CHAPTER 30

# Numerical Differentiation & Integration

1. Numerical differentiation. 2. Formulae for derivatives. 3. Maxima and minima of a tabulated function. 4. Numerical integration. 5. Newton-Cotes quadrature formula. 6. Trapezoidal rule. 7. Simpson's 1/3rd rule. 8. Simpson's 3/8th rule. 9. Boole's rule. 10. Weddle's rule. 11. Objective Type of Questions.

## 30.1 NUMERICAL DIFFERENTIATION

It is the process of calculating the value of the derivative of a function at some assigned value of $x$ from the given set of values $(x_i, y_i)$. To compute $dy/dx$, we first replace the latter as many times as we desire. The choice of the interpolation formula to be used, will depend on the assigned value of $x$ at which $dy/dx$, is desired.

If the values of $x$ are equi-spaced and $dy/dx$, is required near the beginning of the table, we employ Newton's forward formula. If it is required near the end of the table, we use Newton's backward formula. For values near the middle of the table, $dy/dx$, is calculated by means of Stirling's or Bessel's formula. If the values of $x$ are not equi-spaced, we use Newton's divided difference formula to represent the function.

## 30.2 FORMULAE FOR DERIVATIVES

Consider the function $y = f(x)$ which is tabulated for the values $x_i\ (= x_0 + ih)$, $i = 0, 1, 2, \ldots n$.

**(1) Derivatives using forward difference formula.** Newton's forward interpolation formula (p. 958) is

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots$$

Differentiating both sides w.r.t. $p$, we have

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \ldots$$

Since $p = \dfrac{(x-x_0)}{h}$, therefore $\dfrac{dp}{dx} = \dfrac{1}{h}\cdot$

Now
$$\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \frac{4p^3-18p^2+22p-6}{4i}\Delta^4 y_0 + \cdots\right] \quad \ldots(1)$$

At $x = x_0, p = 0$. Hence putting $p = 0$,

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 + \cdots\right] \quad \ldots(2)$$

Again differentiating (1) w.r.t. $x$, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dp}\left(\frac{dy}{dp}\right)\frac{dp}{dx}$$

$$= \frac{1}{h}\left[\frac{2}{2!}\Delta^2 y_0 + \frac{6p-6}{3!}\Delta^3 y_0 + \frac{12p^2-36p+22}{4!}\Delta^4 y_0 + \cdots\right]\frac{1}{h}$$

Putting $p = 0$, we obtain

$$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 + \cdots\right] \quad ...(3)$$

Similarly $$\left(\frac{d^3y}{dx^3}\right)_{x_0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \cdots\right] \quad ...(4)$$

**Otherwise :** We know that $1 + \Delta = E = e^{hD}$

$\therefore$ $$hD = \log(1+\Delta) = \Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \ldots$$

or $$D = \frac{1}{h}\left[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \cdots\right]$$

and $$D^2 = \frac{1}{h^2}\left[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \cdots\right]^2 = \frac{1}{h^2}\left[\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 + \cdots\right]$$

and $$D^3 = \frac{1}{h^2}\left[\Delta^3 - \frac{3}{2}\Delta^4 + \cdots\right]$$

Now applying the above identities to $y_0$, we get

$Dy_0$, *i.e.*, $$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 + \cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 - \cdots\right]$$

and $$\left(\frac{d^3y}{dx^3}\right)_{x_0} = \frac{1}{h^3}\left[\Delta^2 y_0 - \frac{3}{2}\Delta^4 y_0 + \cdots\right]$$

which are the same as (2), (3) and (4) respectively.

**(2) Derivatives using backward difference formula.** Newton's backward interpolation formula (p. 958) is

$$y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots$$

Differentiating both sides w.r.t. $p$, we get

$$\frac{dy}{dp} = \nabla y_n + \frac{2p+1}{2!}\nabla^2 y_n + \frac{3p^2+6p+2}{3!}\nabla^3 y_n + \cdots$$

Since $p = \dfrac{x-x_n}{h}$, therefore $\dfrac{dp}{dx} = \dfrac{1}{h}\cdot$

Now $$\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx} = \frac{1}{h}\left[\nabla y_n \frac{2p+1}{2!}\nabla^2 y_n + \frac{3p^3+6p+2}{3!}\nabla^3 y_n + \cdots\right] \quad ...(5)$$

At $x = x_n$, $p = 0$. Hence putting $p = 0$, we get

$$\left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + \frac{1}{6}\nabla^6 y_n + \cdots\right] \quad ...(6)$$

Again differentiating (5) w.r.t. $x$, we have

$$\frac{d^2y}{dx^2} = \frac{d}{dp}\left(\frac{dy}{dx}\right)\frac{dp}{dx} = \frac{1}{h^2}\left[\nabla^2 y_n + \frac{6p+6}{3!}\nabla^3 y_n + \frac{6p^2+18p+11}{12}\nabla^4 y_n + \cdots\right]$$

Putting $p = 0$, we obtain

$$\left(\frac{d^2y}{dx^2}\right)_{x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \cdots\right] \quad ...(7)$$

Similarly, $$\left(\frac{d^3y}{dx^3}\right)_{x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \cdots\right] \quad ...(8)$$

**Otherwise :** We know that $1 - \nabla = E^{-1} = e^{-hD}$

$$\therefore \quad -hD = \log(1-\nabla) = -\left[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{1}{4}\nabla^4 + \cdots\right]$$

or $$D = \frac{1}{h}\left[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{1}{4}\nabla^4 + \cdots\right]$$

$$\therefore \quad D^2 = \frac{1}{h^2}\left[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \cdots\right]^2 = \frac{1}{h^2}\left[\nabla^2 + \nabla^3 + \frac{11}{12}\nabla^4 + \cdots\right]$$

Similarly, $$D^3 = \frac{1}{h^3}\left[\nabla^3 + \frac{3}{2}\nabla^4 + \cdots\right]$$

Applying these identities to $y_n$, we get

$Dy_n$, *i.e.*, $$\left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y^n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + \frac{1}{6}\nabla^6 y_n + \cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x_n} = \frac{1}{h^2}\left(\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \cdots\right)$$

and $$\left(\frac{d^3y}{dx^3}\right)_{x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \cdots\right]$$

which are the same as (6), (7) and (8).

**(3) Derivatives using central difference formulae.** Stirling's formula (p. 964) is

$$y_p = y_0 + \frac{p}{1!}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1^2)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{p^2(p^2-1^2)}{4!}\Delta^4 y_{-2} + \ldots$$

Differentiating both sides w.r.t. $p$., we get

$$\frac{dy}{dp} = \left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{2p}{2!}\Delta^2 y_{-1} + \frac{3p^2-1}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{4p^3-2p}{4!}\Delta^4 y_{-2} + \ldots$$

Since $p = \dfrac{x - x_0}{h}$, $\therefore \quad \dfrac{dp}{dx} = \dfrac{1}{h}$.

Now $$\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx} = \frac{1}{h}\left[\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + p\Delta^2 y_{-1} + \frac{3p^2-1}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{2p^3-p}{12}\Delta^4 y_{-2} + \cdots\right]$$

At $x = x_0$, $p = 0$. Hence putting $p = 0$, we get

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} - \frac{1}{6}\,\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} + \frac{1}{30}\,\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2} + \cdots\right] \quad ...(9)$$

Similarly $$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3} - \cdots\right] \quad ...(10)$$

**Obs.** We can similarly use any other interpolation formula for computing the derivatives.

**Example 30.1.** *Given that*

| $x$ : | 1.0 | 1.1 | 1.2 | 1.3 | 1.4 | 1.5 | 1.6 |
|---|---|---|---|---|---|---|---|
| $y$ : | 7.989 | 8.403 | 8.781 | 9.129 | 9.451 | 9.750 | 10.031 |

*find* $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ *at (a)* $x = 1.1$ (V.T.V., 2006; Madras, 2003 S)

(b) $x = 1.6$. (Rohtak, 2006; J.N.T.U., 2004 S)

**Solution.** (*a*) The difference table is :

| $x$ | $y$ | $\Delta$ | $\Delta^2$ | $\Delta^3$ | $\Delta^4$ | $\Delta^5$ | $\Delta^6$ |
|---|---|---|---|---|---|---|---|
| 1.0 | 7.989 | | | | | | |
| | | 0.414 | | | | | |
| 1.1 | 8.403 | | – 0.036 | | | | |
| | | 0.378 | | 0.006 | | | |
| 1.2 | 8.781 | | – 0.030 | | – 0.002 | | |
| | | 0.348 | | 0.004 | | 0.001 | |
| 1.3 | 9.129 | | – 0.026 | | – 0.001 | | 0.002 |
| | | 0.322 | | 0.003 | | 0.003 | |
| 1.4 | 9.451 | | – 0.023 | | 0.002 | | |
| | | 0.299 | | 0.005 | | | |
| 1.5 | 9.750 | | – 0.018 | | | | |
| | | 0.281 | | | | | |
| 1.6 | 10.031 | | | | | | |

We have

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^2 y_0 - \frac{1}{6}\Delta^6 y_0 + \cdots\right] \quad ...(i)$$

and

$$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 - \cdots\right] \quad ...(ii)$$

Here $h = 0.1$, $x_0 = 1.1$, $\Delta y_0 = 0.378$, $\Delta^2 y_0 = -0.03$ etc.

Substituting these values in (*i*) and (*ii*), we get

$$\left(\frac{dy}{dx}\right)_{1.1} = \frac{1}{0.1}\left[0.378 - \frac{1}{2}(-0.03) + \frac{1}{3}(0.004) - \frac{1}{4}(-0.001) + \frac{1}{5}(0.003)\right] = 3.952$$

$$\left(\frac{d^2y}{dx^2}\right)_{1.1} = \frac{1}{(0.1)^2}\left[-0.03 - (0.004) + \frac{11}{12}(-0.001) - \frac{5}{6}(0.003)\right] = -3.74$$

(*b*) We use the above difference table and the backward difference operator $\nabla$ instead of $\Delta$.

$$\left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^2 y_n + \frac{1}{6}\nabla^6 y_n + \cdots\right] \quad ...(i)$$

and

$$\left(\frac{d^2y}{dx^2}\right)_{x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \cdots\right] \quad ...(ii)$$

Here $h = 0.1$, $x_n = 1.6$, $\nabla y_n = 0.281$, $\nabla^2 y_n = -0.018$ etc.

Putting these values in (*i*) and (*ii*), we get

$$\left[\frac{dy}{dx}\right]_{1.6} = \frac{1}{0.1}\left[0.281 + \frac{1}{2}(-0.018) + \frac{1}{3}(0.005) + \frac{1}{4}(0.002) + \frac{1}{5}(0.003) + \frac{1}{6}(0.002)\right] = 2.75$$

$$\left(\frac{d^2y}{dx^2}\right)_{1.6} = \frac{1}{(0.1)^2}\left[-0.018 + 0.005 + \frac{11}{12}(0.002) + \frac{5}{6}(0.003) + \frac{137}{180}(0.002)\right]$$

$$= -0.715.$$

**Example 30.2.** *The following data gives the velocity of a particle for 20 seconds at an interval of 5 seconds. Find the initial acceleration using the entire data :*

| | | | | | | |
|---|---|---|---|---|---|---|
| *Time t (sec)* | : | 0 | 5 | 10 | 15 | 20 |
| *Velocity v (m/sec)* | : | 0 | 3 | 14 | 69 | 228 |

(Anna, 2004)

**Solution.** The difference table is :

| $t$ | $v$ | $\Delta v$ | $\Delta^2 v$ | $\Delta^3 v$ | $\Delta^4 v$ |
|---|---|---|---|---|---|
| 0 | 0 | | | | |
| | | 3 | | | |
| 5 | 3 | | 8 | | |
| | | 11 | | 36 | |
| 10 | 14 | | 44 | | 24 |
| | | 55 | | 60 | |
| 15 | 69 | | 104 | | |
| | | 159 | | | |
| 20 | 228 | | | | |

An initial acceleration $\left(i.e. \dfrac{dv}{dt}\right)$ at $t = 0$ is required, we use Newton's forward formula :

$$\left(\frac{dv}{dt}\right)_{t=0} = \frac{1}{h}\left(\Delta v_0 - \frac{1}{2}\Delta^2 v_0 + \frac{1}{3}\Delta^3 v_0 - \frac{1}{4}\Delta^4 v_0 + \cdots\right)$$

$$\therefore \quad \left(\frac{dv}{dt}\right)_{t=0} = \frac{1}{5}\left[3 - \frac{1}{2}(8) + \frac{1}{3}(36) - \frac{1}{4}(24)\right] = \frac{1}{5}\ (3 - 4 + 12 - 6) = 1$$

Hence the initial acceleration is 1 m/sec$^2$.

**Example 30.3.** *A slider in a machine moves along a fixed straight rod. Its distance x cm. along the rod is given below for various values of the time t seconds. Find the velocity of the slider and its acceleration when t = 0.3 seconds.*

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $t =$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 |
| $x =$ | 30.13 | 31.62 | 32.87 | 33.64 | 33.95 | 33.81 | 33.24 |

(V.T.U, 2009)

**Solution.** The difference table is :

| $t$ | $x$ | $\Delta$ | $\Delta^2$ | $\Delta^3$ | $\Delta^4$ | $\Delta^5$ | $\Delta^6$ |
|---|---|---|---|---|---|---|---|
| 0 | 30.13 | | | | | | |
| | | 1.49 | | | | | |
| 0.1 | 31.62 | | – 0.24 | | | | |
| | | 1.25 | | – 0.24 | | | |
| 0.2 | 32.87 | | – 0.48 | | 0.26 | | |
| | | 0.77 | | 0.02 | | – 0.27 | |
| 0.3 | 33.64 | | – 0.46 | | – 0.01 | | 0.29 |
| | | 0.31 | | 0.01 | | 0.02 | |
| 0.4 | 33.95 | | – 0.45 | | 0.01 | | |
| | | – 0.14 | | 0.02 | | | |
| 0.5 | 33.81 | | – 0.43 | | | | |
| | | – 0.57 | | | | | |
| 0.6 | 33.24 | | | | | | |

As the derivatives are required near the middle of the table, we use Stirling's formulae :

$$\left(\frac{dx}{dt}\right)_{t_0} = \frac{1}{h}\left(\frac{\Delta x_0 + \Delta x_{-1}}{2}\right) - \frac{1}{6}\left(\frac{\Delta^3 x_{-1} + \Delta^3 x_{-2}}{2}\right) + \frac{1}{30}\left(\frac{\Delta^5 x_{-2} + \Delta^5 x_{-3}}{2}\right) + \ldots \quad \ldots(i)$$

$$\left(\frac{d^2x}{dt^2}\right)_{t_0} = \frac{1}{h^2}\left[\Delta^2 x_{-1} - \frac{1}{12}\Delta^4 x_{-2} + \frac{1}{90}\Delta^6 x_{-3} \cdots\right] \quad ...(ii)$$

Here $h = 0.1$, $t_0 = 0.3$, $\Delta x_0 = 0.31$, $\Delta x_{-1} = 0.77$, $\Delta^2 x_{-1} = -0.46$ etc.

Putting these values in $(i)$ and $(ii)$, we get

$$\left(\frac{dx}{dt}\right)_{0.3} = \frac{1}{0.1}\left[\frac{0.31+0.77}{2} - \frac{1}{6}\left(\frac{0.01+0.02}{2}\right) + \frac{1}{30}\left(\frac{0.02-0.27}{2}\right) - \cdots\right] = 5.33$$

$$\left(\frac{d^2x}{dt^2}\right)_{0.3} = \frac{1}{(0.1)^2}\left[-0.46 - \frac{1}{12}(-0.01) + \frac{1}{90}(0.29) - \cdots\right] = -45.6$$

Hence the required velocity is 5.33 cm/sec and acceleration is $-45.6$ cm/sec$^2$.

**Example 30.4.** *Using Bessel's formula, find f′ (7.5) from the following table :*

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $x$ : | 7.47 | 7.48 | 7.49 | 7.50 | 7.51 | 7.52 | 7.53 | |
| $f(x)$ : | 0.193 | 0.195 | 0.198 | 0.201 | 0.203 | 0.206 | 0.208 | (J.N.T.U., 2006) |

**Solution.** Taking $x_0 = 7.50$, $h = 0.1$, we have $p = \dfrac{x - x_0}{h} = \dfrac{x - 7.50}{0.01}$

The difference table is :

| $x$ | $p$ | $y_p$ | $\Delta$ | $\Delta^2$ | $\Delta^3$ | $\Delta^4$ | $\Delta^5$ | $\Delta^6$ |
|---|---|---|---|---|---|---|---|---|
| 7.47 | −3 | 0.193 | | | | | | |
| | | | 0.002 | | | | | |
| 7.48 | −2 | 0.195 | | 0.001 | | | | |
| | | | 0.003 | | −0.001 | | | |
| 7.49 | −1 | 0.198 | | 0.000 | | 0.000 | | |
| | | | 0.003 | | −0.001 | | 0.003 | |
| 7.50 | 0 | 0.201 | | −0.001 | | 0.003 | | −0.01 |
| | | | 0.002 | | 0.002 | | −0.007 | |
| 7.51 | 1 | 0.203 | | 0.001 | | −0.004 | | |
| | | | 0.003 | | −0.002 | | | |
| 7.52 | 2 | 0.206 | | −0.001 | | | | |
| | | | 0.002 | | | | | |
| 7.53 | 3 | 0.208 | | | | | | |

Bessel's formula (p. 550) is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\cdot\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{\left(p-\frac{1}{2}\right)p(p-1)}{3!}\cdot\Delta^3 y_{-1}$$
$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\cdot\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \frac{\left(p-\frac{1}{2}\right)(p+1)p(p-1)(p-2)}{5!}\cdot\Delta^5 y_{-2}$$
$$+ \frac{(p+2)p(p+1)p(p-1)(p-2)(p-3)}{6!}\cdot\frac{\Delta^6 y_{-3} + \Delta^6 y_{-2}}{2} + \cdots \quad ...(i)$$

Since $p = \dfrac{x - x_0}{h}$, $\therefore \dfrac{dp}{dx} = \dfrac{1}{h}$ and $\dfrac{dy}{dx} = \dfrac{dy}{dp}\cdot\dfrac{dp}{dx} = \dfrac{1}{h}\dfrac{dy}{dp}$

Differentiating $(i)$ w.r.t. $p$ and putting $p = 0$, we get

$$\left(\frac{dy}{dx}\right)_{7.5} = \frac{1}{h}\left(\frac{dy}{dp}\right)_{p=0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{h}(\Delta^2 y_{-1} + \Delta^2 y_0) + \frac{1}{12}\Delta^3 y_{-1} + \frac{1}{24}(\Delta^4 y_{-2} + \Delta^4 y_{-1})\right.$$
$$\left. - \frac{1}{120}\Delta^5 y_{-2} - \frac{1}{240}(\Delta^6 y_{-3} + \Delta^6 y_{-2})\right]$$

$$\left(\frac{dy}{dx}\right)_{7.5} = \frac{1}{0.01}\left[0.002 - \frac{1}{4}(-0.001 + 0.001) + \frac{1}{12}(0.002)^1 + \frac{1}{24}(-0.004 + 0.003) - \frac{1}{120}(-0.007)\frac{-1}{240}(0.010 + 0)\right]$$

$[\because \quad \Delta^6 y_{-2} = 0]$

$$= 0.2 + 0 + 0.01666 - 0.00583 + 0.00416 = 0.223.$$

**Example 30.5.** *Find $f'(0)$ from the following data :*

| | | | | | |
|---|---|---|---|---|---|
| $x$ : | 3 | 5 | 11 | 27 | 34 |
| $f(x)$ : | – 13 | 23 | 899 | 17315 | 35606 |

**Solution.** As the values of $x$ are not equi-spaced, we shall use Newton's divided difference formula. The divided difference table is

| $x$ | $f(x)$ | 1st div. diff. | 2nd div. diff. | 3rd div. diff. | 4th div. diff. |
|---|---|---|---|---|---|
| 3 | – 13 | | | | |
| | | 18 | | | |
| 5 | 23 | | 16 | | |
| | | 146 | | 0.998 | |
| 11 | 899 | | 39.96 | | 0.0002 |
| | | 1025 | | 1.003 | |
| 27 | 17315 | | 69.04 | | |
| | | 2613 | | | |
| 34 | 35606 | | | | |

Fifth difference being zero, Newton's divided difference formula is

$$f(x) = f(x_0) + (x - x_0) f(x_0 - x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2)$$
$$+ (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) + (x - x_0)(x - x_1)$$
$$\times (x - x_2)(x - x_3) f(x_0, x_1, x_2\, x_3, x_4)$$

Differentiating it w.r.t. $x$, we get

$$f'(x) = f(x_0, x_1) + (2x - x_0 - x_1) f(x_0, x_1, x_2)$$
$$+ [3x^2 - 2x(x_0 + x_1 + x_2) + (x_0x_1 + x_1x_2 + x_2x_0)] \times f(x_0, x_1, x_2, x_3)$$
$$+ [4x^3 - 3x^2(x_0 + x_1 + x_2 + x_3) + 2x(x_0x_1 + x_1x_2 + x_2x_3 + x_3x_0 + x_1x_3 + x_0x_2)$$
$$- x_0x_1x_2 + x_1x_2x_3 + x_2x_3x_0 + x_0x_1x_3] f(x_0, x_1, x_2, x_3, x_4)$$

Putting $x_0 = 3, x_1 = 5, x_2 = 11, x_3 = 27$ and $x = 10$, we obtain

$$f'(x) = 18 + 12 \times 16 + 23 \times 0.998 - 426 \times 0.0002 = 232.869.$$

## 30.3 MAXIMA AND MINIMA OF A TABULATED FUNCTION

Newton's forward interpolation formula is

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots$$

Differentiating it w.r.t. $p$, we get

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2 - 6p + 2}{6}\Delta^3 y_0 + \ldots \qquad \ldots(1)$$

For maxima or minima, $dy/dp = 0$. Hence equating the right hand side of (1) to zero and retaining only upto third differences, we obtain

$$\Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2 - 6p + 2}{6}\Delta^3 y_0 = 0$$

*i.e.,*

$$\left(\frac{1}{2}\Delta^3 y_0\right)p^2 + (\Delta^2 y_0 - \Delta^3 y_0)p + \left(\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0\right) = 0$$

Substituting the values of $\Delta y_0$, $\Delta^2 y_0$, $\Delta^3 y_0$ from the difference table, we solve this quadratic for $p$. Then the corresponding values of $x = x_0 + ph$ at which $y$ is maximum or minimum.

**Example 30.6.** *Find the maximum and minimum value of y from the following data :*

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$ : | −2 | −1 | 0 | 1 | 2 | 3 | 4 |
| $y$ : | 2 | −0.25 | 0 | −0.25 | 2 | 15.75 | 56 |

(Anna, 2004)

**Solution.** The difference table is :

| $x$ | $y$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ | $\Delta^5 y$ |
|---|---|---|---|---|---|---|
| −2 | 2 | | | | | |
| | | −2.25 | | | | |
| −1 | −0.25 | | 2.5 | | | |
| | | 0.25 | | −3 | | |
| 0 | 0 | | −0.5 | | 6 | |
| | | −0.25 | | 3 | | 0 |
| 1 | −0.25 | | 2.5 | | 6 | |
| | | 0.25 | | 9 | | 0 |
| 2 | 2 | | 11.5 | | 6 | |
| | | 13.75 | | 15 | | |
| 3 | 15.75 | | 26.5 | | | |
| | | 40.25 | | | | |
| 4 | 56 | | | | | |

Taking $x_0 = 0$, we have $y_0 = 0$, $\Delta y_0 = -0.25$, $\Delta^2 y_0 = 2.5$, $\Delta^3 y_0 = 9$, $\Delta^4 y_0 = 6$.

Newton's forward difference formula for the first derivative gives

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 - \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 - \frac{4p^3-18p^2+22p-6}{4!}\Delta^4 y_0 - \cdots\right]$$

$$= \frac{1}{1}\left[-0.25 + \frac{2x-1}{2}(2.5) + \frac{1}{6}(3x^2-6x+2)(9) + \frac{1}{24}(4x^2-18x^2+22x-6)(6)\right]$$

$$= \frac{1}{1}[-0.25 + 2.5x - 1.25 + 4.5x^2 - 9x + 3 + x^3 - 4.5x^2 + 5.5x - 1.5] = x^3 - x$$

For $y$ to be maximum or minimum, $\frac{dy}{dx} = 0$ *i.e.*, $x^3 - x = 0$

*i.e.*, $x = 0, 1, -1$

Now $\frac{d^2y}{dx^2} = 3x^2 - 1 = -\text{ve for } x = 0$

$= +\text{ve for } x = 1$

$= +\text{ve for } x = -1$

Since $y = y_0 + x\Delta y_0 + \frac{x(x-1)}{2!}\Delta^2 y_0 + \ldots, y(0) = 0$

Thus $y$ is maximum for $x = 0$, and maximum value $= y(0) = 0$.

Also $y$ is minimum for $x = 1$ and minimum value $= y(0) = -0.25$.

## PROBLEMS 30.1

1. Find $y'(0)$ and $y''$ from the following table :

| | | | | | | |
|---|---|---|---|---|---|---|
| $x$ : | 0 | 1 | 2 | 3 | 4 | 5 |
| $y$ : | 4 | 8 | 15 | 7 | 6 | 2 |

2. Find the first and second derivatives of $f(x)$ at $x = 1.5$ if

| | | | | | | |
|---|---|---|---|---|---|---|
| $x$ : | 1.5 | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 |
| $f(x)$ : | 3.375 | 7.000 | 13.625 | 24.000 | 38.875 | 59.000 |

(S.V.T.U., 2007)

3. Find the first and second derivatives of the function tabulated below, at the point $x = 1.1$ :

| | | | | | | |
|---|---|---|---|---|---|---|
| $x$ : | 1.0 | 1.2 | 1.4 | 1.6 | 1.8 | 2.0 |
| $y$ : | 0 | 0.128 | 0.544 | 1.296 | 2.432 | 4.000 |

(U.P.T.U., 2010; Bhopal, 2009)

4. Given the following table of values of $x$ and $y$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$ : | 1.00 | 1.05 | 1.10 | 1.15 | 1.20 | 1.25 | 1.30 |
| $y$ : | 1.000 | 1.025 | 1.049 | 1.072 | 1.095 | 1.118 | 1.140 |

find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at (a) $x = 1.05$ (b) $x = 1.25$ (c) $x = 1.15$. (V.T.U., 2008)

5. For the following values of $x$ and $y$, find the first derivative at $x = 4$.

| | | | | | |
|---|---|---|---|---|---|
| $x$ : | 1 | 2 | 4 | 8 | 10 |
| $y$ : | 0 | 1 | 5 | 21 | 27 |

(J.N.T.U., 2009)

6. From the following table, find the values of $dy/dx$ and $d^2y/dx^2$ at $x = 2.03$.

| | | | | | |
|---|---|---|---|---|---|
| $x$ : | 1.96 | 1.98 | 2.00 | 2.02 | 2.04 |
| $y$ : | 0.7825 | 0.7739 | 0.7651 | 0.7563 | 0.7473 |

(Anna, 2005)

7. Find the value of cos 1.74 from the following table :

| | | | | | |
|---|---|---|---|---|---|
| $x$ : | 1.7 | 1.74 | 1.78 | 1.82 | 1.86 |
| $\sin x$ : | 0.9916 | 0.9857 | 0.9781 | 0.9691 | 0.9584 |

(J.N.T.U., 2009)

8. The distance covered by an athlete for the 50 metre is given in the following table :

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| *Time (sec)* : | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| *Distance (metre)* : | 0 | 2.5 | 8.5 | 15.5 | 24.5 | 36.5 | 50 |

Determine the speed of the athlete at $t = 5$ sec. correct to two decimals. (U.P.T.U., 2009)

9. The following data gives corresponding values of pressure and specific volume of a superheated stream.

| | | | | | |
|---|---|---|---|---|---|
| $v$ : | 2 | 4 | 6 | 8 | 10 |
| $p$ : | 105 | 42.7 | 25.3 | 16.7 | 13 |

Find the rate of change of

(i) pressure with respect to volume when $v = 2$,

(ii) volume with respect to pressure when $p = 105$.

10. The table below reveals the velocity $v$ of a body during the specific time $t$, find its acceleration at $t = 1.1$ ?

| | | | | | |
|---|---|---|---|---|---|
| $t$ : | 1.0 | 1.1 | 1.2 | 1.3 | 1.4 |
| $v$ : | 43.1 | 47.7 | 52.1 | 56.4 | 60.8 |

(J.N.T.U., 2009)

11. The elevation above a datum line of 7 points of a road is given below :

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$ : | 0 | 300 | 600 | 900 | 1200 | 1500 | 1800 |
| $y$ : | 135 | 149 | 157 | 183 | 201 | 205 | 193 |

Find the gradient of the road at the middle point.

12. A rod is rotating in a plane. The following table gives the angle θ (radians) through which the rod has turned for various values of the time $t$ second.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $t$ : | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 | 1.2 |
| θ : | 0 | 0.12 | 0.49 | 1.12 | 2.02 | 3.20 | 4.67 |

Calculate the angular velocity and the angular acceleration of the rod, when $t = 0.6$ second. (V.T.U., 2004)

13. Find the value of $f'(x)$ at $x = 0.4$ from the following table using Bessel's formula

| | | | | | | |
|---|---|---|---|---|---|---|
| $x$ : | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 |
| $f(x)$ : | 0.1023 | 0.1047 | 0.1071 | 0.1096 | 0.1122 | 0.1148 |

14. If $y = f(x)$ and $y_n$ denotes $f(x_0 + nh)$, prove that, if powers of $h$ above $h^6$ be neglected.

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{3}{4h}\left[(y_1 - y_{-1}) - \frac{1}{5}(y_2 - y_{-2}) + \frac{1}{45}(y_3 - y_{-3})\right]$$ (U.P.T.U., 2006)

[**Hint:** Differentiate Stirling's formula w.r.t. $x$, and put $x = 0$]

15. Find the value of $f'(8)$ from the table given below :

| | | | | |
|---|---|---|---|---|
| $x$ : | 6 | 7 | 9 | 12 |
| $f(x)$ : | 1.556 | 1.690 | 1.908 | 2.158 |

(Anna, 2007)

16. Find the $f'(6)$ from the following data :

| | | | | | | |
|---|---|---|---|---|---|---|
| $x$ : | 0 | 2 | 3 | 4 | 7 | 8 |
| $f(x)$ : | 4 | 26 | 58 | 112 | 466 | 922 |

(J.N.T.U., 2009 ; U.P.T.U., 2008)

17. Find the maximum and minimum values of $y$ from the following table :

| | | | | | | |
|---|---|---|---|---|---|---|
| $x$ : | 0 | 1 | 2 | 3 | 4 | 5 |
| $f(x)$ : | 0 | 0.25 | 0 | 2.25 | 16 | 56.25 |

18. Find the value of $x$ for which $f(x)$ is minimum, using the table

| | | | | | | |
|---|---|---|---|---|---|---|
| $x$ : | 9 | 10 | 11 | 12 | 13 | 14 |
| $f(x)$ : | 1330 | 1340 | 1320 | 1250 | 1120 | 930 |

Also find the maximum value of $f(x)$ ?

## 30.4 NUMERICAL INTEGRATION

The process of evaluating a definite integral from a set of tabulated values of the integrand $f(x)$ is called *numerical integration*. This process when applied to a function of a single variable, is known as *quadrature*.

The problem of numerical integration, like that of numerical differentiation, is solved by representing $f(x)$ by an interpolation formula and then integrating it between the given limits. In this way, we can derive quadrature formula for approximate integration of a function defined by a set of numerical values only.

## 30.5 NEWTON-COTES QUADRATURE FORMULA

Let $$I = \int_a^b f(x)\,dx$$

where $f(x)$ takes the values $y_0, y_1, y_2, \ldots y_n$ for $x = x_0, x_1, x_2, \ldots x_n$. (Fig. 30.1)

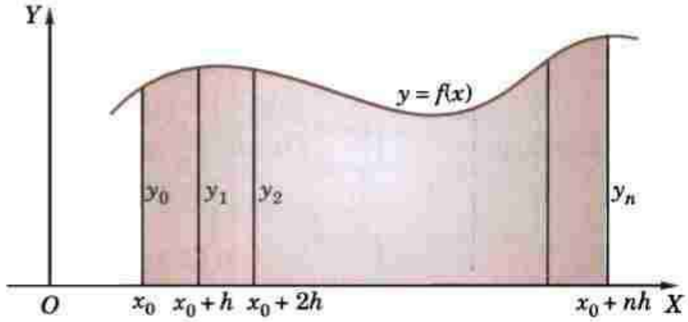


Fig. 30.1

Let us divide the interval $(a, b)$ into $n$ sub-intervals of width $h$ so that $x_0 = a$, $x_1 = x_0 + h$, $x_2 = x_0 + 2h, \ldots x_n = x_0 + nh = b$. Then

$$I = \int_{x_0}^{x_0+nh} f(x)\,dx = h\int_0^n f(x_0 + rh)\,dr,$$

putting $x = x_0 + rh$, $dx = hdr$

$$= h\int_0^n \Big[ y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0 + \frac{r(r-1)(r-2)(r-3)(r-4)}{5!}\Delta^5 y_0 + \frac{r(r-1)(r-2)(r-3)(r-4)(r-5)}{6!}\Delta^6 y_0 + \cdots \Big] dr$$

[By Newton's forward interpolation formula]

Integrating term by, we obtain

$$\int_{x_0}^{x_0+nh} f(x)\,dx = nh\Big[ y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \left(\frac{n^4}{5} - \frac{3n^3}{2} + \frac{11n^2}{3} - 3n\right)\frac{\Delta^4 y_0}{4!} + \left(\frac{n^5}{6} - 2n^4 + \frac{35n^3}{4} - \frac{50n^2}{3} + 12n\right)\frac{\Delta^5 y_0}{5!} + \left(\frac{n^6}{7} - \frac{15n^5}{6} + 17n^4 - \frac{225n^3}{4} + \frac{274n^2}{3} - 60n\right)\frac{\Delta^6 y_0}{6!} + \cdots \Big] \quad \ldots(A)$$

This is known as *Newton-Cotes quadrature formula*. From this general formula, we deduce the following important quadrature rules by taking $n = 1, 2, 3 \ldots$

## 30.6 TRAPEZOIDAL RULE

Putting $n = 1$ in $(A)$ § 30.5 and taking the curve through $(x_0, y_0)$ and $(x_1, y_1)$ as a straight line *i.e.* a polynomial of first order so that differences of order higher than first become zero, we get

$$\int_{x_0}^{x_0+h} f(x)\,dx = h\left(y_0 + \frac{1}{2}\Delta y_0\right) = \frac{h}{2}(y_0 + y_1)$$

Similarly $$\int_{x_0}^{x_0+2h} f(x)\,dx = h\left(y_1 + \frac{1}{2}\Delta y_1\right) = \frac{h}{2}(y_1 + y_2)$$

..................................................

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx = \frac{h}{2}(y_{(n-1)} + y_n)$$

Adding these $n$ integrals, we obtain

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})]$$

This is known as the ***trapezium rule.***

**Obs.** The area of each strip (trapezium) is found separately. Then the area under the curve and the *ordinates* at $x_0$ and $x_0 + nh$ is approximately equal to the areas of the trapeziums.

## 30.7 SIMPSON'S ONE-THIRD RULE

Putting $n = 2$ in (A) above and taking the curve through $(x_0, y_0)$, $(x_1, y_1)$ and $(x_2, y_2)$ as a parabola *i.e.*, a polynomial of second order so that differences of order higher than second vanish, we get

$$\int_{x_0}^{x_0+2h} f(x)\,dx = 2h\left(y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right)\frac{h}{3}(y_0 + 4y_1 + y_2)$$

Similarly, $$\int_{x_0+2h}^{x_0+nh} f(x)\,dx = \frac{h}{3}(y_2 + 4y_3 + y_4) \qquad \text{when}$$

..................................................

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)\,dx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n),\ n \text{ being even.}$$

Adding all these integrals, we have (when $n$ is even)

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3) + \dots + y_{n-1}) + 2(y_2 + y_4 + \dots + y_{n-2})]$$

This is known as the *Simpson's one-third rule* or simply *Simpson's rule* and is most commonly used.

**Obs.** While applying *Simpson's 1/3rd rule, the given interval must be divided into even number of equal subintervals,* since we find the area of two strips at a time.

## 30.8 SIMPSON'S THREE-EIGHTH RULE

Putting $n = 3$ in (A) above and taking the curve through $(x_i, y_i)$: $i = 0, 1, 2, 3$ as a polynomial of third order so that differences above the third order vanish, we get

$$\int_{x_0}^{x_0+3h} f(x)\,dx = 3h\left(y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right)$$

$$= \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3)$$

Similarly,

$$\int_{x_0+3h}^{x_0+nh} f(x)\,dx = \frac{3h}{8}(y_3 + 3y_4 + 3y_5 + y_6) \text{ and so on.}$$

Adding all such expressions from $x_0$ to $x_0 + nh$, where $n$ is a multiple of 3, we obtain

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \dots + y_{n-1}) + 2(y_3 + y_6 + \dots + y_{n-3}]$$

which is known as *Simpson's three-eighth rule.*

**Obs.** While applying *Simpson's 3/8th rule, the number of sub-intervals should be taken as multiple of 3.*

## 30.9 BOOLE'S RULE

Putting $n = 4$ in (A) above and neglecting all differences above the fourth, we obtain

$$\int_{x_0}^{x_0+4h} f(x)\,dx = 4h\left(y_0 + 2\Delta y_0\,\frac{5}{3}\,\Delta^2 y_0 + \frac{2}{3}\,\Delta^3 y_0 + \frac{7}{90}\,\Delta^4 y_0\right)$$

$$= \frac{2h}{45}(7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_8)$$

Similarly

$$\int_{x_0+4h}^{x_0+8h} f(x)\,dx = \frac{2h}{45}(7y_4 + 32y_5 + 12y_6 + 32y_7 + 7y_8) \text{ and so on.}$$

Adding all these integrals from $x_0$ to $x_0 + nh$, where $n$ is a multiple of 4, we get

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{2h}{45}(7y_0 + 32y_1 + 12y_2 + 32y_3 + 14y_4 + 32y_5 + 12y_6 + 32y_7 + 14y_8 + \ldots$$

This is known as *Boole's rule*.

**Obs.** While applying *Boole's rule*, the number of sub-intervals should be taken as a multiple of 4.

## 30.10 WEDDLE'S RULE

Putting $n = 6$ in (A) above and neglecting all differences above the sixth, we obtain

$$\int_{x_0}^{x_0+6h} f(x)\,dx = \left(y_0 + 3\Delta y_0 + \frac{9}{2}\,\Delta^2 y_0 + 4\Delta^3 y_0 + \frac{123}{60}\,\Delta^4 y_0 + \frac{11}{20}\,\Delta^5 x_0 + \frac{1}{6}\cdot\frac{41}{140}\,\Delta^6 y_0\right)$$

If we replace $\frac{41}{140}\,\Delta^6 y_0$ by $\frac{3}{10}\,\Delta^6 y_0$, the error made will be negligible.

$$\therefore \qquad \int_{x_0}^{x_0+6h} f(x)\,dx = \frac{3h}{10}(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6)$$

Similarly

$$\int_{x_0+6h}^{x_0+12h} f(x)\,dx = \frac{3h}{10}(y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}) \text{ and so on.}$$

Adding all these integrals from $x_0$ to $x_0 + nh$, where $n$ is a multiple of 6, we get

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{10}(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + \ldots)$$

This is known as *Weddle's rule*.

**Obs.** While applying *Weddle's rule the number of sub-intervals should be taken as a multiple of 6*. Weddle's rule is generally more accurate than any of the others. Of the two Simpson rules, the 1/3 rule is better.

**Example 30.7.** *Evaluate* $\int_0^6 \frac{dx}{1+x^2}$ *by using (i) Trapezoidal rule,*

*(i) Simpson's 1/3 rule,* *(Mumbai, 2005)*
*(ii) Simpson's 3/8 rule,* *(J.N.T.U., 2008)*
*(iii) Weddle's rule and compare the results with its actual value.* *(V.T.U., 2008)*

**Solution.** Divide the interval (0, 6) into six parts each of width $h = 1$. The values of $f(x) = \frac{1}{1+x^2}$ are given below :

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $f(x)$ $= y$ | 1 $y_0$ | 0.5 $y_1$ | 0.2 $y_2$ | 0.1 $y_3$ | 0.05884 $y_4$ | 0.0385 $y_5$ | 0.027 $y_6$ |

(*i*) By Trapezoidal rule,

$$\int_0^6 \frac{1}{1+x^2} = \frac{h}{2}[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{1}{2}[(1 + 0.027) + 2(0.5 + 0.2 + 0.1 + 0.0588 + 0.0385)] = 1.4108.$$

(*ii*) By Simpson's 1/3 rule,

$$\int_0^6 \frac{1}{1+x^2} = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{1}{3}[(1 + 0.027) + 4(0.5 + 0.1 + 0.0385) + 2(0.2 + 0.0588)] = 1.3662.$$

(*iii*) By Simpson's 3/8 rule,

$$\int_0^6 \frac{1}{1+x^2} = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3}{8}[(1 + 0.027) + 3(0.5 + 0.2 + 0.0588 + 0.0385) + 2(0.1)] = 1.3571.$$

(*iv*) By Weddle's rule,

$$\int_0^6 \frac{1}{1+x^2} = \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4\ 5y_5 + y_6]$$

$$= 0.3[1 + 5(0.5) + 0.2 + 6(0.1) + 0.0588 + 5(0.0385) + 0.027] = 1.3735.$$

Also, $\int_0^6 \frac{dx}{1+x^2} = \left|\tan^{-1} x\right|_0^6 = 1.4056$

**Obs.** This shows that the value of the integral found by Weddle's rule is the nearest to the actual value followed by its value given by Simpson's 1/3rd.

**Example 30.8.** *Use the Trapezoidal rule to estimate the integral* $\int_0^2 e^{x^2}\,dx$ *taking 10 intervals.*

*(U.P.T.U., 2008)*

**Solution.** Let $y = e^{x^2}$, $h = 0.2$ and $n = 10$.

The values of $x$ and $y$ are as follows :

| $x$ : | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 | 1.2 | 1.4 | 1.6 | 1.8 | 2.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $y$ : | 1 | 1.0408 | 1.1735 | 1.4333 | 1.8964 | 2.1782 | 4.2206 | 7.0993 | 12.9358 | 25.5337 | 54.5981 |
| | $y_0$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $y_6$ | $y_7$ | $y_8$ | $y_9$ | $y_{10}$ |

By Trapezoidal rule, we have

$$\int_0^1 e^{x^2} dx = \frac{h}{2}[(y_0 + y_{10}) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9)]$$

$$= \frac{0.2}{2}[(1 + 54.5981) + 2(1.0408 + 1.1735 + 1.4333 + 1.8964$$
$$+ 2.178 + 4.2206 + 7.0993 + 12.9358 + 25.5337)]$$

Hence $\int_0^2 e^{x^2}\,dx = 17.0621.$

**Example 30.9.** *Use Simpson's 1/3rd rule to find* $\int_0^{0.6} e^{-x^2}\,dx$ *by taking seven ordinates.*

*(V.T.U., 2011 ; Bhopal, 2009)*

**Solution.** Divide the interval (0, 0.6) into six parts each of width $h = 0.1$. The values of $y = f(x) = e^{-x^2}$ are given below :

| $x$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 |
|---|---|---|---|---|---|---|---|
| $x^2$ | 0 | 0.01 | 0.04 | 0.09 | 0.16 | 0.25 | 0.36 |
| $y$ | 1 | 0.9900 | 0.9608 | 0.9139 | 0.8521 | 0.7788 | 0.6977 |
| | $y_0$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $y_6$ |

By Simpson's 1/3rd rule, we have

$$\int_0^{0.6} e^{-x^2} dx = \frac{h}{3} [(y_0 + y_6) + 4(y_1 + y_3 + y_5 + 2(y_2 + y_4)]$$

$$= \frac{0.1}{3} [(1 + 0.6977) + 4(0.99 + 0.9139 + 0.7788 + 2(0.9608 + 0.8521)]$$

$$= \frac{0.1}{3} [1.6977 + 10.7308 + 3.6258] = \frac{0.1}{3} (16.0543) = 0.5351.$$

**Example 30.10.** *Compute the value of* $\int_{0.2}^{1.4} (\sin x - \log x + e^x)\, dx$ *using Simpson's* $\frac{3}{8}$*th rule.*

*(Mumbai, 2005)*

**Solution.** Let $y = \sin x - \log_e x + e^x$ and $h = 0.2$, $n = 6$.

The values of $y$ are as given below :

| $x$ : | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 | 1.2 | 1.4 |
|---|---|---|---|---|---|---|---|
| $y$ : | 3.0295 | 2.7975 | 2.8976 | 3.1660 | 3.5597 | 4.0698 | 4.4042 |
| | $y_0$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $y_6$ |

By Simpson's $\frac{3}{8}$th rule, we have

$$\int_{0.2}^{1.4} y\, dx = \frac{3h}{8} [(y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5)]$$

$$= \frac{3}{8}(0.2) [7.7336 + 2(3.1660) + 3\,(13.3247)] = 4.053$$

Hence $\int_{0.2}^{1.4} (\sin x - \log_e x + e^x)\, dx = 4.053.$

**Obs. Applications of Simpson's rule.** If the various ordinates in §30.5 represent equispaced cross-sectional areas, then Simpson's rule gives the volume of the solid. As such, Simpson's rule is very useful to civil engineers for calculating the amount of earth that must be moved to fill a depression or make a dam. Similar if the ordinates denote velocities at equal intervals of time, the Simpson's rule gives the distance travelled. The following examples illustrate these applications.

**Example 30.11.** *The velocity v (km/min) of a moped which starts from rests, is given at fixed intervals of time t (min) as follows :*

| *t* : | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|
| *y* : | 10 | 18 | 25 | 29 | 32 | 20 | 11 | 5 | 2 | 0 |

*Estimate approximately the distance covered in 20 minutes.*

**Solution.** If $s$ km be the distance covered in $t$ (min), then $\frac{ds}{dt} = v$

$$\therefore \qquad \Big| s \Big|_{t=0}^{20} = \int_0^{20} v\, dt = \frac{h}{3} [X + 4.O + 2E], \text{ by Simpson's rule}$$

Hence $h = 2$, $v_0 = 0$, $v_1 = 10$, $v_2 = 18$, $v_3 = 25$ etc.

$$\therefore \qquad X = v_0 + v_{10} = 0 + 0 = 0$$

$$O = v_1 + v_3 + v_5 + v_7 + v_9 = 10 + 25 + 32 + 11 + 2 = 80$$
$$E = v_2 + v_4 + v_6 + v_8 = 18 + 29 + 20 + 5 = 72$$

Hence the required distance $= \Big| s \Big|_{t=0}^{20} = \frac{2}{3}(0 + 4 \times 80 + 2 \times 72)$

$= 309.33$ km.

**Example 30.12.** *The velocity v of a particle at distance s from a point on its linear path is given by the following table :*

| *s (m) :* | *0* | *2.5* | *5.0* | *7.5* | *10.0* | *12.5* | *15.0* | *17.5* | *20.0* |
|---|---|---|---|---|---|---|---|---|---|
| *v (m/sec) :* | *16* | *19* | *21* | *22* | *20* | *17* | *13* | *11* | *9* |

*Estimate the time taken by the particle to traverse the distance of 20 metres, using Boole's rule.*
*)* *(U.P.T.U. 2007)*

**Solution.** If $t$ sec be the time taken to traverse a distance $s$ (m) then $\frac{ds}{dt} = v$

or $$\frac{dt}{ds} = \frac{1}{v} = y \text{ (say)},$$

$\therefore$ then $$\Big| t \Big|_{s=0}^{s=20} = \int_0^{20} y\,ds$$

Here $h = 2.5$ and $n = 8$

Also $y_0 = \frac{1}{16}, y_1 = \frac{1}{19}, y_2 = \frac{1}{4}, y_3 = \frac{1}{22}, y_4 = \frac{1}{20}, y_5 = \frac{1}{17}, y_6 = \frac{1}{13}, y_7 = \frac{1}{11}, y_8 = \frac{1}{9}$

$\therefore$ by Boole's Rules, we have

$$\Big| t \Big|_{s=0}^{s=20} = \int_0^{20} y\,ds = \frac{2h}{45}\,[7y_0 + 32y_1 + 12y_2 + 32y_3 + 14y_4 + 32y_5 + 12y_6 + 32y_7 + 14y_8]$$

$$= \frac{2(2.5)}{45}\left[7\left(\frac{1}{16}\right) + 32\left(\frac{1}{19}\right) + 12\left(\frac{1}{21}\right) + 32\left(\frac{1}{22}\right) + 14\left(\frac{1}{20}\right) + 32\left(\frac{1}{17}\right)\right.$$
$$\left. + 12\left(\frac{1}{13}\right) + 32\left(\frac{1}{11}\right) + 14\left(\frac{1}{9}\right)\right]$$

$$= \frac{1}{9}(12.11776) = 1.35$$

Hence the required time = 1.35 sec.

**Example 30.13.** *A solid of revolution is formed by rotating about the x-axis, the area between the x-axi the lines x = 0 and x = 1 and a curve through the points with the following co-ordinates*

| | | | | | |
|---|---|---|---|---|---|
| *x :* | *0.00* | *0.25* | *0.50* | *0.75* | *1.00* |
| *y :* | *1.0000* | *0.9896* | *0.9589* | *0.9089* | *0.8415* |

*Estimate the volume of the solid formed using Simpson's rule.* *(Raipur, 200*

**Solution.** Here $h = 0.25, y_0 = 1, y_1 = 0.9896, y_2 = 0.9589$, etc.

$\therefore$ Required volume of the solid generated

$$= \int_0^1 \pi y^2\,dx = \pi \cdot \frac{h}{3}\,[(y_0^2 + y_4^2) + 4(y_1^2 + y_3^2) + 2y_2^2]$$

$$= 0.25\,\frac{\pi}{3}\,[\{1 + (0.8415)^2\} + 4\{(0.9896)^2 + (0.9089)^2\} + 2(0.0589)^2]$$

$$= \frac{0.25 \times 3.1416}{3}\,[1.7081 + 7.2216 + 1.839] = 0.2618\,(10.7687) = 2.8192.$$

## PROBLEMS 30.2

1. Evaluate $\int_0^1 \frac{dx}{1+x}$ applying
   (*i*) Trapezoidal rule (*J.N.T.U., 2009*) (*ii*) Simpson's 1/3rd rule
   (*iii*) Simpson's 3/8th rule. (*Mumbai, 2004*)

2. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using (*i*) Trapezoidal rule taking $h = 1/4$
   (*ii*) Simpson's 1/3rd rule taking $h = 1/4$. (*J.N.T.U., 2008*)
   (*iii*) Simpson's 3/8th rule taking $h = 1/6$. (*U.P.T.U, 2010 ; V.T.U., 2007*)
   (*iv*) Weddle's rule taking $h = 1/6$. (*Bhopal, 2009*)
   Hence compute an approximate value of $\pi$ in each case.

3. Find an approximate value of $\log_e 5$ by calculating to 4 decimal places, by Simpson's 1/3 rule, $\int_0^5 \frac{dx}{4x+5}$, dividing the range into 10 equal parts. (*Anna., 2005*)

4. Evaluate $\int_0^6 x \sec x \, dx$ using eight intervals by Trapezoidal rule. (*U.P.T.U., 2009*)

5. Evaluate using Simpson's $\frac{1}{3}$rd rule (*i*) $\int_0^6 \frac{e^x}{1+x} dx$ (*U.P.T.U., 2006*)
   (*ii*) $\int_0^2 e^{-x^2} dx$ (Take $h = 0.25$). (*J.N.T.U., 2007*)

6. Evaluate using Simpson's 1/3rd rule $\int_0^1 \frac{dx}{x^3+x+1}$, choose step length 0.25. (*U.P.T.U., 2009*)

7. Evaluate using Simpson's 1/3rd rule, (*i*) $\int_0^\pi \sin x \, dx$ using 11 ordinates.
   (*ii*) $\int_0^{\pi/2} \sqrt{\cos\theta} \, d\theta$ taking 9 ordinates. (*V.T.U., 2009*)

8. Evaluate correct to 4 decimal places, by Simpson's $\frac{3}{8}$th rule
   (*i*) $\int_0^9 \frac{dx}{1+x^3}$ (*U.P.T.U., M. Tech., 2010*) (*ii*) $\int_0^{\pi/2} e^{\sin x} dx$ (*U.P.T.U., 2007*)

9. Given that

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$ : | 4.0 | 4.2 | 4.4 | 4.6 | 4.8 | 5.0 | 5.2 |
| $\log x$ : | 1.3863 | 1.4351 | 1.4816 | 1.5261 | 1.5686 | 1.6094 | 1.6487 |

   evaluate $\int_{[illegible]}^{5.2} \log x$ [illegible] by
   (*a*) Trapez[illegible]idal r[illegible] (*b*) Simpson's 1/3rd rule, (*Kerala, 2003*)
   (*c*) Simp[illegible] 3/8 [illegible] [illegible]T.U., 2006) (*d*) Weddle's rule (*V.T.U., 2008*)

10. Use Boo[illegible] compute $\int_0^{\pi/2} \sqrt{\sin x} \, dx$. (*U.P.T.U., 2008*)

11. The [illegible] $(t)$ as a function of time :

| | | | | |
|---|---|---|---|---|
| $t$ | [illegible] | 5 | 6 | 7 |
| $f(t)$ | [illegible] | 78 | 70 | 60 |

   Using Sim[illegible] (*J.N.T.U., 2007*)

12. A curve i[illegible] the following table :

| | | | |
|---|---|---|---|
| $x$ : | [illegible] | 3.5 | 4 |
| $y$ : | [illegible] | 2.6 | [illegible]1 |

   Estimate the area bounded by the cu[illegible] [illegible] = 4. (*Bhopal, 2007*)

13. A river is 80 ft wide. The depth $d$ in feet at a distance $x$ ft. from one bank is given by the following table :

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $x$ : | 0 | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 |
| $y$ : | 0 | 4 | 7 | 9 | 12 | 15 | 14 | 8 | 3 |

Find approximately the area of the cross-section. (*Rohtak, 2005*)

14. A curve is drawn to pass through the points given by following table :

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$ : | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 |
| $y$ : | 2 | 2.4 | 2.7 | 2.8 | 3 | 2.6 | 2.1 |

Using Weddle's rule, estimate the area bounded by the curve, the $x$-axis and the lines $x = 1, x = 4$.(*V.T.U., 2011 S*)

15. A body is in the form of a solid of revolution. The diameter $D$ in cms of its sections at distances $x$ cm. from the one end are given below. Estimate the volume of the solid.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$ : | 0 | 2.5 | 5.0 | 7.5 | 10.0 | 12.5 | 15.0 |
| $D$ : | 5 | 5.5 | 6.0 | 6.75 | 6.25 | 5.5 | 4.0 |

16. The velocity $v$ of a particle at distances $s$ from a point on its path is given by the table :

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $s$ ft : | 0 | 10 | 20 | 30 | 40 | 50 | 60 |
| $v$ ft/sec : | 47 | 58 | 64 | 65 | 61 | 52 | 38 |

Estimate the time taken to travel 60 ft. by using Simpson's 1/3 rule. (*U.P.T.U., 2007*)
Compare the result with Simpson's 3/8 rule. (*Madras, 2003*)

17. The following table gives the velocity $v$ of a particle at time $t$ :

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $t$ (second) : | 0 | 2 | 4 | 6 | 8 | 10 | 12 |
| $v$ (m/sec) : | 4 | 6 | 16 | 34 | 60 | 94 | 136 |

Find the distance moved by the particle in 12 seconds and also the acceleration at $t = 2$ sec. (*S.V.T.U., 2007*)

18. A rocket is launched from the ground. Its acceleration is registered during the first 80 seconds and is given in the table below. Using Simpson's $\frac{1}{3}$ rd rule, find the velocity of the rocket at $t = 80$ seconds.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $t$ sec : | 0 | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 |
| $f$ (cm/sec$^2$) : | 30 | 31.63 | 33.34 | 35.47 | 37.75 | 40.33 | 43.25 | 46.69 | 50.67 |

(*Mumbai, 2004*)

19. A reservoir discharging water through sluices at a depth $h$ below the water surface has a surface area $A$ for various values of $h$ as given below :

| | | | | | |
|---|---|---|---|---|---|
| $h$ (ft.) : | 10 | 11 | 12 | 13 | 14 |
| $A$ (sq.ft.) : | 950 | 1070 | 1200 | 1350 | 1530 |

If $t$ denotes time in minutes, the rate of fall of the surface is given by $dh/dt = -48\sqrt{h}/A$.
Estimate the time taken for the water level to fall from 14 to 10 ft. above the sluices.

## 30.12 OBJECTIVE TYPE OF QUESTIONS

### PROBLEMS 30.3

*Select the correct answer or fill up the blanks in the following questions :*

1. The value of $\int_0^1 \frac{dx}{1+x}$ by Simpson's rule is

(*a*) 0.96315 (*b*) 0.63915 (*c*) 0.69315 (*d*) 0.69..

2. Using forward differences, the formula for $f'(a) = ...$
3. In application of Simpson's 1/3 rule, the interval $h$ for closer approximation should be ...
4. $f(x)$ is given by

| | | | |
|---|---|---|---|
| $x$ : | 0 | 0.5 | 1 |
| $f(x)$ : | 1 | 0.8 | 0.5, |

then using Trapezoidal rule, the value of $\int_0^1 f(x)\, dx$ is ...

5. If

| | | | | | |
|---|---|---|---|---|---|
| $x$ : | 0 | 0.5 | 1 | 1.5 | 2 |
| $f(x)$ : | 0 | 0.25 | 1 | 2.25 | 4 |

then the value of $\int_0^2 f(x)\,dx$ by Simpson's 1/3rd rule is ...

6. Simpson's 3/8th rule states that ...
7. For the data :

| | | | | |
|---|---|---|---|---|
| $t$ : | 3 | 6 | 9 | 12 |
| $y(t)$ : | −1 | 1 | 2 | 3, |

the value of $\int_3^{12} y(t)\,dt$ when computed by Simpson's $\frac{1}{3}$rd rule is

(*a*) 15 (*b*) 10 (*c*) 0 (*d*) 5.

8. While evaluating a definite integral by Trapezoidal rule, the accuracy can be increased by taking ...
9. The value of $\int_0^1 \frac{dx}{1+x^2}$ by Simpson's 1/3rd rule (taking $n = 1/4$) is ...
10. For the data:

| | | | | |
|---|---|---|---|---|
| $x$ : | 2 | 4 | 6 | 8 |
| $f(x)$ : | 3 | 5 | 6 | 7, |

$\int_2^8 f(x)\,dx$ when found by the Trapezoidal rule is

(*a*) 18 (*b*) 25 (*c*) 16 (*d*) 32.

11. The expression for $\left(\frac{dy}{dx}\right)_{x=x_0}$ using backward differences is ...
12. The number of strips required in Weddle's rule is ...
13. The number of strips required in Simpson's 3/8th rule is a multiple of

(*a*) 1 (*b*) 2 (*c*) 3 (*d*) 6.

14. If $y_0 = 1, y_1 = \frac{16}{17}, y_2 = \frac{4}{5}, y_3 = \frac{16}{25}, y_4 = \frac{1}{2}$ and $h = \frac{1}{4}$, then using Trapezoidal rule, $\int_0^4 ydx$ = ...
15. Using Simpson's $\frac{1}{3}$rd rule, $\int_0^1 \frac{dx}{x}$ = ... (taking $n = 4$).
16. If $y_0 = 1, y_1 = 0.5, y_2 = 0.2, y_3 = 0.1, y_4 = 0.06, y_5 = 0.04$ and $y_6 = 0.03$, then $\int_0^4 ydx$ by Simpson's $\frac{3}{8}$th, rule is = ...
17. If $f(0) = 1, f(1) = 2.7, f(2) = 7.4, f(3) = 20.1, f(4) = 54.6$ and $h = 1$, then $\int_0^4 f(x)\,dx$ by Simpson's $\frac{1}{3}$rd rule = ...
18. Simpson's 1/3rd rule and direct integration give the same result if ...
19. To evaluate $\int_{x_0}^{x_n} ydx$ by Simpson's 1/3rd rule as well as Simpson's 3/8th rule, the number of intervals should be ....... and ....... respectively.
20. Whenever Trapezoidal rule is applicable, Simpson's 1/3rd rule can also be applied. (True or False)

CHAPTER 31

# Difference Equations



## 31.1 INTRODUCTION

Difference calculus also forms the basis of Difference equations. These equations arise in all situations in which sequential relation exists at various discrete values of the independent variable. The need to work with discrete functions arises because there are physical phenomena which are inherently of a discrete nature. In control engineering, it often happens that the input is in the form of discrete pulses of short duration. The radar tracking devices receive such discrete pulses from the target which is being tracked. As such differences equations arise in the study of electrical networks, in the theory of probability, in statistical problems and many other fields.

Just as the subject of Differential equations grew out Differential calculus to become one of the most powerful instruments in the hands of a practical mathematician when dealing with continuous processes in nature, so the subject of Difference equations is forcing its way to the fore for the treatment of discrete processes. Thus the difference equations may be thought of as the discrete counterparts of the differential equations.

## 31.2 DEFINITION

**(1) A difference equation** *is a relation between the differences of an unknown function at one or more general values of the argument.*

Thus $\Delta y_{(n+1)} + y_{(n)} = 2$ ...(1) and $\Delta y_{(n+1)} + \Delta^2 y_{(n-1)} = 1$ ...(2)

are difference equations.

An alternative way of writing a difference equation is as under :

Since $\Delta y_{(n+1)} = y_{(n+2)} - y_{(n+1)}$, therefore (1) may be written as

$$y_{(n+2)} - y_{(n+1)} + y_{(n)} = 2 \qquad ...(3)$$

Also since, $\Delta^2 y_{(n-1)} = y_{(n+1)} - 2y_{(n)} + y_{(n-1)}$, therefore (2) takes the form :

$$y_{(n+2)} - 2y_{(n)} + y_{(n-1)} = 1 \qquad ...(4)$$

Quite often, difference equations are met under the name of *recurrence relations.*

**(2) Order of a difference equation** *is the difference between the largest and the smallest arguments occuring in the difference equation divided by the unit of increment.*

Thus (3) above is the *second order*, for

$$\frac{\text{largest argument} - \text{smallest argument}}{\text{unit of increment}} = \frac{(n+2)-n}{1} = 2,$$

and (4) is of the *third order*, for $\dfrac{(n+2)-(n-1)}{1} = 3.$

**Obs.** *While finding the order of a difference equation, it must always be expressed in a form free of* $\Delta$*s, for the highest* power of $\Delta$ does not give order of the difference equation.

**(3) Solution** *of a difference equation is an expression for* $y_{(n)}$ *which satisfies the given difference equation.*

*The* general solution *of a difference equation is that in which the number of arbitrary constants is equal to the order of the difference equation.*

*A particular solution* or **particular integral** *is that solution which is obtained from the general solution by giving particular values to the constants.*

## 31.3 FORMATION OF DIFFERENCE EQUATIONS

The following examples illustrate the way in which difference equations arise and are formed.

**Example 31.1.** *Form the difference equation corresponding to the family of curves*

$$y = ax + bx^2 \qquad ...(i)$$

**Solution.** We have $\Delta y = a\Delta(x) + b\Delta(x^2) = a\,(x + 1 - x) + b[(x + 1)^2 - x^2]$

$$= a + b(2x + 1) \qquad ...(ii)$$

and $\Delta^2 y = 2b[(x + 1) - x] = 2b \qquad ...(iii)$

To eliminate $a$ and $b$, we have from $(iii)$, $b = \frac{1}{2}\Delta^2 y$

and form $(ii)$, $a = \Delta y - b\,(2x + 1) = \Delta y - \frac{1}{2}\Delta^2 y\,(2x + 1)$

Substituting these values of $a$ and $b$ in $(i)$, we get

$$y = \left[\Delta y - \frac{1}{2}\Delta^2 y\,(2x+1)\right]x + \frac{1}{2}\Delta^2 y\,.\,x^2$$

or $$(x^2 + x)\,\Delta^2 y - 2x\,\Delta y + 2y = 0$$

This is the desired difference equation which may equally well be written in terms of $E$ as

or $$(x^2 + x)\,y_{x+2} - (2x^2 + 4x)y_{x+1} + (x^2 + 3x + 2)y_x = 0.$$

**Example 31.2.** *From* $y_n = A2^n + B(-3)^n$*, derive a difference equation by eliminating the constants.*

**Solution.** We have $y_n = A.2^n + B(-3)^n,\ y_{n+1} = 2A.2^n - 3B(-3)^n$

and $y_{n+2} = 4A.2^n + 9B(-3)^n.$

Eliminating $A$ and $B$, we get

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 2 & -3 \\ y_{n+2} & 4 & 9 \end{vmatrix} = 0 \quad \text{or} \quad y_{n+2} + y_{n+1} - 6y_n = 0$$

which is the desired difference equation.

### PROBLEMS 31.1

1. Write the difference equation $\Delta^3 y_x + \Delta^2 y_x + \Delta y_x + y_x = 0$ in the subscript notation.
2. Assuming $\frac{\log(1-z)}{1+z} = y_0 + y_1 z + y_2 z^2 + ...... + y_n z^n, ......$, find the difference equations satisfied by $y_n$.
3. Form a difference equation by eliminating arbitrary constant from $u_n = a2^{n+1}$. *(Anna, 2008)*
4. Find the difference equation satisfied by
   (*i*) $y = a/x + b$ *(Tiruchirapalli, 2001)* (*ii*) $y = ax^2 - bx$.
5. Derive the difference equations in each of the following cases :
   (*i*) $y_n = A.3^n + B.5^n$ (*ii*) $y_n = (A + Bx)\,2^x$. *(Madras, 2001)*
6. Form the difference equations generated by
   (*i*) $y_n = ax + b2^x$ (*ii*) $y_n = a2^n + b(-2)^n$ (*iii*) $y_x = a2^x + b3^x + c$.

## 31.4 LINEAR DIFFERENCE EQUATIONS

**(1) Def.** *A **linear difference equation** is that in which $y_{n+1}, y_{n+2}$ etc. occur to the first degree only and are not multiplied together.*

*A **linear difference equation with constant coefficient** is of the form*

$$y_{n+r} + a_1 y_{n+r-1} + a_2 y_{n+r-2} + \dots + a_r y_n = f(n) \quad \dots(1)$$

where $a_1, a_2, \dots a_r$ are constants.

*Now we shall deal with linear difference equations with constant coefficients only.* Their properties are analogous to those of linear differential equations with constant co-efficients.

**(2) Elementary properties.** If $u_1(n), u_2(n), \dots, u_r(n)$ be $r$ independent solution of the equation

$$y_{n+r} + a_1 y_{n+r-1} + \dots + a_r y_n = 0 \quad \dots(2)$$

then its complete solution is $U_n = c_1 u_1(n) + \dots + c_r u_r(n)$

where $c_1, c_2, \dots, c_r$ are arbitrary constants.

If $V_n$ is a particular solution of (1), then the complete solution of (1) is $y_n = U_n + V_n$. *The part $U_n$ is called the **complementary function (C.F.)** and the part $V_n$ is called the **particular integral (P.I.)** of (1).*

*Thus the **complete solution (C.S.)** of (1) is* $\boldsymbol{y_n}$ **= C.F. + P.I.**

## 31.5 RULES FOR FINDING THE COMPLEMENTARY FUNCTION

(*i.e., rules to solve a linear difference equation with constant coefficients having right hand side zero*).

**(1)** To begin with, consider the first order linear equation $y_{n+1} - \lambda y_n = 0$, where $\lambda$ is a constant.

Rewriting it as $\dfrac{y_{n+1}}{\lambda^{n+1}} - \dfrac{y_n}{\lambda^n} = 0$, we have $\Delta\left(\dfrac{y_n}{\lambda^n}\right) = 0$, which gives $y_n/\lambda^n = c$, a constant.

*Thus the solution of* $(E - \lambda)\, y_n = 0$ is $\boldsymbol{y_n = c.\lambda^n}$.

**(2)** Now consider the *second order linear equation* $y_{n+2} + ay_{n+1} + by_n = 0$ which in *symbolic form is*

$$(E^2 + aE + b)y_n = 0 \quad \dots(1)$$

Its symbolic co-efficient equated to zero *i.e.*, $E^2 + aE + b = 0$

is called the *auxiliary equation*. Let its roots be $\lambda_1, \lambda_2$.

**Case I.** *If these roots are real and distinct,* then (1) is to equivalent to

$$(E - \lambda_1)(E - \lambda_2)\, y_n = 0 \quad \dots(2)$$

$$(E - \lambda_2)(E - \lambda_1)\, y_n = 0 \quad \dots(3)$$

If $y_n$ satisfies the subsidiary equation $(E - \lambda_1)y_n = 0$, then it will also satisfy (3).

Similarly, if $y_n$ satisfies the subsidiary equation $(E - \lambda_2)y_n = 0$, then it will also satisfy (2).

$\therefore$ it follows that we can derive two independent solutions of (1), by solving the two subsidiary equations

$$(E - \lambda_1)\, y_n = 0 \quad \text{and} \quad (E - \lambda_2)\, y_n = 0$$

Their solutions are respectively, $y_n = c_1(\lambda_1)^n$ and $y_n = c_2(\lambda_2)^n$

where $c_1$ and $c_2$ are arbitrary constants.

Thus the general solution of (1) is $y_n = c_1(\lambda_1)^n + c_2(\lambda_2)^n$

**Case II.** *If the roots are real and equal* (*i.e.*, $\lambda_1 = \lambda_2$), then (2) becomes

$$(E - \lambda_1)^2\, y_n = 0 \quad \dots(4)$$

Let $\qquad y_n = (\lambda_1)^n z_n$

where $z_n$ is a new dependent variable. Then (4) takes the form

$$(\lambda_1)^{n+2} z_{n+2} - 2\lambda_1(\lambda_1)^{n+1} z_{n+1} + \lambda_1^2 . (\lambda_1)^n z_n = 0$$

or $$z_{n+2} - 2z_{n+1} + z_n = 0 \quad i.e., \quad \Delta^2 z_n = 0$$

$\therefore \qquad z_n = c_1 + c_2 n$, where $c_1, c_2$ are arbitrary constants.

Thus the solution of (1) becomes $y_n = (c_1 + c_2 n)(\lambda_1)^n$.

**Case III.** *If the roots are imaginary,* (*i.e.* $\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta$) then the solution of (1) is

$$y_n = c_1(\alpha + i\beta)^n + c_2(\alpha - i\beta)^n \qquad [\text{Put } \alpha = r\cos\theta \text{ and } \beta = r\sin\theta]$$

$$= r^n\, [c_1(\cos n\theta + i\sin n\theta) + c_2(\cos n\theta - i\sin n\theta)]$$

$$= r^n [A_1 \cos n\theta + A_2 \sin n\theta]$$

where $A_1, A_2$ are arbitrary constants are $r = \sqrt{(\alpha^2 + \beta^2)}$, $\theta = \tan^{-1}(\beta/\alpha)$.

**(3)** *In general, to solve the equation* $y_{n+r} + a_1 y_{n+r-2} + \dots + a_r y_n = 0$ where $a$'s are constants :

(*i*) Write the equation in the *symbolic form* $(E^r + a_1 E^{r-1} + \dots + a_r)y_n = 0$.

(*ii*) Write down the *auxiliary equation i.e.,* $E^r + a_1 E^{r-1} \dots + a_r = 0$ and solve it for $E$.

(*iii*) Write the solution as follows :

| *Roots of A.E.* | *Solution, i.e. C.F.* |
|---|---|
| **1.** $\lambda_1, \lambda_2, \lambda_3, \dots\dots$ (real and distinct roots) | $c_1(\lambda_1)^n + c_2(\lambda_2)^n + c_3(\lambda_3)^n + \dots\dots$ |
| **2.** $\lambda_1, \lambda_1, \lambda_3, \dots\dots$ (2 real and equal roots) | $(c_1 + c_2 n)(\lambda_1)^n + c_3(\lambda_3)^n + \dots\dots$ |
| **3.** $\lambda_1, \lambda_1, \lambda_1, \dots\dots$ (3 real and equal roots) | $(c_1 + c_2 n + c_3 n^2)(y_1)^n + \dots\dots$ |
| **4.** $\alpha + i\beta, \alpha - i\beta, \dots\dots$ (a pair of imaginary roots) | $r^n (c_1 \cos\theta + c_2 \sin n\theta)$ where $r = \sqrt{(\alpha^2+\beta^2)}$ and $\theta = \tan^{-1}(\beta/\alpha)$ |

**Example 31.3.** *Solve the difference equation* $u_{n+3} - 2u_{n+2} - 5u_{n+1} + 6u_n = 0$.

**Solution.** Given equation in *symbolic form* is $(E^3 - 2E^2 - 5E + 6)u_n = 0$

$\therefore$ its auxiliary equation is $E^3 - 2E^2 - 5E + 6 = 0$

or $$(E-1)(E+2)(E-3) = 0. \qquad \therefore \quad E = 1, -2, 3$$

Thus the complete solution is $u_n = c_1(1)^n + c_2(-2)^n + c_3(3)^n$.

**Example 31.4.** *Solve* $u_{n+2} - 2u_{n+1} + u_n = 0$.

**Solution.** Given difference equation in *symbolic form* is $(E^2 - 2E + 1)\, u_n = 0$.

$\therefore$ its auxiliary equation is $E^2 - 2E + 1 = 0$

or $$(E-1)^2 = 0. \qquad \therefore \quad E = 1, 1$$

Thus the required solution is $u_n = (c_1 + c_2 n)(1)^n$, *i.e.*, $u_n = c_1 + c_2 n$.

**Example 31.5.** *Solve* $y_{n+1} - 2y_n \cos\alpha + y_{n-1} = 0$.

**Solution.** This is a second order difference equation in $y_{n-1}$; which in *symbolic form* is

$$(E^2 - 2E\cos\alpha + 1)\, y_n = 0$$

The *auxiliary equation* is $E^2 - 2E\cos\alpha + 1 = 0$

$$E = \frac{2\cos\alpha \pm \sqrt{(4\cos^2\alpha - 4)}}{4} = \cos\alpha \pm i\sin\alpha$$

Thus the solution is $y_{n-1} = (1)^{n-1}[c_1 \cos(n-1)\alpha + c_2 \sin(n-1)\alpha]$

or $$y_n = c_1 \cos n\alpha + c_2 \sin n\alpha.$$

**Example 31.6.** *The integers 0, 1, 1, 2, 3, 5, 8, 13, 21, .... are said to form a Fibonacci sequence. Form the Fibonacci difference equation and solve it.*

**Solution.** In this sequence, each number beyond the second, is the sum of its two previous number. If $y_n$ be the $n$th number then $y_n = y_{n-1} + y_{n-2}$ for $n > 2$.

or $$y_{n+2} - y_{n+1} - y_n = 0 \text{ (for } n > 0)$$

or $$(E^2 - E - 1)y_n = 0 \text{ is the difference equation.}$$

Its A.E. is $E^2 - E - 1 = 0$ which gives $E = \frac{1}{2}(1 \pm \sqrt{5})$.

Thus the solution is $$y_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^2 + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n, \text{ for } n > 0$$

When $n = 1, y = 0$

$$\therefore \qquad c_1\left(\frac{1+\sqrt{5}}{2}\right)+c_2\left(\frac{1-\sqrt{5}}{2}\right)=0 \qquad ...(i)$$

When $n = 2, y_2 = 0$

$$\therefore \qquad c_1\left(\frac{1+\sqrt{5}}{2}\right)^2+c_2\left(\frac{1-\sqrt{5}}{2}\right)^2=0 \qquad ...(ii)$$

Solving (*i*) and (*ii*), we get

$$c_1=\frac{5-\sqrt{5}}{10} \text{ and } c_2=\frac{5+\sqrt{5}}{10}$$

Hence the complete solution is

$$y_n=\frac{5-\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n+\frac{5+\sqrt{5}}{2}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

## PROBLEMS 31.2

Solve the following difference equations :

1. $u_{x+2}-6u_{x+1}+9u_x=0.$
2. $y_{n+2}+y_{n+1}+2y_n=0.$
3. $\Delta^2u_n+2\Delta u_n+u_n=0.$
4. $(\Delta^2-3\Delta+2)y_n=0.$
5. $4y_n-y_{n+2}=0$ given that $y_0=0, y_1=2.$
6. $u_{k+3}-3u_{k+2}+4u_k=0.$
7. $f(x+3)-3f(x+1)-2f(x)=0.$
8. $u_{n+3}-3u_{n+1}+2u_n=0$, given $u_1=0, u_2=8$ and $u_3=-2.$
9. $(E^3-5E^2+8E-4)y_n=0$, given that $y_0=3, y_1=2, y_3=22.$
10. $u_{n+1}-2u_n+2u_{n-1}=0.$
11. $y_{m+3}+16y_{m-1}=0.$
 [**Hint.** $E^4=-16=16\,[\cos(2n+1)\pi+i\sin(2n+1)\pi]$ ; use De Moivre's theorem.]
12. Show that the difference equation $I_{m+1}-(2+r_o/r)\,I_m+I_{m-1}=0$ has the solution.
 $I_m=I_o\sinh(n-m)\,\alpha/\sinh(n-1)\,\alpha$, if $I=I_0$ and $I_n=0$, $\alpha$ being $=2\sinh^{-1}\frac{1}{2}\,(r_o/r)^{1/2}$.
13. A series of values of $y_n$ satisfy the relation, $y_{n+2}+ay_{n+1}+by_n$.
 Given that $y_0=0, y_1=1, y_2=y_3=2$. Show that $y_n=2^{n/2}\sin n\pi/4$.
14. A plant is such that each of its seeds when one year old produces 8-fold and produces 18-fold when two years old or more. A seed is planted and as soon as a new seed is produced it is planted. Taking $y_n$ to be the number of seeds produced at the end of the *n*th year, show that $y_{n+1}=8y_n+18(y_1+y_2+...+y_{n-1})$.
 Hence show that $y_{n+2}-9y_{n+1}-10y_n=0$ and find $y_n$.

## 31.6 RULES FOR FINDING THE PARTICULAR INTEGRAL

Consider the equation $y_{n+r}+a_1y_{n+r-1}+...+a_ry_n=f(n)$

which in *symbolic form* is $\phi(E)y_n=f(n)$ ...(1)

where $\qquad \phi(E)=E^r+a_1E^{r-1}+...+a_r$

Then the particular integral is given by P.I. $=\dfrac{1}{\phi(E)}f(n).$

**Case I.** When $f(n)=a^n$

$$\text{P.I.}=\frac{1}{\phi(E)}a_n, \text{ put } E=a$$

$$=\frac{1}{\phi(a)}a^n, \text{ provided } \phi(a)\neq 0$$

If $\phi(a)=0$, *then for the equation*

(*i*) $(E-a)y_n=a^n$, $\qquad$ P.I. $=\dfrac{1}{E-a}a^n=na^{n-1}$

(ii) $(E-a)^2 y_n = a^n$, $\quad$ P.I. $= \dfrac{1}{(E-a)^2} a^n = \dfrac{n(n-1)}{2!} a^{n-2}$

(iii) $(E-a)^3 y^n = a^n$, $\quad$ P.I. $= \dfrac{1}{(E-a)^3} a^n = \dfrac{n(n-1)(n-2)}{3!} a^{n-3}$

and so on.

**Example 31.7.** *Solve* $y_{n+2} - 4y_{n+1} + 3y_n = 5^n$.

**Solution.** Given equation in *symbolic form* is $(E^2 - 4E + 3)y_n = 5^n$

$\therefore$ The auxiliary equation is $E^2 - 4E + 3 = 0$

or $\quad (E-1)(E-3) = 0. \quad \therefore \quad E = 1, 3$

$\therefore \quad$ C.F. $= c_1(1)^n + c_2(3)^n = c_1 + c_2 \cdot 3^n$

and $\quad$ P.I. $= \dfrac{1}{E^2 - 4E + 3} 5^n$ $\quad$ [Put $E = 5$]

$$= \frac{1}{25 - 4.5 + 3} 5^n = \frac{1}{8} \cdot 5^n$$

Thus the complete solution is $\quad y_n = c_1 + c_2 \cdot 3^n + 5^n/8$.

**Example 31.8.** *Solve* $u_{n+2} - 4u_{n+1} + 4u_n = 2^n$.

**Solution.** Given equation in *symbolic form* is $(E^2 - 4E + 4)u_n = 2^n$.

The auxiliary equation is $E^2 - 4E + 4 = 0. \quad \therefore \quad E = 2, 2$.

C.F. $= (c_1 + c_2 n)2^n$

P.I. $= \dfrac{1}{(E-2)^2} \cdot 2^n = \dfrac{n(n-1)}{2!} \cdot 2^{n-2} = n(n-1)\, 2^{n-3}$

Hence the complete solution is $u_n = (c_1 + c_2 n)\, 2^n + n(n-1)\, 2^{n-3}$.

**Case II.** *When* $f(n) = \sin kn$.

$$\text{P.I.} = \frac{1}{\phi(E)} \sin kn = \frac{1}{\phi(E)} \left( \frac{e^{ikn} - e^{-ikn}}{2i} \right) = \frac{1}{2i} \left[ \frac{1}{\phi(E)} a^n - \frac{1}{\phi(E)} b^n \right]$$

where $a = e^{ik}$ and $b = e^{-ik}$.

Now proceed as in case I.

(2) *When* $f(n) = \cos kn$ $\quad$ P.I. $= \dfrac{1}{\phi(E)} \cos kn = \dfrac{1}{\phi(E)} \left( \dfrac{e^{ikn} + e^{-ikn}}{2} \right)$

$$= \frac{1}{2} \left[ \frac{1}{\phi(E)} a^n + \frac{1}{\phi(E)} b^n \right] \text{ as before}$$

Now proceed as in case I.

**Example 31.9.** *Solve* $y_{n+2} - 2\cos\alpha \cdot y_{n+1} + y_n = \cos \alpha n$. *(Nagpur, 2008)*

**Solution.** Given equation in *symbolic form* is $(E^2 - 2\cos\alpha \cdot E + 1)y_n = \cos \alpha n$.

The *auxiliary equation is* $E^2 - 2\cos\alpha \cdot E + 1 = 0$.

$$\therefore \quad E = \frac{2\cos\alpha \pm \sqrt{(4\cos^2\alpha - 4)}}{2} = \cos\alpha \pm i \sin\alpha$$

$\therefore \quad$ C.F. $= (1)^n [c_1 \cos \alpha n + c_2 \sin \alpha n]$ *i.e.*, $c_1 \cos \alpha n + c_2 \sin \alpha n$

$$\text{P.I.} = \frac{1}{E^2 - 2E\cos\alpha + 1} \cos \alpha n$$

$$= \frac{1}{E^2 - E(e^{i\alpha} + e^{-i\alpha}) + 1} \left( \frac{e^{i\alpha n} + e^{-i\alpha n}}{2} \right)$$

$$= \frac{1}{2}\left[\frac{1}{(E-e^{i\alpha})(E-e^{-i\alpha})}e^{i\alpha n} + \frac{1}{(E-e^{i\alpha})(E-e^{-i\alpha})}e^{-i\alpha n}\right]$$

[Put $E = e^{i\alpha}$] [Put $E = e^{-i\alpha}$]

$$= \frac{1}{2}\left[\frac{1}{(E-e^{i\alpha})}\cdot\frac{1}{e^{i\alpha}-e^{-i\alpha}}e^{i\alpha n} + \frac{1}{E-e^{-i\alpha}}\cdot\frac{1}{e^{-i\alpha}-e^{i\alpha}}e^{-i\alpha n}\right]$$

$$= \frac{1}{4i\sin\alpha}\left[\frac{1}{E-e^{i\alpha}}e^{i\alpha n} - \frac{1}{E-e^{-i\alpha}}e^{-i\alpha n}\right] = \frac{1}{4i\sin\alpha}\,[n\,.\,e^{i\alpha(n-1)} - n\,e^{-i\alpha(n-1)}]$$

$$= \frac{n}{2\sin\alpha}\left[\frac{e^{i\alpha(n-1)} - e^{-i\alpha(n-1)}}{2i}\right] = \frac{n\sin(n-1)\alpha}{2\sin\alpha}$$

Hence the complete solution is

$$y_n = c_1\cos\alpha n + c_2\sin\alpha n + \frac{n\sin(n-1)\alpha}{2\sin\alpha}.$$

**Case III.** *When* $f(n) = n^p$. P.I. $= \dfrac{1}{\phi(E)}n^p = \dfrac{1}{\phi(1+\Delta)}n^p$

(1) Expand $[\phi(1+\Delta)]^{-1}$ in ascending powers of $\Delta$ by the Binomial theorem as far as the term in $\Delta^p$.

(2) Expand $n^p$ in the factorial form (p. 950) and operate on it with each term of the expansion.

**Example 31.10.** *Solve* $y_{n+2} - 4y_n = n^2 + n - 1$. *(Madras, 1999)*

**Solution.** Given equation is $(E^2 - 4)y_n = n^2 + n - 1$.

The *auxiliary equation is* $E^2 - 4 = 0$, $\therefore$ $E = \pm 2$.

$\therefore$ C.F. $= c_1(2)^n + c_2(-2)^n$.

$\therefore$ P.I. $= \dfrac{1}{E^2-4}(n^2+n-1) = \dfrac{1}{(1+\Delta)^2-4}[n(n-1)+2n-1]$.

$$= \frac{1}{\Delta^2+2\Delta-3}([n]^2+2[n]-1) = -\frac{1}{3}\left[1-\left(\frac{2}{3}\Delta+\frac{\Delta^2}{3}\right)\right]^{-1}\{[n]^2+2[n]-1\}$$

$$= -\frac{1}{3}\left[1+\left(\frac{2}{3}\Delta+\frac{\Delta^2}{3}\right)+\left(\frac{2}{3}\Delta+\frac{\Delta^2}{3}\right)+\ldots\right]\{[n]^2+2[n]-1\}$$

$$= -\frac{1}{3}\left\{1+\frac{2}{3}\Delta+\frac{7}{9}\Delta^2+\ldots\right\}\{[n]^2+2[n]-1\} = -\frac{1}{3}\left\{[n]^2+2[n]-1+\frac{2}{3}(2[n]+2)+\frac{7}{9}\times 2\right\}$$

$$= -\frac{1}{3}\left\{[n]^2+\frac{10}{3}[n]+\frac{17}{9}\right\} = -\frac{n^2}{3}-\frac{7}{9}n-\frac{17}{27}.$$

Hence the *complete solution* is $y_n = c_1 2^n + c_2(-2)^n - \dfrac{n^2}{3} - \dfrac{7}{9}n - \dfrac{17}{27}$.

**Case IV.** *When* $f(n) = a^n F(n)$, $F(n)$, being a polynomial of finite degree in $n$.

$$\text{P.I.} = \frac{1}{\phi(E)}a^n F(n) = a^n\frac{1}{\phi(aE)}F(a)$$

Now $F(n)$ being a polynomial in $n$, proceed as in case III.

**Example 31.11.** *Solve* $y_{n+2} - 2y_{n+1} + y_n = n^2 . 2^n$. *(Nagpur, 2008)*

**Solution.** Given equation is $(E^2 - 2E + 1)y_n = n^2 . 2^n$.

Its C.F. $= c_1 + c_2 n$

and P.I. $= \dfrac{1}{(E-1)^2}2^n . n^2 = 2^n\dfrac{1}{(2E-1)^2}n^2 = 2^n\dfrac{1}{(1+2\Delta)^2}n^2$

$$= 2^n (1 + 2\Delta)^{-2}\, n(n-1) + n = 2^n (1 - 4\Delta + 12\Delta^2 - \ldots)([n]^2 + [n])$$
$$= 2^n \{[n]^2 + [n] - 4(2[n] + 1) + 12 \times 2\}$$
$$= 2^n ([n]^2 - 7[n] + 20) = 2^n (n^2 - 8n + 20)$$

Hence the *complete solution* is $y_n = c_1 + c_2 n + 2^n (n^2 - 8n + 20)$.

## PROBLEMS 31.3

Solve the following difference equations :

1. $y_{n+2} - 5y_{n+1} - 6y_n = 4^n, y_0 = 0, y_1 = 1.$ *(Madras, 2003)*
2. $y_{n+2} + 6y_{n+1} + 9y_n = 2^n, y_0 = y_1 = 0.$ *(V.T.U., 2009)*
3. $y_{p+3} - 3y_{p+2}\, 3y_{p+1} - y_p = 1.$ *(Kottayam, 2005)*
4. $y_{n+2} - 2y_{n+1} + 4y_n = 6$, given that $y_0 = 0$ and $y_1 = 2$.
5. $(E^2 - 4E + 3)y = 3^x.$
6. $y_{x+2} - 4y_{x+1} + 4y_x = 3.2^x + 5.4^x.$
7. $u_{n+2} - u_n = \cos n/2.$ *(Madras, 2001 S)*
8. $y_{p+2} - \left(2\cos\frac{1}{2}\right) y_{p+1} + y_p = \sin p/2.$
9. $(E^2 - 4)y_x = x^2 - 1.$
10. $y_{n+3} + y_n = n^2 + 1 y_0 = y_1 = y_2 = 0.$ *(Tirchirapalli, 2001)*
11. $y_{n+3} - 5y_{n+2} + 3y_{n+1} + 9y_n = 2^n + 3n.$ *(Nagpur, 2009)*
12. $(4E^2 - 4E + 1)y = 2^n + 2^{-n}.$ *(Madras, 2001)*
13. $y_{n+2} + 5y_{n+1} + 6y_n = n + 2^n.$ *(Nagpur, 2006)*
14. $u_{x+2} + 6u_{x+1} + 9u_x = x2^x + 3^x + 7.$
15. $y_{n+3} + 8y_n = (2n + 3)\, 2^n$ *(Nagpur, 2005)*
16. $u_{n+2} - 4u_{n+1}\, 4u_n = n^2 2^n.$
17. $(E^2 - 5E + 6)y_k = 4^k (k^2 - k + 5).$
18. $(E^2 - 2E + 4)y_n = -2^n \left\{6\cos\frac{n\pi}{3} + 2\sqrt{3}\sin\frac{n\pi}{3}\right\}.$
19. A beam of length $l$, supported at $n$ points carries a uniform load $w$ per unit length. The bending moments $M_1, M_2 \ldots\ldots M_n$ at the supports satisfy the Clapeyron's equation :
$$M_{r+2} + 4M_{r+1} + M_r = -\frac{1}{2} wl^2$$
If a beam weighing 30 kg is supported at its ends and at two other supports dividing the beam into three equal parts of 1 metre length, show that the bending moment at each of the two middle supports is 1 kg metre.

## 31.7 SIMULTANEOUS DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS

The method used for solving simultaneous differential equations with constant coefficients also applies to simultaneous difference equations with constants coefficients. The following example illustrates the technique.

**Example 31.12.** *Solve the simultaneous difference equations*
$$u_{x+1} + v_x - 3u_x = x,\ 3u_x + v_{x+1} - 5v_x = 4^x$$
*subject to the conditions* $u_1 = 2, v_1 = 0$.

**Solution.** Given equation in symbolic form, are
$$(E - 3)\, u_x + v_x = x \qquad \ldots(i)$$
$$3u_x + (E - 5)v_x = 4^x \qquad \ldots(ii)$$
Operating the first equation with $E - 5$ and subtracting the second from it, we get
$$[(E - 5)(E - 3) - 3]u_x = (E - 5)x - 4^x$$
or
$$(E^2 - 8E + 12)u_x = 1 - 4x - 4^x$$
Its solution is
$$u_x = c_1 2^x + c_2 6^x - \frac{4}{5}x - \frac{19}{25} + \frac{4^x}{4} \qquad \ldots(iii)$$
Substituting the value of $u_x$ from $(iii)$ in $(i)$, we get
$$v_x = c_1 2^x - 3c_2 6^x - \frac{3x}{5} - \frac{34}{25} - \frac{4^x}{4} \qquad \ldots(iv)$$
Taking $u_1 = 2, v_1 = 0$, in $(iii)$ and $(iv)$, we obtain
$$2c_1 + 6c_2 = \frac{64}{25},\ 2c_1 - 18c_2 = \frac{74}{25}$$
when
$$c_1 = 1.33,\ c_2 = -0.0167$$

Hence $u_x = 1.33.2^x - 0.0167.6^x - 0.8x - 0.76 + 4^{x-1}$

$u_x = 1.33.2^x - 0.05.6^x - 0.6x - 1.36 - 4^{x-1}$.

## PROBLEMS 31.4

Solve the following simultaneous difference equations :

1. $y_{x+1} - z_x = 2(x+1), z_{x+1} - y_x = -2(x+1)$.
2. $y_{n+1} - y_n + 2z_{n+1} = 0, z_{n+1} - z_n - 2y_n = 2^n$.
3. $u_{n+1} + n = 3u_n + 2v_n, v_{n+1} - n = u_n + 2v_n$, given $u_0 = 0, v_0 = 3$.
4. $u_{x+1} + v_x + w_x = 1, u_x + v_{x+1} + w_x = x, u_x + v_x + w_{x+1} = 2x$.

## 31.8 APPLICATION TO DEFLECTION OF A LOADED STRING

Consider a light string of length $l$ stretched tightly between $A$ and $B$. Let the forces $P_i$ be acting at its equispaced points $x_i$ $(i = 1, 2, ..., n-1)$ and perpendicular to $AB$ resulting in small transverse displacements $y_i$ at these points (Fig. 31.1). Assuming the angle $\theta_i$ made by the portion between $x_i$ and $x_{i+1}$ with the horizontal, to be small, we have

$$\sin \theta_i = \tan \theta_i = \theta_i \text{ and } \cos \theta_i = 1$$

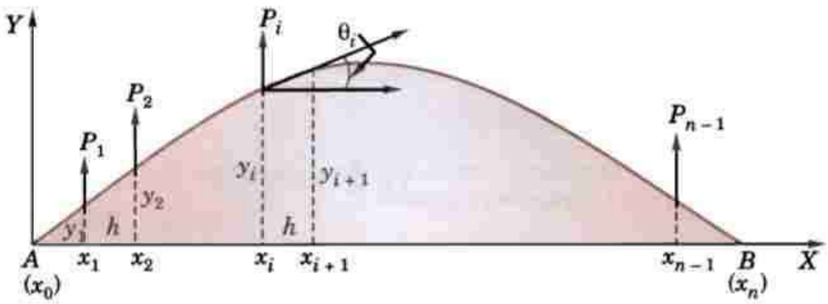


Fig. 31.1

If $T$ be the tension of the string at $x_i$, then $T \cos \theta_i = T$

*i.e.*, the tension may be taken as uniform.

Taking $x_{i+1} - x_i = h$, we have

$$y_{i+1} - y_i = h \tan \theta_i = h\theta_i \quad ...(1)$$

$$y_i - y_{i-1} = h \tan \theta_{i-1} = h\theta_{i-1} \quad ...(2)$$

Also resolving the forces in equilibrium at $(x_i, y_i) \perp$ to $AB$, we get

$$T \sin \theta_i - T \sin \theta_{i-1} + P_i = 0 \text{ i.e. } T(\theta_i - \theta_{i-1}) + P_i = 0 \quad ...(3)$$

Eliminating $\theta_i$ and $\theta_{i-1}$ from (1), (2) and (3), we obtain

$$y_{i+1} - 2y_i + y_{i-1} = -\frac{hP_i}{T} \quad ...(4)$$

which is a difference equation and its solution gives the displacements $y_i$. To obtain the arbitrary constants in the solution, we take $y_0 = y_n = 0$ as the boundary conditions, since the ends $A$ and $B$ of the string are fixed.

**Example 31.13.** *A light string stretched between two fixed nails 120 cm apart, carries 11 loads of weight 5 gm each at equal intervals and the resulting tension is 500 gm weight. Show that the sag at the mid-point is 1.8 cm.*

**Solution.** Taking $h = 10$ cm, $P_i = 5$ gm and $T = 500$ gm wt.,

the above equation (4) becomes $y_{i+1} - 2y_i + y_{i-1} = -1/10$

*i.e.*, $$y_{i+2} - 2y_{i+1} + y_i = -\frac{1}{10}$$

Its A.E. is $(E-1)^2 = 0$ *i.e.* $E = 1, 1$. $\therefore$ C.F. $= c_1 + c_2 i$

and $$\text{P.I.} = \frac{1}{(E-1)^2}\left(-\frac{1}{10}\right) = -\frac{1}{10}\frac{1}{(E-1)^2}(1)^i = -\frac{1}{10}\frac{i(i-1)}{2} = \frac{1}{20}(i-i^2)$$

Thus the C.S. is $y_i = c_1 + c_2 i + \frac{1}{20}(i - i^2)$

Since $y_0 = 0, \quad \therefore \quad c_1 = 0$

and $y_{12} = 0, \quad \therefore \quad c_2 = \frac{11}{20}$.

Hence $y_i = \frac{11}{20} i + \frac{1}{20}(i - i^2)$

At the mid-point $i = 6$, we get $y_6 = 1.8$ cm.

**PROBLEMS 31.5**

1. A light string of length $(n+1)l$ is stretched between two fixed points with a force $P$. It is loaded with $n$ equal masses $m$ at distance $l$. If the system starts rotating with angular velocity $\omega$, find the displacement $y_i$ of the $i$th mass.

## 31.9 OBJECTIVE TYPE OF QUESTIONS

**PROBLEMS 31.6**

*Select the correct answer or fill up the blanks in the following questions :*

1. $y_n = A\,2^n + B\,3^n$, is the solution of the difference equation ...... .
2. The solution of $(E-1)^3 u_n = 0$ is ...... .
3. The solution of the difference equation $u_{n+3} - 2u_{n+2} - 5u_{n+1} + 6u_n = 0$ is ...... .
4. The solution of $y_{n+1} - y_n = 2^n$ is ...... .
5. The difference equation $y_{n+1} - 2y_n = n$ has $y_n = ...$ as its solution.
6. The difference equation corresponding to the family of curves $y = ax^2 + bx$ is ...... .
7. The particular integral of the equation $(E-2)\,y_n = 1$.
8. The solution of $4y_n = y_{n+2}$ such that $y_0 = 0, y_1 = 2$, is ...... .
9. The equation $\Delta^2 u_{n+1} + \frac{1}{2}\Delta^2 u_n = 0$ is of order ...... .
10. The difference equation satisfied by $y = a + b/x$ is ...... .
11. The order of the difference equation $y_{n+2} - 2y_{n+1} + y_n = 0$ is ...... .
12. The solution of $y_{n+2} - 4y_{n+1} + 4y_n = 0$ is ...... .
13. The particular integral of $u_{x+2} - 6u_{x+1} + 9u_x = 3$ is ...... .
14. The difference equation generated by $u_n = (a + bn)\,3^n$ is ...... .
15. Solution of $6y_{n+2} + 5y_{n+1} - 6y_n = 2^n$ is $y_n = A(2/3)^n + B(-3/2)^n + 2^n/28$. (True or False)

CHAPTER 32

# Numerical Solution of Ordinary Differential Equations



## 32.1 INTRODUCTION

The methods of solution so far presented are applicable to a limited class of differential equations. Frequently differential equations appearing in physical problems do not belong to any of these familiar types and one is obliged to resort to numerical methods. These methods are of even greater importance when we realise that computing machines are now available which reduce numerical work considerably.

A number of numerical methods are available for the solution of first order differential equations of the form :

$$\frac{dy}{dx} = f(x, y), \text{ given } y(x_0) = y_0 \qquad ...(1)$$

These methods yield solutions either as a power series in $x$ from which the values of $y$ can be found by direct substitution, or as a set of values of $x$ and $y$. The methods of Picard and Taylor series belong to the former class of solutions whereas those of Euler, Runge-Kutta, Milne, Adams-Bashforth etc. belong to the latter class. In these later methods, the values of $y$ are calculated in short steps for equal intervals of $x$ and are therefore, termed as *step-by-step methods.*

Euler and Runge-Kutta methods are used for computing $y$ over a limited range of $x$-values whereas Milne and Adams-Bashforth methods may be applied for finding $y$ over a wider range of $x$-values. These later methods require starting values which are found by Picard's or Taylor series or Runge-Kutta methods.

The initial condition in (1) is specified at the point $x_0$. *Such problems in which all the initial conditions are given at the initial point only are called* ***initial value problems.*** *But there are problems involving second and higher order differential equations in which the conditions may be given at two or more points. These are known as* ***boundary value problems.*** In this chapter, we shall first explain methods for solving initial value problems and then give a method of solving boundary value problems.

## 32.2 PICARD'S METHOD*

Consider the first order equation $dy/dx = f(x, y)$ ...(1)

* Called after the French mathematician *Emile Picard* (1856—1941) who was professor in Paris since 1881 and is famous for his researches in the theory of functions.

It is required to find that particular solution of (1) which assumes the value $y_0$ when $x = x_0$. Integrating (1) between limits, we get

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)\,dx \quad \text{or} \quad y = y_0 + \int_{x_0}^{x} f(x, y)\,dx \qquad ...(2)$$

This is an integral equation equivalent to (1), for it contains the unknown $y$ under the integral sign.

*As a first approximation $y_1$ to the solution*, we put $y = y_0$ in $f(x, y)$ and integrate (2), giving

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

*For a second approximation $y_2$*, we put $y = y_1$ in $f(x, y)$ and integrate (2), giving

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx.$$

Similarly, *a third approximation is* $y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\,dx.$

Continuing this process, a sequence of functions of $x$, *i.e.*, $y_1, y_2, y_3$ ... is obtained each giving a better approximation of the desired solution than the preceding one.

**Obs.** Picard's method is of considerable theoretical value, but can be applied only to a limited class of equations in which the successive integrations can be performed easily. The method can be extended to simultaneous equations and equations of higher order (See § 32.11 and 32.12).

**Example 32.1.** *Using Picard's process of successive approximation, obtain a solution upto the fifth approximation of the equation $dy/dx = y + x$, such that $y = 1$ when $x = 0$. Check your answer by finding the exact particular solution.*

**Solution.** (*a*) We have $y = 1 + \int_0^x (y + x)\,dx.$

*First approximation.* Put $y = 1$, in $y + x$, giving

$$y_1 = 1 + \int_0^x (1 + x)\,dx = 1 + x + x^2/2.$$

*Second approximation.* Put $y = 1 + x + x^2/2$ in $y + x$, giving

$$y_2 = 1 + \int_0^x (1 + 2x + x^2/2)\,dx = 1 + x + x^2 + x^3/6.$$

*Third approximation.* Put $y = 1 + x + x^2 + x^3/6$ in $y + x$, giving

$$y_3 = 1 + \int_0^x (1 + 2x + x^2 + x^3/6)\,dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}.$$

*Fourth approximation.* Put $y = y_3$ in $y + x$, giving

$$y_4 = 1 + \int_0^x \left(1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}\right) dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}.$$

*Fifth approximation.* Put $y = y_4$ in $y + x$, giving

$$y_5 = 1 + \int_0^x \left(1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}\right) dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{720} \qquad ...(i)$$

(*b*) Given equation :

$\dfrac{dy}{dx} - y = x$ is a Leibnitz's linear in $x$.

Its I.F. being $e^{-x}$, the solution is

$$ye^{-x} = \int xe^{-x}\,dx + c = -xe^{-x} - \int (-e^{-x})\,dx + c = -xe^{-x} - e^{-x} + c \qquad \text{[Integrate by parts]}$$

$\therefore \quad y = ce^x - x - 1.$

Since $y = 1$, when $x = 0$, $\quad \therefore \quad c = 2.$

Thus the desired particular solution is $y = 2e^x - x - 1 \qquad ...(ii)$

Or using the series : $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \infty,$

we get $$y = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{360} + \dots \infty \quad \dots(iii)$$

Comparing $(i)$ and $(iii)$, it is clear that $(i)$ approximates to the exact particular solution $(ii)$ upto the term in $x^5$.

**Obs.** At $x = 1$, the fourth approximation $y_4 = 3.433$ and the fifth approximation $y_5 = 3.434$ whereas exact value is 3.44.

**Example 32.2.** *Find the value of y for x = 0.1 by Picard's method, given that*

$$\frac{dy}{dx} = \frac{y - x}{y + x}, \; y(0) = 1. \qquad (P.T.U., 2002)$$

**Solution.** We have $y = 1 + \int_0^x \frac{y - x}{y + x}\, dx$

*First approximation.* Put $y = 1$ in the integrand, giving

$$y_1 = 1 + \int_0^x \frac{1 - x}{1 + x}\, dx = 1 + \int_0^x \left(-1 + \frac{2}{1 + x}\right) dx$$

$$= 1 + \left[-x + 2 \log (1 + x)\right]_0^x = 1 - x + 2 \log (1 + x) \quad \dots(i)$$

*Second approximation.* Put $y = 1 - x + 2 \log (1 + x)$ in the integrand, giving

$$y_2 = 1 + \int_0^x \frac{1 - x + 2 \log (1 + x) - x}{1 - x + 2 \log (1 + x) + x}\, dx = 1 + \int_0^x \left[1 - \frac{2x}{1 + 2 \log (1 + x)}\right] dx$$

which is very difficult to integrate.

Hence we use the first approximation and taking $x = 0.1$ in $(i)$ we obtain

$$y(0.1) = 1 - (.1) + 2 \log 1.1 = 0.9828.$$

## 32.3 TAYLOR'S SERIES METHOD*

Consider the first order equation $dy/dx = f(x, y)$ ...(1)

Differentiating (1), we have

$$\frac{d^2y}{dx^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \quad i.e., \quad y'' = f_x + f_y f' \quad \dots(2)$$

Differentiating this successively, we can get $y'''$, $y^{iv}$ etc. Putting $x = x_0$ and $y = 0$, the values of $(y')_0$, $(y'')_0$, $(y''')_0$ can be obtained. Hence the Taylor's series

$$y(x) = y_0 + (x - x_0)(y')_0 + \frac{(x - x_0)^2}{2!}(y'')_0 + \frac{(x - x_0)^3}{3!}(y''')_0 + \dots \quad \dots(3)$$

gives the values of $y$ for every value of $x$ for which (3) converges.

On finding the value $y_1$ for $x = x_1$ from (3), $y', y''$ can be evaluated at $x = x_1$ by means of (1), (2) etc. Then $y$ can be expanded about $x = x_1$. In this way, the solution can be extended beyond the range of convergence of series (3).

**Example 32.3.** *Find by Taylor's series method the value of y at x = 0.1 and x = [illegible] to five places of decimals from dy/dx = $x^2y - 1$, y(0) = 1.* (V.T.U., 2009, [illegible]htak, 2005)

**Solution.** Here $(y)_0 = 1, y' = x^2y - 1, (y')_0 = -1$

$\therefore$ Differentiating successively and substituting, we get

$$y'' = 2xy + x^2y', \qquad (y'')_0 = 0$$
$$y''' = 2y + 4xy' + x^2y'', \qquad (y''')_0 = 2$$
$$y^{iv} = 6y' + 6xy'' + x^2y''', \qquad (y^{iv})_0 = -6 \text{ etc.}$$

*See footnote p. 145.

Putting these values in the Taylor's series,

$$y(x) = y_0 + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{iv}(0) + ...,$$

we have $$y(x) = 1 + x(-1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + ... = 1 - x + \frac{x^3}{3} - \frac{x^4}{4} + ...$$

Hence $y(0.1) = 0.90033$ and $y(0.2) = 0.80227$.

**Example 32.4.** *Employ Taylor's method to obtain approximate value of y at x = 0.2 for the differential equation $dy/dx = 2y + 3e^x$, y(0) = 0. Compare the numerical solution obtained with the exact solution.* *(V.T.U., 2009 ; P.T.U., 2003)*

**Solution.** (*a*) We have $y' = 2y + 3e^x$ $\qquad y'(0) = 2y(0) + 3e^0 = 3.$

Differentiating successively and substituting $x = 0, y = 0$, we get

$$\begin{aligned} y'' &= 2y' + 3e^x, & y''(0) &= 2y'(0) + 3 = 9 \\ y''' &= 2y'' + 3e^x, & y'''(0) &= 2y''(0) + 3 = 21 \\ y^{iv} &= 2y''' + 3e^x, & y^{iv}(0) &= 2y'''(0) + 3 = 45 \text{ etc.} \end{aligned}$$

Putting these values in the Taylor's series, we have

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}\,y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{iv}(0) + ...$$

$$= 0 + 3x + \frac{9}{2}x^2 + \frac{21}{6}x^3 + \frac{45}{24}x^4 + ... = 3x + \frac{9}{2}x^2 + \frac{7}{2}x^3 + \frac{15}{8}x^4 + ...$$

Hence $\quad y(0.2) = 3(0.2) + 4.5(0.2)^2 + 3.5(0.2)^3 + 1.875(0.4)^4 + ... = 0.8110$ ...(*i*)

(*b*) Now $\dfrac{dy}{dx} - 2y = 3e^x$ is a Leibnitz's linear in $x$.

Its I.F. being $e^{-2x}$, the solution is

$$ye^{-2x} = \int 3e^x \,.\, e^{-2x}\,dx + c = -3e^{-x} + c \quad \text{or} \quad y = -3e^x + ce^{2x}$$

Since $y = 0$ when $x = 0$, $\quad \therefore \quad c = 3.$

Thus the exact solution is $y = 3(e^{2x} - e^x)$

When $x = 0.2$, $y = 3\,(e^{0.4} - e^{0.2}) = 0.8112$ ...(*ii*)

Comparing (*i*) and (*ii*), it is clear that (*i*) approximates to the exact value upto 3 decimal places.

**Example 32.5.** *Solve by Taylor's series method the equation $\dfrac{dy}{dx} = \log(xy)$ for y(1.1) and y(1.2), given y(1) = 2.* *(Hazaribagh, 2009)*

**Solution.** We have $y' = \log x + \log y$ ; $\quad y'(1) = \log 2$

Differentiating w.r.t. $x$ and substituting $x = 1, y = 2$, we get

$$y'' = \frac{1}{x} + \frac{1}{y}y' \,;\, y''(1) = 1 + \frac{1}{2}\log 2$$

$$y''' = -\frac{1}{x^2} + \frac{1}{y} + y'' + y'\left(-\frac{1}{y1}\right)y' \,;\, y'''(1) = -1 + \frac{1}{2}\left(1 + \frac{1}{2}\log 2\right) - \frac{1}{4}(\log 2)^2$$

Substituting these values in the Taylor's series about $x = 1$, we have

$$y(x) = y(1) + (x-1)\,y'(1) + \frac{(x-1)^2}{2!}\,y''(1) + \frac{(x-1)^3}{3!}\,y'''(1) + ...$$

$$= 2 + (x-1)\log 2 + \frac{1}{2}(x-1)^2\left(1 + \frac{1}{2}\log 2\right) + \frac{1}{6}(x-1)^3\left[-\frac{1}{2} + \frac{1}{4}\log 2 - \frac{1}{4}(\log 2)^2\right]$$

$$\therefore \quad y(1.1) = 2 + (0.1)\log 2 + \frac{(0.1)^2}{2}\left(1 + \frac{1}{2}\log 2\right) + \frac{(0.1)^3}{6}\left[-\frac{1}{2} + \frac{1}{4}\log 2 - \frac{1}{4}(\log 2)^2\right] = 2.036$$

$$y(1.2) = 2 + (0.2)\log 2 + \frac{(0.2)^2}{2}\left(1 + \frac{1}{2}\log 2\right) + \frac{(0.2)^3}{6}\left[-\frac{1}{2} + \frac{1}{4}\log 2 - \frac{1}{4}(\log 2)^2\right] = 2.081.$$

**PROBLEMS 32.1**

1. Using Picard's method, solve $dy/dx = -xy$ with $x_0 = 0, y_0 = 1$ upto third approximation. *(Mumbai, 2005)*
2. Employ Picard's method to obtain, correct to four places of decimal, solution of the differential equation $dy/dx = x^2 + y^2$ for $x = 0.4$, given that $y = 0$ when $x = 0$. *(J.N.T.U., 2009)*
3. Obtain Picard's second approximate solution of the initial value problem : $y' = x^2/(y^2 + 1), y(0) = 0$. *(Marathwada, 2008)*
4. Find an approximate value of y when $x = 0.1$, if $dy/dx = x - y^2$ and $y = 1$ at $x = 0$, using
   (*a*) Picard's method (*b*) Taylor's series. *(V.T.U., 2010 ; Madras, 2006)*
5. Solve $y' = x + y$ given $y(1) = 0$. Find $y(1.1)$ and $y(1.2)$ by Taylor's method. Compare the result with its exact value. *(J.N.T.U., 2008 ; Anna, 2005)*
6. Evaluate $y(0.1)$ correct to six places of decimals by Taylor's series method if $y(x)$ satisfies
   $$y' = xy + 1, y(0) = 1.$$
7. Solve $y' = 3x + y^2, y(0) = 1$ using Taylor's series method and computer $y(0.1)$. *(Mumbai, 2007)*
8. Using Taylor series method, find $y(0.1)$ correct to 3-decimal places given that
   $$dy/dx = e^x - y^2, y(0) = 1.$$

## 32.4 EULER'S METHOD*

Consider the equation $\dfrac{dy}{dx} = f(x, y)$ ...(1)

given that $y(x_0) = y_0$. Its curve of solution through $P(x_0, y_0)$ is shown dotted in Fig. 32.1. Now we have to find the ordinate of any other point $Q$ on this curve.

Let us divide $LM$ into $n$ sub-intervals each of width $h$ at $L_1, L_2, \ldots$ so that $h$ is quite small. In the interval $LL_1$, we approximate the curve by the tangent at $P$. If the ordinate through $L_1$ meets this tangent in $P_1(x_0 + h, y_1)$, then

$$y_1 = L_1P_1 = LP + R_1P_1$$
$$= y_0 + PR_1 \tan\theta = y_0 + h\left(\frac{dy}{dx}\right)_P$$
$$= y_0 + h\, f(x_0, y_0)$$

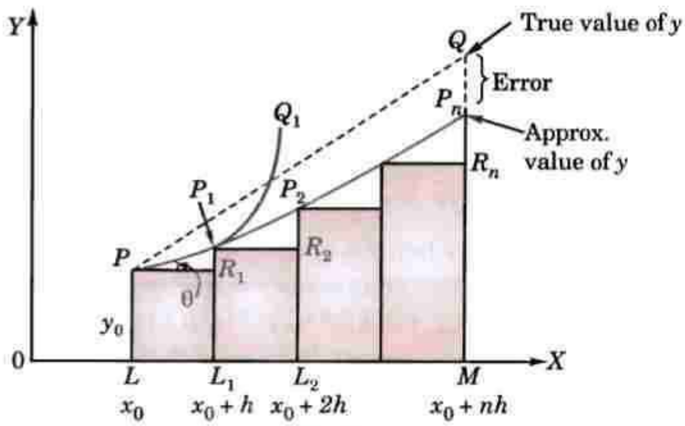

Fig. 32.1

Let $P_1Q_1$ be the curve of solution of (1) through $P_1$ and let its tangent at $P_1$ meet the ordinate through $L_2$ in $P_2(x_0 + 2h, y_2)$. Then

$$y_2 = y_1 + h\, f(x_0 + h, y_1) \quad \text{...(2)}$$

Repeating this process $n$ times, we finally reach an approximation $MP_n$ of $MQ$ given by

$$y_n = y_{n-1} + h\, f(x_0 + \overline{n-1}\, h, y_{n-1})$$

This is *Euler's method* of finding an approximate solution of (1).

**Obs.** In Euler's method, we approximate the curve of solution by the tangent in each interval, *i.e.*, by a sequence of short lines. Unless $h$ is small, the error is bound to be quite significant. This sequence of lines may also deviate considerably from the curve of solution. Hence there is a modification of this method which is given in the next section.

**Example 32.6.** *Using Euler's method, find an approximate value of y corresponding to $x = 1$, given that $dy/dx = x + y$ and $y = 1$ when $x = 0$.* *(Mumbai, 2005 ; Rohtak, 2003)*

*See footnote p. 302.

**Solution.** We take $n = 10$ and $h = 0.1$ which is sufficiently small. The various calculations are arranged as follows :

| $x$ | $y$ | $x + y = dy/dx$ | *Old* $y + 0.1\ (dy/dx) =$ *new* $y$ |
|---|---|---|---|
| 0.0 | 1.00 | 1.00 | 1.00 + 0.1(1.00) = 1.10 |
| 0.1 | 1.10 | 1.20 | 1.10 + 0.1(1.20) = 1.22 |
| 0.2 | 1.22 | 1.42 | 1.22 + 0.1(1.42) = 1.36 |
| 0.3 | 1.36 | 1.66 | 1.36 + 0.1(1.66) = 1.53 |
| 0.4 | 1.53 | 1.93 | 1.53 + 0.1(1.93) = 1.72 |
| 0.5 | 1.72 | 2.22 | 1.72 + 0.1(2.22) = 1.94 |
| 0.6 | 1.94 | 2.54 | 1.94 + 0.1(2.54) = 2.19 |
| 0.7 | 2.19 | 2.89 | 2.19 + 0.1(2.89) = 2.48 |
| 0.8 | 2.48 | 3.89 | 2.48 + 0.1(3.89) = 2.81 |
| 0.9 | 2.81 | 3.71 | 2.81 + 0.1(3.71) = 3.18 |
| 1.0 | 3.18 | | |

Thus the required approximate value of $y = 3.18$.

**Obs.** In example 32.1, the true value of $y$ from its exact solution at $x = 1$ is 3.44 whereas by Euler's method $y = 3.18$ and by Picard's method $y = 3.434$. In the above solution, had we chosen $n = 20$, the accuracy would have been considerably increased but at the expense of double the labour of computation. Euler's method is no doubt very simple but cannot be considered as one of the best.

**Example 32.7.** *Given* $\dfrac{dy}{dx} = \dfrac{y - x}{y + x}$ *with initial condition* $y = 1$ *at* $x = 0$ *; find* $y$ *for* $x = 0.1$ *by Euler's method.*

*(P.T.U., 2001)*

**Solution.** We divide the interval (0, 0.1) into five steps *i.e.* we take $n = 5$ and $h = 0.02$. The various calculations are arranged as follows :

| $x$ | $y$ | $(y - x)/(y + x) = dy/dx$ | *Old* $y + 0.02\ (dy/dx) =$ *new* $y$ |
|---|---|---|---|
| 0.00 | 1.0000 | 1.0000 | 1.0000 + 0.02 (1.0000) = 1.0200 |
| 0.02 | 1.0200 | 0.9615 | 1.0200 + 0.02 (.9615) = 1.0392 |
| 0.04 | 1.0392 | 0.926 | 1.0392 + 0.02 (.926) = 1.0577 |
| 0.06 | 1.0577 | 0.893 | 1.0577 + 0.02 (.893) = 1.0756 |
| 0.08 | 1.0756 | 0.862 | 1.0756 + 0.02 (.862) = 1.0928 |
| 0.10 | 1.0928 | | |

Hence the required approximate value of $y = 1.0928$.

## 32.5 MODIFIED EULER'S METHOD

In the Euler's method, the curve of solution in the interval $LL_1$ is approximated by the tangent at $P$ (Fig. 32.1) such that at $P_1$, we have

$$y_1 = y_0 + h\,f(x_0, y_0) \qquad ...(1)$$

Then the slope of the curve of solution through $P_1$ [*i.e.* $(dy/dx)_{P_1} = f(x_0 + h, y_1)$] is computed and the tangent at $P_1$ to $P_1Q_1$ is drawn meeting the ordinate through $L_2$ in $P_2(x_0 + 2h, y_2)$.

Now we find a better approximation $y_1^{(1)}$ of $y(x_0 + h)$ by taking the slope of the curve as the mean of the slopes of the tangents at $P$ and $P_1$, *i.e.*

$$y_1^{(1)} = y_0 + \frac{h}{2}\,[f(x_0, y_0) + f(x_0 + h, y_1)] \qquad ...(2)$$

As the slope of the tangent at $P_1$ is not known, we take $y_1$ as found in (1) by Euler's method and insert it on R.H.S. of (2) to obtain the first modified value $y_1^{(1)}$. The equation (1) is therefore, called the *predictor* while (2) serves as the *corrector of* $y_1$.

Again the corrector is applied and we find a still better value $y_1^{(2)}$ corresponding to $L_1$ as

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_0 + h, y_1^{(1)})]$$

We repeat this step, till two consecutive values of $y$ agree. This is then taken as the starting point for the next interval $L_1L_2$.

Once $y_1$ is obtained to desired degree of accuracy, $y$ corresponding to $L_2$ is found from the *predictor*

$$y_2 = y_1 + hf(x_0 + h, y_1)$$

and a better approximation $y_2^{(1)}$ is obtained from the *corrector*

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_0 + h, y_1) + f(x_0 + 2h, y_2)].$$

We repeat this step until $y_2$ becomes stationary. Then we proceed to calculate $y_3$ as above and so on.

This is the *modified Euler's method* which is a predictor-corrector method.

**Example 32.8.** *Using modified Euler's method, find an approximate value of y when $x = 0.3$, given that $dy/dx = x + y$ and $y = 1$ when $x = 0$.* (Rohtak, 2005 ; Bhopal, 2002 S ; Delhi, 2002)

**Solution.** Taking $h = 0.1$, the various calculations are arranged as follows :

| $x$ | $x + y = y'$ | *Mean slope* | *Old y + 0.1 (mean slope) = new y* |
|---|---|---|---|
| 0.0 | 0 + 1 | — | 1.00 + 0.1 (1.00) = 1.10 |
| 0.1 | .1 + 1.1 | $\frac{1}{2}(1 + 1.2)$ | 1.00 + 0.1 (1.1) = 1.11 |
| 0.1 | .1 + 1.11 | $\frac{1}{2}(1 + 1.21)$ | 1.00 + 0.1 (1.105) = 1.1105 |
| 0.1 | .1 + 1.1105 | $\frac{1}{2}(1 + 1.2105)$ | 1.00 + 0.1 (1.1052) = 1.1105 |
| 0.1 | 1.2105 | — | 1.1105 + 0.1 (1.2105) = 1.2316 |
| 0.2 | .2 + 1.2316 | $\frac{1}{2}(1.2105 + 1.4316)$ | 1.1105 + 0.1 (1.3211) = 1.2426 |
| 0.2 | .2 + 1.2426 | $\frac{1}{2}(1.2105 + 1.4426)$ | 1.1105 + 0.1 (1.3266) = 1.2432 |
| 0.2 | .2 + 1.2432 | $\frac{1}{2}(1.2105 + 1.4432)$ | 1.1105 + 0.1 (1.3268) = 1.2432 |
| 0.2 | 1.4432 | — | 1.2432 + 0.1 (1.4432) = 1.3875 |
| 0.3 | .3 + 1.3875 | $\frac{1}{2}(1.4432 + 1.6875)$ | 1.2432 + 0.1 (1.5654) = 1.3997 |
| 0.3 | .3 + 1.3997 | $\frac{1}{2}(1.4432 + 1.6997)$ | 1.2432 + 0.1 (1.5715) = 1.4003 |
| 0.3 | .3 + 1.4003 | $\frac{1}{2}(1.4432 + 1.7003)$ | 1.2432 + 0.1 (1.5718) = 1.4004 |
| 0.3 | .3 + 1.4004 | $\frac{1}{2}(1.4432 + 1.7004)$ | 1.2432 + 0.1 (1.5718) = 1.4004 |

Hence $y(0.3) = 1.4004$ approximately.

**Obs.** In example 32.6, the approximate value of $y$ for $x = 0.3$ would be 1.53 whereas by modified Euler's method the corresponding value is 1.4004 which is nearer its true value 1.3997, obtained from its exact solution $y = 2e^x - x - 1$ by putting $x = 0.3$.

**Example 32.9.** *Using modified Euler's method, find y(0.2) and y(0.4) given*

$$y' = y + e^x, \; y(0) = 0.$$

(J.N.T.U., 2009)

**Solution.** We have $y' = y + e^x = f(x, y)$ ; $x = 0, y = 0$ and $h = 0.2$

The various calculations are arranged as under :

To calculate $y(0.2)$ :

| $x$ | $y + e^x = y'$ | *Mean slope* | *Old y + h (mean slope) = new y* |
|---|---|---|---|
| 0.0 | 1 | — | $0 + 0.2\,(1) = 0.2$ |
| 0.2 | $0.2 + e^{0.2} = 1.4214$ | $\frac{1}{2}(1 + 1.4214) = 1.2107$ | $0 + 0.2\,(1.2107) = 0.2421$ |
| 0.2 | $0.2421 + e^{0.2} = 1.4635$ | $\frac{1}{2}(1 + 1.4635) = 1.2317$ | $0 + 0.2\,(1.2317) = 0.2463$ |
| 0.2 | $0.2463 + e^{0.2} = 1.4677$ | $\frac{1}{2}(1 + 1.4677) = 1.2338$ | $0 + 0.2\,(1.2338) = 0.2468$ |
| 0.2 | $0.2468 + e^{0.2} = 1.4682$ | $\frac{1}{2}(1 + 1.4682) = 1.2341$ | $0 + 0.2\,(1.2341) = 0.2468$ |

Since the last two values of $y$ are equal, we take $y(0.2) = 0.2468$.

To calculate $y(0.4)$.

| $x$ | $y + e^x = y'$ | *Mean slope* | *Old y + h (Mean slope) = new y* |
|---|---|---|---|
| 0.2 | $0.2468 + e^{0.2} = 1.4682$ | — | $0.2468 + 0.2\,(1.4682) = 0.5404$ |
| 0.4 | $0.5404 + e^{0.4} = 2.0322$ | $\frac{1}{2}(1.4682 + 2.0322) = 1.7502$ | $0.2468 + 0.2\,(1.7502) = 0.5968$ |
| 0.4 | $0.5968 + e^{0.4} = 2.0887$ | $\frac{1}{2}(1.4682 + 2.0887) = 1.7784$ | $0.2468 + 0.2\,(1.7784) = 0.6025$ |
| 0.4 | $0.6025 + e^{0.4} = 2.0943$ | $\frac{1}{2}(1.4682 + 2.0943) = 1.78125$ | $0.2468 + 0.2\,(1.78125) = 0.6030$ |
| 0.4 | $0.6030 + e^{0.4} = 2.0949$ | $\frac{1}{2}(1.4682 + 2.0949) = 1.7815$ | $0.2468 + 0.2\,(1.7815) = 0.6031$ |
| 0.4 | $0.6031 + e^{0.4} = 2.0949$ | $\frac{1}{2}(1.4682 + 2.0949) = 1.7816$ | $0.2468 + 0.2\,(1.7815) = 0.6031$ |

Since the last two value of $y$ are equal, we take $y(0.4) = 0.6031$.

Hence $y(0.2) = 0.2468$ and $y(0.4) = 0.6031$ approximately.

**Example 32.10.** *Solve the following by Euler's modified method :*

$$\frac{dy}{dx} = \log(x + y),\; y(0) = 2.$$

*at $x = 1.2$ and $1.4$ with $h = 0.2$.* *(Bhopal, 2009 ; U.P.T.U., 2007)*

**Solution.** The various calculations are arranged as follows :

| $x$ | $\log(x + y) = y'$ | *Mean slope* | *Old y + 0.2 (mean slope) = new y* |
|---|---|---|---|
| 0.0 | $\log(0 + 2)$ | — | $2 + 0.2\,(0.301) = 2.0602$ |
| 0.2 | $\log(0.2 + 2.0602)$ | $\frac{1}{2}(0.301 + 0.3541)$ | $2 + 0.2\,(0.3276) = 2.0655$ |
| 0.2 | $\log(0.2 + 2.0655)$ | $\frac{1}{2}(0.301 + 0.3552)$ | $2 + 0.2\,(0.3281) = 2.0656$ |
| 0.2 | 0.3552 | — | $2.0656 + 0.2\,(0.3552) = 2.1366$ |
| 0.4 | $\log(0.4 + 2.1366)$ | $\frac{1}{2}(0.3552 + 0.4042)$ | $2.0656 + 0.2\,(0.3797) = 2.1415$ |
| 0.4 | $\log(0.4 + 2.1415)$ | $\frac{1}{2}(0.3552 + 0.4051)$ | $2.0656 + 0.2\,(0.3801) = 2.1416$ |

| $x$ | $\log(x+y) = y'$ | *Mean slope* | *Old y + 0.2 (mean slope) = new y* |
|---|---|---|---|
| 0.4 | 0.4051 | — | 2.1416 + 0.2 (0.4051) = 2.2226 |
| 0.6 | log (0.6 + 2.2226) | $\frac{1}{2}$(0.4051 + 0.4506) | 2.1416 + 0.2 (0.4279) = 2.2272 |
| 0.6 | log (0.6 + 2.2272) | $\frac{1}{2}$(0.4051 + 0.4514) | 2.1416 + 0.2 (0.4282) = 2.2272 |
| 0.6 | 0.4514 | — | 2.2272 + 0.2 (0.4514) = 2.3175 |
| 0.8 | log (0.8 + 2.3175) | $\frac{1}{2}$(0.4514 + 0.4938) | 2.2272 + 0.2 (0.4726) = 2.3217 |
| 0.8 | log (0.8 + 2.3217) | $\frac{1}{2}$(0.4514 + 0.4943) | 2.2272 + 0.2 (0.4727) = 2.3217 |
| 0.8 | 0.4943 | — | 2.3217 + 0.2 (0.4943) = 2.4206 |
| 1.0 | log (1 + 2.4206) | $\frac{1}{2}$(0.4943 + 0.5341) | 2.3217 + 0.2 (0.5142) = 2.4245 |
| 1.0 | log (1 + 2.4245) | $\frac{1}{2}$(0.4943 + 0.5346) | 2.3217 + 0.2 (0.5144) = 2.4245 |
| 1.0 | 0.5346 | — | 2.4245 + 0.2 (0.5346) = 2.5314 |
| 1.2 | log (1.2 + 2.5314) | $\frac{1}{2}$(0.5346 + 0.5719) | 2.4245 + 0.2 (0.5532) = 2.5351 |
| 1.2 | log (1.2 + 2.5351) | $\frac{1}{2}$(0.5346 + 0.5723) | 2.4245 + 0.2 (0.5534) = 2.5351 |
| 1.2 | 0.5723 | — | 2.5351 + 0.2 (0.5723) = 2.6496 |
| 1.4 | log (1.4 + 2.6496) | $\frac{1}{2}$(0.5723 + 0.6074) | 2.5351 + 0.2 (0.5898) = 2.6531 |
| 1.4 | log (1.4 + 2.6531) | $\frac{1}{2}$(0.5723 + 0.6078) | 2.5351 + 0.2 (0.5900) = 2.6531 |

Hence $y(1.2) = 2.5351$ and $y(1.4) = 2.6531$ approximately.

**Example 32.11.** *Using Euler's modified method, obtain a solution of the equation $dy/dx = x + |\sqrt{y}|$, with initial conditions $y = 1$ at $x = 0$, for the range $0 \le x \le 0.6$ in steps of 0.2.* (V.T.U., 2007)

**Solution.** The various calculations are arranged as follows :

| $x$ | $x + \lvert\sqrt{y}\rvert = y'$ | *Mean slope* | *Old y + .2 (mean slope) = new y* |
|---|---|---|---|
| 0.0 | 0 + 1 = 1 | — | 1 + 0.2 (1) = 1.2 |
| 0.2 | $0.2 + \lvert\sqrt{(1.2)}\rvert = 1.2954$ | $\frac{1}{2}(1 + 1.2954) = 1.1477$ | 1 + 0.2 (1.1477) = 1.2295 |
| 0.2 | $0.2 + \lvert\sqrt{(1.2295)}\rvert = 1.3088$ | $\frac{1}{2}(1 + 1.3088) = 1.1544$ | 1 + 0.2 (1.1544) = 1.2309 |
| 0.2 | $0.2 + \lvert\sqrt{(1.2309)}\rvert = 1.3094$ | $\frac{1}{2}(1 + 1.3094) = 1.1547$ | 1 + 0.2 (1.1547) = 1.2309 |
| 0.2 | 1.3094 | — | 1.2309 + 0.2 (1.3094) = 1.4927 |
| 0.4 | $0.4 + \lvert\sqrt{(1.4927)}\rvert = 1.6218$ | $\frac{1}{2}(1.3094 + 1.6218) = 1.4654$ | 1.2309 + 0.2 (1.4654) = 1.5240 |
| 0.4 | $0.2 + \lvert\sqrt{(1.524)}\rvert = 1.6345$ | $\frac{1}{2}(1.3094 + 1.6345) = 1.4718$ | 1.2309 + 0.2 (1.4718) = 1.5253 |
| 0.4 | $0.4 + \lvert\sqrt{(1.5253)}\rvert = 1.6350$ | $\frac{1}{2}(1.3094 + 1.6350) = 1.4721$ | 1.2309 + 0.2 (1.4721) = 1.5253 |

| $x$ | $x + \lvert\sqrt{y}\rvert = y'$ | Mean slope | Old $y$ + .2 (mean slope) = new $y$ |
|---|---|---|---|
| 0.4 | 1.6350 | — | 1.5253 + 0.2 (1.635) = 1.8523 |
| 0.6 | $0.6 + \lvert\sqrt{(1.8523)}\rvert = 1.9610$ | $\frac{1}{2}(1.635 + 1.961) = 1.798$ | 1.5253 + 0.2 (1.798) = 1.8849 |
| 0.6 | $0.6 + \lvert\sqrt{(1.8849)}\rvert = 1.9729$ | $\frac{1}{2}(1.635 + 1.9729) = 1.8040$ | 1.5253 + 0.2 (1.804) = 1.8861 |
| 0.6 | $0.6 + \lvert\sqrt{(1.8861)}\rvert = 1.9734$ | $\frac{1}{2}(1.635 + 1.9734) = 1.8042$ | 1.5253 + 0.2 (1.8042) = 1.8861 |

Hence $y(0.6) = 1.8861$ approximately.

## PROBLEMS 32.2

1. Apply Euler's method to solve $y' = x + y$, $y(0) = 0$, choosing the step length = 0.2. (Carry out 6 steps). (*Kottayam, 2005*)
2. Using simple Euler's method solve for $y$ at $x = 0.1$ from $dy/dx = x + y + xy$, $y(0) = 1$, taking step size $h = 0.025$.
3. Using Euler's method, find the approximate value of $y$ when $dy/dx = x^2 + y^2$ and $y(0) = 1$ in five steps (*i.e.* $h = 0.2$). (*Mumbai, 2006*)
4. Solve $y' = 1 - y$, $y(0) = 0$ by modified Euler's method and obtain $y$ at $x = 0.1, 0.2, 0.3$. (*Anna, 2005*)
5. Given $y' = x + \sin y$, $y(0) = 1$. Compute $y(0.2)$ and $y(0.4)$ with $h = 0.2$ using Euler's modified method. (*J.N.T.U., 2007*)
6. Given that $dy/dx = x^2 + y$ and $y(0) = 1$. Find an approximate value of $y(0.1)$ taking $h = 0.05$ by modified Euler's method. (*V.T.U., 2010*)
7. Given $\dfrac{dy}{dx} = \dfrac{y - x}{y + x}$ with boundary conditions $y = 1$ when $x = 0$, find approximately $y$ for $x = 0.1$, by Euler's modified method (5 steps). (*V.T.U., 2007*)
8. Given that $dy/dx = 2 + \sqrt{(xy)}$ and $y = 1$ when $x = 1$. Find approximate value of $y$ at $x = 2$ in steps of 0.2, using Euler's modified method. (*Anna, 2004*)

## 32.6 RUNGE'S METHOD*

Consider the differential equation,

$$\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0 \qquad ...(1)$$

Clearly the slope of the curve through $P(x_0, y_0)$ is $f(x_0, y_0)$ (Fig. 32.2).

Integrate both sides of (1) from $(x_0, y_0)$ to $(x_0 + h, y_0 + k)$, we have

$$\int_{y_0}^{y_0 + k} dy = \int_{x_0}^{x_0 + h} f(x, y)\ dx \qquad ...(2)$$

To evaluate the integral on the right, we take $N$ as the mid-point of $LM$ and find the values of $f(x, y)$ (*i.e.* $dy/dx$) at the points $x_0, x_0 + h/2, x_0 + h$. For this purpose, we first determine the values of $y$ at these points.

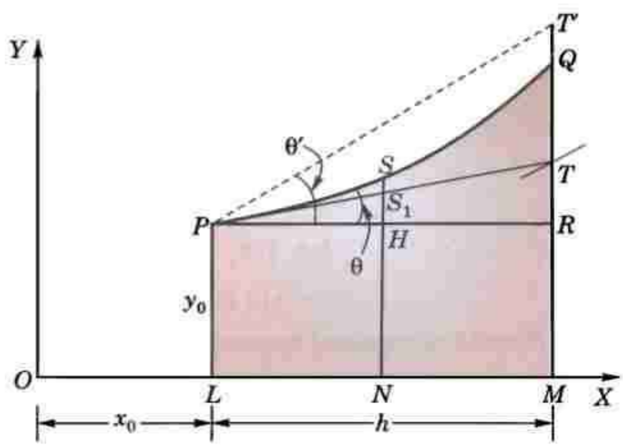

Fig. 32.2

Let the ordinate through $N$ cut the curve $PQ$ in $S$ and the tangent $PT$ in $S_1$. The value of $y_s$ is given by the point $S_1$.

$\therefore \qquad y_s = NS = LP + HS_1 = y_0 + PH \tan\theta$

* Called after the German mathematician *Carl Runge* (1856–1927) who was professor at Gottingen.

$$= y_0 + \frac{h}{2}(dy/dx)_P = y_0 + \frac{h}{2} f(x_0, y_0) \quad ...(3)$$

Also $\quad y_T = MT = LP + RT = y_0 + PR \tan\theta = y_0 + hf(x_0, y_0)$.

Now the value of $y_Q$ at $x_0 + h$ is given by the point $T'$ where the line through $P$ drawn with slope at $T(x_0 + h, y_T)$ meets $MQ$.

$\because$ Slope at $\quad T = \tan\theta' = f(x_0 + h, y_T) = f[x_0 + h, y_0 + hf(x_0, y_0)]$

$$\therefore \quad y_Q = MR + RT' = y_0 + PT \tan\theta' = y_0 + hf[x_0 + h, y_0 + hf(x_0, y_0)] \quad ...(4)$$

Thus the value of $f(x, y)$ at $P = f(x_0, y_0)$,
the value of $f(x, y)$ at $S = f(x_0 + h/2, y_s)$
and the value of $f(x, y)$ at $Q = f(x_0 + h, y_Q)$
where $y_S$ and $y_Q$ are given by (3) and (4).

Hence from (2), we obtain

$$k = \int_{x_0}^{x_0+h} f(x, y)\, dx = \frac{h}{6}[f_P + 4f_S + f_Q] \quad \text{[By Simpsons' rule (p. 1106)]}$$

$$= \frac{h}{6}[f(x_0, y_0) + 4f(x_0 + h/2, y_S) + f(x_0 + h, y_Q)] \quad ...(5)$$

which gives a sufficiently accurate value of $k$ and also of $y = y_0 + k$.

The repeated application of (5) gives the values of $y$ for equispaced points.

**Working rule** *to solve (1) by Runge's method :*

*Calculate successively*

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_1\right)$$

$$k' = hf(x_0 + h, y_0 + k_1)$$

and $$k_3 = hf(x_0 + h, y_0 + k')$$

*Finally compute,* $$k = \frac{1}{6}(k_1 + 4k_2 + k_3).$$

*(Note that $k$ is the weighted mean of $k_1$, $k_2$ and $k_3$)*

**Example 32.12.** *Apply Runge's method to find an approximate value of $y$ when $x = 0.2$, given that $dy/dx = x + y$ and $y = 1$ when $x = 0$.*

**Solution.** Here we have $x_0 = 0, y_0 = 1, h = 0.2, f(x_0, y_0) = 1$

$$\therefore \quad k_1 = hf(x_0, y_0) = 0.2\,(1) = 0.200$$

$$k_2 = hf\left(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_1\right) = 0.2\, f(0.1, 1.1) = 0.240$$

$$k' = hf(x_0 + h, y_0 + k_1) = 0.2\, f(0.2, 1.2) = 0.280$$

and $$k_3 = hf(x_0 + h, y_0 + k') = 0.2\, f(0.1, 1.28) = 0.296$$

$$\therefore \quad k = \tfrac{1}{6}(k_1 + 4k_2 + k_3)$$

$$= \tfrac{1}{6}(0.200 + 0.960 + 0.296) = 0.2426$$

Hence the required approximate value of $y$ is 1.2426.

## 32.7 RUNGE-KUTTA METHOD*

The Taylor's series method of solving differential equations numerically is restricted by the labour involved in finding the higher order derivatives. However there is a class of methods known as Runge-Kutta methods which do not require the calculations of higher order derivatives. These methods agree with Taylor's series solution upto the terms in $h^r$, where $r$ differs from method to method and is called the *order of that method. Euler's method, Modified Euler's method and Runge's method are the Runge-Kutta methods of the first, second and third order respectively.*

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* See footnote p. 1017. Named after *Wilhelm Kutta* (1867—1944).

*The fourth-order Runge-Kutta method is most commonly used and is often referred to as 'Runge-Kutta method' only.*

**Working rule** *for finding the increment k of y corresponding to an increment h of x by Runge-Kutta method from*

$$\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0 \textit{ is as follows :}$$

*Calculate successively*

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}k_1\right)$$

$$k_3 = hf\left(x_0 + \tfrac{1}{2}h,\ y + \tfrac{1}{2}k_2\right)$$

and $$k_4 = hf(x_0 + h, y_0 + k_3)$$

*Finally compute* $$k = \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

which gives the required approximate value $y_1 = y_0 + k$.

*(Note that $k$ is the weighted mean of $k_1$, $k_2$, $k_3$ and $k_4$)*

**Obs.** One of the advantages of these methods is that the operation is identical whether the differential equation is linear or non-linear.

**Example 32.13.** *Apply Runge-Kutta fourth order method, to find an approximate value of y when x = 0.2, given that dy/dx = x + y and y = 1 when x = 0.* (V.T.U., 2009 ; P.T.U., 2007 ; S.V.T.U., 2007)

**Solution.** Here $x_0 = 0, y_0 = 1, h = 0.2, f(x_0, y_0) = 1$

$\therefore$ $$k_1 = hf(x_0, y_0) = 0.2 \times 1 = 0.2000$$

$$k_2 = hf\left(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}k_1\right) = 0.2 \times f(0.1, 1.1) = 0.2400$$

$$k_3 = hf\left(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}k_2\right) = 0.2 \times f(0.1, 1.12) = 0.2440$$

and $$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2 \times f(0.2, 1.244) = 0.2888$$

$\therefore$ $$k = \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \tfrac{1}{6}(0.2000 + 0.4800 + 0.4880 + 0.2888) = \tfrac{1}{6} \times (1.4568) = 0.2468.$$

Hence the required approximate value of $y$ is 1.2428.

**Example 32.14.** *Using Runge-Kutta method of fourth order, solve $\dfrac{dy}{dx} = \dfrac{y^2 - x^2}{y^2 + x^2}$ with y(0) = 1 at x = 0.2, 0.4.* (U.P.T.U., 2010 ; J.N.T.U., 2009 ; V.T.U., 2008)

**Solution.** We have $f(x, y) = \dfrac{y^2 - x^2}{y^2 + x^2}$

*To find y(0.2) :*

Here $x_0 = 0, y_0 = 1, h = 0.2$

$$k_1 = h\,f(x_0, y_0) = 0.2\,f(0, 1) = 0.2000$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_1\right) = 0.2\,f(0.1, 1.1) = 0.19672$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_2\right) = 0.2\,f(0.1, 1.09836) = 0.1967$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2\,f(0.2, 1.1967) = 0.1891$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \frac{1}{6}[0.2 + 2(0.19672) + 2(0.1967) + 0.1891]$$

$$= 0.19599$$

Hence $$y(0.2) = y_0 + k = 1.196.$$

*To find y(0.4) :*

Here $x_1 = 0.2, y_1 = 1.196, h = 0.2$

$$k_1 = h\,f(x_1, y_1) \qquad = 0.1891$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = 0.2\,f(0.3, 1.2906) \qquad = 0.1795$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = 0.2\,f(0.3, 1.2858) \qquad = 0.1793$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.2\,f(0.4, 1.3753) \qquad = 0.1688$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[0.1891 + 2(0.1795) + 2(0.1793) + 0.1688] \qquad = 0.1792$$

Hence $y(0.4) = y_1 + k = 1.196 + 0.1792 = 1.3752.$

**Example 32.15.** *Apply Runge-Kutta method to find an approximate value of y for x = 0.2 in steps of 0.1, if $dy/dx = x + y^2$, given that y = 1, where x = 0.* *(V.T.U., 2009 ; Osmania, 2007 ; Madras, 2000)*

**Solution.** Here we take $h = 0.1$ and carry out the calculations in two steps.

**Step I.** $x_0 = 0, y_0 = 1, h = 0.1$

$$\therefore \quad k_1 = hf(x_0, y_0) = 0.1\,f(0, 1) \qquad = 0.1000$$

$$k_2 = hf\left(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_1\right) = 0.1\,f(0.05, 1.1) \qquad = 0.1152$$

$$k_3 = hf\left(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_2\right) = 0.1\,f(0.05, 1.1152) \qquad = 0.1168$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1\,f(0.1, 1.1168) \qquad = 0.1347$$

$$\therefore \quad k = \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \tfrac{1}{6}(0.1000 + 0.2304 + 0.2336 + 0.1347) \qquad = 0.1165$$

giving $y(0.1) = y_0 + k = 1.1165.$

**Step II.** $x_1 = x_0 + h = 0.1, y_1 = 1.1165, h = 0.1$

$$\therefore \quad k_1 = hf(x_1, y_1) = 0.1\,f(0.1, 1.1165) \qquad = 0.1347$$

$$k_2 = hf\left(x_1 + \tfrac{1}{2}h, y_1 + \tfrac{1}{2}k_1\right) = 0.1\,f(0.15, 1.1838) \qquad = 0.1551$$

$$k_3 = hf\left(x_1 + \tfrac{1}{2}h, y_1 + \tfrac{1}{2}k_2\right) = 0.1\,f(0.15, 1.194) \qquad = 0.1576$$

$$k_4 = hf(x_1 + h, y_2 + k_3) = 0.1\,f(0.2, 1.1576) \qquad = 0.1823$$

$$\therefore \quad k = \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad = 0.1571$$

Hence $y(0.2) = y_1 + k = 1.2736.$

**Example 32.16.** *Using Runge-Kutta method of fourth order, solve for y at x = 1.2, 1.4 from $\dfrac{dy}{dx} = \dfrac{2xy + e^x}{x^2 + xe^x}$ given $x_0 = 1, y_0 = 0$.* *(Mumbai, 2008)*

**Solution.** We have $f(x, y) = \dfrac{2xy + e^x}{x^2 + xe^2}$

*To find y(1.2) :*

Here $x_0 = 1, y_0 = 0, h = 0.2$

$$\therefore \quad k_1 = h\,f(x_0, y_0) = 0.2\,\frac{0 + e}{1 + e} = 0.1462$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2\left\{\frac{2(1 + 0.1)(0 + 0.073) + e^{1+0.1}}{(1 + 0.1)^2 + (1 + 0.1)\,e^{1+0.1}}\right\} = 0.1402$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2 \left\{\frac{2(1+0.1)(0+0.07)+e^{1.1}}{(1+0.1)^2+(1+0.1)\,e^{1.1}}\right\} = 0.1399$$

$$k_4 = hf(x_0 + k, y_0 + k_3) = 0.2 \left\{\frac{2(1.2)(0.1399)+e^{1.2}}{(1.2)^2+(1.2)\,e^{1.2}}\right\} = 0.1348$$

and $$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \frac{1}{6}[0.1462 + 0.2804 + 0.2798 + 0.1348]$$

$$= 0.1402.$$

Hence $y(1.2) = y_0 + k = 0 + 0.1402 = 0.1402.$

*To find y(1.4) :*

Here $x_1 = 1.2, y_1 = 0.1402, h = 0.2$

$$k_1 = hf(x_1, y_1) = 0.2\,f(1.2, 0) = 0.1348$$
$$k_2 = hf(x_1 + h/2, y_1 + k_1/2) = 0.2\,f(1.3, 0.2076) = 0.1303$$
$$k_3 = hf(x_1 + h/2, y_1 + k_1/2) = 0.2\,f(1.3, 0.2053) = 0.1301$$
$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.2\,f(1.3, 0.2703) = 0.1260$$

$\therefore$ $$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \frac{1}{6}[0.1348 + 0.2606 + 0.2602 + 0.1260] = 0.1303$$

Hence $y(1.4) = y_1 + k = 0.1402 + 0.1303 = 0.2705.$

## PROBLEMS 32.3

1. Use Runge's method to approximate $y$ when $x = 1.1$, given that $y = 1.2$ when $x = 1$ and $dy/dx = 3x + y^2$.
2. Using Runge-Kutta method of order 4, find $y(0.2)$ given that $dy/dx = 3x + \frac{1}{2}y$, $y(0) = 1$, taking $h = 0.1$. *(V.T.U., 2004)*
3. Using Runge-Kutta method of order 4, compute $y(.2)$ and $(.4)$ from $10\frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$, taking $h = 0.1$. *(Rohtak, 2003 ; Bhopal, 2002)*
4. Use Runge-Kutta method to find $y$ when $x = 1.2$ in steps of 0.1, given that :
$dy/dx = x^2 + y^2$ and $y(1) = 1.5$. *(Mumbai, 2007)*
5. Find $y(0.1)$ and $y(0.2)$ using Runge-Kutta 4th order formula, given that $y' = x^2 - y$ and $y(0) = 1$. *(J.N.T.U., 2006)*
6. Using 4th order Runge-Kutta method, solve the following equation, taking each step of $h = 0.1$, given $y(0) = 3$, $dy/dx = (4x/y - xy)$. Calculate $y$ for $x = 0.1$ and 0.2. *(Anna, 2007)*
7. Use fourth order Runge-Kutta method to find $y$ at $x = 0.1$, given that $\frac{dy}{dx} = 3e^x + 2y$, $y(0) = 0$ and $h = 0.1$. *(V.T.U., 2006)*
8. Find by Runge-Kutta method an approximate value of $y$ for $x = 0.8$, given that $y = 0.41$ when $x = 0.4$ and $dy/dx = \sqrt{(x+y)}$. *(S.V.T.U., 2007 S)*
9. Using Runge-Kutta method of order 4, find $y(0.2)$ for the equation $\frac{dy}{dx} = \frac{y-x}{y+x}$, $y(0) = 1$. Take $h = 0.2$. *(V.T.U., 2011 S)*
10. Given that $dy/dx = (y^2 - 2x)/(y^2 + x)$ and $y = 1$ at $x = 0$ ; find $y$ for $x = 0.1, 0.2, 0.3, 0.4$ and 0.5. *(Delhi, 2002)*

## 32.8 PREDICTOR-CORRECTOR METHODS

If $x_{i-1}$ and $x_i$ be two consecutive mesh points, we have $x_i = x_{i-1} + h$. In the Euler's method (§ 32.4), we have

$$y_i = y_{i-1} + hf(x_0 + \overline{i-1}\,h, y_{i-1})\ ;\ i = 1, 2, 3, \ldots \qquad \ldots(1)$$

The modified Euler's method (§ 32.5), gives

$$y_i = y_{i-1} + \frac{h}{2}\,[f(x_{i-1}, y_{i-1}) + f(x_i, y_i)] \qquad ...(2)$$

The value of $y_i$ is first estimated by using (1), then this value is inserted on the right side of (2), giving a better approximation of $y_i$. This value of $y_i$ is again substituted in (2) to find a still better approximation of $y_i$. This step is repeated till two consecutive values of $y_i$ agree. *This technique of refining an initially crude estimate of $y_i$ by means of a more accurate formula is known as* **predictor-corrector method.** The equation (1) is therefore called the *predictor* while (2) serves as a *corrector* of $y_i$.

In the methods so far explained, to solve a differential equation over an interval $(x_i, x_{i+1})$ only the value of $y$ at the beginning of the interval was required. In the *predictor-corrector* methods, four prior values are required for finding the value of $y$ at $x_{i+1}$. A predictor formula is used to predict the value of $y$ at $x_{i+1}$ and then a corrector formula is applied to improve this value.

We now describe two such methods, namely : Milne's method and Adams-Bashforth method.

## 32.9 MILNE'S METHOD

*Given $dy/dx = f(x, y)$ and $y = y_0, x = x_0$ ; to find an approximate value of $y$ for $x = x_0 + nh$ by Milne's method, we proceed as fo... :*

The value $y_0 = y(x_0)$ being given, we compute

$$y_1 = y(x_0 + h),\ y_2 = y(x_0 + 2h),\ y_3 = y(x_0 + 3h),$$

by Picard's or Taylor's series method.

Next we calculate,

$$f_0 = f(x_0, y_0),\ f_1 = f(x_0 + h, y_1),\ f_2 = f(x_0 + 2h, y_2),\ f_3 = f(x_0 + 3h, y_3)$$

Then to find $y_4 = y(x_0 + 4h)$, we substitute Newton's forward interpolation formula

$$f(x, y) = f_0 + n\Delta f_0 + \frac{n(n-1)}{2}\,\Delta^2 f_0 + \frac{n(n-1)(n-2)}{6}\,\Delta^3 f_0 + ...$$

in the relation $y_4 = y_0 + \int_{x_0}^{x_0+4h} f(x, y)\,dx$

$$\therefore \quad y_4 = y_0 + \int_{x_0}^{x_0+4h}\left(f_0 + n\Delta f_0 + \frac{n(n-1)}{2}\Delta^2 f_0 + ...\right)dx \qquad [\text{Put } x = x_0 + nh,\ dx = h\,dn]$$

$$= y_0 + h\int_0^4\left(f_0 + n\Delta f_0 + \frac{n(n-1)}{2}\Delta^2 f_0 + ...\right)dn$$

$$= y_0 + h\left(4f_0 + 8\Delta f_0 + \frac{20}{3}\Delta^2 f_0 + \frac{8}{3}\Delta^3 f_0 + ...\right)$$

Neglecting fourth and higher order differences and expressing $\Delta f_0$, $\Delta^2 f_0$ and $\Delta^3 f_0$ in terms of the function values, we get

$$y_4^{(p)} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3) \text{ which is called a } \textit{predictor}.$$

Having found $y_4$, we obtain a first approximation to $f_4 = f(x_0 + 4h, y_4)$.

Then a better value of $y_4$ is found by Simpson's rule (p. 1106) as

$$y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4) \text{ which is called a } \textit{corrector}.$$

Then an improved value of $f_4$ is computed and again the corrector is applied to find a still better value of $y_4$. We repeat this step until $y_4$ remains unchanged.

Once $y_4$ and $f_4$ are obtained to desired degree of accuracy, $y_5 = y(x_0 + 5h)$ is found from the *predictor* as

$$y_5^{(p)} = y_1 + \frac{4h}{3}(2f_2 - f_3 + 2f_4)$$

and $f_5 = f(x_0 + 5h, y_5)$ is calculated. Then a better approximation to the value of $y_5$ is obtained from the *corrector* as

$$y_5^{(c)} = y_3 + \frac{h}{3}(f_3 + 4f_4 + f_5).$$

We repeat this step till $y_5$ becomes stationary and we, then proceed to calculate $y_6$ as before.

This is *Milne's predictor-corrector method.* To ensure greater accuracy, we must first improve the accuracy of the starting values and then sub-divide the intervals.

**Example 32.17.** *Apply Milne's method, to find a solution of the differential equation $y' = x - y^2$ in the range $0 \le x \le 1$ for the boundary conditions $y = 0$ at $x = 0$.* (V.T.U., 2009, Anna, 2005, Rohtak, 2005)

**Solution.** Using Picard's method, we have

$$y = y(0) + \int_0^x f(x, y)\, dx, \text{ where } f(x, y) = x - y^2.$$

To get the first approximation, we put $y = 0$ in $f(x, y)$,

giving $$y_1 = 0 + \int_0^x x\, dx = \frac{x^2}{2}$$

To find the second approximation, we put $y = x^2/2$ in $f(x, y)$,

giving $$y_2 = \int_0^x \left(x - \frac{x^4}{4}\right) dx = \frac{x^2}{2} - \frac{x^5}{20}$$

Similarly, the third approximation is

$$y_3 = \int_0^x \left[x - \left(\frac{x^2}{2} - \frac{x^5}{20}\right)^2\right] dx = \frac{x^2}{2} - \frac{x^5}{20} + \frac{x^8}{160} - \frac{x^{11}}{4400} \quad ...(i)$$

Now let us determine the starting values of the Milne's method from $(i)$, by choosing $h = 0.2$.

$\therefore$

| | | |
|---|---|---|
| $x_0 = 0.0,$ | $y_0 = 0.0000,$ | $f_0 = 0.0000$ |
| $x_1 = 0.2,$ | $y_1 = 0.020,$ | $f_1 = 0.1996$ |
| $x_2 = 0.4,$ | $y_2 = 0.0795,$ | $f_2 = 0.3937$ |
| $x_3 = 0.6,$ | $y_3 = 0.1762,$ | $f_3 = 0.5689$ |

Using the *predictor*, $$y_4^{(p)} = y_0 \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$x = 0.8,$ $\quad y_4^{(p)} = 0.3049,$ $\quad f_4 = 0.7070$

and the *corrector*, $$y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4), \text{ yields}$$

$$y_4^{(c)} = 0.3046, \quad f_4 = 0.7072 \quad ...(ii)$$

Again using the *corrector*, $y_4^{(c)} = 0.3046$, which is same as in $(ii)$

Now using the *predictor*, $$y_5^{(p)} = y_1 + \frac{4h}{3}(2f_2 - f_3 + 2f_4),$$

$x = 1.0,$ $\quad y_5^{(p)} = 0.4554,$ $\quad f_5 = 0.7926$

and the *corrector*, $$y_5^{(c)} = y_3 + \frac{h}{3}(f_3 + 4f_4 + f_5), \text{ gives}$$

$$y_5^{(c)} = 0.4555, \quad f_5 = 0.7925$$

Again using the *corrector*,

$$y_5^{(c)} = 0.4555, \text{ a value which is the same as before.}$$

Hence, $\quad y(1) = 0.4555.$

**Example 32.18.** *Given $y' = x(x^2 + y^2)\, e^{-x}$, $y(0) = 1$, find $y$ at $x = 0.1, 0.2$ and $0.3$ by Taylor's series method and compute $y(0.4)$ by Milne's method.* (Anna, 2007)

**Solution.** Given $y(0) = 1$ and $h = 0.1$

We have $y'(x) = x(x^2 + y^2)e^{-x}$; $y'(0) = 0$

$$y''(x) = [(x^3 + xy^2)(-e^{-x}) + 3x^2 + y^2 + x(2y)y']e^{-x}$$

$$= e^{-x}[-x^3 - xy^2 + 3x^2 + y^2 + 2xyy']; \qquad y''(0) = 1$$

$$y'''(x) = -e^{-x}[-x^3 - xy^2 + 3x^2 + y^2 + 2xyy' + 3x^2 + y^2 + 2xyy' - 6x - 2yy' - 2xy'^2 - 2xyy'] \qquad y'''(0) = -2$$

Substitute these values in the Taylor's series,

$$y(x) = y(0) + \frac{x}{1!}y'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \ldots$$

$$y(0.1) = 1 + (0.1)(0) + \frac{1}{2}(0.1)^2(1) + \frac{1}{6}(0.1)^3(-2) + \ldots$$

$$= 1 + 0.005 - 0.0003 = 1.0047 \quad i.e., \quad 1.005$$

Now taking $x = 0.1, y(0.1) = 1.005, h = 0.1$

$$y'(0.1) = 0.092, y''(0.1) = 0.849, y'''(0.1) = -1.247$$

Substituting these values in the Taylor's series about $x = 0.1$,

$$y(0.2) = y(0.1) + \frac{0.1}{1!}y'(0.1) + \frac{(0.1)^2}{2!}y''(0.1) + \frac{(0.1)^3}{3!}y'''(0.1) + \ldots$$

$$= 1.005 + (0.1)(0.092) + \frac{(0.1)^2}{2}(0.849) + \frac{(0.1)^3}{3}(-1.247) + \ldots$$

$$= 1.018$$

Now taking $x = 0.2, y(0.2) = 1.018, h = 0.1$

$$y'(0.2) = 0.176, y''(0.2) = 0.77, y'''(0.2) = 0.819$$

Substituting these values in the Taylor's series

$$y(0.3) = y(0.2) + \frac{0.1}{1!}y''(0.2) + \frac{(0.1)^2}{2!}y''(0.2) + \frac{(0.1)^3}{3!}y'''(0.2) + \ldots$$

$$= 1.018 + 0.0176 + 0.0039 + 0.0001 = 1.04$$

Thus the starting values of the Milne's method with $h = 0.1$ are

| | | |
|---|---|---|
| $x_0 = 0.0$ | $y_0 = 1$ | $f_0 = y'_0 = 0$ |
| $x_1 = 0.1$ | $y_1 = 1.005$ | $f_1 = 0.092$ |
| $x_2 = 0.2$ | $y_2 = 1.018$ | $f_2 = 0.176$ |
| $x_3 = 0.3$ | $y_3 = 1.04$ | $f_3 = 0.26$ |

Using the predictor, $y_4^{(p)} = y_0 + \dfrac{4h}{3}(2f_1 - f_2 + 2f_3)$

$$= 1 + \frac{4(0.1)}{3}[2(0.092) - (0.176) + 2(0.26)] = 1.09$$

$\therefore$ $x = 0.4$ $\quad y_4^{(p)} = 1.09 \quad f_4 = y'(0.4) = 0.362$

Using the corrector, $y_4^{(c)} = y_2 + \dfrac{h}{3}(f_2 + 4f_3 + f_4)$

$$\therefore \quad y_4^{(c)} = 0.018 + \frac{0.1}{3}(0.176 + 4(0.26) + 0.362) = 1.071$$

Hence $y(0.4) = 1.071$.

**Example 32.19.** *Using Runge-Kutta method of order 4, find y for x = 0.1, 0.2, 0.3 given that dy/dx = xy + $y^2$, y(0) = 1. Continue the solution at x = 0.4 using Milne's method.*

*(V.T.U., 2008 ; S.V.T.U., 2007 ; Madras, 2006)*

**Solution.** We have $f(x, y) = xy + y^2$.

*To find y(0.1) :*

Here $x_0 = 0, y_0 = 1, h = 0.1$.

$\therefore \quad k_1 = h\,f(x_0, y_0) = (0.1)\,f(0.1) \qquad = 0.1000$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = (0.1)\,f(0.05, 1.05) \qquad = 0.1155$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = (0.1)\,f(0.05, 1.0577) \qquad = 0.1172$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = (0.1)\,f(0.1, 1.1172) \qquad = 0.13598$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.1 + 0.231 + 0.2348 + 0.13598) \qquad = 0.11687$$

Thus $y(0.1) = y_1 = y_0 + k = 1.1169.$

*To find y(0.2) :*

Here $x_1 = 0.1, y_1 = 1.1169, h = 0.1.$

$$k_1 = h\,f(x_1, y_1) = (0.1)\,f(0.1, 1.1169) \qquad = 0.1359$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = (0.1)\,f(0.15, 1.1848) \qquad = 0.1581$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = (0.1)\,f(0.15, 1.1959) \qquad = 0.1609$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = (0.1)\,f(0.2, 1.2778) \qquad = 0.1888$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad = 0.1605$$

Thus $y(0.2) = y_2 = y_1 + k = 1.2773.$

*To find y(0.3) :*

Here $x_2 = 0.2, y_2 = 1.2773, h = 0.1.$

$$k_1 = hf(x_2, y_2) = (0.1)\,f(0.2, 1.2773) \qquad = 0.1887$$

$$k_2 = hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_1\right) = (0.1)\,f(0.25, 1.3716) \qquad = 0.2224$$

$$k_3 = hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_2\right) = (0.1)\,f(0.25, 1.3885) \qquad = 0.2275$$

$$k_4 = hf(x_2 + h, y_2 + k_3) = (0.1)\,f(0.3, 1.5048) \qquad = 0.2716$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad = 0.2267$$

Thus $y(0.3) = y_3 = y_2 + k = 1.504.$

Now the starting values of the Milne's method are :

| | | |
|---|---|---|
| $x_0 = 0.0$ | $y_0 = 1.0000$ | $f_0 = 1.0000$ |
| $x_1 = 0.1$ | $y_1 = 1.1169$ | $f_1 = 1.3591$ |
| $x_2 = 0.2$ | $y_2 = 1.2773$ | $f_2 = 1.8869$ |
| $x_3 = 0.3$ | $y_3 = 1.5049$ | $f_3 = 2.7132$ |

Using the *predictor*,

$$y_4^{(p)} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$$x_4 = 0.4 \qquad y_4^{(p)} = 1.8344 \qquad f_4 = 4.0988$$

and the *corrector*,

$$y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4) \text{ yields}$$

$$y_4^{(c)} = 1.2773 + \frac{0.1}{3}[1.8869 + 4\,(2.7132) + 4.098]$$

$$= 1.8386 \qquad f_4 = 4.1159$$

Again using the *corrector*,

$$y_4^{(c)} = 1.2773 + \frac{0.1}{3}\,[1.8869 + 4\,(2.7132) + 4.1159]$$
$$= 1.8391 \qquad\qquad f_4 = 4.1182 \qquad ...(i)$$

Again using the *corrector*

$$y_4^{(c)} = 1.2773 + \frac{0.1}{3}\,[1.8869 + 4\,(2.7132) + 4.1182]$$
$$= 1.8392 \text{ which is same as } (i).$$

Hence $y\,(0.4) = 1.8392$.

## PROBLEMS 32.4

1. Given $\frac{dy}{dx} = x^3 + y$, $y(0) = 2$. The value of $y(0.2) = 2.073$, $y(0.4) = 2.452$, and $y(0.6) = 3.023$ are got by R.K. Method of 4th order. Find $y(0.8)$ by Milne's predictor-corrector method taking $h = 0.2$. *(Anna, 2004)*
2. Given $2\,dy/dx = (1 + x^2)y^2$ and $y(0) = 1$, $y(0.1) = 1.06$, $y(0.2) = 1.12$, $y(0.3) = 1.21$, evaluate $y(0.4)$ by Milne's predictor-corrector method. *(V.T.U., 2011 S ; Madras, 2003)*
3. From the data given below, find $y$ at $x = 1.4$, using Milne's predictor-corrector formula :

$$\frac{dy}{dx} = x^2 + \frac{y}{2}$$

| | | | | |
|---|---|---|---|---|
| $x$ : | 1 | 1.1 | 1.2 | 1.3 |
| $y$ : | 2 | 2.2156 | 2.4549 | 2.7514 |

*(V.T.U., 2007)*

4. Using Milne's method, find $y(4.5)$ given $5xy' + y^2 - 2 = 0$ given $y(4) = 1$, $y(4.1) = 1.0049$, $y(4.2) = 1.0097$, $y(4.3) = 1.0143$, $y(4.4) = 1.0187$. *(Anna, 2007)*
5. If $\frac{dy}{dx} = 2e^x - y$, $y(0) = 2$, $y(0.1) = 2.010$, $y(0.2) = 2.04$ and $y(0.3) = 2.09$ ; find $y(0.4)$ using Milne's predictor-corrector method. *(V.T.U., 2010)*
6. Using Runge-Kutta method, calculate $y(0.1)$, $y(0.2)$, and $y(0.3)$ given that $\frac{dy}{dx} - \frac{2xy}{1+x^2} = 1$, $y(0) = 0$. Taking these values as starting values, find $y(0.4)$ by Milne's method.

## 32.10 ADAMS-BASHFORTH METHOD

Given $\frac{dy}{dx} = f\,(x, y)$ and $y_0 = y(x_0)$, we compute

$$y_{-1} = y(x_0 - h),\ y_{-2} = y(x_0 - 2h),\ y_{-3} = y(x_0 - 3h)$$

by Taylor's series of Euler's method or Runge-Kutta method.

Next we calculate $f_{-1} = f\,(x_0 - h, y_{-1})$, $f_{-2} = f\,(x_0 - 2h, y_{-2})$, $f_{-3} = f\,(x_0 - 3h, y_{-3})$.

Then to find $y_1$, we substitute Newton's backward interpolation formula

$$f\,(x, y) = f_0 + n\,\nabla f_0 + \frac{n(n+1)}{2}\,\nabla^2 f_0 + \frac{n(n+1)\,(n+2)}{6}\,\nabla^3 f_0 + \cdots$$

in

$$y_1 = y_0 + \int_{x_0}^{x_0+h} f(x, y)\,dx \qquad ...(1)$$

$$\therefore \qquad y_1 = y_0 + \int_{x_0}^{x_1}\left(f_0 + n\nabla f_0 + \frac{n(n+1)}{2}\,\nabla^2 f_0 + \cdots\right) dx \qquad [\text{Put } x = x_0 + nh,\ dx = hdn]$$

$$= y_0 + h\int_0^1\left(f_0 + n\nabla f_0 + \frac{n(n+1)}{2}\,\nabla^2 f_0 + \cdots\right) dn$$

$$= y_0 + h\left(f_0 + \frac{1}{2}\,\nabla f_0 + \frac{5}{12}\,\nabla^2 f_0 + \frac{3}{8}\,\nabla^3 f_0 + \cdots\right)$$

Neglecting fourth and higher order differences and expressing $\nabla f_0, \nabla^2 f_0$ and $\nabla^3 f_0$ in terms of function values, we get

$$y_1^{(p)} = y_0 + \frac{h}{24}\,(55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3}) \qquad ...(2)$$

This is called *Adams-Bashforth predictor formula.*

Having found $y_1$, we find $f_1 = f(x_0 + h_1, y_1)$.

Then to find a better value of $y_1$, we derive a *corrector formula* by substituting Newton's backward formula at $f_1$ *i.e.*,

$$f(x, y) = f_1 + n\nabla f_1 + \frac{n(n+1)}{2}\nabla^2 f_1 + \frac{n(n+1)(n+2)}{6}\nabla^3 f_1 + \cdots \text{ in (1).}$$

$$\therefore \qquad y_1 = y_0 + \int_{x_0}^{x_1}\left(f_1 + n\nabla f_1 + \frac{n(n+1)}{2}\nabla^2 f_1 + \cdots\right) dx \quad [\text{Put } x = x_1 + nh, dx = hdn]$$

$$= y_0 + \int_{-1}^{0}\left(f_1 + n\nabla f_1 + \frac{n(n+1)}{2}\nabla^2 f_1 + \cdots\right) dn$$

$$= y_0 + h\left(f_1 - \frac{1}{2}\nabla f_1 - \frac{1}{12}\nabla^2 f_1 - \frac{1}{24}\nabla^3 f_1 - \cdots\right)$$

Neglecting fourth and higher order differences and expressing $\nabla f_1, \nabla^2 f_1$ and $\nabla^3 f_1$ in terms of function values, we obtain

$$y_1^{(c)} = y_0 + \frac{h}{24}\,(9f_1 + 19f_0 - 5f_{-1} + f_{-2}) \qquad ...(3)$$

which is called a *Adams-Moulton corrector formula.*

Then an improved value of $f_1$ is calculated and again the corrector (3) is applied to find a still better value of $y_1$. This step is repeated till $y_1$ remains unchanged and then proceed to calculate $y_2$ as above.

**Obs.** To apply both Milne and Adams-Bashforth methods, we require four starting values of $y$ which are calculated by means of Picard's method or Taylor's series method or Euler's method or Runge-Kutta method. In practice, the Adams formulae (2) and (3) above together with fourth order Runge-Kutta formulae have been found to be most useful.

**Example 32.20.** *Given* $\frac{dy}{dx} = x^2(1 + y)$ *and* $y(1) = 1$, $y(1.1) = 1.233$, $y(1.2) = 1.548$, $y(1.3) = 1.979$, *evaluate* $y(1.4)$ *by Adams-Bashforth method.* (V.T.U., 2010 ; J.N.T.U., 2009 ; Anna, 2004)

**Solution.** Here $f(x, y) = x^2(1 + y)$.

Starting values of the Adams-Bashforth method with $h = 0.1$, are

$x = 1.0, y_{-3} = 1.000, f_{-3} = (1.0)^2(1 + 1.000) = 2.000$

$= 1.1, y_{-2} = 1.233, f_{-2} = 2.702$

$= 1.2, y_{-1} = 1.548, f_{-1} = 3.669$

$= 1.3, y_0 = 1.979, f_0 = 5.035$

Using th

$\frac{h}{}\,(55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3})$

$73, f_1 = 7.004$

U:

$5f_{-1} + f_{-2})$

$19 \times 5.035 - 5 \times 3.669 + 2.702) = 2.575$

Hence 5.

**Example 32.21.** *If* $\frac{dy}{dx} = 2e^x y, y(0) = 0$, *find* $y(4)$ *using Adams predictor-corrector formula by calculating* $y(1), y(2)$ *and* $y(3)$ *using Euler's modified formula.* (J.N.T.U., 2006)

**Solution.** We have $f(x, y) = 2e^x y$.

*To find 0.1 :*

| $x$ | $2e^x y = y'$ | *Mean slope* | *Old y + h (Mean slope) = new y* |
|---|---|---|---|
| 0.0 | 4 | — | $2 + 0.1\,(4) = 2.4$ |
| 0.1 | $2e^{0.1}\,(2.4) = 5.305$ | $\frac{1}{2}(4 + 5.305) = 4.6524$ | $2 + 0.1\,(4.6524) = 2.465$ |
| 0.1 | $2e^{0.1}\,(2.465) = 5.449$ | $\frac{1}{2}(4 + 5.449) = 4.7244$ | $2 + 0.1\,(4.7244) = 2.472$ |
| 0.1 | $2e^{0.1}\,(2.4724) = 5.465$ | $\frac{1}{2}(4 + 5.465) = 4.7324$ | $2 + 0.1\,(4.7324) = 2.473$ |
| 0.1 | $2e^{0.1}\,(2.473) = 5.467$ | $\frac{1}{2}(4 + 5.467) = 4.7333$ | $2 + 0.1\,(4.7333) = 2.473$ |
| 0.1 | 5.467 | — | $2 + 0.1\,(5.467) = 3.0199$ |
| 0.2 | $2e^{0.2}\,(3.0199) = 7.377$ | $\frac{1}{2}(5.467 + 7.377) = 6.422$ | $2.473 + 0.1\,(6.422) = 3.1155$ |
| 0.2 | 7.611 | $\frac{1}{2}(5.467 + 7.611) = 6.539$ | $2.473 + 0.1\,(6.539) = 3.127$ |
| 0.2 | 7.639 | $\frac{1}{2}(5.467 + 7.639) = 6.553$ | $2.473 + 0.1\,(6.553) = 3.129$ |
| 0.2 | 7.643 | $\frac{1}{2}(5.467 + 7.643) = 6.555$ | $2.473 + 0.1\,(6.455) = 3.129$ |
| 0.2 | 7.643 | — | $3.129 + 0.1\,(7.643) = 3.893$ |
| 0.3 | $2e^{0.3}\,(3.893) = 10.51$ | $\frac{1}{2}(7.643 + 10.51) = 9.076$ | $3.129 + 0.1\,(9.076) = 4.036$ |
| 0.3 | 10.897 | $\frac{1}{2}(7.643 + 10.897) = 9.266$ | $3.129 + 0.1\,(9.2696) = 4.056$ |
| 0.3 | 10.949 | $\frac{1}{2}(7.643 + 10.949) = 9.296$ | $3.129 + 0.1\,(9.296) = 4.058$ |
| 0.3 | 10.956 | $\frac{1}{2}(7.643 + 10.956) = 9.299$ | $3.129 + 0.1\,(9.299) = 4.0586$ |

*To find y(0.4) by Adam's method,* the starting values with $h = 0.1$ are

$x = 0.0 \qquad y_{-3} = 2.4 \qquad f_{-3} = 4$

$x = 0.1 \qquad y_{-2} = 2.473 \qquad f_{-2} = 5.467$

$x = 0.2 \qquad y_{-1} = 3.129 \qquad f_{-1} = 7.643$

$x = 0.3 \qquad y_0 = 4.059 \qquad f_{-0} = 10.956$

Using the predictor formula

$$y_1^{(p)} = y_0 + \frac{h}{24}\,(55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3})$$

$$= 4.059 + \frac{0.1}{24}\,(55 \times 10.957 - 59 \times 7.643 + 37 \times 5.467 - 9 \times 4)$$

$$= 5.383$$

Now $x = 0.4 \qquad y_1 = 5.383 \qquad f_1 = 2e^{0.4}\,(5.383) = 16.061$

Using the corrector formula,

$$y_1^{(c)} = y_0 + \frac{h}{24}(9f_1 + 19f_0 - 5f_{-1} + f_{-2})$$

$$= 4.0586 + \frac{0.1}{24}(9 \times 6.061 + 19 \times 10.956 - 5 \times 7.643 + 5.467) = 5.392$$

Hence $y(0.4) = 5.392$.

**Example 32.22.** *Solve the initial value problem $dy/dx = x - y^2$, $y(0) = 1$ to find $y(0.4)$ by Adam's method. Starting solutions required are to be obtained using Runge-Kutta method of order 4 using step value $h = 0.1$.* *(P.T.U., 2003)*

**Solution.** We have $f(x, y) = x - y^2$.

*To find $y(0.1)$:*

Here $x_0 = 0, y_0 = 1, h = 0.1$

$\therefore$ $k_1 = hf(x_0, y_0) = (0.1) f(0, 1) = -0.1000$

$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = (0.1) f(0.05, 0.95) = -0.08525$

$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = (0.1) f(0.05, 0.9574) = -0.0867$

$k_4 = hf(x_0 + h, y_0 + k_3) = (0.1) f(0.1, 0.9137) = -0.07341$

$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = -0.0883$

Thus $y(0.1) = y_1 = y_0 + k = 1 - 0.0883 = 0.9117$

*To find $y(0.2)$:*

Here $x_1 = 0.1, y_1 = 0.9117, h = 0.1$.

$\therefore$ $k_1 = hf(x_1, y_1) = (0.1) f(0.1, 0.9117) = -0.0731$

$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = (0.1) f(0.15, 0.8751) = -0.0616$

$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = (0.1) f(0.15, 0.8809) = -0.0626$

$k_4 = hf(x_1 + h, y_1 + k_3) = (0.1) f(0.2, 0.8491) = -0.0521$

$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = -0.0623$

Thus $y(0.2) = y_2 = y_1 + k = 0.8494$.

*To find $y(0.3)$:*

Here $x_2 = 0.2, y_2 = 0.8494, y = 0.1$

$k_1 = hf(x_2, y_2) = (0.1) f(0.2, 0.8494) = -0.0521$

$k_2 = hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_1\right) = (0.1) f(0.25, 0.8233) = -0.0428$

$k_3 = hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_2\right) = (0.1) f(0.25, 0.828) = -0.0436$

$k_4 = hf(x_2 + h, y_2 + k_3) = (0.1) f(0.3, 0.8058) = -0.0349$

$k = \frac{1}{6}(k_1 + 2y_2 + 2k_3 + k_4) = -0.0438$

Thus $y(0.3) = y_3 = y_2 + k = 0.8061$

Now the starting values of Adam's method with $h = 0.1$ are:

| | | | |
|---|---|---|---|
| $x = 0.0$ | $y_{-3} = 1.0000$ | $f_{-3} = 0.0 - (1.0)^2$ | $= -1.0000$ |
| $x = 0.1$ | $y_{-2} = 0.9117$ | $f_{-2} = 0.1 - (0.9117)^2$ | $= -1.7312$ |
| $x = 0.2$ | $y_{-1} = 0.8494$ | $f_{-1} = 0.2 - (0.8494)^2$ | $= -0.5215$ |
| $x = 0.3$ | $y_0 = 0.8061$ | $f_0 = 0.3 - (0.8061)^2$ | $= -0.3498$ |

Using the *predictor*,

$$y_1^{(p)} = y_0 + \frac{h}{24}(55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3})$$

$x = 0.4$ $\quad y_1^{(p)} = 0.8061 + \frac{0.1}{24}[55(-0.3498) - 59(-0.5215) + 37(-0.7312) - 9(-1)]$

$= 0.7789 \qquad f_1 = -0.2067$

Using the *corrector*,

$$y_1^{(c)} = y_0 + \frac{h}{24}(9f_1 + 19f_0 - 5f_{-1} + f_{-2})$$

$$y_1^{(c)} = 0.8061 + \frac{0.1}{24}[9(-0.2067) + 19(-0.3498) - 5(-0.5215) - 0.7312] = 0.7785$$

Hence $\quad y(0.4) = 0.7785.$

**PROBLEMS 32.5**

1. Using Adams-Bashforth method, obtain the solution of $dy/dx = x - y^2$ at $x = 0.8$, given the values

| | | | | |
|---|---|---|---|---|
| $x$: | 0 | 0.2 | 0.4 | 0.6 |
| $y$: | 0 | 0.0200 | 0.0795 | 0.1762 |

*(Bhopal, 2002)*

2. Using Adams-Bashforth formulae, determine $y(0, 4)$ given the differential equation $dy/dx = \frac{1}{2}xy$ and the data

| | | | | |
|---|---|---|---|---|
| $x$: | 0 | 0.1 | 0.2 | 0.3 |
| $y$: | 1 | 1.0025 | 1.0101 | 1.0228 |

3. Given $y' = x^2 - y$, $y(0) = 1$ and the starting values $y(0.1) = 0.90516$, $y(0.2) = 0.82127$, $y(0.3) = 0.74918$, evaluate $y(0, 4)$ using Adams-Bashforth method. *(S.V.T.U., 2007)*
4. Using Adams-Bashforth method, find $y(4.4)$ given $5xy' + y^2 = 2$, $y(4) = 1$, $y(4, 1) = 1.0049$, $y(4, 2) = 1.0097$ and $y(4.3) = 1.0143$.
5. Given the differential equation $dy/dx = x^2y + x^2$ and the data :

| | | | | |
|---|---|---|---|---|
| $x$: | 1 | 1.1 | 1.2 | 1.3 |
| $y$: | 1 | 1.233 | 1.548488 | 1.978921 |

*(Indore, 2003 S)*

6. Using Adams-Bashforth method, evaluate $y(1.4)$, if $y$ satisfies $dy/dx + y/x = 1/x^2$ and $y(1) = 1$, $y(1.1) = 0.996$, $y(1.2) = 0.986$, $y(1.3) = 0.972$. *(Madras, 2003)*

## 32.11 SIMULTANEOUS FIRST ORDER DIFFERENTIAL EQUATIONS

The simultaneous differential equations of the type

$$\frac{dy}{dx} = f(x, y, z) \qquad ...(1)$$

and

$$\frac{dz}{dx} = \phi(x, y, z) \qquad ...(2)$$

with initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$ can be solved by the methods discussed in the preceding sections, especially by Picard's or Runge-Kutta methods.

(*i*) *Picard's method gives*

$$y_1 = y_0 + \int f(x, y_0, z_0)\,dx, \quad z_1 = z_0 + \int \phi(x, y_0, z_0)\,dx$$

$$y_2 = y_0 + \int f(x, y_1, z_1)\,dx, \quad z_2 = z_0 + \int \phi(x, y_1, z_1)\,dx$$

$$y_3 = y_0 + \int f(x, y_2, z_2)\,dx, \quad z_3 = z_0 + \int \phi(x, y_2, z_2)\,dx$$

and so on.

**(*ii*) *Taylor's series* method is used as follows :**

**If $h$ be the step-size, $y_1 = y(x_0 + h)$ and $z_1 = z(x_0 + h)$.** Then Taylor's algorithm for (1) and (2) gives

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \dots \quad \dots(3)$$

$$z_1 = z_0 + hz_0' + \frac{h^2}{2!} z_0'' + \frac{h^3}{3!} + z_0''' + \dots \quad \dots(4)$$

Differentiating (1) and (2) successively, we get $y''$, $z''$, etc. So the values $y_0', y_0'', y_0''' \dots$ and $z_0', z_0'', z_0''' \dots$ are known. Substituting these in (3) and (4), we obtain $y_1, z_1$ for the next step.

Similarly, we have the algorithms

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \dots \quad \dots(5)$$

$$z_2 = z_1 + hz_1' + \frac{h^2}{2!} z_1'' + \frac{h^3}{3!} z_1''' + \dots \quad \dots(6)$$

Since $y_1$ and $z_1$ are known, we can calculate $y_1', y_1'', \dots$ and $z_1', z_1'' \dots$. Substituting these in (5) and (6), we get $y_2$ and $z_2$.

Proceeding further, we can calculate the other values of $y$ and $z$ step by step.

(*iii*) *Runge-Kutta method* is applied as follows :

Starting at $(x_0, y_0, z_0)$ and taking the step-sizes for $x, y, z$ to be $h, k, l$ respectively, the Runge-Kutta method gives,

$$k_1 = hf(x_0, y_0, z_0) \qquad l_1 = h\phi(x_0, y_0, z_0)$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right) \qquad l_2 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right) \qquad l_3 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3) \qquad l_4 = h\phi(x_0 + h, y_0 + k_3, z_0 + l_3)$$

Hence $\quad y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad$ and $\quad z_1 = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

To compute $y_2$ and $z_2$, we simply replace $x_0, y_0, z_0$ by $x_1, y_1, z_1$ in the above formulae.

**Example 32.23.** *Using Picard's method find approximate values of $y$ and $z$ corresponding to $x = 0.1$, given that $y(0) = 2$, $z(0) = 1$ and $dy/dx = x + z$, $dz/dx = x - y^2$.*

**Solution.** Here $x_0 = 0, y_0 = 2, z_0 = 1$,

$$\frac{dy}{dx} = f(x, y, z) = x + z; \qquad \text{and} \qquad \frac{dz}{dx} = \phi(x, y, z) = x - y^2$$

$$\therefore \qquad y = y_0 + \int_{x_0}^{x} f(x, y\, z)\, dx \qquad \text{and} \qquad z = z_0 + \int_{x_0}^{x} \phi(x, y\, z)\, dx.$$

*First approximations* $\quad y_1 = y_0 + \int_{x_0}^{x} \phi(x, y_0, z_0)\, dx = 2 + \int_0^x (x + 1)\, dx = 2 + x + \frac{1}{2}x^2$

$$z_1 = z_0 + \int_{x_0}^{x} \phi(x, y_0, z_0)\, dx = 1 + \int_0^x (x - 4)\, dx = 1 - 4x + \frac{1}{2}x^2$$

*Second approximations* $\quad y_2 = y_0 + \int_{x_0}^{x} f(x, y_1, z_1)\, dx = 2 + \int_0^x \left(x + 1 - 4x + \frac{1}{2}x^2\right) dx$

$$= 2 + x - \frac{3}{2}x^2 + \frac{x^3}{6}$$

$$z_2 = z_0 + \int_{x_0}^{x} \phi(x, y_1, z_1)\, dx$$

$$= 1 + \int_{x_0}^{x} \left[x - \left(2 + x + \frac{1}{2}x^2\right)^2\right] dx = 1 - 4x + \frac{3}{2}x^2 - x^3 - \frac{x^4}{4} - \frac{x^5}{20}.$$

*Third approximations* $y_3 = y_0 + \int_{x_0}^{x} f(x, y_2, z_2)\, dx$

$$= 2 + x - \frac{3}{2}x^2 - \frac{1}{2}x^3 - \frac{1}{4}x^4 - \frac{1}{20}x^5 - \frac{1}{120}x^6$$

$$z_3 = z_0 + \int_{x_0}^{x} \phi(x, y_2, z_2)\, dx$$

$$= 1 - 4x - \frac{3}{2}x^2 + \frac{5}{3}x^3 + \frac{7}{12}x^4 - \frac{31}{60}x^5 + \frac{1}{12}x^6 - \frac{1}{252}x^7$$

and so on.

When $x = 0.1$,

$$y_1 = 2.105,\ y_2 = 2.08517,\ y_3 = 2.08447$$

$$z_1 = 0.605,\ z_2 = 0.58397,\ z_3 = 0.58672.$$

Hence $y(0.1) = 2.0845,\ z(0.1) = 0.5867$

correct to four decimal places.

**Example 32.24.** *Solve the differential equations*

$$\frac{dy}{dx} = 1 + xz,\quad \frac{dz}{dx} = -xy \text{ for } x = 0.3,$$

*using fourth order Runge-Kutta method. Initial values are $x = 0$, $y = 0$, $z = 1$.*

**Solution.** Here $f(x, y, z) = 1 + xz$, $\phi(x, y, z) = -xy$

$x_0 = 0, y_0 = 0, z_0 = 1$. Let us take $h = 0.3$.

$\therefore$ $k_1 = h f(x_0, y_0, z_0) = 0.3 f(0, 0, 1) = 0.3(1 + 0) = 0.3$

$l_1 = h\,\phi(x_0, y_0, z_0) = 0.3(-0 \times 0) = 0$

$$k_2 = h f\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_1,\ z_0 + \frac{1}{2}l_1\right)$$

$$= (0.3) f(0.15, 0.15, 1) = 0.3(1 + 0.15) = 0.345$$

$$l_2 = h\,\phi\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_1,\ z_0 + \frac{1}{2}l_1\right)$$

$$= 0.3[-(0.15)(0.15)] = -0.00675.$$

$$k_3 = h f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2},\ z_0 + \frac{l_2}{2}\right)$$

$$= (0.3) f(0.15, 0.1725, 0.996625)$$

$$= 0.3[1 + 0.996625 \times 0.15] = 0.34485$$

$$l_3 = h\,\phi\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2},\ z_0 + \frac{l_2}{2}\right)$$

$$= 0.3[-(0.15)(0.1725)] = -0.007762$$

$$k_4 = h f(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$= (0.3) f(0.3, 0.34485, 0.99224) = 0.3893$$

$$l_4 = h\,\phi(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$= 0.3[-(0.3)(0.34485)] = -0.03104$$

Hence $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

*i.e.,* $y(0.3) = 0 + \frac{1}{6}[0.3 + 2(0.345) + 2(0.34485) + 0.3893] = 0.34483$

and $z(x_0 + h) = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

*i.e.,* $z(0.3) = 1 + \frac{1}{6}[0 + 2 + (-0.00675) + 2(-0.0077625) + (-0.03104)] = 0.98999$

## 32.12 SECOND ORDER DIFFERENTIAL EQUATIONS

Consider the second order differential equation $\dfrac{d^2y}{dx^2} = f\left(x, y, \dfrac{dy}{dx}\right)$

By writing $dy/dx = z$, it can be reduced to two first order simultaneous differential equations

$$\frac{dy}{dx} = z, \qquad \frac{dz}{dx} = f(x, y, z)$$

These equations can be solved as explained above.

**Example 32.25.** *Using Runge-Kutta method, solve $y'' = xy'^2 - y^2$ for $x = 0.2$ correct to 4 decimal places. Initial conditions are $x = 0$, $y = 1$, $y' = 0$.* (Delhi, 2002)

**Solution.** Let $dy/dx = z = f(x, y, z)$. Then $dz/dx = xz^2 - y^2 = \phi(x, y, z)$

We have $x_0 = 0, y_0 = 1, z_0 = 0, h = 0.2$.

Using $k_1, k_2 \ldots$ for $f(x, y, z)$ and $l_1, l_2, \ldots$ for $\phi(x, y, z)$, Runge-Kutta formulae become

$$k_1 = hf(x_0, y_0, z_0) = 0.2\,(0) = 0$$

$$k_2 = hf(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_1, z_0 + \tfrac{1}{2}l_1) = 0.2\,(-0.1) = -0.02$$

$$k_3 = hf(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_2, z_0 + \tfrac{1}{2}l_2) = 0.2\,(-0.0999) = -0.02$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3) = 0.2\,(-0.1958) = -0.0392$$

$$\therefore \quad k = \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = -0.0199$$

$$l_1 = h\phi(x_0, y_0, z_0) = 0.2(-1) = -0.2$$

$$l_2 = h\phi(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_1, z_0 + \tfrac{1}{2}l_1) = 0.2(-0.999) = -0.1998$$

$$l_3 = h\phi(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_2, z_0 + \tfrac{1}{2}l_2) = 0.2(-0.9791) = -0.1958$$

$$l_4 = h\phi(x_0 + h, y_0 + k_3, z_0 + l_3) = 0.2(0.9527) = -0.1905$$

$$l = \tfrac{1}{6}(l_1 + 2l_2 + 2l_3 + 2l_4) = -0.1970$$

Hence at $x = 0.2$,

$$y = y_0 + k = 1 - 0.0199 = 0.9801$$

and

$$y' = z = z_0 + l = 0 - 0.1970 = -0.1970.$$

**Example 32.26.** *Given $y'' + xy' + y = 0$, $y(0) = 1$, $y'(0) = 0$, obtain $y$ for $x = 0(0.1)\,0.3$ by any method. Further, continue the solution by Milne's method to calculate $y(0.4)$.* (Anna, 2004; Madras, 2003 S)

**Solution.** Putting $y' = z$, the given equation reduces to the simultaneous equations

$$z' + xz + y = 0, y' = z \qquad \ldots(i)$$

We employ Taylor's series method to find $y$.

Differentiating the given equation $n$ times, we get

$$y_{n+2} + xy_{n+1} + ny_n + y_n = 0$$

At $\quad x = 0, (y_{n+2})_0 = -(n+1)(y_n)_0$

$\therefore \quad y(0) = 1$, gives $y_2(0) = -1, y_4(0) = 32, y_6(0) = -5 \times 3, \ldots$

and $\quad y_1(0) = 0$ yields $y_3(0) = y_5(0) = \ldots = 0.$

Expanding $y(x)$ by Taylor's series, we have

$$y(x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \ldots$$

$$\therefore \quad y(x) = 1 - \frac{x^2}{2!} + \frac{3}{4!}x^4 - \frac{5 \times 3}{6!}x^6 + \ldots \qquad \ldots(ii)$$

and

$$z(x) = y'(x) = -x + \frac{1}{2}x^3 - \frac{1}{8}x^5 = \ldots = -xy \qquad \ldots(iii)$$

From (*ii*), we have

$$y(0.1) = 1 - \frac{(0.1)^2}{2} + \frac{1}{8}(0.1)^4 - \ldots = 0.995$$

$$y(0.2) = 1 - \frac{(0.2)^2}{2} + \frac{(0.2)^4}{8} - \dots = 0.9802$$

$$y(0.3) = 1 - \frac{(0.3)^2}{2} + \frac{(0.3)^4}{8} - \frac{(0.3)^6}{48} + \dots = 0.956$$

From (*iii*), we have

$$z(0.1) = -0.0995, z(0.2) = -0.196, z(0.3) = -0.2863.$$

Also from (*i*), $z'(x) = -(xz + y) \quad \therefore \quad z'(0.1) = 0.985, z'(.2) = -0.941, z'(0.3) = -0.87.$

Applying Milne's predictor formula, first to $z$ and then to $y$, we obtain

$$z(0.4) = z(0) + \frac{4}{3}(0.1)\{2z'(0.1) - z'(0.2) + 2z'(0.3)\}$$

$$= 0 + \left(\frac{0.4}{3}\right)\{-1.79 + 0.941 - 1.74\} = -0.3692$$

and

$$y(0.4) = y(0) + \frac{4}{3}(0.1)\{2y'(0.1) - y'(0.2) + 2y'(0.3)\}$$

$$= 0 + \left(\frac{0.4}{3}\right)\{-0.199 + 0.196 - 0.5736\} = 0.9231 \qquad [\because \; y' = z]$$

Also $z'(0.4) = -\{x(0.4)z(0.4) + y(0.4)\} = \{0.4(-0.3692) + 0.9231\} = -0.7754.$

Now applying Milne's corrector formula, we get

$$z(0.4) = z(0.2) + \frac{h}{3}\{z'(0.2) + 4z'(0.3) + z'(0.4)\}$$

$$= -0.196 + \left(\frac{0.1}{3}\right)\{-0.941 - 3.48 - 0.7754\} = -0.3692$$

and

$$y(0.4) = y(0.2) + \frac{h}{3}\{y'(0.2) + 4y'(0.3) + y'(0.4)\}$$

$$= 0.9802 + \left(\frac{0.1}{3}\right)\{-0.196 - 1.1452 - 0.3692\} = 0.9232$$

Hence $\qquad y(0.4) = 0.9232$ and $z(0.4) = -0.3692.$

## PROBLEMS 32.6

1. Apply Picard's method to find the third approximation to the values of $y$ and $z$, given that $dy/dx = z$, $dz/dx = x^3(y + z)$, given $y = 1$, $z = \frac{1}{2}$ when $x = 0$.
2. Solve the following differential equations using Taylor series method of the 4th order, for $x = 0.1$ and $0.2$, $\frac{dy}{dx} = xz + 1$, $\frac{dz}{dy} = -xy$ ; $y(0) = 0$ and $z(0) = 1$.
3. Find $y(0.1)$, $z(0.1)$, $y(0.2)$ and $z(0.2)$ from the system of equations $y' = x + z$, $z' = x - y^2$ given $y(0) = 0$, $z(0) = 1$ using Runge-Kutta of 4th order. (*J.N.T.U., 2009*)
4. Using Picard's method, obtain the second approximation to the solution of
$$\frac{d^2y}{dx^2} = x^3\frac{dy}{dx} + x^3y \text{ so that } y(0) = 1, y'(0) = \frac{1}{2}.$$
5. Use Picard's method to approximate $y$ when $x = 0.1$, given that $\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + y = 0$ and $y = 0.5$, $\frac{dy}{dx} = 0.1$, when $x = 0$.
6. Using Runge-Kutta method of order four, solve $y'' = y + xy'$, $y(0) = 1$, $y'(0) = 0$ to find $y(0.2)$ and $y'(0.2)$.
7. Consider the second order value problem $y'' - 2y' + 2y = e^{2t}\sin t$ with $y(0) = -0.4$ and $y'(0) = -0.6$. Using the fourth order Runge-Kutta method, find $y(0.2)$. (*Anna, 2003*)

8. The angular displacement $\theta$ of a simple pendulum is given by the equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{l} \sin \theta = 0$$

where $l = 98$ cm and $g = 980$ cm/sec$^2$. If $\theta = 0$ and $d\theta/dt = 4.472$ at $t = 0$, use Runge-Kutta method to find $\theta$ and $d\theta/dt$ when $t = 0.2$ sec.

## 32.13 BOUNDARY VALUE PROBLEMS

Such a problem requires the solution of a differential equation in a region $R$ subject to the various conditions on the boundary of $R$. Practical applications give rise to many such problems. We shall discuss two-point linear boundary value problems of the following types :

(i) $\dfrac{d^2y}{dx^2} + \lambda(x)\dfrac{dy}{dx} + \mu(x)y = \gamma(x)$ with the conditions $y(x_0) = a$, $y(x_n) = b$.

(ii) $\dfrac{d^4y}{dx^4} + \lambda(x)y = \mu(x)$ with the conditions $y(x_0) = y'(x_0) = a$ and $y(x_n) = y'(x_n) = b$.

While there exist many numerical methods for solving such boundary value problems, the method of finite-differences is most commonly used. We shall explain this method in the next section.

## 32.14 FINITE-DIFFERENCE METHOD

In this method, the derivatives appearing in the differential equation and the boundary conditions are replaced by their finite-difference approximations and the resulting linear system of equations are solved by any standard procedure. These roots are the values of the required solution at the pivotal points.

*The finite-difference approximations to the various derivatives are derived as under :*

If $y(x)$ and its derivatives are single-valued continuous functions of $x$ then by Taylor's expansion, we have

$$y(x + h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \frac{h^2}{3!}y'''(x) + \ldots \quad \ldots(1)$$

and
$$y(x - h) = y(x) - hy'(x) + \frac{h^2}{2!}y''(x) - \frac{h^3}{3!}y'''(x) + \ldots \quad \ldots(2)$$

Equation (1) gives $y'(x) = \dfrac{1}{h}[y(x + h) - y(x)] - \dfrac{h}{2}y''(x) - \ldots$

*i.e.*,
$$y'(x) = \frac{1}{h}[y(x + h) - y(x)] + O(h)$$

which is the *forward difference approximation of* $y'(x)$ with an error of the order $h$.

Similarly (2) gives $y'(x) = \dfrac{1}{h}[y(x) - y(x - h)] + O(h)$

which is the *backward difference approximation of* $y'(x)$ with an error of the order $h$.

Subtracting (2) from (1), we obtain

$$y'(x) = \frac{1}{2h}[y(x + h) - y(x - h)] + O(h^2)$$

which is the *central-difference approximation of* $y'(x)$ with an error of the order $h^2$. Clearly this central difference approximation to $y'(x)$ is better than the forward or backward difference approximations and hence should be preferred.

Adding (1) and (2), we get

$$y''(x) = \frac{1}{h^2}[y(x + h) - 2y(x) + y(x - h)] + O(h^2)$$

which is the central difference approximation of $y''(x)$. Similarly we can derive central difference approximations to higher derivatives.

Hence *the working expressions for the central difference approximations to the first four derivatives of* $y_i$ *are as under :*

$$y'_i = \frac{1}{2h}(y_{i+1} - y_{i-1}) \qquad \text{...(3)}$$

$$y''_i = \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1}) \qquad \text{...(4)}$$

$$y'''_i = \frac{1}{2h^3}(y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}) \qquad \text{...(5)}$$

$$y_i^{iv} = \frac{1}{h^4}(y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i+1} + y_{i-2}) \qquad \text{... (6)}$$

**Obs.** The accuracy of this method depends on the size of the sub-interval $h$ and also on the order of approximation. As we reduce $h$, the accuracy improves but the number of equations to be solved also increases.

**Example 32. 27.** *Solve the equation $y'' = x + y$ with the boundary conditions $y(0) = y(1) = 0$. (Calicut, 1999)*

**Solution.** We divide the interval (0, 1) into four sub-intervals so that $h = 1/4$ and the pivot points are $x_0 = 0, x_1 = 1/4, x_2 = 1/2, x_3 = 3/4$ and $x_4 = 1$.

The differential equation is approximated as

$$\frac{1}{h^2}[y_{i+1} - 2y_i + y_{i-1}] = x_i + y_i$$

or
$$16y_{i+1} - 33y_i + 16y_{i-1} = x_i,\ i = 1, 2, 3.$$

Using $y_0 = y_4 = 0$, we get the system of equations

$$16y_2 - 33y_1 = \frac{1}{4}$$

$$16y_3 - 33y_2 + 16y_1 = \frac{1}{2}$$

$$-33y_3 + 16y_2 = \frac{3}{4}$$

Their solution gives

$$y_1 = -0.03488,\ y_2 = -0.05632,\ y_3 = -0.05003.$$

**Obs.** The exact solution being $y(x) = \dfrac{\sinh x}{\sinh 1} - x$, the error at each nodal point is given in the table:

| $x$ | *Computed value $y(x)$* | *Exact value $y(x)$* | *Error* |
|---|---|---|---|
| 0.25 | − 0.03488 | − 0.03505 | 0.00017 |
| 0.5 | − 0.05632 | − 0.05659 | 0.00027 |
| 0.75 | − 0.05003 | − 0.05028 | 0.00025 |

**Example 32.28.** *Determine values of y at the pivotal points of the interval (0, 1), if y satisfies the boundary value problem $y^{iv} + 81y = 81x^2$, $y(0) = y(1) = y''(0) = y''(1) = 0$. (Take $n = 3$).*

**Solution.** Here $h = 1/3$ and the pivotal points are $x_0 = 0$, $x_1 = 1/3$, $x_2 = 2/3$, $x_3 = 1$. The corresponding $y$-values are $y_0 (= 0), y_1, y_2, y_3 (= 0)$.

Replacing $y^{iv}$ by its central difference approximation, the differential equation becomes

$$\frac{1}{h^4}(y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}) + 81y_i = 81x_i^2$$

or
$$y_{i+2} - 4y_{i+1} + 7y_i - 4y_{i-1} + y_{i-2} = x_i^2,\ i = 1, 2$$

At $i = 1$, $\quad y_3 - 4y_2 + 7y_1 - 4y_0 + y_{-1} = 1/9$

At $i = 2$, $\quad y_4 - 4y_3 + 7y_2 - 4y_1 + y_0 = 4/9$

Using $y_0 = y_3 = 0$, we get $\quad -4y_2 + 7y_2 + y_{-1} = 1/9 \qquad$ ...(*i*)

$$y_4 + 7y_1 - 4y_1 = 4/9 \qquad \text{...(ii)}$$

Regarding the conditions $y''_0 = y''_3 = 0$, we know that

$$x_i'' = \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1})$$

At $i = 0$, $\quad y_0'' = 9(y_1 - 2y_0 + y_{-1})$

or $\quad y_{-1} = -y_1 \quad [\because\ y_0 = y_0'' = 0] \quad ...(iii)$

At $i = 3$, $\quad y_3'' = 9(y_4 - 2y_3 + y_2)$

$$y_4 = -y_2 \quad [\because\ y_3 = y_3'' = 0] \quad ...(iv)$$

Using (*iii*), the equation (*i*) becomes

$$-4y_2 + 6y_1 = 1/9 \quad ...(v)$$

Using (*iv*), the equation (*ii*) reduces to

$$6y_2 - 4y_1 = 4/9 \quad ...(vi)$$

Solving (*v*) and (*vi*), we obtain

$$y_1 = 11/90 \text{ and } y_2 = 7/45.$$

Hence $y(1/3) = 0.1222$ and $y(2/3) = 0.1556$.

**Example 32.29.** *The deflection of a beam is governed by the equation*

$$\frac{d^4y}{dx^4} + 81y = \phi(x)$$

*where $\phi(x)$ is given by the table*

| | | | |
|---|---|---|---|
| $x$ : | 1/3 | 2/3 | 1, |
| $\phi(x)$ : | 81 | 162 | 243, |

*and boundary condition $y(0) = y'(0) = y''(1) = y'''(1) = 0$. Evaluate the deflection at the pivotal points of the beam using three sub-intervals.*

**Solution.** Here $h = 1/3$ and the pivotal points are $x_0 = 0, x_1 = 1/3, x_2 = 2/3, x_3 = 1$. The corresponding $y$-values are $y_0(= 0), y_1, y_2, y_3$.

The given differential equation is approximated to

$$\frac{1}{h^4}(y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}) + 81y_i = \phi(x_i)$$

At $i = 1$, $\quad y_3 - 4y_2 + 7y_1 - 4y_0 + y_{-1} = 1 \quad ...(i)$

At $i = 2$, $\quad y_4 - 4y_3 + 7y_2 - 4y_1 + y_0 = 2 \quad ...(ii)$

At $i = 3$, $\quad y_5 - 4y_4 + 7y_3 - 4y_2 + y_1 = 3 \quad ...(iii)$

We have $\quad y_0 = 0 \quad ...(iv)$

Since $\quad y_i' = \frac{1}{2h}(y_{i+1} - y_{i-1})$

$\therefore$ for $i = 0$, $\quad 0 = y_0' = \frac{1}{2h}(y_1 - y_{-1})$ *i.e.* $y_{-1} = y_1 \quad ...(v)$

Since $\quad y_i'' = \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1})$

$\therefore$ for $i = 3$, $\quad 0 = y_3'' = \frac{1}{h^2}(y_4 - 2y_3 + y_2)$, *i.e.* $y_4 = 2y_3 - y_2 \quad ...(vi)$

Also $\quad y_i''' = \frac{1}{2h^3}(y_{i+2} - 2y_{i+1}\ 2y_{i-1} - y_{i-2})$

$\therefore$ for $i = 3$, $\quad 0 = y_3''' = \frac{1}{2h^3}(y_5 - 2y_4 + 2y_2 - y_1)$

$$y_5 = 2y_4 - 2y + y_1 \quad ...(vii)$$

Using (*iv*) and (*v*), the equation (*i*) reduces to

$$y_3 - 4y_2 + 8y_1 = 1 \quad ...(viii)$$

Using (*iv*) and (*vi*), the equation (*ii*) becomes

$$-y_3 + 3y_2 - 2y_1 = 1 \quad ...(ix)$$

Using (*vi*) and (*vii*), the equation (*iii*) reduces to

$$3y_3 - 4y_2 + 2y_1 = 3 \qquad ...(x)$$

Solving (*viii*), (*ix*) and (*x*), we get

$$y_1 = 8/13,\ y_2 = 22/13,\ y_3 = 37/13.$$

Hence $\qquad y(1/3) = 0.6154,\ y(2/3) = 1.6923,\ y(1) = 2.8462.$

### PROBLEMS 32.7

1. Solve the boundary value problem for $x = 0.5$ :
$$\frac{d^2y}{dx^2} + y + 1 = 0,\ y(0) = y(1) = 0. \qquad \text{(Take } n = 4)$$
2. Find an approximate solution of the boundary value problem :
$$y'' + 8(\sin^2 \pi y)y = 0,\ 0 \le x \le 1,\ y(0) = y(1) = 1. \qquad \text{(Take } n = 4)$$
3. Solve the boundary value problem
$$xy'' + y = 0,\ y(1) = 1,\ y(2) = 2. \qquad \text{(Take } n = 4)$$
4. Solve the equation
$$y'' - 4y' + 4y = e^{3x}, \text{ with the conditions } y(0) = 0,\ y(1) = -2, \text{ taking } n = 4.$$
5. Solve the boundary value problem $y'' - 64y + 10 = 0$ with $y(0) = y(1) = 0$ by the finite difference method. Compute the value of $y(0.5)$ and compare with the true value.
6. Solve the boundary value problem
$$y'' + xy' + y = 3x^2 + 2,\ y(0) = 0,\ y(1) = 1.$$
7. The boundary value problem governing the deflection of a beam of length 3 metres is given by
$$\frac{d^4y}{dx^4} + 2y = \frac{1}{9}x^2 + \frac{2}{3}x + 4,\ y(0) = y'(0) = y(3) = y'(3) = 0.$$
The beam is built-in at the left end ($x = 0$) and simply supported at the right end ($x = 3$). Determine $y$ at the pivotal points $x = 1$ and $x = 2$.
8. Solve the boundary value problem,
$$\frac{d^4y}{dx^4} + 81y = 729x^2,\ y(0) = y'(0) = y''(1) = y'''(1) = 0. \qquad \text{(Use } n = 3)$$
9. Solve the equation $y^{iv} - y''' + y = x^2$ subject to the boundary conditions
$$y(0) = y'(0) = 0 \text{ and } y(1) = 2,\ y'(1) = 0. \qquad \text{(Take } n = 5)$$

## 32.15 OBJECTIVE TYPE OF QUESTIONS

### PROBLEMS 32.8

*Select the correct answer or fill up the blanks in the following questions :*

1. Which of the following is a step by step method :
(*a*) Taylor' s (*b*) Adams-Bashforth (*c*) Picard's (*d*) None.
2. The finite difference scheme for the equation $2y'' + y = 5$ is ...... .
3. If $y' = x + y$, $y(0) = 1$ and $y^{(1)}(x) = 1 + x + x^2/2$, then by Picard's method, the value of $y^{(2)}(x)$ is ...... .
4. The iterative formula of Euler's method for solving $y' = f(x, y)$ with $y(x_0) = y_0$, is ...... .
5. Taylor's series for solution of first order ordinary differential equations is ...... .
6. Using Runge-Kutta method of order four, the value of $y(0.1)$ for $y' = x - 2y$, $y(0) = 1$ taking $h = 0.1$ is
(*a*) 0.813 (*b*) 0.825 (*c*) 0.0825 (*d*) none.
7. Given $y_0, y_1, y_2, y_3$, Milne's corrector formula to find $y_4$ for $dy/dx = f(x, y)$, is ...... .
8. The second order Runge-Kutta formula is ...... .
9. Adams-Bashforth predictor formula to solve $y' = f(x, y)$ given $y_0 = y(x_0)$ is ...... .
10. The multi-step methods available for solving ordinary differential equations are ...... .
11. To predict Adam's method atleast ...... values of $y$, prior to the desired value, are required.
12. Taylor's series solution of $y' = -xy$, $y(0) = 1$ upto $x^4$ is ...... .

13. Using modified Euler's method, the value of $y(0.1)$ for $\frac{dy}{dx} = x - y, y(0) = 1$ is ...... .
    (*a*) 0.809 (*b*) 0.909 (*c*) 0.0809 (*d*) none.
14. Milne's Predictor formula is ...... .
15. Adam's corrector formula is ...... .
16. Using Euler's method, $dy/dx = (y - 2x)/y$, $y(0) = 1$ ; gives $y(0.1) =$ ...... .
17. $\frac{d^2y}{dx^2} + y^2 \frac{dy}{dx} + y = 0$ is equivalent to a set of two first order differential equations ...... and ...... .
18. The formula for the 4th order Runge-Kutta method is ...... .
19. Taylor's series method will be useful to give some ....... of Milne's method.
20. The name of two self-starting methods to solve $y' = f(x, y)$ given $y(x_0) = y_0$ are ......
21. In the derivation of fourth order Runge-Kutta formula, it is called fourth order because ......
22. If $y' = x$, $y(0) = 1$ then by Picard's method, the value of $y(1)$ is ......
    (*a*) 0.915 (*b*) 0.905 (*c*) 0.981 (*d*) none.
23. The finite difference scheme of the differential equation $y'' + 2y = 0$ is ......
24. If $y' = -y$, $y(0) = 1$, the Euler's method, the value of $y(1)$ is
    (*a*) 0.99 (*b*) 0.999 (*c*) 0.981 (*d*) none.
25. In Euler's method if $h$ is small the method is too slow, if $h$ is large, it gives inaccurate value. (True or False)
26. Runge-Kutta method is a self-starting method. (True or False)
27. Predictor-corrector methods are self-starting methods. (True or False)

CHAPTER

33

# Numerical Solution of Partial Differential Equations



## 33.1 INTRODUCTION

There are many boundary value problems which involve partial differential equations. Only a few of these equations can be solved by analytical methods. In most cases, we depend on the numerical solution of such partial differential equations. Of the various numerical methods available for solving these equations, the method of finite differences is most commonly used.

In this method, the derivatives appearing in the equation and the boundary conditions are replaced by their finite-difference approximations. Then the given equation is changed to a difference equation which is solved by iterative procedures. This process is slow but gives good results of boundary value problems. An added advantage of this method is that the computation can be done by electronic computer. Here we shall apply this method to the solution of important applied partial differential equations. For a detailed study, the reader should refer to author's book '*Numerical Methods in Engineering and Science*'.

## 33.2 CLASSIFICATION OF SECOND ORDER EQUATIONS

The general linear partial differential equation of the second order in two independent variables is of the form.

$$A(x, y)\frac{\partial^2 u}{\partial x^2} + B(x, y)\frac{\partial^2 u}{\partial x \partial y} + C(x, y)\frac{\partial^2 u}{\partial y^2} + F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

*Such a partial differential equation is said to be*

(*i*) **elliptic,** *if* $B^2 - 4AC < 0$,

(*ii*) **parabolic,** *if* $B^2 - 4AC = 0$,

and (*iii*) **hyperbolic,** *if* $B^2 - 4AC > 0$.

**Example 33.1.** *Classify the following equations :*

(*i*) $\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 4\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$ (*ii*) $x^2\frac{\partial^2 u}{\partial x^2} + (1 - y^2)\frac{\partial^2 u}{\partial y^2} = 0, -\infty < x < \infty, -1 < y < 1$

(*Madras, 2003*)

(*iii*) $(1 + x^2)\frac{\partial^2 u}{\partial x^2} + (5 + 2x^2)\frac{\partial^2 u}{\partial x \partial t} + (4 + x^2)\frac{\partial^2 u}{\partial t^2} = 0.$

**Solution.** (*i*) Comparing this equation with (1) above, we find that

$$A = 1, B = 4, C = 4$$

$\therefore \quad B^2 - 4AC = (4)^2 - 4 \times 1 \times 4 = 0$

So the equation is parabolic.

(*ii*) Here $\quad A = x^2, B = 0, C = 1 - y^2$

$\therefore \quad B^2 - 4AC = 0 - 4x^2 (1 - y^2) = 4x^2 (y^2 - 1)$

For all $x$ between $-\infty$ and $\infty$, $x^2$ is positive

For all $y$ between $-1$ and $1$, $y^2 < 1 \quad \therefore \quad B^2 - 4AC < 0$

Hence the equation is elliptic.

(*iii*) Here $A = 1 + x^2, B = 5 + 2x^2, C = 4 + x^2$

$\therefore \quad B^2 - 4AC = (5 + 2x^2)^2 - 4(1 + x^2)(4 + x^2) = 9$ *i.e.* $> 0$

So the equation is hyperbolic.

## PROBLEMS 33.1

1. What is the classification of the equation $f_{xx} + 2f_{xy} + f_{yy} = 0$.
2. Determine whether the following equation is elliptic or hyperbolic ?
$$(x + 1)u_{xx} - 2(x + 2)u_{xy} + (x + 3)u_{yy} = 0.$$
3. Classify the equations (*i*) $y^2 u_{xx} - 2xy\, u_{xy} + x^2 u_{yy} + 2u_x - 3u = 0$ (*Madras, 2000 S*)

   (*ii*) $x^2 \dfrac{\partial^2 u}{\partial x^2} + y^2 \dfrac{\partial^2 u}{\partial y^2} = x \dfrac{\partial u}{\partial x} - y \dfrac{\partial u}{\partial y}$ (*P.T.U., 2009 S*)

   (*iii*) $\dfrac{3\partial^2 u}{\partial x^2} + 4 \dfrac{\partial^2 u}{\partial x \partial y} + 6 \dfrac{\partial^2 u}{\partial y^2} - 2 \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} - u = 0.$ (*Anna, 2008*)
4. In which parts of the $(x, y)$ plane is the following equation elliptic ?
$$\partial^2 u / \partial x^2 + \partial^2 u / \partial x \partial y + (x^2 + 4y^2)\, \partial^2 u / \partial y^2 = 2 \sin (xy).$$

## 33.3 FINITE-DIFFERENCE APPROXIMATIONS TO DERIVATIVES

Consider a rectangular region $R$ in the $x$-$y$ plane. Divide this region into a rectangular network of sides $\Delta x = h$ and $\Delta y = k$ as shown in Fig. 33.1. The points of intersection of the dividing lines are called *mesh points, nodal points* or *grid points.*

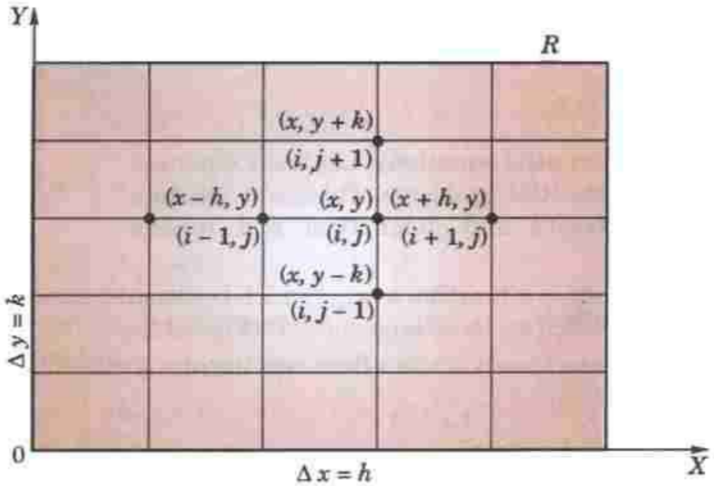



Fig. 33.1

Then we have the finite difference approximations for the partial derivatives in $x$-direction (§ 32.12) :

$$\frac{\partial u}{\partial x} = \frac{u(x + h, y) - u(x, y)}{h} + O(h)$$

$$= \frac{u(x, y) - u(x-h, y)}{h} + O(h) = \frac{u(x+h, y) - u(x-h, y)}{2h} + O(h^2)$$

and
$$\frac{\partial^2 u}{\partial x^2} = \frac{u(x-h, y) - 2u(x, y) + u(x+h, y)}{h^2} + O(h^2)$$

Writing $u(x, y) = u(ih, jk)$ as simply $u_{i, j}$, the above approximations become

$$u_x = \frac{u_{i+1, j} - u_{i, j}}{h} + O(h) \qquad ...(1)$$

$$= \frac{u_{i, j} - u_{i-1, j}}{h} + O(h) \qquad ...(2)$$

$$= \frac{u_{i+1, j} - u_{i-1, j}}{2h} + O(h^2) \qquad ...(3)$$

and
$$u_{xx} = \frac{u_{i-1, j} - 2u_{i, j} + u_{i+1, j}}{h^2} + O(h^2) \qquad ...(4)$$

Similarly we have the approximations for the derivatives w.r.t. $y$ :

$$u_y = \frac{u_{i, j+1} - u_{i, j}}{k} + O(k) \qquad ...(5)$$

$$= \frac{u_{i, j} - u_{i, j-1}}{k} + O(k) \qquad ...(6)$$

$$= \frac{u_{i, j+1} - u_{i, j-1}}{2k} + O(k^2) \qquad ...(7)$$

and
$$u_{yy} = \frac{u_{i, j-1} - 2u_{i, j} + u_{i, j+1}}{k^2} + O(k^2) \qquad ...(8)$$

Replacing the derivatives in any partial differential equation by their corresponding difference approximations (1) to (8), we obtain the finite-difference analogues of the given equations.

## 33.4 ELLIPTIC EQUATIONS

The Laplace's equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and the Poisson's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

are examples of elliptic partial differential equations. Laplace's equation arises in steady-state flow and potential problems. Poisson's equation arises in fluid mechanics, electricity and magnetism and torsion problems.

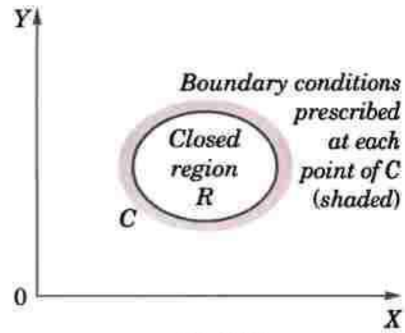



Fig. 33.2

The solution of these equations is a function $u(x, y)$ which is satisfied at every point of a region $R$ subject to certain boundary conditions specified on the closed curve $C$ (Fig. 33.2).

In general, problems concerning steady viscous flow, equilibrium stresses in elastic structures etc., lead to elliptic type of equations.

## 33.5 SOLUTION OF LAPLACE'S EQUATION*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad ...(1)$$

---

* See p. 619

Consider a rectangular region $R$ for which $u(x, y)$ is known at the boundary. Divide this region into a network of square mesh of side $h$, as shown in Fig. 33.3 (assuming that an exact sub-division of $R$ is possible). Replacing the derivatives in (1) by their difference approximations, we have

$$\frac{1}{h^2}[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}] + \frac{1}{h^2}[u_{i,j-1} - 2u_{i,j} + u_{i,j+1}] = 0$$

or

$$u_{i,j} = \frac{1}{4}[u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1}] \quad ...(2)$$

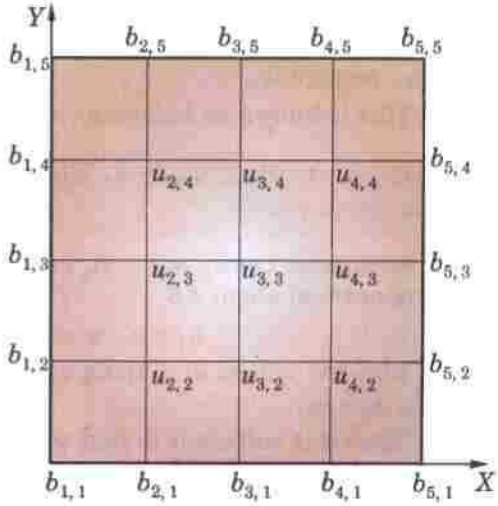


Fig. 33.3

This shows that the value of $u_{i,j}$ at any interior mesh point is the average of its values at four neighbouring points to the left, right, above and below. (2) is called the **standard 5-point formula** which is exhibited in Fig. 33.4.

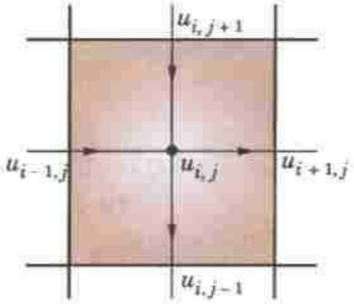


Fig. 33.4

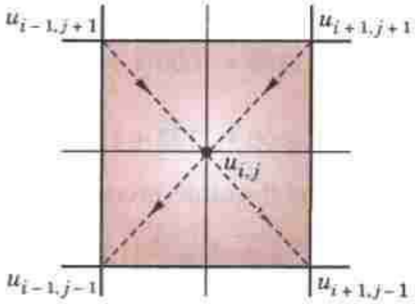


Fig. 33.5

Sometimes a formula similar to (2) is used which is given by

$$u_{i,j} = \frac{1}{4}(u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1}) \quad ...(3)$$

This shows that the value of $u_{i,j}$ is the average of its values at the four neighbouring diagonal mesh points. (3) is called the **diagonal 5-point formula** which is represented in Fig. 33.5. Although (3) is less accurate than (2), yet it serves as a reasonably good approximation for obtaining the starting values at the mesh points.

Now to find the initial values of $u$ at the interior mesh points, we first use diagonal five point formula (3) and compute $u_{3,3}, u_{2,4}, u_{4,4}, u_{4,2}$ and $u_{2,2}$, in this order. Thus we get,

$$u_{3,3} = \frac{1}{4}(b_{1,5} + b_{5,1} + b_{5,5} + b_{1,1})\,;\; u_{2,4} = \frac{1}{4}(b_{1,5} + u_{3,3} + b_{3,5} + b_{1,3})$$

$$u_{4,4} = \frac{1}{4}(b_{3,5} + b_{5,3} + b_{5,5} + u_{3,3})\,;\; u_{4,2} = \frac{1}{4}(u_{3,3} + b_{5,1} + b_{3,1} + b_{5,3})$$

$$u_{2,2} = \frac{1}{4}(b_{1,3} + b_{3,1} + u_{3,3} + b_{1,1})$$

The values at the remaining interior points *i.e.* $u_{2,3}, u_{3,4}, u_{4,3}$ and $u_{3,2}$ are computed by the standard five-point formula (2). Thus, we obtain

$$u_{2,3} = \frac{1}{4}(b_{1,3} + u_{3,3} + u_{2,4} + u_{2,2}),\; u_{3,4} = \frac{1}{4}(u_{2,4} + u_{4,4} + b_{3,5} + u_{3,3})$$

$$u_{4,3} = \frac{1}{4}(u_{3,3} + b_{5,3} + u_{4,4} + u_{4,2}),\; u_{3,2} = \frac{1}{4}(u_{2,2} + u_{4,2} + u_{3,3} + u_{3,1})$$

Having found all the nine values of $u_{i,j}$ once, their accuracy is improved by repeated application of (2) in the form

$$u^{(n+1)}_{i,j} = \frac{1}{4}[u^{(n+1)}_{i-1,j} + u^{(n)}_{i+1,j} + u^{(n+1)}_{i,j+1} + u^{(n)}_{i,j-1}]$$

This formula utilises the latest iterative value available and scans the mesh points symmetrically from left to right along successive rows. This process is repeated till the difference of values in one round and the next becomes negligible.

This is known as *Liebmann's iteration process.*

**Example 33.2.** *Solve the elliptic equation $u_{xx} + u_{yy} = 0$ for the square mash of Fig. 33.6 with boundary values as shown.* *(Rohtak, 2005 ; V.T.U., 2005)*

**Solution.** Let $u_1, u_2, ..., u_9$ be the values of $u$ at the interior mesh-points. Since the boundary values of $u$ are symmetrical about $AB$.

$\therefore \quad u_7 = u_1, u_8 = u_2, u_9 = u_3.$

Also the values of $u$ being symmetrical about $CD$, $u_3 = u_1$, $u_6 = u_4, u_9 = u_7$.

Thus it is sufficient to find the values $u_1, u_2, u_4$ and $u_5$.

Now we find their initial value in the following order :

$$u_5 = \frac{1}{4}(2000 + 2000 + 1000 + 1000) = 1500 \quad \text{(Std. formula)}$$

$$u_1 = \frac{1}{4}(0 + 1500 + 1000 + 2000) = 1125 \quad \text{(Diag. formula)}$$

$$u_2 = \frac{1}{4}(1125 + 1125 + 1000 + 1500) \approx 1188 \quad \text{(Std. formula)}$$

$$u_4 = \frac{1}{4}(2000 + 1500 + 1125 + 1125) \approx 1438 \quad \text{(Std. formula)}$$

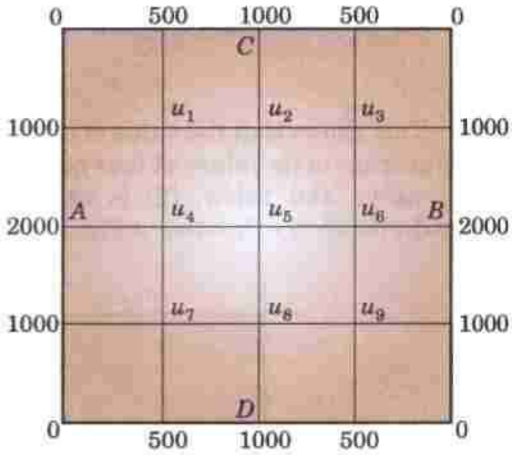



Fig. 33.6

We carry out the iteration process using the formulae :

$$u_1^{(n+1)} = \frac{1}{4}[1000 + u_2^{(n)} + 500 + u_4^{(n)}]$$

$$u_2^{(n+1)} = \frac{1}{4}[u_1^{(n+1)} + u_1^{(n)} + 1000 + u_5^{(n)}]$$

$$u_4^{(n+1)} = \frac{1}{4}[2000 + u_5^{(n)} + u_1^{(n+1)} + u_1^{(n)}]$$

$$u_5^{(n+1)} = \frac{1}{4}[u_4^{(n+1)} + u_4^{(n)} + u_2^{(n+1)} + u_2^{(n)}]$$

*First iteration* : (put $n = 0$)

$$u_1^{(1)} = \frac{1}{4}(1000 + 1188 + 500 + 1438) \approx 1032$$

$$u_2^{(1)} = \frac{1}{4}(1032 + 1032 + 1000 + 1500) = 1141$$

$$u_4^{(1)} = \frac{1}{4}(2000 + 1500 + 1032 + 1032) = 1391$$

$$u_5^{(1)} = \frac{1}{4}(1391 + 1391 + 1141 + 1141) = 1266$$

*Second iteration* : (put $n = 1$)

$$u_1^{(2)} = \frac{1}{4}(1000 + 1141 + 500 + 1391) = 1008$$

$$u_2^{(2)} = \frac{1}{4}(1008 + 1008 + 1000 + 1266) = 1069$$

$$u_4^{(2)} = \frac{1}{4}(2000 + 1266 + 1008 + 1008) = 1321$$

$$u_5^{(2)} = \frac{1}{4}(1321 + 1321 + 1069 + 1069) = 1195$$

*Third iteration :*

$$u_1^{(3)} = \frac{1}{4}(1000 + 1069 + 500 + 1321) = 973$$

$$u_2^{(3)} = \frac{1}{4}(973 + 973 + 1000 + 1195) = 1035$$

$$u_4^{(3)} = \frac{1}{4}(2000 + 1195 + 973 + 973) = 1288$$

$$u_5^{(3)} = \frac{1}{4}(1288 + 1288 + 1035 + 1035) = 1162$$

*Fourth iteration :*

$$u_1^{(4)} = \frac{1}{4}(1000 + 1135 + 500 + 1288) = 956$$

$$u_2^{(4)} = \frac{1}{4}(956 + 956 + 1000 + 1162) = 1019$$

$$u_4^{(4)} = \frac{1}{4}(2000 + 1162 + 956 + 956) = 1269$$

$$u_5^{(4)} = \frac{1}{4}(1269 + 1269 + 1019 + 1019) = 1144$$

*Fifth iteration :*

$$u_1^{(5)} = \frac{1}{4}(1000 + 1019 + 500 + 1269) = 947$$

$$u_2^{(5)} = \frac{1}{4}(947 + 947 + 1000 + 1144) \approx 1010$$

$$u_4^{(5)} = \frac{1}{4}(2000 + 1144 + 947 + 947) \approx 1260$$

$$u_5^{(5)} = \frac{1}{4}(1260 + 1260 + 1010 + 1010) = 1135$$

Similarly, $u_1^{(6)} = 942, u_2^{(6)} = 1005, u_4^{(6)} = 1255, u_5^{(6)} = 1130$
$u_1^{(7)} = 940, u_2^{(7)} = 1003, u_4^{(7)} = 1253, u_5^{(7)} = 1128$
$u_1^{(8)} = 939, u_2^{(8)} = 1002, u_4^{(8)} = 1252, u_5^{(8)} = 1127$
$u_1^{(9)} = 939, u_2^{(6)} = 1001, u_4^{(9)} = 1251, u_5^{(9)} = 1126$

Thus there is negligible difference between the values obtained in the eighth and ninth iterations.
Hence $u_1 = 939$, $u_2 = 1001$, $u_4 = 1251$ and $u_5 = 1126$.

**Example 33.3.** *Given the values of u(x, y) on the boundary of the square in the Fig. 33.7, evaluate the function u(x, y) satisfying the Laplace equation $\nabla^2 u = 0$ at the pivotal points of this figure.*
*(Bhopal, 2009 ; Madras, 2003)*

**Solution.** To get the initial values of $u_1, u_2, u_3, u_4$, we assume $u_4 = 0$. Then

$$u_1 = \frac{1}{4}(1000 + 0 + 1000 + 2000) = 1000 \qquad \text{(Diag. formula)}$$

$$u_2 = \frac{1}{4}(1000 + 500 + 1000 + 0) = 625 \qquad \text{(Std. formula)}$$

$$u_3 = \frac{1}{4}(2000 + 0 + 1000 + 500) = 875 \qquad \text{(Std. formula)}$$

$$u_4 = \frac{1}{4}(875 + 0 + 625 + 0) = 375 \qquad \text{(Std. formula)}$$

We carry out the successive iterations, using the formulae

$$u_1^{(n+1)} = \frac{1}{4}[2000 + u_2^{(n)} + 1000 + u_3^{(n)}]$$

$$u_2^{(n+1)} = \frac{1}{4}[u_1^{(n+1)} + 500 + 1000 + u_4^{(n)}]$$

$$u_3^{(n+1)} = \frac{1}{4}[2000 + u_4^{(n)} + u_1^{(n+1)} + 500]$$

$$u_4^{(n+1)} = \frac{1}{4}[u_3^{(n+1)} + 0 + u_2^{(n+1)} + 0]$$

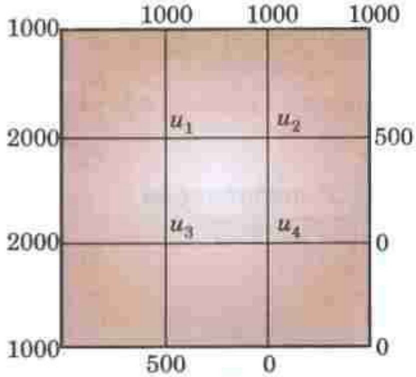


Fig. 33.7

*First iteration* : (put $n = 0$)

$$u_1^{(1)} = \frac{1}{4}(2000 + 625 + 1000 + 875) = 1125$$

$$u_2^{(1)} = \frac{1}{4}(1125 + 500 + 1000 + 375) = 750$$

$$u_3^{(1)} = \frac{1}{4}(2000 + 375 + 1125 + 500) = 1000$$

$$u_4^{(1)} = \frac{1}{4}(1000 + 0 + 750 + 0) \approx 438$$

*Second iteration* : (put $n = 1$)

$$u_1^{(2)} = \frac{1}{4}(2000 + 750 + 1000 + 1000) \approx 1188$$

$$u_2^{(2)} = \frac{1}{4}(1188 + 500 + 1000 + 438) \approx 782$$

$$u_3^{(2)} = \frac{1}{4}(2000 + 438 + 1188 + 500) \approx 1032$$

$$u_4^{(2)} = \frac{1}{4}(1032 + 0 + 782 + 0) \approx 454$$

*Third iteration* : (put $n = 2$)

$$u_1^{(3)} = \frac{1}{4}(2000 + 782 + 1000 + 1032) \approx 1204$$

$$u_2^{(3)} = \frac{1}{4}(1204 + 500 + 1000 + 454) \approx 789$$

$$u_3^{(3)} = \frac{1}{4}(2000 + 454 + 1204 + 500) \approx 1040$$

$$u_4^{(3)} = \frac{1}{4}(1040 + 0 + 789 + 0) \approx 458$$

Similarly, $u_1^{(4)} \approx 1207,\ u_2^{(4)} \approx 791,\ u_3^{(4)} \approx 1041,\ u_4^{(4)} = 458$

and $u_1^{(5)} = 1208,\ u_2^{(5)} = 791.5,\ u_3^{(5)} = 1041.5,\ u_4^{(5)} = 458.25$

Thus there is no significant difference between the fourth and fifth iteration values.

Hence $u_1 = 1208$, $u_2 = 792$, $u_3 = 1042$ and $u_4 = 458$.

**Example 33.4.** *Solve the Laplace equation $u_{xx} + u_{yy} = 0$ given that (Fig. 33.8).*

**Solution.** We first find the initial values in the following order :

$$u_5 = \frac{1}{4}(0 + 17 + 21 + 12.1) = 12.5 \quad \text{(Std. formula)}$$

$$u_1 = \frac{1}{4}(0 + 12.5 + 0 + 17) = 7.4 \quad \text{(Diag. formula)}$$

$$u_3 = \frac{1}{4}(12.5 + 18.6 + 17 + 21) = 17.28 \quad \text{(Diag. formula)}$$

$$u_7 = \frac{1}{4}(12.5 + 0 + 0 + 12.1) = 6.15 \quad \text{(Diag. formula)}$$

$$u_9 = \frac{1}{4}(12.5 + 9 + 21 + 12.1) = 13.65 \quad \text{(Diag. formula)}$$

$$u_2 = \frac{1}{4}(17 + 12.5 + 7.4 + 17.3) = 13.55 \quad \text{(Std. formula)}$$

$$u_4 = \frac{1}{4}(7.4 + 6.2 + 0 + 12.5) = 6.52 \quad \text{(Std. formula)}$$

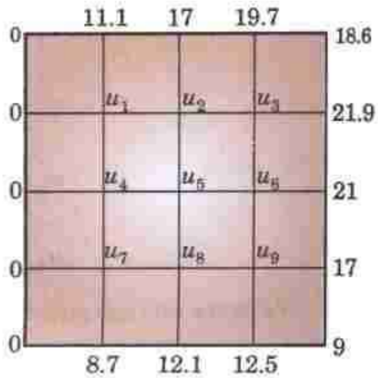


Fig. 33.8

$$u_6 = \frac{1}{4}(17.3 + 13.7 + 12.5 + 21) = 16.12 \quad \text{(Std. formula)}$$

$$u_8 = \frac{1}{4}(12.5 + 12.1 + 6.2 + 13.7) = 11.12 \quad \text{(Std. formula)}$$

Now we carry out the iteration process using the Standard formula :

$$u_1^{(n+1)} = \frac{1}{4}\left[(0 + 11.1 + u_4^{(n)} + u_2^{(n)})\right]$$

$$u_2^{(n+1)} = \frac{1}{4}\left[(u_1^{(n+1)} + 17 + u_5^{(n)} + u_3^{(n)})\right]$$

$$u_3^{(n+1)} = \frac{1}{4}\left[(u_2^{(n+1)} + 19.7 + u_6^{(n)} + 21.9)\right]$$

$$u_4^{(n+1)} = \frac{1}{4}\left[(u_1^{(n+1)} + 19.7 + u_7^{(n)} + u_5^{(n)})\right]$$

$$u_5^{(n+1)} = \frac{1}{4}\left[(u_4^{(n+1)} + u_2^{(n+1)} + u_8^{(n)} + u_6^{(n)})\right]$$

$$u_6^{(n+1)} = \frac{1}{4}\left[(u_5^{(n+1)} + u_3^{(n+1)} + u_9^{(n)} + 21)\right]$$

$$u_7^{(n+1)} = \frac{1}{4}\left[0 + (u_4^{(n+1)} + 8.7 + u_8^{(n)})\right]$$

$$u_8^{(n+1)} = \frac{1}{4}\left[(u_7^{(n+1)} + u_5^{(n+1)} + 12.1 + u_9^{(n)})\right]$$

$$u_9^{(n+1)} = \frac{1}{4}\left[(u_8^{(n+1)} + u_6^{(n)} + 12.8 + 17)\right]$$

*First iteration* (put $n = 0$, in the above results)

$$u_1^{(1)} = \frac{1}{4}\left(0 + 11.1 + u_4^{(o)} + u_2^{(o)}\right) = \frac{1}{4}\ (0 + 11.1 + 6.52 + 13.55) = 7.79$$

$$u_2^{(1)} = \frac{1}{4}\ (7.79 + 17 + 12.5 + 17.28) = 13.64$$

$$u_3^{(1)} = \frac{1}{4}\ (13.64 + 19.7 + 16.12 + 21.9) = 12.84$$

$$u_4^{(1)} = \frac{1}{4}\ (0 + 7.79 + 6.15 + 12.5) = 6.61$$

$$u_5^{(1)} = \frac{1}{4}\ (6.61 + 13.64 + 11.12 + 16.12) = 11.88$$

$$u_6^{(1)} = \frac{1}{4}\ (11.88 + 17.84 + 13.65 + 21) = 16.09$$

$$u_7^{(1)} = \frac{1}{4}\ (0 + 6.61 + 8.7 + 11.12) = 6.61$$

$$u_8^{(1)} = \frac{1}{4}\ (6.61 + 11.88 + 12.1 + 13.65) = 11.06$$

$$u_9^{(1)} = \frac{1}{4}\ (11.06 + 16.09 + 12.8 + 17) = 12.238$$

*Second iteration* (put $n = 1$)

$$u_1^{(2)} = \frac{1}{4}\ (0 + 11.1 + 6.61 + 13.64) = 7.84$$

$$u_2^{(2)} = \frac{1}{4}\ (7.84 + 17 + 11.88 + 17.84) = 16.64$$

$$u_3^{(2)} = \frac{1}{4}\ (13.64 + 19.7 + 16.09 + 21.9) = 17.83$$

$$u_4^{(2)} = \frac{1}{4}\ (0 + 7.84 + 6.61 + 11.88) = 6.58$$

$$u_5^{(2)} = \frac{1}{4}\ (6.58 + 13.64 + 11.06 + 16.09) = 11.84$$

$$u_6^{(2)} = \frac{1}{4}\ (11.84 + 17.83 + 14.24 + 21) = 16.23$$

$$u_7^{(2)} = \frac{1}{4}\ (0 + 6.58 + 8.7 + 11.06) = 6.58$$

$$u_8^{(2)} = \frac{1}{4}\ (6.58 + 11.84 + 12.1 + 14.24) = 11.19$$

$$u_9^{(2)} = \frac{1}{4}\ (11.19 + 16.23 + 12.8 + 17) = 14.30$$

*Third iteration* (put $n = 2$)

$$u_1^{(3)} = \frac{1}{4}\ (0 + 11.1 + 6.58 + 13.64) = 7.83$$

$$u_2^{(3)} = \frac{1}{4}\ (7.83 + 17 + 11.84 + 17.83) = 13.637$$

$$u_3^{(4)} = \frac{1}{4}\ (13.63 + 19.7 + 16.23 + 21.9) = 17.86$$

$$u_4^{(3)} = \frac{1}{4}\ (0 + 7.83 + 6.58 + 11.84) = 6.56$$

$$u_5^{(3)} = \frac{1}{4}\ (6.56 + 13.63 + 11.19 + 16.23) = 11.90$$

$$u_6^{(3)} = \frac{1}{4}\ (11.90 + 17.86 + 14.30 + 21) = 16.27$$

$$u_7^{(3)} = \frac{1}{4}\ (0 + 6.56 + 8.7 + 11.19) = 6.61$$

$$u_8^{(3)} = \frac{1}{4}\ (6.61 + 11.90 + 12.1 + 14.30) = 11.23$$

$$u_9^{(3)} = \frac{1}{4}\ (11.23 + 16.27 + 12.8 + 17) = 14.32$$

Similarly, $u_1^{(4)} = 7.82, u_2^{(4)} = 13.65, u_3^{(4)} = 17.88, u_4^{(4)} = 6.58, u_5^{(4)} = 11.94, u_6^{(4)} = 16.28,$

$u_7^{(4)} = 6.63, u_8^{(4)} = 11.25, u_9^{(4)} = 14.33$

$u_1^{(5)} = 7.83, u_2^{(5)} = 13.66, u_3^{(5)} = 17.89, u_4^{(5)} = 6.50, u_5^{(5)} = 11.95, u_6^{(5)} = 16.29,$

$u_7^{(5)} = 6.64, u_8^{(5)} = 11.25, u_9^{(5)} = 14.34$

## 33.6 SOLUTION OF POISSON'S EQUATION*

$$\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{u}}{\partial \mathbf{y}^2} = \mathbf{f(x, y)} \qquad ...(1)$$

This is an *elliptic equation* which can be solved numerically at the interior mesh points of a square network when the boundary values are known. The standard 5-point formula for (1) takes the form

$$u_{i-1,j} + u_{i+1,j} + u_{i,\ j+1} + u_{i,\ j-1} - 4u_{i,j} = h^2 f(ih, jh) \qquad ...(2)$$

By applying (2) at each mesh-point, we arrive at linear equations in the pivotal values $i, j$. These equations can be solved by Gauss-Seidal iteration method (p. 938).

**Example 33.5.** *Solve the Poisson equation $u_{xx} + u_{yy} = -81xy$, $0 < x < 1$, $0 < y < 1$ given that $u(0, y) = 0$, $u(x, 0) = 0$, $u(1, y) = 100$, $u(x, 1) = 100$ and $h = 1/3$.* (Anna, 2005)

**Solution.** Here $h = 1/3$ $u_{i,j-1}$ $u_{i,j}$ (Fig. 33.9)

* See p. 882.

The standard 5-point formula for the given equation is

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f(ih, jh) = h^2 [-81 (ih \cdot jh)] = h^4 (-81)\, ij = -ij \quad ...(i)$$

For $u_1$ ($i = 1, j = 2$), (*i*) gives $0 + u_2 + u_3 + 100 - 4u_1 = -2$

*i.e.*, $$-4u_1 + u_2 + u_3 = -102 \quad ...(ii)$$

For $u_2$ ($i = 2, j = 2$), (*i*) gives $u_1 + 100 + u_4 + 100 - 4u_2 = -4$

*i.e.*, $$u_1 - 4u_2 + u_4 = -204 \quad ...(iii)$$

For $u_3$ ($i = 1, j = 1$), (*i*) gives $0 + u_4 + 0 + u_1 - 4u_3 = -1$

*i.e.*, $$u_1 - 4u_3 + u_4 = -1 \quad ...(iv)$$

For $u_4$ ($i = 2, j = 1$), gives $u_3 + 100 + u_2 - 4u_4 = -2$

$$u_2 + u_3 - 4u_4 = -102 \quad ...(v)$$

Subtracting (*v*) from (*ii*), $-4u_1 + 4u_4 = 0$ *i.e.* $u_1 = u_4$

Then (*iii*) becomes $2u_1 - 4u_2 = -204$ ...(*vi*)

and (*iv*) becomes $2u_1 - 4u_3 = -1$ ...(*vii*)

Now (4) × (*ii*) + (*vi*) gives $-14u_1 + 4u_3 = -612$ ...(*viii*)

(*vii*) + (*viii*) gives $-12u_1 = -613$

Thus $u_1 = 613/12 = 51.0833 = u_4$.

From (*vi*), $$u_2 = \frac{1}{2}(u_1 + 102) = 76.5477$$

From (*vii*), $$u_3 = \frac{1}{2}\left(u_1 + \frac{1}{2}\right) = 25.7916.$$

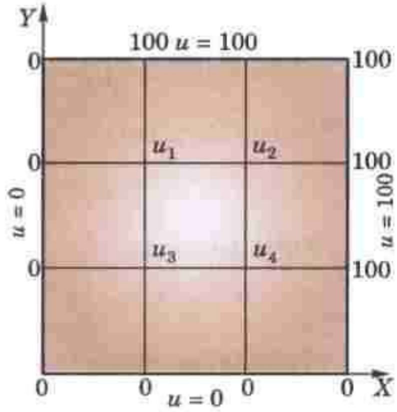


Fig. 33.9

**Example 33.6.** *Solve the partial differential equation $\nabla^2 u = -10(x^2 + y^2 + 10)$ over the square with sides $x = 0 = y$, $x = 3 = y$ with $u = 0$ on the boundary and mesh length = 1.*

(*Anna, 2007 ; P.T.U., 2007 ; Delhi, 2002*)

**Solution.** Here $h = 1$ (Fig. 33.10).

∴ The standard 5-point formula for the given equation is

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = -10(i^2 + j^2 + 10) \quad ...(i)$$

*For* $u_1$($i = 1, j = 2$), (*i*) gives

$$0 + u_2 + 0 + u_3 - 4u_1 = -10(1 + 4 + 10)$$

*i.e.*, $$u_1 = \frac{1}{4}(u_2 + u_3 + 150) \quad ...(ii)$$

*For* $u_2$($i = 2, j = 2$), (*i*) gives $u_2 = \frac{1}{4}(u_1 + u_4 + 180)$ ...(*iii*)

*For* $u_3$($i = 1, j = 1$), we have $u_3 = \frac{1}{4}(u_1 + u_4 + 120)$ ...(*iv*)

*For* $u_4$($i = 2, j = 1$), we have $u_4 = \frac{1}{4}(u_2 + u_3 + 150)$ ...(*v*)

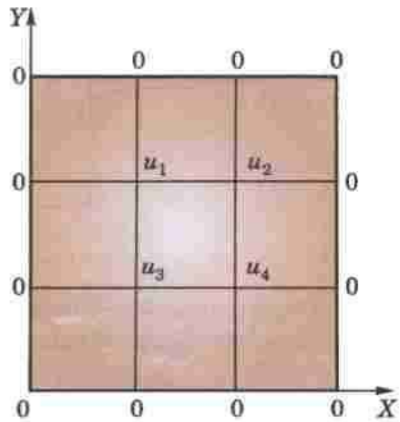


Fig. 33.10

Equations (*ii*) and (*v*) show that $u_4 = u_1$. Thus the above equations reduce to

$$u_1 = \frac{1}{4}(u_2 + u_3 + 150) \quad ...(vi)$$

$$u_2 = \frac{1}{2}(u_1 + 90) \quad ...(vii)$$

$$u_3 = \frac{1}{2}(u_1 + 60) \quad ...(viii)$$

Now let us solve these equations by Gauss-Seidal iteration method.

*First iteration :* Starting from the approximations $u_2 = 0$, $u_3 = 0$, we obtain

$$u_1^{(1)} = 37.5. \text{ Then } u_2^{(1)} = \frac{1}{2}(37.5 + 90) \approx 64\ ;\ u_3^{(1)} = \frac{1}{2}(37.5 + 60) \approx 49$$

*Second iteration :*

$$u_1^{(2)} = \frac{1}{4}(64 + 49 + 150) = 66 \; ; \; u_2^{(2)} = \frac{1}{2}(66 + 90) = 78 \; ; \; u_3^{(2)} = \frac{1}{2}(66 + 60) = 63$$

*Third iteration :*

$$u_1^{(3)} = \frac{1}{4}(78 + 63 + 150) = 73 \; ; \; u_2^{(3)} = \frac{1}{2}(73 + 90) = 82 \; ; \; u_3^{(2)} = \frac{1}{2}(73 + 60) \approx 67$$

*Fourth iteration :*

$$u_1^{(4)} = \frac{1}{4}(82 + 67 + 150) = 75 \; ; \; u_2^{(4)} = \frac{1}{2}(75 + 90) = 82.5 \; ; \; u_3^{(4)} = \frac{1}{2}(75 + 60) = 67.5$$

*Fifth iteration :*

$$u_1^{(5)} = \frac{1}{4}(82.5 + 67.5 + 150) = 75 \; ; \; u_2^{(5)} = \frac{1}{2}(75 + 90) = 82.5 \; ; \; u_3^{(5)} = \frac{1}{2}(75 + 60) = 67.5$$

Since these values are the same as those of fourth iteration, we have

$$u_1 = 75, u_2 = 82.5, u_3 = 67.5 \text{ and } u_4 = 75.$$

## PROBLEMS 33.2

1. Solve the equation $u_{xx} + u_{yy} = 0$ for the following square mesh with boundary values as shown in Fig. 33.11. Iterate until the maximum difference between the successive values at any point is less than 0.001. *(Delhi, 2002)*
2. Solve $\nabla^2 u = 0$ under the conditions $(h = 1, k = 1)$, $u(0, y) = 0$, $u(4, y) = 12 + y$ for $0 \le y \le 4$ ; $u(x, 0) = 3x$, $u(x, 4) = x^2$ for $0 \le x \le 4$. *(Cusat, 2008)*
3. Solve the elliptic equation $u_{xx} + u_{yy} = 0$ for the square mesh with boundary values as shown in Fig. 33.12. Iterate until the maximum difference between successive values at any point is less than 0.005.

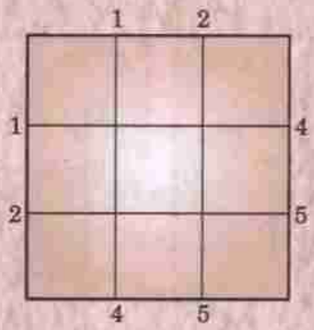



Fig. 33.11

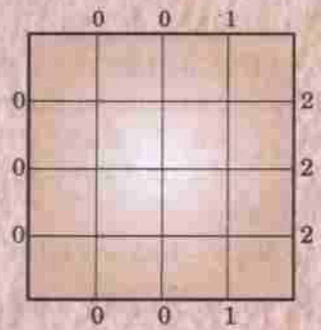



Fig. 33.12

4. Using central-difference approximation solve $\nabla^2 u = 0$ at the nodal points of the square grid of Fig. 33.13 using the boundary values indicated. *(V.T.U., 2000)*

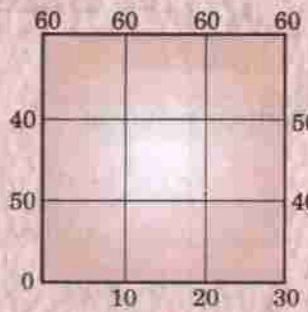



Fig. 33.13

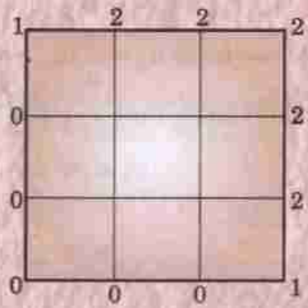



Fig. 33.14

5. Solve $u_{xx} + u_{yy} = 0$ for the square mesh with boundary values as shown in Fig. 33.14. Iterate till the mesh values are correct to two decimal places.

6. Solve the Laplace's equation $u_{xx} + u_{yy} = 0$ in the domain of Fig. 33.15.

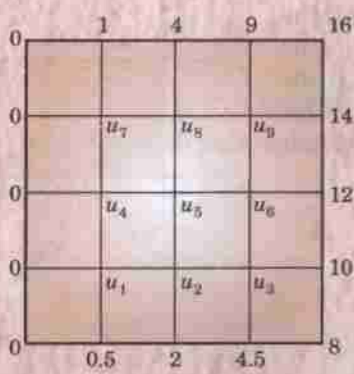


Fig. 33.15

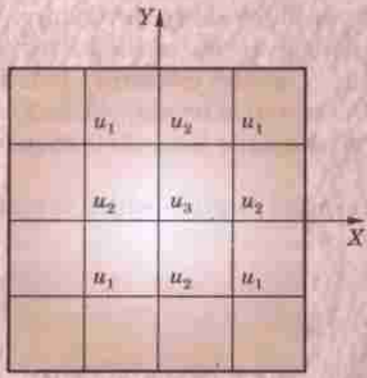


Fig. 33.16

7. Solve the Poisson's equation $\nabla^2 u = 8x^2y^2$ for the square mash of Fig. 33.16 with $u(x, y) = 0$ on the boundary and mesh length = 1. *(J.N.T.U., 2004 S)*

**Note.** Solution of elliptic equations by *Relaxation method* is given in author's book *'Numerical Methods in Engineering and Science'*.

## 33.7 PARABOLIC EQUATIONS

The one-dimensional heat conduction equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

is a well-known example of parabolic partial differential equations. The solution of this equation is a temperature function $u(x, t)$ which is defined for values of $x$ from 0 to $l$ and for values of time $t$ from 0 to $\infty$. The solution is not defined in a closed domain but advances in an open-ended region from initial values, satisfying the prescribed boundary conditions. (Fig. 33.17).

In general, the study of pressure waves in a fluid, propagation of heat and unsteady state problems lead to parabolic type of equations.

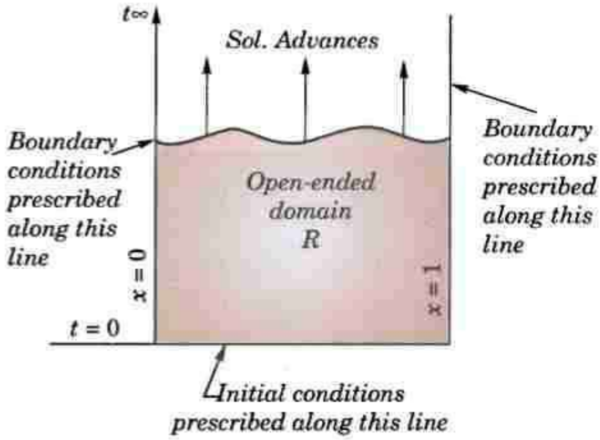


Fig. 33.17

## 33.8 SOLUTION OF HEAT EQUATION

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad ...(1)$$

where $c^2 = k/s\rho$ is the diffusivity of the substance (cm²/sec.)

Consider a rectangular mesh in the $x$-$t$ plane with spacing $h$ along $x$ direction and $k$ along time $t$ direction. Denoting a mesh point $(x, t) = (ih, jk)$ as simply $i, j$, we have

$$\frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j}}{k} \qquad \text{[By (5) § 33.3]}$$

and

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} \qquad \text{[By (4) § 33.3]}$$

Substituting these in (1), we obtain

$$u_{i,j+1} - u_{i,j} = \frac{kc^2}{h^2}\,[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}]$$

or $$u_{i,j+1} = \alpha u_{i-1,j} + (1-2\alpha)\, u_{i,j} + \alpha u_{i+1,j} \qquad ...(2)$$
where $\alpha = kc^2/h^2$ is the mesh ratio parameter.

This formula enables us to determine the value of $u$ at the $(i, j + 1)$th mesh point in terms of the known function values at the points $x_{i-1}$, $x_i$ and $x_{i+1}$ at the instant $t_j$. It is a relation between the function values at the two time levels $j + 1$ and $j$ and is therefore, called a *2-level formula*. In schematic form (2) is shown in Fig. 33.18 which is called the *Schmidt explicit formula* which is valid only for $0 < \alpha \leq \frac{1}{2}$.

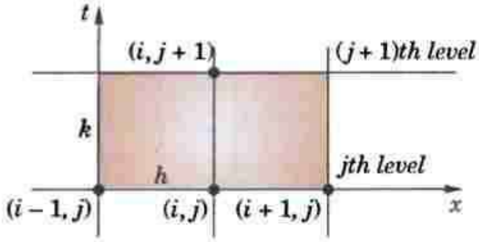


Fig. 33.18

**Obs.** In particular when $\alpha = \frac{1}{2}$, (2) reduces to
$$u_{i,j+1} = \frac{1}{2}(u_{i-1,j} + u_{i+1,j}) \qquad ...(3)$$
which shows that the *value of u at $x_i$ at time $t_{j+1}$ is the mean of the u-values at $x_{i-1}$ and $x_{i+1}$ at time $t_j$*. This relation, known as *Bendre-Schmidt recurrence relation*, gives the values of $u$ at the internal mesh points with the help of boundary conditions.

**Note.** The other formulae (*i.e. Crank-Nicolson formula* and *Du Fort-Frankel formula*) are given in author's book *'Numerical Methods in Engineering and Science'*.

**Example 33.7.** *Solve* $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial^2 x^2}$ *in* $0 < x < 5$, $t \geq 0$ *given that* $u(x, 0) = 20$, $u(0, t) = 0$, $u(5, t) = 100$. *Compute u for the time-step with h = 1 by Crank-Nicholson method.* (Anna, 2006)

**Solution.** Here $c^2 = 1$ and $h = 1$.

Taking $\alpha$ (*i.e.*, $c^2k/h$) = 1, we get $k = 1$

Also we have

| j \ i | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 20 | 20 | 20 | 20 | 100 |
| 1 | 0 | $u_1$ | $u_2$ | $u_3$ | $u_4$ | 100 |

Then Crank-Nicholson formula becomes

$$4u_{i,j+1} = u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}$$
$$4u_1 = 0 + 20 + 0 + u_2 \quad i.e. \quad 4u_1 - u_2 = 20 \qquad ...(1)$$
$$4u_2 = 20 + 20 + u_1 + u_3 \quad i.e. \quad 4u_1 - 4u_2 + u_3 = -40 \qquad ...(2)$$
$$4u_3 = 20 + 20 + u_2 + u_4 \quad i.e. \quad u_2 - 4u_3 + u_4 = -40 \qquad ...(3)$$
$$4u_4 = 20 + 100 + u_3 + 100 \quad i.e. \quad 4u_3 - 4u_4 + = -220 \qquad ...(4)$$

Now (1) – 4(2) gives $15u_2 - 4u_3 = 180$ ...(5)

4(3) + (4) gives $4u_2 - 15u_3 = -380$ ...(6)

Then 15(5) – 4(6) gives $209u_2 = 4220$ *i.e.* $u_2 = 20.2$

From (5), we get $4u_3 = 15 \times 20.2 - 180$ *i.e.* $u_3 = 30.75$

From (1), $4u_1 = 20 + 20.2$ *i.e.* $u_1 = 10.05$

From (4), $4u_4 = 220 + 30.75$ *i.e.* $u_4 = 62.69$

Thus the required values are 10.05, 20.2, 30.75 and 62.68.

**Example 33.8.** *Solve the boundary value problem $u_t = u_{xx}$ under the conditions $u(0, t) = u(1, t) = 0$ and $u(x, 0) = \sin \pi x$, $0 \leq x \leq 1$ using Schmidt method (Take h = 0.2 and α = 1/2).* (Rohtak, 2003)

**Solution.** Since $h = 0.2$ and $\alpha = 1/2$

$\therefore$ $\alpha = \frac{k}{h^2}$ gives $k = 0.02$

Since $\alpha = 1/2$, we use Bendre-Schmidt relation

$$u_{i, j+1} = \frac{1}{2}\,(i_{i-1, j} + u_{i+1, j}) \qquad ...(i)$$

We have $u(0, 0) = 0$, $u(0.2, 0) = \sin \pi/5 = 0.5875$

$u(0.4, 0) = \sin 2\pi/5 = 0.95111$, $u(0.6, 0) = \sin 3\pi/5 = 0.9511$

$u(0.8, 0) = \sin 4\pi/5 = 0.5875$, $u(1, 0) = \sin \pi = 0$

The value of $u$ at the mesh points can be obtained by using the recurrence relation $(i)$ as shown in table below :

| $x \to$ | | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|---|
| $t \downarrow$ | $j \backslash i$ | 0 | 1 | 2 | 3 | 4 | 5 |
| 0 | 0 | 0 | 0.5878 | 0.9511 | 0.9511 | 0.5878 | 0 |
| 0.02 | 1 | 0 | 0.4756 | 0.7695 | 0.7695 | 0.4756 | 0 |
| 0.04 | 2 | 0 | 0.3848 | 0.6225 | 0.6225 | 0.3848 | 0 |
| 0.06 | 3 | 0 | 0.3113 | 0.5036 | 0.5036 | 0.3113 | 0 |
| 0.08 | 4 | 0 | 0.2518 | 0.4074 | 0.4074 | 0.2518 | 0 |
| 0.1 | 5 | 0 | 0.2037 | 0.3296 | 0.3296 | 0.2037 | 0 |

**Example 33.9.** *Find the values of $u(x, t)$ satisfying the parabolic equation $\dfrac{\partial u}{\partial t} = 4\dfrac{\partial^2 u}{\partial x^2}$ and the boundary conditions $u(0, t) = 0 = u(8, t)$ and $u(x, 0) = 4x - \dfrac{1}{2}x^2$ at the points $x = i : i = 0, 1, 2, ..., 8$ and $t = \dfrac{1}{8}j : j = 0, 1, 2, ..., 5$.*

**Solution.** Here $c^2 = 4$, $h = 1$ and $k = 1/8$. Then $\alpha = c^2k/h^2 = 1/2$.

$\therefore$ we have Bendre-Schmidt's recurrence relation

$$u_{i, j+1} = \frac{1}{2}(u_{i-1, j} + u_{i+1, j}) \qquad ...(i)$$

Now since $u(0, t) = 0 = u(8, t)$

$\therefore$ $u_{0, i} = 0$ and $u_{8, j} = 0$ for all values of $j$, *i.e.*, the entries in the first and last column are zero.

Since $u(x, 0) = 4x - \dfrac{1}{2}x^2$

$$\therefore \qquad u_{i, 0} = 4i - \frac{1}{2}i^2 = 0, 3.5, 6, 7.5, 8, 7.5, 6, 3.5$$

for $\qquad i = 0, 1, 2, 3, 4, 5, 6, 7$ at $t = 0$

These are the entries of the first row.

| $j \backslash i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 3.5 | 6 | 7.5 | 8 | 7.5 | 6 | 3.5 | 0 |
| 1 | 0 | 3 | 5.5 | 7 | 7.5 | 7 | 5.5 | 3 | 0 |
| 2 | 0 | 2.75 | 5 | 6.5 | 7 | 6.5 | 5 | 2.75 | 0 |
| 3 | 0 | 2.5 | 4.625 | 6 | 6.5 | 6 | 4.625 | 2.5 | 0 |
| 4 | 0 | 2.3125 | 4.25 | 5.5625 | 6 | 5.5625 | 4.25 | 2.3125 | 0 |
| 5 | 0 | 2.125 | 3.9375 | 5.125 | 5.5625 | 5.125 | 3.9375 | 2.125 | 0 |

Putting $j = 0$ in (*i*), we have

$$u_{i,1} = \frac{1}{2}(u_{i-1,0} + u_{i+1,0})$$

Taking $i = 1, 2, \ldots, 7$ successively, we get

$$u_{1,1} = \frac{1}{2}(u_{0,0} + u_{2,0}) = \frac{1}{2}(0 + 6) = 3$$

$$u_{2,1} = \frac{1}{2}(u_{1,0} + u_{3,0}) = \frac{1}{2}(3.5 + 7.5) = 5.5$$

$$u_{3,1} = \frac{1}{2}(u_{2,0} + u_{4,0}) = \frac{1}{2}(6 + 8) = 7$$

$$u_{4,1} = 7.5,\ u_{5,1} = 7,\ u_{6,1} = 5.5,\ u_{7,1} = 3.$$

These are the entries in the second row.

Putting $j = 1$ in (*i*), the entries of the third row are given by

$$u_{i,2} = \frac{1}{2}(u_{i-1,1} + u_{i+1,1})$$

Similarly putting $j = 2, 3, 4$ successively in (*i*), the entries of the fourth, fifth and sixth rows are obtained. Hence the values of $u_{i,j}$ are as given in the above table.

**Example 33.10.** *Solve the equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

*subject to the conditions* $u(x, 0) = \sin \pi x,\ 0 \le x \le 1$ ; $u(0, t) = u(1, t) = 0$. *Carry out computations for two levels, taking* $h = 1/3$, $k = 1/36$. *(V.T.U., 2005)*

**Solution.** Here $c^2 = 1$, $h = 1/3$, $k = 1/36$ so that

$$\alpha = kc^2/h^2 = 1/4.$$

Also $u_{1,0} = \sin \pi/3 = \sqrt{3}/2$, $u_{2,0} = \sin 2\pi/3 = \sqrt{3}/2$ and all other boundary values are zero as shown in Fig. 33.19.

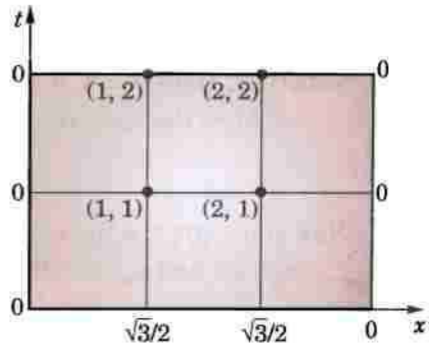



Fig. 33.19

*Schmidt's formula* [(2) of § 33.8]

$$u_{i,j+1} = \alpha u_{i-1,j} + (1 - 2\alpha)\, u_{i,j} + \alpha u_{i+1,j}$$

becomes
$$u_{i,j+1} = \frac{1}{4}[u_{i-1,j} + 2u_{i,j} + u_{i+1,j}]$$

For $i = 1, 2$ ; $j = 0$ :

$$u_{1,1} = \frac{1}{4}[u_{0,0} + 2u_{1,0} + u_{2,0}] = \frac{1}{4}(0 + 2 \times \sqrt{3/2} + \sqrt{3/2}) = 0.65$$

$$u_{2,1} = \frac{1}{4}[u_{1,0} + 2u_{2,0} + u_{3,0}] = \frac{1}{4}(\sqrt{3/2} + 2 \times \sqrt{3/2} + 0) = 0.65$$

For $i = 1, 2, j = 1$ :

$$u_{1,2} = \frac{1}{4}(u_{0,1} + 2u_{1,1} + u_{2,1}) = 0.49$$

$$u_{2,2} = \frac{1}{4}(u_{1,1} + 2u_{2,1} + u_{3,1}) = 0.49.$$

## PROBLEMS 33.3

1. Find the solution of the parabolic equation $u_{xx} = 2u_t$ when $u(0, t) = u(4, t) = 0$ and $u(x, 0) = x(4 - x)$, taking $h = 1$. Find the values upto $t = 5$. *(Madras, 2001)*
2. Solve the equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$ with the conditions $u(0, t) = 0$, $u(x, 0) = x(1 - x)$ and $u(1, t) = 0$. Assume $h = 0.1$. Tabulate $u$ for $t = k$, $2k$ and $3k$ choosing an appropriate value of $k$. *(Anna, 2004)*

3. Given $\frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial t} = 0$ ; $f(0, t) = f(5, t) = 0$, $f(x, 0) = x^2(25 - x^2)$ ; find the values of $f$ for $x = ih (i = 0, 1, ..., 5)$ and $t = jk (j = 0, 1, ..., 6)$ with $h = 1$ and $k = \frac{1}{2}$, using the explicit method.

4. Solve the heat equation $\partial u / \partial t = \partial^2 u / \partial x^2$ subject to the conditions $u(0, t) = u(1, t) = 0$ and

$$u(x, 0) = 2x \text{ for } 0 \le x \le \frac{1}{2}$$
$$= 2(1 - x) \text{ for } \frac{1}{2} \le x \le 1.$$

Take $h = 1/4$ and $k$ according to Bendre-Schmidt equation.

## 33.9 HYPERBOLIC EQUATIONS

The wave equation : $\frac{\partial^2 u}{dt^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ is the simplest example of hyperbolic partial differential equations. Its solution is the displacement function $u(x, t)$ defined for values of $x$ from 0 to $l$ and for $t$ from 0 to $\infty$, satisfying the initial and boundary conditions. The solution as for parabolic equations, advances in an open-ended region (Fig. 33.17). In the case of hyperbolic equations however, we have two initial conditions and two boundary conditions.

Such equations arise from convective type of problems in vibrations, wave mechanics and gas dynamics.

## 33.10 SOLUTION OF WAVE EQUATION

$$\frac{\partial^2 u}{dt^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad ...(1)$$

subject to the initial conditions : $u = f(x)$, $\frac{\partial u}{\partial t} = g(x)$, $0 \le x \le 1$ at $t = 0$ ...(2)

and the boundary conditions : $u(0, t) = \phi(t)$, $u(1, t) = \psi(t)$. ...(3)

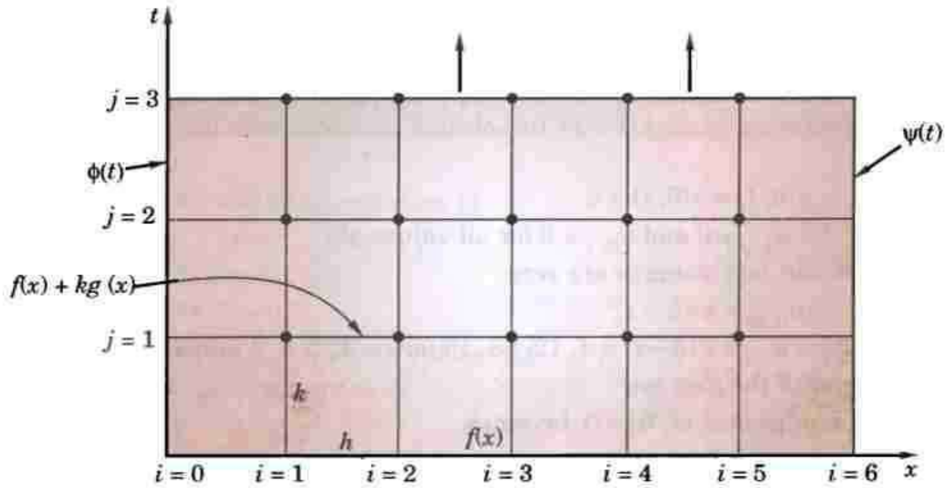



Fig. 33.20

Consider a rectangular mesh in the $x$-$t$ plane spacing $h$ along $x$ direction and $k$ along time $t$ direction. Denoting a mesh point $(x, t) = (ih, jk)$ as simply $i, j$, we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} \quad \text{and} \quad \frac{\partial^2 u}{\partial t^2} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2}$$

Replacing the derivatives in (1) by their above approximations, we obtain

$$u_{i,j-1} - 2u_{i,j} + u_{i,j+1} = \frac{c^2k^2}{h^2}(u_{i-1,j} - 2u_{i,j} + u_{i,j+1})$$

or $$u_{i,j+1} = 2(1-\alpha^2c^2)\,u_{i,j} + \alpha^2c^2\,(u_{i-1,j} + u_{i+1,j} - u_{i,j-1}) \qquad ...(4)$$

where $\alpha = k/h$.

Now replacing the derivative in (2) by its central difference approximation, we get

$$\frac{u_{i,j+1} - u_{i,j-1}}{2k} = \frac{\partial u}{\partial t} = g(x)$$ [See (7) p. 1042]

or $$u_{i,j+1} = u_{i,j-1} + 2k\,g(x) \quad \text{at } t=0 \quad i.e.\ u_{i,1} = u_{i,-1} + 2kg(x) \text{ for } j=0 \qquad ...(5)$$

Also initial condition $u = f(x)$ at $t = 0$ becomes $u_{i,0} = f(x)$ ...(6)

Combining (5) and (6), we have $u_{i,1} = f(x) + 2kg(x)$ ...(7)

Also (3) gives $u_{0,j} = \phi(t)$ and $u_{1,j} = \psi(t)$

Hence (4) gives the values of $u_{i,j+1}$ at the $(j+1)$th level when the nodal values at $(j-1)$th and $j$th levels are known from (6) and (7) as shown in Fig. 32.20. Thus (4) gives an ***implicit scheme*** *for the solution of the wave equation.*

**A special case :** The coefficient of $u_{i,j}$ in (4) will vanish if $\alpha = 1/c$ or $k = h/c$. Then (4) reduces to the *simple form*

$$\mathbf{u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}} \qquad ...(8)$$

*This result provides an* ***explicit scheme*** *for the solution of the wave equation.*

**Obs. 1.** *For $\alpha = 1/c$,* the solution of (4) is stable and coincides with the solution of (1).
*For $\alpha < 1/c$,* the solution is stable but inaccurate.
*For $\alpha > 1/c$,* the solution is unstable.

**Obs. 2.** *The formula (4) converges for $\alpha \le 1$ i.e. for $k \le h$.*

**Example 33.11.** *Evaluate the pivotal values of the equation $u_{tt} = 16u_{xx}$, taking $h = 1$ upto $t = 1.25$. The boundary conditions are $u(0, t) = u(5, t) = 0$, $u_t(x, 0) = 0$ and $u(x, 0) = x^2(5-x)$.* (Madras, 2006)

**Solution.** Here $c^2 = 16$.

∴ The difference equation for the given equation is

$$u_{i,j+1} = 2(1-16\alpha^2)\,u_{i,j} + 16\alpha^2\,(u_{i-1,j} + u_{i+1,j}) - u_{i,j-1} \quad \text{where } \alpha = k/h \qquad ...(i)$$

Taking $h = 1$ and choosing $k$ so that the coefficient of $u_{i,j}$ vanishes, we have $16\alpha^2 = 1$, *i.e.*, $k = h/4, = 1/4$.

∴ (*i*) reduces to $$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1} \qquad ...(ii)$$

which gives a convergent solution (since $k/h < 1$). Its solution coincides with the solution of the given differential equation.

Now since $u(0, t) = u(5, t) = 0$.

∴ $u_{0,j} = 0$ and $u_{5,j} = 0$ for all values of $j$

*i.e.* the entries in the first and last columns are zero.

Since $u_{(x,0)} = x^2(5-x)$

∴ $u_{i,0} = i^2(5-i) = 4, 12, 18, 16$ for $i = 1, 2, 3, 4$ at $t = 0$.

These are the entries of the *first row.*

Finally the initial condition $u_t(x, 0) = 0$, becomes

$$\frac{u_{i,j+1} - u_{i,j-1}}{2k} = 0, \text{ when } j = 0, \text{ giving } u_{i,1} = u_{i,-1} \qquad ...(iii)$$

*Putting $j = 0$ in* (*ii*), $$u_{i,1} = u_{i-1,0} + u_{i+1,0} - u_{i,-1}$$
$$= u_{i-1,0} + u_{i+1,0} - u_{i,1} \quad \text{using } (iii)$$

or $$u_{i,1} = \frac{1}{2}(u_{i-1,0} + u_{i+1,0}) \qquad ...(iv)$$

Taking $i = 1, 2, 3, 4$ successively, we obtain

$$u_{1,1} = \frac{1}{2}(u_{0,0} + u_{2,0}) = \frac{1}{2}(0 + 12) = 6$$

$$u_{2,1} = \frac{1}{2}(u_{1,0} + u_{3,0}) = \frac{1}{2}(4 + 18) = 11$$

$$u_{3,1} = \frac{1}{2}(u_{2,0} + u_{4,0}) = \frac{1}{2}(12 + 16) = 14$$

$$u_{4,1} = \frac{1}{2}(u_{3,0} + u_{5,0}) = \frac{1}{2}(18 + 0) = 9$$

These are the entries of the *second row.*

*Putting $j = 1$* in (*ii*), we get

$$u_{i,2} = u_{i-1,1} + u_{i+1,1} - u_{i,0}$$

Taking $i = 1, 2, 3, 4$ scccessively, we obtain

$$u_{1,2} = u_{0,1} + u_{2,1} - u_{1,0} = 0 + 11 - 4 = 7$$
$$u_{2,2} = u_{1,1} + u_{3,1} - u_{2,0} = 6 + 14 - 12 = 8$$
$$u_{3,2} = u_{2,1} + u_{4,1} - u_{3,0} = 11 + 9 - 18 = 2$$
$$u_{4,2} = u_{3,1} + u_{5,1} - u_{4,0} = 14 + 0 - 16 = -2$$

These are the entries of the *third row.*

Similarly putting $j = 2, 3, 4$ successively in (*ii*), the entries of the fourth, fifth and sixth rows are obtained.

Hence the values of $u_{i,j}$ are as shown in the table below :

| $j$ \ $i$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 4 | 12 | 18 | 16 | 0 |
| 1 | 0 | 6 | 11 | 14 | 9 | 0 |
| 2 | 0 | 7 | 8 | 2 | −2 | 0 |
| 3 | 0 | 2 | −2 | −8 | −7 | 0 |
| 4 | 0 | −9 | −14 | −11 | −6 | 0 |
| 5 | 0 | −16 | −18 | −12 | −4 | 0 |

**Example 33.12.** *The transverse displacement u of a point at a distance x from one end and at any time t of a vibrating string satisfies the equation $\partial^2u/\partial t^2 = 4\partial^2u/\partial x^2$, with boundary conditions $u = 0$ at $x = 0$, $t > 0$ and $u = 0$ at $x = 4$, $t > 0$ and initial conditions $u = x(4 - x)$ and $\partial u/\partial t = 0$ at $t = 0$, $0 \le x \le 4$. Solve this equation numerically for one half period of vibration, taking $h = 1$ and $k = 1/2$.*

**Solution.** Here, $h/k = 2 = c$.

$\therefore$ the difference equation for the given equation is

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1} \quad ...(i)$$

which gives a convergent solution (since $k < h$).

Now since $u(0, t) = u(4, t) = 0$,

$\therefore$ $u_{0,j} = 0$ and $u_{4,j} = 0$ for all values of $j$.

*i.e.*, the entries in the first and last columns are zero.

Since $u_{(x,0)} = x(4 - x)$,

$\therefore$ $u_{i,0} = i(4 - i) = 3, 4, 3$ for $i = 1, 2, 3$ at $t = 0$.

These are the entries of the *first row.*

Also $u_t(x, 0) = 0$ becomes

$$\frac{u_{i,j+1} - u_{i,j-1}}{2k} = 0 \text{ when } j = 0, \text{ giving } u_{i,1} = u_{i,-1} \quad ...(ii)$$

*Putting $j = 0$* in (*i*), $u_{i,1} = u_{i-1,0} + u_{i+1,0} - u_{i,-1}$

$= u_{i-1,0} + u_{i+1,0} - u_{i,1}$, using (*ii*)

or $$u_{i,1} = \frac{1}{2}(u_{i-1,0} + u_{i+1,0}) \quad ...(iii)$$

Taking $i = 1, 2, 3$ successively, we obtain

$$u_{1,1} = \frac{1}{2}(u_{0,0} + u_{1,0}) = 2 \,;\, u_{2,1} = \frac{1}{2}(u_{1,0} + u_{3,0}) = 3$$

$$u_{3,1} = \frac{1}{2}(u_{2,0} + u_{4,0}) = 2$$

These are the entries of the *2nd row.*

*Putting j = 1* in $(i)$, $u_{i,2} = u_{i-1,1} + u_{i+1,1} - u_{i,0}$

Taking $i = 1, 2, 3$, successively, we get

$$u_{1,2} = u_{0,1} + u_{2,1} - u_{1,0} = 0 + 3 - 3 = 0$$
$$u_{2,2} = u_{1,1} + u_{3,1} - u_{2,0} = 2 + 2 - 4 = 0$$
$$u_{3,2} = u_{2,1} + u_{4,1} - u_{3,0} = 3 + 0 - 3 = 0$$

These are the entries of the *3rd row* and so on.

Now the equation of the vibrating string of length $l$ is $u_{tt} = c^2 u_{xx}$.

$\therefore$ Its period of vibration $= \frac{2l}{c} = \frac{2 \times 4}{2} = 4$ sec. $[\because \; l = 4 \text{ and } c = 2]$

This shows that we have to compute $u_{(x,t)}$ upto $t = 2$.

*i.e.*, similarly we obtain the values of $u_{i,2}$ (4th row) and $u_{i,3}$ (5th row).

Hence the values of $u_{i,j}$ are as shown in the table below :

| $j$ \ $i$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 3 | 4 | 3 | 0 |
| 1 | 0 | 2 | 3 | 2 | 0 |
| 2 | 0 | 0 | 0 | 0 | 0 |
| 3 | 0 | −2 | −3 | −2 | 0 |
| 4 | 0 | −3 | −4 | −3 | 0 |

**Example 33.13.** *Find the solution of the initial boundary value problem ;* $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$, $0 \le x \le 1$ *; subject to the initial conditions* $u(x, 0) = \sin \pi x$, $0 \le x \le 1$ *and the boundary conditions* $u(0, t) = 0$, $u(1, t) = 0$, $t > 0$ *; by using in the (a) the explicit scheme.*

*(b) the implicit scheme.* *(Anna, 2007)*

**Solution.** (*a*) *Explicit scheme*

Take $h = 0.2$, $k = \frac{h}{c} = 0.2$ $[\because \; c = 1]$

$\therefore$ We use the formula, $u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$ ...(*i*)

Since $u(0, t) = 0$, $u(1, t) = 0$, $u_{0,j} = 0$, $u_{1,j} = 0$ for all values of $j$

*i.e.*, the entries in the first and last columns are zero.

Since $u(x, 0) = \sin \pi x$, $u_{i,0} = \sin \pi x$

$\therefore$ $u_{1,0} = 0$, $u_{2,0} = \sin (0.2\pi) = 0.5878$, $u_{3,0} = \sin (0.4\pi) = 0.9511$, $u_{4,0} = \sin (0.6\pi) = 0.5878$.

These are the entries of the first row.

Since $u_t(x, 0) = 0$ we have $\frac{1}{2}(u_{i,j+1} - u_{i,j-1} = 0$, when $j = 0$

*i.e.*, $u_{i,1} = u_{i,-1}$ ...(*ii*)

Putting $j = 0$ in (*i*), $u_{i,1} = u_{i-1,0} + u_{i+1,0} - u_{i,-1}$

Using (*ii*) $u_{i,1} = \frac{1}{2}(u_{i-1,0} + u_{i+1,} u_{i+1,0})$

Taking $i = 1, 2, 3, 4$ successively, we obtain the entries of the second row.

Putting $j = 1$ in (*i*), $u_{i,2} = u_{i-1,1} + u_{i+1,1} - u_{i,0}$

Now taking $i = 1, 2, 3, 4$ successively, we get the entries of the third row.

Similarly taking $j = 2, j = 3, j = 4$ successively, we obtain the entries of the fourth, fifth and sixth rows respectively.

| j \ i | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0.5878 | 0.9511 | 0.9511 | 0.5878 | 0 |
| 1 | 0 | 0.4756 | 0.7695 | 0.9511 | 0.7695 | 0 |
| 2 | 0 | 0.1817 | 0.4756 | 0.5878 | 0.3633 | 0 |
| 3 | 0 | 0 | 0.0001 | – 0.1122 | – 0.1816 | 0 |
| 4 | 0 | – 0.1816 | – 0.5878 | – 0.7694 | 0.4755 | 0 |
| 5 | 0 | – 0.5878 | – 0.9511 | – 0.9511 | – 0.5878 | 0 |

(*b*) *Implicit scheme*

We have the formula :

$$u_{i,j+1} = 2(1 - \alpha^2 c^2)\, u_{i,j} + \alpha^2 c^2 (u_{i-1,j} + u_{i+1,j}) - u_{i,j-1}, \text{ where } \alpha = k/h \quad ...(i)$$

Here $c^2 = 1$. Take $h = 0.25$ and $k = 0.5$ so that $\alpha = k/h = 2$.

$\therefore$ (*i*) reduces to

$$u_{i,j+1} = -6u_{i,j} + 4(u_{i-1,j} + u_{i+1,j}) - u_{i,j-1} \quad ...(ii)$$

Since $u_{i,0} = \sin \pi x$

$\therefore$ $u_{(1,0)} = 0.7071, u_{(2,0)} = 0.5, u_{3,0} = 0.7071$

There are the entries of the first row.

Since $u_t(x, 0) = 0$, we have $\frac{1}{2}(y_{i,i+1} - y_{i,i-1}) = 0$, where $j = 0$

$$\therefore \quad y_{i,1} = y_{i,-1} \quad ...(iii)$$

Put $j = 0$ and using (*iii*), (*ii*) reduces to

$$u_{i,\ +1} = -3\,u_{i,0} + 2(u_{i-1,0} + 2u_{i+1,0})$$

Now taking
$$i = 1, u_{1,1} = -3\,u_{1,0} + 2(u_{0,0} + u_{2,0}) = 0.1213$$
$$i = 2, u_{2,1} = -3\,u_{2,0} + 2(u_{1,0} + u_{3,0}) = 0.1716$$
$$i = 3, u_{3,1} = -3\,u_{3,0} + 2(u_{2,0} + u_{4,0}) = 0.1213$$

These are the entries of the second row.

Putting $j = 1$, (*ii*) reduces to

$$u_{i,2} = -6u_{i,1} + 4(u_{i-1,2} + u_{i+1,1})$$

Now taking
$$i = 1, u_{1,2} = -6\,u_{1,1} + 4(u_{0,1} + u_{2,1}) = 0.414$$
$$i = 2, u_{2,2} = -6\,u_{2,1} + 4(u_{1,1} + u_{3,1}) = 0.0592$$
$$i = 3, u_{3,2} = -6\,u_{3,1} + 4(u_{2,1} + u_{4,1}) = 0.414$$

These are the entries of the third row.

Putting $j = 2$, (*ii*) reduces to

$$u_{i,3} = -6u_{i,2} + 4(u_{i-1,2} + u_{i+1,2}) - u_{i,1}$$

Now taking
$$i = 1, u_{1,3} = -6\,u_{1,2} + 4(u_{0,2} + u_{2,2}) - u_{1,1} = 0.1097$$
$$i = 2, u_{2,3} = -6\,u_{2,2} + 4(u_{1,2} + u_{3,2}) - u_{2,1} = 0.1476$$
$$i = 3, u_{3,3} = -6\,u_{3,2} + 4(u_{2,2} + u_{4,2}) - u_{4,2} = 0.1097$$

These are the entries of the third row.

Hence the value of $u_{i,j}$ are as tabulated below :

| j \ i | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0.7071 | 0.5 | 0.7071 | 0 |
| 1 | 0 | – 0.1213 | – 0.1716 | – 0.1213 | 0 |
| 2 | 0 | 0.0414 | 0.0592 | 0.0414 | 0 |

## PROBLEMS 33.4

1. Solve the boundary value problem $u_{tt} = u_{xx}$ with the conditions $u(0, t) = u(1, t) = 0$, $u(x, 0) = \frac{1}{2}x(1-x)$ and $u_t(x, 0) = 0$, taking $h = k = 0.1$ for $0 \le t \le 0.4$. Compare your solution with the exact solution at $x = 0.5$ and $t = 0.3$. *(V.T.U., 2000)*
2. The transverse diaplacement $u$ of a point at a distance $x$ from one end and at any time $t$ of a vibrating string satisfies the equation $\partial^2u/\partial t^2 = 25\, \partial^2u/\partial x^2$, with the boundary conditions $u(0, t) = u(5, t) = 0$ and the initial conditions $u(x, 0) = \begin{cases} 20x & \text{for } 0 \le x \le 1 \\ 5(5-x) & \text{for } 1 \le x \le 5 \end{cases}$ and $u_t(x, 0) = 0$. Solve this equation numerically for one half period of vibration, taking $h = 1$, $k = 0.2$.
3. Solve $y_{tt} = y_{xx}$ upto $t = 0.5$ with a spacing of 0.1 subject to $y(0, t) = 0$, $y(1, t) = 0$, $y_t(x, 0) = 0$, and $y(x, 0) = 10 + x(1-x)$. *(Anna, 2004)*
4. The function $u$ satisfies the equation
$$\partial^2u/\partial t^2 = \partial^2u/\partial x^2$$
and the conditions : $u(x, 0) = \frac{1}{8}\sin \pi x$, $u_t(x, 0) = 0$ for $0 \le x \le 1$,
$$u(0, t) = u(1, t) = 0 \text{ for } t \ge 0.$$
Use the explicit scheme to calculate $u$ for $x = 0(0.1)\,1$ and $t = 0(0.1)\,0.5$.

## 33.11 OBJECTIVE TYPE OF QUESTIONS

## PROBLEMS 33.5

*Fill up the blanks or select the correct answer from each of the following questions :*

1. Which of the following equations is parabolic :
(a) $f_{xy} - f_x = 0$ (b) $f_{xx} + 2f_{xy} + f_{yy} = 0$ (c) $f_{xx} + 2f_{xy} + 4f_{yy} = 0$ (d) none
2. $u_{ij} = \frac{1}{4}(u_{i+1,j} - u_{i-1,j} + u_{i,j+1} - u_{i,j-1})$ is Leibmann's five point formula. (True or False)
3. $u_{xx} + 3u_{xy} + u_{yy} = 0$ is classified as ......
4. $\nabla^2 u = f(x, y)$ is known as ......
5. The simplest formula to solve $u_{tt} = \alpha^2 u_{xx}$ is ......
6. The finite difference form of $\partial^2 u/\partial x^2$ is ......
7. Schmidt's finite difference scheme to solve $u_t = c^2 u_{xx}$ is ......
8. The 5-point diagonal formula gives $u_{ij}$ = ......
9. The partial differential equation $(x+1)u_{xx} - 2(x+2)u_{xy} + (x+3)u_{yy} = 0$ is classified as ......
10. $u_{i,j+1} = \frac{1}{2}(u_{i+1,j} + u_{i-1,j})$ is called ...... recurrence relation.
11. Interms of difference quotients $4u_{xx} = u_{tt}$ is ......
12. Bendre-Schmidt recurrence relation for one dimensional heat equation is ......
13. The diagonal 5-point formula to solve the Laplace equation $u_{xx} + u_{yy} = 0$ is ......
14. In the parabolic equation $u_t = \alpha^2 u_{xx}$ if $\lambda = k\alpha^2/h^2$, where $k = \Delta t$, and $h = \Delta x$, then explicit method is stable if $\lambda$ = ......
15. $2\frac{\partial^2 u}{\partial t^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 3\frac{\partial^2 u}{\partial x^2} = 0$ is classified as ...... *(P.T.U., 2007)*
16. The boundary conditions of one-dimensional wave equation are ......
17. The explicit formula for one-dimensional wave equation with $1 - \lambda^2\alpha^2 = 0$ and $\lambda = k/h$ is ......
18. The general form of Poisson's equation in partial derivations is ......
19. If $u$ satisfies Laplace equation and $u = 100$ on the boundary of a square, the value of $u$ at an interior grid point is ......
20. The Laplace equation $u_{xx} + u_{yy} = 0$ in difference quotients is ......
21. The equation $yu_{xx} + u_{yy} = 0$ is hyperbolic in the region ......
22. To solve $\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}$ by Bendre-Schmidt method with $h = 1$, the value of $k$ is ......

CHAPTER 34

# Linear Programming

1. Introduction. 2. Formulation of the problem. 3. Graphical method. 4. Some exceptional cases. 5. General linear programming problem. 6. Canonical and standard forms of L.P.P. 7. Simplex method. 8. Working procedure of the simplex method. 9. Artificial variable techniques—M-method, Two phase method. 10. Exceptional cases—Degeneracy. 11. Duality concept. 12. Duality principle. 13. Dual simplex method. 14. Transportation problem. 15. Working procedure for transportation problems. 16. Degeneracy in transportation problems. 17. Assignment problem. 18. Working procedure to solve assignment problems. 19. Objective Type of Questions.

## 34.1 INTRODUCTION

Linear programming deals with the optimization (maximization or minimization) of linear functions subject to linear constraints. This technique has found its applications to important areas of product mix, blending problems and diet problems. Oil refineries, chemical industries, steel industries and food processing industry are also using linear programming with considerable success.

In this chapter, our purpose is to present the principles of linear programming and the techniques of its application in a manner that will suit the engineering students. Beginning with the graphical method which provides a great deal of insight into the basic concepts, the simplex method of solving linear programming problems is developed. Then the reader is introduced to the Duality concept. Finally a special class of linear programming problems namely : Transportation and Assignment problems, is taken up. For a detailed study, the student should refer to author's book '*Numerical Methods in Engineering and Science*'.

## 34.2 FORMULATION OF THE PROBLEM

To begin with, a problem is to be presented in a linear programming form which requires defining the variables involved, establishing relationships between them and formulating the objective function and the constraints. We illustrate this through a few examples.

**Example 34.1.** *A manufacturer produces two types of models $M_1$ and $M_2$. Each $M_1$ model requires 4 hours of grinding and 2 hours of polishing ; whereas each $M_2$ model requires 2 hours of grinding and 5 hours of polishing. The manufacturer has 2 grinders and 3 polishers. Each grinder works for 40 hours a week and each polisher works for 60 hours a week. Profit on an $M_1$ model is ₹ 3 and on an $M_2$ model is ₹ 4. Whatever is produced in a week is sold in the market. How should the manufacturer allocate his production capacity to the two types of models so that he may make the maximum profit in a week.*

**Solution.** Let $x_1$ be the number of $M_1$ models and $x_2$, the number of $M_2$ models produced per week. Then the weekly profit (in ₹) is

$$Z = 3x_1 + 4x_2 \qquad ...(i)$$

To produce these number of models, the total number of grinding hours needed per week

$$= 4x_1 + 2x_2$$

and the total number of polishing hours required per week

$$= 2x_1 + 5x_2$$

Since the number of grinding hours available is not more than 80 and the number of polishing hours is not more than 180, therefore,

$$4x_1 + 2x_2 \leq 80 \quad ...(ii)$$

$$2x_1 + 5x_2 \leq 180 \quad ...(iii)$$

Also since the negative number of models are not produced, obviously we must have

$$x_1 \geq 0 \text{ and } x_2 \geq 0 \quad ...(iv)$$

Hence this allocation problem is, to find $x_1$, $x_2$ which

maximize $Z = 3x_1 + 4x_2$

subject to $4x_1 + 2x_2 \leq 80,\ 2x_1 + 5x_2 \leq 180,\ x_1, x_2 \geq 0.$

**Obs.** *The variables that enter into the problem are called* **decision variables.**

*The expression (i) showing the relationship between the manufacturer's goal and the decision variables, is called the* **objective function.**

*The inequalities (ii), (iii) and (iv) are called the* **constraints**.

The objective function and the constraints being all linear, it is a *linear programming problem (L.P.P)*. This is an example of a real situation from industry.

**Example 34.2.** *A firm making castings uses electric furnace to melt iron with the following specifications :*

| | *Minimum* | *Maximum* |
|---|---|---|
| *Carbon* | *3.20%* | *3.40%* |
| *Silicon* | *2.25%* | *2.35%* |

*Specifications and costs of various raw materials used for this purpose are given below :*

| *Material* | *Carbon %* | *Silicon %* | *Cost (₹)* |
|---|---|---|---|
| *Steel scrap* | *0.4* | *0.15* | *850/tonne* |
| *Cast iron scrap* | *3.80* | *2.40* | *900/tonne* |
| *Remelt from foundry* | *3.50* | *2.30* | *500/tonne* |

*If the total charge of iron metal required is 4 tonnes, find the weight in kg of each raw material that must be used in the optimal mix at minimum cost.* (*J.N.T.U., 1999 S*)

**Solution.** Let $x_1, x_2, x_3$ be the amounts (in kg) of these raw materials. The objective is to minimize the cost *i.e.*,

minimize $$Z = \frac{850}{1000} x_1 + \frac{900}{1000} x_2 + \frac{500}{1000} x_3 \quad ...(i)$$

For iron melt to have a minimum of 3.2% carbon,

$$0.4x_1 + 3.8x_2 + 3.5x_3 \geq 3.2 \times 4{,}000 \quad ...(ii)$$

For iron melt to have a maximum of 3.4% carbon,

$$0.4x_1 + 3.8x_2 + 3.5x_3 \leq 3.4 \times 4{,}000 \quad ...(iii)$$

For iron melt to have a minimum of 2.25% silicon,

$$0.15x_1 + 2.41x_2 + 2.35x_3 \geq 2.25 \times 4{,}000 \quad ...(iv)$$

For iron melt to have a maximum of 2.35% silicon,

$$0.15x_1 + 2.41x_2 + 2.35x_3 \leq 2.35 \times 4{,}000 \quad ...(v)$$

Also, since the materials added up must be equal to the full charge weight of 4 tonnes.

$$\therefore \quad x_1 + x_2 + x_3 = 4{,}000 \quad ...(vi)$$

Finally since the amounts of raw material cannot be negative ...(vii)

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0 \quad ...(vii)$$

Thus the linear programming problem is to find $x_1, x_2, x_3$ which

minimize $Z = 0.85x_1 + 0.9x_2 + 0.5x_3$

subject to $0.4x_1 + 3.8x_2 + 3.5x_3 \geq 12{,}800$

$0.4x_1 + 3.8x_2 + 3.5x_3 \leq 13{,}600$

$$0.15x_1 + 2.41x_2 + 2.35x_3 \geq 9{,}000$$
$$0.15x_1 + 2.41x_2 + 2.35x_3 \leq 9{,}400$$
$$x_1 + x_2 + x_3 = 4{,}000$$
$$x_1, x_2, x_3 \geq 0.$$

## PROBLEMS 34.1

1. A firm manufactures two items. It purchases castings which are then machined, bored and polished. Castings for items $A$ and $B$ cost ₹ 3 and ₹ 4 each and are sold at ₹ 6 and ₹ 7 each respectively. Running costs of these machines are ₹ 20, ₹ 14 and ₹ 17.50 per hour respectively. Formulate the problem so that the product mix maximizes the profit? Capacities of the machines are

| | *Part A* | *Part B* |
|---|---|---|
| *Machining capacity* | 25 per hr. | 40 per hr. |
| *Boring capacity* | 28 per hr. | 35 per hr. |
| *Polishing capacity* | 35 per hr. | 25 per hr. |

2. A firm manufactures 3 products $A$, $B$ and $C$. The profits are ₹ 3, ₹ 2 and ₹ 4 respectively. The firm has two machines $M_1$ and $M_2$ and below is the required processing time in minutes for each machine on each product.

| | | *Product* | | |
|---|---|---|---|---|
| | | A | B | C |
| *Machine* | $M_1$ | 4 | 3 | 5 |
| | $M_2$ | 2 | 2 | 4 |

Machines $M_1$ and $M_2$ have 2000 and 2500 machine-minutes respectively. The firm must manufacture 100 $A$'s, 200 $B$'s and 50 $C$'s but not more than 150 $A$'s. Set up an L.P.P. to maximize profit. *(Kurukshetra, 2009 S)*

3. Three products are processed through three different operations. The time (in minutes) required per unit of each product, the daily capacity of the operations (in minutes per day) and the profit per unit sold for each product (in rupees) are as follows:

| *Operation* | *Time per unit* | | | *Operation capacity* |
|---|---|---|---|---|
| | *Product I* | *Product II* | *Product III* | |
| 1 | 3 | 4 | 3 | 42 |
| 2 | 5 | 0 | 3 | 45 |
| 3 | 3 | 6 | 2 | 41 |
| Profit (₹) | 3 | 2 | 1 | |

The zero time indicates that the product does not require the given operation. The problem is to determine the optimum daily production for three products that maximize the profit. Formulate this production planning problem as a linear programming problem assuming that all units produced are sold.

4. An aeroplane can carry a maximum of 200 passengers. A profit of ₹ 400 is made on each first class ticket and a profit of ₹ 300 is made on each economy class ticket. The airline reserves at least 20 seats for first class. However, at least 4 times as many passengers prefer to travel by economy class than by the first class. How many tickets of each class must be sold in order to maximize profit for the airline? Formulate the problem as an L.P. model. *(Rohtak, 2006)*

5. A firm manufactures headache pills in two sizes $A$ and $B$. Size $A$ contains 2 grains of asprin, 5 grains of bicarbonate and 1 grain of codeine. Size $B$ contains 1 grain of asprin, 8 grains of bicarbonate and 6 grains of codeine. It is found by users that it requires at least 12 grains of aspirin, 74 grains of bicarbonate and 24 grains of codeine for providing immediate effect. It is required to determine the least number of pills a patient should take to get immediate relief. Formulate the problem as a standard L.P.P.

6. Consider the following problem faced by a production planner in a soft-drink plant. He has two bottling machines $A$ and $B$. $A$ is designed for 8-ounce bottles and $B$ for 16-ounce bottles. However, each be can used on both types with some loss of efficiency. The following data is available:

| *Machine* | *8-ounce bottles* | *16-ounce bottles* |
|---|---|---|
| *A* | 100/minute | 40/minute |
| *B* | 60/minute | 75/minute |

The machines can be run 8 hour per day, 5 days per week. Profit on a 8-ounce bottle is 15 paise and on a 16-ounce bottle is 25 paise. Weekly production of the drink cannot exceed 300,000 ounces and the market can absorb 25,000

8-ounce bottles and 7,000 16-ounce bottles per week. The planner wishes to maximise his profit subject, of course, to all the production and marketing restrictions. Formulate this as a L.P.P.

7. A dairy feed company may purchase and mix one or more of three types of grains containing different amounts of nutritional elements. The data is given in the table below. The production manager specifies that any feed mix for his live stock must meet at least minimum nutritional requirements and seeks the least costly among all three mixes.

| | Item | *One unit weight of* | | | *Minimum requirement* |
|---|---|---|---|---|---|
| | | *Grain 1* | *Grain 2* | *Grain 3* | |
| | *A* | 2 | 3 | 7 | 1,250 |
| *Nutritional* | *B* | 1 | 1 | 0 | 250 |
| *ingredients* | *C* | 5 | 3 | 0 | 900 |
| | *D* | 6 | 25 | 1 | 232.5 |
| *Cost per weight of* | | 41 | 35 | 96 | |

Formulate the problem as a L.P. model.

8. A firm produces an alloy with the following specifications:

(*i*) specific gravity $\leq 0.97$; (*ii*) chromium content $\geq 15\%$; (*iii*) melting temperature $\geq 494°C$

The alloy requires three raw materials *A*, *B* and *C* whose properties are as follows:

| *Property* | *Properties of raw material* | | |
|---|---|---|---|
| | *A* | *B* | *C* |
| *Sp. gravity* | 0.94 | 1.00 | 1.05 |
| *Chromium* | 10% | 15% | 17% |
| *Melting pt.* | 470°C | 500°C | 520°C |

Find the values of *A*, *B*, *C* to be used to make 1 tonne of alloy of desired properties, keeping the raw material costs at the minimum when they are ₹ 105/tonne for *A*, ₹ 245/tonne for *B* and ₹ 165/tonne for *C*. Formulate an L.P. model for the problem.

## 34.3 GRAPHICAL METHOD

Linear programming problems involving only two variables can be effectively solved by a graphical technique which provides a pictorial representation of the solution and one gets insight into the basic concepts used in solving large L.P.P.

**Working procedure** *to solve a linear programming problem graphically:*

*Step 1.* Formulate the given problem as a linear programming problem.

*Step 2.* Plot the given constraints as equalities on $x_1 - x_2$ coordinate plane and determine the convex region* formed by them.

*Step 3.* Determine the vertices of the convex region and find the value of the objective function at each vertex. The vertex which gives the optimal (maximum or minimum) value of the objective function gives the desired optimal solution to the problem.

*Otherwise.* Draw the dotted line through the origin representing the objective function with $Z = 0$. As $Z$ is increased from zero, this line moves to the right remaining parallel to itself. We go on sliding this line (parallel to itself), till it is *farthest* away from the origin and passes through only one vertex of the convex region. This is the vertex where the maximum value of $Z$ is attained.

* *A* **region** *or a* **set** *of points is said to be* **convex** *if the line joining any two of its points lies completely in the region (or the set).* Figs. 34.1 and 34.2 represent convex regions while Figs. 34.3 and 34.4 do not form convex sets.

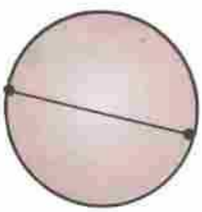


Fig. 34.1

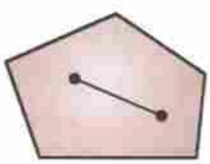


Fig. 34.2

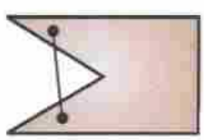


Fig. 34.3

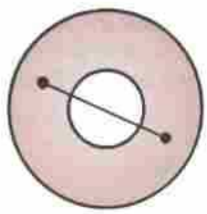


Fig. 34.4

When it is required to minimize $Z$, value of $Z$ is increased till the dotted line passes through the *nearest* vertex of the convex region.

**Example 34.3.** *Solve the L.P.P. of Ex. 34.1 graphically.* (*V.T.U., 2003*)

**Solution.** The problem is:

Maximize $Z = 3x_1 + 4x_2$ ...(*i*)

subject to $4x_1 + 2x_2 \leq 80$ ...(*ii*)

$2x_1 + 5x_2 \leq 180$ ...(*iii*)

$x_1, x_2 \geq 0$ ...(*iv*)

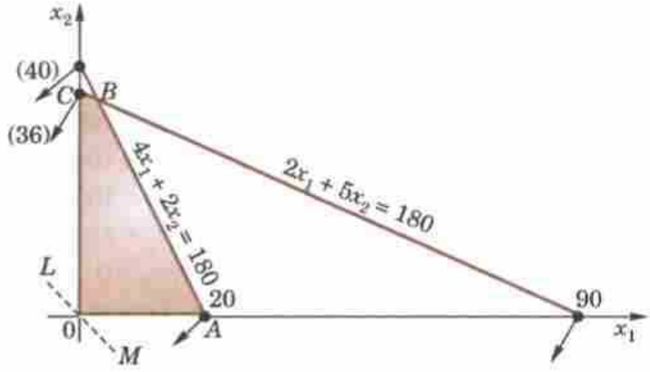


Fig. 34.5

Consider $x_1 - x_2$ coordinate system as shown in Fig. 34.5. The non-negativity restrictions (*iv*) imply that the values of $x_1, x_2$ lie in the first quadrant only.

We plot the lines $4x_1 + 2x_2 = 80$ and $2x_1 + 5x_2 = 180$. Then any point on or below $4x_1 + 2x_2 = 80$ satisfies (*ii*) and any point on or below $2x_1 + 5x_2 = 180$ satisfies (*iii*). This shows that the desired point $(x_1, x_2)$ must be somewhere in the shaded convex region $OABC$. This region is called the *solution space* or *region of feasible solutions* for the given problem. Its vertices are $O(0, 0)$, $A(20, 0)$ $B(2.5, 35)$ and $C(0, 36)$.

The values of the objective function (*i*) at these points are

$Z(O) = 0$, $Z(A) = 60$, $Z(B) = 147.5$, $Z(C) = 144$.

Thus the maximum value of $Z$ is 147.5 and it occurs at $B$. Hence the optimal solution to the problem is

$$x_1 = 2.5, x_2 = 35 \text{ and } Z_{max} = 147.5.$$

**Otherwise.** Our aim is to find the point (or points) in the solution space which maximizes the profit function $Z$. To do this, we observe that on making $Z = 0$, (*i*) becomes $3x_1 + 4x_2 = 0$ which is represented by the dotted line $LM$ through $O$. As the value of $Z$ is increased, the line $LM$ starts moving parallel to itself towards the right. Larger the value of $Z$, more will be the company's profit. In this way, we go on sliding $LM$ till it is farthest away from the origin and passes through one of the corners of the convex region. This is the point where the maximum value of $Z$ is attained. Just possible, such a line may be one of the edges of the solution space. In that case every point on that edge gives the same maximum value of $Z$.

Here $Z_{max}$ is attained at $B(2.5, 35)$. Hence the optical solution is $x_1 = 2.5$, $x_2$ 35 and $Z_{max} = 147.5$.

**Example 34.4.** *Find the maximum value of $Z = 2x + 3y$ subject to the constraints: $x + y \leq 30$, $y \geq 3$, $0 \leq y \geq 12$, $x - y \geq 0$, and $0 \leq x \leq 20$.* (*Rohtak, 2006*)

**Solution.** Any point $(x, y)$ satisfying the conditions $x \geq 0, y \geq 0$ lies in the first quadrant only. Also since $x + y \leq 30$, $y \geq 3$, $y \leq 12$, $x \geq y$ and $x \leq 20$, the desired point $(x, y)$ lies within the convex region $ABCDE$ (shown shaded in Fig. 34.6). Its vertices are $A(3, 3)$, $B(20, 3)$, $C(20, 10)$, $D(18, 12)$, and $E(12, 12)$.

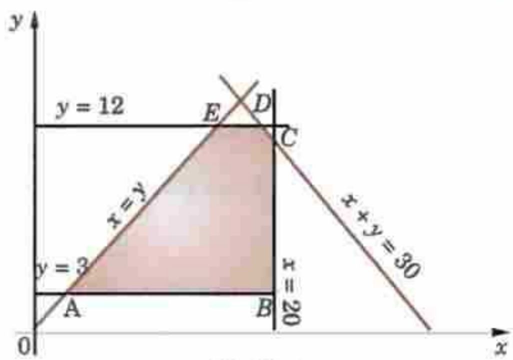


Fig. 34.6

The values of $Z$ at these five vertices are $Z(A) = 15$, $Z(B) = 49$, $Z(C) = 70$, $Z(D) = 72$, and $Z(E) = 60$.

Since the maximum value of $Z$ is 72 which occurs at the vertex $D$, the solution to the L.P.P. is $x = 18$, $y = 12$ and maximum $Z = 72$.

**Example 34.5.** *A company manufactures two types of cloth, using three different colours of wool. One yard length of type a cloth requires 4 oz of red wool, 5 oz of green wool and 3 oz of yellow wool. One yard length of type B cloth requires 5 oz of red wool, 2 oz of green wool and 8 oz of yellow wool. The wool available for manufacturer is 1000 oz of red wool, 1000 oz of green wool and 1200 oz of yellow wool. The manufacturer can make a profit of ₹ 5 on one yard of type A cloth and ₹ 3 on one yard of type B cloth. Find the best combination of the quantities of type A and type B cloth which gives him maximum profit by solving the L.P.P. graphically.*

**Solution.** Let the manufacturer decide to produce $x_1$ yards of type $A$ cloth and $x_2$ yards of type $B$ cloth. Then the total income in rupees, from these units of cloth is given by

$$Z = 5x_1 + 3x_2 \quad ...(i)$$

To produce these units of two types of cloth, he requires

$$\text{red wool} = 4x_1 + 5x_2 \text{ oz,}$$
$$\text{green wool} = 5x_1 + 2x_2 \text{ oz,}$$

and
$$\text{yellow wool} = 3x_1 + 8x_2 \text{ oz.}$$

Since the manufacturer does not have more than 1000 oz of red wool, 1000 oz of green wool and 1200 oz of yellow wool, therefore

$$4x_1 + 5x_2 \leq 1000 \quad ...(ii)$$
$$5x_1 + 2x_2 \leq 1000 \quad ...(iii)$$
$$3x_1 + 8x_2 \leq 1200 \quad ...(iv)$$

Also
$$x_1 \geq 0, x_2 \geq 0. \quad ...(v)$$

Thus the given problem is to maximize $Z$ subject to the constraints (*ii*) to (*v*). (*V.T.U., 2004*)

Any point satisfying the condition (*v*) lies in the first quadrant only. Also the desired point satisfying the constraints (*ii*) to (*iv*) lies in the convex region $OABCD$ (Fig. 34.7). Its vertices are $O(0, 0)$ $A(200, 0)$, $B(3000/17, 1000/17)$, $C(2000/17, 1800/17)$ and $D(0, 150)$.

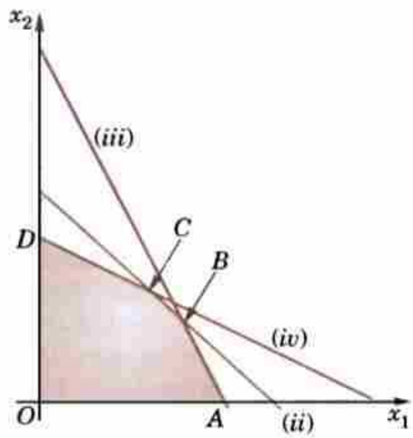


Fig. 34.7

The values of $Z$ at these vertices are given by $Z(O) = 0$, $Z(A) = 1000$, $Z(B) = 1057.6$, $Z(C) = 905.8$ and $Z(D) = 450$.

Since the maximum value of $Z$ is 1058.8 which occurs at the vertex $B$, the solution to the given problem is $x_1 = 3000/17$, $x_2 = 1000/17$ and max. $Z = 1058.8$.

Hence the manufacturer should produce 176.5 yards of type $A$ cloth, 58.8 yards of type $B$ cloth, so as to get the maximum profit of ₹ 1058.8.

**Example 34.6.** *A company making cold drinks has two bottling plants located at towns $T_1$ and $T_2$. Each plant produces three drinks A, B and C and their production capacity per day is shown below:*

| *Cold drinks* | *Plant at* | |
|---|---|---|
| | $T_1$ | $T_2$ |
| A | 6,000 | 2,000 |
| B | 1,000 | 2,500 |
| C | 3,000 | 3,000 |

*The marketing department of the company forecasts a demand of 80,000 bottles of A, 22,000 bottles of B and 40,000 bottles of C during the month of June. The operating costs per day of plants at $T_1$ and $T_2$ are ₹6,000 and ₹ 4,000 respectively. Find (graphically) the number of days for which each plant must be run in June so as to minimize the operating costs while meeting the market demand.*

**Solution.** Let the plants at $T_1$ and $T_2$ be run for $x_1$ and $x_2$ days. Then the objective is to minimize the operation costs *i.e.*,

$$\text{min. } Z = 6000x_1 + 4000x_2 \quad ...(i)$$

Constraints on the demand for the three cold drinks are:

for $A$, $6{,}000x_1 + 2{,}000x_2 \geq 80{,}000$ or $3x_1 + x_2 \geq 40$ ...(*ii*)

for $B$, $1{,}000x_1 + 2{,}500x_2 \geq 22{,}000$ or $x_1 + 2.5x_2 \geq 22$ ...(*iii*)

for $C$, $3{,}000x_1 + 3{,}000x_2 \geq 40{,}000$ or $x_1 + x_2 \geq 40/3$ ...(*iv*)

Also $x_1, x_2 \geq 0$ ...(*v*)

Thus the L.P.P. is to minimize (*i*) subject to constraints (*ii*) to (*v*). (*V.T.U., 2000 S*)

The solution space satisfying the constraints (*ii*) to (*v*) is shown shaded in Fig. 34.8. As seen from the direction of the arrows, the solution space is unbounded. The constraint (*iv*) is dominated by the constraints (*ii*) and (*iii*) and hence does not affect the solution space. Such a constraint as (*iv*) is called the *redundant constraint.*

The vertices of the convex region $ABC$ are $A(22, 0)$, $B(12, 4)$ and $C(0, 40)$.

Values of the objective function (*i*) at these vertices are

$$Z(A) = 132{,}000,\ Z(B) = 88{,}000,\ Z(C) = 160{,}000.$$

Thus the minimum value of $Z$ is ₹ 88,000 and it occurs at $B$. Hence the solution to the problem is $x_1 = 12$ days, $x_2 = 4$ days, $Z_{min}$ = ₹ 88,000.

**Otherwise.** Making $Z = 0$, (*i*) becomes $3x_1 + 2x_2 = 0$ which is represented by the dotted line $LM$ through $O$. As $Z$ is increased, the line $LM$ moves parallel to itself, to the right. Since we are interested in finding the minimum value of $Z$, value of $Z$ is increased till $LM$ passes through the vertex nearest to the origin of the shaded region, *i.e.* $B(12, 4)$.

Thus the operating cost will be minimum for $x_1 = 12$ days, $x_2$ = 4 days and

$$Z_{min} = 6000 \times 12 + 4000 \times 4 = ₹\ 88{,}000.$$

**Obs.** The dotted line parallel to the line $LM$ is called the *iso-cost line* since it represents all possible combinations of $x_1, x_2$ which produce the same total cost.

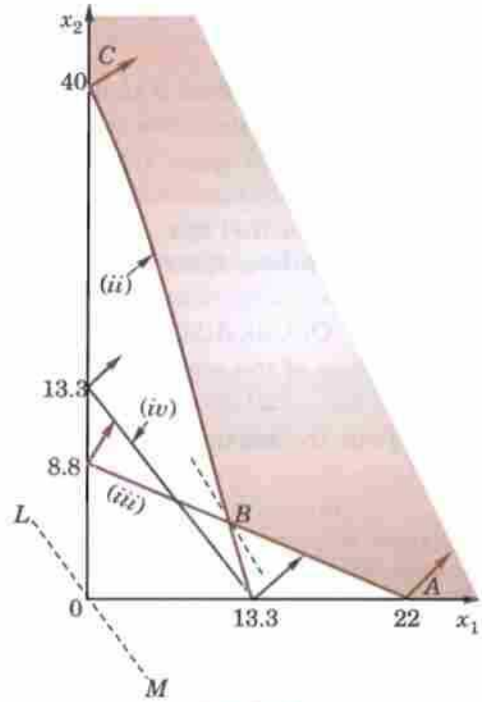


Fig. 34.8

## 34.4 SOME EXCEPTIONAL CASES

The constraints generally, give region of feasible solution which may be bounded or unbounded. In problems involving two variables and having a finite solution, we observed that the optimal solution existed at a vertex of the feasible region. In fact, this is true for all *L.P.* problems for which solutions exist. Thus it may be stated that *if there exists an optimal solution of an L.P.P., it will be at one of the vertices of the solution space.*

In each of the above exmples, the optimal solution was unique. But it is not always so. In fact, *L.P.P. may have*

(*i*) *a unique optimal solution*, or (*ii*) *an infinite number of optimal solutions*, or (*iii*) *an unbounded solution*, or (*iv*) *no solution.*

We now give below a few examples to illustrate the exceptional cases (*ii*) to (*iv*).

**Example 34.7.** *A firm uses milling machines, grinding machines and lathes to produce two motor parts. The machining times required for each part, the machining times available on different machines and the profit on each motor part are given below:*

| *Type of machine* | *Machining time reqd. for the motor part (mts)* | | *Max. time available per week (minutes)* |
|---|---|---|---|
| | *I* | *II* | |
| *Milling machines* | 10 | 4 | 2,000 |
| *Grinding machines* | 3 | 2 | 900 |
| *Lathes* | 6 | 12 | 3,000 |
| *Profit/unit (₹)* | 100 | 40 | |

*Determine the number of parts I and II to be manufactured per week to maximize the profit.*

**Solution.** Let $x_1, x_2$ be the number of parts I and II manufactured per week. Then *objective* being to maximize the profit, we have maximize $Z = 100x_1 + 40x_2$ ...(*i*)

*Constraints* being on the time available on each machine, we obtain

for milling machines, $10x_1 + 4x_2 \le 2{,}000$ ...(*ii*)

for grinding machines, $3x_1 + 2x_2 \le 900$ ...(*iii*) for lathes, $6x_1 + 12x_2 \leq 3,000$ ...(iv)

Also $x_1, x_2 \geq 0$ ...(v)

Thus the problem is to determine $x_1, x_2$ which maximize (i) subject to the constraints (ii) to (v).

The solution space satisfying (ii), (iii), (iv) and meeting the non-negativity restrictions (v) is shown shaded in Fig. 34.9.

Note that (iii) is a redundant constraint as it does not affect the solution space. The vertices of the convex region *OABC* are

$$O(0, 0), A(200, 0), B(125, 187.5), C(0, 250).$$

Values of the objective function (i) at these vertices are $Z(O) = 0$, $Z(A) = 20,000$, $Z(B) = 20,000$ and $Z(C) = 10,000$.

Thus the maximum value of $Z$ occurs at two vertices $A$ and $B$.

$\therefore$ Any point on the line joining $A$ and $B$ will also give the same maximum value of $Z$ *i.e.*, there are infinite number of feasible solutions which yield the same maximum value of $Z$.

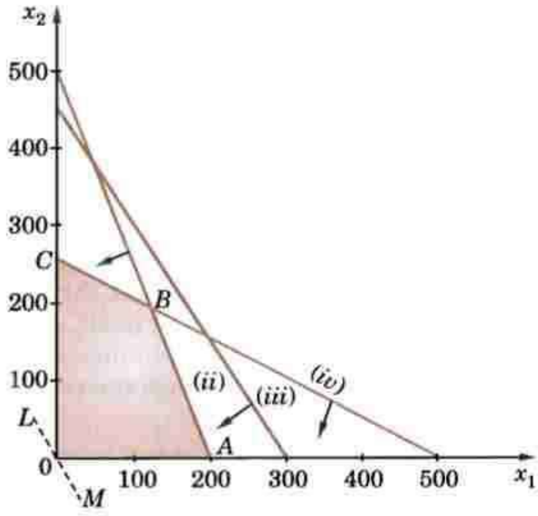



Fig. 34.9

Thus there is no unique optimal solution to the problem and any point on the line $AB$ can be taken to give the profit of ₹ 20,000.

**Obs.** An L.P.P. having more than one optimal solution, is said to have alternative or multiple optimal solutions. It implies that the resources can be combined in more than one way to maximize the profit.

**Example 34.8.** *Using graphical method, solve the following L.P.P.:*

*Maximize* $Z = 2x_1 + 3x_2$ ...(i)

*subject to* $x_1 - x_2 \leq 2$ ...(ii)

$x_1 + x_2 \geq 4$ ...(iii)

$x_1, x_2 \geq 0.$ (*Kurukshetra, 2005; V.T.U., 2003 S*) ... (iv)

**Solution.** Consider $x_1 - x_2$ coordinate system. Any point $(x_1, x_2)$ satisfying the restrictions (iv) lies in the first quadrant only. The solution space satisfying the constraints (ii) and (iii) is the convex region shown shaded in Fig. 34.10.

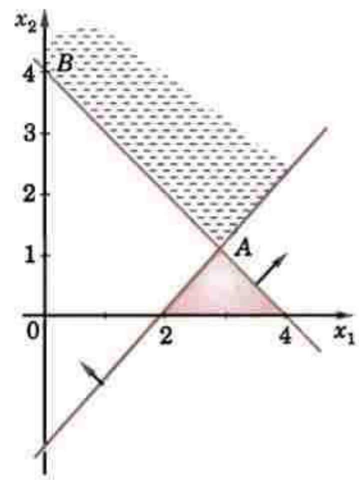



Fig. 34.10

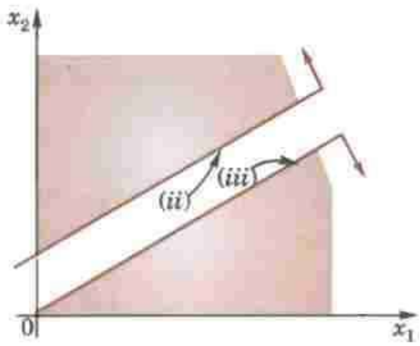



Fig. 34.11

Here the solution space is unbounded. The vertices of the feasible region (in the finite plane) are $A(3, 1)$ and $B(0, 4)$.

Values of the objective function (i) at these vertices are $Z(A) = 9$ and $Z(B) = 12$.

But there are points in this convex region for which $Z$ will have much higher values. For instance, the point (5, 5) lies in the shaded region and the value of $Z$ thereat is 25. In fact, the maximum value of $Z$ occurs at infinity. Thus the problem has an unbounded solution.

**Example 34.9.** *Solve graphically the following L.P.P.:*

*Maximize* $Z = 4x_1 + 3x_2$ ...(*i*)

*subject to* $x_1 - x_2 \le -1$, ...(*ii*)

$-x_1 + x_2 \le 0$ ...(*iii*)

*and* $x_1, x_2 \ge 0$ ...(*iv*)

**Solution.** Consider $x_1 - x_2$ coordinate system. Any point $(x_1, x_2)$ satisfying (*iv*) lies in the first quadrant only. The two solution spaces, one satisfying (*ii*) and the other satisfying (*iii*) are shown shaded in Fig. 34.11.

There being no point $(x_1, x_2)$ common to both the shaded regions, the problem cannot be solved. Hence the solution does not exist since the constraints are inconsistent.

**Obs.** The above problem had no solution because the constraints were incompatible. There may be cases in which the constraints are compatible but the problem may still have no feasible solution.

## PROBLEMS 34.2

Using graphical method, solve the following L.P. problems:

1. Max. $Z = 3x_1 + 5x_2$
   subject to $x_1 + 2x_2 \le 200$, $x_1 + x_2 \le 150$, $x_1 \le 60$, $x_1, x_2 \ge 0$ (*Rajasthan, 2003*)
2. Max. $Z = 5x_1 + 7x_2$
   subject to $x_1 + x_2 \le 4$, $5x_1 + 8x_2 \le 24$, $10x_1 + 7x_2 \le 35$, and $x_1, x_2 \ge 0$.
3. Min. $Z = 20x_1 + 30x_2$
   subject to $x_1 + 2x_2 \le 40$, $3x_1 + x_2 \ge 30$, $4x_1 + 3x_2 \ge 60$, $x_1, x_2 \ge 0$.
   (*Kurukshetra, 2009 S; Mumbai, 2004; V.T.U., 2004*)
4. Max. $z = 3x + 5x_2$ subject to $x_1 + 2x_2 \le 2000$, $x_1 + x_2 \le 1500$, $x_2 \le 600$ and $x_1 \ge 0$, $x_2 \ge 0$. (*Rohtak, 2004*)
5. A firm manufactures two products *A* and *B* on which the profits earned per unit are ₹ 3 and ₹ 4 respectively. Each product is processed on two machines $M_1$ and $M_2$. Product *A* requires one minute of processing time on $M_1$ and 2 minutes on $M_2$ while *B* requires one minute on $M_1$ and one minute on $M_2$. Machine $M_1$ is available for not more than 7 hours and 30 minutes while $M_2$ is available for 10 hours during any working day. Find the number of units of products *A* and *B* to be manufactured to get maximum profit.
6. Two spare parts *X* and *Y* are to be produced in a batch. Each one has to go through two processes *A* and *B*. The time required in hours per unit and total time available are given below:

| | *X* | *Y* | *Total hours available* |
|---|---|---|---|
| *Process A* | 3 | 4 | 24 |
| *Process B* | 9 | 4 | 36 |

   Profits per unit of *X* and *Y* are ₹ 5 and ₹ 6 respectively. Find how many number of spare parts of *X* and *Y* are to be produced in this batch to maximize the profit. (Each batch is complete in all respects and one cannot produce fractional units and stop the batch).
7. A manufacturer has two products I and II both of which are made in steps by machines *A* and *B*. The process times per hundred for the two products on the two machines are:

| *Product* | *M/c. A* | *M/c. B* |
|---|---|---|
| I | 4 hrs. | 5 hrs. |
| II | 5 hrs. | 2 hrs. |

   Set-up times are negligible. For the coming period machine *A* has 100 hrs, and *B* has 80 hrs. The contribution for product I is ₹ 10 per 100 units and for product II is ₹ 5 per 100 units. The manufacturer is in a market which can absorb both products as much as he can produce for the immediate period ahead. Determine graphically, how much of products I and II, he should produce to maximize his contribution.
8. A production manager wants to determine the quantity to be produced per month of products *A* and *B* manufactured by his firm. The data on resources required and availability of resources are given below:

| *Resources* | *Requirements* | | *Available per month* |
|---|---|---|---|
| | *Product A* | *Product B* | |
| Raw material (kg.) | 60 | 120 | 12,000 |
| Machine hrs/piece | 8 | 5 | 600 |
| Assembly man hrs. | 3 | 4 | 500 |
| Sale price/piece | ₹ 30 | ₹ 40 | |

Formulate the problem as a standard L.P.P. Find product mix that would give maximum profit by graphical technique.

9. A pineapple firm produces two products: canned pineapple and canned juice. The specific amounts of material, labour and equipment required to produce each product and the availability of each of these resources are shown in the table given below:

| | *Canned Juice* | *Pineapple* | *Available Resources* |
|---|---|---|---|
| Labour (man hrs) | 3 | 2.0 | 12.0 |
| Equipment (m/c hrs) | 1 | 2.3 | 6.9 |
| Material (unit) | 1 | 1.4 | 4.9 |

Assuming one unit each of canned juice and canned pineapple has profit margins of ₹ 2 and ₹ 1 respectively. Formulate this as L.P. problem and solve it graphically.

Solve the following L.P. problems graphically:

10. Maximize $Z = 6x + 4y$ subject to $2x + y \geq 1$, $3x + 4y \geq 1.5$ and $x, y \geq 0$. *(Bombay, 2004)*

11. Minimize $Z = 8x_1 + 12x_2$ subject to $60x_1 + 30x_2 \geq 240$, $30x_1 + 60x_2 \geq 300$, $30x_1 + 180x_2 \geq 540$.
and $x_1, x_2 \geq 0$.

12. G.J. Breweries Ltd. have two bottling plants one located at '*G*' and other at '*J*'. Each plant produces three drinks: whiskey, beer and brandy. The number of bottles produced per day are as follows:

| *Drink* | *Plant at G* | *Plant at J* |
|---|---|---|
| Whiskey | 1500 | 1500 |
| Beer | 3000 | 1000 |
| Brandy | 2000 | 5000 |

A market survey indicates that during the month of July, there will be a demand of 20,000 bottles of whiskey, 40,000 bottles of beer and 44,000 bottles of brandy. The operating cost per day for plants at *G* and *J* are ₹ 600 and ₹ 400. For how many days each plant be run in July so as to minimize the production cost, while still meeting the market demand. Solve graphically.

## 34.5 GENERAL LINEAR PROGRAMMING PROBLEM

Any L.P.P. problem involving more than two variables may be expressed as follows:

Find the values of the variables $x_1, x_2, \ldots, x_n$ which maximize (or minimize) the objective function

$$Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \qquad \ldots(i)$$

subject to the constraints

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &\leq b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots& \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &\leq b_m \end{aligned}\right\} \qquad \ldots(ii)$$

and meet the non-negativity restrictions

$$x_1, x_2, \ldots, x_n \geq 0. \qquad \ldots(iii)$$

**Def. 1.** *A set of values $x_1, x_2, \ldots, x_n$ which satisfies the constraints of the L.P.P. is called its* **solution**.

**Def. 2.** *Any solution to a L.P.P. which satisfies the non-negativity restrictions of the problem is called its* **feasible solution**.

**Def. 3.** *Any feasible solution which maximizes (or minimizes) the objective function of the L.P.P. is called its* **optimal solution**.

Some of the constraints in (*ii*) may be equalities, some others may be inequalities of (≤) type and remaining ones inequalities of (≥) type. The inequality constraints are changed to equalities by adding (or subtracting) non-negative variables to (from) the left hand side of such constraints.

**Def. 4.** *If the constraints of a general L.P.P. be*

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i$$

*($i$ = 1, 2, ... $k$) then the non-negative variables $s_i$ which satisfy*

$$\sum_{j=1}^{n} a_{ij}x_j + s_i = b_i \ (i = 1, 2, \dots k), \text{ are called } \textbf{slack variables.}$$

**Def. 5.** *If the constraints of a general L.P.P. be*

$$\sum_{j=1}^{n} a_{ij}x_i \geq b_i, \ (i = k, k+1, \dots) \text{ then the non-negative variables } s_i \text{ which satisfy}$$

$$\sum_{j=1}^{n} a_{ij}x_i - s_i = b_i, \ (i = k, k+1, \dots), \text{ are called } \textbf{surplus variables.}$$

## 34.6 CANONICAL AND STANDARD FORMS OF L.P.P.

After the formulation of L.P.P., the next step is to obtain its solution. But before any method is used to find its solution, the problem must be presented in a suitable form. As such, we explain its following two forms:

**(1) Canonical form.** The general L.P.P. can always be expressed in the following form:

Maximize $Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$

subject to the constraints $a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \leq b_i$; $i = 1, 2, \dots m$

$$x_1, x_2, \dots x_n \geq 0,$$

by making some elementary transformations. This form of the L.P.P. is called its **canonical form** and has the following characteristics:

(*i*) Objective function is of maximization type,

(*ii*) All constraints are of (≤) type,

(*iii*) All variables $x_i$ are non-negative.

The canonical form is a format for a L.P.P. which finds its use in the Duality theory.

**(2) Standard form.** The general L.P.P. can also be put in the following form:

Maximize $Z = c_1x_1 + c_2x_2 + \dots c_nx_n$

subject to the constraints $a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i$; $i = 1, 2, \dots m$

$$x_1, x_2, \dots x_n \geq 0,$$

This form of the L.P.P. is called its **standard form** and has the following characteristics:

(*i*) Objective function is of maximization type;

(*ii*) All constraints are expressed as equations;

(*iii*) Right hand side of each constraint is non-negative;

(*iv*) All variables are non-negative.

**Obs.** *Any L.P.P. can be expressed in the standard form.*

As minimize $Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$

is equivalent to maximize $Z'(= -Z) = -c_1x_1 - c_2x_2 \dots - c_nx_n$,

the objective function can always be expressed in the maximization form.

The inequality constraints can always be converted to equalities by adding (or substracting) the slack (or surplus) variables to the left hand sides of such constraints.

So far, the decision variables $x_1, x_2, \dots, x_n$ have been assumed to be all non-negative. In actual practice, these variables could also be zero or negative. If a variable is negative, it can always be expressed as the difference of two non-negative variables *e.g.* a variable $x_i$ can be written as

$$x_i = x_i' - x_i'' \qquad \text{where } x_i' \geq 0, x_i'' \geq 0.$$

**Example 34.10.** *Convert the following L.P.P. to the standard form:*

*Maximize* $Z = 3x_1 + 5x_2 + 7x_3$,

*subject to* $6x_1 - 4x_2 \leq 5$, $3x_1 + 2x_2 + 5x_3 \geq 11$, $4x_1 + 3x_3 \leq 2$, $x_1, x_2 \geq 0$.

**Solution.** As $x_3$ is unrestricted, let $x_3 = x_3' - x_3''$ where $x_3', x_3'' \geq 0$. Now the given constraints can be expressed as

$$6x_1 - 4x_2 \leq 5,$$

$$3x_1 + 2x_2 + 5x_3' - 5x_3'' \geq 11$$

$$4x_1 + 3x_3' - 3x_3'' \le 2$$
$$x_1, x_2, x_3', x_3'' \ge 0.$$

Introducing the slack/surplus variables, the problem in standard form becomes:

Maximize $\quad Z = 3x_1 + 5x_2 + 7x_3' - 7x_3''$

subject to $\quad 6x_1 - 4x_2 + s_1 = 5,$

$$3x_1 + 2x_2 + 5x_3' - 5x_3'' - s_2 = 11,$$
$$4x_1 + 3x_3' - 3x_3'' + s_3 = 2,$$
$$x_1, x_2, x_3', x_3'', s_1, s_2, s_3 \ge 0.$$

**Example 34.11.** *Express the following problem in the standard form:*

*Minimize* $\quad Z = 3x_1 + 4x_2$

*subject to* $\quad 2x_1 - x_2 - 3x_3 = -4,\ 3x_1 + 5x_2 + x_4 = 10,\ x_1 - 4x_2 = 12,\ x_1, x_3, x_4 \ge 0.$

**Solution.** Here $x_3$, $x_4$ are the slack/surplus variables and $x_1$, $x_2$ are the decision variables. As $x_2$ is unrestricted, let $x_2 = x_2' - x_2''$ where $x_2', x_2'' \ge 0$.

$\therefore$ The problem in standard form is

Maximize $\quad Z' (= -Z) = -3x_1 - 4x_2' + 4x_2''$

subject to $\quad -2x_1 + x_2' - x_2'' + 3x_3 = 4$

$$3x_1 + 5x_2' - 5x_2'' + x_4 = 10$$
$$x_1 - 4x_2' - 4x_2'' = 12$$
$$x_1, x_2', x_2'', x_3, x_4 \ge 0.$$

## 34.7 SIMPLEX METHOD

**(1)** While solving an L.P.P. graphically, the region of feasible solutions was found to be convex, bounded by vertices and edges joining them. The optimal solution occurred at some vertex. If the optimal solution was not unique, the optimal points were on an edge. These observations also hold true for the general L.P.P. Essentially the problem is that of finding the particular vertex of the convex region which corresponds to the optimal solution. The most commonly used method for locating the optimal vertex is the **simplex method**. This method consists in moving step by step from one vertex to the adjacent one. Of all the adjacent vertices, the one giving better value of the objective function over that of the preceding vertex, is chosen. This method of jumping from one vertex to the other is then repeated. Since the number of vertices is finite, the simplex method leads to an optimal vertex in a finite number of steps.

**(2)** *In simplex method, an infinite number of solutions is reduced to a finite number of promising solutions by using the following facts:*

(*i*) When there are $m$ constraints and $(m + n)$ variables ($m$ being $\le n$), the starting solution is found by setting $n$ variables equal to zero and then solving the remaining $m$ equations, provided the solution exists and is unique. *The $n$ zero variables are known as* **non-basic variables** *while the remaining $m$ variables are called* **basic variables** *and they form a* **basic solution**.

(*ii*) In an L.P.P., the variables must always be non-negative. Some of the basic solutions may contain negative variables. Such solutions are called *basic infeasible solutions* and should not be considered. To achieve this, we start with a basic solution which is non-negative. The next basic solution must always be non-negative. This is ensured by feasibility condition. Such a solution is known as **basic feasible solution**.

*If all the variables in the basic feasible solution are positive, then it is called* **non-degenerate solution** *and if some of the variables are zero, it is called* **degenerate solution.**

(*iii*) A new basic feasible solution may be obtained from the previous one by equating one of the basic variables to zero and replacing it by a new non-basic variable. *The eliminated variable is called the* **outgoing variable** *while the new variable is known as the* **incoming variable.**

*The incoming variable must improve the value of the objective function* which is ensured by the optimality condition. *This process is repeated till no further improvement is possible. The resulting solution is called the* **optimal basic feasible solution** or simply **optimal solution.**

**(3)** *The simplex method is, therefore, based on the following two conditions:*

**I.** *Feasibility condition.* It ensures that if the starting solution is basic feasible, the subsequent will also be basic feasible.

**II.** *Optimality condition.* It ensures that only improved solutions will be obtained.

**(4)** *Now, we shall elaborate the above terms in relation to the general linear programming problem in standard form, i.e.,*

Maximize $$Z = c_1x_1 + c_2x_2 + \dots + c_nx_n \quad \dots(1)$$

subject to $$\sum_{j=1}^{n} a_{ij}x_j + s_i = b_i\ , i = 1, 2, \dots m \quad \dots(2)$$

and $$x_j \geq 0, s_i \geq 0, j = 1, 2, \dots n \quad \dots(3)$$

(*i*) *Solution.* $x_1, x_2, \dots x_n$ is a solution of the general L.P.P. if it satisfies the constraints (2).

(*ii*) *Feasible solution,* $x_1, x_2, \dots x_n$ is a feasible solution of the general L.P.P. if it satisfies both the constraints (2) and the non-negativity restrictions (3). The set $S$ of all feasible solutions is called the feasible region. A linear programme is said to be *infeasible* when the set $S$ is empty.

(*iii*) *Basic solution* is the solution of the $m$ basic variables when each of the $n$ non-basic variables is equated to zero.

(*iv*) *Basic feasible solution* is that *basic solution* which also satisfies the non-negativity restrictions (3).

(*v*) *Optimal solution* is that basic feasible solution which also optimizes the objective function (1) while satisfying the conditions (2) and (3).

(*vi*) *Non-degenerate basic feasible solution* is that basic feasible solution which contains exactly $m$ non-zero basic variables. If any of the basic variables becomes zero, it is called a *degenerate basic feasible solution.*

**Example 34.12.** *Find all the basic solutions of the following system of equations identifying in each case the basic and non-basic variables:* $2x_1 + x_2 + 4x_3 = 11$, $3x_1 + x_2 + 5x_3 = 14$. (Mumbai, 2004 ; V.T.U., 2003 S)

*Investigate whether the basic solutions are degenerate basic solutions or not. Hence find the basic-feasible solution of the system.*

**Solution.** Since there are $m + n = 3$ variables and there are $m = 2$ constraints in this problem, a basic solution can be obtained by setting any one variable equal to zero and then solving the resulting equations. Also the total number of basic solutions $= {}^{m+n}C_m = {}^3C_2 = 3$.

The characteristics of the various basic solutions are as given below:

| No. of basic sol. | Basic variables | Nonbasic variables | Values of basic variables | Is the sol. feasible ? (Are all $x_j > 0$ ?) | Is the sol. degenerate ? |
|---|---|---|---|---|---|
| 1. | $x_1, x_2$ | $x_3$ | $2x_1 + x_2 = 11$<br>$3x_1 + x_2 = 14$<br>$\therefore\ x_1 = 3, x_2 = 5$ | Yes | No |
| 2. | $x_2, x_3$ | $x_1$ | $x_2 + 4x_3 = 11$<br>$x_2 + 5x_3 = 14$<br>$\therefore\ x_2 = 3, x_3 = -1$ | No | Yes |
| 3. | $x_1, x_3$ | $x_2$ | $2x_1 + 4x_3 = 11$<br>$3x_1 + 5x_3 = 14$<br>$\therefore\ x_1 = 1/2, x_3 = 5/2$ | Yes | No |

The basic feasible solutions are:

(*i*) $x_1 = 3, x_2 = 5, x_3 = 0$; (*ii*) $x_1 = 1/2, x_2 = 0, x_3 = 5/2$

which are also non-degenerate basic solutions.

**Example 34.13.** *Find an optimal solution to the following L.P.P. by computing all basic solutions and then finding one that maximizes the objective function:*

$$2x_1 + 3x_2 - x_3 + 4x_4 = 8,\ x_1 - 2x_2 + 6x_3 - 7x_4 = -3,\ x_1, x_2, x_3, x_4 \geq 0,$$
$$\text{Max. } Z = 2x_1 + 3x_2 + 4x_3 + 7x_4.$$

**Solution.** Since there are four variables and two constraints, a basic solution can be obtained by setting any two variables equal to zero and then solving the resulting equations. Also the total number of basic solutions $= {}^4C_2 = 6$.

The characteristics of the various basic solutions are as given below:

| *No. of basic sol.* | *Basic variables* | *Non-basic variables* | *Values of basic variables* | *Is the sol. feasible ? (Are all $x_j \geq 0$ ?)* | *Value of Z* | *Is the sol. optimal ?* |
|---|---|---|---|---|---|---|
| 1. | $x_1, x_2$ | $x_3, x_4 = 0$ | $2x_1 + 3x_2 = 8$<br>$x_1 - 2x_2 = -3$<br>$\therefore\ x_1 = 1, x_2 = 2$ | Yes | 8 | No |
| 2. | $x_1, x_3$ | $x_2, x_4 = 0$ | $2x_1 - x_3 = 8$<br>$x_1 + 6x_3 = -3$<br>$\therefore\ x_1 = -14/13,$<br>$x_3 = -67/13$ | No | — | — |
| 3. | $x_1, x_4$ | $x_2, x_3 = 0$ | $2x_1 + 4x_4 = 8$<br>$x_1 - 7x_4 = -3$<br>$\therefore\ x_1 = 22/9,$<br>$x_4 = 7/9$ | Yes | 10.3 | No |
| 4. | $x_2, x_3$ | $x_1, x_4 = 0$ | $3x_2 - x_3 = 8$<br>$-2x_2 + 6x_3 = -3$<br>$\therefore\ x_2 = 45/16,$<br>$x_3 = 7/16$ | Yes | 10.2 | No |
| 5. | $x_2, x_4$ | $x_1, x_3 = 0$ | $3x_2 + 4x_4 = 8$<br>$-2x_2 - 7x_4 = -3$<br>$\therefore\ x_2 = 132/39,$<br>$x_4 = -7/13$ | No | — | — |
| 6. | $x_3, x_4$ | $x_1, x_2 = 0$ | $-x_3 + 4x_4 = 8$<br>$6x_3 - 7x_4 = -3$<br>$\therefore\ x_3 = 44/17,$<br>$x_4 = 45/17$ | Yes | 28.9 | Yes |

Hence the optimal basic feasible solution is

$$x_1 = 0, x_2 = 0, x_3 = 44/17, x_4 = 45/17 \text{ and the maximum value of } Z = 28.9.$$

## PROBLEMS 34.3

1. Reduce the following problem to the standard form:
   Determine $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$ so as to
   Maximize $Z = 3x_1 + 5x_2 + 8x_3$
   subject to the constraints $2x_1 - 5x_2 \leq 6, 3x_1 + 2x_2 + x_3 \geq 5, 3x_1 + 4x_3 \leq 3$.
2. Express the following L.P.P. in the standard form
   Maximize $Z = 3x_1 + 2x_2 + 5x_3$
   subject to $-5x_1 + 2x_2 \leq 5, 2x_1 + 3x_2 + 4x_3 \geq 7, 2x_1 + 5x_3 \leq 3, x_1, x_2, x_3 \geq 0$. *(Kurukshetra, 2009)*
3. Convert the following L.P.P. to standard form:
   Maximize $Z = 3x_1 - 2x_2 + 4x_3$
   subject to $x_1 + 2x_2 + x_3 \leq 8, 2x_1 - x_2 + x_3 \geq 2, 4x_1 - 2x_2 - 3x_3 = -6, x_1, x_2 \geq 0$. *(Kurukshetra, 2007 S)*
4. Obtain all the basic solutions to the following system of linear equations:
   $$x_1 + 2x_2 + x_3 = 4, 2x_1 + x_2 + 5x_3 = 5.$$
5. Show that the following system of linear equations has two degenerate feasible basic solutions and the non-degenerate basic solution is not feasible:
   $$2x_1 + x_2 - x_3 = 2, 3x_1 + 2x_2 + x_3 = 3.$$ *(Kurukshetra, 2007 S)*

6. Find all the basic solutions to the following problem:

Maximize $Z = x_1 + 3x_2 + 3x_3$,

subject to $x_1 + 2x_2 + 3x_3 = 4$, $2x_1 + 3x_2 + 5x_3 = 7$ and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$.

Which of the basic solutions are (*a*) non-degenerate basic feasible, (*b*) optimal basic feasible ?

(*Kurushetra, 2009 S; Mumbai, 2003*)

## 34.8 WORKING PROCEDURE OF THE SIMPLEX METHOD

*Assuming the existence of an initial basic feasible solution, an optimal solution to any L.P.P. by simplex method is found as follows:*

*Step 1. (i) Check whether the objective function is to be maximized or minimized.*

If $$Z = c_1x_1 + c_2x_2 + c_3x_3 + \dots + c_nx_n$$

is to be minimized, then convert it into a problem of maximization, by writing

Minimize $Z$ = Maximize $(-Z)$

*(ii) Check whether all b's are positive.*

If any of the $b_i$'s is negative, multiply both sides of that constraint by –1 so as to make its right hand side positive.

*Step 2. Express the problem in the standard form.*

Convert all inequalities of constraints into equations by introducings slack/surplus variables in the constraints giving equations of the form

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + s_1 + 0s_2 + 0s_3 + \dots = b_1.$$

*Step 3. Find an initial basic feasible solution.*

If there are $m$ equations involving $n$ unknowns, then assign zero values to any $(n - m)$ of the variables for finding a solution. Starting with a basic solution for which $x_j : j = 1, 2, \dots, (n - m)$ are each zero, find all $s_i$. If all $s_i$ are $\geq 0$, the basic solution is feasible and non-degenerate. If one or more of the $s_i$ values are zero, then the solution is degenerate.

The above information is conveniently expressed in the following simplex table:

| | $c_j$ | $c_1$ | $c_2$ | $c_3 \dots 0$ | 0 | $0 \dots$ |
|---|---|---|---|---|---|---|
| $c_B$ | Basis | $x_1$ | $x_2$ | $x_3 \dots s_1$ | $s_2$ | $s_3 \dots b$ |
| 0 | $s_1$ | $a_{11}$ | $a_{12}$ | $a_{13} \dots 1$ | 0 | $0 \dots b_1$ |
| 0 | $s_2$ | $a_{21}$ | $a_{22}$ | $a_{23} \dots 0$ | 1 | $0 \dots b_2$ |
| 0 | $s_3$ | $a_{31}$ | $a_{32}$ | $a_{33} \dots 0$ | 0 | $1 \dots b_3$ |
| : | : | : | : | : : | : | : : |
| | | | Body matrix | | | Unit matrix |

[The variables $s_1$, $s_2$, $s_3$ etc. are called *basic variables* and variables $x_1$, $x_2$, $x_3$ etc. are called non-basic variables. *Basis* refers to the basic variables $s_1$, $s_2$, $s_3$ ... $c_j$ row denotes the coefficients of the variables in the objective function, while $c_B$-column denotes the coefficients of the basic variables only in the objective function. $b$-column denotes the values of the basic variables while remaining variables will always be zero. The coefficients of $x$'s (decision variables) in the constraint equations constitute the *body matrix* while coefficients of slack variables constitute the *unit matrix*].

*Step 4. Apply optimality test.*

Compute $C_j = c_j - Z_j$ ; where $Z_j = \Sigma\, c_B a_{ij}$

[$C_j$-row is called *net evaluation row* and indicates the per unit increase in the objective function if the variable heading the column is brought into the solution.]

If all $C_j$ are negative, then the initial basic feasible solution is *optimal*.

If even one $C_j$ is positive, then the current feasible solution is not optimal (*i.e.*, can be improved) and proceed to the next step.

*Step 5. (i) Identify the incoming and outgoing variables.*

If there are more than one positive $C_j$, then the *incoming variable* is the one that heads the column containing maximum $C_j$. The column containing it is known as the *key column* which is shown marked with an arrow at the bottom. If more than one variable has the same maximum $C_j$, any of these variables may be selected arbitrarily as the incoming variable.

Now divide the elements under $b$-column by the corresponding elements of key column and choose the row containing the minimum positive ratio $\theta$. Then replace the corresponding basic variable (by making its value zero). It is termed as the *outgoing variable*. The corresponding row is called the *key row* which is shown marked with an arrow on its right end. The element at the intersection of the key row and key column is called the *key element* which is shown bracketted. If all these ratios are $\leq 0$, the incoming variable can be made as large as we please without violating the feasibility condition. Hence the problem has an *unbounded solution* and no further iteration is required.

(*ii*) *Iterate towards an optimal solution.*

Drop the outgoing variable and introduce the incoming variable alongwith its associated value under $c_B$ column. Convert the key element to unity by dividing the key row by the key element. Then make all other elements of the key column zero by subtracting proper multiples of key row from the other rows.

[This is nothing but the sweep-out process used to solve the linear equations. The operations performed are called *elementary row operations*.]

*Step 6. Go to step 4 and repeat the computational procedure* until either an optimal (or an unbounded) solution is obtained.

**Example 34.14.** *Using simplex method*

*Maximize* $Z = 5x_1 + 3x_2$

*subject to* $x_1 + x_2 \leq 2,\ 5x_1 + 2x_2 \leq 10,\ 3x_1 + 8x_2 \leq 12,\ x_1, x_2 \geq 0.$ (V.T.U., 2003 S)

**Solution.** Consists of the following steps :

*Step 1. Check whether the objective function is to be maximized and all b's are positive.*

The problem being of maximization type and all $b$'s being $\geq 0$, this step is not necessary.

*Step 2. Express the problem in the standard form.*

By introducing the slack variables $s_1, s_2, s_3$, the problem in standard form becomes

Max. $Z = 5x_1 + 3x_2 + 0s_1 + 0s_2 + 0s_3$

subject to
$$x_1 + x_2 + s_1 + 0s_2 + 0s_3 = 2 \quad ...(i)$$
$$5x_1 + 2x_2 + 0s_1 + s_2 + 0s_3 = 10 \quad ...(ii)$$
$$3x_1 + 8x_2 + 0s_1 + 0s_2 + s_3 = 12 \quad ...(iii)$$
$$x_1,\ x_2, s_1, s_2, s_3 \geq 0.$$

*Step 3. Find an initial basic feasible solution.*

There are three equations involving five unknowns and for obtaining a solution, we assign zero values to any two of the variables. We start with a basic solution for which we set $x_1 = 0$ and $x_2 = 0$. (This basic solution corresponds to the origin in the graphical method). Substituting $x_1 = x_2 = 0$ in (*i*), (*ii*) and (*iii*), we get the basic solution

$$s_1 = 2, s_2 = 10, s_3 = 12$$

Since all $s_1, s_2, s_3$ are positive, the basic solution is also feasible and non-degenerate.

$\therefore$ The basic feasible solution is

$$x_1 = x_2 = 0 \text{ (non-basic) and } s_1 = 2, s_2 = 10, s_3 = 12 \text{ (basic)}$$

$\therefore$ *Initial basic feasible solution* is given by the following table :

| | $c_j$ | 5 | 3 | 0 | 0 | 0 | | |
|---|---|---|---|---|---|---|---|---|
| $c_B$ | Basis | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $b$ | $\theta$ |
| 0 | $s_1$ | (1) | 1 | 1 | 0 | 0 | 2 | 2/1← |
| 0 | $s_2$ | 5 | 2 | 0 | 1 | 0 | 10 | 10/5 |
| 0 | $s_3$ | 3 | 8 | 0 | 0 | 1 | 12 | 12/3 |
| | $Z_j = \Sigma c_B a_{ij}$ | 0 | 0 | 0 | 0 | 0 | 0 | |
| | $C_j = c_j - Z_j$ | 5 | 3 | 0 | 0 | 0 | | |
| | | ↑ | | | | | | |

[For $x_1$-column ($j = 1$), $Z_j = \Sigma c_B a_{i1} = 0(1) + 0(5) + 0(3) = 0$

and for $x_2$-column ($j = 2$), $Z_j = \Sigma c_B a_{i2} = 0(1) + 0(2) + 0(8) = 0$

Similarly $Z_j(b) = 0(2) + 0(10) + 0(12) = 0.$]

*Step 4. Apply optimality test.*

As $C_j$ is positive under some columns, the initial basic feasible solution is not optimal (*i.e.* can be improved) and we proceed to the next step.

*Step 5. (i) Identify the incoming and outgoing variables.*

The above table shows that $x_1$ is the *incoming variable* as its incremental contribution $C_j$ (= 5) is maximum and the column in which it appears is the *key column* (shown marked by an arrow at the bottom).

Dividing the elements under $b$-column by the corresponding elements of key-column, we find minimum positive ratio $\theta$ is 2 in two rows. We, therefore, arbitrarily choose the row containing $s_1$ as the *key row* (shown marked by an arrow on its right end). The element at the intersection of key row and the key column *i.e.*, (1), is the *key element*. $s_1$ is therefore, the *outgoing basic variable* which will now become non-basic.

Having decided that $x_1$ is to enter the solution, we have tried to find as to what maximum value $x_1$ could have without violating the constraints. So removing $s_1$, the new basis will contain $x_1$, $s_2$ and $s_3$ as the basic variables.

*(ii) Iterate towards the optimal solution.*

To transform the initial set of equations with a basic feasible solution into an equivalent set of equations with a different basic feasible solution, we make the key element unity. Here the key element being unity, we retain the key row as it is. Then to make all other elements in key column zero, we subtract proper multiples of key row from the other rows. Here we subtract 5 times the elements of key row from the second row and 3 times the elements of key row from the third row. These become the second and the third rows of the next table. We also change the corresponding value under $c_B$ column from 0 to 5, while replacing $s_1$ by $x_1$ under the basis. Thus the *second basic feasible solution* is given by the following table :

| | $c_j$ | 5 | 3 | 0 | 0 | 0 | | |
|---|---|---|---|---|---|---|---|---|
| $c_B$ | *Basis* | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $b$ | $\theta$ |
| 5 | $x_1$ | 1 | 1 | 1 | 0 | 0 | 2 | |
| 0 | $s_2$ | 0 | – 3 | – 5 | 1 | 0 | 0 | |
| 0 | $s_3$ | 0 | 5 | – 3 | 0 | 1 | 6 | |
| | $Z_j = \Sigma c_B a_{ij}$ | 5 | 5 | 5 | 0 | 0 | 10 | |
| | $C_j = c_j - Z_j$ | 0 | – 2 | – 5 | 0 | 0 | | |

As $C_j$ is either zero or negative under all columns, the above table gives the optimal basic feasible solution. This optimal solution is $x_1 = 2$, $x_2 = 0$ and maximum $Z = 10$.

**Example 34.15.** *A firm produces three products which are processed on three machines. The relevant data is given below :*

| *Machine* | *Time per unit (minutes)* | | | *Machine capacity* |
|---|---|---|---|---|
| | *Product A* | *Product B* | *Product C* | *(minutes/day)* |
| $M_1$ | 2 | 3 | 2 | 440 |
| $M_2$ | 4 | — | 3 | 470 |
| $M_3$ | 2 | 5 | — | 430 |

*The profit per unit for products A, B, and C is ₹ 4, ₹ 3 and ₹ 6 respectively. Determine the daily number of units to be manufactured for each product. Assume that all the units produced are consumed in the market.*

**Solution.** Let the firm decide to produce $x_1, x_2, x_3$ units of products $A$, $B$, $C$, respectively. Then the L.P. model for this problem is :

Max. $Z = 4x_1 + 3x_2 + 6x_3$

subject to $2x_1 + 3x_2 + 2x_3 \le 440,\ 4x_1 + 3x_3 \le 470,\ 2x_1 + 5x_2 \le 430,\ x_1, x_2, x_3 \ge 0.$ (*V.T.U., 2004*)

*Step 1. Check whether the objective function is to be maximized and all b's are non-negative.*

The problem being of maximization type and $b$'s being $\ge 0$, this step is not necessary.

*Step 2. Express the problem in the standard form.*

By introducing the slack variables $s_1, s_2, s_3$, the problem in standard form becomes :

Max. $Z = 4x_1 + 3x_2 + 6x_3 + 0s_1 + 0s_2 + 0s_3$

subject to
$$2x_1 + 3x_2 + 2x_3 + s_1 + 0s_2 + 0s_3 = 440$$
$$4x_1 + 0x_2 + 3x_3 + 0s_1 + s_2 + 0s_3 = 470$$
$$2x_1 + 5x_2 + 0x_3 + 0s_1 + 0s_2 + s_3 = 430$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.$$

*Step 3. Find an initial basic feasible solution.*

The basic (non-degenerate) feasible solution is
$$x_1 = x_2 = x_3 = 0 \text{ (non-basic)}$$
$$s_1 = 440, s_2 = 470, s_3 = 430 \text{ (basic)}$$

$\therefore$ *Initial basic feasible solution* is given by the following table :

| | $c_j$ | 4 | 3 | 6 | 0 | 0 | 0 | | |
|---|---|---|---|---|---|---|---|---|---|
| $c_B$ | *Basis* | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | $b$ | $\theta$ |
| 0 | $s_1$ | 2 | 3 | 2 | 1 | 0 | 0 | 440 | 440/2 |
| 0 | $s_2$ | 4 | 0 | (3) | 0 | 1 | 0 | 470 | 470/3← |
| 0 | $s_3$ | 2 | 5 | 0 | 0 | 0 | 1 | 430 | 430/0 |
| $Z_j = \Sigma c_B a_{ij}$ | | 0 | 0 | 0 | 0 | 0 | 0 | | |
| $C_j = c_j - Z_j$ | | 4 | 3 | 6 | 0 | 0 | 0 | | |

*Step 4. Apply optimality test.*

As $C_j$ is positive under some columns, the initial basic feasible solution is not optimal and we proceed to the next step.

*Step 5. (i) Identify the incoming and outgoing variables.*

The above table shows that $x_3$ is the incoming variable while $s_2$ is the outgoing variable and (3) is the key element.

*(ii) Iterate towards the optimal solution.*

Drop $s_2$ and introduce $x_3$ with its associated value 6 under $c_B$ column. Convert the key element to unity and make all other elements of key column zero. Then the *second feasible solution* is given by the table below :

| | $c_j$ | 4 | 3 | 6 | 0 | 0 | 0 | | |
|---|---|---|---|---|---|---|---|---|---|
| $c_B$ | *Basis* | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | $b$ | $\theta$ |
| 0 | $s_1$ | – 2/3 | (3) | 0 | 1 | – 2/3 | 0 | 380/3 | 380/9← |
| 6 | $s_2$ | 4/3 | 0 | 1 | 0 | 1/3 | 0 | 470/3 | ∞ |
| 0 | $s_3$ | 2 | 5 | 0 | 0 | 0 | 1 | 430 | 86 |
| | $Z_j$ | 8 | 0 | 6 | 0 | 2 | 0 | 940 | |
| | $C_j$ | – 4 | 3 | 0 | 0 | – 2 | 0 | | |
| | | | ↑ | | | | | | |

*Step 6. As $C_j$ is positive under the second column, the solution is not optimal and we proceed further. Now $x_2$ is the incoming variable and $s_1$ is the outgoing variable and (3) is the key element for the next iteration.*

Drop $s_1$ and introduce $x_2$ with its associated value 3 under $c_B$ column. Convert the key element to unity and make all other elements of the key column zero. Then the *third basic feasible solution* is given by the following table :

| | $c_j$ | 4 | 3 | 6 | 0 | 0 | 0 | | |
|---|---|---|---|---|---|---|---|---|---|
| $c_B$ | *Basis* | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | $b$ | $\theta$ |
| 3 | $x_2$ | – 2/9 | 1 | 0 | 1/3 | – 2/9 | 0 | 380/9 | |
| 6 | $x_3$ | 4/3 | 0 | 1 | 0 | 1/3 | 0 | 470/3 | |
| 0 | $s_3$ | 28/9 | 0 | 0 | – 5/3 | 10/9 | 0 | 1970/9 | |
| | $Z_i$ | 22/3 | 3 | 6 | 1 | 4/3 | 0 | 3200/3 | |
| | $C_j$ | – 10/3 | 0 | 0 | – 1 | – 4/3 | 0 | | |

Now since each $C_j \le 0$, therefore it gives the optimal solution

$$x_1 = 0, x_2 = 380/9, x_3 = 470/3 \text{ and } Z_{max} = 3200/3 \text{ i.e., } 1066.67 \text{ rupees.}$$

**Example 34.16.** *Solve the following L.P.P. by simplex method :*

*Minimize* $Z = x_1 - 3x_2 + 3x_3$

*subject to* $3x_1 - x_2 + 2x_3 \le 7,\ 2x_1 + 4x_2 \ge -12,\ -4x_1 + 3x_2 + 8x_3 \le 10,\ x_1, x_2, x_3 \ge 0.$

(*Mumbai, 2004 ; V.T.U., 2003*)

**Solution.** Consists of the following steps :

*Step 1. Check whether objective function is to be maximized and all b's are non-negative.*

As the problem is that of minimizing the objective function, converting it to the maximization type, we have Max. $Z' = -x_1 + 3x_2 - 3x_3$.

As the right hand side of the second constraint is negative, we write it as

$$-2x_1 - 4x_2 \le 12$$

*Step 2. Express the problem in the standard form.*

By introducing the slack variables $s_1, s_2, s_3$, the problem in the standard form becomes

Max. $Z' = -x_1 + 3x_2 - 3x_3 + 0s_1 + 0s_2 + 0s_3$

subject to
$$3x_1 - x_2 + 2x_3 + s_1 + 0s_2 + 0s_3 = 7$$
$$-2x_1 - 4x_2 + 0x_3 + 0s_1 + s_2 + 0s_3 = 12$$
$$-4x_1 + 3x_2 + 8x_3 + 0s_1 + 0s_2 + s_3 = 10$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \ge 0.$$

*Step 3. Find initial basic feasible solution.*

The basic (non-degenerate) feasible solution is

$$x_1 = x_2 = x_3 = 0 \text{ (non-basic)} ;\ s_1 = 7, s_2 = 12, s_3 = 10 \text{ (basic)}$$

∴ *Initial basic feasible solution* is given by the table below :

| | $c_j$ | $-1$ | $3$ | $-3$ | $0$ | $0$ | $0$ | | |
|---|---|---|---|---|---|---|---|---|---|
| $c_B$ | *Basis* | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | $b$ | $\theta$ |
| 0 | $s_1$ | 3 | $-1$ | 2 | 1 | 0 | 0 | 7 | $7/(-1)$ |
| 0 | $s_2$ | $-2$ | $-4$ | 0 | 0 | 1 | 0 | 12 | $12/(-4)$ |
| 0 | $s_3$ | $-4$ | (3) | 8 | 0 | 0 | 1 | 10 | $10/3\leftarrow$ |
| $Z_j = \Sigma c_B a_{ij}$ | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| $C_j = c_j - Z_j$ | | $-1$ | 3 | $-3$ | 0 | 0 | 0 | | |
| | | | ↑ | | | | | | |

*Step 4. Apply optimality test.*

As $C_j$ is positive under second column, the initial basic feasible solution is not optimal and we proceed further.

*Step 5. (i) Identify the incoming and outgoing variables.*

The above table shows that $x_2$ is the incoming variable, $s_3$ is the outgoing variable and (3) is the key element.

*(ii) Iterate towards the optimal solution.*

∴ Drop $s_3$ and introduce $x_2$ with its associated value 3 under $c_B$ column. Convert the key element to unity and make all other elements of the key column zero. Then the *second basic feasible solution* is given by the following table :

| | $c_j$ | $-1$ | $3$ | $-3$ | $0$ | $0$ | $0$ | | |
|---|---|---|---|---|---|---|---|---|---|
| $c_B$ | *Basis* | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | $b$ | $\theta$ |
| 0 | $s_1$ | (5/3) | 0 | 14/3 | 1 | 0 | 1/3 | 31/3 | $31/5\leftarrow$ |
| 0 | $s_2$ | $-22/3$ | 0 | 32/3 | 0 | 1 | 4/3 | 76/3 | $-38/11$ |
| 3 | $x_2$ | $-4/3$ | 1 | 8.3 | 0 | 0 | 1/3 | 10/3 | $-5/2$ |
| | $Z_j$ | $-4$ | 3 | 8 | 0 | 0 | 1 | 10 | |
| | $C_j$ | 3 | 0 | $-11$ | 0 | 0 | $-1$ | | |
| | | ↑ | | | | | | | |

*Step 6.* As $C_j$ is positive under first column, the solution is not optimal and we proceed further $x_1$ is the incoming variable, $s_1$ is the outgoing variable and (5/3) is the key element.

∴ Drop $s_1$ and introduce $x_1$ with its associated value – 1 under $c_B$ column. Convert the key element to unity and make all other elements of the key column zero. Then the *third basic feasible solution* is given by the table below :

| $c_B$ | $c_j$ Basis | $-1$ $x_1$ | $3$ $x_2$ | $-3$ $x_3$ | $0$ $s_1$ | $0$ $s_2$ | $0$ $s_3$ | $b$ |
|---|---|---|---|---|---|---|---|---|
| $-1$ | $x_1$ | 1 | 0 | 14/5 | 3/5 | 0 | 1/5 | 31/5 |
| 0 | $s_2$ | 0 | 0 | 156/5 | 22/5 | 1 | 14/5 | 354/5 |
| 3 | $x_2$ | 0 | 1 | 32/5 | 4/5 | 0 | 3/5 | 58/5 |
| | $Z_j$ | $-1$ | 3 | 82/5 | 9/5 | 0 | 8/5 | 143/5 |
| | $C_j$ | 0 | 0 | $-97/5$ | $-9/5$ | 0 | $-8/5$ | |

Now since each $C_j \le 0$, therefore it gives the optimal solution

$$x_1 = 31/5,\ x_2 = 58/5,\ x_3 = 0 \text{ (non-basic) and } Z'_{\max} = 143/5$$

Hence $\qquad Z_{\min} = -143/5.$

**Example 34.17.** *Maximize* $Z = 107x_1 + x_2 + 2x_3$,
*subject to the constraints :* $14x_1 + x_2 - 6x_3 + 3x_4 = 7$,

$$16x_1 + \frac{1}{2}x_2 - 6x_3 \le 5,\ 3x_1 - x_2 - x_3 \le 0,\ x_1, x_2, x_3, x_4 \ge 0.$$

**Solution.** Consists of the following steps :

*Step 1. Check whether objective function is to be maximized and all b's are non-negative.*

This step is not necessary.

*Step 2. Express the problem in the standard form.*

Here $x_4$ is a slack variable. By introducing other slack variables $s_1$ and $s_2$ the problem in standard form becomes

Max. $\quad Z = 107x_1 + x_2 + 2x_3 + 0x_4 + 0s_1 + 0s_2$

subject to
$$\frac{14}{3}x_1 + \frac{1}{3}x_2 - 2x_3 + x_4 + 0s_1 + 0s_2 = \frac{7}{3}$$
$$16x_1 + \frac{1}{2}x_2 - 6x_3 + 0x_4 + s_1 + 0s_2 = 5$$
$$3x_1 - x_2 - x_3 + 0x_4 + 0s_1 + s_2 = 0$$
$$x_1, x_2, x_3, x_4, s_1, s_2 \ge 0.$$

*Step 3. Find initial basic feasible solution.*

The basic feasible solution is

$$x_1 = x_2 = x_3 = 0 \text{ (non-basic) };\ x_4 = 7/3,\ s_1 = 5,\ s_2 = 0 \text{ (basic)}$$

∴ *Initial basic feasible solution* is given in the table below :

| $c_B$ | $c_j$ Basis | $107$ $x_1$ | $1$ $x_2$ | $2$ $x_3$ | $0$ $x_4$ | $0$ $s_1$ | $0$ $s_2$ | $b$ | $\theta$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | $x_4$ | $\frac{14}{3}$ | $\frac{1}{3}$ | $-2$ | 1 | 0 | 0 | $\frac{7}{3}$ | $\frac{7}{3}/\frac{14}{3}$ |
| 0 | $s_1$ | 16 | $\frac{1}{2}$ | $-6$ | 0 | 1 | 0 | 5 | 5/16 |
| 0 | $s_2$ | (3) | $-1$ | $-1$ | 0 | 0 | 1 | 0 | 0/3← |
| $Z_j = \Sigma c_B a_{ij}$ | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| $C_j = c_j - Z_j$ | | 107 ↑ | 1 | 2 | 0 | 0 | 0 | | |

*Step 4. Apply optimality test.*

As $C_j$ is positive under some columns, the initial basic feasible solution is not optimal and we proceed further.

*Step 5. (i) Identify the incoming and outgoing variables.*

The above table shows that $x_1$ is the incoming variable, $s_2$ is the outgoing variable and (3) is the key element.

*(ii) Iterate towards the optimal solution.*

Drop $s_2$ and introduce $x_1$ with its associated value 107 under $c_B$ column. Convert key element to unity and make all other elements of the key column zeros. Then the *second basic feasible solution* is given by the following table :

| | $c_j$ | 107 | 1 | 2 | 0 | 0 | 0 | | |
|---|---|---|---|---|---|---|---|---|---|
| $c_B$ | Basis | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $s_1$ | $s_2$ | $b$ | $\theta$ |
| 0 | $x_4$ | 0 | 17/9 | – 4/9 | 1 | 0 | 14/9 | 7/3 | – 21/4 |
| 0 | $s_1$ | 0 | 35/6 | – 2/3 | 0 | 1 | – 16/3 | 5 | – 15/2 |
| 107 | $x_1$ | 1 | – 1/3 | – 1/3 | 0 | 0 | 1/3 | 0 | 0 |
| | $Z_j$ | 107 | – 107/3 | – 107/3 | 0 | 0 | 107/3 | | |
| | $C_j$ | 0 | 110/3 | 113/3 | 0 | 0 | – 107/3 | | |
| | | | | ↑ | | | | | |

As $C_j$ is positive under some columns, the solution is not optimal. Here 113/3 being the largest positive value of $C_j$, $x_3$ is the incoming variable. But all the values of $\theta$ being $\leq 0$, $x_3$ will not enter the basis. This indicates that the solution to the problem is unbounded.

[**Remember that** *(i) the incoming variable is the non-basic variable corresponding to the largest positive value of* $C_j$ *and*

*(ii) the outgoing variable is the basic-variable corresponding to the least positive ratio* $\theta$*, obtained by dividing the b-column elements by the corresponding key-column elements.*]

## PROBLEMS 34.4

Using simplex method, solve the following *L.P.P.* (1–8) :

1. Maximize $Z = x_1 + 3x_2$,
   subject to $x_1 + 2x_2 \leq 10$, $0 \leq x_1 \leq 5$, $0 \leq x_2 \leq 4$. *(Kurushetra, 2009 ; V.T.U., 2003)*
2. Maximize $Z = 4x_1 + 10x_2$
   subject to $2x_1 + x_2 \leq 50$, $2x_1 + 5x_2 \leq 100$, $2x_1 + 3x_2 \leq 90$, $x_1, x_2 \geq 0$. *(Kurushetra, 2006)*
3. Maximize $Z = 4x_1 + 5x_2$,
   subject to $x_1 - 2x_2 \leq 2$, $2x_1 + x_2 \leq 6$, $x_1 + 2x_2 \leq 5$, $-x_1 + x_2 \leq 2$, $x_1, x_2 \geq 0$.
4. Maximize $Z = 10x_1 + x_2 + 2x_3$
   subject to $x_1 + x_2 - 2x_3 \leq 10$, $4x_1 + x_2 + x_3 \leq 20$, $x_1, x_2, x_3 \geq 0$.
5. Maximize $Z = 3x_1 + 2x_2 + 5x_3$, subject to $x_1 + 2x_2 + x_3 \leq 430$, $3x_1 + 2x_3 \leq 460$, $x_1 + 4x_2 \leq 420$, $x_1, x_2, x_3 \geq 0$. *(Mumbai, 2004)*
6. Minimize $Z = 3x_1 + 5x_2 + 4x_3$
   subject to $2x_1 + 3x_2 \leq 8$, $2x_2 + 5x_3 \leq 10$, $3x_1 + 2x_2 + 4x_3 \leq 15$, $x_1, x_2, x_3 \geq 0$. *(Mumbai, 2004 S)*
7. Minimize $Z = x_1 - 3x_2 + 2x_3$,
   subject to $3x_1 - x_2 + 2x_3 \leq 7$, $-2x_1 + 4x_2 \leq 12$, $-4x_1 + 3x_2 + 8x_3 \leq 10$, $x_1, x_2, x_3 \geq 0$. *(Madras, 2006)*
8. Maximize $Z = 4x_1 + 3x_2 + 4x_3 + 6x_4$
   subject to $x_1 + 2x_2 + 2x_3 + 4x_4 \leq 80$, $2x_1 + 2x_3 + x_4 \leq 60$, $3x_1 + 3x_2 + x_3 + x_4 \leq 80$, $x_1, x_2, x_3, x_4 \geq 0$.
9. A firm produces products *A* and *B* and sells them at a profit of ₹ 2 and ₹ 3 each respectively. Each product is processed on machines *G* and *H*. Product *A* requires 1 minute on *G* and 2 minutes on *H* whereas product *B* requires 1 minute on each of the machines. Machine *G* is not available for more than 6 hrs. 40 min/day whereas the time constraint for machine *H* is 10 hrs. Solve this problem *via* simplex method for maximizing the profit.
10. A company makes two types of products. Each product of the first type requires twice as much labour time as the second type. If all products are of second type only, the company can produce a total of 500 units a day. The market limits daily sales of the first and the second type to 150 and 250 units respectively. Assuming that the profits per

unit are ₹ 8 for type I and ₹ 5 for type II, determine the number of units of each type to be produced to maximize profit ?

11. The owner of a dairy is trying to determine the correct blend of two types of feed. Both contain various percentages of four essential ingredients. With the following data determine the least cost blend ?

| *Ingredient* | *% per kg of feed* | | *Min requirement in kg.* |
|---|---|---|---|
| | *Feed 1* | *Feed 2* | |
| 1 | 40 | 20 | 4 |
| 2 | 10 | 30 | 2 |
| 3 | 20 | 40 | 3 |
| 4 | 30 | 10 | 6 |
| Cost (₹/kg.) | 5 | 3 | |

12. A manufacturing firm has discontinued production of a certain unprofitable product line. This created considerable excess production capacity. Management is considering to devote their excess capacity to one or more of three product 1, 2, and 3. The available capacity on machines and the number of machine-hours required for each unit of the respective product, is given below :

| *Machine Type* | *Available Time (hrs/week)* | *Productivity (hrs/unit)* | | |
|---|---|---|---|---|
| | | *Product 1* | *Product 2* | *Product 3* |
| Milling machine | 250 | 8 | 2 | 3 |
| Lathe | 150 | 4 | 3 | [illegible] |
| Grinder | 50 | 2 | – | 1 |

The unit profit would be ₹ 20, ₹ 6 and ₹ 8 respectively for products 1, 2 and 3. Find how much of each product the firm should produce in order to maximize profit.

13. The following table gives the various vitamin contents of three types of food and daily requirements of vitamins alongwith cost per unit. Find the combination of food for minimum cost.

| *Vitamin (mg)* | *Food F* | *Food G* | *Food H* | *Minimum daily requirement (mg)* |
|---|---|---|---|---|
| *A* | 1 | 1 | 10 | 1 |
| *C* | 100 | 10 | 10 | 50 |
| *D* | 10 | 100 | 10 | 10 |
| Cost/unit (₹) | 10 | 15 | 5 | |

14. A farmer has 1,000 acres of land on which he can grow corn, wheat or soyabeans. Each acre of corn costs ₹ 100 for preparation, requires 7 man-days of work and yields a profit of ₹ 30. An acre of wheat costs ₹ 120 to prepare, requires 10 man-days of work and yields a profit of ₹ 40. An acre of soyabeans costs ₹70 to prepare, requires 8 man-days of work and yields a profit of ₹ 20. If the farmer has ₹ 100,000 for preparation and can count on 8,000 man-days of work, how many acres should be allocated to each crop to maximize profits ?

## 34.9 ARTIFICIAL VARIABLE TECHNIQUES

So far we have seen that the introduction of slack/surplus variables provided the initial basic feasible solution. But there are many problems wherein at least one of the constraints is of (≥) or (=) type and slack variables fail to give such a solution. There are two similar methods for solving such problems which we explain below :

**(1) M-method or Method of Penalties.** This method is due to *A. Charnes* and consists of the following steps :

*Step 1.* Express the problem in standard form.

*Step 2.* Add non-negative variables to the left hand side of all those constraints which are of (≥) or (=) type. Such new variables are called *artificial variables* and the purpose of introducing these is just to obtain an initial basic feasible solution. But their addition causes violation of the corresponding constraints. As such, we would like to get rid of these variables and would not allow them to appear in the final solution. For this purpose, we assign a very large penalty ($-M$) to these artificial variables in the objective function.

*Step* 3. Solve the modified L.P.P. by simplex method.

At any iteration of simplex method, one of the following three cases may arise :

(*i*) There remains no artificial variable in the basis and the optimality condition is satisfied. Then the solution is an optimal basic feasible solution to the problem.

(*ii*) There is at least one artificial variable in the basis at zero level (with zero value in $b$-column) and the optimality condition is satisfied. Then the solution is a degenerate optimal basic feasible solution.

(*iii*) There is at least one artificial variable in the basis at non-zero level (with positive value in $b$-column) and the optimality condition is satisfied. Then the problem has no feasible solution. The final solution is not optimal, since the objective function contains an unknown quantity $M$. Such a solution satisfies the constraints but does not optimize the objective function and is therefore, called *pseudo optimal solution*.

*Step 4.* Continue the simplex method until either an optimal basic feasible solution is obtained or an unbounded solution is indicated.

**Obs.** The artificial variables are only a computational device for getting a starting solution. Once an artificial variable leaves the basis, it has served its purpose and we forget about it *i.e.*, the column for this variable is omitted from the next simplex table.

**Example 34.18.** *Use Charne's penalty method to*

*Minimize* $Z = 2x_1 + x_2$

*subject to* $3x_1 + x_2 = 3,\ 4x_1 + 3x_2 \geq 6,\ x_1 + 2x_2 \leq 3,\ x_1, x_2 \geq 0.$ (*Anna, M. Tech, 2006 ; V.T.U., 2000 S*)

**Solution.** Consists of the following steps :

*Step 1. Express the problem in standard form.*

The second and third inequalities are converted into equations by introducing the surplus and slack variables $s_1, s_2$ respectively.

Also the first and second constraints being of (=) and (≥) type, we introduce two artificial variables $A_1, A_2$.

Converting the minimization problem to the maximization form, the L.P.P. can be rewritten as

Max. $Z' = -2x_1 - x_2 + 0s_1 + 0s_2 - MA_1 - MA_2$

subject to
$$3x_1 + x_2 + 0s_1 + 0s_2 + A_1 + 0A_2 = 3$$
$$4x_1 + 3x_2 - s_1 + 0s_2 + 0A_1 + A_2 = 6$$
$$x_1 + 2x_2 + 0s_1 + s_2 + 0A_1 + 0A_2 = 3$$
$$x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$$

*Step 2. Obtain an initial basic feasible solution.*

Surplus variable $s_1$ is not a basic variable since its value is $-6$. As negative quantities are not feasible, $s_1$ must be prevented from appearing in the initial solution. This is done by taking $s_1 = 0$. By setting the other non-basic variables $x_1, x_2$ each = 0, we obtain the initial basic feasible solution as

$$x_1 = x_2 = 0,\ s_1 = 0\ ;\ A_1 = 3,\ A_2 = 6,\ s_2 = 3$$

Thus the initial simplex table is

| | $c_j$ | $-2$ | $-1$ | 0 | 0 | $-M$ | $-M$ | | |
|---|---|---|---|---|---|---|---|---|---|
| $c_B$ | *Basis* | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $A_1$ | $A_2$ | $b$ | θ |
| $-M$ | $A_1$ | (3) | 1 | 0 | 0 | 1 | 0 | 3 | 3/3← |
| $-M$ | $A_2$ | 4 | 3 | $-1$ | 0 | 0 | 1 | 6 | 6/4 |
| 0 | $s_2$ | 1 | 2 | 0 | 1 | 0 | 0 | 3 | 3/1 |
| $Z_j = \Sigma c_B a_{ij}$ | | $-7M$ | $-4M$ | $M$ | 0 | $-M$ | $-M$ | $-9M$ | |
| $C_j = c_j - Z_j$ | | $7M - 2$ | $4M - 1$ | $-M$ | 0 | 0 | 0 | | |
| | | ↑ | | | | | | | |

Since $C_j$ is positive under $x_1$ and $x_2$ columns, this is not an optimal solution.

*Step 3. Iterate towards optimal solution.*

Introduce $x_1$, and drop $A_1$ from basis.

$\therefore$ The new simplex table is

| $c_B$ | $c_j$ Basis | $-2$ $x_1$ | $-1$ $x_2$ | $0$ $s_1$ | $0$ $s_2$ | $-M$ $A_2$ | $b$ | $\theta$ |
|---|---|---|---|---|---|---|---|---|
| $-2$ | $x_1$ | 1 | 1/3 | 0 | 0 | 0 | 1 | 3 |
| $-M$ | $A_2$ | 0 | (5/3) | $-1$ | 0 | 1 | 2 | 6/5 $\leftarrow$ |
| 0 | $s_2$ | 0 | 5/3 | 0 | 1 | 0 | 2 | 6/5 |
| | $Z_j$ | $-2$ | $-\frac{2}{3}-\frac{5M}{3}$ | $M$ | 0 | $-M$ | $-2-2M$ | |
| | $C_j$ | 0 | $-\frac{1}{3}+\frac{5M}{3}$ $\uparrow$ | $-M$ | 0 | 0 | | |

Since $C_j$ is positive under $x_2$ column, this is not an optimal solution.

$\therefore$ Introduce $x_2$ and drop $A_2$.

Then the revised simplex table is

| $c_B$ | $c_j$ Basis | $-2$ $x_1$ | $-1$ $x_2$ | $0$ $s_1$ | $0$ $s_2$ | $b$ |
|---|---|---|---|---|---|---|
| $-2$ | $x_1$ | 1 | 0 | 1/5 | 0 | 3/5 |
| $-1$ | $x_2$ | 0 | 1 | $-3/5$ | 0 | 6/5 |
| 0 | $s_2$ | 0 | 0 | 1 | 1 | 0 |
| | $Z_j$ | $-2$ | $-1$ | 1/5 | 0 | $-12/5$ |
| | $C_j$ | 0 | 0 | $-1/5$ | 0 | |

Since none of $C_j$ is positive, this an optimal solution. Thus, an optimal basic feasible solution to the problem is

$$x_1 = 3/5, x_2 = 6/5, \text{Max. } Z' = -12/5.$$

Hence the optimal value of the objective function is

$$\text{Min. } Z = -\text{Max. } Z' = -(-12/5) = 12/5$$

**Example 34.19.** *Maximize* $Z = 3x_1 + 2x_2$
*subject to the constraints* : $2x_1 + x_2 \le 2$, $3x_1 + 4x_2 \ge 12$, $x_1, x_2 \ge 0$.

**Solution.** Consists of the following steps :

*Step 1. Express the problem in standard form.*

The inequalities are converted into equations by introducing the slack and surplus variables $s_1$, $s_2$ respectively. Also the second constraint being of ($\ge$) type, we introduce the artificial variable $A$. Thus the L.P.P. can be rewritten as

Max. $Z = 3x_1 + 2x_2 + 0s_1 + 0s_2 - MA$

subject to
$$2x_1 + x_2 + s_1 + 0s_2 + 0A = 2,$$
$$3x_1 + 4x_2 + 0s_1 - s_2 + A = 12,$$
$$x_1, x_2, s_1, A \ge 0$$

*Step 2. Find an initial basic feasible solution.*

Surplus variable $s_2$ is not a basic variable since its value is $-12$. Since a negative quantity is not feasible, $s_2$ must be prevented from appearing in the initial solution. This is done by letting $s_2 = 0$. By taking the other non-basic variables $x_1$ and $x_2$ each $= 0$, we obtain the initial basic feasible solution as

$$x_1 = x_2 = s_2 = 0, s_1 = 2, A = 12$$

$\therefore$ The initial simplex table is

| $c_B$ | $c_j$ Basis | $3$ $x_1$ | $2$ $x_2$ | $0$ $s_1$ | $0$ $s_2$ | $-M$ $A$ | $b$ | $\theta$ |
|---|---|---|---|---|---|---|---|---|
| 0 | $s_1$ | 2 | (1) | 1 | 0 | 0 | 2 | 2 $\leftarrow$ |
| $-M$ | $A$ | 3 | 4 | 0 | $-1$ | 1 | 12 | 3 |
| $Z_j = \Sigma c_B a_{ij}$ | | $-3M$ | $-4M$ | 0 | $M$ | $-M$ | $-12M$ | |
| $C_j = c_j - Z_j$ | | $3+3M$ | $2+4M$ $\uparrow$ | 0 | $-M$ | 0 | | |

Since $C_j$ is positive under some columns, this is not an optimal solution.

*Step 3. Iterate towards optimal solution.*

Introduce $x_2$ and drop $s_1$.

$\therefore$ The new simplex table is

| | $c_j$ | 3 | 2 | 0 | 0 | $-M$ | |
|---|---|---|---|---|---|---|---|
| $c_B$ | Basis | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $A$ | $b$ |
| 2 | $x_2$ | 2 | 1 | 1 | 0 | 0 | 2 |
| $-M$ | $A$ | $-5$ | 0 | $-4$ | $-1$ | 1 | 4 |
| $Z_j$ | | $4+5M$ | 2 | $2+4M$ | $M$ | $-M$ | $4-4M$ |
| $C_j$ | | $-(1+5M)$ | 0 | $-(2+4M)$ | $-M$ | 0 | |

Here each $C_j$ is negative and an artificial variable appears in the basis at non-zero level. Thus there exists a *pseudo optimal solution* to the problem.

**(2) Two-phase method.** This is another method to deal with the artificial variables wherein the L.P.P. is solved in two phases.

**Phase I.** *Step 1.* Express the given problem in the standard form by introducing slack, surplus and artificial variables.

*Step 2.* Formulate an artificial objective function

$$Z^* = -A_1 - A_2 \dots - A_m$$

by assigning $(-1)$ cost to each of the artificial variables $A_i$ and zero cost to all other variables.

*Step 3.* Maximize $Z^*$ subject to the constraints of the original problem using the simplex method. Then three cases arise :

(*a*) *Max.* $Z^* < 0$ *and at least one artificial variable appears in the optimal basis at a positive level*

In this case, the original problem dosn't possess any feasible solution and the procedure comes to an end.

(*b*) *Max.* $Z^* = 0$ *and no artificial variable appears in the optimal basis.*

In this case, a basic feasible solution is obtained and we proceed to phase II for finding the optimal basic feasible solution to the original problem.

(*c*) *Max.* $Z^* = 0$ *and at least one artificial variable appears in the optimal basis at zero level.*

Here a feasible solution to the auxiliary L.P.P. is also a feasible solution to the original problem with all artificial variables set = 0.

To obtain a basic feasible solution, we prolong phase I for pushing all the artificial variables out of the basis (without proceeding on to phase II).

**Phase II.** The basic feasible solution found at the end of phase I is used as the starting solution for the original problem in this phase *i.e.*, the final simplex table of phase I is taken as the initial simplex table of phase II and the artificial objective function is replaced by the original objective function. Then we find the optimal solution.

**Example 34.20.** *Use two-phase method to*

*Minimize* $Z = 7.5x_1 - 3x_2$

*subject to the constraints* $3x_1 - x_2 - x_3 \geq 3,\ x_1 - x_2 + x_3 \geq 2,$

$x_1, x_2, x_3 \geq 0.$

**Phase I.** *Step 1. Express the problem in standard form.*

**Solution.** Introducing surplus variables $s_1$, $s_2$ and artificial variables $A_1$, $A_2$, the phase I problem in standard form becomes

Max. $\quad Z^* = 0x_1 + 0x_2 + 0x_3 + 0s_1 + 0s_2 - A_1 - A_2$

subject to
$$3x_1 - x_2 - x_3 - s_1 + 0s_2 + A_1 + 0A_2 = 3$$
$$x_1 - x_2 + x_3 + 0s_1 - s_2 + 0A_1 + A_2 = 2$$
$$x_1, x_2, x_3, s_1, s_2, A_1, A_2 \geq 0.$$

*Step 2. Find an initial basic feasible solution.*

Setting $x_1 = x_2 = x_3 = s_1 = s_2 = 0$,

we have $\quad A_1 = 3, A_2 = 2$ and $Z^* = -5$

∴ Initial simplex table is

| | $c_j$ | 0 | 0 | 0 | 0 | 0 | −1 | −1 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $c_B$ | Basis | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $A_1$ | $A_2$ | $b$ | θ |
| −1 | $A_1$ | (3) | −1 | −1 | −1 | 0 | 1 | 0 | 3 | 1← |
| −1 | $A_2$ | 1 | −1 | 1 | 0 | −1 | 0 | 1 | 2 | 2 |
| $Zj^* = \Sigma c_B a_{ij}$ | | −4 | 2 | 0 | 1 | 1 | −1 | −1 | −5 | |
| $C_j = c_j - Z_j^*$ | | 4 ↑ | −2 | 0 | −1 | −1 | 0 | 0 | | |

As $C_j$ is positive under $x_1$ column, this solution is not optimal.

*Step 3. Iterate towards an optimal solution.*

Making key element (3) unity and replacing $A_1$ by $x_1$, we have the new simplex table :

| | $c_j$ | 0 | 0 | 0 | 0 | 0 | −1 | −1 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $c_B$ | Basis | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $A_1$ | $A_2$ | $b$ | θ |
| 0 | $x_1$ | 1 | $-\frac{1}{3}$ | $-\frac{1}{3}$ | $-\frac{1}{3}$ | 0 | $\frac{1}{3}$ | 0 | 1 | −3 |
| −1 | $A_2$ | 0 | $-\frac{2}{3}$ | $\left(\frac{4}{3}\right)$ | $\frac{1}{3}$ | −1 | $-\frac{1}{3}$ | 1 | 1 | $\frac{3}{4}$← |
| | $Z_j^*$ | 0 | $\frac{2}{3}$ | $-\frac{4}{3}$ | $-\frac{1}{3}$ | 1 | $\frac{1}{3}$ | −1 | −1 | |
| | $C_j$ | 0 | $-\frac{2}{3}$ | $\frac{4}{3}$ ↑ | $\frac{1}{3}$ | −1 | $-\frac{1}{3}$ | 0 | | |

Since $C_j$ is positive under $x_3$ and $s_1$ columns, this solution is not optimal.

Making key element (4/3) unity and replacing $A_2$ by $x_3$, we obtain the revised simplex table :

| | $c_j$ | 0 | 0 | 0 | 0 | 0 | −1 | −1 | |
|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | Basis | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $A_1$ | $A_2$ | $b$ |
| 0 | $x_1$ | 1 | −1/2 | 0 | −1/4 | −1/4 | 1/4 | 1/4 | 5/4 |
| 0 | $x_2$ | 0 | −1/2 | 1 | 1/4 | −3/4 | −1/4 | 3/4 | 3/4 |
| $Z_j^*$ | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $C_j$ | | 0 | 0 | 0 | 0 | 0 | −1 | −1 | |

Since all $C_j \le 0$, this table gives the optimal solution. Also $Z^*_{\max} = 0$ and no artificial variable appears in the basis. Thus an optimal basic feasible solution to the auxiliary problem and therefore to the original problem, has been attained.

**Phase II.** Considering the actual costs associated with the original variables, the objective function is

Max. $Z' = -15/2x_1 + 3x_2 + 0x_3 + 0s_1 + 0s_2 - 0A_1 - 0A_2$

subject to $3x_1 - x_2 - x_3 - s_1 + 0s_2 + A_1 + 0A_2 = 3,$

$x_2 - x_2 + x_3 + 0s_1 - s_2 + 0A_1 + A_2 = 2,$

$x_2, x_2, x_3, s_1, s_2, A_1, A_2 \ge 0$

The optimal initial feasible solution thus obtained, will be an optimal basic feasible solution to the original L.P.P.

Using final table of phase I, the initial simplex table of phase II is as follows :

| | $c_j$ | −15/2 | 3 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|
| $c_B$ | Basis | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $b$ |
| −15/2 | $x_1$ | 1 | −1/2 | 0 | −1/4 | −1/4 | 5/4 |
| 0 | $x_3$ | 0 | −1/2 | 1 | 1/4 | −3/4 | 3/4 |
| $Z_j$ | | −15/2 | 15/4 | 0 | 15/8 | 15/8 | −75/8 |
| $C_j$ | | 0 | −3/4 | 0 | −15/8 | −15/8 | |

Since all $C_j \le 0$, this solution is optimal.

Hence an optimal basic feasible solution to the given problem is

$$x_1 = 5/4, x_2 = 0, x_3 = 3/4 \quad \text{and} \quad \text{min. } Z = 75/8.$$

## 34.10 EXCEPTIONAL CASES

**(1) Tie for the incoming variable.** When more than one variable has the same largest positive value in $C_j$ row (in maximization problem), a tie for the choice of incoming variable occurs. As there is no method to break this tie, we choose any one of the prospective incoming variables arbitrarily. Such an arbitrary choice does not in any way affect the optimal solution.

**(2) Tie for the outgoing variable.** When more than one variable has the same least positive ratio under then θ-column, a tie for the choice of outgoing variable occurs. If the equal values of said ratio are > 1, choose any one of the prospective leaving variables arbitrarily. Such an arbitrary choice doesn't affect the optimal solution.

If the equal values of ratios are zero, the simplex method fails and we make use of the following degeneracy technique.

**(3) Degeneracy.** We know that a basic feasible solution is said to be degenerate if any of the basic variables vanishes. This phenomenon of getting a degenerate basic feasible solution is called *degeneracy* which may arise

(*i*) *at the initial state,* when atleast one basic variable is zero in the initial basic feasible solution

or (*ii*) *at any subsequent stage,* when the least positive ratios under θ-column are equal for two or more rows.

In this case, an arbitrary choice of one of these basic variables may result in one or more basic variables becoming zero in the next iteration. At times, the same sequence of simplex iterations is repeated endlessly without improving the solution. These are termed as *cycling* type of problems. Cycling occurs very rarely. In fact, cycling has seldom occurred in practical problems.

*To avoid cycling, we apply the following* **perturbation procedure :**

(*i*) Divide each element in the tied rows by the *positive coefficients* of the key column in that row.

(*ii*) Compare the resulting ratios (from left to right) first of unit matrix and then of the body matrix, column by column.

(*iii*) The outgoing variable lies in that row which first contains the smallest algebraic ratio.

**Example 34.21.** *Maximize* $Z = 5x_1 + 3x_2$
*subject to* $x_1 + x_2 \le 2$, $5x_1 + 2x_2 \le 10$, $3x_1 + 8x_2 \le 12$ ; $x_1, x_2 \ge 0$.

**Solution.** Consists of the following steps :

*Step 1. Express the problem in the standard form.*

Introducing the slack variables $s_1, s_2, s_3$, the problem in the standard form is

$$\text{Max. } Z = 5x_1 + 3x_2 + 0s_1 + 0s_2 + 0s_3$$
$$x_1 + x_2 + s_1 + 0s_2 + 0s_3 = 2$$
$$5x_1 + 2x_2 + 0s_1 + s_2 + 0s_3 = 10$$
$$3x_1 + 8x_2 + 0s_1 + 0s_2 + s_3 = 12$$
$$x_1, x_2, s_1, s_2, s_3 \ge 0.$$

*Step 2. Find the initial basic feasible solution.*

The initial basic feasible solution is

$$x_1 = x_2 = 0 \text{ (non-basic)}$$
$$s_1 = 2, s_2 = 10, s_3 = 12 \text{ (basic) and } Z = 0.$$

∴ Initial simplex table is

| | $c_j$ | 5 | 3 | 0 | 0 | 0 | | |
|---|---|---|---|---|---|---|---|---|
| $c_B$ | *Basis* | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $b$ | θ |
| 0 | $s_1$ | 1 | 1 | 1 | 0 | 0 | 2 | 2/1 |
| 0 | $s_2$ | (5) | 2 | 0 | 1 | 0 | 10 | 10/5← |
| 0 | $s_3$ | 3 | 8 | 0 | 0 | 1 | 12 | 12/3 |
| $Z_j = \Sigma c_B a_{ij}$ | | 0 | 0 | 0 | 0 | 0 | 0 | |
| $C_j = c_j - Z_j$ | | 5 | 3 | 0 | 0 | 0 | | |
| | | ↑ | | | | | | |

As $C_j$ is positive under $x_2$ columns, this solution is not optimal.

*Step 3. Iterate towards optimal solution.*

$x_1$ is the incoming variable. But the first two rows have the same ratio under θ-column. Therefore we apply *perturbation* method.

First column of the unit matrix has 1 and 0 in the tied rows. Dividing these by the corresponding elements of the key columns, we get 1/1 and 0/5, $s_2$-row gives the smaller ratio and therefore $s_2$ is the first outgoing variable and (5) is the key element.

Thus the new simplex table is

| | $c_j$ | 5 | 3 | 0 | 0 | 0 | | |
|---|---|---|---|---|---|---|---|---|
| $c_B$ | *Basis* | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $b$ | θ |
| 0 | $s_1$ | 0 | (3/5) | 1 | – 1/5 | 0 | 0 | 0 |
| 5 | $x_1$ | 1 | 2/5 | 0 | 1/5 | 0 | 2 | 5← |
| 0 | $s_3$ | 0 | 34/5 | 0 | – 3/5 | 1 | 6 | 15/17 |
| $Z_j$ | | 5 | 2 | 0 | 1 | 0 | 10 | |
| $C_j$ | | 0 | 1 | 0 | – 1 | 0 | | |
| | | | ↑ | | | | | |

As $C_j$ is positive under $x_2$ column, this solution is not optimal.

Making key element (3/5) unity and replacing $s_1$ by $x_2$, we obtain the revised simplex table :

| | $c_j$ | 5 | 3 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|
| $c_B$ | *Basis* | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $b$ |
| 3 | $x_2$ | 0 | 1 | 5/3 | – 1/3 | 0 | 0 |
| 5 | $x_1$ | 1 | 0 | – 2/3 | 1/3 | 0 | 2 |
| 0 | $s_3$ | 0 | 0 | – 34/3 | 5/3 | 1 | 6 |
| $Z_j$ | | 5 | 3 | 5/3 | 2/3 | 0 | 10 |
| $C_j$ | | 0 | 0 | – 5/3 | – 2/3 | 0 | |

As $C_j \le 0$ under all columns, this table gives the optimal solution. Hence an optimal basic feasible solution is $x_1 = 2, x_2 = 0$ and $Z_{max} = 10$.

## PROBLEMS 34.5

Solve the following L.P. problems using *M*-method :

1. Maximize $Z = 3x_1 + 2x_2 + 3x_3$
   subject to : $2x_1 + x_2 + x_3 \le 2,\ 3x_1 + 4x_2 + 2x_3 \ge 8,\ x_1, x_2, x_3 \ge 0.$
2. Maximize $Z = 2x_1 + x_2 + 3x_3$
   subject to : $x_1 + x_2 + 2x_3 \le 5,\ 2x_1 + 3x_2 + 4x_3 = 12,\ x_1, x_2, x_3 \ge 0.$
3. Maximize $Z = 8x_2$,
   subject to : $x_1 - x_2 \ge 0,\ 2x_1 + 3x_2 \le -6,\ x_1, x_2$ unrestricted.
4. Maximize $Z = 5x_1 - 2x_2 + 3x_3$
   subject to : $2x_1 + 2x_2 - x_3 \ge 2,\ 3x_1 - 4x_2 \le 3,\ x_2 + 3x_3 \le 5,\ x_1, x_2, x_3 \ge 0.$ *(Mumbai, 2004)*
5. Maximize $Z = x_1 + 2x_2 + 3x_3 - x_4$
   subject to : $x_1 + 2x_2 + 3x_3 = 15,\ 2x_1 + x_2 + 5x_3 = 20,$
   $x_1 + 2x_2 + x_3 + x_4 = 10,\ x_1, x_2, x_3, x_4 \ge 0.$ *(Madras, 2003)*

Use two phase method to solve the following L.P. problems :

6. Minimize $Z = x_1 + x_2$
   subject to : $2x_1 + x_2 \ge 4,\ x_1 + 7x_2 \ge 7,$
   $x_1, x_2 \ge 0.$ *(Rajasthan, 2005)*
7. Maximize $Z = 5x_1 + 3x_2$
   subject to : $2x_1 + x_2 \le 1,\ x_1 + 4x_2 \ge 6,$
   $x_1, x_2 \ge 0.$ *(Kottayam, 2005)*

8. Maximize $Z = 5x_1 - 4x_2 + 3x_3$,
subject to : $2x_1 + 2x_2 - x_3 \geq 2$,
$3x_1 - 4x_2 \leq 3, x_2 + x_3 \leq 5$,
$x_1, x_2, x_3 \geq 0$. *(Mumbai, 2009)*

9. Maximize $Z = 5x_1 - 4x_2 + 3x_3$,
subject to : $2x_1 + x_2 - 6x_3 = 20$,
$6x_1 + 5x_2 + 10x_3 \leq 76$,
$8x_1 - 3x_2 + 6x_3 \leq 50$,
$x_1, x_2, x_3 \geq 0$.

Solve the following degenerate L.P. problems :

10. Maximize $Z = 9x_1 + 3x_2$
subject to : $4x_1 + x_2 \leq 8, 2x_1 + x_2 \leq 4$,
$x_1, x_2 \geq 0$.

11. Maximize $Z = 2x_1 + 3x_2 + 10x_3$
subject to : $x_1 + 2x_3 = 0, x_2 + x_3 = 1$,
$x_1, x_2, x_3 \geq 0$.

## 34.11 (1) DUALITY CONCEPT

One of the most interesting concepts in linear programming is the *duality* theory. Every linear programming problem has associated with it, another linear programming problem involving the same data and closely related optimal solutions. Such two problems are said to be *duals* of each other. While one of these is called the *primal*, the other the *dual*.

The importance of the duality concept is due to two main reasons. Firstly, if the primal contains a large number of constraints and a smaller number of variables, the labour of computation can be considerably reduced by converting it into the dual problem and then solving it. Secondly, the interpretation of the dual variables from the cost or economic point of view proves extremely useful in making future decisions in the activities being programmed.

**(2) Formulation of dual problem.** Consider the following L.P.P. :

Maximize $Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$,

subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &\leq b_1, \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &\leq b_2, \\ &\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &\leq b_m \\ x_1, x_2, \dots, x_n &\geq 0. \end{aligned}$$

To construct the dual problem, we adopt the following guidelines :

(*i*) The maximization problem in the primal becomes the minimization problem in the dual and *vice versa*.

(*ii*) ($\leq$) type of constraints in the primal become ($\geq$) type of constraints in the dual and *vice versa*.

(*iii*) The coefficients $c_1, c_2, \dots, c_n$ in the objective function of the primal become $b_1, b_2, \dots, b_m$ in the objective function of the dual.

(*iv*) The constants $b_1, b_2, \dots, b_m$ in the constraints of the primal become $c_1, c_2, \dots, c_n$ in the constraints of the dual.

(*v*) If the primal has $n$ variables and $m$ constraints, the dual will have $m$ variables and $n$ constraints *i.e.* the transpose of the body matrix of the primal problem gives the body matrix of the dual.

(*vi*) The variables in both the primal and dual are non-negative.

Then the dual problem will be

Minimize $W = b_1y_1 + b_2y_2 + \dots + b_my_m$

subject to the constraints

$$\begin{aligned} a_{11}y_1 + a_{21}y_2 + \dots + a_{m1}y_m &\geq c_1, \\ a_{12}y_1 + a_{22}y_2 + \dots + a_{m2}y_m &\geq c_2, \\ &\dots\dots\dots \\ a_{1n}y_1 + a_{2n}y_2 + \dots + a_{mn}y_m &\geq c_n, \\ y_1, y_2, \dots, y_m &\geq 0. \end{aligned}$$

**Example 34.22.** *Write the dual of the following L.P.P.:*

*Minimize* $Z = 3x_1 - 2x_2 + 4x_3$,

*subject to* $3x_1 + 5x_2 + 4x_3 \geq 7, 6x_1 + x_2 + 3x_3 \geq 4, 7x_1 - 2x_2 - x_3 \leq 10$,
$x_1 - 2x_2 + 5x_3 \geq 3, 4x_1 + 7x_2 - 2x_3 \geq 2, x_1, x_2, x_3 \geq 0$.

**Solution.** Since the problem is of minimization, all constraints should be of $\geq$ type. We multiply the third constraint throughout by $-1$ so that $-7x_1 + 2x_2 + x_3 \geq -10$.

Let $y_1, y_2, y_3, y_4$ and $y_5$ be the dual variables associated with the above five constraints. Then the dual problem is given by

Maximize $\quad W = 7y_1 + 4y_2 - 10y_3 + 3y_4 + 2y_5$

subject to $\quad 3y_1 + 6y_2 - 7y_3 + y_4 + 4y_5 \leq 3,\ 5y_1 + y_2 + 2y_3 - 2y_4 + 7y_5 \leq -2,$

$4y_1 + 3y_2 + y_3 + 5y_4 - 2y_5 \leq 4,\ y_1, y_2, y_3, y_4, y_5 \geq 0.$

**(3) Formulation of dual problem when the primal has equality constraints.** Consider the problem

Maximize $\quad Z = c_1x_1 + c_2x_2$

subject to $\quad a_{11}x_1 + a_{12}x_2 = b_1,\ a_{21}x_1 + a_{22}x_2 \leq b_2,\ x_1, x_2 \geq 0.$

The equality constraint can be written as

$$a_{11}x_1 + a_{12}x_2 \leq b_1 \text{ and } a_{11}x_1 + a_{12}x_2 \geq b_1$$

or
$$a_{11}x_1 + a_{12}x_2 \leq b_1 \text{ and } -a_{11}x_1 - a_{12}x_2 \leq -b_1,$$

Then the above problem can be restated as

Maximize $\quad Z = c_1x_1 + c_2x_2$

subject to $\quad a_{11}x_1 + a_{12}x_2 \leq b_1, -a_{11}x_1 - a_{12}x_2 \leq -b_1,$

$a_{21}x_1 + a_{22}x_2 \leq b_2,\ x_1, x_2 \geq 0.$

Now we form the dual using $y_1', y_1'', y_2$ as the dual variables. Then the dual problem is

Minimize $\quad W = b_1(y_1' - y_1'') + b_2y_2,$

subject to $\quad a_{11}(y_1' - y_1'') + a_{21}y_2 \geq c_1,\ a_{12}(y_1' - y_1'') + a_{22}y_2 \geq c_2,\ y_1' y_1'', y_2 \geq 0.$

The term $(y_1' - y_1'')$ appears in both the objective function and all the constraints of the dual. This will always happen whenever there is an equality constraint in the primal. Then the new variable $y_1' - y_1''$ $(= y_1)$ becomes unrestricted in sign being the difference of two non-negative variables and the above dual problem takes the form.

Minimize $\quad W = b_1y_1 + b_2y_2,$

subject to $\quad a_{11}y_1 + a_{21}y_2 \geq c_1,\ a_{12}y_1 + a_{22}y_2 \geq c_2,\ y_1$ unrestricted in sign, $y_2 \geq 0$.

In general, if the primal problem is

Maximize $\quad Z = c_1x_1 + c_2x_2 + \dots + c_nx_n,$

subject to
$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$
$$x_1, x_2, \dots, x_n \geq 0,$$

then the dual problem is

Minimize $\quad W = b_1y_1 + b_2y_2 + \dots + b_my_m$

subject to
$$a_{11}y_1 + a_{21}y_2 + \dots + a_{m1}y_m \geq c_1,$$
$$a_{12}y_1 + a_{22}y_2 + \dots + a_{m2}y_m \geq c_2,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$a_{1n}y_1 + a_{2n}y_2 + \dots + a_{mn}y_m \geq c_n$$
$$y_1, y_2, \dots, y_m \text{ all unrestricted in sign.}$$

Thus *the dual variables corresponding to equality constraints are unrestricted in sign.* Conversely *when the primal variables are unrestricted in sign, corresponding dual constraints are equalities.*

**Example 34.23.** *Construct the dual of the L.P.P. :*

*Maximize* $\quad Z = 4x_1 + 9x_2 + 2x_3,$

*subject to* $\quad 2x_1 + 3x_2 + 2x_3 \leq 7,\ 3x_1 - 2x_2 + 4x_3 = 5,\ x_1, x_2, x_3 \geq 0.$

**Solution.** Let $y_1$ and $y_2$ by the dual variables associated with the first and second constraints. Then the dual problem is

Minimize $\quad W = 7y_1 + 5y_2,$

subject to $\quad 2y_1 + 3y_2 \leq 4,\ 3y_1 - 2y_2 \leq 9,\ 2y_1 + 4y_2 \leq 2,\ y_1 \geq 0,\ y_2$ is unrestricted in sign.

**PROBLEMS 34.6**

Write the duals of the following problems (1 – 4) :

1. Maximize $Z = 10x_1 + 13x_2 + 19x_3$
   subject to $6x_1 + 5x_2 + 3x_3 \le 26,\ 4x_1 + 2x_2 + 5x_3 \le 7,\ x_1, x_2, x_3 \ge 0.$
2. Minimize $Z = 2x_1 + 4x_2 + 3x_3$,
   subject to $3x_1 + 4x_2 + x_3 \ge 11,\ -2x_1 - 3x_2 + 2x_3 \le -7,\ x_1 - 2x_2 - 3x_3 \le -1$
   $3x_1 + 2x_2 + 2x_3 \ge 5,\ x_1, x_2, x_3 \ge 0.$
3. Maximize $Z = 3x_1 + 16x_2 + 7x_3$
   subject to $x_1 - x_2 + x_3 \ge 3,\ -3x_1 + 2x_3 \le 1,\ 2x_1 + x_2 - x_3 = 4,\ x_1, x_2, x_3 \ge 0.$
4. Minimize $Z = 3x_1 - 3x_2 + x_3$
   subject to $2x_1 - 3x_2 + x_3 \le 5,\ 4x_1 - 2x_2 \ge 9,\ -8x_1 + 4x_2 + 3x_3 = 8,$
   $x_1, x_2 \ge 0$ and $x_3$ is unrestricted. (*Mumbai, 2004*)
5. Obtain the dual problem of the following L.P.P.
   Maximize $f(x) = 2x_1 + 5x_2 + 6x_3$
   subject to $5x_1 + 6x_2 - x_3 \le 3,\ -2x_1 + x_2 + 4x_3 \le 4,\ x_1 - 5x_2 + 3x_3 \le 1,$
   $-3x_1 - 3x_2 + 7x_3 \le 6,\ x_1, x_2, x_3 \ge 0.$
   Also verify that the dual of the dual problem is the primal problem.

## 34.12 (1) DUALITY PRINCIPLE

*If the primal and the dual problems have feasible solutions then both have optimal solutions and the optimal value of the primal objective function is equal to the optimal value of the dual objective function i.e.,*

$$\text{Max. } Z = \text{Min. } W$$

This is the fundamental theorem of duality. It suggests that an optimal solution to the primal problem can directly be obtained from that of the dual problem and *vice-versa*.

**(2) Working rules for obtaining an optimal solution to the primal (dual) problem from that of the dual (primal) :**

Suppose we have already found an optimal solution to the dual (primal) problem by simplex method.

*Rule I.* If the primal variable corresponds to a slack starting variable in the dual problem, then its optimal value is directly given by the coefficient of the slack variable with changed sign, in the $C_j$ row of the optimal dual simplex table and *vice-versa*.

*Rule II.* If the primal variable corresponds to an artificial variable in the dual problem, then its optimal value is directly given by the coefficient of the artificial variable, with changed sign, in the $C_j$ row of the optimal dual simplex table, after deleting the constant $M$ and *vice-versa*.

On the other hand, if the primal has an unbounded solution, then the dual problem will not have a feasible solution and *vice-versa*.

Now we shall workout two examples to demonstrate the primal dual relationships.

**Example 34.24.** *Construct the dual of the following problem and solve both the primal and the dual :*
*Maximize* $Z = 2x_1 + x_2$,
*subject to* $-x_1 + 2x_2 \le 2\ x_1 + x_2 \le 4,\ x_1 \le 3,\ x_1, x_2 \ge 0.$ (*Rohtak, 2005*)

**Solution.** *Using the primal problem.* Since only two variables are involved, it is convenient to solve the problem graphically.

In the $x_1, x_2$-plane, the five constraints show that the point $(x_1, x_2)$ lies within the shaded region $OABCD$ of Fig. 34.12. Values of the objective function $Z = 2x_1 + x_2$ at these corners are $Z(0) = 0$, $Z(A) = 6$, $Z(B) = 7$, $Z(C) = 6$ and $Z(D) = 1$. Hence the optimal solution is $x_1 = 3, x_2 = 1$ and max. $(Z) = 7$.

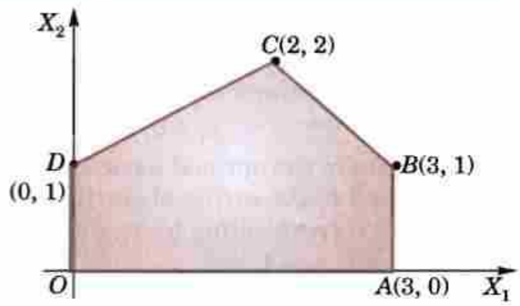


Fig. 34.12

**Solution.** *Using the dual problem.* The dual problem of the given primal is :

Minimize $W = 2y_1 + 4y_2 + 3y_3$

subject to $-y_1 + y_2 + y_3 \geq 2,\ 2y_1 + y_2 \geq 1,\ y_1, y_2 \geq 0.$

*Step 1. Express the problem in the standard form.*

Introducing the slack and the artificial variables, the dual problem in the standard form is

Max. $W' = -2y_1 - 4y_2 - 3y_3 + 0s_1 + 0s_2 - MA_1 - MA_2$

subject to $-y_1 + y_2 + y_3 - s_1 + 0s_2 + A_1 + 0A_2 = 2,$

$2y_1 + y_2 + 0y_3 + 0s_1 - s_2 + 0A_1 + A_2 = 1$

*Step 2. Find an initial basic feasible solution.*

Setting the non-basic variables $y_1, y_2, y_3, s_1, s_2$, each equal to zero, we get the initial basic feasible solution as

$$y_1 = y_2 = y_3 = s_1 = s_2 = 0 \text{ (non-basic)};\ A_1 = 2, A_2 = 1. \quad \text{(basic)}$$

$\therefore$ Initial simplex table is

| | $c_j$ | $-2$ | $-4$ | $-3$ | $0$ | $0$ | $-M$ | $-M$ | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $c_B$ | *Basis* | $y_1$ | $y_2$ | $y_3$ | $s_1$ | $s_2$ | $A_1$ | $A_2$ | $b$ | $\theta$ |
| $-M$ | $A_1$ | $-1$ | $1$ | $1$ | $-1$ | $0$ | $1$ | $0$ | $2$ | $2/1$ |
| $-M$ | $A_2$ | $2$ | $(1)$ | $0$ | $0$ | $-1$ | $0$ | $1$ | $1$ | $1/1 \leftarrow$ |
| | $Z_j$ | $-M$ | $-2M$ | $-M$ | $M$ | $M$ | $-M$ | $-M$ | $-3M$ | |
| | $C_j$ | $M-2$ | $2M-4$ | $M-3$ | $-M$ | $-M$ | $0$ | $0$ | | |
| | | | $\uparrow$ | | | | | | | |

As $C_j$ is positive under some columns, the initial solution is not optimal.

*Step 3. Iterate towards an optimal solution.*

(*i*) Introduce $y_2$ and drop $A_2$. Then the new simplex table is

| | $c_j$ | $-2$ | $-4$ | $-3$ | $0$ | $0$ | $-M$ | $-M$ | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $c_B$ | *Basis* | $y_1$ | $y_2$ | $y_3$ | $s_1$ | $s_2$ | $A_1$ | $A_2$ | $b$ | $\theta$ |
| $-M$ | $A_1$ | $-3$ | $0$ | $(1)$ | $-1$ | $1$ | $1$ | $-1$ | $1$ | $1/1 \leftarrow$ |
| $-4$ | $y_2$ | $2$ | $1$ | $0$ | $0$ | $-1$ | $0$ | $1$ | $1$ | $1/0$ |
| | $Z_j$ | $3M-8$ | $-4$ | $-M$ | $M$ | $4-M$ | $-M$ | $M-4$ | $-M-4$ | |
| | $C_j$ | $6-3M$ | $0$ | $M-3$ | $-M$ | $M-4$ | $0$ | $4-2M$ | | |
| | | | | $\uparrow$ | | | | | | |

As $C_j$ is positive under some columns, this solution is not optimal.

(*ii*) Now introduce $y_3$ and drop $A_1$. Then the revised simplex table is

| | $c_j$ | $-2$ | $-4$ | $-3$ | $0$ | $0$ | $-M$ | $-M$ | |
|---|---|---|---|---|---|---|---|---|---|
| $c_B$ | *Basis* | $y_1$ | $y_2$ | $y_3$ | $s_1$ | $s_2$ | $A_1$ | $A_2$ | $b$ |
| $-3$ | $y_3$ | $-3$ | $0$ | $1$ | $-1$ | $1$ | $1$ | $-1$ | $1$ |
| $-4$ | $y_2$ | $2$ | $1$ | $0$ | $0$ | $-1$ | $0$ | $1$ | $1$ |
| | $Z_j$ | $1$ | $-4$ | $-3$ | $3$ | $1$ | $-3$ | $-1$ | $-7$ |
| | $C_j$ | $-3$ | $0$ | $0$ | $-3$ | $-1$ | $3-M$ | $1-M$ | |

As all $C_j \leq 0$, the optimal solution is attained.

Thus an optimal solution to the dual problem is

$$y_1 = 0, y_2 = 1, y_3 = 1, \text{ Min. } W = -\text{Max. } (W') = 7.$$

To derive the optimal basic feasible solution to the primal problem, we note that the primal variables $x_1$, $x_2$ correspond to the artificial starting dual variables $A_1, A_2$ respectively. In the final simplex table of the dual problem, $C_j$ corresponding to $A_1$, and $A_2$ are 3 and 1 respectively after ignoring $M$. Thus by rule II, we get opt. $x_1 = 3$ and opt. $x_2 = 1$.

Hence an optimal basic feasible solution to the given primal is

$$x_1 = 3, x_2 = 1 \text{ ; max. } Z = 7.$$

**Obs.** The validity of the duality theorem is therefore, checked since max. $Z$ = min. $W$ = 7 from both the methods.

**Example 34.25.** *Using duality solve the following problem :*
*Minimize* $Z = 0.7x_1 + 0.5x_2$
*subject to* $x_1 \geq 4, x_2 \geq 6, x_1 + 2x_2 \geq 20, 2x_1 + x_2 \geq 18, x_1, x_2 \geq 0.$ (V.T.U., 2004)

**Solution.** The dual of the given problem is Max. $W = 4y_1 + 6y_2 + 20y_3 + 18y_4$,
subject to $y_1 + y_3 + 2y_4 \leq 0.7, y_2 + 2y_3 + y_4 \leq 0.5, y_1, y_2, y_3, y_4 \geq 0.$

*Step 1. Express the problem in the standard form.*

Introducing slack variables, the dual problem in the standard form becomes

Max. $W = 4y_1 + 6y_2 + 20y_3 + 18y_4 + 0s_1 + 0s_2$,
subject to $y_1 + 0y_2 + y_3 + 2y_4 + s_1 + 0s_2 = 0.7,$
$0y_1 + y_2 + 2y_3 + y_4 + 0s_1 + s_2 = 0.5, y_1, y_2, y_3, y_4 \geq 0.$

*Step 2. Find an initial basic feasible solution.*

Setting non-basic variables $y_1, y_2, y_3, y_4$ each equal to zero, the basic solution is

$y_1 = y_2 = y_3 = y_4 = 0$ (non-basic) ; $s_1 = 0.7, s_2 = 0.5$ (basic)

Since the basic variables $s_1, s_2 > 0$, the initial basic solution is feasible and non-degenerate.

Initial simplex table is

| | $c_j$ | 4 | 6 | 20 | 18 | 0 | 0 | | |
|---|---|---|---|---|---|---|---|---|---|
| $c_B$ | *Basis* | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $s_1$ | $s_2$ | $b$ | θ |
| 0 | $s_1$ | 1 | 0 | 1 | 2 | 1 | 0 | 0.7 | 0.7/1 |
| 0 | $s_2$ | 0 | 1 | (2) | 1 | 0 | 1 | 0.5 | 0.5/2← |
| | $Z_j$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| | $C_j$ | 4 | 6 | 20 | 18 | 0 | 0 | | |
| | | | | ↑ | | | | | |

As $C_j$ is positive in some columns, the initial basic solution is not optimal.

*Step 3. Iterate towards an optimal solution.*

(*i*) Introduce $y_3$ and drop $s_2$. Then the new simplex table is

| | $c_j$ | 4 | 6 | 20 | 18 | 0 | 0 | | |
|---|---|---|---|---|---|---|---|---|---|
| $c_B$ | *Basis* | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $s_1$ | $s_2$ | $b$ | θ |
| 0 | $s_1$ | 1 | – 1/2 | 0 | (3/2) | 1 | – 1/2 | 9/20 | 3/10← |
| 20 | $y_3$ | 0 | 1/2 | 1 | $\frac{1}{2}$ | 0 | 1/2 | 1/4 | 1/2 |
| | $Z_j$ | 0 | 10 | 20 | 10 | 0 | 10 | 5 | |
| | $C_j$ | 4 | – 4 | 0 | 8 | 0 | – 10 | | |
| | | | | | ↑ | | | | |

As $C_j$, is positive under some of the columns, this solution is not optimal.

(*ii*) Introduce $y_4$ and drop $s_1$. Then the revised simplex table is

| | $c_j$ | 4 | 6 | 20 | 18 | 0 | 0 | |
|---|---|---|---|---|---|---|---|---|
| $c_B$ | *Basis* | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $s_1$ | $s_2$ | $b$ |
| 18 | $y_4$ | 2/3 | – 1/3 | 0 | 1 | 2/3 | – 1/3 | 3/10 |
| 20 | $y_3$ | – 1/3 | 2/3 | 1 | 0 | – 1/3 | 2/3 | 1/10 |
| | $Z_j$ | 16/3 | 22/3 | 20 | 18 | 16/3 | 22/3 | 74/10 |
| | $C_j$ | – 4/3 | – 4/3 | 0 | 0 | – 16/13 | – 22/3 | |

As all $C_j \le 0$, this table gives the optimal solution.

Thus the optimal basic feasible solution is $y_1 = 0, y_2 = 0, y_3 = 20, y_4 = 18$, max. $W = 7.4$

*Step 4. Derive optimal solution to the primal.*

We note that the primal variables $x_1, x_2$ correspond to the slack starting dual variables $s_1, s_2$ respectively. In the final simplex table of the dual problem, $C_j$ values corresponding to $s_1$ and $s_2$ are $-16/3$ and $-22/3$ respectively.

Thus, by rule I, we conclude that opt. $x_1 = 16/3$ and opt. $x_2 = 22/3$.

Hence an optimal basic feasible solution to the given primal is

$$x_1 = 16/3, x_2 = 22/3; \text{ min. } Z = 7.4.$$

**Obs.** To check the validity of the duality theorem, the student is advised to solve the given L.P.P. directly by simplex method and see that min. $Z$ = max. $W = 7.4$.

### PROBLEMS 34.7

Using duality solve the following problems (1 – 4) :

1. Minimize $Z = 2x_1 + 9x_2 + x_3$,
   subject to $x_1 + 4x_2 + 2x_3 \ge 5, 3x_1 + x_2 + 2x_3 \ge 4$ and $x_1, x_2 \ge 0$. *(J.N.T.U., 2001)*
2. Maximize $Z = 2x_1 + x_2$,
   subject to $x_1 + 2x_2 \le 10, x_1 + x_2 \le 6, x_1 - x_2 \le 2, x_1 - 2x_2 \le 1, x_1, x_2 \ge 0$. *(Andhra M. Tech., 2006)*
3. Maximize $Z = 3x_1 + 2x_2$,
   subject to $x_1 + x_2 \ge 1, x_1 + x_2 \le 7, x_1 + 2x_2 \le 10, x_2 \le 3, x_1, x_2 \ge 0$.
4. Maximize $Z = 3x_1 + 2x_2 + 5x_3$
   subject to $x_1 + 2x_2 + x_3 \le 430, 3x_1 + 2x_3 \le 460, x_1 + 4x_2 \le 420, x_1, x_2, x_3 \ge 0$.

## 34.13 (1) DUAL SIMPLEX METHOD

In § 34.9, we have seen that a set of basic variables giving a feasible solution can be found by introducing artificial variables and using $M$-method or Two phase method. Using the primal-dual relationships for a problem, we have another method (known as *Dual simplex method*) for finding an initial feasible solution. Whereas the regular simplex method starts with a basic feasible (but non-optimal) solution and works towards optimality, the dual simplex method starts with a basic unfeasible (but optimal) solution and works towards feasibility. The dual simplex method is quite similar to the regular simplex method, the only difference lies in the criterion used for selecting the incoming and outgoing variables. In the dual simplex method, we first determine the outgoing variable and then the incoming variable while in the case of regular simplex method reverse is done.

**(2) Working procedure for dual simplex method :**

*Step 1.* (*i*) *Convert the problem to maximization form*, if it is not so.

(*ii*) *Convert* ($\ge$) *type constraints, if any to* ($\le$) *type* by multiplying such constraints by –1.

(*iii*) *Express the problem in standard form* by introducing slack variables.

*Step 2. Find the initial basic solution and express this information in the form of dual simplex table.*

*Step 3. Test the nature of* $C_j = c_j - Z_j$ :

(*a*) If all $C_j \le 0$ and all $b_i \ge 0$, then optimal basic feasible solution has been attained.

(*b*) If all $C_j \le 0$ and at least one $b_i < 0$, then go to step 4.

(*c*) If any $C_j \ge 0$, the method fails.

*Step 4. Mark the outgoing variable.* Select the row that contains the most negative $b_i$. This will be the key row and the corresponding basic variable is the outgoing variable.

*Step 5. Test the nature of key row elements :*

(*a*) If all these elements are $\ge 0$, the problem does not have a feasible solution.

(*b*) If at least one element < 0, find the ratios of the corresponding elements of $C_j$-row to these elements. Choose the smallest of these ratios. The corresponding column is the key column and the associated variable is the *incoming variable.*

*Step 6. Iterate towards optimal feasible solution.* Make the key element unity. Perform row operations as in the regular simplex method and repeat iterations until either an optimal feasible solution is attained or there is an indication of non-existence of a feasible solution.

**Example 34.26.** *Using dual simplex method :*
*maximize* $-3x_1 - 2x_2$
*subject to* $x_1 + x_2 \geq 1, x_1 + x_2 \leq 7, x_1 + 2x_2 \geq 10, x_2 \geq 3, x_1 \geq 0, x_2 \geq 0.$ *(Mumbai, 2004)*

**Solution.** Consists of the following steps :

*Step 1. (i) Convert the first and third constraints into (≤) type. These constraints become*

$$-x_1 - x_2 \leq -1, -x_1 - 2x_2 \leq -10.$$

*(ii) Express the problem in standard form*

Introducing slack variables $s_1, s_2, s_3, s_4$ the given problem takes the form

Max. $Z = -3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

subject to $-x_1 - x_2 + s_1 = -1, x_1 + x_2 + s_2 = 7, -x_1 - 2x_2 + s_3 = -10,$
$x_2 + s_4 = 3, x_1, x_2, s_1, s_2, s_3, s_4 \geq 0.$

*Step 2. Find the initial basic solution*

Setting the decision variables $x_1, x_2$ each equal to zero, we get the basic solution

$$x_1 = x_2 = 0, s_1 = -1, s_2 = 7, s_3 = -10, s_4 = 3 \text{ and } Z = 0.$$

∴ Initial solution is given by the table below :

| | $c_j$ | $-3$ | $-2$ | $0$ | $0$ | $0$ | $0$ | |
|---|---|---|---|---|---|---|---|---|
| $c_B$ | *Basis* | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $b$ |
| 0 | $s_1$ | $-1$ | $-1$ | 1 | 0 | 0 | 0 | $-1$ |
| 0 | $s_2$ | 1 | 1 | 0 | 1 | 0 | 0 | 7 |
| 0 | $s_3$ | $-1$ | $(-2)$ | 0 | 0 | 1 | 0 | $-10 \leftarrow$ |
| 0 | $s_4$ | 0 | 1 | 0 | 0 | 0 | 1 | 3 |
| | $Z_j = \Sigma c_B a_{ij}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | $C_j = c_j - Z_j$ | $-3$ | $-2$ | 0 | 0 | 0 | 0 | |
| | | | $\uparrow$ | | | | | |

*Step 3. Test nature of $C_j$.*

Since all $C_j$ values are $\leq 0$ and $b_1 = -1$, $b_3 = -10$, the initial solution is optimal but infeasible. We therefore, proceed further.

*Step 4. Mark the outgoing variable.*

Since $b_3$ is negative and numerically largest, the third row is the key row and $s_3$ is the outgoing variable.

*Step 5. Calculate ratios of elements in $C_j$-row to the corresponding negative elements of the key row.*

These ratios are $-3/-1 = 3, -2/-2 = 1$ (neglecting ratios corresponding to +ve or zero elements of key row). Since the smaller ratio is 1, therefore, $x_2$-column is the key column and $(-2)$ is the key element.

*Step 6. Iterate towards optimal feasible solution.*

*(i)* Drop $s_3$ and introduce $x_2$ alongwith its associated value $-2$ under $c_B$ column. Convert the key element to unity and make all other elements of the key column zero. Then the second solution is given by the table below :

| | $c_j$ | $-3$ | $-2$ | $0$ | $0$ | $0$ | $0$ | |
|---|---|---|---|---|---|---|---|---|
| $c_B$ | *Basis* | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $b$ |
| 0 | $s_1$ | $-\frac{1}{2}$ | 0 | 1 | 0 | $-\frac{1}{2}$ | 0 | 4 |
| 0 | $s_2$ | $\frac{1}{2}$ | 0 | 0 | 1 | $\frac{1}{2}$ | 0 | 2 |
| $-2$ | $x_2$ | $\frac{1}{2}$ | 1 | 0 | 0 | $-\frac{1}{2}$ | 0 | 5 |
| 0 | $s_4$ | $\left(-\frac{1}{2}\right)$ | 0 | 0 | 0 | $\frac{1}{2}$ | 1 | $-2 \leftarrow$ |
| | $Z_j = \Sigma c_B a_{ij}$ | $-1$ | $-2$ | 0 | 0 | 1 | 0 | $-10$ |
| | $C_j = c_j - Z_j$ | $-2$ | 0 | 0 | 0 | $-1$ | 0 | |
| | | $\uparrow$ | | | | | | |

Since all $C_j$ values are $\leq 0$ and $b_4 = -2$, this solution is optimal but infeasible. We therefore proceed further.

*(ii) Mark the outgoing variable.*

Since $b_4$ is negative, the fourth row is the key row and $s_4$ is the outgoing variable.

*(iii) Calculate ratios of elements in $C_j$-row to the corresponding negative elements of the key row.*

This ratios is $-2/-\frac{1}{2} = 4$ (neglecting other ratios corresponding to + ve or 0 elements of key row).

$\therefore$ $x_1$-column is the key column and $\left(-\frac{1}{2}\right)$ is the key element.

*(iv) Drop $s_4$ and introduce $x_1$* with its associated value $-3$ under the $c_B$ column. Convert the key element to unity and make all other elements of the key column zero. Then the third solution is given by the table below :

| | $c_j$ | $-3$ | $-2$ | 0 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|---|
| $c_B$ | *Basis* | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $b$ |
| 0 | $s_1$ | 0 | 0 | 1 | 0 | $-1$ | $-1$ | 6 |
| 0 | $s_2$ | 0 | 0 | 0 | 1 | 1 | 1 | 0 |
| $-2$ | $x_2$ | 0 | 1 | 0 | 0 | 0 | 1 | 3 |
| $-3$ | $x_1$ | 1 | 0 | 0 | 0 | $-10$ | $-2$ | 4 |
| $Z_j$ | | $-3$ | $-2$ | 0 | 0 | 3 | 4 | $-18$ |
| $C_j$ | | 0 | 0 | 0 | 0 | $-3$ | $-4$ | |

Since all $C_j$ values are $\leq 0$ and all $b$'s are $\geq 0$, therefore this solution is optimal and feasible. Thus the *optimal solution* is $x_1 = 4$, $x_2 = 3$ and $Z_{max} = -18$.

**Example 34.27.** *Using dual simplex method, solve the following problem :*

*Minimize* $Z = 2x_1 + 2x_2 + 4x_3$

*subject to* $2x_1 + 3x_2 + 5x_3 \geq 2$, $3x_1 + x_2 + 7x_3 \leq 3$, $x_1 + 4x_2 + 6x_3 \leq 5$, $x_1, x_2, x_3 \geq 0$.

*(Kurukshetra, 2009 ; Kerala, 2005)*

**Solution.** Consists of the following steps :

*Step 1. (i) Convert the given problem to maximization form* by writing

Maximize $Z' = -2x_1 - 2x_2 - 4x_3$.

*(ii) Convert the first constraint into (≤) type.* Thus it is equivalent to

$$-2x_1 - 3x_2 - 4x_3 \leq -2$$

*(iii) Express the problem in standard form.*

Introducing slack variables, $s_1, s_2, s_3$, the given problem becomes

Max. $Z' = -2x_1 - 2x_2 - 4x_3 + 0s_1 + 0s_2 + 0s_3$

subject to
$$-2x_1 - 3x_2 - 5x_3 + s_1 + 0s_2 + 0s_3 = -2,$$
$$3x_1 + x_2 + 7x_3 + 0s_1 + s_2 + 0s_3 = 3,$$
$$x_1 + 4x_2 + 6x_3 + 0s_1 + 0s_2 + s_3 = 5,$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.$$

*Step 2. Find the initial basic solution.*

Setting the decision variables $x_1, x_2, x_3$ each equal to zero, we get the basic solution

$$x_1 = x_2 = x_3 = 0, s_1 = -2, s_2 = 3, s_3 = 5 \text{ and } Z' = 0.$$

$\therefore$ Initial solution is given by the table below :

| | $c_j$ | $-2$ | $-2$ | $-4$ | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|---|
| $c_B$ | *Basis* | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | $b$ |
| 0 | $s_1$ | $-2$ | $(-3)$ | $-5$ | 1 | 0 | 0 | $-2 \leftarrow$ |
| 0 | $s_2$ | 3 | 1 | 7 | 0 | 1 | 0 | 3 |
| 0 | $s_3$ | 1 | 4 | 6 | 0 | 0 | 1 | 5 |
| | $Z_j$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | $C_j$ | $-2$ | $-2$ | $-4$ | 0 | 0 | 0 | |
| | | | $\uparrow$ | | | | | |

*Step 3. Test nature of $C_j$.*

Since all $C_j$ values are $\le 0$ and $b_1 = -2$, the initial solution is optimal but infeasible.

*Step 4. Mark the outgoing variable.*

Since $b_1 < 0$, the first row is the key row and $s_1$ is the outgoing variable.

*Step 5. Calculate the ratio of elements of $C_j$-row to the corresponding negative elements of the key row.*

These ratios are $-2/-2 = 1, -2/-3 = 0.67, -4/-5 = 0.8$.

Since 0.67 is the smallest ratio, $x_2$-column is the key column and $(-3)$ is the key element.

*Step 6. Iterate towards optimal feasible solution.*

Drop $s_1$ and introduce $x_2$ with its associated value $-2$ under $c_B$ column. Then the revised dual simplex table is

| | $c_j$ | $-2$ | $-2$ | $-4$ | $0$ | $0$ | $0$ | |
|---|---|---|---|---|---|---|---|---|
| $c_B$ | *Basis* | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | $b$ |
| $-2$ | $x_2$ | $2/3$ | $1$ | $5/3$ | $-1/3$ | $0$ | $0$ | $2/3$ |
| $0$ | $s_2$ | $7/3$ | $0$ | $16/3$ | $1/3$ | $1$ | $0$ | $7/3$ |
| $0$ | $s_3$ | $-5/3$ | $0$ | $-2/3$ | $4/3$ | $0$ | $1$ | $7/3$ |
| | $Z_j$ | $-4/3$ | $-2$ | $-10/3$ | $2/3$ | $0$ | $0$ | $-4/3$ |
| | $C_j$ | $-2/3$ | $0$ | $-2/3$ | $-2/3$ | $0$ | $0$ | |

Since all $C_j \le 0$ and all $b_i$'s are $> 0$, this solution is optimal and feasible. Thus the optimal solution is

$$x_1 = 0, x_2 = 2/3, x_3 = 0 \text{ and max. } Z' = -4/3 \text{ i.e., min. } Z = 4/3.$$

**PROBLEMS 34.8**

Using dual simplex method, solve the following problems :

1. Maximize $Z = -3x_1 - x_2$
   subject to $x_1 + x_2 \ge 1, 2x_1 + 3x_2 \ge 2; x_1, x_2 \ge 0.$
2. Minimize $Z = 2x_1 + x_2$,
   subject to $3x_1 + x_2 \ge 3, 4x_1 + 3x_2 \ge 6, x_1 + 2x_2 \le 3, x_1, x_2 \ge 0.$ *(Kurukshetra, 2007 S)*
3. Minimize $Z = x_1 + 2x_2 + 3x_3$,
   subject to $2x_1 - x_2 + x_3 \ge 4, x_1 + x_2 + 2x_3 \le 8, x_2 - x_3 \ge 2; x_1, x_2, x_3 \ge 0.$
4. Minimize $Z = x_1 + 2x_2 + x_3 + 4x_4$
   subject to $2x_1 + 4x_2 + 5x_3 + x_4 \ge 10, 3x_1 - x_2 + 7x_3 - 2x_4 \ge 2$
   $5x_1 + 2x_2 + x_3 + 6x_4 \ge 15, x_1, x_2, x_3, x_4 \ge 0.$

## 34.14 (1) TRANSPORTATION PROBLEM

This is a special class of linear programming problems in which the objective is to transport a single commodity from various origins to different destinations at a minimum cost.

**(2) Formulation of a transportation problem.** *There are m plant locations (origins) and n distribution centres (destinations). The production capacity of the ith plant is $a_i$ and the number of units required at the jth destination is $b_j$. The transportation cost of one unit from the ith plant to the jth destination is $c_{ij}$. Our objective is to determine the number of units to be transported from the ith plant to jth destination so that the total transportation cost is minimum.*

Let $x_{ij}$ be the number of units shipped from *i*th plant to *j*th destination, then *the general transportation problem is :*

$$\text{Minimize } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

subject to the constraints

$$x_{i1} + x_{i2} + \dots + x_{in} = a_i, \text{ for } i\text{th origin } (i = 1, 2, \dots m)$$
$$x_{1j} + x_{2j} + \dots + x_{mj} = b_j, \text{ for } j\text{th destination } (j = 1, 2, \dots n)$$
$$x_{ij} \geq 0.$$

**Def. 1.** The two sets of constraints will be consistent if

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

which is the condition for a transportation problem to have a *feasible solution*. Problems satisfying this condition are called *balanced transportation problems*.

**2.** A feasible solution to a transportation problem is said to be a *basic feasible solution* if it contains at the most $(m + n - 1)$ strictly positive allocations, otherwise the solution will *degenerate*. If the total number of positive (non-zero) allocations is exactly $(m + n - 1)$, then the basic feasible solution is said to be *non-degenerate*.

**3.** A feasible solution which minimizes the transportation cost is called an *optimal solution*. This problem is explicitly represented in the following *transportation table :*

*Distribution centres (Destinations)*

*Plants (origins)*

| | 1 | 2 | | $j$ | | $n$ | *Supply* |
|---|---|---|---|---|---|---|---|
| 1 | $c_{11}$ | $c_{12}$ | | $c_{1j}$ | | $c_{1n}$ | $a_1$ |
| 2 | $c_{21}$ | $c_{22}$ | | $c_{2j}$ | | $c_{2n}$ | $a_2$ |
| | | | | | | | $a_i$ |
| $i$ | $c_{i1}$ | $c_{i2}$ | | $c_{ij}$ | | $c_{in}$ | |
| | | | | | | | $a_m$ |
| $m$ | $c_{m1}$ | $c_{m2}$ | | $c_{mj}$ | | $c_{mn}$ | |
| *Demand* | $b_1$ | $b_2$ | | $b_j$ | | $b_n$ | $\Sigma a_i = \Sigma b_j$ |

The $mn$ squares are called *cells*. The per unit cost $c_{ij}$ of transporting from the $i$th origin to the $j$th destination is displayed in the *lower right side of the (i, j)th cell*. Any feasible solution is shown in the table by entering the value of $x_{ij}$ in the *small square at the upper left side of the (i, j)th cell*. The various $a$'s and $b$'s are called *rim requirements*. The feasibility of a solution can be verified by summing the values of $x_{ij}$ along the rows and down the columns.

**Obs. 1.** The special features of a transportation problem are that
(*i*) the coefficients of all $x_{ij}$ in the constraints are unity, and
(*ii*) the total supply $\Sigma a_i$ = total demand $\Sigma b_j$.

**Obs. 2.** The objective function and the constraints being all linear, the problem can be solved be simplex method. But the number of variables being large, there will be too many calculations. However, the coefficients of all $x_{ij}$ in the constraints being unity, we can look for some technique which would be simpler than the simplex method.

## 34.15 WORKING PROCEDURE FOR TRANSPORTATION PROBLEMS

*Step 1. Construct transportation table.* Express the supply from the origins $a_i$, demand at destinations $b_j$ and the unit shipping cost $c_{ij}$ in the form of a matrix, know as transportation table. If the supply and demand are equal, the problem is *balanced*.

*Step 2. Find the initial basic feasible solution.* We find an initial allocation which satisfies the demand at each project site without violating the capacities of the plants (origins) and also meeting the non-negativity restrictions. There are several methods for initial allocations *e.g.*, North-West corner rule, Row minima method, Least cost method, Vogel's approximation method. *The Vogel's approximation method (VAM) takes into account not only the least cost $c_{ij}$ but also the costs that just exceed the least cost $c_{ij}$ and therefore yields a better initial solution than obtained from other methods. As such we shall confine ourselves to VAM only which consists of the following steps :*

(*i*) Display the difference between the least and the next to least costs in each row, by enclosing them in brackets to the right of the row. Similarly display the differences for each column within brackets below that column.

(*ii*) Identify the row or column with the largest difference among all the rows and columns and allocate as much as possible under the rim requirements, to the lowest cost cell in that row or column. In case of a tie allocate to the cell associated with the lower cost.

If the greatest difference corresponds to $i$th row and $c_{ij}$ is the lowest cost in the $i$th row, allocate as much as possible *i.e.*, min $(a_i, b_j)$ in the $(i, j)$th cell and cross off the $i$th row or the $j$th column.

(*iii*) Recalculate the row and column differences for the reduced table and go to the previous step.

(*iv*) Repeat the procedure till all the rim requirements are satisfied. Note the solution in the upper left corner small squares of the basic cells.

*Step 3. Apply optimality check*

In the above solution, the number of allocations must be '$m + n - 1$', otherwise the basic solution degenerates.

*Now to test for optimality, we apply the modified distribution (MODI) method* and examine each unoccupied cell to determine whether making an allocation in it reduces the total transportation cost and then repeat this procedure until lowest possible transportation cost is obtained. This method consists of the following steps :

(*i*) Note the numbers $u_i$ along the left and $v_j$ along the top of the cost matrix such that their sums equal to the original costs of occupied cells *i.e.*, solve the equations $[u_i + v_j = c_{ij}]$ starting initially with some $u_i = 0$.

(*ii*) *Compute the net evaluations* $w_{ij} = u_i + v_j - c_{ij}$ for all the empty cells and enter them in upper right hand corners of the corresponding cells.

(*iii*) *Examine the sign of each* $w_{ij}$. If all $w_{ij} \le 0$, then the current basic feasible solution is optimal. If even one $w_{ij} > 0$, this solution is not optimal and we proceed further.

*Step 4. Iterate towards optimal solution*

(*i*) Choose the unoccupied cell with the largest $w_{ij}$ and mark $\theta$ in it.

(*ii*) Draw a closed path consisting of horizontal and vertical lines beginning and ending at $\theta$-cell and having its other corners at the allocated cells.

(*iii*) Add and subtract $\theta$ alternately to and from the transition cells of the loop subject to rim requirements. Assign a maximum value to $\theta$ so that one basic variable becomes zero and the other basic variables remain non-negative. Now the basic cell whose allocation has been reduced to zero leaves the basis.

*Step 5. Return to step 3 and repeat the process until an optimal basic feasible solution is obtained.*

**Example 34.28.** *Solve the following transportation problem :*

| | | *Destination* A | B | C | D | |
|---|---|---|---|---|---|---|
| *Source* | *I* | 21 | 16 | 25 | 13 | 11 |
| | *II* | 17 | 18 | 14 | 23 | 13 *Availability* |
| | *III* | 33 | 27 | 18 | 41 | 19 |
| | *Requirement* | 6 | 10 | 12 | 15 | 43 |

**Solution.** *Consists of the following steps :*

*Step 1. Transportation table.* Here the total availability and the total requirement being the same *i.e.* 43, the problem is balanced.

*Step 2. Find the initial basic feasible solution.* Following VAM, the differences between the smallest and next to the smallest costs in each row and each column are computed and displayed within brackets against the respective rows and columns (Table 1). The largest of these differences is (10) which is associated with the fourth column.

Table 1

| | | | | |
|---|---|---|---|---|
| 21 | 16 | 25 | $\boxed{11}$ 13 | 11(3) |
| 17 | 18 | 14 | 23 | 13(3) |
| 32 | 27 | 18 | 41 | 19(9) |
| 6 (4) | 10 (2) | 12 (4) | 15 (10) | |

Table 2

| | | | + | |
|---|---|---|---|---|
| 17 | 18 | 14 | $\boxed{4}$ 23 | 13(3) |
| 32 | 27 | 18 | 41 | 19(9) |
| 6 (15) | 10 (9) | 12 (4) | 4 + (18) | |

Since $c_{14}$ (= 13) is the minimum cost, we allocate $x_{14}$ = min (11, 15) = 11. This exhausts the availability of first row and therefore we cross it.

Table 3

| | | | |
|---|---|---|---|
| $\boxed{6}$ 17 | 18 | 14 | 9(3) |
| 32 | 27 | 18 | 19(9) |
| 6 (15) | 10 (9) | 12 (4) | |

Table 4

| | | |
|---|---|---|
| $\boxed{3}$ 18 | 14 | 3(4) |
| 27 | 18 | 19(9) |
| 10 (9) | 12 (4) | |

Table 5

| | | |
|---|---|---|
| $\boxed{7}$ 27 | $\boxed{12}$ 18 | 19 |
| 7 | 12 + | |

The row and column differences are now computed for reduced table 2 and displayed within brackets. The largest of these is (18) which is against the fourth column. Since $c_{14}$ (= 23) is the minimum cost, we allocate $x_{14}$ = min (13, 4) = 4.

This exhausts the availability of fourth column which we cross off. Proceeding in this way, the subsequent reduced transportation tables and differences for the remaining rows and columns are shown in Tables 3, 4 and 5.

Finally the initial basic feasible solution is as shown in Table 6.

Table 6

| | | | |
|---|---|---|---|
| 21 | 16 | 25 | $\boxed{11}$ 13 |
| $\boxed{6}$ 17 | $\boxed{3}$ 18 | 14 | $\boxed{4}$ 23 |
| 32 | $\boxed{7}$ 27 | $\boxed{12}$ 18 | 41 |

Table 7

| $u_i \backslash v_j$ | 17 | 18 | 9 | 23 |
|---|---|---|---|---|
| – 10 | (–) 21 | (–) 16 | (–) 25 | $\boxed{11}$ 13 |
| 0 | $\boxed{6}$ 17 | $\boxed{3}$ 18 | (–) 14 | $\boxed{4}$ 23 |
| 9 | (–) 32 | $\boxed{7}$ 27 | $\boxed{12}$ 18 | (–) 41 |

*Step 3. Apply optimality check*

As the number of allocations = $m + n - 1$ (*i.e.*, 6), we can apply MODI method.

(*i*) We have $u_2 + v_1 = 17$, $u_2 + v_2 = 18$, $u_3 + v_2 = 27$

$$u_3 + v_3 = 18,\ u_1 + v_4 = 13,\ u_2 + v_4 = 23$$

Let $u_2 = 0$, then $v_1 = 17$, $v_2 = 18$, $u_3 = 9$, $v_3 = 9$, $v_4 = 23$, $u_1 = -10$.

(*ii*) Net evaluations $w_{ij} = (u_i + v_j) - c_{ij}$ for all empty cells are

$$w_{11} = -14,\ w_{12} = -8,\ w_{13} = -26,\ w_{23} = -5,\ w_{31} = -6,\ w_{34} = -9.$$

(*iii*) Since all the net evaluations are negative, the current solution is optimal. Hence the optimal allocation is given by

$$x_{14} = 11,\ x_{21} = 6,\ x_{22} = 3,\ x_{24} = 4,\ x_{32} = 7 \text{ and } x_{33} = 12.$$

∴ The optimal (minimum) transportation cost

$$= 11 \times 13 + 6 \times 17 + 3 \times 18 + 4 \times 23 + 7 \times 27 + 12 \times 18 = ₹\ 796.$$

**Example 34.29.** *A company has three cement factories located in cities 1, 2, 3 which supply cement to four projects located in towns 1, 2, 3, 4. Each plant can supply 6, 1, 10 truck loads of cement daily respectively and the daily cement requirements of the projects are respectively 7, 5, 3, 2 truck loads. The transportation costs per truck load of cement (in hundreds of rupees) from each plant to each project site are as follows :*

| | | *Project sites* | | | |
|---|---|---|---|---|---|
| | | *1* | *2* | *3* | *4* |
| *Factories* | *1* | *2* | *3* | *11* | *7* |
| | *2* | *1* | *0* | *6* | *1* |
| | *3* | *5* | *8* | *15* | *9* |

*Determine the optimal distribution for the company so as to minimize the total transportation cost.*

**Solution.** Consists of the following steps :

*Step 1. Construct transportation table.* Express the supply from the factories, demands at sites and the unit shipping cost in the form of the following transportation table (Table 1). Here the supply being equal to the demand, the problem is balanced.

Table 1

| | | Project sites | | | | |
|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 | Supply |
| Factories | 1 | 2 | 3 | 11 | 7 | 6 |
| | 2 | 1 | 0 | 6 | 1 | 1 |
| | 3 | 5 | 8 | 15 | 9 | 10 |
| Demand | | 7 | 5 | 3 | 2 | 17 |

*Step 2. Find the initial basic feasible solution.*

Using VAM, the initial basic feasible solution is as shown in Table 2. The transportation cost according to this route is given by

$$Z = ₹\,(1 \times 2 + 5 \times 3 + 1 \times 1 + 6 \times 5 + 3 \times 15 + 1 \times 9) \text{ times } 100 = ₹\,10{,}200.$$

*Step 3. Apply optimality check.*

As the numbers of allocations = $(m + n - 1)$ *i.e.*, 6, we can apply MODI method.

We now compute the net evaluations $w_{ij} = (u_i + v_j) - c_{ij}$ which are exhibited in Table 3. Since the net evaluations in two cells are positive, a better solution can be found.

Table 2

| | | | | |
|---|---|---|---|---|
| [1] 2 | [5] 3 | 11 | 7 | 6 |
| 1 | 0 | 6 | [1] 1 | 1 |
| [6] 5 | 8 | [3] 15 | [1] 9 | 10 |
| 7 | 5 | 3 | 2 | |

Table 3

| $u_i$ \ $v_j$ | 2 | 3 | 12 | 6 |
|---|---|---|---|---|
| 0 | [1] 2 | [5] 3 | (+) 11 | (–) 7 |
| – 5 | (–) 1 | (–) 0 | (+) 6 | [1] 1 |
| 3 | [6] 5 | (–) 8 | [3] 15 | [1] 9 |

*Step 4. Iterate towards optimal solution.*

*First iteration :*

(*a*) *Next basic feasible solution.*

(*i*) Choose the unoccupied cell with the maximum $w_{ij}$. In case of a tie, select the one with lower original cost. In Table 3, cells (1, 3) and (2, 3) each have $w_{ij} = 1$ and out of these all (2, 3) has lower original cost 6, therefore we take this as the next basic cell and note θ in it.

(*ii*) Draw a closed path beginning and ending at θ-cell. Add and subtract θ, alternately to and from the transition cells of the loop subject the rim requirements. Assign a maximum value to θ so that one basic variable becomes zero and the other basic variables remain ≥ 0. Now the basic cell whose allocation has been reduced to zero leaves the basis. This gives the *second basic feasible solution* (Table 5).

Table 4

| | | | |
|---|---|---|---|
| $\boxed{1}$ 2 | $\boxed{5}$ 3 | 11 | 7 |
| 1 | 0 | $\boxed{\theta}$ 6 | $\boxed{1}$ $-\theta$ 1 |
| $\boxed{6}$ 5 | 8 | $\boxed{3}$ $-\theta$ 15 | $\boxed{1}$ $+\theta$ 9 |

Table 5

| | | | |
|---|---|---|---|
| $\boxed{1}$ 2 | $\boxed{5}$ 3 | 11 | 7 |
| 1 | 0 | $\boxed{\theta = 1}$ 6 | $\boxed{1-1}$ 1 |
| $\boxed{6}$ 5 | 8 | $\boxed{3-1}$ 15 | $\boxed{1+1}$ 9 |

∴ Total transportation cost of this revised solution.

$= ₹\,(1 \times 2 + 5 \times 3 + 1 \times 6 + 6 \times 5 + 2 \times 15 + 2 \times 9)$ *times* $100 = ₹\,10{,}100.$

(*b*) *Optimality check.* As the number of allocations in table 5 = $m + n - 1$ (*i.e.*, 6), we can apply MODI method. We compute the net evaluations which are shown in Table 6. Since the cell (1, 3) has a positive value, the second basic feasible solution is not optimal.

Table 6

| $u_i \backslash v_j$ | 2 | 3 | 12 | 6 |
|---|---|---|---|---|
| 0 | $\boxed{1}$ 2 | $\boxed{5}$ 3 | (+) 11 | (–) 7 |
| – 6 | (–) 1 | (–) 0 | $\boxed{1}$ 6 | (–) 1 |
| 3 | $\boxed{6}$ 5 | (–) 8 | $\boxed{2}$ 15 | $\boxed{2}$ 9 |

Table 7

| | | | |
|---|---|---|---|
| $\boxed{1-1}$ 2 | 5 3 | $\boxed{\theta = 1}$ 11 | 7 |
| 1 | 0 | $\boxed{1}$ 6 | 1 |
| $\boxed{6+1}$ 5 | 8 | $\boxed{2-1}$ 15 | $\boxed{2}$ 9 |

*Second iteration :*

(*a*) *Next basic feasible solution.* In the second basic feasible solution introduce the cell (1, 3) taking $\theta = 1$ and drop the cell (1, 1) giving Table 7. Thus we obtain the third basic feasible solution (Table 8).

Table 8

| | | | |
|---|---|---|---|
| 2 | $\boxed{5}$ 3 | $\boxed{1}$ 11 | 7 |
| 1 | 0 | $\boxed{1}$ 6 | 1 |
| $\boxed{7}$ 5 | 8 | $\boxed{1}$ 15 | $\boxed{2}$ 9 |

Table 9

| $u_i \backslash v_j$ | 1 | 3 | 11 | 5 |
|---|---|---|---|---|
| 0 | (–) 2 | $\boxed{5}$ 3 | $\boxed{1}$ 11 | (–) 7 |
| – 5 | (–) 1 | (–) 0 | $\boxed{1}$ 6 | (–) 1 |
| 4 | $\boxed{7}$ 5 | (–) 8 | $\boxed{1}$ 15 | $\boxed{2}$ 9 |

(*b*) *Optimality Check.* As the number of allocations in Table 8 = $m + n - 1$ (*i.e.*, 6), we can apply MODI method.

We compute the net evaluations which are shown in Table 9. Since all the net evaluations are $\le 0$, this basic feasible solution is optimal.

Thus the optimal transportation policy is as shown in Table 9 and the optimal transportation cost

$= ₹\,[5 \times 3 + 1 \times 11 + 1 \times 6 + 7 \times 5 + 1 \times 15 + 2 \times 9]$ *times* $100 = ₹\,10{,}000.$

## 34.16 DEGENERACY IN TRANSPORTATION PROBLEMS

*When the number of basic cells in a non-transportation table, is less than '$m + n - 1$' the basic solution degenerates.* To remove the degeneracy, we assign a small positive value $\varepsilon$ to as many zero-valued variables as may be necessary to complete '$m + n - 1$' basic variables. The cells containing $\varepsilon$ are then treated like other basic cells and the problem is solved in the usual way. The $\varepsilon$'s are kept till the optimum solution is attained. Then we let each $\varepsilon \to 0$.

**Example 34.30.** *Solve the following transportation problem :*

| | To | | | | | | |
|---|---|---|---|---|---|---|---|
| From | 9 | 12 | 9 | 6 | 9 | 10 | 5 |
| | 7 | 3 | 7 | 7 | 5 | 5 | 6 |
| | 6 | 5 | 9 | 11 | 3 | 11 | 2 |
| | 6 | 8 | 11 | 2 | 2 | 10 | 9 |
| | 4 | 4 | 6 | 2 | 4 | 2 | 22 |

**Solution.** Consists of the following steps :

*Step 1. Transporation table.* The total supply and total demand being equal, the transportation problem is balanced.

*Step 2. Find the initial basic feasible solution.*

Using VAM, the initial basic feasible solution is as shown in Table 1.

*Step 3. Apply optimality check.* Since the number of basic cells is 8 which is less than $m + n - 1 = 9$, the basic solution degenerates. In order to complete the basis and thereby remove degeneracy, we require only one more positive basic variable. We select the variable $x_{23}$ and allocate a small positive quantity $\varepsilon$ to the cell (2, 3).

Table 1

| | | | | | | + |
|---|---|---|---|---|---|---|
| 9 | 12 | [5] 9 | 6 | 9 | 10 | 5 |
| 7 | [4] 3 | [$\varepsilon$] 7 | 7 | 5 | [2] 5 | $6 + \varepsilon = 6$ |
| [1] 6 | 5 | [1] 9 | 11 | 3 | 11 | 2 |
| [3] 6 | 8 | 11 | [2] 2 | [4] 2 | 10 | 9 |
| 4 | 4 | $6 + \varepsilon = 6$ | 2 | 4 | 2 + | |

We now compute the net evaluations $w_{ij} = (u_i + v_j) - c_{ij}$ which are exhibited in Table 2. Since all the net evaluations are $\leq 0$, the current solutions is optimal. Hence the optimal allocation is

$$x_{13} = 5, x_{22} = 4, x_{26} = 2, x_{31} = 1, x_{33} = 1, x_{41} = 3, x_{44} = 2 \text{ and } x_{45} = 4.$$

$\therefore$ The minimum (optimal) transportation cost

$$= 5 \times 9 + 4 \times 3 + \varepsilon \times 7 + 2 \times 5 + 1 \times 6 + 1 \times 9 + 3 \times 6 + 2 \times 2 + 4 \times 2$$
$$= 112 + 7\varepsilon = ₹\ 112 \text{ as } \varepsilon \to 0.$$

Table 2

| $u_i$ \ $v_j$ | 4 | 3 | 7 | 0 | 0 | 5 |
|---|---|---|---|---|---|---|
| 2 | (–) 9 | (–) 12 | [5] 9 | (–) 6 | (–) 9 | (–) 10 |
| 0 | (–) 7 | [4] 3 | [$\varepsilon$] 7 | (–) 7 | (–) 5 | [2] 5 |
| 2 | [1] 6 | (0) 5 | [1] 9 | (–) 11 | (–) 3 | (–) 11 |
| 2 | [3] 6 | (–) 8 | (–) 11 | [2] 2 | [4] 2 | (–) 10 |

## PROBLEMS 34.9

1. Obtain an initial basic feasible solution to the following transportation problem :

| From \ To | D | E | F | G | |
|---|---|---|---|---|---|
| A | 11 | 13 | 17 | 14 | 250 |
| B | 16 | 18 | 14 | 10 | 300 |
| C | 21 | 24 | 13 | 10 | 400 |
| | 200 | 225 | 275 | 250 | |

2. Solve the following transportation problem :

| Suppliers / Consumers | A | B | C | Available |
|---|---|---|---|---|
| I | 6 | 8 | 4 | 14 |
| II | 4 | 9 | 8 | 12 |
| III | 1 | 2 | 6 | 5 |
| Required | 6 | 10 | 15 | 31 |

3. Consider four bases of operations $B_i$ and three targets $T_j$. The tons of bombs per aircraft from any base that can be delivered to any target are given in the following table :

| $B_i$ \ $T_j$ | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 8 | 6 | 5 |
| 2 | 6 | 6 | 6 |
| 3 | 10 | 8 | 4 |
| 4 | 8 | 6 | 4 |

The daily sortie capability of each of the four bases is 150 sorties per day. The daily requirement in sorties over each target is 200. Find the allocation of sorties from each base to each target which maximizes the total tonnage over all the three targets.

4. A company has factories $F_1, F_2, F_3$ which supply warehouses at $W_1$, $W_2$ and $W_3$. Weekly factory capacities, weekly warehouse requirements and unit shipping costs (in rupees) are as follows :

| *Factories* | *Warehouses* | | | *Supply* |
|---|---|---|---|---|
| | $W_1$ | $W_2$ | $W_3$ | |
| $F_1$ | 16 | 20 | 12 | 200 |
| $F_2$ | 14 | 8 | 18 | 160 |
| $F_3$ | 26 | 24 | 16 | 90 |
| *Demand* | 180 | 120 | 150 | 450 |

Determine the optimal distribution for this company to minimize shipping costs.

5. A company is spending ₹ 1,000 on transportation of its units from plants to four distribution centres. The supply and demand of units, with unit cost of transportation are given below :

| *Plants* | *Distribution centres* | | | | *Availabilities* |
|---|---|---|---|---|---|
| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | |
| $P_1$ | 19 | 30 | 50 | 12 | 7 |
| $P_2$ | 70 | 30 | 40 | 60 | 10 |
| $P_3$ | 40 | 10 | 60 | 20 | 18 |
| *Requirements* | 5 | 8 | 7 | 15 | |

What can be the maximum saving by optimal scheduling.

6. A departmental store wishes to stock the following quantities of a popular product in three types of containers :

| | | | |
|---|---|---|---|
| *Container type* : | 1 | 2 | 3 |
| *Quantity* : | 170 | 200 | 180 |

Tenders are submitted by four dealers who undertake to supply not more than the quantities shown below :

| | | | | |
|---|---|---|---|---|
| *Dealer* : | 1 | 2 | 3 | 4 |
| *Quantity* : | 150 | 160 | 110 | 130 |

The store estimates that profit per unit will vary with the dealer as shown below :

| *Dealers* → <br> *Container type* ↓ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 8 | 9 | 6 | 3 |
| 2 | 6 | 11 | 5 | 10 |
| 3 | 3 | 8 | 7 | 9 |

Find the maximum profit of the store.

7. Obtain an optimum basic feasible solution to the following transportation problem :

| | To | | | |
|---|---|---|---|---|
| *From* | 7 | 3 | 4 | 2 |
| | 2 | 1 | 3 | 3 *Available* |
| | 3 | 4 | 6 | 5 |
| *Demand* | 4 | 1 | 5 | 10 |

8. A company has three plants at locations $A$, $B$ and $C$ which supply to warehouses located as $D$, $E$, $F$, $G$ and $H$. Monthly plant capacities are 800, 500 and 900 units respectively. Monthly warehouse requirements are 400, 400, 500, 400 and 800 units respectively. Unit transportation costs in rupees are given below :

| | | To | | | | |
|---|---|---|---|---|---|---|
| | | *D* | *E* | *F* | *G* | *H* |
| | *A* | 5 | 8 | 6 | 6 | 3 |
| *From* | *B* | 4 | 7 | 7 | 6 | 6 |
| | *C* | 8 | 4 | 6 | 6 | 3 |

Determine an optimum distribution for the company in order to minimize the total transportation cost.

## 34.17 (1) ASSIGNMENT PROBLEM

An assignment problem is a special type of transportation problem in which the objective is to assign a number of origins to an *equal* number of destinations at a minimum cost (or maximum profit).

**(2) Formulation of an assignment problem.** *There are n new machines $M_i$ ($i = 1, 2, \ldots n$) which are to be installed in a machine shop. There are n vacant spaces $S_j$ ($j = 1, 2, \ldots n$) available. The cost of installing the machine $M_i$ at space $S_j$ is $c_{ij}$ rupees. Let us formulate the problem of assigning machines to spaces so as to minimize the overall cost.*

Let $x_{ij}$ be the assignment of machine $M_i$ to space $S_j$ *i.e.*, let $x_{ij}$ be a variable such that

$$x_{ij} = \begin{cases} 1, \text{ if } i\text{th machine is installed at } j\text{th space} \\ 0, \text{ otherwise} \end{cases}$$

Since one machine can only be installed at each space, we have

$$x_{i1} + x_{i2} + \ldots + x_{in} = 1, \text{ for machine } M_i \ (i = 1, 2, \ldots n)$$
$$x_{1j} + x_{2j} + \ldots + x_{nj} = 1, \text{ for space } S_j \ (j = 1, 2, \ldots n)$$

Also the total installation cost is $\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$ .

*Thus the assignment problem can be stated as follows :*

Determine $x_{ij} \geq 0$ ($j = 1, 2, \ldots n$) so as to

$$\text{minimize} \qquad Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij}$$

subject to the constraints $\sum_{i=1}^{n} x_{ij} = 1, j = 1, 2, \ldots n$ and $\sum_{j=1}^{n} x_{ij} = 1, i = 1, 2, \ldots n$.

This problem is explicitly represented by the following $n \times n$ *cost matrix* :

| | | Spaces | | | | |
|---|---|---|---|---|---|---|
| | | $S_1$ | $S_2$ | $S_3$ | ... | $S_n$ |
| | $M_1$ | $c_{11}$ | $c_{12}$ | $c_{13}$ | ... | $c_{1n}$ |
| | $M_2$ | $c_{21}$ | $c_{22}$ | $c_{23}$ | ... | $c_{2n}$ |
| *Machines* | $M_3$ | $c_{31}$ | $c_{32}$ | $c_{33}$ | ... | $c_{3n}$ |
| | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ |
| | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ |
| | $M_n$ | $c_{n1}$ | $c_{n2}$ | $c_{n3}$ | | $c_{nn}$ |

**Obs.** This assignment problem constitutes $n$ ! possible ways of installing $n$ machines at $n$ spaces. If we enumerate all these $n$ ! alternatives and evaluate the cost of each one of them and select the one with the minimum cost, the problem would be solved. But this method would be very slow and time consuming, even for small value of $n$ and hence it is not at all suitable. However, a much more efficient method of solving such problems is available. This is the **Hungarian method** for solution of assignment problems which we describe below.

## 34.18 WORKING PROCEDURE TO SOLVE AN ASSIGNMENT PROBLEM

*Step 1. Reduce the matrix.* Subtract the smallest element of each row (of the given cost matrix) from all elements of that row. See if each row contains at least one zero. If not, subtract the smallest element of each column (not containing zero) from all the elements of that column. This gives the *reduced matrix.*

*Step 2. Assign the zeros*

(*a*) Examine rows (of the reduced matrix) successively until a row with exactly one unmarked zero is found. Make an assignment to this single zero by encircling it. Cross all other zeros in the column of this encircled zero, as these will not be considered for any future assignment. Continue in this way until all the rows have been examined.

(*b*) Now examine columns successively until a column with exactly one unmarked zero is found. Encircle this zero and make an assignment there. Then cross any other zero in its row. Continue in this way until all the columns have been examined.

In case, some rows or columns contain mor  than one unmarked zeros, encircle any unmarked zero arbitrarily and cross all other zeros in its row or column. Proceed in this way, till no zero is left unmarked.

*Step 3. Apply optimality check.*

Repeat step 2 (*a*) and (*b*) until one of the following occurs :

(*i*) If no row or no column is without assignment (encircled zero), then the current assignment is optimal.

(*ii*) If there is some row and/or column without an assignment, then the current assignment is not optimal and we go to next step.

*Step 4. Find minimum number of lines crossing all zeros.*

(*a*) Tick (✓) the rows which do not have assignments.

(*b*) Tick (✓) the columns (not already marked) which have zeros in the ticked row.

(*c*) Tick (✓) the rows (not already marked) which have assignments in ticked columns.

Repeat (*b*) and (*c*) until no more marking is required.

(*d*) Draw lines through all unticked rows and ticked columns. If the number of these lines is equal to the order of the matrix then it is an optimal solution otherwise not.

*Step 5. Iterate towards optimal solution.*

Select the smallest element and subtract it from all uncovered elements. Add this smallest element to every element lying at the intersection of two lines. The resulting matrix is the second basic feasible solution.

*Step 6.* Go to step 2 and repeat the procedure until the optimal solution is attained.

**Example 34.31.** *Four jobs are to be done on four different machines. The cost (in rupees) of producing ith job on the jth machine is given below :*

| | | Machines | | | |
|---|---|---|---|---|---|
| | | $M_1$ | $M_2$ | $M_3$ | $M_4$ |
| | $J_1$ | 15 | 11 | 13 | 15 |
| | $J_2$ | 17 | 12 | 12 | 13 |
| Jobs | $J_3$ | 14 | 15 | 10 | 14 |
| | $J_4$ | 16 | 13 | 11 | 17 |

*Assign the jobs to different machines so as to minimize the total cost.*

**Solution.** Consists of the following steps :

*Step 1. Reduce the matrix.* Subtract the smallest element 11 of row 1 from all its elements. Similarly subtract 12, 10 and 11 from rows 2, 3 and 4 respectively. The resulting matrix is as shown in Table 1. Columns 1 and 4 do not have any zero element. Subtract the smallest element 4 of Col. 1 from all its elements and element 1 from all elements of Col. 4. The *reduced matrix* is as given in Table 1.

Table 1

| | $M_1$ | $M_2$ | $M_3$ | $M_4$ |
|---|---|---|---|---|
| $J_1$ | 4 | 0 | 2 | 4 |
| $J_2$ | 5 | 0 | 0 | 1 |
| $J_3$ | 4 | 5 | 0 | 4 |
| $J_4$ | 5 | 2 | 0 | 6 |

Table 2

| | $M_1$ | $M_2$ | $M_3$ | $M_4$ |
|---|---|---|---|---|
| $J_1$ | 0 | 0 | 2 | 3 |
| $J_2$ | 1 | $\cancel{0}$ | $\cancel{0}$ | 0 |
| $J_3$ | 0 | 5 | $\cancel{0}$ | 3 |
| $J_4$ | 1 | 2 | 0 | 5 |

*Step 2. Assign the zeros.* Row 4 has a single unmarked zero in Col. 3. Encircle it and cross all other zeros in Col. 3. Row 3 has a single unmarked zero in Col. 1. Encircle it and cross the other zero in col. 1. Row 1 has a single unmarked zero in Col. 2. Encircle it and cross the other zero in Col. 2. Finally row 2 has a single unmarked zero in Col. 4. Encircle it (Table 2).

*Step 3. Apply optimality check.* Since we have one encircled zero in each row and in each column, this gives the optimal solution.

∴ The optimal assignment policy is

Job 1 to machine 2, Job 2 to machine 4, Job 3 to machine 1, Job 4 to machine 3,

and the minimum assignment cost = ₹ (11 + 13 + 14 + 11) = ₹ 49.

**Example 34.32.** *A marketing manager has 5 salesmen and 5 sales districts. Considering the capabilities of the salesmen and the nature of districts, the marketing manager estimates that sales per month (in hundred rupees) for each salesman in each district would be as follows :*

| | | Sales districts | | | | |
|---|---|---|---|---|---|---|
| | | A | B | C | D | E |
| | 1 | 32 | 38 | 40 | 28 | 40 |
| | 2 | 40 | 24 | 28 | 21 | 36 |
| Salesman | 3 | 41 | 27 | 33 | 30 | 37 |
| | 4 | 22 | 38 | 41 | 36 | 36 |
| | 5 | 29 | 33 | 40 | 35 | 39 |

*Find the assignment of salesmen to districts that will result in maximum sales.* (Madras, 2000)

**Solution.** Consists of the following steps :

*Step 1. Reduce the matrix.* Convert the given maximization problem into a minimization problem, by making all the profits negative, since max. $Z$ = min. $(-Z)$. Then subtract the smallest element of each row from the elements of that row. Now subtract the smallest element of each col. (not containing zero) from the elements of that column. This gives the *reduced matrix* (Table 1).

Table 1

| | | | | |
|---|---|---|---|---|
| 8 | 0 | 0 | 7 | 0 |
| 0 | 14 | 12 | 14 | 4 |
| 0 | 12 | 8 | 6 | 4 |
| 19 | 1 | 0 | 0 | 5 |
| 11 | 5 | 0 | 0 | 1 |

Table 2

| | | | | |
|---|---|---|---|---|
| 12 | 0 | 0 | 7 | 0 |
| 0 | 10 | 8 | 10 | 0 |
| 0 | 8 | 4 | 2 | 0 |
| 23 | 1 | 0 | 0 | 5 |
| 15 | 5 | 0 | 0 | 1 |

*Step 2. Assign the zeros.* Rows 2 and 3 have each a single unmarked zero in Col. 1. Encircle these. Columns 2 and 5 have each a single unmarked zero in row 1. Encircle these and cross the zero in row 1. Columns 3 and 4 have each unmarked zeros. Encircle the zeros in each of the rows 4 and 5 as shown in Table 1 and cross other zeros.

*Step 3. Apply optimality check.* As col. 4 is without assignment, this solution is not optimal. Therefore we go to next step.

*Step 4. Find minimum number of lines crossing all zeros.* Draw the least number of horizontal and vertical (dotted) lines which cover all the zeros. Since there are four dotted lines which are less than the order of the cost matrix (= 5), we got to step 5.

*Step 5. Iterate towards optimal solution.* Select the smallest element in the Table 1, not covered by the dotted lines. Such an element is 4 which lies at two different positions. Selecting the elements that lies at position (3, 5) arbitrarily, subtract it from all the uncovered elements of the cost matrix (Table 1) and add the same to the elements lying at the intersection of two dotted lines. Now draw more minimum number of dotted lines so as to cover the new zero. Here we draw such a line in Col. 5 (Table 2).

Table 3

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | | 0 | 0 | | |
| 2 | 0 | | | | 0 |
| 3 | 0 | | | | 0 |
| 4 | | | 0 | 0 | |
| 5 | | | 0 | 0 | |

Now, since the number of dotted lines is equal to the order for the cost matrix, the optimal solution is attained.

Finally, to determine this optimal assignment, we consider only the zero elements (Table 3) :

(*i*) Examine successively the rows with exactly one zero. There is no such row.

(*ii*) Examine successively the columns with exactly one zero. Col. 2 has one zero, encircle it and cross all zeros of row 1.

(*iii*) Encircle arbitrarily the zero in position (2, 1) and cross all zeros in row 2 and Col. 1. Then encircle the unmarked zero in row 3. Now encircle arbitrarily the zero in position (4, 3) and cross all zeros in row 4 and Col. 3. Finally encircle the remaining unmarked zero in row 5.

Now each row and each column has one encircled zero, therefore the optimal assignment policy is :

Salesman 1 to district $B$, 2 to $A$, 3 to $E$, 4 to $C$ and 5 to $D$.

Hence the maximum sales = ₹ (38 + 40 + 37 + 41 + 35) × 100 = ₹ 19,100.

## PROBLEMS 34.10

1. A firm plans to begin production of three new products on its three plants. The unit cost of producing $i$ at plant $j$ is as given below. Find the assignment that minimizes the total unit cost.

| | | Plant 1 | Plant 2 | Plant 3 |
|---|---|---|---|---|
| Product | 1 | 10 | 8 | 12 |
| | 2 | 18 | 6 | 14 |
| | 3 | 6 | 4 | 2 |

2. Solve the following assignment problem :

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| *A* | 10 | 12 | 19 | 11 |
| *B* | 5 | 10 | 7 | 8 |
| *C* | 12 | 14 | 13 | 11 |
| *D* | 8 | 15 | 11 | 9 |

3. A machine tool company decides to make four sub-assemblies through four contractors. Each contractor is to receive only one sub-assembly. The cost of each sub-assembly is determined by the bids submitted by each contractor and is shown in table below (in hundreds of rupees). Assign different assemblies to contractors so as to minimize the total cost.

| | | Contractor *A* | *B* | *C* | *D* |
|---|---|---|---|---|---|
| Sub-assembly | I | 15 | 13 | 14 | 17 |
| | II | 11 | 12 | 15 | 13 |
| | III | 18 | 12 | 10 | 11 |
| | IV | 15 | 17 | 14 | 16 |

4. Four professors are each capable of teaching any one of the four different courses. Class preparations time in hours for different topics varies from professor to professor and is given in the table below. Each professor is assigned only one course. Find the assignment policy schedule so as to minimize the total course preparation time for all courses.

| *Prof.* | *L.P.* | *Queuing Theory* | *Dynamic Programming* | *Regression analysis* |
|---|---|---|---|---|
| *A* | 2 | 10 | 9 | 7 |
| *B* | 15 | 4 | 14 | 8 |
| *C* | 13 | 14 | 16 | 11 |
| *D* | 3 | 15 | 13 | 8 |

5. Consider the problem of assigning four working labour units to four jobs. The assignment costs in thousands of rupees are given by the following matrix.

| *Labour unit* | *Job* | | | |
|---|---|---|---|---|
| ↓ | *I* | *II* | *III* | *IV* |
| $L_1$ | 42 | 35 | 28 | 21 |
| $L_2$ | 30 | 25 | 20 | 15 |
| $L_3$ | 30 | 25 | 20 | 15 |
| $L_4$ | 24 | 20 | 16 | 12 |

Find the optimal assignment.

6. A company has six jobs to be processed by six mechanics. The following table gives the return in rupees when the $i$th job is assigned to the $j$th mechanic. How should the jobs be assigned to the mechanics so as to maximize the over all return ?

| Mechanic ↓ | Job I | II | III | IV | V | VI |
|---|---|---|---|---|---|---|
| 1 | 9 | 22 | 58 | 11 | 19 | 27 |
| 2 | 43 | 78 | 72 | 50 | 63 | 48 |
| 3 | 41 | 28 | 91 | 37 | 45 | 33 |
| 4 | 74 | 42 | 27 | 49 | 39 | 32 |
| 5 | 36 | 11 | 57 | 22 | 25 | 18 |
| 6 | 13 | 56 | 53 | 31 | 17 | 28 |

## 34.19 OBJECTIVE TYPE OF QUESTIONS

### PROBLEMS 34.11

*Fill up the blanks in the following questions :*

1. Infeasibility in a linear programming problem means ......
2. The significance of the $(Z_j - C_j)$ row in the simplex solution procedure is that ......
3. The duality principle states that ......
4. The difference between the transportation problem and the assignment problem is ......
5. The special features of a transportation problem are ......
6. The canonical form of an L.P.P. is such that ......
7. The dual problem of the L.P.P. :
   Max. $Z = 4x_1 + 9x_2 + 2x_3$,
   subject to $2x_1 + 3x_2 + 2x_3 \le 7$, $3x_1 - 2x_2 + 4x_3 = 5$, $x_1, x_2, x_3 \ge 0$, is ......
8. The optimality and feasibility conditions related with Dual simplex method are ......
9. Feasible and basic solutions related with a transportation problem are ......
10. A transportation problem is

|  |  |  |  |  | Supply |
|---|---|---|---|---|---|
|  | 2 | 3 | 11 | 4 | 15 |
|  | 5 | 6 | 8 | 7 | 20 |
| Demand | 10 | 5 | 12 | 8 |  |

    Its linear programming problem is ......
11. The basic feasible solutions of $2x_1 + x_2 + 4x_3 = 11$, $3x_1 + x_2 + 5x_3 = 14$ are ......
12. A slack variable is defined as ......
13. The advantage of dual simplex method is ......
14. If the total availability is equal to the total requirements, the transportation problem is called ......
15. An artificial variable is that ......
16. Two conditions on which the simplex method is based are ......
17. A feasible solution which minimizes the transportation cost is called an ...... solution.
18. The dual problem of : Maximize $5x_1 + 6x_2$ subject to $x_1 + 2x_2 = 5$, $-x_1 + 5x_2 \ge 3$, $x_1$ unrestricted and $x_2 \ge 0$, is ......
19. For a balanced transportation problem with 3 rows and 3 columns, the number of basic variables will be ......
20. Using graphical method, Max. $Z = 5x_1 + 3x_2$ subject to $5x_1 + 2x_2 \le 10$, $3x_1 + 5x_2 \le 15$, $x_1, x_2 \ge 0$, is ......
21. In a L.P. problem, unbounded solution is that ......
22. Degeneracy in a transportation problem is resolved by ...... .
23. A basic solution is said to be non-degenerate in L.P.P. when ......
24. The dual of the problem Max. $Z = 2x_1 + x_2$ subject to $-x_1 + 2x_2 \le 2$, $x_1 + x_2 \le 4$, $x_1 \le 3$, $x_1, x_2 \ge 0$ is ......
25. The two methods used to find the initial solution of a transportation problem are ...... .
26. Constraints involving 'equal to sign' do not require use of ...... or ...... variables.

CHAPTER 35

# Calculus of Variations



## 35.1 INTRODUCTION

The calculus of variations is a powerful technique for the solution of problems in dynamics of rigid bodies, optimization of orbits and vibration problems. The subject primarily concerns with finding maximum or minimum value of a definite integral involving a certain function. It is something beyond finding stationary values of a given function. Only an elementary exposition of the subject is given here with the sole aim of introducing the student to a topic whose importance is fast growing in science and engineering.

Before proceeding further, the student should revise § 5.12 concerning maxima and minima of functions of several variables.

## 35.2 FUNCTIONALS

Consider the problem of finding a curve through two points $(x_1, y_1)$ and $(x_2, y_2)$ whose length is a minimum (Fig. 35.1). It is same as determining the curve $y = y(x)$ for which $y(x_1) = y_1$, $y(x_2) = y_2$ such that $\int_{x_1}^{x_2} \sqrt{(1 + y'^2)}\, dx$ is a minimum.

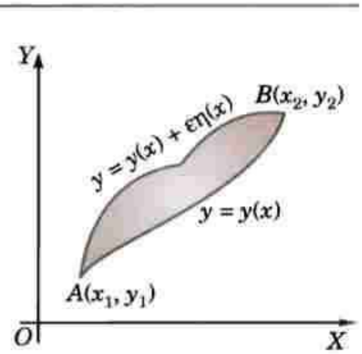



Fig. 35.1

In general terms, we wish to find the curve $y = y(x)$ where $y(x_1) = y_1$ and $y(x_2) = y_2$ such that for a given function $f(x, y, y')$,

$$\int_{x_1}^{x_2} f(x, y, y')\, dx \text{ is a stationery value or an extremum.} \qquad ...(1)$$

*An integral such as (1), which assumes a definite value for functions of the type $y = y(x)$ is called a* **functional.**

In differential calculus, we deal with the problems of maxima and minima of functions. The calculus of variations is however, concerned with maximizing or minimizing functionals.

## 35.3 EULER'S EQUATION

*A necessary condition for*

$$I = \int_{x_1}^{x_2} f(x, y, y')\, dx \textit{ to be an extremum is that}$$

$$\frac{\partial \mathbf{f}}{\partial \mathbf{y}} - \frac{\mathbf{d}}{\mathbf{dx}}\left(\frac{\partial \mathbf{f}}{\partial \mathbf{y}'}\right) = \mathbf{0}$$

This is called *Euler's equation.*

*Proof.* Let $y = y(x)$ be the curve joining points $A(x_1, y_1)$, $B(x_2, y_2)$ which makes $I$ an extremum. Let

$$y = y(x) + \varepsilon\eta(x) \quad ...(1)$$

be a neighbouring curve joining these points so that at $A$, $\eta(x_1) = 0$ and at $B$, $\eta(x_2) = 0$. ...(2)

The value of $I$ along (1) is $I = \int_{x_1}^{x_2} f[x, y(x) + \varepsilon\eta(x), y'(x) + \varepsilon\eta'(x)]\, dx$

This being a function of $\varepsilon$, is a maximum or minimum for $\varepsilon = 0$, when

$$\frac{dI}{d\varepsilon} = 0 \text{ at } \varepsilon = 0 \quad ...(3)$$

$\therefore$ Differentiating $I$ under the integral sign by Leibnitz's rule (p. 139), we have

$$\frac{dI}{d\varepsilon} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial \varepsilon} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \varepsilon} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \varepsilon}\right) dx \quad ...(4)$$

But $\varepsilon$ being independent of $x$, $\frac{\partial x}{\partial \varepsilon} = 0$. Also from (1), $\frac{\partial y}{\partial \varepsilon} = \eta(x)$ and $\frac{\partial y'}{\partial \varepsilon} = \eta'(x)$.

Substituting these values in (4), we get $\frac{dI}{d\varepsilon} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\eta(x) + \frac{\partial f}{\partial y'}\eta'(x)\right] dx$

Integrating the second term on the right by parts, we have

$$\frac{dI}{d\varepsilon} = \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\eta(x)\, dx + \left[\left|\frac{\partial f}{\partial y'}\eta(x)\right|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta(x)\, dx\right]$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] \eta(x)\, dx \quad \text{[By (2)]}$$

Since this has to be zero by (3),

$$\therefore \quad \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \quad ...(I)$$

which is the desired *Euler's equation.*

**Obs. 1. Other forms of Euler's equation.**

(*a*) Since $f$ is a function of $x, y, y'$, we have $\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx}$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \quad ...(5)$$

and
$$\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{\partial f}{\partial y'} y'' \quad ...(6)$$

Subtracting (6) from (5), we get $\frac{df}{dx} - \frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' - y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$

or
$$\frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial x} = y'\left\{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right\} = 0 \quad \text{[By (I)]}$$

Hence
$$\frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial x} = 0 \quad ...(II)$$

which is another form of (*I*).

(*b*) Again since $\frac{\partial f}{\partial y'}$ is also a function of $x, y, y'$, say : $\psi(x, y, y')$.

$$\therefore \quad \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{d\psi}{dx} = \frac{\partial \psi}{\partial y} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} + \frac{\partial \psi}{\partial y'}\frac{dy'}{dx}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y'}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y'}\right) y' + \frac{\partial}{\partial y'}\left(\frac{\partial f}{\partial y'}\right) y'' = \frac{\partial^2 f}{\partial x \partial y'} + y' \frac{\partial^2 f}{\partial y \partial y'} + y'' \frac{\partial^2 f}{\partial y'^2}$$

Substituting this in (*I*), we get $\frac{\partial f}{\partial y} - \frac{\partial^2 f}{\partial x \partial y'} - y' \frac{\partial^2 f}{\partial y \partial y'} - y'' \frac{\partial^2 f}{\partial y'^2} = 0$ (*V.T.U., 2001*) ...(*III*)

which is an extended form of (*I*).

**Obs. 2.** The above problem can easily be extended to the integral

$$= \int_{x_1}^{x_2} f(x, y_1, y_2 \ldots y_n, y_1', y_2', \ldots y_n')\, dx$$

involving $n$ functions $y_1, y_2, \ldots y_n$ of $x$. Then the necessary condition for this integral to be stationary is

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right) = 0, \quad i = 1, 2, \ldots, n \qquad \ldots(IV)$$

These are Euler's equations for the $n$ functions.

## 35.4 SOLUTIONS OF EULER'S EQUATION

Every solution of the Euler's equation which satisfies the boundary conditions, is called an *extremal* or a *stationary function* of the problem. The extremal can easily be obtained in the following cases :

**(1)** *When f is independent of x*

We have $\partial f/\partial x = 0$ and Euler's equation (*II*) above becomes $\frac{d}{dx}\left(f - y' \frac{\partial f}{\partial y'}\right) = 0$

Integrating, we get $f - y' \frac{\partial f}{\partial y'}$ = constant. This directly gives a solution of Euler's equation.

**(2)** *When f is independent of y*

We have $\partial f/\partial y = 0$ and Euler's equation (*I*) reduces to $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

Integrating $\frac{\partial f}{\partial y}$ = constant which gives a solution directly.

**(3)** *When f is independent of y'.*

We have $\partial f/\partial y' = 0$ and the equation (*I*) becomes $\frac{\partial f}{\partial y} = 0$ which gives the desired solution.

**(4)** *When f is independent of x and y*

We have $\quad \frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$ and $\frac{\partial^2 f}{\partial x \partial y'} = 0, \frac{\partial^2 f}{\partial y \partial y'} = 0.$

Then the equation (*III*) above becomes $y'' \frac{\partial^2 f}{\partial y'^2} = 0.$

If $\frac{\partial^2 f}{\partial y'^2} \neq 0$, it reduces to $y'' = 0$ which gives a solution of the form $y = ax + b$.

**Example 35.1.** *Find the extremals of the functional* $\int_{x_0}^{x_1} (y'^2/x^3)\, dx.$ (*V.T.U., 2003*)

**Solution.** We have $f = y'^2/x^3$ which is independent of $y$ *i.e.*, $\partial f/\partial y = 0$.

Also $\quad \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{d}{dx}\left(\frac{2y'}{x^3}\right) = 2\,\frac{x^3 y'' - y' \cdot 3x^2}{x^6} = \frac{2}{x^4}(xy'' - 3y')$

$\therefore$ Euler's equation reduces to $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

*i.e.*, $\quad \frac{2}{x^4}(xy'' - 3y') = 0$ or $y''/y' = 3/x$

Integrating both sides, $\int \frac{y''}{y'} dy = 3 \int \frac{dx}{x} + \log c$

*i.e.,* $\log y' = 3 \log x + \log c$ or $y' = cx^3$

Hence $y = cx^4/4 + c'$ or $y = c_1 x^4 + c_2$

This is the required extremal.

**Example 35.2.** *Prove that the shortest distance between two points in a plane is a straight line.* *(V.T.U., 2003 S ; Bhopal, 2003)*

**Solution.** Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the given points and $s$ the arc length of a curve connecting them (Fig. 35.2). Then

$$s = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} \sqrt{(1 + y'^2)}\ dx$$

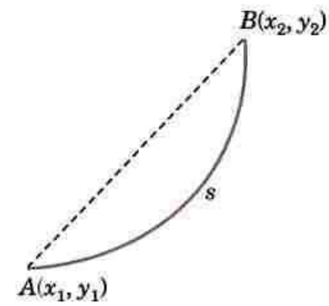


Fig. 35.2

Now $s$ will be minimum if it satisfies Euler's equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

Here $f = \sqrt{(1 + y'^2)}$ which is independent of $y$ *i.e.,* $\partial f/\partial y = 0$

$$\therefore \quad \frac{d}{dx}\left\{\frac{\partial}{\partial y'}\sqrt{(1 + y'^2)}\right\} = 0 \quad \text{or} \quad \frac{d}{dx}\left\{\frac{y'}{\sqrt{(1 + y'^2)}}\right\} = 0.$$

$\therefore$ On integration, we have $y'/\sqrt{(1 + y'^2)}$ = constant $y'$ = constant, $m$ say.

Integrating, we get $y = mx + c$, which is a straight line, the constants $m$ and $c$ are determined from the fact that the straight line passes through $A$ and $B$.

**Example 35.3.** *Find the curve passing through the points $(x_1, y_1)$ and $(x_2, y_2)$ which when rotated about the x-axis gives a minimum surface area.* *(V.T.U., 2009)*

**Solution.** In Fig. 35.3, the surface area $= \int_{x_1}^{x_2} 2\pi y\ ds$

$$= 2\pi \int_{x_1}^{x_2} y\sqrt{(1 + y'^2)}\, dx. \text{ This has to be minimum.}$$

Since $f = y\sqrt{(1 + y'^2)}$ is independent of $x$, therefore, Euler's equation reduces to

$$f - y'\frac{\partial f}{\partial y'} = \text{constant, } c : \text{say} \qquad \text{[By § 35.4 (1)]}$$

$$\therefore \quad y\sqrt{(1 + y'^2)} - y'\frac{\partial}{\partial y'}\left\{y\sqrt{(1 + y'^2)}\right\} = c$$

*i.e.,*
$$y\sqrt{(1 + y'^2)} - y'\left\{\frac{y}{2}(1 + y'^2)^{-1/2} \cdot 2y'\right\} = c$$

or
$$y/\sqrt{(1 + y'^2)} = c \quad \text{or} \quad y' = \frac{dy}{dx} = \frac{\sqrt{(y^2 - c^2)}}{c}$$

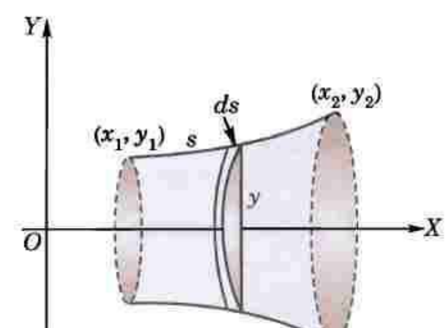


Fig. 35.3

Separating the variables and integrating, we have

$$\int \frac{dy}{\sqrt{(y^2 - c^2)}} = \int \frac{dx}{c} + c' \quad \text{or} \quad \cosh^{-1}\left(\frac{y}{c}\right) = \frac{x + a}{c}$$

*i.e.,*
$$y = c \cosh\left(\frac{x + a}{c}\right)$$

which is a *catenary*. The constants $a$ and $c$ are determined from the points $(x_1, y_1)$ and $(x_2, y_2)$.

**Obs.** This problem is also important in connection with *soap films* which are known to have shapes with minimum surface areas.

**Example 35.4.** *Find the path on which a particle in the absence of friction, will slide from one point to another in the shortest time under the action of gravity.* (V.T.U., 2004)

**Solution.** Let the particle start sliding on the curve $OP_1$ from $O$ with zero velocity (Fig. 35.4). At time $t$, let the particle be $P(x, y)$ such that arc $OP = s$.

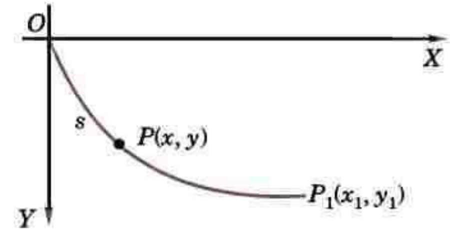


Fig. 35.4

By the principle of work and energy, we have

K.E. at $P$ – K.E. at $O$ = Work done in moving the particle from $O$ to $P$.

$$\frac{1}{2} m \left(\frac{ds}{dt}\right)^2 - 0 = mgy$$

or $$ds/dt = \sqrt{(2gy)} \qquad ...(i)$$

Thus the time taken by the particle to move from $O$ to $P_1$ is

$$T = \int_0^T dt = \int_0^{x_1} \frac{ds}{\sqrt{(2gy)}} = \frac{1}{\sqrt{(2g)}} \int_0^{x_1} \frac{\sqrt{(1+y'^2)}}{\sqrt{y}}\, dx$$

Here $f = \sqrt{[(1+y'^2)/y]}$ is independent of $x$.

$\therefore$ Euler's equation reduces to $f - y'\, \partial f/\partial y'$ = constant, $c$ : say

i.e., $$\frac{\sqrt{(1+y'^2)}}{\sqrt{y}} - y' \frac{\partial}{\partial y'}\left\{\frac{\sqrt{(1+y'^2)}}{\sqrt{y}}\right\} = c \quad \text{or} \quad \frac{\sqrt{(1+y'^2)}}{\sqrt{y}} - y'\left\{\frac{y'}{\sqrt{(1+y'^2)}\sqrt{y}}\right\} = c$$

or $$\sqrt{[y(1+y'^2)]} = 1/c = \sqrt{a}, \text{ say.}$$

Solving for $y'$, we have $$y' = \frac{dy}{dx} = \sqrt{\left(\frac{a-y}{y}\right)}$$

Separating the variables and integrating, we get

$$\int_0^x dx = \int_0^y \sqrt{\left(\frac{y}{a-y}\right)}\, dy \qquad [\text{Put } y = a \sin^2\theta] \quad ...(i)$$

or $$x = \int_0^\theta \sqrt{\left(\frac{a\sin^2\theta}{a - a\sin^2\theta}\right)}\, 2a \sin\theta\cos\theta\, d\theta$$

$$= a\int_0^\theta 2\sin^2\theta\, d\theta = a\int_0^\theta (1-\cos 2\theta)\, d\theta = \frac{a}{2}(2\theta - \sin 2\theta) \qquad ...(ii)$$

Writing $a/2 = b$ and $2\theta = \phi$, equations (*ii*) and (*i*) become $x = b(\phi - \sin\phi)$, $y = b(1-\cos\phi)$ which is a *cycloid*. The constant $b$ is found from the fact that the curve goes through $(x_1, y_1)$.

**Obs.** This is the well-known **brachistochrone problem** which derives its name from the Greek words 'brachistos' meaning shortest and 'chronos' meaning time. It was proposed by John Bernoulli in 1696 and its solution formed the basis for the study of the 'Calculus of Variations'. (V.T.U., 2006)

**Example 35.5.** *Find the curves on which the functional* $\int_0^1 [(y')^2 + 12xy]\, dx$ *with* $y(0) = 0$ *and* $y(1) = 1$ *can be extremised.* (V.T.U., 2010)

**Solution.** We have $$f = y'^2 + 12xy$$

$\therefore$ $$\partial f/\partial y = 12x\,;\ \frac{\partial f}{\partial y'} = 2y'\,;\ \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 2y''$$

Hence the Euler's equation $\dfrac{\partial f}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'}\right) = 0$ becomes

$$12x - 2y'' = 0 \quad i.e., \quad y'' = 6x \qquad ...(i)$$

Integrating (*i*), $y' = 3x^2 + C$ ...(*ii*) and $y = x^3 + Cx + C'$ ...(*iii*)

Using the boundary conditions, when $x = 0, y = 0$ (*iii*) gives $C' = 0$.

When $x = 1, y = 1$, (*iii*) again gives $C = 0$.

Hence (*iii*) reduces to $y = x^3$ which is the only curve on which extremum can be attained.

**Example 35.6.** *On which curve the functional* $\int_0^{\pi/2} (y'^2 - y^2 + 2xy)dy$ *with* $y(0) = 0$ *and* $y(\pi/2) = 0$, *be extremized?* (V.T.U., 2006)

**Solution.** Let $f = y'^2 - y^2 + 2xy$ so that $\dfrac{\partial f}{\partial y} = 0 - 2y + 2x$

and
$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{d}{dx}(2y') = 2y''$$

$\therefore$ Euler's equation $\dfrac{\partial f}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'}\right) = 0$ becomes

$$-2y + 2x - 2y'' = 0 \quad \text{or} \quad y'' + y = x \quad \text{or} \quad (D^2 + 1)\, y = x$$

Its A.E. $D^2 + 1 = 0$ gives $D = \pm i$.

$\therefore$
$$\text{C.F.} = c_1 \cos x + c_2 \sin x$$

and
$$\text{P.I.} = \frac{1}{D^2 + 1} x = (1 + D^2)^{-1} x = (1 - D^2)\, x = x$$

Thus
$$y = c_1 \cos x + c_2 \sin x + x \qquad ...(i)$$

Using boundary conditions : when $x = 0, y = 0$, (*i*) gives $c_1 = 0$ ;

when $x = \pi/2, y = 0$, (*i*) gives $0 = c_2 + \pi/2$, *i.e.*, $c_2 = -\pi/2$.

Hence (*i*) reduces to $y = x - \dfrac{\pi}{2} \sin x$, which is the only curve on which the given functional can be extremized.

**Example 35.7.** *Solve the variational problem*

$$\delta \int_1^2 [x^2 (y')^2 + 2y(x + y)]\, dx = 0, \text{ given } y(1) = y(2) = 0. \qquad \text{(V.T.U., 2006)}$$

**Solution.** Let $f = x^2 (y')^2 + 2xy + 2y^2$ so that $\dfrac{\partial f}{\partial y} = 2x + 4y$, $\dfrac{\partial f}{\partial y} = 2x^2 y'$

$\therefore$ Euler's equation $\dfrac{\partial f}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'}\right) = 0$ becomes

$$2x + 4y - \frac{d}{dx}(2x^2 y') = 0 \quad \text{or} \quad 2x + 4y - (2x^2 y'' + 4xy') = 0$$

or $x^2 y'' + 2xy' - 2y = x$. This is Cauchy's homogeneous linear (§ 13.9)

Putting $x = e^t$, it reduces to $(D^2 + D - 2)\, y = e^t$.

Its solution is
$$y = c_1 e^t + c_2 e^{-2t} + \frac{1}{3} te^t \quad \text{or} \quad y = c_1 x + \frac{c_2}{x^2} + \frac{1}{3} x \log x \qquad ...(i)$$

Since $y(1) = 0$, we have $c_1 + c_2 = 0$

and
$$y(2) = 0 \text{ gives } 0 = 2\, c_1 + \frac{1}{4} c_2 + \frac{2}{3} \log 2$$

Solving these equations, we get $c_1 = -c_2 = \dfrac{-8}{21} \log 2$.

Putting the values of $c_1$ and $c_2$ in (*i*), we get

$$y = \frac{1}{21} \{8 \log 2\, (x^{-2} - x) + 7x \log x\}$$

which is the required solution.

## 35.5 GEODESICS

*A geodesic on a surface is a curve along which the distance between any two points of the surface is a minimum.* To find the geodesics on a surface is a variational problem involving the conditional extremum. This problem was first studied by *Jacob Bernoulli* in 1698 and its general method of solution was given by *Euler*.

**Example 35.8.** *Show that the geodesics on a plane are straight lines.* (*V.T.U., 2009*)

**Solution.** Let $y = y(x)$ be a curve joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the $xy$-plane. Then the length of a curve joining $A$ and $B$ is given by

$$s = \int_A^B \frac{ds}{dx}\,dx = \int_{x_1}^{x_2} \sqrt{[1 + (dy/dx)^2]}\,dx \quad \textit{i.e.,} \quad s = \int_{x_1}^{x_2} \sqrt{(1 + y'^2)}\,dx$$

The geodesic on the $xy$-plane is the curve $y = y(x)$ for which $s$ is minimum.

We have $f(x, y, y') = \sqrt{(1 + y'^2)}$ which depends on $y'$ only. Hence the Euler's equation.

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \text{ yields}$$

$$\frac{d}{dx}\left\{\frac{2y'}{2\sqrt{(1 + y'^2)}}\right\} = 0 \quad \textit{i.e.,} \quad y''\sqrt{(1 + y'^2)} - \frac{y'_2 2y'y''}{2\sqrt{(1 + y'^2)}} = 0$$

*i.e.*
$$y''(1 + y'^2) - y'^2 y'' = 0 \quad \textit{i.e.,} \quad \frac{d^2y}{dx^2} = 0$$

Integrating twice, we get $y = c_1 x + c_2$
which is a straight line.

Hence *the geodesics on a plane are straight lines.*

**Example 35.9.** *Find the geodesics on a right circular cylinder of radius a.*

**Solution.** In cylindrical coordinates $(\rho, \phi, z)$ we have

$$x = r\cos\phi, \; y = r\sin\phi, \; z = z. \qquad \text{(}p. 357\text{)}$$

$\therefore$ The element of arc on a right circular cylinder of radius $a$, is given by

$$ds^2 = (dx)^2 + (dy)^2 + (dz)^2 = (d\rho)^2 + (\rho d\phi)^2 + dz^2 = a^2\,d\phi^2 + dz^2 \qquad [\because \; \rho = a \text{ and } d\rho = 0]$$

or
$$ds = \sqrt{\{a^2 + (dz/d\phi)^2\}}\,.\,d\phi \quad \text{or} \quad s = \int_{\phi_1}^{\phi_2} \sqrt{(a^2 + z'^2)}\,d\phi \qquad ...(i)$$

Now the geodesic for the given cylinder is the curve for which $s$ is minimum. Here $f = \sqrt{(a^2 + z'^2)}$, which is a function of $\phi$ and $z'$ while $z$ is absent.

$\therefore$ Euler's equation for the functional (*i*) reduces to

$$\frac{\partial f}{\partial z'} = \text{constant} \quad \text{or} \quad \frac{z'}{\sqrt{(a^2 + z'^2)}} = c, \text{ say.}$$

This simplifies to $(z')^2$ = constant or $dz/d\phi = c_1$, say.

Integrating, $z = c_1\phi + c_2$

*This is the desired geodesics on a circular cylinder which is a circular helix.* (Example 8.3, *p.* 318)

**Example 35.10.** *Show that the geodesics on a sphere of radius a are its great circles.*

**Solution.** In spherical coordinates $(r, \theta, \phi)$, we have

$$x = r\sin\theta\cos\phi, \; y = r\sin\theta\sin\phi, \; z = r\cos\theta. \qquad (p.\ 359) \quad ...(i)$$

$\therefore$ The arc element on a sphere of radius $a$, is given by

$$ds^2 = dr^2 + (r\,d\theta)^2 + (r\sin\theta\,d\phi)^2 = a^2\,d\theta^2 + (a\sin\theta)^2\,d\phi^2 \qquad [\because \; r = a, dr = 0$$

or
$$ds = a\sqrt{[1 + \sin^2\theta\,(d\phi/d\theta)^2]}\,d\theta, \quad \text{or} \quad s = a\int_{\theta_1}^{\theta_2} \sqrt{[1 + \sin^2\theta\,.\,(\phi')^2]}\,d\theta$$

Now the geodesic on the sphere $r = a$ is the curve for which $s$ is minimum. Here $f = a\sqrt{(1+\sin^2\theta\,.\,\phi'^2)}$ which is a function of $\theta$ and $\phi'$ while $\phi$ is absent.

$\therefore$ Euler's equation reduces to $\partial f/\partial\phi'$ = constant.

$$\frac{\partial f}{\partial \phi'} = \frac{a\sin^2\theta\,.\,\phi'}{\sqrt{(1+\sin^2\theta\,.\,\phi'^2)}} = \text{constant.}$$

or $$\frac{\sin^2\theta\,.\,\phi'}{\sqrt{(1+\sin^2\theta\,.\,\phi'^2)}} = c \text{ (say)} \quad \text{or} \quad \sin^2\theta\,(\sin^2\theta - c^2)\,\phi'^2 = c^2$$

or $$\frac{d\phi}{d\theta} = \frac{c}{\sin\theta\sqrt{(\sin^2\theta - c^2)}} = \frac{c\,\text{cosec}^2\,\theta}{\sqrt{(1-c^2\,\text{cosec}^2\,\theta)}}$$

Integrating $$\phi = \int \frac{c\,\text{cosec}^2\,\theta\,.\,d\theta}{\sqrt{[(1-c^2)-(c\cot\theta)^2]}} + c' = -\sin^{-1}\left\{\frac{c\cot\theta}{\sqrt{(1-c^2)}}\right\} + c'$$

or $$\cot\theta = A\cos\phi + B\sin\phi \quad \text{or} \quad a\cos\theta = Aa\sin\theta\cos\phi + Ba\sin\theta\sin\phi$$

or $$z = Ax + By \qquad \text{[By } (i) \text{ when } r = a]$$

This is a plane through the centre (0, 0, 0) of the sphere which cuts the sphere along a great circle. Hence the required *geodesics are the arcs of the great circles.*

## PROBLEMS 35.1

1. Solve the Euler's equation for the following functionals :

   (*i*) $\int_{x_0}^{x_1} (x+y')\,y'\,dx$ (*ii*) $\int_{x_0}^{x_1} (1+x^2y')\,y'\,dx.$ (*V.T.U., 2004*)

2. Show that the general solution of the Euler's equation for the integral

$$\int_a^b \frac{1}{y}\sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\,dx \text{ is } (x-h)^2 + y^2 = k^2.$$

Find the extremals of the following functionals :

3. $\int_{x_1}^{x_2} (y^2 + y'^2 + 2ye^x)\,dx.$ (*V.T.U., 2004*)

4. $\int_0^{\pi/2} (y^2 + y'^2 - 2y\sin x)\,dx,\ y(0) = y\,(\pi/2) = 0.$ (*V.T.U., 2008*)

5. $\int_0^{\pi} (y'^2 - y^2 + 4y\cos x)\,dx;\ y(0) = 0,\ y\,(\pi) = 0.$ (*V.T.U., 2008 S*)

6. $\int_{x_0}^{x_1} \frac{1+y^2}{y'^3}\,dx.$

7. $\int_1^2 \frac{x^3}{y'^2}\,dx\,;\ y\,(1) = 0,\ y(2) = 3.$

8. $\int_1^2 \frac{\sqrt{(1+y'^2)}}{x}\,dx\,;\ y\,(1) = 0,\ y(2) = 1.$ (*Madurai, M.E., 2000 S*)

9. Solve the variational problem $\delta\int_0^{\pi/2} [y^2 - (y')^2]\,dx$ under the conditions $y(0) = 0,\ y(\pi/2) = 2.$ (*V.T.U., 2010*)

10. A heavy cable hangs freely under gravity between two fixed points. Show that the shape of the cable is a catenary.

11. A particle is moving with a force perpendicular to and proportional to its distance from the line of zero velocity. Show that the path of quickest descent (brachistochrone) is a circle.

12. Find the geodesics on a *right circular cone* of semi-vertical angle $\alpha$.

## 35.6 ISOPERIMETRIC PROBLEMS

In certain problems, it is necessary to make a given integral.

$$I = \int_{x_1}^{x_2} f(x, y, y')\, dx \qquad ...(1)$$

maximum or minimum while keeping another integral

$$J = \int_{x_1}^{x_2} g(x, y, y')\, dx \qquad ...(2)$$

constant. Such problems involve one or more constraint conditions, just as $J$ = a constant. A typical example of this type is that of finding a closed curve of a given perimeter and maximum area. This being one of the earliest problems to engage attention, we often refer to problems of this type as *isoperimetric problems.*

Such problems are generally solved by the method of Lagrange multipliers. To extremize (1), we multiply (2) by $\lambda$ and add to (1) where $\lambda$ is the Lagrange multiplier. Then the necessary condition for the integral $\int_{x_1}^{x_2} H\, dx$ to be an extremum is $\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0$ where $H = f + \lambda g$. The values of the two constants of integration and the parameter $\lambda$ are determined from the three conditions namely : the two boundary conditions and the integral $J$ having given constant value.

**Example 35.11.** *Find the plane curve of fixed perimeter and maximum area.* (V.T.U., 2000 S)

**Solution.** Let $l$ be the fixed perimeter of a plane curve between the points with abscissae $x_1$ and $x_2$ (Fig. 35.5). Then

$$l = \int_{x_1}^{x_2} \sqrt{(1 + y'^2)}\, dx \qquad ...(i)$$

Also the area between the curve and the $x$-axis is

$$A = \int_{x_1}^{x_2} y\, dx \qquad ...(ii)$$

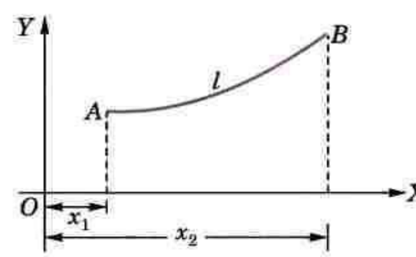


Fig. 35.5

We have to maximize (*ii*) subject to constraint (*i*).

$\therefore$ Taking $\quad f = y$ and $g = \sqrt{(1 + y'^2)}$, we write $H = f + \lambda g = y + \lambda\sqrt{(1 + y'^2)}$

Now $H$ must satisfy the Euler's equation

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0 \quad \therefore \quad 1 - \frac{d}{dx}\left[\frac{\lambda y'}{\sqrt{(1 + y'^2)}}\right] = 0$$

Integrating w.r.t. $x$, we have $x - \lambda y'/\sqrt{(1 + y'^2)} = 0$

Solving for $y'$, we get $\quad y' = \dfrac{x - a}{\sqrt{[\lambda^2 - (x - a)^2]}}$

Integrating again, $\quad y = \sqrt{[\lambda^2 - (x - a)^2]} + b$ *i.e.* $(x - a)^2 + (y - b)^2 = \lambda^2$ which is a circle.

**Example 35.12.** *Prove that the sphere is the solid figure of revolution which, for a given surface area, has maximum volume.* (V.T.U., 2006 ; Madras, 2000 S)

**Solution.** Consider the arc $OPA$ of the curve which rotates about the $x$-axis as shown in Fig. 35.6.

Then surface area $\quad S = \int_{x=0}^{a} 2\pi y\, ds = \int_0^a 2\pi y \sqrt{(1 + y'^2)}\, dx$

and volume of the solid so formed $\quad V = \int_0^a \pi y^2\, dx.$

Here we have to maximize $V$ subject to fixed $S$. Taking $f = \pi y^2$ and $g = 2\pi y\sqrt{(1 + y'^2)}$, we write $H = f + \lambda g = \pi y^2 + 2\pi\lambda y\sqrt{(1 + y'^2)}$.

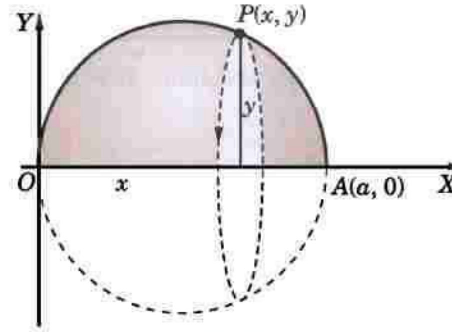



Fig. 35.6

Now $H$ has to satisfy Euler's equation. But it does not contain $x$ explicitly.

$$\therefore \qquad H - y'\frac{\partial H}{\partial y'} = \text{constant, } c : \text{say}$$

*i.e.*,
$$\pi y^2 + 2\pi\lambda y\sqrt{(1 + y'^2)} - y' \,.\, 2\pi\lambda y\,\frac{y'}{\sqrt{(1 + y'^2)}} = c$$

or
$$\pi y^2 + \frac{2\pi\lambda y}{\sqrt{(1 + y'^2)}} = c \qquad ...(i)$$

Since the curve passes through $O$ and $A$ for which $y = 0$, $(i)$ gives $c = 0$.

$$\therefore \qquad y + 2\lambda/\sqrt{(1 + y'^2)} = 0$$

Solving for $y'$, $\qquad y'\left(= \frac{dy}{dx}\right) = \frac{\sqrt{(4\lambda^2 - y^2)}}{y}$

Separating the variables and integrating, we get

$$\int dx = \int \frac{y\,dy}{\sqrt{(4\lambda^2 - y^2)}} + k \quad \text{or} \quad x = k - \sqrt{(4\lambda^2 - y^2)} \qquad ...(ii)$$

When $x = 0, y = 0 \quad \therefore \quad k = 2\lambda$

$\therefore \quad (ii)$ becomes $(x - 2\lambda)^2 + y^2 = (2\lambda)^2$ which is a circle with centre $(2\lambda, 0)$ and radius $2\lambda$.

Hence the figure formed by the revolution of given arc is a sphere.

## PROBLEMS 35.2

1. Find a function $y(x)$ for which $\int_0^1 (x^2 + y'^2)\,dx$ is stationary, given that $\int_0^1 y^2\,dx = 2$; $y(0) = 0$, $y(1) = 0$. *(Madras, 2000 S)*
2. Find the extremals of the isoperimetric problem $v[y(x)] = \int_{x_0}^{x_1} y'^2\,dx$ given that $\int_{x_0}^{x_1} y\,dx = c$, a constant.
3. Show that the curve $c$ of given length $l$ which minimizes the curved surface area of the solid generated by the revolution of $c$ about the $x$-axis is a catenary. *(V.T.U., 2000 S)*
4. Find the extremal of the functional $I = \int_0^\pi [(y')^2 - y^2]\,dx$ under the conditions $y(0) = 0$, $y(\pi) = 1$ and subject to the constraint $\int_0^\pi y\,dx = 1$.
5. Prove that the extremal of the isoperimetric problem $v[y(x)] = \int_1^4 y_2'\,dx$, $y(1) = 3$, $y(4) = 24$, subject to the condition $\int_1^4 y\,dx = 36$ is a parabola. *(Madras M.E., 2000)*

## 35.7 SEVERAL DEPENDENT VARIABLES

We now extend the variational problem of § 35.3 to a problem with several variables as functions of a single independent variable *i.e.*, *A necessary condition for*

$$I = \int_{x_1}^{x_2} f(x, y_1, y_2, ..., y_n, y_1', y_2', ..., y_n')\,dx \qquad ...(1)$$

*to be an extremum is that* $\dfrac{\partial \mathbf{f}}{\partial \mathbf{y}_i} - \dfrac{\mathbf{d}}{\mathbf{dx}}\left(\dfrac{\partial \mathbf{f}}{\partial \mathbf{y}_i'}\right) = \mathbf{0}, i = 1, 2, ..., n$ ...(2)

Let $y_1, y_2, ..., y_n$ satisfy the boundary conditions $y_i(x_1) = y_{i1}, y_i(x_2) = y_{i2}$

Consider arbitrary functions $\eta_1(x), \eta_2(x), ..., \eta_n(x)$ which are all zero on the boundary *i.e.*,

$$\eta_i(x_1) = 0 = \eta_i(x_2)$$

Replacing $y_1, y_2, ...$ by $y_1 + \varepsilon_1\eta_1, y_2 + \varepsilon_2\eta_2, ...$ in (1), we get

$$I(\varepsilon) = \int_{x_1}^{x_2} f\,(x, y_1 + \varepsilon_1\eta_1, y_2 + \varepsilon_2\eta_2, ..., y_1' + \varepsilon_1\eta_1', y_2' + \varepsilon_2\eta_2', ...)\, dx \quad ...(3)$$

This is a function of the parameters $\varepsilon_1, \varepsilon_2, ...$ and reduces to (1) for $\varepsilon_1 = \varepsilon_2 = ... = 0$.

To find the stationary value of (1), we find the stationary value of $I(\varepsilon)$ for $\varepsilon_1 = \varepsilon_2 = ... = 0$, $I(\varepsilon)$ will have a stationary value when

$$\frac{\partial I(\varepsilon)}{\partial \varepsilon_1} = 0, \ \frac{\partial I(\varepsilon)}{\partial \varepsilon_2} = 0, ...$$

Writing $\quad f = f(x, y_1, y_2, ..., y_1', y_2', ...)$

and $\quad F = f(x, y_1 + \varepsilon_1\eta_1, y_2 + \varepsilon_2\eta_2, ..., y_1' + \varepsilon_1\eta_1', y_2' + \varepsilon_2\eta_2', ...)$.

(3) becomes $I(\varepsilon) = \int_{x_1}^{x_2} F\, dx$.

$x_1, x_2$ being constants independent of $\varepsilon_1$, differentiating under the integral sign, we get

$$\frac{\partial I(\varepsilon)}{\partial \varepsilon_1} = \int_{x_1}^{x_2} \frac{\partial F}{\partial \varepsilon_1}\, dx = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y_1}\eta_1 + \frac{\partial F}{\partial y_1}\eta_1'\right) dx$$

$\therefore \quad \dfrac{\partial I(\varepsilon)}{\partial \varepsilon_1} = 0$, when $\varepsilon_1 = \varepsilon_2 = ... = 0$ gives

$$\int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y_1}\eta_1 + \frac{\partial f}{\partial y_1'}\eta_1'\right) = 0$$

Integrating by parts the second term, we get

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y_1}\eta_1 + \left|\frac{\partial f}{\partial y_1'}\eta_1\right|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y_1'}\right)\eta_1(x)\, dx = 0$$

*i.e.*, $$\int_{x_1}^{x_2} \left\{\frac{\partial f}{\partial y_1} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_1'}\right)\right\}\eta_1(x)\, dx = 0 \qquad [\because \ \eta_1(x_1) = 0 = \eta_1(x_2)]$$

Since this equation must hold good for all values of $\eta_1(x)$, we get

$$\frac{\partial f}{\partial y_1} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_1'}\right) = 0$$

Similarly $\dfrac{\partial I(\varepsilon)}{\partial \varepsilon_2} = 0$ when $\varepsilon_1 = \varepsilon_2 = ... = 0$, will give

$$\frac{\partial f}{\partial y_2} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_2'}\right) = 0 \text{ and so on.}$$

All these conditions give a system of Euler's equation (2). A solution of these equations leads to the desired curves.

**Example 35.13.** *Show that the functional* $\int_0^{\pi/2} \left\{2xy + \left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2\right\} dt$ *such that* $x(0) = 0$, $x(\pi/2) = -1$, $y(0) = 0$, $y(\pi/2) = 1$ *is stationary for* $x = -\sin t$, $y = \sin t$.

**Solution.** Euler's equations are $\frac{\partial f}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial x'}\right) = 0$ ...(i)

$$\frac{\partial f}{\partial y} - \frac{d}{dt}\left(\frac{\partial f}{\partial y'}\right) = 0 \qquad ...(ii)$$

Here $f = 2xy + x'^2 + y'^2$. $\therefore$ $\frac{\partial f}{\partial x} = 2y,\ \frac{\partial f}{\partial x'} = 2x',\qquad \frac{\partial f}{\partial y} = 2x,\ \frac{\partial f}{\partial y'} = 2y'$

(i) becomes $2y - \frac{d}{dt}(2x') = 0$ *i.e.*, $2y - 2\frac{d^2x}{dt^2} = 0$ or $\frac{d^2x}{dt^2} = y$ ...(iii)

(ii) becomes $2x - \frac{d}{dt}(2y') = 0$ *i.e.*, $2x - 2\frac{d^2y}{dt^2} = 0$ or $\frac{d^2y}{dt^2} = x$ ...(iv)

Now to solve these simultaneous differential equations, we differentiate (iii) twice,

$$\frac{d^4x}{dt^4} = \frac{d^2y}{dt^2} = x \qquad \text{[By (iv)]}$$

or $(D^4 - 1)\,x = 0$ which is a linear equation with constant coefficients.

Its solution is $x = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$ ...(v)

From (iii), $y = x'' = c_1 e^x + c_2 e^{-x} - c_3 \cos x - c_4 \sin x$ ...(vi)

Since $x = 0$ when $t = 0$ $\therefore$ $0 = c_1 + c_2 + c_3$ ...(vii)

$y = 0$ when $t = 0$ $\therefore$ $0 = c_1 + c_2 - c_3$ ...(viii)

Also $x = -1$ when $t = \pi/2$ $\therefore$ $-1 = c_1 e^{\pi/2} + c_2 e^{-\pi/2} + c_4$ ...(ix)

$y = 1$ when $t = \pi/2$ $\therefore$ $1 = c_1 e^{\pi/2} + c_2 e^{-\pi/2} - c_4$ ...(x)

Adding (vii) and (viii), $c_1 + c_2 = 0$

Adding (ix) and (x), $c_1 e^{\pi/2} + c_2 e^{-\pi/2} = 0$

Solving these equations, we get $c_1 = c_2 = 0$.

From (viii), $c_3 = c_1 + c_2 = 0$. From (ix), $c_4 = -1$.

Hence from (v), $x = -\sin x$ and from (vi), $y = \sin x$.

## 35.8 FUNCTIONALS INVOLVING HIGHER ORDER DERIVATIVES

*A necessary condition for*

$$I = \int_{x_1}^{x_2} f(x, y, y', y'')\, dx \qquad ...(1)$$

*to be extremum is* $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) = 0.$

Let $y(x)$ be the function which makes (1) stationary and satisfies the boundary conditions

$$y(x_1) = y_1,\ y(x_2) = y_2,\ y'(x_1) = y_1' \text{ and } y'(x_2) = y_2'.$$

Consider the differentiable function $\eta(x)$ such that

$$\eta(x_1) = 0 = \eta(x_2) \quad \text{and} \quad \eta'(x_1) = 0 = \eta'(x_2) \qquad ...(2)$$

Replacing $y$ by $y + \varepsilon\eta$ in (1), we get

$$I(\varepsilon) = \int_{x_1}^{x_2} f(x, y + \varepsilon\eta, y' + \varepsilon\eta', y'' + \varepsilon\eta'')\, dx \qquad ...(3)$$

This is a function of the parameter $\varepsilon$ and reduces to (1) for $\varepsilon = 0$.

To find the stationary value of (1), we find the stationary value of $I(\varepsilon)$ for $\varepsilon = 0$. But $I(\varepsilon)$ will have a stationary value when $dI(\varepsilon)/d\varepsilon = 0$.

Writing $f = f(x, y, y', y'')$ and $F = f(x, y + \varepsilon\eta, y' + \varepsilon\eta', y'' + \varepsilon\eta'')$.

(3) becomes $I(\varepsilon) = \int_{x_1}^{x_2} F\ dx$

$x_1, x_2$ being constants independent of $\varepsilon$, differentiating under the integral sign, we get

$$\frac{dI(\varepsilon)}{d\varepsilon} = \int_{x_1}^{x_2} \frac{dF}{d\varepsilon}\, dx = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta' + \frac{\partial F}{\partial y''}\eta''\right) dx$$

$\therefore$ $\dfrac{dI(\varepsilon)}{d\varepsilon} = 0$ when $\varepsilon = 0$ gives $\displaystyle\int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta' + \frac{\partial f}{\partial y''}\eta''\right) dx = 0$

Integrating by parts once the second term and twice the third term, we get

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y}\eta\, dx + \left|\frac{\partial f}{\partial y'}\eta\right|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right).\eta\, dx + \left|\frac{\partial f}{\partial y''}\eta' - \frac{d}{dx}\left(\frac{\partial f}{\partial y''}\right).\eta\right|_{x_1}^{x_2}$$

$$+ \int_{x_1}^{x_2} \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right).\eta\, dx = 0$$

or $$\int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right)\right]\eta(x)\, dx = 0 \qquad \text{[By (2)]}$$

Since this equation must hold good for all values of $\eta(x)$, we get

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) = 0$$

In general, *a necessary condition for the functional* $I = \displaystyle\int_{x_1}^{x_2} f(x, y, y', y'', \ldots y^{(n)})\, dx$ *to be stationary will be*

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) - \ldots + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial f}{\partial y^{(n)}}\right) = 0$$

which is called the *Euler-Poisson equation* and its solutions are called *extremals.*

**Example 35.14.** *Show that the curve which extremizes the functional* $I = \displaystyle\int_0^{\pi/4} (y''^2 - y^2 + x^2)\, dx$ *under the conditions* $y(0) = 0,\ y'(0) = 1,\ y(\pi/4) = y'(\pi/4) = 1/\sqrt{2}$ *is* $y = \sin x$. *(Madras M.E., 2000 S)*

**Solution.** The Euler-Poisson equation is

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) = 0 \qquad ...(i)$$

Here $$f = (y'')^2 - y^2 + x^2,\quad \frac{\partial f}{\partial y} = -2y,\quad \frac{\partial f}{\partial y'} = 0,\quad \frac{\partial f}{\partial y''} = 2y''$$

$\therefore$ (*i*) becomes $-2y + \dfrac{d^2}{dx^2}(2y'') = 0$ or $y^{iv} - y = 0$ or $(D^4 - 1)\,y = 0$

Its solution is $y(x) = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$ ...(*ii*)

$\therefore$ $y'(x) = c_1 e^x - c_2 e^{-x} - c_3 \sin x + c_4 \cos x$ ...(*iii*)

Applying the given boundary conditions to (*ii*) and (*iii*), we get

$$0 = y(0) = c_1 + c_2 + c_3,\quad 1 = y'(0) = c_1 - c_2 + c_4$$

$$\frac{1}{\sqrt{2}} = y(\pi/4) = c_1 e^{\pi/4} + c_2 e^{-\pi/4} + \frac{1}{\sqrt{2}}c_3 + \frac{1}{\sqrt{2}}c_4 \qquad ...(iv)$$

$$\frac{1}{\sqrt{2}} = y'(\pi/4) = c_1 e^{\pi/4} - c_2 e^{-\pi/4} - \frac{1}{\sqrt{2}}c_3 + \frac{1}{\sqrt{2}}c_4$$

Solving the equation (*iv*), we get $c_1 = c_2 = c_3 = 0$, $c_4 = 1$. Hence the required curve is $y = \sin x$.

## PROBLEMS 35.3

1. Show that the functional $\int_0^1 \left\{2x + \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\} dt$, such that $x(0) = 1, y(0) = 1; x(1) = 1.5, y(1) = 1$ is stationary for $x = 1 + t^2/2, y = t$.
2. Find the extremals of the functional $v(y, z) = \int_{x_0}^{x_1} (2yz - 2y^2 + y'^2 - z'^2)\, dx$ (V.T.U., M.E., 2006)
3. Find a function $y(x)$ such that $\int_0^\pi y^2\, dx = 1$ which makes $\int_0^\pi y''^2\, dx$ a minimum if $y(0) = 0 = y(\pi), y''(0) = 0 = y''(\pi)$.
4. Find the extremals of the functional $\int_0^{\pi/2} (y''^2 - y^2 + x^2)\, dx$ that satisfies the conditions $y(0) = 1, y'(0) = 0, y(\pi/2) = 0, y'(\pi/2) = -1$. (Madras, M.E. 2000 S)
5. Find the extremals of the functional $\int_{-a}^{a} \left(\lambda y + \frac{1}{2}\mu y''^2\right) dx$ which satisfies the boundary conditions $y(-a) = 0, y'(-a) = 0, y(a) = 0, y'(a) = 0$.
6. Find the extremals of the following functionals :

   (i) $v[y(x)] = \int_{x_0}^{x_1} (16y^2 - y''^2 + x^2)\, dx$ (Nagpur, 1997) (ii) $v[y(x)] = \int_{x_0}^{x_1} (2xy + y'''^2)\, dx$.

## 35.9 APPROXIMATE SOLUTION OF BOUNDARY VALUE PROBLEMS — Rayleigh-Ritz Method

In § 35.3, we have seen that the solution of Euler's differential equation alongwith boundary conditions amounts to extremising a certain definite integral. This fact provides a technique of solving a boundary value problem approximately by assuming a trial solution satisfying the given boundary conditions and then extremising the integral whose integrand is found from the given differential equation.

To solve a boundary value problem of Rayleigh-Ritz method, we try to write the given differential equation as the Euler's equation of some variational problem. Then we reduce this variational problem to a minimizing problem assuming an approximate solution in the form

$$\bar{y}(x) = y_0(x) + \Sigma c_i\, \phi_i(x) \qquad \text{...(1)}$$

where the *trial functions* $\phi_i(x)$ satisfy the boundary conditions and $\phi_i(x) = 0$ on the boundary $C$ of its region $R$.

Let the integral to be extremised be $I = \int_a^b f(y, y', x)\, dx$ ...(2)

such that $y(a) = A$ and $y(b) = B$.

Substituting (1) in (2) by replacing $y$ in $\bar{y}$ in $I$, giving $\bar{I}$ as a function of the unknowns $c_i$. Then $c$'s become parameters which are so determined as to extremise $\bar{I}$. This requires

$$\frac{\partial \bar{I}}{\partial c_i} = 0, \qquad i = 1, 2, \ldots$$

Solving these equations, we get the values of $c_i$, which when substituted in (1) give the desired solution.

**Example 35.15.** *Solve the boundary value problem $y'' - y + x = 0$ $(0 \le x \le 1)$, $y(0) = y(1) = 0$ by Rayleigh-Ritz method.*

**Solution.** Given differential equation is $y'' - y + x = 0$ ...(*i*)

Its solution is equivalent to extremising the integral $I = \int_0^1 F(x, y, y')\, dx$

where $$F(x, y, y') = 2xy - y^2 - y'^2, \qquad \text{...(ii)}$$

since the Euler's equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$ gives (*i*).

Assume that the *trial function* is $\bar{y}(x) = c_0 + c_1 x + c_2 x^2$ ...(*iii*)

To satisfy $y(0) = 0, y(1) = 0$, we require $c_0 = 0, c_2 = -c_1$.

$\therefore$ (*iii*) becomes $\bar{y}(x) = c_1 x(1-x)$ ...(*iv*)

Substituting $\bar{y}$ and $\bar{y}'$ in $I$, we have

$$\bar{I} = \int_0^1 [2x\bar{y} - \bar{y}^2 - (\bar{y}')^2]\, dx = \int_0^1 [2c_1(x^2 - x^3) - c_1^2(x - x^2)^2 - c_1^2(1-x)^2]\, dx$$

$$= \frac{1}{6}c_1 - \frac{11}{30}c_1^2$$

Its stationary values are given by $dI/dc_1 = 0$. $\therefore \quad \frac{1}{6} - \frac{11}{15}c_1 = 0$ *i.e.*, $c_1 = \frac{5}{22}$.

Thus the approximate solution is $\bar{y}(x) = \frac{5}{22}x(1-x)$ ...(*v*)

## 35.10 WEIGHTED RESIDUAL METHOD—Galerkin's Method

The starting point of this method is to guess a solution to the differential equation which satisfies the boundary conditions. This *trial solution* will contain certain parameters which can be adjusted to minimize the errors so that the trial solution is as close to the exact solution as possible.

Consider the boundary value problem

$$y'' = f(y, y', x) \text{ with } y(a) = A \text{ and } y(b) = B \quad ...(1)$$

We write the differential equation as $R = \bar{y}'' - f(\bar{y}, \bar{y}', x)$ ...(2)

where $R$ is the residual of the equation ($R = 0$ for the exact solution $y(x)$ only which will satisfy the boundary conditions).

Consider the trial solution as $\bar{y}(x) = c_1\,\phi_1(x) + c_2\,\phi_2(x) + \ldots$

where $\bar{y}(a) = A$ and $\bar{y}(b) = B$. The trial solution is differentiated twice and is substituted in (2).

To find $c_1, c_2, \ldots$, we weight the residual by trial functions $\phi_1(x), \phi_2(x), \ldots$ and set the integrals to zero. Thus we have $\int_c R\,\phi_1(x)\,dx = 0, \int_c R\,\phi_2(x)\,dx = 0, \ldots$

These lead to simultaneous equations in the unknowns.

Having found $c_1, c_2, \ldots$, the approximate solution $\bar{y}(x)$ is obtained.

**Example 35.16.** *Use Galerkin's method to solve the boundary value problem of Example 35.14. Compare your approximate solution with the exact solution.*

**Solution.** The residual is $\quad R = \bar{y}'' - \bar{y} + x$ ...(*i*)

To find the trial solution which satisfies the boundary conditions, we derive a Lagrangian polynomial (§ 28.8) which passes through the points :

| | | | | |
|---|---|---|---|---|
| $x$ | : | 0 | 1/2 | 1 |
| $y$ | : | 0 | $c$ | 0 |

The resulting polynomial is $\bar{y}(x) = 4cx(1-x)$, so that $\phi(x) = x(1-x)$.

Substituting $\bar{y}(x), \bar{y}''(x)$ in (*i*), we get $R = 4cx^2 + (1-4c)x - 8c$

Thus $\int R\,\phi(x)\,dx = 0$ gives $\int_0^1 [4cx^2 + (1-4c)x - 8c]\,x(1-x)\,dx = 0$ whence $c = 5/88$.

Hence the approximate solution is $\bar{y}(x) = \frac{5}{22}x(1-x)$ which is same as found in Example 34.14.

*Exact solution.* Rewriting the given equation as $(D^2 - 1)y = -x$,

we find its solution as $y = c_1 e^x + c_2 e^{-x} + x$

Since $y(0) = 0$ and $y(1) = 0$, therefore $c_2 = -c_1 = 1/(e - e^{-1})$.

Hence the exact solution is $y = x - \dfrac{e^x - e^{-x}}{e - e^{-1}}$

The approximate and the exact solutions for some values of $x$ are given below for comparison :

| $x$ | *Approx. Sol.* | *Exact Sol.* |
|---|---|---|
| 0.25 | 0.043 | 0.035 |
| 0.50 | 0.057 | 0.057 |
| 0.75 | 0.043 | 0.05 |

**Obs.** To obtain a trial solution containing two unknown parameters, we derive a Lagrangian polynomial which passes through the points :

| | | | | | |
|---|---|---|---|---|---|
| $x$ | : | 0 | 1/3 | 2/3 | 1 |
| $y$ | : | 0 | $c_1$ | $c_2$ | 0 |

More the undetermined parameters, the more accurate is the solution, but it involves more labour to find their values.

**Example 35.17.** *Find the approximate deflection of a simply supported beam under a uniformly distributed load w Fig. 35.7, using Galerkin's method.*

**Solution.** The differential equation governing the deflection $y(x)$ of the beam is

$$EI \frac{d^4 y}{dx^4} - w = 0, \; 0 < x < l \qquad (i) \; [\S\ 14.8]$$

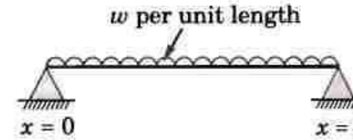



Fig. 35.7

The boundary conditions to be satisfied are

$$y\ (x = 0) = y\ (x = l) = 0 \quad \text{(deflection is zero at ends)} \qquad ...(ii)$$

$$\frac{d^2 y}{dx^2}(x = 0) = \frac{d^2 y}{dx^2}(x = l) = 0 \quad \text{(bending moment zero at ends)} \qquad ...(iii)$$

We assume the trial solution $\bar{y}(x) = c_1 \sin(\pi x/l) + c_2 \sin(3\pi x/l)$, which satisfies the boundary conditions $(ii)$ and $(iii)$.

Substituting the trial solution in $(i)$, we obtain the residual

$$R = EIc_1 (\pi/l)^4 \sin(\pi x/l) + EIc_2 (3\pi/l)^4 \sin(3\pi x/l) - w$$

Thus $\quad \int_0^l R \,.\, \sin(\pi x/l) = 0 \quad \text{and} \quad \int_0^l R \,.\, \sin(3\pi x/l) = 0$

Computing these integrals, we get

$$EIc_1 (\pi/l)^4\, l/2 - w \,.\, 2l/\pi = 0, \; EIc_2 (3\pi/l)^4\, l/2 - w \,.\, 2l/3\pi = 0$$

Solving these, we obtain $\quad c_1 = \dfrac{4wl^4}{\pi^5 EI}$ and $c_2 = \dfrac{4wl^4}{243\pi^5 EI}$.

Hence the deflection of the beam is given by

$$\bar{y}(x) = \frac{4wl^4}{\pi^5 EI}\left\{\sin\left(\frac{\pi x}{l}\right) + \frac{1}{243}\sin\left(\frac{3\pi x}{l}\right)\right\}.$$

## PROBLEMS 35.4

1. Solve the boundary value problem :
   $y'' + y + x = 0 \; (0 \le x \le 1), \; y(0) = y(1) = 0$ by
   $(i)$ Rayleigh-Ritz method, $\quad (ii)$ Galerkin's method. Compare your solution with the exact solution.
2. Using Galerkin's method, solve the boundary value problem $y'' = 3x + 4y$ ; $y(0) = 0, y(1) = 1$.
3. *Apply Galerkin's method to the boundary value problem $y'' + y + x = 0$ $(0 \le x \le 1)$ ; $y(0) = y(1) = 0$, to find the coefficients of the approximate solution $\bar{y}(x) = c_1 x(1 - x) + c_2 x^2(1 - x)$.*

[**Hint.** Substituting $\bar{y}(x)$, $\bar{y}''(x)$ in the given equation replacing $y, y''$ by $\bar{y}, \bar{y}''$, we get the residual $R = -2c_1 + c_2(2-6x) + x(1-x)(c_1 + c_2 x) + x$

Thus $\int_0^1 R \, . \, x(1-x)\,dx = 0$ and $\int_0^1 R \, . \, x^2(1-x)\,dx = 0.$

Computing these integrals, we get

$$\frac{3}{10}c_1 + \frac{3}{20}c_2 = \frac{1}{12}, \frac{3}{20}c_1 + \frac{13}{305}c_2 = \frac{1}{20}.$$

Solving these, we obtain $c_1 = 71/369$, $c_2 = 7/41$.]

4. Using Ritz method, find an approximate solution of the problem $y'' - y + 4xe^x = 0$, $y'(0) - y(0) = 1$, $y'(1) + y(1) = -e$.
5. Solve the boundary value problem : $y'' + (1 + x^2)y + 1 = 0$, $y(-1) = y(1) = 0$, by taking the approximate solution $\bar{y}(x) = c_1(1 - x^2) + c_2 x^2(1 - x^2)$ and using (*i*) Ritz method, (*ii*) Galerkin's method.
6. Given the boundary value problem : $y'' + \pi^2 y = x$, $y(0) = 1$, $y(1) = -0.9$.
   Use Galerkin's method to estimate $y(0.5)$, taking the trial solution :
   $$y = 1 - 1.9x + c_1 x(1-x) + c_2 x^2(1-x).$$
7. Using Galerkin's method, obtain an approximate solution of the boundary value problem :
   $$\frac{d}{dx}\left(x\frac{dy}{dx}\right) + y = x,\ y(0) = 0,\ y(1) = 1,$$
   in the form $\bar{y}(x) = x + x(1-x)(c_1 + c_2 x)$.
8. Of all the parabolas which pass through (0, 0) and (1, 1), determine the one, which when rotated about the $x$-axis, generates a solid of revolution with least possible volume between $x = 0$ and $x = 1$.
   [**Hint.** Take the parabola as $y = x + cx(1-x)$.]
9. *Using Rayleigh-Ritz method, find the potential at any point due to a charged sphere of radius a.*
   [**Hint.** Potential at a distance $r$ from the centre of the sphere is $\phi = \phi_0 (r/a)^k$, where $\phi_0$ is the value of $\phi$ for $r = a$ and $k < 0$ so that $\phi \to 0$ as $r \to \infty$.
   Electrostatic field due to charged sphere being *conservative*, electrostatic intensity $E = \nabla\phi$.
   Also potential energy for unit volume $= \frac{1}{8\pi}E^2$
   $\therefore$ Total potential energy of the field in the entire region $R$ exterior to the given sphere is
   $$V = \frac{1}{8\pi}\iiint_R E^2\,dx\,dy\,dz = \frac{1}{8\pi}\iiint_R \left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right] dx\,dy\,dz$$
   $$= \frac{1}{8\pi}\int_a^\infty \int_0^\pi \int_0^{2\pi} \left(\frac{\partial\phi}{\partial r}\right)^2 r^2 \sin\theta\, d\phi\, d\theta\, dr.$$
   Electrostatic field will be in stable equilibrium if $V$ is minimum, *i.e.*, $dV/dp = 0$ and $d^2V/dp^2 > 0$.
   This gives $k = -1$. Hence $\phi = \phi_0\, a/r$.]

## 35.11 HAMILTON'S PRINCIPLE*

An important concept of mathematical physics is *Hamilton's principle* which states that *the time integral of the difference between the kinetic and potential energies of a dynamical system is stationary*

Consider a particle of mass $m$ moving from a fixed origin $O$ under the effect of a force $F$ (Fig. 35.8). At any time $t$, let its position vector be $\mathbf{R}$. Then by Newton's second law,

$$\frac{md^2\mathbf{R}}{dt^2} = \mathbf{F} \qquad ...(1)$$

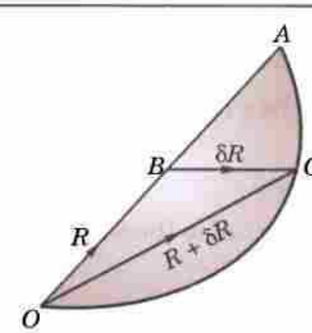



Fig. 35.8

* Named after the Irish mathematician *William Rowan Hamilton* (1805–1865) who is known for his contributions to dynamics.

Let the natural path $OBA$ of the particle be changed to another path $OCA$, end points remaining the same. Let this variation in path, often called *virtual displacement*, be $\delta\mathbf{R}$. Then the work done during this displacement is

$$\delta W = \mathbf{F} \cdot \delta\mathbf{R} = \frac{md^2\mathbf{R}}{dt^2} \cdot \delta\mathbf{R} \qquad \text{[By (1)]}$$

Also the kinetic energy of the particle is

$$T = \frac{1}{2} m \left(\frac{d\mathbf{R}}{dt}\right)^2$$

$$\therefore \qquad \delta T = m \frac{d\mathbf{R}}{dt} \cdot \delta\left(\frac{d\mathbf{R}}{dt}\right) = m \frac{d\mathbf{R}}{dt} \cdot \frac{d}{dt}(\delta\mathbf{R})$$

Thus $$\delta T + \delta W = m \frac{d\mathbf{R}}{dt} \cdot \frac{d}{dt}(\delta\mathbf{R}) + m \frac{d^2\mathbf{R}}{dt^2} \cdot \delta\mathbf{R} = m \frac{d}{dt}\left(\frac{d\mathbf{R}}{dt} \cdot \delta\mathbf{R}\right)$$

Integrating both sides w.r.t. $t$ from $t_0$ to $t_1$, we get

$$\int (\delta T + \delta W)\, dt = m \left| \frac{d\mathbf{R}}{dt} \cdot \delta\mathbf{R} \right|_{t_0}^{t_1} = 0 \qquad \text{...(2)} \quad [\because \ \delta\mathbf{R} = 0 \text{ at } t_0 \text{ and } t_1]$$

If the force field is conservative, there exists a potential $V$ such that $W = -V$. Then (2) takes the form

$$\int (\delta T - \delta V)\, dt = 0 \quad \text{or} \quad \delta \int (T - V)\, dt = 0 \quad i.e., \quad \int (T - V)\, dt \qquad \text{...(3)}$$

is stationary. This proves the Hamilton's principle for a particle. Its derivation can be extended to a system of particles by summation and to a rigid body by integration. Hence the principle is true for any dynamical system.

**Obs.** It can be easily shown that *the integral (3) is a minimum along the natural path* as compared to that along any other path joining $O$ to $A$.

**Def.** *The energy difference $T - V = L$ is called the* **kinetic potential** *or the* **Lagrangian function.**

## 35.12 LAGRANGE'S EQUATION

In a dynamical system, the position of a body can be specified by the quantities $q_1, q_2, \ldots q_n$ which are called the *generalised coordinates.*

The potential energy $V$, being a function of position only depends on these generalised coordinates $q_i$. The kinetic energy $T$, however, depends upon $q_i$ and the velocities $dq_i/dt$ (*i.e.*, $q_i$) $i = 1, 2, \ldots n$. Therefore, Lagrangian function $L = T - V$ is also a function of $q_1$ and $q_i$, $i = 1, 2, \ldots n$.

Thus by Hamilton's principle, the system moves so that $\int_{t_0}^{t_1} L\, dt$ is stationary.

$\therefore$ Euler's equation must hold good, *i.e.*, $\dfrac{\partial L}{\partial q_i} - \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial q_i}\right) = 0, i = 1, 2, \ldots n.$

These are called *Lagrange's equations* which determine the motion of the system.

**Example 35.18.** *A mass, suspended at the end of a light spring having spring constant k, is set into vertical motion. Use Lagrange's equation, to find the equation of motion of the mass.*

**Solution.** At any time $t$, let the displacement of $m$ from the equilibrium position $O$ be $x$ (Fig. 35.9). Then the kinetic energy of $P$ is

$$T = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2$$

Also the work done during its fall from $O$ to $P$ is

$$W = \int_0^x (mg - kx)\, dx = mgx - \tfrac{1}{2} kx^2.$$

If $V$ is the potential energy of the mass at $P$, then

$$V = -W = \tfrac{1}{2} kx^2 - mgx$$

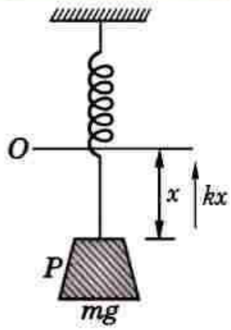


Fig. 35.9

$\therefore$ Lagrangian $\qquad L = T - V = \frac{1}{2}m\dot{x}^2 + mgx - \frac{1}{2}kx^2.$

Thus the Lagrange's equation

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0$$

becomes $\qquad (mg - kx) - \frac{d}{dt}(m\dot{x}) = 0 \quad \text{or} \quad m\frac{d^2x}{dt^2} = mg - kx$

which is the required equation of motion.

**Example 35.19.** *Apply Lagrange's equations, to show that the equations of motion of the double pendulum of Fig. 35.10 are given by*

$$(m_1 + m_2)\, l_1 \ddot{\theta}_1 + m_2\, l_2 \ddot{\theta}_2 + (m_1 + m_2)\, g\theta_1 = 0$$

*and* $\qquad l_1\ddot{\theta}_1 + l_2\ddot{\theta}_2 + g\theta_2 = 0$

*for small angles* $\theta_1, \theta_2$. *(Punjab, M.E., 1997)*

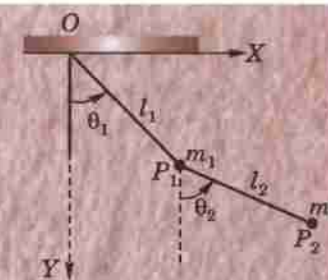



Fig. 35.10

**Solution.** At any time $t$, let the masses $m_1, m_2$ be at $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ (Fig. 35.10) so that

$$\left.\begin{aligned} x_1 &= l_1 \sin\theta_1,\ y_1 = l_1\cos\theta_1 \\ x_2 &= l_1\sin\theta_1 + l_2\sin\theta_2,\ y_2 = l_1\cos\theta_1 + l_2\cos\theta_2 \end{aligned}\right\} \qquad ...(i)$$

Then total kinetic energy is

$$T = \frac{1}{2}m_1(\dot{x}_1^{\,2} + \dot{y}_1^{\,2}) + \frac{1}{2}m_2(\dot{x}_2^{\,2} + \dot{y}_2^{\,2})$$

$$= \frac{1}{2}(m_1 + m_2)\, l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2) \qquad \text{[Using (i)]}$$

Also total potential energy is

$$V = m_1 g l_1 (1 - \cos\theta_1) + m_2 g(l_1 + l_2 - l_1\cos\theta_1 - l_2\cos\theta_2)$$

$\therefore$ Lagrangian $\qquad L = T - V = \frac{1}{2}(m_1 + m_2)\, l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2$

$$+ m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2) - (m_1 + m_2)\, g l_1(1 - \cos\theta_1) - m_2 g l_2(1 - \cos\theta_2) \qquad ...(ii)$$

Thus the Lagrange's equation corresponding to $\theta_1$, is

$$\frac{\partial L}{\partial \theta_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = 0, \text{ becomes}$$

$$-m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2 \sin(\theta_1 - \theta_2) - (m_1 + m_2)\, g l_1 \sin\theta_1 - \frac{d}{dt}\left[(m_1 + m_2)\, l_1^2\,\dot{\theta}_1 + m_2 l_1 l_2\,\dot{\theta}_2\cos(\theta_1 + \theta_2)\right] = 0$$

or $\qquad (m_1 + m_2)\, l_1\ddot{\theta}_1 + m_2\, l_2\ddot{\theta}_2\cos(\theta_1 - \theta_2) + m_2 l_2\,\dot{\theta}_2^{\,2}\sin(\theta_1 - \theta_2) + (m_1 + m_2)\, g\sin\theta_1 = 0 \qquad ...(iii)$

Similarly from *(ii)*, Lagrange's equation corresponding to $\theta_2$, *i.e.*,

$$\frac{\partial L}{\partial \theta_2} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = 0$$

becomes $\qquad m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) - m_2 g l_2 \sin\theta_2 - \frac{d}{dt}\left[m_2 l_2^2\,\dot{\theta}_2 + m_2 l_1 l_2\,\dot{\theta}_1\cos(\theta_1 - \theta_2)\right] = 0$

or $\qquad l_2\ddot{\theta}_2 + l_1\ddot{\theta}_1\cos(\theta_1 - \theta_2) - l_1\dot{\theta}_1^2\sin(\theta_1 - \theta_2) + g\sin\theta_2 = 0 \qquad ...(iv)$

Now $\dot{\theta}_1$ and $\dot{\theta}_2$ being small, retaining first order terms only, *(iii)* and *(iv)* reduce to

$$(m_1 + m_2)\, l_1\ddot{\theta}_1 + m_2\, l_2\ddot{\theta}_2 + (m_1 + m_2)\, g\theta_1 = 0 \quad \text{and} \quad l_1\ddot{\theta}_1 + l_2\ddot{\theta}_2 + g\theta_2 = 0.$$

## PROBLEMS 35.5

1. In a single pendulum, a mass $m$ is suspended by a light rod of length $l$ and the system vibrates in a plane. Using Lagrange's equation, show that $\ddot{\theta} + (g/l) \sin \theta = 0$.

   Show that if $\theta$ is small, the period of oscillation is $2\pi \sqrt{(l/g)}$.
2. Two masses $m_1$ and $m_2$ are connected by an inextensible string which passes over a fixed pulley. Using Lagrange's equations, show that the acceleration of either mass is numerically
$$= (m_1 - m_2)\, g/(m_1 + m_2).$$
3. A perfectly flexible rope of uniform density per unit length is suspended with its end points fixed. Show that it assumes the shape of a catenary.
4. A bead of mass $m$ from rest slides without friction under gravity along a wire inclined at an angle $\alpha$ to the vertical and rotating with constant angular velocity $\omega$. Show that in times $t$, the bead has slided through a distance
$$\frac{g \cos \alpha}{\omega^2 \sin^2 \alpha} \cosh (\omega t \sin \alpha - 1).$$

CHAPTER 36

# Integral Equations



## 36.1 INTRODUCTION

Integral equations play an effective role in the study of boundary value problems. Such equations also occur in many fields of mechanics and mathematical physics. Integral equations may be obtained directly from physical problems *e.g.*, radiation transfer problem and neutron diffusion problem etc. They also arise as representation formulae for the solutions of differential equations. A differential equation can be replaced by an integral equation with the help of initial and boundary conditions.

Integral equations were first encountered in the theory of Fourier integrals. In 1826, another integral equation was obtained by *Abel*. Actual development of the theory of integral equations began with the works of the Italian mathematician V. Volterra (1896) and the Swedish mathematician I. Fredholm (1900).

## 36.2 DEFINITION

*An integral equation is an equation in which an unknown function appears under the integral sign.* We shall take up integral equations in which only linear functions of the unknown function are involved. The general type of *linear integral equation* is of the form

$$y(x) = F(x) + \lambda \int_a^b K(x, t)\, y(t)\, dt$$

where $F(x)$ and $K(x, t)$ are known functions while $y(x)$ is to be determined. The function $K(x, t)$ is called the *Kernel of the integral equation.*

If $a$ and $b$ are constants, the equation is known as *Fredholm integral equation.*

If $a$ is a constant while $b$ is a variable, it is called a *Volterra integral equation.*

## 36.3 CONVERSION OF A LINEAR DIFFERENTIAL EQUATION TO AN INTEGRAL EQUATION AND VICE VERSA

*To make this transformation, the use of the following formula is necessary :*

$$\int_a^x \int_a^x f(x)\, dx\, dx = \int_a^x (x - t)\, f(t)\, dt \qquad ...(I)$$

In general, $$\int_a^x \int_a^x ... \int_a^x f(x)\, dx^n = \frac{1}{(n-1)!} \int_a^x (x - t)^{n-1}\, f(t)\, dt \qquad ...(II)$$

**Proof.** Let $$I_n = \int_a^x (x-t)^{n-1} f(t)\, dt \qquad \text{...(1)}$$

where $n$ is a positive integer and $a$ is a constant.

Differentiating both sides w.r.t. $x$, using Leibnitz's rule (p. 139)

$$\frac{dI_n}{dx} = \int_a^x \frac{\partial}{\partial x}(x-t)^{n-1} f(t)\, dt + [(x-t)^{n-1} f(t)]_{t=x} \cdot 1 - [(x-t)^{n-1} f(t)]_{t=a} \cdot 0$$

$$= \int_a^x (n-1)(x-t)^{n-2} f(t)\, dt = (n-1)\, I_{n-1}(x) \qquad \text{...(2)}$$

Again differentiating (2) w.r.t. $x$

$$\frac{d^2 I_n}{dx^2} = (n-1)\frac{d}{dx}\,[I_{n-1}, (x)] = (n-1)(n-2)\, I_{n-2}, \text{ using (1)}$$

Proceeding in this way, we get

$$\frac{d^{n-1} I_n}{dx^{n-1}} = (n-1)(n-2) \ldots 1 \,.\, I_1(x) = (n-1)!\, I_1(x)$$

Now taking $n = 1$ in (1), we get

$$I_1 = \int_a^x f(t)\, dt = \int_a^x f(x_1)\, dx_1 \qquad \text{...(3)}$$

Putting $x = a$ in (1), we obtain

$$I_n(a) = 0 \text{ for all } n$$

Taking $n = 2$ in (2), we get $\dfrac{dI_2}{dx} = I_1(x)$

$$\therefore \quad I_2 = \int_a^x I_1(x_2)\, dx_2 \qquad [\because\ I_2(a) = 0]$$

$$= \int_a^x \int_a^{x_2} f(x_1)\, dx_1\, dx_2 \qquad \text{[Using (3)] ...(4)}$$

Putting $n = 3$ in (2), we have $\dfrac{dI_3}{dx} = 2I_2(x)$

$$\therefore \quad I_3 = 2\int_a^x I_2(x)\, dx \qquad [\because\ I_3(a) = 0]$$

$$= 2\int_a^x \int_a^{x_3} \int_a^{x_2} f(x_1) dx_1\, dx_2\, dx_3 \qquad \text{[Using (4)]}$$

Proceeding in this way, we get

$$I_n = (n-1)!\, \int_a^x \int_a^{x_n} \cdots \int_a^{x_2} F(x_1)\, dx_1\, dx_2 \ldots dx_n$$

*i.e.,* $$\int_a^x \int_a^{x_n} \cdots \int_a^{x_2} f(x_1)\, dx_1\, dx_2 \ldots dx_n = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)\, dt$$

If $x_2, x_3, \ldots x_n$ be the same as $x$, we get the formula (II) above.

**Example 36.1.** *Convert the differential equation $y''(x) - 3y'(x) + 2y(x) = 5 \sin x$, $y(0) = 1$, $y'(0) = -2$ into an integral equation.*

**Solution.** Integrating both sides of the given differential equation, we get

$$[y'(x) - y'(0)] - 3[y(x) - y(0)] + 2\int_0^x y(t)\ dt = 5(1 - \cos x)$$

Since $y'(0) = -2$ and $y(0) = 1$, it becomes

$$y'(x) + 2 - 3y(x) + 3 + 2\int_0^x y(t)\ dt = 5 - 5\cos x$$

*i.e.,* $$y'(x) - 3y(x) + 2\int_0^x y(t)\,dt = -5\cos x$$

Integrating again as before, we have

$$[y(x) - y(0)] - 3\int_0^x y(t)\,dt + 2\int_0^x \int_0^x y(t)\,dt = -5\sin x$$

or $$y(x) - 1 - 3\int_0^x y(t)\,dt + 2\int_0^x (x-t)\,y(t)\,dt = -5\sin x \qquad \text{[Using (I) above]}$$

or $$y(x) + \int_0^x [2(x-t) - 3]\,y(t)\,dt = 1 - 5\sin x$$

which is the desired integral equation.

**Example 36.2.** *Show that the integral equation*

$$y(x) = \int_0^x (x+t)\,y(t)\,dt + 1 \qquad ...(i)$$

*is equivalent to the differential equation*

$$y''(x) - 2x\,y'(x) - 3y(x) = 0,\; y(0) = 1,\; y'(0) = 0. \qquad \textit{(Kerala, M. Tech., 2005)}$$

**Solution.** Differentiating $(i)$ by Leibnitz's rule (p. 139), we have

$$\frac{dy}{dx} = \int_0^x \frac{\partial}{\partial x}(x+t)\,y(t)\,dt + (x+x)\,y(x)\frac{d}{dx}(x) - (x+0)\,y(0)\frac{d}{dx}(0)$$

$$= \int_0^x y(t)\,dt + 2xy(x) = \int_0^x y(x)\,dx + 2xy(x) \qquad ...(ii)$$

Differentiating again w.r.t. $x$, we get

$$\frac{d^2y}{dx^2} = y(x) + 2[xy'(x) + 1\,.\,y(x)] = 2xy'(x) + 3y(x)$$

or $$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 3y(x) = 0 \qquad ...(iii)$$

Putting $x = 0$ in $(i)$, we obtain

$$y(0) = \int_0^0 (x+t)\,y(t)\,dt + 1 \quad i.e., \quad y(0) = 1$$

and putting $x = 0$ in $(ii)$, we get $y'(0) = 0$.

Hence $(i)$ is equivalent to $(iii)$ with initial conditions $y(0) = 1$, $y'(0) = 0$.

**Example 36.3.** *Show that the integral equation*

$$y(x) = \int_0^x t(t-x)\,y(t)\,dt + \frac{1}{2}x^2, \qquad ...(i)$$

*is equivalent to the differential equation*

$$\frac{d^2y}{dx^2} + xy = 1 \text{ and the conditions } y(0) = y'(0) = 0.$$

**Solution.** Differentiating $(i)$ w.r.t. $x$ by Leibnitz's rule (p. 139), we have

$$\frac{dy}{dx} = \int_0^x \frac{\partial}{\partial x}[t\,(t-x)]\,y(t)\,dt + [t\,(t-x)\,y(t)]_{t=x}\,.\,1 + x - [t\,(t-x)\,y(t)]_{t=0}\,.\,0$$

$$= \int_0^x t(-1)\,y(t)\,dt + x = x - \int_0^x ty(t)\,dt \qquad ...(ii)$$

Differentiating $(ii)$ w.r.t. $x$, we get

$$\frac{d^2y}{dx^2} = 1 - \left\{\int_0^x \frac{\partial}{\partial x}[t\,y(t)]\,dt - [t\,y(t)]_{t=0}\cdot 0 + [ty(t)]_{t=x}\cdot 1\right\} = 1 - xy(x)$$

or $$\frac{d^2y}{dx^2} + xy = 1 \text{ which is the differential equation corresponding to } (i).$$

Also $y(0) = 0$ and $y'(0) = 0$.

**Example 36.4.** *Find the integral equation corresponding to the boundary value problem*

$$y''(x) + \lambda y(x) = 0, \qquad y(0) = y(1) = 0.$$

**Solution.** Integrating both sides of the given differential equation w.r.t. $x$ over $(0, x)$, we get

$$y'(x) - y'(0) + \lambda \int_0^x y(x)\, dx = 0$$

or $$y'(x) = c - \lambda \int_0^x y(x)\, dx, \text{ taking } y'(0) = c \quad ...(i)$$

Again integrating $(i)$ w.r.t. $x$ in $(0, x)$, we obtain

$$y(x) - y(0) = cx - \lambda \int_0^x \int_0^x y(x)\, dx$$

$$y(x) = cx - \lambda \int_0^x (x-t)\, y(t)\, dt \quad ...(ii)$$

[Using (I) of § 36.3 and noting that $y(0) = 0$]

Putting $x = 1$ in $(ii)$, we get

$$y(1) = c - \lambda \int_0^1 (1-t)\, y(t)\, dt \qquad [\because \ y(1) = 0]$$

$$\therefore \qquad c = \lambda \int_0^1 (1-t)\, y(t)\, dt \quad ...(iii)$$

Substituting the value of $c$ from $(iii)$ in $(ii)$, we get

$$y(x) = \lambda x \int_0^1 (1-t)\, y(t)\, dt - \lambda \int_0^x (x-t)\, y(t)\, dt$$

$$= \lambda x \left\{ \int_0^x (1-t)\, y(t)\, dt + \int_x^1 (1-t)\, y(t)\, dt \right\} - \lambda \int_0^x (x-t)\, y(t)\, dt$$

$$= \lambda \int_0^x t(1-x)\, y(t)\, dt + \lambda \int_x^1 (1-t)\, y(t)\, dt$$

$$= \lambda \left\{ \int_0^x K(x,t)\, y(t)\, dt + \int_x^1 K(x,t)\, y(t)\, dt \right\}$$

where $$K(x,t) = \begin{cases} t(1-x) & \text{when } t < x \\ x(1-t) & \text{when } t > x \end{cases}$$

Hence $$y(x) = \lambda \int_0^1 K(x,t)\, y(t)\, dt$$

which is a *Fredholm integral equation* with a symmetric kernel $K(x, t)$.

## PROBLEMS 36.1

Transform each of the following boundary value problems into corresponding integral equations :

1. $y'' + xy' + y = 0$, given that $y(0) = 1$, $y'(0) = 0$. *(Madras, M.E., 2000)*
2. $y'' - \sin xy' + e^y = x$, given that $y = 1$, $\dfrac{dy}{dx} = -1$ when $x = 0$.
3. $y'' + xy' = 1$, given that $y(0) = y'(0) = 0$. *(Madras, 2000 S)*
4. $y'' + (1-x)\,y' + e^{-x} y = x^3 - 5x$, given that $y = -3$, $\dfrac{dy}{dx} = 4$ when $x = 0$.
5. $\dfrac{d^3y}{dx^3} + y = \cos x$ given that $y = 0$, $y' = 1$, $y'' = 2$ at $x = 0$. *(Kerala, M. Tech, 2005)*

6. $\dfrac{d^3y}{dx^3}+\dfrac{d^2y}{dx^2}-xy=\sin x$ given that $y=1, y'=-1, y''=\dfrac{1}{2}$ at $x=0$.

7. $\dfrac{d^4y}{dx^4}-4\dfrac{d^3y}{dx^3}+6\dfrac{d^2y}{dx^2}-4\dfrac{dy}{dx}+y=4\cos 2x$ given that
$y=-1, y'=4, y''=0, y'''=2$ when $x=0$.

Convert each of the following integral equations into differential equations alongwith initial conditions :

8. $y(x)=\int_0^x (x+t)\,y(t)\,dt-1.$

9. $y(x)=\int_0^x (x-t)\,y(t)\,dt+3\sin x.$

10. $y(x)+3\int_0^x (x-t)^2\,y(t)\,dt=x^2-3x+4.$

11. $y(x)+\int_0^x [(x-t)^2+4(x-t)-3]\,y(t)\,dt=e^{-x}.$

12. If $y''(x)=f(x)$ ; $y(0)=y(l)=0$, show that $y(x)=\int_0^l K(x,t)\,f(t)\,dt$,

where $K(x,t)=\begin{cases}\dfrac{t}{l}(x-l) & \text{when } t<x\\ \dfrac{x}{l}(t-l) & \text{when } t>x\end{cases}$

## 36.4 CONVERSION OF BOUNDARY VALUE PROBLEMS TO INTEGRAL EQUATIONS USING GREEN'S FUNCTION

Consider the linear homogeneous differential equation

$$L(y)+\phi(x)=0 \qquad ...(1)$$

where $$L(y)=\left[\frac{d}{dx}\left(p\frac{d}{dx}\right)+q\right]y=py''+p'y'+qy \qquad ...(2)$$

together with the homogeneous boundary conditions of the form

$$\alpha y+\beta\frac{dy}{dx}=0 \qquad ...(3)$$

Now let us find a function $G(x,t)$ which for a given number $t$, is given by $G_1(x)$ when $x<t$ and by $G_2(x)$ when $x>t$ and which has the following properties :

I. $G_1$ and $G_2$ satisfy the equation $L(G)=0$ in their defined intervals *i.e.*, $L(G_1)=0$ when $x<t$ ; $L(G_2)=0$ when $x>t$.

II. $G_1$ and $G_2$ satisfy the boundary conditions at the end points $x=a$ and $x=b$ respectively.

III. $G(x,t)$ is continuous at $x=t$ *i.e.*, $G_1(t)=G_2(t)$.

IV. The derivative of $G$ is continuous at every point within the range of $x$ except at $x=t$ *i.e.*, $G_2'(t)-G_1'(t)=-1/p(t)$

**Def.** *$G(x,t)$ as defined above is called the* **Green's function.**

*If $G(x,t)$ exists, then the solution of the given problem can be transformed to the integral equation*

$$y(x)=\int_a^b G(x,t)\,\phi(t)\,dt \qquad ...(4)$$

Let $y=y_1(x)$ and $y=y_2(x)$ be the non-trivial solutions of the equations $L(y)=0$ which satisfy the homogeneous conditions at $x=a$ and $x=b$ respectively.

The above conditions $I$ and $II$ are satisfied if we write

$$G=\begin{cases}C_1y_1(x), & \text{when } x<t\\ C_2y_2(x), & \text{when } x>t\end{cases} \qquad ...(5)$$

Imposing the condition III on (5), we get

$$C_2y_2(t)-C_1y_1(t)=0 \qquad ...(6)$$

Imposing the condition IV on (5), we have

$$C_2y_2'(t)-C_1y_1'(t)=-1/p(t) \qquad ...(7)$$

Equations (6) and (7) give a unique solution, if

$$\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)\, y_2'(t) - y_2(t)\, y_1'(t) \neq 0$$

By Abel's formula*, we find that

$$y_1(t)\, y_2'(t) - y_2(t)\, y_1'(t) = C/p(t) \qquad \text{...(8)}$$

Now $[(7) \times y_2(t) - (6) \times y_1(t)]$ gives on using (8),

$$C_1 = -\frac{1}{C}\, y_2(t) \text{ and } C_2 = -\frac{1}{C}\, y_1(t)$$

Thus (5) reduces to
$$G(x, t) = \begin{cases} -\dfrac{1}{C}\, y_1(x)\, y_2(t), & x < t \\ -\dfrac{1}{C}\, y_1(t)\, y_2(x), & x > t \end{cases} \qquad \text{...(9)}$$

*Conversely it can be shown that the integral equation.*

$$y(x) = \int_a^b G(x, t)\, \phi(t)\, dt$$

*where $G(x, t)$ is as defined by (9), satisfies the differential equation $L(y) + \phi(x) = 0$ together with the prescribed boundary conditions.*

**Example 36.5.** *Transform the differential equation $y'' + y = x$, $y(0) = 1$, $y'(1) = 0$ to a Fredholm integral equation, finding the corresponding Green's function.* (Madras M.E., 2000 S)

**Solution.** Given equation is $\quad L(y) + \phi(x) = 0 \qquad$ ...(*i*)

with the conditions $y(0) = 0$, $y'(1) = 0$, where $L(y) = y''$ and $\phi(x) = y - x$

The associated equation $\quad L(y) = 0$ is $y''(y) = 0 \qquad$ ...(*ii*)

Its solution is $\quad y = C_1 x + C_2 \qquad$ ...(*iii*)

Now $y_1(x)$ is a particular solution of (*ii*) satisfying the condition $y(0) = 0$.

$\therefore$ Taking $C_2 = 0$ and $C_1 = 1$, we get $y_1(x) = x \qquad$ ...(*iv*)

$y_2(x)$ is another solution of (*ii*) satisfying the condition $y'(1) = 0$.

$\therefore$ From (*iii*), $y'(1) = C_1 = 0$. Then $y(x) = C_2$

Taking $C_2 = 1$, a particular solution is $y_2(x) = 1$.

The constant $C$ is found from $y_1(x)\, y_2'(x) - y_2(x) y_1'(x) = \dfrac{C}{p(x)}$

Since $L(y) = py'' + p'y' + qy = y''$ (given), $\quad \therefore \quad p = 1$.

Thus $C = y_1(x)\, y_2'(x) - y_2(x)\, y_1'(x) = x \,.\, 0 - 1.1 = -1$.

$\therefore$ Green's function is given by

$$G(x, t) = \begin{cases} \dfrac{-y_1(x)\, y_2(t)}{C}, & x < t \\ \dfrac{-y_1(t)\, y_2(x)}{C}, & x > t \end{cases} = \begin{cases} x, & x < t \\ t, & x > t \end{cases} \qquad \text{...(v)}$$

Hence the equation (*i*) is equivalent to the integral equation

$$y(x) = \int_0^1 G(x, t)\, \phi(t)\, dt = \int_0^1 G(x, t) \,.\, (y - t)\, dt$$

$$= \int_0^1 G(x, t)\, y(t)\, dt - \left\{ \int_0^x x \,.\, t\, dt + \int_x^1 t \,.\, t\, dt \right\}$$

---

* The conditions that $y_1(x)$ and $y_2(x)$ satisfy the equation

$L(y) = 0$ are $(p\, y_1')' + q y_1 = 0 \qquad$ ...(*i*), $\qquad (p y_2')' + q y_2 = 0 \qquad$ ...(*ii*)

$[(i) \times y_2 - (ii) \times y_1]$ gives $y_2 (p y_1')' - y_1 (p y_2')' = 0$

or $\qquad [p(y_1 y_2' - y_2 y_1')]' = 0 \quad$ or $\quad y_1 y_2' - y_2 y_1' = C/p$

$$= \int_0^1 G(x,t)\,y(t)\,dt - \left\{ x \left| \frac{t^2}{2} \right|_0^x + \left| \frac{t^3}{3} \right|_x^1 \right\}$$

$$= \int_0^1 G(x,t)\,y(t)\,dt - \frac{1}{6}(x^3 + 2)$$

where $G(x, t)$ is given by $(v)$.

**Example 36.6.** *Find the Green's function for the boundary value problem*

$$d^2y/dx^2 + \mu^2 x = 0, \qquad y(0) = 0 = y(1).$$

**Solution.** We observe that the solution $y_1 = \sin \mu x$ satisfies the boundary condition $y(0) = 0$ and the solution $y_2 = \sin \mu(x - 1)$ satisfies the second condition $y(1) = 0$. Also both these solutions are linearly independent.

The constant $C$ is found from $y_1 y_2' - y_2 y_1' = C/p\,(x)$.

Since $L(y) = py'' + p'y' + qy = y''$ (given), $\quad \therefore \; p = 1$

$\therefore \quad C = y_1 y_2' - y_2 y_1' = \mu \sin \mu x \cos \mu(x-1) - \mu \sin \mu(x-1) \cos \mu x = \mu \sin \mu$

Hence the Green's function is given by

$$G(x,t) = \begin{cases} -\dfrac{y_1(x)\,y_2(t)}{C}, & x < t \\ -\dfrac{y_1(t)\,y_2(x)}{C}, & x > t \end{cases} = \begin{cases} -\dfrac{\sin \mu x \sin \mu(t-1)}{\mu \sin \mu}, & x < t \\ -\dfrac{\sin \mu t \sin \mu(x-1)}{\mu \sin \mu}, & x > t \end{cases}$$

## PROBLEMS 36.2

1. Transform the problem $d^2y/dx^2 + xy = 1$ ; $y(0) = 0 = y(1)$ to an integral equation, finding the corresponding Green's function.
2. Transform the problem $y'' + y = x$ ; $y(0) = 1, y'(1) = 0$ to an integral equation using Green's function.
3. Construct an integral equation corresponding to the boundary value problem

$$\frac{d^2u}{dx^2} + e^u\,u = x\;;\; u(0) = 0,\, u(1) = 1.$$

4. Find the Green's function for the boundary value problem $d^2y/dx^2 - y = 0$ with $y(0) = y(1) = 0$.
5. Transform the boundary value problem $x^2 \dfrac{d^2u}{dx^2} + x\dfrac{du}{dx} + (\lambda x^2 - 1)\,u = 0$ ; $u(0) = u(1) = 0$, to an integral equation.

[**Hint.** $u_1(x) = x$, $u_2(x) = \dfrac{1}{x} - x$ and $C = -2$]

## 36.5 SOLUTION OF AN INTEGRAL EQUATION

*The solution of the integral equation* $y(x) = F(x) + \lambda \int_a^b K(x,t)\; y(t)\,dt$ *is a function* $y(x)$ *which when substituted in the equation reduces it to an identity w.r.t. x.*

**Example 36.7.** *Show that* $y(x) = 2 - x$ *is a solution of the integral equation*

$$\int_0^x e^{x-t}\,y(t)\,dt = e^x + x - 1. \qquad \text{...(i)}$$

**Solution.** Since $\qquad y(t) = 2 - t$

$$\therefore \quad \int_0^x e^{x-t}\,y(t)\,dt = \int_0^x e^{x-t}\,(2-t)\,dt$$

$$= 2e^x \int_0^x e^{-t}\,dt - e^x \int_0^x t\,e^{-t}\,dt$$

$$= 2e^x \Big| -e^{-t} \Big|_0^x - e^x \left\{ \Big| t.(-e^{-t}) \Big|_0^x - \int_0^x 1.(-e^{-t})\,dt \right\}$$

$$= 2e^x(-e^{-x}+1) + e^x(xe^{-x}) - e^x \Big| -e^{-t} \Big|_0^x$$

$$= -2 + 2e^x + x + e^x(e^{-x}-1) = e^x + x - 1.$$

Thus ($i$) is identically satisfied by $y(x) = 2 - x$. Hence $y(x) = 2 - x$ is a solution of ($i$).

**Example 36.8.** *Show that the function $y(x) = (1 + x^2)^{-3/2}$ is a solution of the Volterra integral equation :*

$$y(x) = \frac{1}{1+x^2} - \int_0^x \frac{t}{1+x^2}\, y(t)\, dt. \qquad ...(i)$$

**Solution.** Substituting $y(x) = (1 + x^2)^{-3/2}$ in the R.H.S. of ($i$), we have

$$\frac{1}{1+x^2} - \int_0^x \frac{t}{1+x^2} \cdot \frac{1}{(1+t^2)^{3/2}}\, dt$$

$$= \frac{1}{1+x^2} + \frac{1}{1+x^2}\left\{\frac{1}{(1+t^2)^{1/2}}\right\}_0^x$$

$$= \frac{1}{1+x^2} + \frac{1}{1+x^2} \cdot \frac{1}{(1+x^2)^{1/2}} - \frac{1}{1+x^2} = \frac{1}{(1+x^2)^{3/2}} = y(x)$$

Thus $y(x) = (1 + x^2)^{-3/2}$ is a solution of the integral equation ($i$).

**Example 36.9.** *Show that the function $y(x) = xe^x$ is a solution of the integral equation*

$$y(x) = \sin x + 2\int_0^x \cos(x-t)\, y(t)\, dt. \qquad ...(i)$$

**Solution.** Substituting $y(x) = xe^x$ in the R.H.S. of ($i$), we have

$$\sin x + 2\int_0^x \cos(x-t)\,.\,te^t\, dt$$

$$= \sin x + 2\left\{\cos x \int_0^x t.e^t \cos t\, dt + \sin x \int_0^x te^t \sin t\, dt\right\} \qquad \text{(Integrating by parts)}$$

$$= \sin x + 2\cos x \left\{\frac{1}{2}\Big| t.e^t(\cos t + \sin t)\Big|_0^x - \frac{1}{2}\int_0^x e^t(\cos t + \sin t)\, dt\right\}$$

$$+ 2\sin x \left\{\frac{1}{2}\Big| t.e^t(\sin t - \cos t)\Big|_0^x - \frac{1}{2}\int_0^x e^t(\sin t - \cos t)\, dt\right\}$$

$$= \sin x + xe^x(\cos^2 x + \cos x \sin x + \sin^2 x - \sin x \cos x)$$

$$- \cos x \int_0^x e^t(\cos t + \sin t)\, dt - \sin x \int_0^x t(\sin t - \cos t)\, dt$$

$$= \sin x + xe^x - \cos x \Big| e^t \sin t \Big|_0^x + \sin x \Big| e^t \cos t \Big|_0^x$$

$$= \sin x + xe^x - e^x \cos x \sin x + e^t \sin x \cos x - \sin x = xe^x$$

Thus $y(x) = xe^x$ is a solution of the integral equation ($i$).

## 36.6 INTEGRAL EQUATIONS OF THE CONVOLUTION TYPE

$$y(x) = F(x) + \int_0^x K(x-t)\, y(t)\, dt$$

is an *integral equation of convolution type* and can be written as

$$y(x) = F(x) + K(x) * y(x)$$ [See p. 748]

It is a special integral equation of importance in applications.

Taking Laplace transform of both sides, assuming that $L\, F(x) = f(s)$ and $L[K(x)] = k(s)$ both exist, and using convolution theorem

$$\bar{y}(s) = f(s) + k(s)\,.\,y(s) \quad \text{or} \quad \bar{y}(s) = \frac{f(s)}{1-k(s)}$$

Now taking the inverse transform of both sides, we get the required solution.

**Example 36.10.** *Solve the integral equation*

$$y(x) = 3x^2 + \int_0^x y(t) \sin(x-t)\, dt.$$

**Solution.** Given integral equation can be written as

$$y(x) = 3x^2 + y(x) * \sin(x-t) \qquad \text{...}(i)$$

Taking Laplace transform of both sides and using the convolution theorem (p. 748), we get

$$\bar{y} = \frac{6}{s^3} + y \cdot \frac{s}{s^2+1} \quad \text{or} \quad \bar{y} = \frac{6(s^2+1)}{s^5} = 6\left(\frac{1}{s^3} + \frac{1}{s^5}\right)$$

On inversion, we get $\quad y = 6\left(\dfrac{x^2}{2!} + \dfrac{x^4}{4!}\right) = 3x^2 + x^4/4$

which is the required solution of $(i)$.

**Obs.** The above solution can also be verified by direct substitution in the given integral equation.

**Example 36.11.** *Solve* $y(x) = x + 2\int_0^x \cos(x-t)\, y(t)\, dt.$

**Solution.** The given equation can be written as

$$y(x) = x + 2\cos(x) * y(x)$$

Taking Laplace transform of both sides and using convolution theorem, we get

$$\bar{y}(s) = \frac{1}{s^2} + 2\frac{s}{s^2+1} \cdot \bar{y}(s) \quad \text{or} \quad \bar{y}\left(1 - \frac{2s}{s^2+1}\right) = \frac{1}{s^2}$$

or
$$\bar{y} = \frac{s^2+1}{s^2(s-1)^2} = \frac{2}{s} + \frac{1}{s^2} - \frac{2}{s-1} + \frac{2}{(s-1)^2}$$

On inversion, we obtain $\quad y = 2 + x - 2e^x + 2xe^x$

Hence $y = x + 2 + 2(x-1)\,e^x$ is the desired solution.

**Example 36.12.** *Solve the integral equation* $\int_0^x y(t)\, y(x-t)\, dt = 4 \sin 9x.$

**Solution.** The given integral equation can be written as

$$y(x) * y(x) = 4 \sin 9x \qquad \text{...}(i)$$

Taking Laplace transform of both sides and using the convolution theorem, we get

$$\{\bar{y}(s)\}^2 = \frac{36}{s^2+81} \quad \text{or} \quad \bar{y} = \pm\frac{6}{\sqrt{s^2+81}}$$

On inversion and noting that $L^{-1}\dfrac{1}{\sqrt{(s^2+a^2)}} = J_0(ax)$, we get

$$y = \pm 6\, L^{-1}\left\{\frac{1}{\sqrt{(s^2+9^2)}}\right\} = \pm 6\, J_0(9x)$$

Thus $y = 6J_0(9x)$ and $y = -6\,J_0(9x)$ are both solutions of $(i)$.

## 36.7 ABEL'S INTEGRAL EQUATION

The integral equation $\int_0^x \frac{y(t)}{(x-t)^\alpha}\,dx = G(x)$

such that $G(x)$ is given and $\alpha$ is a constant $(0 < \alpha < 1)$, is known as *Abel's integral equation*. This is an important integral equation of the convolution type. An application of this equation is in finding the shape of a smooth wire lying in a vertical plane such that a bead placed anywhere on the wire slides to the lowest point in the same time. This is the well known *tautochrone problem* and the shape of the wire is a *cycloid*.

**Example 36.13.** *Solve the Abel's integral equation* $\int_0^x \frac{y(t)}{\sqrt{(x-t)}}\,dt = 1 + 2x - x^2$.

**Solution.** The given equation can be written as

$$y(x) * x^{-1/2} = 1 + 2x - x^2$$

Taking the Laplace transform of both sides and using convolution theorem, we get

$$\bar{y}\ .\ L(x^{-1/2}) = L(1 + 2x - x^2)$$

or $$\bar{y}\frac{\Gamma(1/2)}{s^{1/2}} = \frac{1}{s} + \frac{2}{s^2} - \frac{2}{s^3} \quad \text{or} \quad \bar{y} = \frac{1}{\Gamma(1/2)}\left(\frac{1}{s^{1/2}} + 2\cdot\frac{1}{s^{3/2}} - 2\frac{1}{s^{5/2}}\right)$$

On inversion, and noting that $L^{-1}\frac{1}{s^{n+1}} = \frac{x^n}{\Gamma(n+1)}$, we have

$$y = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\frac{x^{-1/2}}{\Gamma\left(\frac{1}{2}\right)} + 2.\frac{x^{1/2}}{\Gamma\left(\frac{3}{2}\right)} - 2\frac{x^{3/2}}{\Gamma\left(\frac{5}{2}\right)}$$

$$= \frac{1}{\pi}\left(x^{-1/2} + 4x^{1/2} - \frac{8}{3}x^{3/2}\right)$$

Hence $$y = \frac{1}{3\pi\sqrt{x}}(3 + 12x - 8x^3) \text{ is the desired solution.}$$

## 36.8 INTEGRO-DIFFERENTIAL EQUATIONS

An integral equation in which various derivatives of the unknown function $y(x)$ are also present, is called an *integro-differential equation*. An example of such an equation is

$$y'(x) = y(x) - \cos x + \int_0^x \sin(x-t)\, y(t)\, dt$$

The solution of integro-differential equation subject to given initial conditions can also be found by Laplace transforms as illustrated below :

**Example 36.14.** *Solve* $\frac{dy}{dx} + 3y + 2\int_0^x y\,dx = x$, *given* $y(0) = 1$.

**Solution.** Given equation can be written as $y'(x) + 3y(x) + 2\int_0^x y\,dx = x$

Taking Laplace transform of both sides, we get

$$L[y'(x)] + 3\,L[y(x)] + 2L\left\{\int_0^x y(x)\,dx\right\} = L(x)$$

or $$\{s\,\bar{y}(s) - y(0)\} + 3\,\bar{y}(s) + 2\frac{1}{s}\bar{y}(s) = \frac{1}{s^2} \qquad \text{[Using § 21.6]}$$

$$s\bar{y} + 3\bar{y} + 2\frac{1}{s}\bar{y} = 1 + \frac{1}{s^2} \qquad [\because\ y(0) = 1]$$

or $$\bar{y} = \frac{1+s^2}{s(s^2+3s+2)} = \frac{1+s^2}{s(s+1)(s+2)} = \frac{1}{2s} - \frac{2}{s+1} + \frac{5}{2(s+2)}$$

On inversion, we obtain $y = \frac{1}{2} L^{-1}\left(\frac{1}{s}\right) - 2L^{-1}\left(\frac{1}{s+1}\right) + \frac{5}{2} L^{-1}\left(\frac{1}{s+2}\right)$

Hence $y = \frac{1}{2} - 2e^{-x} + \frac{5}{2} e^{-2x}$ is the required solution.

**Example 36.15.** *Solve* $\frac{dy}{dx} = 3\int_0^x \cos 2(x-t)\, y(t)\, dt + 2$ *given* $y(0) = 1$.

**Solution.** Given equation can be written as

$$y'(x) = 3 \cos 2x * y(x) + 2$$

Taking Laplace transform of both sides, we get

$$L[(y'(x)] = 3L(\cos 2x) \,.\, \bar{y}(s) + \frac{2}{s}$$

$$s\,\bar{y}(s) - y(0) = \frac{3s\bar{y}(s)}{s^2+4} + \frac{2}{s} \quad \text{or} \quad \bar{y} = \frac{(s+2)(s^4+4)}{s^2(s^2+1)} \qquad [\because\ y(0) = 1]$$

$$= \frac{4}{s} + \frac{8}{s^2} - 3\frac{s}{s^2+1} - 6\frac{1}{s^2+1}$$

On inversion, we obtain $y = 4 + 8x - 3\cos x - 6\sin x$
which is the required solution.

**Obs.** The given integro-differential equation can be converted into the following integral equation by integrating it from 0 to $x$ and using $y(0) = 1$.

$$y(x) = 2x + 1 + 3\int_0^x (x-t)\cos 2(x-t)\, y(t)\, dt.$$

## PROBLEMS 36.3

1. Show that $y(x) = 1 - x$ is a solution of the integral equation $\int_0^x e^{x-t}\, y(t)\, dt = x$.
2. Show that $y(x) = 1$ is a solution of the Fredholm integral equation
$$y(x) + \int_0^1 x(e^{tx} - 1)\, y(t)\, dt = e^x - x.$$
3. Show that $y(x) = \frac{1}{\pi\sqrt{x}}$ is a solution of the integral equation $\int_0^x \frac{y(t)}{\sqrt{(x-t)}}\, dt = 1$.
4. Show that $y(x) = e^x(2x - 2/3)$ is a solution of the Fredholm integral equation
$$y(x) + \int_0^1 e^{x-t}\, y(t)\, dt = 2xe^x.$$

Solve each of the following integral equations :

5. $y(x) = x + \frac{1}{6}\int_0^x (x-t)^3\, y(t)\, dt.$
6. $y(x) = x^2 + \int_0^x y(t)\sin(x-t)\, dt.$
7. $\int_0^x y(t)\, y(x-t)\, dt = 2y(x) + x - 2.$
8. $\int_0^x y(t)\, y(x-t)\, dt = 9\sin 4x.$
9. Find a solution of the integral equation $y(x) = \frac{1}{2}\sin 2x + \int_0^x y(t)\, y(x-t)\, dt.$

Solve the following integral equations :

10. $\frac{dy}{dx} + 4y + 5\int_0^x y(t)\, dt = e^{-x}, y(0) = 0.$
11. $\frac{dy}{dx} + 2y + \int_0^x y(t)\, dt = \sin x, y(0) = 1.$ *(Mumbai, 2006)*

12. $y'(x) = x + \int_0^x y(x-t)\cos t\, dt,\ y(0) = 1.$

13. $y'(x) = \int_0^x y(t)\cos(x-t)\, dt,\ y(0) = 1.$

14. $\int_0^x \frac{y(t)}{\sqrt{(x-t)}}\, dt = \sqrt{x}.$

15. $\int_0^x \frac{y(t)}{(x-t)^{1/3}}\, dt = x(x+1).$

## 36.9 INTEGRAL EQUATIONS WITH SEPARABLE KERNELS

A kernel $K(x, t)$ of Fredholm integral equation is said to be *separable* (or *degenerate*) if it can be expressed as the sum of a finite number of terms, each of which is the product of a function of $x$ alone and a function of $t$ alone *i.e.*, if it is of the form $K(x, t) = \sum_{n=1}^{m} f_n(x)\, g_n(t)$.

Also since $\cos(x + t) = \cos x \cos t - \sin x \sin t$, $\cos(x + t)$ is a separable kernel.

## 36.10 SOLUTION OF FREDHOLM EQUATIONS WITH SEPARABLE KERNELS

Consider the integral equation

$$y(x) = \lambda \int_a^b K(x, t) \cdot y(t)\, dt + F(x) \qquad ...(1)$$

where
$$K(x, t) = f_1(x) \cdot g_1(t) + f_2(x) \cdot g_2(t) + \ldots + f_m(x) \cdot g_m(t) \qquad ...(2)$$

Substituting (2) in (1), we get

$$y(x) = \lambda \int_a^b \left[ \sum_{n=1}^{m} f_n(x) \cdot g_n(t) \right] y(t)\, dt + F(x)$$

$$= \lambda \sum_{n=1}^{m} f_n(x) \left\{ \int_a^b g_n(t)\, y(t)\, dt \right\} + F(x) \qquad ...(3)$$

Evidently $\int_a^b g_n(t)\, y(t)\, dt = C_n$ (say), is a constant and will be different for different values of $n$. Then (3) takes the form

$$y(x) = \lambda \sum_{n=1}^{m} C_n\, f_n(x) + F(x) \qquad ...(4)$$

This is a solution of (1) in which $m$ constants $C_1, C_2, \ldots, C_m$ are to be determined.

Now multiplying both sides of (4) by $g_k(x)$ and integrating from $a$ to $b$, we obtain

$$\int_a^b y(x)\, g_k(x)\, dx = \lambda \int_a^b \sum_{n=1}^{m} C_n\, f_n(x)\, g_k(x)\, dx + \int_a^b F(x)\, g_k(x)\, dx$$

Writing $\int_a^b g_k(x)\, f_n(x)\, dx = \alpha_{kn}$ and $\int_a^b g_k(x)\, F(x)\, dx = \beta_k$, the above equation becomes

$$C_k = \lambda \sum_{n=1}^{m} C_n \alpha_{kn} + \beta_k$$

$$= \lambda(C_1\alpha_{k1} + C_2\alpha_{k2} + \ldots + C_m\alpha_{km}) + \beta_k$$

Taking $k = 1, 2, \ldots, m$, we get the following $m$ equations which determine $C_1, C_2, \ldots C_m$ :

$$\left.\begin{array}{r} (1 - \lambda\alpha_{11})\, C_1 - \lambda\alpha_{12}\, C_2 - \ldots - \lambda\, \alpha_{1m} C_m = \beta_1 \\ -\lambda a_{21}\, C_1 + (1 - \lambda a_{22})\, C_2 - \ldots - \lambda\, a_{2m}\, C_m = \beta_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ -\lambda\alpha_{m1} C_1 - \lambda\alpha_{m2}\, C_2 - \ldots + (1 - \lambda\alpha_{mm})\, C_m = \beta_m \end{array}\right\} \qquad ...(5)$$

Equations (5) will give a unique solution if the determinant $\Delta$ of the coefficients of $C_1, C_2, \ldots C_m$ is not zero.

Now the following cases arise :

*I. When $F(x) = 0$*, (1) is said to be a *homogeneous integral equation* and all β's are zero.

(*i*) *If* $\Delta \neq 0$, the only solution of (4) is the trivial solution $C_1 = C_2 = \ldots = C_m = 0$. Then $y(x) = 0$ is the trivial solution of (1).

(*ii*) *If* $\Delta = 0$, at least one of the $C$'s can be assigned any value and the remaining $C$'s can be found accordingly. Then (4) gives infinitely many solutions. The values of λ for which $\Delta = 0$ are known as the *eigen values*. Any non-trivial solution of the homogeneous integral equation for a certain value of λ is called the corresponding *eigen function*. The solutions corresponding to eigen values of λ can be expressed as arbitrary multiples of eigen functions.

*II. When $F(x) \neq 0$.* Let us assume that $\int_a^b g_m(x) F(x)\, dx = 0$ so that $\beta_m = 0$

(*i*) *If* $\Delta \neq 0$, the only solution of (*i*) is the trivial solution $C_1 = C_2 = \ldots = C_m = 0$.

Then $y = F(x)$ is the desired solution of (1).

(*ii*) *If* $\Delta = 0$, atleast one of the $C$'s can be given any value and the remaining $C$'s can be found.

Then (4) gives infinitely many solutions.

*III. When atleast one of the β's ≠ 0*

(*i*) *If* $\Delta \neq 0$, then equations (5) give a unique solution of the constants $C$.

Hence there is a unique solution of (1).

(*ii*) *If* $\Delta = 0$, then equations (5) will be inconsistent.

∴ Either there is no solution or infinitely many solutions of (*i*) exist.

**Example 36.16.** *Find the eigen values and eigen functions of the following homogeneous integral equations with degenerate kernels :*

(*i*) $y(x) = \lambda \int_0^1 (2xt - 4x^2)\, y(t)\, dt$ (*ii*) $y(x) = \dfrac{1}{e^2 - 1} \int_0^1 e^{x+t}\, y(t)\, dt.$

**Solution.** (*i*) Given equation may be written as

$$y(x) = \lambda \left\{ 2x \int_0^1 t y(t)\, dt - 4x^2 \int_0^1 y(t)\, dt \right\}$$

or
$$y(x) = (2\lambda x)\, C_1 - (4\lambda x^2)\, C_2 \qquad \ldots(i)$$

where $C_1 = \int_0^1 t y(t)\, dt,\ C_2 = \int_0^1 y(t)\, dt$

Substituting $y(x)$ in $C_1, C_2$ we get

$$C_1 = \int_0^1 t\{(2\lambda t)\, C_1 - (4\lambda t^2)\, C_2\}\, dt$$

$$C_2 = \int_0^1 \{(2\lambda t)\, C_1 - (4\lambda t^2)\, C_2\}\, dt$$

or
$$C_1 \left\{1 - 2\lambda \int_0^1 t^2 dt\right\} + C_2 \left\{4\lambda \int_0^1 t^3\, dt\right\} = 0$$

$$C_1 \left\{-2\lambda \int_0^1 t\, dt\right\} + C_2 \left\{1 + 4\lambda \int_0^1 t^2\, dt\right\} = 0$$

or
$$C_1 \left(1 - \frac{2\lambda}{3}\right) + \lambda C_2 = 0\ ;\ -\lambda C_1 + C_2\, (1 + 4\lambda/3) = 0 \qquad \ldots(ii)$$

∴ The determinant of eigen values will be

$$\begin{vmatrix} 1 - 2\lambda/3 & \lambda \\ -\lambda & 1 + 4\lambda/3 \end{vmatrix} = 0 \text{ or } (\lambda + 3)^2 = 0$$

∴ The eigen values are $\lambda_1 = -3,\ \lambda_2 = -3$

For $\lambda_1 = \lambda_2 = -3$, the equations (*ii*) reduce to

$$3C_1 - 3C_2 = 0\ ;\ 3C_1 - 3C_2 = 0 \text{ i.e., } C_1 = C_2$$

∴ From (*i*) $y(x) = -6C_1(x - 2x^2) = x - 2x^2$ if $C_1 = -1/6$

Hence the eigen function corresponding to $\lambda_1 = \lambda_2 = -3$, is

$$y(x) = x - 2x^2$$

(*ii*) Given integral equation may be written as

$$y(x) = \frac{e^x}{e^2 - 1} \int_0^1 e^5 e^t \cdot y(t)\, dt = \frac{e^x}{e^2 - 1} C, \quad ...(i)$$

where $C = \int_0^1 e^t\, y(t)\, dt$

Substituting the value of $y(t)$ from (*i*) in $C$, we get

$$C = \int_0^1 e^t \left(\frac{e^t C}{e^2 - 1}\right) dt = \frac{C}{e^2 - 1} \int_0^1 e^{t^2}\, dt$$

or
$$C\left[1 - \frac{1}{e^2 - 1} \int_0^1 e^{t^2}\, dt\right] = 0 \quad i.e. \quad C = 0$$

Hence from (*i*), $y(x) = 0$

which shows that the given integral equation has only trivial solution.

Thus it does not contain any eigen values or eigen functions.

**Example 36.17.** *Solve the integral equation*

$$y(x) = \cos x + \lambda \int_0^\pi \sin (x - t)\, y(t)\, dt.$$

**Solution.** Writing the given equation in the following form :

$$y(x) = \cos x + \lambda \left\{\sin x \int_0^\pi \cos t\, y(t)\, dt - \cos x \int_0^\pi \sin t\, y(t)\, dt\right\}$$

or
$$y(x) = \cos x + (\lambda \sin x)\, C_1 - (\lambda \cos x)\, C_2 \quad ...(i)$$

where $C_1 = \int_0^\pi \cos t \cdot y(t)\, dt$, $\quad C_2 = \int_0^\pi \sin t \cdot y(t)\, dt$

Substituting $y(x)$ in $C_1$ and $C_2$, we get

$$C_1 = \int_0^\pi \cos t\, \{\cos t + (\lambda \sin t)\, C_1 - (\lambda \cos t)\, C_2\}\, dt$$

$$C_2 = \int_0^\pi \sin t\, \{\cos t + (\lambda \sin t)\, C_1 - (\lambda \cos t)\, C_2\}\, dt$$

or
$$\left.\begin{aligned} &C_1 \left\{1 - \lambda \int_0^\pi \cos t \sin t\, dt\right\} + C_2 \left(\lambda \int_0^\pi \cos^2 t\, dt\right) = \int_0^\pi \cos^2 t\, dt \\ &C_1 \left\{-\lambda \int_0^\pi \sin^2 t\, dt\right\} + C_2 \left\{1 + \lambda \int_0^\pi \sin t \cos t\, dt\right\} = \int_0^\pi \sin t \cos t\, dt \end{aligned}\right\} \quad ...(ii)$$

By evaluating each of the integrals in (*ii*), we get

$$C_1 + \frac{1}{2} C_2\, \lambda\pi = \frac{1}{2}\pi \,; -\frac{1}{2} C_1\, \lambda\pi + C_2 = 0 \quad ...(iii)$$

The determinant of the equations (*iii*) is given by

$$\begin{vmatrix} 1 & \frac{1}{2}\lambda\pi \\ -\frac{1}{2}\lambda\pi & 1 \end{vmatrix} = 1 + \frac{1}{4}\lambda^2\pi^2 \neq 0$$

Thus the equations (*iii*) have a unique solution

$$C_1 = \frac{2\pi}{4 + \lambda^2\pi^2} \,; C_2 = \frac{\lambda\pi^2}{4 + \lambda^2\pi^2}$$

Substituting these values of $C_1$ and $C_2$ in $(i)$, we obtain the required solution

$$y(x) = \cos x + \frac{\lambda}{4+\lambda^2\pi^2}(2\pi \sin x - \lambda\pi^2 \cos x)$$

or
$$y(x) = \frac{2}{4+\lambda^2\pi^2}(2\cos x + \pi\lambda \sin x).$$

## PROBLEMS 36.4

Determine the eigen values and eigen functions for the following homogeneous integral equations with degenerate kernels :

1. $y(x) = \lambda \int_0^1 (3x-2)\ t \,.\, y(t)\, dt.$
2. $y(x) = \lambda \int_{-1}^1 (5xt^3 + 4x^2 t + 3xt)\, y(t)\, dt.$
3. $y(x) = \lambda \int_0^{\pi/4} \sin^2 x \cdot y(t)\, dt.$
4. $y(x) - \lambda \int_0^{2\pi} \sin x \sin t\, y(t)\, dt = 0.$ (*Madras M.E., 2000 S*)
5. $y(x) = \lambda \int_0^{\pi} \sin x \cos t \,.\, y(t)\, dt.$
6. $y(x) = \lambda \int_0^{2\pi} \sin (x+t) \,.\, y(t)\, dt.$ (*Madras M.E., 2000*)

Solve the following integral equations :

7. $y(x) = x + \lambda \int_0^1 (x-t)\, y(t)\, dt.$
8. $y(x) = x + \lambda \int_0^1 (1+x+t)\, y(t)\, dt.$
9. $y(x) = x + \lambda \int_0^{\pi} (1+\sin x \sin t)\, y(t)\, dt.$
10. $y(x) = (2x-\pi) + 4\int_0^{\pi/2} \sin^2 x \cdot y(t)\, dt.$
11. $y(x) = \sin x + \lambda \int_0^{\pi/2} \sin x \cos t \cdot y(t)\, dt.$
12. $y(x) = x + \lambda \int_{-\pi}^{\pi} (x\cos t + t^2 \sin x + \cos x \sin t)\, y(t)\, dt.$
13. Obtain the solution of $y(x) = 1 + \lambda \int_0^1 xt \cdot y(t)\ dt$ in the form $y(x) = 1 + \dfrac{3\lambda x}{2(3-\lambda)}$ $(\lambda \neq 3)$.

    What happens when $\lambda = 3$ ?
14. For the integral equation
$$y(x) = F(x) + \lambda \int_0^1 (1-3xt)\ y(t)\, dt,$$
    find the eigen values of $\lambda$ and the corresponding eigen functions.
15. Obtain the most general solution of the equation
$$y(x) = F(x) + \lambda \int_0^{2\pi} \sin (x+t)\ y(t)\, dt$$
    where $(i)$ $F(x) = x$ $(ii)$ $F(x) = 1$, under the assumption that $\lambda \neq \pm 1/\pi$.

## 36.11 SOLUTION OF FREDHOLM INTEGRAL EQUATION BY THE METHOD OF SUCCESSIVE APPROXIMATIONS

Consider the Fredholm equation $y(x) = F(x) + \lambda \int_a^b K(x,t)\, y(t)\, dt$ ...(1)

where $F(x)$ is continuous in $a \le x \le b$ and $K(x, t)$ is finite and continuous in the rectangle $a \le x \le b$ and $a \le t \le b$.

Replacing $y$ under the integral sign by an initial approximation $y(0)$, we get the first approximation

$$y^{(1)}(x) = F(x) + \lambda \int_a^b K(x,t)\, y^{(0)}(t)\, dt \quad ...(2)$$

Replacing $y$ under the integral sign in (1) by $y^{(1)}$, we get the next approximation

$$y^{(2)}(x) = F(x) + \lambda \int_a^b K(x,t)\, y^{(1)}(t)\, dt \quad ...(3)$$

Proceeding in this manner, we get the general formula for successive approximations as

$$y^{(n)}(x) = F(x) + \lambda \int_a^b K(x, t)\, y^{(n-1)}(t)\, dt \qquad ...(4)$$

We now, obtain the condition for the convergency of the sequence $y^{(n)}(x)$.

Replacing $x$ by $t$ and $t$ by a dummy variable $t_1$, (2) becomes

$$y^{(1)}(t) = F(t) + \lambda \int_a^b K(t, t_1)\, y^{(0)}(t_1)\, dt_1$$

Then (3) takes the form

$$y^{(2)}(x) = F(x) + \lambda \int_a^b K(x, t)\left\{F(t) + \lambda \int_a^b K(t, t_1)\, y^{(0)}(t_1)\, dt_1\right\} dt$$

$$= F(x) + \lambda \int_a^b K(x, t)\, F(t)\, dt + \lambda^2 \int_a^b K(x, t) . \int_a^b K(t, t_1) y^{(0)}(t_1)\, dt_1\, dt \qquad ...(5)$$

Writing $K^* \phi(x) = \int_a^b K(x, t)\ \phi(t)\, dt$, the equations (1), (2) and (5) become

$$y(x) = F(x) + \lambda K^* y(x)$$
$$y^{(1)}(x) = F(x) + \lambda K^* y^{(0)}(x)$$
$$y^{(2)}(x) = F(x) + \lambda K^* F(x) + \lambda^2 K^{*2} y^{(0)}(x)$$

Similarly $\quad y^{(3)}(x) = F(x) + \lambda K^* F(x) + \lambda^2 K^{*2} F(x) + \lambda^3 K^{*3} y^{(0)} x$

In general $\quad y^{(n)}(x) = F(x) + \lambda K^* F(x) + \lambda^2 K^{*2} F(x) + \lambda^3 K^{*3} F(x) + \ldots + \lambda^n K^{*n} y^{(0)}(x)$

As $n \to \infty$, we get

$$y(x) = \underset{n \to \infty}{\text{Lt}}\ y^n(x) = F(x) + \underset{n \to \infty}{\text{Lt}}\ \{\lambda K^* F(x) + \lambda^2 K^{*2} F(x) + \ldots \infty\} \qquad ...(6)$$

Now $F(x)$ and $K(x, t)$ being continuous for all values of $x$ and $t$ in $(a, b)$, $F(x) \le M$ and $|K(x, t)| \le m$ where $M$, $m$ are their respective maximum values in $(a, b)$.

$$\therefore \qquad |K^* F(x)| = \left| \int_a^b K(x, t)\ F(x)\, dt \right|$$

$$\le mM \left| \int_a^b dt \right| \le mM(b - a)$$

Similarly $\qquad K^{*r} F(x) \le m^r M . (b - a)^r$

Then $\qquad |\lambda^r K^{*r} F(x)| \le |\lambda^r| . m^r M(b - a)^r$

$$\le M\{|\lambda|\ m(b - a)\}^r$$

$\therefore$ In (6), $\qquad \sum_1^\infty \lambda^r K^{*r}\ F(x) \le M \sum_1^\infty [|\lambda| m(b - a)]^r$

Now the series on R.H.S. being a geometric series, converges for $|\lambda|\ m(b - a) < 1$.

Thus by comparison test, $\sum_1^\infty \lambda^r K^{*r}\ F(x)$ also converges for $|\lambda|\ m(b - a) < 1$

*i.e.*, for $$|\lambda| < \frac{1}{m(b - a)} \qquad ...(7)$$

Hence the given integral equation (1) will have a continuous solution when the condition (7) is satisfied.

**Obs. 1.** To evaluate the successive terms in the series (6) conveniently, we choose $y^{(0)}(x) = F(x)$.

**Obs. 2.** Volterra integral equations can also be solved by following exactly similar procedure as above (See Example 36.19).

**Example 36.18.** *Solve, by using the method of successive approximations, the integral equation*

$$y(x) = 1 + \lambda \int_0^1 xt\ y(t)\, dt. \qquad ...(i)$$

**Solution.** Taking the initial approximation $y^{(0)}(x) = 1$ and substituting it in the R.H.S. of $(i)$, we get

$$y^{(1)}(x) = 1 + \lambda \int_0^1 xt \cdot 1\, dt = 1 + \lambda x \left| \frac{t^2}{2} \right|_0^1 = 1 + \frac{\lambda x}{2}$$

Substituting $y^{(1)}(x)$ in the R.H.S. of $(i)$, we have

$$y^{(2)}(x) = 1 + \lambda \int_0^1 xt\left(1 + \lambda \frac{t}{2}\right) dt = 1 + \lambda x \int_0^1 \left(t + \frac{\lambda t^2}{2}\right) dt$$

$$= 1 + \lambda x \left| \frac{t^2}{2} + \frac{\lambda}{2} \cdot \frac{t^3}{3} \right|_0^1 = 1 + \frac{\lambda x}{2} + \frac{\lambda^2 x}{6}$$

Substituting $y^{(2)}(x)$ in the R.H.S. of $(i)$, we get

$$y^{(3)}(x) = 1 + \lambda \int_0^1 xt\left(1 + \frac{\lambda t}{2} + \frac{\lambda^2 t}{6}\right) dt = 1 + \lambda x \left| \frac{t^2}{2} + \left(\frac{\lambda}{2} + \frac{\lambda^2}{6}\right)\frac{t^3}{3} \right|_0^1$$

$$= 1 + \frac{\lambda x}{2} + \frac{\lambda^2 x}{6} + \frac{\lambda^3 x}{18} = 1 + \frac{\lambda x}{2}\left(1 + \frac{\lambda}{3} + \frac{\lambda^2}{3^2} + \cdots\right)$$

Hence the solution of $(i)$ is

$$y(x) = 1 + \frac{\lambda x}{2}\left(1 + \frac{\lambda}{3} + \left(\frac{\lambda}{3}\right)^2 + \left(\frac{\lambda}{3}\right)^3 + \cdots\right)$$

As the number of terms tends to infinity, the exact solution is

$$y(x) = 1 + \frac{\lambda x}{2}\left(1 + \frac{\lambda}{3} + \left(\frac{\lambda}{3}\right)^2 + \left(\frac{\lambda}{3}\right)^3 + \cdots \infty\right)$$

$$= 1 + \frac{\lambda x}{2} \frac{1}{1 - \lambda/3}$$ [Summing up the G.P. which converges for $\lambda/3 < 1$]

or $$y(x) = 1 + \frac{3\lambda x}{2(3 - \lambda)}$$ only if $\lambda < 3$.

**Example 36.19.** *Using the method of successive approximations, solve the Volterra integral equation*

$$y(x) = 1 + x + \int_0^x (x - t)\, y(t)\, dt.$$

**Solution.** Taking the initial approximation $y^{(0)}(x) = 1 + x$ and substituting it in the R.H.S. of $(i)$, we get

$$y^{(1)}(x) = 1 + x + \int_0^x (x - t)(1 + t)\, dt$$

$$= 1 + x + x\left(x + \frac{x^2}{2}\right) - \left(\frac{x^2}{2} + \frac{x^3}{3}\right) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

Substituting $y^{(1)}(x)$ in the R.H.S. of $(i)$, we obtain

$$y^{(2)}(x) = 1 + x + \int_0^x (x - t)\left(1 + t + \frac{t^2}{2} + \frac{t^3}{6}\right) dt$$

$$= 1 + x + x\left(x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}\right) - \left(\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{30}\right)$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}$$

Proceeding in this manner, we get

$$y^{(n)}(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!}$$

As $n \to \infty$, the exact solution of $(i)$ is

$$y(x) = \underset{n \to \infty}{\text{Lt}} \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\right) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \infty = e^x.$$

## PROBLEMS 36.5

Apply the method of successive approximations, to solve the following Fredholm integral equations : (1 to 3)

1. $y(x) = 1 - \lambda \int_0^1 xt\, y(t)\, dt.$
2. $y(x) = \sin x + \lambda \int_0^{2\pi} \cos(x+t)\, y(t)\, dt.$
3. $y(x) = 1 + \lambda \int_0^1 (1 - 3xt)\, y(t)\, dt.$

Using the iterative method, solve the following Volterra integral equations : (4 to 6)

4. $y(x) = 1 + \int_0^x y(t)\, dt.$ (*Madras M.E., 2000 S*)
5. $y(x) = 1 + x - \int_0^x y(t)\, dt.$
6. $y(x) = x + \int_0^x (t - x)\, y(t)\, dt.$
7. $y(x) = 2(1 + x^2) - \int_0^x xy\,(t)\, dt.$
8. Choosing the initial approximation $y^{(0)}(x) = 0$, for the solution of the integral equation

$$y(x) = \int_0^x t(t - x)\, y(t)\, dt + \frac{x^2}{2}, \text{ show that } y^{(2)}(x) = \frac{x^2}{2} - \frac{x^5}{40}.$$

9. Starting with the initial approximation $y^{(0)}(x) = 1$, for the solution of the integral equation

$$y(x) = 1 + \int_0^x (x + t)\, y(t)\, dt,$$

show that $y^{(3)}(x) = 1 + \dfrac{3x^2}{2} + \dfrac{7x^4}{8} + \dfrac{77x^6}{240}.$ (*Madras M.E., 2000*)

CHAPTER 37

# Discrete Mathematics



## I. SET THEORY

### 37.1 SETS

**(1)** The concept and the language of sets play a very important role in expressing mathematical ideas concisely and precisely. It was Cantor* who first introduced and developed the notion of sets in mathematical investigations. It is, therefore, essential for a student of engineering to grasp the basic ideas of *Set Theory*.

**Def.** *A collection of objects defined by some property, is called a* **set**. *The objects belonging to a set are called its* **elements** *or* **members**.

*Examples* of a set are (*i*) the set of positive integers less than 25, (*ii*) set of pages in a book, and (*iii*) set of women students in a college.

A set is denoted by a single capital letter *e.g.* $A, B, ..., S, X, Y$ and the elements of a set are generally denoted by small letters $a, b, c, ..., x, y, z$.

*When e is an element of a set S, we write $e \in S$ and read as 'e belongs to S'. When e is not an element of S, we write $e \notin S$.*

If $S$ be a set of odd integers, $3, 7, 11 \in S$ but $4, 6 \notin S$.

**(2) Representation of a set**

(*i*) *Tabular form of a set.* In this, the elements are enclosed in curly brackets after separating them by commas, *e.g.*, the set of positive even integers less than 9 is written as $S = \{2, 4, 6, 8\}$ and the set of prime numbers between 4 and 14 is $T = \{5, 7, 11, 13\}$.

(*ii*) *Symbolic form of a set.* In this, the set is written as $\{x/P(x)\}$ where $x$ is a typical element of the set and $P(x)$ is the property satisfied by this element. In symbolic form, the above two sets are

$$S = \{x/x = a \text{ positive even number} < 9\}$$
$$T = \{x/x = a \text{ prime number between 4 and 14}\}.$$

*The great German mathematician George Cantor (1845–1918), the creator of Set theory.

**(3) Empty set or null set.** *A set which has no elements is called an* **empty set** *or the* **null set** *and is denoted by the symbol* $\phi$.

**(4) Finite** *and* **infinite sets**. *A set is said to be* **finite** *if it has a finite number of elements. Otherwise a set is said to be* **infinite**.

*The number of distinct elements in a finite set A is called its* **cardinality** *and is denoted by* $|A|$.

*For instance*, the set of days in a year is *finite*, the set of points in a line is an *infinite set*.

**(5) Subset.** *If every element of a set A is also an element of set B, then A is called a* **subset** *of B* and this relationship is denoted by $A \subset B$ or $B \supset A$ ; which is read as '$A$ is contained in $B$'.

*Another definition : If A and B are two sets such that*

$$x \in A \quad \Rightarrow \quad x \in B,$$

*then A is called a subset of B.*

The notation $\Rightarrow$ stands for the word '*implies*'.

*For instance*, the set $V$ of vowels is a subset of the set $A$ of the English alphabet and we write $V \subset A$.

**(6) Power set.** *For a set A, collection of all subsets of A is called the power set of A and is denoted by P(A).*

If $A = [1, 2, 3]$ then $P(A)$ consists of $2^3$ *i.e.* 8 elements $\phi$, [1], [2], [3], [1, 2], [2, 3], [3, 1] and [1, 2, 3].

In general, if $A$ has $n$ elements, then $P(A)$ has $2^n$ elements.

**(7) Equality of sets.** *Two sets A and B are said to be equal if the elements of both are the same i.e.*, if each element of $A$ is also an element of $B$ and *vice versa*, and we write $A = B$.

In other words, *if A and B are two sets such that*

$$A \subset B \quad \text{and} \quad B \subset A \Leftrightarrow A = B.$$

Here $\Leftrightarrow$ stands for '*implies and is implied by*' or '*if and only if*'.

*For instance*, $\{2, 3, 5\} = \{3, 2, 5, 3\} = \{2, 5, 3, 2\}$, since the change in the order of elements or the repetition of an element is immaterial and all these contain the same elements 2, 3, 5.

**(8) Proper and improper subsets.** *When the set B contains all the elements of A and some others, A is said to be a* **proper subset** *of B* and is denoted by $A \subseteq B$.

*i.e.*, if $A \subset B$ and $A \neq B$ then $A \subseteq B$.

*If* $A \subset B$ *and every element of B is also an element of A i.e.*, $B \subset A$, *then A is said to be an* **improper subset** *of B i.e.*, $A = B$.

*For instance*, the set of positive odd integers and the set of positive even integers are both proper subsets of the set of natural numbers.

**(9) Universal set** *is that which has all the sets under investigation as its subsets. It is generally denoted by 'U'.*

*For instance* the set of all letters of English alphabet is a universal set of the sets of the form $\{a, i, e, u\}$, $\{b, x, u, m\}$ etc.

**Example 37.1.** *If A, B, C are sets such that* $A \subseteq B$ *and* $B \subseteq C$, *then show that* $A \subseteq C$.

**Solution.** Let $x$ be any element of $A$.

Since $A \subseteq B$ *i.e.*, all the elements of $A$ belong to $B$,

so $$x \in A \quad \Rightarrow \quad x \in B \qquad ...(i)$$

Again as $B \subseteq C$ *i.e.*, all elements of $B$ belong to $C$,

so $$x \in B \quad \Rightarrow \quad x \in C \qquad ...(ii)$$

$\therefore$ It follows from ($i$) and ($ii$) that $x \in A \;\; \Rightarrow \;\; x \in C$

*i.e.*, $$A \subseteq C.$$

**Example 37.2.** *Which of the following sets are equal ?*

$$S_1 = \{1, 2, 2, 3\},\ S_2 = \{x : x^2 - 2x + 1 = 0\}$$
$$S_3 = \{3, 2, 1\} \text{ and } S_4 = \{x : x^3 - 6x^2 + 11x - 6 = 0\}.$$

**Solution.** Here $S_1 = \{1, 2, 2, 3\} = \{1, 2, 3\}$

$$S_2 = \{x : (x-1)^2 = 0\} = \{1\},\ S_3 = \{1, 2, 3\}$$
$$S_4 = \{x : (x-1)(x-2)(x-3) = 0\} = \{1, 2, 3\}$$

From these we find that $S_1$, $S_3$, $S_4$ are equal.

## 37.2 SET OPERATIONS

**(1) Union** *of two sets A and B is the set of all elements which belong to A or to B or to both. It is denoted by* $A \cup B$ read as '$A$ union $B$' and is represented by the shaded portion in Fig. 37.1.

Symbolically $A \cup B = \{x/x \in A \text{ or } x \in B\}$.

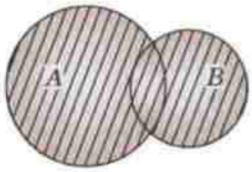



Fig. 37.1

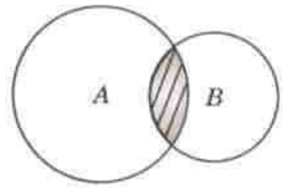



Fig. 37.2

**(2) Intersection** *of two sets A and B is the set of elements which are common to both A and B.* It is denoted by $A \cap B$ read as '$A$ intersection $B$' and is represented by the shaded portion in Fig. 37.2.

Symbolically $\qquad A \cap B = \{x/x \in A \text{ and } x \in B\}$

*Such diagrams as Figs. 37.1 and 37.2 which exhibit the various relations between the sets are known as* **Venn diagrams**.

**(3) Disjoint sets.** *If the sets A and B have no common elements, they are called* **disjoint sets**. Their intersection is an empty set.

*For instance*, if $A$ be a set of boys in a college and $B$ the set of girls in the same college, then $A$ and $B$ are disjoint sets *i.e.* $A \cap B = \phi$.

**(4) Complement of a set.** *If $B \subset A$, the set of elements of A which are not in B is called the* **complement of B in A** *and is denoted by $B^c$ in A.* It is also known as the difference $A - B$ of sets $A$ and $B$. Thus

$$B^c \text{ in } A = \{x/x \in A \text{ and } x \notin B\}$$

which is shown shaded in Fig. 37.3 (*i*).

*If U be a universal set, then the set 'U – A' is called the complement of A and is denoted by $A^c$*, which is shown shaded in Fig. 37.3 (*ii*).

*For instance*, if $U = \{1, 2, 3, 4, 5, \ldots\}$ and $A = (1, 3, 5, 7, \ldots)$, then $A^c = \{2, 4, 6, 8 \ldots\}$.

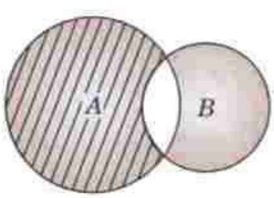



Fig. 37.3 (i)

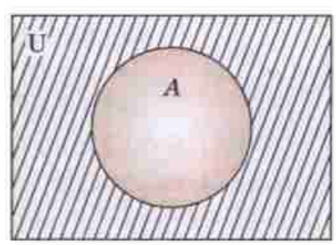



Fig. 37.3 (ii)

**(5) Cartesian product** *of two sets A and B denoted by $A \times B$ is defined to be set of all ordered pairs (a, b), where $a \in A$ and $b \in B$ i.e.,*

$$A \times B = \{(a, b) : a \in A \quad \text{and} \quad b \in B\}$$

*For instance* if $A = \{1, 2\}$, $B = \{1, 2, 3\}$, then $A \times B = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}$, $B \times A = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 2)\}$ $\therefore$ $A \times B \neq B \times A$.

**Example 37.3.** *If $A = \{2, 5, 6, 7\}$, $B = \{0, 2, 5, 7, 8\}$, $C = \{1, 2, 3, 5, 6\}$, show that*
$$A \cup (B \cup C) = (A \cup B) \cap (A \cup C).$$

**Solution.** Here $\qquad B \cap C = \{2, 5\}$

$\therefore \qquad A \cup (B \cap C) = \{2, 5, 6, 7\} \qquad \ldots(i)$

Again $\qquad A \cup B = \{0, 2, 5, 6, 7, 8\}$,

$\qquad A \cup C = \{1, 2, 3, 5, 6, 7\}$

$\therefore \qquad (A \cup B) \cap (A \cup C) = [2, 5, 6, 7] \qquad \ldots(ii)$

Hence from (*i*) and (*ii*), we get

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

**Example 37.4.** *With the help of Venn diagram, show that*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

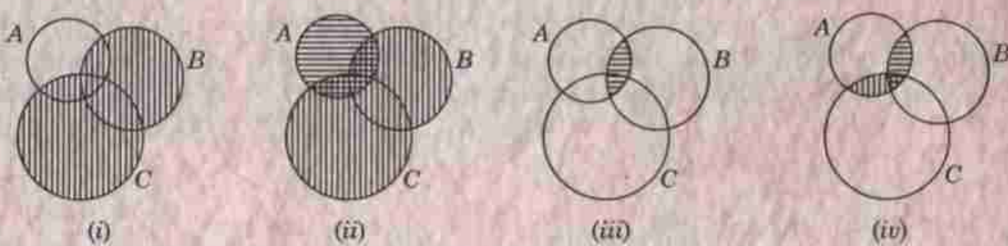



Fig. 37.4

**Solution.** First we draw vertical lines in the whole areas of $B$ and $C$ so as to represent $B \cup C$. [Fig. 37.4 (*i*)].

Now draw horizontal lines in the whole area $A$. Thus the double hatched area in [Fig. 37.4 (*ii*)] represents area common to $A$ and $B \cup C$ *i.e.*, $A \cap (B \cup C)$.

Again we draw horizontal lines in the area common to $A$ and $B$ so as to represent $A \cap B$ [Fig. 37.4 (*iii*)].

Now draw vertical lines in the area common to $A$ and $C$, so as to represent $A \cap C$. Then the whole hatched area in [Fig. 37.4 (*iv*)] represents $(A \cap B) \cup (A \cap C)$.

Hence we observe that the double hatched area in Fig. 37.4 (*ii*) is equal to the total hatched area in Fig. 37.4 (*iv*).

**Example 37.5.** *Prove that* (*i*) $A - (B \cap C) = (A - B) \cup (A - C)$.

(*ii*) $A \times (B \cap C) = (A \times B) \cap (A \times C)$. *(Tiruputi, 2001)*

**Solution.** (*i*) Let $x$ be an arbitrary element of the set $A - (B \cap C)$, then

$$\begin{aligned} x \in A - (B \cap C) &\Rightarrow x \in A \text{ and } x \notin (B \cap C) \qquad [\because\ x \notin (A - B) \Rightarrow x \in A \text{ and } x \notin B] \\ &\Rightarrow x \in A \text{ and } (x \notin B \text{ or } x \notin C) \\ &\Rightarrow (x \in A \text{ and } x \notin B) \text{ or } (x \in A \text{ and } x \notin C) \\ &\Rightarrow x \in (A - B) \text{ or } x \in (A - C) \\ &\Rightarrow x \in (A - B) \cup (A - C). \end{aligned}$$

$$\therefore \quad A - (B \cap C) \subset (A - B) \cup (A - C) \qquad \text{...}(i)$$

Again if $x$ be an arbitrary element of the set $(A - B) \cup (A - C)$, then

$$\begin{aligned} x \in (A - B) \cup (A - C) &\Rightarrow x \in (A - B) \text{ or } x \in (A - C) \\ &\Rightarrow (x \in A \text{ and } x \notin B) \text{ or } (x \in A \text{ and } x \notin C) \\ &\Rightarrow x \in A \text{ and } (x \notin B \text{ or } x \notin C) \\ &\Rightarrow x \in A \text{ and } x \notin B \cap C \\ &\Rightarrow x \in A - (B \cap C) \end{aligned}$$

$$\therefore \quad (A - B) \cup (A - C) \subset A - (B \cap C) \qquad \text{...}(ii)$$

From (*i*) and (*ii*), we get $A - (B \cap C) = (A - B) \cup (A - C)$.

(*ii*) $(x, y) \in A \times (B \cap C)$

$$\begin{aligned} &\Rightarrow x \in A \text{ and } y \in (B \cap C) \\ &\Rightarrow (x \in A \text{ and } y \in B) \text{ and } (x \in A \text{ and } y \in C) \\ &\Rightarrow (x, y) \in (A \times B) \text{ and } (x, y) \in (A \times C) \\ &\Rightarrow (x, y) \in (A \times B) \cap (A \times C) \end{aligned}$$

Hence $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

## 37.3 LAWS OF SET THEORY

**1.** *Commutative Law*

$$A \cup B = B \cup A \; ; A \cap B = B \cap A.$$

**2.** *Associative Law*

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

**3.** *Distributive Law*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

**4.** *Complement Law*

$$A \cup A^c = U \; ; A \cap A^c = \phi.$$

**5.** *Identity Law*

$$A \cup \phi = A = \phi \cup A$$
$$A \cap U = A = U \cap A.$$

**6.** *Absorption Law*

$$A \cup (A \cap B) = A \; ; A \cap (A \cap B) = A$$

**7.** *De Morgan's Law*

$$(A \cup B)^c = A^c \cap B^c \; ; (A \cap B)^c = A^c \cup B^c$$

**8.** *Involution Law*

$$(A^c)^c = A.$$

## 37.4 PRINCIPLE OF INCLUSION

**(1)** *If A and B be sets with cardinalities* $|A|$ *and* $|B|$, *then*

$$|A \cup B| = |A| + |B| - |A \cup B|$$

*Proof.* The number of common elements in $A$ and $B$ is $|A \cap B|$. Each of these elements is counted twice in $|A| + |B|$, once in $|A|$ and once in $|B|$. This should be adjusted by subtracting the term $|A \cap B|$ from $|A| + |B|$.

Hence $$|A \cup B| = |A| + |B| - |A \cap B|.$$

**Obs.** Using the distributive law, we can extend the above result for three sets $A, B, C$ so that

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$

(*V.T.U., 2002*)

For
$$\begin{aligned} |A \cup B \cup C| &= |(A \cup B) \cup C| \\ &= |A \cup B| + |C| - |(A \cup B) \cap C| \\ &= |A| + |B| - |A \cap B| + |C| - |(A \cap C) \cup (B \cap C)| \\ &= |A| + |B| + |C| - |A \cap B| - [|A \cap C| + |B \cap C| - |A \cap B \cap C|] \end{aligned}$$

whence follows the result.

**(2) Duality.** *If S be any identity involving sets and operations* (*e.g.* complement, intersection $\cap$ and union $\cup$ etc.) *and a new set $S^*$ is obtained by replacing $\cap$ by $\cup$, $\cup$ by $\cap$, $\phi$ by U and U by $\phi$ in S, then the statement $S^*$ is true and is called the* **dual** *of the statement S.*

*For instance*, the dual of $A \cap (B \cup A) = A$ is $A \cup (B \cap A) = A$.

**Example 37.6.** *In a survey conducted on 250 persons, it was found that 180 drink tea and 70 drink coffee and 50 take both tea and coffee. How many drink atleast one bevarage and how many drink neither ?*

**Solution.** Let $A$ be the set of tea drinkers and $B$ the set of coffee drinkers. Then

$$|A \cup B| = |A| + |B| - |A \cap B| = 180 + 70 - 50 = 200$$

Hence 200 persons drink at least one beverage and 250 – 200 = 50 persons drink neither tea nor coffee.

**Example 37.7.** *How many integers between 1 and 468 are divisible by 3 but not by 5.*

**Solution.** Number of integers between 1 and 468 which are divisible by 3 $= \left[\frac{468}{3}\right] = 156$

Number of integers between 1 and 468 which are divisible by 3 and 5 $= \left[\frac{468}{3 \times 5}\right] = 31$

Hence the number of integers between 1 and 468 divisible by 3 but not by 5 = 156 – 31 = 125.

**Example 37.8.** *How many integers are between 1 and 200 which are divisible by any one of the integers 2, 3 and 5 ?*

**Solution.** Let $A_1$, $A_2$, $A_3$ denote the set of integers between 1 and 200 which are divisible by 2, 3, 5 respectively.

$$[A_1] = \left[\frac{200}{2}\right] = 100,\ [A_2] = \left[\frac{200}{3}\right] = 66,\ [A_3] = \left[\frac{200}{5}\right] = 40$$

$$[A_1 \cap A_2] = \left[\frac{200}{2 \times 3}\right] = 33,\ [A_1 \cap A_3] = \left[\frac{200}{2 \times 5}\right] = 20$$

$$[A_2 \cap A_3] = \left[\frac{200}{3 \times 5}\right] = 13,\ [A_1 \cap A_2 \cap A_3] = \left[\frac{200}{2 \times 3 \times 5}\right] = 6$$

Hence
$$\begin{aligned}[A_1 \cup A_2 \cup A_3] &= |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| \\ &\quad - |A_2 \cap A_3| - |A_3 \cap A_1| + [A_1 \cap A_2 \cap A_3] \\ &= 100 + 66 + 40 - 33 - 13 - 20 + 6 = 146.\end{aligned}$$

## PROBLEMS 37.1

1. Show that the following sets are equal :
$A = \{2, 1\}$, $B = \{1, 2, 1, 2, 1, 2\}$, $C = \{x : x^2 - 3x + 2 = 0\}$.
2. Which of the following statements are true ? Give reason to support your answer.
(*i*) $\{a\} \subset \{a, b, c\}$ (*ii*) $a \subset \{a, b, c\}$ (*iii*) $a \subseteq \{a, b, c\}$
(*iv*) $\{a, b\} \subset \{a, b, c\}$ (*v*) $\{a, b\} \in \{a, b, c\}$ (*vi*) $\phi \subset \{a, b, c\}$.
3. Prove that
(*i*) $B - A$ is a subset of $A^c$. (*ii*) $B - A^c = B \cap A$ *(Andhra, 2004)*
(*iii*) $\{A \subset B, B \subset C, C \subset A\} \Rightarrow A = C$.
4. If $A = \{a, b, c, d, e\}$, $B = \{a, c, e, g\}$ and $C = \{b, e, f, g\}$, prove that
(*i*) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ (*ii*) $(A \cap B) \cup (A \cap C) = A \cap (B \cup C)$.
5. If $A = A \cup B$ then prove that $B = A \cap B$.
6. Prove that $A \cup B = B \Leftrightarrow A \subset B$.
7. With the help of the Venn-diagram, prove that
(*i*) $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$. *(Andhra, 2004)*
(*ii*) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. *(V.T.U., 2001 S)*
8. If $B \subset A$, prove that
(*i*) $B \cup C \subset A \cup C$ (*ii*) $B \cap C \subset A \cap C$.
9. (*i*) If $A \cup B = A \cap B$, show that $A = B$.
(*ii*) If $A \cup B = A \cup C$ and $A \cap B = A \cap C$, show that $B = C$.
10. (*i*) Prove that (*i*) $A - B = A - A \cap B$. *(V.T.U., 2001 ; Madras, 2000)*
(*ii*) $A - (B \cup C) = (A - B) \cap (A - C)$.
11. Show that for any two sets $A$ and $B$
(*i*) $A - B = A \cap B^c$ (*ii*) $A \subseteq B \Leftrightarrow B^c \leq A^c$
(*iii*) $A \cup B = (A \cap \overline{B}) \cup (B \cap \overline{A}) \cup (A \cap B)$.
12. If $A$, $B$, $C$ be sets such that $A \subset B$, $B \cap C = \phi$, show that $A \cap C = \phi$.
13. Show that $A \cup (B \cup C)^c = (A \cup B^c) \cap (A \cup C^c)$.

14. Prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$. *(Andhra, 2001)*
15. If S is any set and $P(s)$ is its power set and $A$ and $B$ belong to $P(s)$, prove that $B \cap (A - B) = \phi$.
16. If $A$ and $B$ are finite sets then prove that $A \cup B$ and $A \cap B$ are finite sets and
$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$ *(Andhra, 2004)*
17. In survey conducted on 200 people, it was found that 140 are smokers while 80 are alcoholic and 40 are both smokers and alcoholics. Find how many are neither smokers not alcoholics.
18. How many integers between 1 and 789 are divisible by 5 but not by 7.
19. How many integers are between 1 and 250 while are divisible by any of the integers 3, 5, and 7.
20. Out of a class of 153 students, 54 have taken History, 63 have taken Geography, 62 have taken Economics, and 43 have taken Geography and History, 45 have taken History and Economics, 46 have taken Geography and Economics and 37 have taken all the three subjects. How many of the students have not taken any of these three subjects ? Use a Venn diagram.

# II. ALGEBRA OF LOGIC

## 37.5 INTRODUCTION

**(1)** Logic is concerned with all types of reasoning such as valid statements, mathematical proofs, valid conclusions etc. Logical reasoning is used to prove theorems, to verify the correctness of computer programs and to draw conclusions from experiments. Later on, we shall observe that the algebra of sets and logic is analogous to the algebra of switching circuits which is similar to 'Boolean Algebra'.

**(2) Propositions and Statements.** A *proposition* is a declarative sentence which is either true (1) or false (0). Some authors use $T$ and $F$ respectively for 1 and 0. The truth or falsity of a proposition is defined as its *truth value*.

All the declarative sentences to which it is possible to assign one and only one of the two possible truth values are called *statements*.

**Example 37.9.** *Which of the following are statements : (a) Agra is in India (b) $3 + 4 = 5$ (c) Where do you live ? (d) Do you speak Hindi ?*

**Solution.** (*a*) and (*b*) are statements that happen (*a*) is true and (*b*) is false.

(*c*) and (*d*) are questions so they are not statements.

**(3) Compound statements.** The statement which is composed of sub-statements and logical connectives is called a *compound statement.*

*e.g.*, 'It is raining and it is cold' is a compound statement as it is comprised of two sub-statements 'It is raining' and 'it is cold'.

**(4) Truth table.** The truth value of a compound statement is completely determined by the truth value of its substatements. *A convenient way to represent a compound statement is by means of the* **truth table** *wherein the values of a compound statement are specified for all possible choices of the values of the sub statements.*

We shall use the numbers 0 and 1 to denote the false and true statements. Also we use letters $p, q, r, \ldots$ to represent a proposition or logical variable.

## 37.6 LOGICAL OPERATORS

**(1) Conjunction.** *If p and q are two statements then their conjunction p and q written as $p \wedge q$, is defined by the truth table 1.*

Table 1

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 0 | 0 |
| 0 | 1 | 0 |
| 0 | 0 | 0 |

Table 2

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 0 | 1 |
| 0 | 1 | 1 |
| 0 | 0 | 0 |

*For example,* the conjunction of $p$ : it is raining and $q$ : I am cold is $p \wedge q$ : It is raining and I am cold.

**(2) Disconjunction.** *If p and q are two statements, then their disconjunction p or q written as $p \vee q$ is defined by the truth table 2.*

*For example,* the disjunction of $p$ and $q$ for $p$ : it is raining today ; $q$ : 3 is an odd integer is

$p \vee q$ : it is raining today or 3 is an odd integer.

**(3) Negation.** *If p is a given statement and its negative 'not p', written as $\sim p$ (or Np or $\rceil p$) is defined by the following truth table :*

| $p$ | $\sim p$ |
|---|---|
| 0 | 1 |
| 1 | 0 |

*For example,* the negation of the following statement

(*a*) $p : 2 + 3 > 1$ is $\sim p : 2 + 3 \leq 1$

(*b*) $q$ : it is hot is $\sim q$ : it is cold.

**Example 37.10.** *If p be 'it is hot' and q be 'it is raining', describe each of the following statements by a sentence :*

(*a*) $q \vee \sim p$ (*b*) $\sim p \wedge \sim q$ (*c*) $\sim (\sim p \vee q)$.

**Solution.** (*a*) It is raining or it is not hot. (*b*) It is not hot and it is not raining.

(*c*) It is hot but not raining.

**(4) Conditional operator.** *The conditional statement 'if p then q' written as $p \to q$ is defined by the truth table 4 :*

Table 4

| $p$ | $q$ | $p \to q$ |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 0 | 0 |
| 0 | 1 | 1 |
| 0 | 0 | 1 |

Table 5

| $p$ | $q$ | $p \leftrightarrow q$ |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 0 | 0 |
| 0 | 1 | 0 |
| 0 | 0 | 1 |

**Obs.** The contrapositive of conditional statement $p \to q$ is the statement $\sim p \to \sim q$.

**(5) Biconditional operator.** If $p$ and $q$ be two statements, then the statement '$p$ if and only if $q$' denoted by $p \leftrightarrow q$ and abbreviated as '$p$ if $q$' is called a *biconditional statement*. The truth table for biconditional statement is table 5.

**Example 37.11.** *Construct the truth tables for*

(*a*) $p \wedge \sim q$ (*b*) $(p \vee q) \vee \sim p$

(*c*) $(p \to q) \wedge (q \to p)$ (*d*) $(p \to q) \vee \sim (p \leftrightarrow \sim q)$.

**Solution.** (*a*) The truth table is

| $p$ | $q$ | $\sim q$ | $p \wedge \sim q$ |
|---|---|---|---|
| 1 | 1 | 0 | 0 |
| 1 | 0 | 1 | 1 |
| 0 | 1 | 0 | 0 |
| 0 | 0 | 1 | 0 |

(*b*) The truth table is

| $p$ | $q$ | $p \vee q$ | $\sim p$ | $(p \vee q) \vee \sim p$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 1 |
| 1 | 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 1 | 1 |
| 0 | 0 | 0 | 1 | 1 |

(*c*) The truth table is

| $p$ | $q$ | $p \to q$ | $q \to p$ | $(p \to q) \wedge (q \to p)$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 | 0 |
| 0 | 1 | 1 | 0 | 0 |
| 0 | 0 | 1 | 1 | 1 |

(*d*) The truth table in this case is

| $p$ | $q$ | $p \to q$ | $\sim q$ | $p \to \sim q$ | $(p \to \sim q)$ | $p \to q \vee \sim (p \to \sim q)$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 0 | 1 | 1 |
| 1 | 0 | 0 | 1 | 1 | 0 | 0 |
| 0 | 1 | 1 | 0 | 1 | 0 | 1 |
| 0 | 0 | 1 | 1 | 0 | 1 | 1 |

## 37.7 STATEMENTS GENERATED BY A SET

**(1)** *If S be a set of statements, then any valid combination of statements in S with conjunction, disconjunction or negation is a statement generated by S.*

A statement generated by a set $S$ need not include each element of $S$ in its expression.

*For example, if p, q, r are statements in S then*

(*a*) $(p \wedge q) \wedge r$ (*b*) $\sim q \wedge r$ (*c*) $(p \wedge q) \vee (\sim q \wedge r)$

*are statements generated by S. Their truth tables are :*

| $p$ | $q$ | $r$ | $p \wedge q$ | $(p \wedge q) \wedge r$ (*a*) | $\sim q$ | $\sim q \wedge r$ (*b*) | (*c*) |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 0 | 0 | 1 |
| 1 | 1 | 0 | 1 | 0 | 0 | 0 | 1 |
| 1 | 0 | 1 | 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 | 1 | 1 | 1 |
| 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |

**(2) Tautology** *is an expression involving logical variables which is true for all cases in its truth table.* It is also called a *logical truth.*

**(3) Contradiction** is an expression involving logical variables which is false for all cases in its truth table. Obviously, the negation of a contradiction is a tautology.

In other words, a statement formula which is a tautology is identically true, while a formula which is a contradiction is identically false.

**Obs.** The conjunction of two tautologies is also a tautology.

**Example 37.12.** *Show that (a)* $p \vee \sim p$ *is a tautology (b)* $p \to q \leftrightarrow (\sim p \vee q)$.

**Solution.** (*a*) The truth table is

| $p$ | $\sim p$ | $p \vee \sim p$ |
|---|---|---|
| 1 | 0 | 1 |
| 0 | 1 | 1 |

Hence $p \vee \sim p$ is a tautology.

(*b*) The truth table is

| $p$ | $q$ | $p \to q$ | $\sim p$ | $\sim p \vee q$ | $(p \to q) \leftrightarrow (\sim p \vee q)$ |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 0 | 0 | 1 |
| 0 | 1 | 1 | 1 | 1 | 1 |
| 0 | 0 | 1 | 1 | 1 | 1 |

Hence $(p \to q) \leftrightarrow (\sim p \vee q)$ is true.

## 37.8 EQUIVALENCE

**(1)** *If p and q be statements generated by the set of statements S, then p and q are equivalent if* $p \leftrightarrow q$ *is a tautology which is denoted by* $p \Leftrightarrow q$.

If $p \to q$ is a tautology, then we say that $p$ *implies* $q$ and write it as $p \Rightarrow q$.

**Obs.** All tautologies are equivalent to each other and all contradictions are equivalent to each other.

**(2) Equivalent formulae.** Some basic equivalent formulae are given below which can be proved by using truth tables :

| | | |
|---|---|---|
| 1. $p \vee p \Leftrightarrow p$ | $p \wedge p \Leftrightarrow p$ | *Idempotent laws* |
| 2. $p \vee q \Leftrightarrow q \wedge p$ | $p \wedge q \Leftrightarrow q \wedge p$ | *Commutative laws* |
| 3. $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$ | $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$ | *Associative laws* |
| 4. $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$ | $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ | *Distributive laws* |
| 5. $p \vee \sim p \Leftrightarrow 1$ | $p \wedge \sim p \Leftrightarrow 0$ | *Negation law* |
| 6. $p \vee 0 \Leftrightarrow p$ | $p \wedge 1 \Leftrightarrow p$ | *Identity laws* |
| 7. $p \vee 1 \Leftrightarrow 1$ | $p \wedge 0 \Leftrightarrow 0$ | *Null laws* |
| 8. $p \vee (p \wedge q) \Leftrightarrow p$ | $p \wedge (p \vee q) \Leftrightarrow p$ | *Absorption laws* |
| 9. $\sim (p \vee q) \Leftrightarrow \sim p \wedge \sim q$ | $\sim (p \vee q) \Leftrightarrow \sim p \vee \sim q$ | *De Morgan's laws* |
| 10. $p \Rightarrow p \vee q$ | $q \Rightarrow p \vee q$ | *Disjunctive addition* |
| 11. $p \wedge q \Rightarrow q$ | $p \wedge q \Rightarrow q$ | |
| 12. $(p \vee q) \wedge \sim p \Rightarrow q$ | $(p \vee q) \wedge \sim q \Rightarrow p$ | |
| 13. $(p \to q) \wedge (q \to r) \Rightarrow (p \to r)$ | | *Chain rule* |
| 14. $p \to q \Leftrightarrow \sim p \vee q$ | | *Conditional equivalence* |
| 15. $p \leftrightarrow q \Leftrightarrow (p \to q) \wedge (q \to p)$ | | *Biconditional equivalence* |

**Example 37.13.** *Show that*

(*a*) $p \to (q \to r) \Leftrightarrow p \to (\sim q \vee r) \Leftrightarrow (p \wedge q) \to r$

(*b*) $[\sim p \wedge (\sim q \wedge r)] \vee (q \wedge r) \vee (p \wedge r) \Leftrightarrow r$

**Solution.** (*a*) By conditional equivalence $q \to r \Leftrightarrow \sim q \vee r$

Replacing $q \to r$ by $\sim q \vee r$, we get $p \to (\sim q \vee r)$ which is equivalent to $\sim p \vee (\sim q \vee r)$ by the same rule.

Thus
$$\begin{aligned} \sim p \vee (\sim q \vee r) &\Leftrightarrow (\sim p \vee \sim q) \vee r && \text{[By (3)]} \\ &\Leftrightarrow \sim (p \wedge q) \vee r && \text{[By (9)]} \\ &\Leftrightarrow (p \wedge q) \to r && \text{[By (14)]} \end{aligned}$$

(*b*) $[\sim p \wedge (\sim q \wedge r)] \vee (q \wedge r) \vee (p \wedge r)$
$$\begin{aligned} &\Leftrightarrow [\sim p \wedge (\sim q \wedge r)] \vee (q \vee p) \wedge r && \text{[By (4)]} \\ &\Leftrightarrow [\sim p \wedge \sim q \wedge r] \vee (q \vee p) \wedge r && \text{[By (3)]} \\ &\Leftrightarrow [\sim (p \vee q) \wedge r] \vee (q \vee p) \wedge r && \text{[By (9)]} \\ &\Leftrightarrow [\sim (p \vee q) \vee (p \vee q)] \wedge r && \text{[By (4)]} \\ &\Leftrightarrow 1 \wedge r && \text{[By (5)]} \\ &\Leftrightarrow r && \text{[By (6)]} \end{aligned}$$

## 37.9 DUALITY LAW

**(1)** *Two formulae A and A* are said to be duals of each other if either one can be obtained from the other by replacing* $\wedge$ *by* $\vee$ *and* $\vee$ *by* $\wedge$.

If the formula A contains special variables 1 or 0, then its dual A* is obtained by replacing 1 by 0 and 0 by 1.

*e.g.,* (*i*) Dual of $(p \vee q) \wedge r$ is $(p \wedge q) \vee r$

(*ii*) Dual of $(p \wedge q) \vee 0$ is $(p \vee q) \wedge 1$.

**(2) Tautology implications.** *A statement A is said to tautologically imply a statement B if and only if* $A \to B$ *is a tautology which is read as "A implies B".*

*The implications listed below have important applications* which can be proved by truth tables :

1. $p \wedge q \Rightarrow p$ $\qquad$ $p \Rightarrow p \vee q$
2. $\sim p \Rightarrow p \to q$ $\qquad$ $q \Rightarrow p \to q$
3. $\sim (p \to q) \Rightarrow p$ $\qquad$ $\sim (p \to q) \Rightarrow \sim q$
4. $p \wedge (p \to q) \Rightarrow q$ $\qquad$ $\sim p \wedge (p \vee q) \Rightarrow q$
5. $(p \to q) \wedge (q \to r) \Rightarrow p \to r$

## 37.10 ARGUMENTS

**(1)** *An argument is an assertion that a given set of propositions* $p_1, p_2, ..., p_n$ (called *premises*) *yields another proposition* $q$ (called *conclusion*). The argument is symbolically written as "$p_1, p_2, ..., p_n \vdash q$".

An argument $p_1, p_2, ..., p_n \vdash q$ is true provided $q$ is true whenever all the premises $p_1, p_2, ..., p_n$ are true. An argument which is true is said to be '*valid argument*'. Otherwise it is called a *fallacy*.

**Example 37.14.** *Show that*

(*a*) *the argument* $p \leftrightarrow q, q \vdash p$ *is valid.*

(*b*) *the argument* $p \to q, q \vdash p$ *is a fallacy.*

**Solution.** (*a*) Let us first prepare the truth table as follows :

| $p$ | $q$ | $p \leftrightarrow q$ |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 0 | 0 |
| 0 | 1 | 0 |
| 0 | 0 | 1 |

Since $p \leftrightarrow q$ is true in cases (rows) 1 and 4, and $q$ is true in cases 1 and 3, therefore $p \leftrightarrow q$ and $q$ both are true in case 1 only when $p$ is also true. This shows that the given argument is valid.

(*b*) Let us first prepare the truth table below :

| $p$ | $q$ | $p \to q$ |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 0 | 0 |
| 0 | 1 | 1 |
| 0 | 0 | 1 |

This table shows that $p \to q$ and $q$ both are true in case 3 only while the conclusion $p$ is false. Hence the given argument is a fallacy.

**(2) Theorem.** *The argument* $p_1, p_2, ..., p_n \vdash q$ *is valid if and only if the proposition* $(p_1 \wedge p_2 \wedge p_3 \wedge ..., \wedge p_n) \to q$ *is a tautology.*

The proportions $p_1, p_2, ..., p_n$ are simultaneously true if and only if the proposition $p_1 \wedge p_2 \wedge ..., \wedge p_n$ is true *i.e.*, if the proposition $(p_1 \wedge p_2 \wedge ..., \wedge p_n) \to q$ is a tautology.

**Obs.** The validity of an argument depends upon the particular form of the argument, not on the truth values of the statement appearing in the argument.

**Example 37.15.** *Test the validity of the following argument :*

$S_1$ : *If 5 is less than 3, then 6 is not a prime number*

$S_2$ : *5 is not less than 6*

$S_3$ : *5 is a prime number.*

**Solution.** Let '5 is less than 3' be $p$ and '5 is a prime number' be $q$. Then the given argument is of the form $p \to \sim q, \sim p \vdash q$.

Since in the last line of the truth table, the premises $p \to \sim q$ and $\sim q$ are true but the conclusion $q$ is false, therefore the argument is fallacy.

| $p$ | $q$ | $\sim q$ | $p \to \sim q$ | $\sim p$ |
|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 | 0 |
| 0 | 1 | 0 | 1 | 1 |
| 0 | 0 | 1 | 1 | 1 |

## PROBLEMS 37.2

1. If $p \equiv$ Sam is a teacher, $q \equiv$ John is an honest boy, then translate the following into logical sentences :
   (*a*) $\sim (p \wedge q)$, (*b*) $p \vee \sim q$, (*c*) $\sim p \Leftrightarrow q$, (*d*) $p \Rightarrow \sim q$.
2. Change the following sentence into symbols :
   (*a*) 'If I do not have car or I do not wear good dress then I am not a millionaire'.
   (*b*) Everyone who is healthy can do all kinds of work. (*Anna, 2004 S*)
3. Prepare truth tables for the following statements (*a*) $(p \Rightarrow q) \wedge \sim q$, (*b*) $(p \Leftrightarrow q) \wedge (r \vee q)$.
4. Write down the truth table of
   (*a*) $p \vee q$ (*Madras, 1997*) (*b*) $p \wedge (p \wedge q)$ (*Madras, 2005 S*)
5. Verify that the following are tautologies :
   (*a*) $p \to [q \to (p \wedge q)]$ (*Anna, 2005*)
   (*b*) $(p \wedge q \Rightarrow r) \Leftrightarrow (p \Rightarrow r) \vee (q \Rightarrow r)$ (*c*) $(p \Rightarrow q \wedge r) \Rightarrow (\sim r \Rightarrow \sim p)$.
6. Show that $Q \vee (P \wedge \sim Q) \vee (\sim P \wedge \sim Q)$ is a tautology. (*Anna, 2004 S ; Madras, 2003 S*)
7. Over the universe of positive integers
   $p(n)$ : $n$ is prime and $n < 32$.
   $q(n)$ : $n$ is a power of 3.
   $r(n)$ : $n$ is a divisor of 27.
   (*i*) What are the truth sets of these propositions ?
   (*ii*) Which of the three propositions implies one of the others ?
8. Given the propositions over the natural numbers
   $p : n < 4$, $q : 2n > 17$ and $r$ : $n$ is a divisor of 18, what are the truth sets of
   (*i*) $q$, (*ii*) $p \wedge q$,
   (*iii*) $r$, (*iv*) $q \to r$. (*Madras, 1999*)
9. Show that (*a*) $\sim Q, P \to Q \Rightarrow \sim P$. (*Madras, 2003*)
   (*b*) $(P \to R) \wedge (Q \to R) \Leftrightarrow (P \vee Q) \to R$. (*Madras, 2001*)
10. Construct the truth table for (*i*) $(\sim p \to q) \wedge (q \rightleftarrows p)$. (*Bharthian, MSc. 2001*)
    (*ii*) $\rceil [P \vee (Q \wedge R)] \rightleftarrows (P \vee Q) \wedge (P \vee R)$. (*Andhra, 2004*)
11. Prove that the following statement is a contradiction :
    $S = [(p \vee q) \wedge (p \vee \sim q) \wedge (\sim p \vee q) \wedge (\sim p \vee \sim q)]$.
12. If $p, q, r$ are three statements then prove that
    (*a*) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ (*b*) $(p \Rightarrow q) \vee r \equiv (p \vee r) \Rightarrow (q \vee r)$
    (*c*) $\sim (p \vee q) \equiv \sim p \wedge \sim q$.
13. Define conjunction, conditional, biconditional and negation, with examples.

14. Show that (*i*) $p \wedge q$ logically implies $p \leftrightarrow q$.
 (*ii*) $p \leftrightarrow \sim q$ does not logically imply $p \rightarrow q$. (*U.P.T.U., 2001*)
15. Write the duals of $(p \vee q) \wedge r$ and $(p \wedge q) \vee t$.
16. Show that $s \vee r$ is tautologically implied by $(p \vee q) \wedge (q \rightarrow r) \wedge (q \rightarrow s)$. (*Andhra, 2004 ; Bharathian, 2001*)
17. Let $P(n)$ be '$8^n - 3^n$ is a multiple of 5'. Prove that $P(n)$ is a tautology on $n$.
18. Prove that $P \rightarrow \rceil Q, R \rightarrow Q, R \vdash \rceil P$ is a valid argument. (*Madras, 2003*)
19. Prove that $p \rightarrow (q \vee r) \Leftrightarrow (p \wedge \rceil q) \rightarrow r$. (*Anna, 2005*)
20. Show that $R \vee S$ follows logically from the premises $C \vee D, C \vee D \rightarrow -H, H \rightarrow A \wedge \sim B, A \wedge \sim B \rightarrow R \vee S$.

## 37.11 PREDICATES

Statements involving variables such as '$x > 7$' and '$x = y + 7$' are neither true nor false so long as the values of the variable $x$, $y$ are not specified. We now, discuss the ways that propositions can be evolved from such statements. The statement '$x > 7$' has two parts : First part—'the variable $x$' is the *subject* of the statement ; Second part–'is greater than 7' is the *predicate* which refers to the property that the subject of statement can have. If $P(x)$ denotes the statement '$x > 7$' then $P$ is the *predicate* and $x$ is the *variable*. The statement $P(x)$ is also known as the value of the propositional function $P$ at $x$. The *predicates* are denoted by capital letters and the *objects* by the variables (denoted by small letters in brackets).

Thus *predicates are simple statements which turn out to be propositions involving variables whose values are not well specified.*

In other words, *predicate is a variable statement which becomes specific when particular values are assigned to the variables.*

There are statements which involves more than one variable consider the statement '$x = y + 7$' which is denoted by $Q(x, y)$ where $Q$ is the predicate and $x, y$ are the variables. When values are assigned to the variables $x, y$, the statement $Q(x, y)$ has the truth value.

Similarly $R(x, y, z)$ denotes the statement of the type '$x + y = z$'. When values are assigned to $x, y, z$, this statement has a truth value.

*For example, consider the statements (i) Ram is fair ; (ii) Sham is fair.*

Here in (*i*) and (*ii*) 'is fair' is the predicate while Ram and Sham are the objects. If we denote the predicate by $F$ and the objects by $r$ and $s$, then the above statements can be symbolically expressed as (*i*) $F(r)$ ; (*ii*) $F(s)$.

*Now consider the statement Ram is fair and the house is pink.*

Writing 'the house is pink' as $P(h)$, the given statement can be expressed as $F(r) \wedge P(h)$.

## 37.12 QUANTIFIERS

**(1)** In a propositional function, when all the variables are assigned values, the resulting statement has a truth value. However, there is another method to create a proposition from a propositional function which is called *quantification*. It is of two types : *Universal quantification* and *Existential quantification*.

**(2) Universal quantification.** Many statements assert that a property is true for all values of a variable in a certain domain. This domain is termed as the universe of discourse and such a statement is expressed using universal quantification.

Thus *the universal quantification of P(x) is the proposition 'P(x) is true for all values of x in the universe of discourse'.*

The universal quantification of $P(x)$ is denoted by $\forall\, xP(x)$. The symbol $\forall$ is called the *universal quantifier*.

**Obs.** When it is possible to list all the elements in the universe of discourse say : $x_1, x_2, ..., x_n$, then the universal quantification $\forall\, x\, P(x)$ is same as the conjunction $P(x_1) \wedge P(x_2) \wedge ..., P(x_n)$ for this conjunction is true if and only if $P(x_1)$, $P(x_2)$, ..., $P(x_n)$ are all true.

**Example 37.16.** *What is the truth value of the quantification $\forall\, x\, P(x)$ where*
*(a) $P(x)$ is the statement '$x < 5$' and universe of discourse is the set of real numbers.*
*(b) $P(x)$ is the statement '$x^2 < 18$' and the universe of discourse consists of positive integers not exceeding 5 ?*

**Solution.** (*a*) For instance, $P(6)$ is false ; therefore $P(x)$ is not true for all real numbers $x$.
Thus $\forall\, xP(x)$ is false.

(*b*) The statement $\forall\, xP(x)$ is same as the conjunction $P(1) \wedge P(2) \wedge P(3) \wedge P(4) \wedge P(5)$

Here the universe of discourse is 1, 2, 3, 4, 5 and $P(5)$ is the statement '$5^2 < 18$' which is false. Hence $\forall\, xP(x)$ is also false.

**(3) Existential quantification.** Many statements assert that there is an element with a certain property. To express such statements we use *existential quantification*. In such cases, we form a proposition which is true if and only if $P(x)$ is true for at least one value of $x$ in the universe of discourse.

Thus *the existential quantification of $P(x)$ is the proposition 'these exists an element $x$ in the universe of discourse such that $P(x)$ is true'.*

The notation $\exists xP(x)$ is used for the *existential quantification* wherein $\exists$ is called the *existential quantifier*.

**Obs.** When it is possible to list all the elements in the universe say : $x_1, x_2, ..., x_n$, then the existential quantification $\exists x\, P(x)$ is same as the disjunction $P(x_1) \vee P(x_2) \vee ..., P(x_n)$.

**Example 37.17.** *What is the truth value of the quantification $\exists xP(x)$ where*

*(a) $P(x)$ is the statement '$x > 5$' and universe of discourse is the set of real numbers.*

*(b) $P(x)$ is the statement '$x^2 > 18$' and the universe of discourse consists of positive integers not exceeding 5 ?*

**Solution.** (*a*) Since '$x > 5$' is true, say : for $x = 6$, 8 etc.

$\therefore \quad \exists xP(x)$ is true.

(*b*) The statement $\exists xP(x)$ is same as the disjunction $P(1) \vee P(2) \vee P(3) \vee P(4) \vee P(5)$

The universe of discourse is 1, 2, 3, 4, 5 and $P(5)$ is the statement $5^2 > 18$, which is true.

Hence $\exists xP(x)$ is also true.

**Example 37.18.** *Find the truth value of each of the following statements :*

*(a) $\exists x,\, x^2 = 1$ (b) $\forall x,\, |\, x\, | = x$ (c) $\exists x,\, x + 4 = x$.*

**Solution.** (*a*) If $x_0 = 1$, then $x_0^{\,2} = 1$, therefore the given statement is true.

(*b*) If $x_0 = -3$, then $|\, x_0\, | \neq x_0$, therefore the given statement is false.

(*c*) As there is no solution to $x + 4 = x$, the given statement is false.

**Example 37.19.** *Given $P = \{2, 3, 4, 5, 6\}$, state the truth values of each of the following statements :*

*(a) $(\forall\, x \in P)\ (x + 3 < 10)$ (b) $(\forall\, x \in P)\ (x + 2 \le 7)$*

*(c) $(\exists x \in P)\ (x + 3 = 10)$ (d) $(\exists x \in P)\ (x + 2 < 7)$.*

**Solution.** (*a*) True, for each number in $P$ satisfies $x + 3 < 10$.

(*b*) False, for if $x_0 = 6$, then $x_0 + 2$ is not $\le 7$.

(*c*) False, for no number in $P$ is a solution to $x + 3 = 10$.

(*d*) True, for if $x_0 = 2$ then $x_0 + 2 < 7$ is a solution.

**Example 37.20.** *Negate each of the following statements :*

*(a) $\forall x,\, x^2 = x$, (b) $\forall\, x,\, x + 4 > x$ (c) $\forall x,\, |\, x\, | = x$.*

**Solution.** (*a*) $\sim \forall\, x,\, x^2 = x \equiv \forall\, x \sim (x^2 = x) \equiv \forall\, x,\, x^2 \neq x$.

(*b*) $\sim \forall\, x,\, x + 4 > x \equiv \exists\, x \sim (x + 4 > x) \equiv \rightarrow x\, ,\, x + 4 \le x$.

(*c*) $\sim \forall\, x,\, |\, x\, | = x \equiv \exists\, x \sim (|\, x\, | = x) \equiv \exists\, x,\, |\, x\, | \neq x$.

**Example 37.21.** *Symbolise using quantifiers :*

*(i) Every even number is divisible by 2.*

*(ii) There is no prime number between 23 and 29.* (*V.T.U., MCA, 2001 S*)

**Solution.** (*i*) $E(x)$ : $x$ is even number ; $D(x)$ : $x$ is divisible by 2.

$(\forall\, x)\ [E(x) \rightarrow D(x)]$

(*ii*) If $p$ denotes the set of prime numbers, then $(\nexists_n)_p\ (23 < n < 29)$.

**Example 37.22.** *Symbolise the expression 'All the world loves a lover.'* (Madras, 2001)

**Solution.** Let $p(x) : x$ is a person ;

$L(x) : x$ is a lover.

and $Q(x, y) : x$ loves $y$

Then the required expression is

$$(\forall x)\ [p(x) \to (y)\ (p(y) \wedge L(y)] \to Q(x, y).$$

**Summary.** (*i*) $\forall\, Q(x)$ means that the predicate $Q(x)$ is true *for all values* in the universe of $x$.

(*ii*) $\exists Q(x)$ means that the predicate $Q(x)$ is satisfied if there is *at least one value* in the universe of $x$.

## 37.13 NORMAL FORMS

**(1)** For the given variables $p_1, p_2, ..., p_n$, we may form a statement $S\ (p_1, p_2, ..., p_n)$. The truth table for $S$ will contain $2^n$ rows for all possible truth values of the $n$ variables. The expression $S$ may have the truth value 1 in all cases or may have the truth value 0 in all cases or have the truth value 1 for at least one combination of truth values assigned to the $n$ variables. (Here $S$ is said to be satisfiable). The problem of finding in a finite number of steps whether a given expression is a tautology or a contradiction or at least satisfiable is known as a *decision problem*. As the formation of a truth table is quite cumbersome, we go for an alternate approach called *normal form.*

In this approach, we use the word 'sum' in place of disjunction and 'product' in place of conjunction.

A sum of the variables and their negations is called an *elementary sum*. Similarly a product of the variables and their negation is called an *elementary product*.

**(2) Disjunctive normal form** *of a given formula is the formula which is equivalent to the given formula and which contains the sum of the elementary products.* The disjunctive normal form of a given form is not unique. In fact, several disjunctive normal forms can be obtained for a given formula by applying the distributive laws in different ways.

A given formula is however, identically false if every elementary product appearing in its disjunctive normal form is identically false.

**Example 37.23.** *Obtain the disjunctive normal forms of*

(*i*) $p \wedge (p \to q)$ (*ii*) $\sim (p \vee q) \leftrightarrow (p \wedge q)$.

**Solution.** (*i*) $p \wedge (p \to q) \Leftrightarrow p \wedge (\sim p \vee q) \Leftrightarrow (p \wedge \sim p) \vee (p \wedge q)$.

which is the desired disjunctive normal form.

(*ii*) $\sim (p \vee q) \leftrightarrow (p \wedge q) \Leftrightarrow \sim (p \vee q) \wedge (p \wedge q) \vee (p \vee q) \wedge \sim (p \wedge q)$ $[\because\ E \leftrightarrow F \Leftrightarrow (E \wedge F) \vee (\sim E \wedge \sim F)$

$$\Leftrightarrow (\sim p \wedge \sim q \wedge p \wedge q) \vee [(p \vee q) \wedge (\sim p \vee \sim q)]$$
$$\Leftrightarrow (\sim p \wedge \sim q \wedge p \wedge q) \vee [(p \vee q) \wedge \sim p] \vee [(p \vee q) \wedge \sim q]$$
$$\Leftrightarrow (\sim p \wedge \sim q \wedge p \wedge q) \vee (p \wedge \sim p) \vee (q \wedge \sim p) \vee (p \wedge \sim q) \vee (q \wedge \sim q)$$

which is the desired disjunctive normal form.

**(3) Conjunctive normal form** *of a given formula is that formula which is equivalent to the given formula and contains the product of elementary sums.*

**Example 37.24.** *Find a conjunctive normal form of* $\sim (p \vee q) \leftrightarrow (p \wedge q)$.

**Solution.** $\sim (p \vee q) \leftrightarrow (p \wedge q) \Leftrightarrow [\sim (p \vee q) \to (p \wedge q)] \wedge [p \wedge q \to \sim (p \vee q)]$

$$\Leftrightarrow [(p \vee q) \vee (p \wedge q)] \vee [\sim (p \wedge q) \vee \sim (p \vee q)]$$ [By conditional equivalence]

$$\Leftrightarrow (p \vee q \vee p) \wedge (p \vee q \vee q) \wedge [(\sim p \vee \sim q) \vee (\sim p \wedge \sim q)]$$
$$\Leftrightarrow p \vee q \vee p \wedge (p \vee q \vee q) \wedge (\sim p \vee \sim q \vee \sim p) \wedge (\sim p \vee \sim q \vee \sim q)$$

which is the required conjunctive normal form.

**(4) Principal disjunctive normal form.** Consider a formula for the propositions $p$ and $q$ using conjunction as $p \wedge q, p \wedge \sim q, \sim p \wedge q, \sim p \wedge \sim q$. These terms are called *minterms* or *Boolean conjunction* of $p$ and $q$.

*An equivalent formula for a given formula, consisting of disjunctions of minterms only is called the* **principal disjunctive normal form** (*pdnf*) or **sum of products canonical form**.

**Procedure** *to obtain the principle disjunctive normal form* : (*i*) Replace the conditions and biconditions by their equivalent formulae containing $\wedge, \vee, \sim$ only.

(*ii*) Using DeMorgans laws, apply negations to the variables.

(*iii*) Apply the distribution laws.

(*iv*) Introduce the missing factors to obtain minterms in the disjunctions.

(*v*) Delete identical minterms appearing in the disjunctions.

**Example 37.25.** *Obtain the pdnf for*
(*i*) $p \vee q$ (*ii*) $\sim(p \wedge q)$ (*iii*) $\sim p \vee q$ *i.e.*, $p \to q$.

**Solution.** (*i*)
$$\begin{aligned} p \vee q &\Leftrightarrow [p \wedge (q \vee \sim q)] \vee [q \wedge (p \vee \sim p)] \\ &\Leftrightarrow (p \wedge q) \vee (p \wedge \sim q) \vee (q \wedge p) \vee (q \wedge \sim p) \\ &\Leftrightarrow (p \wedge q) \vee (p \wedge \sim q) \vee (q \wedge \sim p) \end{aligned}$$

(*ii*)
$$\begin{aligned} \sim (p \wedge q) &\Leftrightarrow (\sim p \vee \sim q) \Leftrightarrow [\sim p \wedge (\sim q \vee q)] \vee [\sim q \wedge (p \vee \sim p)] \\ &\Leftrightarrow (\sim p \wedge \sim q) \vee (\sim p \wedge q) \vee (\sim q \wedge p) \vee (\sim q \wedge \sim p) \\ &\Leftrightarrow (\sim p \wedge \sim q) \vee (\sim p \wedge q) \vee (\sim q \wedge p) \end{aligned}$$

(*iii*)
$$\begin{aligned} \sim p \vee q &\Leftrightarrow \sim p \wedge (q \vee \sim q) \vee [q \wedge (p \vee \sim p)] \\ &\Leftrightarrow (\sim p \wedge q) \vee (\sim p \wedge \sim q) \vee (q \wedge p) \vee (q \wedge \sim p) \\ &\Leftrightarrow (\sim p \wedge q) \vee (\sim p \wedge \sim q) \vee (q \wedge p). \end{aligned}$$

**(5) Principal conjunctive normal form.** Consider a formula for the propositions $p$ and $q$ using disconjunction as $p \vee q$, $\sim p \vee q$, $p \vee \sim q$, $\sim p \vee \sim q$. These terms are called *max terms.*

*An equivalent formula for a given formula, consisting of conjunctions of the max terms only is called the* **principal conjunctive normal form** (*pcnf*) or **product of sum canonical form**.

**Procedure** *for obtaining pcnf* for a given formula is similar to the one for *pdnf* as all assertions made for *pdnf* can be made for *pcnf* using duality principle.

**Example 37.26.** *Obtain the principal disjunctive and conjunctive normal forms of*
$p \to [(p \to q) \wedge \sim (\sim q \vee \sim p)]$. (*Bharathiar, 2001*)

**Solution.** (*i*) $p \to [(p \to q) \wedge \sim (\sim q \vee \sim p)]$
$$\Leftrightarrow \sim p \wedge [(\sim p \vee q) \wedge (q \wedge p)]$$
[Using DeMorgan's law and equivalence $p \to q \Leftrightarrow \sim p \vee q$.]
$$\begin{aligned} &\Leftrightarrow \sim p \vee [\sim p \wedge (q \wedge p)] \vee [q \wedge (q \wedge p)] \\ &\Leftrightarrow \sim p \vee (q \wedge p) \\ &\Leftrightarrow [\sim p \wedge (q \vee \sim q)] \vee (q \wedge p) \\ &\Leftrightarrow (\sim p \wedge q) \vee (\sim p \wedge \sim q) \vee (q \wedge p) \end{aligned}$$

This is the desired *pdnf.*

(*ii*) $p \to [(p \to q) \wedge \sim (\sim q \vee \sim p)]$
$$\begin{aligned} &\Leftrightarrow \sim p \vee [(\sim p \vee q) \wedge (q \wedge p)] \\ &\Leftrightarrow [\sim p \vee (\sim p \vee q)] \wedge [\sim p \vee (q \wedge p)] \\ &\Leftrightarrow (\sim p \vee q) \wedge (\sim p \vee q) \wedge (\sim p \vee p) \\ &\Leftrightarrow \sim p \vee q \end{aligned}$$

This is the desired *pcnf.*

**Example 37.27.** *Obtain the principal conjunctive normal form for* $(Q \to P) \wedge (\sim P \wedge Q)$. (*Andhra, 2004*)

**Solution.**
$$\begin{aligned} (Q \to P) \wedge (\sim P \wedge Q) &\Leftrightarrow (\sim Q \vee P) \wedge (\sim P \wedge Q) \\ &\Leftrightarrow (\sim Q \vee P) \wedge [\sim P \vee (Q \wedge \sim Q) \wedge Q \vee (P \wedge \sim P)] \\ &\Leftrightarrow (\sim Q \vee P) \wedge (\sim P \vee Q) \wedge (\sim P \vee \sim Q) \wedge (Q \vee P) \wedge (Q \vee \sim P) \\ &\Leftrightarrow (\sim Q \vee P) \wedge (\sim P \vee Q) \wedge (\sim P \vee \sim Q) \wedge (Q \vee P). \end{aligned}$$

## 37.14 INFERENCE THEORY

Inferring the conclusions from certain premises is known as the *inference theory*. When conclusion is reached from a set of premises by using the accepted rules of reasoning, then this process is called a *deduction* or a *formal proof*. A proof of a theorem is a valid argument.

The criteria for finding 'whether an argument is valid' are called *rules* which are expressed in terms of premises and conclusions or in terms of statement formulae.

*The proofs are of two types : Direct or Indirect.*

(*i*) If in a proof the truth of the premises directly shows the truth of the conclusions, then it is called a *direct proof*.

(*ii*) An *indirect proof* proceeds by assuming that $p$ is true and also $C$ is false, and then deduce a contradiction using $p$ and $\sim C$, along with other premises.

The only difference between assumptions in a direct proof or an indirect proof is the negated conclusion.

*Rules for deriving direct and indirect proofs* :

(*i*) the proof should have finite steps only ;

(*ii*) each step must be either a premise or a proposition which is implied from previous steps using valid equivalence or implication ;

(*iii*) the last step for a direct proof must be conclusion while for an indirect proof it must be a contradiction.

**Example 37.28.** *State whether the conclusion C follows logically from the premises R and S*

(*a*) $R : p \to q,$ $S : p,$ $C : \sim q$

(*b*) $R : p,$ $S : p \leftrightarrow q,$ $C : \sim (p \wedge q)$

(*c*) $R : p \to q,$ $S : p,$ $C : q$

**Solution.** Let us first form the following truth table :

| $p$ | $q$ | $\sim p$ | $\sim q$ | $p \to q$ | $\sim (p \wedge q)$ | $p \leftrightarrow q$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 1 | 0 | 1 |
| 1 | 0 | 0 | 1 | 0 | 1 | 0 |
| 0 | 1 | 1 | 0 | 1 | 0 | 0 |
| 0 | 0 | 1 | 1 | 1 | 1 | 1 |

(*a*) Here only the first row of premises $R$ and $S$ contains 1 but not the conclusion $C$.

Hence $C$ is not valid.

(*b*) Only the first row of both $R$ and $S$ contains 1 but not the conclusion $C$.

Hence $C$ is not valid.

(*c*) Here both $R$ and $S$ contain '1' only in the first row and the conclusion $C$ also has '1' in that row. Hence our conclusion is valid.

**Example 37.29.** *Find the direct and indirect proofs of* $p \to (q \to r), \sim r \vee p, q \Rightarrow r \to s.$

**Solution.** *Direct proof*

| | | | |
|---|---|---|---|
| (*i*) $\sim r \vee p$ | (premise) | (*ii*) $r$ | (another premise) |
| (*iii*) $p$ | [By (*i*) and (*ii*)] | (*iv*) $p \to (q \to r)$ | (premise) |
| (*v*) $q \to r$ | [By (*iii*) and (*iv*)] | (*vi*) $q$ | (premise) |
| (*vii*) $r$ | [By (*v*) and (*vi*)], which is a conclusion. | | |

*Indirect proof*

| | |
|---|---|
| (*i*) $\sim (s \to r)$ | (negative of conclusion) |
| (*ii*) $s \wedge \sim r$ | (By conditional equivalence) |
| (*iii*) $\sim r \vee p$ | (premise) |
| (*iv*) $\sim r$ and (*v*) $s$ | [By conjunctive simplification] |
| (*vi*) $p$ | [By disconjunction of (*iii*) and (*iv*)] |
| (*vii*) $p \to (q \to r)$ | (premise) |

(viii) $q \to r$ [By (vi) and (vii)]
(ix) $q$ (premise)
(x) $r$ [By (viii) and (ix)]
(xi) $r \wedge \sim r$ [By (x) and (iv)]

This is a contradiction.

### PROBLEMS 37.3

1. If $A = \{1, 2, 3, 4, 5\}$ be the universal set, determine the truth values of each of the following statements :
   (a) $(\forall\, x \in A)\,(x + 2 < 10)$ (b) $\exists\, x \in A\,(x + 2 = 10)$
2. Negate each of the following statements :
   (a) $\forall\, x, x^3 = x$ ; (b) $\forall\, x,\ x + 5 > x$
   (c) Some students are 26 or older. (d) All students live in the hostels.
3. What is the truth value of $\forall\, x\, P(x)$ where $P(x)$ is a statement '$x^2 < 10$' and the universe of discourse consists of positive integers not exceeding 4.
4. Use universal quantifier to state 'the sum of any two rational numbers is rational'.
5. Over the universe of real numbers, use quantifier to say that the equation $a + x = b$ has a solution for all values of $a$ and $b$.
6. Translate the following statements involving quantifiers, into formulae :
   (a) All rationals are reals. (b) No rationals are reals.
   (c) Some rationals are reals. (d) Some rationals are not reals.
7. Show that $Q \vee (P \wedge \sim Q) \vee (\sim P \wedge \sim Q)$ is a tautology.
8. Convert $\sim A \wedge (\sim B \to C) \Rightarrow 0$ into CNF. (V.T.U., MCA, 2001)
9. Without constructing truth tables, obtain the product of sums canonical form of the formula $(\neg P \to R) \wedge (Q \leftrightarrow P)$. Hence find the sum of products canonicals form. (Anna, 2004 S)
10. Find the direct proof of $p \to r, q \to s, p \vee q \Rightarrow s \vee r$.
11. Prove that $P \to Q, Q \to R, P \vee R \Rightarrow R$ by using indirect method. (Anna, 2004 S)
12. Using quantifier say $\sqrt{2}$ is not a real number ?
13. State whether the conclusion $C$ follows logically from the premises $R$ and $S$
    (a) $R : p \to q,\ \ S : \sim q,\ \ C : q$ (b) $R : p \to q,\ \ S : q,\ \ C : \sim p$.

# III. BOOLEAN ALGEBRA

## 37.15 INTRODUCTION

**(1)** The concept of Boolean algebra was first introduced by George Boole* in 1854 through his paper 'An investigation of the laws of thought'. It is basically two values *i.e.*, (0, 1) set. Earlier it had applications to statements and sets which are either true or false. In 1938 Claude Shannon showed that basic rules given by Boole could be used to design circuits. These days however, Boolean algebra has wide applications to switching circuits, electrical networks and electronic computers.

Basically there are three operations in the Boolean algebra (*i*) *AND*, (*ii*) *OR*, and (*iii*) *NOT*, which are symbolically represented by $\wedge$, $\vee$ and / respectively. Some authors, use the symbols (+), (.) and (/) for the same operations.

Here ′ denotes the complement of an element and is defined by $0' = 1$ and $1' = 0$.

The operator $\wedge$ (*i.e.*, '*AND*') has the following values $1 \wedge 1 = 1, 1 \wedge 0 = 0, 0 \wedge 1 = 0, 0 \wedge 0 = 0$ ; while the operator $\vee$ (*i.e.*, *OR*) has the values $1 \vee 1 = 1, 1 \vee 0 = 1, 0 \vee 1 = 1, 0 \vee 0 = 0$.

**Def.** *Any non-empty set B with the binary operations '$\wedge$' and '$\vee$' and the uninary operation '/' is called the Boolean algebra [B, $\wedge$, $\vee$, /] if the following axioms hold where a, b, c are elements in B :*

1. *Commutative law* : $a \wedge b = b \wedge a$ ; $a \vee b = b \vee a$
2. *Associative law* : $a \wedge (b \wedge c) = (a \wedge b) \wedge c$
   $a \vee (b \vee c) = (a \vee b) \vee c$

*A British mathematician George Boole (1813–1864) who created Boolean algebra.

3. *Distributive law* : $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

$a \wedge b \vee c = (a \wedge b) \vee (a \wedge c)$

4. *Complement law* : $a \wedge a' = 0, a \vee a' = 1$

The operations $\wedge$, $\vee$ and / are called sum, product, and complement respectively. We shall follow the usual practice that / has precedence over $\vee$, and $\vee$ has precedence over $\wedge$, unless guided by brackets. *e.g.*, $a \wedge b \vee c$ means $a \wedge (b \vee c)$ not $(a \wedge b) \vee c$ while $a \vee b'$ implies $a \vee b'$ not $(a \vee b)'$.

**(2) Boolean function.** The variable $x$ is called a *Boolean variable* if it assumes values only from $B$ {0, 1}.

**Def.** *A function from the set* $\{(x_1, x_2, ..., x_n) : x_i \in B, 1 \le i \le n\}$ *is called a Boolean function of degree n.* Boolean functions can be represented by expressions comprised of variables and Boolean operations.

*e.g.*, 0, 1, $x_1, x_2, ..., x_n$ are Boolean expressions in the variables $x_i$ $(1 \le i \le n)$. If $p$ and $q$ are Boolean expressions then $p \wedge q$, $p \vee q$ and $p'$ are also Boolean expressions and each represents a Boolean function.

By substituting 0 and 1 for the variables in the expression, the values of this function can be found.

**(3)** *If f and g be Boolean functions of degree n, then*

(*i*) *Complement of f* is the function $f'$ where

$$f'(x_1, x_2, ..., x_n) = [f(x_1, x_2, ..., x_n)]'$$

(*ii*) *f and g are equal* if $f(x_1, x_2, ..., x_n) = g(x_1, x_2, ..., x_n)$

(*iii*) *Boolean sum* $f \vee g$ *is*

$$(f \vee g)(x_1, x_2, ..., x_n) = f(x_1, x_2, ..., x_n) \vee g(x_1, x_2, ..., x_n)$$

(*iv*) *Boolean product* $f \wedge g$ *is*

$$(f \wedge g)(x_1, x_2, ..., x_n) = f(x_1, x_2, ..., x_n) \wedge g(x_1, x_2, ..., x_n).$$

**(4) Power in a Boolean function :** $x^2 = x \vee x = x$

$$x^3 = x^2 \vee x = x \vee x = x, ..., x^n = x$$

Similarly, $2x = x$, $3x = x$ etc.

**Example 37.30.** *Find the values of the Boolean function* $f = (x \wedge y') \vee z'$.

**Solution.** $f$ being third degree Boolean function has $2^3$ *i.e.* 8 values which are shown in the following table :

| $x$ | $y$ | $z$ | $y'$ | $z'$ | $x \wedge y'$ | $(x \wedge y') \vee z'$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 | 0 | 1 | 1 |
| 0 | 0 | 1 | 1 | 0 | 0 | 0 |
| 1 | 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 0 | 0 | 1 | 0 | 1 |
| 0 | 0 | 0 | 1 | 1 | 0 | 1 |

## 37.16 DUALITY

**(1)** The dual of any Boolean function is obtained by interchanging Boolean sums and Boolean products along with the interchange of zeros and ones.

*For example* the dual of $x \vee (y \wedge 0)$ is $x \wedge (y \vee 1)$.

The dual of any theorem of a Boolean algebra is also its theorem. This implies that the dual of any theorem in Boolean algebra is always true.

**(2) Principle of duality.** *The dual of any theorem (or property) in Boolean algebra is also a theorem (or property).*

## 37.17 BOOLEAN IDENTITIES

There are many identities in Boolean algebra which are quite useful in simplifying electrical circuits. Some of the important ones are given below :

1. *Identity law* : $x \vee 0 = x$ ; $x \wedge 1 = x$
2. *Dominance laws* : $x \vee 1 = 1$ ; $x \wedge 0 = 0$
3. *Complement law* : $x \vee x' = 1$ ; $x \wedge x' = 0$
4. *Idempotent law* : $x \vee x = x$ ; $x \wedge x = x$
5. *Double complement law* : $(x')' = x$
6. *Commutative law* : $x \vee y = y \vee x$ ; $x \wedge y = y \wedge x$
7. *Associative law* : $x \vee (y \vee z) = (x \vee y) \vee z$
   $x \wedge (y \wedge z) = (x \wedge y) \wedge z$
8. *Distributive law* : $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
   $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
9. *De-Morgan's law* : $x \wedge y = x' \vee y'$ ; $(x \vee y)' = x' \wedge y'$ (*Bhopal, 2008*)
10. *Absorption law* : $x \wedge (x \vee y) = x$

**Example 37.31.** *Let B be a Boolean algebra. Show that for all $a \in B$, there exists a unique complement $a'$.* (*Andhra, 2004*)

**Solution.** Let $b$ and $c$ be two complements of $a$.

Then
$$\begin{aligned} b &= b \wedge 1 && [\because \ 0 \text{ is an additive identity}] \\ &= b \wedge (a \vee c) && [\because \ c \text{ is complement of } a] \\ &= (b \wedge a) \vee (b \wedge c) = (a \wedge b) \vee (b \wedge c) \\ &= 0 \vee (b \wedge c) && [\because \ a \wedge b = a \wedge a' = 0] \\ &= b \wedge c && \ldots(i) \end{aligned}$$

Similarly
$$\begin{aligned} c &= c \wedge 1 = c \wedge (a \vee b) && [\because \ a \vee b = a \vee a' = 1] \\ &= (c \wedge a) \vee (c \wedge b) \\ &= (a \wedge c) \vee (b \wedge c) && [\because \ a \wedge c = a \wedge a' = 0] \\ &= 0 \vee (b \wedge c) \\ &= b \wedge c && \ldots(ii) \end{aligned}$$

From (*i*) and (*ii*), we find that $b = c$.

Thus the complement of $a$ is unique.

**Example 37.32.** *In a Boolean algebra, show that*

(*i*) $x + (x \,.\, y) = x$ (*ii*) $x \,.\, (x + y) = x$. (*Bhopal, 2008*)

**Solution.** (*i*)
$$\begin{aligned} x + (x \,.\, y) &= x \wedge (x \vee y) = (x \vee 0) \wedge (x \vee y) && [\because \ x \vee 0 = x] \\ &= x \vee (0 \wedge y) && [\text{By distributive law}] \\ &= x \vee (y \wedge 0) && [\text{By commutative law}] \\ &= x \vee 0 && [\because \ y \wedge 0 = 0 \\ &= x. \end{aligned}$$

(*ii*)
$$\begin{aligned} x \,.\, (x + y) &= x \vee (x \wedge y) \\ &= (x \wedge 1) \vee (x \wedge y) && [\because \ x \wedge 1 = x \\ &= x \wedge (1 \vee y) && [\text{By distributive law}] \\ &= x \wedge (y \vee 1) && [\text{By commutative law}] \\ &= x \wedge 1 && [\because \ y \vee 1 = 1 \\ &= x. \end{aligned}$$

**Example 37.33.** *Simplify the following :*

(*i*) $(x + y) \,.\, x' \,.\, y'$ (*ii*) $x \vee y \wedge y \vee z \wedge y \vee z'$ (*iii*) $x \vee y \wedge [(x \wedge y') \vee y]'$.

**Solution.** (*i*) $(x \wedge y) \vee x' \vee y' = (x \wedge y) \vee (x' \vee y')$

$= (x \wedge y) \vee (x \wedge y)'$ [By De Morgan's law]

$= 1.$ [$\because \quad p \vee p' = 1$]

(*ii*) $x \vee y \wedge y \vee z \wedge y \vee z' = (y \vee x) \wedge (y \vee z) \wedge (y \vee z')$ [By commutative law]

$= [y \vee (x \wedge z)] \wedge (y \vee z')$ [By distributive law]

$= y \vee [x \wedge z \wedge z']$

$= y \vee [x \wedge (z \wedge z')]$ [By associative law]

$= y \vee (x \wedge 0)$ [$\because \quad z \wedge z' = 0$]

$= y \vee 0$ [$\because \quad x \wedge 0 = 0$]

$= y.$

(*iii*) $x \vee y \wedge [(x \wedge y') \vee y]' = x \vee y \wedge [y \vee (x \wedge y')]'$ [By commutative law]

$= x \vee y \wedge [(y \vee x) \wedge (y \vee y')]'$ [By distributive law]

$= (x \vee y) \wedge [(x \vee y) \wedge 1]'$ [$\because \quad y \vee y' = 1$]

$= (x \vee y) \wedge (x \vee y)' = 0.$

**Example 37.34.** *Show that*

*(i)* $x \vee y \wedge y \vee z \wedge z \vee x = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$

*(ii)* $(x \wedge y) \vee (x' \wedge z) = (x' \vee y) \wedge (x \vee z).$ *(Bhopal, 2008)*

**Solution.** (*i*) R.H.S. $= (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$

$= (x \wedge y) \vee (z \wedge y) \vee (z \wedge x)$ [By commutative law]

$= (x \wedge y) \vee (z \wedge y \vee x)$ [By distributive law]

$= (x \vee z) \wedge (y \vee z) \wedge [x \vee (y \vee x)] \wedge [y \vee (y \wedge x)]$

$= (x \vee z) \wedge (y \vee z) \wedge [(x \vee y) \wedge (y \vee x)]$ [$\because \quad x \vee x = x$ etc.]

$= (x \vee z) \wedge (y \vee z) \wedge [(x \vee y) \wedge (x \vee y)]$

$= (x \vee z) \wedge (y \vee z) \wedge (x \vee y)$ [$\because \quad p \wedge p = p$

$= (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$ (By commutative law)

$=$ L.H.S.

(*ii*) L.H.S. $= (x \wedge y) \vee (x' \wedge z)$

$= [x \vee (x' \wedge z)] \wedge [y \vee (x' \wedge z)]$ [By distributive law]

$= [(x \vee x') \vee (x \vee z)] \wedge [(y \vee x') \wedge (y \vee z)]$

$= [1 \vee (x \vee z)] \wedge [(y \vee x') \wedge (y \vee z)]$

$= (x \vee z) \wedge (y \vee x') \wedge (y \vee z) \vee 1$

$= (x \vee z) \wedge (y \vee x') \wedge [(y \vee z) \vee (x \wedge x')]$

$= (x \vee z) \wedge (y \vee x') \wedge [(y \vee z) \vee x] \vee (y \vee z) \vee x']$

$= (x \vee z) \wedge [y \vee z \vee x] \wedge (x' \vee y) \vee [x' \wedge (y \vee z)]$

$= (x \vee z) \vee (1 \wedge y) \wedge (x' \vee y) \vee (1 \wedge z)$

$= [(x \vee z) \wedge 1] \wedge [(x' \vee y) \vee 1]$

$= (x \vee z) \wedge (x' \vee y) =$ R.H.S.

**Example 37.35.** *Show that*

*(i)* $x \vee y \wedge x' \vee y' = (x' \wedge y) \vee (x \wedge y')$

*(ii)* $[x \wedge (x' \vee y)] \vee [x' \wedge (x \vee y)] = y.$

**Solution.** (*i*) $(x \vee y) \wedge (x' \vee y') = [(x \vee y) \wedge x'] \vee [(x \vee y) \wedge y']$

$= [(x \wedge x') \vee (y \wedge x')] \vee \{(x \wedge y') \vee (y \wedge y')\}$ [By distributive law]

$= \{0 \vee (x' \wedge y)\} \vee \{(x \wedge y') \vee 0\}$ [$\because \quad x \wedge x' = 0$

$= (x' \wedge y) \vee (x \wedge y').$

(*ii*) $[x \wedge (x' \vee y)] \vee [x' \wedge (x \vee y)]$

$= [(x \wedge x') \vee (x \wedge y)] \vee [(x' \wedge x) \vee (x' \wedge y)]$

$= [0 \vee (x \wedge y)] \vee [0 \vee (x' \wedge y)]$ [$\because \quad x \wedge x' = 0$

$= (x \vee x') \wedge y = y \wedge 1 = y.$

**Example 37.36.** *If $a \vee x = b \vee x$ and $a \vee x' = b \vee x'$, then prove that $a = b$.*

**Solution.** Since $a \vee x = b \vee x$ and $a \vee x' = b \vee x'$

$\therefore$ $(a \vee x) \wedge (a \vee x') = (b \vee x) \wedge (b \vee x')$

*i.e.,* $a \vee (x \wedge x') = b \vee (x \wedge x')$ [By distributive law]

or $a \vee 0 = b \vee 0$ [$\because$ $x \wedge x' = 0$]

or $a = b$.

**Example 37.37.** *In Boolean algebra $[B, +, ., /]$, show that*

$(x . y' + y . z) . (x . z + y . z') = x . z$ *(Bhopal, 2008 ; M.P.T.U., 2001)*

**Solution.** $\{(x \vee y') \wedge (y \vee z)\} \vee \{(x \vee z) \wedge (y \vee z')\}$

$$= \{(x \vee y') \wedge y\} \vee \{(x \vee y') \wedge z\} \vee \{(x \wedge z) \wedge y\} \vee \{(x \vee z) \wedge z'\}$$
$$= \{(x \wedge y) \vee (y' \wedge y)\} \vee \{(x \wedge z) \vee (y' \wedge z)\} \vee \{(x \wedge y) \vee (z \wedge y)\} \vee \{(x \wedge z') \vee (z \wedge z')\}$$
$$= \{(x \wedge y) \vee 0\} \vee \{(x \wedge z) \vee (y' \wedge z)\} \vee \{(x \wedge y) \vee (z \wedge y)\} \vee \{(x \wedge z') \vee 0\}$$
$$= \{(x \wedge y) \vee (x \wedge y)\} \vee \{(x \wedge z) \vee (x \wedge z')\} \vee \{(z \wedge y) \vee (z \wedge y')\}$$
$$= (x \wedge y) \vee (x \wedge 1) \vee (z \wedge 1) = (x \wedge y) \vee x \vee z$$
$$= (x \vee x \vee z) \wedge (y \vee x \vee z) = (x \vee z) \wedge (y \vee x \vee z)$$
$$= (x \vee z) \wedge (1 \vee y) = (x \vee z) \wedge 1$$
$$= x \vee z.$$

## PROBLEMS 37.4

1. Find the truth table for the Boolean function $f(x, y, z) = (x \wedge y) \vee (y \wedge z')$.
2. Write the dual of the Boolean expression $x + x' . y = x + y$. *(Andhra, 2004)*
3. Simplify the following :
   (*i*) $(x \wedge y \wedge z)'$ (*ii*) $(x \vee y \vee z) \wedge (x' \wedge y' \wedge z')$.
4. In a Boolean algebra $[B, \wedge, \vee, /]$, prove that
   (*i*) $(x \wedge y) \vee (x \wedge y') = x$. *(Anna, 2005)* (*ii*) $x' \wedge (x \vee y) = x' \wedge y$.
5. If $a \wedge x = b \wedge x$ and $a \wedge x' = b \wedge x'$, then show that $a = b$.
6. In a Boolean algebra $[B, \wedge, \vee, /]$, show that
   (*i*) $x \wedge (x \wedge y) = x \wedge y$ (*ii*) $x \vee (x \vee y) = x \vee y$.
7. In Boolean algebra, prove that
   (*i*) $x \wedge (x' \vee y) = x \wedge y$ (*ii*) $x' \wedge y = x' \wedge (x \vee y)$.
8. Show that $(x \wedge y') \vee (x' \wedge y) \vee (x' \wedge y') = x' \vee y'$.
9. If $B$ be a Boolean algebra and $x, y, z \in B$, prove that
   $$(x \vee y) \wedge (x \vee y') \wedge (x' \vee y) = x \wedge y$$
10. Prove that $(x \vee y) \wedge [z \vee (x' \wedge y')] = (x \vee y) \wedge z$.
11. In a Boolean algebra $B$, prove that $(a + b)' = a' . b' \ \forall \ a, b \in B$. *(Bhopal, 2009)*
12. In any Boolean algebra, show that $a = b$ if and only if $a . b' + a' . b = 0$. *(Madras, 2001)*
13. In Boolean algebra, show that
   (*i*) $(a + b) . (a' + c) = a . c + a' . b + b . c$.
   (*ii*) $(a + b') . (b + c') . (c + a') = (a' + b) . (b' + c) . (c' + a)$. *(Andhra, 2004)*
14. Give the truth table for the Boolean function
   $f : B_3 \to B$ determined by the polynomial
   $$P(x_1, x_2, x_3) = (x_1 \vee x_3)' \wedge (x_1 \wedge (x_2 \vee x'_3)).$$ *(V.T.U., 2001)*

## 37.18 MINIMAL BOOLEAN FUNCTION

**Def.** *A minimal Boolean function in n variables is the product of $x_1, x_2, ..., x_n$.* It is also called *minterm.*

If $x, y$ are two variables and $x', y'$ are their complementary variables respectively, then $x \vee y, x' \vee y, x' \wedge y, x \wedge y', x' \vee y'$ are each a minimal Boolean function.

Similarly there are $2^3$ *i.e.* 8 minimal Boolean functions in the three variables $x, y, z$ *i.e.*, $x \vee y \vee z, x' \vee y \vee z, x \vee y' \vee z, x \vee y \vee z', x \vee y' \vee z, x' \vee y \vee z', x' \vee y' \vee z, x' \vee y' \vee z'$.

In general, *there are $2^n$ minimal Boolean functions (or minterms) in n variables.*

Similarly the join of the variables $x_1, x_2, .... x_n$ is called a *maxterm* and there will be $2^n$ *maxterms*.

## 37.19 DISJUNCTIVE NORMAL FORM

**(1) Def.** *A Boolean function which can be expressed as sum of minimal Boolean functions is called a* **Disjunctive normal form** *or* **minterm normal form** or **Canonical form.**

**(2)** *If the number of distinct terms in a disjunctive normal form of Boolean function in n variables are $2^n$, then it is called a* **complete disjunctive normal form.**

**(3) Complement function of a disjunctive normal form** *function f is the sum of all those terms of a complete disjunctive normal form which are not present in the disjunctive normal form of f.* The complement of $f$ is denoted by $f'$.

*For example,* if $f = (x \vee y) \wedge (x \vee y')$

then its complete disjunctive normal form in variables $x$ and $y$ is

$$(x \vee y) \wedge (x' \vee y) \wedge (x \wedge y') \wedge (x' \vee y')$$

$\therefore$ The complement function of this disjunctive normal form is $f'(x, y) = (x' \wedge y) \wedge (x' \vee y')$.

**Example 37.38.** *Find the value of the complete disjunctive normal form in three variables x, y, z.*

**Solution.** The complete disjunctive normal form in three variables $x, y, z$ is

$$\begin{aligned} f(x, y, z) &= (x \vee y \vee z) \wedge (x \vee y \vee z') \wedge (x \vee y' \vee z) \wedge (x' \vee y \vee z) \\ &\quad \wedge (x \vee y' \vee z') \wedge (x' \vee y \vee z') \wedge (x' \vee y' \vee z) \wedge (x' \vee y' \vee z') \\ &= [(x \vee y) \vee (z \wedge z')] \wedge [(x \wedge y') \vee (z \wedge z')] \wedge [(x' \wedge y) \vee (z \wedge z')] \wedge [(x' \vee y') \vee (z \wedge z')] \\ &= [(x \vee y) \vee 0] \wedge [(x \wedge y') \vee 0] \wedge [(x' \wedge y) \vee 0] \wedge [(x' \vee y') \vee 0] \\ &= [x \vee (y \wedge y')] \wedge [x' \vee (y \wedge y')] \\ &= (x \vee 0) \wedge (x' \vee 0) = x \wedge x' = 0. \end{aligned}$$

## 37.20 CONJUNCTIVE NORMAL FORM

**Def.** *If a Boolean function $f(x_1, x_2, ..., x_n)$ is expressed in the form of factors and each factor is the sum of all the n-variables, then such a function is called a* **conjective normal form** or **maxterm normal form** or **dual canonical form.**

**(2)** *If a conjective normal of a function of n variables contains all the $2^n$ distinct factors, then such a function is called a* **complete conjective normal form.**

**(3) Complement function of a conjective normal form** *function f is a Boolean function which is the product of all those terms of complete conjective normal form which are not present in conjective normal form of f.*

The complement of conjective normal form $f$ is denoted by $f'$.

*For example,* if $f(x, y) = (x \wedge y) \vee (x \wedge y')$

then its complete conjunctive normal form in $x$ and $y$ is

$$(x \wedge y) \vee (x \wedge y') \vee (x' \wedge y) \vee (x' \wedge y')$$

$\therefore$ The complement function of this conjunctive normal form is

$$f'(x, y) = (x' \wedge y) \vee (x' \wedge y')$$

**Example 37.39.** *Given Boolean expression f, where $f(x_1, x_2, x_3) = (x_3' \wedge x_2) \vee (x_1' \wedge x_3) \vee (x_2 \wedge x_3)$, simplify this expression stating the laws used and obtain the minterm normal form.* (*Bharathiar, 1997*)

**Solution.** Given

$$\begin{aligned} f(x_1, x_2, x_3) &= (x_3' \wedge x_2) \vee (x_1' \wedge x_3) \vee (x_2 \wedge x_3) \\ &= (x_3' \wedge x_2) \vee (x_2 \wedge x_3) \vee (x_1' \wedge x_3) && \text{[By commutative law]} \\ &= (x_2 \wedge x_3') \vee (x_2 \wedge x_3) \vee (x_1' \wedge x_3) \\ &= [x_2 \wedge (x_3' \vee x_3)] \vee (x_1' \wedge x_3) && \text{[By distributive law]} \end{aligned}$$

$= (x_2 \wedge 1) \vee (x_1' \wedge x_3)$ $[\because \ p \vee p' = 1]$

$= x_2 \vee x_1' \wedge x_3$ [By identity law]

Minterm normal form of $f(x_1, x_2, x_3)$

$$= [(x_3' \wedge x_2) \wedge (x_1 \vee x_1')] \vee [(x_1' \wedge x_3) \wedge (x_2 \vee x_2')] \vee [(x_2 \wedge x_3) \wedge (x_1 \vee x_1')]$$

$$= (x_3' \wedge x_2 \wedge x_1) \vee (x_3' \wedge x_2 \wedge x_1') \vee (x_1' \wedge x_3 \wedge x_2) \vee (x_1' \wedge x_3 \wedge x_2') \vee (x_2 \wedge x_3 \wedge x_1) \vee (x_2 \wedge x_3 \wedge x_1')$$

$$= (x_3' \wedge x_2 \wedge x_1) \vee (x_3' \wedge x_2 \wedge x_1') \vee (x_1' \wedge x_3 \wedge x_2) \vee (x_1' \wedge x_3 \wedge x_2') \vee (x_2 \wedge x_3 \wedge x_1).$$

**Example 37.40.** *Express the following functions into conjunctive normal forms :*

*(i)* $x' \wedge y$ *(ii)* $(x \wedge y) \vee (x' \wedge y')$.

**Solution.** *(i)* $x' \wedge y = x' \wedge y \wedge (z \vee z') = (x' \wedge y \wedge z) \vee (x' \wedge y \wedge z')$.

*(ii)* $(x \wedge y) \vee (x' \wedge y') = [x \wedge y \wedge (z \vee z')] \vee [x' \wedge y' \wedge (z \vee z')]$

$$= (x \wedge y \wedge z) \vee (x \wedge y \wedge z') \vee (x' \wedge y' \wedge z) \vee (x' \wedge y' \wedge z').$$

**Example 37.41.** *The function* $f = (x \wedge y \wedge z) \vee (x \wedge y' \wedge z) \vee (x \wedge y \wedge z') \vee (x' \wedge y' \wedge z)$ *is in conjunctive normal form. Write its complement ?*

**Solution.** The complete conjunctive normal form in three variables $x, y, z$ is $(x \wedge y \wedge z) \vee (x' \wedge y \wedge z) \vee (x \wedge y' \wedge z) \vee (x \vee y \vee z') \vee (x \wedge y' \wedge z') \vee (x' \wedge y \wedge z') \vee (x' \wedge y' \wedge z) \vee (x' \wedge y' \wedge z')$.

$\therefore$ The complement of the given function $F$ is

$$F' = (x' \wedge y \wedge z) \vee (x \wedge y' \wedge z') \vee (x' \wedge y \wedge z') \vee (x' \wedge y' \wedge z').$$

## PROBLEMS 37.5

1. Find the value of a complete disjunctive normal form in
   *(i)* two variables $x, y$. *(ii)* three variables $x, y, z$.
2. Express the following functions into disjunctive normal form :
   *(i)* $x \vee y$ *(ii)* $x \wedge (x' \vee y)$.
3. Express the Boolean function $F = A \vee (B' \wedge C)$ in a sum of minterms.
4. Convert the function $x \wedge y'$ to disjunctive normal form in three variables $x, y, z$.
5. Express the function $f = (x \vee y') \wedge (x \vee z) \wedge (x \vee y)$ into conjunctive normal form in which maximum number of variables are used.
6. Write the complement of the conjunctive normal form function $(x \wedge y \wedge z') \vee (x \wedge y' \wedge z') \vee (x \wedge y \wedge z')$.

## 37.21 SWITCHING CIRCUITS

**(1)** *A switching network is an arrangement of wires and switches* (or gates) *which connect two terminals.* A switch can be either closed or open. A closed switch permits and an open switch stops flow of current

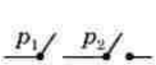
**(2)** If $p$ denotes a switch, then $p'$ denotes that switch which is open when $p$ is closed and $p'$ is closed when $p$ is open.

If $x$ denotes the state of the switch $p$, then $x'$ represents the state of the switch $p'$. $x$ is called the Boolean variable which is a binary variable.

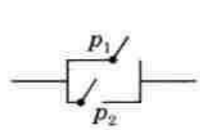
If $x = 1$ denotes the switch is closed or current flows, then $x = 0$ denotes that the switch is open or current stops.

**(3)** *Two switches* $p_1$ *and* $p_2$ *are either connected in series* (represented by $\wedge$) *or connected in parallel* (represented by $\vee$).

These are shown as follows :

*(i)* 


($p_1$ & $p_2$ in series : $p_1 \wedge p_2$)

*(ii)* 

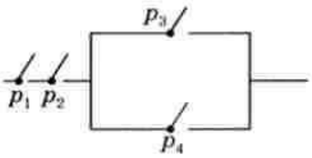

($p_1$ & $p_2$ in parallel : $p_1 \vee p_2$)

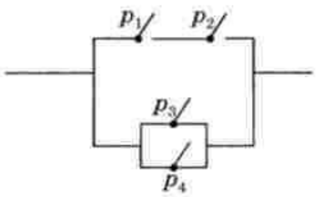
If $B$ [0, 1] is non-empty set and $\wedge$, $\vee$, $/$ are the operations on $B$, then the system $[\{0, 1\}, \wedge, \vee, 1]$ is usually called *Boolean switching algebra.*

**(4) Simplification of circuits.** The simplification of a circuit means the least complicated circuit with minimum cost and best results. This depends on the cost of the equipment, number of switches and the type of the material used. Thus the simplification of circuits implies the use of lesser number of switches which can be achieved by using different properties of Boolean algebra. In other words, *the simplification of switching circuits is equivalent to simplification of the corresponding Boolean function.*

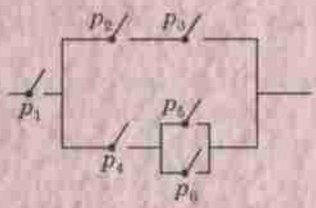
**Example 37.42.** *Draw the circuit which represents the Boolean function :*
*(i)* $(p_1 \wedge p_2) \wedge (p_3 \vee p_4)$ *(ii)* $(p_1 \wedge p_2) \vee (p_3 \vee p_4)$.

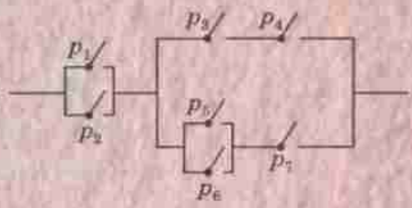
**Solution.** Here $p_1 \wedge p_2$ is a series circuit while $p_3 \vee p_4$ is a parallel circuit.
The required circuits are as follows :

(*i*)



(*ii*)



**Example 37.43.** *Write the Boolean functions representing the following circuits :*

(*i*)



(*ii*)



*Also draw the circuit diagram which would be the complement of the circuit in (ii).*

**Solution.** (*i*) The given circuit is represented by the Boolean function :

$$f = p_1 \wedge (p_2 \wedge p_3) \vee [p_4 \wedge (p_5 \vee p_6)]$$

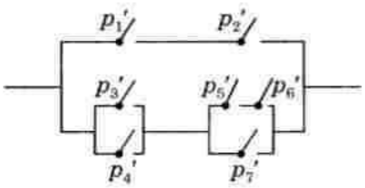
(*ii*) The Boolean function for the given circuit is

$$f = (p_1 \vee p_2) \wedge [(p_3 \wedge p_4) \vee [(p_5 \vee p_6) \wedge p_7]$$

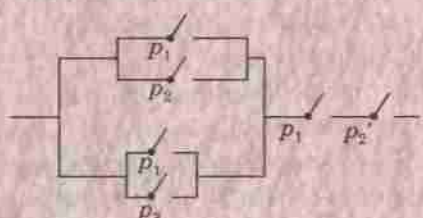
The complement of $f$ *i.e.,*

$$f' = (p_1 \vee p_2)' \vee [(p_3 \wedge p_4) \vee ((p_5 \vee p_6) \wedge p_7)]'$$
$$= (p_1' \wedge p_2') \vee [(p_3 \wedge p_4)' \wedge ((p_5 \vee p_6) \wedge p_7)']$$
$$= (p_1' \wedge p_2') \vee [(p_3' \vee p_4') \wedge ((p_5' \wedge p_6') \vee p_7')]$$

Its circuit diagram is as follows :

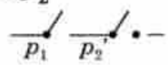



**Example 37.44.** *Simplify the following circuit and draw the diagram of the resulting circuit :*



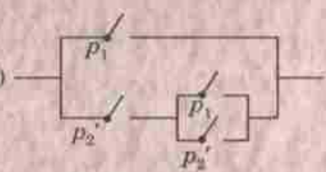
**Solution.** The given circuit is represented by the Boolean function $f$

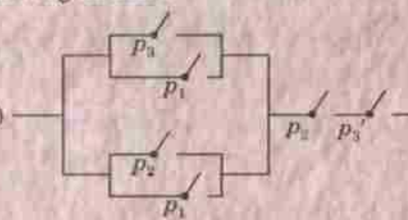
$$= [(p_1 \vee p_2) \vee (p_1 \vee p_3)] \wedge (p_1 \wedge p_2') = (p_1 \vee p_2 \vee p_3) \wedge (p_1 \wedge p_2')$$
$$= (p_1 \wedge p_1 \wedge p_2') \vee (p_2 \wedge p_1 \wedge p_2') \vee (p_3 \wedge p_1 \wedge p_2') \qquad \text{(By distributive law)}$$
$$= (p_1 \wedge p_2') \vee (p_3 \wedge p_1 \wedge p_2') = p_1 \wedge [p_2' \vee (p_3 \wedge p_2')]$$
$$= p_1 \wedge p_2' \qquad \text{[By absorption law]}$$

Its circuit diagram is 

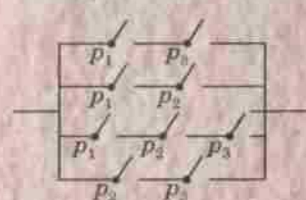

**PROBLEMS 37.6**

1. Draw the circuit diagram represented by the Boolean functions :
   (*i*) $[p_1 \wedge (p_1 \vee p_2)] \vee [p_2 \wedge (p_1' \vee p_2)]$ (*ii*) $p_1 \wedge [(p_2 \vee p_4') \vee (p_3' \wedge (p_1 \vee p_4 \vee p_3'))] \wedge p_2$.
2. Write the Boolean functions representing the following circuits :

(*i*) 


(*ii*) 


3. Simplify the Boolean functions, $p \vee (p' \wedge q) \vee (p \wedge q)$
4. Simplify the following circuit and draw the diagram of the resulting circuit :




5. Draw the simplified network of $f(x, y, z) = (x \vee y \vee z) \wedge (x \vee y' \vee z) \wedge (x' \vee y' \vee z)$. (*M.P.T.U., 2001*)
6. Consider the function $f(x_1, x_2, x_3) = [(x_1 \wedge x_2) \wedge (x_1 \wedge x_3)] \vee (x_1 \vee x_2')$
   (*a*) Simplify $f$ algebraically. (*b*) Draw the switching circuit of $f$.
   (*c*) Also find the minterm normal form of $f$. (*Madras, 1998*)

# IV. FUZZY SETS

## 37.22 FUZZY LOGIC

We have so far dealt with the fundamentals of classical logic. Besides this, we have *crisp logic* which deals with propositions that are required to be either true or false. There is however another type of logic which includes not only the crisp values but all the values between true (1) and false (0). But there is some degree of vagueness about the exact value between [0, 1]. *The logic to infer a definite outcome from such vague inputs is called* **fuzzy logic**.

**(2) Fuzzy set.** To provide a mathematical modelling to fuzzy logic, L.A. Zudeh introduced the concept of 'Fuzzy sets' in 1965 on the basis of a *membership function*. The theory of 'fuzzy sets' is now fully developed.

**Def.** *A fuzzy set $F$ of a non-zero set $X(x)$ is defined as $F = \{x, \mu_F(x)\} : x \in X$.*

Here $\mu_F : X \to [0, 1]$ is a function called the *membership function* of $F$ and $\mu_F(x)$ is the degree of membership of $x \in X$ in $F$.

In particular $\quad \mu(x) = 1$ implies full membership

$\mu(x) = 0$ implies non-membership

and $0 < \mu(x) < 1$ means intermediate membership.

A fuzzy set $F$ is, therefore, a set of pairs consisting of a particular element of the universe $X$ and its degree of membership *i.e.*, each $x$ is assigned a value in the range (0, 1) indicating the extent to which $x$ has the attribute $F$. It can also be represented as $F = \{[x_1, \mu_F(x_1)], [x_2, \mu_F(x_2)], ..., [x_n, \mu_F(x_n)]\}$.

*For example,* if $x$ is the number of cars in a lane, 'small' may be taken as a particular value of the fuzzy variable $x$ and to each $x$ is assigned a number in the range (0, 1) then $\mu_{\text{small}}(x) \in (0, 1)$ is the *membership function.*

**Example 37.45.** *In a car-race, all the cars complete the race in four time-groups : shortest time, moderate time, long time and longest time.,If we note the time taken by each car in a group, it will give rise to a distribution of times. Now let us find the outcome of the race based on engine power, car speed and road conditions. Each of these variables may further be divided into :*

(i) *low, medium and high for the variable engine power,*

(ii) *slow, moderate and fast for the variable car speed,*

(iii) *rough, bumpy and smooth for the variable road conditions.*

Now we try to predict on some basis, in which of the four groups the car will finish, if it has low engine power, moderate speed and rough road.

Then the distribution for the engine power would correspond to the membership function for low, medium and high. Similarly the distribution for the speed would depend on the membership function for slow, moderate and fast, while the distribution for road conditions would depend on membership function for rough, bumpy and smooth.

## 37.23 FUZZY SET OPERATIONS

**(1)** A fuzzy set is said to be *normalised* when the largest element of the set (called *supremum*) is unity.

*For instance,* the set of members [5, 10 , 15, 20, 25] is normalised to [0.2, 0.4, 0.6, 0.8, 1] by dividing each member by 25, the supremum in the set.

*The normalization of a fuzzy set F is expressed as* $\underset{x \in X}{Sup}\, F(x) = 1.$

**(2) Complement.** *The complement of a fuzzy set F is the set* $F^c$ *with degree of membership of an element in* $F^c$ *equal to one minus degree of membership of this element in F. (Fig. 37.5)*

*For example,* if $F$ = [0.4 Ram, 0.6 Sham, 0.8 Jyoti, 0.9 Ritu]
be a set of intelligent students, then $F^c$ = [0.6 Ram, 0.4 Sham, 0.2 Jyoti, 0.1 Ritu]
is a set of non-intelligent students.


Fig. 37.5

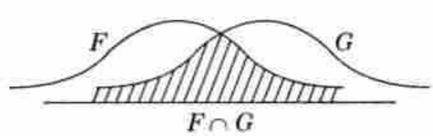



Fig. 37.6

**(3) Intersection.** *The intersection of two fuzzy sets F and G is the set* $F \cap G$*, where the degree of membership of an element in* $F \cap G$ *is the minimum of the degrees of membership of this element in F and G. (Fig. 37.6)*

**(4) Union.** *The union of two fuzzy sets F and G is the set* $F \cup G$*, where the degree of membership of an element in* $F \cup G$*, is the maximum of this element in F and G. (Fig. 37.7)*

*For example,* if

$F$ = [0.5 Rani, 0.2 Suman, 0.4 Anita, 0.8 Sunita]

be a set of fat girls, and

$G$ = [0.1 Rani, 0.6 Suman, 0.9 Anita, 0.5 Sunita]

be a set of tall girls, then

$F \cap G$ = [0.1 Rani, 0.2 Suman, 0.4 Anita, 0.5 Sunita]

and $F \cup G$ = [0.5 Rani, 0.6 Suman, 0.9 Anita, 0.8 Sunita]

**(5) Equality.** Two fuzzy sets $F$ and $G$ are said to be equal if and only if $F(x) = G(x)$ for all $x$ in $X$.

**(6) Subset.** The fuzzy set $F$ is said to be a subset of the fuzzy set $G$ (*i.e.*, $F \subseteq G$) if and only if $F(x) \le G(x)$ for all $x \in X$.

**(7) Double negation.** If $F$ is a fuzzy set, then $(F^c)^c = 1$.

**(8) De Morgan's laws.** If $F$ and $G$ are two fuzzy sets then

$$(F \cup G)^c = F^c \cap G^c \;;\; (F \cap G)^c = F^c \cup G^c.$$

**Example 37.46.** *Let the membership functions for the fuzzy sets F and G be as in the following table :*

| $X$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $F$ | 0 | 0 | 0.1 | 0.5 | 0.8 | 1 | 0.3 | 0.5 | 0 | 0 |
| $G$ | 0 | 0 | 0 | 0 | 0.1 | 0.3 | 0.5 | 0.8 | 1 | 1 |
| $F^c$ | 1 | 1 | 0.9 | 0.5 | 0.2 | 0 | 0.7 | 0.5 | 1 | 1 |

Then the corresponding

$$F \cap G = [0, 0.1, 0.3, 0.5]$$
$$F \cup G = [0.1, 0.5, 0.8, 1, 0.5, 0.8, 1, 1]$$

Clearly $F$ is not a subset of $G$ and $G$ is not a subset of $F$.

## 37.24 TRUTH VALUE

**(1) Truth value** *of the negation of a proposition in fuzzy logic is one minus the truth value of the proposition.*

*For example,* if the truth value of the statement 'Ram is happy' is 0.8, then the truth value of 'Ram is not happy' is 0.2.

**(2) Truth value** *of the conjunction of two propositions in the fuzzy logic is the minimum of the truth values of the two propositions.*

**(3) Truth value of the disconjunction of two propositions** *in the fuzzy logic is the maximum of the truth values of two propositions.*

*For example,* if the truth value of 'Ram is fat' is 0.6, and the truth value of 'John is fat' is 0.9, then the truth value of the statement

(*a*) '*Ram and John are fat*' is 0.6.

(*b*) '*Ram or John is fat*' is 0.9.

(*c*) '*neither Ram nor John is fat*' is negation of minimum of negation of 'Ram is fat' (*i.e.*, 0.4) and negation of 'John is, fat' (*i.e.*, 0.1) = 0.1.

(*d*) '*Ram is not fat or John is not fat*' is maximum of 0.4 and 0.1 *i.e.*, 0.4.

## 37.25 ALGEBRAIC OPERATIONS ON FUZZY SETS

**(1) Algebraic sum** *of two fuzzy sets F and G is defined by the membership function*

$$\mu_{F+G}(x) = \mu_F(x) + \mu_G(x) - \mu_F(x)\, \mu_G(x)$$

and is written as $F + G$.

**Algebraic product** *of two fuzzy sets F and G is defined by the membership function.*

$$\mu_{F.G}(x) = \mu_F(x).\mu_G(x)$$

and is written as $F.G$.

**(2) Properties of fuzzy set operations** which are common to crisp set operations, are as under

1. *Idempotent* : $F \cup F = F, \quad F \cap F = F$
2. *Identity* : $F \cup \phi = F, F \cap U = F$
3. *Commutative* : $F \cup G = G \cup F, \quad F \cap G = G \cap F$
4. *Distributive* : $F \cap (G \cap H) = (F \cap G) \cup (F \cap H), \quad F \cup (G \cap H) = (F \cup G) \cap (F \cup H)$
5. *Associative* : $(F \cup G) \cup H = F \cup (G \cup H), \quad (F \cap G) \cap H = F \cap (G \cap H)$
6. *Absorption* : $F \cup (F \cap G) = F, \quad F \cap (F \cup G) = F.$

## 37.26 GENERATION OF RULES FOR FUZZY PROBLEMS

We should know before hand all possible input-output relations while dealing with problems concerning fuzzy engines or fuzzy controls. These input-output rules are then expressed with 'if ... then' statements.

*For instance,* if $F$'s and $G$'s are inputs of fuzzy problems and $R$'s are the actions taken for each rule, then the set of 'if ... then' rules with two input variables $F_1$ and $G_1$ and the actions taken are shown in table 1.

*i.e.,* if $F_1$ and or $G_1$, then $R_{11}$, else
if $F_2$ and or $G_1$, then $R_{21}$, else
if $F_1$ and or $G_2$, then $R_{12}$, else
if $F_2$ and or $G_2$, then $R_{22}$.

Table 1

| | | |
|---|---|---|
| $F_1$ | $R_{11}$ | $R_{12}$ |
| $F_2$ | $R_{21}$ | $R_{22}$ |
| | $G_1$ | $G_2$ |

In case, the fuzzy statements have more variables, then 'if ... then' rules becomes more complicated to tabulate.

However such a tabulation can be simplified by following a *decomposition process* as follows :

The decomposition process of three fuzzy variables $F$, $G$ and $H$ with actions $R$'s taken is shown below :

Table 2

| | | |
|---|---|---|
| $F_1$ | $R_{11}$ | $R_{12}$ |
| $F_2$ | $R_{21}$ | $R_{22}$ |
| | $G_1$ | $G_2$ |

Table 3

| | | | | |
|---|---|---|---|---|
| $H_1$ | $R_{111}$ | $R_{121}$ | $R_{211}$ | $R_{221}$ |
| $H_2$ | $R_{112}$ | $R_{122}$ | $R_{212}$ | $R_{222}$ |
| | $R_{11}$ | $R_{12}$ | $R_{21}$ | $R_{22}$ |

Here the statements $F_2$ and $G_1$ and $H_1$ then $R_{211}$ is decomposed into 'if $F_2$ and $G_1$ then $R_{21}$' and 'if $R_{21}$ and $H_1$, then $R_{211}$.

Similarly $R_{21}$ and $H_2$ then $R_{212}$.

This decomposition process can easily be extended to any number of input variables.

## 37.27 FUZZY PROPOSITIONS

**(1)** *A fuzzy number is a fuzzy set $R \to [0, 1]$.* We can easily extend classical two-valued logic to three-valued logic. Fuzzy logic, however is an extension of multi-valued logic. It provides foundations for approximate reasoning with imprecise fuzzy propositions using fuzzy set theory.

The classical propositions are statements which are either true or false. In fuzzy logic, the truth or falsity of fuzzy propositions is assigned different degrees *i.e.*, the truth and falsity are expressed by numbers in [0, 1].

A variable whose values are 'words' or 'sentences' is called a *linguistic variable*. For example 'height' is a linguistic variable and its values are tall, very tall, quite tall, not tall, short, not very short, not quite tall etc.

**(2) Classification of fuzzy propositions.** The classification propositions are statements which are either true or false. In fuzzy logic, the truth or falsity of fuzzy propositions is assigned different degrees *i.e.*, the truth and falsity are expressed by numbers in [0, 1].

The fuzzy propositions of simple nature can be classified into the following four types. In each case, we introduce the relevant canonical form and then discuss its interpretation.

**Type I.** *Unconditional and unqualified propositions.*

The standard canonical form of this type of proposition is expressed as $p : u$ if $F$ ...(1)

Here $u$ is the variable that takes value $u$ from some universal set $U$ and $F$ is a fuzzy set on $U$ which represents a fuzzy predicate such as young, tall, low, high etc. Given a particular value $u$ (say $v$), this value belongs to $F$ with membership grade $F(v)$. This membership grade is then interpreted as the degree of truth $T(p)$ of proposition $p$

*i.e.,* $$T(p) = F(v) \quad ...(2)$$

Here $T$ is a fuzzy set on [0, 1] which assigns the membership grade $F(v)$ to each value $v$ of $u$.

In some fuzzy propositions, values of variable $u$ in (1) are assigned to individuals in a given set $I$ *i.e.*, variable $u$ becomes a function $u : I \to u$ where $u(i)$ is the value of $v$ for individual $i$ in $U$. Accordingly the canonical form (2) is modified to the form

$$p : u\,(i) \text{ is } F \text{ where } i \in I \quad ...(3)$$

**Example 37.47.** *Consider a set I of persons, each person is characterized by his 'age' and a fuzzy set expressing the predicate 'young' is given. Denoting our variable by 'age' and fuzzy set by 'young', the canonical form is*

*p : age (i) is young.*

**Solution.** The degree of truth of this proposition $T(p)$ is then determined for each person $i$ in $I$ by means of the equation $T(p)$ = young [age ($i$)].

**Example 37.48.** *At a particular place on the earth, consider the air temperature u (in °C). Let Fig. 37.8 represent the membership function as predicate 'high'. Assuming that all relevant temperature readings are given, the corresponding fuzzy proposition is expressed as,*

$$p : \text{temp } (u) \text{ is high } (°C)$$

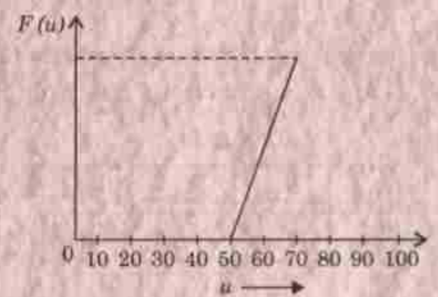



Fig. 37.8

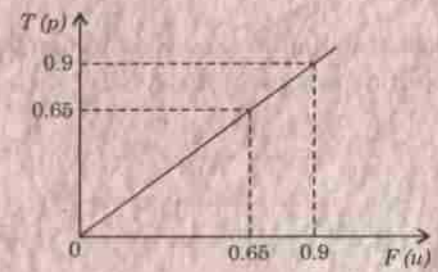



Fig. 37.9

**Solution.** The degree of truth $T(p)$ depends upon the actual value of the temperature and on the nature of predicate 'high' which is defined by the membership function $T$ in Fig. 37.9.

*e.g.*, if $u = 75$ then $F(75) = 0.65$ and $T(p) = 0.65$.

**Type II.** *Conditional and unqualified propositions.*

A proposition $p$ of this type is expressed by the canonical form

$$p : \text{If } x \text{ is } F \text{ then } y \text{ is } G \qquad ...(4)$$

where $x, y$ are variables whose values are in the sets $X$, $Y$ and $F$, $G$ are fuzzy sets on $X$ and $Y$ respectively. These propositions may also be viewed as propositions of the form

$$\{x, y\} \text{ is } R \qquad ...(5)$$

where $R$ is a fuzzy set on $X \times Y$ which is determined for each $x \in X$ and each $y \in Y$ by the formula

$$R(x, y) = B[F(x), G(y)], \qquad ...(6)$$

where $B$ is a binary operation as [0, 1] representing a suitable *fuzzy implication.*

**Type III.** *Unconditional and qualified propositions*

A proposition of this type is expressed by either of the following canonical forms :

$$p : u \text{ is } F \text{ is } S \qquad ...(7)$$

or

$$p : \text{Prob } (U \text{ is } F) \text{ is } P \qquad ...(8)$$

where $u$ is a variable that takes value $v$ from some universal set $U$ and $F$ is a fuzzy set on $U$ which represents a fuzzy predicate such as small, young, daughter etc.

Prob. ($U$ is $F$) is the probability of fuzzy set event '$u$ is $F$' ; $S$ is the fuzzy truth qualifier and $P$ is the fuzzy probability qualifier. $S$ and $P$ are both represented by fuzzy set on [0, 1].

**Example 37.49.** *Mary is 'young' is 'very true' where the predicate 'young' and the truth qualifier 'very true' are represented by the respective fuzzy sets shown in Fig. 37.10.*

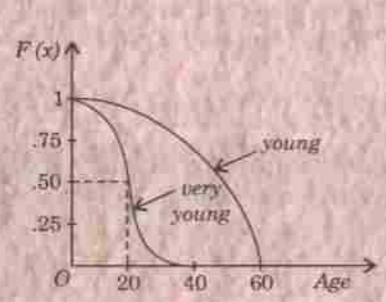



Fig. 37.10

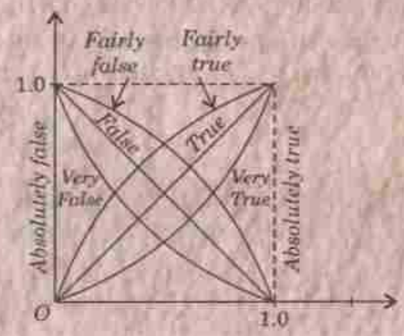



Fig. 37.11

**Solution.** The degree of truth $T(p)$ of any truth-qualified proposition $p$ is given for each $u \in U$ by $T(p) = S\{F(u)\}$.

Assuming that Mary's age is 20, she belongs to the set representing the predicate 'very young' with membership grade 0.50, our proposition belongs to the set of propositions which are 'very true' with membership grade 0.50 as shown in Fig. 37.11. This implies that the degree of truth of our truth qualified proposition is 0.50.

If the proposition be modified by changing the predicate to 'young' or the truth qualifier to 'fairly true', we would obtain the corresponding degree of truth of such propositions by the same method.

**Type IV.** *Conditional and qualified propositions*

This type of propositions can either be expressed in the canonical form

$$p : \text{If } x \text{ is } F, \text{ then } y \text{ is } G \text{ in } S$$

or

$$p : \text{Prob } \{x \text{ is } F/y \text{ is } G\} \text{ in } P,$$

where Prob $\{x$ is $F/y$ is $G\}$ is conditional probability.

## 37.28 APPLICATIONS OF FUZZY SETS

The concept of fuzzy sets has already influenced all engineering disciplines to various degrees.

*Electrical engineering* is the first such discipline where the utility of fuzzy logic and fuzzy sets has been recognized by developing controllers. Electronic circuits for fuzzy image processing have also been developed.

Some ideas regarding the application of fuzzy sets *in civil engineering* emerged around 1970. In the construction of bridges, dams, buildings etc. a designer has to take into account the safety factor for which the fuzzy theory has an effective role to play. Fuzzy set theory has also proved quite useful for assessing the life of existing constructions.

In *mechanical engineering* design problems, the utility of Fuzzy set theory was realised during mid 1980's. The membership function is expressed in terms of thermal expansion or corrosion or cost of different materials etc.

When the utility of fuzzy controllers was increasingly felt around mid 1980's, the need for computer hardware to implement the various operations involving fuzzy logic, had been recognized. In digital mode, fuzzy sets have been expressed as vectors of (0, 1) members.

Fuzzy control and fuzzy decision making are two well-developed areas of fuzzy set theory. These are directly relevant to *industrial engineering problems.* The utility of fuzzy sets has also been recognized for estimating the service life of given equipment under various conditions.

*Modern Reliability theory* has also been developed on the assumption of fuzzy sets. At any given time, an engineering product may be in functioning state to some degree or in failed state to another degree. The behaviour of an engineering product with respect to its functioning state and failed state has been characterized as based on fuzzy set theory.

The use of fuzzy set theory in *Robotics* includes approximate reasoning, fuzzy controllers, fuzzy pattern recognition and fuzzy data bases.

### PROBLEMS 37.7

1. Given fuzzy sets $F_1 = \{0.6 \text{ Sonu}, 0.9 \text{ Renu}, 0.7 \text{ Paul}, 0.3 \text{ Sham}\}$
$F_2 = \{0.3 \text{ Sham}, 0.8 \text{ Paul}, 0.9 \text{ Renu}, 0.5 \text{ Sonu}\}$
and $F_3 = \{0.8 \text{ Paul}, 0.3 \text{ Sham}, 0.5 \text{ Sonu}, 0.9 \text{ Renu}\}$
Which of the above two sets are equal ?
2. Write the complement set of the fuzzy set $F$, if $F = \{0.8 \text{ Ram}, 0.3 \text{ Sham}, 0.6 \text{ John}, 0.7 \text{ Charu}\}$.
3. If $F = \{0.3x_1, 0.7x_2, 0.5x_3, 0.8x_4\}$ and $G = \{0.4x_1, 0.6x_2, 0.1x_3, 0.9x_4\}$ been two fuzzy sets, then write down $F \cup G$ and $F \cap G$.
4. State the truth values of the negation of the following propositions :
(*i*) Truth value of '$F$ is rich' is 0.8
(*ii*) Truth value of '$G$ is fat' is 0.6
(*iii*) Truth value of 'Mary is beautiful' is 0.7.
5. Let the membership functions of fuzzy sets $F$ and $G$ be as follows :
$X : \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$F : [0, 0, 0, 0, 0.1, 0.3, 0.5, 0.9, 1, 1]$

$G : [0, 0, 0.1, 0.5, 0.9, 1, 0.9, 0.5, 0.0]$

State whether (*i*) $F = G$ (*ii*) $F$ is a subset of $G$.

Also write down $F'$, $F \cap G$ and $F \cup G$.

6. The truth values of the statements

   'Latif is a good player' is 0.7

   and 'John is a good player' is 0.6.

   What is the truth value of

   (*i*) the conjunction of the above two prepositions.

   (*ii*) the disjunction of the above prepositions.

7. Define a Fuzzy set and the standard operations on Fuzzy sets. (*Bhopal, 2009*)
8. State the constituents of the pair in a fuzzy set.
9. Write a note on 'Fuzzy logic affects many disciplines' ? (*Bhopal, 2001*)

CHAPTER 38

# Tensor Analysis



## 38.1 INTRODUCTION

Some physical quantities are specified by their magnitude only while others by their magnitude and direction. But certain quantities are associated with two or more directions. Such a quantity is called a *tensor*. The stress at a point of an elastic solid is an example of a tensor which depends on two directions—one normal to the area and other that of the force on it.

The properties of tensors are independent of the frames of reference used to describe them. That is why *Einstein* found tensors as a convenient tool for formulation of his Relativity theory. Since then, the subject of tensor analysis shot into prominence and is of great use in the study of Riemannian geometry, mechanics, elasticity, electro-magnet theory and numerous other fields of science and engineering. The emergence of tensor calculus as a symmetric subject is due to *Ricci* and his student *Levi-Cita*.

## 38.2 SUMMATION CONVENTION

Consider a sum of the type

$$a_1x_1 + a_2x_2 + \dots + a_nx_n \quad i.e., \quad \sum_{i=1}^{n} a_ix_i \qquad \dots(1)$$

In tensor analysis, the subscripts of the symbols $x_1, x_2, \dots x_n$ are replaced by superscripts and we write these as $x^1, x^2, \dots, x^n$. The superscripts do not stand for the various powers of $x$ but act as labels to distinguish different symbols. The power of a symbol (say : $x^i$) will be indicated as $(x^i)^2$, $(x^i)^3$ etc. Hence (1) is written as

$$\sum_{i=1}^{n} a_ix^i \ . \qquad \dots(2)$$

A still simpler notation is to drop the Σ sign and write (2) as $a_ix^i$ ...(3)

In this the repeated index $i$ successively takes up the values 1, 2, ..., $n$ and the expression (3) represents the sum of all such terms. The repeated index $i$ over which the summation is to be done, is called a *dummy* index since it doesn't appear in the final result. This notation, known as *summation convention*, is due to Einstein. We shall adopt this convention throughout this chapter and take the sum whenever a letter appears in a term once as a subscript and once as superscript.

**Example 38.1.** *Write the terms contained in $S = a_{ij}x^i x^j$ taking $n = 3$.*

**Solution.** Since the index $i$ occurs both as a subscript and as a superscript, we first sum on $i$ from 1 to 3.

$\therefore$ $$S = a_{1j}x^1x^j + a_{2j}x^2x^j + a_{3j}x^3x^j$$

Now each term in $S$ has to be summed up w.r.t. repeated index $j$ from 1 to 3.

$\therefore$ $$S = a_{11}x^1x^1 + a_{12}x^1x^2 + a_{13}x^1x^3 + a_{21}x^2x^1 + a_{22}x^2x^2 + a_{23}x^2x^3 + a_{31}x^3x^1 + a_{32}x^3x^2 + a_{33}x^3x^3$$
$$= a_{11}(x^1)^2 + a_{22}(x^2)^2 + a_{33}(x^3)^2 + (a_{12} + a_{21})x^1x^2 + (a_{13} + a_{31})x^1x^3 + (a_{23} + a_{32})x^2x^3.$$

**Example 38.2.** *If $f$ is a function of $n$ variables $x^i$, write the differential of $f$.*

**Solution.** Since $f = f(x^1, x^2, ..., x^n)$

$\therefore$ From Calculus, we have

$$df = \frac{\partial f}{\partial x^1}dx^1 + \frac{\partial f}{\partial x^2}dx^2 + ... + \frac{\partial f}{\partial x^n}dx^n = \frac{\partial f}{\partial x^i}dx^i.$$

## 38.3 (1) TRANSFORMATION OF COORDINATES

In a 3-dimensional space, the coordinates of a point are $(x^1, x^2, x^3)$ referred to a particular frame of reference. Similarly in an $n$-dimensional space, the coordinates of a point are $n$ independent variables $(x^1, x^2, ..., x^n)$ with respect to a certain frame of reference. Let $(\bar{x}^1, \bar{x}^2, ..., \bar{x}^n)$ be the coordinates of the same point referred to another frame of reference. Suppose, $\bar{x}^1, \bar{x}^2, ..., \bar{x}^n$ are independent single-valued functions of $x^1, x^2, ..., x^n$ so that

$$\bar{x}^1 = \phi^1(x^1, x^2, ..., x^n)$$
$$\bar{x}^2 = \phi^2(x^1, x^2, ..., x^n)$$
$$\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots$$
$$\bar{x}^n = \phi^n(x^1, x^2, ..., x^n)$$

or more briefly $$\bar{x}^i = \phi^i(x^1, x^2, ..., x^n) \quad ...(1)$$

We can solve the equations (1) and express $x^i$ as functions of $\bar{x}^i$ so that

$$x^i = \psi^i(\bar{x}^1, \bar{x}^2, ..., \bar{x}^n) \quad ...(2)$$

*The equations (1) and (2) are said to define a transformation of the coordinates from one frame of reference to another.*

**(2) Scalars or invariants.** *A function $\phi(x^1, x^2, x^3)$ is called a scalar or an invariant if its original value does not change upon transformation of coordinates from $x^1, x^2, x^3$ to $\bar{x}^1, \bar{x}^2, \bar{x}^3$.*

*i.e.,* $$\phi(x^1, x^2, x^3) = \psi(\bar{x}^1, \bar{x}^2, \bar{x}^3)$$

A scalar or invariant is also called a *tensor of order (or rank) zero.*

## 38.4 KRONECKER DELTA*

The quantity $\delta_i^j$ defined by the relations

$$\delta_i^j = 0, \quad \text{when } j \neq i$$

and $$\delta_i^j = 1, \quad \text{when } j = i,$$ is called *Kronecker delta.*

Evidently $$\delta_1{}^1 = \delta_2{}^2 = \delta_3{}^3 = ... = \delta_n{}^n = 1$$

while $$\delta_1{}^2 = \delta_2{}^3 = \delta_3{}^2 = ... = 0$$

*Called after the German mathematician *Leopold Kronecker* (1823–91) who made important contributions to number theory, algebra and group theory.

We note that by summing up w.r.t. the repeated index $j$,

$$\delta_{3j}\delta_2^j = a_{31}\delta_2^{\ 1} + a_{32}\delta_2^{\ 2} + a_{33}\delta_2^{\ 3} + a_{34}\delta_2^{\ 4} + \ldots$$
$$= 0 + a_{32} + 0 + 0 = a_{32}$$

In general,
$$a_{ij}\delta_k^j = a_{i1}\delta_k^{\ 1} + a_{i2}\delta_k^{\ 2} + \ldots + a_{ik}\delta_k^{\ k} + \ldots + a_{in}\delta_k^{\ n}$$
$$= 0 + 0 + \ldots + a_{ik} \,.\, 1 + \ldots + 0 = a_{ik}.$$

**Example 38.3.** *Show that $a_{ij}A^{kj} = \Delta\delta_i^k$, where $\Delta$ is a determinant of order three and $A^{ij}$ are cofactors of $a^{ij}$.* *(Delhi, 2002)*

**Solution.** By expansion of determinants, we have

$$a_{11}A^{11} + a_{12}A^{12} + a_{13}A^{13} = \Delta$$
$$a_{11}A^{21} + a_{12}A^{22} + a_{13}A^{23} = 0$$
$$a_{11}A^{31} + a_{12}A^{32} + a_{13}A^{33} = 0$$

which can be compactly written as

$$a_{1j}A^{1j} = \Delta,\ a_{1j}A^{2j} = 0,\ a_{1j}A^{3j} = 0$$

Using Kronecker delta notation, these can be combined into a single equation

$$a_{1j}A^{kj} = \Delta\delta_1^{\ k}$$

Similarly
$$a_{2j}A^{kj} = \Delta\delta_2^{\ k},\ a_{3j}A^{kj} = \Delta\delta_3^{\ k}$$

All these nine equations are included in $a_{ij}A^{kj} = \Delta\delta_i^{\ k}$.

**Example 38.4.** *If $x^i$ and $\bar{x}^i$ are independent coordinates of a point, show that*

$$\frac{\partial x^j}{\partial \bar{x}^k}\frac{\partial \bar{x}^k}{\partial x^i} = \delta_i^j.$$

**Solution.** The partial derivatives of $\phi$ in the two coordinate systems are different and are connected by the following formula of Differential Calculus :

$$\frac{\partial \phi}{\partial x^i} = \frac{\partial \phi}{\partial \bar{x}^1}\cdot\frac{\partial \bar{x}^1}{\partial x^i} + \frac{\partial \phi}{\partial \bar{x}^2}\cdot\frac{\partial \bar{x}^2}{\partial x^i} + \frac{\partial \phi}{\partial \bar{x}^3}\frac{\partial \bar{x}^3}{\partial x^i} = \frac{\partial \phi}{\partial \bar{x}^k}\frac{\partial \bar{x}^k}{\partial x^i}$$

In particular, when $\phi = x^j$, we have $\dfrac{\partial x^j}{\partial x^i} = \dfrac{\partial x^j}{\partial \bar{x}^k}\dfrac{\partial \bar{x}^k}{\partial x^i}$ ...(*i*)

Since $x^j$ is independent of $x^i$, $\partial x^j/\partial x^i = 0$, when $j \neq i$
$= 1$, when $j = i$ ...(*ii*)

Hence the result follows from (*i*) and (*ii*).

## 38.5 (1) CONTRAVARIANT VECTORS

Let $A^1, A^2, \ldots, A^n$ (*i.e.*, $A^i$) *be a set of n functions of the coordinate system* $x^1, x^2, \ldots, x^n$ (*i.e.*, $x^i$). *If these transform in another system of coordinates* $\bar{x}^1, \bar{x}^2, \ldots\ldots \bar{x}^n$ (*i.e.*, $\bar{x}^i$) *according to the law*

$$\bar{A}^i = \frac{\partial \bar{x}^i}{\partial x^j}A^j \quad \ldots(1)$$

*then $A^i$ are called components of a* **contravariant vector** *or* **contravariant tensor of order one.**

*An example of a contravariant vector.* Let us transform the coordinates of a point $x^i$ to $\bar{x}^i$ in a $n$-dimensional space.

Since $x$ is a function of $x^i$ (*i.e.*, $x_1, x_2, \ldots, x_n$), therefore,

$$d\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^1}dx^1 + \frac{\partial \bar{x}^i}{\partial x^2}dx^2 + \ldots + \frac{\partial \bar{x}^i}{\partial x^n}dx^n$$

$$= \frac{\partial \bar{x}^i}{\partial x^j}dx^j, \text{ using the summation convention.}$$

Comparing this with (1), it follows that *the set of differentials* $dx^1, dx^2, ..., dx^n$ *is an example of a contravariant vector.* That is why the coordinates of a point are numbered by superscripts and not by subscripts.

**(2) Covariant vectors.** *Let* $A_i$ *be a set of n functions of the coordinate system* $x^i$. *If these transform in another system of coordinates* $\bar{x}^i$ *according to the law*

$$\bar{A}_i = \frac{\partial x^j}{\partial \bar{x}^i} A_j \quad ...(2)$$

*then* $A_i$ *are called the components of a* **covariant vector** *or* **covariant tensor of order one.**

*An example of a covariant vector.* Let $\phi$ be a function which has a fixed value at each point of space independent of the coordinate system employed. Therefore, $\phi$ is a function of the coordinates $x^i$ in the first system and a function of the coordinates $\bar{x}^i$ in the second system. By the chain rule

$$\frac{\partial \phi}{\partial \bar{x}^i} = \frac{\partial \phi}{\partial x^1}\frac{\partial x^1}{\partial x^i} + \frac{\partial \phi}{\partial x^2}\frac{\partial x^2}{\partial x^i} + ... + \frac{\partial \phi}{\partial x^n}\frac{\partial x^n}{\partial x^i} = \frac{\partial \phi}{\partial x^1}\frac{\partial x^1}{\partial \bar{x}^i} + \frac{\partial \phi}{\partial x^2}\frac{\partial x^2}{\partial \bar{x}^i} + ... + \frac{\partial \phi}{\partial x^n}\frac{\partial x^n}{\partial \bar{x}^i}$$

Comparing this equation with (2), it follows that *the set of derivatives,*

$$\partial\phi/\partial x^1, \partial\phi/\partial x^2, ..., \partial\phi/\partial x^n$$

*form a covariant vector.*

**Example 38.5.** *A covariant tensor has components xy,* $2y - z^2$, *xz in rectangular coordinates. Find its covariant components in spherical coordinates.*

**Solution.** Here $x^1 = x, x^2 = y, x^3 = z$

and $\bar{x}^1 = r, \bar{x}^2 = \theta, \bar{x}^3 = \phi$ ...(i)

Let $A_1 = xy, A_2 = 2y - z^2, A_3 = xz$ ...(ii)

According to the law of transformation, we have $\bar{A}_i = \frac{\partial x^j}{\partial \bar{x}^i} A_j \quad (i = 1, 2, 3)$

and we wish to evaluate $\bar{A}_1, \bar{A}_2, \bar{A}_3$ where $A_1, A_2, A_3$ are known.

We know that $x = r \sin\theta \cos\phi, y = r \sin\theta \sin\phi, z = r = \cos\theta$ ...(iii)

Now $\bar{A}_1 = \frac{\partial x^1}{\partial \bar{x}^1} A_1 + \frac{\partial x^2}{\partial \bar{x}^1} A_2 + \frac{\partial x^3}{\partial \bar{x}^1} A_3 = \frac{\partial x}{\partial r} xy + \frac{\partial y}{\partial r}(2y - z^2) + \frac{\partial z}{\partial r} xz$ [From (i) and (ii)]

$= \sin\theta \cos\phi \,.\, r \sin\theta \cos\phi \,.\, r \sin\theta \sin\phi + \sin\theta \sin\phi\,(2r \sin\theta \sin\phi - r^2 \cos^2\theta)$

$+ \cos\theta \,.\, r \sin\theta \cos\phi \,.\, r\cos\theta$ [From (iii)]

Similarly $\bar{A}_2 = \frac{\partial x^1}{\partial \bar{x}^2} A_1 + \frac{\partial x^2}{\partial \bar{x}^2} A_2 + \frac{\partial x^3}{\partial \bar{x}^2} A_3 = r\cos\theta\cos\phi \,.\, r \sin\theta \cos\phi \,.\, r \sin\theta \sin\phi$

$+ r\cos\theta \sin\phi\,(2r \sin\theta \sin\phi - r^2 \cos^2\theta) + (-r\sin\theta)\, r \sin\theta \cos\phi \,.\, r\cos\theta$

and $\bar{A}_3 = \frac{\partial x^1}{\partial \bar{x}^3} A_1 + \frac{\partial x^2}{\partial \bar{x}^3} A_2 + \frac{\partial x^3}{\partial \bar{x}^3} A_3$

$= - r\sin\theta\sin\phi \,.\, r \sin\theta \cos\theta \,.\, r\sin\theta\sin\phi$

$+ r \sin\theta \cos\phi\,(2r\sin\theta\sin\phi - r^2\cos^2\theta) + 0$

## 38.6 TENSORS OF HIGHER ORDER

Let $i$ and $j$ be each given values 1 to $n$, then the symbol $A^{ij}$ will give rise to $n^2$ functions.

**(1)** *If* $A^{ij}$ *be a set of* $n^2$ *functions of the coordinates* $x^1, x^2, ..., x^n$ *which transforms to* $\bar{A}^{ij}$ *in another system of coordinates* $\bar{x}^1, \bar{x}^2, ..., \bar{x}^n$, *according to the law*

$$\bar{A}^{ij} = \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial \bar{x}^i}{\partial x^l} A^{kl} \quad ...(1)$$

*then the functions* $\bar{A}^{ij}$ *are said to be components of a* **contravariant tensor of the second order.**

**(2)** *If $A_{ij}$ be a set of $n^2$ functions of $x^1, x^2, ..., x^n$ which transform to $\bar{A}_{ij}$ in another system of coordinates, $\bar{x}^1, \bar{x}^2, ..., \bar{x}^n$ according to the law*

$$\bar{A}_{ij} = \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} A_{kl}, \qquad ...(2)$$

*then the functions $\bar{A}_{ij}$ are said to be the components of a* **covariant tensor of the second order.**

**(3)** *If $A^i_{\ j}$ be a set of $n^2$ functions of $x^1, x^2, ..., x^n$ which transform to $\bar{A}^i_j$ in another system of coordinates, $\bar{x}^1, \bar{x}^2, ..., \bar{x}^n$, according to the law*

$$\bar{A}^i_j = \frac{\partial \bar{x}^i}{\partial x^k} \cdot \frac{\partial x^l}{\partial \bar{x}^j} A^k_{\ l}, \qquad ...(3)$$

*then $\bar{A}^i_j$ are said to be the components of a* **mixed tensor of the second order.** It transforms like a contravariant vector with respect to the index $i$ and like a covariant vector with regard to the index $j$. That is why $i$ is placed as a superscript and $j$ as subscript.

We can similarly define tensors of the orders higher than two.

**Obs.** Each of the above laws of transformation (1) to (3), give rise to $n^2$ equations as $i$ and $j$ are each given the value 1 to $n$.

**Example 38.6.** *Show that the Kronecker delta is a mixed tensor of order two.*

**Solution.** If $\delta_i^j$ transforms to $\bar{\delta}_i^j$ in the coordinate system $\bar{x}^1, \bar{x}^2, ..., \bar{x}^n$ by the law for mixed tensors of order two, then

$$\bar{\delta}_i^j = \frac{\partial x^l}{\partial \bar{x}^i} \frac{\partial \bar{x}^j}{\partial x^m} \delta_l^m = \frac{\partial \bar{x}^j}{\partial x^m} \cdot \frac{\partial x^m}{\partial \bar{x}^i} = \frac{\partial \bar{x}^j}{\partial \bar{x}^i} = \delta_i^{\ j} \qquad [\because \ \delta_l^m = 0 \text{ for } l \neq m]$$

Hence $\delta_i^j$ is a mixed tensor of order two, having the same components in every coordinate system.

**Example 38.7.** *Show that the velocity of a fluid at any point is a contravariant tensor of rank one.*

**Solution.** Let $dx^1/dt, dx^2/dt, dx^3/dt$ be the components of fluid velocity of the point $(x^1, x^2, x^3)$, *i.e.*, $dx^i/dt$ be the components of velocity in the coordinate system $x^i$. Suppose the corresponding components of velocity in the coordinate system $\bar{x}^j$ are $d\bar{x}^j/dt$. Then $\bar{x}^1, \bar{x}^2, \bar{x}^3$ being the functions of $x^1, x^2, x^3$ which in turn are functions of $t$, we can write

$$\frac{d\bar{x}^j}{dt} = \frac{\partial \bar{x}^j}{\partial x^1} \frac{dx^1}{dt} + \frac{\partial \bar{x}^j}{\partial x^2} \frac{dx^2}{dt} + \frac{\partial \bar{x}^j}{\partial x^3} \frac{dx^3}{dt}$$

or

$$\frac{d\bar{x}^j}{dt} = \frac{\partial \bar{x}^j}{\partial x^i} \frac{dx^i}{dt} \qquad ...(i)$$

Now according to the law of tensor transformation, (*i*) shows that the velocity of a fluid is a contravariant tensor of rank one.

**Example 38.8.** *Prove that there is no distribution between contravariant and covariant vectors if the transformation law is of the form $\bar{x}^i = a^i_m x^m + b^i$, where a's and b's are constants such that $a^i_{\ r} a^i_{\ m} = \delta^r_m$.* (Bhopal, 2003)

**Solution.** Given transformation $\quad \bar{x}^i = a^i_m x^m + b^i \qquad ...(i)$

yields

$$\frac{\partial \bar{x}^i}{\partial x^m} = a^i_{\ m} \qquad ...(ii)$$

Also from (*i*),

$$a^i_{\ r} \bar{x}^i = a^i_{\ r} a^i_m x^m + a^i_{\ r} b^i$$
$$= \delta^r_m x^m + a^i_r b^i = x^r + a^i_{\ r} b^i.$$

$\therefore$

$$\frac{\partial x^r}{\partial \bar{x}^i} = a^i_r \ \text{ i.e., } \ \frac{\partial x^m}{\partial \bar{x}^i} = a^i_m \qquad ...(iii)$$

From (*ii*) and (*iii*), it is clear that any vector with components $a, b, c$ will on transformation give the same components whether transformed as a contravariant vector or as a covariant vector. Thus in this case, there is no distinction between the two.

## 38.7 SYMMETRIC AND SKEW-SYMMETRIC TENSORS

(1) *A tensor is said to be* **symmetric** *with respect to two contravariant (or two covariant) indices if its components remain unchanged on an interchange of the two indices.*

Thus the tensor $A^{ij}$ is symmetric if $A^{ij} = A^{ji}$, for every $i$ and $j$.

(2) *A tensor is said to be* **skew-symmetric** *with respect to two contravariant (or covariant) indices, if its components change sign on interchange of the two indices.*

Thus the tensor $A^{ij}$ is skew-symmetric if $A^{ji} = -A^{ij}$ for every $i$ and $j$.

In general, the tensor $A^{ijk}_{lm}$ is said to be symmetric or skew symmetric in $i$ and $j$ according as

$$A^{ijk}_{lm} = A^{jik}_{lm} \quad \text{or} \quad -A^{jik}_{lm}$$

**Example 38.9.** *Show that (i) a symmetric tensor of the second order has only $\frac{1}{2}n(n+1)$ different components.*

*(ii) A skew-symmetric tensor of the second order has only $\frac{1}{2}n(n-1)$ different non-zero components.*

**Solution.** (*i*) Let $A^{ij}$ be a symmetric tensor of order two so that $A^{ij} = A^{ji}$.

If each of the indices $i$ and $j$ take the values 1 to $n$, then $A^{ij}$ will have $n^2$ components. Out of these $n^2$ components, $n$ components $A_{11}, A_{22}, \ldots A_{nn}$ are independent.

Thus the remaining components are $(n^2 - n)$ which can be taken in pairs ($\because \; A_{12} = A_{21}, A_{31} = A_{13}$ etc.)

Hence the total number of independent components

$$= n + \frac{1}{2}(n^2 - n) = \frac{1}{2}n(n+1)$$

(*ii*) Let $A^{ij}$ be a skew-symmetric tensor of order two so that $A^{ij} = -A^{ji}$. As above, $A^{ij}$ will have $n^2$ components. Out of these, $n$ components $A^{11}, A^{22}, \ldots A^{nn}$ are all zero. [$\because \; A^{11} = -A^{11}$].

Omitting these, there are $(n^2 - n)$ components. Since $A^{12} = -A^{21}, A^{13} = -A^{31}$ etc., therefore ignoring the sign, $(n^2 - n)$ components can be taken in pairs.

Hence the total number of independent non-zero components

$$= \frac{1}{2}(n^2 - n) = \frac{1}{2}n(n-1).$$

### PROBLEMS 38.1

1. Write the following using the summation convention :

   (*i*) $\dfrac{d\phi}{dt} = \dfrac{\partial\phi}{\partial x^1}\dfrac{dx^1}{dt} + \dfrac{\partial\phi}{\partial x^2}\dfrac{dx^2}{dt} + \ldots + \dfrac{\partial\phi}{\partial x^n}\dfrac{dx^n}{dt}$ (*ii*) $(x^1)^2 + (x^2)^2 + (x^3)^2 + \ldots + (x^n)^2$.

2. Write out in full the following :

   (*i*) $a_{ij}x^i x^j \; (i, j = 1, 2, 3)$ (*ii*) $g_{ij}dx^i dx^j (i, j = 1, 2, 3)$

   (*iii*) $g_{lm}g^{mp}$.

3. (*a*) Shows that $\delta^i_j$ is an invariant. (*Bhopal, 2003*)

   (*b*) Evaluate (*i*) $\delta^i_j \, \delta^j_k$ (*ii*) $\delta^p_q \, \delta^q_r \, \delta^r_s$.

4. Show that (*i*) $\dfrac{\partial x^p}{\partial x^q} = \delta^p_q$ (*ii*) $\dfrac{\partial x^p}{\partial \bar{x}^q}\dfrac{\partial \bar{x}^p}{\partial x^r} = \delta^p_r$.

5. If the $\bar{x}$'s are $n$ independent functions of $x$'s and $i, j, k, l$ each take values from 1 to $n$, show that

$$\frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial x^l}{\partial \bar{x}^j} \cdot \delta^k_l = \delta^i_j.$$

6. Write down the law of transformation for the tensors

(i) $A_l^{jk}$ (ii) $C_{mn}$.

7. A quantity $A(i, j, k, l, m)$ which is a function of the coordinates $x^p$ transforms to another coordinate system $\bar{x}^p$ according to the law :

$$\bar{A}(r, s, t, u, v) = \frac{\partial \bar{x}^r}{\partial x^i}\frac{\partial \bar{x}^s}{\partial x^j}\frac{\partial x^k}{\partial \bar{x}^t}\frac{\partial x^l}{\partial \bar{x}^u}\frac{\partial \bar{x}^v}{\partial x^m} A(i, j, k, l, m)$$

Is this quantity a tensor ? If so express it suitably and state its nature and rank ?

8. If the components of two tensors are equal in one coordinate system, show that they are equal in all coordinate systems.

9. A covariant tensor has components $2x - z$, $x^2y$, $yz$ in cartesian coordinate system. Find its components in
(a) cylindrical coordinates (*Punjab, M.E., 1989*) (b) spherical coordinates.

10. If $g_{(i)}$ denotes the components of a covariant tensor of rank two, show that the product $g_{ij}\, dx^i\, dx^j$ is an invariant scalar. (*Delhi, 2002*)

11. A contravariant tensor has components $a$, $b$, $c$ in rectangular coordinates ; find the components in spherical coordinates.

12. Prove that $A_{ij}B^iC^j$ is an invariant, if $B^i$ and $C^j$ are contravariant vectors and $A_{ij}$ is a covariant tensor. (*Madras, M.E., 2000*)

13. Show that $\partial A_p/\partial x^q$ is not a tensor even though $A_p$ is a covariant tensor of rank one. (*Bhopal, 2003*)

14. If a tensor $A^{pqr}{}_{st}$ is a skew-symmetric with respect to the indices $p$ and $q$ in one coordinate system, show that it remains skew-symmetric with respect to $p$ and $q$ in any coordinate system.

## 38.8 ADDITION OF TENSORS

*The sum (or difference) of two tensors of the same order and type is another tensor of the same order and type.*

Let $A_{ij}$ and $B_{ij}$ be two tensors of the same order and same type. Their components in the coordinates system $\bar{x}^1, \bar{x}^2, ..., \bar{x}^n$ are $A_{ij}$ and $B_{ij}$, such that

$$\bar{A}_{ij} = \frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial x^l}{\partial \bar{x}^j} A_{kl} \qquad \text{and} \qquad \bar{B}_{ij} = \frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial x^l}{\partial \bar{x}^j} B_{kl}$$

$$\therefore \qquad \bar{A}_{ij} \pm \bar{B}_{ij} = \frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial x^l}{\partial \bar{x}^j}(A_{kl} \pm B_{kl}) \quad \text{i.e.,} \quad \bar{C}_{ij} = \frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial x^l}{\partial \bar{x}^j} C_{kl}$$

Thus $C_{ij}$ transforms in exactly the same manner as $A_{ij}$ and $B_{ij}$ and is, therefore, a tensor of the same order and same type.

## 38.9 OUTER PRODUCT OF TWO TENSORS

If $A^{ij}$ is a contravariant tensor of order two and $B_{kl}$ is a covariant tensor of order two then their product is a mixed tensor $C^{ij}{}_{kl}$ of order four such that

$$C^{ij}_{kl} = \bar{A}^{ij}\,\bar{B}_{kl} = \left(\frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial \bar{x}^j}{\partial x^q} A^{pq}\right)\left(\frac{\partial x^r}{\partial \bar{x}^k}\frac{\partial x^s}{\partial \bar{x}^l} B_{rs}\right) = \frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial \bar{x}^j}{\partial x^q}\frac{\partial x^r}{\partial \bar{x}^k}\frac{\partial x^s}{\partial \bar{x}^l} A^{pq} B_{rs} = \frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial \bar{x}^j}{\partial x^q}\frac{\partial x^r}{\partial \bar{x}^k}\frac{\partial x^s}{\partial \bar{x}^l} C^{pq}_{rs}$$

But this is the law of transformation of a mixed tensor of order four. Therefore, $C^{ij}{}_{kl}$ is a mixed tensor of order four. Such products are called *outer products of two tensors.*

## 38.10 CONTRACTION OF A TENSOR

Consider a mixed tensor $A_l^{ijk}$ of order four. By the law of transformation, we have

$$\bar{A}_l^{ijk} = \frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial \bar{x}^j}{\partial x^q}\frac{\partial \bar{x}^k}{\partial x^r}\frac{\partial x^s}{\partial \bar{x}^l} A_s^{pqr}$$

In this, put the covariant index $l = a$ contravariant index $i$, so that

$$\bar{A}_i^{ijk} = \frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial \bar{x}^j}{\partial x^q}\frac{\partial \bar{x}^k}{\partial x^r}\frac{\partial x^s}{\partial \bar{x}^i}A_s^{pqr} = \frac{\partial \bar{x}^j}{\partial x^q}\frac{\partial \bar{x}^k}{\partial x^r}\frac{\partial x^s}{\partial x^p}A_s^{pqr}$$

$$= \frac{\partial \bar{x}^j}{\partial x^q}\frac{\partial \bar{x}^k}{\partial x^r}\delta_p^s A_s^{pqr} = \frac{\partial \bar{x}^j}{\partial x^q}\frac{\partial \bar{x}^k}{\partial x^r}A_p^{pqr}$$

This shows that $A_i^{ijk}$ is a contravariant tensor of order two.

*The process of getting a tensor of lower order (reduced by 2) by putting a covariant index equal to a contravariant index and performing the summation indicated is known as* **contraction.**

The tensors $A_i^{ijk}$ and $A_j^{ijk}$ obtained from contraction of the same tensor $A_l^{ijk}$ are generally different from each other unless the tensor $A_l^{ijk}$ is symmetric with respect to $i$ and $j$ (*i.e.*, $A_j^{ijk} = A_i^{ijk}$).

## 38.11 INNER PRODUCT OF TWO TENSORS

Given the tensors $A_k^{ij}$ and $B_{qr}^p$, if we first form their outer product $A_k^{ij}\, B_{qr}^p$ and contract this by putting $p = k$, then the result is $A^{ij}{}_k\, B_{qr}^k$ which is also a tensor, called the *inner product of the given tensors.*

*Hence the inner product of two tensors is obtained by first taking their outer product and then by contracting it.* We can get several inner products for the same two tensors by contracting in different ways.

**Example 38.10.** *Show that any inner product of the tensors $A_r^p$ and $B_t^{qs}$ is a tensor of rank three.*

**Solution.** The transformation laws for $A_r^p$ and $B_t^{qs}$ are

$$\bar{A}_r^p = \frac{\partial \bar{x}^p}{\partial x^i}\frac{\partial x^k}{\partial \bar{x}^r}A_k^i \quad ...(i) \qquad \text{and} \qquad \bar{B}^{qs}{}_t = \frac{\partial \bar{x}^q}{\partial x^j}\frac{\partial \bar{x}^s}{\partial x^l}\frac{\partial x^m}{\partial \bar{x}^t}B^{jl}{}_m \quad ...(ii)$$

$\therefore$ Inner product of $\bar{A}_q^p$ and $\bar{B}_t^{qs}$ is

$$\bar{A}_q^p\, \bar{B}_t^{qs} = \left(\frac{\partial \bar{x}^p}{\partial x^i}\frac{\partial x^k}{\partial \bar{x}^q}\right)\left(\frac{\partial \bar{x}^q}{\partial x^j}\frac{\partial \bar{x}^s}{\partial x^l}\frac{\partial x^m}{\partial \bar{x}^t}\right)A_k^i\, B_m^{jl}$$

$$= \frac{\partial \bar{x}^p}{\partial x^i}\frac{\partial \bar{x}^s}{\partial x^l}\frac{\partial x^m}{\partial \bar{x}^t}\delta_j^k\, A_k^l\, B_m^{jl} \qquad \left[\because \frac{\partial x^k}{\partial \bar{x}^q}\frac{\partial \bar{x}^q}{\partial x^j} = \frac{\partial x^k}{\partial x^j} = \delta_j^k\right.$$

$$= \frac{\partial \bar{x}^p}{\partial x^i}\frac{\partial \bar{x}^s}{\partial x^l}\frac{\partial x^m}{\partial \bar{x}^t}A_j^i\, B_m^{jl}$$

Hence the inner product of $\bar{A}_q^p$ and $\bar{B}_t^{qs}$ is a tensor of rank 3.

Similarly putting $p = t$ in the product of (*i*) and (*ii*) and noting that

$$\frac{\partial \bar{x}^p}{\partial x^i}\frac{\partial x^m}{\partial \bar{x}^p} = \frac{\partial x^m}{\partial x^i} = \delta^m{}_i,$$

$A_r^p B_p^{qs}$ is found to be a tensor of rank 3.

Similarly, $A_r^p B_t^{qr}$ can also be shown to be a tensor of rank 3.

## 38.12 QUOTIENT LAW

To ascertain that a set of given functions forms the components of a tensor, we have to verify if the functions obey the tensor transformation laws. But this is a very tedious job. A simple test is provided by the quotient law which states that *if the inner product of a set of functions with an arbitrary tensor is a tensor, then these set of functions are the components of a tensor.*

The proof of this law is given below for a particular case.

**Example 38.11.** *Show that the expression $A(i, j, k)$ is a tensor if its inner product with an arbitrary tensor $B_k^{jl}$ is a tensor.*

**Solution.** Let $$A(i, j, k)\, B_k^{jl} = C_i^l \qquad ...(i)$$

where $C_i^l$ is a tensor. In the coordinate system $\bar{x}^i$, let (*i*) transform to

$$\bar{A}(p, q, r)\bar{B}_r^{qs} = \bar{C}_p^s \qquad ...(ii)$$

where $\bar{B}_r^{qs}$ and $\bar{C}_p^s$ are the components of the tensors $B_k^{\ jl}$ and $C_i^{\ l}$. Expressing $B_r^{\ qs}$ in terms of $\bar{B}_k^{\ jl}$ and $\bar{C}_p^{\ s}$ in terms of $\bar{C}_i^{\ l}$, (*ii*) takes the form

$$\bar{A}(p, q, r)\frac{\partial \bar{x}^q}{\partial x^i}\frac{\partial \bar{x}^s}{\partial \bar{x}^l}\frac{\partial x^k}{\partial \bar{x}^r}B_k^{\ jl} = \frac{\partial \bar{x}^s}{\partial x^l}\frac{\partial x^l}{\partial \bar{x}^p}C_i^{\ l} \qquad ...(iii)$$

Multiplying (*i*) by $\dfrac{\partial \bar{x}^s}{\partial x^l}\dfrac{\partial x^i}{\partial \bar{x}^p}$ and subtracting from (*iii*), we get

$$\left\{\bar{A}(p, q, r)\frac{\partial \bar{x}^q}{\partial x^i}\frac{\partial \bar{x}^s}{\partial x^l}\frac{\partial x^k}{\partial \bar{x}^r} - A(i, j, k)\frac{\partial \bar{x}^s}{\partial x^l}\frac{\partial x^i}{\partial \bar{x}^p}\right\} B_k^{jl} = 0$$

Now $B_k^{jl}$ being an arbitrary tensor, the quantity within the brackets must be identically zero, *i.e.*,

$$\bar{A}(p, q, r)\frac{\partial \bar{x}^q}{\partial x^j}\frac{\partial \bar{x}^s}{\partial x^l}\frac{\partial x^k}{\partial \bar{x}^r} = A(i, j, k)\frac{\partial \bar{x}^s}{\partial x^l}\frac{\partial x^j}{\partial \bar{x}^p}$$

or $$\bar{A}(p, q, r) = \frac{\partial x^i}{\partial \bar{x}^p}\frac{\partial x^j}{\partial \bar{x}^q}\frac{\partial \bar{x}^r}{\partial x^k} = A(i, j, k)$$

But this is the law of tensor transformation. Hence $A(i, j, k)$ is a tensor of order three, with $i, j$ as covariant indices and $k$ as contravariant index.

## PROBLEMS 38.2

1. Prove that if a tensor equation is true for one coordinate system, it is true for all coordinate systems.
2. Show that every tensor can be expressed as the sum of two tensors, one of which is symmetric and the other skew-symmetric.
3. If $A^{pq}{}_r$ and $B^{pq}{}_r$ are tensors, prove that their sum and differences are also tensors.
4. Show that $A_{ij}$ is a tensor if its inner product with an arbitrary mixed tensor $B^j{}_k$ is a tensor.
5. Prove that (*a*) the contraction of the tensor $A^p{}_q$ is an invariant.
   (*b*) the contraction of the outer product of the tensors $A^p$ and $B_q$ is also an invariant.
6. Let $A^{pq}{}_{rst}$ be a tensor ; choose $p = t$ and $q = s$ and show that $A^{pq}{}_{rqp}$ is also a tensor. What is its rank ?

## 38.13 (1) RIEMANNIAN SPACE

The distance $ds$ between two adjacent points whose rectangular Cartesian coordinates are $(x, y, z)$ and $(x + dx, y + dy, z + dz)$ is given by $ds^2 = dx^2 + dy^2 + dz^2$.

Riemann extended the concept of distance to a space of $n$ dimensions and defined the distance $ds$ between two adjacent points $x^i$ and $x^i + dx^i$ $(i = 1, 2, ..., n)$ by the relation

$$ds^2 = a_{11}(dx^1)^2 + a_{22}(dx^2)^2 + ... + a_{nn}(dx^n)^2 + a_{12}dx^1dx^2 + ... + a_{lm}dx^ldx^m + ...$$
$$= a_{ij}dx^idx^j, \text{ using summation convention.} \qquad ...(1)$$

The coefficients $a_{ij}$ are the functions of the coordinates $x^i$. The quadratic form (1) is called a Riemannian metric and any space in which the distance is given by a such a metric is called a *Riemannian space.**

If in a particular coordinate system $X^i$, the quadratic form (1) reduces to the form

$$ds^2 = (dX^1)^2 + (dX^2)^2 + ... (dX^n)^2,$$

then it is called a Euclidean metric and the corresponding space is called the *Euclidean space.*

* See footnote on p. 673

**Obs.** The geometry based on the Riemannian metric is called the *Riemannian geometry* and that based on the Euclidean metric is called the *Euclidean geometry.*

**(2) Metric tensor.** As in the physical space, the distance $ds$ in the $n$-dimensional space is assumed to be independent of the coordinate system, *i.e.* a scalar invariant or a tensor of order zero. In the relation (1), $dx^i$ and $dx^j$ being displacements are components of a contravariant vector or a tensor of order one. Therefore, their outer product $dx^i dx^j$ is a contravariant tensor of order two. By the quotient law, the functions $a_{ij}$ must be components of a covariant tensor of order two.

Let us write $\quad a_{ij} = g_{ij} + h_{ij} \quad$ where $g_{ij} = \frac{1}{2}(a_{ij} + a_{ji})$ and $h_{ij} = \frac{1}{2}(a_{ij} - a_{ji})$.

Interchanging $i$ and $j$, we have $g_{ji} = \frac{1}{2}(a_{ji} + a_{ij}) = g_{ij}$ and $h_{ji} = \frac{1}{2}(a_{ji} - a_{ij}) = -h_{ij}$

$\therefore$ $g_{ij}$ is symmetric and $h_{ij}$ is skew-symmetric. Thus (1) take the form

$$ds^2 = a_{ij} dx^i dx^j = (g_{ij} + h_{ij})\, dx^i dx^j$$

Now $h_{ij} dx^i dx^j$ is zero, since $h_{ij}$ is skew-symmetric. Hence $ds^2 = g_{ij} dx^i dx^j$ where $g_{ij}$ is a covariant symmetric tensor of order two. It is called the *metric tensor* or the *first fundamental tensor.*

## 38.14 CONJUGATE TENSOR

Let $g$ be the determinant $|g_{ij}|$ and $G_{ij}$ denote the cofactors of $g_{ij}$ in $g$. Define the function of $g^{ij}$ by the relation $g^{ij} = G_{ij}/g$ ...(1)

Since the functions $g_{ij}$ and $G_{ij}$ are symmetric in the subscripts, the functions $g^{ij}$ will be symmetric in the superscripts. Now

$$g_{ij} g^{ij} = g_{ij} \frac{G_{ij}}{g} = \frac{g}{g} = 1 \quad \text{and} \quad g_{ij} g^{lj} = g_{ij} \frac{G_{lj}}{g}$$

$$= 1, \text{ if } l = i, \ = 0, \text{ if } l \neq i$$

Thus
$$g_{ij} g^{lj} = \delta_i^l \qquad \text{...(2)}$$

If $u^j$ be an arbitrary contravariant tensor, then its inner product with the tensor $g_{ij}$ will be an arbitrary covariant tensor due to contraction, *i.e.*,

$$g_{ij} u^j = v_i \qquad \text{...(3)}$$

$$\therefore \quad g^{lj} v_l = g^{lj} g_{lj} u^j = u^j,$$

which is a contravariant tensor of order one. Therefore by quotient law, $g^{lj}$ are the components of a contravariant tensor of order two. Hence $g^{ij}$ is a symmetric contravariant tensor which is called the *conjugate tensor* or the *second fundamental tensor.*

**Obs.** In view of (2), the relation between $g_{ij}$ and $g^{ij}$ is reciprocal. As such the *first and second fundamental tensors are also called* **reciprocal tensors.**

**Example 38.12.** *Find the components of the first and second fundamental tensors in spherical coordinates.*

**Solution.** Let $(x^1, x^2, x^3)$ be the rectangular cartesian coordinates and $\bar{x}^1, \bar{x}^2, \bar{x}^3$ be the spherical coordinates of a point so that

$$x^1 = x, x^2 = y, x^3 = z, \text{ and } \bar{x}^1 = r, \bar{x}^2 = \theta, \bar{x}^3 = \phi \qquad \text{...(i)}$$

and

We know that
$$x = r \sin\theta \cos\phi, y = r \sin\theta \sin\phi, z = r\cos\theta \qquad \text{...(ii)}$$

Let $g_{pq}$ and $\bar{g}_{ij}$ be the metric tensors in cartesian and spherical coordinates respectively.

Then
$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 = g_{pq} dx^p dx^q$$

$$\therefore \quad g_{11} = 1 = g_{22} = g_{33}, \text{ and } g_{12} = 0 = g_{13} = g_{23} \text{ etc.} \qquad \text{...(iii)}$$

On transformation

$$\bar{g}_{ij} = \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^q}{\partial \bar{x}^j} g_{pq} = \frac{\partial x^1}{\partial \bar{x}^i} \frac{\partial x^1}{\partial \bar{x}^j} g_{11} + \frac{\partial x^2}{\partial \bar{x}^i} \frac{\partial x^2}{\partial \bar{x}^j} g_{22} + \frac{\partial x^3}{\partial \bar{x}^i} \frac{\partial x^3}{\partial \bar{x}^j} g_{33} \qquad \text{...(iv)}$$

Putting $i = j = 1$ in *(iv)*, we have

$$\bar{g}_{11} = \left(\frac{\partial x^1}{\partial \bar{x}^1}\right)^2 g_{11} + \left(\frac{\partial x^2}{\partial \bar{x}^1}\right)^2 g_{22} + \left(\frac{\partial x^3}{\partial \bar{x}^1}\right)^2 g_{33}$$

$$= \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2 \qquad \text{[By } (i) \text{ and } (iii)\text{]}$$

$$= (\sin\theta\cos\phi)^2 + (\sin\theta\sin\phi)^2 + (\cos\theta)^2 = \sin^2\theta + \cos^2\theta = 1 \qquad \text{[By } (ii)\text{]}$$

Putting $i = j = 2$ in *(iv)*, we have

$$\bar{g}_{22} = \left(\frac{\partial x^1}{\partial \bar{x}^2}\right)^2 g_{11} + \left(\frac{\partial x^2}{\partial \bar{x}^2}\right)^2 g_{22} + \left(\frac{\partial x^3}{\partial \bar{x}^2}\right)^2 g_{33}$$

$$= \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 \qquad \text{[By } (i) \text{ and } (iii)\text{]}$$

$$= (r\cos\theta\cos\phi)^2 + (r\cos\theta\sin\phi)^2 + (-r\sin\theta)^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$$

Similarly $$\bar{g}_{33} = \left(\frac{\partial x^1}{\partial \bar{x}^3}\right)^2 g_{11} + \left(\frac{\partial x^2}{\partial \bar{x}^3}\right)^2 g_{22} + \left(\frac{\partial x^3}{\partial \bar{x}^3}\right)^2 g_{33} = r^2\sin^2\theta$$

and $$\bar{g}_{12} = 0 = \bar{g}_{13} = \bar{g}_{21} = \bar{g}_{23} = \bar{g}_{31} = \bar{g}_{32}$$

Hence the first fundamental tensor, written in matrix form, is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{bmatrix}$$

$\therefore$ $$g = \begin{vmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{vmatrix} = r^4\sin^2\theta$$

and the cofactors in $g$ are given by

$$G_{11} = r^4\sin^2\theta,\ G_{22} = r^2\sin^2\theta,\ G_{33} = r^2\ ;\ G_{12} = 0 = G_{13} = G_{21} = G_{23} = G_{31} = G_{32}$$

The components of the second fundamental tensor are given by $g^{ij} = G_{ij}/g$. Hence the second fundamental tensor in matrix form, is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1/r^2\sin^2\theta \end{bmatrix}$.

**Example 38.13.** *Find the components of the metric tensor and the conjugate tensor in cylindrical coordinates.*

**Solution.** Let $(x^1, x^2, x^3)$ be the cartesian coordinates and $(\bar{x}^1, \bar{x}^2, \bar{x}^3)$ be the cylindrical coordinates of a point so that

$$x^1 = x,\ x^2 = y,\ x^3 = z \quad \text{and} \quad \bar{x}^1 = \rho,\ \bar{x}^2 = \phi,\ \bar{x}^3 = z \qquad ...(i)$$

We know that $$x = \rho\cos\phi,\ y = \rho\sin\phi,\ z = z \qquad ...(ii)$$

Let $g_{pq}$ and $\bar{g}_{ij}$ be the metric tensors in cartesian and cylindrical coordinates respectively.

Then $$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 = g_{pq}dx^p$$

$\therefore$ $$g_{11} = 1 = g_{22} = g_{33} \quad \text{and} \quad g_{12} = 0 = g_{13} = g_{23} \text{ etc.} \qquad ...(iii)$$

On transformation,

$$\bar{g}_{ij} = \frac{\partial x^p}{\partial \bar{x}^i}\frac{\partial x^q}{\partial \bar{x}^j} g_{pq} = \frac{\partial x^1}{\partial \bar{x}^i}\frac{\partial x^1}{\partial \bar{x}^j} g_{11} + \frac{\partial x^2}{\partial \bar{x}^i}\frac{\partial x^2}{\partial \bar{x}^j} g_{22} + \frac{\partial x^3}{\partial \bar{x}^i}\frac{\partial x^3}{\partial \bar{x}^j} g_{33} \qquad ...(iv)$$

Putting $i = j = 1$ in (*iv*), we have

$$\bar{g}_{11} = \left(\frac{\partial x^1}{\partial \bar{x}^1}\right)^2 g_{11} + \left(\frac{\partial x^2}{\partial \bar{x}^1}\right)^2 g_{22} + \left(\frac{\partial x^3}{\partial \bar{x}^1}\right)^2 g_{33} = \left(\frac{\partial x}{\partial \rho}\right)^2 + \left(\frac{\partial y}{\partial \rho}\right)^2 + \left(\frac{\partial z}{\partial \rho}\right)^2$$

$$= \cos^2 \phi + \sin^2 \phi + 0 = 1 \qquad \text{[By } (i) \text{ and } (ii)]$$

Putting $i = j = 2$ in (*iv*), we have

$$\bar{g}_{22} = \left(\frac{\partial x^1}{\partial \bar{x}^2}\right)^2 g_{11} + \left(\frac{\partial x^2}{\partial \bar{x}^2}\right)^2 g_{22} + \left(\frac{\partial x^3}{\partial \bar{x}^2}\right)^2 g_{33} = \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2$$

$$= (-\rho \sin \phi)^2 + (\rho \cos \phi)^2 + 0 = \rho^2 \qquad \text{[By } (i) \text{ and } (ii)]$$

Similarly $$\bar{g}_{33} = \left(\frac{\partial x^1}{\partial \bar{x}^3}\right)^2 g_{11} + \left(\frac{\partial x^2}{\partial \bar{x}^3}\right)^2 g_{22} + \left(\frac{\partial x^3}{\partial \bar{x}^3}\right)^2 g_{33} = 0 + 0 + 1 = 1$$

and $$\bar{g}_{12} = 0 = \bar{g}_{13} = \bar{g}_{21} = \bar{g}_{23} = \bar{g}_{31} = \bar{g}_{32}$$

Hence the metric tensor, written in matrix form, is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \therefore \qquad g = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho^2$$

Also cofactors in $g$ are given by

$$G_{11} = \rho^2, G_{22} = 1, G_{33} = \rho^2 ; G_{12} = 0 = G_{13} = G_{21} = G_{23} = G_{31} = G_{32}$$

The components of the conjugate tensor are given by $g^{ij} = G_{ij}/g$.

Hence the conjugate tensor in matrix form is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\rho^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

## 38.15 ASSOCIATED TENSORS

From (3) of § 38.14, we have $\quad u^j . g_{ij} = v_i \qquad$ ...(1)

*i.e.*, the inner product of the tensor $u^j$ with the fundamental tensor $g_{ij}$ is another tensor $v_i$ which is called the *associated tensor* of $u^j$.

Similarly, we have $\quad v_l . g^{lj} = u^j \qquad$ ...(2)

Hence $u^j$ is the associated tensor of $v_l$.

Thus the indices of any tensor can be lowered or raised by forming its inner product with either of the fundamental tensors $g_{ij}$ or $g^{ij}$ as in (1) or (2) above.

## 38.16 (1) LENGTH OF A VECTOR

The vector **A** is given by

$$\mathbf{A} = A^i g_i \quad \text{or} \quad \mathbf{A} = A_i g^i \qquad ...(1)$$

Also we have the associated vectors $A_i = g_{ij} A^j \qquad$ ...(2)

or $\quad A^i = g^{ij} A_j \qquad$ ...(3)

$\therefore$ Length of vector $\quad \mathbf{A} = (\mathbf{A} . \mathbf{A})^{1/2} = (A^i g_i . A^j g_j)^{1/2} \qquad$ [By (1)]

$= (g_{ij} A^i A^j)^{1/2} \qquad [\because \; g_i \cdot g_k = g_{ij}]$

$= (A_i A^i)^{1/2} \qquad$ [By (2)]

Also length of vector $\mathbf{A} = (\mathbf{A} . \mathbf{A})^{1/2} = (A_i g^i . A_j g^j)^{1/2} = (g^{ij} A_i A_j)^{1/2} = (A_i A^i)^{1/2} \qquad$ [By (3)]

Hence the magnitude or length of the vector $A = \sqrt{(g_{ij} A^i A^j)} = \sqrt{(g^{ij} A_i A_j)} = \sqrt{(A_i A^i)} \qquad$ ...(4)

which is an invariant.

Obs. The length of a vector $A^1, 0, 0$ (in 3-dimensions) is $\sqrt{(g_{11}A^1A^1)}$, *i.e.* $\sqrt{g_{11}}A^1$. Similarly the length of the vector $0, A^2, 0$ is $\sqrt{g_{22}}A^2$ and the length of the vector $0, 0, A^3$ is $\sqrt{g_{33}}A^3$. Hence the physical components of a vector $A^i$ are $\sqrt{g_{11}}A^1$, $\sqrt{g_{22}}A^2$, $\sqrt{g_{33}}A^3$.

**(2) Angle between two vectors.** Let **A** and **B** be the given vectors such that

$$\mathbf{A} = A^i g_i \text{ and } B = B^j g_j \quad ...(5)$$

$\therefore$ $$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}|\ |\mathbf{B}| \cos\theta$$

or $$\cos\theta = \frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \frac{A^i B^j g_{ij}}{\sqrt{(g_{ij}A^iA^j)}\sqrt{(g_{ij}B^iB^j)}} \quad \text{[Using (4) and (5)]}$$

In terms of associated vectors, we have

$$\cos\theta = \frac{A^i B_i}{\sqrt{(A^iA_i)}\sqrt{(B^iB_i)}}.$$

PROBLEMS 38.3

1. If $ds^2 = 5(dx^1)^2 + 3(dx^2)^2 + 4(x^3)^2 - 6dx^1dx^2 + 4dx^2dx^3$, find the values of $g_{ij}$ and $g^{ij}$.
2. Find $g$ and $g^{ij}$ corresponding to the metric $ds^2 = \dfrac{dr^2}{1 - r^2/a^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$.
3. The contravariant components of a vector **A** in plane cartesian coordinates $x = x^1$, $y = x^2$ are (33, 56). Using the tensor law of transformation, obtain the new components in polar coordinates $r = \bar{x}^1$ and $\theta = \bar{x}^2$.
4. Prove that the angles $\theta_{12}$, $\theta_{23}$ and $\theta_{31}$ between the coordinate curves in a 3-dimensional coordinate system are given by $\cos\theta_{12} = \dfrac{g_{12}}{\sqrt{(g_{11}g_{22})}}$, $\cos\theta_{23} = \dfrac{g_{23}}{\sqrt{(g_{22}g_{23})}}$, $\cos\theta_{31} = \dfrac{g_{31}}{\sqrt{(g_{33}g_{11})}}$.
5. Prove that for an orthogonal coordinate system
   (*i*) $g_{12} = g_{23} = g_{31} = 0$ (*ii*) $g^{11} = 1/g_{11}$, $g_{22} = 1/g^{22}$, $g_{33} = 1/g^{33}$.

## 38.17 CHRISTOFFEL SYMBOLS

*Christoffel symbol of the first kind* is denoted by $[ij, k]$ and is defined by

$$[ij, k] = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right) \quad ...(1)$$

where $g_{ij}$ are the components of the metric tensor.

*Christoffel symbol of the second kind* is denoted by $\begin{Bmatrix} k \\ ij \end{Bmatrix}$ and is defined by

$$\begin{Bmatrix} k \\ ij \end{Bmatrix} = g^{kl}\,[ij, l] \quad ...(2)$$

Some authors write Christoffel symbol of the second kind as $\{ij, k\}$ or $\Gamma_{ij}^{\ k}$.

Obs. 1. No summation is indicated in the Christoffel symbol of the first kind, but summation is to be made over $l$ in the Christoffel symbol of the second kind.

Obs. 2. It is evident from (1) and (2) that the Christoffel symbols of both kinds are symmetric in the indices $i$ and $j$.

$$[ij, k] = [ji, k] \text{ and } \begin{Bmatrix} k \\ ij \end{Bmatrix} = \begin{Bmatrix} k \\ ji \end{Bmatrix}.$$

**Example 38.14.** *If $(ds)^2 = (dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta\,(d\phi)^2$, find the values of*

(*a*) $[22, 1]$ *and* $[13, 3]$ (*b*) $\begin{Bmatrix} 1 \\ 22 \end{Bmatrix}$ *and* $\begin{Bmatrix} 3 \\ 13 \end{Bmatrix}$.

**Solution.** It is a 3-dimensional space in spherical coordinates such that

$$x^1 = r,\ x^2 = \theta \text{ and } x^3 = \phi$$

Clearly $\qquad g_{11} = 1, g_{22} = r^2, g_{33} = r^2 \sin^2\theta$ and $g_{ij} = 0$ for $i \neq j$. ...(*i*)

Also $g^{11} = 1, g^{22} = 1/r^2, g^{33} = 1/r^2 \sin^2\theta$ (See Ex. 38.12) ...(*ii*)

(*a*) Christoffel symbols of the first kind are given by

$$[ij, k] = \frac{1}{2}\left[\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right]_{i,\,j,\,k\,=\,1,\,2,\,3} \qquad \text{...(iii)}$$

Taking $i = 2, j = 2$, and $k = 1$ in (*iii*), we get

$$[22, 1] = \frac{1}{2}\left[\frac{\partial g_{21}}{\partial x^2} + \frac{\partial g_{21}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^1}\right] = \frac{1}{2}\left[\frac{\partial(0)}{\partial\theta} + \frac{\partial(0)}{\partial\theta} - \frac{\partial(r^2)}{\partial r}\right] = -r \qquad \text{...(iv)}$$

Putting $i = 1, j = 3$ and $k = 3$ in (*iii*), we obtain

$$[13, 3] = \frac{1}{2}\left[\frac{\partial g_{33}}{\partial x^1} + \frac{\partial g_{13}}{\partial x^3} - \frac{\partial g_{13}}{\partial x^3}\right] = \frac{1}{2}\left[\frac{\partial(r^2\sin^2\theta)}{\partial r} + \frac{\partial(0)}{\partial\phi} - \frac{\partial(0)}{\partial\phi}\right] = r\sin^2\theta \qquad \text{...(v)}$$

(*b*) Christoffel symbols of the second kind are defined by

$$\begin{Bmatrix} k \\ ij \end{Bmatrix} = g^{kl}\,[ij, l] = g^{k1}\,[ij, 1] + g^{k2}\,[ij, 2] + g^{k3}\,[ij, 3] \qquad \text{...(vi)}$$

$$\therefore \qquad \begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = g^{11}[22, 1] + g^{12}[22, 2] + g^{13}[22, 3] = [22, 1] + 0[22, 2] + 0[22, 3]$$

$$= -r \qquad \text{[By (iv)]}$$

$$\begin{Bmatrix} 3 \\ 13 \end{Bmatrix} = g^{31}[13, 1] + g^{32}[13, 2] + g^{33}[13, 3]$$

$$= 0[13, 1] + 0[13, 2] + \frac{1}{r^2\sin^2\theta}[13, 3] \qquad \text{[By (ii)]}$$

$$= \frac{1}{r^2\sin^2\theta} \cdot r\sin^2\theta = \frac{1}{r}. \qquad \text{[By (v)]}$$

**Example 38.15.** *Prove that*

(*a*) $\dfrac{\partial g_{ij}}{\partial x^k} = [ik, j] + [jk, i]$ $\qquad$ (*b*) $\dfrac{\partial g^{ij}}{\partial x^k} = -g^{jl}\begin{Bmatrix} i \\ lk \end{Bmatrix} - g^{im}\begin{Bmatrix} j \\ mk \end{Bmatrix}$.

**Solution.** (*a*) By definition of Christoffel symbol of the first kind, we have

$$[ik, j] = \frac{1}{2}\left[\frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^j}\right] \qquad \text{...(i)}$$

and

$$[jk, i] = \frac{1}{2}\left[\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{ji}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^i}\right] \qquad \text{...(ii)}$$

Since $g_{ij}$ is a symmetric tensor, $\therefore\ g_{ij} = g_{ji}, g_{jk} = g_{kj}, g_{ki} = g_{ik}$

Adding (*i*) and (*ii*), we get the required result.

(*b*) We know that $g^{ij}g^{lj} = \delta^i_{\ l}$. [Refer to § 38.14 (2)]

Differentiating w.r.t. $x^k$, we get

$$g^{ij}\frac{\partial g_{lj}}{\partial x^k} + g_{lj}\frac{\partial g^{ij}}{\partial x^k} = 0 \qquad [\because\ \delta^i_{\ l} = 1 \text{ or } 0]$$

Multiplying by $g^{lm}$ and transposing, we have

$$g^{lm}g_{lj}\frac{\partial g^{ij}}{\partial x^k} = -g^{lm}g^{ij}\frac{\partial g_{lj}}{\partial x^k} \quad \text{or} \quad \delta^m_j\frac{\partial g^{ij}}{\partial x^k} = -g^{lm}g^{ij}\{[lk, j] + [jk, l]\} \qquad \text{[From (a)]}$$

or $$\frac{\partial g^{im}}{\partial x^k} = -g^{lm}\left(g^{ij}[lk, j]\right) - g^{ij}\left(g^{lm}[jk, l]\right) = -g^{lm}\begin{Bmatrix} i \\ lk \end{Bmatrix} - g^{ij}\begin{Bmatrix} m \\ jk \end{Bmatrix}$$

Interchanging $m$ and $j$, we obtain the desired result.

## 38.18 TRANSFORMATION OF CHRISTOFFEL SYMBOLS

The fundamental tensors $g_{ij}, g^{ij}$ and also $[ij, k]$ are functions of the coordinates $x^i$. Let these transform to $\bar{g}_{ij}, \bar{g}^{ij}$ and $\overline{[ij, k]}$ in another coordinate system $\bar{x}^i$.

(1) *Law of transformation of Christoffel symbol of first kind.*

Let $[ij, k]$ which is a function of $x^i$, transform to $\overline{[ij, k]}$ in another coordinate system $\bar{x}^i$. Then

$$\overline{[ij, k]} = \frac{1}{2}\left(\frac{\partial \bar{g}_{jk}}{\partial \bar{x}^i} + \frac{\partial \bar{g}_{ik}}{\partial \bar{x}^j} - \frac{\partial \bar{g}_{ij}}{\partial \bar{x}^k}\right) \qquad ...(1)$$

Since $\bar{g}_{ij}$ is a covariant tensor of order two, we have

$$\bar{g}_{ij} = \frac{\partial x^p}{\partial \bar{x}^i}\frac{\partial x^q}{\partial \bar{x}^j} g_{pq} \qquad ...(2)$$

Differentiating both sides w.r.t. $\bar{x}^k$, we get

$$\frac{\partial \bar{g}_{ij}}{\partial \bar{x}^k} = \left(\frac{\partial^2 x^p}{\partial \bar{x}^k \partial \bar{x}^i}\frac{\partial x^q}{\partial \bar{x}^j} + \frac{\partial x^p}{\partial \bar{x}^i}\frac{\partial^2 x^q}{\partial \bar{x}^k \partial \bar{x}^j}\right) g_{pq} + \frac{\partial x^p}{\partial \bar{x}^i}\frac{\partial x^q}{\partial \bar{x}^j}\frac{\partial g_{pq}}{\partial x^r}\frac{\partial x^r}{\partial \bar{x}^k} \qquad ...(3)$$

[Note that $g_{pq}$ is in terms of original coordinates $x$ and to differentiate it w.r.t. $\bar{x}^k$, first we differentiate it w.r.t. $x^r$ and then differentiate $x^r$ w.r.t. $\bar{x}^k$.]

Interchanging $i, k$ and also $p, r$ in the last term of (3), we have

$$\frac{\partial \bar{g}_{ik}}{\partial \bar{x}^i} = \left(\frac{\partial^2 x^p}{\partial \bar{x}^i \partial \bar{x}^k}\frac{\partial x^q}{\partial \bar{x}^j} + \frac{\partial x^p}{\partial \bar{x}^k}\frac{\partial^2 x^q}{\partial \bar{x}^i \partial \bar{x}^j}\right) g_{pq} + \frac{\partial x^p}{\partial \bar{x}^k}\frac{\partial x^q}{\partial \bar{x}^j}\frac{\partial x^p}{\partial \bar{x}^i}\frac{\partial g_{qr}}{\partial x^p} \qquad ...(4)$$

Similarly interchanging $j, k$ and also $q, r$ in the last term of (3), we get

$$\frac{\partial \bar{g}_{ik}}{\partial \bar{x}^j} = \left(\frac{\partial^2 x^p}{\partial \bar{x}^j \partial \bar{x}^i}\frac{\partial x^q}{\partial \bar{x}^k} + \frac{\partial x^p}{\partial \bar{x}^i}\frac{\partial^2 x^q}{\partial \bar{x}^k \partial \bar{x}^j}\right) g_{pq} + \frac{\partial x^p}{\partial \bar{x}^i}\frac{\partial x^r}{\partial \bar{x}^k}\frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial g_{pr}}{\partial \bar{x}^q} \qquad ...(5)$$

Substituting the values from (3), (4) and (5) in (1), we obtain

$$\overline{[ij, k]} = \frac{\partial^2 x^p}{\partial \bar{x}^i \partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k} g_{pq} + \frac{\partial x^p}{\partial \bar{x}^i}\frac{\partial x^q}{\partial \bar{x}^j}\frac{\partial x^r}{\partial \bar{x}^k}[pq, r] \qquad ...(6)$$

This is the desired law of transformation of Christoffel symbol of the first kind.

(2) *Law of transformation of Christoffel symbol of the second kind.*

Let $g^{kl}[ij, l]$ transform to $\bar{g}^{kl}\ \overline{[ij, l]}$.

Since $\bar{g}^{kl}$ is a contravariant tensor of order two.

$$\therefore \qquad \bar{g}^{kl} = \frac{\partial \bar{x}^k}{\partial x^s}\frac{\partial \bar{x}^l}{\partial x^t} g^{st} \qquad ...(7)$$

From (6), we have $\overline{[ij, l]} = \dfrac{\partial^2 x^p}{\partial \bar{x}^i \partial \bar{x}^j}\dfrac{\partial x^q}{\partial \bar{x}^l} g_{pq} + \dfrac{\partial x^p}{\partial \bar{x}^i}\dfrac{\partial x^q}{\partial \bar{x}^j}\dfrac{\partial x^r}{\partial \bar{x}^l}[pq, r] \qquad ...(8)$

Multiplying the respective sides of (7) and (8), we get

$$\overline{\begin{Bmatrix} l \\ ij \end{Bmatrix}} = \frac{\partial \bar{x}^t}{\partial \bar{x}^t}\delta^q{}_s \frac{\partial^2 x^p}{\partial \bar{x}^i \partial \bar{x}^j} g^{st} g_{pq} + \frac{\partial \bar{x}^l}{\partial x^t}\delta^r{}_s \frac{\partial x^p}{\partial \bar{x}^i}\frac{\partial x^q}{\partial \bar{x}^j} g^{st}\,[pq, r] \qquad ...(9)$$

Since $\qquad \delta^q{}_s g^{st} g_{pq} = g^{qt} g_{pq} = \delta_p{}^t$

and $$\delta_s^r g^{st}[pq, r] = g^{rt}[pq, r] = \begin{Bmatrix} t \\ pq \end{Bmatrix}$$

$$\therefore \qquad \begin{Bmatrix} \bar{l} \\ ij \end{Bmatrix} = \frac{\partial^2 x^p}{\partial \bar{x}^i \partial \bar{x}^j} \delta^t{}_p \frac{\partial \bar{x}^l}{\partial x^t} + \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^q}{\partial \bar{x}^j} \frac{\partial \bar{x}^l}{\partial x^t} \begin{Bmatrix} t \\ pq \end{Bmatrix}$$

or $$\begin{Bmatrix} \bar{l} \\ ij \end{Bmatrix} = \frac{\partial^2 x^t}{\partial \bar{x}^i \partial \bar{x}^j} \frac{\partial \bar{x}^l}{\partial x^t} + \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^q}{\partial \bar{x}^j} \frac{\partial \bar{x}^l}{\partial x^t} \begin{Bmatrix} t \\ pq \end{Bmatrix} \qquad ...(10)$$

**This is the law of transformation of Christoffel symbol of the second kind.**

**Obs. 1.** From (10), we obtain the following important relation :

$$\frac{\partial^2 x^t}{\partial \bar{x}^i \partial \bar{x}^j} = \frac{\partial x^t}{\partial \bar{x}^l} \begin{Bmatrix} \bar{l} \\ ij \end{Bmatrix} - \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^q}{\partial \bar{x}^j} \begin{Bmatrix} t \\ pq \end{Bmatrix} \qquad ...(11)$$

**Obs. 2.** It is evident from (6) and (10) that the Christoffel 3-index symbols are not tensors. These symbols transform like tensors only for linear transformation of coordinates.

**Example 38.16.** *Prove that* $\begin{Bmatrix} i \\ ij \end{Bmatrix} = \frac{\partial}{\partial x^j} (\log \sqrt{g})$.

**Solution.** Let $G_{jk}$ be the co-factor of $g_{ik}$ in $g$ so that $g = g_{ik} G_{ik}$ (summation over $k$ only)

$$\therefore \qquad \frac{\partial g}{\partial g_{ik}} = G_{ik} \qquad [\because \; G_{ik} \text{ does not contain } g_{ik} \text{ implicitly}]$$

Also $$\frac{\partial g}{\partial g^j} = \frac{\partial g}{\partial g_{ik}} \frac{\partial g_{ik}}{\partial x^j} = G_{ik} \frac{\partial g_{ik}}{\partial x^j} \qquad \text{(summation over } i \text{ and } k) \qquad (i)$$

We know that $$g^{ik} = \frac{G_{ik}}{g} \qquad ...(ii)$$

Substituting the value of $G_{ik}$ from (*ii*) in (*i*), we get

$$\frac{\partial g}{\partial x^j} = g g^{ik} \frac{\partial g_{ik}}{\partial x^j} \quad \text{or} \quad \frac{1}{g} \frac{\partial g}{\partial x^j} = g^{ik} \frac{\partial g_{ik}}{\partial x^j}$$

or $$\frac{\partial}{\partial x^j} (\log g) = g^{ik}([jk, i] + [ij, k]) \qquad \text{[By Ex. 38.15 (a)]}$$

$$= g^{ik}[jk, i] + g^{ik}[ji, k] = \begin{Bmatrix} k \\ jk \end{Bmatrix} + \begin{Bmatrix} i \\ ji \end{Bmatrix} = 2 \begin{Bmatrix} i \\ ji \end{Bmatrix}$$

Hence $$\begin{Bmatrix} i \\ ij \end{Bmatrix} = \frac{1}{2} \frac{\partial}{\partial x^j} (\log g) = \frac{\partial}{\partial x^j} \log \sqrt{g}. \qquad ...(iii)$$

## 38.19 (1) COVARIANT DIFFERENTIATION OF A COVARIANT VECTOR

Let $A_i$ and $\bar{A}_i$ be the components of a covariant vector (*i.e.*, a tensor of first order) in the coordinate system $x^i$ and $\bar{x}^i$ respectively. Let us investigate the tensor character of the partial derivatives of $A_i$ w.r.t. the variables $x^j$. From the law of transformation. $\bar{A}_i = \frac{\partial x^p}{\partial \bar{x}^i} A_p$

Differentiating w.r.t. $\bar{x}^j$, we have $$\frac{\partial \bar{A}_i}{\partial \bar{x}^j} = \frac{\partial^2 x^p}{\partial \bar{x}^j \partial \bar{x}^i} A_p + \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial A_p}{\partial x^q} \frac{\partial x^q}{\partial \bar{x}^j} \qquad ...(1)$$

(Note that $A_p$ is not directly a function of $\bar{x}^j$).

Due to presence of the first term on the R.H.S. of (1), it is evident that $\frac{\partial A_p}{\partial x^q}$ is not a tensor.

On replacing this term by $\dfrac{\partial^2 x^s}{\partial \bar{x}^i \partial \bar{x}^j} A_s$ and substituting for the second derivative from (11) of § 35.18, we get

$$\frac{\partial \bar{A}_i}{\partial \bar{x}^j} = \frac{\partial x^s}{\partial \bar{x}^l} A_s \begin{Bmatrix} \bar{l} \\ i \ j \end{Bmatrix} - \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^q}{\partial \bar{x}^j} A_s \begin{Bmatrix} s \\ pq \end{Bmatrix} + \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^q}{\partial \bar{x}^j} \frac{\partial A_p}{\partial \bar{x}^q}$$

or
$$\frac{\partial \bar{A}_i}{\partial \bar{x}^j} - \bar{A}_l \begin{Bmatrix} \bar{l} \\ i \ j \end{Bmatrix} = \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^q}{\partial \bar{x}^j} \left[ \frac{\partial A_p}{\partial \bar{x}^q} - A_s \begin{Bmatrix} s \\ pq \end{Bmatrix} \right]$$

This shows that the expression $\dfrac{\partial \bar{A}_i}{\partial \bar{x}^j} - \bar{A}_l \begin{Bmatrix} \bar{l} \\ i \ j \end{Bmatrix}$

is a covariant tensor of the second order. This is called the covariant derivative of $\bar{A}_i$ w.r.t $\bar{x}^j$ and is denoted by $\bar{A}_{i,j}$.

**(2) Covariant differentiation of a contravariant vector.** Let $A^i$ and $\bar{A}^i$ be the components of a contravariant vector in the coordinate systems $x^i$ and $\bar{x}^i$. From the law of transformation $A^s = \dfrac{\partial x^s}{\partial \bar{x}^i} \bar{A}^i$.

Differentiating w.r.t. $\bar{x}^j$, we have $\dfrac{\partial A^s}{\partial \bar{x}^j} = \dfrac{\partial^2 x^s}{\partial \bar{x}^j \partial \bar{x}^i} \bar{A}^i + \dfrac{\partial x^s}{\partial \bar{x}^i} \dfrac{\partial \bar{A}^i}{\partial \bar{x}^j}$

Substituting for the second derivative from (11) of § 35.18, we get

$$\frac{\partial A^s}{\partial x^q} \frac{\partial x^q}{\partial \bar{x}^j} = \frac{\partial x^s}{\partial x^l} \begin{Bmatrix} l \\ i \ j \end{Bmatrix} - \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^q}{\partial \bar{x}^j} \begin{Bmatrix} s \\ p \ q \end{Bmatrix} \bar{A}^i + \frac{\partial x^s}{\partial \bar{x}^i} \frac{\partial \bar{A}^i}{\partial \bar{x}^j}$$

Interchanging the dummy indices $i, l$ in the first term on the R.H.S. and putting

$\dfrac{\partial x^p}{\partial \bar{x}^i} \bar{A}^i = A^p$ in the second term, we obtain

$$\frac{\partial x^q}{\partial \bar{x}^j} \frac{\partial A^s}{\partial x^q} = \frac{\partial x^s}{\partial \bar{x}^i} \begin{Bmatrix} \bar{i} \\ l \ j \end{Bmatrix} \bar{A}^l - \frac{\partial x^q}{\partial \bar{x}^j} A^p \begin{Bmatrix} s \\ p \ q \end{Bmatrix} + \frac{\partial x^s}{\partial \bar{x}^i} \frac{\partial \bar{A}^i}{\partial \bar{x}^j}$$

Transposing the second term on the R.H.S. to the L.H.S., we get

$$\frac{\partial x^q}{\partial \bar{x}^j} \left[ \frac{\partial A^s}{\partial x^q} + A^p \begin{Bmatrix} s \\ p \ q \end{Bmatrix} \right] = \frac{\partial x^s}{\partial \bar{x}^i} \left[ \frac{\partial \bar{A}^i}{\partial \bar{x}^j} + \bar{A}^l \begin{Bmatrix} \bar{i} \\ l \ j \end{Bmatrix} \right]$$

or
$$\frac{\partial \bar{A}^i}{\partial \bar{x}^j} + \bar{A}^l \begin{Bmatrix} \bar{i} \\ l \ j \end{Bmatrix} = \frac{\partial \bar{x}^i}{\partial x^s} \frac{\partial x^q}{\partial \bar{x}^j} \left[ \frac{\partial A^s}{\partial x^q} + A^p \begin{Bmatrix} s \\ p \ q \end{Bmatrix} \right]$$

This shows that $\dfrac{\partial A^i}{\partial x^j} + A^l \begin{Bmatrix} i \\ l \ j \end{Bmatrix}$ is a mixed tensor of the second order. This is called the *covariant derivative of $A^i$ w.r.t. $x^j$ and is denoted by $A^i_{,j}$.*

**Obs.** *The following laws hold good for covariant differentiation :*

*(i) Covariant derivative of the sum (or difference) of two tensors = sum (or difference) of their covariant derivatives.*

*(ii) Covariant derivative of the product of two tensors = covariant derivative of first tensor × second tensor + covariant derivative of second tensor × first tensor.*

**Example 38.17.** *Prove that the covariant derivative of $g^{ij}$ is zero.*

**Solution.** Let $A_i$ denote a covariant vector which moves parallel to itself so that

$$A_{i,k} \text{ or } \frac{\partial A_i}{\partial x^k} - A_l \begin{Bmatrix} l \\ i \ k \end{Bmatrix} = 0 \qquad ...(i)$$

Let $\phi = g^{ij} A_i A_j$ so that $\phi$ is a scalar invariant. Differentiating it w.r.t. $x^k$, we have

$$\frac{\partial \phi}{\partial x^k} = \frac{\partial g^{ij}}{\partial x^k} A_i A_j + g^{ij} \frac{\partial A_i}{\partial x^k} A_j + g^{ij} A_i \frac{\partial A_j}{\partial x^k}$$

$$= \frac{\partial g^{ij}}{\partial x^k} A_i A_j + g^{ij} \begin{Bmatrix} l \\ i \ k \end{Bmatrix} A_l A_j + g^{ij} \begin{Bmatrix} l \\ j \ k \end{Bmatrix} A_i A_l \qquad \text{[By (i)]}$$

Interchanging $i$ and $l$ in the second term and $j$ and $l$ in the last term on the right, we get

$$\frac{\partial \phi}{\partial x^k} = \left[ \frac{\partial g^{ij}}{\partial x^k} + g^{lj} \begin{Bmatrix} i \\ l \ k \end{Bmatrix} + g^{il} \begin{Bmatrix} j \\ l \ k \end{Bmatrix} \right] A_i A_j$$

Since $\partial\phi/\partial x^k$ is a covariant vector, the expression

$$\frac{\partial g^{ij}}{\partial x^k} + g^{lj} \begin{Bmatrix} i \\ l \ k \end{Bmatrix} + g^{il} \begin{Bmatrix} j \\ l \ k \end{Bmatrix}$$

is a tensor of the third order by quotient law. Thus it is the covariant derivative $g^{ij}{}_{,k}$.

$$\therefore \qquad g^{ij}{}_{,k} = \frac{\partial g^{ij}}{\partial x^k} + g^{lj} \begin{Bmatrix} i \\ l \ k \end{Bmatrix} + g^{il} \begin{Bmatrix} j \\ l \ k \end{Bmatrix}$$

$$= - g^{jl} \begin{Bmatrix} i \\ l \ k \end{Bmatrix} - g^{im} \begin{Bmatrix} j \\ m \ k \end{Bmatrix} + g^{lj} \begin{Bmatrix} i \\ l \ k \end{Bmatrix} + g^{il} \begin{Bmatrix} j \\ l \ k \end{Bmatrix} \qquad \text{[By Ex. 38.15 (b)]}$$

$$= - g^{im} \begin{Bmatrix} j \\ m \ k \end{Bmatrix} + g^{il} \begin{Bmatrix} j \\ l \ k \end{Bmatrix} = - g^{il} \begin{Bmatrix} j \\ l \ k \end{Bmatrix} + g^{il} \begin{Bmatrix} j \\ l \ k \end{Bmatrix} = 0$$

[Changing the dummy index $m$ to $l$]

## 38.20 (1) GRADIENT

If $\phi$ be a scalar function of the coordinates, then the gradient of $\phi$ is denoted by grad $\phi = \dfrac{\partial \phi}{\partial x^i}$ which is a *covariant vector.*

**(2) Divergence.** *The divergence of the contravariant vector $A^i$ is defined by*

$$div\ A^i = \frac{\partial A^i}{\partial x^i} + A^k \begin{Bmatrix} l \\ k \ i \end{Bmatrix} \text{ which is sometimes written as } A^i{}_{,i}.$$

*The divergence of the covariant vector $A_i$ is defined by div $A_i = g^{ik} A_{ik}$.*

**(3) Curl.** Let $A_i$ be a covariant vector, then

$$A_{i,j} = \frac{\partial A_i}{\partial x^j} - A_k \begin{Bmatrix} k \\ i \ j \end{Bmatrix} \text{ and } A_{j,i} = \frac{\partial A_j}{\partial x^i} - A_k \begin{Bmatrix} k \\ j \ i \end{Bmatrix} \text{ are covariant tensors.}$$

$$\therefore \qquad A_{i,j} - A_{j,i} = \frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}$$

*is a covariant tensor of second order, which is called curl of $A_i$.*

Thus $\qquad curl\ A_i = A_{i,j} - A_{j,i}$.

**Obs 1.** *Curl $A_i$ is a skew-symmetric tensor.*
Since $\qquad A_{j,i} - A_{i,j} = -(A_{i,j} - A_{j,i})$.

**Obs. 2.** *Curl is a tensor and not a vector.* In a 3-dimensional space, however, curl has only three independent non-zero components and it can, therefore, be taken as a vector.

**Example 38.18.** *Prove that*

(a) $div\ A_i = \dfrac{1}{\sqrt{g}} \cdot \dfrac{\partial}{\partial x^k} (\sqrt{g}\, A^k)$ $\qquad$ (b) $\nabla^2 \phi = \dfrac{1}{\sqrt{g}} \dfrac{\partial}{\partial x^k} \left( \sqrt{g}\, g^{kr} \dfrac{\partial \phi}{\partial x^r} \right)$.

**Solution.** (*a*) Using $\begin{Bmatrix} i \\ k\ i \end{Bmatrix} = \dfrac{\partial}{\partial x^k} \log \sqrt{g} = \dfrac{1}{\sqrt{g}} \dfrac{\partial \sqrt{g}}{\partial x^k}$ [By Ex. 38.16]

$$\text{div } A^i = \frac{\partial A^i}{\partial x^i} + A^k \begin{Bmatrix} i \\ k\ i \end{Bmatrix} = \frac{\partial A^i}{\partial x^i} + \frac{A^k}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^k} = \frac{\partial A^k}{\partial x^k} + \frac{A^k}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^k}$$

$$= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^k} (\sqrt{g}\, A^k) \qquad ...(i)$$

(*ii*) We have $\nabla^2 \phi = \text{div grad } \phi$ ...(*ii*)

and $\text{grad } \phi = \dfrac{\partial \phi}{\partial x^r}$, which is a covariant vector.

The contravariant vector associated with $\partial\phi/\partial x^r$ is

$$A^k = g^{kr}\, \partial\phi/\partial x^r.$$

Then from (*i*) and (*ii*), $\nabla^2 \phi = \text{div} \left( g^{kr} \dfrac{\partial \phi}{\partial x^r} \right) = \dfrac{1}{\sqrt{g}} \dfrac{\partial}{\partial x^k} \left( \sqrt{g}\, g^{kr} \dfrac{\partial \phi}{\partial x^r} \right).$

## PROBLEMS 38.4

1. Determine Christoffel symbols of the first and second kind for an orthogonal curvilinear coordinate system.
2. Determine the Christoffel symbols of the first kind in (*a*) rectangular, (*b*) cylindrical and (*c*) spherical coordinates.
3. Evaluate the Christoffel symbols of the second kind in (*a*) rectangular, (*b*) cylindrical and (*c*) spherical coordinates.
4. Prove that (*a*) $[pq, r] = [qp, r]$, (*b*) $[pq, r] = g_{rs} \begin{Bmatrix} s \\ pq \end{Bmatrix}$
5. If $(ds)^2 = r^2 (d\theta)^2 + r^2 \sin^2 \theta\, (d\phi)^2$, find the values of
   (*a*) $[22, 1]$ and $[12, 2]$ (*b*) $\begin{bmatrix} 1 \\ 22 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 12 \end{bmatrix}$.
6. If $(ds)^2 = (dr)^2 + r^2 (d\theta)^2 + r^2 \sin^2 \theta\, (d\phi)^2$, find the values of
   (*a*) $[33, 1]$ and $[23, 3]$ (*b*) $\begin{bmatrix} 1 \\ 33 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 23 \end{bmatrix}$.
7. Show that the tensors $g_{ij}$, $g^{ij}$ and $\delta^i_j$ are constants with respect to covariant differentiation.
8. Write the covariant derivative w.r.t. $x^k$ of the tensors $u^{ij}$ and $A^k_{ij}$.
9. Show that the covariant derivative of $g_{ij}$ is zero. (*Madras M.E., 2000*)
10. Find the covariant derivative $A^j_k\, B^{lm}_n$ with respect to $x^q$.
11. Evaluate div $A^i$ in (*a*) cylindrical, (*b*) spherical coordinates.
12. Obtain the Laplace's equation in (*a*) cylindrical, (*b*) spherical coordinates.
13. If $A_{ij,\,k}$ is the curl of a covariant vector, prove that
    $$A_{ij,\,k} + A_{jk,\,i} + A_{ki,\,j} = 0. \qquad \text{(Madras M.E., 2000 S)}$$
14. Using tensor notation, show that
    (*a*) div curl $A^r = 0$ (*b*) curl grad $\phi = 0$.

APPENDIX 1

# Useful Information

## I. BASIC DATA

### 1. Basic Constants

| | | | |
|---|---|---|---|
| $e = 2.7183$ | $1/e = 0.3679$ | $\log_e 2 = 0.6931$ | $\log_e 3 = 1.0986$ |
| $\pi = 3.1416$ | $1/\pi = 0.3183$ | $\log_e 10 = 2.3026$ | $\log_{10} e = 0.4343$ |
| $\sqrt{2} = 1.4142$ | $\sqrt{3} = 1.732$ | 1 rad. $= 57° 17' 45''$ | $1° = 0.0174$ rad. |

### 2. Conversion Factors

| | |
|---|---|
| 1 ft. = 30.48 cm = 0.3048 m | 1 m = 100 cm = 3.2804 ft. |
| 1 $ft^2$ = 0.0929 $m^2$ | 1 acre = 4840 $yd^2$ = 4046.77 $m^2$ |
| 1 $ft^3$ = 0.0283 $m^3$ | 1 $m^3$ = 35.32 $ft^3$ |
| 1 m/sec = 3.2804 ft/sec. | 1 mile/h = 1.609 km/h. |

### 3. Systems of Units

| *Quantity* | *F.P.S. system* | *C.G.S. system* | *M.K.S. system* |
|---|---|---|---|
| Length | foot (ft) | centrimetre (cm) | metre (m) |
| Mass | pound (lb) | gram (gm) | kilogram (kg) |
| Time | second (sec) | second (sec) | second (sec) |
| Force | lb. wt. | dyne | newton (nt) |

**Note.** The M.K.S. system is also known as the *International system of units (SI system).*

### 4. Greek Letters Used

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $\alpha$ | alpha | $\theta$ | theta | $\kappa$ | kappa | $\tau$ | tau |
| $\beta$ | beta | $\phi$ | phi | $\mu$ | mu | $\chi$ | chi |
| $\gamma$ | gamma | $\psi$ | psi | $\nu$ | nu | $\omega$ | omega |
| $\delta$ | delta | $\xi$ | xi | $\pi$ | pi | $\Gamma$ | cap. gamma |
| $\varepsilon$ | epsilon | $\eta$ | eta | $\rho$ | rho | $\Delta$ | cap. delta |
| $\iota$ | iota | $\zeta$ | zeta | $\sigma$ | sigma | $\Sigma$ | cap. sigma |
| | | $\lambda$ | lambda | | | | |

### 5. Some Notations

| | | | |
|---|---|---|---|
| $\in$ | belongs to | $\cup$ | union |
| $\notin$ | does not belong to | $\cap$ | intersection |
| $\Rightarrow$ | implies | $\ni$ | such that |
| $\Leftrightarrow$ | implies & implied by | | |

*Factorial n i.e.,* $n\,! = n(n-1)(n-2) \ldots\ldots 3\,.\,2\,.\,1.$

*Double factorials :* $(2n)!! = 2n\,(2n-2)\,(2n-4) \ldots\ldots 6\,.\,4\,.\,2.$

$$(2n-1)\,!! = (2n-1)\,(2n-3)\,(2n-5)\;\ldots\ldots\; 5\,.\,3\,.\,1.$$

*Stirling's approximation.* When $n$ is large $n\,! \sim \sqrt{2\pi n}\,.\,n^n\,e^{-n}$.

## II. ALGEBRA

**1. Quadratic equation :** $ax^2 + bx + c = 0$ has roots

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac)}}{2a},\ \beta = \frac{-b - \sqrt{(b^2 - 4ac)}}{2a}$$

$$\alpha + \beta = -\frac{b}{a},\ \alpha\beta = \frac{c}{a}.$$

*Roots are equal if* $b^2 - 4ac = 0$

*Roots are real and distinct if* $b^2 - 4ac > 0$

*Roots are imaginary if* $b^2 - 4ac < 0$

**2. Cubic equation :** $x^3 + lx^2 + mx + n = 0$

*Cardan's method :*

(*i*) Remove $x^2$ term by putting $y = x - (-l/3)$

(*ii*) Equate coeffs. in the new equation and $y^3 - 3uvy - (u^3 + v^3) - 0$ [$\because\ y = u + v$

(*iii*) Find $u^3$ and $v^3$. Then find $u$ and $v$.

(*iv*) Get $y = u + v$ and $x = y - l/3$.

**3. Biquadratic equation :** $x^4 + kx^3 + lx^2 + mx + n = 0$

I. *Ferrari's method :*

(*i*) Combine $x^4$ and $x^3$ terms into a perfect square by adding a term in $\lambda$.

(*ii*) Make R.H.S. a perfect square to find $\lambda$.

(*iii*) Solve resulting quadratic equations.

II. *Descarte's method :*

(*i*) Remove $x^3$ term by putting $y = x - (-k/4)$

(*ii*) Equate transformed expression to $(y^2 + py + q)\,(y^2 - py + q')$

(*iii*) Equate coeffs. of like powers from both sides.

(*iv*) Find $p$, $q$ and $q'$ and solve resulting quadratics.

**4. Cross-multiplication :**

$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0$$

Then $$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1}$$

**5. Method of least squares :**

(*i*) *Straight line of best fit* $y = a + bx$.

*Normal equations :* $\Sigma y = na + b\Sigma x,\ \Sigma xy = a\Sigma x + b\Sigma x^2$.

*To find a, b, solve these equations.*

(*ii*) *Parabola of best fit* $y = a + bx + cx^2$

*Normal equations :* $\Sigma y = na + b\Sigma x + c\Sigma x^2$,

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3,\ \Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$$

To find $a, b, c$, solve these equations.

**6. Progressions :**

(*i*) Numbers $a, a + d, a + 2d, \ldots$ are said to be in *Arithmetic progression (A.P.)*

Its $n$th term $T_n = a + \overline{n-1}\,d$ and *sum* $S_n = \dfrac{n}{2}\,(2a + \overline{n-1}\,d)$

(*ii*) Numbers $a, ar, ar^2, \ldots\ldots$, are said to be in *Geometric progression (G.P.)*

Its $n$th term $T_n = ar^{n-1}$ and *sum* $S_n = \dfrac{a(1-r^n)}{1-r},\ S_\infty = \dfrac{a}{1-r}\ (r < 1)$

(*iii*) Numbers $1/a$, $1/(a+d)$, $1/(a+2d)$, ...... are said to be in *Harmonic progression (H.P.)* (*i.e.*, a sequence is said to be in H.P. if its reciprocals are in A.P.)

Its $n$th term $T_n = 1/(a + \overline{n-1}\, d)$

(*iv*) If $a$ and $b$ be two numbers then their

*Arithmetic mean* $= \frac{1}{2}(a+b)$, *Geometric mean* $= \sqrt{ab}$, *Harmonic mean* $= 2ab/(a+b)$

(*v*) *Natural numbers* are 1, 2, 3, ......, $n$.

$$\Sigma n = \frac{n(n+1)}{2},\ \Sigma n^2 = \frac{n(n+1)(2n+1)}{6},\ \Sigma n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$$

**7. Permutations and Combinations**

$$^nP_r = \frac{n!}{(n-r)!};\ ^nC_r = \frac{n!}{r!(n-r)!} = \frac{^nP_r}{r!};\ ^nC_{n-r} = {^nC_r};\quad ^nC_0 = 1 = {^nC_n}.$$

**8. Binomial theorem**

(*i*) *When n is a positive integer*

$$(1+x)^n = 1 + {^nC_1}\,x + {^nC_2}\,x^2 + {^nC_3}\,x^3 + \ldots\ldots + {^nC_n}\,x^n.$$

(*ii*) *When n is a negative integer or a fraction*

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \ldots\ldots \infty.$$

(*iii*) *Binomial coefficients* : $^nC_r = \dfrac{n!}{r!(n-r)!}$

**9. Logarithms**

(*i*) *Naturals logarithm* $\log x$ has base $e$ and is inverse of $e^x$.

*Common logarithm* $\log_{10} x = M \log x$ where $M = \log_{10} e = 0.4343$

(*ii*) $\log_a 1 = 0$; $\log_a 0 = -\infty\ (a > 1)$ ; $\log_a a = 1$.

(*iii*) $\log(mn) = \log m + \log n$ ; $\log(m/n) = \log m - \log n$ ; $\log(m^n) = n \log m$.

**10. Partial Fractions**

A fraction of the form $\dfrac{a_0x^m + a_1x^{m-1} + \ldots\ldots + a_m}{b_0x^n + b_1x^{n-1} + \ldots\ldots + b_n}$

in which $m$ and $n$ are positive integers, is called a *rational algebraic fraction*. When the numerator is of a lower degree than the denominator, it is called a *proper fraction*.

*To resolve a given fraction into partial fractions*, we first factorise the denominator into real factors. These will be either linear or quadratic, and some factors repeated. Then the proper fraction is resolved into a sum of partial fractions such that

(*i*) to a *non-repeated linear factor* $x - a$ in the denominator corresponds a partial fraction of the form $A/(x-a)$;

(*ii*) to a *repeated linear factor* $(x-a)^r$ in the denominator corresponds the sum of $r$ partial fractions of the form $\dfrac{A_1}{x-a} + \dfrac{A_2}{(x-a)^2} + \dfrac{A_3}{(x-a)^3} + \ldots + \dfrac{A_r}{(x-a)^r}$;

(*iii*) to a *non-repeated quadratic factor* $(x^2 + ax + b)$ in the denominator, corresponds a partial fraction of the form $\dfrac{Ax+B}{x^2+ax+b}$;

(*iv*) to a *repeated quadratic factor* $(x^2 + ax + b)^r$ in the denominator, corresponds the sum of $r$ partial fractions of the form $\dfrac{A_1x+B_1}{x^2+ax+b} + \dfrac{A_2x+B_2}{(x^2+ax+b)^2} + \ldots\ldots + \dfrac{A_rx+B_r}{(x^2+ax+b)^r}$.

Then we have to determine the unknown constants $A, A_1, B_1$ etc.

*To obtain the partial fraction corresponding to the non-repeated linear factor $x - a$ in the denominator, put $x = a$ everywhere in the given fraction except in the fac tor $x - a$ itself.*

In all other cases, equate the given fraction to a sum of suitable partial fractions in accordance with (*i*) to (*iv*) above, having found the partial fractions corresponding to the non-repeated linear factors by the above rule. Then multiply both sides by the denominator of the given fraction and equate the coefficients of like powers of $x$ or substitute convenient numerical values of $x$ on both sides. Finally solve the simplest of the resulting equations to find the unknown constants.

**11. Matrices**

(*i*) *A system of mn numbers arranged in a rectangular array of m rows and n columns is called a* **matrix** *of order* $m \times n$.

In particular if $m = n$, it is called a *square matrix of order n.*

(*ii*) *Two matrices of the same order can be added or subtracted* by adding or subtracting the corresponding elements.

(*iii*) *Product of a matrix A by a scalar k* is a matrix whose each element is $k$ times the corresponding elements of $A$.

(*iv*) *Two matrices can be multiplied* only when the number of columns in the first is equal to the number of rows in the second. If $A$ is of order $m \times n$ and $B$ is of order $n \times p$, then the product $AB$ is a matrix of order $m \times p$, obtained by multiplying and adding the row elements of $A$ with the corresponding column elements of $B$.

(*v*) *Transpose of a matrix A* is the matrix obtained by interchanging its rows and columns and is denoted by $A'$.

A square matrix $A$ is said to be *symmetric* if $A = A'$ and *skew symmetric* if $A = -A'$.

(*vi*) If $A$ and $B$ are two square matrices such that $AB = I$ (*i.e.*, a unit matrix), then $B$ is called the *inverse* of $A$ and is denoted by $A^{-1}$. Then $AA^{-1} = A^{-1}A = I$.

(*vii*) *Rank of a matrix* is the largest order of any non-vanishing minor of the matrix.

(*viii*) *Consistency of a system of equations in n unknowns.*

If the rank of the coefficient matrix $A$ be $r$ and that of the augmented matrix $K$ be $r'$, then

(*a*) the equations are inconsistent (*i.e.* there is no solution) when $r \neq r'$,

(*b*) the equations are consistent when $r = r'$.

(*c*) the equations are consistent and there are infinite number of solutions when $r = r' < n$.

(*ix*) *Eigen values:* If $A$ is any square matrix of order $n$, then the determinant of the matrix $A - \lambda I_n$ equated to zero is called the *Characterist equation* of $A$ and its roots are called the *eigen values of A.*

(*x*) *Cayley Hamilton theorem*: Every square matrix satisfies its own characteristic equation.

**12. Determinants**

(*i*) A **determinant** is defined for a square matrix $A$ and is denoted by $|A|$. Unlike a matrix it has a single value *e.g.*,

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1 (b_2c_3 - b_3c_2) - b_1 (a_2c_3 - a_3c_2) + c_1 (a_2b_3 - a_3b_2)$$

In this way, determinant can be expanded in terms of any row or column.

(*ii*) **Properties :**

I. A determinant remains unaltered if its rows and columns are interchanged.

II. A determinant vanishes if two of its rows (or columns) are identical or proportional.

III. If each elements of a row (or column) consists of $m$ terms, the determinant can be expressed as the sum of $m$ determinants.

IV. If to each elements of a row (or column) be added equi-multiples of the corresponding elements of two or more rows (or columns), the determinant remains unaltered.

V. If the elements of a determinant $\Delta$ are functions of $x$ and two parallel lines become identical when $x = a$, then $x - a$ is a factor of $\Delta$.

## III. GEOMETRY

**1. Coordinates of a point** : *Cartesian* $(x, y)$ *and polar* $(r, \theta)$.

$$x = r\cos\theta, \qquad y = r\sin\theta$$

or

$$r = \sqrt{(x^2 + y^2)}, \qquad \theta = \tan^{-1}(y/x). \text{ (Fig. 0.1).}$$

*Distance between two points* $(x_1, y_1)$ and $(x_2, y_2) = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2]}$

*Point of division* of the line joining $(x_1, y_1)$ and $(x_2, y_2)$ in the ratio $m_1 : m_2$ is

$$\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right)$$

*In a triangle* having vertices $(x_1, y_1)$, $(x_2, y_2)$ and $(x_3, y_3)$

(*i*) *Area* $= \dfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

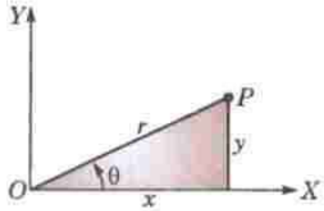


Fig. AP-1.1

(*ii*) *Centroid* (point of intersection of medians) is

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

(*iii*) *Incentre* (point of intersection of the internal bisectors of the angles) is

$$\left(\frac{ax_1 + bx_2 + cx_3}{a + b + c}, \frac{ay_1 + by_2 + cy_3}{a + b + c}\right)$$

where $a, b, c$ are the lengths of the sides of the triangle.

(*iv*) *Circumcentre* is the point of intersection of the right bisectors of the sides of the triangle.

(*v*) *Orthocentre* is the point of intersection of the perpendiculars drawn from the vertices to the opposite sides of the triangle.

**2. Straight Line**

(*i*) *Slope of the line* joining the points $(x_1, y_1)$ and $(x_2, y_2) = \dfrac{y_2 - y_1}{x_2 - x_1}$

Slope of the line $ax + by + c = 0$ is $-\dfrac{a}{b}$ *i.e.*, $-\dfrac{\text{coeff. of } x}{\text{coeff. of } y}$

(*ii*) *Equation of a line*

(*a*) having slope $m$ and cutting an intercept $c$ on $y$-axis is $y = mx + c$.

(*b*) cutting intercepts $a$ and $b$ from the axes is $\dfrac{x}{a} + \dfrac{y}{b} = 1$.

(*c*) passing through $(x_1, y_1)$ and having slope $m$ is $y - y_1 = m(x - x_1)$

(*d*) passing through $(x_1, y_1)$ and making an $\angle\theta$ with the $x$-axis is $\dfrac{x - x_1}{\cos\theta} = \dfrac{y - y_1}{\sin\theta} = r$

(*e*) through the point of intersection of the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is $a_1x + b_1y + c_1 + k(a_2x + b_2y + c_2) = 0$

(*iii*) *Angle between two lines* having slopes $m_1$ and $m_2$ is $\tan^{-1}\dfrac{m_1 - m_2}{1 + m_1m_2}$

| | | | |
|---|---|---|---|
| Two lines are parallel if | $m_1 = m_2$ | | |
| Two lines are perpendicular if | $m_1m_2 = -1$ | | |
| Any line parallel to the line | $ax + by + c = 0$ | is | $ax + by + k = 0$ |
| Any line perpendicular to | $ax + by + c = 0$ | is | $bx - ay + k = 0$ |

(*iv*) *Length of the perpendicular* from $(x_1, y_1)$ to the line $ax + by + c = 0$, is $\dfrac{ax_1 + by_1 + c}{\sqrt{(a^2 + b^2)}}$.

**3. Circle**

(*i*) *Equation of the circle* having centre $(h, k)$ and radius $r$ is $(x - h)^2 + (y - k)^2 = r^2$

(*ii*) *Equation* $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle having centre $(-g, -f)$ and radius $= \sqrt{(g^2 + f^2 - c)}$.

(iii) *Equation of the tangent* at the point $(x_1, y_1)$ to the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$.

(iv) *Condition for the line* $y = mx + c$ *to touch the circle* $x^2 + y^2 = a^2$ is $c = a\sqrt{(1 + m^2)}$.

(v) *Length of the tangent* from the point $(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $\sqrt{(x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c)}$.

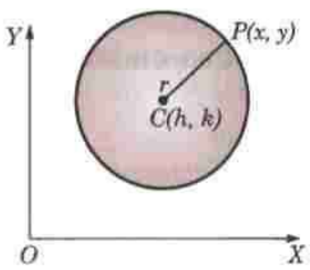


Fig. AP-1.2

## 4. Parabola

(i) *Standard equation of the parabola* $y^2 = 4ax$.

*Its parametric equations are* $x = at^2, y = 2at$.

*Latus-rectum* $LL' = 4a$, *Focus is* $S(a, 0)$

*Directrix* $ZM$ is $x + a = 0$.

*Focal distance* of any point $P(x_1, y_1)$ on the parabola

$$y^2 = 4ax \text{ is } SP = x_1 + a$$

*Equation of the tangent* at $(x_1, y_1)$ to the parabola $y^2 = 4ax$ is

$$yy_1 = 2a(x + x_1)$$

*Condition for the line* $y = mx + c$ *to touch the parabola*

$$y^2 = 4ax \text{ is } c = a/m.$$

*Equation of the normal to the parabola* $y^2 = 4ax$ *in terms of its slope* $m$ is

$$y = mx - 2am - am^3.$$

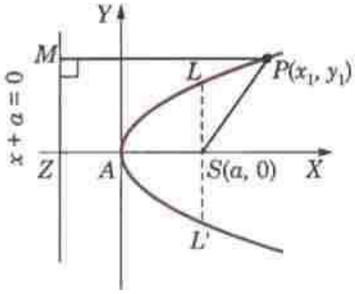
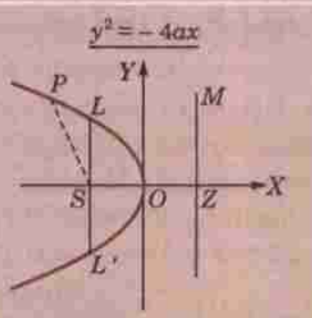
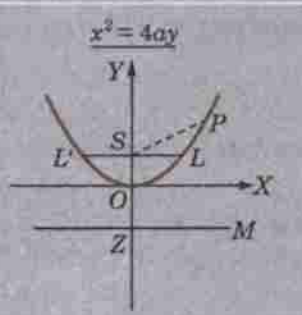
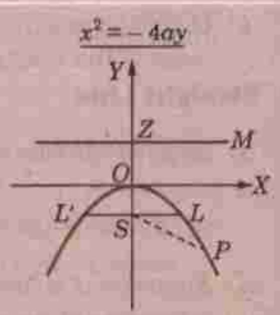


Fig. AP-1.3

(ii) *Other standard forms of parabola*

| $y^2 = -4ax$ | $x^2 = 4ay$ | $x^2 = -4ay$ |
|---|---|---|
|  |  |  |
| *Latus rectum* : $LL' = 4a$ | $LL' = 4a$ | $LL' = 4a$ |
| *Focus* : $S(-a, 0)$ | $S(0, a)$ | $S(0, -a)$ |
| Directrix $ZM : x = a$ | $ZM : y = -a$ | $ZM : y = a$ |
| *Focal distance* : $SP = a - x$ | $SP = a + y$ | $SP = a - y$ |

## 5. Ellipse

(i) *Standard equation of the ellipse* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ $(a > b > 0)$.

*It parametric equations are* $x = a\cos\theta, y = b\sin\theta$.

*Eccentricity* $e = \sqrt{(1 - b^2/a^2)}$,

*Latus-rectum* $LSL' = 2b^2/a$.

Foci $S(-ae, 0)$ and $S'(ae, 0)$

*Directrices* $ZM$ $(x = -\bar{a}/e)$ and $Z'M'$ $(x = a/e)$.

*Sum of the focal distances of any point on the ellipse is equal to the major axis i.e.,*

$$SP + S'P = 2a.$$

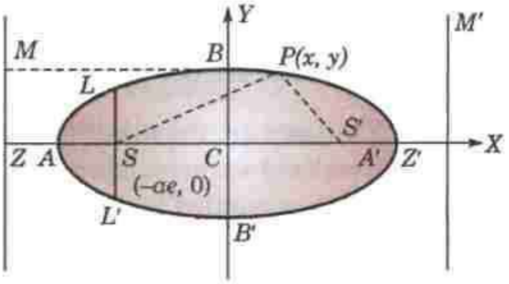


Fig. AP-1.4

*Equation of the tangent* at the point $(x_1, y_1)$ to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is } \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$$

*Condition for the line* $y = mx + c$ *to touch the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $c = \sqrt{(a^2 m^2 + b^2)}$

(*ii*) *Another standard form of ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b > 0)$

*Vertices:* $A$ (0o, $a$) ; $A'$ $(0, -a)$

*Foci* : $S$ $(0, ae)$ ; $S'$ $(0, -ae)$

*Directrices* : $ZM$ : $y = a/e$, $Z'M'$ : $y = -a/e$

*Latus rectum* : $LSL' = 2b^2/a$

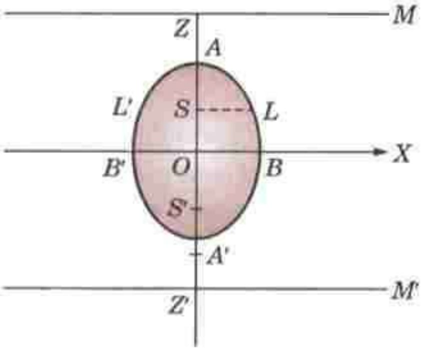



Fig. AP-1.4 (a)

### 6. Hyperbola

(*i*) *Standard equations of the hyperbola is*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Its parametric equations are

$x = a \sec \theta$, $\quad y = b \tan \theta$.

*Eccentricity* $e = \sqrt{(1 + b^2/a^2)}$,

*Latus-rectum* $LSL' = 2b^2/a$.

*Directrices*

$ZM$ $(x = a/e)$ and $Z'M'$ $(x = -a/e)$.

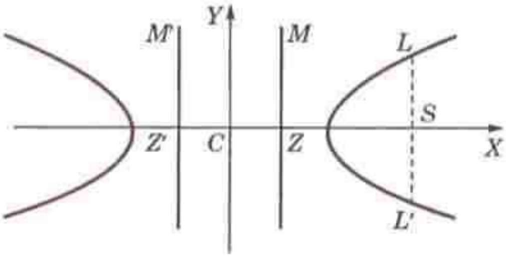



Fig. AP-1.5

(*ii*) *Equation of the tangent at the point* $(x_1, y_1)$ to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is } \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

(*iii*) *Condition for the line* $y = mx + c$ *to touch the hyperbola*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is } c = \sqrt{(a^2 m^2 - b^2)}$$

(*iv*) *Asymptotes of the hyperbola* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $\frac{x}{a} + \frac{y}{b} = 0$ and $\frac{x}{a} - \frac{y}{b} = 0$.

(*v*) *Equation of the rectangular hyperbola with asymptotes as axes is* $xy = c^2$.

Its parametric equations are $x = ct$, $y = c/t$.

### 7. Nature of a conic

*The equation* $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ *represents*

(*i*) *a pair of lines, if* $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ $(= \Delta) = 0$

(*ii*) *a circle, if* $a = b$, $h = 0$, $\Delta \neq 0$

(*iii*) *a parabola, if* $ab - h^2 = 0$, $\Delta \neq 0$.

(*iv*) *an ellipse, if* $ab - h^2 > 0$, $\Delta \neq 0$.

(*v*) *a hyperbola, if* $ab - h^2 < 0$, $\Delta \neq 0$,

*and a rectangular hyperbola if* in addition, $a + b = 0$.

## IV. SOLID GEOMETRY

1. (*i*) If $l, m, n$ be the *direction cosines* of a line then $l^2 + m^2 + n^2 = 1$.

If $a, b, c$ be the *direction ratios* of a line then $l = \frac{a}{\sqrt{\Sigma a^2}}$; $m = \frac{b}{\sqrt{\Sigma a^2}}$; $n = \frac{c}{\sqrt{\Sigma a^2}}$

(*ii*) If $\theta$ be the *angle between the lines having* d.c.'s $l, m, n$ and $l', m', n'$, then

$$\cos \theta = ll' + mm' + nn'$$

*Lines are perpendicular* if, $ll' + mm' + nn' = 0$

*Lines are parallel if* $l = l'$, $m = m'$, $n = n'$

(iii) *Projection of the line joining the points* $(x_1, y_1, z_1)$ *and* $(x_2, y_2, z_2)$ *on a line having d.c.'s* $l, m, n = l(x_1 - x_2) + m(y_1 - y_2) + n(z_1 - z_2)$.

**2. Plane**

(i) *Different forms of equation of a plane*

— *General form :* $ax + by + cz = d$

where $a, b, c$ are the d.r.s of a normal to the plane.

— *Normal form :* $lx + my + nz = p$

— *Intercept form :* $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$.

— *Any plane passing through the point* $(x_1, y_1, z_1)$ is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

(ii) *Angle* $\theta$ *between the planes* $ax + by + cz = d$ and $a'x + b'y + c'z = d'$ is given by

$$\cos\theta = \frac{aa' + bb' + cc'}{\sqrt{(a^2 + b^2 + c^2)}\sqrt{(a'^2 + b'^2 + c'^2)}}$$

*Planes are perpendicular if* $aa' + bb' + cc' = 0$

*Planes are parallel if* $a/a' = b/b' = c/c'$

(iii) *Any plane parallel to the plane* $ax + by + cz = d$ is $ax + by + cz = k$.

**3. Straight line**

(i) *Equation of the line* through the point $(x_1, y_1, z_1)$ having d.r.s $a, b, c$ is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \qquad \textit{(Symmetrical form)}$$

(ii) *Equation of the line* through the points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \qquad \textit{(Two point form)}$$

(iii) *Angle* $\theta$ *between the plane* $ax + by + cz = d$ *and the line*

$$\frac{x - x_1}{a'} = \frac{y - y_1}{b'} = \frac{z - z_1}{c'}$$

is
$$\sin\theta = \frac{aa' + bb' + cc'}{\sqrt{(a^2 + b^2 + c^2)}\sqrt{(a'^2 + b'^2 + c'^2)}}$$

*Line is parallel to the plane* if $aa' + bb' + cc' = 0$

*Line is perpendicular to the plane* if $a/a' = b/b' = c/c'$

(iv) *Coplanar lines*

Two lines $\dfrac{x - x_1}{l_1} = \dfrac{y - y_1}{m_1} = \dfrac{z - z_1}{n_1}$ and $\dfrac{x - x_2}{l_2} = \dfrac{y - y_2}{m_2} = \dfrac{z - z_2}{n_2}$

are coplanar if
$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

and equation of the plane containing these lines is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

(v) *Shortest distance* between two skew lines

$$\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1} \text{ and } \frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$$

is
$$l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$$

where $l, m, n$ are given by $ll_1 + mm_1 + nn_1 = 0$ and $ll_2 + mm_2 + nn_2 = 0$

*Equation of the line of S.D.* is given by

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 \text{ and } \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix} = 0$$

**4. Sphere**

(*i*) *Equation of the sphere* having centre $(a, b, c)$ and radius $r$ is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

(*ii*) Equation $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ represents a sphere having *centre* $(-u, -v, -w)$ and *radius* $= \sqrt{(u^2 + v^2 + w^2 - d)}$

(*iii*) *Equation of the sphere* having the points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ as the ends of a diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$

(*iv*) *Equation of a circle* (*i.e.*, section of a sphere $S = 0$ by the plane $U = 0$) is given by $S = 0$ and $U = 0$ taken together.

(*v*) *Equation of any sphere* through the circle of intersection of the sphere $S = 0$ and the plane $U = 0$ is $S + kU = 0$.

(*vi*) *Tangent plane* at any point $(x_1, y_1, z_1)$ of the sphere

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \text{ is}$$

$$xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0$$

(*vii*) Two spheres $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ and

$$x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z + d' = 0 \text{ cut orthogonally if } 2uu' + 2vv' + 2ww' = d + d'.$$

**5. Cone**

(*i*) *Equation of a cone* with vertex at the origin is a homogeneous equation of the second degree in $x, y, z$.

(*ii*) *Enveloping cone* of the sphere $S = 0$ with vertex $(x_1, y_1, z_1)$ is $SS_1 = T^2$ where $S = x^2 + y^2 + z^2 - a^2$, $S_1 = x_1^2 + y_1^2 + z_1^2 - a^2$, $T = xx_1 + yy_1 + zz_1 - a^2$

**6. Quadric surfaces**

(*i*) *Ellipoid:* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

(*ii*) *Hyperboloid* of one sheet: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$

*Hyperboloid* of two sheets: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = -1$

(*iii*) *Cone:* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 0$

(*iv*) *Elliptic paraboloid:* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{2z}{c}$

*Hyperbolic paraboloid:* $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = \dfrac{2z}{c}$

**7. Volumes and surface areas**

| *Solid* | *Volume* | *Curved surface area* | *Total surface area* |
|---|---|---|---|
| **Cube** (side $a$) | $a^3$ | $4a^2$ | $6a^2$ |
| **Cuboid** (length $l$, breadth $b$, height $h$) | $lbh$ | $2(l + b)h$ | $2(lb + bh + hl)$ |
| **Sphere** (radius $r$) | $\frac{4}{3}\pi r^3$ | — | $4\pi r^2$ |
| **Cylinder** (base radius $r$, height $h$) | $\pi r^2 h$ | $2\pi rh$ | $2\pi r(r + h)$ |
| **Cone** (base radius $r$, height $h$) | $\frac{1}{3}\pi r^2 h$ | $\pi rl$ | $\pi r(r + l)$ |
| where slant height $l$ is given by $l = \sqrt{(r^2 + h^2)}$. | | | |

## V. TRIGONOMETRY

1.

| $\theta° =$ | 0 | 30 | 45 | 60 | 90 | 180 | 270 | 360 |
|---|---|---|---|---|---|---|---|---|
| $\sin\theta$ | 0 | 1/2 | $1/\sqrt{2}$ | $\sqrt{3}/2$ | 1 | 0 | $-1$ | 0 |
| $\cos\theta$ | 1 | $\sqrt{3}/2$ | $1/\sqrt{2}$ | 1/2 | 0 | $-1$ | 0 | 1 |
| $\tan\theta$ | 0 | $1/\sqrt{3}$ | 1 | $\sqrt{3}$ | $\infty$ | 0 | $-\infty$ | 0 |

2. **Signs and variations of t-ratios**

| *Quadrant* | *sin θ* | *cos θ* | *tan θ* |
|---|---|---|---|
| I | + (0 to 1) | + (1 to 0) | + (0 to ∞) |
| II | + (1 to 0) | − (0 to − 1) | − (− ∞ to 0) |
| III | − (0 to − 1) | − (− 1 to 0) | + (0 to ∞) |
| IV | − (− 1 to 0) | + (0 to 1) | − (− ∞ to 0) |

3. *Any t-ratio of* $(n \,.\, 90° \pm \theta) = \pm$ *same ratio of* $\theta$*, when n is even.*

$= \pm$ *co-ratio of* $\theta$*, when n is odd.*

*The sign + or − is to be decided from the quadrant in which* $n \,.\, 90° \pm \theta$ *lies.*

*e.g.,* $\sin 570° = \sin(6 \times 90° + 30°) = -\sin 30° = -\dfrac{1}{2}$,

$\tan 315° = \tan(3 \times 90° + 45°) = -\cot 45° = -1$

4. $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$$\sin 2A = 2\sin A\cos A = 2\tan A/(1 + \tan^2 A)$$

$$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1 = \frac{1-\tan^2 A}{1+\tan^2 A}$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B};\ \tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

5. $\sin A \cos B = \dfrac{1}{2}[\sin(A + B) + \sin(A - B)]$

$\cos A \sin B = \dfrac{1}{2}[\sin(A + B) - \sin(A - B)]$

$\cos A \cos B = \dfrac{1}{2}[\cos(A + B) + \cos(A - B)]$

$\sin A \sin B = \dfrac{1}{2}[\cos(A - B) - \cos(A + B)]$

6. $\sin C + \sin D = 2\sin\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$

$\sin C - \sin D = 2\cos\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

$\cos C + \cos D = 2\cos\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$

$\cos C - \cos D = -2\sin\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

7. $\sin 3A = 3 \sin A - 4 \sin^3 A$, $\cos 3A = 4 \cos^3 A - 3 \cos A$; $\tan 3A = \dfrac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$

8. $$a \sin x + b \cos x = r \sin (x + \theta)$$
$$a \cos x + b \sin x = r \cos (x - \theta)$$
where $a = r \cos \theta$, $b = r \sin \theta$ so that $r = \sqrt{(a^2 + b^2)}$, $\theta = \tan^{-1} (b/a)$

9. *In any* $\Delta ABC$ :

(*i*) $a/\sin A = b/\sin B = c/\sin C$ (*Sine formula*)

(*ii*) $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ (*Cosine formula*)

(*iii*) $a = b \cos C + c \cos B$ (*Projection formula*)

(*iv*) *Area of* $\Delta ABC = \dfrac{1}{2} bc \sin A = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \dfrac{1}{2}(a + b + c)$.

10. **Series**

(*i*) *Exponential series* : $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots\ldots \infty$

(*ii*) *Sin, cos, sinh, cosh series*

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots \infty, \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots \infty$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots \infty, \qquad \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots \infty$$

(*iii*) *Log series*

$$\log (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots \infty, \qquad \log (1 - x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\ldots \infty\right)$$

(*iv*) *Gregory series*

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\ldots \infty, \qquad \tanh^{-1} x = \frac{1}{2} \log \frac{1+x}{1-x} = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots \infty$$

11. (*i*) **Complex number :** $z = x + iy = r (\cos \theta + i \sin \theta) = r e^{i\theta}$ [see Fig. AP-1.1]

(*ii*) **Euler's theorem :** $\cos \theta + i \sin \theta = e^{i\theta}$

(*iii*) **Demoivre's theorem :** $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$

12. (*i*) **Hyperbolic functions :** (*i*) $\sinh x = \dfrac{e^x - e^{-x}}{2}$; $\cosh x = \dfrac{e^x + e^{-x}}{2}$;

$$\tanh x = \frac{\sinh x}{\cosh x};\ \coth x = \frac{\cosh x}{\sinh x};\ \text{sech}\ x = \frac{1}{\cosh x};\ \text{cosech}\ x = \frac{1}{\sinh x}$$

(*ii*) *Relations between hyperbolic and circular functions :*

$$\sin ix = i \sinh x;\ \cos ix = \cosh x;\ \tan ix = i \tanh x.$$

(*iii*) *Inverse hyperbolic functions:*

$$\sinh^{-1} x = \log [x + \sqrt{x^2 + 1}];\ \cosh^{-1} x = \log [x + \sqrt{(x^2 - 1)}];\ \tanh^{-1} x = \frac{1}{2} \log \frac{1+x}{1-x}.$$

## VI. DIFFERENTIAL CALCULUS

1. **Standard limits :**

(*i*) $\underset{x \to a}{Lt} \dfrac{x^n - a^n}{x - a} = na^{n-1}$, $n$ any rational number

(*ii*) $\underset{x \to 0}{Lt} \dfrac{\sin x}{x} = 1$

(*iii*) $\underset{x \to 0}{Lt} (1 + x)^{1/x} = e$

(*iv*) $\underset{x \to \infty}{Lt} x^{1/x} = 1$

(*v*) $\underset{x \to 0}{Lt} \dfrac{a^x - 1}{x} = \log_e a$

**2. Differentiation**

(i) $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$ $\qquad$ $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\,du/dx - u\,dv/dx}{v^2}$

$\frac{du}{dx} = \frac{du}{dy}\cdot\frac{dy}{dx}$ *(Chain Rule)* $\qquad$ $\frac{d}{dx}(ax+b)^n = n(ax+b)^{n-1}\,.\,a$

(ii) $\frac{d}{dx}(e^x) = e^x$ $\qquad$ $\frac{d}{dx}(a^x) = a^x \log_e a$

$\frac{d}{dx}(\log_e x) = 1/x$ $\qquad$ $\frac{d}{dx}(\log_a x) = \frac{1}{x\log a}$

(iii) $\frac{d}{dx}(\sin x) = \cos x$ $\qquad$ $\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\tan x) = \sec^2 x$ $\qquad$ $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$ $\qquad$ $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

(iv) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{(1-x^2)}}$ $\qquad$ $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{(1-x^2)}}$

$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ $\qquad$ $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{(x^2-1)}}$ $\qquad$ $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{x\sqrt{(x^2-1)}}$

(v) $\frac{d}{dx}(\sinh x) = \cosh x$ $\qquad$ $\frac{d}{dx}(\cosh x) = \sinh x$.

$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$ $\qquad$ $\frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x$

(vi) $D^n(ax+b)^m = m(m-1)(m-2)\ldots\ldots(m-n+1)(ax+b)^{m-n}\,.\,a^n$

$D^n \log(ax+b) = (-1)^{n-1}(n-1)!\,a^n/(ax+b)^n$

$D^n(e^{mx}) = m^n e^{mx}$ $\qquad$ $D^n(a^{mx}) = m^n(\log a)^n\,.\,a^{mx}$

$$D^n\begin{bmatrix}\sin(ax+b)\\ \cos(ax+b)\end{bmatrix} = a^n\begin{bmatrix}\sin(ax+b+n\pi/2)\\ \cos(ax+b+n\pi/2)\end{bmatrix}$$

$$D^n e^{ax}\begin{bmatrix}\sin(bx+c)\\ \cos(bx+c)\end{bmatrix} = (a^2+b^2)^{n/2} e^{ax}\begin{bmatrix}\sin(bx+c+n\tan^{-1} b/a)\\ \cos(bx+c+n\tan^{-1} b/a)\end{bmatrix}$$

(vii) *Leibnitz theorem:* $(uv)_n = u_n + {}^nC_1 u_{n-1}v_1 + {}^nC_2 u_{n-2}v_2 + \ldots\ldots + {}^nC_r u_{n-r}v_r + \ldots\ldots + {}^nC_n v_n$

**3.** (i) *Maclaurin's series* : $f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$

(ii) *Taylor's series* : $f(x+a) = f(a) + \frac{x}{1!}f'(a) + \frac{x^2}{2!}f''(a) + \frac{x^3}{3!}f'''(a) + \ldots$

**4. Curvature**

(i) *Radius of curvature* $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$, $\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$ ; $\rho = r\frac{dr}{dp}$.

(ii) *Centre of curvature* : $\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$, $\bar{y} = y + \frac{1}{y_2}(1+y_1^2)$.

(iii) **Evolute** *is the locus of the centre of curvature of a curve.* The curve is called the *involute* of the evolute.

(iv) **Envelope** *of a curve $f(x, y, \alpha) = 0$ is the '$\alpha$' eliminant from*

$$f(x,y,\alpha) = 0 \text{ and } \frac{\partial f}{\partial \alpha}(x,y,\alpha) = 0.$$

*The envelope of the normals to a curve is its* ***evolute.***

**5. Asymptotes**

(*i*) *Asymptotes parallel to x-axis are obtained by equating to zero the coefficient of the highest power of x in the equation,* provided this is not merely a constant.

*Asymptotes parallel to y-axis are obtained by equating to zero the coefficient of highest power of y in the equation,* provided this is not merely a constant.

(*ii*) *Oblique asymptotes are obtained as follows:*

Put $x = 1, y = m$ in the highest degrees terms getting $\phi_n(m)$

Put $\phi_n(m) = 0$ and find the values of $m$.

Find $c$ from $c = -\phi_{n-1}(m)/\phi_n'(m)$

If two values of $m$ are equal, then find $c$ from

$$\frac{c^2}{2}\phi_n''(m) + c\phi'_{n-1}(m) + \phi_{n-2}(m) = 0$$

The asymptotes is $y = mx + c$.

(*iii*) *Asymptotes of polar curve* $1/r = f(\theta)$ is $r \sin(\theta - \alpha) = 1/f'(\alpha)$ where $\alpha$ is a root of $f(\theta) = 0$.

**6. Curve tracing**

(*i*) *A curve is symmetrical about x-axis,* if only even powers of $y$ occur in its equation.

(*ii*) *A curve is symmetrical about y-axis,* if only even powers of $x$ occur in its equation.

(*iii*) *A curve is symmetrical about the line $y = x$,* if on interchanging $x$ and $y$, its equation remains unchanged.

(*iv*) *A curve passes through the origin,* if there is no constant term in its equation.

(*v*) *Tangents to curve at the origin* are found by equating to zero the lowest degree terms.

**7. Partial Differentiation**

(*i*) *Euler's theorem.* If $u$ is a homogeneous function in $x$ and $y$ of degree $n$, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu\ ; \quad x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)\,u$$

(*ii*) *Chain rule* : $\dfrac{du}{dt} = \dfrac{\partial u}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial u}{\partial y}\dfrac{dy}{dt}$, if $u = f(x, y), x = \phi(t), y = \psi(t)$.

(*iii*) $\dfrac{dy}{dx} = -\dfrac{\partial\phi}{\partial x}\Big/\dfrac{\partial\phi}{\partial y}$, if $\phi(x, y) = c$

(*iv*) *Jacobian* $J\left(\dfrac{u, v}{x, y}\right) = \dfrac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}$

If $J = \partial(u, v)/\partial(x, y)$ and $J' = \partial(x, y)/\partial(u, v)$, then $JJ' = 1$

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(r, s)} \cdot \frac{\partial(r, s)}{\partial(x, y)}$$

(*v*) *Taylor's series* : $f(a + h, b + k) = f(a, b) + \left(h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y}\right)f + \dfrac{1}{2!}\left(h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y}\right)^2 f + \ldots\ldots$

(*vi*) *Maxima Minima* (*a*) $\dfrac{\partial f}{\partial x} = 0, \dfrac{\partial f}{\partial y} = 0$

(*b*) $\dfrac{\partial^2 f}{\partial x^2}\dfrac{\partial^2 f}{\partial y^2} > \dfrac{\partial^2 f}{\partial x \partial y}$ ; $\dfrac{\partial^2 f}{\partial x^2} < 0$ for maximum $\dfrac{\partial^2 f}{\partial x^2} > 0$ for minimum.

(*vii*) *Leibnitz's Rule* $\dfrac{d}{d\alpha}\left\{\displaystyle\int_a^b f(x, \alpha)\, d\alpha\right\} = \displaystyle\int_a^b \dfrac{\partial f(x, \alpha)}{\partial \alpha}\, d\alpha$

where $f(x, \alpha)$ and $\dfrac{\partial f(x, \alpha)}{\partial \alpha}$ are continuous functions of $x$ and $\alpha$ and $a, b$ are constants.

## VII. INTEGRAL CALCULUS

### 1. Integration

(i) $\int x^n\,dx = \dfrac{x^{n+1}}{n+1}\ (n \neq -1)$ $\quad\int \dfrac{1}{x}\,dx = \log_e x$

$\int e^x\,dx = e^x$ $\quad\int a^x\,dx = a^x/\log_e a$

(ii) $\int \sin x\,dx = -\cos x$ $\quad\int \cos x\,dx = \sin x$

$\int \tan x\,dx = -\log\cos x$ $\quad\int \cot x\,dx = \log\sin x$

$\int \sec x\,dx = \log(\sec x + \tan x)$ $\quad\int \operatorname{cosec} x\,dx = \log(\operatorname{cosec} x - \cot x)$

$\int \sec^2 x\,dx = \tan x$ $\quad\int \operatorname{cosec}^2 x\,dx = -\cot x$

(iii) $\int \dfrac{dx}{a^2+x^2} = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a}$ $\quad\int \dfrac{dx}{\sqrt{(a^2-x^2)}} = \sin^{-1}\dfrac{x}{a}$

$\int \dfrac{dx}{a^2-x^2} = \dfrac{1}{2a}\log\dfrac{a+x}{a-x}$ $\quad\int \dfrac{dx}{\sqrt{(a^2+x^2)}} = \sinh^{-1}\dfrac{x}{a}$

$\int \dfrac{dx}{x^2-a^2} = \dfrac{1}{2a}\log\dfrac{x-a}{x+a}$ $\quad\int \dfrac{dx}{\sqrt{(x^2-a^2)}} = \cosh^{-1}\dfrac{x}{a}$

(iv) $\int \sqrt{(a^2-x^2)}\,dx = \dfrac{x\sqrt{(a^2-x^2)}}{2} + \dfrac{a^2}{2}\sin^{-1}\dfrac{x}{a}$

$\int \sqrt{(a^2+x^2)}\,dx = \dfrac{x\sqrt{(a^2+x^2)}}{2} + \dfrac{a^2}{2}\sinh^{-1}\dfrac{x}{a}$

$\int \sqrt{(x^2-a^2)}\,dx = \dfrac{x\sqrt{(x^2-a^2)}}{2} - \dfrac{a^2}{2}\cosh^{-1}\dfrac{x}{a}$

(v) $\int e^{ax}\sin bx\,dx = \dfrac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

$\int e^{ax}\cos bx\,dx = \dfrac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

(vi) $\int \sinh x\,dx = \cosh x$ $\quad\int \cosh x\,dx = \sinh x$

$\int \tanh x\,dx = \log\cosh x$ $\quad\int \coth x\,dx = \log\sinh x$

$\int \operatorname{sech}^2 x\,dx = \tanh x$ $\quad\int \operatorname{cosech}^2 x\,dx = -\coth x$

(vii) $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2}\cos^n x\,dx = \dfrac{(n-1)(n-3)(n-5)\ldots\ldots}{n(n-2)(n-4)\ldots\ldots}\times\left(\dfrac{\pi}{2}, \text{only if } n \text{ is even}\right)$

$\int_0^{\pi/2} \sin^m x\cos^n x\,dx = \dfrac{(m-1)(m-3)\ldots\ldots\times(n-1)(n-3)\ldots\ldots}{(m+n)(m+n-2)(m+n-4)\ldots\ldots}\times\left(\dfrac{\pi}{2}, \text{only if both } m \text{ and } n \text{ are even}\right)$

(viii) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx,$ if $f(x)$ is an *even function*

$= 0,$ if $f(x)$ is an *odd function.*

$\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx,$ if $f(2a-x) = f(x)$

$= 0,$ if $f(2a-x) = -f(x).$

**2. Lengths of curves**

(*i*) *Lenth of curve* $y = f(x)$ *between* $x = a$, $x = b$ *is* $\int_a^b \sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}\,dx$

(*ii*) *Length of curve* $x = f(y)$ *between* $y = a$, $y = b$ *is* $\int_a^b \sqrt{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}}\,dy$

(*iii*) *Length of curve* $x = f(t)$, $y = \phi(t)$ *between* $t = t_1$, $t = t_2$ *is* $\int_{t_1}^{t_2} \sqrt{\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right\}}\,dt$

(*iv*) *Length of curve* $r = f(\theta)$ *between* $\theta = \alpha$, $\theta = \beta$ *is* $\int_\alpha^\beta \sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}\,d\theta$

**3. Areas of curves**

(*i*) *Area bounded by* $y = f(x)$, *x-axis and* $x = a$, $x = b$ *is* $\int_a^b y\,dx$

(*ii*) *Area bounded by* $x = f(y)$, *y-axis* and $y = a$, $y = b$ *is* $\int_a^b x\,dy$

(*iii*) *Area bounded by* $r = f(\theta)$ and *lines* $\theta = \alpha$, $\theta = \beta$ *is* $\frac{1}{2}\int_\alpha^\beta r^2\,d\theta$

**4. Volumes of revolution**

(*i*) *Volume of revolution about x-axis of area bounded by* $y = f(x)$, *x-axis* and $x = a$, $x = b$ *is*

$$\int_a^b \pi y^2\,dx$$

(*ii*) *Volume of revolution about y-axis of area bounded by* $x = f(y)$, *y-axis and* $y = a$, $y = b$ *is*

$$\int_a^b \pi x^2\,dy$$

(*iii*) *Volume of revolution bounded by* $r = f(\theta)$ and $\theta = \alpha$, $\theta = \beta$

(*a*) *about OX* $= \int_\alpha^\beta \frac{2\pi}{3} r^3 \sin\theta\,d\theta$ (*b*) *about OY* $= \int_\alpha^\beta \frac{2\pi}{3} r^3 \cos\theta\,d\theta$

**5. Surface areas of revolution**

(*i*) *Surface area of revolution about x-axis of curve* $y = f(x)$ *from* $x = a$ *to* $x = b$ *is*

$$S = \int_{x=a}^{x=b} 2\pi y\,ds$$

*Cartesian form :* $S = \int_a^b 2\pi y \frac{ds}{dx}\,dx$ where $\frac{ds}{dx} = \sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}$

*Parametric form :* $S = \int 2\pi y \frac{ds}{dt}\,dt$ where $\frac{ds}{dt} = \sqrt{\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right\}}$

*Polar form :* $S = \int 2\pi y \frac{ds}{d\theta}\,d\theta$ where $\frac{ds}{d\theta} = \sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}$

(*ii*) *Surface area of revolution about y-axis is* $\int 2\pi x\,ds$.

**6. Multiple integrals**

(*i*) *Area* $= \int_{x_1}^{x_2}\int_{y_1}^{y_2} dx\,dy$; *Volume* $= \int_{x_1}^{x_2}\int_{y_1}^{y_2} z\,dx\,dy$ *or* $\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} dx\,dy\,dz$

(*ii*) *C.G. of a plane lamina:* $\bar{x} = \frac{\iint x\rho\,dx\,dy}{\iint \rho\,dx\,dy}$, $\bar{y} = \frac{\iint y\rho\,dx\,dy}{\iint \rho\,dx\,dy}$

*C.G. of a solid* $\bar{x} = \dfrac{\iiint x\rho \, dx \, dy \, dz}{\iiint \rho \, dx \, dy \, dz}, \ \bar{y} = \dfrac{\iiint y\rho \, dx \, dy \, dz}{\iiint \rho \, dx \, dy \, dz}, \ \bar{z} = \dfrac{\iiint z\rho \, dx \, dy \, dz}{\iiint \rho \, dx \, dy \, dz}$

(*iii*) *Centre of pressure* $\bar{x} = \dfrac{\iint_A px \, dx \, dy}{\iint_A p \, dx \, dy}, \ \bar{y} = \dfrac{\iint_A py \, dx \, dy}{\iint_A p \, dx \, dy}$

(*iv*) *M.I. about x-axis i.e.,* $I_x = \iiint_V \rho (y^2 + z^2) \, dx \, dy \, dz$

*M.I. about y-axis i.e.,* $I_y = \iiint_V \rho (z^2 + x^2) \, dx \, dy \, dz$

*M.I. about z-axis i.e.,* $I_z = \iiint_V \rho (x^2 + y^2) \, dx \, dy \, dz$

7. *Gamma function* $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} \, dx = (n-1)!, \Gamma(n+1) = n\,\Gamma(n) = n!, \Gamma(1/2) = \sqrt{\pi}$

*Beta function* $\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx = \dfrac{\Gamma(m) \ \Gamma(n)}{\Gamma(m+n)} \ (m > 0, n > 0)$

## VIII. VECTORS

1. (*i*) If $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$ then $r = |\mathbf{R}| = \sqrt{(x^2 + y^2 + z^2)}$

   (*ii*) $\vec{PQ}$ = Position vector of $Q$ – position vector of $P$.

2. If $\mathbf{A} = a_1\mathbf{I} + a_2\mathbf{J} + a_3\mathbf{K}, \mathbf{B} = b_1\mathbf{I} + b_2\mathbf{J} + b_3\mathbf{K}$, then

   (*i*) *Scalar product:* $\mathbf{A} \cdot \mathbf{B} = ab \cos\theta = a_1 b_1 + a_2 b_2 + a_3 b_3$

   (*ii*) *Vector product:* $\mathbf{A} \times \mathbf{B} = ab \sin\theta \, \hat{\mathbf{N}}$ = Area of the parallelogram having **A** and **B** as sides

$$= \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

   (*iii*) $\mathbf{B} \perp \mathbf{A}$ if $\mathbf{A} \cdot \mathbf{B} = 0$ and **A** is parallel to **B** if $\mathbf{A} \times \mathbf{B} = \mathbf{0}$

3. (*i*) *Scalar triple product* $[\mathbf{A}\,\mathbf{B}\,\mathbf{C}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ = Volume of parallelopiped

   (*ii*) If $[\mathbf{A}\,\mathbf{B}\,\mathbf{C}] = 0$, then **A, B, C** are coplanar.

   (*iii*) *Vector triple product* $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\,\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\,\mathbf{C}$

   $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{C} \cdot \mathbf{A})\,\mathbf{B} - (\mathbf{C} \cdot \mathbf{B})\,\mathbf{A}$

4. (*i*) *grad* $f = \nabla f = \dfrac{\partial f}{\partial x}\mathbf{I} + \dfrac{\partial f}{\partial y}\mathbf{J} + \dfrac{\partial f}{\partial z}\mathbf{K}$

   *div* $\mathbf{F} = \nabla F = \dfrac{\partial f_1}{\partial x} + \dfrac{\partial f_2}{\partial y} + \dfrac{\partial f_3}{\partial z}$

   *curl* $\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$ where $\mathbf{F} = f_1\mathbf{I} + f_2\mathbf{J} + f_3\mathbf{K}$

   (*ii*) If *div* $\mathbf{F} = 0$, then **F** is called a *solenoidal vector.*

   (*iii*) If *curl* $\mathbf{F} = \mathbf{0}$ then **F** is called an *irrotational vector*

5. *Velocity* = $d\mathbf{R}/dt$ ; *Acceleration* = $d^2\mathbf{R}/dt^2$ ; *Tangent vector* = $d\mathbf{R}/dt$ ; *Normal vector* = $\nabla \phi$

6. **Green's theorem:** $\int_C (\phi \, dx + \psi \, dy) = \iint_C \left( \dfrac{\partial \psi}{\partial x} - \dfrac{\partial \phi}{\partial y} \right) dxdy$

**Stoke's theorem:** $\int_C \mathbf{F} \cdot d\mathbf{R} = \int_S curl\, \mathbf{F} \cdot \mathbf{N}\, ds$

**Gauss divergence theorem:** $\int_S \mathbf{F} \cdot \mathbf{N}\, ds = \int_v div\, \mathbf{F}\, dv$

**7. Coordinate systems**

| | *Polar coordinates* $(r, \theta)$ | *Cylindrical coordinates* $(\rho, \phi, z)$ | *Spherical polar coordinates* $(r, \theta, \phi)$ |
|---|---|---|---|
| *Coordinate transformations* | $x = r\cos\theta$<br>$y = r\sin\theta$ | $x = \rho\cos\phi$<br>$y = \rho\sin\phi$<br>$z = z$ | $x = r\sin\theta\cos\phi$<br>$y = r\sin\theta\sin\phi$<br>$z = r\cos\theta$ |
| *Jacobian* | $\frac{\partial(x, y)}{\partial(r, \theta)} = r$ | $\frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \rho$ | $\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2\sin\theta$ |
| *(Arc-length)*$^2$ | $(ds)^2 = (dr)^2 + r^2(d\theta)^2$<br>$dxdy = r\, d\theta\, dr$ | $(ds)^2 = (d\rho)^2 + \rho^2(d\phi)^2 + (dz)^2$ | $(ds)^2 = (dr)^2 + r^2(d\theta)^2 + (r\sin\theta)^2(d\phi)^2$ |
| *Volume-element* | | $dV = \rho\, d\rho\, d\phi\, dz$ | $dV = r^2\sin\theta\, dr\, d\theta\, d\phi$ |

## IX. DIFFERENTIAL EQUATIONS

**1. Equations of first order**

(*i*) *Variables separable* : $f(y)\, dy/dx = \phi(x)$, $\int f(y)\, dy = \int \phi(x)\, dx + c$.

(*ii*) *Homogeneous equation* $dy/dx = f(x, y)/\phi(x, y)$ where $f(x, y)$ and $\phi(x, y)$ are of the same degree.
Put $y = vx$ so that $dy/dx = v + x\, dv/dx$.

(*iii*) *Equations reducible to homogenous form* : $\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'}$

*When* $a/a' \neq b/b'$, put $x = X + h$, $y = Y + k$

*When* $a/a' = b/b'$, put $ax + by = t$.

(*iv*) *Leibnitz's linear equation* : $\frac{dy}{dx} + Py = Q$ where $P$, $Q$ are functions of $x$.

$I.F. = e^{\int P\, dx}$, then *solution* is $y\,(I.F.) = \int Q\,(I.F.)\, dx + c$.

(*v*) *Bernoulli's equation* : $dy/dx + Py = Qy^n$, reducible to Leibnitz's equation by writing it as

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q \text{ and putting } y^{1-n} = z.$$

(*vi*) *Exact equation* : $M(x, y)\, dx + N(x, y)\, dy = 0$

*Solution* is $\int_{(y\ cons.)} M\, dx + \int$ (terms of $N$ not containing $x$) $dy = c$, provided $\partial M/\partial y = \partial N/\partial x$.

(*vii*) *Clairaut's equation* : $y = px + f(p)$ where $p = dy/dx$.
Solution is obtained on replacing $p$ by $c$.

**2. Linear equations with constant coefficients**

$$\frac{d^n y}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots + k_{n-1}\frac{dy}{dx} + k_n y = X$$

*Symbolic form* : $(D^n + k_1 D^{n-1} + \ldots. + k_{n-1}D + k_n)\, y = X$.

**I. To find C.F.**

| Roots of A.E. | C.F. |
|---|---|
| (i) $m_1, m_2, m_3, \ldots\ldots$ | $c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \ldots$ |
| (ii) $m_1, m_1, m_3, \ldots\ldots$ | $(c_1 + c_2 x)\, e^{m_1 x} + c_3 e^{m_3 x} + \ldots$ |
| (iii) $\alpha + i\beta, \alpha - i\beta, m_3, \ldots\ldots$ | $e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_3 x} + \ldots$ |
| (iv) $\alpha \pm i\beta, \alpha \pm i\beta, m_5, \ldots\ldots$ | $e^{\alpha x}\, [(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x] + c_5 e^{m_5 x} + \ldots$ |

**II. To find P.I.**

(i) $X = e^{ax}$, $\quad$ P.I. $= \dfrac{1}{f(D)} e^{ax}$, put $D = a$, $\qquad [f(a) \neq 0$

$= x \dfrac{1}{f'(D)} e^{ax}$, put $D = a$, $\qquad [f(a) = 0, f'(a) \neq 0$

$= x^2 \dfrac{1}{f''(D)} e^{ax}$, put $D = a$, $\qquad [f'(a) = 0, f''(a) \neq 0$

(ii) $X = \sin (ax + b)$ *or* $\cos (ax + b)$

P.I. $= \dfrac{1}{\phi(D^2)} \sin (ax + b)$ [or $\cos (ax + b)$], put $D^2 = -a^2$, $\qquad [\phi(-a)^2 \neq 0]$

$= x \dfrac{1}{\phi'(D^2)} \sin (ax + b)$ [or $(\cos ax + b)$], put $D^2 = -a^2$, $\qquad [\phi(-a^2) = 0, \phi'(-a^2) \neq 0]$

$= x^2 \dfrac{1}{\phi''(D^2)} \sin (ax + b)$ [or $\cos (ax + b)$], put $D^2 = -a^2$, $\qquad [\phi'(-a^2) = 0, \phi''(-a^2) \neq 0$

(iii) $X = x^m$, P.I. $= \dfrac{1}{f(D)} x^m = [f(D)]^{-1} x^m$. Expand $[f(D)]^{-1}$ in ascending powers of $D$ as for as $D^m$ and operate on $x^m$ term by term.

(iv) $X = e^{ax} V$, P.I. $= \dfrac{1}{f(D)} e^{ax} V = e^{ax} \dfrac{1}{f(D + a)} V$.

**III. Complete Solution :** C.S. is $y =$ C.F. + P.I.

**3. Homogeneous linear equation :** $x^3 \dfrac{d^3 y}{dx^3} + k_1 x^2 \dfrac{d^2 y}{dx^2} + k_2 x \dfrac{dy}{dx} + k_3 y = X$

reduces to linear equation with constant coefficients by putting

$$x = e^t, x \frac{dy}{dx} = Dy, x^2 \frac{d^2 y}{dx^2} = D(D - 1)\, y, x^3 \frac{d^3 y}{dx^3} = D(D - 1)(D - 2)\, y$$

**4. Lagrange's linear partial differential equation**

$Pp + Qq = R$, $P, Q, R$ being functions of $x, y, z$.

*To solve it* (i) form the subsidiary equations $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

(ii) solve these equations giving $u = a, v = b$.

(iii) Complete solution is $\phi(u, v) = 0$ or $u = f(v)$.

**5. Homogeneous linear partial differential equations with constant coefficients**

$$\frac{\partial^n z}{\partial x^n} + k_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + \ldots + k_n \frac{\partial^n z}{\partial y^n} = F(x, y)$$

*Symbolic form :* $(D^n + k_1 D^{n-1} D' + \ldots + k_n D'^n) z = F(x, y)$

**To find C.F.**

| Roots of A.E. | C.F. |
|---|---|
| (i) $m_1, m_2, m_3, ......$ | $f_1(y + m_1x) + f_2(y + m_2x) + f_3(y + m_3x) + ......$ |
| (ii) $m_1, m_1, m_3, ......$ | $f_1(y + m_1x) + xf_2(y + m_1x) + f_3(y + m_3x) + ......$ |
| (iii) $m_1, m_1, m_1, ......$ | $f_1(y + m_1x) + xf_2(y + m_1x) + x^2f_3(y + m_1x) + ......$ |

**To find P.I.**

(i) $F(x, y) = e^{ax+by}$, $P.I. = \dfrac{1}{f(D, D')} e^{ax+by}$, put $D = a$, $D' = b$.

(ii) $F(x, y) = \sin(mx + ny)$ or $\cos(mx + ny)$

$$P.I. = \frac{1}{f(D^2, DD', D'^2)} \sin \text{ or } \cos(mx + ny), \text{ put } D^2 = -m^2, DD' = -mn, D^2 = -n^2$$

(iii) $F(x, y) = x^m y^n$, $P.I. = [f(D, D')]^{-1} x^m y^n$. Expand $[f(D, D')]^{-1}$ and operate on $x^m y^n$.]

(iv) $F(x, y)$ is any function of $x$ and $y$, $P.I. = \dfrac{1}{f(D, D')}$, $F(x, y)$.

Resolve $1/f(D, D')$ into partial fractions considering $f(D, D')$ as a function of $D$ alone and operate each partial fraction on $F(x, y)$ remembering that $\dfrac{1}{D - mD'} F(x, y) = \int F(x, c - mx)\, dx$.

*Complete solution:* C.S. is $y = C.F. + P.I.$

## X. INFINITE SERIES

1. *Basic test:* If $\underset{n\to\infty}{Lt}\, u_n \neq 0$ then the series $\Sigma u_n$ diverges.
2. *G.P. Series:* $1 + r + r^2 + r^3 + ...... \infty$ converge if $|r| < 1$; diverges if $r \geq 1$ and oscillates if $r \leq -1$.
3. *p-series:* $\dfrac{1}{1^p} + \dfrac{1}{2^p} + \dfrac{1}{3^p} + ...... \infty$ converge for $p > 1$; diverges for $p \leq 1$.
4. *Comparison test:* If two positive term series $\Sigma u_n$ and $\Sigma v_n$ be such that $\underset{n\to\infty}{Lt} \dfrac{u_n}{v_n}$ = *finite quantity* $(\neq 0)$, then $\Sigma u_n$ and $\Sigma v_n$ converge or diverge together.
5. *Ratio test:* In the positive term series $\Sigma u_n$, if $\underset{n\to\infty}{Lt} \dfrac{u_n}{u_{n+1}} = k$, then the series converges for $k > 1$, diverges for $k < 1$ and fails for $k = 1$.
6. *Raabes' test:* In the positive term series $\Sigma u_n$, if $\underset{n\to\infty}{Lt}\, n\left(\dfrac{u_n}{u_{n+1}} - 1\right) = k$, then the series converges for $k > 1$, diverges for $k < 1$ and fails for $k = 1$.
7. *Logarithmic test:* In the positive term series $\Sigma u_n$, if $\underset{n\to\infty}{Lt}\left(n \log \dfrac{u_n}{u_{n+1}}\right) = k$, then the series converges for $k > 1$, diverges for $k < 1$ and fails for $k = 1$.
8. If $u_n/u_{n+1}$ does not involve $n$ as an exponent or a logarithm, then the series $\Sigma u_n$ diverges.
9. *Cauchy's root test:* In a positive term series $\Sigma u_n$ if $\underset{n\to\infty}{Lt}(u_n)^{1/n} = \lambda$, then the series converges for $\lambda < 1$, diverges for $\lambda > 1$ and fails for $\lambda = 1$.
10. *Integral test:* A positive term series $\Sigma f(n)$ converges or diverges according as $\int_1^\infty f(x)\, dx$ is finite or infinite where $f(n)$ is continuous in $1 < x < \infty$ and decreases as $n$ increases.
11. *Leibnitz's test for alternating series:* An alternating series $u_1 - u_2 + u_3 - u_4 + ...... \infty$ converges if each term is numerically less than the previous term and $\underset{n\to\infty}{Lt}\, u_n = 0$.

    if $\underset{n\to\infty}{Lt} \neq 0$, then the given series is oscillatory.

12. *General Ratio test:* In an arbitrary term series $\Sigma u_n$ if $\underset{n \to \infty}{Lt} \frac{|u_{n+1}|}{|u_n|} = |k|$, then $\Sigma u_n$ is absolutely convergent if $|k| < 1$ and divergent if $|k| > 1$ and the test fails if $|k| = 1$.

## XI. FOURIER SERIES

1. $f(x) = \frac{1}{2} a_0 + a_1 \cos x + a_2 \cos 2x + \ldots + b_1 \sin x + b_2 \sin 2x + \ldots\ldots$ in $(0\ 2\pi)$.

   where $\alpha_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx,\ a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx,\ b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$

2. $f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \cos nx$ in any interval $(0, 2c)$,

   where $a_0 = \frac{1}{c}\int_0^{2c} f(x)\,dx,\ a_n = \frac{1}{c}\int_0^{2c} f(x)\cos\frac{n\pi x}{c}\,dx,\ b_n = \frac{1}{c}\int_0^{c} f(x)\sin\frac{n\pi x}{c}\,dx$

3. *For even function f(x), Fourier expansion contains only cosine terms.*

   *i.e.,* $a_0 = \frac{2}{c}\int_0^{c} f(x)\,dx,\ a_n = \frac{2}{c}\int_0^{c} f(x)\cos\frac{n\pi x}{c}\,dx,\ b_n = 0.$

   *For odd function f (x), Fourier expansion contains only sine terms.*

   *i.e.,* $a_0 = 0, a_n = 0, b_n = \frac{2}{c}\int_0^{c} f(x)\sin\frac{n\pi x}{c}\,dx.$

## XII. TRANSFORMS

1. **Laplace Transforms.** $L(f(t)) = \int_0^{\infty} e^{-st} f(t)\,dt$

(*i*) $L(1) = \frac{1}{s}$

(*ii*) $L(t^n) = \frac{n!}{s^{n+1}}$

(*iii*) $L(e^{at}) = \frac{1}{s-a}$

(*iv*) $L(\sin at) = \frac{a}{s^2+a^2}$

(*v*) $L(\cos at) = \frac{s}{s^2+a^2}$

(*vi*) $L(\sinh at) = \frac{a}{s^2-a^2}$

(*vii*) $L(\cosh at) = \frac{s}{s^2-a^2}$

(*viii*) $L\, e^{at} f(t) = F(s-a)$

(*ix*) $L f'(t) = sL f(t) - f(0)$

(*x*) $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[F(s)]$

(*xi*) $L\left[\frac{1}{t} f(t)\right] = \int_s^{\infty} F(s)\,ds$

(*xii*) $u(t-a) = \begin{cases} 0 \text{ when } t < a \\ 1 \text{ when } t > a \end{cases}$

(*xiii*) $L[u(t-a)] = \frac{e^{-as}}{s}$

(*xiv*) $L\delta(t-a) = e^{-as}$

(*xv*) $L f(t) = \frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}$ where $f(t)$ is a periodic function of period $T$.

2. **Inverse Laplace Transforms**

(*i*) $L^{-1}\left(\frac{1}{s}\right) = 1$

(*ii*) $L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!}$

(*iii*) $L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

(*iv*) $L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at$

(*v*) $L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$

(*vi*) $L^{-1}\left(\frac{s}{s^2-a^2}\right) = \frac{1}{a}\sinh at$

(vii) $L^{-1}\left(\dfrac{s}{s^2-a^2}\right) = \cosh at.$ (viii) $L^{-1}\left(\dfrac{s}{(s^2+a^2)^2}\right) = \dfrac{1}{2a}\, t \sin at$

(ix) $L^{-1}\left[\dfrac{s^2-a^2}{(s^2+a^2)^2}\right] = t \cos at.$

3. **Fourier Transforms** : $F(s) = \int_{-\infty}^{\infty} f(t)\, e^{ist}\, .\, dt$

*Fourier sine transform* : $F_s(s) = \int_0^{\infty} f(t) \sin st\, dt$

*Fourier cosine transform* : $F_c(s) = \int_0^{\infty} f(t) \cos st\, dt$

$$F\left(\frac{\partial^2 u}{\partial x^2}\right) = -s^2 F(u).$$

4. **Z-Transforms** : $Z(u_n) = \sum_{n=0}^{\infty} u_n z^{-n}$

(i) $Z(1) = \dfrac{z}{z-1}$ (ii) $Z(n) = \dfrac{z}{(z-1)^2}$

(iii) $Z(n^2) = \dfrac{z^2+z}{(z-1)^3}$ (iv) $Z(a^n) = \dfrac{z}{z-a}$

(v) $Z(na^n) = \dfrac{az}{(z-a)^2}$ (vi) $Z(\sin n\theta) = \dfrac{z \sin\theta}{z^2 - 2z\cos\theta + 1}$

(vii) $Z(\cos n\theta) = \dfrac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}$ (viii) $Z(\sinh n\theta) = \dfrac{z\sinh\theta}{z^2 - 2z\cosh\theta + 1}$

(ix) $Z(\cosh n\theta) = \dfrac{z(z-\cosh\theta)}{z^2 - 2z\cosh\theta + 1}$

## XIII. STATISTICS AND PROBABILITY

1. A.M. $\bar{x} = \dfrac{\Sigma f_i x_i}{\Sigma f_i}$ 2. S.D. $\sigma = \sqrt{\dfrac{\Sigma f_i (x_i - \bar{x})^2}{\Sigma f_i}}$

3. *Moments about the mean* : $\mu_0 = 1, \mu_1 = 0, \mu_2 = \sigma^2, \mu_r = \dfrac{\Sigma f_i (x_i - \bar{x})^r}{\Sigma f_i}$

4. *Coeff. of skewness* = (mean – mode)/$\sigma$ which lies between – 1 and 1.

5. Kurtosis : $\beta_2 = \mu_4/\mu_2^2$.

6. *Coeff. of correlation* $r = \dfrac{n\,\Sigma d_x d_y - \Sigma d_x \Sigma d_y}{\sqrt{[\{n\,\Sigma d_x^2 - (\Sigma d_x)^2\}\{n\Sigma d_y^2 - (\Sigma d_y)^2\}]}}; -1 < r < 1$

7. *Line of regression of y on x* : $y - \bar{y} = r\,\dfrac{\sigma_y}{\sigma_x}(x - \bar{x})$

*Line of regression of x on y* : $x - \bar{x} = \dfrac{r\,\sigma_x}{\sigma_y}(y - \bar{y})$

8. **Probability** $p(A) = \dfrac{\text{No. of ways favourable to } A}{\text{Total no. of equally likely ways}}, p + q = 1.$

(i) $p(A \text{ or } B) = p(A) + p(B)$, (ii) $p(A \text{ and } B) = p(A)\,.\,p(B)$

9. **Binomial distribution** : $p(r) = {}^nC_r\, p^r q^{n-r}$

*Mean* = $np$, *Variance* ($\sigma^2$) = $npq$

**10. Poisson distribution** : $p(r) = \dfrac{m^r}{r!} e^{-m}$

*Mean* $= m$, *Variance* $(\sigma^2) = m$.

**11. Normal distribution** : $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, *Standard variate* $= \dfrac{x-\mu}{\sigma}$

(*i*) *Probable error* $\lambda = 0.6745\ \sigma$.

(*ii*) 68% of values lie between $x = \mu - \sigma$ and $x = \mu + \sigma$.
95% of values lie between $x = \mu - 1.96\ \sigma$ and $x = \mu + 1.96\ \sigma$
99% of values lie between $x = \mu - 2.58\ \sigma$ and $x = \mu + 2.58\ \sigma$

## XIV. NUMERICAL TECHNIQUES

**1. Solution of equations**

(*i*) *Bisection method* : $x_n = \dfrac{1}{2}(x_{n-1} + x_{n-2})$

(*ii*) *Method of False position* : $x_2 = x_0 - \dfrac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$.

(*iii*) *Newton-Raphson method* : $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

(*iv*) *Iterative formula to find* $1/N$ is $x_{n+1} = x_n(2 - Nx_n)$

(*v*) *Iterative formula to find* $\sqrt{N}$ is $x_{n+1} = \dfrac{1}{2}(x_n + N/x_n)$

**2. Solution of Linear Simultaneous equations**

(*i*) *Matrix inversion method.* For the equations

$$a_1x + b_1y + c_1z = d_1,\ a_2x + b_2y + c_2z = d_2,\ a_3x + b_3y + c_3z = d_3$$

if
$$A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

then
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{|A|} \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \times \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

where $A_1, B_1$, etc., are the co-factors of $a_1, b_1$, etc., in the determinant $|A|$.

(*ii*) *Gauss-elimination method.* In this method the coefficient matrix is transformed to *upper triangular matrix.*

(*iii*) *Gauss-Jordan method.* In this method the coefficient matrix is transformed to *diagonal matrix.*

(*iv*) *Gauss-Jordan method of finding the inverse of a matrix A.* The matrices $A$ and $I$ are written side by side and the same row transformations are performed on both till $A$ is reduced to $I$. Then the other matrix represents $A^{-1}$.

**3. Finite differences and Interpolation**

(*i*) *Forward differences:* $\Delta y_r = y_{r+1} - y_r$
*Backward differences:* $\nabla y_r = y_r - y_{r-1}$
*Central differences:* $\delta y_{n-1/2} = y_n - y_{n-1}$

(*ii*) *Relations between operations* :

$$\Delta = E - 1;\ \nabla = 1 - E^{-1};\ \delta = E^{1/2} - E^{-1/2}$$

$$\mu = \frac{1}{2}(E^{1/2} + E^{-1/2});\ \Delta = E\nabla = \nabla E = \delta E^{1/2};\ E = e^{hD}$$

(*iii*) *Factorial notation* $\quad |\,x\,|^r = x(x-1)(x-2)\ldots(x-r+1)$.
*Factorial polynomial* $\quad [x]^n = x(x-h)(x-2h)\ldots(x-h-1h)$

(iv) *Newton's forward interpolation formula*

$$y_p = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\,\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\,\Delta^3 y_0 + \ldots \text{ where } p = (x - x_0)/h.$$

(v) *Newton's backward interpolation formula:*

$$y_p = y_n + p\,\nabla y_n + \frac{p(p+1)}{2!}\,\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\,\nabla^3 y_n + \ldots\ldots \text{ where } p = (x - x_n)/h.$$

(vi) *Stirling's formula:*

$$y_p = y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\,\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{p^2(p^2-1)}{4!}\,\Delta^4 y_{-2} + \ldots\ldots$$

(vii) *Bessel's formula:*

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\,\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{\left(p - \frac{1}{2}\right)p\,(p-1)}{3!}\,\Delta^3 y_{-1} + \frac{(p+1)\,p(p-1)(p-2)}{4!}\,\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \ldots$$

(viii) *Lagrange's interpolation formula:*

$$y = \frac{(x-x_1)(x-x_2)\ldots\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots\ldots(x_0-x_n)}\,y_0 + \frac{(x-x_0)(x-x_2)\ldots\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots\ldots(x_1-x_n)}\,y_1 + \ldots\ldots + \frac{(x-x_0)(x-x_1)\ldots\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots\ldots(x_n-x_{n-1})}\,y_n$$

(ix) *Newton's divided difference formul*

$$y = f(x) = y_0 + (x - x_0)\,[x_0, x_1] + (x - x_0)(x - x_1)\,[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)\,[x_0, x_1, x_2, x_3] + \ldots$$

where $[x_0, x_1]\ \dfrac{y_1 - y_0}{x_1 - x_0}$, $[x_0, x_1, x_2] = \dfrac{[x_1, x_2] - [x_0, x_1]}{[x_2 - x_0]}$ and so on.

## 4. Numerical differentiation

(i) *Forward difference formulae:*

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\ldots\right]$$

$$\left(\frac{d^2 y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \ldots\ldots\right] \text{ and so on.}$$

(ii) *Backward difference formulae:*

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\nabla y_0 + \frac{1}{2}\nabla^2 y_0 + \frac{1}{3}\nabla^3 y_0 + \frac{1}{4}\nabla^4 y_0 + \ldots\ldots\right]$$

$$\left(\frac{d^2 y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\nabla^2 y_0 + \nabla^3 y_0 + \frac{11}{12}\nabla^4 y_0 + \ldots\ldots\right] \text{ and so on.}$$

(iii) *Central difference formulae:*

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} - \frac{1}{6}\,\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} + \frac{1}{30}\,\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2} + \ldots\ldots\right]$$

$$\left(\frac{d^2 y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3} + \ldots\ldots\right]$$

## 5. Numerical integration

(i) *Trapezoidal rule:*

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\,[(y_0 + y_n) + 2\,(y_1 + y_2 + \ldots\ldots + y_{n-1})]$$

(ii) *Simpon's 1/3 th rule:*

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + \ldots\ldots + y_{n-1}) + 2(y_2 + y_4 + \ldots\ldots + y_{n-2})]$$

(Number of sub intervals should be taken as even)

(iii) *Simpson's 3/8 th rule:*

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \ldots\ldots + y_{n-1}) + 2(y_3 + y_5 + \ldots\ldots + y_{n-2})]$$

(Number of sub-intervals should be taken as a multiple of 3)

(iv) *Weddle's rule:*

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + \ldots\ldots]$$

(Number of sub-intervals should be taken as multiple of (6)

**6. Numerical solution of ordinary differential equations**

(i) *Picard's method:* $y_1 = y_0 + \int_{x_0}^{x} (x, y_0)\,dx$

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx \text{ etc.}$$

(ii) *Taylor's method:*

$$y = y_0 + (x - x_0)(y)_0 + \frac{(x - x_0)^2}{2!}(y'')_0 + \frac{(x - x_0)^3}{3!}(y''')_0 + \ldots$$

(iii) *Euler's method:* $y_2 = y_1 + h\,f(x_0 + h, y_1)$

Repeat this process till $y_2$ is stationary. Then calculate $y_3$ and so on.

(iv) *Modified Euler's method:* $y_2 = y_1 + \frac{1}{2}[f(x_0 + h, y_1) + f(x_0 + 2h, y_2)]$

Repeat this step till $y_2$ is stationary. Then calculate $y_3$ and so on.

(v) *Runge Kutta method:* $y_1 = y_0 + h$ where $h = \frac{1}{6}(k_1 + 2k_2, 2k_3 + k_4)$

such that $k_1 = h\,f(x_0, y_0)$; $k_2 = h\,f(x_0 + h/2, y_0 + k_1/2)$

$k_3 = h\,f(x_0 + h/2, y_0 + k_2/2)$; $k_4 = hf(x_0 + h, y_0 + k_3)$

(vi) *Milne's method*

*Predictor formula:* $y_4 = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$

*Corrector formula:* $y_4 = y_2 + \frac{h}{3}(f_2 - 4f_3 + f_4)$

(Four prior values are required to find the next values)

(vii) *Adams-Bashforth method:*

*Predictor formula:* $y_1 = y_0 + \frac{h}{24}(55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3})$

*Corrector formula:* $y_1 = y_0 + \frac{h}{24}(9f_1 + 19f_0 - 5f_{-1} - f_{-2})$

(Four prior values are required to find the next values)

**7. Numerical solution of partial differential equations**

(i) *Classification of a second order equations:*

$$A(x, y)\frac{\partial^2 u}{\partial x^2} + B(x, y)\frac{\partial^2 u}{\partial x \partial y} + C(x, y)\frac{\partial^2 u}{\partial y^2} + \left(F(x, y, u,)\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right) = 0$$

is said to be *elliptic* if $B^2 - 4AC < 0$

*parabolic* if $B^2 - 4AC = 0$

*hyperbolic* if $B^2 - 4AC > 0$

(*ii*) *Laplace equation:* $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

*Standard 5-point formula:* $u_{i,j} = \dfrac{1}{4}[u_{i-1,j} + u_{i+1,j}\, u_{i,j+1} + u_{i,j-1}]$

*Diagonal 5-point formula:* $u_{i,j} = \dfrac{1}{4}[u_{i-1,j+1} + u_{i+1,j-1}\, u_{i+1,j+1} + u_{i-1,j-1}]$

(*iii*) *Poisson's equation:* $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f(x, y).$

*Standard 5-point formula:*

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f(ih, jh)$$

(*iv*) *One-dimensional heat equation:* $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$

*Schmidt formula:* $u_{i,j+1} = \alpha\, u_{i-1,j} + (1 - 2d)\, u_{i,j} + \alpha\, u_{i+1,j}$ where $\alpha = kc^2/h^2$
when $\alpha = 1/2$, it reduces to

*Bendre-Schmidt relation:* $u_{i,j+1} = \dfrac{1}{2}(u_{i-1,j} + u_{i+1,j})$

(*v*) *Wave equation:* $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$

*Explict formula for solution is*

$$u_{i,j+1} = 2(1 - \alpha^2 c^2)\, u_{i,j} + \alpha^2 c^2\, (u_{i-1,j} + u_{i+1,j}) - u_{i,j-1} \text{ where } \alpha = k/h$$

If $\alpha$ is so chosen that the coefficient of $u_{i,j}$ is zero *i.e.*, $k = h/c$ then the above explicit formula takes the *simplified form*

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$

APPENDIX 2

# Tables

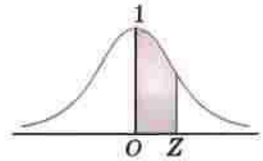
**Table I : Gamma Function, $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$**

| $\alpha$ | $\Gamma(\alpha)$ | $\alpha$ | $\Gamma(\alpha)$ | $\alpha$ | $\Gamma(\alpha)$ | $\alpha$ | $\Gamma(\alpha)$ |
|---|---|---|---|---|---|---|---|
| 1.00 | 1.000000 | 1.26 | 0.904397 | 1.52 | 0.887039 | 1.78 | 0.926227 |
| 1.01 | 0.994326 | 1.27 | 0.902503 | 1.53 | 0.887568 | 1.79 | 0.928767 |
| 1.02 | 0.988844 | 1.28 | 0.900719 | 1.54 | 0.888178 | 1.80 | 0.931384 |
| 1.03 | 0.983550 | 1.29 | 0.899042 | 1.55 | 0.888869 | 1.81 | 0.934076 |
| 1.04 | 0.978438 | 1.30 | 0.897471 | 1.56 | 0.889639 | 1.82 | 0.936845 |
| 1.05 | 0.973504 | 1.31 | 0.896004 | 1.57 | 0.890490 | 1.83 | 0.939690 |
| 1.06 | 0.968744 | 1.32 | 0.894640 | 1.58 | 0.891420 | 1.84 | 0.942612 |
| 1.07 | 0.964152 | 1.33 | 0.893378 | 1.59 | 0.892428 | 1.85 | 0.945611 |
| 1.08 | 0.959725 | 1.34 | 0.892215 | 1.60 | 0.893516 | 1.86 | 0.948687 |
| 1.09 | 0.955459 | 1.35 | 0.891151 | 1.61 | 0.894681 | 1.87 | 0.951840 |
| 1.10 | 0.951351 | 1.36 | 0.890184 | 1.62 | 0.895924 | 1.88 | 0.955071 |
| 1.11 | 0.947395 | 1.37 | 0.889313 | 1.63 | 0.897244 | 1.89 | 0.958380 |
| 1.12 | 0.943590 | 1.38 | 0.888537 | 1.64 | 0.898642 | 1.90 | 0.961766 |
| 1.13 | 0.939931 | 1.39 | 0.887854 | 1.65 | 0.900117 | 1.91 | 0.965231 |
| 1.14 | 0.936416 | 1.40 | 0.887264 | 1.66 | 0.901668 | 1.92 | 0.968774 |
| 1.15 | 0.933041 | 1.41 | 0.886764 | 1.67 | 0.903296 | 1.93 | 0.972397 |
| 1.16 | 0.929803 | 1.42 | 0.886356 | 1.68 | 0.905001 | 1.94 | 0.976099 |
| 1.17 | 0.926700 | 1.43 | 0.886036 | 1.69 | 0.906782 | 1.95 | 0.979881 |
| 1.18 | 0.923728 | 1.44 | 0.885805 | 1.70 | 0.908639 | 1.96 | 0.983742 |
| 1.19 | 0.920885 | 1.45 | 0.885661 | 1.71 | 0.910572 | 1.97 | 0.987685 |
| 1.20 | 0.918169 | 1.46 | 0.885604 | 1.72 | 0.912580 | 1.98 | 0.991708 |
| 1.21 | 0.915577 | 1.47 | 0.885633 | 1.73 | 0.914665 | 1.99 | 0.995813 |
| 1.22 | 0.913106 | 1.48 | 0.885747 | 1.74 | 0.916826 | 2.00 | 1.000000 |
| 1.23 | 0.910755 | 1.49 | 0.885945 | 1.75 | 0.919062 | | |
| 1.24 | 0.908521 | 1.50 | 0.886227 | 1.76 | 0.921375 | | |
| 1.25 | 0.906403 | 1.51 | 0.886592 | 1.77 | 0.923763 | | |

## Table II : Bessel Functions

| $x$ | $J_0(x)$ | $J_1(x)$ | $x$ | $J_0(x)$ | $J_1(x)$ |
|---|---|---|---|---|---|
| 0.0 | 1.0000 | 0.0000 | 3.0 | − 0.2601 | 0.3991 |
| 0.1 | 0.9975 | 0.0499 | 3.1 | − 0.2921 | 0.3009 |
| 0.2 | 0.9900 | 0.0995 | 3.2 | − 0.3202 | 0.2613 |
| 0.3 | 0.9776 | 0.1483 | 3.3 | − 0.3443 | 0.2207 |
| 0.4 | 0.9604 | 0.1960 | 3.4 | − 0.3643 | 0.1792 |
| 0.5 | 0.9385 | 0.2423 | 3.5 | − 0.3801 | 0.1374 |
| 0.6 | 0.9120 | 0.2867 | 3.6 | − 0.3918 | 0.0955 |
| 0.7 | 0.8812 | 0.3290 | 3.7 | − 0.3992 | 0.0538 |
| 0.8 | 0.8463 | 0.3688 | 3.8 | − 0.4026 | 0.0128 |
| 0.9 | 0.8075 | 0.4059 | 3.9 | − 0.4018 | − 0.0272 |
| 1.0 | 0.7652 | 0.4401 | 4.0 | − 0.3971 | − 0.0660 |
| 1.1 | 0.7196 | 0.4709 | 4.1 | − 0.3887 | − 0.1033 |
| 1.2 | 0.6711 | 0.4983 | 4.2 | − 0.3766 | − 0.1386 |
| 1.3 | 0.6201 | 0.5220 | 4.3 | − 0.3610 | − 0.1719 |
| 1.4 | 0.5669 | 0.5419 | 4.4 | − 0.3423 | − 0.2028 |
| 1.5 | 0.5118 | 0.5579 | 4.5 | − 0.3205 | − 0.2311 |
| 1.6 | 0.4554 | 0.5699 | 4.6 | − 0.2961 | − 0.2566 |
| 1.7 | 0.3980 | 0.5778 | 4.7 | − 0.2693 | − 0.2791 |
| 1.8 | 0.3400 | 0.5815 | 4.8 | − 0.2404 | − 0.2985 |
| 1.9 | 0.2818 | 0.5812 | 4.9 | − 0.2097 | − 0.3147 |
| 2.0 | 0.2239 | 0.5767 | 5.0 | − 0.1776 | − 0.3276 |
| 2.1 | 0.1666 | 0.5683 | 5.1 | − 0.1443 | − 0.3371 |
| 2.2 | 0.1104 | 0.5560 | 5.2 | − 0.1103 | − 0.3432 |
| 2.3 | 0.0555 | 0.5399 | 5.3 | − 0.0758 | − 0.3460 |
| 2.4 | 0.0025 | 0.5202 | 5.4 | − 0.0412 | − 0.3453 |
| 2.5 | − 0.0484 | 0.4971 | 5.5 | − 0.0068 | − 0.3414 |
| 2.6 | − 0.0968 | 0.4708 | 5.6 | 0.0270 | − 0.3343 |
| 2.7 | − 0.1424 | 0.4416 | 5.7 | 0.0599 | − 0.3241 |
| 2.8 | − 0.1850 | 0.4097 | 5.8 | 0.0917 | − 0.3110 |
| 2.9 | − 0.2243 | 0.3754 | 5.9 | 0.1220 | − 0.2951 |

**Table III : Area under the Normal curve**




| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.0000 | 0.0040 | 0.0080 | 0.0120 | 0.0160 | 0.0199 | 0.0239 | 0.0279 | 0.0319 | 0.0359 |
| 0.1 | 0.0398 | 0.0438 | 0.0478 | 0.0517 | 0.0557 | 0.0596 | 0.0636 | 0.0675 | 0.0714 | 0.0754 |
| 0.2 | 0.0793 | 0.0832 | 0.0871 | 0.0910 | 0.0948 | 0.0987 | 0.1026 | 0.1064 | 0.1103 | 0.1141 |
| 0.3 | 0.1179 | 0.1217 | 0.1255 | 0.1293 | 0.1331 | 0.1368 | 0.1406 | 0.1443 | 0.1480 | 0.1517 |
| 0.4 | 0.1554 | 0.1591 | 0.1628 | 0.1664 | 0.1700 | 0.1736 | 0.1772 | 0.1808 | 0.1844 | 0.1879 |
| 0.5 | 0.1915 | 0.1950 | 0.1985 | 0.2019 | 0.2054 | 0.2088 | 0.2123 | 0.2157 | 0.2190 | 0.2224 |
| 0.6 | 0.2258 | 0.2291 | 0.2324 | 0.2357 | 0.2389 | 0.2422 | 0.2454 | 0.2486 | 0.2518 | 0.2549 |
| 0.7 | 0.2580 | 0.2612 | 0.2642 | 0.2673 | 0.2704 | 0.2734 | 0.2764 | 0.2794 | 0.2823 | 0.2852 |
| 0.8 | 0.2881 | 0.2910 | 0.2939 | 0.2967 | 0.2996 | 0.3023 | 0.3051 | 0.3078 | 0.3106 | 0.3133 |
| 0.9 | 0.3159 | 0.3186 | 0.3212 | 0.3238 | 0.3264 | 0.3289 | 0.3315 | 0.3340 | 0.3365 | 0.3389 |
| 1.0 | 0.3413 | 0.3438 | 0.3461 | 0.3485 | 0.3508 | 0.3531 | 0.3554 | 0.3577 | 0.3599 | 0.3621 |
| 1.1 | 0.3643 | 0.3665 | 0.3686 | 0.3708 | 0.3729 | 0.3749 | 0.3770 | 0.3790 | 0.3810 | 0.3830 |
| 1.2 | 0.3849 | 0.3869 | 0.3888 | 0.3907 | 0.3925 | 0.3944 | 0.3962 | 0.3980 | 0.3997 | 0.4015 |
| 1.3 | 0.4032 | 0.4049 | 0.4066 | 0.4082 | 0.4099 | 0.4115 | 0.4131 | 0.4147 | 0.4162 | 0.4177 |
| 1.4 | 0.4192 | 0.4207 | 0.4222 | 0.4236 | 0.4251 | 0.4265 | 0.4279 | 0.4292 | 0.4306 | 0.4319 |
| 1.5 | 0.4332 | 0.4345 | 0.4357 | 0.4370 | 0.4382 | 0.4394 | 0.4406 | 0.4418 | 0.4429 | 0.4441 |
| 1.6 | 0.4452 | 0.4463 | 0.4474 | 0.4484 | 0.4495 | 0.4505 | 0.4515 | 0.4525 | 0.4535 | 0.4545 |
| 1.7 | 0.4554 | 0.4564 | 0.4573 | 0.4582 | 0.4591 | 0.4599 | 0.4608 | 0.4616 | 0.4625 | 0.4633 |
| 1.8 | 0.4641 | 0.4649 | 0.4656 | 0.4664 | 0.4671 | 0.4678 | 0.4686 | 0.4693 | 0.4699 | 0.4706 |
| 1.9 | 0.4713 | 0.4719 | 0.4726 | 0.4732 | 0.4738 | 0.4744 | 0.4750 | 0.4756 | 0.4761 | 0.4767 |
| 2.0 | 0.4772 | 0.4778 | 0.4783 | 0.4788 | 0.4793 | 0.4798 | 0.4803 | 0.4808 | 0.4812 | 0.4817 |
| 2.1 | 0.4821 | 0.4826 | 0.4830 | 0.4834 | 0.4838 | 0.4842 | 0.4846 | 0.4850 | 0.4854 | 0.4857 |
| 2.2 | 0.4861 | 0.4864 | 0.4868 | 0.4871 | 0.4875 | 0.4878 | 0.4881 | 0.4884 | 0.4887 | 0.4890 |
| 2.3 | 0.4893 | 0.4896 | 0.4898 | 0.4901 | 0.4904 | 0.4906 | 0.4909 | 0.4911 | 0.4913 | 0.4916 |
| 2.4 | 0.4918 | 0.4920 | 0.4922 | 0.4925 | 0.4927 | 0.4929 | 0.4931 | 0.4932 | 0.4934 | 0.4936 |
| 2.5 | 0.4938 | 0.4940 | 0.4941 | 0.4943 | 0.4945 | 0.4946 | 0.4948 | 0.4949 | 0.4951 | 0.4952 |
| 2.6 | 0.4953 | 0.4955 | 0.4956 | 0.4957 | 0.4959 | 0.4960 | 0.4961 | 0.4962 | 0.4963 | 0.4964 |
| 2.7 | 0.4965 | 0.4966 | 0.4967 | 0.4968 | 0.4969 | 0.4970 | 0.4971 | 0.4972 | 0.4973 | 0.4974 |
| 2.8 | 0.4974 | 0.4975 | 0.4976 | 0.4977 | 0.4977 | 0.4978 | 0.4979 | 0.4979 | 0.4980 | 0.4981 |
| 2.9 | 0.4981 | 0.4982 | 0.4982 | 0.4983 | 0.4984 | 0.4984 | 0.4985 | 0.4985 | 0.4986 | 0.4986 |
| 3.0 | 0.4987 | 0.4987 | 0.4987 | 0.4988 | 0.4988 | 0.4989 | 0.4989 | 0.4989 | 0.4990 | 0.4990 |
| 3.1 | 0.4990 | 0.4991 | 0.4991 | 0.4991 | 0.4992 | 0.4992 | 0.4992 | 0.4992 | 0.4993 | 0.4993 |

**Table IV : Values of | *t* | with probability *P* and degrees of freedom ν**

| ν \ P | 0.50 | 0.10 | 0.05 | 0.02 | 0.01 |
|---|---|---|---|---|---|
| 1 | 1.000 | 6.34 | 12.71 | 31.82 | 63.66 |
| 2 | 0.816 | 2.92 | 4.30 | 6.96 | 9.92 |
| 3 | 0.765 | 2.35 | 3.18 | 4.54 | 5.84 |
| 4 | 0.741 | 2.13 | 2.78 | 3.75 | 4.60 |
| 5 | 0.727 | 2.02 | 2.57 | 3.36 | 4.03 |
| 6 | 0.718 | 1.94 | 2.45 | 3.14 | 3.71 |
| 7 | 0.711 | 1.90 | 2.36 | 3.00 | 3.50 |
| 8 | 0.706 | 1.86 | 2.31 | 2.90 | 3.36 |
| 9 | 0.703 | 1.83 | 2.26 | 2.82 | 3.25 |
| 10 | 0.700 | 1.81 | 2.23 | 2.76 | 3.17 |
| 11 | 0.697 | 1.80 | 2.20 | 2.72 | 3.11 |
| 12 | 0.695 | 1.78 | 2.18 | 2.68 | 3.06 |
| 13 | 0.694 | 1.77 | 2.16 | 2.65 | 3.01 |
| 14 | 0.692 | 1.76 | 2.14 | 2.62 | 2.98 |
| 15 | 0.691 | 1.75 | 2.13 | 2.60 | 2.95 |
| 16 | 0.690 | 1.75 | 2.12 | 2.58 | 2.92 |
| 17 | 0.689 | 1.74 | 2.11 | 2.57 | 2.90 |
| 18 | 0.688 | 1.73 | 2.10 | 2.55 | 2.88 |
| 19 | 0.688 | 1.73 | 2.09 | 2.54 | 2.86 |
| 20 | 0.687 | 1.72 | 2.09 | 2.53 | 2.84 |
| 21 | 0.686 | 1.72 | 2.08 | 2.52 | 2.83 |
| 22 | 0.686 | 1.72 | 2.07 | 2.51 | 2.82 |
| 23 | 0.685 | 1.71 | 2.07 | 2.50 | 2.81 |
| 24 | 0.685 | 1.71 | 2.06 | 2.49 | 2.80 |
| 25 | 0.684 | 1.71 | 2.06 | 2.48 | 2.79 |
| 26 | 0.684 | 1.71 | 2.06 | 2.48 | 2.78 |
| 27 | 0.684 | 1.70 | 2.05 | 2.47 | 2.77 |
| 28 | 0.683 | 1.70 | 2.05 | 2.47 | 2.76 |
| 29 | 0.683 | 1.70 | 2.04 | 2.46 | 2.76 |
| 30 | 0.683 | 1.70 | 2.04 | 2.46 | 2.75 |

**Table V : Values of $\chi^2$ with probability $P$ and df $\nu$**

| $\nu$ \ $P$ | 0.99 | 0.95 | 0.50 | 0.30 | 0.20 | 0.10 | 0.05 | 0.01 |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.0002 | 0.004 | 0.46 | 1.07 | 1.64 | 2.71 | 3.84 | 6.64 |
| 2 | 0.020 | 0.103 | 1.39 | 2.41 | 3.22 | 4.60 | 5.99 | 9.21 |
| 3 | 0.115 | 0.35 | 2.37 | 3.66 | 4.64 | 6.25 | 7.82 | 11.34 |
| 4 | 0.30 | 0.71 | 3.36 | 4.88 | 5.99 | 7.78 | 9.49 | 13.28 |
| 5 | 0.55 | 1.14 | 4.35 | 6.06 | 7.29 | 9.24 | 11.07 | 15.09 |
| 6 | 0.87 | 1.64 | 5.35 | 7.23 | 8.56 | 10.64 | 12.59 | 16.81 |
| 7 | 1.24 | 2.17 | 6.35 | 8.38 | 9.80 | 12.02 | 14.07 | 18.48 |
| 8 | 1.65 | 2.73 | 7.34 | 9.52 | 11.03 | 13.36 | 15.51 | 20.09 |
| 9 | 2.09 | 3.32 | 8.34 | 10.66 | 12.24 | 14.68 | 16.92 | 21.67 |
| 10 | 2.56 | 3.94 | 9.34 | 11.78 | 13.44 | 15.99 | 18.31 | 23.21 |
| 11 | 3.05 | 4.58 | 10.34 | 12.90 | 14.63 | 17.28 | 19.68 | 24.72 |
| 12 | 3.57 | 5.23 | 11.34 | 14.01 | 15.81 | 18.55 | 21.03 | 26.22 |
| 13 | 4.11 | 5.89 | 12.34 | 15.12 | 16.98 | 19.81 | 22.36 | 27.69 |
| 14 | 4.66 | 6.57 | 13.34 | 16.22 | 18.15 | 21.06 | 23.68 | 29.14 |
| 15 | 5.23 | 7.26 | 14.34 | 17.32 | 19.31 | 22.31 | 25.00 | 30.58 |
| 16 | 5.81 | 7.96 | 15.34 | 18.42 | 20.46 | 23.54 | 26.30 | 32.00 |
| 17 | 6.41 | 8.67 | 16.34 | 19.51 | 21.62 | 24.77 | 27.59 | 33.41 |
| 18 | 7.02 | 9.39 | 17.34 | 20.60 | 22.76 | 25.99 | 28.87 | 34.80 |
| 19 | 7.63 | 10.12 | 18.34 | 21.69 | 23.90 | 27.20 | 30.14 | 36.19 |
| 20 | 8.26 | 10.85 | 19.34 | 22.78 | 25.04 | 28.41 | 31.41 | 37.57 |
| 21 | 8.90 | 11.59 | 20.34 | 23.86 | 26.17 | 29.62 | 32.67 | 38.93 |
| 22 | 9.54 | 12.34 | 21.34 | 24.94 | 27.30 | 30.81 | 33.92 | 40.29 |
| 23 | 10.20 | 13.09 | 22.34 | 26.02 | 28.43 | 32.01 | 35.17 | 41.64 |
| 24 | 10.86 | 13.85 | 23.34 | 27.10 | 29.55 | 33.20 | 36.42 | 42.98 |
| 25 | 11.52 | 14.61 | 24.34 | 28.17 | 30.68 | 34.68 | 37.65 | 44.31 |
| 26 | 12.20 | 15.38 | 25.34 | 29.25 | 31.80 | 35.56 | 38.88 | 45.64 |
| 27 | 12.88 | 16.15 | 26.34 | 30.32 | 32.91 | 36.74 | 40.11 | 46.96 |
| 28 | 13.56 | 16.93 | 27.34 | 31.39 | 34.03 | 37.92 | 41.34 | 48.28 |
| 29 | 14.26 | 17.71 | 28.34 | 32.46 | 35.14 | 39.09 | 42.56 | 49.59 |
| 30 | 14.95 | 18.49 | 29.34 | 33.53 | 36.25 | 40.26 | 43.77 | 50.89 |

**Table VI : 5% and 1% points of F**

| $v_2$ \ $v_1$ | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 12 | 24 | ∞ |
|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 18.51 | 19.00 | 19.16 | 19.25 | 19.30 | 19.32 | 19.37 | 19.41 | 19.45 | 19.50 |
| | 98.49 | 99.00 | 99.17 | 99.25 | 99.30 | 99.33 | 99.36 | 99.42 | 99.46 | 99.50 |
| 3 | 10.13 | 9.55 | 9.28 | 9.12 | 9.01 | 8.94 | 8.84 | 8.74 | 8.64 | 8.53 |
| | 34.12 | 30.82 | 29.46 | 28.71 | 28.24 | 27.91 | 27.49 | 27.05 | 26.60 | 26.12 |
| 4 | 7.71 | 6.94 | 6.59 | 6.39 | 6.26 | 6.16 | 6.04 | 5.91 | 5.77 | 5.63 |
| | 21.20 | 18.00 | 16.69 | 15.98 | 15.52 | 15.21 | 14.80 | 14.37 | 13.93 | 13.46 |
| 5 | 6.61 | 5.79 | 5.41 | 5.19 | 5.05 | 4.95 | 4.82 | 4.68 | 4.53 | 4.36 |
| | 16.26 | 13.27 | 12.06 | 11.39 | 10.97 | 10.67 | 10.27 | 9.89 | 9.47 | 9.02 |
| 6 | 5.99 | 5.14 | 4.76 | 4.53 | 4.39 | 4.28 | 4.15 | 4.00 | 3.84 | 3.67 |
| | 13.74 | 10.92 | 9.78 | 9.15 | 8.75 | 8.47 | 8.10 | 7.72 | 7.31 | 6.88 |
| 7 | 5.59 | 4.74 | 4.35 | 4.12 | 3.97 | 3.87 | 3.73 | 3.57 | 3.41 | 3.23 |
| | 12.25 | 9.55 | 8.45 | 7.85 | 7.46 | 7.19 | 6.84 | 6.47 | 6.07 | 5.65 |
| 8 | 5.32 | 4.46 | 4.07 | 3.84 | 3.69 | 3.58 | 3.44 | 3.28 | 3.12 | 2.93 |
| | 11.26 | 8.65 | 7.59 | 7.01 | 6.63 | 6.37 | 6.03 | 5.67 | 5.28 | 4.86 |
| 9 | 5.12 | 4.26 | 3.86 | 3.63 | 3.48 | 3.37 | 3.23 | 3.07 | 2.90 | 2.71 |
| | 10.56 | 8.02 | 6.99 | 6.42 | 6.06 | 5.80 | 5.47 | 5.11 | 4.73 | 4.31 |
| 10 | 4.96 | 4.10 | 3.71 | 3.48 | 3.33 | 3.22 | 3.07 | 2.91 | 2.74 | 2.54 |
| | 10.04 | 7.56 | 6.55 | 5.99 | 5.64 | 5.39 | 5.06 | 4.71 | 4.33 | 3.91 |
| 12 | 4.75 | 3.88 | 3.49 | 3.26 | 3.11 | 3.00 | 2.85 | 2.69 | 2.50 | 2.30 |
| | 9.33 | 6.93 | 5.95 | 5.41 | 5.06 | 4.82 | 4.50 | 4.16 | 3.78 | 3.36 |
| 14 | 4.60 | 3.74 | 3.34 | 3.11 | 2.96 | 2.85 | 2.70 | 2.53 | 2.35 | 2.13 |
| | 8.86 | 6.51 | 5.56 | 5.03 | 4.69 | 4.46 | 4.14 | 3.80 | 3.43 | 3.00 |
| 16 | 4.49 | 3.63 | 3.24 | 3.01 | 2.85 | 2.74 | 2.59 | 2.42 | 2.24 | 2.01 |
| | 8.53 | 6.23 | 5.29 | 4.77 | 4.44 | 4.20 | 3.89 | 3.55 | 3.18 | 2.75 |
| 18 | 4.41 | 3.55 | 3.16 | 2.93 | 2.77 | 2.66 | 2.51 | 2.34 | 2.15 | 1.92 |
| | 8.28 | 6.01 | 5.09 | 4.58 | 4.25 | 4.01 | 3.71 | 3.37 | 3.01 | 2.57 |
| 20 | 4.35 | 3.49 | 3.10 | 2.87 | 2.71 | 2.60 | 2.45 | 2.28 | 2.08 | 1.84 |
| | 8.10 | 5.85 | 4.94 | 4.43 | 4.10 | 3.87 | 3.56 | 3.23 | 2.86 | 2.42 |
| 25 | 4.24 | 3.38 | 2.99 | 2.76 | 2.60 | 2.49 | 2.34 | 2.16 | 1.96 | 1.71 |
| | 7.77 | 5.57 | 4.68 | 4.18 | 3.86 | 3.63 | 3.32 | 2.99 | 2.62 | 2.17 |
| 30 | 4.17 | 3.32 | 2.92 | 2.69 | 2.53 | 2.42 | 2.27 | 2.09 | 1.89 | 1.62 |
| | 7.56 | 5.39 | 4.51 | 4.02 | 3.70 | 3.47 | 3.17 | 2.84 | 2.47 | 2.01 |
| 40 | 4.08 | 3.23 | 2.84 | 2.61 | 2.45 | 2.34 | 2.18 | 2.00 | 1.79 | 1.51 |
| | 7.31 | 5.18 | 4.31 | 3.83 | 3.51 | 3.29 | 2.99 | 2.66 | 2.29 | 1.81 |
| 60 | 4.00 | 3.15 | 2.76 | 2.52 | 2.37 | 2.25 | 2.10 | 1.92 | 1.70 | 1.39 |
| | 7.08 | 4.98 | 4.13 | 3.65 | 3.34 | 3.12 | 2.82 | 2.50 | 2.12 | 1.60 |

APPENDIX 3

# Answers to Problems

## Problems 1.1, page 5

**1.** $x^4 - 6x^3 + 3x^2 + 42x - 70 = 0$ **2.** (*i*) $-2, 1 + 3i, 1 - 3i$ (*ii*) $2 \pm \sqrt{3}, 3, -5$

**5.** Two roots between (1, 2) and (– 3, – 4)

**6.** $2, 2, -\frac{1}{3}$ **7.** 3 **8.** $a = 2, b = 1$

**9.** 6, 4, – 1 **10.** – 4, 2, 6 **11.** 1, 1, 2, 2

**12.** $\frac{1}{2}(3 \pm \sqrt{5}\,;\ \frac{1}{2}(5 \pm \sqrt{5})$ **13.** 1, 4, 7 **14.** $1, \frac{1}{2}, \frac{1}{4}$

**16.** (*i*) – 5, – 2, 1, 4 (*ii*) 1, – 2, 4, – 8 **17.** (*i*) $m^2 - 2ln$, (*ii*) $lm - n$

**18.** 36 **19.** (*i*) 4/3, (*ii*) 16/9.

## Problems 1.2, page 8

**1.** $x^3 + 6x^2 - 36x + 27 = 0$ **2.** $6x^5 - 7x^4 - 13x^3 + 4x^2 - 2 = 0$ **3.** $10x^4 + 9x^3 + 8x^2 - 7x + 1 = 0$

**4.** $-\frac{1}{2}, \frac{1}{3}, 2$ **5.** (*i*) $x^3 - 9x^2 + 26x - 24 = 0$ ; (*ii*) $x^4 + 13x^3 + 60x^2 + 116x + 80 = 0$ ;
(*iii*) $x^5 + 7 = 0$

**6.** $x^3 + 15x^2 + 52x - 36 = 0$ **7.** $y^3 + (p^3 + 3q)\,y^2 + 3q^2y + q^3 = 0$ **8.** $3x^3 - 11x^2 + 9x - 2 = 0$

**9.** $y^3 - qy^2 + pry - r^2 = 0$

**10.** (*a*) $y^3 + 4my - 8n = 0$ ; (*b*) $nx^3 + m^2x^2 - 2mnx + n^2 = 0$ ; (*c*) $x(nx + m)^2 = n$

**11.** $y^3 - 30y^2 + 225y - 68 = 0$

**12.** (*i*) $\dfrac{-5 \pm \sqrt{21}}{2}, \dfrac{5 \pm \sqrt{91}i}{12}$ (*ii*) 2, 2, 1/2, 1/2 (*iii*) 1, – 2, 4, – 1/2, 1/4 ;
(*iv*) – 1, – 2, 3, – 1/2, 1/3 ; (*v*) $\pm 1, -3, -1/3, \dfrac{3 \pm \sqrt{5}}{2}$

**13.** $\frac{1}{2}(5 \pm \sqrt{21})$ ; $\frac{1}{2}(-3 \pm \sqrt{5})$ **14.** – 1, – 2, – 6, – 7.

## Problems 1.3, page 11

**1.** – 6, 3, 3 **2.** $5, \frac{1}{2}(-5 \pm i\sqrt{3})$ **3.** $6, -3 \pm 2\sqrt{(-3)}$

**4.** $-1, -2, \frac{1}{2}$ **5.** $\dfrac{1}{2}, -\dfrac{1}{6}(3 \pm i\sqrt{3})$ **6.** $5, \dfrac{1}{2}(1 + i\sqrt{3})$

**7.** $2\cos\dfrac{2\pi}{9}, 2\cos\dfrac{8\pi}{9}, 2\cos\dfrac{14\pi}{9}$ **8.** $\dfrac{1}{2}, \dfrac{-7 \pm 9i\sqrt{3}}{6}$.

## Problems 1.4, page 12

**1.** 1, 2, 3, 4
**2.** $-3, 1, \pm 2$
**3.** $4, -2, -1 \pm i$
**4.** $-1, 3, 3 \pm \sqrt{30}$
**5.** $1 \pm \sqrt{7}, 2 \pm \sqrt{3}$
**6.** $1 \pm 2i, -1 \pm \sqrt{2}$
**7.** $2 \pm \sqrt{3}, -2 \pm i\sqrt{3}$
**8.** $2, 4, 2 \pm 2i\sqrt{2}$.

## Problems 1.5, page 15

**1.** 1.32
**2.** 2.29
**3.** 0.45
**4.** (*i*) 0.71 rad
**5.** 1.81 rad
**6.** 0.26.

## Problems 1.6, page 15

**1.** (*d*)
**2.** (*c*)
**3.** (*c*)
**4.** (*c*)
**5.** (*c*)
**6.** (*a*)
**7.** (*d*)
**8.** (*b*)
**9.** (*c*)
**10.** (*a*)
**11.** (*c*)
**12.** minus
**13.** § 15.1 (v)
**14.** $p/q$
**15.** 21
**16.** $-3$ and $-2$
**17.** Conjugate pairs
**18.** $f(x)$ is continuous in $(a, b)$
**19.** $x^3 - 9x^2 + 29x - 24 = 0$
**20.** $3, 6, -2$
**21.** $x^3 - 200x - 7000 = 0$
**22.** $p/r$
**23.** $x^4 + 2x^3 - x^2 - 6x - 6 = 0$
**24.** 6
**25.** minus
**26.** $pq = r$
**27.** $1, \frac{1}{2}(-1 \pm \sqrt{3}i)$
**28.** (*iii*)
**29.** $1, 1, -2$
**30.** $x^3 - 7x^2 + 12x - 10 = 0$
**31.** Zero and 2
**32.** 21
**33.** True
**34.** True.

## Problems 2.1 page 25

**5.** (*i*) 1 (*ii*) 0
**13.** $(a-b)(b-c)(c-a)$
**14.** $(a-b)(b-c)(c-a)(ab+bc+ca)$
**15.** $(b-c)(c-a)(a-b)(a-1)(b-1)(c-1)$
**16.** $(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$
**17.** $x = 0, \pm\sqrt{(a^2+b^2+c^2-ab-bc-ca)}$
**18.** $0, -\dfrac{1}{2}$.

## Problems 2.2 page 31

**1.** $x = 0, 3$
**2.** $x = -3, y = -2, z = -4, a = 3$
**3.** $x = 3, y = 8$
**4.** $-2\begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$
**5.** $AB = \begin{bmatrix} 4 & 4 & 7 & 10 \\ 10 & 7 & 11 & 21 \\ 4 & 3 & 3 & 9 \end{bmatrix}$
**7.** (*i*) $[ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy]$

(*ii*) $\begin{bmatrix} 8 & 7 \\ 122 & 104 \\ -365 & -131 \end{bmatrix}$ (*iii*) $\begin{bmatrix} 9 & 6 \\ -18 & -12 \\ 27 & 18 \end{bmatrix}$

**10.** $3I$
**11.** $\begin{bmatrix} -6 & 1 & 2 \\ 5 & 4 & 4 \\ 2 & 8 & -3 \end{bmatrix}$
**15.** $\begin{bmatrix} 1 & 0 & 0 \\ 7/5 & 1 & 0 \\ 3/5 & 41/19 & 1 \end{bmatrix}\begin{bmatrix} 5 & -2 & 1 \\ 0 & 19/5 & -32/5 \\ 0 & 0 & 327/19 \end{bmatrix}$

## Problems 2.3, page 35

**2.** (i) $\begin{bmatrix} 3 & 0 & 5.5 \\ 0 & 7 & 1.5 \\ 5.5 & 1.5 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -2 & 5 \\ 2 & 0 & -2.5 \\ -0.5 & 2.5 & 0 \end{bmatrix}$ (ii) $\begin{vmatrix} a & \frac{1}{2}(a+c) & \frac{1}{2}(b+c) \\ \frac{1}{2}(a+c) & b & \frac{1}{2}(a+b) \\ \frac{1}{2}(b+c) & \frac{1}{2}(a+b) & c \end{vmatrix} + \begin{vmatrix} 0 & \frac{1}{2}(a-c) & \frac{1}{2}(b-c) \\ \frac{1}{2}(c-a) & 0 & \frac{1}{2}(b-a) \\ \frac{1}{2}(c-b) & \frac{1}{2}(a-b) & 0 \end{vmatrix}$

**4.** (i) $\begin{bmatrix} 2 & 4/5 & 9/5 \\ 3 & -4/5 & -14/5 \\ -1 & 1/5 & 6/5 \end{bmatrix}$ (ii) $\begin{bmatrix} 1/33 & -4/33 & 2/11 \\ -4/33 & 14/33 & 13/33 \\ 2/11 & 13/33 & -1/33 \end{bmatrix}$

**5.** $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **6.** (i) $\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$.

## Problems 2.4, page 40

**1.** 3 **2.** 2 **3.** 3 **4.** 2 **5.** 3

**6.** No value of $p$ is possible.

**7.** (i) $\begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$ (ii) $\frac{1}{21}\begin{bmatrix} 1 & 10 & -7 \\ 1 & -11 & 14 \\ -3 & 12 & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} -1/9 & 2/9 & 2/9 \\ 2/9 & -1/9 & 2/9 \\ 2/9 & 2/9 & -1/9 \end{bmatrix}$ (iv) $\begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -1/2 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$

**8.** $P = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & 2/3 & -7/24 & 5/6 \\ 0 & -1/3 & 0 & 1/3 \\ 0 & 0 & -5/24 & 1/2 \\ 0 & 0 & -1/12 & 0 \end{bmatrix}$, Rank $(A) = 3$

**9.** $\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$ **10.** (i) $P = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix}, Q = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$;

(ii) $P = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{4}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{6} & -\frac{5}{6} & \frac{7}{6} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

**11.** (i) 3 (ii) 3 (iii) 2 (iv) 3.

## Problems 2.5, Page 43

**1.** $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ **2.** $\frac{1}{8}\begin{bmatrix} 2 & 2 & -2 \\ -9 & 11 & 5 \\ 5 & -7 & -1 \end{bmatrix}$

**3.** $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 1 & -2 & 2 & -3 \\ 0 & 1 & -1 & 1 \\ -2 & 3 & -2 & 3 \end{bmatrix}$ **4.** $\begin{bmatrix} 7 & -3 & 0 & -5 \\ 8 & 1 & -2 & -11 \\ -5 & 0 & 1 & 6 \\ 19 & 5 & -6 & -28 \end{bmatrix}$

## Problems 2.6, page 45

**1.** $x = 1, y = 2, z = 1$ **2.** $x = 2, y = -1, z = 1/2$

**3.** $x = 1.2, y = 2.2, z = 3.2$ **4.** $x = y = z = e^2$

**5.** $u = 1, v = 1/2, w = 1/3$

**6.** $x_1 = 1, x_2 = -5, x_3 = 5$

**7.** $x = 2, y = 1, z = 0$

**8.** $x = 1/7, y = 10/7, z = 1/7$

**9.** $x = y = z = 2$

**10.** $x_1 = 2, x_2 = 1, x_3 = -1, x_4 = 3$

**11.** $x = 1, y = -1, z = 1$

**12.** $i_1 = 1.5, i_3 = 2.5$

**13.** $i_1 = 369/175, i_2 = 24/25, i_3 = 72/175$

**14.** $x_1 = 2, x_2 = 1/5, x_3 = 0, x_4 = 4/5.$

## Problems 2.7, page 50

**1.** Consistent ; $x = 1, y = 3k - 2, z = k$ for all $k$

**2.** $k = 1, x = -3z, y = 2z + 1$ ; $k = 2, x = 1 - 3z, y = 2z$

**3.** (*i*) $\lambda = 3, \mu \neq 10$ ; (*ii*) $\lambda \neq 3$ ; (*iii*) $\lambda = 3, \mu = 10$

**4.** (*i*) Equations are inconsistent ; (*ii*) consistent ; $x = -1, y = 1, z = 2$ ;
(*iii*) Equations are inconsistent ; (*iv*) Consistent : $x = 2, y = 1, z = -4$

**5.** If $a = -1, b = 6$, equations will be consistent and have infinite number of solutions
If $a = -1, b \neq 6$, equations will be inconsistent ;
If $a \neq -1$, $b$ has any value, equations will be consistent and have a unique solution

**6.** $\lambda \neq -5, x = 4/7, y = -9/7, z = 0$ ; $\lambda = -5, x = \frac{1}{7}(4 - 5k), y = \frac{1}{7}(13k - 9), z = k$ for all $k$

**9.** $\lambda = -1, 1, 12$ ; $x = -1/11, y = -15/11$ ; $x = -5, y = 1$ ; $x = \frac{1}{2}, y = 1$

**11.** $k = 3$ is the only real value for which $x = y = z$

**12.** $\lambda = 1, x_1 = 2t - s, x_2 = t, x_3 = s$ ; $\lambda = -3, x_1 = -t, x_2 = -2t, x_3 = t$

**13.** $\lambda = 1, -9$. For $\lambda = 1$, sol. is $x = k, y = -k, z = 2k$
For $\lambda = -9$, sol. is $x = 3k, y = 9k, z = -2k$

**14.** (*i*) Have infinite number of non-trivial solutions ; $x = \lambda - 5\mu/3, y = \lambda - 4\mu/3, z = \lambda, w = \mu$ for all values of $\lambda$ and $\mu$. (*ii*) $x = 11k_2 + 6k_1, y = -8k_2 - 3k_1, z = k_2, w = k_1$ where $k_1, k_2$ are arbitrary constants.

## Problems 2.8, page 54

**1.** $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 11 & -2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

**2.** $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}, A^{-1} = A' = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

**3.** $Z = (BA)X$, where $BA = \begin{bmatrix} 1 & 4 & -1 \\ -1 & 9 & -1 \\ -3 & 14 & -1 \end{bmatrix}$

**4.** $x_1 = 19y_1 - 9y_2 + 2y_3$ ; $x_2 = -4y_1 + 2y_2 - y_3$ ; $x_3 = -2y_1 + y_2$

**6.** $a = \pm \frac{1}{\sqrt{2}}, b = \pm \frac{1}{\sqrt{6}}, c = \pm \frac{1}{\sqrt{3}}$

**9.** (*i*) No. (*ii*) No. (*iii*) Yes, $9x_1 - 12x_2 + 5x_3 - 5x_4 = 0.$

## Problems 2.9, page 60

**1.** 10 ; 30

**2.** (*a*) 10, 3 ; (1, 2), (3, – 1) ; (*b*) – 1, 6 ; (1, 1), (2, – 5)

**3.** (*a*) 0, 3, 5 ; (1, 2, 2), (2, 1, – 2), (2, – 2, 1) (*b*) 1, 2, 3 ; (1, 0, – 1), (0, 1, 0), (1, 0, 1)
(*c*) 5, – 3, – 3 ; (1, 2, – 1), (2, – 1, 0), (3, 0, 1)
(*d*) 8, 2, 2 ; (2, – 1, 1), (1, 0, – 2), (1, 2, 0) (*e*) 2, 3, – 1 ; (3, 1, 1), (– 4, 1, – 3), (0, 5, 5)

**5.** (*i*) 8, 12, 6 (*ii*) 49, 121, 25

**6.** 1, 1, 1/5

**9.** (*i*) $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ ; (*ii*) $\frac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix}$ ; (*iii*) $\frac{1}{4}\begin{bmatrix} 12 & 4 & 6 \\ -5 & -1 & -3 \\ -1 & -1 & -1 \end{bmatrix}$ ; (*iv*) $\begin{bmatrix} 1 & 1/2 & -2/3 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$

**10.** $\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0, \frac{1}{35}\begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}$

**11.** (*i*) $\frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$ (*ii*) $\frac{1}{3}\begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$ (*iii*) $\frac{1}{27}\begin{bmatrix} 1 & 10 & -8 \\ -8 & 1 & 10 \\ 10 & -8 & 1 \end{bmatrix}$

**12.** $625I$

**13.** $\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0, \begin{bmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{bmatrix}$

**14.** $\begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}$

**15.** $A^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 6 & 6 \\ -1 & 6 & 2 \\ 1 & -10 & -6 \end{bmatrix}, A^{-2} = \frac{1}{16}\begin{bmatrix} 1 & -18 & -18 \\ -5 & 10 & -6 \\ 5 & 6 & 22 \end{bmatrix},$

$A^{-3} = \frac{1}{64}\begin{bmatrix} 1 & 78 & 78 \\ -21 & 90 & 26 \\ 21 & -154 & -90 \end{bmatrix}$

**16.** $\begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$.

## Problems 2.10, page 67

**3.** $\begin{bmatrix} -5 & 0 \\ 0 & 2 \end{bmatrix}$

**4.** $A^n = \begin{bmatrix} 2^n + 3.6^n & -3.2^n + 3.6^n \\ -2^n + 6^n & 3.2^n + 6^n \end{bmatrix}; A^4 = \begin{bmatrix} 976 & 960 \\ 320 & 336 \end{bmatrix}$

**5.** $\begin{bmatrix} 251 & -405 & 235 \\ -405 & 891 & -405 \\ 235 & -405 & 251 \end{bmatrix}$

**7.** $(1, 1, -1), (1, 1, -1), (2, -1, 1)\ ;\ 4x^2 + y^2 + z^2$

**8.** $x^2 + y^2 - 2z^2$

**10.** (*a*) 1, 2, 4 ; (1, 0, 0), (0, 1, 1), (0, 1, − 1) ; $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$
(*b*) $x_1^2 + 2x_2^2 + 4x_3^2$

**10.** (*i*) $x_1^2 + 4x_2^2 + 4x_3^2, \begin{bmatrix} -1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$, positive definite ;

(*ii*) $3y^2 + 15z^2, \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$, positive semidefinite

**11.** 2, 1

**12.** Indefinite.

## Problems 2.11, Page 71

**8.** $\begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + i\begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 3 \\ -1 & 3 & 0 \end{bmatrix}$

## Problems 2.12, page 72

**1.** (*b*) **2.** (*a*) **3.** (*c*) **4.** (*c*) **5.** (*c*) **6.** (*a*)
**7.** (*a*) **8.** (*b*) **9.** (*d*) **10.** (*a*) **11.** (*d*) **12.** (*c*)
**13.** (*b*) **14.** (*d*) **15.** (*c*) **16.** 2 **17.** sum **18.** 0, 8
**19.** 2 **20.** $\begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}$ **21.** 0

**22.** All the eigen values are ≥ 0 and at least one eigen value is zero.

**23.** (*a*) $n = p$, (*b*) $m = p, n = q$ **24.** 8 **25.** (*b*)

**26.** $\begin{bmatrix} 2 & 3 & 1 \\ 4 & 6 & 2 \\ 6 & 9 & 3 \end{bmatrix}$ **27.** $x^2 + 4xy - 4y^2$ **28.** $x = y = z = 0$ **29.** 2, 2, 8

**30.** $A^2 = A$ **31.** 2 **32.** 1, 4, 9 **33.** (*iv*)

**34.** 4 **35.** zero **36.** Indefinite **37.** 1 – 1

**38.** The elements of its leading diagonal **39.** 2 **40.** $\lambda_i$, $i = 1, 2, ..., n$

**41.** (*c*) **42.** $A$ or $A^T$ **43.** 1, 1/2, 1/3 **44.** $\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$

**45.** 1, 1/3 **46.** $x = 3 - 1$ **47.** Symmetric ; skew-symmetric **48.** 7 ; 5

**49.** $\begin{bmatrix} \cos 3\theta & \sin 3\theta \\ -\sin 3\theta & \cos 3\theta \end{bmatrix}$ **50.** its determinant **51.** $\lambda^2 - 6\lambda + 3 = 0$ **52.** Augmented matrix

**53.** $\begin{bmatrix} 4 & -1 & 3 \\ -1 & -2 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ **54.** $\lambda_1^3, \lambda_2^3, \lambda_3^3$ **55.** $1/\lambda$ **56.** $\begin{bmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 1 & -0.5 \\ -0.5 & -0.5 & 1 \end{bmatrix}$

**57.** 38 **58.** 2 **59.** Index = 2, Signature = 1

**60.** False **61.** False **62.** False **63.** True

**64.** False **65.** True **66.** False **67.** True

**68.** False **69.** True **70.** True **71.** True.

## Problems 3.1, page 80

**1.** (*i*) $\sqrt{159}$ ; $6/\sqrt{159}, 1/\sqrt{159}, 11/\sqrt{159}$ ; (*ii*) $\sqrt{131}$ ; $9/\sqrt{131}, -7/\sqrt{131}, 1/\sqrt{131}$

**3.** 90° **5.** $x = 1, y = -1$ **11.** 2 : 1.

## Problems 3.2, page 88

**1.** 5 **2.** (*ii*) $A, B, C$ form a $\Delta$, rt. $\angle ed$ at $C$ **4.** $\cos^{-1}(.62)$

**6.** 2.11 **7.** 13 ; 12/13, 4/13, 3/13 **13.** 60°

**14.** $\cos^{-1}\left(-\dfrac{2}{3\sqrt{21}}\right)$.

## Problems 3.3, page 89

**1.** $\mathbf{I} - 10\mathbf{J} - 18\mathbf{K}$, $\dfrac{1}{5\sqrt{17}}\mathbf{I} - \dfrac{2}{\sqrt{17}}\mathbf{J} - \dfrac{18}{5\sqrt{17}}\mathbf{K}$, $\sin^{-1}\left(\dfrac{5\sqrt{17}}{21}\right)$ **3.** $(2\mathbf{J} + \mathbf{K})/\sqrt{5}$

**5.** $\frac{1}{2}\sqrt{(94)}$ **6.** (*b*) $10\sqrt{3}$ **7.** $-2/\sqrt{26}$.

## Problems 3.4, page 92

**1.** 40 **2.** 17 ; $-24\mathbf{I} + 13\mathbf{J} + 4\mathbf{K}$ **3.** 3.33

**4.** 70.5 **5.** $2\mathbf{I} - 7\mathbf{J} - 2\mathbf{K}$ ; $\sqrt{(57)}$ **6.** (1, 2, 2)

**7.** 6 **8.** 8.25 **9.** $\frac{5}{6}(-3\mathbf{I} + 2\mathbf{J} + 10\mathbf{K})$, $\frac{5}{6}\sqrt{113}$.

## Problems 3.5, page 96

**1.** 7 **2.** – 4 **3.** (*ii*) Yes **4.** Not linearly dependent

**5.** 5/6 **6.** (*i*) 15 ; (*ii*) $1\frac{1}{3}$ **11.** (*i*) $-7\mathbf{I} - 11\mathbf{J} + 5\mathbf{K}$ ; (*ii*) $-30\mathbf{I} - 15\mathbf{J} + 15\mathbf{K}$

**15.** (*b*) $\frac{1}{6}abc\begin{vmatrix} 1 & \cos\psi & \cos\phi \\ \cos\psi & 1 & \cos\theta \\ \cos\phi & \cos\theta & 1 \end{vmatrix}$.

## Problems 3.6, page 101

**1.** $2x - y + 3z = 9$ **2.** $\mathbf{R} \,.\, (2\mathbf{I} + 2\mathbf{J} + \mathbf{K}) = 5$ **3.** $\frac{1}{3}(2\mathbf{I} + 2 + \mathbf{K})$

**4.** 3 **5.** $4x - 3y + 2z = 3$ **7.** $x - 5y - 2z + 7 = 0$

**8.** $2$ ; $2x + 2y - 3z = 6$ **9.** $\Sigma(x_1 - x_2)\left\{x - \frac{1}{2}(x_1 + x_2)\right\} = 0$ **10.** $y = 2$

**11.** $3x + 4y - 5z = 9$ **12.** $k = 10.2$ ; $5x - 15y - 21z = 34$ **13.** $1/6$ . $2x - 3y + 6z + 5 = 0$

**14.** $\cos^{-1}(\sqrt{2}/3)$ **15.** (*i*) $\frac{5}{\sqrt{83}}, \frac{-7}{\sqrt{83}}, \frac{-3}{\sqrt{83}}$ (*ii*) 83.7° (*iii*) $5x - 7y - 3z + 7 = 0$

**17.** $6x + 3y - 2z = 18$ ; $2x - 3y - 6z = 6$ **20.** $x^{-2} + y^{-2} + z^{-2} = 9p^{-2}$

**21.** $xyz = 6k^3$

**22.** (*i*) $25x + 17y + 62z - 78 = 0$ ; (*ii*) $x + 35y - 10z - 156 = 0$ ; (*iii*) bisects the acute angle.

## Problems 3.7, page 105

**1.** $\frac{x-3}{1} = \frac{y-2}{3} = \frac{z-4}{-2}$ **2.** 43°3′ **3.** 90°

**4.** $x + 2 = \frac{y-3}{2} = \frac{z-4}{2}$ **5.** 3 **6.** $\frac{x-1}{1} = \frac{y-2}{-2} = \frac{z+1}{1}$

**7.** $\frac{x-1}{1} = \frac{y+2}{-19} = \frac{z-3}{3.5}$ ; $\frac{x-1}{11} = \frac{y+1}{13} = \frac{z-3}{-21.5}$ **8.** (3, 4, 5)

**9.** 4.1 **10.** 8.57

**11.** (3, 4, 5) ; (*ii*) (26/7, – 15/7, 17/7) **12.** 40°15′

**13.** $29x - 27y - 22z = 85$ **14.** $2 - x = y + 1 = (z + 1)/3$.

## Problems 3.8, page 107

**1.** $7x - 2y - 3z = 0$ **2.** $2x + 3y + 6z = 38$ **3.** $11x + 12y - 8z = 5$

**4.** $3y - z = 2$ **5.** $\frac{x-4}{7} = \frac{y-6}{-13} = \frac{z+2/3}{9}$ **6.** $x + y + 2z = 1, x + y + (2/5)z = 1$

**7.** $\frac{x+4/15}{-11} = \frac{y-2/5}{9} = \frac{z}{15}$ **8.** $\frac{x-2}{3} = \frac{y-1}{-1} = \frac{z-1}{1}$.

## Problems 3.9, page 110

**1.** $x - 2y + z = 0$ **2.** (5, – 7, 6)

**3.** $\frac{a\alpha + b\beta + c\gamma + d}{al + bm + cn} = \frac{a'\alpha + b'\beta + c'\gamma + d'}{a'l + b'm + c'n}$ **4.** (0, 1, 2) ; $4x + y - 2z + 3 = 0$

**7.** $-\frac{1}{6}(x - 5) = y - 3 = \frac{1}{2}(z - 13)$ **8.** $\frac{x-2}{7} = \frac{y-3}{4} = \frac{z-1}{-5}$

**9.** (2, 8, – 3) ; (0, 1, 2) ; 8.83.

## Problems 3.10, page 113

**1.** $1/\sqrt{6}$ ; $11x + 2y - 7z + 0 = 0$ ; $7x + y - 5z + 7 = 0$

**2.** 10.77 ; $\frac{1}{2}(x - 3) = \frac{1}{3}(y - 5) = \frac{1}{4}(z - 7)$ ; (3, 5, 7) ; (– 1, – 1, – 1)

**3.** $\frac{1}{\sqrt{5}}$ ; $3x - 10y + 6z - 1 = 0 = x + 2z$.

## Problems 3.11, page 115

**4.** First and second planes cut along $x - 36 = -\frac{1}{2}(y + 22) = z$.

## Problems 3.12, page 118

**1.** $x^2 + y^2 + z^2 - 4x + 6y - 2z + 5 = 0$ ; $(2, -3, -1)$ ; 3

**2.** $x^2 + y^2 + z^2 - 2x + 2y - 4z = 0$ ; $(1, -1, 2)$ ; $\sqrt{6}$

**3.** (*a*) $x^2 + y^2 + z^2 - 4x - 4y - 4z + 3 = 0$ (*b*) $3(x^2 + y^2 + z^2) - 2(x + y + z) - 1 = 0$

**7.** (*i*) $x^2 + y^2 + z^2 - ax - by - cz = 0$,

(*ii*) $x^2 + y^2 + z^2 - ax - by - cz = 0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ ; $\left[\frac{a(b^{-2} + c^{-2})}{2\Sigma a^{-2}}, \frac{b(c^{-2} + a^{-2})}{2\Sigma a^{-2}}, \frac{c(a^{-2} + b^{-2})}{2\Sigma a^{-2}}\right]$

**8.** $(1, 3, 4)$ ; $\sqrt{7}$

**10.** $x^2 + y^2 + z^2 + 2(x + y + z + 1) = 0$

**11.** $13(x^2 + y^2 + z^2) - 35x - 21y + 43z + 176 = 0$

**12.** $3(x^2 + y^2 + z^2) - 7x - 8y + z + 10 = 0$

**13.** $x^2 + y^2 + z^2 + 7y - 8z + 24 = 0$.

## Problems 3.13, page 120

**1.** (*i*) $x + 3 = 0, x - 7 = 0$ (*ii*) $x + 2y + 2z = 9,\ 2x + y - 2z = 9$

**2.** $x^2 + y^2 + z^2 + 2x + 4y + 6z - 11 = 0$ and $5(x^2 + y^2 + z^2) - 4x - 8y - 12z - 13 = 0$

**3.** (*i*) $x^2 + y^2 + z^2 - 10y - 10z - 31 = 0$

**4.** (*ii*) $x^2 + y^2 + z^2 - 14(x + y + z) + 98 = 0$

**6.** $\frac{x - 0.6}{-2} = \frac{y - 2.4}{7} = \frac{z}{5}$

**8.** $3\sqrt{6}, \sqrt{6}$

**9.** $3x + y + z + 6 = 0$

**10.** $(12/5, 4, 9/5)$.

## Problems 3.14, page 124

**1.** $(\beta z - \gamma y)^2 = 4a(\alpha z - \gamma x)(z - \gamma)$

**2.** $528x^2 + 363y^2 + 76z^2 - 528xy - 264yz + 353zx + 704x + 1352z - 4436 = 0$

**3.** $5x^2 + 3y^2 + z^2 - 2xy - 6yz - 4zx + 6x + 8y + 10z - 26 = 0$

**4.** $x^2 + y^2 - 3z^2 - 2x - 2y + 6z - 1 = 0$

**5.** $x^2 + y^2 = z^2 \tan^2 \alpha$

**6.** $x^2 + 7y^2 + z^2 + 8xy + 8yz - 16zx = 0$

**7.** $4x^2 + 40y^2 + 19z^2 - 48xy - 72yz + 36zx = 0$

**8.** $x^2 - y^2 + z^2 + 4y - 4z = 0$

**9.** $yz \pm zx \pm xy = 0$, $\cos^{-1}(1/\sqrt{3})$ ; $x = y/\pm 1 = z/\pm 1$

**10.** $\cos^{-1} 4/\sqrt{(41)}$ ; $25x^2 - 16y^2 - 16z^2 = 0$

**11.** $4x^2 + 4y^2 - z^2 + 20z - 100 = 0$

**12.** $x = y/2 = z/-1$ ; $x/-2 = y = z$

**14.** $-2x^2 + y^2 - 2z^2 + 4xy - 8xz + 4yz + 8x - 10y + 8z - 3 = 0$.

## Problems 3.15, page 126

**1.** $5x^2 + 8y^2 + 5z^2 + 4yz + 8xz - 4xy - 144 = 0$

**2.** $3x^2 + 6y^2 + 3z^2 + 8yz - 2zx + 6x - 24y - 18z + 24 = 0$

**3.** $45x^2 + 40y^2 + 13z^2 + 12xy + 36yz - 24zx - 42x - 112y - 126z - 392 = 0$

**4.** $x^2 + y^2 + z^2 - yz - zx - xy = a^2$

**5.** $9x^2 + 5y^2 + 9z^2 + 12xy + 6yz - 36x - 30y - 18z + 36 = 0$ ; $\pi$ units

**6.** $x^2 + y^2 - 2x - 4y - 11 = 0$

**7.** $a(nx - lz)^2 + 2h(nx - lz)(ny - mz) + b(ny - mz)^2 + 2gn(nx - lz) + 2fn(ny - mz) + cn^2 = 0$.

## Problems 3.16, page 131

**1.** Ellipsoid, 33.51

**2.** Hyperboloid of revolution of one sheet ; Hyperbola $5x^2 - y^2 = 6$. No area

3. Right circular cylinder with axis along $z$-axis
4. Hyperbolic paraboloid
5. Hyperboloid of two sheets
6. Parabolic cylinder
7. Right circular cylinder
8. Cone with vertex at the origin
9. Hyperbolic paraboloid
10. Hyperboloid of two sheets.

## Problems 3.17, page 131

1. (*b*) 2. (*a*) 3. (*c*) 4. (*b*) 5. (*b*) 6. (*a*)
7. (*c*) 8. (*d*) 9. (*c*) 10. (*c*) 11. (*d*) 12. (*c*)
13. (*b*) 14. (*c*) 15. (*c*) 16. (*c*) 17. (*c*) 18. (*b*)
19. (*c*) 20. (*c*) 21. (*b*) 22. (*b*) 23. (*c*) 24. (*c*)
25. (*b*) 26. (*a*) 27. $1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$ 28. 0, 0, 1 29. $x = 0, y = 0$

30. $(-3, 2, -1)$
31. 8 or $-10$
32. $\begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} = 0$

33. $y^2 + z^2 = (bx/a)^2$
34. $12x + 31y - 20z = 66$
35. 523.6

36. $\cos^{-1}(6/\sqrt{42})$
37. $\dfrac{x^2}{4} + \dfrac{y^2}{9} + z^2 = (x + y + z)^2$

38. $(3, 5, 7), (5, 8, 11)$
38. $-(x - 1) = y - 2 = \frac{1}{6}(z - 3)$
40. $x^2 + y^2 + z^2 + x - 6y - 7z + 9 = 0$

41. $al + bm + cn = 0, ax_1 + by_1 + cz_1 + d = 0$
42. $2/\sqrt{26}$

43. $(3/2, -2, 2), 3\sqrt{5}/2$
44. $\sqrt{44}/3$
45. $\dfrac{x + 1/3}{1} = \dfrac{y + 2/3}{-2} = \dfrac{z}{1}$

46. Parabolic cylinder
47. Hyperboloid of two sheets

48. $\cos^{-1}\left(\dfrac{7}{\sqrt{84}}\right)$
49. $6, -4, 12$
50. 6

51. True
52. True
53. True

54. Elliptic cylinder
55. $4(x^2 + y^2 + z^2) + 9(xy + yz + zx) = 0$

56. $(\mathbf{I} - 2\mathbf{J} - 8\mathbf{K})/\sqrt{69}$

## Problems 4.1, page 135

6. $8t^3/(1 - t^2)^3$
7. $-\dfrac{3}{2}$
8. $\sin t/a \cos^4 t$.

## Problems 4.2, page 138

1. $(-1)^{n-1}(n-1)!\,2^n\,[(2x+1)^{-n} + (2x-1)^{-n}]$
2. $(-1)^n \left\{\dfrac{n!}{(x+1)^{n+1}} - \dfrac{(n-1)!}{(x+2)^n} + \dfrac{(n-1)!}{(x+1)^n}\right\}$

3. $\dfrac{1}{16}\{2\sin(x + n\pi/2) + 3^n \sin(3x + n\pi/2) - 5^n \sin(5x + n\pi/2)\}$

4. $\dfrac{1}{256}[9^n \cos(9\theta + n\,\pi/2) + 9.7^n \cos(7\theta + n\pi/2) + 36.5^n \cos(5\theta + n\pi/2) + 84.3^n \cos(3\theta + n\pi/2) + 126\cos\theta]$

5. $\dfrac{(20)^{n/2}}{2}[e^{2x}\sin(2x + n\tan^{-1}2) - e^{-2x}\sin(4x - n\tan^{-1}2)]$

6. $\dfrac{1}{2}e^{5x}\,S\,(41)^{n/2}\cos[4x + n\tan^{-1}(0.8)] + (29)^{n/2}\cos[2x + n\tan^{-1}(0.4)]$

**7.** $\dfrac{(-1)^n\, n!}{3}\left\{\dfrac{4}{(x-1)^{n+1}}\;\dfrac{1}{(x+2)^{n+1}}\right\}$ **8.** $(-1)^n\, n!\left\{\dfrac{9(2)^{n-1}}{(2x+3)^{n+1}}-\dfrac{4}{(x+2)^{n+1}}\right\}$

**9.** $\dfrac{(-1)^n\, n!}{3}\left\{\dfrac{4}{(x+1)^{n+1}}+\dfrac{i-1}{4}\,\dfrac{1}{(x+i)^{n+1}}-\dfrac{i+1}{4}\,\dfrac{1}{(x-i)^{n+1}}\right\}$

**10.** $\dfrac{(-1)^n\, n!\cos(n+1)\theta}{(x^2+a^2)^{(n+1)/2}}$ where $\theta=\tan^{-1}(a/x)$

**11.** $2(-1)^{n-1}(n-1)!\sin n\alpha\sin^n\alpha$ where $\alpha=\cot^{-1}x$.

## Problems 4.3, page 141

**1.** (*i*) $\dfrac{(-1)^{n-3}(n-3)!}{x^n}\{(n-1)(n-2)+n(3-n)x^2\}$ ;

(*ii*) $\dfrac{1}{256}\{(\log 2)^n 2^x(\cos 9\theta+9\cos 7\theta+3\cos 5\theta+84\cos 3\theta+126\cos\theta)\}+{}^nC_1(\log 2)^{n-1}2^x\{\cos 9\theta+\pi/2)$
$+9\cos(7\theta+\ n\pi/2)+36\cos(5\theta+\pi/2)+84\cos(3\theta+\pi/2)+126\cos\theta+\pi/2\}+\ldots\ldots+2^x\{\cos(9\theta+n\pi/2)$
$+9\cos(7\theta+n\pi/2)+36\cos(5\theta+n\pi/2)+84\cos(3\theta+n\pi/2)+126\cos(\theta+n\pi/2)\}]$

**5.** $y_{2m}(0)=0,\ y_{2m+1}(0)=(-1)^m\,.\,(2m)!$

**7.** $(y_n)_o=0$, if $n$ is even
$=m(1^2-m^2)(3^2-m^2)\ldots\ldots[(2n-1)^2-m^2]$, if $n$ is odd

**8.** $(y_{2n})_0=e^{m\pi/2}\,m^2(2^2+m^2)(4^2+m^2)\ldots\ldots[(2n-2)^2+m^2]$
$(y_{2n+1})_o=-e^{m\pi/2}\,m\,(1^2+m^2)(3^2+m^2)\ldots\ldots[(2n-1)^2+m^2]$

**17.** $\{m^2-(n-2)^2\}\{m^2-(n-4)^2\}\ldots\ldots(m^2-2^2)\,m^2$, $n$ even
$\{m^2-(n-2)^2\}\{m^2-(n-4)^2\}\ldots\ldots(m^2-1^2)\,m$, $n$ odd.

## Problems 4.4, page 146

**2.** $x=(2m-1)\,a/(2m+2n-1)$

**3.** (*i*) $c=3.154,\ 0.846$ ; (*ii*) $c=\pi/2$ ; (*iii*) $c=e-1$. (*iv*) $c=0.5413$

**6.** 0.36 **12.** $\theta=0.25$.

## Problems 4.5, page 150

**1.** $x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+\ldots\ldots\infty$ **2.** $x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}+\ldots\ldots\infty$

**3.** $1+x-\dfrac{x^2}{2}+\dfrac{x^3}{6}+\dfrac{x^4}{24}+\ldots\ldots$ **4.** $x+\dfrac{1}{2}\cdot\dfrac{x^3}{3}+\dfrac{1.3}{2.4}\cdot\dfrac{x^5}{5}+\dfrac{1.3.5}{2.4.6}\cdot\dfrac{x^7}{7}+\ldots\ldots\infty$

**5.** $x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\dfrac{x^7}{7}+\ldots\ldots\infty$ **6.** $\dfrac{x^2}{2}+\dfrac{x^4}{12}+\dfrac{x^6}{45}+\ldots\ldots$

**24.** $\dfrac{m\sin\theta}{1!}-\dfrac{m(m^2-1^2)}{3!}\sin^3\theta+\dfrac{m(m^2-1^2)(m^2-3^2)}{5!}\sin^5\theta-\ldots\ldots$

**25.** $4+21(x-1)+13(x-1)^2+2(x-1)^3$ **26.** (*i*) $e\left\{1+\dfrac{(x-1)}{1!}+\dfrac{(x-1)^2}{2!}+\dfrac{(x-1)^3}{3!}+\ldots\ldots\right\}$

(*ii*) $\dfrac{\pi}{4}+\dfrac{1}{2}(x-1)-\dfrac{1}{4}(x-1)^2+\dfrac{1}{12}(x-1)^3$ **27.** $1-\dfrac{(x-\pi/2)^2}{2!}+\dfrac{(x-\pi/2)^4}{4!}-\ldots\ldots$ ; .9998

**29.** $\log(0.5) - \sqrt{3}(x - \pi/3) - 2(x - \pi/3)^2 - \frac{4\sqrt{3}}{3}(x - \pi/3)^3 + \ldots\ldots$

**30.** 0.8482 **31.** (*i*) 2.6121. (*ii*) 1.12.

## Problems 4.6, page 154

**1.** $\log_e(a/b)$ **2.** $-1/3$ **3.** 1/3 **4.** $a \log a$ **5.** 1 **6.** 1/18
**7.** 1/2 **8.** 1/12 **9.** 3/2 **10.** 0 **11.** 1/30 **12.** 1
**13.** 1/3 **14.** 2 **15.** 1 **16.** 1 **17.** 2 **18.** $11e/24$
**19.** $a = 2 ; 1$ **20.** $a = 5, b = -5$ **21.** $a = 1, b = 2, c = 1$.

## Problems 4.7, page 156

**1.** $-1/3$ **2.** 1/2 **3.** $-2$ **4.** $-1/3$ **5.** 2/3 **6.** $1/e$
**7.** $ae$ **8.** 1 **9.** $e$ **10.** $1/\sqrt{e}$ **11.** $1/e$ **12.** 0
**13.** 0 **14.** 1 **15.** $e^{-1/6}$ **16.** $e$ **17.** $e^{2/\pi}$ **18.** $e^{1/12}$
**19.** $-\frac{1}{2}$ **20.** $(6)^{1/3}$.

## Problems 4.8, page 160

**1.** $x - 20y = 7 ; 20x + y = 140$ **2.** $(a, b)$ **10.** $\pi/4$
**14.** $T = 2a \sin t/2 ; N = 2a \tan t/2. \sin t/2 ; S.T. = a \sin t ; S.N. = 2a \sin^2 t/2. \tan t/2$
**15.** $a \sin^3 \theta \ \tan \theta$.

## Problems 4.9, page 162

**7.** (*i*) $\pi/2$ ; (*ii*) $\pi/2$.

## Problems 4.10, page 166

**4.** $r^3 = 2ap^2$ **5.** $r^3 = a^2p$ **6.** $pa^m = r^{m+1}$ **7.** $r^{m+1} = \sqrt{2}\, a^m p$
**8.** $(1 + m^2)\, p^2 = r^2$ **9.** (*i*) $\sqrt{(1 + 9x/4a)}$ ; (*ii*) $\cosh x/c$ **10.** $a\theta$
**11.** (*i*) $2a \sin \theta/2$ ; (*ii*) $a\sqrt{(\sec 2\theta)}$ ; (*iii*) $r\sqrt{(8r - 3)}$.

## Problems 4.11, page 172

**1.** (*i*) $2a(1 + t^2)^{3/2}$ ; (*ii*) $y^2/c$ (*iii*) $(1 + a^3)^{3b}/6a^2$
**5.** (*i*) $(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{3/2}/ab$ ; (*ii*) $4a \sin \theta/2$ ; (*iii*) $at$
**11.** (*i*) 3/2 ; (*ii*) 1 ; (*iii*) $\sqrt{2a}$
**12.** (*i*) $\frac{4a}{3} \sin \frac{\theta}{2}$ ; (*ii*) $a^n/(n + 1)\, r^{n-1}$ **14.** $2\sqrt{(r^3/a)}$

## Problems 4.12, page 176

**1.** $a(2 + 3t)t^2, -4\sqrt{2}\, at^{3/2}$ **4.** (*i*) $x = a(t - \sin t), y - 2a = a(1 + \cos t)$, (*ii*) $x = a \cos \theta, y = a \sin \theta$
**5.** $(x + y)^{2/3} + (x - y)^{2/3} = 2a^{2/3}$
**7.** (*i*) $(x - 3a/4)^2 + (y + 3a/4)^2 = a^2/2$ (*ii*) $x^2 + y^2 - \frac{21}{8}(x + y) + \frac{432}{128} = 0$
**11.** $y^2 = 4ax$ **12.** $(x/a)^2 + (y/b)^2 = 1$ **13.** $27ay^2 = 4(x - 2a)^3$.

**14.** $(x/a)^2 + (y/b)^2 = 1$ **15.** $y = \dfrac{u^2}{2g} - \dfrac{gx^2}{2u^2}$

**16.** (*i*) $\sqrt{x} + \sqrt{y} = \sqrt{c}$ ; (*ii*) $4xy = c^2$ ; (*iii*) $x^{2/3} + y^{2/3} = c^{2/3}$ **17.** $x^{2/3} + y^{2/3} = c^{2/3}$

## Problems 4.13, page 181

**2.** $a = 1, b = 1/4$, Point of minima **4.** $x = 0.42l$ **5.** $v = (aw^2/3b)^{1/4}$

**8.** $\theta = \dfrac{\pi}{4} - \dfrac{\alpha}{2}$; $(1 - \sin\alpha)/(1 + \sin\alpha)$ **10.** Sq. with side $\sqrt{2}a$

**13.** $8 + 2\sqrt{7}, 2 + 2\sqrt{7}, 5 - \sqrt{7}$ **14.** Depth is half the width **15.** $(a^{2/3} + b^{2/3})^{3/2}$

**16.** $3\sqrt{3}a/4$ **25.** 2.5 km/hr.

## Problems 4.14, page 185

**1.** $x + y + a = 0$ **2.** $x = \pm a, y = \pm b$ **3.** $x = \pm a,\ y = \pm b$

**4.** $y = 0$ ; $x + 1 = 0$ ; $x + y = 0$ **5.** $y = x, y + 2x = 0, y + 2x + 1 = 0$

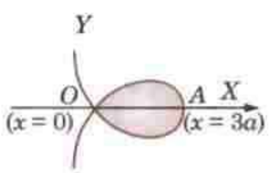
**6.** $x + a = 0$ ; $x - a = 0$ ; $x - y + \sqrt{2}a = 0$ ; $x - y - \sqrt{2}a = 0$

**7.** $x + 2y + 2 = 0, x + y = \pm 2\sqrt{2}$ **10.** $r\cos\theta = a$ ; $r\cos\theta = -a$ **11.** $r\cos\theta = 0$ ; $r\cos\theta = 2a$

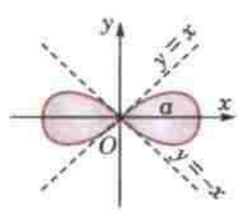
**12.** $r\sin\theta = 2$ **13.** $r\sin(\theta - m\pi/n) = a/n \cos m\pi$.

## Problems 4.15, page 194

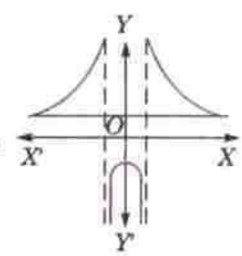
**1.**



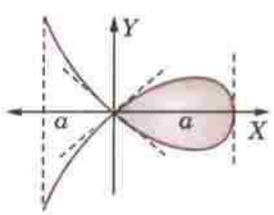

**2.**



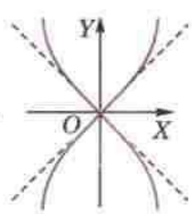

**3.**



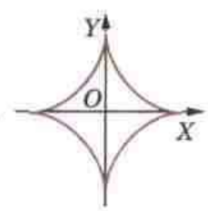

**4.**



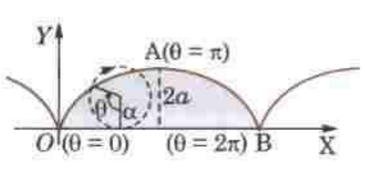

**5.**



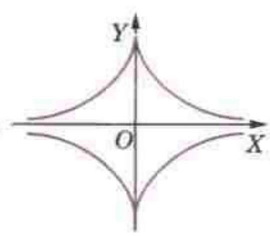

**6.**



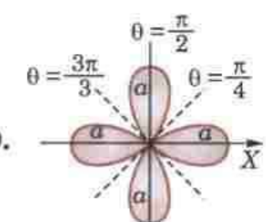

**7.**



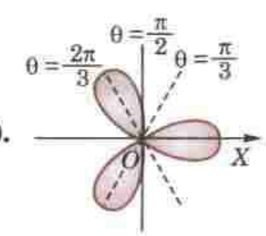

**8.**



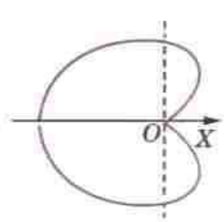

**9.**



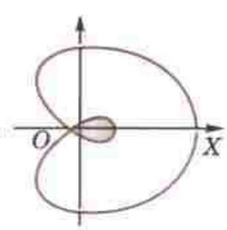

**10.**




**11.**




**12.**



## Problems 4.16, page 194

**1.** $c$ **2.** $x^2 + 4ay = 0$ **3.** 1/5 **4.** $(c)$ **5.** $(a)$ **6.** $(b)$
**7.** $(c)$ **8.** $(b)$ **9.** $(b)$ **10.** $(c)$ **11.** $(b)$ **12.** $(c)$
**13.** $(c)$ **14.** $(b)$ **15.** $x^2 = 4y$ **16.** of constant length

**17.** $x + \dfrac{x^3}{3\,!} + \dfrac{x^5}{5\,!} + \ldots$ **18.** $\dfrac{1}{4}\left[2^n \cos\left(2x + n\,\dfrac{\pi}{2}\right) + 4^n \cos\left(4x + n\,\dfrac{\pi}{2}\right) + 6^n \cos\left(6x + n\,\dfrac{\pi}{2}\right)\right]$

**19.** $-32/3a$ **20.** True **21.** $-a$ **22.** $2a(1 + t^2)^{3/2}$
**23.** $(x - a)^2 + (y - b)^2 = k^{-2}$ **24.** envelope **25.** $xy = c^2$
**26.** $\alpha$ **27.** $2a$ **28.** $e^x(x^3 - 12x^2 - 36x - 24)$
**29.** $(iii)$ **30.** $(x/a)^2 + (y/b)^2 = 1$ **31.** (B)
**32.** $c = 2.5$ **33.** $x = y$ **34.** node
**35.** Four loops of $r = a \sin 2\theta$ and three loops of $r = a \cos 3\theta$.
**36.** $y = \pm x$ **37.** $x = 4$ **38.** $4b$
**39.** (A) **40.** $r > a$ **41.** (D)
**42.** (D) **43.** (C).

## Problems 5.1, page 198

**1.** 2/3 **2.** Does not exist **3.** Zero **4.** Does not exist
**7.** Discontinuous.

## Problems 5.2, page 202

**1.** $(i)$ $xy(2 - \cos xy) - \sin xy$ ; $x^2(1 - \cos xy)$ ;
$(ii)$ $2x/(x^2 + y^2)$, $2y/(x^2 + y^2)$ ;
$(iii)$ $(x^2 + 2xy - y^2)/\{(x^2 + y^2)^2 + (x + y)^2\}$ ; $(y^2 + 2xy - x^2)/\{(x^2 + y^2)^2 + (x + y)^2\}$ ;
$(iv)$ $\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial y} = \dfrac{z}{1 - z}$

**11.** $n = 2, -3$ **18.** $e^{xyz}(x^2y^2z^2 + 3xyz + 1)$.

## Problems 5.4, page 208

**13.** $2u$.

## Problems 5.5, page 211

**2.** $4a^2t(t^2 + 2)$ **2.** $-2/(e^{2t} + e^{-2t})$ **3.** zero
**4.** 6.5 sq. ft./sec **6.** $8e^{4t}$.

## Problems 5.6, page 214

**9.** 0 ; 0.

## Problems 5.7, page 218

**6.** zero **7.** $x(yv + 1 - w) + z - 2uv$ **10.** 0 ; $u = \tan v$
**11.** $u^2 - v^2 = 8\,w$

## Problems 5.8, page 220

**1.** $4X + Y + Z = 6$ ; $\dfrac{X-2}{4} = Y - 1 = Z + 3$ **2.** $3Y + 2Z - X - 3 = 0$, $1 - X = \dfrac{Y-2}{3} = \dfrac{Z+1}{2}$

**3.** $\dfrac{X}{x_1} + \dfrac{Y}{y_1} + \dfrac{Z}{z_1} = 3$ ; $x_1(X - x_1) = y_1(Y - y_1) = z_1(Z - z_1)$

**4.** $7X - 3Y + 8Z = 26$ ; $\dfrac{X-1}{7} = \dfrac{Y+1}{-3} = \dfrac{Z-2}{8}$

**5.** $(-1, 2, 2/3)$ **7.** $\dfrac{X-x}{x} = \dfrac{Y-y}{y} = \dfrac{Z-z}{z}$.

## Problems 5.9, page 226

**1.** (*i*) $x - \dfrac{1}{6}(x^3 + 3xy^2)$

(*ii*) $$\frac{1}{2\sqrt{2}}\Big[1 + \{(x+1) + (y - \pi/4)\} + \frac{1}{2}\{(x+1)^2 - 2(x+1)(y-\pi/4) + (y-\pi/4)^2\} + \frac{1}{6}\{(x+1)^3 + 3(x+1)^2(y-\pi/4) - 3(x+1)(y-\pi/4)^2 - (y-\pi/4)^3\} + \ldots\Big]$$

(*iii*) $1 + x + \dfrac{1}{2!}(x^2 - y^2) + \dfrac{1}{6}(y^3 - 3xy^2) + \ldots$

**2.** $1 + (x-1) + (x-1)(y-1) + \dfrac{1}{2}(x-1)^2(y-1) + \ldots$

**3.** $-0.8232$ **4.** $-4500$ units **5.** 2% **7.** 2%

**11.** $\dfrac{\alpha}{2}\cot\alpha + 2$ **12.** Rs. 43.20 **13.** $(p - 3q - 4r)\%$ **14.** $-1\frac{1}{3}\%$ **15.** $5r$.

## Problems 5.10, page 233

**1.** (*i*) $(a, a)$ gives maximum if $a < 0$ and minimum if $a > 0$

(*ii*) Min. at $(a, a)$ (*iii*) Max. at $(4, 0)$, Min. at $(6, 0)$

(*iv*) Max. at $(\pm 1, 0)$ ; Min. at $(0, \pm 1)$ (*v*) Max. at $(\pi/3, \pi/3)$ ; Min. at $(2\pi/3, 2\pi/3)$

**2.** 4, 2, 1 **3.** (*i*) $3a^2$ ; (*ii*) $p^2/(a^2 + b^2 + c^2)$ ; (*iii*) $3a^2$ **4.** $12 \times 12 \times 6$ cm

**6.** $(0, 0, \pm 1)$ **8.** 4, 1 **9.** 50

**10.** 4, 8, 12 **11.** Two stationary values of $u$ are given by $\dfrac{l^2}{au-1} + \dfrac{m^2}{bu-1} + \dfrac{n^2}{cu-1} = 0$.

## Problems 5.11, page 236

**1.** $\dfrac{1}{2a^3}\tan^{-1}\dfrac{x}{a} + \dfrac{x}{2a^2(x^2+a^2)}$ **2.** $\dfrac{(-1)^n\, n!}{(m+1)^{n+1}}$

**3.** $\pi \log\left[\dfrac{1}{2} + \dfrac{1}{2}\sqrt{(1-a^2)}\right]$ **4.** $-\pi/(a^2-1)^{3/2}$.

## Problems 5.12, page 236

**1.** zero **2.** (*a*) **3.** 1 **4.** (*b*) **5.** (*b*) **6.** (*b*)

**7.** (*c*) **8.** (*c*) **9.** (*d*) **10.** (*d*) **11.** (*b*) **12.** (*d*)

**13.** (*a*) **14.** (*d*) **15.** (*c*) **16.** (*b*) **17.** zero **18.** $2/(x+y)$

**19.** $rt - s^2 < 0$ **20.** $(d)$ **21.** $4u$ **22.** $\partial/(u, v)/\partial(x, y)$

**23.** $f_x(a, b) = 0, f_y(a, b) = 0$ **24.** $(c)$ **25.** $\frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$

**26.** $(c)$ **27.** $-1$ **28.** $(c)$ **29.** equal

**30.** False.

## Problems 6.1, page 244

**1.** (*i*) 128/315 ; (*ii*) 8/45 **2.** (*i*) 128/315 ; (*ii*) $11\pi/192$

**3.** (*i*) $\frac{(2n-3)(2n-5)\ldots\ldots 3.1}{(2n-2)(2n-4)\ldots\ldots 4.2} \frac{\pi}{2}$ (*ii*) $\frac{1}{8}\left(\frac{\pi}{8} + \frac{1}{6}\right)$ **4.** $35\pi/10240$

**5.** (*i*) $3\pi/512$ (*ii*) 1/144 **6.** (*i*) $5\pi/256$ ; (*ii*) 1/15 **7.** (*i*) $35\pi a^4/8$ ; (*ii*) $5\pi a^3/2$

**8.** (*i*) $5\pi/8$ ; (*ii*) $28\pi$.

## Problems 6.2, page 247

**1.** (*i*) $\frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x$ (*ii*) $-\frac{1}{4}\cot^4 x + \frac{1}{2}\cot x + \log\sin x$

**3.** $\frac{1}{2}\log 2 - \frac{1}{4}$ **4.** $\frac{\pi}{4} - \frac{2}{3}$ **5.** $I_n = \frac{(2)^{(n-2)/2}}{n-1} + \frac{n-2}{n-1} + I_{n-2}$

**6.** (*i*) $\frac{1}{5}\sec^4 x \,.\, \tan x + \frac{4}{15}(\sec^2 x + 2)\tan x$ (*ii*) $\frac{11\sqrt{3}}{4} + \frac{3}{8}\log(2 + \sqrt{3})$

(*iii*) $-\frac{1}{4}\cot x \operatorname{cosec} x - \frac{3}{8}\cot x \operatorname{cosec} x + \frac{3}{8}\log(\operatorname{cosec} x - \cot x)$

**7.** $\left\{\frac{67\sqrt{2}}{48} + \frac{5}{16}\log(1 + \sqrt{2})\right\} a^6$ **8.** $\frac{t^5}{5} - \frac{t^3}{3} + t - \tan^{-1} t$.

## Problems 6.3, page 250

**1.** $e^x(1 - x + x^2 - x^3 + x^4)$

**3.** $\int x^m (\log x)^n \, dx = \frac{x^{m+1}}{m+1}(\log x)^n - \frac{n}{m+1}\int x^m (\log x)^{n-1} \, dx$ ; $\int_0^1 x^5 (\log x)^3 \, dx = -1/216$

**5.** 149/225 **6.** $3\pi^2/64 - 1/4$ **7.** $\frac{5}{16}\pi^4 - 15\pi^2 + 120$

**11.** $I_n = \frac{e^{ax}\cos^{n-1} x(a\cos x + n\sin x)}{a^2 + n^2} + \frac{n(n-1)}{a^2 + n^2} I_{n-2}$ ; $\int_0^{\pi/2} e^{2x}\cos^3 x \, dx = \frac{2}{65}(3e^{\pi} - 11)$

**12.** 24/85.

## Problems 6.4, page 254

**7.** (*i*) $3\pi/8$ ; (*ii*) $5\pi/8$ ; (*iii*) $3\pi/256$ ; (*iv*) $15\pi/640$ **8.** (*i*) $16\pi/35$ ; (*ii*) $8\pi/315$.

## Problems 6.5, page 256

**1.** log 2 **2.** $\frac{1}{3}\log 2$ **3.** 1/3 **4.** $\pi/2$ **5.** $\frac{1}{4}\log 2$ **6.** $2e^{(\pi-4)/2}$.

## Problems 6.6, page 260

**1.** (*i*) $\pi ab$ ; (*ii*) $8a^2/3$ **2.** $21\frac{1}{12}$ **3.** $2a^2/5$

**4.** (*i*) $8a^2/15\sqrt{3}$ ; (*ii*) $(2-\pi/2)a^2$ **5.** (*i*) $\pi a^2$ ; (*ii*) $\dfrac{\sqrt{2}}{3}a^2$ **6.** (*i*) $\pi a^2$ ; (*ii*) $4a^2$

**8.** $3\pi a^2$ **9.** $a^2/6$ **10.** $3\pi a^2$ ; $\pi a^2$

**11.** $\dfrac{11\pi}{3}-2\sqrt{3}$ **13.** $(3\pi-8)\,a/6$ **14.** $64a^2/3$

**15.** $1\frac{1}{8}$.

## Problems 6.7, page 262

**1.** (*i*) $3\pi a^2/2$ ; (*ii*) $a^2$ **2.** (*i*) $\pi a^2/8$ ; (*ii*) $\pi a^2/12$ **5.** $(1-\pi/4)a^2$ **6.** $\pi a^2/2$.

## Problems 6.8, page 265

**1.** $12\dfrac{11}{27}a$ **2.** (*i*) $\log(2+\sqrt{3})$, (*ii*) $\log_e(e+1/e)$

**3.** (*i*) $a\left[\sqrt{2}+\log(1+\sqrt{2})\right]$ ; (*ii*) $(15/16+\log 2)\,a$ **4.** (*i*) $4a/\sqrt{3}$; (*ii*) $4\sqrt{3}$

**5.** 37.85 **7.** (*i*) $8a$ **8.** $6a$

**9.** $4\sqrt{3}$ **11.** $2+\dfrac{1}{2}\log 3$ **12.** $8a$

**13.** $\sqrt{2}\,\pi a\left\{1+\left(\dfrac{1}{2}\right)^2+\left(\dfrac{1.3}{2.4}\right)^2.\left(\dfrac{1}{2}\right)^2+\left(\dfrac{1.3.5}{2.4.6}\right)^2.\left(\dfrac{1}{2}\right)^2+\ldots\ldots\right\}$ **14.** $2a\left[\sqrt{2}+\log(\sqrt{2}+1)\right]$.

## Problems 6.9, page 269

**1.** $\pi c^3(1+\sinh 1\cosh 1)$ **2.** $\pi h^2(a-h/3)$ **3.** $2\pi a^3$

**4.** $\pi a^3/12$ **5.** (*i*) $\dfrac{4}{3}\pi ab^2$ ; (*ii*) $\dfrac{4}{3}\pi a^2 b$ **6.** $\dfrac{\pi h}{3}(r^2+rR+R^2)$

**7.** $48\,\pi a^3$ **8.** (*i*) $2\pi a^3(\log 2-2/3)$ ; (*ii*) $\pi a^3/24$ ; (*iii*) $\pi/48$

**9.** (*i*) $5\pi^2a^3$ ; (*ii*) $5\pi^2a^3$ **10.** $32\,\pi a^3/105$ **11.** $4\pi^2a^3$

**13.** (*i*) $\dfrac{4}{3}\pi a^3$ ; (*ii*) $\dfrac{8}{3}\pi a^3$ **14.** $\dfrac{4}{3}\pi a(a^2+b^2)$ **15.** $\dfrac{\pi a^3}{4}\left\{\dfrac{1}{\sqrt{2}}\log(\sqrt{2}+1)-\dfrac{1}{3}\right\}$.

## Problems 6.10, page 271

**1.** $\dfrac{\pi a^2}{2}(2+\sinh 2)$ **2.** $\dfrac{8\pi a^2}{3}(2\sqrt{2}-1)$ **3.** $2\pi ab\left\{\dfrac{b}{a}+\dfrac{a}{\sqrt{(a^2-b^2)}}\sin^{-1}[\sqrt{(a^2-b^2)/a}]\right\}$

**4.** $\dfrac{1}{2}\pi r^2 h$ ; $\pi r\sqrt{(r^2+h^2)}$, where $r$ is the base-radius and $h$ the height of the cone

**5.** $4\pi a^2$ **8.** $\dfrac{64}{3}\pi a^2$ **9.** $\dfrac{64}{3}\pi a^2$ **12.** $4\pi a^2$

**13.** $\dfrac{32}{5}\pi a^2$ **14.** $4\pi a^2(1-1/\sqrt{2})$ **15.** $\pi a^2\,[3\sqrt{2}-\log(\sqrt{2}+1)]$.

## Problems 6.11, page 271

**1.** (*b*) **2.** (*c*) **3.** (*b*) **4.** (*c*) **5.** (*b*) **6.** (*c*)

**7.** (*c*) **8.** (*d*) **9.** (*b*) **10.** (*d*) **11.** (*c*) **12.** (*c*)

**13.** (*a*) **14.** $\dfrac{3\pi a^2}{2}$ **15.** (*iii*) **16.** $\pi a^2/12$ **17.** 1 **18.** $7\pi/8$

**19.** (*iii*) **20.** (*iii*) **21.** $\pi h(r_1^2 + r_1 r_2 + r_2^2)$ **22.** (*c*) **23.** (*a*)
**24.** (*b*) or (*c*) **25.** (*a*).

## Problems 7.1, page 280

**1.** 13 **2.** 3/35 **3.** $\frac{1}{2}(e-1)$ **4.** $\frac{1}{4}\pi \log(1+\sqrt{2})$

**5.** $a^4/8$ **6.** $\frac{\pi}{4}ab(a^2+b^2)$ **7.** 3/56 **8.** $\pi a/4$

**9.** 241/60 **10.** $1 - 1/\sqrt{2}$ **11.** $\frac{\pi a^2}{4}(\log e - \frac{1}{2})$ **12.** 1/24

**13.** $\frac{2}{3}a^4$ **14.** $\pi a^2/b$ **15.** 1

**16.** (*i*) $\int_0^{2a}\int_{y\sqrt{2a}}^{2a} f(x,y)\,dx\,dy - \int_0^a \int_{a-\sqrt{(a^2-y^2)}}^{a+\sqrt{(a^2-y^2)}} f(x,y)\,dx\,dy,$ (*ii*) $\int_0^{\pi/2}\int_a^{ae^{\theta/2}} f(r,\theta)\,r\,dr\,d\theta$

**18.** $4a^2/3$ **19.** $45\pi/2$.

## Problems 7.2, page 283

**1.** 4.5 **2.** 7/6 **3.** $\pi a^2$ **4.** $\frac{3}{2}\log_e 3 - \frac{2}{3}$
**5.** $a^2$ **6.** $a^2(1-\pi/4)$ **7.** 4/3 **8.** $a^2/4$.

## Problems 7.3, page 284

**1.** $\frac{abc}{3}(a^2+b^2+c^2)$ **2.** $\frac{8}{3}abc(a^2+b^2+c^2)$ **3.** 4/35

**4.** $\frac{1}{8}e^{4a} - \frac{3}{4}e^{2a} + e^a - \frac{3}{8}$ **5.** $\frac{8}{3}\log 2 - \frac{19}{9}$ **6.** $\frac{1}{4}(13 - 8e + e^2)$

**7.** $5\pi a^3/64$.

## Problems 7.4, page 291

**1.** $\pi/8$ **2.** $\pi/2$ **3.** $8\left(\frac{\pi}{2} - \frac{5}{3}\right)a^2$ **4.** $\frac{2^{n+3}}{n+4}$

**5.** $\frac{\pi}{4} - \frac{1}{2}$ **6.** 0 **7.** $\frac{15\pi a^4}{64}$ **9.** $\pi$

**10.** $\pi^2/8$ **11.** $4\pi a$ **12.** $\frac{1}{2}\left(\log 2 - \frac{5}{8}\right)$ **14.** $\pi a^8/12$

**15.** $3\pi$ **16.** $4\pi$ **17.** $\pi a^3(2-\sqrt{2})/3$ **18.** $16a^3/3$

**19.** $\pi a^3/8$ **20.** $128a^3/15$ **21.** $3\pi a^3$ **22.** $4\sqrt{3}\,\pi$

**23.** $8a^4/3$ **25.** $\frac{1}{4}$ **26.** $\frac{1}{6}abc$.

## Problems 7.5, page 293

**2.** 64 **3.** $2(\pi-2)a^2$ **4.** $2\pi a^2$ **5.** $\frac{3\pi a^2}{4}$.

## Problems 7.6, page 297

1. $182\frac{7}{24}\lambda$
2. $21\,\pi\mu a^4/32$
3. 30.375
4. $\left(\frac{3a}{20}, \frac{3a}{16}\right)$
5. $\left[\frac{a(4a+3b)}{6(a+b)}, \frac{b(3a+b)}{6(a+b)}\right]$
6. $\left(\frac{\pi a\sqrt{2}}{8}, 0\right)$
8. $\bar{x} = 3a/5,\ \bar{y} = 9a/40$ where $a = OA$
9. (1/5, 1/5, 2/5).
10. $\bar{x} = 3/4$
11. $\left(\frac{16a}{35}, \frac{16b}{35}, \frac{16c}{35}\right)$
12. $\frac{27}{26}$ metres
13. $\left(\frac{3a}{8}, \frac{3\pi a}{16}\right)$
14. Divides the diagonal in the ratio 7 : 5
15. $\left(\frac{a}{2}, \frac{2}{3}h\right)$ where $a$ is the base, $h$ the depth
16. C.P. lies on the radius $\perp$ to the bounding diameter at a depth $32a/(15\pi)$ from the centre.

## Problems 7.7, page 301

1. $ab^3/12$
2. $5Ma^2/4$
3. $2M/9$
4. $\frac{1}{3}M(a^2 + b^2)$
5. $(21/32)\,\pi\rho a^4$
6. $\frac{2}{5}MR^2$
7. $\frac{1}{2}Mr^2$ ; $\frac{1}{12}M(3r^2 + 4h^2)$
8. (*i*) $\frac{3Mr^2}{10}$ ; (*ii*) $\frac{3M}{20}(r^2 + 4h^2)$ ; (*iii*) $\frac{M}{20}(3r^2 + 2h^2)$
9. $104803770\rho$
10. $\frac{1}{30}$
11. $\frac{\pi\rho abc(a^2 + b^2)}{30}$
12. $\frac{\rho a^2 b^2}{8}$.

## Problems 7.8, page 309

1. (*i*) 3.323, (*ii*) 11.629 ; (*iii*) $\pi\sqrt{2}$ ; (*iv*) 0.1964 ; (*v*) 0.1227
2. (*i*) $\sqrt{\pi}/2$ ; (*ii*) $\Gamma(5/4)$ ; (*iii*) $\sqrt{\pi}/3$ ; (*iv*) $2^{p+q-1}\beta(p, q)$
4. $\pi/4\sqrt{2}$
7. $-3/8$
9. $\dfrac{\Gamma\left(\frac{m+1}{n}\right)\Gamma(p+1)}{n\Gamma\left(\frac{m}{n} + p + 1 + \frac{1}{n}\right)}$, (*i*) $\frac{1}{396}$ (*ii*) $\frac{\sqrt{\pi}}{n}\cdot\dfrac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n} + \frac{1}{2}\right)}$
10. 16/35
15. $\frac{ka^2b^2c^2}{48}$.

## Problems 7.9, page 312

2. $\frac{1}{2}K(\sqrt{3}/2)$
3. $\frac{2}{\sqrt{3}}\left\{F\left(\sqrt{\frac{2}{3}}, \frac{1}{2}\pi\right) - F\left(\sqrt{\frac{2}{3}}, \frac{1}{4}\pi\right)\right\}$
4. $2\sqrt{2}E(1/\sqrt{2}) - \sqrt{2}K(1/\sqrt{2})$
5. $erf(x) = \frac{2}{\pi}\left(x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \ldots\ldots\right)$; $erf(o) = 0$
6. (*i*) 0.3248 ; (*ii*) 0.5204.

## Problems 7.10, page 313

1. 4
2. Area of the triangle having vertices (0, 0), (0, 1), (1, 0)
3. $\sqrt{\pi}/2$
4. 3.1416
5. $15\sqrt{\pi}/8$
6. $92\pi$
6. 26
8. $-1/3$
9. $1/2\ \beta(4, 3/4)$
10. 1

**11.** 3/4 **12.** $\frac{\pi a^3}{6}$ **13.** (*d*) **14.** 27/4

**15.** $\pi a^2/12$ **16.** $-1/3$ **17.** $\infty$ **18.** 44/105

**19.** $r^2 \sin\theta \, dr \, d\theta \, d\phi$ **20.** $e^2 - 1$ **21.** $\frac{1}{4}\pi ab(a^2 + b^2)$. **22.** $\frac{1}{4}\pi \log(1 + \sqrt{2})$

**23.** 3/256 **24.** (*c*) **25.** $\frac{6}{25} + \frac{1}{2}\sin\frac{3}{5}$ **26.** 48/5

**27.** 1/6 **28.** 16/3 **29.** $\int_0^a \int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} f(x, y)\, dy\, dx$

**30.** $\sqrt{2}\,\pi$ **31.** $3\pi/256$ **32.** 1/2 **33.** $\int_0^{\pi/2} \int_0^{2a\cos\theta} r\, d\theta\, dr$

**34.** $\frac{7h}{12}$ **35.** $\left(\frac{3}{20}, \frac{3}{16}\right)$ **36.** 16 **37.** $3Mr^2/10$

**38.** 1 **39.** log 2 **40.** $\frac{1}{2}\sqrt{\pi}$ **41.** (*c*) **42.** (*b*).

## Problems 8.1, page 318

**3.** (*i*) $t^3 \sin t + 7t^2 \cos t + 20t \sin t - 10t$ ; (*ii*) $(20t^3 + t\sin t - \cos t)\mathbf{I}$
$-(2t\cos t + 2\sin t + 75t^2)\mathbf{J} - t(t\sin t + 2t^2\cos t + 10\cos t)\mathbf{K}$

**5.** $-4(\mathbf{I} + 2\mathbf{J})$ **6.** (*i*) $(ua^2 \sec\alpha)$. (*ii*) $a^3 \tan\alpha$ ; $(\cos t\,\mathbf{J} - \sin t\,\mathbf{I})\cos\alpha + \sin\alpha\,\mathbf{K}$

**7.** $\{t\mathbf{I} + 2\mathbf{J} + (2t-3)\mathbf{K}\}/\sqrt{(5t^2 - 12t + 13)}$ ; $\frac{1}{3}(2\mathbf{I} + 3\mathbf{J} + \mathbf{K})$

**8.** $(x - a/\sqrt{2}) = y - a/\sqrt{2} = \left(z - \frac{a\pi}{4}\tan\alpha\right)/\sqrt{2}\tan\alpha$

**9.** (*i*) $ab/(a^2\sin^2 t + b^2\cos^2 t)^{3/2}$ ; (*ii*) $1/4\sqrt{2}$

**10.** (*i*) $\mathbf{R} = (p+q)\mathbf{I} + q\mathbf{J} + 2q\mathbf{K}$ ; $\frac{2\mathbf{J} - \mathbf{K}}{\sqrt{5}}$

(*ii*) $\mathbf{R} = p\mathbf{I} + (p+2q)\mathbf{J} + (p+q)\mathbf{J} + (p+q)\mathbf{K}$ ; $(2\mathbf{K} - \mathbf{I} - \mathbf{J})/\sqrt{6}$ .

## Problems 8.2 page 321

**1.** $v(\text{at } t = 0) = \sqrt{37}$, $a(\text{at } t = 0) = \sqrt{325}$ **2.** $\alpha = \pm 1/\sqrt{6}$

**3.** $8\sqrt{14/7}$ ; $-\sqrt{14/7}$ **6.** (*a*) $d^2s/dt^2$ ; $v^2/\rho$ ; (*b*) 0 ; 3

**7.** $\sqrt{(17)}$ m.p.h. in the direction $\tan^{-1}(0.25)$ North of East

**8.** 21.29 knots/hr. in the direction 74°47′ South of East.

## Problems 8.3, page 325

**1.** (*a*) $2(x\mathbf{I} + y\mathbf{J} + z\mathbf{K})/(x^2 + y^2 + z^2)$. (*b*) $\frac{6}{3} + c$ **2.** $(-\mathbf{I} + 3\mathbf{J} + 2\mathbf{K})/\sqrt{14}$

**3.** $12\frac{1}{3}$ **4.** $15/\sqrt{17}$ **5.** $a = 6, b = 24, c = -8$

**6.** $-260/(69)$ ; $\sqrt{(1056)}$ **7.** $a = \pm\frac{20}{9}, b = \pm\frac{55}{9}, c = \pm\frac{55}{9}$ **8.** $96(\mathbf{I} + 3\mathbf{J} - 3\mathbf{K})$ ; $96\sqrt{(19)}$

**9.** 9 **10.** $\frac{1}{3}(2\mathbf{I} + 2\mathbf{J} - \mathbf{K})$ **11.** $\cos^{-1}(1/\sqrt{22})$

**12.** $\cos^{-1}(-1/\sqrt{30})$ **13.** $a = -6, b = -10$.

## Problems 8.4, page 333

**1.** (*i*) 12 ; $5\mathbf{I}-16\mathbf{J}+9\mathbf{K}$ ; (*ii*) 278 ; $5(27\,I-54\,J+8\,K)$ ; (*iii*) $-32$ ; 0

**4.** $a=-2$ ; $4x\,(z-xy)\,\mathbf{I}+(y-2yz+4xy^2)\,\mathbf{J}+(2x^2+y^2-z^2-z)\,\mathbf{K}$

**13.** (*i*) 0 ; (*ii*) $2(x+z)\,\mathbf{J}+2y\,\mathbf{K}$ **14.** (*a*) $2n(2n-1)/x^2+y^2+z^2)^{n+1}$ ; $n=1/2$

**16.** (*i*) $2(y^3+3x^2y-6xy^2)z\mathbf{I}+2(3xy^2+x^3-6x^2y)\,z\mathbf{J}+2(xy^2+x^3-3x^2y)\,y\mathbf{K}$ ; (*ii*) Zero

**17.** $1724/\sqrt{21}$ .

## Problems 8.5, page 335

**1.** $75\frac{1}{3}\mathbf{I}+360\,\mathbf{J}-42\,\mathbf{K}$ **2.** $(t^3-t+2)\,\mathbf{I}+(1-t^4)\mathbf{J}+(4-4\cos t-3t)\,\mathbf{K}$

**3.** $\mathbf{V}=6\sin 2t\mathbf{I}+4(\cos 2t-1)\mathbf{J}+8t^2\mathbf{K}$ ; $\mathbf{R}=3(1-\cos 2t)\mathbf{I}+2\sin 2t\mathbf{J}+\dfrac{8t^3}{3}\mathbf{K}.$

## Problems 8.6, page 336

**1.** 0 **2.** 35 **3.** $-2/3$ **4.** 5 **5.** $\dfrac{\pi^3\sqrt{2}}{3}$ **6.** zero

**7.** 303 **8.** $8\dfrac{8}{35}$ **9.** $9\pi$ **10.** $\left(2-\dfrac{\pi}{4}\right)\mathbf{I}-(\pi-\frac{1}{2})\,\mathbf{J}.$

## Problems 8.7, page 339

**2.** $3\dfrac{1}{3}$ **3.** 8.

## Problems 8.8, page 341

**3.** $\pi ab$ **4.** $\pi a^2$ **5.** Zero **6.** 128/5 **7.** $35\pi a^4/16$.

## Problems 8.9, page 345

**3.** $-2ab^2$ **5.** $\dfrac{19}{2}\pi$ **6.** Zero **10.** 2 **11.** 0 **12.** $\pi$.

## Problems 8.10, page 350

**4.** $108\pi$ **7.** (*i*) $\frac{12}{5}\pi a^5$ (*ii*) $12(e-e^{-1})$ **8.** $4\pi a^3$

**9.** $\dfrac{\pi a^6}{12}$ **10.** $\dfrac{5}{4}\pi a^4 b$ **11.** $-4\pi$ **12.** 8/3.

## Problems 8.11, page 354

**3.** $14\frac{2}{3}$ **4.** $x^3y-y^2z^2+z^3$

**5.** (*i*) $\frac{1}{3}(x^3+y^3+z^3-3xyz)$ ; (*ii*) $x^2y+y^2z+z$ ;
(*iii*) $xz^3-yz+3x^2y$. (*iv*) $x^2y^2+y^2z^2+xyz=0$

**6.** (*i*) Yes, $\dfrac{a}{2}(x^2+y^2-2z^2)$ ; (*ii*) Yes

**7.** $xy\sin z+\cos x+y^2z+c$ **8.** $x^2y+xz^3$ ; 202

**9.** $a=4, b=2, c=-1$ **10.** $a=4$ ; $2x^2y-xz^3$ ; 47.

## Problems 8.12, page 362

1. (i) $(\rho \sin 2\phi - z \sin \phi)\, \mathbf{T}_\rho - (2\rho \sin^2 \phi + z \cos \phi)\, \mathbf{T}_\phi + 3\rho \cos \phi\, \mathbf{T}_z$
   (ii) $(2p \cos^2 \phi - 3p^2 \sin^3 \phi)\, \mathbf{T}_p - (\rho \sin 2\phi + 3\rho^2 \sin^2 \phi \cos \phi)\, \mathbf{T}_\phi + \rho z \cos \phi\, \mathbf{T}_z$

2. (i) $r \sin \theta \Big[ \{\sin \theta (1 + \sin^2 \phi) + r \cos^2 \theta \sin \phi\}\, \mathbf{T}_r$
$$+ \{\cos \theta (1 + \sin^2 \phi) - r \sin \theta \cos \theta \sin \phi\}\, \mathbf{T}_\theta + \sin \phi \cos \phi\, \mathbf{T}_\phi \Big]$$
   (ii) $r^2 \sin \theta \{(\sin^2 \theta \cos^2 \phi \sin \phi + \sin \theta \cos \theta \sin^2 \phi + \cos^2 \theta \cos \phi)\, \mathbf{T}_r$
$$+ (\sin \theta \cos \theta \cos^2 \phi \sin \phi + \cos^2 \theta \sin^2 \phi - \sin \theta \cos \theta \cos \phi)\, \mathbf{T}_\theta$$
$$+ (\cos \theta \sin \phi \cos \phi - \sin \theta \sin^2 \phi \cos \phi)\, \mathbf{T}_\phi\}$$

3. $\rho z \sin 2\phi\, \mathbf{T}_\rho + \rho z \cos 2\phi\, \mathbf{T}_\phi + \frac{1}{2} \rho^2 \sin 2\phi\, \mathbf{T}_z$.

## Problems 8.13, page 363

1. $1/\sqrt{14}$, $2/\sqrt{14}$, $3/14$
2. $\frac{1}{4}(x-2) = y - 1 = z + 3$
3. $du\,dv = \frac{1}{h_1 h_2}\, dx\,dy$
4. $4x - 3z + 2xz$
5. zero
6. $\frac{1}{2} \int_C (x\,dy - y\,dx)$
7. 3V
8. 3 ; 0
9. Irrotational
10. $4\pi$
11. solenoidal
12. $-28/\sqrt{5}$
13. zero.
14. $-(y\mathbf{I} + z\mathbf{J} + x\mathbf{K})$.
15. $\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$.
16. zero
17. $-(12\mathbf{I} + 5\mathbf{J} + 8\mathbf{K})$
18. zero
19. zero
20. zero
21. zero
22. $\mathbf{R}/r^2$ ; $nr^{n-2}\,\mathbf{R}$
23. §8.5(2)
24. $\frac{1}{\sqrt{21}}(2\mathbf{I} + 4\mathbf{J} - \mathbf{K})$.
25. 2, – 2, 2
26. $6\frac{\sqrt{7}}{3}$
27. $2/r$
28. 7/3
29. zero
30. (c)
31. (c)
32. (b)
33. (c)
34. (a)
35. (a)
36. $5u$
37. zero
38. irrotational field
39. (a)
40. the rate at which fluid is originating at $P$ per unit volume.
41. (a)
42. it gives the maximum rate of change of $\phi$.
43. (iv)
44. (a)
45. (a)
46. (b)
47. (a)
48. (b)
49. zero
50. True
51. True.

## Problems 9.1, page 366

1. Convergent
2. Convergent
3. Convergent
4. Divergent
5. Convergent
6. Convergent
7. Convergent
8. Divergent.

## Problems 9.2, page 367

1. Convergent
2. Convergent
3. Oscillatory
4. Convergent
5. 15 m.

## Problems 9.3, page 372

1. Convergent
2. Convergent
3. Divergent
4. Divergent
5. Convergent
6. Conv. for $p > 2$ ; Div. for $p \le 2$.
7. Divergent
8. Convergent
9. Convergent
10. Convergent
11. Convergent

**12.** Divergent **13.** Divergent **14.** Convergent **15.** Convergent
**16.** Convergent **17.** Divergent **18.** Convergent.

## Problems 9.4, page 376

**1.** Conv. for $x < 1$ ; Div. for $x \geq 1$ **2.** Conv. for $x < 1$ ; Div. for $x \geq 1$
**3.** Conv. for $x \leq 1$ ; Div. for $x > 1$ **4.** Conv. for $\geq 1$ ; Div. for $x < 1$
**5.** Convergent for all values of $p$ **6.** Convergent **7.** Convergent
**8.** Convergent **9.** Conv. for $x < 1$ ; Div. for $x \geq 1$ **10.** Convergent
**11.** Convergent **12.** Convergent **13.** Divergent
**14.** Convergent **15.** Conv. for $x < 1$, Div. for $x > 1$ ; Conv. for $p > 1$ and Div. for $p \leq 1$
**16.** Divergent **17.** Conv. if $\beta > \alpha > 0$ ; Div. if $\alpha \geq \beta > 0$.

## Problems 9.5, page 379

**1.** Conv. for $x \leq 1$ ; Div. for $x > 1$ **2.** Conv. for $x \leq 1$ ; Div. for $x > 1$
**3.** Conv. for $x < 1$ ; Div. for $x \geq 1$ **4.** Conv. for $x < 2$ ; Div. for $x \geq 2$
**5.** Conv. for $x < e$ ; Div. for $x \geq e$ **6.** Conv. for $x \leq 1$ ; Div. for $x > 1$
**7.** Conv. for $x \leq 1$ ; Div. for $x > 1$ **8.** Conv. for $x^2 \leq 1$ ; Div. for $x^2 > 1$
**9.** Conv. for $x^2 < 4$ ; Div. for $x^2 \geq 4$ **10.** Convergent
**11.** Conv. for $x < 1/e$ ; Div. for $x \geq 1/e$ **12.** Conv. for $x < 1$ ; Div. for $x \geq 1$
**13.** Diverges
**14.** Conv. for $x < 1$ ; Div. for $x > 1$. When $x = 1$, Conv. for $b - a > 1$, Div. for $b - a \leq 1$.

## Problems 9.6, page 381

**1.** Convergent **2.** Convergent **3.** Convergent
**4.** Convergent **5.** Conv. for $x < 1$ ; Div. for $x \geq 1$
**6.** Conv. for $x < \frac{1}{2}$ ; Div. for $x \geq \frac{1}{2}$ **7.** Convergent.

## Problems 9.7, page 383

**1.** Oscillatory **2.** Convergent **3.** Convergent **4.** Convergent
**5.** Convergent **6.** Oscillatory **7.** Convergent **8.** Convergent
**9.** Convergent **10.** Oscillatory.

## Problems 9.8, page 387

**1.** (*i*) and (*ii*) conditionally convergent
**3.** (*i*) Conditionally convgt. for $0 < p \leq 1$ ; (*ii*) Conditionally convgt
**4.** Absolutely convergent for (*i*) $0 < x < 1$ ; (*ii*) $-1 < x \leq 1$ ; (*iii*) $|x| \leq 1$.
**5.** Convergent for $x \leq 1$ and not convergent for $x > 1$
**6.** (*i*) $-1 < x \leq 1$ ; (*ii*) $-1 < x \leq 1$ ;
**7.** $-e < x \leq e$ **8.** (*i*) Absolutely convergent (*ii*) convergent
**9.** Absolutely convergent.

## Problems 9.9, page 388

**1.** Conv. for $x < 1$ ; Div. for $x \geq 1$ **2.** Convergent **3.** Divergent
**4.** Convergent **5.** Divergent **6.** Conv. for $x < 1$ ; Div. for $x \geq 1$
**7.** Conv. for $x < 1$ ; Div. for $x \geq 1$ **8.** Conv. for $x < 1$ ; Div. for $x \geq 1$
**9.** Conv. for $x < 1/4$ ; Div. for $x \geq 1/4$ **10.** Conv. for $x < 2$ ; Div. for $x \geq 2$

**11.** Convergent for all $x$ **12.** Conv. for $x < 1$ ; Div. for $x \geq 1$ **13.** Convergent
**14.** Absolutely convergent **15.** Convergent
**16.** Convergent for $p > 1$ ; divergent for $p \leq 1$.

## Problems 9.10, page 391

**1.** Uniformly convergent for $0 \leq x \leq 1$. **2.** to
**5.** Uniformly convergent for all real values of $x$ **6.** Uniformly convergent for $0 \leq x \leq 1/a$
**10.** (*i*) and (*ii*) Both converge uniformly for all real values of $x$.

## Problems 9.11, page 392

**1.** (*c*) **2.** (*d*) **3.** (*a*) **4.** (*b*) **5.** (*c*) **6.** (*d*)
**7.** (*a*) **8.** (*b*) **9.** (*b*) **10.** (*d*) **11.** (*c*)
**12.** (*a*) $(-1, 1)$ (*b*) $(-1/2, 1/2)$ **13.** $-1 < x \leq 1$ **14.** $k > 1$ **15.** $a_n < k$ **16.** Oscillatory
**17.** All values of $x$ **18.** $k < 1$ **19.** Convergent. **20.** Divergent. **21.** $q - p > 1$
**22.** Divergent **23.** Convergent. **24.** $0 < x < 4$ **25.** yes **26.** True **27.** Convergent
**28.** Divergent **29.** $x > 1$ **30.** $0 \leq x \leq 1$ **31.** (*b*) **32.** (*c*) **33.** (*d*)
**34.** (*b*) **35.** True.

## Problems 10.1, page 400

**1.** $\dfrac{2\sinh a\pi}{\pi}\left\{\left(\dfrac{1}{2a} - \dfrac{a\cos x}{1^2+a^2} + \dfrac{a\cos 2x}{2^2+a^2} - \dots\right) + \left(\dfrac{\sin x}{1^2+a^2} - \dfrac{2\sin 2x}{2^2+a^2} + \dfrac{3\sin 3x}{3^2+a^2} - \dots\right)\right\}$

$\dfrac{\pi}{\sinh \pi} = 2\left[\dfrac{1}{2^2+1} - \dfrac{1}{3^2+1} + \dfrac{1}{4^2+1} - \dots\right]$.

## Problems 10.2, page 401

**1.** No **2.** No **3.** Yes.

## Problems 10.3, page 404

**1.** $\dfrac{1}{2}\pi - \dfrac{4}{\pi}\left(\cos x + \dfrac{\cos 3x}{3^2} + \dfrac{\cos 5x}{5^2} + \dots\dots\right)$ **2.** $\dfrac{I_0}{\pi} + \dfrac{1}{2}I_0 \sin x - \dfrac{2I_0}{\pi}\sum_{n=1}^{\infty}\dfrac{\cos 2nx}{4n^2-1}, \dfrac{1}{2}$

**3.** $\dfrac{\pi^2}{6} - 2\left(\cos x - \dfrac{\cos 2x}{2^2} + \dfrac{\cos 3x}{3^2} - \dots\dots\right) - \dfrac{1}{\pi}\left\{\left(\dfrac{2}{1^3} - \dfrac{\pi^2}{1}\right)\sin x - \left(\dfrac{2}{2^3} - \dfrac{\pi^2}{2}\right)\sin 2x + \dots\right\}$

**4.** $2\left(\pi - \dfrac{4}{\pi}\right)\sin x - \pi \sin 2x + \dfrac{2}{3}\left(\pi - \dfrac{4}{9\pi}\right)\sin 3x - \dfrac{\pi}{2}\sin 4x + \dots\dots$

**5.** $\dfrac{4}{\pi}\left(\sin x + \dfrac{\sin 3x}{3} + \dfrac{\sin 5x}{5} + \dots\dots \infty\right)$.

## Problems 10.4, page 408

**1.** $-\dfrac{2}{\pi}\sum_{n=1}^{\infty}\dfrac{\sin n\pi x}{n}$

2. (i) $\frac{a^2}{3}\frac{a^2}{11^2}\left\{\frac{1}{12}\cos\frac{2\pi x}{a}+\frac{1}{2^2}\cos 4\pi\frac{x}{a}+......\right\}\ \frac{a^2}{\pi}\left\{\frac{1}{1}\sin\frac{2\pi x}{a}\frac{1}{2}\sin\frac{4\pi x}{a}+......\right\}$

(ii) $f(t)=\frac{2}{3}+\frac{4}{\pi^2}\left(\cos\pi t-\frac{\cos 2\pi t}{2^2}+\frac{\cos 3\pi t}{3^2}-......\right)$

4. $\frac{3}{2}-\frac{12}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\left\{\frac{\cos(2n-1)\pi x}{3}\right\}$

5. $\frac{E}{\pi}+\frac{E}{2}\sin\omega t-\frac{2E}{\pi}\left[\frac{1}{1.3}\cos 2\omega t+\frac{1}{3.5}\cos 4\omega t+\frac{1}{5.7}\cos 6\omega t+...\right]$

6. $f(x)=\frac{\pi}{4}+2\sum_{n=1}^{\infty}(-1)^{n+1}\sin n\pi x$ ; put $x = 1/2$.

## Problems 10.5, page 412

1. $\frac{a^2}{3}+\sum_{n=1}^{\infty}\frac{4a^2}{n^2\pi^2}(-1)^n\cos\frac{n\pi x}{a}$

3. $1-\frac{1}{2}\cos x-\frac{2}{1.3}\cos 2x+\frac{2}{2.4}\cos 3x-\frac{2}{3.5}\cos 4x-...$

5. $\frac{1}{2}\pi-\frac{4}{\pi}\left(\cos x+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+......\right)$

6. (i) $\frac{2}{\pi}-\frac{4}{\pi}\left(\frac{\cos 2x}{3}+\frac{\cos 4x}{15}+......+\frac{\cos 2nx}{4n^2-1}+......\right)$ (ii) $\frac{2}{\pi}+\sum_{n=1}^{\infty}\frac{4(-1)^{m+1}}{(4m^2-1)\pi}\cos\frac{2m\pi x}{l}$

7. $\frac{\pi}{2}+1-\frac{4}{\pi}\left(\cos x+\frac{1}{3^2}\cos 3x+\frac{1}{5^2}\cos 5x+...\right);\frac{\pi^2}{8}$

8. $\frac{4k}{\pi}\left(\sin x+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+......\infty\right)$.

## Problems 10.6, page 416

2. $\frac{\pi}{2}-\frac{4}{\pi}\left[\cos x+\frac{1}{3^2}.\cos 3x+\frac{1}{5^2}.\cos 5x+......\right]$ ; $2\left[\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-......\right]$

3. $\frac{\pi^2}{3}-4\left[\cos x-\frac{1}{2^2}\cos 2x+\frac{1}{3^2}\cos 3x-\frac{1}{4^2}\cos 4x+...\infty\right]$

4. $\sum_{n=2}^{\infty}\frac{1}{n}\sin 2nx$

5. $\frac{1}{3}+\frac{4}{\pi^2}\sum_{1}^{\infty}\frac{1}{n^2}\cos n\pi x$

6. $\frac{8}{\pi^3}\left(\frac{\sin\pi t}{1^3}+\frac{\sin 3\pi t}{3^3}+\frac{\sin 5\pi t}{5^3}+......\right)$

7. $\frac{1}{n\pi}[1-(-1)^n]$

8. $-\frac{1}{2}\sin x+\sum_{n=2}^{\infty}\frac{(-1)^n.2n}{n^2-1}\sin nx$

9. $\sum_{n=1}^{\infty}\frac{2n\pi}{1+n^2\pi^2}(1-e\cos n\pi)\sin n\pi x$

10. $\sum_{n=1}^{\infty}\frac{4}{\pi n^3}[1-(-1)^n]\sin nx$

12. $\frac{l}{4}+\sum_{n=1}^{\infty}\frac{2l}{(n\pi)^2}\left\{2\cos\frac{n\pi r}{2}-1-(-1)^n\right\}\cos\frac{n\pi x}{l}$

13. $\frac{8}{\pi}\cos\frac{\pi}{4}\left[\frac{\sin 2x}{1.3}-\frac{\sin 6x}{5.7}+\frac{\sin 10x}{9.11}+......\right]$

14. $\frac{2l^2h}{a(l-a)\pi^2}\left[\sin\frac{\pi a}{l}\sin\frac{\pi x}{l}+\frac{1}{2^2}\sin\frac{2\pi a}{l}\sin\frac{2\pi x}{l}+\frac{1}{3^2}\sin\frac{3\pi a}{l}\sin\frac{3\pi x}{l}+......\right]$.

## Problems 10.7, page 419

**3.** $\pi^4/96$.

## Problems 10.8, page 420

**1.** $\sum_{-\infty}^{\infty} \frac{(\sinh al \cos n\pi l - i \cosh al \sin n\pi l)(a + in\pi)}{(a^2 + n^2\pi^2)} e^{in\pi x/l}$

**2.** $\frac{2}{\pi}\left\{1 - \frac{e^{2it} + e^{-2it}}{1.3} - \frac{e^{4it} + e^{-4it}}{3.5} - \frac{e^{6it} + e^{-6it}}{5.7} - \ldots\ldots\right\}$

**3.** $\frac{a}{\pi} \sin a\pi \sum_{-\infty}^{\infty} \frac{(-1)^n e^{inx}}{a^2 - n^2}$

**4.** $\sin h\, 9 \sum_{-\infty}^{\infty} \frac{(-1)^n (9 + n\pi i)}{81 + (n\pi)^2} e^{n\pi i x/3}$

**5.** $\frac{a}{2} - \frac{a}{\pi}\left[(e^u - e^{-u}) + \frac{1}{3}(e^{3u} - e^{-3u}) + \frac{1}{5}(e^{5u} - e^{-5u}) + \ldots\ldots\right]$ where $u = i\pi x/l$.

## Problems 10.9, page 423

**1.** $11.733 - 7.733 \cos 2x - 2.833 \cos 4x + \ldots\ldots - 1.566 \sin 2x - 0.116 \sin 4x + \ldots\ldots$

**2.** $1.45 + (-0.37 \cos x + 0.17 \sin x) - (0.1 \cos 2x + 0.06 \sin 2x)$

**3.** $a_0 = 41.66, a_1 = -8.33, b_1 = -1.15$

**4.** $-0.0731$

**5.** $y = 2.102 + 0.558 \cos x + 1.531 \sin x + 0.354 \cos 2x + 0.145 \sin 2x$

**6.** $7.8 \sin \theta + 1.5 \sin 2\theta - 9.2 \sin 3\theta + 11.6 \sin 4\theta - \ldots\ldots$ .

## Problems 10.10, page 424

**1.** $2\pi/3$

**2.** $\frac{1}{2}\{f(c-0) + f(c+0)\}$

**3.** $(-1, 1)$ such that $f(x) = -f(-x)$

**4.** $f(x) = A$ when $0 < x < \pi$ and $f(x) = -A$ when $\pi < x < 2\pi$

**5.** Sine

**6.** § 10.11 (3)

**7.** Zero

**8.** not defined

**9.** odd

**10.** Cosine

**11.** even

**12.** $x = k/n$

**13.** Zero

**14.** Cosine

**15.** Zero

**16.** $\int_0^2 x^2 \cos \frac{n\pi x}{2}\, dx$

**17.** $\frac{1}{T} \int_\alpha^{\alpha + 2T} f(x) \sin \frac{n\pi x}{T}\, dx$

**18.** § 10.3

**19.** Zero

**20.** $a_0 = \frac{2}{l} \int_0^l f(x)\, dx,\ a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l}\, dx$

**21.** $\frac{4}{\pi}\left\{\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots\right\}$

**22.** $\pi$

**23.** Zero

**24.** $2l$

**25.** $\sum_{n=-\infty}^{\infty} (-1)^n \frac{(1 - in\pi)}{1 + n^2\pi^2} \sinh 1 e^{in\pi x}$

**26.** False

**27.** $-\pi/2$

**28.** odd

**29.** Zero

**30.** 3.5355

**31.** zero

**32.** $-1/2$

**33.** $\frac{1}{2} a_n$

**34.** $\frac{\pi^2}{8}$

**35.** $\frac{\pi}{2} - \frac{4}{\pi}\left\{\cos x + \frac{1}{3^2} \cos 3x + \frac{1}{5^2} \cos 5x + \ldots\right\}$

**36.** $f(x) = \frac{3}{8} + \frac{1}{2} \cos 2x + \frac{1}{4} \cos 4x$

**37.** $x^2 - x$

**38.** $x(l + x)$

**39.** True

**40.** False

**41.** False.

## Problems 11.1, page 429

1. $x^2 \dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + 6y = 0$
2. $\dfrac{d^2y}{dx^2} + 4y = 0$
3. $x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - xy = 0$
4. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = 0$
5. $\dfrac{d^3y}{dx^3} - 7\dfrac{dy}{dx} + 6y = 0$
6. $2xy\dfrac{dy}{dx} + x^2 - y^2 = 0$
7. $(x^2 - 25)\left(\dfrac{dy}{dx}\right)^2 + x^2 = 0$
8. $y\dfrac{dy}{dx} = 2a.$
10. $y''' - 3y'' + 3y' - y = 0.$

## Problems 11.2, page 431

1. $\sqrt{(1-x^2)} + \sqrt{(1-y^2)} = c$
2. $\log\dfrac{x}{y} - \dfrac{1}{x} - \dfrac{1}{y} = c$
3. $\tan x \tan y = c$
4. $(1+x^2)^{3/2} - 3\sqrt{(1+y^2)} = c$
5. $\tan y = c(1-e^x)$
6. $2e^{-y} = e^{-x^2} + 1$
7. $x = 2\cos y$
8. $(x^2+1)(y^2+1) = c$
9. $3e^{2x} - 2e^{3y} + 8x^3 = c$
10. $(1-ay)(a+x) = cy$
11. $(x+1)(2-e^y) = c$
12. $a\log\left(\dfrac{x-y-a}{x-y+a}\right) = 2y + c.$
13. $y = \tan^{-1}(x+y+1) + c$
14. $\tan(x+y) = \sec(x+y) + x + c$
15. $x = \operatorname{cosec}(x+y+1) - \cot(x+y+1) + c$
16. $\log\sin(y-x) = \dfrac{1}{2}x^2 + cz$
17. $\cos xy + \dfrac{1}{2x^2} = c.$

## Problems 11.3, page 432

1. $x(x^2 - 3y^2) = c$
2. $cy^3 = x^2e^{-x/y}$
3. $(x/y)^3 = 3\log cy$
4. $y + \sqrt{(x^2+y^2)} = c$
5. $y^2 = 2x[y + x\log(cx)]$
6. $x(c+y) = ay^2$
7. $y = 2x\tan^{-1}(cx)$
8. $e^{x/y} = y + c$
9. $\log y - \dfrac{x^2}{4y^2}\left(z\log\dfrac{y}{x} + 1\right) = c$
10. $\log x = \dfrac{1}{2}\left[\dfrac{y}{x} - \dfrac{1}{2}\sin\left(\dfrac{2y}{x}\right)\right] + c$
11. $xy\cos(y/x) = c.$

## Problems 11.4, page 434

1. $(X^2 + 2Y^2)^2 = c\left(\dfrac{\sqrt{2}Y - X}{\sqrt{2}Y + X}\right)$ where $X = x+1,\ Y = y-1$
2. $(y-x)^3 = c(y+x-2)$
3. $(x+y)^7 = c(x-y-2/3)^3$
4. $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$
5. $3(2y-x) + \log(3x+3y+4) = c$
6. $x - y + \dfrac{3}{4}\log(8x - 12y - 5) = c$
7. $\log(x + y + \frac{1}{3}) + \frac{3}{2}(y-x) = c.$

## Problems 11.5, page 437

1. $y = ce^{-\tan x} + \tan x - 1$
2. $y = \log x + c/\log x$
3. $y\sec^2 x = \sec x - 2$
4. $y\cosh x = c + \dfrac{2}{3}\cosh^3 x$
5. $y\sqrt{(1-x^2)} = \sin^{-1}x + c$
6. $y = c(1-x)^2 + (1-x^2)$
7. $y(1+\sin x) = c - x^2/2$
8. $2r\sin^2\theta + \sin^4\theta = c$
9. $ye^{x^2} = 2x + c$

10. $x = y^3 + cy$
11. $x = \sin^{-1} y - 1 + ce^{-\sin^{-1} y}$
12. $xy^{-2} = c - e^{-y}$
13. $xe^{\tan^{-1} y} = \tan^{-1} y + c$
14. $xe^y = c + \tan y$.

## Problems 11.6, page 439

1. $y^{-1} \sec x = \tan x + c$
2. $1/r = \sin\theta + c\cos\theta$
3. $x^2 + (4x^5 + c)y^4 = 0$
4. $1/y = x^2 - 2 + ce^{-x^2/2}$
5. $y^2 = x^2 + cx - 1$
6. $y/x = \log y + c$
7. $\sin y = (1 + x)(e^x + c)$
8. $e^{x+y} = \frac{1}{2}e^{2x} + c$
9. $\tan y = x^3 - 3x^2 + 6x - 6 + ce^{-x}$
10. $\cos y = \cos x (\sin x + c)$
11. $\sqrt{x} = \sqrt{y}(\log\sqrt{y} + c)$
12. $y^{-1} = \frac{1}{2}\log x + \frac{1}{4} + cx^2$.

## Problems 11.7, page 442

1. $x^3 + y^3 - 3axy = c$
2. $x^4 + 2x^2y^2 - y^4 - 2a^2x^2 - 2b^2y^2 = c$
3. $x^3 - 6x^2y - 6xy^2 + y^3 = c$
4. $\frac{x^5}{5} - x^2y^2 + xy^4 + \cos y = c$
5. $e^{xy} + y^2 = c$
6. $x^5 + x^3y^2 - x^2y^3 - y^5 = c$
7. $x^3 + 3x^2y^2 + y^4 = c$
8. $x^2 - y^2 = cy^3$
9. $3y\cos 2x + 6y + 2y^3 = c$
10. $e^x = \sec x \tan y + c$
11. $x^2y + xy - x\tan y + \tan y = c$.

## Problems 11.8, page 445

1. $ax + \tan^{-1} y/x = c$
2. $x^2 + y^2 - 2a^2\tan^{-1}(y/x) = c$
3. $y + cx + \log x + 1 = 0$
4. $3\log x - (y/x)^3 = c$
5. $\log(y/x) + \frac{1}{2}x^2y^2 = c$
6. $xy + \log(x/y) - (1/xy) = c$
7. $(y + 2/y^2)x + y^2 = c$
8. $4x^4y + 4x^3y^2 - x^4 = c$
9. $2\cos(xy) + x^{-2} = c$
10. $\log(x/y) = c + xy$
11. $(x/y) + e^{x^3} = c$
12. $4(xy)^{1/3} - \frac{2}{3}(x/y)^{3/2} = c$
13. $4y\log x = y^2 + c$.

## Problems 11.9, page 446

1. $(x - y + c)(x^2 + y^2 + c) = 0$
2. $(2y - x^2 + c)(y + x + ce^{-x} - 1) = 0$
3. $x^2 + y^2 = cx$
4. $(y - cx)(y^2 - x^2 - c) = 0$
5. $(y - c)(y + x^2 - c)(xy + cy + 1) = 0$.

## Problems 11.10, page 448

1. $x + c = \frac{a}{2}\left[\log\frac{p-1}{\sqrt{(1+p^2)}} - \tan^{-1} p\right]$, with the given relation
2. $xy = c^2x + c$
3. $y = 2\sqrt{(xc)} + c^2$
4. $2cy = c^2x^2 + 1$
5. $x = (\log p - p + c)(p - 1)^2$, with the given relation
6. $x = \sin p + c$, with the given relation.

## Problems 11.11, page 449

1. $y = c(x - c)^2$
2. $y^2 = 2cx + c^3$
3. $(y + \alpha p)\sqrt{(p^2 - 1)} + a\cosh^{-1} p = c$, with the given relation
4. $y + (1 + p^2)^{-1} = c$, with the given relation.

## Problems 11.12, page 450

1. (i) Gen. sol. : $y = cx + a/c^2$ ; Singular sol. : $2ax^2 = (2ac + x)^3$
 (ii) Gen. sol. : $c = \log(cx - y)$ ; Singular sol. : $y = x(\log x - 1)$
 (iii) Gen. sol. $y = cx + \sqrt{(a^2c^2 + b^2)}$ ; Singular sol. $y + \sqrt{1 - x^2} = 0$
 (iv) Gen. sol. $y = cx - \sin^{-1} c$ ; Singular sol. $y = \sqrt{x^2 - 1} - \sin^{-1} \frac{\sqrt{x^2 - 1}}{x}$

2. $y = cx + (c - 2c^2)$
3. $(y - cx)(c - 1) = c$
4. $(y - cx)(c + 1) + ac^2 = 0$
5. $y^2 = cx^2 + c^2$ [**Hint :** Put $x^2 = u, y^2 = v$]
6. $xy = cy - c^2$ [**Hint :** Put $u = y, v = xy$]
7. $y^2 = cx^2 - \frac{2c}{1 + c}$.

## Problems 11.13, page 450

1. (i)
2. (ii)
3. (iii)
4. (i)
5. $\log y + c = x^2/2y^2$
6. $yx^2 = x^3 + c$
7. $e^x + x^2y + cy = 0$.
8. (iii)
9. $x^2 + y^2 + 2 \tan^{-1} y/x = c$
10. $\log x + c = y^3/3x^3$
11. (i)
12. $y^2 + 1/x + ce^{-y^2/2} = 2$
13. $y = cx + a/c^2$
14. $c = \log(cx - y)$
15. $xy = c$ or $x^2 - y^2 = c$
16. 2
17. $xy = c$
18. (b)
19. (b)
20. $(1 + x^2)^{3/2} + (1 + y^2)^{3/2} = c$
21. $y = 5e^{-x}$
22. $x + y = u$
23. $x^{-5}$
24. § 11.11 (3)
25. $5x^4y^2 + 2(x^5 + y^5) = c$
26. $\sin(y/x) = cx$
27. (a)
28. (c)
29. $x + y\, dy/dx = 0$
30. $e^{-x^2} + 2\cos y = c$
31. (c)
32. False
33. False.

## Problems 12.1, page 454

1. (i) $9y + 4x^2 = 0$ ; (ii) $3(x + 3y) = 2(1 - e^{3x})$
2. $y + 1 = 2e^{x^2/2}$
3. $x^2 + y^2 = cx$
4. $y = \sqrt{(a^2 - x^2)} + a \log\left(\frac{\alpha - \sqrt{a^2 - x^2}}{x}\right) + c$
5. $y^2 = 4x$
6. $y = ae^{cx}$
7. $y = ax + b$
8. $x = 3y^2$
9. (i) $r(\theta - \alpha) = c$ ; (ii) $r = a + b\cos\theta$
10. $r^2 = a^2 \sin 2\theta$
11. $c^2x^2 = 2cy + 1$
12. $r = ae^{\theta \cot \alpha}$.

## Problems 12.2, page 457

1. $2x^2 + y^2 = c$
2. $x^2 + 2y^2 = c^2$
3. $3y^2 + 2x^2 = c^2$
4. $x^2 + y^2 + 2\mu y - c = 0$
5. This system is self-orthogonal
6. $r = c(1 - \cos\theta)$
7. $r = b(\cos\theta - \sin\theta)$
8. $r = 2b/(1 - \cos\theta)$
9. $r^2 = c^2 \sin 2\theta$
10. $r^n n \sin\theta = b$
13. $x^2 + y^2 + cx + 1 = 0$
14. $y = cx$.

## Problems 12.3, page 462

1. $V = \sqrt{\left(\frac{mg}{k}\right)} \tanh\left(\frac{9k}{m}t + c\right)$
3. $\frac{1}{k} \log_e 2$
5. $2\sqrt{v_o}/k$
6. $v^2 = 2gx - \frac{\lambda}{m} x^2$
10. $y = (\sqrt{150} - 0.001328t)^2$ ; $t_1$ = 45 min. 1 sec., $t_2$ = 1 hr. 16 min. 51 sec., $t_3$ = 1 hr. 38 min. 13 sec
11. 17 min. 4 sec.

## Problems 12.4, page 465

**1.** 0.0006931 sec

**2.** $\frac{10}{L^2 + R^2}(R \sin t - L \cos t + Le^{-Rt/L})$

**3.** $i = \frac{1}{5}(1 - e^{-100})$

**4.** $i = ke^{-t/RC} + \frac{\omega C E_m}{\sqrt{(1 + R^2C^2\omega^2)}} \sin(\omega t + \theta)$ where $\theta = \cot^{-1}(RC\omega)$.

## Problems 12.5, page 467

**1.** 52.5 mts
**2.** 48°C
**3.** $B$ drinks hotter coffee
**4.** 490,000 cal
**5.** 2.16 cm.

## Problems 12.6, page 469

**1.** 604.9
**2.** 2 log 3/log 2
**3.** $(1 - 1/p)^{21}$ times the original amount
**4.** 64.5 days
**5.** 21.5 gm
**6.** $t = 300 - 5 \log 2 + 5 \log \frac{0.7 - x}{0.5 - x}$
**7.** 3 hr. 50 min. 16 sec
**8.** $100(2 - e^{-t/20})$ ; 13.9 min.

## Problems 12.7, page 469

**1.** $6(1 - e^{-3})$
**2.** 54 m
**3.** 90.25%
**4.** $r(\theta - \alpha) = c$
**5.** $y = ae^{cx}$
**6.** rectangular hyperbola
**7.** $x^2 - y^2 = c$
**8.** The system is self-orthogonal
**9.** $2\sqrt{v_0/k}$
**10.** 2 log 3/log 2
**11.** Sunil
**12.** (*d*)
**13.** (*c*)
**14.** (*d*)
**15.** 2.21
**16.** (*c*)
**17.** (*c*)
**18.** (*a*)
**19.** False
**20.** True.

## Problems 13.1, page 474

**1.** $\frac{2}{3} e^{2t} \sin 3t$

**2.** $y = e^x(4 \cos 3x - \sin 3x)$

**3.** $y = c_1 + (c_2 + c_3 x)e^{-x/2}$

**4.** $y = c_1 e^{-x} + e^{x/2}\left(c_2 \cos \frac{\sqrt{3}x}{2} + c_3 \sin \frac{\sqrt{3}x}{2}\right)$

**5.** $y = (c_1 + c_2 x + c_3 x^2)e^x$

**6.** $y = (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x$

**7.** $y = c_1 e^{-x} + c_2 e^{2x} + e^{x/2}(c_3 \cos x/\sqrt{2} + c_4 \sin x/\sqrt{2})$

**8.** $y = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x + c_5 e^x$.

## Problems 13.2, page 486

**1.** $y = (c_1 + c_2 x)e^{3x} + 3x^2 e^{3x} + \frac{7}{25} e^{-2x} - \frac{1}{9} \log 2$

**2.** $y = \frac{3}{5} e^{-2x}(\cos x + 3 \sin x) - \frac{e^x}{10} - \frac{e^{-x}}{2}$

**3.** $x = c_1 \cos nt + c_2 \sin nt + \frac{kx}{2n} \sin(nt + \alpha)$

**4.** $x = e^{-t}(c_1 \cos \sqrt{2}t + c_2 \sin \sqrt{2}t) + \frac{1}{4}(\sin t - \cos t)$

**5.** $y = c_1 e^{-x} + c_2 e^{-2x} + 1 + \frac{1}{10}(3 \sin 2x - \cos 2x)$

**6.** $y = c_1e^x + c_2e^{3x} + \dfrac{1}{884}(10 \cos 5x - 11 \sin 5x) + \dfrac{1}{20}(\sin x + 2 \cos x)$

**7.** $y = c_1 + (c_2 + c_3x)\, e^{-x} - \dfrac{x^2}{2}e^{-x} + \dfrac{3}{50} \cos 2x - \dfrac{2}{25} \sin 2x$

**8.** $y = (c_1 + c_2x)\, e^{-x} + \dfrac{1}{2} + \dfrac{1}{5}(2 \sin 2x + \cos 2x)$

**9.** $y = (c_1 + c_2x)e^x + c_3\, e^{3x} + \dfrac{1}{8}(xe^{3x} - x^2e^x)$ **10.** $y = c_1e^x + c_2e^{-x} + \dfrac{e^x}{12}(2x^3 - 3x^2 + 9x)$

**11.** $y = c_1 + c_2e^x + c_3e^{-x} + xe^x - (x^2 + x) - 2 \sin x$

**12.** $y = e^{3x}(c_1 \cos 4x + c_2 \sin 4x) + \dfrac{1}{17}e^{2x} + \dfrac{1}{565}(23 \sin x + 6 \cos x) + \dfrac{x}{25} + \dfrac{6}{625}$

**13.** $y = (c_1 + c_2x) \cos x + (c_3 + c_4x) \sin x + x^4 - 24x^2 + 72 + \dfrac{1}{225} \sin 4x - \dfrac{1}{9} \sin 2x$

**14.** $y = c_1\, e^{-2x} + c_2e^{-3x} - \dfrac{e^{-2x}}{10}(\cos 2x + 2 \sin 2x)$

**15.** $y = e^{-x/2}\left\{(c_1 + x/4) \cos (x\sqrt{3/2}) + (c_2 + x/4\sqrt{3}) \sin (x\sqrt{3/2}\right\} + e^{x/2}\left\{c_3 \cos \sqrt{3x/2} + c_4 \sin \sqrt{3x/2}\right\}$

**16.** $y = e^{-x}(c_1 \cos \sqrt{2x} + c_2 \sin \sqrt{2x}\,) + \dfrac{e^x}{41}(4 \sin x + 5 \cos x).$

**17.** $y = c_1\, e^{-x} + c_2e^{-3x} - \dfrac{e^{-x}}{5}(\sin x + 2 \cos x) + \dfrac{e^{3x}}{22}\left(x - \dfrac{5}{11}\right)$

**18.** $y = c_1 \cos \sqrt{2x} + c_2 \sin \sqrt{2x} + \dfrac{e^{3x}}{11}\left(x^2 - \dfrac{12}{11}x + \dfrac{50}{121}\right) + \dfrac{e^x}{17}(4 \sin 2x - \cos 2x)$

**19.** $y = c_1e^x + c_2e^{-x} + c_3 \cos x + c_4 \sin x - (1/5) \cos x \cosh x$

**20.** $y = c_1 + (c_2 + c_3\, x)\, e^{-x} + \dfrac{e^{2x}}{18}\left(x^2 - \dfrac{7x}{8} + \dfrac{11}{6}\right) + \dfrac{1}{100}(3 \sin 2x + 4 \cos 2x)$

**21.** $y = c_1 \cos 4x + c_2 \sin 4x + \dfrac{1}{7}\left(x \sin 3x - \dfrac{6}{7} \cos 3x\right)$

**22.** $y = (c_1 + c_2x)\, e^{-x} + \dfrac{1}{2} \cos x + \dfrac{1}{2}(x - 1) \sin x$

**23.** $y = c_1e^x + c_2e^{-x} - \dfrac{1}{2}(x \sin x + \cos x) + (xe^x/12)\,(2x^2 - 3x + 9)$

**24.** $y = c_1e^{-x} + c_2e^{-2x} + c_3e^{-3x} + e^{-2x}\,.\, e^{e^x}$

**25.** $y = c_1 \cos ax + c_2 \sin ax - \dfrac{1}{a^2} \cos ax \log (\sec ax + \tan ax).$

## Problems 13.3, page 490

**1.** $y = (c_1 - x/a) \cos ax + [c_2 + (1/a^2) \log \sin ax] \sin ax$

**2.** $y = c_1 \cos x + c_2 \sin x + \cos x \log (\cos x) + x \sin x$

**3.** $y = c_1 \cos x + c_2 \sin x - \cos x \log (\sec x + \tan x)$

**4.** $y = c_1 \cos x + c_2 \sin x + \dfrac{x}{2} \sin x - \dfrac{x^2}{4} \cos x$ **5.** $y = (c_1 + c_2x)\, e^x + xe^x \log x$

**6.** $y = (e^x + e^{2x}) \log (1 + e^x) + (c_1 - 1 - x)\, e^x + (c_2 - x)\, e^{2x}$

**7.** $y = e^x\,(c_1 \cos x + c_2 \sin x) - e^x \cos x \log (\sec x + \tan x)$

**8.** $y = c_1 + c_2e^{2x} - \dfrac{1}{2}e^x \sin x$ **9.** $y = c_1 \cos x + c_2 \sin x + \sin x \log (1 + \sin x) - x \cos x - 1$

**10.** $y = c_1\, e^x + c_2\, e^{2x} + \dfrac{1}{2}(x^2 + 3x + 3.5 - 2xe^x)$ **11.** $y = c_1 \cos x + c_2 \sin x - x \sin x$

**12.** $y = c_1e^{2x} + c_2e^{3x} + xe^{3x} + \frac{1}{10}(\sin x + \cos x)$ **13.** $y = c_1e^x + c_2e^{-2x} - \frac{1}{4}(2x+1) - \frac{1}{10}(\cos x + 3\sin x)$

**14.** $y = e^x(c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x) + \frac{1}{27}(9x^3 + 18x^2 + 6x - 8) + \frac{1}{4}(\cos x - \sin x)$

**15.** $y = c_1 + c_2e^{2x} - \frac{1}{2}e^x \sin x.$

## Problems 13.4, page 495

**1.** $y = c_1x^2 + c_2x^3 - x^2 \log x$ **2.** $y = c_1 x^4 + c_2 x^{-1} + \frac{x^4}{5} \log x$

**3.** $y = (c_1 + c_2 \log x)x^2 + \frac{1}{4} + 2x + \frac{1}{2}x^2(\log x)^2$

**4.** $y = c_1x^2 + c_2x^{-1} + \frac{1}{3}(x^2 - 1/x)\log x$ **5.** $u = \frac{kr}{8}(a^2 - r^2)$

**6.** $c_1x^{-1} + c_2x^{-2} + \frac{1}{2}\log x - \frac{3}{4}$

**7.** $y = c_1x^{-1} + \sqrt{x}\,\{c_2 \cos[(\sqrt{3}/2)\log x] + c_3 \sin[(\sqrt{3}/2)\log x]\} + \frac{1}{2}x + \log x$

**8.** $y = c_1x^{-2} + x[c_2 \cos(\sqrt{3}\log x) + c_3 \sin(\sqrt{3}\log x)] + 8\cos(\log x) - \sin(\log x)$

**9.** $y = c_1x^{-1} + \{c_2 \cos(\log x) + c_3 \sin(\log x)\}x + 5x + 10\log x/x$

**10.** $y = \frac{1}{x}(c_1 + c_2 \log x) + \frac{1}{x}\log\frac{x}{1-x}$ **11.** $y = x^{-2}(c_1 + c_2 \log x) + \frac{x}{9}\left(\log x - \frac{2}{3}\right)$

**12.** $y = c_1x^3 + c_2x^{-4} + \frac{x^3}{98}\log x(7\log x - 2)$

**13.** $y = c_1(2x+3)^a + c_2(2x+3)^b - \frac{3}{14}(2x+3) + \frac{3}{4}$ where $a, b = \frac{3 \pm \sqrt{57}}{4}$

**14.** $y = c_1(x-1) + c_2(x-1)^2 + c_3(x-1)^{-2} + \log(x+1) + 1$

**15.** $y = c_1 \cos\log(1+x) + c_2 \sin\log(1+x) - \frac{1}{3}\sin[2\log(1+x)]$

**16.** $y = c_1(3x+2)^{1/3} + c_2(3x+2)^{-1} + \frac{1}{27}\left[\frac{1}{15}(3x+2)^2 + \frac{1}{4}(3x+2) - 7\right]$

## Problems 13.6, page 499

**1.** $x = (c_1 + c_2x)e^{3x}$ ; $y = [(1-2x)(c_2 - 2c_1)]e^{3x}$

**2.** $x = e^t + e^{-t}, y = e^{-t} - e^t + \sin t$ **3.** $x = c_1e^t + c_2e^{-5t} + \frac{6}{7}e^{2t}$ ; $y = c_2e^{-5t} - c_1e^t + \frac{8}{7}e^{2t}$

**4.** $x = e^{6t}(c_1 \cos t + c_2 \sin t), y = e^{6t}[(c_1 - c_2)\cos t + (c_1 + c_2)\sin t]$

**5.** $x = \frac{1}{5}e^t + \frac{2}{5}e^{-t} - c_1 \sin 2t + c_2 \cos 2t, y = \frac{2}{5}e^t + \frac{1}{5}e^{-t} + c_1 \cos 2t + c_2 \sin 2t$

**6.** $x = c_1e^t + c_2e^{-5t} + \frac{3}{7}e^{2t} - \frac{2}{5}t - \frac{13}{25}, y = c_1e^t - c_2e^{-5t} - \frac{4}{7}e^{2t} - \frac{3t}{5} - \frac{12}{25}$

**7.** $x = -t - \frac{2}{3}, y = \frac{1}{2}t^2 + \frac{4}{3}t + c$ **8.** $y = c_1e^x + c_2e^{-2x} + 2e^{-x}, z = 3c_1e^x + 2c_2e^{-2x} + 3e^{-x}$

**9.** $x = c_1e^{-t} + c_2e^{3t} - \frac{1}{5}(\cos t - 2\sin t)$ ; $y = 2c_1e^{-t} - 2c_2e^{3t} + \frac{1}{5}(\sin t + 2\cos t)$

**10.** $x = \frac{1}{2}\left(t + \frac{1}{t}\right), y = \frac{1}{2}\left(-t + \frac{1}{t}\right)$

**11.** $x = c_1e^{-t} + c_2e^{3t} - \frac{1}{5}(\cos t - 2\sin t), y = 2c_1e^{-t} - 2c_2e^{3t} + \frac{2}{5}\cos t + \frac{1}{5}\sin t$

**12.** $x = (c_1 + c_2t)e^{-t} + (c_3 + c_4t)e^t, y = -\frac{1}{2}[c_1 + c_2(1+t)]e^{-t} + \frac{1}{2}[c_4(1-t) - c_3]e^t$

**13.** $x = c_1e^t + c_2e^{-t} + c_3 \cos t + c_4 \sin t - \frac{t}{4} \cos t + \frac{t}{4} \sin t$

$y = -c_1e^t - c_2e^{-t} + c_3 \cos t + c_4 \sin t + \frac{1}{4}(2 + t)(\sin t - \cos t)$

**14.** $x = \frac{8}{9}\left(1 - \cos \frac{3}{2}t\right), y = \frac{4}{3}t - \frac{8}{9} \sin \frac{3}{2}t.$

## Problems 13.7, page 500

**1.** $y = c_1e^{ax} + c_2e^{-ax} + c_3 \cos ax + c_4 \sin ax$

**2.** $-\frac{1}{25}(3 \sin 2x + 4 \cos 2x)$ **3.** 1/6 **4.** $e^x(x - 1)$

**5.** (*b*) **6.** $y = c_1 + (c_2 + c_3 x + c_4 x^2) e^{2x}$ **7.** (*a*)

**8.** $y = e^x (c_1 + c_2 x) \cos 2x + (c_3 + c_4 x) \sin 2x$ **9.** $y = \cos x + 2 \sin x$

**10.** (*ii*) **11.** $y = (c_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x$

**12.** $\frac{1}{10} \cosh 3x$ **13.** $y = a \log x + 6$

**14.** $y = (c_1 + c_2 x) e^{\sqrt{2}x} + (c_3 + c_4x) e^{-\sqrt{2}x}$ **15.** $\frac{1}{6}x^3e^{-x}$

**16.** $\frac{1}{2}e^{2x}$ **17.** $\sin 2x$ **18.** $\frac{1}{2}x^2 e^{-x}$

**19.** $y = (c_1 + c_2 x) e^{-x/2} + c_3$ **20.** (*c*) **21.** (*a*)

**22.** $x e^{-t}$ **23.** $\frac{d^2y}{dt^2} + 7y = 2e^t.$ **24.** (*c*)

**25.** (*a*) **26.** (*b*)

**27.** $y = (c_1 + c_2 \log x) x$ **28.** $x^2y_2 + xy_1 - y = 0$ **29.** $\frac{1}{9} \log 2$

**30.** $e^t$ **31.** (*d*)

**32.** $y = c_1e^{-x} + c_2 e^{2(1+\sqrt{2})x} + c_3 e^{2(1-\sqrt{2})x}$ **33.** $\frac{d^3y}{dx^3} + 2$

**34.** $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$ **35.** False **36.** False.

## Problems 14.1, page 506

**1.** 38 sec **4.** $x = \frac{ue^{-\lambda nt}}{n\sqrt{1 - \lambda^2}} \sin [nt \sqrt{(1 - \lambda^2)}]$

**6.** It must be shortened by 1/8640 of its length

**7.** It must be increased by 0.0074 ft./sec$^2$ **8.** 4321/4319

**9.** $k^2 > 4\mu$, $\theta = c_1e^{\frac{-k+\lambda}{2}t} + c_2e^{\frac{-k-\lambda}{2}t}$ where $\lambda = \sqrt{(k^2 - 4\mu)}$

$k^2 = 4\mu$, $\theta = (c_1 + c_2t) e^{-kt/2}$

$k^2 < 4\mu$, $\theta = c_1 e^{-kt/2} \cos \left(\frac{\sqrt{4\mu - k^2}}{2}t + c_2\right)$

**10.** $x = \frac{F_0}{2n^2}(\sin nt - n \cos nt).$

## Problems 14.2, page 513

**2.** 1 ft. ; $\pi/2\sqrt{2}$ sec ; $4\sqrt{2}$ ft./sec

**4.** $\pi/\sqrt{7}$

**5.** $x = e^{-5t}\left\{\cos\sqrt{220t} + (5/\sqrt{220})\sin\sqrt{220}\,t\right\}$.

**6.** 0.45 sec. ; 1.15 sec

**8.** $x = \dfrac{10}{21}e^{-t} - \dfrac{17}{27}e^{-8t} - \dfrac{\sqrt{2}}{3}\sin\left(9t + \dfrac{\pi}{4}\right)\dfrac{\sqrt{2}}{3}, \dfrac{2\pi}{9}$ sec., $9/2\pi$ cycles/sec

**9.** $0.8\,(2\sin 4t - \cos 4t)$

**10.** (*i*) $x = Ae^{-kt}\cos\left\{t\sqrt{(b^2 - k^2)} + B\right\} + \{e^{-kt}/(b^2 + k^2 - n^2)\}\sin nt$

(*ii*) $x = Ae^{-kt}\cos\left\{t\sqrt{(b^2 - k^2)} + B\right\} - (te^{-kt}/2n)\cos nt$

## Problems 14.3, page 517

**2.** $i = I\sin(T/\sqrt{LC})$

**3.** $i = 2Eke^{-RT/2L}\sin(kt/2L)$, where $k = \sqrt{\left(\dfrac{4L - CR^2}{C}\right)}$

**4.** $R^2 > 4L/C$ for over damping ; $R^2 = 4L/C$ for critical damping ; $R^2 < 4L/C$ for under damping ; critical resistance $= 2\sqrt{L/C}$

**5.** $q = e^{-500t}(0.002\cos 1323t + 0.0008\sin 1323t)$

**8.** (*i*) $i = Ae^{-\alpha t}\cosh(\beta t + \gamma)$ ; (*ii*) $i = Ae^{-\alpha t}\cos(\beta t + \gamma) + \dfrac{E}{R}\cos\phi\sin(pt + \phi)$

where $\alpha = -\dfrac{R}{2L}, \beta = \pm\left\{\left(\dfrac{R}{2L}\right)^2 - \dfrac{1}{CL}\right\}$ and $\phi = \tan^{-1}\{(1 - CLp^2)/CRp\}$.

## Problems 14.4, page 525

**4.** $y = \dfrac{wl}{2Pn}\operatorname{cosec}\dfrac{nl}{2}\cos\left(nx - \dfrac{nl}{2}\right) - \dfrac{wl}{2nP}\cot\dfrac{nl}{2} + \dfrac{w}{2P}(x^2 - lx)$

**6.** $y = \dfrac{F}{P}[n\sin nx - l\cos nx + l - x]$

**7.** $\pi^2 EI/4l^2$

**8.** $\dfrac{W}{2a^2}\left(\operatorname{sech}\dfrac{al}{2} - \sec al\right)$

## Problems 14.5, page 528

**1.** $\dfrac{2u\sin\alpha}{g}; \dfrac{u^2\sin 2\alpha}{g}$

**2.** (*i*) $\dfrac{2u^2\sin(\alpha - \beta)\cos\alpha}{g\cos^2\beta}$ ; (*iii*) $\dfrac{u^2}{g(1 + \sin\beta)}$

**4.** $4x^2 + k^2y^2 = 4$

**5.** $i_1 = \dfrac{a}{p + \omega}\sin pt, i_2 = \dfrac{a}{p + \omega}\cos pt.$

**6.** $i_1 = \dfrac{E}{R}\left(\dfrac{2}{3} - \dfrac{1}{2}e^{-Rt/L} - \dfrac{1}{6}e^{-3Rt/L}\right); \; i_2 = \dfrac{E}{R}\left(\dfrac{1}{3} - \dfrac{1}{2}e^{-Rt/L} + \dfrac{1}{6}e^{-3Rt/L}\right)$

**7.** $x = a(nt - \sin nt), y = a(1 - \cos nt)$

**8.** $x = \dfrac{E}{H\omega}(1 - \cos\omega t), y = \dfrac{E}{H\omega}(\omega t - \sin\omega t)$, where $\omega = eH/m$.

## Problems 14.6, page 529

**1.** (*b*) **2.** (*b*) **3.** (*c*) **4.** (*b*)
**5.** (*b*) **6.** (*b*) **7.** (*b*) **8.** 60 sec
**9.** $30/\pi\sqrt{LC}$ **10.** 0.0074 sec **11.** resonance **12.** $EI\dfrac{d^2y}{dx^2} - Py = \dfrac{w}{2}(x^2 - lx)$
**13.** $y = 0$ and $\dfrac{dy}{dx} = 0$

## Problems 15.1, page 531

**1.** $y = -x^2 \sin x - 4x \cos x + c_1x + c_2$
**2.** $y = \dfrac{x^4}{24} + \dfrac{x^3}{6}\log x - \dfrac{11}{35}x^3 + c_1x^2 + c_2x + c_3$

## Problems 15.2, page 533

**1.** $2(y^{1/4} - 1) = x$
**2.** $\sqrt{(y^2 - 8y)} + 4\cosh^{-1}(\tfrac{1}{4}y - 1) = 3x$
**3.** $r = \dfrac{\sqrt{(v^2 - a^2\omega^2)}}{\omega}\sinh\left[\omega t + \sinh^{-1}\dfrac{a\omega}{\sqrt{(v^2 - a^2\omega^2)}}\right]$
**4.** $t = \dfrac{h^{3/2}}{\sqrt{(2g)a}}\left[\cos^{-1}\sqrt{\dfrac{x}{h}} - \sqrt{\left(\dfrac{hx - x^2}{h}\right)}\right]$

## Problems 15.3 page 534

**1.** $y = c_1 - x^2 - c_2/x$
**2.** $y = c_1x + (c_1^2 + 1)\log(x - c_1) + c_2$
**3.** $15c_1^2y = 4(c_1x + a^2)^{5/2} + c_2x + c_3$
**4.** $x^2 + y^2 = a^2$
**5.** $\theta = \dfrac{m}{\mu a}\log\left(1 + \dfrac{\mu a\omega t}{m}\right)$
**6.** $v = \dfrac{1}{r_1 - r_2}\left[v_1r_1 - v_2r_2 - \dfrac{(v_2 - v_1)r_1r_2}{r}\right].$

## Problems 15.4, page 536

**1.** $y = 2x - 2\log(1 - c_1e^{2x}) + c_2$
**2.** $y^2 = x^2 + c_1x + c_2$
**3.** $\log y = c_1e^x + c_2e^{-x}$
**4.** $(\log y - 1)(c_1x + c_2) = 1$
**5.** $(x - a)^2 + y^2 = c^2$, circles whose centres are on the $x$-axis.

## Problems 15.5, page 537

**1.** $y = e^{x^2}(c_1x + c_2)$
**2.** $y = (x^2 - x + c_1x)e^x + c_2x$
**3.** $y = e^x(c_1\log x + x + c_2)$
**4.** $cy = 1 + (k - x)\cot x$
**5.** $y = \left[c_1 - \dfrac{1}{2}\cos x - \dfrac{1}{5}c_2e^{-2x}(\cos x + 2\sin x)\right]e^x.$

## Problems 15.6, Page 539

**1.** $y = c_1\cos(\sin x) + c_2\sin(\sin x)$
**2.** $y = c_1\cos(1/x) + c_2\sin(1/x)$
**3.** $y = c_1e^t + c_2e^{-t} - t$ where $t = \cos x$
**4.** $y = c_1\cos(2\tan^{-1}x) + c_2\sin(2\tan^{-1}x)$
**5.** $y = c_1e^{\sqrt{2}\sin x} + c_2e^{-\sqrt{2}\sin x} + \sin^2 x.$

## Problems 15.7, page 540

**1.** $nx - lz = c(mz - ny)$ **2.** $x^2 + y^2 + z^2 = cx$ **3.** $xy^2 = cz^3$

**4.** $x^2 + y^2 - xz = cz$ **5.** $y(x + z) = c(x + y + z)$ **6.** $x + y + z + \log(xyz) = c.$

## Problems 15.8, page 541

**1.** $x^3 - y^3 = c_1, x^2 - z^2 = c_2$ **2.** $lx + my + nz = c_1, x^2 + y^2 + z^2 = c_2$ **3.** $\dfrac{x-y}{y-z} = c_1, \dfrac{z-x}{y-z} = c_2$

**4.** $x^2 - y - 2xy = c_1,\ x^2 - y^2 - z^2 = c_2$ **5.** $xyz = c_1, x^2 + y^2 + z^2 = c_2$ **6.** $y = c_1 z, x^2 + y^2 + z^2 = c_2 z.$

## Problems 16.1, page 544

**1.** $y = a_1\left(x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dots\right)$

**2.** $$y = a_0\left(1 - \frac{x^4}{4\cdot 3} + \frac{x^8}{8\cdot 7\cdot 4\cdot 3} - \frac{x^{12}}{12\cdot 11\cdot 8\cdot 7\cdot 4\cdot 3} + \dots\dots\right)$$
$$+ a_1\left(x - \frac{x^5}{5\cdot 4} + \frac{x^9}{9\cdot 8\cdot 5\cdot 4} - \frac{x^{13}}{13\cdot 12\cdot 9\cdot 8\cdot 5\cdot 4} + \dots\dots\right)$$

**3.** $y = a_0\left(1 - \dfrac{x^2}{2} + \dfrac{x^4}{2\cdot 4} - \dfrac{x^6}{2\cdot 4\cdot 6} + \dots\dots\right) + a_1\left(x - \dfrac{x^3}{3} + \dfrac{x^5}{3\cdot 5} - \dfrac{x^7}{3\cdot 5\cdot 7} + \dots\dots\right)$

**4.** $y = 4 + 5x - 4x^2 - \dfrac{5}{3}x^3 - \dfrac{x^5}{3} - \dfrac{x^7}{7} - \dots\dots$ **5.** $y = a_0\left(1 + \dfrac{x^2}{2} - \dfrac{x^4}{8} + \dfrac{x^6}{16} - \dfrac{5x^8}{128} + \dots\right) + a_1 x$

**6.** $y = a_0(1 - x^2 + \frac{1}{3}x^4 - \frac{1}{5}x^6 + \dots\dots) + a_1 x.$

## Problems 16.2, page 550

**1.** $y = c_1 \cos\sqrt{x} + c_2 \sin\sqrt{x}$ **2.** $y = a_0\left(1 - x^2 + \dfrac{x^4}{4}\dots\dots\right) + a_1\left(x - \dfrac{x^3}{2} + \dfrac{3x^5}{10}\dots\dots\right)$

**3.** $y = (c_1 + c_2 \log x)\left[1 + x + \dfrac{1}{(2\,!)^2}x^2 + \dfrac{1}{(3\,!)^3}x^3 + \dots\dots\right] - 2c_2\left[x + \dfrac{1}{(2\,!)^2}\left(1 + \frac{1}{2}\right)x^2 + \dfrac{1}{(3\,!)^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right)x^3 + \dots\dots\right]$

**4.** $y = c_1(1 + x + x^2/4 + x^3/4\,.\,7 + \dots) + c_2 x^{2/3}(1 + \frac{1}{3}x + x^2/3\,.\,6 + x^3/3\,.\,6\,.\,9 + \dots)$

**5.** $y = a_0\left(1 - 2x + \dfrac{3}{2!}x^2 - \dfrac{4}{3!}x^3 + \dots\right) + a_1\left[y_1 \log x + a_0\left(3x - \dfrac{13}{4}x^2 + \dots\right)\right]$

**6.** $y = a_0 x\left(1 + \dfrac{x}{5} + \dfrac{x^2}{70} + \dots\right) + a_1 x^{-1/2}\left(1 - x - \dfrac{x^2}{2} + \dots\right)$

**7.** $y = c_1 x^{-\frac{1}{2}}\left(1 + \dfrac{x}{2} + \dfrac{x^2}{40} + \dots\dots\right) + c_2\, x^{1/4}\left(1 + \dfrac{x}{14} + \dfrac{x^2}{616} + \dots\dots\right)$

$+ c_2\sqrt{x}(x + x^2/2\,.\,3 + x^4/2\,.\,4\,.\,3\,.\,7 + x^6/2\,.\,4\,.\,6\,.\,3\,.\,7\,.\,11 + \dots)$

**8.** $y = a_0\sqrt{x}(1 - x) + a_1\left(1 - 3x + \dfrac{3x^2}{1\cdot 3} + \dfrac{3x^3}{3\cdot 5} + \dfrac{3x^4}{5\cdot 7} + \dots\right)$

**9.** $y = a_0(1 - \frac{2}{3}x + \frac{1}{3}x^2 + \ldots) + a_1x^4(1 - 2x + 3x^2 - 4x^3 + \ldots)$

**10.** $y = c_1\left(1 + 3x^2 + \dfrac{3}{5}x^4 \ldots\ldots\right) + c_2x^{3/2}\left(1 + \dfrac{3}{8}x^2 - \dfrac{1.3}{8.16}x^4 + \dfrac{1.3.5}{8.16.24}x^6 + \ldots\ldots\right).$

## Problems 16.3, page 557

**1.** 0.224, 0.44.

## Problems 16.4, page 562

**1.** $y = c_1J_{1/2}(x) + c_2J_{-1/2}(x)$

**2.** $y = c_1J_{2/5}(x) + c_2J_{-2/5}(x)$

**3.** $y = x^n[c_1J_n(kx) + c_2y_n(kx)]$ where $n = \dfrac{1}{2}(1 - \alpha)$

**4.** $y = x[c_1J_1(2x) + c_2Y_1(2x)]$

**5.** $y = c_1\sqrt{x}J_1(2\sqrt{x}) + c_2\sqrt{x}\,Y_1(2\sqrt{x})$

**7.** $y = c_1\sqrt{x}J_n(x) + c_2\sqrt{x}\,J_{-n}(x)$

**11.** $x^2 = \displaystyle\sum_{n=1}^{\infty} \frac{2}{\alpha_n^2}\cdot\frac{1}{J_2^2(3\alpha_n)}(3\alpha_nJ_1(3\alpha_n) - 2J_2(2\alpha_n))$

## Problems 16.5, page 570

**3.** (*i*) $2P_3 + 4P_1$; (*ii*) $\dfrac{2}{5}P_3 + \dfrac{4}{3}P_2 - \dfrac{2}{5}P_1 - \dfrac{7}{3}P_0$;

(*iii*) $\dfrac{8}{5}P_3 - 4P_2 + \dfrac{47}{5}P_1 + 4$ (*iv*) $\dfrac{8}{35}P_4 + \dfrac{6}{5}P_3 - \dfrac{2}{21}P_2 + \dfrac{34}{5}P_1 - \dfrac{224}{105}P_0$

**9.** (*i*) $f(x) = -\dfrac{7}{3}P_0(x) - \dfrac{2}{5}P_1(x) + \dfrac{4}{3}P_2(x) + \dfrac{2}{5}P_3(x)$;

(*ii*) $f(x) = -\dfrac{32}{15}P_0(x) - \dfrac{4}{5}P_1(x) - \dfrac{40}{21}P_2(x) + \dfrac{2}{5}P_3(x) + \dfrac{8}{35}P_4(x)$.

## Problems 16.6, page 572

**1.** $x^3 = \dfrac{1}{4}(3T_1 + T_3)$.

## Problems 16.7, page 575

**1.** $y_n(x) = \sin nx,\ n = 1, 2, \ldots$

**2.** $y_n(x) = \sin[(2n + 1)\pi x/2l],\ n = 0, 1, 2, \ldots$

**3.** $y_n(x) = \cos nx,\ n = 0, 1, 2, \ldots$

**4.** $1, \sin x, \cos x, \sin 2x, \cos 2x, \ldots$

**5.** $y_n(x) = \sin\left[(2n + 1)\dfrac{\pi}{2}\log|x|\right],\ n = 0, 1, 2, \ldots$

**6.** $[xe^{-x}y']' + ne^{-x^2}y = 0,\ p(x) = e^{-x}$

**7.** $[e^{-x^2}y']' + 2ne^{-x^2}y = 0,\ p(x) = e^{-x^2}$.

## Problems 16.8, page 575

**1.** $\dfrac{1}{3}(10 - 9P_1 + 8P_2)$

**2.** $\sqrt{(2/\pi x)}\cos x$

**3.** $\dfrac{2}{(2n + 1)}$

**4.** zero

**5.** zero

**6.** $J_n(x) = \dfrac{x}{2n}(J_{n+1}(x) + J_{n-1}(x))$

**7.** $\displaystyle\int_0^1 xJ_n(\alpha x)J_n(\beta x)\,dx = 0$

**8.** $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + xy = 0$

**9.** $\sqrt{(2/\pi x)}\sin x$

**10.** $x^n J_{n-1}(x)$ **11.** $\frac{1}{2}(3x^2-1)$ **12.** zero

**13.** True **14.** $P_n(x) = \frac{1}{n!2^n}\frac{d^n}{dx^n}(x^2-1)^n$ **15.** $\alpha \neq \beta$

**16.** $2P_3 + 4P_1$ **17.** $(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x)$

**18.** $-J_1(x)$ **19.** False **20.** True **21.** True **22.** True **23.** True
**24.** False **25.** (*b*) **26.** (*c*) **27.** (*iv*) **28.** (*iii*) **29.** (*iii*)
**30.** (*iv*) **31.** (*iii*) **32.** (*iii*) **33.** (*iii*) **34.** (*iii*) **35.** (*iv*)
**36.** 0, 1.

## Problems 17.1, page 579

**1.** $z = px + qy + p^2 + q^2$ **2.** $z^2(p^2+q^2+1) = c^2$ **3.** $p^2 + q^2 = \tan^2 \alpha$
**4.** $p + q = px + qy$ **5.** $z^2(p^2+q^2+1) = 9$ **6.** $py - qx = 0$
**7.** $py + qx = 0$ **8.** $qx - py = x + y$ **9.** $xys = px + py - z$
**10.** $xyr = 2(px + qy - 2z)$
$y$ **11.** $\frac{\partial^2 z}{\partial y^2} = \frac{\partial z}{\partial y}$ **12.** $x(y-z)p + y(z-x)q = z(x-y)$
**13.** $z\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}$ **14.** $p + q = mz$ **15.** $px^2 + qy = 2y^2$
**16.** $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = 0$ **17.** $\frac{\partial^2 v}{\partial t^2} = \frac{a^2}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial v}{\partial r}\right)$ **18.** $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial t} + \frac{\partial^2 z}{\partial t^2} = 0$
**19.** $p(x-2z) + q(2z-y) = y - x$ **20.** $(y-z)p + (z-x)q = x - y$.

## Problems 17.2, page 581

**1.** $z = \frac{x^2}{2}\log x + axy + \phi(x) + \psi(y)$ **2.** $z = \frac{1}{6}x^3 y + xf(y) + \phi(y)$
**3.** $u = -e^{-t}\sin x + \phi(x) + \psi(t)$ **4.** $z = f(x) + x\phi(y) + \psi(y) - \frac{1}{12}\sin(2x+3y)$
**5.** $z = e^x \cosh y + e^{-x}\sinh x$ **6.** $z = \sin x + e^y \cos x$.

## Problems 17.3, page 584

**1.** $x = z^3 f(x/y)$ **2.** $\sqrt{x} - \sqrt{y} = f(\sqrt{x} - \sqrt{z})$ **3.** $x^2 + y^2 + z^2 = f(x+y+z)$
**4.** $[\cos(x+y) + \sin(x+y)]e^{y-x} = \phi\left[z^{\sqrt{2}}\tan\left(\frac{x+y}{2} + \frac{\pi}{8}\right)\right]$ **5.** $x^2 - y^2 = f(y^2 - z^2)$
**6.** $\phi\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$ **7.** $x\log(x+y) - z = f(x+y)$ **8.** $x^2 + y^2 + 2z = [\log(xy)]$
**9.** $x^2 + y^2 - z^2 = f(x+y+z)$ **10.** $x + y + z = f(xyz)$ **11.** $\phi(x^2+y^2+z^2, xyz) = 0$
**12.** $x^2 + y^2 = f(y^2 - yz)$ **13.** $\phi(y/z, x^2+y^2+z^2) = 0$ **14.** $x^2 + y^2 + z^2 = f(y^2 - 2yz - z^2)$
**15.** $f\left(\frac{y}{z}, \frac{z}{x} - \frac{y}{x} + x^2\right) = 0$.

## Problems 17.4, page 587

**1.** $z = ax - ay/(1 + a) + b$

**2.** $z = ax + \sqrt{(1 - a^2)}y + c$

**3.** $4z(1 + a^2) = (x + ay + b)^2$

**4.** $(1 - a + az) = (x + ay + b)^2$

**5.** $2z = ay^2 - [a/(a + 1)]x^2 + b$

**6.** $z = a(x - y) - (\cos x + \cos y) + b$

**7.** $\frac{8}{9}z = (x + a)^{3/2} + (y + a)^{3/2} + b$

**8.** $3z = (x + a)^3 + (y - a)^3 + b$

**9.** $z = \dfrac{a^2}{2}\sinh^{-1}\dfrac{x}{a} + \dfrac{x_1\sqrt{(x^2+a^2)}}{2} + \dfrac{y\sqrt{(y^2-a^2)}}{2} - \dfrac{a^2}{2}\cosh^{-1}\dfrac{y}{a} + b$

**10.** $z = ax + by + \sin(a + b)$

**11.** $z = \dfrac{1}{6}(zx + a)^3 + a^2y + b$

**12.** $z = ax + by - 2\sqrt{(ab)}$

**13.** $z = axy + a^2(x + y) + b$.

## Problems 17.5, page 590

**1.** $z = \{\sqrt{(ax)} + \sqrt{(b+y)}\}^2/(1+a)$

**2.** $z = ax^by^{1/b}$

**3.** $\dfrac{z^2}{2} \pm \left\{\dfrac{z}{2}\sqrt{z^2-4a^2} - 2a^2\log\left(z + \sqrt{z^2-4a^2}\right)\right\} = 2ax + 2y + b$

**4.** $\log(z - ax) = y - a\log(a + y) + b$

**5.** $2\sqrt{(z - a - b)} = \sqrt{ax} + \dfrac{1}{\sqrt{a}}y + c$

**6.** $z = axe^{-y} - \frac{1}{2}a^2e^{-2y} + b$.

## Problems 17.6, page 595

**1.** $z = f_1(y) + f_2(y + 2x) + xf_3(y + 2x)$

**2.** $z = f_1(y - x) + f_2(y + 2x) + xf_3(y + 2x) + \dfrac{e^{x+2y}}{27}$

**3.** $z = f_1(x + y) + xf_2(x + y) + \dfrac{x^2}{2} \times e^{x+y}$

**4.** $z = f_1(y + x) + zf_2(y + x) + f_3(y + 2x) - e^{2x+y}$

**5.** $z = f_1(y + x) + xf_2(y + x) - \sin x$

**6.** $y = f_1(x - at) + f_2(x + at) - \dfrac{E}{p^2}\sin pt$

**7.** $z = f_1(y) + f_2(y + 2x) + xf_3(y + 2x) + 3x\cos(3x + 2y)$

**8.** $f_1(y\,x) + f_2(y - 2x) + f_3(y + 3x) + \dfrac{1}{75}\sin(x + 2y) + \dfrac{2}{3}x^3$.

**9.** $z = f_1(y + x) + f_2(y + 2x) + \dfrac{1}{12}e^{2x-y} - xe^{x+y} - \dfrac{1}{3}\cos(x + 2y)$

**10.** $z = f_1(y) + f_2(y + x) + \frac{1}{3}(\sin x\cos 2y + 2\cos x\sin 2y)$

**11.** $z = f_1(y) + f_2(y + x) + \dfrac{1}{2}[\sin(x + 2y) + \cos(x + 2y)] - \dfrac{1}{6}[\sin(x - 2y) + \cos(x - 2y)]$

**12.** $z = f_1(y + x) + f_2(y - x) + \dfrac{3}{28}e^{x-y}[\sin(x + 2y) - 2\cos(x + 2y)]$

**13.** $z = f_1(y - x) + f_2(y - 2x) + 4x^3y - 3x^4$

**14.** $z = f_1(y - x) + xf_2(y - x) + \dfrac{1}{4}(x^4 - 2x^3y + 2x^2y^2)$

**15.** $z = f_1(y - x) + f_2(y + 2x) + ye^x$

**16.** $z = f_1(y - x) + xf_2(y - x) + f_3(y + x) + \dfrac{e^x}{25}(\cos 2y + 2\sin 2y)$

**17.** $z = f_1(y - x) + xf_2(y - x) + x\sin y$.

## Problems 17.7, page 597

1. $z = e^{-x}\phi_1(y) + e^x\phi_2(y-x) - \dfrac{xe^{-x}}{2}$
2. $z = e^x\phi_1(y+x) + e^{2x}\phi_2(y+x) + \frac{1}{2}e^{2x-y}$
2. $z = e^x\phi_1(y-x) + e^{3x}\phi_2(y-2x) + x + 2y + 6$
4. $z = f_1(y) + e^{-x}f_2(y+x) + \frac{1}{3}x^3 - x^2 + xy^2 + 6x$
5. $z = e^x\phi_1(y) + e^{-x}\phi_2(x+y) + \frac{1}{2}\cos(x+2y)$
6. $z = f_1(x) + e^{3y}f_2(2y-x) + \dfrac{3}{50}[4\cos(3x-2y) + 3\sin(3x-2y)]$

## Problems 17.8, page 598

1. $z = \phi_1(x) + \phi_2(x+y+z)$
2. $z = \phi_1(y + \sin x) + \phi_2(y - \sin x)$
3. $z = \phi_1(xy^2) + \phi_2(x^2y)$
4. $z = \phi_1(y/x) + \phi_2(x^2+y^2) + xy$
5. $y = \phi_1(z) + e^x\phi_2(z)$
6. $y = \phi_1(x+y+z) + x\phi_2(x+y+z)$.

## Problems 17.9, page 598

1. order two & degree two
2. $z = f_1(y+2x) + xf_2(y+2x)$
3. $z = -x^2\sin xy + yf(x) + \phi(x)$
4. $x^2+y^2+z^2 = f(x+y+z)$
5. $-\dfrac{1}{2}\sin(x+y)$
6. $xp + yq = z$
7. $z = ax + (1-\sqrt{a})^2y + c$
8. $\sqrt{x} - \sqrt{y} = f(6/x - \sqrt{z})$
9. $x\log(x+y) = z + f(x+y)$
10. First
11. $z = 2x + y\log x + f(xy)$
12. $\partial z/\partial x = \partial z/\partial y$
13. $z = f_1(y) + f_2(y+x) + f_3(y+2x)$
14. $4y^2p = q^2$
15. $u = \int f(y)\,dy + \phi(x)$
16. $c = 1$
17. $u = \dfrac{1}{6}x^3y + xf(y) + \phi(y)$
18. $f_1(y+x) + f_2(y+6x)$
19. (*iv*)
20. (*iii*)
21. (*iii*)
22. (*iii*)
23. (*ii*)
24. (*iv*)
25. (*iv*)
26. (*i*)
27. False
28. False
29. True
30. True
31. False.

## Problems 18.1, page 601

1. $z = ce^{4ax^3}.e^{-3ay^4}$
2. $u = ce^{k(1/y - 1/x)}$
3. $u = 8e^{-12x-3y}$
4. $u = 3e^{x-y} - e^{2x-5y}$
5. $u = 3e^{-5x-3y} + 2e^{-3x-2y}$
6. $u = \dfrac{1}{\sqrt{2}}\sinh\sqrt{2x} + e^{-3y}\sin x$.

## Problems 18.2, page 610

1. $y(x,t) = \sum_{n=1}^{\infty} b_n \sin\dfrac{n\pi x}{l}\cos\dfrac{n\pi ct}{l}$, when $b_n = \dfrac{2}{l}\int_0^b f(x).\sin\dfrac{n\pi x}{l}dx$
2. $y(x,t) = \sum_{n=1}^{\infty}\left(a_n\cos\dfrac{n\pi ct}{l} + b_n\sin\dfrac{n\pi ct}{l}\right)\sin\dfrac{n\pi x}{l}$ where

$$a_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx,\ b_n = \frac{2}{n\pi c}\int_0^l g(x)\sin\frac{n\pi x}{l}dx$$

3. $y = \dfrac{8k}{\pi^2}\left(\sin\dfrac{\pi x}{l}\cos\dfrac{\pi ct}{l} - \dfrac{1}{3^2}\sin\dfrac{3\pi x}{l}\cos\dfrac{3\pi ct}{l} + \ldots\right)$
4. $y = \dfrac{8h}{\pi^2}\left(\sin\dfrac{\pi x}{l}\cos\dfrac{\pi ct}{l} - \dfrac{1}{3^2}\sin\dfrac{3\pi x}{l}\cos\dfrac{3\pi ct}{l} + \ldots\right)$

**6.** $y(x, t) = \dfrac{4l^2c}{a\pi^3}\left\{\sin\dfrac{\pi x}{l}\sin\dfrac{\pi at}{l} - \dfrac{1}{33}\sin\dfrac{3\pi x}{l}\sin\dfrac{3\pi at}{l}\ ...\right\}$

**7.** (*i*) $y(x, t) = a(x - x^2 - c^2t^2)$; (*ii*) $y(x, t) = \dfrac{a}{2}(1 - \cos 2\pi x \cos 2\pi ct)$.

## Problems 18.3, page 617

**1.** $u(x, t) = \dfrac{400}{\pi^2}\sum_{n=0}^{\infty}\dfrac{(-1)^n}{(2n+1)^2}e^{-[(2n+1)c\pi/100]^2 t}\sin\dfrac{(2n+1)\pi x}{100}$

**2.** $\sum_{n=\text{odd}}\dfrac{8a}{n^3\pi^3}\sin\dfrac{n\pi x}{l}e^{(n\pi c/l)^2 t}$

**3.** $u(x, t) = \dfrac{40}{\pi}\sum_{n=1}^{\infty}\dfrac{[1-4(-1)^n]}{n}\sin\left(\dfrac{n\pi x}{30}\right)\bar{e}\,\dfrac{(an\pi)^2 t}{900}$

**4.** $u(x, t) = -3x + 90 - \dfrac{80}{\pi}\sum_{1}^{\infty}\dfrac{1}{n}\sin\dfrac{n\pi x}{5}\,e^{-c^2n^2\pi^2 t/25}$

**5.** $u(x, t) = \dfrac{5}{2} - \dfrac{4l}{\pi^2}\sum_{1}^{\infty}\dfrac{1}{(2n-1)^2}\cos\dfrac{(2n-1)\pi x}{l}\,.\,e^{-(2n-1)^2c^2\pi^2 t/25}$

**6.** $u(x, t) = 50 - \dfrac{400}{\pi^2}\sum_{1}^{\infty}\dfrac{2}{(2n-1)^2}\cos\dfrac{(2n-1)\pi x}{50}\,.\,e^{-c^2\pi^2(2n-1)^2 t/2500}$

**7.** $\theta = \dfrac{4\theta_0}{\pi}\left[e^{-(\pi/2l)^2 kt}\cos\dfrac{\pi x}{2l} - \dfrac{1}{3}e^{-(3\pi/2l)^2 kt}\cos\dfrac{3\pi}{2l}x + \dfrac{1}{5}e^{-(5\pi/2l)^2 kt}\cos\dfrac{5\pi}{2l}x - ......\right]$

**8.** $V = V_0e^{-\sqrt{(n/2k)}x}\sin[nt - \sqrt{(n/2k)}\,x]$.

## Problems 18.4, page 623

**1.** $u = -\dfrac{8}{\pi}\sum_{n=1,3,5...}^{\infty}\dfrac{\sin nx\sinh n(\pi - y)}{n(n^2-4)\sinh n\pi}$

**5.** $u(x, y) = \dfrac{3200}{\pi^3}\sum_{1}^{\infty}\dfrac{\sin\dfrac{(2n-1)\pi x}{20}\sinh\dfrac{(n-1)\pi y}{20}}{(2n-1)^2\sinh(2n-1)\pi}$

**8.** $u(x, y) = u_0\cosh\dfrac{\pi x}{a}\cosh\dfrac{\pi}{a}(b - y)\,\text{sech}\,\dfrac{\pi b}{a}$.

## Problems 18.5, page 626

**1.** $u(r, \theta) = \dfrac{8k}{\pi}\sum_{1}^{\infty}\left(\dfrac{r}{a}\right)^{2n-1}\dfrac{\sin(2n-1)\theta}{(2n-1)^3}$

**2.** $u(r, \theta) = \dfrac{3200}{\pi^2}\sum_{1}^{\infty}\left(\dfrac{r}{10}\right)^{2n-1}\dfrac{\sin(2n-1)\theta}{(2n-1)^3}$

**3.** $u(r, \pi) = \dfrac{2}{\pi}\sum_{n=1,3,5...}^{\infty}\left(\dfrac{a}{r}\right)^{2n}\dfrac{r^{4n}-b^{4n}}{a^{4n}-b^{4n}}\cdot\dfrac{\sin 2n\theta}{n^3}$

**4.** $u(r, \theta) = \sum\dfrac{2k}{n\pi}\left(\dfrac{r}{a}\right)^{4n}(1 - \cos n\pi)\sin 4n\theta$

**5.** $u(r, \theta) = 50 - \dfrac{200}{\pi}\sum_{1}^{\infty}\dfrac{1}{2n-1}\left(\dfrac{r}{a}\right)^{2n-1}\sin(2n-1)\theta$

**6.** $u(r, \theta) = \cos\theta\left(\dfrac{200}{r} - \dfrac{r}{2}\right) + \sin\theta\left(2r - \dfrac{200}{r}\right)$. **7.** $u(r, \theta) = 4\cos\theta\,(r - 1/r) + 4\sin\theta\,(r + 1/r)$.

## Problems 18.6, page 630

**1.** $u = \sum_{m=1}^{\infty}\dfrac{J_2(\alpha_m)}{\alpha_m^2 J_1^2(\alpha_m)}\cos\alpha_m t\,J(\alpha_m r)$.

## Problems 18.7, page 634

**1.** $e = e_0 \sin \frac{\pi x}{l} \cos \frac{\pi t}{l\sqrt{(LC)}}$; $i = i_0 - e_0 \sqrt{\left(\frac{C}{L}\right)} \cos \frac{\pi x}{l} \sin \frac{\pi t}{l\sqrt{CL}}$

**3.** $v = \frac{20(l-x)}{l} + \frac{24}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l} \exp(-n^2\pi^2 t / RCl^2)$

$i = \frac{20}{lR} + \frac{24}{lR} \sum_{n=1}^{\infty} (-1)^n \frac{n\pi x}{l} \exp(-n^2\pi^2 t / RCl^2)$

**4.** $v = V_0 \cos \{pt - px\sqrt{(LC)}\}$.

## Problems 18.9, page 638

**1.** $\frac{\partial^2 v}{\partial x^2} = LC \frac{\partial^2 v}{\partial t^2}, \frac{\partial^2 i}{\partial t^2} = LC \frac{\partial^2 i}{\partial t^2}$

**2.** $\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$

**3.** If $u(x, t)$ is the temperature, then temperature gradient at a point is $\partial u/\partial x$ for all $t$.

**4.** elliptic

**5.** $\partial^2 y/\partial t^2 = c^2 \partial^2 y/\partial x^2$

**6.** $u = \frac{10}{l} x + 30$

**7.** parabolic partial differential equation

**8.** $r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0$

**9.** $y = \frac{2h}{l} x, 0\ x < \frac{l}{2}; y = \frac{l}{2h}(2h - y), \frac{l}{2} < x < l$

**10.** $y(0, t) = 0, y(l, t) = 0, \left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$

**11.** zero

**12.** $u(0, y) = 0, v(a, y) = 0, 0 < y < a; u_x(x, 0) = 0$ for all $t$ and $u\ (x, a) = u$ for $0 < x < a$

**13.** $\frac{\partial u\ (0, t)}{\partial x} = 0, \frac{\partial u(l, t)}{\partial x} = 0$ for all $t$

**14.** $y(x, t) = f(x + ct) + f(x - ct)$

**15.** $\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$ where $c^2$ is the diffusivity

**16.** $u(x) = x^2 + 20$

**17.** § 18.8-(6), (7), (8)

**18.** $u = 8e^{-12x - 3y}$

**19.** $z = 4e^{3x + t}$

**20.** $\alpha^2$ (= $k/s\rho$) is called the diffusivity of the substance ($cm^2$/sec)

**21.** $\frac{\partial^2 v}{\partial x^2} = RC \frac{\partial v}{\partial t}, \frac{\partial^2 i}{\partial x^2} = RC \frac{\partial c}{\partial t}$

**22.** $u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t / l^2}$

**23.** § 18.7 – (3), (4), (5)

**25.** False.

## Problems 19.1, page 646

**1.** (*i*) $\sqrt{2}\left(\cos \frac{\alpha}{2} + \sin \frac{\alpha}{2}\right) \cos\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)$. (*ii*) $-8i/25$

**2.** $\frac{-y}{x^2 + y^2 - 2x + 1}$

**3.** $x = \pm 1.5, y = \pm 2$

**5.** A circle: centre (–1, 1) radius $\sqrt{2}$

**8.** $-1 + i\sqrt{3}, -1 - i\sqrt{3}, 1 - i\sqrt{3}; 4\sqrt{3}$

**9.** $-2 + 0.\ i, 1 - i\sqrt{3}$

**10.** $-1 - i, \sqrt{2}(\mp \sin 15° \pm i \cos 15°), \sqrt{2}(\mp \cos 15° \pm i \sin 15°)$

**11.** (*i*) Annular region between the circles of radii 2 and 4 with centre $(-3, 0)$ including boundary of inner circle; (*ii*) Region of complex plane above the line $y = 2$; (*iii*) Infinite region bounded by the lines $\theta = \pi/3$ and $\theta = \pi/2$; (*iv*) Real axis and region above it between $x = \pm 2$

**13.** (*i*) Ellipse with foci at $z = \pm 1$ and major for axis $= 3$

(*ii*) (*a*) Right bisector of the line joining $z = 3$ and $z = -1$;

(*b*) Circle through the points $z = 3$ and $z = -1$;

**14.** (*i*) Right bisector of the points 0 and 2; (*ii*) Circle through the points $\frac{1}{3}$ and 3

**15.** (*i*) A straight line; (*ii*) Circle with centre $(1, 1/2)$ and radius $\sqrt{5/2}$.

## Problems 19.2, page 650

**5.** $4m\pi/n(n+1)$

**7.** $2^{n+1}\sin^n \dfrac{\alpha-\beta}{2}\cos n\left(\dfrac{\pi+\alpha+\beta}{2}\right)$.

## Problems 19.3, page 653

**1.** (*i*) $(2)^{1/8}\,[0.98 \pm i(0.195)]6$; $(2)^{1/2}\,[-0.195 \pm i\,(0.98)]$

(*ii*) $(2)^{1/5}\cos\dfrac{4n+3}{10}\pi$, where $n = 0, 1, 2, 3, 4$; (*iii*) $\pm 2\sqrt{2}$

(*iv*) $2^3\sqrt{2}\,\cos r\pi/9$, where $r = 1, 7$ or $13$

**3.** $\pm i$, $\frac{1}{2}\left(\sqrt{3} \pm i\right), \frac{1}{2}\left(-\sqrt{3} \pm i\right)$

**5.** (*i*) $1, -1, \cos(\pm\pi/5), \cos(\pm 3\pi/5)$

(*ii*) $-1, \frac{1}{2}\left(1 \pm i\sqrt{3}\right), \pm(1+i)/\sqrt{2}, \pm(-1+i)/\sqrt{2}$

(*iii*) $\dfrac{1 \pm i}{\sqrt{2}}, \dfrac{-1 \pm i}{\sqrt{2}}; 1, \cos\left(\pm\dfrac{2\pi}{5}\right), \cos\left(\pm\dfrac{4\pi}{5}\right)$.

(*iv*) $\cos(2n+1)\pi/6$, where $n = 0, 1, 2, 3, 4, 5$; $2\,(2m+1)\pi/3$, where $m = 0, 1, 2$

**6.** $(-1+i)/\sqrt{2}, (1-i)/\sqrt{2}$

**7.** $\pm 1, \pm i, \pm\left(\cos\dfrac{\pi}{6} \pm i\sin\dfrac{\pi}{6}\right), \pm\left(\cos\dfrac{\pi}{3} \pm i\sin\dfrac{\pi}{3}\right)$; Last four values

**9.** $x^3 + x^2 - 2x - 1 = 0$.

## Problems 19.4, page 655

**1.** $32\cos^5\theta - 24\cos^3\theta + 6\cos\theta$

**13.** $-(2)^{-11}(\sin 12\theta - 2\sin 10\theta - 4\sin 8\theta + 10\sin 6\theta + 5\sin 4\theta - 20\sin 2\theta)$

**14.** $\sin^5\theta = A\sin\theta - B\sin 3\theta + C\sin 5\theta$.

## Problems 19.5, page 661

**1.** (*i*) $e^{x^2-y^2}\cos 2xy, e^{x^2-y^2}\sin 2xy$ (*ii*) $ie^5$. (*iii*) $e^{16}\cos 30, e^{16}\sin 30$

**5.** $[(pq' - p'q)(qr' - q'r)]^2 = [(pq' - p'q)^2 + (qr' - q'r)^2](pr' - p'r)^2$

**10.** $\dfrac{1}{64}(\cosh 7\theta + 7\cosh 5\theta + 21\cosh 3\theta + 35\cosh\theta)$

**17.** $-\log^3$

**18.** $-13/12$.

## Problems 19.6, page 664

**10.** $\pm \frac{\pi}{4} + \frac{i}{4} \log \frac{1+\sin\theta}{1-\sin\theta}$ according as $\cos\theta$ is + ve or – ve

**11.** $\sin^{-1}\left(\sqrt{\sin\theta}\right) + i \log\left[\sqrt{(1+\sin\theta)} - \sqrt{\sin\theta}\right]$.

## Problems 19.7, page 667

**1.** (*i*) $\log_e 10 + i\,[\tan^{-1}(4/3) \pm 2n\pi]$; (*ii*) $\log_e 1 + i\,(\pi + 2n\pi)$

**4.** (*i*) $\sqrt{2}e^{-(2n-\frac{1}{4})\pi}, (2n-\frac{1}{4})\pi + \log\sqrt{2}$; (*ii*) $e^{-\pi^2/8}, (\pi/4)\log_e 2$

**9.** $\sqrt{[\frac{1}{2}(\cos 2x + \cosh 2y)]} - i\tan^{-1}(\tan x \tanh y)$

**10.** (*i*) $2n\pi \pm i\log(2+\sqrt{3})$; (*ii*) $-\frac{1}{2}\log 3 + (n+\frac{1}{2})i\pi$.

## Problems 19.8, page 669

**1.** $e^{\sin\theta\cos\theta}\cos(\theta + \sin^2\theta)$

**2.** $\sin\alpha\cos(\cos\beta)\cosh(\sin\beta) - \cos\alpha\sin(\cos\beta)\sinh(\sin\beta)$

**3.** $\tan^{-1}\frac{x\sin\alpha}{1+x\cos\alpha}$, except when $x = 1$ and $\alpha = (2n+1)\pi$

**4.** $\log(2\cos\theta/2)$

**5.** $-\frac{1}{2}\tan^{-1}(\cos\beta\,\text{cosech}\,\alpha)$

**6.** $\frac{1}{2}\tan^{-1}\frac{2c\sin\alpha}{1-c^2}$

**7.** $(2\cos\theta)^{-1/2}\cos\theta/2$

**8.** $\sin\frac{n(\pi-\alpha)}{2}/(2\sin\alpha/2)^n$

**9.** $\sin\left(\alpha + \frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}\,\text{cosec}\,\frac{\beta}{2}$

**10.** $\frac{\cos\{\alpha + \frac{1}{2}(n-1)\beta\}\sin n\frac{\beta}{2}}{\sin\frac{1}{2}\beta}$

**11.** $\frac{\sin\alpha(\cos\alpha - \sin\alpha)}{1-\sin 2\alpha + \sin^2\alpha}$

**12.** $\frac{1 - x\cos\theta - x^n\cos n\theta + x^{n+1}\cos\overline{n-1}\,\theta}{1-2x\cos\theta + x^2}$.

## Problems 19.9, page 670

**2.** 0.053 radians **3.** 1°59′ **4.** 39.7.

## Problems 19.10, page 670

**1.** (*b*) **2.** (*c*) **3.** (*b*) **4.** (*c*)
**5.** (*b*) **6.** (*d*) **7.** odd **8.** $32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$
**9.** $6(1-2i)$ **10.** $2i\sin n\theta$ **11.** $\frac{1}{25}(-6+17i)$ **12.** $\cosh x\cos y$
**13.** $\frac{1}{19}$ radians **14.** 1 **15.** $-\cos x\sinh y$ **16.** $e^{-\pi/4\sqrt{2}}$ **17.** $\alpha+\beta+\gamma$
**18.** $2i\,n\pi$ **19.** real **20.** $\sinh\phi$ **21.** $\sinh 2\phi/(\cosh 2\theta + \cosh 2\phi)$
**22.** $16\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha$ **23.** an equilateral **24.** $\pi/2$
**25.** $x = \pm 1, y = -4$. **26.** a circle **27.** True **28.** True
**29.** True **30.** True **31.** False **32.** True
**33.** False.

## Problems 20.1, page 682

**4.** $a = 1, b = -6, c = 1, d = 2, e = 4$

**6.** (*i*) and (*ii*) Not analytic. (*iii*) Analytic

**7.** $p = -1$

**11 & 12.** $f(z)$ is not analytic at origin although C-R equations are satisfied there

**14.** (*i*) $z^3 + 3z^2 + 1 + ic$ ; (*ii*) $\cos z$ ; (*iii*) $\log z$ ; (*iv*) $i/z$ ; (*v*) $e^z + i(c - z)$ ; (*vi*) $ze^{2z} + ic$ ; (*vii*) $z \sin z$ ; (*viii*) $x^2e^z + ic$

**15.** (*i*) $(1 + i)/z + c$ ; (*ii*) $\cos z + c$ ; (*iii*) $e^z$ ; (*iv*) $\bar{z}e^{-\bar{z}} + c$ ; (*v*) $1 + ize^{-z}$ ; (*vi*) $(2 \cos x \cosh y)/(\cos 2x + \cosh 2y)$

**16.** (*i*) $f(z) = c - iz^3$ ; (*ii*) $f(z) = \left(\frac{1}{2} - \cot \frac{z}{2}\right)$ (*iii*) $\frac{\cot z}{1+c} + c\left(\frac{1+3i}{5}\right)e^z + c$

**17.** $\psi = 3xy^2 - x^3 + c$

**18.** $2 \tan^{-1} (y/x)$ ; $2 \log z + c$

**20.** $v = C - e^{-2xy} \cos (x^2 - y^2)$ ; $f(z) = C - i\, e^{iz^2}$

**22.** (*i*) $x^4 - 6x^2y^2 + y^4 = c$ (*ii*) $x^2 - y^2 + 2e^x \sin y = c$ (*iii*) $r^2 \sin 2\theta = c'$

**23.** (*i*) $x/(x^2 + y^2)$ ; (*ii*) $\frac{1}{2} \log (x^2 + y^2)$

**24.** (*i*) $a\,(1 + \cos \theta + i \sin \theta \log r)$ ; (*ii*) $(r + 1/r) \cos \theta + (r - 1/r) \sin \theta + c$

**27.** $-2 \tan^{-1} [(y - 2)/(x - 1)]$, $2i \log (z - 1 - 2i)$.

## Problems 20.2, page 687

**1.** $z = \pm i$

**2.** $w = -1/z$

**3.** $w = (2i - 6z)/(iz - 3)$ ; fixed points $z = i, 2i$ ; no critical points

**4.** $w = \dfrac{(20 + 18i) - (32 + 12i)z}{(29 + 17i) - (11 - 3i)z}$

**5.** (*i*) $w = i(1 - z)/(1 + z)$ ; (*ii*) $w = \dfrac{(4i - 2) + (5 - 3i)z}{2i + (1 + i)z}$ (*iii*) $w = (1 - z)/(1 + z)$.

## Problems 20.3, page 692

**1.** (*i*) $I(w) > 0$ ; (*ii*) Region bounded by the parabolas $v^2 = 4(1 \pm u)$ ; and $u^2 = 1 - 2v$ ; (*iii*) Region bounded by parabolas $v^2 = \frac{1}{4} \pm u$, $v^2 = 4(1 \pm u)$ ; (*iv*) Region boundary $\rho = 2\sqrt{\rho} \cos \frac{\phi}{2} + 3$.

**2.** $w = z^6$

**3.** Lines parallel to $x$ and $y$ axes map into two families of rectangular hyperbolas in the $w$-plane which cut each other at right angles. Lines parallel to $u$ and $v$ axes map into two families of parabolas in the $z$ plane which cut each other orthogonally. It is conformal at $z = 0$

**4.** (*a*) Line $4v + 1 = 0$

**5.** (*b*) Every circle through the origin ($z = 0$) transforms into a st. line not passing through the origin ($w = 0$). If a line passes through $z = 0$, its image is a line through $w = 0$. (*b*) Circle with centre (1/2, 1/2) and radius ($1/\sqrt{2}$) (*c*) Lemniscate $p^2 = \cos 2\phi$

**10.** $z = \pm a$

**11.** See § 20.10 (3)

**14.** See § 20.10(4).

## Problems 20.4, page 694

**1.** $z = \cosh w$

**3.** $w = \sin z$

**4.** $w = \log z$.

## Problems 20.5, page 696

**1.** (*a*) $(5 - i)/6$ (*b*) $(5 + i)/6$

**2.** (*i*) $4 + (25/3)i$ ; (*ii*) $4 + 8i$

**3.** $\frac{1}{6}(64i - 103)$

**6.** (*i*) $i$ ; (*ii*) $2i$ ; (*iii*) 0

**7.** (*a*) $\frac{1}{3}(i - 1)$, (*b*) $\frac{1}{6}(5i - 3)$

**9.** (*i*) $\frac{2}{3}$; (*ii*) $-\frac{2}{3}$.

## Problems 20.6, page 702

**1.** (*i*) 0 ; (*ii*) $2\pi i$
**2.** 0
**3.** (*a*) zero ; (*b*) zero
**4.** (*i*) $5\pi i$ ; (*ii*) $\pi i/2$ (*iii*) $4\pi i$
**5.** (*a*) $-10\pi i$ (*b*) $2\pi ie$
**6.** (*i*) $4\pi i$ ; (*ii*) $2\pi ie^{-1}$ ; (*iii*) $-\pi i$
**7.** (*i*) $8\pi i$ ; (*ii*) 0
**8.** (*i*) 0 ; (*ii*) $2\pi(6+13i)$ ; (*iii*) $12\pi i$
**9.** zero.

## Problems 20.7, page 709

**1.** $\sum_{n=1}^{\infty}(-1)^{n+1}\, n(z-1)^n$ ; Convgt. in $|z-1|<1$

**2.** $f(z)=\frac{1}{3}+\frac{u}{9}(z+i)-\frac{7}{27}(z+1)^2+\ldots\ldots$ Region of convergence is $|z+i|<1$

**3.** (*i*) $\frac{1}{2}(z-1)-\frac{1}{2^2}(z-1)^2+\frac{1}{2^3}(z-1)^3-\ldots\ldots$ (*ii*) $-(z-\pi/2)+(z-\pi/2)^3/3!-(z-\pi/2)^5/5!+\ldots$

(*iii*) (*a*) $f(z)=-\frac{1}{5}\sum_{n=0}^{\infty}(-1)^n(z+1)^n-\frac{1}{20}\sum_{n=0}^{\infty}\left(\frac{z+1}{4}\right)^n$ *in the region* $|z+1|<1$. Also, $(z+1)<4$

(*b*) $f(z)=-\frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{z-1}{5}\right)^n\frac{1}{10}\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n$ in the region $|z-1|<2$. Also $|z-1|<3$

**4.** (*i*) $\frac{1}{z+1}+\frac{1}{(z+1)^2}+\frac{1}{(z+1)^3}+\ldots+\frac{1}{2}\left[1+\frac{z+1}{2}+\frac{(z+1)^2}{2^2}+\frac{(z+1)^3}{2^3}\right]$

(*ii*) $\frac{1}{4z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\ldots\right)-\frac{1}{12}\left(1-\frac{z}{3}+\frac{z^2}{9}-\ldots\right)$. (*iii*) $-\frac{1}{2(z-1)}-3\sum_{n=1}^{\infty}\frac{(z-1)^{n-1}}{2^{n+1}}$

**5.** (*i*) $e\left[(z-1)^{-2}+(z-1)^{-1}+\frac{1}{2!}+\frac{1}{3!}(z-1)+\frac{1}{4!}(z-1)^2+\ldots\right.$

(*ii*) $e^2(z-1)^{-3}+2e^2(z-1)^{-2}+2e^2(z-1)^{-1}+\frac{4e^2}{3}+\frac{2e^2}{3}(z-1)+\ldots$

**6.** (*i*) $\sum_{n=1}^{\infty}(-1)^{n-1}n\,.\,(z-1)^{-n}$ for $|z-1|>1$. (*ii*) $-\sum_{n=2}^{\infty}\frac{z^{2n-5}}{2(n-1)!}$

**7.** (*i*) $1+\frac{3}{z}\left(1-\frac{2}{z}+\frac{2^2}{z^2}-\frac{2^3}{z^3}+\ldots\ldots\right)-\frac{8}{3}\left(1-\frac{z}{3}+\frac{z^2}{3^2}-\frac{z^3}{3^3}+\ldots\ldots\right)$

(*ii*) $\frac{2}{z+2}+\frac{3}{(z+2)^2}+\frac{3^2}{(z+2)^3}+\ldots+\frac{1}{5}\left(1+\frac{z+2}{5}+\frac{(z+2)^2}{5^2}+\frac{(z+2)^3}{5^3}+\ldots\right)$

(*iii*) (*a*) $\frac{7}{z}-\frac{9}{z^2}-\frac{45}{z^3}-\frac{81}{z^4}+\ldots\ldots$ (*b*) $\frac{5}{2(z-3)}+\frac{7}{12}-\frac{z-3}{24}-\frac{5(z-3)^2}{432}+\frac{7(z-3)^2}{864}+\ldots\ldots$

**8.** (*a*) $\frac{z}{4}-\frac{5z^2}{16}+\frac{21}{64}z^5-\ldots\ldots$; (*b*) $\frac{1}{3}\left(\frac{1}{z^5}-\frac{1}{z^3}-\frac{z}{4}+\frac{z^3}{16}-\frac{z^5}{64}+\ldots\ldots\right)$; (*c*) $\frac{1}{z^3}-\frac{3}{z^5}+\ldots\ldots$

**9.** $z=0, z=2$ are the isolated singularities
**10.** $z=0$ is an isolated essential singularity
**11.** $z=0$, is a non-isolated essential singularity
**12.** $z=1$ is a pole of order 2
**13.** $z=1$ is a pole of order 4
**14.** $z=a$ is a double pole and $z=0,\pm 1,\pm 2,\ldots\ldots$ are simple poles.

## Problems 20.8, page 715

**1.** $-\frac{1}{t}-2i+3t+4i\,t^2+\ldots$ where $t=z-i$ ; $-1$

2. (i) $3e/2$ ; (ii) Res $(z = -ai/2)$

3. (i) Res $f(0) = -1/2$, Res $f(2) = 2\frac{1}{2}$ ;

(ii) Res $f(-1) = 0$, Res $f(i) = \dfrac{1+2i}{2(1-i)}$, Res $f(-i) = \dfrac{2i-1}{2(i-1)}$

(iii) Res $f(-1) = 1$, Res $f(i) = \dfrac{2+i}{-1+i}$, Res $f(-i) = \dfrac{-2+i}{1+i}$

4. (i) Res $f(0) = -4/3$ ; (ii) Res $f(i) = \frac{1}{2}e^{-1}$ ; Res $f(-i) = \frac{1}{2}e$ ; (iii) $-i\,\pi/4$, Res $f(n\pi) = 1$, $n$ an integer.

5. (i) 0 ; (ii) $16\pi i/(2+3i)$

6. (i) $\dfrac{\pi i e^{-4}}{5}$ (ii) 0 ; (iii) $i\pi/4$

7. (i) $\pi i$ ; (ii) $\pi/2\,(3+2i)$, (iii) zero.

8. (i) $-2\pi i$ ; (ii) $\pi i$ ; (iii) $\pi/16$

9. (i) $-2\pi i$ ; (ii) $8\pi i/3e^2$ ; (iii) $\dfrac{\pi e^{2i}}{2}$ ; (iv) $\dfrac{8\pi}{3}ie^{-2}$

10. (i) $\dfrac{21\pi i}{16}$ ; (ii) $2\pi i$ sec $1(1 + \tan 1)$ ; (iii) $-2\pi i$

11. $2\pi i$

12. $\dfrac{1}{z^3} - \dfrac{1}{6z} + \dfrac{7z}{360} - \dfrac{31z^6}{15120} + \ldots\ldots ; -\dfrac{1}{3}\pi i.$

## Problems 20.10, page 723

1. (i) 2. (iii) 3. (a) 4. (iii) 5. (ii) 6. (ii)
7. (ii) 8. (ii) 9. (iii) 10. (ii) 11. (ii) 12. (i)
13. (ii) 14. (iii)

15. $v(x, y) = x^2 - y^2 + 2y + c$

16. $3x^2y - y^3 + c$

17. $u + iv$ is an analytic function

18. $2i/3$

19. 1

20. $z = \dfrac{1}{2}(a + b)$

21. $2u + 1 = 0$

22. $z = 1, \dfrac{1}{2}(1 \pm \sqrt{3})$

23. $z = 0$

24. $-1$

25. $z - \dfrac{z^2}{2} + \dfrac{z^3}{3} - \dfrac{z^4}{4} + \ldots$

26. zero

27. (iii)

28. $\dfrac{1}{2}\left[\dfrac{d^2}{dz^2}\{(z-a)^3 f(z)\}\right] z = a.$

29. zero

30. $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \dfrac{\partial u}{\partial y} = -\dfrac{\partial u}{\partial x}$

31. zero

32. $e^x \sin y$

33. zero

34. $z = 2$

35. no point in the $z$-plane

36. $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}, \dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$

37. zero

38. A simply connected region is one in which any closed curve, lying entirely within it, can be shrunk to a point without going out of the region

39. which is analytic or regular

40. $z = 1, 2$

41. $(z_1 - z_2)(z_3 - z_4)/(z_1 - z_4)(z_3 - z_2)$

42. $i$

43. $\dfrac{1}{2} + \dfrac{3}{4}z + \dfrac{7}{8}z^2 + \dfrac{15}{16}z^3 + \ldots$

44. $\pm i$

45. 1

46. Magnification and rotation

47. The coefficient of $(z-a)^{-1}$ in the expansion of $f(z)$ around an isolated singularity $(z = a)$ is called the residue of $f(z)$ at that point.

48. $2\pi i$

49. zero

50. $z = 1, 2$

51. $-\dfrac{1}{2}\left\{1 + \dfrac{z}{2} + \dfrac{z^2}{4} + \dfrac{z^3}{8} + \ldots\right\}$

52. $i/2$

53. $z = 0, 2$

54. § 20.14

55. $\pm i$

56. zero

57. Zeroes are at $z = \pm 1$, singularity is at $z = 0$

58. $-1$

**59.** essential singularity **60.** zero

**61.** $\sin z = \dfrac{1}{\sqrt{2}}\left\{1 + (z - \pi/4) - \dfrac{(z-\pi/4)^2}{2!} - \dfrac{(z-\pi/4)^3}{3!} + \dfrac{(z-\pi/4)^4}{4!} + \ldots\right\}$

**62.** a circle with centre (3, 2) and radius 2 in $w$-plane. **63.** $n\pi$, $n$ an integer

**64.** $\phi(a)/\psi'(a)$ **65.** circles **66.** True **67.** False **68.** True **69.** True
**70.** False **71.** True **72.** True **73.** True **74.** True **75.** True
**76.** True **77.** False **78.** False **79.** False **80.** True **81.** True
**82.** True.

## Problems 21.1, page 732

**1.** $\dfrac{1}{s-2} + \dfrac{24}{s^4} + \dfrac{3(s-2)}{s^4+9}$. **2.** $\dfrac{1}{s} + \dfrac{\sqrt{\pi}}{s^{3/2}} + \sqrt[3]{\left(\dfrac{\pi}{s}\right)}$ **3.** $\dfrac{3s-20}{s^2-25}$

**4.** $\dfrac{s\cos b - a\sin b}{s^2+a^2}$ **5.** $\dfrac{s^2-2s+4}{s(s^2+4)}$ **6.** $\dfrac{2(s^2-5)}{(s^2+1)(s^2+25)}$

**7.** $\dfrac{\sqrt{\pi} - e^{1(1/4s)}}{2s^3/2}$ **8.** $\dfrac{5}{4}\left\{\dfrac{1}{s^2+1} - \dfrac{3/2}{s^2+9} + \dfrac{1/2}{s^2+25}\right\}$ **9.** $\dfrac{s(s^2+28)}{(s^2+4)(s^2+36)}$

**10.** $\dfrac{b}{(s+a)^2-b^2}$ **11.** $\dfrac{60}{s-2} - \dfrac{s-2}{s^2-4s+20}$ **12.** $\dfrac{30(s+3)}{(s^2+6s+13)(s^2+6s+73)}$

**13.** $\dfrac{2}{(s+1)(s^2+2s+5)}$ **14.** $\dfrac{1}{8}\left\{\dfrac{3}{s-2} - \dfrac{4(s-2)}{s^2-4s+8} + \dfrac{s-4}{s^2-8s+32}\right\}$

**15.** $\dfrac{a(s^2+2a^2)}{s^4+4a^4}$ **16.** $\dfrac{3}{2}\left[\dfrac{1}{s^2-9} + \dfrac{s^2-13}{s^4-10s^2+169}\right]$ **17.** $\dfrac{2}{(s+2)^3}$

**18.** $\dfrac{1}{n} + \dfrac{3}{(s+1)^2} + \dfrac{6}{(s+2)^3} + \dfrac{6}{(s+3)^4}$ **19.** $4\dfrac{(4s^2+4s-1)}{(4s^2+1)^2}$

**20.** $\dfrac{4}{s} - \dfrac{e^{-s}}{s}$ **21.** $\dfrac{1+e^{-\pi s}}{s^2+1}$ **22.** $\dfrac{e^{-\pi s/3}}{s^2+1}$

**23.** $e^{-2\pi s/3}\dfrac{s}{s^2+1}$ **24.** $\dfrac{2}{s^3} - \dfrac{e^{-2s}}{s^3}(2+3s+3s^2) + \dfrac{e^{-3s}}{s^2}(5s-1)$

**25.** $\dfrac{4}{(s-1)(s^2-2s+5)}$.

## Problems 21.2, page 734

**1.** $(1/s^2T) - e^{-sT}/s(1-e^{sT})$ **2.** $[Ew/(s^2+w^2)]\coth(\pi s/2w)$ **3.** $(a/s)\tanh(as/2)$

**4.** $(1/s^2)\tanh\frac{1}{2}as$ **5.** $\dfrac{1}{\sqrt{(s^2+a^2)}}$ **6.** $(s^2-2as+a^2+b^2)^{-1/2}$

**7.** $\dfrac{2}{(s-2)\sqrt{(s+1)}}$.

## Problems 21.3, page 740

**1.** $\dfrac{s+1}{s(s^2+2s+2)}$ **4.** $\dfrac{2(3s^2+4)}{s^2(s^2+4)^2}$ **5.** $\dfrac{16}{(s^2+4)^2}$

**6.** $\dfrac{2s^3 - 6a^2 s}{(s^2 + a^2)^3}$ **7.** $\dfrac{2as}{(s^2 - a^2)^2}$ **8.** $\dfrac{6(s-2)}{(s^2 - 4s + 13)^2}$

**9.** $\dfrac{8(s+2)}{s^2 + 4s + 20}$ **10.** $\dfrac{2(s^3 + 6s^2 + 9s + 2)}{(s^2 + 4s + 5)^3}$ **11.** $\log\{(s+b)/(s+a)\}$

**12.** $\cot^{-1}(s)$ **13.** $\frac{1}{2}\log\{(s^2+36)/(s^2+16)\}$ **14.** $\dfrac{1}{2}\log\left(\dfrac{s^2+b^2}{(s-a)^2}\right)$

**15.** $\cot^{-1}(s+1)$ **16.** $\dfrac{1}{2}\log\left(\dfrac{s^2+9}{s^2}\right)$ **17.** $\cot^{-1} s - \frac{1}{2}s\log(1+s^{-2})$

**18.** $\dfrac{1}{s-\log 2} + \dfrac{2s}{(s^2+1)^2} + \dfrac{1}{2}\log\left(\dfrac{s^2+9}{s^2+4}\right)$ **19.** (*i*) $\log 2/3$ ; (*ii*) $\pi/8$ ; (*iii*) $12/169$ ; (*iv*) $\dfrac{8(s+1)}{s(s^2+2s+17)}$

**21.** (*i*) $\dfrac{1}{s}\cot^{-1}(s)$ ; (*ii*) $\dfrac{1}{s}\cdot\dfrac{s+1}{s^2+2s+2}$ ; (*iii*) $\dfrac{\cot^{-1}(s-1)}{s}$.

## Problems 21.4, page 743

**1.** $\dfrac{1}{2}\left(\cos\dfrac{5t}{2} - \sin\dfrac{5t}{2}\right) - 4\cosh 3t + 6\sinh 3t$ **2.** $e^{3t} - e^{2t}$

**3.** $3e^{t/2} + 2e^{t/3}$ **4.** $e^{2t} + 2e^{-4t}$ **5.** $\dfrac{1}{3}(8e^{2t} - e^{-t})$

**6.** $\cosh t$ **7.** $e^{t} + e^{-2t} - 2e^{3t}$ **8.** $2e^{3t} - \frac{3}{5}e^{2t} - \frac{2}{5}e^{7t}$

**9.** $\dfrac{1}{2}e^{t} - e^{2t} + \dfrac{5}{2}e^{3t}$ **10.** $\frac{1}{2}t\sinh t$ **11.** $\frac{1}{3}t(e^{t} - e^{-2t})$

**12.** $\frac{1}{13}(3e^{3t} - 3\cos 2t + 2\sin 2t)$ **13.** $\frac{1}{2}(\sin t - te^{-t})$ **14.** $\frac{1}{2}[\cos at + \cosh at]$

**15.** $(\frac{1}{3}a^2)[e^{at} - e^{-at/2}\{\cos(\sqrt{3}at/2) + \sqrt{3}\sin(\sqrt{3}at/2)\}]$ **16.** $\frac{1}{3}(5\sin t - \sin 2t)$

**17.** $\frac{1}{3}e^{-2t}(6\cos 3t - 7\sin 3t)$ **18.** $\dfrac{1}{5}(1+e^{-t})\sin t + \dfrac{3}{5}(1-e^{-t})\cos t$

**19.** $\frac{1}{3}e^{-t}(\sin t + \sin 2t)$ **20.** $(2/\sqrt{3})\sinh(\frac{1}{2}t)\sin(\frac{1}{2}\sqrt{3}t)$ **21.** $\cos at\sinh at$.

## Problems 21.5, page 750

**1.** $\dfrac{1}{25}(e^{-5t} + 5t - 1)$ **2.** $\dfrac{1}{8} - \dfrac{1}{4}\left(t^2 + t + \dfrac{1}{2}\right)e^{-2t}$ **3.** $\dfrac{1}{a^2}\cos\left(\dfrac{bt}{a}\right)$

**4.** $\dfrac{1}{a^2}\left(t - \dfrac{1}{a}\sin at\right)$ **5.** $\dfrac{1}{2}t^2 + \cos t + 1$ **6.** $\dfrac{1}{2}te^{-2t}\sin 2t$

**7.** $t\sin at$ **8.** $\dfrac{1}{2}(a^2t^2 - 4at + 2)e^{-at}$ **9.** $\dfrac{1-e^{t}}{t}$

**10.** $\dfrac{1}{t}(e^{-bt} - e^{-at})$ **11.** $e^{-t} - e^{-2t} - e^{-3t}$ **12.** $\dfrac{1}{t}(\cos at - \cos bt)$

**13.** $\dfrac{2}{t}(1 - \cosh at)$ **14.** $\dfrac{2}{t}(e^{t} - \cos t)$ **15.** $\dfrac{\sin 2t}{t}$

**16.** $\dfrac{\sin t}{t}$ **17.** $\dfrac{2(\sinh t - t\cosh t)}{t^2}$ **18.** $\dfrac{e^{-bt} - e^{-at}}{a-b}$

**19.** $\frac{1}{2a^3}(\sin at - at\cos at)$ **20.** $\frac{1}{a^3}(at - \sin at)$ **21.** $t(e^{-t}+1)+2(e^{-t}-1)$

**22.** $\frac{1}{16}(e^{2t}-e^{-2t}-4te^{-2t})$ **23.** $\frac{1}{13}(3\sin 3t + 2\cos 2t - 2e^{-2t})$ **24.** $\frac{t^2}{2}+\cos t - 1$

**25.** $\frac{e^{-2t}}{54}(\sin 3t - 3t\cos 3t)$.

## Problems 21.6, page 754

**1.** $y = \frac{7}{4}e^{-t} - \frac{3}{4}e^{-3t} - \frac{1}{2}te^{-t}$ **2.** $x = \frac{at}{2}\sinh t$ **3.** $y = t - 3\sin t + \cos t$

**4.** $y = 2t + 3 + \frac{1}{2}(e^{3t}-e^{t}) - 2e^{2t}$ **5.** $y = 4e^{2t}(1+t) - 7e^t$ **6.** $y = 2\cos 5t + t\sin 5t$

**7.** $y = \frac{1}{2\omega}\sin \omega t$ **8.** $y = \frac{1}{8}e^t - \frac{1}{40}e^{-3t} - \frac{1}{10}(2\sin t + \cos t)$

**9.** $y = \frac{1}{2}(\cos kt + \cosh kt)$ **10.** $y = \frac{1}{8}[(3-t^2)\sin t - 3t\cos t]$ **11.** $y = \frac{11}{3}e^{-t}(\sin t + \sin 2t)$

**12.** $y = \frac{-12}{5} + \frac{12}{5}e^{-t}\cos 2t + \frac{7}{10}e^{-t}\sin 2t$ **13.** $y = e^{2t}(x^2-6x+12) - e^t(15x^2+7x+11)$

**14.** $x = \frac{4}{9}\sin 2t - \frac{5}{9}\sin t - \frac{1}{3}t\cos 2t$ **15.** $y = \frac{1}{2}\left(\frac{3\sin t}{t} - \cos t\right)$

**16.** $y = e^{2t}$ **17.** $y = t$ **18.** $y = 3J_0(2t)$

**21.** $(n\sin at - a\sin nt)F_0/mn(n^2-a^2)$, where $n^2 = k/m$.

## Problems 21.7, page 756

**1.** $x = \frac{1}{2}(e^t + \cos t + 2\sin t - t\cos t)$, $y = \frac{1}{2}(t\sin t - e^t + \cos t - \sin t)$

**2.** $x = e^t + e^{-t}$, $y = e^{-t} - e^t + \sin t$ **3.** $x = 2 + t^2/2$, $y = -1 - t^2/2$

**4.** $x = \frac{1}{10}(5 - 2e^{-t} - 3e^{-6t/11})$, $y = \frac{1}{5}(e^{-t} - e^{-6t/11})$ **5.** $x = e^6(1+2t) + 2e^{3t}$, $y = \sinh t + \cosh t - e^{-3t} - te^t$

**6.** $i_1 = \frac{a}{p+\omega}(\sin \omega t + \sin pt)$ ; $i_2 = \frac{a}{p-\omega}(\cos \omega t - \cos pt)$.

## Problems 21.8, page 762

**1.** $\frac{2}{s^2+4}(e^{-2\pi s} - e^{-4\pi s})$

**2.** (*i*) $(1-2t)\,u(t-\pi) + 2tu(t)$, $\frac{2}{s^2} + \left(\frac{1-2\pi}{s} - \frac{2}{s^2}\right)e^{-as}$

(*ii*) $t^2[u(t) - u(t-2)]$, $\frac{2(1-e^{-2s})}{s^3} - \frac{4e^{-2s}(1+s)}{s^2}$

(*iii*) $\{u(t) - u(t-T)\}\cos(\omega t + \phi)$ ; $[(s\cos\phi - \omega\sin\phi) - e^{-sT}\times\{s\cos(\phi+\omega T) - \omega\sin(\phi+\omega T)\}]/(s^2+\omega^2)$

**3.** (*i*) $\frac{s}{s^2+1} + \left(\frac{1}{s} + \frac{s}{s^2+1}\right)e^{-\pi s} - \left(\frac{1}{s} - \frac{1}{s^2+1}\right)e^{-2\pi s}$

(*ii*) $\frac{1}{s^2+1} + e^{-\pi s}\left(\frac{s}{s^2+4} - \frac{s}{s^2+1}\right) + e^{-2\pi s}\left(\frac{s}{s^2+9} - \frac{s}{s^2+4}\right)$

(*iii*) $\frac{2}{s^3}\{1 + e^{-2s}(2s^2-1) - 2e^{-4s}(1+4s)\}$

**4.** (*i*) $e^{-s}/(s-1)$ ; (*ii*) $2e^{-s}/s^3$ ; (*iii*) $e^{-2s}\left\{\frac{24}{s^4}+\frac{42}{s^3}+\frac{28}{s^2}+\frac{25}{s}\right\}$ ; (*iv*) $e^{-s}(2+2s+s^2+s^3)/s^3$

**5.** $20e^{-2}$

**6.** (*i*) $-\sin t \,.\, u(t-\pi)$ ; (*ii*) $\frac{1}{3}e^{-4(t-2)}\sin 3(t-2)\,.\,u(t-2)$ ;

(*iii*) $\frac{1}{2}e^{-(t-1)}(t-1)^2\,u(t-1)$ (*iv*) $3-4(t-1)\,u(t-1)+4(t-3)\,u(t-3)$

**7.** $y=\frac{1}{2}\sin 2t+\frac{1}{4}(1-\cos 2t)-\left\{\frac{1}{4}1-\cos(t-1)\,u(t-1)\right\}$

**8.** $x=3-2\cos t+2\{t-4-\sin(t-4)\}\,u(t-4)$.

**9.** $$y(x)=\begin{cases}\dfrac{2Wx^2(3l-5x)}{81EI}, & 0<x<l/3\\[2mm] \dfrac{2Wx^2(3l-5x)}{81EI}+\dfrac{W}{6EI}\left(x-\dfrac{l}{3}\right)^3, & \dfrac{l}{3}<x<l\end{cases}$$

**10.** $y(x)=\frac{wl^2}{16EI}x^2-\frac{wl}{12EI}x^3+\frac{w}{24EI}x^4-\frac{w}{24EI}(x-l/2)^4\,u(x-l/2)$

**11.** $x=\frac{I}{mn}e^{-\mu t/2m}\sin nt$ ; $\frac{dx}{dt}=\frac{I}{m}e^{-\mu t/2m}\left(\cos nt-\frac{\mu}{2mn}\sin nt\right)$, where $n^2=\frac{k}{m}-\frac{\mu^2}{4m^2}$.

## Problems 21.9, page 764

**1.** $\frac{2s}{(s^2+1)^2}$ **2.** (*d*) **3.** (*b*)

**4.** $\frac{1}{s^2-4s+5}$ **5.** $te^{-2t}$ **6.** $\frac{1}{3}e^{-2t}\sin 3t$

**7.** $s\bar{f}(s)-f(0)$ **8.** $\frac{1}{8}(2-3t)e^{-3t/2}$ **9.** $\frac{1}{2}\left[\frac{1}{s}+\frac{s}{s^2-16}\right]$

**10.** $\frac{1}{s-\log 2}$ **11.** $\frac{k!}{(s+1)^{k+1}}$ **12.** $\frac{2}{13}$

**13.** $e^{-at/s}$ **14.** $\Gamma(3/2)/s^{3/2}$ **15.** $s^2f(s)-sf(0)-f'(0)$

**16.** $\frac{1}{4}\left[\frac{3s}{s^2+16}+\frac{s}{s^2+144}\right]$ **17.** $\cot^{-1}(s/a)$ **18.** $\frac{e^{-3t}t^4}{24}$

**19.** $\frac{s\cos 3-2\sin 3}{s^2+4}$ **20.** (*c*) **21.** $f(t-a)\,u(t-a)$

**22.** $e^{-as}$ **23.** $1-3t+2t^2$ **24.** $\int_0^t f(t)\,dt$

**25.** $\int_0^T \frac{e^{-st}f(t)\,dt}{(1-e^{-st})}$ **26.** $\frac{1}{(s+1)^2(s+2)}$ **27.** $\frac{2(s-3)}{s^2-6s+34}+\frac{12}{s^2-6s+25}$

**28.** $1/(s-\log 4)$ **29.** $\frac{e^{-3t}}{\sqrt{(\pi t)}}$. **30.** (*d*)

**31.** $\frac{t}{2}\sin\frac{t}{2}$ **32.** (*c*) **33.** (*c*) **34.** (*ii*)

**35.** (*iii*) **36.** (*ii*) **37.** (*iv*) **38.** (*i*)

**39.** $\int_a^b f(x)dx$ **40.** (*ii*) **41.** (*iii*) **42.** (*d*)

**43.** True **44.** False **45.** False.

## Problems 22.1, page 776

**1.** $\frac{2}{\pi}\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}d\lambda$ ; $\frac{\pi}{2}$ for $|x| < 1$, $\frac{\pi}{4}$ for $|x| = 1$, 0 for $|x| > 1$

**2.** (*i*) $\frac{4}{\pi}\int_0^\infty \frac{\sin\omega - \omega\cos\omega}{\omega^3}\cos\omega x\,d\omega$ (*ii*) $\frac{2}{\pi}\int_0^\infty \cos\omega x\frac{a}{a+\omega^2}\,d\omega$

**4.** (*i*) $\frac{2\sin as}{s}$, $\pi$ ; (*ii*) $2\{(a^2s^2-2)\sin as + 2as\cos as\}/s^3$

**5.** $\frac{4}{s^3}(\sin sa - sa\cos sa)$ **6.** (*i*) $\sqrt{(3\pi)}\,e^{-3s^2/4}$ (*ii*) $\frac{\sqrt{\pi}}{2}e^{(3is - s^2/4)}$

**7.** $\frac{1-\cos 2s}{s}, \frac{\sin 2s}{s}$ **8.** $\sqrt{(\pi/2)}\,e^{-as}$ **9.** $\frac{a}{a^2+s^2}$ ; $\frac{\pi}{2a}e^{-a\lambda}$

**10.** 1 **13.** (*i*) $\frac{\pi}{2a^2}(1-e^{-as})$ (*ii*) $\tan^{-1}(s/a)$

**11.** $\frac{1}{a\sqrt{2}}e^{-s^2/4a^2}$ ; $\frac{1}{2a^3\sqrt{2}}e^{-s^2/4a^2}$

**14.** (*i*) $\frac{1}{2}\left\{\frac{\sin[a(1-s)]}{1-s} - \frac{\sin[a(1+s)]}{1+s}\right\}$ (*ii*) $(2\cos s - \cos 4s - 1)/s^2 - (2\sin s)/s$

**15.** $2/(\pi s^2)$ **16.** $F_s(p) = -32(-1)^p/p\pi$ ; $F_c(p) = 32\frac{(-1)^p - 1}{p^2\pi^2}$

**17.** $(\pi/s)\cos s/c$ **19.** $f(x) = (2 + 2\cos x - 4\cos 2x)/\pi x$ **20.** $2/\pi(1+x^2)$.

## Problems 22.2, page 780

**2.** $\frac{1}{4}\int_{-\infty}^{\infty} e^{-\{|x-t| + |t|\}}\,dt$ **5.** $2\left(\frac{1-\cos s}{s^2}\right)$.

## Problems 22.3, page 783

**1.** $\frac{1}{9}[e^{-t} + e^{2t}(3t-1)]$ **2.** $\frac{1}{5}(e^{-2t} - 2\sin t - \cos t)$ **3.** $(\sinh at - at)/a^2$

**4.** $\frac{1}{2}[e^x(x-1) + \cos x]$ **5.** $\frac{1}{2}(\sin t - t\cos t)$.

## Problems 22.4, page 791

**1.** $y = 30\,e^{-75t}\cos 5x$ **2.** $\sum_{n=1}^{\infty}\frac{V_o}{n\pi}(1-\cos n\pi)\,e^{-n^2 t\sin nx_0}$

**3.** $u(x,t) = \frac{2}{\pi}\int_0^\infty e^{-s^2 t}\left\{\frac{\sin(1+x)s + \sin(1-x)s}{s}\right\}ds$

**6.** $\theta(x,t) = \theta_0\,erf\left(\frac{x}{2\sqrt{(kt)}}\right) + \theta_0\sum_{n=1}^{\infty}(-1)^n\left\{erf\frac{nl-x}{2\sqrt{(kt)}} - erf\frac{nl+x}{2\sqrt{(kt)}}\right\}$.

## Problems 22.5, page 792

**1.** $F_c(s) = \int_0^\infty f(t) \cos st\, dt$ **2.** $s^2/2$

**3.** The Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transforms.

**4.** $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$ **5.** $\int_{-\infty}^{\infty} t^n f(t)\, e^{ist}\, dt$ **6.** $e^{isa} F(s)$

**7.** $\frac{2}{\pi} \int_0^\infty \sin \lambda x \int_0^\infty f(t) \sin \lambda t\, dt\, d\lambda$ **8.** $\frac{1}{a}$ **9.** $-s^2 [F(u)]$

**10.** $\frac{2}{\pi} \int_0^\infty \sin(\lambda x)\, d\lambda \int_0^\pi \sin(\lambda t)\, dt$ **11.** $-\frac{n\pi}{l} \int_0^1 f(x) \cdot \cos \frac{n\pi x}{l}\, dx$ **12.** $\frac{1}{a} F\left(\frac{\lambda}{a}\right)$

**13.** $f(x) = \frac{2}{\pi^3} \sum_{p=1}^{\infty} \left(\frac{1 - \cos p\pi}{p^2}\right) \sin px$ **14.** $\frac{1}{2} F(s/2)$

**15.** $1/(s^2 + 1)$ **16.** True **17.** False **18.** True

**19.** True **20.** False **21.** True.

## Problems 23.1, page 800

**1.** (i) $e^{a/z}$; (ii) $z(e^{-z} - 1)$; (iii) $\frac{z}{z - e^{i\theta}}$

**2.** (i) $\frac{2z}{(z-1)^2} + \frac{z/\sqrt{2}}{z^2 - \sqrt{2z} + 1} + \frac{z}{z-1}$; (ii) $\frac{z^3 - 3z^2 + 4z}{(z-1)^3}$; (iii) $\frac{z(z+1)}{(z-1)^3} + \frac{3z}{(z-1)^2} + \frac{2z}{z-1}$

**6.** (i) $\frac{z^2 \sin\theta}{z^2 - 2z\cos\theta + 1}$ (ii) $\cos\alpha \frac{z(z - \cos\pi/8)}{z^2\ 2z\cos\pi/8 + 1} - \sin\alpha \frac{z \sin\pi/8}{z^2 - 22\cos\pi/8 + 1}$

**7.** $\frac{z(z - \cos\theta)}{z^2 - 2z\cos\theta + 1}, |z| > 1; \frac{z(z^2\cos\theta - 2z + \cos\theta)}{(z^2 - 2z\cos\theta + 1)^2}$ **8.** $\frac{z^2}{z^2+1}, |z| > 1; \frac{z^2}{z^2 + a^2}$

**9.** (i) $\frac{z}{z - e^{-a}}$; (ii) $\frac{z\, e^{-a}}{(z - e^{-a})^2}$; (iii) $\frac{(z + e^{-a})\, e^{-a}}{(z - e^{-a})^3}$

**12.** $\frac{z^2(1 + 3z^2)}{(1-z)(1+z^2)}$ **13.** $u_2 = 2, u_3 = 11.$

## Problems 23.2 page 804

**1.** $\frac{z}{z-4}$; $|z| > 4$ **2.** $\frac{z}{2-z}$; $|z| < 2$ **3.** $\frac{3z}{(4-z)(z-3)}$; $3 < |z| < 4$.

**4.** $\frac{5z}{(4-5)^2}$; $|z| > 5$ **5.** $-\log(1 - 3/z)$; $|z| > 3$ **6.** $e^{3/z}$, ROC is $z$-plane

**7.** $(1 - e^a/z)^{-1}$; $|z| > |e^a|$.

## Problems 23.3, page 807

**1.** $\frac{1}{2}(3^{n+1} - 1)$ **2.** $(n+1)\, a^n$ **3.** $\frac{1}{2} n(n-1)$

**5.** $4a^n$ **6.** $(1/3)^n - 2^n$ **7.** $n$

**8.** $\frac{3}{4}\left\{\frac{1}{(2)^{n-1}} + \frac{1}{(-4)^n}\right\}$ **9.** $(n^2 + 7n + 4)(4)^{n-1}$

**13.** $u_n = (2)^{n-1} + (3)^{n-1} + (4)^{n-1}\ (n > 0)$

**11.** $2 + (2)^n + 3(n-1)\,2^n, (n \geq 1)$

**12.** $(-1)^{n+1} - 2n + \cos n\pi/2$

**13.** $1 - e^{-at}$

**14.** (i) $\left(-\frac{1}{3} - \frac{z}{3^2} - \frac{z^2}{3^3} - \frac{z^3}{3^4} \cdots\right) + \left(\frac{1}{2} + \frac{z}{2^2} + \frac{z^2}{2^3} + \frac{z^3}{2^4} + \cdots\right)$; (ii) $(-2^{n-1})\,z^{-n}, n > 0$

(iii) $(3^{n-1} - 2^{n-1})\,z^{-n}, n \geq 1, 0, n \leq 0$

**15.** $\frac{1}{2}\,(n-1)\,(n-2)5^{n-3}, n \geq 3$ and $= 0, n < 3$

**16.** $2(-i)^{n-1} - (-2)^{n-1}$

**17.** $\frac{1}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^n$

**18.** $\frac{1}{3} - \frac{1}{3}\left(-\frac{1}{2}\right)^{n-1}$

**19.** $u_n = 1 + \frac{1}{2}\,[(i)^{n-2} + (-i)^{n-2}]$

**20.** $2n \sin(n\pi/2), n = 0, 1, 2, \ldots$

## Problems 23.4, page 811

**1.** $y_k = \frac{8}{5}\left(\frac{1}{2}\right)^k - \frac{3}{5}\left(\frac{-1}{3}\right)^k$

**2.** $y(n) = (n-1)\,(-1)^{n-2}\,y(n-2) - 2^n$

**3.** $y_n = 2^{n-1} + (-2)^{n-1}$

**4.** $f(n) = 2 + (-4)^n$

**5.** $y_n = \frac{4}{3}\left[2\,(-1)^n + (2)^n\right]$

**6.** $36\left[\frac{1}{2} - (2)^n + \left(\frac{1}{2}\right)^n\right]$

**7.** $y_n = (c_1 + c_2 n)\,3^n + \frac{1}{2}\,n(n-1)\,3^{n-2}$

**8.** $y_n = c_1 + c_2 \,.\, 3^n + 5^n/8$

**9.** $y_n = 2\left[\left(\frac{1}{4}\right)^n - \left(-\frac{1}{4}\right)^n\right]$

**10.** $5.2^n$

**11.** $y_n = \frac{1}{3}(-1)^n - \frac{2}{5}(-3)^n + \frac{1}{15}(2)^n$

**12.** $y_k = 1 - 2k + 2^k$

**13.** $y_n = c_1\,4^n + \left(c_2 - \frac{n}{4}\right)\,2^n + 2n - \frac{8}{3}$

**14.** $y_k = \frac{1}{2}\,(k+2)\,\frac{1}{5^k}\cos\frac{k\pi}{2}$

**15.** $y_n = \left[\frac{1}{4} - \frac{9}{4}(-3)^n\right]\,\mu(n)$

**16.** $y_n = (-2)^{n-1}, (n \geq 1)$.

## Problems 23.5, page 811

**1.** $z/(z-1)$

**2.** $\sum_{n=0}^{\infty} u_n\, z^{-n}$

**3.** $z/(z-1)^2$

**4.** $az/(z-a)^2$

**5.** $\frac{z \sin\theta}{z^2 - 2z\cos\theta + 1}$

**6.** $e^{1/z}$

**7.** $(z^2 + z)/(z-1)^3$

**8.** $Z(au_n + bv_n + cw_n) = aZ\,(u_n) + bZ(v_n) + cZ(w_n)$

**9.** $2^{n-1}$

**10.** $(-1)^{n-1}\,n$

**11.** $u_0 = \underset{z \to \infty}{Lt}\,\{Z(u_n)\}$

**12.** False

**13.** False

**14.** True

**15.** True

**16.** False

**17.** False

**18.** False.

## Problems 24.1, page 815

**1.** $a = 2.28, b = 6.1879, p = 30.46$

**2.** $a = 1120, b = 55.1$

**3.** $a = 0.2, b = 0.0044$

**4.** $a = 0.5012, n = 0.5$

**5.** $a = 0.115, b = 11.8$

**6.** $a = 4.1, b = 0.43$

**7.** $a = 0.0498, b = -0.02$.

## Problems 24.2, page 819

1. $y = 13.6x$
2. 15.2 thousand tons
3. $Y = 0.004P + 0.048$
4. $R = 70.052 + 0.292t$
5. $a = 0.545$, $b = 0.636$
6. (*a*) $y = 4.193 + 1.117x$ (*b*) $y = 8 - 0.5\,x$
7. $y = 1.243 - 0.004x + 0.22x^2$
8. $y = 0.34 - 0.78x + 0.99x^2$
9. $y = 18.866 + 66.158x - 4.333$
10. $R = 3.48 - 0.002V + 0.0029V^2$
11. $V = 2.593 - 0.326T + 0.023\,T^2$.

## Problems 24.3, page 823

1. 6.32, $b = 0.0095$
2. $a = 1.52$, $b = 0.49$
3. $a = 3$, $b = 2$
4. $y = 7.187 - 5.16\,\dfrac{1}{x}$ ; 4.894
5. $a = 0.988$, $b = 3.275$
6. $y = 2.978\,x^{0.5143}$ ; 5.8769
7. $a = 0.1839$, $b = 0.0221$
8. $f(t) = 0.678\,e^{-3t} + 0.312\,e^{-2t}$
9. $a = 146.3$, $k = -0.412$
10. $a = 99.86$, $b = 1.2$.

## Problems 24.4, page 826

1. $a = 11.1$, $b = 0.71$
2. $y = 46.05 + 6.1\,x$
3. $a = 0.0028$, $b = 0.01$, $c = 4.18$
4. $a = 15.8$, $b = 2.1$, $c = -0.5$
5. $a = 1.459$, $b = 0.062$.

## Problems 24.5, page 828

1. $y = 0.12 + 0.47x$
2. $y = 1.184 + .523x$
3. $y = 1.53 + 0.063x + 0.074x^2$
4. $y = 0.485 + 0.397x + 0.124x^2$.

## Problems 24.6, page 829

1. $Y = aX + b$ where $X = x$, $Y = y/x$
2. $Y = A + BX$, where $X = \log_{10} p$, $Y = \log_{10} v$, $A = \dfrac{1}{r}\log b$, $B = -1/r$
3. § 24.4
4. $y = aX + c$, where $X = x^b$
5. (*ii*)
6. $\Sigma y = nA + B\Sigma x$, $\Sigma xy = A\Sigma x + B\Sigma x^2$ where $y = \log_{10} y$, $A = \log_{10} a$, $B = \log_{10} b$
7. § 24.12
8. Zero
9. $y = aX + b$ where $X = x^2/\log_{10} z$, $Y = y/\log_{10} x$
10. $a = 0.0167$, $b = 1.05$.
11. The moments of the observed values of $y$ are respectively equal to the moments of the calculated values of $y$
12. $a = 1.7$, $b = 1.26$
13. $y = a + bx$ where $x = 1/x$, $y = 1/y$
14. (*r*)
15. (*b*)

## Problems 25.1, page 837

1. 336.79
2. 64% get more than 50 marks ; median 54.7, $Q_1 = 46$, $Q_3 = 61.5$
3. Mean = 27.9 ; Median = 25.66 ; Mode = 21.85
4. Mean = 32.58 ; Median = 32.6 ; Mode = 35.1
5. 3.1%
6. 1.3%
7. 192 km/hr
8. 60 km/hr
9. 38.6 ; 36.2
10. Median = 12.2 days ; Mode = 11.4 days

## Problems 25.2, page 842

1. 4.45, 0.39
3. 4, 7
4. 10.04, 10.13, 11.69, 5.54, 2.35

**5.** 32, 32.6, 12.4 **6.** Q.D. = 10.9, S.D. = 15.26 **7.** 1.845 ; 1.8175
**8.** 0.55, 1.24 ; first, yes **9.** Height **10.** *A*
**11.** *B* is a better player and more consistent
**12.** *A* is more efficient, *B* is more consistent **13.** 161.3, 5.68.

## Problems 25.3, page 845

**1.** $\mu_1 = \mu_3 = 0, \mu_2 = 2, \mu_4 = 11 ; \beta_1 = 0, \beta_2 = 2.75$ **2.** 8.85 ; 5.25 ; 0.32 ; 1.09
**3.** – 0.2064 **4.** 0.22 ; 1.157 **6.** 0 ; 2.9
**7.** $\beta_1 = 0.493 ; \beta_2 = 0.655$ ; platykurtic.

## Problems 25.4, page 854

**1.** $r = 0.81 ; x = 0.5y + 0.5, y = 1.3x + 1.1$ **2.** $r = 0.96$
**3.** $r = 0.92$ **4.** $r = -0.055$ **5.** 0.7291
**6.** $r = 0.4517$ **7.** $r = 0.632 ; y = 0.467 + 0.8x, x = 0.167 + 0.5y$
**9.** $m = (\beta - b)/(a - \alpha)$ **10.** $\bar{x} = 4, \bar{y} = 7, r = -0.5$ **11.** $\bar{x} = 9.06, \bar{y} = 5.52, r = 0.46$
**12.** $r = 0.7395 ; \bar{x} = -0.1034 ; \bar{y} = 0.5172$ **13.** 134.5
**14.** 1.28 inch **15.** 0.8545 **16.** 0.932.

## Problems 25.5, page 855

**1.** (*d*) **2.** (*d*) **3.** (*a*) **4.** (*a*)
**5.** (*b*) **6.** (*a*) **7.** (*b*) **8.** Coefficient of correlation
**9.** No **10.** 13.83 **11.** $\bar{x} = 2, \bar{y} = 3, r = \sqrt{3}$
**12.** Zero **13.** $\frac{1}{2}(Q_3 - Q_1)$ **14.** –1
**15.** § 25.9 **16.** $\frac{\Sigma XY}{n\,\sigma_x\,\sigma_y}$ **17.** $(\bar{x}, \bar{y})$
**18.** Reliability or consistency **19.** $\sqrt{\beta_1}$ **20.** degree of peakedness
**21.** $y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})$ **22.** $100\,\sigma/\bar{x}$ **23.** $\tan^{-1}\left\{\frac{1-r^2}{r}\cdot\frac{\sigma_x\,\sigma_y}{\sigma_x^2 + \sigma_y^2}\right\}$
**24.** 3 **25.** Two regression coefficients **26.** perpendicular
**27.** greater **28.** ± 1 **29.** $1 - \frac{6\,\Sigma\,d_i^2}{n^3 - n}$
**30.** Coefficient of standard deviation **31.** zero
**32.** – 1 and 1 **33.** – 0.6 **34.** False
**35.** True **36.** False

## Problems 26.1, page 858

**1.** 4096 **2.** 360 ; 120 **3.** 120 ; 115 **4.** (*a*) 676000 (*b*) 468000

## Problems 26.2, page 868

**1.** (*i*) 7/12 ; (*ii*) $P(A/B) = 3/4, P(A \cup B) = 7/12, P(A'\,B') = 3/8$
**2.** (*a*) 1/36 ; (*b*) 1/6, Yes **3.** 36 : 30 : 25
**4.** 1/7 **5.** (*i*) 1/4, (*ii*) 7/8, (*iii*) 11/16 **6.** 15/1024

**7.** 3/28 **8.** 20/81 **9.** (*i*) 2816/4165 ; (*ii*) 2197/2025,
**10.** 0.11 **11.** (*a*) 6.739 ; (*b*) 0.024 **12.** (*a*) 1/114 ; (*b*) 685/1140
**13.** 2/801 **14.** 10/21 **15.** $1-(1-p_1)(1-p_2)\ldots(1-p_n)$ ; 0.518
**16.** 15/17 **17.** 1/2 **18.** 5/12 **19.** (*i*) 83/110 (*ii*) 25/83
**20.** $1-2/(n-1)$ **21.** 7/20 **22.** (*a*) 1/6 ; (*b*) 3/4 **23.** 61/90
**24.** 0.72 **25.** 0.2223 **26.** 0.88

## Problems 26.3, page 871

**1.** 3/11 **2.** 25/69, 28/69, 16/69 **3.** 0.3175, 0.254 **4.** 15/59.

## Problems 26.4, page 878

**1.** $k = 1$ ; $\mu = .8$, $\sigma^2 = 2.232$ **2.** $2\sqrt{5}$

**3.** $F(x) = 0, -\infty < x < 0$
$= 1/8, 0 \le x \le 1$
$= 1/2, 1 \le x \le 2$
$= 7/8, 2 \le x \le 3$
$= 1, 3 \le x \le \infty$

**5.** ₹ 32

**6.** 2 **7.** $f(x)$ is a p.d.f. $\bar{x} = \frac{1}{2}, \sigma^2 = \frac{1}{20}$ **8.** (*i*) 9/16, 7/16 ; (*ii*) $k = 0.45$
**9.** (*i*) 0.37, (*ii*) 0.63 **10.** 4/9
**12.** $y_0 = 3/4$ ; Mean = 1 ; Variance = 1/5. **13.** 0.2
**14.** 1/3, 2/9 **15.** $F(x) = 0, x < x_1$ ; $(x - x_1)/(x_2 - x_1), x_1 \le x < x_2$ ; $1, x \ge x_2$.

## Problems 26.5, page 881

**1.** $n = 4, p = q = \frac{1}{2}; \frac{15}{16}$ **2.** ${}^4C_r(1/6)^{4-r}(5/6)^r$ ; $r = 0, 1, 2, 3, 4$
**3.** (*a*) 0.02579 ; (*b*) 0.04571 ; (*c*) $1.024 \times 10^{-7}$ **4.** 0.3456
**5.** 45927/50000 **6.** (*i*) 0.246 ; (*ii*) 0.345
**7.** (*i*) ${}^{20}C_1(1/20)(19/20)^{19}$ ; (*ii*) $\sum_{r=0}^{5} {}^{20}C_r(1/20)^r(19/20)^{20-r}$ (*iii*) 19
**8.** (*a*) 0.08 ; (*b*) 0.26 ; (*c*) 0.92 **9.** (*a*) 250 ; (*b*) 25 ; (*c*) 500
**10.** (*i*) 0.59049 ; (*ii*) 0.32805 ; (*iii*) 0.08146 **11.** 11
**12.** 99.83 **13.** 0.3585, 0.3773, 0.1887 ; 0.0596
**14.** 600 **15.** $100\,(.432 + .568)^5$
**16.** $200\,(0.554 + 0.446)^6$.

## Problems 26.6, page 884

**1.** (*i*) 2 ; (*ii*) $\frac{2}{3}e^{-2}$ **2.** $P(0) = 0.2636$, $P(3) = 0.1041$, $P(>3) = 0.1506$
**4.** $(10)^{15}e^{-10}/15! = 0.035$ **5.** 0.08
**6.** (*i*) 0.2231 (*ii*) 0.1913 **7.** 0.0008
**8.** $m = 0.51 = \sigma^2$ ; Poisson frequencies of 0, 1, 2, 3, 4 accidents are 180.1, 91.9, 23.4, 4, 0.6
**9.** 0.6 **11.** Theoretical frequencies are 44, 43, 21, 7, 1
**12.** Theoretical frequencies are 109, 142, 92, 40, 13, 3, 1, 0, 0, 0, 0.

## Problems 26.7, page 890

2. (*i*) 0.1644 ; (*ii*) 0.7686
3. (*i*) 0.095 ; (*ii*) – 0.995
4. 36.4
5. (*i*) 16, (*ii*) 2
6. 294
7. 543
8. (*i*) 79 ; (*ii*) 35% ; (*iii*) 11
10. 52
11. 67
12. ₹ 866
13. $y = \dfrac{100}{\sqrt{(3.4\pi)}} e^{-\frac{(x-2)^2}{3.4}}$
14. $\mu = 13.64$. $\sigma = 3.98$.

## Problems 26.8, page 892

1. (*a*) 0.0287, (*b*) 0.9672, (*c*) 0.5111
2. (*a*) 0.7854, (*b*) 0.1815, (*c*) 0.1815
3. (*a*) 0.97815, (*b*) 0.00595, (*c*) 0.01209.

## Problems 26.9, page 893

3. $\frac{1}{2}(n+1), \frac{1}{12}(n^2-1)$
5. Mean = $a + b$, variance = $b^2$
6. $[(1-e^t)/t]^2$
8. $(1-t)^{3/4}$.

## Problems 26.10, page 894

1. (*a*)
2. (*b*)
3. (*d*)
4. (*b*)
5. 1/7
6. 1/2
7. (*b*)
8. (*b*)
9. (*a*)
10. (*c*)
11. 0.1288
12. 2
13. 0.21
14. 0.24
15.

| $X$: | 0 | 1 | 2 |
|---|---|---|---|
| $p(x)$: | 1/4 | 2/4 | 1/4 |

16. § 25.5
17. 0.7837
18. zero
19. equal
20. $P(A) + P(B)$
21. $\beta_1 = 0, \beta_2 = 3$
22. 120
23. 0.2646
24. 1/9
25. 0.2222
26. 1/6
27. $e^{-3}$
28. 5/36
29. 2
30. symmetrical
31. $1 - e^{-m}$
32. six
33. 0.6915
34. $(q + pe^t)^n$
35. 4 : 5
36. 1/6
37. 3.5
38. $\sqrt{2}$
39. unity
40. $n \to \infty, p \to 0$ such that $np$ is fixed
41. $P(A) + P(B)$
42. ${}^yC_x / {}^{y+z}C_x$
43. $np$
44. $P(A \cup B) = 0.72, P(A \cap B') = 0.1653$
45. $\left(\frac{1}{3} + \frac{2}{3}\right)^{18}$
46. 1
47. $\mu_r' = \left[\dfrac{d^r}{dt^r}(\Sigma\, p_i\, e^{tx_i})\right]_{t=0}$
48. 1/6
49. 3/4
50. 1/2
51. 0.2
52. 2/7
53. $(q + p)^n$
54. 2/3
55. Mean and S.D.
56. 1/13
57. 2
58. § 26.6
59. $e^{-4/3}$
60. 1/3
61. True
62. False
63. True
64. False
65. False
66. True
67. False
68. False
69. $k = 2$
70. 8
71. $f(x) = \lambda e^{-\lambda x}$ for $x > 0$, $\lambda$ is a parameter
72. 25/12
73. 2
74. $l$
75. 4
76. 8
77. $n, m$ degrees of freedom
78. $\int_{-\infty}^{\infty} e^{t(x-a)} f(x)\, dx$

**79.** $-5/7$ **80.** $\dfrac{e^{-x}x^{l-1}}{\Gamma(l)}, 0 < x < \infty$ **81.** $\dfrac{1}{2}(b+a)$

**82.** 50% **83.** 3/8 **84.** 1 **85.** 3/4

**86.** (*iii*) **87.** $F(x) = \int_{-\infty}^{x} f(x)\,dx$ **88.** $\dfrac{\lambda^{2r}e^{-\lambda}}{(2r)!}$ **89.** $-\infty < t < \infty$

**90.** 6 **91.** 1/9 **92.** False **93.** 2

**94.** $4(1-x)^3$ **95.** 0.264

## Problems 27.1, page 901

**1.** Die is biased **2.** No **3.** Yes

**4.** 8.91% and 15.07% **5.** Consistent **6.** $p = 65/500$, S.E. = 0.015

**7.** 37.5% ; 30.3 and 44.7 respectively **8.** No

**9.** Difference is not significant

**10.** $Z \sim 6.56$ so that the difference is significant **11.** No.

## Problems 27.2, page 904

**1.** No **2.** Mean weight lies between 64.6 and 69.4 lbs.

**3.** 0.0774 **5.** 62 **6.** 2.696 **7.** No

**8.** No **9.** (*i*) Yes, (*ii*) No **10.** Yes.

## Problems 27.3, page 910

**1.** 0.25 **2.** $t = 0.62$, Yes **3.** 11.887 and 12.113 cm

**4.** Refute the claim **5.** Process is not under control **6.** No

**7.** Sample mean = 575.2 kg., S.E. = 2.75 kg **8.** Accept null hypothesis

**9.** Yes with 75% confidence. **10.** No **11.** No

**12.** No

## Problems 27.4, page 914

**1.** 0.41 **2.** Hypothesis is correct

**3.** Significant at 5% level **4.** Yes

**5.** $f_e$ : 33.82 161.78 315.98 308.48 150.54 29.4

$\chi^2 = 7.97$. Binomial distribution gives a good fit at 5% level

**6.** $f_o$ : 305 365 210 80 18 12

$f_e$ : 301.2 361.4 216.8 86.7 26 7.9 ; $\chi^2 = 3.097$

Poisson distribution gives a good fit at 5% level

**7.** $f_o$ : 314 355 204 85 29 12

$f_e$ : 301 362.2 217.3 86.9 26.1 6.5

Poisson distribution can be fitted to the data

**8.** $\chi^2 = 1.2$. The fit is quite good at 5% level.

## Problems 27.5, page 917

**1.** First variance cannot be regarded as significantly greater than the second

**2.** Not significant as $F = 2.1$ and $F_{0.05} = 4.15$ **4.** Not significant as $F = 2.4$ and $F_{0.01} = 3.2$

**5.** Product of firm $B$ cannot be said to be of better quality than those of firm $A$.

**6.** Not significant at 1% level and just significant at 5% level as $F = 2$, $F_{0.01} = 2.62$ and $F_{0.05} = 1.98$
**7.** $F = 1.49$, Not-significant
**8.** $F = 1.025$ ; Yes

## Problems 27.6, page 917

**1.** § 27.3 (3)
**2.** § 27.15
**3.** § 27.3 (2)
**4.** We are testing the hypothesis that one process is better than another
**5.** § 27.11
**6.** 1
**7.** 50
**8.** $(0, \infty)$
**9.** II
**10.** 8 ; 16
**11.** $r = n - 1$
**12.** False
**13.** Less than 30
**14.** § 27.17
**15.** $-\infty < t < \infty$
**16.** (*ii*)
**17.** True

## Problems 28.1, page 926

**1.** (*i*) 2.687, (*ii*) 1.46, (*iii*) 2.375, (*iv*) 2.875
**2.** (*i*) 0.519, (*ii*) 2.875, (*iii*) 1.146, (*iv*) 0.367
**3.** (*i*) – 0.686, (*ii*) 2.7065, (*iii*) 0.686, (*iv*) 1.4036
**4.** (*i*) 0.853, (*ii*) 0.607, (*iii*) 2.798, (*iv*) 3.789, (*v*) – 0.134
**5.** 1.861
**6.** (*i*) 1.532, (*ii*) 2.095, (*iii*) 1.834, (*iv*) 1.226
**7.** (*i*) 1.855 (*ii*) 2.198 (*iii*) 1.662
**8.** – 16.56
**9.** (*i*) 0.853, (*ii*) – 1.9338, (*iii*) 2.7985, (*iv*) 4.545
**10.** (*i*) 0.518, (*ii*) 0.052, (*iii*) 0.695, (*iv*) 2.911
**11.** $x_{n+1} = \frac{1}{2}(x_n + N/x_n)$ ; (*i*) 3.605 (*ii*) 3.162
**12.** 3.4482
**13.** 2.3784
**14.** (*i*) 0.055 (*ii*) 0.258 (*iii*) 0.4347

## Problems 28.2, page 929

**1.** (*i*) 1.532, (*ii*) 0.684, (*iii*) 3.18, (*iv*) 1.168
**2.** 1.674
**3.** 2.231
**4.** – 1.328
**5.** 2.924

## Problems 28.3, page 936

**1.** $x = 7, y = -9, z = 5$
**2.** $x = -51/4, y = 115/8, z = 35/4$
**3.** $x = 1, y = 2, z = 3$
**4.** $x_1 = 2, x_2 = -1, x_3 = 3$
**5.** $x_1 = 1, x_2 = 2, x_3 = -1, x_4 = -2$
**6.** $x = 1, y = 3, z = 5$
**7.** $x = 8.7, y = 5.7, z = -1.3$
**8.** $x = 1, y = 2, z = 3$
**9.** $x = 7, y = -9, z = 5$
**10.** $x_1 = 2, x_2 = 1/5, x_3 = 0, x_4 = 4/5$
**11.** $x = y = z = 1$
**12.** $x = 1, y = 2, z = 3$
**13.** $x = 35/18, y = 29/18, z = 5/18$
**14.** $x_1 = -1, x_2 = 0, x_3 = 1, x_4 = 2$
**15.** $\begin{bmatrix} 1.2 & -0.4 & 0.2 \\ -0.2 & -0.1 & 0.3 \\ -0.4 & 0.3 & 0.1 \end{bmatrix}$.

## Problems 28.4, page 942

**1.** $x = 2.556, y = 1.722, z = -1.055$
**2.** (*a*) $x = 2.426, y = 3.573, z = 1.926$ (*b*) $x = 2.426, y = 3.573, z = 1.926$
**3.** $x = 1, y = 1, z = 1$
**4.** $x = 0.998, y = 1.723, z = 2.024$
**6.** $x = 1.052, y = 1.369, z = 1.962$
**8.** $x = -13.223, y = 16.766, z = -2.306$
**9.** $x = 1, y = 2, z = 3, u = 4$
**10.** $x = 1.36, y = 2.103, z = 2.845$
**11.** $x = y = z = 1$
**12.** $x = 52.5, y = 44.5, z = 59.7$
**13.** $x = 1.93, y = 3.57, z = 2.43$

## Problems 28.5, page 943

1. $x = 2, y = 1$
2. $x = -1.853, y = -1.927$
3. $x = 0.7974, y = 0.4006$
4. $x = 3.162, y = 6.45$
5. $x = -3.131, y = 2.362$

## Problems 28.6, page 945

1. (*a*) 5.38, $\begin{bmatrix} 0.46 \\ 1 \end{bmatrix}$; (*b*) 4.418, $\begin{bmatrix} 1 \\ 0.618 \end{bmatrix}$
2. 3.41 ; [0.74, – 1, 0.67]′
3. (*a*) 6, [1, 1, – 1]′ (*b*) 8, [1, – 0.5, 0.5]′
4. (*a*) 7 ; [2.099/7, 0.467/7, 1] (*b*) 25.182, [1, 0.045, 0.068]′ (*d*) 11.66 [0.025, 0.422, 1.000].

## Problems 28.7, page 945

1. Newton-Raphson method
2. $x_n = x_{n-1} - \dfrac{f(x_{n-1})}{f'(x_{n-1})}$
3. Chord AB
4. $x_{n+1} = \dfrac{1}{2}(x_n + N/x_n)$
5. initial approximation $x_0$ is chosen sufficiently close to the root
6. diagonal
7. (*c*)
8. (*a*)
9. $x_{n+1} = \dfrac{1}{3}(2x_n + N/x^2_n)$
10. $x_{n+1} = \dfrac{1}{2}(x_n + x_{n-1})$
11. $x_2 = x_0 - \dfrac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$
12. (*a*)
13. $x_{n+1} = x_n(2 - Nx_n)$
14. (*b*)
15. Newton-Raphson method
16. § 28.6
17. Upper triangular matrix
18. False
19. True
20. $x = 1, y = 1$.

## Problems 29.1, page 952

1. 0.4
2. – 7459
5. 239
6. 4.68, 2.68, 55.8, 99.88
8. (*i*) $1 - 2 \sin (x + 1/2) \sin 1/2$ ; (*ii*) $\tan^{-1}(1/2n^2)$ ;
   (*iii*) $192 [x(x + 4)(x + 8)(x + 12)(x + 16)]$ (*iv*) $-2/[(x + 2)(x + 3)(x + 4)]$
9. (*i*) $e^{3x} [e^3 \log (1 + 1/x) + (e^3 - 1) \log 2x]$ (*ii*) $2^x (1 - x)/(1 + x)$
   (*iii*) $(a - 1)^n a^x$ ; (*iv*) $(-1)^n n!/[x(x + 1)(x + 2) \dots (x + n)]$.
12. (*i*) – 36 ; (*ii*) $24 \times 2^{10} \times 10!$
14. $u = [x]^4 - 6[x]^3 + 13[x]^2 + x + 9$
15. $4x^3 - 12x^2 + 8x + 1$ ; $12x(x - 1)$
16. $\dfrac{1}{2}[x]^4 + 3[x]^3 + 4[x] + c$
17. $y(4) = 74, y(6) = 261$
19. 15.

## Problems 29.2, page 957

1. $\left(\dfrac{\Delta^2}{E}\right) u_x = u_{x+h} - 2u_x + u_{x-h}$ ; $\dfrac{\Delta^2 u_x}{E u_x} = \dfrac{u_{x+2h} - 2u_{x+h} + u_x}{u_{x+h}}$
2. (*i*) $2(\cos h - 1) \sin x$ ; (*ii*) $6x$ ; (*iii*) $2(\cos h - 1)[\sin (x + h) + 1]$ ; (*iv*) 8
8. *Error* = 10
9. 31
10. $f(1.5) = 0.222, f(5) = 22.022$
11. $y(4) = 74, y(6) = 261$
12. – 99
13. $y_4 = 1$ approx
15. (*i*) $n(3n^2 + 6n + 1)$ ; (*ii*) $\dfrac{n(n + 1)(n + 2)(n + 3)}{4}$
16. $2/(1 - x)^3$.

## Problems 29.3, page 961

**1.** 5.54 **2.** 6.36 **3.** 1.1312 **4.** 0.788
**5.** ₹ 110.52 **6.** 8666 **7.** 352 ; 219 **8.** 0.9623, 0.2903
**9.** 24 approx **10.** $f(x) = 9x - 4x^2$ **11.** 1.625 **12.** 0.1955
**13.** 4.219 **14.** 2530 **15.** $y_1 = 0.1, y_{10} = 100$ **16.** $u_2 = 42, u_4 = 49$
**17.** 10, 22 **18.** 755.

## Problems 29.4, page 971

**1.** 19.4 **2.** 12.826 **3.** 54000 **4.** 3.2219
**5.** 3.0375 **6.** 395 **7.** 3.347 **8.** 9
**9.** 3250.875 **11.** 2.5283 by all formulae.

## Problems 29.5, page 974

**1.** 14.63 **2.** 2.8168 **3.** 0.89 **4.** 100
**5.** $648 + 30x - x^2$ **6.** $x^3 - 3x^2 + 5x - 6$ **7.** $x^5 - 9x^4 + 18x^3 - x^2 + 19x - 18$
**8.** 3 **9.** $\frac{0.5}{x-1} - \frac{0.5}{x+1} + \frac{1}{x-2}$

## Problems 29.6, page 977

**1.** 1 **2.** 3.09 **3.** 448, 3150 **4.** 133.19
**5.** $f(x) = \frac{1}{24} x^3 - 25x + 24 - \frac{7}{6} x^2 + \frac{557}{60} x - 25$ **6.** $f(x) = \frac{1}{20} x^3$ **7.** $f(x) = x^4 - 3x^3 + 5x^2 - 6$
**8.** 31.

## Problems 29.7, page 978

**1.** 11.5 **2.** 6.5928 **3.** 37.23

## Problems 29.8, page 978

**1.** \$ 7.3 **2.** (*b*)

**3.**

| $x$ | $f(x)$ | I.D.D. | II.D.D. |
|---|---|---|---|
| 5 | 7 | | |
| | | 2.9 | |
| 15 | 36 | | 0.87 |
| 22 | 160 | 17.7 | |

**4.** Intermediate value of the variables. **5.** \$ 7.8
**6.** $\frac{[x_1, x_2, x_3, x_4] - [x_0, x_1, x_2, x_3]}{x_4 - x_0}$ **7.** $\frac{-1}{4}$ and $\frac{1}{4}$ **8.** \$ 7.14

**9.** $f(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} + \frac{(x - x_0)(x - x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$

**10.** $\frac{13}{5}$ **11.** (*c*) **12.** 1.857
**13.** *Extrapolation* is the process of estimating the value of a function outside the given range of values.
**14.** $1/(abc)$ **15.** (*a*) **16.** $x^3 - 7x^2 - 18x - 12$
**17.** (*b*) **18.** $6h^2 (x + h)$.

## Problems 30.1, page 987

**1.** – 27.9, 117.67 **2.** 4.75, 9 **3.** 0.63, 6.6
**4.** (*a*) 0.493, – 1.165 (*b*) 0.4473, – 0.1583 ; (*c*) 0.4662, – 0.2043 **5.** 2.8326
**6.** – 0.06 ; 0.5 **7.** 0.175 **8.** 13.13 m/sec
**9.** (*i*) – 52.4, (*ii*) – 0.0191 **10.** 44.92 **11.** 0.085
**12.** 3.82 rad/sec, 6.75 rad/sec$^2$ **13.** 0.2561 **15.** 0.0186
**16.** 135 **17.** $y_{max}(1) = 0.25, y_{min}(0) = 0$ **18.** *Max* $f(10.04) = 1340.03$

## Problems 30.2, page 995

**1.** (*i*) 0.695 (*ii*) 0.693 (*iii*) 0.693 **2.** (*i*) 0.7854 (*ii*) 0.7854, (*iii*) 0.78535, (*iv*) 0.7854
**3.** 1.61 **4.** – 6.436 **5.** (*i*) 70.16 (*ii*) 0.635
**6.** 0.6305 **7.** (*i*) 2.0009 ; (*ii*) 1.1873 **8.** (*i*) 1.1249 (*ii*) 0.911
**9.** (*a*) 1.8276551, .0001924 ; (*b*) 1.8278472, .0000003 ; (*c*) 1.8278470, 0000005 ; (*d*) 1.8278474, .00000001
**10.** 1.3028 **11.** 403.67 **12.** 7.78
**13.** 710 sq.ft **14.** 3.032 **15.** 408.8 cub. cm.
**16.** 1.063 sec ; 1.064 sec **17.** 552 m ; 3 m/sec$^2$. **18.** 30.87 m/sec.
**19.** 29 min nearly.

## Problems 30.3, page 996

**1.** (*c*) **2.** $\frac{1}{h}\left[\Delta f(a) - \frac{1}{2}\Delta^2 f(a) + \frac{1}{3}\Delta^3 f(a) - \cdots\right]$
**3.** $h$ should be small **4.** 0.775 **5.** $2\frac{2}{3}$
**6.** 30.8 **7.** (*b*)
**8.** larger number of sub-intervals **9.** 0.7854 **10.** (*d*)
**11.** $y'_{x_n}\ \frac{1}{h} = \left[\nabla_{y_n} + \frac{1}{2}\nabla^2{}_{y_n} + \frac{1}{3}\nabla^3{}_{y_n} + \cdots\right]$ **12.** a multiple of 6
**13.** (*c*) **14.** 0.783 **15.** 0.69
**16.** 1.36125 **17.** 1.36
**18.** if the entire curve is itself a parabola
**19.** even and multiples of 3 **20.** False

## Problems 31.1, page 999

**1.** $y_{x+3} - 2y_{x+2} + 2y_{x+1} = 0$ **2.** $\Delta y_n = (-1)^{n+1}/(n+1)$ **3.** $u_{n+1} - 2u_n = 0$
**4.** (*i*) $(x+2)y_{x+2} - 2(2x+1)y_{x+1} + xy_x = 0$ ; (*ii*) $(x^2+x)y_{x+2} - (2x^2+4x)y_{x+1} + (x^2+3x+2)y_x = 0$
**5.** (*ii*) $y_{n+2} - 8y_{n+1} + 15y_n = 0$ ; (*ii*) $y_{n+2} - 6y_{n+1} + 4y_n = 0$
**6.** (*i*) $(x-1)y_{x+2} - (3x-2)y_{x+1} + 2xy_x = 0$ ; (*ii*) $y_{x+2} - 4y_x = 0$ ;
(*iii*) $y_{x+3} - 6y_{x+2} + 11y_{x+1} - 6y_x = 0$.

## Problems 31.2, page 1002

**1.** $u_p = (c_1 + c_2 p)\,3p$ **2.** $y_n = c_1 \cos\frac{2n\pi}{3} + c_2 \sin\frac{2n\pi}{3}$ **3.** $u_n = c_1 \cos n\pi/2 + c_2 \sin n\pi/2$
**4.** $y_n = c_1 . 2^n + c_2 . 3^n$. **5.** $y_n = (2)^{n-1} + (-2)^{n-1}$ **6.** $u_k = c_1(-1)^k + (c_2 + c_3 k)\,2^k$.
**7.** $f(x) = (c_1 + c_2 x)(-1)^x + c_3 . 2^x$ **8.** $u_n = 2n + (-2)^n$. **9.** $y_n = 6 + (n-3)\,2^n$
**10.** $u_n = 2^{n/2}[c_1 \cos n\pi/4 + c_2 \sin n\pi/4]$
**11.** $y_m = 2^m\left\{c_1 \cos\frac{m\pi}{4} + c_2 + \sin\frac{m\pi}{4} + c_3 \cos\frac{3m\pi}{4} + c_4 \sin\frac{3m\pi}{4}\right\}$ **14.** $y_n = c_1(-1)^n + c_2(10)^n$

## Problems 31.3, page 1005

1. $y_n = c_1(-1)^n + c_2(6)^n - 2^n/12$

2. $y_n = \left(\frac{n}{15} - \frac{1}{25}\right)(-3)^n + \frac{2^n}{25}$

3. $y_p = c_1 + c_2 p + c_3 p^2 + \frac{1}{6}\,p(p-1)(p-2)$

4. $y_n = 2^n\left(\frac{2}{\sqrt{3}}\sin\frac{n\pi}{3} - 2\cos\frac{n\pi}{3}\right) + 2$

5. $y = c_1 + c_2 \,.\, 3^x + \frac{1}{2}\,x \,.\, 3^{x-1}$

6. $y_x = (c_1 + c_2 x)\,2^x + 3x(x-1)\,2^{x-3} + 5\,.\,4^{x-1}$

7. $u_n = c_1 + c_2(-1)^n + \frac{1}{2}\,\frac{\cos\left(\frac{n}{2}-1\right) - \cos\frac{n}{2}}{1-\cos 1}$

8. $y_p = c_1\cos\frac{p}{2} + c_2\sin\frac{p}{2} + \frac{p\cos\left(p-\frac{1}{2}\right)}{2\sin\frac{1}{2}}$

9. $y_x = c_1 + 2^x + c_2(-2)^x - \frac{1}{27}\,(9x^2 + 12x + 11)$

10. $y_n = c_1(-1)^n + c_2\cos\frac{n\pi}{3} + c_3 + \sin\frac{n\pi}{3} + \frac{1}{2}\,n(n-3)$

11. $y_n = (c_1 + c_2 n)(3)^n + c_3(-1)^n + \frac{1}{3}(2)^n - \frac{3n}{4}$

12. $y_n = (c_1 + c_2 n)\,2^{-n} + \frac{2^n}{9} + n(n-1)\left(\frac{1}{2}\right)^{n-1}$

13. $y_n = c_1(-2)^n + c_2(-3)^n + \frac{n}{12} - \frac{7}{144}$

14. $u_x + (c_1 + c_2 x)(-3)^x + \frac{2^x}{25}(5x-2) + \frac{2}{4}\,x^{x-2} + \frac{7}{16}$

15. $y_n = c_1(-2)^n + 2^n(c_2\cos n\pi/3 + c_3\sin n\pi/3) + \frac{3}{16}\,(2)^n + 2^{n-4}(2n+3)$

16. $u_n = \left\{c_1 + c_2 n + \frac{1}{48}\,n(n-1)^2(n-2)\right\}2^n.$

17. $y_k = c_1\,.\,2^k + c_2\,.\,3^k + \frac{4^k}{2}(k^2 - 13k + 61)$

18. $y_n = 2^n\left\{(c_1 + n)\cos\frac{n\pi}{3} + c_2 + \sin\frac{n\pi}{3}\right\}.$

## Problems 31.4, page 1006

1. $y_x = a + b(-1)^x + x,\ z_x = a + b(-1)^{x+1} - (x+1)$

2. $y_x = (a + bx)(-1)^x - \frac{1}{9}\,.\,2^{x+2},\ z_x = \frac{2^x}{9} - (-1)^x\left[a + b\left(x - \frac{1}{2}\right)\right]$

3. $u_n = 2.4^n - 2\frac{1}{2}\,n(n-1),\ v_n = 4^n + 2 + \frac{1}{n} + n(n+1)$

4. $u_x = -2a + b(-2)^x - c + \frac{1}{2x}\,(3-x),\ v_x = a + c + b(-2)^x + \frac{1}{2}x(x-1).$

## Problems 31.5, page 1007

1. $y_{i+1} - 2y_i + y_{i-1} = -\frac{l_m}{P}\,y_i$. Solve it for $y_i$.

## Problems 31.6, page 1007

1. $y_{n+2} - 5y_{n+1} + 6y_n = 0$
2. $u_n = c_1 + c_2 n + c_3 n^2$
3. $u_n = c_1 + c_2(-2)^n + c_3(3)^n$
4. $y_n = c + 2^n.$
5. $y_n = c(2)^n - (n+1)$
6. $(x^2 + x)y_{x+2} - (2x^2 + 4x)y_{x+1} - (x^2 + 3x + 2)y_x = 0$
7. $y_n = c(2)^k + 1$
8. $y_n = (2)^{n+2} + (-2)^{n-1}$
9. Third

**10.** $(x+2)y_{n+2} - 2(n+1)y_{n+1} + ny_n = 0$ **11.** Second

**12.** $y_n = (C_1 + C_2 n)2^n$ **13.** $\frac{1}{2}x(x-1)(3)^{x-2}$ **14.** $y_{n+2} - 6y_{n+1} + 9y_n = 0$

**15.** True.

## Problems 32.1, page 1012

**1.** $y = 1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48}$ **2.** 0.0214 **3.** $y = \frac{1}{3}x^3 - \frac{1}{81}x^9 + \ldots$

**4.** (*a*) 0.9138, (*b*) 0.1938

**5.** $y(1.1) = 0.1103$, $y(1.2) = 0.2428$. Exact $y(1.1) = 0.1103$ and $y(1.2) = 0.2428$

**6.** 1.1053425 **7.** 1.1272 **8.** 1.005.

## Problems 32.2, page 1017

**1.** 1.1831808 **2.** 1.1448 **3.** 4.5559

**4.** $y(0.1) = 0.095$, $y(0.2) = 0.181$, $y(0.3) = 0.259$ **5.** $y(0.2) = 1.2046$, $y(0.4) = 1.4644$

**6.** 2.2352 **7.** 1.0928 **8.** 5.051.

## Problems 32.3, page 1021

**1.** 1.7278 **2.** 1.1749 **3.** 1.0207, 1.0438

**4.** 2.5005 **5.** $y(0.1) = 0.9052$, $y(0.2) = 0.8213$

**6.** $y(0.1) = 2.9919$, $y(0.2) = 2.9627$ **7.** 0.3487 **8.** 0.8489

**9.** 1.1678 **10.** 1.0911, 1.1677, 1.2352, 1.2902, 1.338.

## Problems 32.4, page 1026

**1.** 3.795 **2.** 1.2797 **3.** $y(1.4) = 3.0794$

**4.** $y(4.5) = 1.023$ **5.** $y(0.4) = 2.162$ **6.** $y(0.4) = 0.441$.

## Problems 32.5, page 1030

**1.** 0.2416 **2.** 1.0408 **3.** 0.6897

**4.** $y(4.4) = 1.019$ **5.** 2.5751 **6.** $y(1.4) = 0.949$.

## Problems 32.6, page 1034

**1.** $y_3 = 1 + \frac{x}{2} + \frac{3}{40}x^5 + \frac{1}{40}x^6 + \frac{1}{192}x^9$ . $z_3 = \frac{1}{2} + \frac{3}{8}x^4 + \frac{1}{10}x^5 + \frac{3}{34}x^8 + \frac{7}{340}x^9 + \frac{1}{256}x^{12}$

**2.** $y(0.1) = 0.105$, $z(0.1) = 0.999$ ; $y(0.2) = 0.22$, $Z(0.2) = 0.997$

**3.** $y(0.1) = 2.0845$, $z(0.1) = 0.5867$ **4.** $y_2 = 1 + \frac{1}{2}x + \frac{3}{40}x^5$

**5.** $y(0.1) = 0.5075$ **6.** $y(0.2) = 0.9802$, $y(0.2) = -0.196$

**7.** $-0.5159$ **8.** $\theta(0.2) = 0.8367$, $(d\theta/dt)_{0.2} = 3.6545$.

## Problems 32.7, page 1038

**1.** 0.14031 **2.** $y(.25) = y(.75) = 2.4$, $y(.5) = 3.2$

**3.** $y(1.25) = 1.3513$, $y(1.5) = 1.635$, $y(1.75) = 1.8505$

**4.** $y(.25) = -0.3473$, $y(.5) = -0.9508$, $y(.75) = -1.7257$

**5.** $n = 2$, $y(0.5) = 0.1389$, true value $= 0.1505$ ; $n = 4$, $y(0.5) = 0.147$

6. $y(0.25) = 0.062, y(0.5) = 0.25, y(0.75) = 0.562$
7. $y(1) = 1.0171, y(2) = 1.094$
8. $y(1/3) = 1.1539, y(2/3) = 3.9231 ; y(1) = 7.4615$.

## Problems 32.8, page 1038

1. (*b*)
2. $y_{i+1} - 2y_i + y_{i-1} + h^2 y_i = 5h^2$
3. $1 + x + x^2 + x^3/6$
4. § 31.4
5. § 31.3
6. (*b*)
7. $y_4 = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$
8. Modified Euler's method
9. $y_1 = y_0 + \frac{h}{24}(55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3})$
10. Milne's method and Adam-Bashforth method
11. four
12. $y = 1 + \frac{x^2}{2} + \frac{x^4}{8}$
13. (*b*)
14. $y_4 = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$
15. $y_1 = y_0 + \frac{h}{24}(9f_1 + 19f_0 - 5f_{-1} + f_{-2})$
16. 1.1818
17. $\frac{dy}{dx} = z, \frac{dz}{dx} + y(1 + yz) = 0$
18. § 31.7
19. starting values
20. Picards and Runge-Kutta methods
21. It agrees with Taylor's series solution upto the term in $h^4$
22. (*d*)
23. $y_{i+1} + 2(h^2 - 1)\, y_i + y_{i-1} = 0$
24. (*a*)
25. True
26. False
27. False.

## Problems 33.1, page 1041

1. Parabolic
2. Hyperbolic
3. (*i*) Parabolic (*ii*) Elliptic (*iii*) Elliptic
4. Outside the ellipse $(x/0.5)^2 + (y/0.25)^2 = 1$.

## Problems 33.2, page 1050

1. $u_1 = 1.999, u_2 = 2.999, u_3 = 3.999$
2. 2.37, 5.6, 9.87, 2.89, 6.14, 9.89, 3.02, 6.17, 9.51
3. $u_1 = 10.188, u_2 = 0.5, u_3 = 1.188, u_4 = 0.25, u_5 = 0.625, u_6 = 1.25$
4. $u_1 = 26.66, u_2 = 33.33, u_3 = 43.33, u_4 = 46.66$
5. $u_1 = 0.99, u_2 = 1.49, u_3 = 0.49$
6. $u_1 = 1.57, u_2 = 3.71, u_3 = 6.57, u_4 = 2.06, u_5 = 4.69, u_6 = 8.06, u_7 = 2, u_8 = 4.92, u_9 = 9$
7. $u_1 = -3, u_2 = -2, u_3 = -2$.

## Problems 33.3, page 1054

1.

| $j \backslash i$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 3 | 4 | 3 | 0 |
| 1 | 0 | 2 | 3 | 2 | 0 |
| 2 | 0 | 1.5 | 2 | 1.5 | 0 |
| 3 | 0 | 1 | 1.5 | 1 | 0 |
| 4 | 0 | 0.75 | 1 | 0.75 | 0 |
| 5 | 0 | 0.5 | 0.75 | 0.5 | 0 |

**2.**

| j \ i | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0.09 | 0.16 | 0.21 | 0.24 | 0.25 | 0.24 | 0.21 | 0.16 | 0.09 | 0 |
| 1 | 0 | 0.08 | 0.15 | 0.20 | 0.23 | 0.24 | 0.23 | 0.20 | 0.15 | 0.08 | 0 |
| 2 | 0 | 0.075 | 0.14 | 0.19 | 0.22 | 0.23 | 0.22 | 0.19 | 0.14 | 0.075 | 0 |
| 3 | 0 | 0.07 | 0.133 | 0.18 | 0.21 | 0.22 | 0.21 | 0.18 | 0.133 | 0.07 | 0 |

**3.**

| j \ i | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 24 | 84 | 144 | 144 | 0 |
| 1 | 0 | 42 | 84 | 114 | 72 | 0 |
| 2 | 0 | 42 | 78 | 78 | 57 | 0 |
| 3 | 0 | 39 | 60 | 67.5 | 39 | 0 |
| 4 | 0 | 30 | 53.25 | 49.5 | 33.75 | 0 |
| 5 | 0 | 26.6 | 39.75 | 43.5 | 24.75 | 0 |
| 6 | 0 | 19.88 | 35.06 | 32.25 | 21.75 | 0 |

**4.**

| j \ i | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0.5 | 1 | 0.5 | 0 |
| 1 | 0 | 0.5 | 0.5 | 0.5 | 0 |
| 2 | 0 | 0.25 | 0.5 | 0.25 | 0 |
| 3 | 0 | 0.25 | 0.25 | 0.25 | 0 |

## Problems 33.4, page 1060

**1.**

| $t = 0.3$ ; $x =$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|---|---|---|---|---|---|
| Numerical sol. $u =$ | 0.02 | 0.04 | 0.06 | 0.075 | 0.08 |
| Exact sol. $u =$ | 0.02 | 0.04 | 0.06 | 0.075 | 0.08 |

**2.**

| j \ i | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 20 | 15 | 10 | 5 | 0 |
| 1 | 0 | 7.5 | 15 | 10 | 5 | 0 |
| 2 | 0 | – 5 | 2.5 | 10 | 5 | 0 |
| 3 | 0 | – 5 | – 10 | – 2.5 | 5 | 0 |
| 4 | 0 | – 5 | – 10 | – 15 | – 7.5 | 0 |
| 5 | 0 | – 5 | – 10 | – 15 | – 20 | 0 |

3.

| $j$ \ $i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 10.19 | 10.16 | 10.21 | 10.24 | 10.25 | 10.24 | 10.21 | 10.16 | 10.09 | 0 |
| 1 | 0 | 5.08 | 10.15 | 10.20 | 10.23 | 10.24 | 10.23 | 10.20 | 10.15 | 5.08 | 0 |
| 2 | 0 | 0.06 | 5.12 | 10.17 | 10.20 | 10.21 | 10.20 | 10.17 | 10.12 | 0.06 | 0 |
| 3 | 0 | 0.04 | 0.08 | 5.12 | 10.15 | 10.16 | 10.15 | 10.12 | 10.08 | 0.04 | 0 |
| 4 | 0 | 0.02 | 0.04 | 0.06 | 5.08 | 10.09 | 10.08 | 10.06 | 10.04 | 0.02 | 0 |
| 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | – 0.02 | 0 |

4.

| $j$ \ $i$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|---|---|---|---|---|---|---|
| 0.1 | 0 | 0.037 | 0.07 | 0.096 | 0.113 | 0.119 |
| 0.2 | 0 | 0.031 | 0.059 | 0.082 | 0.096 | 0.101 |
| 0.3 | 0 | 0.023 | 0.043 | 0.059 | 0.07 | 0.074 |
| 0.4 | 0 | 0.012 | 0.023 | 0.031 | 0.037 | 0.039 |
| 0.5 | 0 | 0 | 0 | 0 | 0 | 0 |

## Problems 33.5, page 1060

**1.** (*b*)
**2.** False
**3.** a hyperbolic equation
**4.** Poisson's equation
**5.** $u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$.
**6.** $(u_{i-1,j} - 2u_{i,j} + u_{i+1,j})/h^2$
**7.** § 32.8 (2)
**8.** $\frac{1}{4}(u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1})$
**9.** hyperbolic
**10.** Bendre-Schmidt
**11.** $u_{i,j+1} = 2(1-4\alpha^2)\, u_{i,j} + 4\alpha^2 (u_{i-1,j} + u_{i+1,j} - u_{i,j-1})$ where $\alpha = k/h$.
**12.** $u_{i,j+1} = \frac{1}{2}\,(u_{i-1,j} + u_{i+1,j})$
**13.** $33.5(2)
**14.** $\lambda < \frac{1}{2}$
**15.** Elliptic
**16.** $u(0, t) = 0 = u\,(1, t), (t > 0)$ ; $u(x, 0) = f(x), 0 < x < l$ ; $\delta x/\delta t\,(x, 0) = 0, 0 < x < l$
**17.** $u_{i,j+1} = u_{i+1,j} - u_{i-1,j} - u_{i,j-1}$
**18.** $\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = f(x, y)$
**19.** 100
**20.** $\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} = 0$
**21.** $y < 0$
**22.** $k = 1/4$.

## Problems 34.1, page 1063

**1.** Max. $Z = 1.2x_1 + 1.4x_2$ ; subject to $40x_1 + 25x_2 \le 1000$,
$35x_1 + 28x_2 \le 980$, $25x_1 + 35x_2 \le 875$ and $x_1, x_2 \ge 0$
**2.** Max. $Z = 3x_1 + 2x_2 + 4x_3$ ; subject to $4x_1 + 3x_2 + 5x_3 \le 2000$,
$2x_1 + 2x_2 + 4x_3 \le 2500$, $100 \le x_1 \le 150$, $200 \le x_2$ and $50 \le x_3$
**3.** Max. $Z = 3x_1 + 2x_2 + x_3$ ; subject to $3x_1 + 4x_2 + 3x_3 \le 42$,
$5x_1 + 3x_3 \le 45$, $3x_1 + 6x_2 + 2x_3 \le 41$ and $x_1, x_2, x_3 \ge 0$.
**4.** Max. $Z = 400\,x + 300\,y$ ; subject to $x + y \le 200$, $x \ge 20$, $y \ge 4x$, $x \ge 0$, $y \ge 0$
**5.** Min. $Z = x_1 + x_2$ ; subject to $2x_1 + x_2 \ge 12$, $5x_1 + 8x_2 \ge 74$,
$x_1 + 6x_2 \ge 24$ and $x_1, x_2, x_3 \ge 0$.

6. Max. $Z = 0.15x_1 + 0.25x_2$ ; subject to
$2x_1 + 5x_2 \le 480{,}000$, $5x_1 + 4x_2 \le 720{,}000$,
$8x_1 + 16x_2 \le 300{,}000$, $0 \le x_1 \le 25{,}000$ and $0 \le x_2 \le 7{,}000$

7. Min. $Z = 41x_1 + 35x_2 + 96x_3$ ; subject to
$2x_1 + 3x_2 + 7x_3 \ge 1250$, $x_1 + x_2 \ge 250$, $5x_1 + 3x_2 \ge 900$,
$6x_1 + 25x_2 + x_3 \ge 232.5$ and $x_1, x_2, x_3 \ge 0$

8. Min. $Z = 100x_1 + 250x_2 + 160x_3$ ; subject to
$0.94x_1 + x_2 + 1.04x_3 \le 0.98$, $10x_1 + 15x_2 + 17x_3 \ge 14$,
$470x_1 + 500x_2 + 520x_3 \ge 495$, $x_1 + x_2 + x_3 = 1$ and $x_1, x_2, x_3 \ge 0$.

## Problems 34.2, page 1069

1. $x_1 = 100, x_2 = 50$ ; max. $Z = 550$
2. $x_1 = 8/15, x_2 = 12/5$, max. $Z = 24.8$
3. $x_1 = 15, x_2 = 0$ ; max. $Z = 300$
4. $x_1 = 1000, x_2 = 500$ ; max. $Z = 5500$
5. 450 units of product $B$ only ; max. profit = ₹ 1800
6. $X = 2, Y = 4.5$ ; max. profit = ₹ 37
7. $A = 1.18$ units, $B = 0.53$ units ; max. profit = ₹ 14.50 approx.
8. 2000/11 units of product $A$ and 1000/11 units of $B$ ; max. profit = ₹ 10,000
9. $x_1 = 4, x_2 = 0$ ; max. $z = 8$
10. Unbounded solution
11. $x_1 = 2, x_2 = 4$ ; min. $Z = 64$
12. Production cost will be min. if G and J run for 12 and 4 days respectively.

## Problems 34.3, page 1074

1. Max. $Z = 3x_1 + 5x_2 + 8x_3$ ; subject to $2x_1 - 5x_2 + s_1 = 6$,
$3x_1 + 2x_2 + x_3 - s_2 = 5$, $3x_1 + 4x_3 + s_3 = 3$ ;
$x_1, x_2, x_3, s_1, s_2, s_3 \ge 0$

2. Min. $Z = 3x_1 + 2x_2 + 5x_3$, subject to $-5x_1 + 2x_2 + s_1 = 5$,
$2x_1 + 3x_2 + 4x_3 - s_2 = 7$, $2x_1 + 5x_3 + s_3 \ge 3$
$x_1, x_2, x_3, s_1, s_2, s_3 \ge 0$.

3. Max. $Z = 3x_1 - 2x_2 + 4x_4 - 4x_5$ ; subject to
$x_1 + 2x_2 + x_4 - x_5 + s_1 = 8$, $2x_1 - x_2 + x_4 - x_5 - s_2 = 2$,
$-4x_1 + 2x_2 + 3x_4 - 3x_5 = 6$ ; $x_1, x_2, x_4, x_5, s_1, s_2 \ge 0$

4. (*i*) $x_1 = 2, x_3 = 1$ (Basic) ; $x_3 = 0$ (Non-basic). (*ii*) $x_1 = 5$,
$x_3 = -1$ (Basic) ; $x_2 = 0$ (Non-basic) ; (*iii*) $x_2 = 5/3, x_3 = 2/3$ (Basic) ; $x_1 = 0$ (Non-basic). All the three basic solutions are non-degenerate

6. Basic solutions are (*i*) $x_1 = 2, x_2 = 1$ (Basic) and $x_3 = 0$ ;
(*ii*) $x_1 = x_3 = 1$ (Basic) and $x_2 = 0$ ; (*iii*) $x_2 = -1, x_3 = 2$ (Basic) and $x_1 = 0$
(*a*) First two solutions are non-degenerate basic feasible solutions
(*b*) First solution is optimal and $Z_{max} = 5$.

## Problems 34.4, page 1081

1. $x_1 = 2, x_2 = 4$, max. $Z = 14$
2. $x_1 = 0, x_2 = 20$ ; max. $Z = 200$
3. $x_1 = 7/3, x_2 = 4/3$ ; max. $Z = 16$
4. $x_1 = 5, x_2 = x_3 = 0$ ; max $Z = 50$
5. $x_1 = 0, x_2 = 100, x_3 = 230$ ; max. $Z = 1350$
6. $x_1 = 89/41, x_2 = 50/41, x_3 = 62/41$ ; max. $Z$ = ₹ 765/41
7. $x_1 = 4, x_2 = 5, x_3 = 0$ ; min $Z = -11$
8. $x_1 = 280/13, x_2 = 0, x_3 = 20/13, x_4 = 180/13$ ; max. $Z = 2280/13$
9. $x_1 = 0, x_2 = 400$ units ; max. profit = ₹ 1200
10. $x_1 = 125, x_2 = 250$ units ; max profit = ₹ 2250
11. $x_1 = 400$ gms, $x_2 = 0$ ; min. cost = ₹ 2
12. $x_1 = 0, x_2 = x_3 = 50$ ; max. profit = ₹ 700

13. $x_1 = 0.5, x_2 = x_3 = 0.04$ units ; min. cost = ₹ 5.80
14. Averages for corn, wheat, soyabeans are 250, 625, zero respectively to achieve a max. profit of ₹ 32,000.

## Problems 34.5, page 1088

1. $x_1 = 0, x_2 = 2, x_3 = 0$ ; max. $Z = 4$
2. $x_1 = 3, x_2 = 2, x_3 = 0$ ; max. $Z = 8$
3. $x_1 = x_2 = -6/15$ ; max. $Z = -48/5$
4. $x_1 = 23/3, x_2 = 5, x_3 = 0$ ; max $Z = 85/3$
5. $x_1 = x_2 = x_3 = 5/2, x_4 = 0$ ; max. $Z = 15$
6. $x_1 = 21/13, x_2 = 10/13$ ; max. $Z = 31/13$
7. Infeasible
8. $x_1 = 23/3, x_2 = 5, x_3 = 0$ ; max. $Z = 85/3$
9. $x_1 = 55/7, x_2 = 30/7, x_3 = 0$ ; max. $Z = 155/7$
10. $x_1 = 2, x_2 = 0$ ; max. $Z = 18$
11. Degenerate solution : $x_1 = 0$ (non-basic) ; $x_2 = 1, x_3 = 0$ (basic) ; max. $Z = 3$.

## Problems 34.6, page 1091

1. Min. $W = 26y_1 + 7y_2$ ; subject to $6y_1 + 4y_2 \geq 10$
   $5y_1 + 2y_2 \geq 13, 3y_1 + 5y_2 \geq 19$ ; $y_1, y_2, y_3 \geq 0$
2. Max. $W = 11y_1 + 7y_2 + y_3 + 5y_4$ ; subject to $3y_1 + 2y_2 - y_3 + 3y_4 \leq 2$,
   $4y_1 + 3y_2 + 2y_3 + 2y_4 \leq 4, y_1 - 2y_2 + 3y_3 + 2y_4 \leq 3$ ;
   $y_1, y_2, y_3, y_4 \geq 0$
3. Min. $W = -3y_1 + y_2 + 4y_3$ ; subject to $y_1 + 3y_2 - 2y_3 \leq -3$,
   $y_1 + y_3 \geq 16, y_1 - 2y_2 + y_3 \leq -7$ ;
   $y_1, y_2 \geq 0, y_3$, unrestricted in sign
4. Max. $W = -5y_1 + 9y_2 + 8y_3$ ; subject to $-2y_1 + 4y_2 - 8y_3 \leq 3$,
   $3y_1 - 2y_2 + 4y_3 \leq -2, -y_1 + 3y_3 = 1$ ;
   $y_1, y_2 \geq 0, y_3$ unrestricted
5. Min. $y = 3y_1 + 4y_2 + y_3 + 6y_4$ ; subject to $5y_1 - 2y_2 + y_3 - 3y_4 \geq 2$,
   $6y_1 + y_2 - 5y_3 - 3y_4 \geq 5, -y_1 + 4y_2 + 3y_3 + 7y_4 \geq 6, y_1, y_2, y_3, y_4 \geq 0$.

## Problems 34.7, page 1094

1. $x_1 = x_2 = 0, x_3 = 5/2$ ; min. $Z = 2.5$
2. $x_1 = 4, x_2 = 2$ ; max. $Z = 10$.
3. $x_1 = 7, x_2 = 0$, max. $Z = 21$
4. $x_1 = 0, x_2 = 100, x_3 = 230$ ; max. $Z = 1350$.

## Problems 34.8, page 1097

1. $x_1 = 0, x_2 = 1$ ; max. $Z = -1$
2. $x_1 = 3/5, x_2 = 6/5$ ; min. $Z = 12/5$
3. $x_1 = 6, x_2 = 2, x_3 = 0$ ; min. $Z = 10$
4. $x_1 = 65/23, x_2 = 0, x_3 = 20/23, x_4 = 0$ ; min. $Z = 215/23$.

## Problems 34.9, page 1104

1. $x_{11} = 200, x_{12} = 50, x_{22} = 175, x_{24} = 125, x_{33} = 275, x_{34} = 125$; min. cost = ₹ 12075
2. $x_{13} = 14, x_{21} = 6, x_{22} = 5, x_{23} = 1, x_{32} = 5$ ; min. cost = 143
3. $x_{11} = 50, x_{12} = 100, x_{21} = 150, x_{33} = 150, x_{42} = 100, x_{43} = 50$ ; min. tonnage = 3300
4. $x_{11} = 140, x_{13} = 60, x_{21} = 40, x_{22} = 120, x_{33} = 90$ ; min. cost = ₹ 5920
5. $x_{11} = 5, x_{14} = 2, x_{22} = 3, x_{23} = 7, x_{32} = 5, x_{34} = 13$ ;
   min. cost = ₹ 799 and maximum saving = ₹ 201
6. $x_{11} = 150, x_{13} = 20, x_{22} = 160, x_{24} = 40, x_{33} = 90, x_{34} = 90$ ; max. profit = 4920
7. $x_{13} = 2, x_{22} = 1, x_{23} = 2, x_{31} = 4, x_{33} = 1$ ; min. cost = 33
8. $x_{13} = 0, x_{15} = 800, x_{21} = 400, x_{24} = 100, x_{32} = 400, x_{33} = 200$,
   $x_{34} = 300, x_{43} = 300$ ; min. cost = 9200.

## Problems 34.10, page 1109

1. $x_{11} = x_{22} = x_{33} = 1$ ; min. cost = ₹ 18
2. $A \to 2, B \to 3, C \to 4, D \to 1$ ; min. $Z = 38$
3. $I \to B, II \to A, III \to D, IV \to C$ ; min. cost = ₹ 49
4. $A \to$ Dyn. Prog., $B \to$ Queuing Th., $C \to$ Reg. Analysis, $D \to$ L.P. ; min. time = 28 hrs
5. (*i*) $A \to I, B \to II, C \to III, D \to IV$ ; (*ii*) $A \to I, B \to III, C \to II, D \to IV$
6. $1 \to IV, 2 \to II, 3 \to VI, 4 \to I, 5 \to III, 6 \to V$ ; max. profit = ₹ 270

## Problems 34.11, page 1110

1. § 34.5 Def. 2
2. it provides an optimality test
3. § 34.11
4. § 34.16 (1)
5. § 34.13
6. § 34.6 (1)
7. Min. $W = 7y_1 + 5y_2$, subject to $2y_1 + 3y_2 \le 4$,
   $3y_1 - 2y_2 \le 9$, $2y_1 + 4y_2 \le 2$, $y_1 \ge 0$, $y_2$ is unrestricted in sign
8. § 34.12 (2)
9. § 34.14
10. Minimize $Z = (2x_{11} + 3x_{12} + 11x_{13} + 4x_{14}) + (5x_{21} + 6x_{22} + 8x_{23} + 7x_{24})$,
    subject to $x_{11} + x_{12} + x_{13} + x_{14} = a_1 (= 15)$, $x_{21} + x_{22} + x_{23} + x_{24} = a_2 (= 20)$,
    $x_{11} + x_{21} = b_1 (= 10)$, $x_{12} + x_{22} = b_2 (= 5)$ ; $x_{13} + x_{23} = b_3 (= 12)$ ; $x_{14} + x_{24} = b_4 (= 8)$ and $x_{ij} \ge 0$.
    [$\because \quad \Sigma a_i = \Sigma b_j = 35$]
11. (*i*) $x_1 = 3, x_2 = 5, x_3 = 0$ ; (*ii*) $x_1 = 0.5, x_2 = 0, x_3 = 2.5$
12. § 34.5 (Def. 4)
13. § 34.15
14. balanced
15. § 34.9
16. § 34.7 (3)
17. optimal
18. Minimize $y = 5y_4 - 3y_3$, subject to $y_4 + y_3 = 5$, $2y_4 - 5y_3 \ge 6$, $y_3 \ge 0$ and $y_4$ unrestricted
19. 5
20. Max. $Z = 5/19$
21. § 34.7
22. § 34.16
23. § 34.7 [2 (*ii*)]
24. *Min.* $W = 2y_1 + 4y_2 + 3y_3$, subject to $-y_1 + y_2 + y_3 \ge 2$, $2y_1 + y_2 \ge 1$, $y_1, y_2 \ge 0$
25. North west corner rule and Vogeli approximation method
26. Slash or surplus variables.

## Problems 35.1, page 1118

1. (*i*) $y = \dfrac{1}{4}x^2 + c_1x + c_2$ ; (*ii*) $y = c_1x^{-1} + c_2$
3. $y = c_1 e^x + c_2 e^{-x} + \dfrac{xe^x}{2}$
4. $y = -x \cos x/2$
5. $y = (c + x) \sin x$
6. $y = \sinh (c_1x + c_2)$
7. $y = x^2 - 1$
8. $(x + c)^2 + y^2 = k^2$
9. $y = 2 \sin x$
12. The spirals of the family $r = a \sec (\phi \sin \alpha + b)$.

## Problems 35.2, page 1120

1. $y = \pm 2 \sin m\pi x$, where $m$ is an integer
2. $y = \lambda x^2 + ax + b$, where $\lambda, a, b$ are determined from the isoperimetric and boundary conditions.
4. $y(x) = \dfrac{1}{2}(1 - \cos x) + \dfrac{1}{4}(2 - \pi) \sin x$.

## Problems 35.3, page 1124

2. $y = (c_1 + c_2x) \cos x + (c_3 + c_4x) \sin x$,
   $z = (c_1 + c_2x) \cos x + (c_3 + c_4x) \sin x - 2c_2 \sin x + 2c_4 \cos x$
3. $y = a_n \sin nx, n = 1, 2, 3 \ldots$
4. $y = \cos x$
5. $y = -\dfrac{\lambda}{24\mu}(x^2 - a^2)^2$

**6.** (*i*) $y = c_1e^{2x} + c_2e^{-2x} + c_3 \cos 2x + c_4 \sin 2x$ ;
(*ii*) $y = c_1 + c_2x + c_3x^2 + c_4x^3 + c_5x^4 + c_6x^5 + x^7/7\,!$.

## Problems 35.4, page 1126

**1.** (*i*) and (*ii*) $\bar{y} = \dfrac{5}{18}x(1-x)$ **2.** $\bar{y} = \dfrac{x}{4}(5x-1)$ **4.** $\bar{y} = 0.58 + 0.27x$
**5.** (*i*) and (*ii*) $c_1 = 0.93, c_2 = -0.05$ **6.** 0.05 **7.** $c_1 = 3.27, c_2 = -2.69$
**8.** $y = \dfrac{1}{2}(5x^2 - 3x)$.

## Problems 36.1, page 1134

**1.** $y(x) = 6x - 5 + \int_0^x (5 - 6x + 6t)\, y(t)\, dt.$

**2.** $y(x) = x - \sin x + e^x (x-1) + \int_0^x [\sin x - e^x (x-t)]\, y(t)\, dt$

**3.** $y(x) = \int_0^x t(t-x)\, y(t)\, dt + \dfrac{1}{2}x^2$

**4.** $y(x) + \int_0^x [1 + x - 2t + (x-t)\, e^{-t}]\, y(t)\, dt = \dfrac{x^5}{20} - \dfrac{5x^3}{6} + x - 3$

**5.** $y(x) = \cos x - \dfrac{1}{2}x^2 - \dfrac{1}{3}x^3 - \dfrac{1}{2}\int_0^x (x-t)^2\, y(t)\, dt$

**6.** $y(x) = \sin x - \dfrac{1}{2} + \int_0^x \left\{\dfrac{1}{2}x(x-t)^2 - 1\right\} y(t)\, dt$

**7.** $y(x) - \int_0^x \{4 - 6(x-t) + 2(x-t)^2 - \dfrac{1}{6}(x-t)^3\}\, y(t)\, dt = \dfrac{1}{4}\cos 2x - \dfrac{19}{12} + \dfrac{32}{3}x - \dfrac{85}{6}x^2 + \dfrac{20}{3}x^3$

**8.** $y''(x) - 2xy'(x) - 3y(x) = 0\,;\, y(0) = 1, y'(0) = 0$

**9.** $y''(x) - y(x) + 3\sin x = 0\,;\, y(0) = 3, y'(0) = 0$

**10.** $y'''(x) + 6y(x) = 0\,;\, y(0) = 4, y'(0) = -3, y''(0) = 2$

**11.** $y'''(x) - 3y''(x) + 4y'(x) + 2y(x) + e^{-x} = 0\,;\, y(0) = 1, y'(0) = 2, y''(0) = 3.$

## Problems 36.2, page 1137

**1.** $y(x) = \int_0^1 G(x,t)\, ty(t)\, dt + \dfrac{1}{2}x(x-1)$, where $G(x,t) = x(1-t), (x < t)$ and $= t(1-x), x > t$

**2.** $y(x) = \int_0^1 G(x,t)\, y(t)\, dt + \dfrac{1}{6}(x^3 - 3x + 6)$, where $G(x,t) = x, x < t$ and $= t, x > t$

**3.** $u(x) = \int_0^1 G(x,t)\, e^t\, u(t)\, dt + \dfrac{1}{6}(x^3 + x)$, where $G(x,t) = x(1-2t), x < t$ and $= t(1-2x), x > t$

**4.** $G(x,t) = \dfrac{\sinh x \sinh (t-1)}{\sinh 1}, x < t$ and $= \dfrac{\sinh t \sinh (x-1)}{\sinh 1}, x > t$

**5.** $u(x) = \lambda \int_0^1 G(x,t)\, t\,.\, u(t)\, dt$, where $G(x,t) = \dfrac{1}{2}x\left(\dfrac{1}{t-t}\right), x < t$ and $= \dfrac{1}{2}\cdot t\left(\dfrac{1}{x-x}\right), x > t$

## Problems 36.3, page 1141

**5.** $y(x) = \dfrac{1}{2}(\sin x + \sinh x)$ **6.** $y(x) = x^2 + \dfrac{1}{12}x^4$ **7.** $y(x) = 1$
**8.** $y(x) = \pm 6J_0(4x)$ **9.** $y(x) = J_1(2x) \quad (x > 0)$

**10.** $y(x) = \frac{1}{2} e^{-2x} (\cos x + 3 \sin x) - \frac{1}{2} e^{-x}$

**11.** $y(x) = (1 - x) e^{-x} + 1/2 \sin x$

**12.** $y(x) = 1 + x^2 + x^4/24.$

**13.** $y(x) = 1 + x^2/2$

**14.** $y(x) = 1/2$

**15.** $y(x) = \frac{3\sqrt{3}}{4\pi} x^{1/3} (3x + 2).$

## Problems 36.4, page 1145

**1.** Has no eigen values and eigen functions

**2.** Eigen value $\lambda = 1/4$ ; eigen function is $y(x) = x^2 + 3x/2$

**3.** Eigen values $\lambda = 8/(\pi - 2)$ ; eigen function is $y(x) = \sin^2 (x)$

**4.** Eigen value $\lambda = 1/\pi, y = \sin x$

**5.** Has no eigen values or eigen functions

**6.** Eigen values are $\lambda = \pm 1/\pi$ ; eigen functions are $y(x) = \cos x + \sin x, y(x) = \cos x - \sin x$

**7.** $y(x) = x + \frac{\lambda [(6 - \lambda) x - 4]}{12 + \lambda^2}$

**8.** $y(x) = x + \frac{\lambda}{12 (1 - 2\lambda) - \lambda^2} [10 + (6 + \lambda) x]$

**9.** $y(x) = x + \frac{\lambda}{(1 - \lambda\pi)\left(1 - \frac{1}{2}\lambda\pi\right) + 4\lambda^2} \left\{2\lambda\pi + \frac{1}{2}\pi^2\left(1 - \frac{1}{2}\lambda\pi\right) + \pi(1 - 2\lambda\pi) \sin x\right\}$

**10.** $y(x) = 2x - \pi + \frac{\pi^2 \sin^2 x}{\pi - 1}$

**11.** $y(x) = \frac{2}{2 - \lambda} \sin x, \lambda \neq 2$

**12.** $y(x) = x + \frac{2\lambda\pi}{1 + 2\lambda^2\pi^2} (\lambda\pi x - 4\lambda\pi \sin x + \cos x)$

**13.** There is no solution to the integral equation when $\lambda = 3$

**14.** $\lambda_1 = 2, \lambda_2 = -2$ ; $y_1(x) = 1 - x, y_2(x) = 1 - 3x$

**15.** (*i*) When $F(x) = x$, solution is $y(x) = x + \lambda \left\{\frac{2\lambda\pi^2}{\lambda^2\pi^2 - 1} \sin x + \frac{2\pi}{\lambda^2\pi^2 - 1} \cos x\right\}$

(*ii*) When $F(x) = 1$, solution is $y(x) = 1$.

## Problems 36.5, page 1148

**1.** $y(x) = 1 - \frac{3\lambda x}{2(3 + \lambda)}$ $(\lambda \neq -3)$

**2.** $y(x) = \frac{\sin x}{1 + \lambda\pi}$ only if $|\lambda| < \frac{1}{\pi}$

**3.** $y(x) = \frac{4 + 2\lambda (2 - 3x)}{4 - \lambda^2}$ $(\lambda \neq 2)$

**4.** $y = e^x$

**5.** $y(x) = 1$

**6.** $y = \sin x$

**7.** $y(x) = 2.$

## Problems 37.1, page 1154

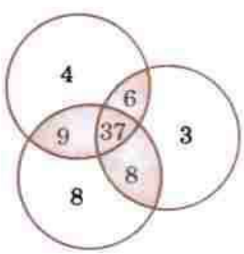
**2.** (*i*) *True*, since $\{a\}$ is a subset of the set $\{a, b, c\}$ ; (*ii*) and (*iii*) *False*, since the element $a$ cannot be a subset of the set $\{a, b, c\}$ ; (*iv*) *True*, since the set $\{a, b\}$ is a subset of the set $\{a, b, c\}$ ; (*v*) *False*, since the set $\{a, b\}$ is not an element of the set $\{a, b, c\}$ ; (*vi*) *True*, since the null set $\phi$ is a subset of every set.

**17.** 20

**18.** 105

**19.** 136

**20.** Number of students not taking any of these courses is 71.



## Problems 37.2, page 1160

**1.** (*a*) It is not true that Sam is a teacher and John is an honest boy ; (*b*) Sam is a teacher and John is not an honest boy ; (*c*) Sam is not a teacher iff John is an honest boy ; (*d*) If Sam is a teacher then John is not an honest boy.

**2.** (*a*) $(p \vee q) \Rightarrow r$ where $p \equiv I$ have no car, $q = I$ do not wear good dress, $r = I$ am not, a millionaire.

**3.**

| $p$ | $q$ | $\sim q$ | $p \Rightarrow q$ | $p \Rightarrow q \wedge \sim q$ |
|---|---|---|---|---|
| 1 | 1 | 0 | 1 | 0 |
| 1 | 0 | 1 | 0 | 0 |
| 0 | 1 | 0 | 1 | 0 |
| 0 | 0 | 1 | 1 | 1 |

(*b*)

| $p$ | $q$ | $r$ | $p \Leftrightarrow q$ | $r \vee q$ | $(p \Leftrightarrow q) \wedge (r \vee q)$ |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 1 | 1 | 0 |
| 1 | 0 | 1 | 0 | 1 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 | 1 | 0 |
| 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 0 | 0 | 1 | 0 | 0 |

**7.** (*i*) $T_p = [2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31]$
$T_q = [1, 3, 9, 27, ...]$, $T_r = [1, 3, 9, 7]$
(*ii*) $T_r \leq T_q$

**10.** (*i*)

| $p$ | $q$ | $q \to p$ | $p \to (q \to p)$ |
|---|---|---|---|
| 0 | 0 | 1 | 1 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 |

**15.** (*i*) Dual of $(p \wedge q) \vee r$ is $(p \wedge q) \vee r$ (*ii*) Dual of $(p \wedge q) \vee t$ is $(p \vee q) \wedge t$

## Problems 37.3, page 1166

**1.** (*a*) $(\forall\, x \in A)\,(x + 2 < 10)$ (*b*) $(1\, x \in A)\,(x + 2 = 10)$

**2.** (*a*) $\forall\, x, (x^3 \neq x)$ (*b*) $\to x, (x + 5 \leq x)$

(*c*) None of the students are 26 or older (*d*) Some students do not live in the hostels.

**3.** $\forall\, x\, P(x)$ is false **4.** $\forall\, (x_1, x_2)\, Q(x_1 + x_2)Q$

**5.** $\forall\, (a, b)\, R(a + x = b)$

**6.** (*a*) $\forall\, x[Q\,(x) \to R(x)]$ (*b*) $\sim \forall\, x[Q(x) - R(x)]$, (*c*) $\exists\, x[Q(x) \wedge R(x)]$, (*d*) $\exists x\, [Q\,(x) \wedge \sim R(x)]$

**8.** $(\sim A \vee \sim A) \wedge (B \vee \sim A) \wedge (\sim A \vee C) \wedge (B \vee C)$

**10.** 1. $p \vee q$ (Premise), 2. $\sim p \to q$ (conditional equivalence)

3. $q \to s$ (Premise) 4. $\sim p \to s$ (2, 3 chain rule)

5. $p \to r$ (Premise) 6. $\sim s \to p$ (4, conditional equivalence)

7. $\sim s \to r$(5, 6 chain rule) 8. $s \vee r$ (7, conditional equivalence)

**12.** $(b\, x)\, R\, (x = \sqrt{Z}\,)$ **13.** (*a*) Conclusion is not valid (*b*) Conclusion is not valid

## Problems 37.4, page 1170

**1.**

| $x$ | $y$ | $z$ | $x \wedge y$ | $z'$ | $y \wedge z'$ | $(x \wedge y) \vee (y \wedge z')$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 | 0 | 0 |
| 1 | 1 | 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 1 | 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 0 | 0 | 1 |

**2.** $x \vee z' \wedge y = x \wedge y$

**3.** (*i*) $x' \vee y' \vee z'$ (*ii*) 0

**14.**

| $x_1$ | $x_2$ | $x_3$ | $x_1 \vee x_3$ | $x_3'$ | $x_2 \vee x_3'$ | $x_1 \wedge (x_2 \vee x_3')$ | $P$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 |
| 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 |
| 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 | 0 | 1 | 0 | 0 |
| 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 |

## Problems 37.5, page 1172

**1.** (*i*) 0 (*ii*) 0

**2.** (*i*) $(x \vee y \vee z) \wedge (x \vee y \vee z')$ (*ii*) $x \vee y \wedge (x \vee y') \wedge (x' \vee y)$

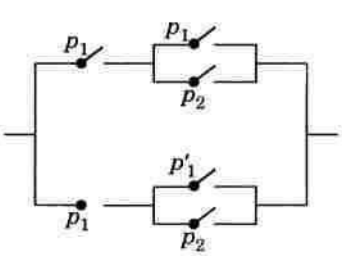
**4.** $(x \vee y \vee z) \wedge (x \vee y \vee z') \wedge (x \vee y' \vee z) \wedge (x \vee y' \vee z') \wedge (x \vee y' \vee z') \wedge (x' \vee y' \vee z) \wedge (x' \vee y' \vee z')$

**5.** $(x \wedge y \wedge z) \vee (x \wedge y \wedge z') \vee (x \wedge y' \wedge z) \vee (x \wedge y' \wedge z')$

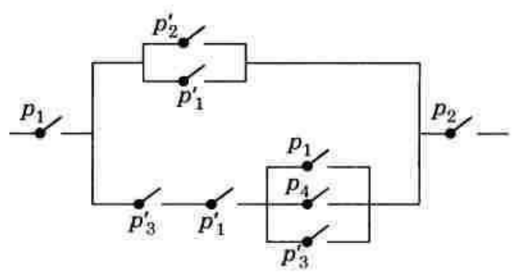
**6.** $F' = (x \wedge y \wedge z) \vee (x \wedge y' \wedge z) \vee (x' \wedge y \wedge z) \vee (x' \wedge y \wedge z') \vee (x' \wedge y' \wedge z)$

## Problems 37.6, page 1174

**1.** (*i*)

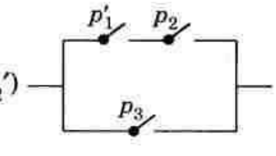



(*ii*)



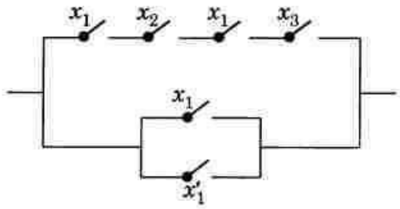

**2.** (*i*) $p_1 \vee [p_2' \wedge (p_1 \vee p_2')]$

(*ii*) $[(p_1 \vee p_2) \vee (p_1 \vee p_3)] \wedge (p_1 \wedge p_2')$




**3.** $x \wedge y$

**4.** $p_1' \wedge p_2 \vee p_3$ ;

**6.** (a) $x_1 \wedge x_2'$ (b) 


(c) $(x_1 \vee x_2' \vee x_3) \wedge (x_1 \vee x_2' \vee x_3') \wedge (x_3 \vee x_2' \vee x_1) \wedge (x_3 \vee x_2' \vee x_1')$

## Problems 37.7, page 1179

**1.** $F_2 = F_3$

**2.** $F^c$ = [0.2 Ram, 0.7 Sham, 0.4 John, 0.3 Charu]

**3.** $F \cup G = [0.4\, x_1, 0.7\, x_2, 0.5\, x_3, 0.9\, x_4]$
$F \cap G = [0.3\, x_1, 0.6\, x_2, 0.1\, x_3, 0.8\, x_4]$

**4.** (i) Truth value of '$F$ is not rich' is 0.2
(ii) Truth value of '$G$ is not fat' is 0.4
(iii) Truth value of 'Mary is not beautiful' is 0.3

**5.** (i) $F \neq G$ (ii) $F$ is not a subset of $G$ ; $G$ is not a subset of $F$.
(iii) $F^c = [1, 1, 1, 1, 0.9, 0.7, 0.5, 0.1, 0, 0]$
$F \cap G = [0, 0.1, 0.3, 0.5]$
$F \cup G = [0.1, 0.5, 0.9, 1, 0.9, 0.9, 1, 1]$

**6.** (i) Truth value of the cojunction of 'Latif and John are good players' is 0.6.
(ii) Truth value of the disjunction of 'Latif and John are good players' is 0.7.

**8.** Members and its degree of membership.

## Problems 38.1, page 1186

**1.** (i) $\dfrac{d\phi}{dt} = \dfrac{d\phi}{\partial x^i} \cdot \dfrac{dx^i}{dt}$ ; (ii) $x^i x^i$

**2.** (i) $a_{11}(x^1)^2 + a_{22}(x^2)^2 + a_{33}(x^3)^2 + (a_{12} + a_{21})\, x^1x^2 + (a_{13} + a_{31})x^1x^3 + (a_{23} + a_{32})x^2x^3$
(ii) $g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2 + 2g_{12}dx^1dx^2 + 2g_{23}dx^2dx^3 + 2g_{31}dx^3\,dx^1$.
(iii) $g_{11} = g_{12}\, g^{2p} + \ldots + g_{ln}\, g^{np}$

**3.** (i) $\delta^i_h$ ; (ii) $\delta^p_s$.

**6.** (i) $\bar{A}_p^{\;qr} = \dfrac{\partial \bar{x}^q}{\partial x^j}\dfrac{\partial \bar{x}^r}{\partial x^k}\dfrac{\partial x^i}{\partial \bar{x}^p} A_i^{jk}$ ; (ii) $\bar{C}_{pq} = \dfrac{\partial x^m}{\partial \bar{x}^p}\dfrac{\partial x^n}{\partial \bar{x}^q} C_{mn}$

**7.** Yes, $A^{ij}{}_{hl}{}^{m}$, contravariant order 3, covariant order 2, Rank 5

**9.** (a) $2\rho \cos^2\phi - z\cos\phi + \rho^3 \sin^2\phi \cos^2\phi$, $-\rho^2 \sin 2\phi + \rho z \sin\phi + \rho^4 \sin\phi \cos^3\phi$, $\rho z \sin\phi$ ;
(b) $2r \sin^2\theta \cos^2\phi - r\sin\theta\cos\theta\cos\phi + r^3 \sin^4\theta \sin^2\phi \cos^2\phi + r^2 \sin\theta \cos^2\theta \sin\phi$ ;
$2r^2 \sin\theta\cos\theta\cos^2\phi - r^2\cos^2\theta\cos\phi + r^4 \sin^3\theta\cos\theta\sin^2\phi\cos^2\phi - r^3\sin^2\theta\cos\theta\sin\phi$ ;
$-2r^2 \sin^2\theta \sin\phi\cos\phi + r^2\sin\theta\cos\theta\sin\phi + r^4\sin^4\theta\sin\phi\cos^3\phi$

**11.** $(a\cos\phi + b\sin\phi)\sin\theta + c\cos\theta$ ; $[(a\cos\phi + b\sin\phi)\cos\theta - c\sin\theta]/r$ ; $(b\cos\phi - a\sin\phi)/r\sin\theta$.

## Problems 38.2, page 1189

**6.** Rank = 1

## Problems 36.3, page 1193

**1.** $g = 4, g^{11} = 2, g^{22} = 5, g^{33} = 1.5, g^{12} = 3, g^{23} = -2.5, g^{13} = -1.5$,

**2.** $g_{11} = 1, g_{22} = \rho^2, g_{33} = 1, g_{ij} = 0\ (i \neq j)$ ; $g^{11} = 1, g^{22} = \rho^{-2}, g^{33} = 1, g^{ij} = 0\ (i \neq j)$

**3.** $g = r^4 \sin^2\theta/(1 - r^2/R^2)$ ; $g^{11} = 1 - r^2/R^2, g^{22} = 1/r^2, g^{33} = (r\sin\theta)^{-2}, g^{ij} = 0\ (i \neq j)$.

## Problems 38.4, page 1199

**1.** (*a*) $[ii, i] = \frac{1}{2}\partial g^{ij}/\partial x^i$, $[ii, k] = -\frac{1}{2}\partial g_{ii}/\partial x^k$,

$[ik, k] = [ki, k] = \frac{1}{2}\partial g_{kk}/\partial x^i$, $[ij, k] = 0$, when $i, j, k$ are all different

(*b*) $\begin{Bmatrix} i \\ ii \end{Bmatrix} = \frac{1}{2} g^{ij} \frac{\partial g_{ii}}{\partial x^i}, \begin{Bmatrix} k \\ ii \end{Bmatrix} = -\frac{1}{2} g^{kk} \frac{\partial g_{ii}}{\partial x^k}$

$\begin{Bmatrix} k \\ ik \end{Bmatrix} = \begin{Bmatrix} k \\ ki \end{Bmatrix} = \frac{1}{2} g^{kk} \frac{\partial g_{kk}}{\partial x^i}$ (no summation over $i$ or $k$)

$\begin{Bmatrix} k \\ ij \end{Bmatrix} = 0$, when $i, j, k$ are all different

**2.** (*a*) All are zero

(*b*) $[21, 2] = \rho = [12, 2]$ ; $[22, 1] = \rho$, all others are zero

(*c*) $[21, 2] = r = [12, 2]$ ; $[31, 3] = r \sin^2 \theta = [13, 3]$ ; $[32, 3]$
$= r^2 \sin \theta \cos \phi = [23, 3]$ ; $[22, 1] = -r$ ; $[33, 1] = -r \sin^2 \theta$ ;
$[33, 2] = -r^2 \sin \theta \cos \phi$ ; all others are zero

**3.** (*a*) All are zero

(*b*) $\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = -\rho, \begin{Bmatrix} 2 \\ 21 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \frac{1}{\rho}$, all others are zero

(*c*) $\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = -r, \begin{Bmatrix} 1 \\ 33 \end{Bmatrix} = -r \sin^2 \theta, \begin{Bmatrix} 2 \\ 21 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \frac{1}{r}$,

$\begin{Bmatrix} 2 \\ 33 \end{Bmatrix} = -\sin \theta \cos \theta, \begin{Bmatrix} 3 \\ 31 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 13 \end{Bmatrix} = \frac{1}{r}$,

$\begin{Bmatrix} 3 \\ 32 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 23 \end{Bmatrix} = \cot \theta$, all others are zero

**5.** (*a*) $r^2 \sin \theta \cos \theta$ ; $r^2 \sin \theta \cos \theta$ (*b*) $-\sin \theta \cos \theta$ ; $\cot \theta$

**6.** (*a*) $-r \sin^2 \theta$ ; $r^2 \sin \theta \cos \theta$ (*b*) $-r \sin^2 \theta$ ; $\cot \theta$

**8.** (*a*) $u^{ij}{}_{,k} = \frac{\partial u^{ij}}{\partial x^k} + \begin{Bmatrix} i \\ ks \end{Bmatrix} u^{sj} + \begin{Bmatrix} i \\ ks \end{Bmatrix} u^{is}$ ; (*b*) $A^h{}_{ij,\,k} = \frac{\partial A^h{}_{ij}}{\partial x^k} - \begin{Bmatrix} s \\ ik \end{Bmatrix} A^h{}_{sj} + \begin{Bmatrix} h \\ ks \end{Bmatrix} A^s{}_{ij}$

**10.** $A^j{}_{k,\,q} B_n{}^{lm} + A^j{}_k B^{lm}{}_{n,\,q}$

**11.** (*a*) $\frac{1}{\rho}\left[\frac{\partial}{\partial p}(\rho A_\rho) + \frac{\partial}{\partial \phi}(A_\phi) + \frac{\partial}{\partial z}(\rho A_z)\right]$ ; (*b*) $\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r \sin \theta}\frac{\partial}{\partial r}(\sin \theta\, A_\theta) + \frac{1}{r \sin \theta}\frac{\partial A_\phi}{\partial \phi}$

**12.** (*a*) $\frac{\partial^2 v}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2 v}{\partial \phi^2} + \frac{\partial^2 v}{\partial z^2} + \frac{1}{\rho}\frac{\partial v}{\partial \rho} = 0$ (*b*) $\frac{\partial^2 v}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{1}{r^2 \sin^2 \theta}\frac{\partial^2 v}{\partial \phi^2} + \frac{2}{r}\frac{\partial v}{\partial r} + \frac{\cot \theta}{r^2}\frac{\partial v}{\partial \theta} = 0.$

# INDEX

## A



## B



## C

## D



## E



## F

## G



## H



## I



## J



## K



## L



## M

## T



## U



## V



## W



## Z

# Higher Engineering Mathematics

Dr. B.S. Grewal

The book provides a clear exposition of essential tools of applied mathematics from a modern point of view and meets complete requirements of engineering and computer science students. Every effort has been made to keep the presentation at once simple and lucid. It is written with the firm conviction that a good book is one that can be read with minimum guidance from the instructor. To achieve this, more than the usual number of solved examples, followed by properly graded problems have been given. Many of the examples and problems have been selected from recent papers of various university and other engineering examinations. Basic Concepts and Useful Information has been given in an Appendix. However, the subject matter has been set in eight main units:

- ***Algebra & Geometry*** : Solution of equations, Linear algebra: Deter-minants, Matrices, Vector algebra and Solid geometry.
- ***Calculus*** : Differential calculus, Partial differentiation, Integral calculus, Multiple integrals, Vector calculus.
- ***Series*** : Infinite series and Fourier series.
- ***Differential Equations*** : Differential equations of first order and their applications, Linear differential equations and their applications, Differential equations of different types, Series solution of differential equations and special functions, Partial differential equations and their applications.
- ***Complex Analysis*** : Complex numbers and functions, Calculus of complex functions.
- ***Transforms*** : Laplace transforms, Fourier transforms and Z-transforms.
- ***Numerical Techniques*** : Empirical Laws and Curve fitting, Statistical methods, Probability and Distributions, Sampling and Inference, Numerical methods, Finite differences and Interpolation, Difference equations, Numerical solution of Ordinary and Partial differential equations, Linear programming.
- ***Special Topics*** : Calculus of variations, Integral equations, Discrete mathematics, Tensors.

An exhaustive list of *'Objective Type of Questions'* has been given at the end of each chapter. *Standard Tables, Answers to Problems,* and a fairly comprehensive *Index* is given at the end.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