance respectively.*

Example 27.23. *Two gauge operations are tested for precision in making measurements. One operator completes a set of 26 readings with a standard deviations of 1.34 and the other does 34 readings with a standard deviations of 0.98. What is the level of significance of this difference.*

(Given that for $\nu_1 = 25$ and $\nu_2 = 33$, $z_{0.05} = 0.305$, $z_{0.01} = 0.432$)

Solution. We have $n_1 = 26$, $\sigma_x = 1.34$; $n_2 = 34$, $\sigma_y = 0.98$

$$\therefore \quad s_1^2 = \frac{n_1}{n_1 - 1} \cdot \sigma_x^2 = \frac{26}{25}(1.34)^2 \approx (1.34)^2 \quad \text{and} \quad s_2^2 = \frac{n_2}{n_2 - 1} \cdot \sigma_y^2 = \frac{34}{33}(0.98)^2 \approx (0.98)^2$$

$$\text{Hence} \quad F = \left(\frac{1.34}{0.98}\right)^2 = 1.8696 \quad \text{and} \quad z = \frac{1}{2}\log_e F = 1.1513 \log_{10} 1.8696 = 0.3129$$

Since the calculated value of z is just greater than $z_{0.05}$ and less than $z_{0.01}$, the difference between the standard deviation is just significant at 5% level and insignificant at 1% level.

PROBLEMS 27.5

1. Two samples of 9 and 7 individuals have variances 4.8 and 9.6 respectively. Is the variance 9.6 significantly greater than the variance 4.6 ?
2. Test for breaking strength were carried out on two lots of 5 and 9 steel wires. The variance of one lot was 230 and that of other was 492. Is there a significant difference in their variability ?
3. Show how you would use Fisher's z-test to decide whether the two sets of observations 17, 27, 18, 25, 27, 29, 27, 23, 17 and 16, 16, 20, 16, 20, 17, 15, 21, indicate samples from the same universe.
4. In two groups of ten children each, the increase in weight due to different diets during the same period, were in pounds

   3, 7, 5, 6, 5, 4, 4, 5, 3, 6

   8, 5, 7, 8, 3, 2, 7, 6, 5, 7.

   Is there a significant difference in their variability ?
5. The mean diameter of rivets produced by two firms A and B are practically the same but their standard deviations are different. For 16 rivets manufactured by firm A, the S.D. is 3.8 mm while for 22 rivets manufactured by firm B is 2.9 mm. Do you think products of firm B are of better quality than those of firm A.
6. The I.Q.'s of 25 students from one college showed a variance of 16 and those of an equal number from the other college had a variance of 8. Discuss whether there is any significant difference in variability of intelligence.
7. Two random samples from two normal populations are given below :

Sample I	:	16	26	27	23	24	22
Sample II	:	33	42	35	32	28	31

   Do the estimates of population variances differ significantly ?

Degrees of freedom	:	(5, 5)	(5, 6)	(6, 5)
5% value of F	:	5.05	4.39	4.95.

8. Two independent samples of sizes 7 and 6 have the following values :

Sample A :	28	30	32	33	33	29	34
Sample B :	29	30	30	24	27	29	

   Examine whether the samples have been drawn from normal populations having the same variance ?

   [Given that the values of F at 5% level for (6, 5) d.f. is 4.95 and for (5, 6) d.f. is 4.39].

27.21 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 27.6

Select the correct answer or fill up the blanks in each of the following questions :

1. The 'null hypothesis' implies that ....
2. The uses of t-distribution are ....
3. Type I and type II errors are such that ....
4. A single-tailed test is used when ....
5. Control limit theorem states that ....
6. A hypothesis is true, but is rejected. Then this is an error of type ....
7. If the standard deviation of a χ^2 distribution is 10, then its degree of freedom is ....
8. Range of F-distribution is ....
9. A hypothesis is false but accepted, then there is an error of type ....
10. The mean and variance of a χ^2 distribution with 8 degrees of freedom are .... and .... respectively.
11. In a t-distribution of sample size n, the degrees of freedom are ....
12. The test statistic $F = \dfrac{s_1^2}{s_2^2}$ is used when

    (*i*) $s_2^2 > s_1^2$ (*ii*) $s_2^2 < s_1^2$ (*iii*) $s_1^2 = s_2^2$ (*iv*) none of these.
13. The t-test is applicable to samples for which n is ....
14. The two main uses of χ^2-test are ....
15. Range of t-distribution is ....
16. If two samples are taken from two populations of unequal variances, we can apply t-test to test the difference of means. (True or False)
17. The Chi-square distribution is continuous. (True or False)

CHAPTER 28

Numerical Solution of Equations

1. Introduction. 2. Solution of algebraic and transcendental equations—Bisection method, Method of false position, Newton's method. 3. Useful deductions from the Newton-Raphson Formula. 4. Approximate solution of equations—Horner's method. 5. Solution of linear simultaneous equations. 6. Direct methods of solution—Gauss elimination method, Gauss-Jordan method, Factorization method. 7. Iterative methods of solution—Jacobi's method, Gauss-Seidal method, Relaxation method. 8. Solution of non-linear simultaneous equations—Newton-Raphson method. 9. Determination of eigen values by iteration. 10. Objective Type of Questions.

28.1 INTRODUCTION

The limitations of analytical methods have led the engineers and scientists to evolve graphical and numerical methods. As seen in § 1.8, the graphical methods, though simple, give results to a low degree of accuracy. Numerical methods can, however, be derived which are more accurate. With the advent of high speed digital computers and increasing demand for numerical answers to various problems, numerical techniques have become indispensible tool in the hands of engineers.

Numerical methods are often, of a repetitive nature. These consist in repeated execution of the same process where at each step the result of the preceding step is used. This is known as *iteration process* and is repeated till the result is obtained to a desired degree of accuracy.

In this chapter, we shall discuss some numerical methods for the solution of algebraic and transcendental equations and simultaneous linear and non-linear equations. We shall close the chapter by describing an iterative method for the solution of eigen-value problem. For a detailed study of these topics, the reader should refer to author's book *'Numerical Methods in Engineering & Science'*.

28.2 SOLUTION OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS

To find the roots of an equation $f(x) = 0$, we start with a known approximate solution and apply any of the following methods :

(1) Bisection method. This method consists in locating the root of the equation $f(x) = 0$ between a and b. If $f(x)$ is continuous between a and b, and $f(a)$ and $f(b)$ are of opposite signs then there is a root between a and b. For definiteness, let $f(a)$ be negative and $f(b)$ be positive. Then the first approximation to the root is $x_1 = \frac{1}{2}(a + b)$.

If $f(x_1) = 0$, then x_1 is a root of $f(x) = 0$. Otherwise, the root lies between a and x_1 or x_1 and b according as $f(x_1)$ is positive or negative. Then we bisect the interval as before and continue the process until the root is found to desired accuracy.

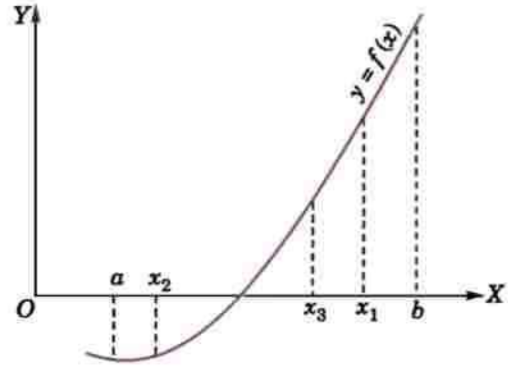

Fig. 28.1

In the Fig. 28.1, $f(x_1)$ is +ve, so that the root lies between a and x_1. Then the second approximation to the root is $x_2 = \frac{1}{2}(a + x_1)$. If $f(x_2)$ is – ve, the root lies between x_1 and x_2. Then the third approximation to the root is $x_3 = \frac{1}{2}(x_1 + x_2)$ and so on.

Example 28.1. *(a) Find a root of the equation $x^3 - 4x - 9 = 0$, using the bisection method correct to three decimal places.* *(Mumbai, 2003)*

(b) Using bisection method, find the negative root of the equation $x^2 - 4x + 9 = 0$. *(J.N.T.U., 2009)*

Solution. (*a*) Let $f(x) = x^3 - 4x - 9$

Since $f(2)$ is –ve and $f(3)$ is +ve, a root lies between 2 and 3

$\therefore$ first approximate to the root is

$$x_1 = \frac{1}{2}(2 + 3) = 2.5$$

Thus $f(x_1) = (2.5)^3 - 4(2.5) - 9 = -3.375$ *i.e.*, – ve

$\therefore$ the root lies between x_1 and 3. Thus the second approximation to the root is

$$x_2 = \frac{1}{2}(x_1 + 3) = 2.75$$

Then $f(x_2) = (2.75)^3 - 4(2.75) - 9 = 0.7969$ *i.e.*, + ve

$\therefore$ the root lies between x_1 and x_2. Thus the third approximation to the root is

$$x_3 = \frac{1}{2}(x_1 + x_2) = 2.625$$

Then $f(x_3) = (2.625)^3 - 4(2.625) - 9 = -1.4121$ *i.e.*, – ve

$\therefore$ the root lies between x_2 and x_3. Thus the fourth approximation to the root is

$$x_4 = \frac{1}{2}(x_2 + x_3) = 2.6875$$

Repeating this process, the successive approximations are

$x_5 = 2.71875$, $x_6 = 2.70313$, $x_7 = 2.71094$
$x_8 = 2.70703$, $x_9 = 2.70508$, $x_{10} = 2.70605$
$x_{11} = 2.70654$, $x_{12} = 2.70642$

Hence the root is 2.7064

(*b*) If α, β, γ are the roots of the given equation, then $-\alpha, -\beta, -\gamma$ are the roots of $(-x)^3 - 4(-x) + 9 = 0$

$\therefore$ the negative root of the given equation is the positive root of $x^3 - 4x - 9 = 0$ which we have found above to be 2.7064.

Hence the negative root for the given equation is – 2.7064.

Example 28.2. *By using the bisection method, find an approximate root of the equation $\sin x = 1/x$, that lies between $x = 1$ and $x = 1.5$ (measured in radians). Carry out computations upto the 7th stage.* *(V.T.U., 2003 S)*

Solution. Let $f(x) = x \sin x - 1$. We know that $1^r = 57.3°$.

Since $f(1) = 1 \times \sin(1) - 1 = \sin(57.3°) - 1 = -0.15849$

and $f(1.5) = 1.5 \times \sin(1.5)^r - 1 = 1.5 \times \sin(85.95)° - 1 = 0.49625$;

a root lies between 1 and 1.5.

$\therefore$ first approximation to the root is $x_1 = \frac{1}{2}(1 + 1.5) = 1.25$.

Then $f(x_1) = (1.25)\sin(1.25) - 1 = 1.25 \sin(71.625°) - 1 = 0.18627$ and $f(1) < 0$.

$\therefore$ a root lies between 1 and $x_1 = 1.25$.

Thus the second approximation to the root is $x_2 = \frac{1}{2}(1 + 1.25) = 1.125$.

Then $f(x_2) = 1.125 \sin(1.125) - 1 = 1.125 \sin(64.46)° - 1 = 0.01509$ and $f(1) < 0$.

$\therefore$ a root lies between 1 and $x_2 = 1.125$.

Thus the third approximation to the root is $x_3 = \frac{1}{2}(1 + 1.125) = 1.0625$

Then $f(x_3) = 1.0625 \sin(1.0625) - 1 = 1.0625 \sin(60.88) - 1 = -0.0718 < 0$

and $f(x_2) > 0$, *i.e.* now the root lies between $x_3 = 1.0625$ and $x_2 = 1.125$.

$\therefore$ fourth approximation to the root is $x_4 = \frac{1}{2}(1.0625 + 1.125) = 1.09375$

Then $f(x_4) = -0.02836 < 0$ and $f(x_2) > 0$,

i.e., the root lies between $x_4 = 1.09375$ and $x_2 = 1.125$.

$\therefore$ fifth approximation to the root is $x_5 = \frac{1}{2}(1.09375 + 1.125) = 1.10937$

Then $f(x_5) = -0.00664 < 0$ and $f(x_2) > 0$.

$\therefore$ the root lies between $x_5 = 1.10937$ and $x_2 = 1.125$.

Thus the sixth approximation to the root is

$$x_6 = \frac{1}{2}(1.10937 + 1.125) = 1.11719$$

Then $f(x_6) = 0.00421 > 0$. But $f(x_5) < 0$.

$\therefore$ the root lies between $x_5 = 1.10937$ and $x_6 = 1.11719$.

Thus the seventh approximation to the root is $x_7 = \frac{1}{2}(1.10937 + 1.11719) = 1.11328$

Hence the desired approximation to the root is 1.11328.

(2) Method of false position *or* **Regula-falsi method.** This is the oldest method of finding the real root of an equation $f(x) = 0$ and closely resembles the bisection method. Here we choose two points x_0 and x_1 such that $f(x_0)$ and $f(x_1)$ are of opposite signs *i.e.*, the graph of $y = f(x)$ crosses the x-axis between these points (Fig. 28.2). This indicates that a root lies between x_0 and x_1 consequently $f(x_0) f(x_1) < 0$.

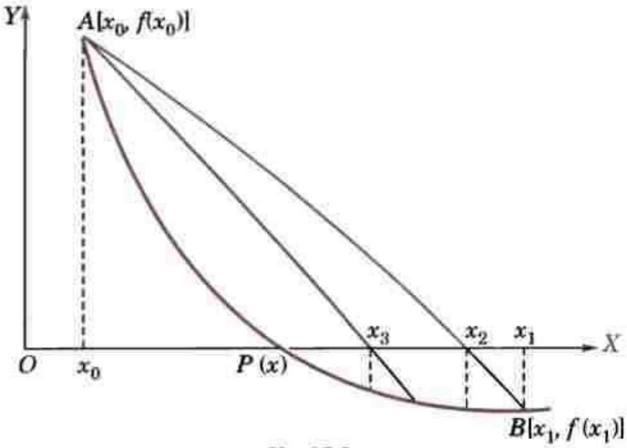

Fig. 28.2

Equation of the chord joining the points $A[x_0, f(x_0)]$ and $B[x_1, f(x_1)]$ is

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) \qquad ...(1)$$

The method consists in replacing the curve AB by means of the chord AB and taking the point of intersection of the chord with the x-axis as an approximation to the root. So the abscissa of the point where the chord cuts the x-axis ($y = 0$) is given by

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \qquad ...(2)$$

which is an approximation to the root.

If now $f(x_0)$ and $f(x_2)$ are of opposite signs, then the root lies between x_0 and x_2. So replacing x_1 by x_2 in (2), we obtain the next approximation x_3. (The root could as well lie between x_1 and x_2 and we would obtain x_3 accordingly). This procedure is repeated till the root is found to desired accuracy. The iteration process based on (1) is known as the *method of false position*.

Example 28.3. *Find a real root of the equation $x^3 - 2x - 5 = 0$ by the method of false position correct to three decimal places.* *(Manipal, 2005)*

Solution. Let $f(x) = x^3 - 2x - 5$

so that $f(2) = -1$ and $f(3) = 16$ *i.e.*, A root lies between 2 and 3.

$\therefore$ taking $x_0 = 2, x_1 = 3, f(x_0) = -1, f(x_1) = 16$, in the method of false position, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 2 + \frac{1}{17} = 2.0588 \qquad ...(i)$$

Now $f(x_2) = f(2.0588) = -0.3908$ *i.e.*, the root lies between 2.0588 and 3.

$\therefore$ taking $x_0 = 2.0588, x_1 = 3, f(x_0) = -0.3908, f(x_1) = 16$, in (*i*), we get

$$x_3 = 2.0588 - \frac{0.9412}{16.3908}(-0.3908) = 2.0813$$

Repeating this process, the successive approximations are

$$x_4 = 2.0862, x_5 = 2.0915, x_6 = 2.0934, x_7 = 2.0941, x_8 = 2.0943 \text{ etc.}$$

Hence the root is 2.094 correct to 3 decimal places.

Example 28.4. *Find the root of the equation* $\cos x = xe^x$ *using the regula-falsi method correct to four decimal places.* (*Bhopal, 2009*)

Solution. Let $f(x) = \cos x - xe^x = 0$

So that $f(0) = 1, f(1) = \cos 1 - e = -2.17798$

i.e., the root lies between 0 and 1.

$\therefore$ taking $x_0 = 0, x_1 = 1, f(x_0) = 1$ and $f(x_1) = -2.17798$ in the regula-falsi method, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 0 + \frac{1}{3.17798} \times 1 = 0.31467 \qquad \text{...(i)}$$

Now $f(0.31467) = 0.51987$

i.e., the root lies between 0.31467 and 1.

$\therefore$ taking $x_0 = 0.31467, x_1 = 1, f(x_0) = 0.51987, f(x_1) = -2.17798$ in (*i*), we get

$$x_3 = 0.31467 + \frac{0.68533}{2.69785} \times 0.51987 = 0.44673$$

Now $f(0.44673) = 0.20356$

i.e., the root lies between 0.44673 and 1.

$\therefore$ taking $x_0 = 0.44673, x_1 = 1, f(x_0) = 0.20356, f(x_1) = -2.17798$ in (*i*), we get

$$x_4 = 0.44673 + \frac{0.55327}{2.38154} \times 0.20356 = 0.49402$$

Repeating this process, the successive approximations are

$$x_5 = 0.50995, \quad x_6 = 0.51520, \quad x_7 = 0.51692$$
$$x_8 = 0.51748, \quad x_9 = 0.51767, \quad x_{10} = 0.51775 \text{ etc.}$$

Hence the root is 0.5177 correct to 4 decimal places.

Example 28.5. *Find a real root of the equation* $x \log_{10} x = 1.2$ *by regula-falsi method correct to four decimal places.* (*V.T.U., 2010 ; J.N.T.U., 2008 ; Kottayam, 2005*)

Solution. Let $f(x) = x \log_{10} x - 1.2$

so that $f(1) = -\text{ve}, f(2) = -\text{ve}$ and $f(3) = +\text{ve}$.

$\therefore$ a root lies between 2 and 3.

Taking $x_0 = 2$ and $x_1 = 3, f(x_0) = -0.59794$ and $f(x_1) = 0.23136$, in the method of false position, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 2.72102 \qquad \text{...(i)}$$

Now $f(x_2) = f(2.72102) = -0.01709$

i.e., the root lies between 2.72102 and 3.

$\therefore$ taking $x_0 = 2.72102, x_1 = 3, f(x_0) = -0.01709$

and $f(x_1) = 0.23136$ in (*i*), we get

$$x_3 = 2.72102 + \frac{0.27898}{0.23136 + 0.01709} \times 0.01709 = 2.74021$$

Repeating this process, the successive approximations are

$$x_4 = 2.74024, x_5 = 2.74063 \text{ etc.}$$

Hence the root is 2.7406 correct to 4 decimal places.

Example 28.6. *Use the method of false position, to find the fourth root of 32 correct to three decimal places.*

Solution. Let $x = (32)^{1/4}$ so that $x^4 - 32 = 0$

Take $f(x) = x^4 - 32$. Then $f(2) = -16$ and $f(3) = 49$, *i.e.*, a root lies between 2 and 3.

$\therefore$ taking $x_0 = 2, x_1 = 3, f(x_0) = -16, f(x_1) = 49$ in the method of false position, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 2 + \frac{16}{65} = 2.2462 \quad ...(i)$$

Now $f(x_2) = f(2.2462) = -6.5438$ *i.e.* the root lies between 2.2462 and 3.

$\therefore$ taking $x_0 = 2.2462, x_1 = 3, f(x_0) = -6.5438, f(x_1) = 49$

in (*i*), we get
$$x_3 = 2.2462 - \frac{3 - 2.2462}{49 + 6.5438}(-6.5438) = 2.335$$

Now $f(x_3) = f(2.335) = -2.2732$ *i.e.* the root lies between 2.335 and 3.

$\therefore$ taking $x_0 = 2.335$ and $x_1 = 3, f(x_0) = -2.2732$ and $f(x_1) = 49$ in (*i*), we obtain

$$x_4 = 2.335 - \frac{3 - 2.335}{49 + 2.2732}(-2.2732) = 2.3645$$

Repeating this process, the successive approximations are $x_5 = 2.3770, x_6 = 2.3779$ etc.

Since $x_5 = x_6$ upto 3 decimal places, we take $(32)^{1/4} = 2.378$.

(3) Newton-Raphson method*. Let x_0 be an approximate root of the equation $f(x) = 0$. If $x_1 = x_0 + h$ be the exact root, then $f(x_1) = 0$.

$\therefore$ expanding $f(x_0 + h)$ by Taylor's series

$$f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \ldots\ldots = 0$$

Since h is small, neglecting h^2 and higher powers of h, we get

$$f(x_0) + hf'(x_0) = 0 \quad \text{or} \quad h = -\frac{f(x_0)}{f'(x_0)} \quad ...(1)$$

$\therefore$ a closer approximation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Similarly, starting with x_1, a still better approximation x_2 is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

In general,
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad ...(2)$$

which is known as the *Newton-Raphson formula* or *Newton's iteration formula*.

Obs. 1. *Newton's method is useful in cases of large values of* $f'(x)$ *i.e.* when the graph of $f(x)$ while crossing the x-axis is nearly vertical.

Obs. 2. **Newton's method has a second order of quadratic convergence.** Suppose x_n differs from the root α by a small quantity ε_n so that $x_0 = \alpha + \varepsilon_n$ and $x_{n+1} = \alpha + \varepsilon_{n+1}$.

Then (2) becomes $\alpha + \varepsilon_{n+1} = \alpha + \varepsilon_n - \dfrac{f(\alpha + \varepsilon_n)}{f'(\alpha + \varepsilon_n)}$

i.e.,
$$\varepsilon_{n+1} = \varepsilon_n - \frac{f(\alpha + \varepsilon_n)}{f'(\alpha + \varepsilon_n)} = \varepsilon_n - \frac{f(\alpha) + \varepsilon_n f'(\alpha) + \frac{1}{2!}\varepsilon_n^2 f''(\alpha) + \ldots}{f'(\alpha) + \varepsilon_n f''(\alpha) + \ldots} \quad \text{[By Taylor's expansion.]}$$

$$= \varepsilon_n - \frac{\varepsilon_n f'(\alpha) + \frac{1}{2}\varepsilon_n^2 f''(\alpha) + \ldots}{f'(\alpha) + \varepsilon_n f''(\alpha) + \ldots} \quad [\because f(\alpha) = 0]$$

$$= \frac{\varepsilon_n^2 f''(\alpha)}{2[f'(\alpha) + \varepsilon_n f''(\alpha)]} = \frac{\varepsilon_n^2}{2} \cdot \frac{f''(\alpha)}{f'(\alpha)}. \quad \left[\begin{array}{l}\text{neglecting third and} \\ \text{higher powers of } \varepsilon_n.\end{array}\right]$$

This shows that the subsequent error at each step, is proportional to the square of the previous error and as such the convergence is quadratic. (*P.T.U., 2005*)

Obs. 3. **Geometrical interpretation.** Let x_0 be a point near the root α of the equation $f(x) = 0$ (Fig. 28.3). Then the equation of the tangent at $A_0\,[x_0, f(x_0)]$ is $y - f(x_0) = f'(x_0)(x - x_0)$.

*See footnote p. 466. Named after the English mathematician *Joseph Raphson* (1648–1715) who suggested a method similar to Newton's method.

It cuts the x-axis at $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ which is a first approximation to the root α. If A_1 is the point corresponding to x_1 on the curve, then the tangent at A_1 will cut the x-axis of x_2 which is nearer to α and is, therefore, a second approximation to the root. Repeating this process, we approach to the root α quite rapidly. Hence *the method consists in replacing the part of the curve between the point A_0 and the x-axis by means of the tangent to the curve at A_0.*]

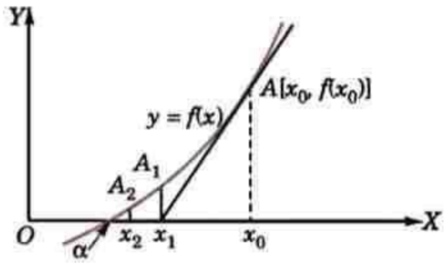

Fig. 28.3

Example 28.7. *Find the positive root of $x^4 - x = 10$ correct to three decimal places, using Newton-Raphson method.* (J.N.T.U., 2008 ; Madras, 2006)

Solution. Let $f(x) = x^4 - x - 10$

So that $f(1) = -10 = -\text{ve}, f(2) = 16 - 2 - 10 = 4 = +\text{ve}$

$\therefore$ a root of $f(x) = 0$ lies between 1 and 2. Let us take $x_0 = 2$

Also $f'(x) = 4x^3 - 1$

Newton-Raphson's formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \ldots(i)$$

Putting $n = 0$, the first approximation x_1 is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{4}{4 \times 2^3 - 1} = 2 - \frac{4}{31} = 1.871$$

Putting $n = 1$ in (i), the second approximation is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.871 - \frac{f(1.871)}{f'(1.871)}$$

$$= 1.871 - \frac{(1.871)^4 - (1.871) - 10}{4(1.871)^3 - 1} = 1.871 - \frac{0.3835}{25.199} = 1.856$$

Putting $n = 2$ in (i), the third approximation is

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.856 - \frac{(1.856)^4 - (1.856) - 10}{4(1.856)^3 - 1}$$

$$= 1.856 - \frac{0.010}{24.574} - 1.856$$

Here $x_2 = x_3$. Hence the desired is 1.856 correct to three decimal places.

Example 28.8. *Find the Newton's method, the real root of the equation $3x = \cos x + 1$.* (V.T.U., 2009 ; S.V.T.U., 2007)

Solution. Let $f(x) = 3x - \cos x - 1$

$f(0) = -2 = -\text{ve}, f(1) = 3 - 0.5403 - 1 = 1.4597 = +\text{ve}.$

So a root of $f(x) = 0$ lies between 0 and 1. It is nearer to 1. Let us take $x_0 = 0.6$.

Also $f'(x) = 3 + \sin x$

$\therefore$ Newton's iteration formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{3x_n - \cos x_n - 1}{3 + \sin x_n}$$

$$= \frac{x_n \sin x_n + \cos x_n + 1}{3 + \sin x_n} \quad \ldots(i)$$

Putting $n = 0$, the first approximation x_1 is given by

$$x_1 = \frac{x_0 \sin x_0 + \cos x_0 + 1}{3 + \sin x_0} = \frac{(0.6) \sin (0.6) + \cos (0.6) + 1}{3 \sin (0.6)}$$

$$= \frac{0.6 \times 0.5729 + 0.82533 + 1}{3 + 0.5729} = 0.6071$$

Putting $n = 1$ in (i), the second approximation is

$$x_2 = \frac{x_1 \sin x_1 + \cos x_1 + 1}{3 + \sin x_1} = \frac{0.6071 \sin (0.6071) + \cos (0.6071) + 1}{3 + \sin (0.6071)}$$

$$= \frac{0.6071 \times 0.57049 + 0.8213 + 1}{3 + 0.57049} = 0.6071 \quad \text{Clearly, } x_1 = x_2.$$

Hence the desired root is 0.6071 correct to four decimal places.

Example 28.9. *Using Newton's iterative method, find the real root of $x \log_{10} x = 1.2$ correct to five decimal places.* (V.T.U., 2005 ; Mumbai, 2004 ; Burdwan, 2003)

Solution. Let $f(x) = x \log_{10} x - 1.2$

$f(1) = -1.2 = -\text{ve}$, $f(2) = 2 \log_{10} 2 - 1.2 = 0.59794 = -\text{ ve}$

and $f(3) = 3 \log_{10} 3 - 1.2 = 1.4314 - 1.2 = 0.23136 = +\text{ ve}$

So a root of $f(x) = 0$ lies between 2 and 3. Let us take $x_0 = 2$

Also $f'(x) = \log_{10} x + x \cdot \frac{1}{x} \log_{10} e = \log_{10} x + 0.43429$

$\therefore$ Newton's iteration formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \frac{0.43429\, x_n + 1.2}{\log_{10} x_n + 0.43429} \qquad ...(i)$$

Putting $n = 0$, the first approximation is

$$x_1 = \frac{0.43429 \times x_0 + 1.2}{\log_{10} x_0 + 0.43429} = \frac{0.43429 \times 2 + 1.2}{\log_{10} 2 + 0.43429} = \frac{0.86858 + 1.2}{0.30103 + 0.43429} = 2.81$$

Similarly putting $n = 1, 2, 3, 4$ in (i), we get

$$x_2 = \frac{0.43429 \times 2.81 + 1.2}{\log_{10} 2.81 + 0.43429} = 2.741$$

$$x_3 = \frac{0.43429 \times 2.741 + 1.2}{\log_{10} 2.81 + 0.43429} = 2.74064$$

$$x_4 = \frac{0.43429 \times 2.74064 + 1.2}{\log_{10} 2.74064 + 0.43429} = 2.74065$$

$$x_5 = \frac{0.43429 \times 2.74065 + 1.2}{\log_{10} 2.74065 + 0.43429} = 2.74065$$

Clearly $x_4 = x_5$.

Hence the required root is 2.74065 correct to five decimal places.

28.3 USEFUL DEDUCTIONS FROM THE NEWTON-RAPHSON FORMULA

(1) *Iterative formula to find $1/N$ is* $\quad \boldsymbol{x_{n+1} = x_n (2 - N x_n)}$

(2) *Iterative formula to find $\sqrt{N}$ is* $\quad \boldsymbol{x_{n+1} = \frac{1}{2}(x_n + N/x_n)}$

(3) *Iterative formula to find $1/\sqrt{N}$ is* $\quad \boldsymbol{x_{n+1} = \frac{1}{2}(x_n + 1/N x_n)}$

(4) *Iterative formula to find $\sqrt[k]{N}$ is* $\quad \boldsymbol{x_{n+1} = \frac{1}{k}[(k-1)x_n + N/x_n^{k-1})]}$

Proofs. (1) Let $x = 1/N$ or $1/x - N = 0$

Taking $f(x) = 1/x - N$, we have $f'(x) = -x^{-2}$

Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(1/x_n - N)}{-x_n^{-2}} = x_n + \left(\frac{1}{x^n} - N\right) x_n^2 = x_n + x_n - N x_n^2 = x_n (2 - N x_n)$$

(2) Let $x = \sqrt{N}$ or $x^2 - N = 0$

Taking $f(x) = x^2 - N$, we have $f'(x) = 2x$

Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - N}{2x_n} = \frac{1}{2}(x_n + N/x_n) \quad \textit{(Madras, 2006)}$$

(3) Let $x = \dfrac{1}{\sqrt{N}}$ or $x^2 - \dfrac{1}{N} = 0$

Taking $f(x) = x^2 - 1/N$, we have $f'(x) = 2x$

Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 1/N}{2x_n} = \frac{1}{2}\left(x_n + \frac{1}{Nx_n}\right)$$

(4) Let $x = \sqrt[k]{N}$ or $x^k - N = 0$

Taking $f(x) = x^k - N$, we have $f'(x) = kx^{k-1}$

Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^k - N}{kx_n^{k-1}} = \frac{1}{k}\left[(k-1)x_n + \frac{N}{x_n^{k-1}}\right].$$

Example 28.10. *Evaluate the following (correct to four decimal places) by Newton's iteration method :*

(i) $1/31$ *(ii)* $\sqrt{5}$ *(Anna, 2007)*

(iii) $1/\sqrt{14}$ *(iv)* $\sqrt[3]{24}$ *(Madras, 2003)* *(v)* $(30)^{-1/5}$.

Solution. (*i*) Taking $N = 31$, the above formula (1) becomes

$$x_{n+1} = x_n(2 - 31x_n)$$

Since an approximate value of $1/31 = 0.03$, we take $x_0 = 0.03$

Then $x_1 = x_0(2 - 31x_0) = 0.03\,(2 - 31 \times 0.03) = 0.0321$

$x_2 = x_1(2 - 31x_1) = 0.0321(2 - 31 \times 0.0321) = 0.032257$

$x_3 = x_2\,(2 - 31x_2) = 0.032257\,(2 - 31 \times 0.032257) = 0.03226$

Since $x_2 = x_3$ upto 4 decimal places, we have $1/31 = 0.0323$.

(*ii*) Taking $N = 5$, the above formula (2), becomes $x_{n+1} = \dfrac{1}{2}(x_n + 5/x_n)$

Since an approximate value of $\sqrt{5} = 2$, we take $x_0 = 2$

Then $x_1 = \dfrac{1}{2}(x_0 + 5/x_0) = \dfrac{1}{2}(2 + 5/2) = 2.25$

$x_2 = \dfrac{1}{2}(x_1 + 5/x_1) = 2.2361$

$x_3 = \dfrac{1}{2}(x_2 + 5/x_2) = 2.2361$

Since $x_2 = x_3$ upto 4 decimal places, we have $\sqrt{5} = 2.2361$.

(*iii*) Taking $N = 14$, the above formula (3), becomes $x_{n+1} = \dfrac{1}{2}[x_n + 1/(14x_n)]$

Since an approximate value of $1/\sqrt{14} = 1/\sqrt{16} = \dfrac{1}{4} = 0.25$, we take $x_0 = 0.25$

Then $x_1 = \dfrac{1}{2}[x_0 + (14x_0)^{-1}] = \dfrac{1}{2}[0.25 + (14 \times 0.25)^{-1}] = 0.26785$

$x_2 = \dfrac{1}{2}[x_1 + (14x_1)^{-1}] = \dfrac{1}{2}[0.26785 + (14 \times 0.26785)^{-1}] = 0.2672618$

$x_3 = \dfrac{1}{2}[x_2 + (14x_2)^{-1}] = \dfrac{1}{2}[0.2672618 + (14 \times 0.2672618)^{-1}] = 0.2672612$

Since $x_2 = x_3$ upto 4 decimal places, we take $1/\sqrt{14} = 0.2673$.

(*iv*) Taking $N = 24$ and $k = 3$, the above formula (4) becomes $x_{n+1} = \dfrac{1}{3}[2x_n + 24/x_n^2]$

Since an approximate value of $(24)^{1/3} = (27)^{1/3} = 3$, we take $x_0 = 3$.

Then $$x_1 = \frac{1}{3}(2x_0 + 24/x_0^2) = \frac{1}{3}(6 + 24/9) = 2.88889$$

$$x_2 = \frac{1}{3}(2x_1 + 24/x_1^2) = \frac{1}{3}[(2 \times 2.88889) + 24/(2.88889)^2] = 2.88451$$

$$x_3 = \frac{1}{3}(2x_2 + 24/x_2^2) = \frac{1}{3}[2 \times 2.88451 + 24/(2.88451)^2] = 2.8845$$

Since $x_2 = x_3$ upto 4 decimal places, we take $(24)^{1/3} = 2.8845$

(*v*) Taking $N = 30$ and $k = -5$, the above formula (4) becomes

$$x_{n+1} = \frac{1}{-5}(-6x_n + 30/x_n^{-6}) = \frac{x_n}{5}(6 - 30x_n^5)$$

Since an approximate value of $(30)^{-1/5} = (32)^{-1/5} = 1/2$, we take $x_0 = 1/2$

Then $$x_1 = \frac{x_0}{5}(6 - 30\,x_0^5) = \frac{1}{10}(6 - 30/2^5) = 0.50625$$

$$x_2 = \frac{x_1}{5}(6 - 30\,x_1^5) = \frac{0.50625}{5}[6 - 30(0.50625)^5] = 0.506495$$

$$x_3 = \frac{x_2}{5}(6 - 30\,x_2^5) = \frac{0.506495}{5}[6 - 30(0.506495)^5] = 0.506496.$$

Since $x_2 = x_3$ upto 4 decimal places, we take $(30)^{-1/5} = 0.5065$.

PROBLEMS 28.1

1. Find a root of the following equations, using the bisection method correct to three decimal places :
   (*i*) $x^3 - 2x - 5 = 0$ (*P.T.U., 2005*) (*ii*) $x^3 - x^2 - 1 = 0$ (*J.N.T.U., 2009*)
   (*iii*) $x^3 - x - 11 = 0$ which lies between 2 and 3 (*iv*) $2x^3 + x^2 - 20x + 12 = 0$.
2. Using the bisection method, find a real root of the following equations correct to three decimal places :
   (*i*) $\cos x = xe^x$ (*Mumbai, 2004*) (*ii*) $x \log_{10} x = 1.2$ lying between 2 and 3
   (*iii*) $e^x - x = 2$ lying between 1 and 1.4 (*iv*) $e^x = 4 \sin x$.
3. Find a real root of the following equations correct to three decimal places by the method of false position :
   (*i*) $x^3 + x - 1 = 0$ (*ii*) $x^3 - 4x - 9 = 0$ (*V.T.U., 2007*)
   (*iii*) $x^3 + x - 1 = 0$ near $x = 1$ (*iv*) $x^6 - x^4 - x^3 - 1 = 0$. (*Nagarjuna, 2001*)
4. Using regula-falsi method, compute the real root of the following equations correct to three decimal places :
   (*i*) $xe^x = 2$ (*S.V.T.U., 2007*) (*ii*) $\cos x = 3x - 1$ (*iii*) $x \tan x - 1 = 0$
   (*iv*) $2x - \log x = 7$ (*J.N.T.U., 2006*) (*v*) $xe^x = \sin x$. (*P.T.U., 2005*)
5. Find the fourth root of 12 correct to three decimal places using the method of false position.
6. Find by Newton's method, a root of the following equations correct to 3 decimal places :
   (*i*) $x^3 - 3x + 1 = 0$ (*Bhopal, 2009*) (*ii*) $x^3 - 2x - 5 = 0$ (*P.T.U., 2005*)
   (*iii*) $x^3 - 5x + 3 = 0$ (*Mumbai, 2004*)
   (*iv*) $3x^3 - 9x^2 + 8 = 0$ lying between 1 and 2. (*Madras, 2003*)
7. Find a root of the following equations correct to three significant figures using Newton's iterative method :
   (*i*) $x^4 + x^3 - 7x^2 - x + 5 = 0$ lying between 2 and 3 (*Madras, 2003*)
   (*ii*) $x^5 - 5x^2 + 3 = 0$.
8. Find the negative root of the equation $x^3 - 21x + 3500 = 0$ correct to two decimal places by Newton's method.
9. Using Newton-Raphson method, find a root of the following equations correct to the three decimal places :
   (*i*) $xe^x - 2 = 0$ (*V.T.U., 2005*) (*ii*) $x^2 + 4 \sin x = 0$ (*Hazaribagh, 2009*)
   (*iii*) $x \tan x + 1 = 0$ which is near $x = \pi$ (*J.N.T.U., 2006 ; V.T.U., 2006*)
   (*iv*) $e^x = x^2 + \cos 25x$ which is near $x = 4.5$. (*V.T.U., 2007*)
10. Find by Newton's method, the root of the equations :
    (*i*) $\cos x = xe^x$ (*J.N.T.U., 2009 ; V.T.U., 2003*) (*ii*) $x \log_{10} x = 12.34$ (*Anna, 2004*)
    (*iii*) $10^x + x - 4 = 0$ (*iv*) $x + \log_{10} x = 3.375$ (*Rohtak, 2003*)
11. Develop a recurrence formula for finding $\sqrt{N}$, using Newton-Raphson method and hence compute to three decimal places
    (*i*) $\sqrt{13}$ (*U.P.T.U., 2008*) (*ii*) $\sqrt{10}$ (*J.N.T.U., 2008*)

12. Find the cube root of 41, using Newton-Raphson method. *(Madras, 2003)*
13. Develop an algorithm using N-R method, to find the fourth root of a positive number N and hence find $(32)^{1/4}$. *(W.B.T.U., 2005)*
14. Evaluate the following (correct to 3 decimal places) by using the Newton-Raphson method :
 (*i*) 1/18 *J.N.T.U., 2004)* (*ii*) $1/\sqrt{15}$ (*iii*) $(28)^{-1/4}$.

28.4 APPROXIMATE SOLUTION OF EQUATIONS—HORNER'S METHOD

This is the best method of finding approximate values of both rational and irrational roots of a numerical equation. Horner's method consists in diminution of the root of an equation by successive digits occurring in the roots.

If the root of an equation lies between a and $a + 1$, then the value of this root will be $a \,.\, bcd$ ..., where b, c, d ... are digits in its decimal part. To obtain these, we proceed as follows :

(*i*) Diminish the roots of the given equation by a so that the root of the new equation is $0 \,.\, bcd$ ...

(*ii*) Then multiply the roots of the transformed equation by 10 so that the root of the new equation is $b \,.\, cd$...

(*iii*) Now diminish the root by b and multiply the roots of the resulting equation by 10 so that the root is $c.\,d$...

(*iv*) Next diminish the root by c and so on. By continuing this process, the root may be evaluated to any desired degree of accuracy digit by digit. The method will be clear from the following example.

Example 28.11. *Find by Horner's method, the positive root of the equation $x^3 + x^2 + x - 100 = 0$ correct to three decimal places.*

Solution. *Step I.* Let $f(x) = x^3 + x^2 + x - 100$

By Descartes' rule of signs, there is only one positive root. Also $f(4) = -$ ve and $f(5) = +$ ve, therefore, the root lies between 4 and 5.

Step II. Diminish the roots of given equation by 4 so that the transformed equation is

$$x^3 + 13x^2 + 57x - 16 = 0 \qquad ...(i)$$

Its root lies between 0 and 1. (We draw a zig-zag line above the set of figures 13, 57, – 16 which are the coefficients of the terms in (*i*) as shown below. Now multiply the roots of (*i*) by 10 for which multiply the second term by 10, the third term by 100 and the fourth term by 1000 (*i.e.* attach one zero to the second term, two zeros to the third term and three zeros to the fourth term). Then we get the equation

$$f_1(x) = x^3 + 130x^2 + 5700x - 16000 = 0 \qquad ...(ii)$$

1	1	1	– 100 (4.264
	4	20	84
	5	21	– 16000
	4	36	11928
	9	5700	– 4072000
	4	264	3788376
	130	5964	– 283624000
	2	268	
	132	623200	
	2	8196	
	134	631396	
	2	8232	
	1360	63962800	
	6		
	1366		
	6		
	1372		
	6		
	13780		

Its root lies between 0 and 10.

Clearly $f_1(2) = -$ ve, $f_1(3) = +$ ve

$\therefore$ the root of (*ii*) lies between 2 and 3 *i.e.*, first figure after decimal is 2.

Step III. Diminish the roots of $f_1(x) = 0$ by 2 so that the next transformed equation is

$$x^3 + 136x^2 + 6232x - 4072 = 0 \qquad ...(iii)$$

Its root lies between 0 and 1. (We draw the second zig-zag line above the set of figures 136, 6232, – 4072). Multiply the roots of (*iii*), by 10, *i.e.* attach one zero to second term, two zeros to third term and three zeros to the fourth term. Then the new equation is

$$f_2(x) = x^3 + 1360x^2 + 623200x - 4072000 = 0$$

Its root lies between 0 and 10, which is nearly $= \dfrac{4072000}{623200} = 6$

Hence second figure after decimal place is 6.

Step IV. Diminish the roots of $f_2(x) = 0$ by 6, so that the transformed equation is

$$x^3 + 1378x^2 + 639628x - 283624 = 0.$$

Its root lies between 0 and 1. (We draw the third zig-zag line above the set of figures 1378, 639628, – 283624.) As before multiply its roots by 10, *i.e.* attach one zero to the second term, two zeros to the third term and three zeros to the fourth term. Then the equation becomes

$$f_3(x) = x^3 + 13780x^2 + 63962800x - 283624000 = 0$$

Its root lies between 0 and 10, which is nearly $= \dfrac{283624000}{63962800} = 4$. Thus the roots of $f_3(x) = 0$ are to be diminished by 4 *i.e.* the third figure after decimal place is 4. But there is no need to proceed further as the root is required correct to three decimal places only. Hence the root is 4.264.

Obs. 1. *After two steps of diminishing, we apply the* **principle of trial divisor** *in which we divide the last coefficient by last but one coefficient to get the next integer by which the roots are to be diminished. These last two coefficients should have opposite signs.*

Obs. 2. *At any stage if the trial divisor suggests the next integer to be zero, then we should again multiply the roots by 10 and write zero in decimal place of the root.*

Example 28.12. *Find the cube root of 30 correct to 3 decimal places, using Horner's method.*

Solution. *Step I.* Let $x = \sqrt[3]{30}$ *i.e.* $f(x) = x^3 - 30 = 0$

Now $f(3) = -3$ (– ve), $f(4) = 34$ (+ ve)

$\therefore$ the root lies between 3 and 4.

Step II. Diminish the roots of the given equation by 3 so that the transformed equation is

$$x^3 + 9x^2 + 27x - 3 = 0 \qquad ...(i)$$

Its roots lies between 0 and 1. (We draw a zig-zag line above the set of numbers 9, 27, – 3 which are the coefficients of the terms in (*i*)). Now multiply the roots of (*i*) by 10 for which attach one zero to the second term, two zeros to the third term and three zeros to the fourth term. Then we get the equation

$$f_1(x) = x^3 + 90x^2 + 2700x - 3000 = 0 \qquad ...(ii)$$

Its roots lies between 0 and 10.

Clearly $f_1(1) = -$ ve, $f_2(2) = +$ ve

$\therefore$ the root of (*ii*) lies between 1 and 2 *i.e.*, first figure after decimal place is 1.

Step III. Diminish the roots of $f_1(x) = 0$ by 1, so that the next transformed equation is

$$x^3 + 93x^2 + 2883x - 209 = 0 \qquad ...(iii)$$

Its root lies between 0 and 1. (We draw a second zig-zag line above the set of figures 93, 2883, – 209). Multiply the roots of (*iii*) by 10 *i.e.*, attach one zero to second term, two zeros to third term and three zeros to the fourth term. Then the new equation is

$$f_2(x) = x^3 + 930x^2 + 288300x - 209000 = 0$$

Its root lies between 0 and 10, which is nearly

$$= 209000/288300 = 0.724 > 0 \text{ and } < 1.$$

Hence second figure after decimal place is 0.

1	0	0	− 30	(3.107
	3	9	27	
	3	9	− 30000	
	3	18	2791	
	6	2700	− 209000000	
	3	91		
	90	2791		
	1	92		
	91	28830000		
	1			
	92			
	1			
	9300			

Step IV. Diminish the root of $f_2(x) = 0$ by 0 and then multiply its roots by 10 so that
$f_3(x) = x^3 + 9300x^2 + 28830000x - 209000000 = 0.$

Its root lies between 0 and 10, which is nearly $= 209000000/28830000 = 7.2 > 7$ and < 8. Thus the roots of $f_3(x) = 0$ are to be diminished by 7 *i.e.*, the third figure after decimal is 7. Hence the required root is 3.107.

PROBLEMS 28.2

1. Find by Horner's method, the root (correct to three decimal places) of the equation
   (*i*) $x^3 - 3x + 1 = 0$ which lies between 1 and 2
   (*ii*) $x^3 + x - 1 = 0$ (*Coimbatore, 1997*) (*iii*) $x^3 - 6x - 13 = 0$
   (*iv*) $x^3 - 3x^2 + 2.5 = 0$ which lies between 1 and 2. (*Madras, 2000 S*)
2. Using Horner's method, find the largest real root of $x^3 - 4x + 2 = 0$ correct to three decimal places.
3. Show that the root of the equation $x^4 + x^3 - 4x^2 - 16 = 0$ lies between 2 and 3. Find its value correct to two decimal places by Horner's method.
4. Find the negative root of the equation $x^3 - 9x^2 + 18 = 0$ correct to two decimal places by Horner's method.
5. Find the cube root of 25 by Horner's method correct to 3 decimal places.

28.5 SOLUTION OF LINEAR SIMULTANEOUS EQUATIONS

Simultaneous linear equations occur in various engineering problems. The student knows that a given system of linear equations can be solved by Cramer's rule or by Matrix method (§ 2.10). But these methods become tedious for large systems. However, there exist other numerical methods of solution which are well-suited for computing machines. We now explain some direct and iterative methods of solution.

28.6 DIRECT METHODS OF SOLUTION

(1) Gauss elimination method*. In this method, the unknowns are eliminated successively and the system is reduced to an upper triangular system from which the unknowns are found by back substitution. The method is quite general and is well-adapted for computer operations. Here we shall explain it by considering a system of three equations for the sake of clarity.

Consider the equations

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \qquad \text{...(1)}$$

Step I. To eliminate x from second and third equations.

Assuming $a_1 \neq 0$, we eliminate x from the second equation by subtracting (a_2/a_1) times the first equation from the second equation. Similarly we eliminate x from the third equation by eliminating (a_3/a_1) times the first equation from the third equation. We thus, get the new system

*See footnote p. 37.

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ b_2'y + c_2'z &= d_2' \\ b_3'y + c_3'z &= d_3' \end{aligned}\right\} \qquad ...(2)$$

Here the first equation is called the *pivotal equation* and a_1 is called the *first pivot*.

Step II. To eliminate y from third equation in (2).

Assuming $b_2' \neq 0$, we eliminate y from the third equation of (2), by subtracting (b_3'/b_2') times the second equation from the third equation. We thus, get the new system

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ b_2'y + c_2'z &= d_2' \\ c_3''z &= d_3'' \end{aligned}\right\} \qquad ...(3)$$

Here the second equation is the *pivotal equation* and b_2' is the *new pivot*.

Step III. To evaluate the unknowns.

The values of x, y, z are found from the reduced system (3) by back substitution.

Obs. 1. On writing the given equations as $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ *i.e.*, $AX = D$, this method consists in ***transforming the coefficient matrix A to upper triangular matrix*** by elementary row transformations only.

Obs. 2. Clearly the method will fail if any one of the pivots a_1, b_2' or c_3'' becomes zero. In such cases, we rewrite the equations in a different order so that the pivots are non-zero.

Obs. 3. *Partial and complete pivoting.* In the first step, the numerically largest coefficient of x is chosen from all the equations and brought as the first pivot by interchanging the first equation with the equation having the largest coefficient of x. In the second step, the numerically largest coefficient of y is chosen from the remaining equations (leaving the first equation) and brought as the *second pivot* by interchanging the second equation with the equation having the largest coefficient of y'. This process is continued till we arrive at the equation with the single variable. This modified procedure is called *partial pivoting.*

If we are not taken about the elimination of x, y, z in a specified order, then we choose at each stage the numerically largest coefficient of the entire matrix of coefficients. This requires not only an interchange of equations but also an interchange of the position of the variables. This method of elimination is called *complete pivoting.* It is more complicated and does not appreciably improve the accuracy.

Example 28.13. *Apply Gauss elimination method to solve the equations $x + 4y - z = -5$; $x + y - 6z = -12$; $3x - y - z = 4$.* *(Mumbai, 2009)*

		Check sum	
Solution. We have	$x + 4y - z = -5$	-1	...(*i*)
	$x + y - 6z = -12$	-16	...(*ii*)
	$3x - y - z = 4$	5	...(*iii*)

Step I. Operate $(ii) - (i)$ and $(iii) - 3(i)$ to eliminate x :

	Check sum	
$-3y - 5z = -7$	-15	...(*iv*)
$-13y + 2z = 19$	8	...(*v*)

Step II. Operate $(v) - \dfrac{13}{3}(iv)$ to eliminate y :

	Check sum	
$\dfrac{71}{3}z = \dfrac{148}{3}$	73	...(*vi*)

Step III. By back-substitution, we get

From (*vi*) : $\quad z = \dfrac{148}{71} = 2.0845$

From (*iv*) : $\quad y = \dfrac{7}{3} - \dfrac{5}{3}\left(\dfrac{148}{71}\right) = -\dfrac{81}{71} = -1.1408$

From (*i*) : $$x = -5 - 4\left(-\frac{81}{71}\right) + \frac{148}{71} = \frac{117}{71} = 1.6479$$

Hence $x = 1.6479, y = -1.1408, z = 2.0845$

Note. A useful check is provided by noting the sum of the coefficients and terms on the right, operating on those numbers as on the equations and checking that the derived equations have the correct sum.

Otherwise : We have $\begin{bmatrix} 1 & 4 & -1 \\ 1 & 1 & -6 \\ 3 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -12 \\ 4 \end{bmatrix}$

Operate $R_2 - R_1$ and $R_3 - 3R_1$, $\begin{bmatrix} 1 & 4 & -1 \\ 0 & -3 & -5 \\ 0 & -13 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \\ 19 \end{bmatrix}$

Operate $R_3 - \frac{13}{3}R_2$, $\begin{bmatrix} 1 & 4 & -1 \\ 0 & -3 & -5 \\ 0 & 0 & 71/3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \\ 148/3 \end{bmatrix}$

Thus, we have $z = 148/71 = 2.0845$,

$$3y = 7 - 5z = 7 - 10.4225 = -3.4225 \quad i.e., \quad y = -1.1408$$

and $$x = -5 - 4y + z = -5 + 4(1.1408) + 2.0845 = 1.6479$$

Hence $x = 1.6479, y = -1.1408, z = 2.0845$.

Example 28.14. *Solve $10x - 7y + 3z + 5u = 6, -6x + 8y - z - 4u = 5, 3x + y + 4z + 11u = 2, 5x - 9y - 2z + 4u = 7$ by Gauss elimination method.* *(S.V.T.U., 2007)*

Solution. We have

	Check sum	
$10x - 7y + 3z + 5u = 6$	17	...(*i*)
$-6x + 8y - z - 4u = 5$	2	...(*ii*)
$3x + y + 4z + 11u = 2$	21	...(*iii*)
$5x - 9y - 2z + 4u = 7$	5	...(*iv*)

Step I. To eliminate x, operate $\left[(ii) - \left(\frac{-6}{10}\right)(i)\right], \left[(iii) - \frac{3}{10}(i)\right], \left[(iv) - \frac{5}{10}(i)\right]$:

	Check sum	
$3.8y + 0.8z - u = 8.6$	12.2	...(*v*)
$3.1y + 3.1z + 9.5u = 0.2$	15.9	...(*vi*)
$-5.5y - 3.5z + 1.5u = 4$	-3.5	...(*vii*)

Step II. To eliminate y, operate $\left[(vi) - \frac{3.1}{3.8}(v)\right], \left[(vii) - \left(\frac{-5.5}{3.8}\right)(v)\right]$:

$$2.4473684z + 10.315789u = -6.8157895 \quad ...(viii)$$

$$-2.3421053z + 0.0526315u = 16.447368 \quad ...(ix)$$

Step III. To eliminate z, operate $\left[(ix) - \left(\frac{-2.3421053}{2.4473684}\right)(viii)\right]$:

$$9.9249319u = 9.9245977$$

Step IV. By back-substitution, we get

$$u = 1, z = -7, y = 4 \text{ and } x = 5.$$

(2) Gauss-Jordan method*. This is a modification of the Gauss elimination method. In this method, elimination of unknowns is performed not in the equations below but in the equations above also, ultimately **reducing the system to a diagonal matrix form *i.e.*,** *each equation involving only one unknown. From these equations the unknowns x, y, z can be obtained readily.*

Thus in this method, the labour of back-substitution for finding the unknowns is saved at the cost of additional calculations.

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*See footnote p. 37.

Example 28.15. *Apply Gauss-Jordan method to solve the equations*
$x + y + z = 9$; $2x - 3y + 4z = 13$; $3x + 4y + 5z = 40$. (V.T.U., 2009 ; P.T.U., 2005)

Solution. We have

$$x + y + z = 9 \quad ...(i)$$
$$2x - 3y + 4z = 13 \quad ...(ii)$$
$$3x + 4y + 5z = 40 \quad ...(iii)$$

Step I. Operate $(ii) - 2(i)$ and $(iii) - 3(i)$ to eliminate x from (ii) and (iii).

$$x + y + z = 9 \quad ...(iv)$$
$$-5y + 2z = -5 \quad ...(v)$$
$$y + 2z = 13 \quad ...(vi)$$

Step II. Operate $(iv) + \frac{1}{5}(v)$ and $(vi) + \frac{1}{5}(v)$ to eliminate y from (iv) and (vi) :

$$x + \frac{7}{5}z = 8 \quad ...(vii)$$
$$-5y + 2z = -5 \quad ...(viii)$$
$$\frac{12}{5}z = 12 \quad ...(ix)$$

Step III. Operate $(vii) - \frac{7}{12}(ix)$ and $(viii) - \frac{5}{6}(ix)$ to eliminate z from (vii) and $(viii)$:

$$x = 1$$
$$-5y = -15$$
$$\frac{12}{5}z = 12$$

Hence the solution is $x = 1, y = 3, z = 5$.

Otherwise : Rewriting the equations as
$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 40 \end{bmatrix}$$

Operate $R_2 - 2R_1, R_3 - 3R_1$,
$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -5 & 2 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \\ 13 \end{bmatrix}$$

Operate $R_3 + \frac{1}{5}R_2$,
$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -5 & 2 \\ 0 & 0 & 12/5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \\ 12 \end{bmatrix}$$

Operate $-R_2, 5R_3$
$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & -2 \\ 0 & 0 & 12 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 60 \end{bmatrix}$$

Operate $R_2 + \frac{1}{6}R_3, \frac{1}{12}R_3$
$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 15 \\ 5 \end{bmatrix}$$

Operate $\frac{1}{5}R_2$
$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ 5 \end{bmatrix}$$

Operate $R_1 - R_2 - R_3$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

Hence, $x = 1, y = 3, z = 5$.

Obs. Here the process of elimination of variables amounts to *reducing the given coefficient metric to a* **diagonal matrix** *by elementary row transformations only.*

Example 28.16. *Solve the equations of example 28.14, by Gauss-Jordan method.*

Solution. We have

$$10x - 7y + 3z + 5u = 6 \quad ...(i)$$
$$-6x + 8y - z - 4u = 5 \quad ...(ii)$$
$$3x + y + 4z + 11u = 2 \quad ...(iii)$$
$$5x - 9y - 2z + 4u = 7 \quad ...(iv)$$

Step I. To eliminate x, operate $\left[(ii) - \left(\frac{-6}{10}\right)(i)\right], \left[(iii) - \left(\frac{3}{10}\right)(i)\right], \left[(iv) - \left(\frac{5}{10}\right)(i)\right]$:

$$10x - 7y + 3z + 5u = 6 \quad ...(v)$$
$$3.8y + 0.8z - u = 8.6 \quad ...(vi)$$
$$3.1y + 3.1z + 9.5u = 0.2 \quad ...(vii)$$
$$-5.5y - 3.5z + 1.5u = 4 \quad ...(viii)$$

Step II. To eliminate y, operate $\left[(v) - \left(\frac{-7}{3.8}\right)(vi)\right], \left[(vii) - \left(\frac{3.1}{3.8}\right)(vi)\right], \left[(viii) - \left(\frac{-5.5}{3.8}\right)(vi)\right]$:

$$10x + 4.4736842z + 3.1578947u = 21.842105 \quad ...(ix)$$
$$3.8y + 0.8z - u = 8.6 \quad ...(x)$$
$$2.4473684z + 10.315789u = -6.8157895 \quad ...(xi)$$
$$-2.3421053x + 0.0526315u = 16.447368 \quad ...(xii)$$

Step III. To eliminate z, operate $\left[(ix) - \left(\frac{4.473684}{2.4473684}\right)(xi)\right]$,

$$\left[(x) - \left(\frac{0.8}{2.4473684}\right)(xi)\right], \left[(xii) - \left(\frac{-2.3421053}{2.4473684}\right)(xi)\right]:$$

$$10x - 15.698923u = 34.301075$$
$$3.8y - 4.3720429u = 10.827957$$
$$2.4473684z + 10.315789u = -6.8157895$$
$$9.9247309u = 9.9245975$$

Step IV. From the last equation $u = 1$ nearly.

Substitution of $u = 1$ in the above three equations gives $x = 5, y = 4, z = -7$.

(3) Factorization method*. This method is based on the fact that every matrix A can be expressed as the product of a lower triangular matrix and an upper triangular matrix, provided all the principal minors of A are non-singular, *i.e.*, if $A = [a_{ij}]$, then

$$a_{11} \neq 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0, \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0, \text{ etc.}$$

Also such a factorization if it exists, is unique.

Now consider the equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

which can be written as $AX = B$ ...(1)

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Let $A = LU,$ ...(2)

where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

*Another name given to this discomposition is **Doolittle's method.**

Then (1) becomes $LUX = B$ ...(3)

Writing $UX = V$, ...(4)

(3) becomes $LV = B$ which is equivalent to the equations

$$v_1 = b_1 \; ; \; l_{21}v_1 + v_2 = b_2 \; ; \; l_{31}v_1 + l_{31}v_2 + v_3 = b_3$$

Solving these for v_1, v_2, v_3, we know V. Then, (4) becomes

$$u_{11}x_1 + u_{12}x_2 + u_{13}x_3 = v_1 \; ; \; u_{22}x_2 + u_{23}x_3 = v_2 \; ; \; u_{33}x_3 = v_3,$$

from which x_3, x_2 and x_1 can be found by *back-substitution.*

To compute the matrices L and U, we write (2) as

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 0 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Multiplying the matrices on the left and equating corresponding elements from both sides, we obtain

(*i*) $u_{11} = a_{11}$, $u_{12} = a_{12}$, $u_{13} = a_{13}$

(*ii*) $l_{21}u_{11} = a_{21}$ or $l_{21} = a_{21}/a_{11}$

$l_{31}u_{11} = a_{31}$ or $l_{31} = a_{31}/a_{11}$

(*iii*) $l_{21}u_{12} + u_{22} = a_{22}$ or $u_{22} = a_{22} - \dfrac{a_{21}}{a_{11}} a_{12}$

$l_{21}u_{13} + u_{23} = a_{23}$ or $u_{23} = a_{23} - \dfrac{a_{21}}{a_{11}} a_{13}$

(*iv*) $l_{31}u_{12} + l_{33}u_{22} = a_{32}$ or $l_{32} = \dfrac{1}{u_{22}}\left[a_{32} - \dfrac{a_{31}}{a_{11}} a_{12}\right]$

(*v*) $l_{31}u_{13} + l_{33}u_{23} + u_{33} = a_{33}$ which gives u_{33}.

Thus we compute the elements of L and U in the following set order :

(*i*) First row of U, (*ii*) First column of L,

(*iii*) Second row of U, (*iv*) Second column of L, (*v*) Third row of U.

This procedure can easily be generalised.

Obs. This method is superior to Gauss elimination method and is often used for the solution of linear systems and **for finding the inverse of a matrix.** Among the direct methods, Factorization method is also preferred as the software for computers.

Example 28.17. *Apply factorization method to solve the equations :*

$3x + 2y + 7z = 4$; $2x + 3y + z = 5$; $3x + 4y + z = 7$. *(Madras, 2000 S)*

Solution. Let $\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix}$ (*i.e.* A),

so that

(*i*) $u_{11} = 3$, $u_{12} = 2$, $u_{13} = 7$.

(*ii*) $l_{21}u_{11} = 2$, $\therefore \; l_{21} = 2/3$

$l_{31}u_{11} = 3$, $\therefore \; l_{31} = 1$.

(*iii*) $l_{21}u_{12} + u_{22} = 3$, $\therefore \; u_{22} = 5/3$,

$l_{21}u_{13} + u_{23} = 1$, $\therefore \; u_{23} = -11/3$.

(*iv*) $l_{31}u_{12} + l_{32}u_{22} = 4$, $\therefore \; l_{32} = 6/5$.

(*v*) $l_{31}u_{13} + l_{32}u_{23} + u_{33} = 1$

$\therefore \quad u_{33} = -8/5$

Thus $$A = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1 & 6/5 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 7 \\ 0 & 5/3 & -11/3 \\ 0 & 0 & -8/5 \end{bmatrix}$$

Writing $UX = V$, the given system becomes $\begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1 & 6/5 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$

Solving this system, we have $v_1 = 4$,

$$\frac{2}{3} v_1 + v_2 = 5 \qquad \text{or} \qquad v_2 = \frac{7}{3}$$

$$v_1 + \frac{6}{5} v_2 + v_3 = 7 \qquad \text{or} \qquad v_3 = \frac{1}{5}$$

Hence the original system becomes

$$\begin{bmatrix} 3 & 2 & 7 \\ 0 & 5/3 & -11/3 \\ 0 & 0 & -8/5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 7/3 \\ 1/5 \end{bmatrix}$$

i.e., $$3x + 2y + 7z = 4 \,;\ \frac{5}{3} y - \frac{11}{3} z = \frac{7}{3} \,;\ -\frac{8}{5} z = \frac{1}{5}$$

By back-substitution, we have $z = -1/8$, $y = 9/8$ and $x = 7/8$.

Example 28.18. *Solve the equations of Example 28.14 by factorization method.*

Solution. Let $\begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{13} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix} = \begin{bmatrix} 10 & -7 & 3 & 5 \\ -6 & 8 & -1 & -4 \\ 3 & 1 & 4 & 11 \\ 5 & -9 & -2 & 4 \end{bmatrix}$ (*i.e.*, A)

so that

(*i*) R_1 of $U : u_{11} = 10, u_{12} = -7, u_{13} = 3, u_{14} = 5$

(*ii*) C_1 of $L : l_{21} = -0.6, l_{31} = 0.3, l_{41} = 0.5$

(*iii*) R_2 of $U : u_{22} = 3.8, u_{23} = 0.8, u_{24} = -1$

(*iv*) C_2 of $L : l_{32} = 0.81579, l_{42} = -1.44737$

(*v*) R_3 of $U : u_{33} = 2.44737, u_{34} = 10.31579$

(*vi*) C_3 of $L : l_{43} = -0.95699$

(*vii*) R_4 of $U : u_{44} = 9.92474$

Thus

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -0.6 & 1 & 0 & 0 \\ 0.3 & 0.81579 & 1 & 0 \\ 0.5 & -1.44737 & -0.95699 & 1 \end{bmatrix} \begin{bmatrix} 10 & -7 & 3 & 5 \\ 0 & 3.8 & 0.8 & -1 \\ 0 & 0 & 2.44737 & 10.31579 \\ 0 & 0 & 0 & 9.92474 \end{bmatrix}$$

Writing $UX = V$, the given system becomes

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -0.6 & 1 & 0 & 0 \\ 0.3 & 0.81579 & 1 & 0 \\ 0.5 & -1.44737 & -0.95699 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 2 \\ 7 \end{bmatrix}$$

Solving this system, we get

$$v_1 = 6, v_2 = 8.6, v_3 = -6.81579, v_4 = 9.92474.$$

Hence the original system becomes

$$\begin{bmatrix} 10 & -7 & 3 & 5 \\ 0 & 3.8 & 0.8 & -1 \\ 0 & 0 & 2.44737 & 10.31579 \\ 0 & 0 & 0 & 9.92474 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 6 \\ 8.6 \\ -6.81579 \\ 9.92474 \end{bmatrix}$$

i.e., $$10x - 7y + 3z + 5u = 6, \quad 3.8y + 0.8z - u = 8.6,$$
$$2.44737z + 10.31579u = -6.81579, u = 1.$$

By back-substitution, we get $u = 1, z = -7, y = 4, x = 5$.

PROBLEMS 28.3

Solve the following equations by Gauss elimination method :

1. $2x + y + z = 10$; $3x + 2y + 3z = 18$; $x + 4y + 9z = 16$. *(P.T.U., 2005)*
2. $2x + 2y + z = 12$; $3x + 2y + 2z = 8$; $5x + 10y - 8z = 10$. *(W.B.T.U., 2004)*
3. $2x - y + 3z = 9$; $x + y + z = 6$; $x - y + z = 2$. *(Bhopal, 2009)*
4. $2x_1 + 4x_2 + x_3 = 3$; $3x_1 + 2x_2 - 2x_3 = -2$; $x_1 - x_2 + x_3 = 6$. *(Marathwada, 2008)*
5. $5x_1 + x_2 + x_3 + x_4 = 4$; $x_1 + 7x_2 + x_3 + x_4 = 12$;
   $x_1 + x_2 + 6x_3 + x_4 = -5$; $x_1 + x_2 + x_3 + 4x_4 = -6$.

Solve the following equations by Gauss-Jordan method :

6. $2x + 5y + 7z = 52$; $2x + y - z = 0$; $x + y + z = 9$. *(V.T.U., 2010)*
7. $2x - 3y + z = -1$; $x + 4y + 5z = 25$; $3x - 4y + z = 2$. *(Kerala, 2003)*
8. $x + 3y + 3z = 16$; $x + 4y + 3z = 18$; $x + 3y + 4z = 19$. *(Anna, 2005)*
9. $2x + y + z = 10$; $3x + 2y + 3z = 18$; $x + 4y + 9z = 16$. *(V.T.U., 2008)*
10. $2x_1 + x_2 + 5x_3 + x_4 = 5$; $x_1 + x_2 - 3x_3 + 4x_4 = -1$;
    $3x_1 + 6x_2 - 2x_3 + x_4 = 8$; $2x_1 + 2x_2 + 2x_3 - 3x_4 = 2$.

Solve the following equations by factorization method :

11. $10x + y + z = 12$; $2x + 10y + z = 13$; $2x + 2y + 10z = 14$. *(Andhra, 2004 ; P.T.U., 2003)*
12. $x + 2y + 3z = 14$; $2x + 3y + 4z = 20$; $3x + 4y + z = 14$.
13. $2x + 3y + z = 9$; $x + 2y + 3z = 6$; $3x + y + 2z = 8$.
14. $2x_1 - x_2 + x_3 = -1$; $2x_2 - x_3 + x_4 = 1$; $x_1 + 2x_3 - x_4 = -1$; $x_1 + x_2 + 2x_4 = 5$.
15. Find the inverse of the matrix $\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & 4 \\ 1 & 2 & 2 \end{bmatrix}$ by Crout's method.

28.7 ITERATIVE METHODS OF SOLUTION

The preceding methods of solving simultaneous linear equations are known as *direct methods* as they yield exact solutions. On the other hand, an iterative method is that in which we start from an approximation to the true solution and obtain better and better approximations from a computation cycle repeated as often as may be necessary for achieving a desired accuracy.

Simple iteration methods can be devised for systems in which the coefficients of the leading diagonal are large compared to others. We now explain three such methods :

(1) Jacobi's iteration method*. Consider the equations

$$\left.\begin{aligned} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{aligned}\right\} \qquad ...(1)$$

If a_1, b_2, c_3 are large as compared to other coefficients, then solving these for x, y, z respectively, the system can be written in the form

$$\left.\begin{aligned} x = k_1 - l_1y - m_1z \\ y = k_2 - l_2x - m_2z \\ z = k_3 - l_3x - m_3y \end{aligned}\right\} \qquad ...(2)$$

Let us start with the initial approximations x_0, y_0, z_0 (each $= 0$) for the values of x, y, z. Substituting these on the right, we get the first approximations $x_1 = k_1, y_1 = k_2, z_1 = k_3$.

Substituting these on the right-hand sides of (2), the second approximations are given by

$$x_2 = k_1 - l_1y_1 - m_1z_1$$
$$y_2 = k_2 - l_2x_1 - m_2z_1$$
$$z_2 = k_3 - l_3x_1 - m_3y_1$$

This process is repeated till the difference between two consecutive approximations is negligible.

*See footnote p. 215.

Example 28.19. *Solve by Jacobi's iteration method, the equations $10x + y - z = 11.19$, $x + 10y + z = 28.08$, $-x + y + 10z = 35.61$, correct to two decimal places.* (Anna, 2007)

Solution. Rewriting the given equations as

$$x = \frac{1}{10}(11.19 - y + z), y = \frac{1}{10}(28.08 - x - z), z = \frac{1}{10}(35.61 + x - y)$$

We start from an approximation, $x_0 = y_0 = z_0 = 0$.

First iteration $\quad x_1 = \frac{11.19}{10} = 1.119, y_1 = \frac{28.08}{10} = 2.808, z_1 = \frac{35.61}{10} = 3.561$

Second iteration $\quad x_2 = \frac{1}{10}(11.19 - y_1 + z_1) = 1.19$

$$y_2 = \frac{1}{10}(28.08 - x_1 - z_1) = 2.24$$

$$z_2 = \frac{1}{10}(35.61 + x_1 - y_1) = 3.39$$

Third iteration $\quad x_3 = \frac{1}{10}(11.19 - y_2 + z_2) = 1.22$

$$y_3 = \frac{1}{10}(28.03 - x_2 - z_2) = 2.35$$

$$z_3 = \frac{1}{10}(35.61 + x_2 - y_2) = 3.45$$

Fourth iteration $\quad x_4 = \frac{1}{10}(11.19 - y_3 + z_3) = 1.23$

$$y_4 = \frac{1}{10}(28.03 - x_3 - z_3) = 2.34$$

$$z_4 = \frac{1}{10}(35.61 + x_3 - y_3) = 3.45$$

Fifth iteration $\quad x_5 = \frac{1}{10}(11.19 - y_4 + z_4) = 1.23$

$$y_5 = \frac{1}{10}(28.08 - x_4 - z_4) = 2.34$$

$$z_5 = \frac{1}{10}(35.61 + x_4 - y_4) = 3.45$$

Hence $x = 1.23, y = 2.34, z = 3.45$.

Example 28.20. *Solve, by Jacobi's iteration method, the equations*

$20x + y - 2z = 17$; $3x + 20y - z = -18$; $2x - 3y + 20z = 25$. (Bhopal, 2009)

Solution. We write the given equations in the form

$$\left.\begin{aligned} x &= \frac{1}{20}(17 - y + 2z) \\ y &= \frac{1}{20}(-18 - 3x + z) \\ z &= \frac{1}{20}(25 - 2x + 3y) \end{aligned}\right\} \quad ...(i)$$

We start from an approximation $x_0 = y_0 = z_0 = 0$.

Substituting these on the right sides of the equations (*i*), we get

$$x_1 = \frac{17}{20} = 0.85 ; y_1 = -\frac{18}{20} = -0.9 ; z_1 = \frac{25}{20} = 1.25$$

Putting these values on the right of the equations (*i*), we obtain

$$x_2 = \frac{1}{20}(17 - y_1 + 2z_1) = 1.02$$

$$y_2 = \frac{1}{20}(-18 - 3x_1 + z_1) = -0.965$$

$$z_2 = \frac{1}{20}(25 - 2x_1 + 3y_1) = 1.1515$$

Substituting these values in the right sides of the equations (*i*), we have

$$x_3 = \frac{1}{20}(17 - y_2 + 2z_2) = 1.0134$$

$$y_3 = \frac{1}{20}(-18 - 3x_2 + z_2) = -0.9954$$

$$z_3 = \frac{1}{20}(25 - 2x_2 + 3y_2) = 1.0032$$

Substituting these values, we get

$$x_4 = \frac{1}{20}(17 - y_3 + 2z_3) = 1.0009$$

$$y_4 = \frac{1}{20}(-18 - 3x_3 + z_3) = -1.0018$$

$$z_4 = \frac{1}{20}(25 - 2x_3 + 3y_3) = 0.9993$$

Putting these values, we have

$$x_5 = \frac{1}{20}(17 - y_4 + 2z_4) = 1.0000$$

$$y_5 = \frac{1}{20}(-18 - 3x_4 + z_4) = -1.0002$$

$$z_5 = \frac{1}{20}(25 - 2x_4 + 3y_4) = 0.9996$$

Again substituting these values, we get

$$x_6 = \frac{1}{20}(17 - y_5 + 2z_5) = 1.0000$$

$$y_6 = \frac{1}{20}(-18 - 3x_5 + z_5) = -1.0000$$

$$z_6 = \frac{1}{20}(25 - 2x_5 + 3y_5) = 1.0000$$

The values in the 5th and 6th iterations being practically the same, we can stop.

Hence the solution is $x = 1, y = -1, z = 1$.

(2) Gauss-Seidel iteration method*. This is a modification of the Jacobi's iteration method. As before, we start with initial approximations x_0, y_0, z_0 (each = 0) for x, y, z respectively. Substituting $y = y_0, z = z_0$ in the first of the equations (2) on page 837, we get

$$x_1 = k_1$$

Then putting $x = x_1, z = z_0$ in the second of the equations (2) on page 837, we have

$$y_1 = k_2 - l_2 x_1 - m_2 z_0$$

Next substituting $x = x_1, y = y_1$ in the third of the equations (2) on page 837, we obtain

$$z_1 = k_3 - l_3 x_1 - m_3 y_1$$

and so on, *i.e.*, as soon as new approximation for an unknown is found, it is immediately used in the next step.

This process of iteration is continued till convergency to the desired degree of accuracy is obtained.

Obs 1. Since the most recent approximation of the unknowns are used while proceeding to the next step, the convergence in the Gauss-Seidel method is faster than in Jacobi's method.

Obs 2. Gauss-Sedial method converges if in each equation, the absolute value of the largest coefficient is greater than the sum of the absolute values of the remaining coefficients.

*See footnote p. 37. *After Philipp Ludwig Von Seidel* (1821–1896) who also suggested a similar method.

Example 28.21. *Apply Gauss-Seidel iteration method to solve the equations of Ex. 28.20.*
(V.T.U., 2011 ; Rohtak, 2005 ; Madras, 2003)

Solution. We write the given equation in the form

$$x = \frac{1}{20}(17 - y + 2z) \; ; \; y = \frac{1}{20}(-18 - 3x + z) \; ; \; z = \frac{1}{20}(25 - 2x + 3y) \qquad ...(i)$$

We start from the approximation $x_0 = y_0 = z_0 = 0$. Substituting $y = y_0$, $z = z_0$ in the right side of the first of equations (*i*), we get

$$x_1 = \frac{1}{20}(17 - y_0 + 2z_0) = 0.8500$$

Putting $x = x_1$, $z = z_0$ in the second of the equations (*i*), we have

$$y_1 = \frac{1}{20}(-18 - 3x_1 + z_0) = -1.0275$$

Putting $x = x_1$, $y = y_1$ in the last of the equations (*i*), we obtain

$$z_1 = \frac{1}{20}(25 - 2x_1 + 3y_1) = 1.0109$$

For the second iteration, we have

$$x_2 = \frac{1}{20}(17 - y_1 + 2z_1) = 1.0025$$

$$y_2 = \frac{1}{20}(-18 - 3x_2 + z_1) = -0.9998$$

$$z_2 = \frac{1}{20}(25 - 2x_2 + 3y_2) = 0.9998$$

For the third iteration, we get

$$x_3 = \frac{1}{20}(17 - y_2 + 2z_2) = 1.0000$$

$$y_3 = \frac{1}{20}(-18 - 3x_3 + z_2) = -1.0000$$

$$z_3 = \frac{1}{20}(25 - 3x_3 + 2y_3) = 1.0000$$

The values in the 2nd and 3rd iterations being practically the same, we can stop.

Hence the solution is $x = 1$, $y = -1$, $z = 1$.

Example 28.22. *Solve the equations :*

$$10x_1 - 2x_2 - x_3 - x_4 = 3$$
$$-2x_1 + 10x_2 - x_3 - x_4 = 15$$
$$-x_1 - x_2 + 10x_3 - 2x_4 = 27$$
$$-x_1 - x_2 - 2x_3 + 10x_4 = -9$$

by Gauss-Seidal iteration method. *(Bhopal, 2009 ; J.N.T.U., 2004)*

Solution. Rewriting the given equations as

$$x_1 = 0.3 + 0.2x_2 + 0.1x_3 + 0.1x_4 \qquad ...(i)$$
$$x_2 = 1.5 + 0.2x_1 + 0.1x_3 + 0.1x_4 \qquad ...(ii)$$
$$x_3 = 2.7 + 0.1x_1 + 0.1x_2 + 0.2x_4 \qquad ...(iii)$$
$$x_4 = -0.9 + 0.1x_1 + 0.1x_2 + 0.2x_3 \qquad ...(iv)$$

First iteration

Putting $x_2 = 0$, $x_3 = 0$, $x_4 = 0$ in (*i*), we get $x_1 = 0.3$

Putting $x_1 = 0.3$, $x_3 = 0$, $x_4 = 0$ in (*ii*), we obtain $x_2 = 1.56$

Putting $x_1 = 0.3$, $x_2 = 1.56$, $x_4 = 0$ in (*iii*), we obtain $x_3 = 2.886$

Putting $x_1 = 0.3$, $x_2 = 1.56$, $x_3 = 2.886$ in (*iv*), we get $x_4 = -0.1368$

Second iteration

Putting $x_2 = 1.56$, $x_3 = 2.886$, $x_4 = -0.1368$ in (*i*), we obtain

$x_1 = 0.8869$

Putting $x_1 = 0.8869$, $x_3 = 2.886$, $x_4 = -0.1368$ in (*ii*), we obtain

$x_2 = 1.9523$

Putting $x_1 = 0.8869$, $x_2 = 1.9523$, $x_4 = -0.1368$ in (*iii*), we have

$x_3 = 2.9566$

Putting $x_1 = 0.8869$, $x_2 = 1.9523$, $x_3 = 2.9566$ in (*iv*), we get

$x_4 = -0.0248.$

Third iteration

Putting $x_2 = 1.9523$, $x_3 = 2.9566$, $x_4 = -0.0248$ in (*i*), we obtain

$x_1 = 0.9836$

Putting $x_1 = 0.9836$, $x_3 = 2.9566$, $x_4 = -0.0248$ in (*ii*), we obtain

$x_2 = 1.9899$

Putting $x_1 = 0.9836$, $x_2 = 1.9899$, $x_4 = -0.0248$ in (*iii*), we get

$x_3 = 2.9924$

Putting $x_1 = 0.9836$, $x_1 = 1.9899$, $x_3 = 2.9924$ in (*iv*), we get

$x_4 = -0.0042.$

Fourth iteration. Proceeding as above

$x_1 = 0.9968$, $x_2 = 1.9982$, $x_3 = 2.9987$, $x_4 = -0.0008.$

Fifth iteration is

$x_1 = 0.9994$, $x_2 = 1.9997$, $x_3 = 2.9997$, $x_4 = -0.0001.$

Sixth iteration is

$x_1 = 0.9999$, $x_2 = 1.9999$, $x_3 = 2.9999$, $x_4 = -0.0001.$

Hence the solution is $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $x_4 = 0$.

(3) **Relaxation method*.** Consider the equations

$$a_1x + b_1y + c_1z = d_1\ ;\ a_2x + b_2y + c_2z = d_2\ ;\ a_3x + b_3y + c_3z = d_3$$

We define the residuals R_x, R_y, R_z by the relations

$$R_x = d_1 - a_1x - b_1y - c_1z\ ;\ R_y = d_2 - a_2x - b_2y - c_2z\ ;\ R_z = d_3 - a_3x - b_3y - c_3z \qquad ...(1)$$

To start with we assume $x = y = z = 0$ and calculate the initial residuals. Then the residuals are reduced step by step by giving increments to the variables. For this purpose, we construct the following *operation table :*

	δR_x	δR_y	δR_z
$\delta x = 1$	$-a_1$	$-a_2$	$-a_3$
$\delta y = 1$	$-b_1$	$-b_2$	$-b_3$
$\delta z = 1$	$-c_1$	$-c_2$	$-c_3$

We note from the equations (1) that if x is increased by 1 (keeping y and z constant), R_x, R_y and R_z decrease by a_1, a_2, a_3 respectively. This is shown in the above table alongwith the effects on the residuals when y and z are given unit increments. (The table is the transpose of the coefficient matrix).

At each step, the numerically largest residual is reduced to almost zero. To reduce a particular residual, the value of the corresponding variable is changed ; *e.g.*, to reduce R_x by p, x should be increased by p/a_1.

When all the residuals have been reduced to almost zero, the increments in x, y, z are added separately to give the desired solution.

Obs. As a check, the computed values of x, y, z are substituted in (1) and the residuals are calculated. If these residuals are not all negligible, then there is some mistake and the entire process should be rechecked.

Example 28.23. *Solve, by Relaxation method, the equations :*

$$9x - 2y + z = 50,\ x + 5y - 3z = 18,\ -2x + 2y + 7z = 19. \qquad \text{(Madras, 2000 S)}$$

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*This method was originally developed by *R.V. Southwell* in 1935, for application to structural engineering problems.

Solution. The residuals are given by

$R_x = 50 - 9x + 2y - z$; $R_y = 18 - x - 5y + 3z$; $R_z = 19 + 2x - 2y - 7z$

The operations table is

	δR_x	δR_y	δR_z
$\delta x = 1$	-9	-1	2
$\delta y = 1$	2	-5	-2
$\delta z = 1$	-1	3	-7

The relaxation table is

	R_x	R_y	R_z	
$x = y = z = 0$	50	18	19	...(*i*)
$\delta x = 5$	5	13	29	...(*ii*)
$\delta z = 4$	1	25	1	...(*iii*)
$\delta y = 5$	11	0	-9	...(*iv*)
$\delta x = 1$	2	-1	-7	...(*v*)
$\delta z = -1$	3	-4	0	...(*vi*)
$\delta y = -0.8$	1.4	0	1.6	...(*vii*)
$\delta z = 0.23$	1.17	0.69	$-.09$	...(*viii*)
$\delta x = 0.13$	0	0.56	0.17	...(*ix*)
$\delta y = 0.112$	0.224	0	-0.054	...(*x*)

$\Sigma\delta x = 6.13$, $\Sigma\delta y = 4.31$, $\Sigma\delta z = 3.23$

Thus $x = 6.13, y = 4.31, z = 3.23.$

[**Explanation.** In (*i*), the largest residual is 50. To reduce it, we give an increment $dx = 5$ and the resulting residuals are shown in (*ii*). Of these $R_x = 29$ is the largest and we given an increment $\delta z = 4$ to get the results in (*iii*). In (*vi*), $R_y = -4$ is the (numerically) largest and we give an increment $\delta y = -4/5 = -0.8$ to obtain the results in (*vii*). Similarly the other steps have been carried out.]

Example 28.24. *Solve by Relaxation method, the equations :*

$10x - 2y - 3z = 205$; $-2x + 10y - 2z = 154$; $-2x - y + 10z = 120$. (V.T.U., 2011 S ; Rohtak, 2005)

Solution. The residuals are given by

$R_x = 205 - 10x + 2y + 3z$; $R_y = 154 + 2x - 10y + 2z$; $R_z = 120 + 2x + y - 10z$.

The operations table is

	δR_x	δR_y	δR_z
$\delta x = 1$	-10	2	2
$\delta y = 1$	2	-10	-1
$\delta z = 1$	3	2	-10

The relaxation table is :

	R_x	R_y	R_z
$x = y = z = 0$	205	154	120
$\delta x = 20$	5	194	160
$\delta y = 19$	43	4	179
$\delta z = 18$	97	40	-1
$\delta x = 10$	-3	60	19
$\delta y = 6$	9	0	25
$\delta z = 2$	15	4	5
$\delta x = 2$	-5	8	9
$\delta z = 1$	-2	10	-1
$\delta y = 1$	0	0	0

$\Sigma\delta x = 32$, $\Sigma y = 26$, $\Sigma z = 21$.

Hence $x = 32, y = 26, z = 21.$

PROBLEMS 28.4

1. Solve by Jacobi's method, the equations : $5x - y + z = 10$; $2x + 4y = 12$; $x + y + 5z = -1$. Start with the solution (2, 3, 0).
2. Solve the equations $27x + 6y - z = 85$; $x + y + 54z = 110$; $6x + 15y + 2z = 72$. by (*a*) Jacobi's method (*b*) Gauss-Seidel method. *(Anna, 2006)*

Solve the following equations by Gauss-Seidel method :

3. $2x + y + 6z = 9$; $8x + 3y + 2z = 13$; $x + 5y + z = 7$.
4. $28x + 4y - z = 32$; $x + 3y + 10z = 24$; $2x + 17y + 4z = 35$. *(Mumbai, 2009)*
5. $10x + y + z = 12$; $2x + 10y + z = 13$; $2x + 2y + 10z = 104$. *(V.T.U., MCA, 2007)*
6. $83x + 11y - 4z = 95$; $7x + 52y + 13z = 104$; $3x + 8y + 29z = 71$. *(Hazaribagh, 2009)*
7. $3x_1 - 0.1x_2 - 0.2x_3 = 7.85$; $0.1x_1 + 7x_2 - 0.3x_3 = -19.3$; $0.3x_1 - 0.2x_2 + 10x_3 = 71.4$ *(Mumbai, 2004)*
8. $1.2x + 2.1y + 4.2z = 9.9$; $5.3x + 6.1y + 4.7z = 21.6$; $9.2x + 8.3y + z = 15.2$.
9. $\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \end{bmatrix}$

Solve by Relaxation method, the following sets of equations :

10. $3x + 9y - 2z = 11$; $4x + 2y + 13z = 24$; $4x - 4y + 3z = -8$. *(Bhopal, 2002)*
11. $10x - 2y - 2z = 6$; $-x + 10y - 2z = 7$; $-x - y + 10z = 8$.
12. $-9x + 3y + 4z + 100 = 0$; $x - 7y + 3z + 80 = 0$; $2x + 3y - 5z + 60 = 0$.
13. $54x + y + z = 110$; $2x + 15y + 6z = 72$; $-x + 6y + 27z = 85$. *(Bhopal, 2003)*

28.8 SOLUTION OF NON-LINEAR SIMULTANEOUS EQUATIONS—NEWTON-RAPHSON METHOD

Consider the equations

$$f(x, y) = 0, g(x, y) = 0 \qquad ...(1)$$

If an initial approximation (x_0, y_0) to a solution has been found by graphical method or otherwise, then a better approximation (x_1, y_1) can be obtained as follows :

Let $x_1 = x_0 + h, y_1 = y_0 + k$, so that $f(x_0 + h, y_0 + k) = 0, g(x_0 + h, y_0 + k) = 0$ ...(2)

Expanding each of the functions in (2) by Taylor's series to first degree terms, we get approximately

$$\left.\begin{aligned} f_0 + h\frac{\partial f}{\partial x_0} + k\frac{\partial f}{\partial y_0} = 0 \\ g_0 + h\frac{\partial g}{\partial x_0} + k\frac{\partial g}{\partial y_0} = 0 \end{aligned}\right\} \qquad ...(3)$$

where $f_0 = f(x_0, y_0), \dfrac{\partial f}{\partial x_0} = \left(\dfrac{\partial f}{\partial x}\right)_{x_0, y_0}$ etc.

Solving the equations (3) for h and k, we get a new approximation to the root as

$$x_1 = x_0 + h, y_1 = y_0 + k$$

This process is repeated till we get the values to the desired accuracy.

Example 28.25. *Solve the system of non-linear equations :*

$$x^2 + y = 11, y^2 + x = 7. \qquad \textit{(Pune, 2000)}$$

Solution. An initial approximation to the solution is obtained from a rough graph of the given equations, as $x_0 = 3.5$ and $y_0 = -1.8$.

We have $f = x^2 + y - 11$ and $g = y^2 + x - 7$ so that

$$\frac{\partial f}{\partial x} = 2x, \frac{\partial f}{\partial y} = 1 \quad \text{and} \quad \frac{\partial g}{\partial x} = 1, \frac{\partial g}{\partial y} = 2y.$$

Then Newton-Raphson's equations (3) above will be

$$7h + k = 0.55, h - 3.6k = 0.26$$

Solving these, we get $h = 0.0855, k = -0.0485$

∴ the better approximation to the root is

$$x_1 = x_0 + h = 3.5855, y_1 = y_0 + k = -1.8485$$

Repeating the above process, replacing (x_0, y_0) by (x_1, y_1), we obtain $x_2 = 3.5844, y_2 = -1.8482$.

PROBLEMS 28.5

1. Solve the equations $x^2 + y = 5$; $y^2 + x = 3$.
2. Solve the non-linear equations $x = 2(y + 1), y^2 = 3xy - 7$ correct to three decimals.
3. Use Newton-Raphson method to solve the equations $x = x^2 + y^2, y = x^2 - y^2$ correct to two decimals, starting with the approximation (0.8, 0.4).
4. Solve the non-linear equations $x^2 - y^2 = 4, x^2 + y^2 = 16$ numerically with $x_0 = y_0 = 2.828$ using N.R. method. Carry out two iterations. *(V.T.U., MCA, 2007)*
5. Solve the equations $2x^2 + 3xy + y^2 = 3$; $4x^2 + 2xy + y^2 = 30$. Correct to three decimal places, using Newton-Raphson method, given that $x_0 = -3$, and $y_0 = 2$.

28.9 DETERMINATION OF EIGEN VALUES BY ITERATION

In § 2.14, we came across equations of the type

$$\left.\begin{aligned}(a_{11} - \lambda)x_1 + a_{12}x_2 + a_{13}x_3 = 0\\ a_{21}x_1 + (a_{22} - \lambda)x_2 + a_{23}x_3 = 0\\ a_{31}x_1 + a_{32}x_2 + (a_{33} - \lambda)x_3 = 0\end{aligned}\right\} \quad ...(1)$$

which in matrix form, may be written as $[A - \lambda I]\, X = 0$ or $AX = \lambda X$ ...(2)

where $A = [a_{ij}]$ and X is the column matrix $[x_i]$.

Equation (1) will have a non-trivial solution if the coefficient matrix vanishes *e.g.*,

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} = 0$$

This gives a cubic in λ whose roots are *eigen values* of (2) and corresponding to each *eigen value*, we have a non-zero solution $X = [x_1, x_2, x_3]$ which is called an *eigen vector*. Such an equation can ordinarily be solved easily.

In some applications, it is required to compute the numerically largest *eigen value* and the corresponding *eigen vector*. In such cases, the following iterative method is more convenient which is also well-suited for computing machines.

If X_1, X_2, X_3 be the eigen vectors corresponding to the eigen values $\lambda_1, \lambda_2, \lambda_3$, then an arbitrary column vector can be written as $X = k_1X_1 + k_2X_2 + k_3X_3$

Then $\quad AX = k_1AX_1 + k_2AX_2 + k_3AX_3 = k_1\lambda_1X_1 + k_2\lambda_2X_2 + k_3\lambda_3X_3$

Similarly $\quad A^2X = k_1\lambda_1^2X_1 + k_2\lambda_2^2X_2 + k_3\lambda_3^2X_3$

and $\quad A^rX = k_1\lambda_1^r X_1 + k_2\lambda_2^rX_2 + k_3\lambda_3^rX_3$

If $|\lambda_1| > |\lambda_2| > |\lambda_3|$, then the contribution of the term $k_1\lambda_1^rX_1$ to the sum on the right increases with r and therefore, every time we multiply a column vector by A, it becomes nearer to the eigen vector X_1. Then we make the largest component of the resulting column vector unity to avoid the factor k_1.

Thus we start with a column vector X which is as near the solution as possible and evaluate AX which is written as $\lambda^{(1)} X^{(1)}$ after normalisation. This gives the first approximation $\lambda^{(1)}$ to the eigen value and $X^{(1)}$ to eigen vector. Similarly we evaluate $AX^{(1)} = \lambda^{(2)} X^{(2)}$ which gives the second approximation. We repeat this process till $[X^{(r)} - X^{(r-1)}]$ becomes negligible. Then $\lambda^{(r)}$ will be the largest eigen value of (1) and $X^{(r)}$, the corresponding eigen vector.

*This iterative procedure for finding the dominant eigen value of a matrix is known as Rayleigh's power method.**

*After the English mathematician and physicist John William Strut known as *Lord Rayleigh* (1842–1919) who made important contributions to the theory of waves, elasticity and hydrodynamics. He was professor at Cambridge and London.

Example 28.26. *Determine the largest eigen value and the corresponding eigen vector of the matrices using the power method :*

(i) $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ (V.T.U., 2007)

Solution. (i) Let the initial approximation to the eigen vector corresponding to the largest eigen value of A be $X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Then $$AX = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0.2 \end{bmatrix} = \lambda^{(1)}X^{(1)}$$

So the first approximation to the eigen value is $\lambda^{(1)} = 5$ and the corresponding eigen vector is $X^{(1)} = \begin{bmatrix} 1 \\ 0.2 \end{bmatrix}$.

Now $$AX^{(1)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 5.8 \\ 1.4 \end{bmatrix} = 5.8\begin{bmatrix} 1 \\ 0.241 \end{bmatrix} = \lambda^{(2)}X^{(2)}$$

Thus the second approximation to the eigen-value is $\lambda^{(2)} = 5.8$ and the corresponding eigen-vector is $X^{(2)} = \begin{bmatrix} 1 \\ 0.241 \end{bmatrix}$, repeating the above process, we get

Now $$AX^{(2)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.241 \end{bmatrix} = 5.966\begin{bmatrix} 1 \\ 0.248 \end{bmatrix} = \lambda^{(3)}X^{(3)}$$

$$AX^{(3)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.249 \end{bmatrix} = 5.994\begin{bmatrix} 1 \\ 0.250 \end{bmatrix} = \lambda^{(4)}X^{(4)}$$

$$AX^{(4)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.250 \end{bmatrix} = 5.999\begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = \lambda^{(5)}X^{(5)}$$

$$AX^{(5)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = \lambda^{(6)}X^{(6)}$$

Clearly $\lambda^{(5)} = \lambda^{(6)}$ and $X^{(5)} = X^{(6)}$ upto 3 decimal places. Hence the largest eigen-value is 6 and the corresponding eigen vector is $\begin{bmatrix} 1 \\ 0.25 \end{bmatrix}$.

(ii) Let the initial approximation to the required eigen vector be $X = [1, 0, 0]'$.

Then $$AX = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \lambda^{(1)}X^{(1)}.$$

So the first approximation to the eigen value is $\lambda^{(1)} = 2$ and the corresponding eigen vector

$$X^{(1)} = [1, -0.5, 0]'.$$

Hence $$AX^{(1)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -2 \\ 0.5 \end{bmatrix} = 2.5\begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix} = \lambda^{(2)}X^{(2)}.$$

Repeating the above process, we get

$$AX^{(2)} = 2.8\begin{bmatrix} 1 \\ -1 \\ 0.43 \end{bmatrix} = \lambda^{(3)}X^{(3)}; \quad AX^{(3)} = 3.43\begin{bmatrix} 0.87 \\ -1 \\ 0.54 \end{bmatrix} = \lambda^{(4)}X^{(4)}$$

$$AX^{(4)} = 3.41\begin{bmatrix} 0.80 \\ -1 \\ 0.61 \end{bmatrix} = \lambda^{(5)}X^{(5)}; \quad AX^{(5)} = 3.41\begin{bmatrix} 0.76 \\ -1 \\ 0.65 \end{bmatrix} = \lambda^{(6)}X^{(6)}; \quad AX^{(6)} = 3.41\begin{bmatrix} 0.74 \\ -1 \\ 0.67 \end{bmatrix} = \lambda^{(7)}X^{(7)}$$

Clearly $\lambda^{(6)} = \lambda^{(7)}$ and $X^{(6)} = X^{(7)}$ approximately.

Hence the largest eigen value is 3.41 and the corresponding eigen vector is $[0.74, -1, 0.67]'$.

PROBLEMS 28.6

1. Find by power method, the larger eigen-value of the matrices :

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (*Anna, 2005*) (b) $\begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}$.

2. Obtain the largest eigen-value and the corresponding eigen-vector for the equations
$(2-\lambda)x_1 - x_2 = 0\ ;\ -x_1 + (2-\lambda)x_2 - x_3 = 0\ ;\ -x_2 + (2-\lambda)x_3 = 0$
by Rayleigh Quotient method.

3. Find the dominant eigen value and the corresponding eigen vector of the following matrices using the power method :

(a) $\begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}$ (*V.T.U., 2011*) (b) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ (*V.T.U., 2011 S*)

4. Find the largest eigen-value and the corresponding eigen-vector of the matrices :

(a) $\begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$ (*Anna, 2005*) (b) $\begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$ (*V.T.U., 2008*)

(c) $\begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}$ with initial approximation $[1, 1, 0]^T$. (*Madras, 2006*)

28.10 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 28.7

Fill up the blanks or select the correct answer to each of the following problems :

1. Out of Regula-falsi method and Newton-Rephson method, the rate of convergence is faster for ....
2. If x_n is the nth iterate, then the Newton-Raphson formula is ....
3. In the Regula-falsi method of finding the real root of an equation, the curve AB is replaced by ....
4. Newton's iterative formula to find the value of $\sqrt{N}$ is ....
5. Newton-Raphson formula converges when ....
6. In solving simultaneous equations by Gauss-Jordan method, the coefficient matrix is reduced to .... matrix.
7. In the case of bisection method, the convergence is
(a) linear (b) quadratic (c) very slow.
8. The order of convergence in Newton-Raphson method is
(a) 2 (b) 3 (c) 0 (d) none.
9. The Newton-Raphson algorithm for finding the cube root of N is ....
10. The bisection method for finding the root of an equation $f(x) = 0$ is ....
11. In Regula-falsi method, the first approximation is given by ....
12. The order of convergence in Newton-Raphson method is
(a) 2 (b) 3 (c) 0 (d) none.
13. The iterative formula for finding the reciprocal of N is $x_{n+1} =$ ....
14. As soon as a new value of a variable is found by iteration, it is used immediately in the following equations, this method is called
(a) Gauss-Jordan method (b) Gauss-Seidal method
(c) Jacobi's method (d) Relaxation method.
15. Out of Regula-falsi method and Newton-Raphson method, the rate of convergence is faster for ....
16. The difference between direct and iterative methods of solving simultaneous linear equations is ....
17. To which form the coefficient matrix is transformed when $AX = B$ is solved by Gauss elimination method.
18. Jacobi's iteration method can be used to solve a system of non-linear equations. (True or False)
19. The convergence in the Gauss-Seidal method is thrice as fast as in Jacobi's method. (True or False)
20. By Gauss elimination method, solve $x + y = 2$ and $2x + 3y = 5$. (*Anna, 2007*)

CHAPTER 29

Finite Differences and Interpolation

1. Finite differences. 2. Differences of a polynomial. 3. Factorial notation 4. Relations between the operators. 5. To find one or more missing terms. 6. Newton's interpolation formulae. 7. Central difference interpolation formulae—Gauss's interpolation formulae ; Stirling's formula ; Bessel's formula ; Everett's formula. 8. Choice of an interpolation formula. 9. Interpolation with unequal intervals. 10. Lagrange's formula. 11. Divided differences. 12. Newton's divided difference formula. 13. Inverse interpolation. 14. Objective Type of Questions.

29.1 FINITE DIFFERENCES

Suppose we are given the following values of $y = f(x)$ for a set of values of x :

x :	x_0	x_1	$x_2 \dots x_n$.
y :	y_0	y_1	$y_2 \dots y_n$.

Then the process of finding the values of y corresponding to any value of $x = x_i$ between x_0 and x_n is called *interpolation*. Thus *interpolation is the technique of estimating the value of a function for any intermediate value of the independent variable* while the process of computing the value of the function outside the given range is called *extrapolation*. The study of the interpolation is based on the concept of differences of a function which we proceed to discuss. For a detailed study, the reader should refer to author's book *'Numerical Methods in Engineering and Science'.*

Suppose that the function $y = f(x)$ is tabulated for the equally spaced values $x = x_0, x_0 + h, x_0 + 2h, \dots, x_0 + nh$ giving $y = y_0, y_1, y_2, \dots, y_n$. To determine the values of $f(x)$ or $f'(x)$ for some intermediate values of x, the following three types of differences are found useful :

(1) Forward differences. The differences $y_1 - y_0, y_2 - y_1, \dots, y_n - y_{n-1}$ when denoted by $\Delta y_0, \Delta y_1, \dots, \Delta y_{n-1}$ respectively are called the *first forward differences* where Δ is the *forward difference operator.* Thus the first forward differences are $\Delta y_r = y_{r+1} - y_r$.

Similarly, the second forward differences are defined by

$$\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r$$

In general, $$\Delta^p y_r = \Delta^{p-1} y_{r+1} - \Delta^{p-1} y_r$$

defines the *pth forward differences.*

These differences are systematically set out as follows in what is called a *Forward Difference Table.*

In a difference table, x is called the *argument* and y the *function* or the *entry* y_0, the first entry is called the *leading term* and $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0$ etc. are called the *leading differences.*

Obs. Any higher order forward difference can be expressed in terms of the entries.

We have $\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0$

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0) = y_3 - 3y_2 + 3y_1 - y_0$$

$$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0 = (y_4 - 3y_3 + 3y_2 - y_1) - (y_3 - 3y_2 + 3y_1 - y_0) = y_4 - 4y_3 + 6y_2 - 4y_1 + y_0$$

The coefficients occurring on the right hand side being the binomial coefficient, we have in general,

$$\Delta^n y_0 = y_n - {}^nc_1 y_{n-1} + {}^nc_2 y_{n-2} - \dots + (-1)^n y_0$$

Forward Difference Table

Value of x	*Value of y*	*1st. diff.*	*2nd diff.*	*3rd diff.*	*4th diff.*	*5th diff.*
x_0	y_0					
		Δy_0				
$x_0 + h$	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
$x_0 + 2h$	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
$x_0 + 3h$	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
$x_0 + 4h$	y_4		$\Delta^2 y_3$			
		Δy_4				
$x_0 + 5h$	y_5					

(2) Backward differences. The differences $y_1 - y_0, y_2 - y_1, ..., y_n - y_{n-1}$ when denoted by $\nabla y_1, \nabla y_2, ..., \nabla y_n$ respectively, are called the *first backward differences* where ∇ is the *backward difference operator.* Similarly we define higher order backward differences. Thus we have

$$\nabla y_r = y_r - y_{r-1}, \nabla^2 y_r = \nabla y_r - \nabla y_{r-1},$$
$$\nabla^3 y_r = \nabla^2 y_r - \nabla^2 y_{r-1} \text{ etc.}$$

The differences are exhibited in the following :

Backward Difference Table

Value of x	*Value of y*	*1st. diff.*	*2nd diff.*	*3rd diff.*	*4th diff.*	*5th diff.*
x_0	y_0					
		∇y_1				
$x_0 + h$	y_1		$\nabla^2 y_2$			
		∇y_2		$\nabla^3 y_3$		
$x_0 + 2h$	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$	
		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$
$x_0 + 3h$	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$	
		∇y_4		$\nabla^3 y_5$		
$x_0 + 4h$	y_4		$\nabla^2 y_5$			
		∇y_5				
$x_0 + 5h$	y_5					

(3) Central differences. Sometimes it is convenient to employ another system of differences known as *central differences.* In this system, the *central difference operator* δ is defined by the relations :

$$y_1 - y_0 = \delta y_{1/2}, y_2 - y_1 = \delta y_{3/2}, ..., y_n - y_{n-1} = \delta y_{n-1/2}$$

Similarly, higher order central differences are defined as

$$\delta y_{3/2} - \delta y_{1/2} = \delta^2 y_1, \delta y_{5/2} - \delta y_{3/2} = \delta^2 y_2, ...,$$
$$\delta^2 y_2 - \delta^2 y_1 = \delta^3 y_{3/2} \text{ and so on.}$$

These differences are shown in the following :

Central Difference Table

Value of x	*Value of y*	*1st. diff.*	*2nd diff.*	*3rd diff.*	*4th diff.*	*5th diff.*
x_0	y_0					
		$\delta y_{1/2}$				
$x_0 + h$	y_1		$\delta^2 y_1$			
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$		
$x_0 + 2h$	y_2		$\delta^2 y_2$		$\delta^4 y_2$	
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$		$\delta^5 y_{5/2}$
$x_0 + 3h$	y_3		$\delta^2 y_3$		$\delta^4 y_3$	
		$\delta y_{7/2}$		$\delta^3 y_{7/2}$		
$x_0 + 4h$	y_4		$\delta^2 y_4$			
		$\delta y_{9/2}$				
$x_0 + 5h$	y_5					

We see from this table that the central differences on the same horizontal line have the same suffix. Also the differences of odd order are known only for half values of the suffix and those of even order for only integral values of the suffix.

It is often required to find the mean of adjacent values in the same column of differences. We denote this mean by μ. Thus

$$\mu\delta y_1 = \tfrac{1}{2}(\delta y_{1/2} + \delta y_{3/2}),\ \mu\delta^2 y_{3/2} = \tfrac{1}{2}(\delta^2 y_1 + \delta^2 y_2) \text{ etc.}$$

Obs. The reader should note that it is only the notation which changes and not the differences.

$$y_1 - y_0 = \Delta y_0 = \nabla y_1 = \delta y_{1/2}.$$

Of all the interpolation formulae, those involving central differences are most useful in practice as the coefficients in such formulae decrease much more rapidly.

Example 29.1. *Evaluate (i) $\Delta \tan^{-1} x$ (ii) $\Delta(e^x \log 2x)$ (iii) $\Delta(x^2/\cos 2x)$ (iv) $\Delta^2 \cos 2x$.* (P.T.U., 2001)

Solution. (*i*) $\Delta \tan^{-1} x = \tan^{-1}(x+h) - \tan^{-1} x$

$$= \tan^{-1}\left\{\frac{x+h-x}{1+(x+h)x}\right\} = \tan^{-1}\left\{\frac{h}{1+hx+x^2}\right\}$$

(*ii*)
$$\Delta(e^x \log 2x) = e^{x+h}\log 2(x+h) - e^x \log 2x$$
$$= e^{x+h}\log 2(x+h) - e^{x+h}\log 2x + e^{x+h}\log 2x - e^x \log 2x$$
$$= e^{x+h}\log\frac{x+h}{x} + (e^{x+h} - e^x)\log 2x$$
$$= e^x\left[e^h \log\left(1+\frac{h}{x}\right) + (e^h - 1)\log 2x\right]$$

(*iii*)
$$\Delta\left(\frac{x^2}{\cos 2x}\right) = \frac{(x+h)^2}{\cos 2(x+h)} - \frac{x^2}{\cos 2x} = \frac{(x+h)^2\cos 2x - x^2\cos 2(x+h)}{\cos 2(x+h)\cos 2x}$$
$$= \frac{[(x+h)^2 - x^2]\cos 2x + x^2[\cos 2x - \cos 2(x+h)]}{\cos 2(x+h)\cos 2x}$$
$$= \frac{(2hx+h^2)\cos 2x + 2x^2\sin(h)\sin(2x+h)}{\cos 2(x+h)\cos 2x}$$

(*iv*)
$$\Delta^2\cos 2x = \Delta\{\cos 2(x+h) - \cos 2x\}$$
$$= \Delta\cos 2(x+h) - \Delta\cos 2x$$
$$= [\cos 2(x+2h) - \cos 2(x+h)] - [\cos 2(x+h) - \cos 2x]$$
$$= -2\sin(2x+3h)\sin h + 2\sin(2x+h)\sin h$$
$$= -2\sin h\,[\sin(2x+3h) - \sin(2x+h)]$$
$$= -2\sin h\,[2\cos(2x+2h)\sin h] = -4\sin^2 h\cos(2x+2h).$$

Example 29.2. *Evaluate (i) $\Delta^2\left(\frac{5x+12}{x^2+5x+16}\right)$* (Mumbai, 2003) *(ii) $\Delta^2(ab^x)$ (iii) $\Delta^n(e^x)$ interval of differencing being unity.* (Rohtak, 2003)

Solution. (*i*)
$$\Delta^2\left(\frac{5x+12}{x^2+5x+6}\right) = \Delta^2\left\{\frac{5x+12}{(x+2)(x+3)}\right\} = \Delta^2\left\{\frac{2}{x+2} + \frac{3}{x+3}\right\}$$
$$= \Delta\left\{\Delta\left(\frac{2}{x+2}\right) + \Delta\left(\frac{3}{x+3}\right)\right\} = \Delta\left\{2\left(\frac{1}{x+3} - \frac{1}{x+2}\right) + 3\left(\frac{1}{x+4} - \frac{1}{x+3}\right)\right\}$$
$$= -2\Delta\left\{\frac{1}{(x+2)(x+3)}\right\} - 3\Delta\left\{\frac{1}{(x+3)(x+4)}\right\}$$
$$= -2\left\{\frac{1}{(x+3)(x+4)} - \frac{1}{(x+2)(x+3)}\right\} - 3\left\{\frac{1}{(x+4)(x+5)} - \frac{1}{(x+3)(x+4)}\right\}$$

$$= \frac{4}{(x+2)(x+3)(x+4)} + \frac{6}{(x+3)(x+4)(x+5)} = \frac{2(5x+16)}{(x+2)(x+3)(x+4)(x+5)}$$

(ii) $\Delta(ab^x) = a\,\Delta(b^x) = a(b^{x+1} - b^x) = ab^x(b-1)$

$\Delta^2(ab^x) = \Delta[\Delta ab^x] = a(b-1)\,\Delta(b^x)$

$= a(b-1)(b^{x+1} - b^x) = a(b-1)^2 - b^x.$

(iii) $\Delta e^x = e^{x+1} - e^x = (e-1)e^x$

$\Delta^2 e^x = \Delta(\Delta e^x) = \Delta[(e-1)\,e^x]$

$= (e-1)\,\Delta e^x = (e-1)(e-1)\,e^x = (e-1)^2\,e^x$

Similarly $\Delta^3 e^x = (e-1)^3\,e^x, \Delta^4 e^x = (e-1)^4\,e^x, \ldots$ and $\Delta^n e^x = (e-1)^n\,e^x.$

29.2 DIFFERENCES OF A POLYNOMIAL

The nth differences of a polynomial of the nth degree are constant and all higher order differences are zero.

Let the polynomial of the nth degree in x, be

$$f(x) = ax^n + bx^{n-1} + cx^{n-2} + \ldots + k(x+h) + l$$

$\therefore$

$$\Delta f(x) = f(x+h) - f(x)$$
$$= a[(x+h)^n - x^n] + b[(x+h)^{n-1} - x^{n-1}] + \ldots + kh$$
$$= anhx^{n-1} + b'x^{n-2} + c'x^{n-3} + \ldots + k'x + l' \qquad \ldots(1)$$

where $b', c', \ldots, l'$ are new constant coefficients.

Thus the first differences of a polynomial of the nth degree is a polynomial of degree $(n-1)$.

Similarly $\Delta^2 f(x) = \Delta[f(x+h) - f(x)] = \Delta f(x+h) - \Delta f(x)$

$= anh[(x+h)^{n-1} - x^{n-1}] + b'\,[(x+h)^{n-2} - x^{n-2}] + \ldots + k'h$

$= an(n-1)\,h^2x^{n-2} + b''x^{n-3} + c''\,x^{n-4} + \ldots + k''$, [by (1)]

$\therefore$ the second differences represent a polynomial of degree $(n-2)$.

Continuing this process, for the nth differences we get a polynomial of degree zero *i.e.*

$$\Delta^n f(x) = an(n-1)(n-2)\ldots 1\,.\,h^n = an\,!\,h^n \qquad \ldots(2)$$

which is a constant. Hence the $(n+1)$th and higher differences of a polynomial of nth degree will be zero.

Obs. The converse of this theorem is also true *i.e. if the nth differences of a function tabulated at equally spaced intervels are constant, the function is a polynomial of degree n.* This fact is important in numerical analysis as it enables us to approximate a function by a polynomial of nth degree, if its nth order differences become nearly constant.

Example 29.3. *Evaluate* $\Delta^{10}[(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)]$.

Solution. $\Delta^{10}[(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)] = \Delta^{10}[abcd\,x^{10} + (\ )\,x^9 + (\ )\,x^8 + \ldots + 1]$

$= abcd\,\Delta^{10}(x^{10})$ [$\because\ \Delta^{10}(x^n) = 0$ for $n < 10$]

$= abcd\,(10\,!).$ [by (2) above]

29.3 (1) FACTORIAL NOTATION

A product of the form $x(x-1)(x-2)\ldots(x-r+1)$ is denoted by $[x]^r$ and is called a **factorial.**

In particular $[x] = x$, $[x]^2 = x(x-1)$

$[x]^3 = x(x-1)(x-2)$, etc.

In general $[x]^n = x(x-1)(x-2)\ldots(x-n+1)$

In case, the interval of differencing is h, then

$$[x]^n = x(x-h)(x-2h)\ldots(x-\overline{n-1}h)$$

which is called a *Factorial polynomial or function.*

The factorial notation is of special utility in the theory of finite differences. It helps in finding the successive differences of a polynomial directly by simple rule of differentiation.

The result of differencing $[x]^r$ is analogous to that of differentiating x^r.

(2) To express a polynomial in the factorial notation

(*i*) *arrange the coefficients of the powers of x in descending order, replacing missing powers by zeros ;*

(*ii*) *using detached coefficients divide by x, x – 1, x – 2, etc. successively.*

Obs. Every polynomial of degree n can be expressed as a factorial polynomial of the same degree and *vice versa*.

Example 29.4. *Express* $y = 2x^3 - 3x^2 + 3x - 10$ *in a factorial notation and hence show that* $\Delta^3 y = 12$.

(*Bhopal, 2007 ; P.T.U., 2005*)

Solution. First method : Let $y = A[x]^3 + B[x]^2 + C[x] + D$.

Then

	x^3	x^2	x	
1	2	– 3	3	– 10 = D
	—	2	– 1	
2	2	– 1	2 = C	
	—	4		
3	2	3 = B		
	—			
	2 = A			

Hence $\quad y = 2[x]^3 + 3[x]^2 + 2[x] - 10$

$\therefore \quad \Delta y = 2 \times 3[x]^2 + 3 \times 2[x] + 2$

$\Delta^2 y = 6 \times 2[x] + 6$

$\Delta^3 y = 12$, which shows that the third differences of y are constant, as they should be.

Obs. *The coefficient of the highest power of x remains unchanged while transforming a polynomial to factorial notation.*

Second method (*Direct method*) :

Let
$$y = 2x^3 - 3x^2 + 3x - 10$$
$$= 2x(x-1)(x-2) + Bx(x-1) + Cx + D$$

Putting $x = 0, -10 = D$

Putting $x = 1, 2 - 3 + 3 - 10 = C + D$

$\therefore \quad C = -8 - D = -8 + 10 = 2$

Putting $x = 2, 16 - 12 + 6 - 10 = 2B + 2C + D$

$$\therefore \quad B = \frac{1}{2}(-2C - D) = \frac{1}{2}(-4 + 10) = 3.$$

Hence $\quad y = 2x(x-1)(x-2) + 3x(x-1) + 2x - 10 = 2[x]^3 + 3[x]^2 + 2[x] - 10$

$\therefore \quad \Delta y = 2 \times 3[x]^2 + 3 \times 2[x] + 2, \Delta^2 y = 6 \times 2[x] + 6, \Delta^3 y = 12.$

Example 29.5. *Find the missing values in the following table :*

x :	45	50	55	60	65
y :	3.0	—	2.0	—	– 2.4

(*Bhopal, 2007 ; V.T.U., 2001*)

Solution. The difference table is as follows :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
45	$y_0 = 3$			
		$y_1 - 3$		
50	y_1		$5 - 2y_1$	
		$2 - y_1$		$3y_1 + y_3 - 9$
55	$y_2 = 2$		$y_1 + y_3 - 4$	
		$y_3 - 2$		$3.6 - y_1 - 3y_3$
60	y_3	$-2.4 - y_3$	$-0.4 - 2y_3$	
65	$y_4 = -2.4$			

As only three entries y_0, y_2, y_4 are given, the function y can be represented by a second degree polynomial.

$$\therefore \qquad \Delta^3 y_0 = 0 \quad \text{and} \quad \Delta^3 y_1 = 0$$

i.e., $$3y_1 + y_3 = 9\,; \qquad y_1 + 3y_3 = 3.6$$

Solving these, we get $y_1 = 2.925$, $y_3 = 0.225$.

Otherwise : As only three entries $y_0 = 3, y_2 = 2, y_4 = -2.4$ are given, the function y can be represented by a second degree polynomial.

$$\therefore \qquad \Delta^3 y_0 = 0 \quad \text{and} \quad \Delta^3 y_1 = 0$$

i.e., $$(E-1)^3 y_0 = 0 \quad \text{and} \quad (E-1)^3 y_1 = 0$$

i.e., $$(E^3 - 3E^2 + 3E - 1)\, y_0 = 0 \quad \text{and} \quad (E^3 - 3E^2 + 3E - 1) y_1 = 0$$

i.e., $$y_3 - 3y_2 + 3y_1 - y_0 = 0$$
$$y_4 - 3y_3 + 3y_2 - y_1 = 0$$

i.e., $$y_3 + 3y_1 = 9\,; \; 3y_3 + y_1 = 3.6$$

Solving these, we get $y_1 = 2.925$, $y_3 = 0.225$.

Example 29.6. *Assuming that the following values of y belong to a polynomial of degree 4, compute the next three values :*

x :	0	1	2	3	4	5	6	7
y :	1	-1	1	-1	1	—	—	—

Solution. We construct the following difference table from the given data :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	$y_0 = 1$				
		-2			
1	$y_1 = -1$		4		
		2		-8	
2	$y_2 = 1$		-4		16
		-2		8	
3	$y_3 = -1$		4		16
		2		$\Delta^3 y_2$,	
4	$y_4 = 1$		$\Delta^2 y_3$		16
		Δy_4		$\Delta^2 y_3$,	
5	y_5		$\Delta^2 y_4$		16
		Δy_5		$\Delta^3 y_4$,	
6	y_6		$\Delta^2 y_5$		
		Δy_6			
7	y_7				

Since the values of y belong to a polynomial of degree 4, the fourth differences must be constant. But $\Delta^4 y = 16$.

$\therefore$ The other fourth order differences must also be 16. Thus

$$\Delta^4 y_1 = 16 = \Delta^3 y_2 - \Delta^3 y_1$$

i.e., $$\Delta^3 y_2 = \Delta^3 y_1 + \Delta^4 y_1 = 8 + 16 = 24$$
$$\Delta^2 y_3 = \Delta^2 y_2 + \Delta^3 y_2 = 4 + 24 = 28$$
$$\Delta y_4 = \Delta y_3 + \Delta^2 y_3 = 2 + 28 = 30$$

and $$y_5 = y_4 + \Delta y_4 = 1 + 30 = 31$$

Similarly starting with $\Delta^4 y_2 = 16$, we get

$$\Delta^3 y_3 = 40, \; \Delta^2 y_4 = 68, \; \Delta y_5 = 98, \; y_6 = 129.$$

Starting with $\Delta^4 y_3 = 16$, we obtain

$$\Delta^3 y_4 = 56, \; \Delta^2 y_5 = 124, \; \Delta y_6 = 222, \; y_7 = 351.$$

PROBLEMS 29.1

1. Construct the table of differences for the data below :

x :	0	1	2	3	4
$f(x)$:	1.0	1.5	2.2	3.1	4.6

Evaluate $\Delta^3 f(2)$.

2. If $u_0 = 3, u_1 = 12, u_2 = 18, u_3 = 2000, u_4 = 100$, calculate Δu_0.
3. Show that $\Delta^3 y_i = y_{i+3} - 3y_{i+2} + 3y_{i+1} - y_i$.
4. Form the table of backward differences of the function

$$f(x) = x^3 - 3x^2 - 5x - 7 \text{ for } x = -1, 0, 1, 2, 3, 4, 5.$$

5. Form a table of differences for the function

$$f(x) = x^3 + 5x - 7 \text{ for } x = -1, 0, 1, 2, 3, 4, 5$$

Continue the table to obtain $f(6)$.

6. Extend the following table to two more terms on either side by constructing the difference table :

x :	$-.2$	0.0	0.2	0.4	0.6	0.8	1.0
y :	2.6	3.0	3.4	4.28	7.08	14.2	29.0

7. Show that

(i) $\Delta\left[\dfrac{1}{f(x)}\right] = \dfrac{-\Delta f(x)}{f(x)\, f(x+1)}$; *(Raipur, 2005)* (ii) $\Delta \log f(x) = \log\left\{1 + \dfrac{\Delta f(x)}{f(x)}\right\}$.

8. Evaluate :

(i) $\Delta(x + \cos x)$ (ii) $\Delta \tan^{-1}\left(\dfrac{n-1}{n}\right)$ (iii) $\Delta\left\{\dfrac{1}{x(x+4)(x+6)}\right\}$ *(Madras, 2001)*

(iv) $\Delta^2\left(\dfrac{1}{x^2+5x+6}\right)$ *(P.T.U., 2001)*

9. Evaluate :

(i) $\Delta(e^{3x}\log 2x)$ (ii) $\Delta(2^x/x\,!)$ (iii) $\Delta^n(a^x)$ *(Burdwan, 2003)* (iv) $\Delta^n\left(\dfrac{1}{x}\right)$.

10. If $f(x) = e^{ax+b}$, show that its leading differences form a geometric progression. *(Mumbai, 2003)*
11. Prove that

(i) $y_3 = y_2 + \Delta y_1 + \Delta^2 y_0 + \Delta^3 y_0$ (ii) $\nabla^2 y_8 = y_8 - 2y_7 + y_6$; $\delta^2 y_5 = y_6 - 2y_5 + y_4$.

12. Evaluate :

(i) $\Delta^3 [(1-x)(1-2x)(1-3x)]$

(ii) $\Delta^{10}[(1-x)(1-2x^2)(1-3x^3)(1-4x^4)]$, if the interval of differencing is 2.

13. Express $x^3 - 2x^2 + x - 1$ into factorial polynomial. Hence show that $\Delta^4 f(x) = 0$ *(P.T.U., 2001)*
14. Express $u = x^4 - 12x^3 + 24x^2 - 30x + 9$ and its successive differences in factorial notation. Hence show that $\Delta^5 u = 0$.
15. Find the first and second differences of $x^4 - 6x^3 + 11x^2 - 5x + 8$ with $h = 1$. Show that the fourth difference is constant.
16. Obtain the function whose first difference is $2x^3 + 3x^2 - 5x + 4$.
17. Find the first term of the series whose second and subsequent terms are 8, 3, 0, -1, 0.
18. If $u(x)$ and $v(x)$ be two functions of x, prove that

(i) $\Delta[u(x)\, v(x)] = u(x)\,\Delta v(x) + v(x+1)\,\Delta u(x)$. (ii) $\Delta\left\{\dfrac{u(x)}{v(x)}\right\} = \dfrac{v(x)\,\Delta u(x) - u(x)\,\Delta v(x)}{v(x)\, v(x+1)}$.

29.4 (1) OTHER DIFFERENCE OPERATORS

We have already introduced the operators Δ, ∇ and δ. Besides these, there are the operators E and μ, which we define below :

(i) **Shift operator E** *is the operation of increasing the argument x by h* so that

$$Ef(x) = f(x+h),\ E^2 f(x) = f(x+2h),\ E^3 f(x) = f(x+3h) \text{ etc.}$$

The inverse operator E^{-1} is defined by $E^{-1} f(x) = f(x-h)$

If y_x is the function $f(x)$, then $Ey_x = y_{x+h}$, $E^{-1}y_x = y_{x-h}$, $E^n y_x = y_{x+nh}$,
where n may be any real number.

(*ii*) **Averaging operator** μ is defined by the equation $\mu y_x = \frac{1}{2}(y_{x+h/2} + y_{x-h/2})$

Obs. *In the difference calculus, Δ and E are regarded as the fundamental operators and ∇, δ, μ can be expressed in terms of these.*

(2) Relations between the operators. We shall now establish the following identities :

(*i*) $\mathbf{\Delta = E - 1}$ (*ii*) $\mathbf{\nabla = 1 - E^{-1}}$

(*iii*) $\mathbf{\delta = E^{1/2} - E^{-1/2}}$ (*iv*) $\mathbf{\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})}$

(*v*) $\mathbf{\Delta = E\nabla = \nabla E = \delta E^{1/2}}$ (*vi*) $\mathbf{E = e^{hD}}$.

Proofs. (*i*) $\Delta y_x = y_{x+h} - y_x = Ey_x - y_x = (E-1)\,y_x$.

This shows that the operators Δ and E are connected by the symbolic relation

$$\Delta = E - 1 \quad \text{or} \quad E = 1 + \Delta$$

(*ii*) $\nabla y_x = y_x - y_{x-h} = y_x - E^{-1}y_x = (1 - E^{-1})y_x$

$\therefore$ $\nabla = 1 - E^{-1}$ or $E = (1-\nabla)^{-1}$

(*iii*) $\delta y_x = y_{x+h/2} - y_{x-h/2} = E^{1/2}y_x - E^{-1/2}y_x = (E^{1/2} - E^{-1/2})y_x$

$\therefore$ $\delta = E^{1/2} - E^{-1/2}$.

(*iv*) $\mu y_x = \frac{1}{2}(y_{x+h/2} + y_{x-y/2}) = \frac{1}{2}(E^{1/2}y_x + E^{-1/2}y_x) = \frac{1}{2}(E^{1/2} + E^{-1/2})\,y_x$

$\therefore$ $\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$.

(*v*) $E\nabla y_x = E(y_x - y_{x-h}) = Ey_x - Ey_{x-h} = y_{x+h} - y_x = \Delta y_x \quad \therefore \quad E\nabla = \Delta$

Also $\nabla E y_x = \nabla y_{x+h} = y_{x+h} - y_x = \Delta y_x \quad \therefore \quad \nabla E = \Delta$

$\delta E^{1/2}y_x = \delta y_{x+h/2} = y_{x+h/2+h/2} - y_{x+h/2-h/2} = y_{x+h} - y_x = \Delta y_x$

$\therefore$ $\delta E^{1/2} = \Delta$

Hence $\Delta = E\nabla = \nabla E = \delta E^{1/2}$.

(*vi*) $Ef(x) = f(x+h)$

$$= f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \ldots \qquad \text{[By Taylor's series]}$$

$$= f(x) + hD\,f(x) + \frac{h^2}{2!}D^2 f(x) + \ldots = \left(1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \ldots\ldots\right) f(x) = e^{hD} f(x)$$

$\therefore$ $E = e^{hD}$

Cor. 1. $E = 1 + \Delta = e^{hD}$.

2. $D = \dfrac{1}{h}\log(1+\Delta) = \dfrac{1}{h}\left(\Delta - \dfrac{1}{2}\Delta^2 + \dfrac{1}{3}\Delta^3 - \ldots\right)$ (*Burdwan, 2003*)

Note. *A table showing the symbolic relations between the various operators is given below for ready reference. To prove such relations between the operators, always express each operator in terms of the fundamental operator E.*

(3) Relations between the various operators

In terms of	E	Δ	∇	δ	hD
E		$\Delta + 1$	$(1+\nabla)^{-1}$	$1 + \frac{1}{2}\delta^2 + \delta\sqrt{(1+\delta^2/4)}$	e^{hD}
Δ	$E - 1$	—	$(1-\nabla)^{-1} - 1$	$\frac{1}{2}\delta^2 + \delta\sqrt{(1+\delta^2/4)}$	$e^{hD} - 1$
∇	$1 - E^{-1}$	$1 - (1+)^{-1} - 1$	—	$-\frac{1}{2}\delta^2 + \delta\sqrt{(1+\delta^2/4)}$	$1 - e^{-hD}$
δ	$E^{1/2} - E^{-1}/2$	$\Delta(1+\Delta)^{-1/2}$	$\nabla(1-\nabla)^{-1/2}$	—	$2\sinh(hD/2)$
μ	$\frac{1}{2}(E^{1/2} + E^{-1/2})$	$(1+\Delta/2)(1+\Delta)^{-1/2}$	$(1+\nabla/2)(1+\nabla)^{-1/2}$	$\sqrt{(1+\delta^2/4)}$	$\cosh(hD/2)$
hD	$\log E$	$\log(1+\Delta)$	$\log(1-)^{-1}$	$2\sinh^{-1}(\delta/2)$	

Example 29.7. *Prove that*

$$e^x = \left(\frac{\Delta^2}{E}\right)e^x \cdot \frac{Ee^x}{\Delta^2 e^x}, \text{ the interval of differencing being } h. \quad \textit{(Bhopal, 2009)}$$

Solution. Since $\left(\frac{\Delta^2}{E}\right)e^x = \Delta^2 . E^{-1}e^x = \Delta^2 e^{x-h} = \Delta^2 e^x . e^{-h} = e^{-h}\Delta^2 e^x$

$\therefore$ $\qquad$ R.H.S. $= e^{-h}\Delta^2 e^x . \dfrac{Ee^x}{\Delta^2 e^x} = e^{-h} Ee^x = e^{-h} . e^{x+h} = e^x.$

Example 29.8. *Prove with the usual notations, that*

(*i*) $hD = \log(1+\Delta) = -\log(1-\nabla) = \sinh^{-1}(\mu\delta)$ *(Rohtak, 2005)*

(*ii*) $(E^{1/2} + E^{-1/2})(1+\Delta)^{1/2} = 2 + \Delta$ *(Bhopal, 2009 ; U.P.T.U., 2009)*

(*iii*) $\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{(1+\delta^2/4)}$ $\qquad$ (*iv*) $\Delta^3 y_2 = \nabla^3 y_5$.

Solution. (*i*) We know that $e^{hD} = E = 1 + \Delta \quad \therefore \quad hD = \log(1+\Delta)$

Also $\qquad hD = \log E = -\log(E^{-1}) = -\log(1-\nabla) \qquad [\because \; E^{-1} = 1 - \nabla]$

We have proved that $\qquad \mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$ and $\delta = E^{1/2} - E^{-1/2}$

$\therefore \qquad \mu\delta = \frac{1}{2}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2}) = \frac{1}{2}(E - E^{-1}) = \frac{1}{2}(e^{hD} - e^{-hD}) = \sinh(hD)$

i.e. $\qquad hD = \sinh^{-1}(\mu\delta).$

Hence $\qquad hD = \log(1+\Delta) = -\log(1-\nabla) = \sinh^{-1}(\mu\delta)$

(*ii*) $(E^{1/2} + E^{-1/2})(1+\Delta)^{1/2} = (E^{1/2} + E^{-1/2})E^{1/2} = E + 1 = 1 + \Delta + 1 = 2 + \Delta.$

(*iii*) $\frac{1}{2}\delta^2 + \delta\sqrt{(1+\delta^2/4)}$

$$= \tfrac{1}{2}(E^{1/2} - E^{-1/2})^2 + (E^{1/2} - E^{-1/2})\sqrt{[1 + (E^{1/2} - E^{-1/2})^2/4]}$$

$$= \tfrac{1}{2}(E + E^{-1} - 2) + (E^{1/2} - E^{-1/2})\sqrt{[(E + E^{-1} + 2)/4]}$$

$$= \tfrac{1}{2}(E + E^{-1} - 2) + \tfrac{1}{2}(E^{1/2} - E^{-1/2})(E^{1/2} + E^{-1/2})$$

$$= \tfrac{1}{2}[(E + E^{-1} - 2) + (E - E^{-1})] = \tfrac{1}{2}(2E - 2) = E - 1 = \Delta.$$

(*iv*) $\qquad \Delta^3 y_2 = (E-1)^3 y_2 \qquad [\because \; \Delta = E - 1]$

$\qquad = (E^3 - 3E^2 + 3E - 1)\,y_2 = y_5 - 3y_4 + 3y_3 - y_2 \qquad$ ...(1)

$\qquad \nabla^3 y_5 = (1 - E^{-1})^3 y_5 \qquad [\because \; \Delta = 1 - E^{-1}]$

$\qquad = (1 - 3E^{-1} + 3E^{-2} - E^{-3})\,y_5 = y_5 - 3y_4 + 3y_3 - y_2 \qquad$ ...(2)

From (1) and (2), $\Delta^3 y_2 = \nabla^3 y_5$.

29.5 TO FIND ONE OR MORE MISSING TERMS

When one or more values of $y = f(x)$ corresponding to the equidistant values of x are missing, we can find these using any of the following two methods :

First method : We assume the missing term or terms as a, b etc. and form the difference table. Assuming the last difference as zero, we solve these equations for a, b. These give the missing term/terms.

Second method : If n entries of y are given, $f(x)$ can be represented by $a(n-1)$th degree polynomial *i.e.*, $\Delta^n y = 0$. Since $\Delta = E - 1$, therefore $(E-1)^n y = 0$. Now expanding $(E-1)^n$ and substituting the given values, we obtain the missing term/terms.

Example 29.9. *Find the missing term in the table :* *(U.P.T.U., 2008)*

x :	2	3	4	5	6
y :	45.0	49.2	54.1	...	67.4

Solution. Let the missing term be a. Then the difference table is as follows :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
2	$45.0\ (= y_0)$				
		4.2			
3	$49.2\ (= y_1)$		0.7		
		49.9		$a - 59.7$	
4	$54.1\ (= y_2)$		$a - 59.0$		$240.2 - 4a$
		$a - 54.1$		$180.5 - 3a$	
5	$a (= y_3)$		$121.5 - a$		
		$67.4 - a$			
6	$67.4\ (= y_4)$				

We know that $\Delta^4 y = 0$ *i.e.*, $240.2 - 4a = 0$.

Hence $a = 60.05$.

Otherwise: As only four entries y_0, y_1, y_2, y_3 are given, therefore $y = f(x)$ can be represented by a third degree polynomial.

$\therefore$ $\Delta^3 y = \text{constant}$ or $\Delta^4 y = 0$ *i.e.*, $(E - 1)^4 = 0$

i.e., $(E^4 - 4E^3 + 6E^2 - 4E + 1) = 0$ or $y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$

Let the missing entry y_3 be a so that

$$67.4 - 4a + 6(54.1) - 4(49.2) + 45 = 0 \text{ or } - 4a = - 240.2$$

Hence $a = 60.05$.

Example 29.10. *Find the missing values in the following data :*

x :	45	50	55	60	65
y :	3.0	...	2.0	...	-2.4

(Bhopal, 2007)

Solution. Let the missing value be a, b. Then the difference table is as follows :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
45	$3 (= y_0)$			
		$a - 3$		
50	$a (= y_1)$		$5 - 2a$	
		$2 - a$		$3a + b - 9$
55	$2 (= y_2)$		$b + a - 4$	
		$b - 2$		$3.6 - a - 36$
60	$b (= y_3)$		$-0.4 - 2b$	
		$-2.4 - b$		
65	$-2.4\ (= y_4)$			

As only three entries y_0, y_2, y_4 are given, y can be represented by a second degree polynomial having third differences as zero.

$\therefore$ $\Delta^3 y_0 = 0$ and $\Delta^3 y_1 = 0$

i.e., $3a + b = 9, a + 3b = 3.6$

Solving these, we get $a = 2.925, b + 0.0225$.

Otherwise. As only three entries $y_0 = 3, y_2 = 2, y_4 = - 2.4$ are given, y can be represented by a second degree polynomial having third differences as zero.

$\therefore$ $\Delta^3 y_0 = 0$ and $\Delta^3 y_1 = 0$

i.e., $(E - 1)^3 y_0 = 0$ and $(E - 1)^3 y_1 = 0$

i.e., $(E^3 - 3E^3 + 3E - 1) y_0 = 0 ; (E^3 - 3E^2 + 3E - 1) . y_1 = 0$

or $y_3 - 3y_2 + 3y_1 - y_0 = 0 ; y_4 - 3y_3 + 3y_2 - y_1 = 0$

or $y_3 + 3y_1 = 9 ; 3y_3 + y_1 = 3.6$

Solving three, we get $y_1 = 2.925, y_2 = 0.225$.

Example 29.11. *If $y_{10} = 3$, $y_{11} = 6$, $y_{12} = 11$, $y_{13} = 18$, $y_{14} = 27$, find y_4.* (Mumbai, 2005)

Solution. Taking y_{14} as u_0, we are required to find y_4 *i.e.*, u_{-10}. Then the difference table is

x	u	Δu	$\Delta^2 u$	
x_{-4}	$y_{10} = u_{-4} = 3$			
		3		
x_{-3}	$y_{11} = u_{-3} = 6$		2	
		5		0
x_{-2}	$y_{12} = u_{-2} = 11$		2	
		7		0
x_{-1}	$y_{13} = u_{-1} = 18$		2	
		9		
x_0	$y_{14} = u_0 = 27$			

Then
$$y_4 = u_{-10} = (E^{-1})^{10}\, u_0 = (1 - \nabla)^{10}\, u_8$$
$$= \left(1 - 10\nabla + \frac{10.9}{2}\nabla^2 - \frac{10.9.8}{1.2.3}\nabla^3 + \ldots\right) u_0$$
$$= u_0 - 10\nabla u_0 + 45\nabla^2 u_0 - 120\nabla^3 u_0$$
$$= 27 - 10 \times 9 + 45 \times 2 - 120 \times 0 = 27.$$

Example 29.12. *If y_x is a polynomial for which fifth difference is constant and $y_1 + y_7 = -7845$, $y_2 + y_6 = 686$, $y_3 + y_5 = 1088$, find y_4.* (Mumbai, 2004)

Solution. Starting with y_1 instead of y_0, we note that $\Delta^6 y_1 = 0$ [$\because$ $\Delta^5 y_1$ is constant.

i.e., $$(E - 1)^6\, y_1 = (E^6 - 6E^5 + 15E^4 - 20E^3 + 15E^2 - 6E + 1)\, y_1 = 0$$

$\therefore$ $$y_7 - 6y_6 + 15y_5 - 20y_4 + 15y_3 - 6y_2 + y_1 = 0$$

or $$(y_7 + y_1) - 6(y_6 + y_2) + 15(y_5 + y_3) - 20y_4 = 0$$

i.e. $$y_4 = \frac{1}{20}\,[(y_1 + y_7) - 6(y_2 + y_6) + 15(y_3 + y_5)]$$
$$= \frac{1}{20}\,[-784 - 6(686) + 15(1088)] = 571.$$

Example 29.13. *Prove the following identities :*

(i) $u_1 x + u_2 x^2 + u_3 x^3 + \ldots = \dfrac{x}{1-x} u_1 + \left(\dfrac{x}{1-x}\right)^2 \Delta u_1 + \left(\dfrac{x}{1-x}\right)^3 \Delta^2 u_1 + \ldots$

(ii) $u_0 + \dfrac{u_1 x}{1!} + \dfrac{u_2 x^2}{2!} + \dfrac{u_3 x^3}{3!} + \ldots = e^x \left(u_0 + x\Delta u_0 + \dfrac{x^2}{2!}\Delta^2 u_0 + \dfrac{x^3}{3!}\Delta^3 u_0 + \ldots\right).$

Solution. (*i*) L.H.S. $= xu_1 + x^2 E u_1 + x^3 E^2 u_1 + \ldots = x(1 + xE + x^2E^2 + \ldots)\, u_1$ [$\because$ $u_{x+h} = E^h u_x$]

$$= x \cdot \frac{1}{1 - xE}\, u_1, \text{ taking sum of infinite G.P.}$$

$$= x\left[\frac{1}{1 - x(1 + \Delta)}\right] u_1 \qquad [\because\ E = 1 + \Delta]$$

$$= x\left(\frac{1}{1 - x - x\Delta}\right) u_1 = \frac{x}{1-x}\left(1 - \frac{x\Delta}{1-x}\right)^{-1} u_1 = \frac{x}{1-x}\left(1 + \frac{x\Delta}{1-x} + \frac{x^2\Delta^2}{(1-x^2)} + \ldots\ldots\right) u_1$$

$$= \frac{x}{1-x} u_1 + \frac{x^2}{(1-x)^2}\Delta u_1 + \frac{x^2}{(1-x)^3}\Delta^2 u_1 + \ldots = \text{R.H.S.}$$

(ii) $$\text{L.H.S.} = u_0 + \frac{x}{1!}Eu_0 + \frac{x^2}{2!}E^2u_0 + \frac{x^3}{3!}E^3u_0 + \ldots$$

$$= \left(1 + \frac{xE}{1!} + \frac{x^2E^2}{2!} + \frac{x^3E^3}{3!} + \ldots\right)u_0 = e^{xE}\,u_0 = e^{x(1+\Delta)}\,u_0$$

$$= e^x \,.\, e^{x\Delta}\,u_0 = e^x\left(1 + \frac{x\Delta}{1!} + \frac{x^2\Delta^2}{2!} + \frac{x^3\Delta^3}{3!} + \ldots\right)u_0$$

$$= e^x\left(u_0 + \frac{x}{1!}\Delta u_0 + \frac{x^2}{2!}\Delta^2 u_0 + \frac{x^3}{3!}\Delta^3 u_0 + \ldots\right) = \text{R.H.S.}$$

PROBLEMS 29.2

1. Explain the difference between $\left(\frac{\Delta^2}{E}\right)u_x$ and $\frac{\Delta^2 u_x}{Eu_x}$. *(Madras, 2003)*

2. Evaluate taking h as the interval of differencing :

   (i) $\frac{\Delta^2}{E}\sin x$ (ii) $\left(\frac{\Delta^2}{E}\right)x^4, (h = 1)$ *(W.B.T.U., 2005)*

   (iii) $\left(\frac{\Delta^2}{E}\right)\sin(x+h) + \frac{\Delta^2\sin(x+h)}{E\sin(x+h)}$ (iv) $(\Delta + \nabla)^2(x^2 + x), (h = 1)$.

3. With the usual notations, show that

   (i) $\nabla = 1 - e^{-hD}$ (ii) $D = \frac{2}{h}\sinh^{-1}\left(\frac{\delta}{2}\right)$

   (iii) $(1 + \Delta)(1 - \nabla) = 1$. (iv) $\Delta - \nabla = \nabla\Delta = \delta^2$. *(Mumbai, 2005)*

4. Prove that

   (i) $\delta = \Delta(1+\Delta)^{-1/2} = \nabla(1-\nabla)^{-1/2}$ (ii) $\mu^2 = 1 + \frac{\delta^2}{4}$ *(U.P.T.U, 2009)*

   (iii) $\delta(E^{1/2} + E^{-1/2}) = \Delta E^{-1} + \Delta$ (iv) $\nabla = \Delta E^{-1} = E^{-1}\Delta = 1 - E^{-1}$

5. Show that (i) $\mu\delta = \frac{1}{2}(\Delta + \nabla)$ (ii) $1 + \delta^2/2 = \sqrt{(1 + \delta^2\mu^2)}$ *(U.P.T.U., MCA, 2008)*

   (iii) $\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$ *(U.P.T.U., 2009)* (iv) $\nabla^2 = h^2D^2 - h^3D^3 + \frac{7}{12}h^4D^4 - \ldots\ldots$

6. Prove that

   (i) $\nabla^r f_k = \Delta^r f_{k-r}$ (ii) $\Delta f_k^{\,2} = (f_k + f_{k+1})\Delta f_k$ *(J.N.T.U., MCA, 2006)*

   (iii) $\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{\left(1 + \frac{\delta^2}{4}\right)}$ (iv) $E^{1/2} = (1 + \delta^2/4)^{1/2} + \delta/2$.

7. Prove that $\nabla y_{n+1} = h\left(1 + \frac{1}{2}\nabla + \frac{5}{12}\nabla^2 + \ldots\right)y'_n$.

8. The following table gives the values of y which is a polynomial of degree five. It is known that $f(3)$ is in error. Correct the error.

x :	0	1	2	3	4	5	6
y :	1	2	33	254	1025	3126	7777

(Mumbai, 2004)

9. Estimate the missing term in the following table :

x :	0	1	2	3	4
$f(x)$:	1	3	9	—	81

(S.V.T.U., 2007)

10. Find the missing terms of the following data :

x :	1	1.5	2	2.5	3	3.5	4
$f(x)$:	6	?	10	20	?	1.5	5

(U.P.T.U., 2010)

11. Find the missing values in the following table :

x :	0	1	2	3	4	5	6
y :	5	11	22	40	...	140	...

(V.T.U., 2006)

12. If $u_{13} = 1$, $u_{14} = -3$, $u_{15} = -1$, $u_{16} = 13$ find u_8. (Mumbai, 2004)

13. Evaluate y_4 from the following data (stating the assumptions you make) :
$$y_0 + y_8 = 1.9243,\ y_1 + y_7 = 1.9590,\ y_2 + y_6 = 1.9823,\ y_3 + y_5 = 1.9956.$$
(Mumbai, 2003)

14. Using the method of separation of symbols, prove that
(i) $u_0 + u_1 + u_2 + \ldots\ldots + u_n = {}^{n+1}C_1 u_0 + {}^{n+1}C_3 \Delta^2 u_0 + \ldots\ldots + {}^{n+1}C_{n+1} \Delta^n u_0$.
(ii) $y_x = y_n - {}^{n-x}C_1 \Delta y_{n-1} + {}^{n-x}C_2 \Delta^2 y_{n-2} - \ldots\ldots + (-1)^{n-x} \Delta^{n-x} y_{n-(n-x)}$.

15. Using the method of finite differences, sum the following series :
(i) 2.5 + 5.8 + 8.11 + 11.14 + ... to n terms.
(ii) 1.2.3 + 2.3.4 + 3.4.5 + ... to n terms.

16. Prove that $u_0 + u_1 x + u_2 x^2 + \ldots\ldots \infty = \dfrac{u_0}{1-x} + \dfrac{x\Delta u_0}{(1-x)^2} + \dfrac{x^2\Delta^2 u_0}{(1-x)^3} + \ldots\ldots \infty$

Hence sum the series $1.2 + 2.3x + 3.4x^2 + \ldots\ldots \infty$.

29.6 NEWTON'S INTERPOLATION FORMULAE*

We now derive two important interpolation formulae by means of the forward and backward differences of a function. These formulae are often employed in engineering and scientific problems.

(1) Newton's forward interpolation formula. Let the function $y = f(x)$ take the values $y_0, y_1, y_2, \ldots$ corresponding to the values $x_0, x_0 + h, x_0 + 2h, \ldots$ of x. Suppose it is required to evaluate $f(x)$ for $x = x_0 + ph$, where p is any real number.

For any real number p, we have defined E such that

$$E^p f(x) = f(x + ph)$$

$$\therefore \quad y_p = f(x_0 + ph) = E^p f(x_0) = (1 + \Delta)^{-p} y_0 \qquad [\because\ E = 1 + \Delta]$$

$$= \left\{1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \ldots\right\} y_0 \qquad \text{[Using Binomial theorem]}$$

i.e.,
$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots \qquad \ldots(1)$$

It is called **Newton's forward interpolation formula** as (1) contains y_0 and the forward differences of y_0.

Obs. *This formula is used for interpolating the values of y near the beginning of a set of tabulated values and extrapolating values of y a little backward (i.e. to the left) of* y_0.

(2) Newton's backward interpolation formula. Let the function $y = f(x)$ take the values $y_0, y_1, y_2, \ldots$ corresponding to the values $x_0, x_0 + h, x_0 + 2h, \ldots$ of x. Suppose it is required to evaluate $f(x)$ for $x = x_0 + ph$, where p is any real number. Then we have

$$y_p = f(x_n + ph) = E^p f(x_n) = (1 - \nabla)^{-p} y_n \qquad [\because\ E^{-1} = 1 - \nabla]$$

$$= \left[1 + p\nabla + \frac{p(p+1)}{2!}\nabla^2 + \frac{p(p+1)(p+2)}{3!}\nabla^2 + \ldots\right] y_n \qquad \text{[Using Binomial theorem]}$$

i.e.,
$$y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \ldots \qquad \ldots(2)$$

It is called *Newton's backward interpolation formula as* (2) contains y_n and backward differences of y_n.

Obs. *This formula is used for interpolating the values of y near the end of a set of tabulated values and also for extrapolating values of y a little ahead* (to the right) *of* y_n.

Example 29.14. *The table gives the distances in nautical miles of the visible horizon for the given heights in feet above the earth's surface :*

x = height :	100	150	200	250	300	350	400
y = distance :	10.63	13.03	15.04	16.81	18.42	19.90	21.27

Find the values of y when (i) x = 218 ft (Madras, 2003 S) *(ii) 410 ft.* (V.T.U., 2002)

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*See foot note p.466.

Solution. The difference table is as under :

x	y	Δ	Δ^2	Δ^3	Δ^4
100	10.63				
		2.40			
150	13.03		-0.39		
		2.01		0.15	
200	**15.04**		-0.24		-0.07
		1.77		0.08	
250	16.81		**-0.16**		-0.05
		1.61		**0.03**	
300	18.42		-0.13		**-0.01**
		1.48		**0.02**	
350	19.90		**-0.11**		
		1.37			
400	**21.27**				

(*i*) If we take $x_0 = 200$, then $y_0 = 15.04$, $\Delta y_0 = 1.77$, $\Delta^2 y_0 = -0.16$, $\Delta^3 y_0 = 0.03$ etc.

Since $x = 218$ and $h = 50$, $\therefore$ $p = \dfrac{x - x_0}{h} = \dfrac{18}{50} = 0.36$

$\therefore$ Using Newton's forward interpolation formula, we get

$$y_{218} = y_0 + p\Delta y_0 + \frac{p(p-1)}{1\,.\,2}\,\Delta^2 y_0 + \frac{p(p-1)(p-2)}{1\,.\,2\,.\,3}\,\Delta^3 y_0 + \ldots$$

$$f(218) = 15.04 + 0.36(1.77) + \frac{0.36(-0.64)}{2}(-0.16) + \frac{0.36(-0.64)(-1.64)}{6}(0.03) + \ldots$$

$$= 15.04 + 0.637 + 0.018 + 0.001 + \ldots = 15.696 \quad \textit{i.e.}, \text{ 15.7 nautical miles}$$

(*ii*) Since $x = 410$ is near the end of the table, we use Newton's backward interpolation formula.

$\therefore$ taking $x_n = 400$, $p = \dfrac{x - x_n}{h} = \dfrac{10}{50} = 0.2$

Using the line of backward differences

$$y_n = 21.27,\ \nabla y_n = 1.37,\ \nabla^2 y_n = -0.11,\ \nabla^3 y_n = 0.02 \text{ etc.}$$

$\therefore$ newton's backward formula gives

$$y_{410} = y_{400} + p\nabla y_{400} + \frac{p(p+1)}{2}\nabla^2 y_{400} + \frac{p(p+1)(p+2)}{1\,.\,2\,.\,3\,.}\,\nabla^3 y_{400} + \ldots$$

$$= 21.27 + 0.2(1.37) + \frac{0.2(1.2)}{2}(-0.11) + \ldots = 21.53 \text{ nautical miles.}$$

Example 29.15. *From the following table, estimate the number of students who obtained marks between 40 and 45 :*

Marks :	*30—40*	*40—50*	*50—60*	*60—70*	*70—80*
No. of Students :	*31*	*42*	*51*	*35*	*31*

(*V.T.U., 2011 S ; S.V.T.U., 2007 ; Madras, 2006*)

Solution. First we prepare the cumulative frequency table, as follows :

Marks less than (x) :	40	50	60	70	80
No. of Students (y_x) :	31	73	124	159	190

Now the difference table is

x	y	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$	$\Delta^4 y_x$
40	**31**				
		42			
50	73		**9**		
		51		**-25**	
60	124		-16		37
		35		12	
70	159		-4		
		31			
80	190				

We shall find y_{45} *i.e.* number of students with marks less than 45.

Taking $x_0 = 40, x = 45$, we have $p = \dfrac{x - x_0}{h} = \dfrac{5}{10} = 0.5$ [$\because$ $h = 10$]

$\therefore$ using Newton's forward interpolation formula, we get

$$y_{45} = y_{40} + p\Delta y_{40} + \frac{p(p-1)}{2}\Delta^2 y_{40} + \frac{p(p-1)(p-2)}{1.2.3}\Delta^3 y_{40} + \ldots$$

$$= 31 + 0.5 \times 42 + \frac{0.5(-0.5)}{2} \times 9 + \frac{0.5(0.5)(-1.5)}{6} \times (-25) + \frac{0.5(-0.5)(-1.5)(-2.5)}{24} \times 37$$

$= 47.87$, on simplification.

$\therefore$ the number of students with marks less than 45 is 47.87 *i.e.*, 48.

But the number of students with marks less than 40 is 31.

Hence the number of students getting marks between 40 and 45 = 48 – 31 = 17.

Example 29.16. *Find the cubic polynomial which takes the following values :*

x :	*0*	*1*	*2*	*3*
f (x) :	*1*	*2*	*1*	*10*

Hence or otherwise evaluate f (4). (*Bhopal, 2009 ; Rohtak, 2005 ; W.B.T.U., 2005*)

Solution. The difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	**1**			
		1		
1	2		**– 2**	
		– 1		**12**
2	1		10	
		9		
3	10			

We take $x_0 = 0$ and $p = \dfrac{x - 0}{h} = x$ [$\because$ $h = 1$]

$\therefore$ using Newton's forward interpolation formula, we get

$$f(x) = f(0) + \frac{x}{1}\Delta f(0) + \frac{x(x-1)}{1.2}\Delta^2 f(0) + \frac{x(x-1)(x-2)}{1.2.3}\Delta^3 f(0)$$

$$= 1 + x(1) + \frac{x(x-1)}{2}(-2) + \frac{x(x-1)(x-2)}{6}(12)$$

$= 2x^3 - 7x^2 + 6x + 1$, which is the required polynomial.

To compute $f(4)$, we take $x_n = 3, x = 4$ so that $p = \dfrac{x - x_n}{h} = 1$ [$\because$ $h = 1$]

Using Newton's backward interpolation formula, we get

$$f(4) = f(3) + p\nabla f(3) + \frac{p(p+1)}{1.2}\nabla^2 f(3) + \frac{p(p+1)(p+2)}{1.2.3}\nabla^3 f(3)$$

$= 10 + 9 + 10 + 12 + 41$.

which is the same value as that obtained by substituting $x = 4$ in the cubic polynomial above.

Obs. The above example shows that *if a tabulated function is a polynomial, then interpolation and extrapolation give the same values.*

Example 29.17. *In the table below, the values of y are consecutive terms of a series of which 23.6 is the 6th term. Find the first and tenth terms of the series :*

x :	*3*	*4*	*5*	*6*	*7*	*8*	*9*
y :	*4.8*	*8.4*	*14.5*	*23.6*	*36.2*	*52.8*	*73.9*

(*Anna, 2007*)

Solution. The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
3	4.8				
		3.6			
4	8.4		2.5		
		6.1		0.5	
5	14.5		3.0		0
		9.1		0.5	
6	23.6		3.5		0
		12.6		0.5	
7	36.2		4.0		0
		16.6		0.5	
8	52.8		4.5		
		21.1			
9	73.9				

To find the first term, use Newton's forward interpolation formula with $x_0 = 3, x = 1, h = 1$ and $p = -2$. We have

$$y(1) = 4.8 + \frac{(-2)}{1} \times 3.6 + \frac{(-2)(-3)}{1.2} \times 2.5 + \frac{(-2)(-3)(-4)}{1.2.3} \times 0.5 = 3.1$$

To obtain the tenth term, use Newton's backward interpolation formula with $x_n = 9, x = 10, h = 1$ and $p = 1$. This gives

$$y(10) = 73.9 + \frac{1}{1} \times 21.1 + \frac{1(2)}{1.2} \times 4.5 + \frac{1(2)(3)}{1.2.3} \times 0.5 = 100.$$

PROBLEMS 29.3

1. Using Newton's forward formula, find the value of $f(1.6)$, if

x :	1	1.4	1.8	2.2
$f(x)$:	3.49	4.82	5.96	6.5

(*J.N.T.U., 2006*)

2. State Newton's interpolation formula and use it to calculate the value of exp (1.85), given the following table :

x :	1.7	1.8	1.9	2.0	2.1	2.2	2.3
$f(x)$:	5.474	6.050	6.686	7.389	8.166	9.025	9.974

(*Kottayam, 2005*)

3. If $f(1.15) = 1.0723$, $f(1.20) = 1.0954$, $f(1.25) = 1.1180$ and $f(1.30) = 1.1401$, find $f(1.28)$.
4. Given sin 45° = 0.7071, sin 50° = 0.7660, sin 55° = 0.8192, sin 60° = 0.8660, find sin 52°, using Newton's forward formula.
5. From the following table of half-yearly premium for policies maturing at different ages, estimate the premium for policies maturing at age of 46 :

Age :	45	50	55	60	65
Premium (in rupees) :	114.84	96.16	83.32	74.48	68.48

(*U.P.T.U., 2010*)

6. The area A of a circle of diameter d is given for the following values :

d :	80	85	90	95	100
A :	5026	5674	6362	7088	7854

(*V.T.U., 2010*)

Calculate the area of a circle of diameter 105.

7. Estimate the value of $f(22)$ and $f(42)$ from the following available data :

x :	20	25	30	35	40	45
$f(x)$:	354	332	291	260	231	204

(*J.N.T.U., 2007*)

8. From the following table :

$x°$:	10	20	30	40	50	60	70	80
cos x :	0.9848	0.9397	0.8660	0.7660	0.6428	0.5000	0.3420	0.1737

Calculate cos 25° and cos 73° using Gregory Newton formulae. (*U.P.T.U., 2006*)

9. Find the number of men getting wages below Rs. 15 from the following data :

Wages in Rs. :	0—10	10—20	20—30	30—40
Frequency :	9	30	35	42

(*Nagarjuna, 2001*)

10. Find the polynomial interpolating the data :

x :	0	1	2
$f(x)$:	0	5	2

(*U.P.T.U., 2008*)

11. Construct Newton's forward interpolation polynomial for the following data :

x :	4	6	8	10
y :	1	3	8	16

(*Madras, 2006*)

Hence evaluate y for $x = 5$.

12. Construct the difference table for the following data :

x :	0.1	0.3	0.5	0.7	0.9	1.1	1.3
$f(x)$:	0.003	0.067	0.148	0.248	0.370	0.518	0.697

Evaluate $f(0.6)$ (*J.N.T.U., 2007*)

13. Estimate from following table $f(3.8)$ to three significant figures using Gregory Newton backward interpolation formula:

x :	0	1	2	3	4
$f(x)$:	1	1.5	2.2	3.1	4.6

(*U.P.T.U., 2009*)

14. The following table gives the population of a town during the last six censuses. Estimate the increase in the population during the period from 1976 to 1978 :

Year :	1941	1951	1961	1971	1981	1991
Population (in thousands) :	12	15	20	27	39	52

(*U.P.T.U., 2009*)

15. In the following table, the values of y are consecutive terms of a series of which 12.5 is the 5th term. Find the first and tenth terms of the series.

x :	3	4	5	6	7	8	9
y :	2.7	6.4	12.5	21.6	34.3	51.2	72.9

(*P.T.U., 2001*)

16. Given $u_1 = 40$, $u_3 = 45$, $u_5 = 54$, find u_2 and u_4. (*Nagarjuna, 2003 S*)
17. If $u_{-1} = 10$, $u_1 = 8$, $u_2 = 10$, $u_4 = 50$, find u_0 and u_3.
18. Given $y_0 = 3, y_1 = 12, y_2 = 81, y_3 = 200, y_4 = 100, y_5 = 8$, without forming the difference table, find $\Delta^5 y_0$.

29.7 CENTRAL DIFFERENCE INTERPOLATION FORMULAE

In the preceding section, we derived Newton's forward and backward interpolation formulae which are applicable for interpolation near the beginning and end of tabulated values. Now we shall develop central difference formulae which are best suited for interpolation near the middle of the table.

If x takes the values $x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h$ and the corresponding values of $y = f(x)$ are $y_{-2}, y_{-1}, y_0, y_1, y_2$, then we can write the difference table in the two notations as follows :

x	y	*1st diff.*	*2nd diff.*	*3rd diff.*	*4th diff.*
$x_0 - 2h$	y_{-2}				
		$\Delta y_{-2} (= \delta y_{-3/2})$			
$x_0 - h$	y_{-1}		$\Delta^2 y_{-2} (= \delta^2 y_{-1})$		
		$\Delta y_{-1} (= \delta y_{-1/2})$		$\Delta^3 y_{-2} (= \delta^3 y_{-1/2})$	
x_0	y_0		$\Delta^2 y_{-1} (\delta^2 y_0)$		$\Delta^4 y_{-2} (= \delta^4 y_0)$
		$\Delta y_0 (= \delta y_{1/2})$		$\Delta^3 y_{-1} (= \delta^3 y_{1/2})$	
$x_0 + h$	y_1		$\Delta^2 y_0 (= \delta^2 y_1)$		
		$\Delta y_1 (= \delta y_{3/2})$			
$x_0 + 2h$	y_2				

(1) Gauss's forward interpolation formula. The Newton's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{1.2} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{1.2.3} \Delta^3 y_0 + \ldots \qquad \ldots(1)$$

We have $\Delta^2 y_0 - \Delta^2 y_{-1} = \Delta^3 y_{-1}$

i.e., $\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$ ...(2)

Similarly $\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$ ...(3)

$\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1}$ etc. ...(4)

Also $\Delta^3 y_{-1} - \Delta^3 y_{-2} = \Delta^4 y_{-2}$

i.e., $\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$

Similarly $\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$ etc. ...(5)

Substituting for $\Delta^2 y_0, \Delta^3 y_0, \Delta^4 y_0$ ... from (2), (3), (4) ... in (1), we get

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{1.2}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{p(p-1)(p-2)}{1.2.3}(\Delta^3 y_{-1} + \Delta^4 y_{-1}) + \frac{p(p-1)(p-2)(p-3)}{1.2.3.4}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \ldots$$

Hence $$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \ldots \text{ [Using (5)]}$$

which is called *Gauss's forward interpolation formula.*

Cor. *In the central differences notation, this formula will be*

$$y_p = y_0 + p\delta y_{1/2} + \frac{p(p-1)}{2!}\delta^2 y_0 + \frac{(p+1)p(p-1)}{3!}\delta^3 y_{1/2} + \frac{(p+1)p(p-1)(p-2)}{4!}\delta^4 y_0 + \ldots$$

Obs. 1. *It employs odd differences just below the central line and even difference on the central line as shown below:*

$y_0 \ldots\ldots \Delta^2 y_{-1} \ldots\ldots \Delta^4 y_{-2} \ldots\ldots \Delta^6 y_{-3} \ldots\ldots$ Central line

$\Delta y_0 \qquad \Delta^3 y_{-1} \qquad \Delta^5 y_{-2} \qquad \Delta^7 y_{-3}$

Obs. 2. This formula is used to interpolate the values of y for p $(0 < p < 1)$ measured forwardly from the origin.

(2) Gauss's backward interpolation formula. The Newton's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{1.2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{1.2.3}\Delta^3 y_0 + \ldots \quad \text{...(1)}$$

We have $\Delta y_0 - \Delta y_{-1} = \Delta^2 y_{-1}$

i.e., $\Delta y_0 = \Delta y_{-1} + \Delta^2 y_{-1}$ ...(2)

Similarly $\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$ ...(3)

$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$ etc. ...(4)

Also $\Delta^3 y_{-1} - \Delta^3 y_{-2} = \Delta^4 y_{-2}$

i.e., $\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$ ...(5)

Similarly $\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$ etc. ...(6)

Substituting for $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0$, ... from (2), (3), (4) in (1), we get

$$y_p = y_0 + p(\Delta y_{-1} + \Delta^2 y_{-1}) + \frac{p(p-1)}{1.2}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{p(p-1)(p-2)}{1.2.3}(\Delta^3 y_{-1} + \Delta^4 y_{-1}) + \frac{p(p-1)(p-2)(p-3)}{1.2.3.4}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \ldots$$

$$= y_0 + p\Delta y_{-1} + \frac{(p+1)p}{1.2}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{1.2.3}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{1.2.3.4}\Delta^4 y_{-1} + \frac{p(p-1)(p-2)(p-3)}{1.2.3.4}\Delta^5 y_{-1} + \ldots$$

$$= y_0 + p\Delta y_{-1} + \frac{(p+1)p}{1.2}\Delta^2 y_{-1} + \frac{(p+1)\,p\,(p-1)}{1.2.3}(\Delta^3 y_{-2} + \Delta^4 y_{-2})$$
$$+ \frac{(p+1)\,p\,(p-1)(p-2)}{1.2.3.4}(\Delta^4 y_{-2} + \Delta^5 y_{-2}) + \ldots \quad \text{[Using (5) and (6)]}$$

Hence $$y_p + y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)\,p\,(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+2)(p+1)\,p\,(p-1)}{4!}\Delta^4 y_{-2} + \ldots$$

which is called *Gauss's backward interpolation formula.*

Cor. *In the central differences notation, this formula will be*

$$y_p = y_0 + p\delta y_{-1/2} + \frac{(p+1)p}{2!}\delta^2 y_0 + \frac{(p+1)\,p(p-1)}{3!}\delta^3 y_{-1/2} + \frac{(p+2)(p+1)\,p\,(p-1)}{4!}\delta^4 y_0 + \ldots$$

Obs. 1. This formula contains odd differences above the central line and even differences on the central line as shown below :

Δy_{-1} $\quad$ $\Delta^3 y_{-2}$ $\quad$ $\Delta^5 y_{-3}$

y_0 .... $\Delta^2 y_{-1}$ .... $\Delta^4 y_{-2}$ .... $\Delta^6 y_{-3}$ .... Central line

Obs. 2. It is used to interpolate the values of y for a negative value of p lying between -1 and 0.

Obs. 3. Gauss's forward and backward formulae are not of much practical use. However, these serve as intermediate steps for obtaining the important formulae of the following sections.

(3) **Stirling's formula.*** Gauss's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p\,(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)\,p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)\,p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \ldots \quad \ldots(1)$$

Gauss's backward interpolation formula is

$$y_p = y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)\,p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+2)(p+1)\,p\,(p-1)}{4!}\Delta^4 y_{-2} + \ldots \quad \ldots(2)$$

Taking the mean of (1) and (2), we obtain

$$y_p = y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p\,(p^2-1)}{3!} \times \left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{p^2\,(p^2-1)}{4!}\Delta^4 y_{-2} + \ldots \quad \ldots(3)$$

which is called *Stirling's formula*.

Cor. *In the central differences notation, (3) takes the form*

$$y_p = y_0 + p\,\mu\delta y_0 + \frac{p^2}{2!}\delta^2 y_0 + \frac{p\,(p^2+1^2)}{3!}\mu\delta^3 y_0 + \frac{p^2\,(p^2-1^2)}{4!}\delta^4 y_0 + \ldots \quad \ldots(4)$$

for $$\frac{1}{2}(\Delta y_0 + \Delta y_{-1}) = \frac{1}{2}(\delta y_{1/2} + \delta y_{-1/2}) = \mu\delta y_0$$

$$\frac{1}{2}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) = \frac{1}{2}(\delta^3 y_{1/2} + \delta^3 y_{-1/2}) = \mu\delta^3 y_0 \text{ etc.}$$

Obs. *This formula involves means of the odd differences just above and below the central line and even differences on this line* as shown below :

$$\ldots y_0 \ldots \begin{pmatrix}\Delta y_{-1}\\ \Delta y_0\end{pmatrix} \ldots \Delta^2 y_{-1} \ldots \begin{pmatrix}\Delta^3 y_{-2}\\ \Delta^3 y_{-1}\end{pmatrix} \ldots \Delta^4 y_{-2} \ldots \begin{pmatrix}\Delta^5 y_{-3}\\ \Delta^5 y_{-2}\end{pmatrix} \ldots \Delta^6 y_{-3} \ldots \text{ Central line}$$

(4) **Bessel's formula.**** Gauss's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p\,(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)\,p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)\,p(p-1)(p-2)}{4!}\Delta^4 y_{-2} \ldots \quad \ldots(1)$$

*Named after the Scotish mathematicians *James Stirling* (1692-1770).

**See footnote p. 550.

We have $\Delta^2 y_0 - \Delta^2 y_{-1} = \Delta^3 y_{-1}$

i.e., $\Delta^2 y_{-1} = \Delta^2 y_0 - \Delta^3 y_{-1}$ ...(2)

Similarly $\Delta^4 y_{-2} = \Delta^4 y_{-1} - \Delta^5 y_{-2}$ etc. ...(3)

Now (1) can be written as

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\left(\frac{1}{2}\Delta^2 y_{-1} + \frac{1}{2}\Delta^2 y_{-1}\right) + \frac{p(p^2-1)}{3!}\Delta^3 y_{-1} + \frac{p(p^2-1)(p-2)}{4!}\left(\frac{1}{2}\Delta^4 y_{-2} + \frac{1}{2}\Delta^4 y_{-2}\right) + \ldots$$

$$= y_0 + p\Delta y_0 + \frac{1}{2}\frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{1}{2}\frac{p(p-1)}{2!}(\Delta^2 y_0 - \Delta^3 y_{-1}) + \frac{p(p^2-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{1}{2}\frac{p(p^2-1)(p-2)}{4!}\Delta^4 y_{-2} + \frac{1}{2}\frac{p(p^2-1)(p-2)}{4!} \times (\Delta^4 y_{-1} - \Delta^5 y_{-2}) + \ldots \quad \text{[Using (2), (3) etc.]}$$

$$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\cdot\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{p(p-1)}{2!} \times \left(\frac{p+1}{3} - \frac{1}{2}\right)\Delta^3 y_{-1} + \frac{p(p^2-1)(p-2)}{4!}\cdot\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \ldots$$

Hence $$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\cdot\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{(p-1/2)\,p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)\,p(p-1)(p-2)}{4!}\cdot\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \ldots \quad \ldots(4)$$

which is known as the *Bessel's formula.*

Cor. *In the central differences notation, (4) becomes*

$$y_p = y_0 + p\delta y_{1/2} + \frac{p(p-1)}{2!}\mu\delta^2 y_{1/2} + \frac{(p-1/2)\,p(p-1)}{3!}\delta^3 y_{1/2} + \frac{(p+1)\,p(p-1)(p-2)}{4!}\mu\delta^4 y_{1/2} + \ldots \quad \ldots(5)$$

for $\frac{1}{2}(\Delta^2 y_{-1} + \Delta^2 y_0) = y\delta^2 y_{1/2}$, $\frac{1}{2}(\Delta^4 y_{-2} + \Delta^4 y_{-1}) = \mu\delta^4 y_{1/2}$ etc.

Obs. *This is a very useful formula for practical purposes. It involves odd differences below the central line and means of even differences of and below his line as shown below :*

$$\begin{array}{cccccccc} y_0 \text{---} & \left\{\begin{matrix}\Delta^2 y_{-1} \\ \\ \Delta^2 y_0\end{matrix}\right\} & \text{---} & \left\{\begin{matrix}\Delta^4 y_{-2} \\ \\ \Delta^4 y_{-1}\end{matrix}\right\} & \text{---} & \left\{\begin{matrix}\Delta^6 y_{-3} \\ \\ \Delta^6 y_{-2}\end{matrix}\right\} & \text{--- Central line} \\ \Delta y_0 & & \Delta^3 y_{-1} & & \Delta^5 y_{-2} & & \end{array}$$

(5) Everett's formula. Gauss's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)\,p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)\,p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \frac{(p+2)(p+1)\,p(p-1)(p-2)}{5!}\Delta^5 y_{-2} + \ldots \quad \ldots(1)$$

We eliminate the odd difference in (1) by using the relations

$$\Delta y_0 = y_1 - y_0,\ \Delta^3 y_{-1} = \Delta^2 y_0 - \Delta^2 y_{-1},\ \Delta^5 y_{-2} = \Delta^4 y_{-1} - \Delta^4 y_{-2} \text{ etc.}$$

Then (1) becomes

$$y_p = y_0 + p(y_1 - y_0) + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)\,p(p-1)}{3!}(\Delta^2 y_0 - \Delta^2 y_{-1})$$

$$+ \frac{(p+1)\,p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \frac{(p+2)(p+1)\,p(p-1)(p-2)}{5!} \times (\Delta^4 y_{-1} - \Delta^4 y_{-2}) + \ldots$$

$$= (1-p)\,y_0 + py_1 - \frac{p(p-1)(p-2)}{3!}\Delta^2 y_{-1} + \frac{(p+1)\,p(p-1)}{3!}\Delta^2 y_0$$

$$- \frac{(p+1)\,p\,(p-1)(p-2)(p-3)}{5!}\,\Delta^4 y_{-2} + \frac{(p+2)(p+1)\,p\,(p-1)(p-2)}{5!}\,\Delta^4 y_{-1} - \ldots$$

To change the terms with negative sign, putting $p = 1 - q$, we obtain

$$y_p = qy_0 + \frac{q(q^2-1^2)}{3!}\Delta^2 y_{-1} + \frac{q(q^2-1^2)(q^2-2^2)}{5!}\Delta^4 y_{-2} + \ldots\ldots + py_1 + \frac{p(p^2-1^2)}{3!}\Delta^2 y_0 + \frac{p(p^2-1^2)(p^2-2^2)}{5!}\Delta^4 y_{-1} + \ldots$$

This is known as *Everett's formula.*

Obs. *This formula is extensively used and involves only even differences on and below the central line* as shown below :

$$\begin{array}{ccccc} y_0 & \Delta^2 y_{-1} & \Delta^4 y_{-2} & \Delta^6 y_{-3} & \text{Central line} \\ y_1 & \Delta^2 y_0 & \Delta^4 y_{-1} & \Delta^6 y_{-2} & \end{array}$$

29.8 CHOICE OF AN INTERPOLATION FORMULA

The coefficients in the central difference formulae are smaller and converge faster than those in Newton's formulae. After a few terms, the coefficients in the Stirling's formula decrease more rapidly than those of the Bessel's formula and the coefficients of Bessel's formula decrease more rapidly than those of Newton's formula. As much, whenever possible, *central difference formulae should be used in preference to Newton's formulae.*

The right choice of an interpolation formula however, depends on the position of the interpolated value in the given data.

The following rules will be found useful :

1. *To find a tabulated value near the beginning of the table, use Newton's forward formula.*
2. *To find a value near the end of the table, use Newton's backward formula.*
3. *To find an interpolated value near the centre of the table, use either Stirling's or Bessel's or Everett's formula.*

If interpolation is required for p lying between $-1/4$ and $1/4$, prefer Stirling's formula.

If interpolation is desired for p lying between $1/4$ and $3/4$, use Bessel's or Everett's formula.

Example 29.18. *Find f(22) from the Gauss forward formula :*

x :	20	25	30	35	40	45
$f(x)$:	354	332	291	260	231	204

(J.N.T.U., 2007)

Solution. Taking $x_0 = 25$, $h = 5$, we have to find the value of $f(x)$ for $x = 22$.

i.e., for $p = \dfrac{x - x_0}{h} = \dfrac{22-25}{5} = -0.6$

The difference table is as follows :

x	p	y_p	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$	$\Delta^4 y_p$	$\Delta^5 y_p$
20	−1	354 $(= y_{-1})$					
			−22				
25	0	332 $(= y_0)$		−19			
			−41		29		
30	1	291 $(= y_1)$		10		−37	
			−31		−8		45
35	2	260 $(= y_2)$		2		8	
			−29		0		
40	3	231 $(= y_3)$		2			
			−27				
45	4	204 $(= y_4)$					

Gauss forward formula is

$$y^p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\,\Delta y_{-1} + \frac{(p+1)\,p\,(p-1)}{3!}\,\Delta^3 y_{-1}$$

$$+ \frac{(p+1)\,p\,(p-1)\,(p-2)}{4!}\,\Delta^4 y_{-2} + (p+1)\,(p-1)\,(p-2)\,(p+2)\,\Delta^5 y_{-2}$$

$$\therefore \qquad f(22) = 332 + (0.6)\,(-41) + \frac{(-0.6)\,(-0.6-1)}{2!}\,(-19) + \frac{(-0.6+1)\,(-0.6)\,(-0.6-1)}{3!}\,(-8)$$

$$+ \frac{(-0.6-1)\,(-0.6)\,(-0.6-1)\,(-0.6-2)}{4!}\,(-37)$$

$$+ \frac{(-0.6+1)\,(-0.6)\,(-0.6-1)\,(-0.6-2)\,(-0.6+2)}{5!}\,(45)$$

$$= 332 + 24.6 - 9.12 + 1.5392 - 0.5241$$

Hence $\qquad f(22) = 347.983.$

Example 29.19. *Interpolate by means of Gauss's backward formula, the population of a town for the year 1974, given that :*

Year	*1939*	*1949*	*1959*	*1969*	*1979*	*1989*
Population (in thousands) :	*12*	*15*	*20*	*27*	*39*	*52*

(Kottayam, 2005 ; Madras, 2003)

Solution. Taking $x_0 = 1969$, $h = 10$, the population of the town is to be found for $p = \dfrac{1974 - 1969}{10} = 0.5.$

The central difference table is

x	p	y_p	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$	$\Delta^4 y_p$	$\Delta^5 y_p$
1939	−3	12					
			3				
1949	−2	15		2			
			5		0		
1959	−1	20		2		3	
			7		3		−10
1969	0	27		5		−7	
			12		−4		
1979	1	39		1			
			13				
1989	2	52					

Gauss's backward formula is

$$y_p = y_0 + p\Delta y_{-1} + \frac{(p+1)\,p}{2!}\,\Delta^2 y_{-1} + \frac{(p+1)\,p\,(p-1)}{3!}\,\Delta^3 y_{-2} + \frac{(p+1)\,(p+1)\,p\,(p-1)}{4!}\,\Delta^4 y_{-2}$$

$$+ \frac{(p+2)\,(p+1)\,p\,(p-1)\,(p-2)}{5!}\,\Delta^5 y_3 + \ldots$$

i.e.,

$$y_5 = 27 + (0.5)(7) + \frac{(1.5)(.5)}{2}\,(5) + \frac{(1.5)\,(.5)\,(-.5)}{6}\,(3) + \frac{(2.5)\,(1.5)\,(-.5)}{24}\,(-7)$$

$$+ \frac{(2.5)\,(1.5)\,(.5)\,(-.5)\,(-1.5)}{120}\,(-10)$$

$$= 27 + 3.5 + 1.875 - 0.1875 + 0.2743 - 0.1172 = 32.345 \text{ thousands approx.}$$

Example 29.20. *Given*

$\theta°$	:	0	5	10	15	20	25	30
$\tan\theta$	:	0	0.0875	0.1763	0.2679	0.3640	0.4663	0.5774

Using Stirling's formula, estimate the value of tan 16°. (Anna, 2005)

Solution. Taking the origin at $\theta° = 15°$, $h = 5°$ and $p = \dfrac{\theta - 15}{5}$, we have the following central difference table :

p						
−3	0.0000					
		0.08575				
−2	0.0875		0.0013			
		0.0888				
−1	0.1763		0.0028			
		0.0916		0.0015		
0	0.2679		0.0045		0.0002	
		0.0961		0.0017		−0.0002
1	0.3640		0.0062		0.0000	
		0.1023		0.0017		0.0009
2	0.4663		0.0088		0.0009	
		0.1111		0.0026		
3	0.5774					

At $\theta = 16°$, $\quad p = \dfrac{16 - 15}{5} = 0.2$

Stirling's formula is

$$y_p = y_0 + \frac{p}{1} \cdot \frac{\Delta y_{-1} + \Delta y_0}{2} + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!} \cdot \frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2} + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \ldots$$

$$\therefore \quad y_{0.2} = 0.2679 + (0.2)\left(\frac{0.0916 + 0.0961}{2}\right) + \frac{(0.2)^2}{2!}\,(0.0045) + \ldots$$

$$= 0.2679 + 0.01877 + 0.00009 + \ldots = 0.28676$$

Hence $\quad \tan 16° = 0.28676.$

Example 29.21. *Employ Stirling's formula to compute $y_{12.2}$ from the following table ($y_x = 1 + \log_{10}\sin x$) :*

$x°$	:	10	11	12	13	14
$10^5 y_x$	:	23,967	28,060	31,788	35,209	38,368

(V.T.U., 2004)

Solution. Taking the origin at $x_0 = 12°$, $h = 1$ and $p = x - 12$, we have the following central table :

p	y_x	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$	$\Delta^4 y_x$
−2	0.23967				
		0.04093			
−1	0.28060		−0.00365		
		0.03728		0.00058	
0	0.31788		−0.00307		−0.00013
		0.034121		−0.00045	
1	0.35209		−0.00062		
		0.03159			
2	0.38368				

At $x = 12.2$, $p = 0.2$. (As p lies between − 1/4 and 1/4, the use of Stirling's formula will be quite suitable.)

Stirling's formula is

$$y_p = y_0 + \frac{p}{1} \cdot \frac{\Delta y_{-1} + \Delta y_{-0}}{2} + \frac{p^2}{2!} \Delta^2 y_{-1} + \frac{p(p^2-1)}{3!} \cdot \frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2} + \frac{p^2(p^2-1)}{4!} \Delta^4 y_{-2} + \ldots$$

When $p = 0.2$, we have

$$\therefore \quad y_{0.2} = 0.31788 + 0.2 \left(\frac{0.03728 + 0.03421}{2}\right) + \frac{(0.2)^2}{2} (-0.00307)$$

$$+ \frac{(0.2)[(0.2)^2 - 1]}{6} \left(\frac{0.00058 - 0.00045}{2}\right) + \frac{(0.2)^2 [(0.2)^2 - 1]}{24} (-0.00013)$$

$$= 0.31788 + 0.00715 - 0.00006 - 0.000002 + 0.0000002 = 0.32497.$$

Example 29.22. *Apply Bessel's formula to obtain y_{25}, given $y_{20} = 2854$, $y_{24} = 3162$, $y_{28} = 3544$, $y_{32} = 3992$.* *(S.V.T.U., 2007 ; V.T.U., 2000 S)*

Solution. Taking the origin at $x_0 = 24$, $h = 4$, we have $p = \frac{1}{4}(x - 24)$.

$\therefore$ The central difference table is

p	y	Δy	$\Delta^2 y$	$\Delta^3 y$
−1	2854			
		308		
0	3162		74	
		382		−8
1	3544		66	
		448		
2	3992			

At $x = 25$, $p = (25 - 24)/4 = 1/4$. (As p lies between 1/4 and 3/4, the use of Bessel's formula will yield accurate esult.)

Bessel's formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{(p - 1/2)\, p\, (p-1)}{3!} \Delta^3 y_{-1} + \ldots \qquad \ldots(1)$$

When $p = 0.25$, we have

$$y_p = 3162 + 0.25 \times 382 + \frac{0.25(-0.75)}{2} \left(\frac{74 + 66}{2}\right) + \frac{(-0.25)\, 0.25\, (-0.75)}{6} (-8)$$

$$= 3162 + 95.5 - 6 - 5625 - 0.0625 = 3250.875 \text{ approx.}$$

Example 29.23. *Apply Bessel's formula to find the value of f(27.5) from the table :*

x :	25	26	27	28	29	30
$f(x)$:	4.000	3.846	3.704	3.571	3.448	3.333

(U.P.T.U., 2009)

Solution. Taking the origin at $x_0 = 27$, $h = 1$, we have $p = x - 27$

The central difference table is

x	p	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
25	−2	4.000				
			−0.154			
26	−1	3.846		0.012		
			−0.142		−0.003	
27	0	3.704		0.009		0.004
			−0.133		−0.001	
28	1	3.571		0.010		−0.001
			−0.123		−0.002	
29	2	3.448		0.008		
			−0.115			
30	3	3.333				

At $x = 27.5$, $p = 0.5$ (As p lies between 1/4 and 3/4, the use of Bessel's formula will yield accurate result)
Bessel's formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{\left(p-\frac{1}{2}\right)p(p-1)}{3!}\Delta^3 y_{-1}$$
$$+ \frac{(p+1)\,p(p-1)\,(p-2)}{4!}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \ldots$$

When $p = 0.5$, we have

$$y_p = 3.704 - \frac{(0.5)\,(0.5-1)}{2}\left(\frac{0.009 + 0.010}{2}\right) + 0$$
$$+ \frac{(0.5+1)\,(0.5)\,(0.5-1)\,(0.5-2)}{24}\left(\frac{-0.001 - 0.004}{2}\right)$$
$$= 3.704 - 0.11875 - 0.00006 = 3.585$$

Hence $f(27.5) = 3.585$.

Example 29.24. *Given the table*

x :	310	320	330	340	350	360
$\log x$:	2.49136	2.50515	2.51851	2.53148	2.54407	2.55630

find the value of log 337.5 by Everett's formula.

Solution. Taking the origin at $x_0 = 330$ and $h = 10$, we have $p = \dfrac{x - 330}{10}$

$\therefore$ The central difference table is

p	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
−2	2.49136					
		0.01379				
−1	2.50515		−0.00043			
		0.01336		0.00004		
0	2.51881		−0.00039		−0.00003	
		0.01297		0.00001		0.00004
1	2.53148		−0.00038		0.00001	
		0.01259		0.00002		
2	2.54407		−0.00036			
		0.01223				
3	2.55630					

To evaluate log 337.5 *i.e.* for $x = 337.5$, $p = \dfrac{337.5 - 330}{10} = 0.75$

(As $p > 0.5$ and $= 0.75$, Everett's formula will be quite suitable)

Everett's formula is

$$y_p = qy_0 + \frac{q(q^2-1^2)}{3!}\Delta^2 y_{-1} + \frac{q(q^2-1^2)(q^2-2^2)}{5!}\Delta^4 y_{-2} + \ldots + py_1 + \frac{p(p^2-1^2)}{3!}\Delta^2 y_0$$
$$+ \frac{p(p^2-1^2)(p^2-2^2)}{5!}\Delta^4 y_{-1} + \ldots$$
$$= 0.25 \times 2.51851 + \frac{0.25\,(0.0625-1)}{6} \times (-0.00039) + \frac{0.25\,(0.0625-1)\,(0.0625-4)}{120}$$
$$\times (-0.00003) + 0.75 \times 2.53148 + \frac{0.75\,(0.5625-1)}{6} \times (-0.00038)$$
$$+ \frac{0.75\,(0.5625-1)\,(0.5625-4)}{120} \times (0.00001)$$
$$= 0.62963 + 0.00002 - 0.0000002 + 1.89861 + 0.00002 + 0.0000001 = 2.52828 \text{ nearly.}$$

PROBLEMS 29.4

1. Using Gauss's forward formula, evaluate $f(3.75)$ from the table :

x :	2.5	3.0	3.5	4.0	4.5	5.0
y :	24.145	22.043	20.225	18.644	17.262	16.047

(Bhopal, 2002 ; Madras, 2000)

2. Using Gauss's backward difference formula, find $y(8)$ from the following table :

x :	0	5	10	15	20	25
y :	7	11	14	18	24	32

(J.N.T.U., 2007)

3. Using Gauss's backward formula, estimate the number of persons earning wages between Rs. 60 and Rs. 70 from the following data :

Wages (₹) :	Below 40	40–60	60–80	80–100	100–120
No. of persons : (in thousands)	250	120	100	70	50

(Madras, 2000)

4. From the following table :

x :	1.00	1.05	1.10	1.15	1.20	1.25	1.30
e^x :	2.7183	2.8577	3.0042	3.1582	3.3201	3.4903	3.6693

(U.P.T.U., 2006)

Find $e^{1.17}$, using Gauss forward formula.

5. The pressure p of wind corresponding to velocity v is given by the following data. Estimate p when $y = 25$.

v :	10	20	30	40
p :	1.1	2	4.4	7.9

6. Using Stirling's formula find y_{35}, given $y_{20} = 512$, $y_{30} = 439$, $y_{40} = 346$, $y_{50} = 243$, where y_x represents the number of persons at age x years in a life table. *(Nagarjuna, 2003 S)*

7. Employ Bessel's formula to find the value of F at $x = 1.95$, given that

x :	1.7	1.8	1.9	2.0	2.1	2.2	2.3
F :	2.979	3.144	3.283	3.391	3.463	3.997	4.491

Which other interpolation formula can be used here ? Which is more appropriate ? Give reasons.

8. Calculate the value of $f(1.5)$ using Bessel's interpolation formula, from the following table :

x :	0	1	2	3
$f(x)$:	3	6	12	15

(U.P.T.U., 2008)

9. Apply Everett's formula to obtain u_{25}, given $u_{20} = 854$, $u_{24} = 3162$, $u_{28} = 3544$, $u_{32} = 3992$. *(S.V.T.U., 2007)*

10. Using Everett's formula, evaluate $f(30)$, if $f(20) = 2854$, $f(28) = 3162$, $f(36) = 7088$, $f(44) = 7984$ *(U.P.T.U., 2006)*

11. Given the table :

x :	310	320	330	340	350	360
$\log x$:	2.4914	2.5052	2.5185	2.5315	2.5441	2.5563

Find the value of log 337.5 by Gauss's, Stirling's and Bessel's formulae.

29.9 INTERPOLATION WITH UNEQUAL INTERVALS

The various interpolation formulae derived so far possess the disadvantages of being applicable only to equally spaced values of the argument. It is, therefore, desirable to develop interpolation formulae for unequally spaced values of x. Now we shall study two such formulae :

(*i*) Lagrange's interpolation formula

(*ii*) Newton's general interpolation formula with divided differences.

29.10 LAGRANGE'S INTERPOLATION FORMULA

If $y = f(x)$ *takes the value* $y_0, y_1, \ldots, y_n$ *corresponding to* $x = x_0, x_1, \ldots, x_n$, *then*

$$f(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}y_1 + \ldots + \frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}y_n \quad \ldots(1)$$

This is known as *Lagrange's interpolation formula for unequal intervals.*

Proof. Let $y = f(x)$ be a function which takes the values $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$. Since there are $n + 1$ pairs of values of x and y, we can represent $f(x)$ by a polynomial in x of degree n. Let this polynomial be of the form

$$y = f(x) = a_0(x - x_1)(x - x_2) \ldots (x - x_n) + a_1(x - x_0)(x - x_2) \ldots (x - x_n)$$
$$+ a_2(x - x_0)(x - x_1)(x - x_3) \ldots (x - x_n) + \ldots + a_n(x - x_0)(x - x_1) \ldots (x - x_{n-1}) \quad \ldots(2)$$

Putting $x = x_0, y = y_0$, in (2), we get

$$y_0 = a_0(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)$$
$$a_0 = y_0/[(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)]$$

Similarly putting $x = x_1, y = y_1$ in (2), we have $a_1 = y_1/[(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)]$

Proceeding the same way, we find $a_2, a_3 \ldots a_n$

Substituting the values of $a_0, a_1, \ldots, a_n$ in (2), we get (1).

Obs. Lagranges interpolation formula (1) for n points is a polynomial of degree $(n - 1)$ which is known as *Lagrangian polynomial* and is very simple to implement on a computer.

This formula can also be used to split the given function into partial fractions.

For on dividing both sides of (1) by $(x - x_0)(x - x_1) \ldots (x - x_n)$, we get

$$\frac{f(x)}{(x_0 - x_0)(x_0 - x_1)\ldots(x_0 - x_n)} = \frac{y_0}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)} \cdot \frac{1}{x - x_0}$$
$$+ \frac{y_0}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)} \cdot \frac{1}{x - x_1} + \ldots + \frac{y_n}{(x_n - x_0)(x_n - x_1)\ldots(x_n - x_{n-1})} \cdot \frac{1}{x - x_n}.$$

Example 29.25. *Given the values*

x :	5	7	11	13	17
$f(x)$:	150	392	1492	2366	5202

evaluate f (9), using (i) Lagrange's formula. (Anna, 2006)

Solution. (*i*) Here $x_0 = 5, x_1 = 7, x_2 = 11, x_3 = 13, x_4 = 17$

and $y_0 = 150, y_1 = 392, y_2 = 1452, y_3 = 2366, y_4 = 5202.$

Putting $x = 9$ and substituting the above values in Lagrange's formula, we get

$$f(9) = \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)} \times 150 + \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)} \times 392$$
$$+ \frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)} \times 1452 + \frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)} \times 2366$$
$$+ \frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)} \times 5202 = -\frac{50}{3} + \frac{3136}{15} + \frac{3872}{3} - \frac{2366}{3} + \frac{578}{5} = 810.$$

Example 29.26. *Find the polynomial f(x) by using Lagrange's formula and hence find f (3) for*

x :	0	1	2	5
$f(x)$:	2	3	12	147

(Anna, 2005)

Solution. Here $x_0 = 0, x_1, x_2 = 2, x_3 = 5$

and $y_0 = 2, y_1 = 3, y_2 = 12, y_3 = 147$

Lagrange's formula is

$$y = \frac{(x - x_1)(x - x_2)\ldots(x - x_3)}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_3)} y_0 + \frac{(x - x_0)(x - x_2)\ldots(x - x_3)}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_3)} y_1$$
$$+ \frac{(x - x_0)(x - x_1)\ldots(x - x_3)}{(x_2 - x_0)(x_2 - x_1)\ldots(x_2 - x_3)} y_2 + \frac{(x - x_0)(x - x_2)\ldots(x - x_5)}{(x_3 - x_0)(x_3 - x_2)\ldots(x_3 - x_2)} y_3$$
$$= \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}(2) + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}(3)$$
$$+ \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}(12) + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}(147)$$

Hence $f(x) = x^3 + x^2 - x + 2$

$\therefore$ $f(3) = 27 + 9 - 3 + 2 = 35.$

Example 29.27. *A curve passes through the point (0, 18), (1, 10), (3, – 18) and (6, 90). Find the slope of the curve at x = 2.* *(J.N.T.U., 2009)*

Solution. Here $x_0 = 0, x_1 = 1, x_2 = 3, x_3 = 6$ and $y_0 = 18, y_1 = 10, y_2 = -18, y_3 = 90$

Since the values of x are unequally spaced, we use the Lagrange's formula :

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$= \frac{(x-1)(x-3)(x-6)}{(0-1)(0-2)(0-6)}(18) + \frac{(x-0)(x-3)(x-6)}{(1-0)(1-3)(1-6)}(10) + \frac{(x-0)(x-1)(x-6)}{(3-0)(3-1)(3-6)}(-18) + \frac{(x-0)(x-1)(x-3)}{(6-0)(6-1)(6-3)}(90)$$

$$= (-x^3 + 10x^2 - 27x + 18) + (x^3 - 9x^2 + 18x) + (x^3 - 7x^2 + 6x) + (x^3 - 4x^2 + 3x)$$

i.e., $y = 2x^3 - 10x^2 + 18$

Thus the slope of the curve at $(x = 2) = \left(\frac{dy}{dx}\right)_{x=2}$

$$= (6x^2 - 20x)_{x=2} = -16.$$

Example 29.28. *Using Lagrange's formula, express the function* $\frac{3x^2+x+1}{(x-1)(x-2)(x-3)}$ *as a sum of partial fractions.*

Solution. Let us evaluate $y = 3x^2 + x + 1$ for $x = 1, x = 2$ and $x = 3$

These values are

x :	$x_0 = 1$	$x_1 = 2$	$x_2 = 3$
y :	$y_0 = 5$	$y_1 = 15$	$y_2 = 31$

Lagrange's formula is

$$y = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2$$

$$= \frac{(x-1)(x-2)}{(0-1)(0-2)}(5) + \frac{(x-1)(x-3)}{(2-1)(2-3)}(15) + \frac{(x-1)(x-2)}{(3-1)(3-1)}(31)$$

Substituting the above values, we get

$$= \frac{(x-2)(x-3)}{(1-2)(1-3)}(5) + \frac{(x-1)(x-3)}{(2-1)(2-3)}(15) + \frac{(x-1)(x-2)}{(3-1)(3-2)}(31)$$

$$= 2.5(x-2)(x-3) - 15(x-1)(x-3) + 15.5(x-1)(x-2)$$

Thus $$\frac{3x^2+x+1}{(x-1)(x-2)(x-3)} = \frac{2.5(x-2)(x-3) - 15(x-1)(x-3) + 15.5(x-1)(x-2)}{(x-1)(x-2)(x-3)}$$

$$= \frac{2.5}{x-1} - \frac{15}{x-2} + \frac{15.5}{x-3}.$$

Example 29.29. *Find the distance moved by a particle and its acceleration at the end of 4 seconds, if the time verses velocity data is as follows :*

t :	*0*	*1*	*3*	*4*
v :	*21*	*15*	*12*	*10*

Solution. Since the values of t are not equispaced, we use Lagrange's formula :

$$v = \frac{(t-t_1)(t-t_2)(t-t_3)}{(t_0-t_1)(t_0-t_2)(t_0-t_3)}v_0 + \frac{(t-t_0)(t-t_2)(t-t_3)}{(t_1-t_0)(t_1-t_2)(t_1-t_3)}v_1$$
$$+ \frac{(t-t_0)(t-t_1)(t-t_3)}{(t_2-t_0)(t_2-t_1)(t_2-t_3)}v_2 + \frac{(t-t_0)(t-t_2)(t-t_5)}{(t_3-t_0)(t_3-t_2)(t_3-t_2)}v_3$$

$$v = \frac{(t-1)(t-3)(t-4)}{(-1)(-2)(-4)}(21) + \frac{t(t-3)(t-4)}{(1)(-2)(-3)}(15) + \frac{t(t-1)(t-4)}{(3)(2)(-1)}(12) + \frac{t(t-1)(t-3)}{(4)(3)(1)}(10)$$

i.e., $$v = \frac{1}{12}(-5t^3 + 38t^2 - 105t + 252)$$

$\therefore$ Distance moved $s = \int_0^4 v\,dt = \frac{1}{12}\int_0^4 (-5t^3 + 38t^2 - 105t + 252)\,dt$ $\left[\because\ v = \frac{ds}{dt}\right]$

$$= \frac{1}{12}\left(-\frac{5t^4}{4} + \frac{38t^3}{3} - \frac{105t^2}{2} + 252t\right)_0^4$$

$$= \frac{1}{12}\left(-320 + \frac{2432}{3} - 840 + 1008\right) = 54.9$$

Also acceleration $= \frac{dv}{dt} = \frac{1}{2}(-15t^2 + 76t - 105 + 0)$

Hence acceleration at $(t = 4) = \frac{1}{12}(-15(16) + 76(4) - 105) = -3.4.$

PROBLEMS 29.5

1. Use Lagrange's interpolation formula to find the value of y when $x = 10$, if the following values of x and y are given :

x :	5	6	9	11
y :	12	13	14	16

(*U.P.T.U., 2009 ; J.N.T.U., 2008*)

2. Given $\log_{10} 654 = 2.8156$, $\log_{10} 658 = 2.8182$, $\log_{10} 659 = 2.8189$, $\log_{10} 661 = 2.8202$, find by using Lagrange's formula, the value of $\log_{10} 656$. (*Hazaribagh, 2009*)
3. The following are the measurements T made on a curve recorded by oscilograph representing a change of current I due to a change in the conditions of an electric current.

T :	1.2	2.0	2.5	3.0
I :	1.36	0.58	0.34	0.20

Using Lagrange's formula, find I at $T = 16$. (*J.N.T.U., 2009*)

4. Using Lagrange's interpolation, calculate the profit in the year 2000 from the following data :

Year :	1997	1999	2001	2002
Profit in Lakhs of ₹ :	43	65	159	248

(*Anna, 2004*)

5. Use Lagrange's formula to find the form of $f(x)$, given

x :	0	2	3	6
$f(x)$:	648	704	729	792

(*Madras, 2003 S*)

6. If $y(1) = -3$, $y(3) = 9$, $y(4) = 30$, $y(6) = 132$, find the Lagrange's interpolation polynomial that takes the same values as y at the given points. (*V.T.U., 2006*)
7. Given $f(0) = -18$, $f(1) = 0$, $f(3) = 0$, $f(5) = -248$, $f(6) = 0$, $f(9) = 13104$, find $f(x)$. (*Nagarjuna, 2003*)
8. Find the missing term in the following table using interpolation

x :	1	2	4	5	6
y :	14	15	5	...	9

9. Using Lagrange's formula, express the function $\dfrac{x^2 + x - 3}{x^3 - 2x^2 - x + 2}$ as sum of partial fractions.

29.11 DIVIDED DIFFERENCES

The Lagrange's formula has the drawback that if another interpolation value were inserted, then the interpolation coefficients are required to be recalculated. The labour of recomputing the interpolation coefficients is saved by using Newton's general interpolation formula which employs what are called 'divided differences'. Before deriving this formula, we shall first define these differences.

If $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots$ be given points, then the *first divided difference* for the arguments, x_0, x_1 is defined by the relation $[x_0, x_1] = \dfrac{y_1 - y_0}{x_1 - x_0}$.

Similarly $[x_1, x_2] = \dfrac{y_2 - y_1}{x_2 - x_1}$ and $[x_2, x_3] = \dfrac{y_3 - y_2}{x_3 - x_2}$ etc.

The *second divided difference* for x_0, x_1, x_2 is defined as $[x_0, x_1, x_2] = \dfrac{[x_1 - x_2] - [x_0, x_1]}{x_2 - x_0}$

The *third divided difference* for x_0, x_1, x_2, x_3 is defined as

$[x_0, x_1, x_2, x_3] = \dfrac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_3 - x_0}$ and so on.

Obs. 1. *The divided differences are symmetrical in their arguments i.e. independent of the order of the arguments.*

For it is easy to write $[x_0, x_1] = \dfrac{y_0}{x_0 - x_1} + \dfrac{y_1}{x_1 - x_0} = [x_1, x_0]$ $[x_0, x_1, x_2]$

$$= \frac{y_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)} \cdot \frac{y_2}{(x_2 - x_0)(x_2 - x_1)}$$

$= [x_1, x_2, x_0]$ or $[x_2, x_0, x_1]$ and so on.

Obs. 2. *The nth divided differences of a polynomial of the nth degree are constant.*

Let the arguments be equally spaced so that, $x_1 - x_0 = x_2 - x_1 = \ldots = x_n - x_{n-1} = h$. Then

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h}$$

$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0} = \frac{1}{2h}\left\{\frac{\Delta y_1}{h} - \frac{\Delta y_0}{h}\right\}$$

$= \dfrac{1}{2!h^2} \Delta^2 y_0$ and in general, $[x_0, x_1, x_2, \ldots x_n] = \dfrac{1}{n!h^n} \Delta^n y_0$.

If the tabulated function is a nth degree polynomial, then $\Delta^n y_0$ will be constant. Hence the nth divided differences will also be constant.

29.12 NEWTON'S DIVIDED DIFFERENCE FORMULA

Let $y_0, y_1, \ldots, y_n$ be the values of $y = f(x)$ corresponding to the arguments $x_0, x_1, \ldots, x_n$. Then from the definition of divided differences, we have

$$[x, x_0] = \frac{y - y_0}{x - x_0}$$

so that $$y = y_0 + (x - x_0)[x, x_0] \qquad \ldots(1)$$

Again $$[x, x_0, x_1] = \frac{[x, x_0] - [x_0, x_1]}{x - x_1}$$

which gives $$[x, x_0] = [x_0, x_1] + (x - x_1) + (x - x_1)[x, x_0, x_1]$$

Substituting this value of $[x, x_0]$ in (1), we get

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x, x_0, x_1] \qquad \ldots(2)$$

Also $$[x, x_0, x_1, x_2] = \frac{[x, x_0, x_1] - [x_0, x_1, x_2]}{x - x_2}$$

which gives $[x, x_0, x_1,] = [x_0, x_1, x_2] + (x, x_2)[x, x_0, x_1, x_2]$

Substituting this value of $[x, x_0, x_1]$ in (2), we obtain

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)[x, x_0, x_1, x_2]$$

Proceeding in this manner, we get

$$y = f(x) = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3] + \ldots + (x - x_0)(x - x_1)\ldots(x - x_n)[x, x_0, x_1, \ldots, x_n] \quad \ldots(3)$$

which is called *Newton's general interpolation formula with divided differences.*

Example 29.30. *Given the values*

x :	5	7	11	13	17
$f(x)$:	150	392	1452	2366	5202,

evaluate f (9), using Newton's divided difference formula. (V.T.U., 2010 ; P.T.U., 2005)

Solution. The divided difference table is

x	y	1st divided differences	2nd divided differences	3rd divided differences
5	150			
		$\frac{392-150}{7-5} = 121$		
7	392		$\frac{265-121}{11-5} = 24$	
		$\frac{1452-392}{11-7} = 265$		$\frac{32-24}{13-5} = 1$
11	1452		$\frac{457-265}{13-7} = 32$	
		$\frac{2366-1452}{13-11} = 457$		$\frac{42-32}{17-7} = 1$
13	2366		$\frac{709-457}{17-11} = 42$	
		$\frac{5202-2366}{17-13} = 709$		
17	5202			

Taking $x = 9$ in the Newton's divided difference formula, we obtain

$$f(9) = 150 + (9-5) \times 121 + (9-5)(9-7) \times 24 + (9-5)(9-7)(9-11) \times 1$$
$$= 150 + 484 + 192 - 16 = 810.$$

Example 29.31. *Determine f (x) as a polynomial in x for the following data :*

x :	− 4	− 1	0	2	5
$f(x)$:	1245	33	5	9	1335

(V.T.U., 2007)

Solution. The divided differences table is

x	$f(x)$	1st divided differences	2nd divided differences	3rd divided differences	4th divided differences
− 4	1245				
		− 404			
− 1	33		94		
		− 28		− 14	
0	5		10		3
		2		13	
2	9		88		
		442			
5	1335				

Applying Newton's divided difference formula

$$f(x) = f(x_0) + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + \ldots$$
$$= 1245 + (x+4)(-404) + (x+4)(x+1)(94)$$
$$+ (x+4)(x+1)(x-0)(-14) + (x+4)(x+1)x(x-2)(3)$$
$$= 3x^4 - 5x^3 + 6x^2 - 14x + 5.$$

PROBLEMS 29.6

1. Find the third divided difference with arguments 2, 4, 9, 10 of the function $f(x) = x^3 - 2x$. *(U.P.T.U., 2005)*

2. Use Newton's divided difference method to compute $f(5.5)$ from the following data :

x :	0	1	4	5	6
$f(x)$:	1	14	15	6	3

(U.P.T.U., 2010)

3. Using Newton's divided difference formula, evaluate $f(8)$ and $f(15)$ given :

x :	4	5	7	10	11	13
$f(x)$:	48	100	294	900	1210	2028

(U.P.T.U., MCA, 2009, V.T.U., 2008)

4. Obtain the Newton's divided difference interpolation polynomial and hence find $f(6)$:

x :	3	7	9	10
$f(x)$:	168	120	72	63

(U.P.T.U., 2007)

5. Using Newton's divided difference interpolation, find the polynomial of the given data :

x :	−1	0	1	3
$f(x)$:	2	1	0	−1

(Anna, 2007)

6. For the following table, find $f(x)$ as a polynomial in x using Newton's divided difference formula:

x :	5	6	9	11
$f(x)$:	12	13	14	16

7. Using the following table, find $f(x)$ as a polynomial in

x :	−1	0	3	6	7
$f(x)$:	3	−6	39	822	1611

(U.P.T.U., 2009)

8. Find the missing term in the following table using Newton's divided difference formula

x :	0	1	2	3	4
y :	1	3	9	...	81

29.13 INVERSE INTERPOLATION

So far, given a set of values of x and y, we have been finding the values of y corresponding to a certain value of x. On the other hand, the process of estimating the value of x for a value of y (which is not in the table) is called the *inverse interpolation.*

Lagrange's formula is merely a relation between two variables either of which may be taken as the independent variable. Therefore, on inter-changing x and y in the Lagrange's formula, we obtain

$$x = \frac{(y-y_1)(y-y_2)\ldots(y-y_n)}{(y_0-y_1)(y_0-y_2)\ldots(y_0-y_n)}x_0 + \frac{(y-y_0)(y-y_2)\ldots(y-y_n)}{(y_1-y_0)(y_1-y_2)\ldots(y_1-y_n)}x_1$$
$$+ \ldots + \frac{(y-y_0)(y-y_1)\ldots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\ldots(y_n-y_{n-1})}x_n \quad \ldots(1)$$

which is used for inverse interpolation.

Example 29.32. *The following table gives the values of x and y :*

x :	1.2	2.1	2.8	4.1	4.9	6.2
y :	4.2	6.8	9.8	13.4	15.5	19.6

Find the value of x corresponding to y = 12, using Lagrange's technique. *(V.T.U., 2009)*

Solution. Here $x_0 = 1.2, x_1 = 2.1, x_2 = 2.8, x_3 = 4.1, x_4 = 4.9, x_5 = 6.2$

and $y_0 = 4.2, y_1 = 6.8, y_2 = 9.8, y_3 = 13.4, y_4 = 15.5, y_5 = 19.6$

Taking $y = 12$, the above formula (1) gives

$$x = \frac{(12-6.8)(12-9.8)(12-13.4)(12-15.5)(12-19.6)}{(4.2-6.8)(4.2-9.8)(4.2-13.4)(4.2-15.5)(4.2-19.6)} \times 1.2$$
$$+ \frac{(12-4.2)(12-9.8)(12-13.4)(12-15.5)(12-19.6)}{(6.8-4.2)(6.8-9.8)(6.8-13.4)(6.8-15.5)(6.8-19.6)} \times 2.1$$
$$+ \frac{(12-4.2)(12-6.8)(12-13.4)(12-15.5)(12-19.6)}{(9.8-4.2)(9.8-6.8)(9.8-13.4)(9.8-15.5)(9.8-19.6)} \times 2.8$$
$$+ \frac{(12-4.2)(12-6.8)(12-9.8)(12-15.5)(12-19.6)}{(13.4-4.2)(13.4-6.8)(13.4-9.8)(13.4-15.5)(13.4-19.6)} \times 4.1$$
$$+ \frac{(12-4.2)(12-6.8)(12-9.8)(12-13.4)(12-19.6)}{(15.5-4.2)(15.5-6.8)(15.5-9.8)(15.5-13.4)(15.5-19.6)} \times 4.9$$
$$+ \frac{(12-4.2)(12-6.8)(12-9.8)(12-13.4)(12-15.5)}{(19.6-4.2)(19.6-6.8)(19.6-9.8)(19.6-13.4)(19.6-15.5)} \times 6.2$$
$$= 0.022 - 0.234 + 1.252 + 3.419 - 0.964 + 0.055 = 3.55.$$

Example 29.33. *Apply Lagrange's formula inversely to obtain a root of the equation $f(x) = 0$, given that $f(30) = -30$, $f(34) = -13$, $f(38) = 3$, and $f(42) = 18$.* (V.T.U., 2009 S)

Solution. Here $x_0 = 30, x_1 = 34, x_2 = 38, x_3 = 42$

and $y_0 = -30, y_1 = -13, y_2 = 3, y_3 = 18$

It is required to find x corresponding to $y = f(x) = 0$.

Taking $y = 0$, the Lagrange's formula gives,

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)} x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y-y_2)(y_1-y_3)} x_1$$
$$+ \frac{(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)} x_2 + \frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)} x_3$$
$$= \frac{13(-3)(-18)}{(-17)(-33)(-48)} \times 30 + \frac{30(-3)(-18)}{17(-16)(-31)} \times 34 + \frac{30(13)(-18)}{33(16)(-15)} \times 38 + \frac{30(13)(-3)}{48(31)(15)} \times 42$$
$$= -0.782 + 6.532 + 33.682 - 2.202 = 37.23$$

Hence the desired root of $f(x) = 0$ is 37.23.

PROBLEMS 29.7

1. Apply Lagrange's method to find the value of x when $f(x) = 15$ from the given data :

x :	5	6	9	11
$f(x)$:	12	13	14	16

(Madras, 2000)

2. Obtain the value of t when $A = 85$ from the following table, using Lagrange's method :

t :	2	5	8	14
A :	94.8	87.9	81.3	68.7

29.14 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 29.8

Select the correct answer or fill up the blanks in the following problems :

1. Newton's backward interpolation formula is ......
2. Bessel's formula is most appropriate when p lies between

(a) – 0.25 and 0.25 (b) 0.25 and 0.75 (c) 0.75 and 1.00.

3. From the divided difference table for the following data :

x :	5	15	22
y :	7	36	160

4. Interpolation is the technique of estimating the value of a function for any ....
5. Bessel's formula for interpolation is ....
6. The 4th divided differences for x_0, x_1, x_2, x_3, x_4 = ...
7. Stirling's formula is best suited for p lying between ....
8. Newton's divided differences formula is ....
9. Given $(x_0, y_0), (x_1, y_1), (x_2, y_2)$, Lagrange's interpolation formula is ....
10. If $f(0) = 1$, $f(2) = 5$, $f(3) = 10$ and $f(x) = 14$, then $x = 0$ ....
11. Gauss forward interpolation formula involves
    (a) even differences above the central line and odd differences on the central line
    (b) even differences below the central line and odd differences on the central line
    (c) odd differences below the central line and even differences on the central line
    (d) odd differences above the central line and even differences on the central line.
12. If $y(1) = 4$, $y(3) = 12$, $y(4) = 19$ and $y(x) = 7$ find x using Lagrange's formula.
13. Extrapolation is defined as ....
14. The second divided difference of $f(x) = 1/x$, with arguments, a, b, c, is....... . *(Anna, 2007)*
15. Gauss-forward interpolation formula is used to interpolate values of y for
    (a) $0 < p < 1$ (b) $-1 < p < 0$
    (c) $0 < p < -\alpha$ (d) $-\alpha < p < 0$.
16. Given

x :	0	1	3	4
y :	-12	0	6	12

Using Lagrange's formula, a polynomial that can be fitted to the data is ....

17. The nth divided difference of a polynomial of degree n is
    (a) zero (b) a constant
    (c) a variable (d) none of these.
18. If h is the interval of differencing, $\Delta^2 x^3$ = ...