6

FINITE DIFFERENCES

6.1. INTRODUCTION

The calculus of finite differences deals with the changes that take place in the value of the function (dependent variable), *due to finite changes in the independent variable*. Through this, we also study the relations that exist between the values assumed by the function, whenever the independent variable changes by finite jumps whether equal or unequal. On the other hand, in infinitesimal calculus, we study those changes of the function which occur when the independent variable changes continuously in a given interval. In this chapter, we shall study the variations in the function when the independent variable changes by equal intervals.

6.2. FINITE DIFFERENCES

Suppose that the function $y = f(x)$ is tabulated for the equally spaced values $x = x_0, x_0 + h, x_0 + 2h, \ldots\ldots, x_0 + nh$ giving $y = y_0, y_1, y_2, \ldots\ldots, y_n$. To determine the values of $f(x)$ or $f'(x)$ for some intermediate values of x, the following three types of differences are found useful :

(1) Forward differences. The differences $y_1 - y_0, y_2 - y_1, \ldots\ldots, y_n - y_{n-1}$ when denoted by $\Delta y_0, \Delta y_1, \ldots\ldots, \Delta y_{n-1}$ respectively are called the *first forward differences* where Δ is the *forward difference operator*. Thus the first forward differences are $\Delta y_r = y_{r+1} - y_r$.

Similarly the second forward differences are defined by $\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r$

In general, $\Delta^p y_r = \Delta^{p-1} y_{r+1} - \Delta^{p-1} y_r$ defines the *pth forward differences*.

These differences are systematically set out in Table 6.1.

In a difference table, x is called the *argument* and y the *function* or the *entry*. y_0, the first entry is called the *leading term* and $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0$ etc. are called the *leading differences*.

Table 6.1. Forward Difference Table

Value of x	*Value of y*	*1st diff.*	*2nd diff.*	*3rd diff.*	*4th diff.*	*5th diff.*
x_0	y_0					
		Δy_0				
x_0+h	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
x_0+2h	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
x_0+3h	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
x_0+4h	y_4		$\Delta^2 y_3$			
		Δy_4				
x_0+5h	y_5					

Obs. 1. *Any higher order forward difference can be expressed in terms of the entries.*

We have
$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0$$
$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0) = y_3 - 3y_2 + 3y_1 - y_0$$
$$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0 = (y_4 - 3y_3 + 3y_2 - y_1) - (y_3 - 3y_2 + 3y_1 - y_0)$$
$$= y_4 - 4y_3 + 6y_2 - 4y_1 + y_0$$

The coefficients occurring on the right hand side being the binomial coefficients, we have in general, $\Delta^n y_0 = y_n - {}^n c_1\, y_{n-1} + {}^n c_2\, y_{n-2} - \ldots\ldots + (-1)^n\, y_0$.

Obs. 2. *The operator Δ obeys the distributive, commutative and index laws*

i.e. (*i*) $\Delta[f(x) \pm \phi(x)] = \Delta f(x) \pm \Delta\phi(x)$

(*ii*) $\Delta[cf(x)] = c\, \Delta f(x)$, c being a constant.

(*iii*) $\Delta^m\, \Delta^n\, f(x) = \Delta^{m+n} f(x)$, m and n being positive integers. In view of (*i*) and (*ii*), Δ is a *linear operator*.

But $\Delta[f(x)\,.\,\phi(x)] \neq f(x)\,.\,\Delta\phi(x)$.

(2) Backward differences. The differences $y_1 - y_0, y_2 - y_1, \ldots\ldots, y_n - y_{n-1}$ when denoted by $\nabla y_1, \nabla y_2, \ldots\ldots, \nabla y_n$ respectively, are called the *first backward differences* where ∇

Table 6.2. Backward Difference Table

Value of x	*Value of y*	*1st diff.*	*2nd diff.*	*3rd diff.*	*4th diff.*	*5th diff.*
x_0	y_0					
		∇y_1				
x_0+h	y_1		$\nabla^2 y_2$			
		∇y_2		$\nabla^3 y_3$		
x_0+2h	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$	
		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$
x_0+3h	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$	
		∇y_4		$\nabla^3 y_5$		
x_0+4h	y_4		$\nabla^2 y_5$			
		∇y_5				
x_0+5h	y_5					

is the *backward difference operator*. Similarly we define higher order backward differences. Thus we have $\nabla y_r = y_r - y_{r-1}$, $\nabla^2 y_r = \nabla y_r - \nabla y_{r-1}$, $\nabla^3 y_r = \nabla^2 y_r - \nabla^2{}_{r-1}$, etc.

These differences are exhibited in the Table 6.2.

(3) Central differences. Sometimes it is convenient to employ another system of differences known as *central differences*. In this system, the *central difference operator* δ is defined by the relations : $y_1 - y_0 = \delta y_{1/2}$, $y_2 - y_1 = \delta y_{3/2}$, ..., $y_n - y_{n-1} = \delta y_{n-1/2}$

Similarly, higher order central differences are defined as

$$\delta y_{3/2} - \delta y_{1/2} = \delta^2 y_1,\ \delta y_{5/2} - \delta y_{3/2} = \delta^2 y_2,\ \ldots\ldots,\ \delta^2 y_2 - \delta^2 y_1 = \delta^3 y_{3/2} \text{ and so on.}$$

These differences are shown in Table 6.3.

Table 6.3. Central Difference Table

Value of x	*Value of y*	*1st diff.*	*2nd diff.*	*3rd diff.*	*4th diff.*	*5th diff.*
x_0	y_0					
		$\delta y_{1/2}$				
$x_0 + h$	y_1		$\delta^2 y_1$			
		$\delta y_{3/2}$ 1.5		$\delta^3 y_{3/2}$		
$x_0 + 2h$	y_2		$\delta^2 y_2$		$\delta^4 y_2$	
		$\delta y_{5/2}$ 2.5		$\delta^3 y_{5/2}$		$\delta^5 y_{5/2}$
$x_0 + 3h$	y_3		$\delta^2 y_3$		$\delta^4 y_3$	
		$\delta y_{7/2}$ 3.5		$\delta^3 y_{7/2}$		
$x_0 + 4h$	y_4		$\delta^2 y_4$			
		$\delta y_{9/2}$ 4.5				
$x_0 + 5h$	y_5					

We see from this table that the central differences on the same horizontal line have the same suffix. Also the differences of odd order are known only for half values of the suffix and those of even order for only integral values of the suffix.

It is often required to find the mean of adjacent values in the same column of differences. We denote this mean by μ. Thus $\mu\delta y_1 = \frac{1}{2}(\delta y_{1/2} + \delta y_{3/2})$, $\mu\delta^2 y_{3/2} = \frac{1}{2}(\delta^2 y_1 + \delta^2 y_2)$ etc.

Obs. *The reader should note that it is only the notation which changes and not the differences. e.g.*

$$y_1 - y_0 = \nabla y_1 = \Delta y_0 = \delta y_{1/2}.$$

Of all the formulae, those involving central differences are most useful in practice as the coefficients in such formulae decrease much more rapidly.

Example 6.1. *Evaluate (i)* $\Delta \tan^{-1} x$ *(ii)* $\Delta (e^x \log 2x)$ *(iii)* $\Delta (x^2 / \cos 2x)$ *(iv)* $\Delta^2 \cos 2x$.

(P.T.U., B. Tech., 2001)

Sol. (*i*) $\Delta \tan^{-1} x = \tan^{-1}(x+h) - \tan^{-1} x$

$$= \tan^{-1}\left\{\frac{x+h-x}{1+(x+h)x}\right\} = \tan^{-1}\left\{\frac{h}{1+hx+x^2}\right\}$$

(*ii*) $\Delta (e^x \log 2x) = e^{x+h} \log 2(x+h) - e^x \log 2x$

$$= e^{x+h} \log 2(x+h) - e^{x+h} \log 2x + e^{x+h} \log 2x - e^x \log 2x$$

6.6. OTHER DIFFERENCE OPERATORS

We have already introduced the operators Δ, ∇ and δ. Besides these, there are the operators E and μ, which we define below :

(1) Shift operator E *is the operation of increasing the argument x by h so that* $Ef(x) = f(x+h)$, $E^2 f(x) = f(x+2h)$, $E^3 f(x) = f(x+3h)$ etc.

The inverse operator E^{-1} is defined by $E^{-1} f(x) = f(x-h)$

If y_x is the function $f(x)$, then $Ey_x = y_{x+h}$, $E^{-1}y_x = y_{x-h}$, $E^n y_x = y_{x+nh}$, where n may be any real number.

(2) Averaging operator μ is defined by the equation $\mu y_x = \frac{1}{2}(y_{x+\frac{1}{2}h} + y_{x-\frac{1}{2}h})$

Obs. *In the difference calculus E is regarded as the fundamental operator and* $\Delta, \nabla, \delta, \mu$ *can be expressed in terms of E.*

6.7. RELATIONS BETWEEN THE OPERATORS

(1) We shall now establish the following identities :

(*i*) $\mathbf{\Delta = E - 1}$ (*ii*) $\mathbf{\nabla = 1 - E^{-1}}$

(*iii*) $\mathbf{\delta = E^{1/2} - E^{-1/2}}$ (*iv*) $\mathbf{\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})}$

(*v*) $\mathbf{\Delta = E\nabla = \nabla E = \delta E^{1/2}}$ (*vi*) $\mathbf{E = e^{hD}}$.

Proofs. (*i*) $\Delta y_x = y_{x+h} - y_x = Ey_x - y_x = (E-1)y_x$

This shows that the operators Δ and E are connected by the symbolic relation

$$\Delta = E - 1 \quad \text{or} \quad \mathbf{E = 1 + \Delta}.$$

Obs. *These relations imply that the effect of operator E on* y_x *is the same as that of the operator* $(1+\Delta)$ *on* y_x. *The operator's E and* Δ *do not have any existence as separate entities.*

(*ii*) $$\nabla y_x = y_x - y_{x-h} = y_x - E^{-1} y_x = (1 - E^{-1})y_x$$

$$\therefore \quad \nabla = 1 - E^{-1}$$

(*iii*) $$\delta y_x = y_{x+\frac{1}{2}h} - y_{x-\frac{1}{2}h} = E^{1/2} y_x - E^{-1/2} y_x = (E^{1/2} - E^{-1/2})y_x$$

$$\delta = E^{1/2} - E^{-1/2}.$$

(*iv*) $$\mu y_x = \frac{1}{2}(y_{x+\frac{1}{2}h} + y_{x-\frac{1}{2}h}) = \frac{1}{2}(E^{1/2}y_x + E^{-1/2}y_x) = \frac{1}{2}(E^{1/2} + E^{-1/2})y_x$$

$$\therefore \quad \mu = \frac{1}{2}(E^{1/2} + E^{-1/2}).$$

(*v*) $$E\nabla y_x = E(y_x - y_{x-h}) = Ey_x - Ey_{x-h} = y_{x+h} - y_x = \Delta y_x$$

$$\therefore \quad E\nabla = \Delta$$

$$\nabla E y_x = \nabla y_{x+h} = y_{x+h} - y_x = \Delta y_x$$

$$\therefore \quad \nabla E = \Delta$$

$$\delta E^{1/2} y_x = \delta y_{x+\frac{1}{2}h} = y_{x+\frac{1}{2}h+\frac{1}{2}h} - y_{x+\frac{1}{2}h-\frac{1}{2}h} = y_{x+h} - y_x = \Delta y_x$$

$$\therefore \quad \delta E^{1/2} = \Delta$$

Hence $$\Delta = E\nabla = \nabla E = \delta E^{1/2}.$$

(*vi*) $$Ef(x) = f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \ldots\ldots$$ [by Taylor's series]

$$= f(x) + hDf(x) + \frac{h^2}{2!} D^2 f(x) + \ldots\ldots$$

$$= \left(1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \ldots\ldots\right) f(x) = e^{hD} f(x)$$

$\therefore \quad E = e^{hD}$

Cor. $\quad E = 1 + \Delta = e^{hD}$.

Note. *A table showing the symbolic relations between the various operators is given below for ready reference. To prove such relations between the operators, always express each operator in terms of the fundamental operator E.*

(2) Relations between the various operators

In terms of	E	Δ	∇	δ	hD
E	—	$\Delta + 1$	$(1 - \nabla)^{-1}$	$1 + \frac{1}{2}\delta^2 + \delta\sqrt{(1 + \delta^2/4)}$	e^{hD}
Δ	$E - 1$	—	$(1 - \nabla)^{-1} - 1$	$\frac{1}{2}\delta^2 + \delta\sqrt{(1 + \delta^2/4)}$	$e^{hD} - 1$
∇	$1 - E^{-1}$	$1 - (1 + \Delta)^{-1}$	—	$-\frac{1}{2}\delta^2 + \delta\sqrt{(1 + \delta^2/4)}$	$1 - e^{-hD}$
δ	$E^{1/2} - E^{-1/2}$	$\Delta(1 + \Delta)^{-1/2}$	$\nabla(1 - \nabla)^{-1/2}$	—	$2 \sinh (hD/2)$
μ	$\frac{1}{2}(E^{1/2} + E^{-1/2})$	$(1 + \Delta/2)(1 + \Delta)^{-1/2}$	$(1 + \nabla/2)(1 + \nabla)^{-1/2}$	$\sqrt{(1 + \delta^2/4)}$	$\cosh (hD/2)$
hD	$\log E$	$\log (1 + \Delta)$	$\log (1 - \nabla)^{-1}$	$2 \sinh^{-1} (\delta/2)$	—

Example 6.11. *Prove that* $e^x = \left(\frac{\Delta^2}{E}\right) e^x \cdot \frac{Ee^x}{\Delta^2 e^x}$*, the interval of differencing being h.*

(Bhopal, B.E., 2009)

Sol. Since $\left(\frac{\Delta^2}{E}\right) e^x = \Delta^2 \cdot E^{-1} e^x = \Delta^2 e^{x-h} = \Delta^2 e^x \cdot e^{-h} = e^{-h} \Delta^2 e^x$

$\therefore$ R.H.S. $\quad = e^{-h} \Delta^2 e^x \cdot \frac{Ee^x}{\Delta^2 e^x} = e^{-h} Ee^x = e^{-h} \cdot e^{x+h} = e^x$.

Example 6.12. *Prove with the usual notations, that*

(i) $hD = \log (1 + \Delta) = -\log (1 - \nabla) = \sinh^{-1} (\mu\delta)$ *(M.D.U., B. Tech., 2005)*

(ii) $(E^{1/2} + E^{-1/2})(1 + \Delta)^{1/2} = 2 + \Delta$ *(Bhopal, B. Tech., 2009)*

(iii) $\Delta - \nabla = \Delta\nabla = \delta^2$ *(Mumbai, B.E., 2005)*

(iv) $\Delta^3 y_2 = \nabla^3 y_5$.

Sol. (i) We know that $\quad e^{hD} = E = 1 + \Delta$

$\therefore \quad hD = \log (1 + \Delta)$

Also $\quad hD = \log E = -\log (E^{-1}) = -\log (1 - \nabla) \quad [\because \; E^{-1} = 1 - \nabla]$

We have proved that $\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$

and $\delta = E^{1/2} - E^{-1/2}$

$\therefore$ $\mu\delta = \frac{1}{2}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2})$

$$= \frac{1}{2}(E - E^{-1}) = \frac{1}{2}(e^{hD} - e^{-hD}) = \sinh(hD)$$

i.e. $hD = \sinh^{-1}(\mu\delta)$.

Hence $hD = \log(1 + \Delta) = -\log(1 - \nabla) = \sinh^{-1}(\mu\delta)$.

(*ii*) $(E^{1/2} + E^{-1/2})(1 + \Delta)^{1/2}$

$$= (E^{1/2} + E^{-1/2})E^{1/2} = E + 1 = 1 + \Delta + 1 = 2 + \Delta.$$

(*iii*) We know that $\Delta = E - 1$, $\nabla = 1 - E^{-1}$ and $\delta = E^{1/2} - E^{-1/2}$

$\therefore$ $\Delta - \nabla = E - 2 + E^{-1} = (E^{1/2} - E^{-1/2})^2 = \delta^2$

Also $\Delta\nabla = (E - 1)(1 - E^{-1}) = E + E^{-1} - 2$

$$= (E^{1/2} - E^{-1/2})^2 = \delta^2.$$

Hence $\Delta - \nabla = \Delta\nabla = \delta^2$.

(*iv*) $\Delta^3 y_2 = (E - 1)^3 y_2$ [$\because$ $\Delta = E - 1$

$$= (E^3 - 3E^2 + 3E - 1)y_2$$

$$= y_5 - 3y_4 + 3y_3 - y_2 \quad ...(1)$$

$\nabla^3 y_5 = (1 - E^{-1})^3 y_5$ [$\because$ $\nabla = 1 - E^{-1}$

$$= (1 - 3E^{-1} + 3E^{-2} - E^{-3})y_5$$

$$= y_5 - 3y_4 + 3y_3 - y_2 \quad ...(2)$$

From (1) and (2), $\Delta^3 y_2 = \nabla^3 y_5$.

Example 6.13. *Prove that*

(*i*) $\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{\left(1 + \frac{\delta^2}{4}\right)}$ *(Mumbai, B.E., 2004)*

(*ii*) $1 + \delta^2\mu^2 = \left(1 + \frac{1}{2}\delta^2\right)$

(*iii*) $\mu = \frac{2 + \Delta}{2\sqrt{(1 + \Delta)}} = \sqrt{\left(1 + \frac{1}{4}\delta^2\right)}$

Sol. (*i*) $\frac{1}{2}\delta^2 + \delta\sqrt{\left(1 + \frac{\delta^2}{4}\right)}$ [$\because$ $\delta = E^{1/2} - E^{-1/2}$]

$$= \frac{1}{2}(E^{1/2} - E^{-1/2})^2 + (E^{1/2} - E^{-1/2})\sqrt{[1 + (E^{1/2} - E^{-1/2})^2/4]}$$

$$= \frac{1}{2}(E + E^{-1} - 2) + (E^{1/2} - E^{-1/2})\sqrt{[(E + E^{-1} + 2)/4]}$$

$$= \frac{1}{2}(E + E^{-1} - 2) + (E^{1/2} - E^{-1/2})(E^{1/2} + E^{-1/2})/2$$

$$= \frac{1}{2}[(E + E^{-1} - 2) + (E - E^{-1})] = E - 1 = \Delta.$$

(*ii*) We know that $\delta = E^{1/2} - E^{-1/2}$ and $\mu = (E^{1/2} + E^{-1/2})/2$.

$\therefore$ L.H.S. $= 1 + \delta^2\mu^2 = 1 + (E^{1/2} - E^{-1/2})^2\,(E^{1/2} + E^{-1/2})^2/4$

$$= \frac{1}{4}\,[4 + (E - E^{-1})^2] = \frac{1}{4}\,(E^2 + E^{-2} + 2) = \frac{1}{4}\,(E + E^{-1})^2$$

R.H.S. $= (1 + \frac{1}{2}\,\delta^2)^2 = [1 + \frac{1}{2}\,(E^{1/2} - E^{-1/2})^2]^2 = [1 + \frac{1}{2}\,(E + E^{-1} - 2)]^2$

$$= \frac{1}{4}\,(E + E^{-1})^2$$

Hence $1 + \delta^2\mu^2 = \left(1 + \frac{1}{2}\delta^2\right)^2$.

(*iii*) Since $\Delta = E - 1$, $\delta = E^{1/2} - E^{-1/2}$ and $\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$

$$\therefore \quad \frac{2+\Delta}{2\sqrt{(1+\Delta)}} = \frac{2+E-1}{2\sqrt{(1+E-1)}} = \frac{E+1}{2\sqrt{E}}$$

$$= \frac{1}{2}(E^{1/2} + E^{-1/2}) = \mu \qquad \text{...(1)}$$

Also $\sqrt{\left(1+\frac{1}{4}\delta^2\right)} = \sqrt{\left[1+\frac{1}{4}(E^{1/2} - E^{-1/2})^2\right]} = \sqrt{\left[1+\frac{1}{4}(E + E^{-1} - 2)\right]}$

$$= \frac{1}{2}\sqrt{(E + E^{-1} + 2)} = \frac{1}{2}(E^{1/2} + E^{-1/2}) = \mu \qquad \text{...(2)}$$

Hence from (1) and (2), we get

$$\mu = \frac{2+\Delta}{2\sqrt{(1+\Delta)}} = \sqrt{\left(1+\frac{1}{4}\delta^2\right)}.$$

Example 6.14. *Prove that* $\nabla y_{n+1} = h\left(1 + \frac{1}{2}\nabla + \frac{5}{12}\nabla^2 + \ldots\ldots\right)y'_n$

Sol. We have $\nabla_{y+1} = y_{n+1} - y_n = (E - 1)\,y_n$

$$= (e^{hD} - 1)\,y_n = \left(1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \ldots\ldots - 1\right)y_n$$

$$= hD\left(1 + \frac{hD}{2!} + \frac{h^2D^2}{3!} + \ldots\ldots\right)y_n$$

$$= h\left(1 + \frac{hD}{2!} + \frac{h^2D^2}{3!} + \ldots\ldots\right)Dy_n$$

Since $E^{-1} = 1 - \nabla = e^{-hD}$, $\therefore$ $hD = -\log(1 - \nabla) = \nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \ldots\ldots$

$$\therefore \quad \nabla y_{n+1} = h\left\{1 + \frac{1}{2}\left(\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \ldots\ldots\right) + \frac{1}{6}\left(\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \ldots\ldots\right)^2 + \ldots\ldots\right\} y'_n$$

Hence $$\nabla y_{n+1} = h\left(1 + \frac{1}{2}\nabla + \frac{5}{12}\nabla^2 + \ldots\ldots\right) y'_n .$$

6.8 TO FIND ONE OR MORE MISSING TERMS

When one or more values of $y = f(x)$ corresponding to the equidistant values of x are missing, we can find these using any of the following two methods :

First method: We assume the missing term or terms as a, b etc. and form the difference table. Assuming the last difference as zero, we solve these equations for a, b. These give the missing term/terms.

Second method: If n entries of y are given, $f(x)$ can be represented by a $(n-1)$th degree polynomial *i.e.*, $\Delta^n y = 0$. Since $\Delta = E - 1$, therefore $(E-1)^n y = 0$. Now expanding $(E-1)^n$ and substituting the given values, we obtain the missing term/terms.

Example 6.15. *Find the missing term is the table :*

x :	2	3	4	5	6
y :	45.0	49.2	54.1	...	67.4

(U.P.T.C., B. Tech., 2008)

Sol. Let the missing value be a. Then the difference table is as follows :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
2	45.0 $(= y_0)$				
		4.2			
3	49.2 $(= y_1)$		0.7		
		4.9		$a - 59.7$	
4	54.1 $(= y_2)$		$a - 59.0$		$240.2 - 4a$
		$a - 54.1$		$180.5 - 3a$	
5	a $(= y_3)$		$121.5 - 2a$		
		$67.4 - a$			
6	67.4 $(= y_4)$				

We know that $\Delta^4 y = 0$ *i.e.*, $240.2 - 4a = 0$.

Hence $a = 60.05$.

Otherwise. As only four entries y_0, y_1, y_2, y_3 are given, therefore $y = f(x)$ can be represented by a third degree polynomial.

$\therefore \quad \Delta^3 y = \text{constant} \quad \text{or} \quad \Delta^4 y = 0$, *i.e.*, $(E-1)^4 y = 0$

i.e., $\quad (E^4 - 4E^3 + 6E^2 - 4E + 1)\, y = 0 \quad \text{or} \quad y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$

Let the missing entry y_3 be a so that

$$67.4 - 4a + 6(54.1) - 4\,(49.2) + 45 = 0 \quad \text{or} \quad -4a = -240.2$$

Hence $a = 60.05$.

Example 6.16. *Find the missing values in the following data :*

x :	45	50	55	60	65
y :	3.0	...	2.0	...	– 2.4

(Bhopal, B.E., 2007)

Sol. Let the missing values be a, b. Then the difference table is as follows :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
45	$3\ (= y_0)$			
		$a - 3$		
50	$a\ (= y_1)$		$5 - 2a$	
		$2 - a$		$3a + b - 9$
55	$2\ (= y_2)$		$b + a - 4$	
		$b - 2$		$3.6 - a - 3b$
60	$b\ (= y_3)$		$-0.4 - 2b$	
		$-2.4 - b$		
65	$-2.4\ (= y_4)$			

As only three entries y_0, y_2, y_4 are given, y can be represented by a second degree polynomial having third differences as zero.

$\therefore$ $$\Delta^3 y_0 = 0 \text{ and } \Delta^3 y_1 = 0$$

i.e., $$3a + b = 9,\ a + 3b = 3.6$$

Solving these, we get $a = 2.925$, $b = 0.225$.

Otherwise. As only three entries $y_0 = 3$, $y_2 = 2$, $y_4 = -2.4$ are given, y can be represented by a second degree polynomial having third differences as zero.

$\therefore$ $$\Delta^3 y_0 = 0 \text{ and } \Delta^3 y_1 = 0$$

i.e., $$(E - 1)^3 y_0 = 0 \text{ and } (E - 1)^3 y_1 = 0$$

i.e., $$(E^3 - 3E^2 + 3E - 1)\, y_0 = 0\ ;\ (E^3 - 3E^2 + 3E - 1)\,.\, y_1 = 0$$

or $$y_3 - 3y_2 + 3y_1 - y_0 = 0\ ;\ y_4 - 3y_3 + 3y_2 - y_1 = 0$$

or $$y_3 + 3y_1 = 9\ ;\ 3y_3 + y_1 = 3.6$$

Solving three, we get $y_1 = 2.925$, $y_2 = 0.225$.

Example 6.17. *The following table gives the values of y which is a polynomial of degree five. It is known that f (3) is in error. Correct the error.*

x :	0	1	2	3	4	5	6
y :	1	2	33	254	1025	3126	7777

(Mumbai, B.E., 2004)

Sol. Let the correct value of y when $x = 3$ be a. Then the difference table is as follows :

x :	y :	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
0	1						
		1					
1	2		30				
		31		$a-94$			
2	33		$a-64$		$1216-4a$		
		$a-33$		$1122-3a$		$-2320-10a$	
3	a		$1058-2a$		$-1104+6a$		$4880-20a$
		$1025-a$		$18+3a$		$2560-10a$	

4	1025		$1076 + a$		$1456 - 4a$		
		2101		$1474 - a$			
5	3126		2550				
		4651					
6	7777						

Since y is a polynomial of fifth degree, the sixth difference $\Delta^6 y = 0$

i.e., $$4880 - 20a = 0$$

Hence $a = 244$.

Otherwise. As y is a polynomial of fifth degree, the sixth difference $\Delta^6 y = 0$

i.e., $$(E - 1)^6 y = 0$$

or $$(E^6 - 6E^5 + 15E^4 - 20\, E^3 + 15E^2 - 6E + 1)\, y_0 = 0$$

or $$y_6 - 6y_5 + 15y_4 - 20y_3 + 15y_2 - 6y_1 + y_0 = 0$$

i.e., $$7777 - 6\,(3126) + 15\,(1025) + 20y_3 + 15\,(33) - 6\,(2) + 1 = 0$$

$$\therefore \quad 4880 = 20y_3 \quad \therefore \quad y_3 = 244.$$

Hence the error $= 254 - 244 = 10$.

Example 6.18. *If $y_{10} = 3$, $y_{11} = 6$, $y_{12} = 11$, $y_{13} = 18$, $y_{14} = 27$, find y_4.* *(Mumbai, B. Tech., 2005)*

Sol. Taking y_{14} as u_0, we are required to find y_4 *i.e.* u_{-10}. Then the difference table is

x	u	Δu	$\Delta^2 u$	$\Delta^3 u$
x_{-4}	$y_{10} = u_{-4} = 3$			
		3		
x_{-3}	$y_{11} = u_{-3} = 6$		2	
		5		0
x_{-2}	$y_{12} = u_{-2} = 11$		2	
		7		0
x_{-1}	$y_{13} = u_{-1} = 18$		2	
		9		
x_0	$y_{14} = u_0 = 27$			

Then $$y_4 = u_{-10} = (E^{-1})^{10}\, u_0 = (1 - \nabla)^{10}\, u_0$$

$$= \left(1 - 10\,\nabla + \frac{10.9}{2}\nabla^2 - \frac{10.9.8}{1.2.3}\nabla^3 + \ldots\right) u_0$$

$$= u_0 - 10\,\nabla u_0 + 45\,\nabla^2 u_0 - 120\,\nabla^3 u_0$$

$$= 27 - 10 \times 9 + 45 \times 2 - 120 \times 0 = 27.$$

Example 6.19. *If y_x is a polynomial for which fifth difference is constant and $y_1 + y_7 = -784$, $y_2 + y_6 = 686$, $y_3 + y_5 = 1088$, find y_4.* *(Mumbai, B. Tech., 2004)*

Sol. Starting with y_1 instead of y_0, we note that $\Delta^6 y_1 = 0$ [$\because$ $\Delta^5 y_1$ is constant.

i.e., $$(E - 1)^6\, y_1 = (E^6 - 6E^5 + 15\, E^4 - 20\, E^3 + 15\, E^2 - 6E + 1)\, y_1 = 0$$

$\therefore$ $$y_7 - 6y_6 + 15y_5 - 20\,y_4 + 15y_3 - 6y_2 + y_1 = 0$$

or $$(y_7 + y_1) - 6(y_6 + y_2) + 15\,(y_5 + y_3) - 20\,y_4 = 0$$

i.e., $$y_4 = \frac{1}{20}\,[(y_1 + y_7) - 6(y_2 + y_6) + 15(y_3 + y_5)]$$

$$= \frac{1}{20}\,[-784 - 6(686) + 15(1088)] = 571.$$

Example 6.20. *Using the method of separation of symbols, prove that*

(i) $$u_1x + u_2x^2 + u_3x^3 + \ldots = \frac{x}{1-x}\,u_1 + \left(\frac{x}{1-x}\right)^2 \Delta u_1 + \left(\frac{x}{1-x}\right)^3 \Delta^2 u_1 + \ldots$$

(ii) $$u_0 + \frac{u_1x}{1!} + \frac{u_2x^2}{2!} + \frac{u_3x^3}{3!} + \ldots = e^x\left(u_0 + x\Delta u_0 + \frac{x^2}{2!}\Delta^2 u_0 + \frac{x^3}{3!}\Delta^3 u_0 + \ldots\right).$$

Sol. *(i)* L.H.S. $= xu_1 + x^2\,Eu_1 + x^3E^2u_1 + \ldots$ $\quad [\because\ u_{x+h} = E^h u_x$

$$= x(1 + xE + x^2E^2 + \ldots)u_1 = x\,.\,\frac{1}{1-xE}\,u_1, \text{ taking sum of infinite G.P.}$$

$$= x\left[\frac{1}{1-x(1+\Delta)}\right]u_1 \qquad [\because\ E = 1 + .$$

$$= x\left(\frac{1}{1-x-x\Delta}\right)u_1 = \frac{x}{1-x}\left(1 - \frac{x\Delta}{1-x}\right)^{-1} u_1$$

$$= \frac{x}{1-x}\left(1 + \frac{x\Delta}{1-x} + \frac{x^2\Delta^2}{(1-x)^2} + \ldots\right)u_1$$

$$= \frac{x}{1-x}\,u_1 + \frac{x^2}{(1-x)^2}\,\Delta u_1 + \frac{x^3}{(1-x)^3}\,\Delta^2 u_1 + \ldots = \text{R.H.S.}$$

(ii) L.H.S. $= u_0 + \frac{x}{1!}\,Eu_0 + \frac{x^2}{2!}\,E^2u_0 + \frac{x^3}{3!}\,E^3u_0 + \ldots$

$$= \left(1 + \frac{xE}{1!} + \frac{x^2E^2}{2!} + \frac{x^3E^3}{3!} + \ldots\right)u_0 = e^{xE}\,u_0 = e^{x(1+\Delta)}u_0 = e^x\,.\,e^{x\Delta}\,u_0$$

$$= e^x\left(1 + \frac{x\Delta}{1!} + \frac{x^2\Delta^2}{2!} + \frac{x^3\Delta^3}{3!} + \ldots\right)u_0$$

$$= e^x\left(u_0 + \frac{x}{1!}\,\Delta u_0 + \frac{x^2}{2!}\,\Delta^2 u_0 + \frac{x^3}{3!}\,\Delta^3 u_0 + \ldots\right) = \text{R.H.S.}$$

PROBLEMS 6.3

1. Explain the difference between $\left(\frac{\Delta^2}{E}\right) u_x$ and $\left(\frac{\Delta^2 u_x}{E u_x}\right)$. (*Madras B. Tech.*, 2003)

2. Evaluate taking h as the interval of differencing :

   (*i*) $\frac{\Delta^2}{E} \sin x$ (*ii*) $(\Delta + \nabla)^2(x^2 + x)$, $(h = 1)$.

   (*iii*) $\frac{\Delta^2 x^3}{E x^3}$ (*iv*) $\frac{\Delta^2}{E} \sin (x + h) + \frac{\Delta^2 \sin (x + h)}{E \sin (x + h)}$.

3. With the usual notations, show that

   (*i*) $\nabla = 1 - e^{-hD}$ (*ii*) $D = \frac{2}{h} \sinh^{-1}\left(\frac{\delta}{2}\right)$ (*Madras, B.Tech.*, 2001)

   (*iii*) $(1 + \Delta)(1 - \nabla) = 1$ (*iv*) $\Delta\nabla = \nabla\Delta = \delta^2$.

4. Prove that

   (*i*) $\delta = \Delta(1 + \Delta)^{-1/2} = \nabla(1 - \nabla)^{-1/2}$ (*Madras, B.E.*, 2001 S)

   (*ii*) $\mu^2 = 1 + \frac{\delta^2}{2}$ (*iii*) $\delta(E^{1/2} + E^{-1/2}) = \Delta E^{-1} + \Delta$.

5. Show that

   (*i*) $\delta = \Delta E^{-1/2} = \nabla E^{1/2}$ (*ii*) $\mu\delta = \frac{1}{2}(\Delta + \nabla)$

   (*iii*) $1 + \delta^2/2 = \sqrt{(1 + \delta^2 \mu^2)}$ (*Madras, B. Tech.*, 2001)

6. Show that

   (*i*) $\Delta = \mu\delta + \frac{\delta^2}{2}$ (*ii*) $E^{1/2} = \left(1 + \frac{\delta^2}{4}\right)^{1/2} + \frac{\delta^2}{2}$

   (*iii*) $E^r = (\mu + \frac{1}{2}\delta)^{2r}$ (*iv*) $\mu = \frac{2 + \Delta}{2\lambda\,(1 + \Delta)} = \frac{2 - \nabla}{2\lambda\,(1 - \nabla)}$

7. Prove that

   (*i*) $\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$ (*ii*) $\nabla = \Delta E^{-1} = E^{-1}\Delta = 1 - E^{-1}$

   (*iii*) $E = \sum_{i=0}^{\infty} \nabla_i$ (*iv*) $\nabla^2 = h^2D^2 - h^3D^3 + \frac{7}{12} h^4D^4 - \ldots$

8. Prove that $\delta^2 y_5 = y_6 - 2y_5 + y_4$.

9. Prove with usual notations, that

   (*i*) $\nabla^r f_k = \Delta^r f_{k-r}$ (*ii*) $\Delta(f_k^2) = (f_k + f_{k+1})\Delta f_k$ (*J.N.T.U., B. Tech.*, 2006)

   (*iii*) $\sum_{k=0}^{n-1} \Delta^2 f_k = \Delta f_n - \Delta f_0$.

10. Estimate the missing term in the following table :

x :	0	1	2	3	4
$f(x)$:	1	3	9	–	81

(S.V.T.U., B. Tech., 2007)

11. Obtain the estimate of the missing figures in the following table :

x :	1	2	3	4	5	6	7	8
y :	2	4	8	...	32	...	128	256

(Mumbai, B.E., 2004)

12. Find the missing values in the following table :

0	1	2	3	4	5	6
5	11	22	40	...	140	...

(V.T.U., B.E., 2003)

13. Estimate the production for 2004 and 2006 from the following data :

Year :	2001	2002	2003	2004	2005	2006	2007
Production :	200	200	260	...	350	...	430

14. If $u_{13} = 1, u_{14} = -3, u_{15} = -1, u_{16} = 13$, find u_8. (Mumbai, B. Tech., 2004)

15. Evaluate y_4 from the following data (stating the assumptions you make) :

$y_0 + y_8 = 1.9243, y_1 + y_7 = 1.9590$

$y_2 + y_6 = 1.9823, y_3 + y_5 = 1.9956$ (Mumbai, B. Tech., 2003)

Using the method of separation of symbols, prove that

16. $u_0 + u_1 + u_2 + ... + u_n = {}^{n+1}C_1 u_0 + {}^{n+1}C_2 \Delta u_0 + {}^{n+1}C_3 \Delta^2 u_0 + ... + {}^{n+1}C_{n+1} \Delta^n u_0$

17. $\Delta^n u_x = u_{x+n} - {}^nC_1 u_{x+n-1} + {}^nC_2 u_{x+n-2} + ... + (-1)^n u_x$.

18. $y_x = y_n - {}^{n-x}C_1 \Delta y_{n-1} + {}^{n-x}C_2 \Delta^2 y_{n-2} - ..... + (-1)^{n-x} \Delta^{n-x} y_{n-(n-x)}$.

6.9. APPLICATION TO SUMMATION OF SERIES

The calculus of finite differences is very useful for finding the sum of a given series. The method is best illustrated by the following examples :

Example 6.21. *Sum the following series $1^3 + 2^3 + 3^3 + ... + n^3$.*

Sol. Denoting $1^3, 2^3, 3^3, ...,$ by $u_0, u_1, u_2, ...$ respectively, the required sum

$$S = u_0 + u_1 + u_2 + ... + u_{n-1}$$

$$= (1 + E + E^2 + ... + E^{n-1})u_0 \qquad [\because \ u_1 = Eu_0,\ u_2 = E^2u_0 \text{ etc.}]$$

$$= \frac{E^n - 1}{E - 1} u_0 = \frac{(1+\Delta)^n - 1}{\Delta} u_0$$

$$= \frac{1}{\Delta}\left[1 + n\Delta + \frac{n(n-1)}{2!}\Delta^2 + \frac{n(n-1)(n-2)}{3!}\Delta^3 + ... + \Delta^n - 1\right] u_0$$

$$= n + \frac{n(n-1)}{2!}\Delta u_0 + \frac{n(n-1)(n-2)}{3!}\Delta^2 u_0 + ...$$

Now $\Delta u_0 = u_1 - u_0 = 2^3 - 1^3 = 7$, $\Delta^2 u_0 = u^2 - 2u_1 + u_0 = 3^3 - 2.2^3 + 1^3 = 12$,

$\Delta^3 u_0 = u_3 - 3u_2 + 3u_1 - u_0 = 4^3 - 3.3^3 + 3.2^3 - 1^3 = 6$

and $\Delta^4 u_0, \Delta^5 u_0, ...$ are all zero as $u_r = r^3$ is a polynomial of third degree.

$$\text{Hence } S = n + \frac{n(n-1)}{2} . 7 + \frac{n(n-1)(n-2)}{6} . 12 + \frac{n(n-1)(n-2)(n-3)}{24} . 6$$

$$= \frac{n^2}{4}(n^2 + 2n + 1) = \left[\frac{n(n+1)}{2}\right]^2.$$

7

INTERPOLATION

7.1. INTRODUCTION

Suppose we are given the following values of $y = f(x)$ for a set of values of x :

x :	x_0	x_1	$x_2 \ldots\ldots\ldots x_n$
y :	y_0	y_1	$y_2 \ldots\ldots\ldots y_n$.

Then the process of finding the value of y corresponding to any value of $x = x_i$ between x_0 and x_n is called *interpolation*. Thus *interpolation is the technique of estimating the value of a function for any intermediate value of the independent variable* while the process of computing the value of the function outside the given range is called *extrapolation*. The term interpolation however, is taken to include extrapolation.

If the function $f(x)$ is known explicitly, then the value of y corresponding to any value of x can easily be found. Conversely, if the form of $f(x)$ is not known (as is the case in most of the applications), it is very difficult to determine the exact form of $f(x)$ with the help of tabulated set of values (x_i, y_i). In such cases, $f(x)$ is replaced by a simpler function $\phi(x)$ which assumes the same values as those of $f(x)$ at the tabulated set of points. Any other value may be calculated from $\phi(x)$ which is known as the *interpolating function* or *smoothing function*. If $\phi(x)$ is a polynomial, then it called the *interpolating polynomial* and the process is called the *polynomial interpolation*. Similarly when $\phi(x)$ is a finite trigonometric series, we have trigonometric interpolation. But we shall confine ourselves to polynomial interpolation only.

The study of interpolation is based on the calculus of finite differences. We begin by deriving two important *interpolation formulae* by means of forward and backward differences of a function. These formulae are often employed in engineering and scientific investigations.

7.2. NEWTON'S FORWARD INTERPOLATION FORMULA

Let the function $y = f(x)$ take the values $y_0, y_1, \ldots, y_n$ corresponding to the values $x_0, x_1, \ldots, x_n$ of x. Let these values of x be equi-spaced such that $x_i = x_0 + ih$ $(i = 0, 1, \ldots)$. Assuming $y(x)$ to be a polynomial of the nth degree in x such that $y(x_0) = y_0, y(x_1) = y_1, \ldots, y(x_n) = y_n$. We can write

$$y(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) + \ldots + a_n(x - x_0)(x - x_1) \ldots (x - x_{n-1}) \quad \ldots(1)$$

Putting $x = x_0, x_1, \ldots, x_n$ successively in (1), we get

$$y_0 = a_0, y_1 = a_0 + a_1 (x_1 - x_0), y_2 = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

and so on.

From these, we find that $a_0 = y_0$, $\Delta y_0 = y_1 - y_0 = a_1(x_1 - x_0) = a_1 h$

$$\therefore \quad a_1 = \frac{1}{h} \Delta y_0$$

Also $\quad \Delta y_1 = y_2 - y_1 = a_1(x_2 - x_1) + a_2(x_2 - x_0)(x_2 - x_1) = a_1 h + a_2.2h.h = \Delta y_0 + 2h^2 a_2$

$$\therefore \quad a_2 = \frac{1}{2h^2} (\Delta y_1 - \Delta y_0) = \frac{1}{2!\,h^2} \Delta^2 y_0$$

Similarly $a_3 = \dfrac{1}{3!\,h^3} \Delta^3 y_0$ and so on.

Substituting these values in (1), we obtain

$$y(x) = y_0 + \frac{\Delta y_0}{h} (x - x_0) + \frac{\Delta^2 y_0}{2!\,h^2} (x - x_0)(x - x_1) + \frac{\Delta^3 y_0}{3!\,h^3} (x - x_0)(x - x_1)(x - x_2) + \ldots \quad \ldots(2)$$

Now if it is required to evaluate y for $x = x_0 + ph$, then

$$x - x_0 = ph, \; x - x_1 = x - x_0 - (x_1 - x_0) = ph - h = (p - 1)h,$$
$$x - x_2 = x - x_1 - (x_2 - x_1) = (p - 1)h - h = (p - 2)h \text{ etc.}$$

Hence, writing $y(x) = y(x_0 + ph) = y_p$, (2) becomes

$$\mathbf{y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \ldots + \frac{p(p-1)\ldots(p-\overline{n-1})}{n!} \Delta^n y_0} \quad \ldots(3)$$

It is called *Newton's forward interpolation formula* as (3) contains y_0 and the forward differences of y_0.

Otherwise : Let the function $y = f(x)$ take the values $y_0, y_1, y_2, \ldots$ corresponding to the values $x_0, x_0 + h, x_0 + 2h, \ldots$ of x. Suppose it is required to evaluate $f(x)$ for $x = x_0 + ph$, where p is any real number.

For any real number p, we have defined E such that

$$E^p f(x) = f(x + ph)$$

$$y_p = f(x_0 + ph) = E^p f(x_0) = (1 + \Delta)^p y_0 \qquad [\because E = 1 + \Delta$$

$$= \left\{1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \ldots\ldots\right\} y_0 \qquad \ldots(4)$$

[using Binomial theorem

i.e. $$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots\ldots$$

If $y = f(x)$ is a polynomial of the nth degree, then $\Delta^{n+1}y_0$ and higher differences will be zero. Hence (4) will become

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots\ldots + \frac{p(p-1)(p-\overline{n-1})}{n!}\Delta^n y_0$$

which is same as (3).

Obs. *This formula is used for interpolating the values of y near the beginning of a set of tabulated values and extrapolating values of y a little backward (i.e. to the left) of y_0.*

7.3. NEWTON'S BACKWARD INTERPOLATION FORMULA

Let the function $y = f(x)$ take the values $y_0, y_1, y_2, \ldots$ corresponding to the values x_0, $x_0 + h, x_0 + 2h, \ldots$ of x. Suppose it is required to evaluate $f(x)$ for $x = x_n + ph$, where p is any real number. Then we have

$$y_p = f(x_n + ph) = E^p f(x_n) = (1 - \nabla)^{-p} y_n \qquad [\because\ E^{-1} = 1 - \nabla$$

$$= \left[1 + p\nabla + \frac{p(p+1)}{2!}\nabla^2 + \frac{p(p+1)(p+2)}{3!}\nabla^3 + \ldots\ldots\right] y_n$$

[using Binomial theorem

i.e. $$\mathbf{y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \ldots} \qquad \ldots(1)$$

It is called *Newton's backward interpolation formula* as (1) contains y_n and backward differences of y_n.

Obs. *This formula is used for interpolating the values of y near the end of a set of tabulated values and also for extrapolating values of y a little ahead* (to the right) *of y_n.*

Example 7.1. *The table gives the distance in nautical miles of the visible horizon for the given heights in feet above the earth's surface :*

x = height :	100	150	200	250	300	350	400
y = distance :	10.63	13.03	15.04	16.81	18.42	19.90	21.27

Find the values of y when

(i) x = 218 ft (Madras, B.E., 2003 S)

(ii) x = 410. (V.T.U., B.E., 2002)

Sol. The difference table is as under :

x	y	Δ	Δ^2	Δ^3	Δ^4
100	10.63				
		2.40			
150	13.03		− 0.39		
		2.01		0.15	
200	**15.04**		− 0.24		− 0.07
218		**1.77**		0.08	
250	16.81		**− 0.16**		− 0.05
		1.61		**0.03**	
300	18.42		− 0.13		− 0.01
		1.48		**0.02**	
350	19.90		**− 0.11**		
		1.37			
400	**21.27**				

(*i*) If we take $x_0 = 200$, then $y_0 = 15.04$, $\Delta y_0 = 1.77$, $\Delta^2 y_0 = -0.16$, $\Delta^3 = 0.03$ etc.

Since $x = 218$ and $h = 50$, $\therefore$ $p = \dfrac{x - x_0}{h} = \dfrac{18}{50} = 0.36$

$\therefore$ Using Newton's forward interpolation formula, we get

$$y_{218} = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \ldots\ldots$$

$$f(218) = 15.04 + 0.36(1.77) + \frac{0.36(-0.64)}{2}(-0.16) + \frac{0.36(-0.64)(-1.64)}{6}(0.03) + \frac{0.36(-0.64)(-1.64)(-2.64)}{24}(-0.01)$$

$$= 15.04 + 0.637 + 0.018 + 0.002 + 0.0004$$

$$= 15.697 \text{ i.e. } 15.7 \text{ nautical miles.}$$

(*ii*) Since $x = 410$ is near the end of the table, we use Newton's backward interpolation formula.

$\therefore$ Taking $x_n = 400$, $p = \dfrac{x - x_n}{h} = \dfrac{10}{50} = 0.2$

Using the line of backward difference

$$y_n = 21.27,\ \nabla y_n = 1.37,\ \nabla^2 y_n = -0.11,\ \nabla^3 y_n = 0.02 \text{ etc.}$$

$\therefore$ Newton's backward formula gives

$$y_{410} = y_{400} + p\nabla y_{400} + \frac{p(p+1)}{2!}\nabla^2 y_{400} + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_{400} + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_{400}$$

$$= 21.27 + 0.2(1.37) + \frac{0.2(1.2)}{2!}(-0.11) + \frac{0.2(1.2)(2.2)}{3!}(0.02) + \frac{0.2(1.2)(2.2)(3.2)}{4!}(-0.01)$$

$$= 21.27 + 0.274 - 0.0132 + 0.0018 - 0.0007$$

$$= 21.53 \text{ nautical miles.}$$

Example 7.2. *From the following table, estimate the number of students who obtained marks between 40 and 45 :*

Marks :	*30—40*	*40—50*	*50—60*	*60—70*	*70—80*
No. of students :	*31*	*42*	*51*	*35*	*31*

(V.T.U., B. Tech., 2007)

Sol. First we prepare the cumulative frequency table, as follows :

Marks less than (x) :	40	50	60	70	80
No. of students (y_x) :	31	73	124	159	190

Now the difference table is

x	y_x	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$	$\Delta^4 y_x$
40	**31**				
		42			
50	73		**9**		
		51		**– 25**	
60	124		**– 16**		**37**
		35		**12**	
70	159		**– 4**		
		31			
80	190				

We shall find y_{45} *i.e.* number of students with marks less than 45. Taking $x_0 = 40$, $x = 45$, we have

$$p = \frac{x - x_0}{h} = \frac{5}{10} = 0.5 \qquad [\because \; h = 10$$

$\therefore$ Using Newton's forward interpolation formula, we get

$$y_{45} = y_{40} + p\,\Delta y_{40} + \frac{p(p-1)}{2!}\Delta^2 y_{40} + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_{40} + \frac{p(p-1)(p-2)(p-3)}{3!}\Delta^4 y_{40}$$

$$= 31 + 0.5 \times 42 + \frac{0.5(-0.5)}{2} \times 9 + \frac{0.5(-0.5)(-1.5)}{6} \times (-25) + \frac{0.5(-0.5)(-1.5)(-2.5)}{24} \times 37$$

$$= 31 + 21 - 1.125 - 1.5625 - 1.4453$$

$$= 47.87, \text{ on simplification.}$$

The number of students with marks less than 45 is 47.87 *i.e.*, 48.

But the number of students with marks less than 40 is 31.

Hence the number of students getting marks between 40 and 45 = 48 – 31 = 17.

Example 7.3. *Find the cubic polynomial which takes the following values :*

x :	0	1	2	3
$f(x)$:	1	2	1	10

Hence or otherwise evaluate f(4). *(Bhopal, B.E., 2009)*

Sol. The difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	1			
		1		
1	2		–2	
		–1		12
2	1		10	
		9		
3	10			

We take $x_0 = 0$ and $p = \dfrac{x-0}{h} = x$ $[\because h = 1]$

$\therefore$ Using Newton's forward interpolation formula, we get

$$f(x) = f(0) + \frac{x}{1}\Delta f(0) + \frac{x(x-1)}{1.2}\Delta^2 f(0) + \frac{x(x-1)(x-2)}{1.2.3}\Delta^3 f(0)$$

$$= 1 + x(1) + \frac{x(x-1)}{2}(-2) + \frac{x(x-1)(x-2)}{6}(12)$$

$$= 2x^3 - 7x^2 + 6x + 1,$$

which is the required polynomial.

To compute $f(4)$, we take $x_n = 3$, $x = 4$ so that $p = \dfrac{x - x_n}{h} = 1$ $[\because h = 1]$

Obs. Using Newton's backward interpolation formula, we get

$$f(4) = f(3) + p\nabla f(3) + \frac{p(p+1)}{1.2}\nabla^2 f(3) + \frac{p(p+1)(p+2)}{1.2.3}\nabla^3 f(3)$$

$$= 10 + 9 + 10 + 12 = 41$$

which is the same value as that obtained by substituting $x = 4$ in the cubic polynomial above.

The above example shows that *if a tabulated function is a polynomial, then interpolation and extrapolation give the same values.*

Example 7.4. *Using Newton's backward difference formula, construct an interpolating polynomial of degree 3 for the data : f (– 0.75) = – 0.0718125, f (– 0.5) = – 0.02475, f (– 0.25) = 0.3349375, f (0) = 1.10100. Hence find f (– 1/3).* *(Anna, B.Tech., 2003)*

Sol. The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
– 0.75	– 0.0718125			
		0.0470625		
– 0.5	– 0.02475		0.312625	
		0.3596875		0.09375
– 0.25	0.3349375		**0.400375**	
		0.7660625		
0	**1.10100**			

We use Newton's backward difference formula

$$y(x) = y_3 + \frac{p}{1!}\nabla y_3 + \frac{p(p+1)}{2!}\nabla^2 y_3 + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_3$$

taking $\quad x_3 = 0, p = \dfrac{x-0}{h} = \dfrac{x}{0.25} = 4x, \qquad [\because \quad h = 0.25]$

$$\therefore \quad y(x) = 1.10100 + 4x\,(0.7660625) + \frac{4x\,(4x+1)}{2}\,(0.400375) + \frac{4x\,(4x+1)\,(4x+2)}{6}\,(0.09375)$$

$$= 1.101 + 3.06425x + 3.251x^2 + 0.81275x + x^3 + 0.75x^2 + 0.125x$$

$$= x^3 + 4.001x^2 + 4.002x + 1.101.$$

Put $\quad x = -\dfrac{1}{3}$, so that

$$y\left(-\frac{1}{3}\right) = \left(-\frac{1}{3}\right)^3 + 4.001\left(-\frac{1}{3}\right)^2 + 4.002\left(-\frac{1}{3}\right) + 1.101$$

$$= 0.1745$$

Example 7.5. *In the table below, the values of y are consecutive terms of a series of which 23.6 is the 6th term. Find the first and tenth terms of the series :*

x :	3	4	5	6	7	8	9
y :	4.8	8.4	14.5	23.6	36.2	52.8	73.9

(Anna, B.E., 2007)

Sol. The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
3	4.8				
		3.6			
4	8.4		2.5		
		6.1		0.5	
5	14.5		3.0		0
		9.1		0.5	

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6	23.6		3.5		0
		12.6		0.5	
7	36.2		4.0		0
		16.6		0.5	
8	52.8		4.5		
		21.1			
9	73.9				

To find the first term, use Newton's forward interpolation formula with $x_0 = 3, x = 1$, $h = 1$ and $p = -2$. We have

$$y(1) = 4.8 + \frac{(-2)}{1} \times 3.6 + \frac{(-2)(-3)}{1.2} \times 2.5 + \frac{(-2)(-3)(-4)}{1.2.3} \times 0.5 = 3.1$$

To obtain the tenth term, use Newton's backward interpolation formula with $x_n = 9$, $x = 10$, $h = 1$ and $p = 1$. This gives

$$y(10) = 73.9 + \frac{1}{1} \times 21.1 + \frac{1(2)}{1.2} \times 4.5 + \frac{1(2)(3)}{1.2.3} \times 0.5 = 100.$$

Example 7.6. *Using Newton's forward interpolation formula, show that*

$$\Sigma n^3 = \left\{\frac{n(n+1)}{2}\right\}^2.$$

Sol. If $s_n = sn^3$, then $s_{n+1} = \Sigma(n+1)^3$

$\therefore$ $\quad \Delta s_n = s_{n+1} - s_n = \Sigma(n+1)^3 - \Sigma n^3 = (n+1)^3.$

Then $\quad \Delta^2 s_n = \Delta s_{n+1} - \Delta s_n = (n+2)^3 - (n+1)^3 = 3n^2 + 9n + 7$

$$\Delta^3 s_n = \Delta^2 s_{n+1} - \Delta^2 s_n$$
$$= [3(n+1)^2 + 9(n+1) + 7] - (3n^2 + 9n + 7) = 6n + 12.$$
$$\Delta^4 s_n = \Delta^3 s_{n+1} - \Delta^3 s_n = [6(n+1) + 12] - [6n + 12] = 6$$

and $\quad \Delta^5 s_n = \Delta^5 \Delta_n = \ldots\ldots = 0.$

Since the first term of the given series is 1, therefore taking $n = 1$, $s_1 = 1$, $\Delta s_1 = 8$, $\Delta^2 s_1 = 19$, $\Delta^3 s_1 = 18$, $\Delta^4 s_1 = 6$.

Substituting these in the Newton's forward interpolation formula *i.e.*

$$s_n = s_1 + (n-1)\Delta s_1 + \frac{(n-1)(n-2)}{2!}\Delta^2 s_1 + \frac{(n-1)(n-2)(n-3)}{3!}\Delta^3 s_1$$
$$+ \frac{(n-1)(n-2)(n-3)(n-4)}{4!}\Delta^4 s_1,$$

$$s_n = 1 + 8(n-1) + \frac{19}{2}(n-1)(n-2) + 3(n-1)(n-2)(n-3)$$
$$+ \frac{1}{4}(n-1)(n-2)(n-3)(n-4) = \frac{1}{4}(n^4 + 2n^3 + n^2) = \left\{\frac{n(n+1)}{2}\right\}^2.$$

PROBLEMS 7.1

1. Using Newton's forward formula, find the value of $f(1.6)$, if

x :	1	1.4	1.8	2.2
$f(x)$:	3.49	4.82	5.96	6.5

(*J.N.T.U., B. Tech., 2006*)

2. From the following table, find y when $x = 1.85$ and 2.4 by Newton's interpolation formulae :

x :	1.7	1.8	1.9	2.0	2.1	2.2	2.3
$y = e^x$:	5.474	6.050	6.686	7.389	8.166	9.025	9.974

(*Kottayam, B.E., 2005*)

3. If $f(1.15) = 1.0723$, $f(1.20) = 1.0954$, $f(1.25) = 1.1180$ and $f(1.30) = 1.1401$, find $f(1.28)$.
4. Given $\sin 45° = 0.7071$, $\sin 50° = 0.7660$, $\sin 55° = 0.8192$, $\sin 60° = 0.8660$, find $\sin 52°$ using Newton's forward formula. (*J.N.T.U., B. Tech., 2006*)
5. From the following table :

x :	0.1	0.2	0.3	0.4	0.5	0.6
$f(x)$:	2.68	3.04	3.38	3.68	3.96	4.21

find $f(0.7)$ approximately. (*V.T.U., B.E., 2001*)

6. The area A of a circle of diameter d is given for the following values :

d :	80	85	90	95	100
A :	5026	5674	6362	7088	7854

Calculate the area of a circle of diameter 105. (*V.T.U., B.E., 2004*)

7. From the following table :

$x°$:	10	20	30	40	50	60	70	80
$\cos x$:	.9848	.9397	.8660	.7660	.6428	.5000	.3420	.1737

Calculate $\cos 25°$ and $\cos 73°$ using Gregory-1 Newton formula. (*U.P.T.U., B. Tech., 2006*)

8. A test performed on *NPN* transistor gives the following result :

Base current f (mA)	0	0.01	0.02	0.03	0.04	0.05
Collector current I_C (mA)	0	1.2	2.5	3.6	4.3	5.34

Calculate (*i*) the value of the collector current for the base current of 0.005 mA.
(*ii*) the value of base current required for a collector correct of 4.0 mA.

(*Pune, B.Tech., 2004*)

9. Find $f(22)$ from the following data using Newten's backward formulae.

x :	20	25	30	35	40	45
$f(x)$:	354	332	291	260	231	204

(*J.N.T.U., B.Tech., 2007*)

10. Find the number of men getting wages between Rs. 10 and 15 from the following data :

Wages in Rs. :	0—10	10—20	20—30	30—40
Frequency :	9	30	35	42

(*Nagarjuna, B.E., 2001*)

11. From the following data, estimate the number of persons having incomes between 2000 and 2500 :

Income	Below 500	500–1000	1000–2000	2000–3000	3000–4000
No. of persons	6000	4250	3600	1500	650

12. Construct Newton's forward interpolation polynomial for the following data :

x :	4	6	8	10
y :	1	3	8	16

(*Madras, B. Tech., 2006*)

Hence evaluate y for $x = 5$.

13. Find the cubic polynomial which takes the following values :

$$y(0) = 1, y(1) = 0, y(2) = 1 \text{ and } y(3) = 10.$$

(*U.P.T.U., B. Tech., 2005*)

Hence or otherwise, obtain $y(4)$.

14. Construct the difference table for the following data :

x :	0.1	0.3	0.5	0.7	0.9	1.1	1.3
$f(x)$:	0.003	0.067	0.148	0.248	0.370	0.518	0.697

(*J.N.T.U., B. Tech., 2007*)

Evaluate $f(0.6)$

15. Apply Newton's backward difference formula to the data below, to obtain a polynomial of degree 4 in x :

x :	1	2	3	4	5
y :	1	– 1	1	– 1	1

16. The following table gives the population of a town during the last six censuses. Estimate the increase in the population during the period from 1976 to 1978 :

Year :	1941	1951	1961	1971	1981	1991
Population : (in thousands)	12	15	20	27	39	52

(*U.P.T.C., B.E., 2009*)

17. In the following table, the values of y are consecutive terms of a series of which 12.5 is the 5th term. Find the first and tenth terms of the series.

x :	3	4	5	6	7	8	9
y :	2.7	6.4	12.5	21.6	34.3	51.2	72.9

(*P.T.U., B. Tech., 2001*)

18. Using a polynomial of the third degree, complete the record given below of the export of a certain commodity during five years :

Year :	1989	1990	1991	1992	1993
Export : (in tons)	443	384	—	397	467

19. Given $u_1 = 40$, $u_3 = 45$, $u_5 = 54$, find u_2 and u_4. (*Nagarjuna, B.E., 2003 S*)

20. If $u_{-1} = 10$, $u_1 = 8$, $u_2 = 10$, $u_4 = 50$, find u_0 and u_3.

21. Given $y_0 = 3, y_1 = 12, y_2 = 81, y_3 = 200, y_4 = 100, y_5 = 8$, without forming the difference table, find $\Delta^5 y_0$.

7.4. CENTRAL DIFFERENCE INTERPOLATION FORMULAE

In the preceding sections, we derived Newton's forward and backward interpolation formulae which are applicable for interpolation near the beginning and end of tabulated values. Now we shall develop central difference formulae which are best suited for interpolation near the middle of the table.

If x takes the values $x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h$ and the corresponding values of $y = f(x)$ are $y_{-2}, y_{-1}, y_0, y_1, y_2$, then we can write the difference table in the two notations as follows :

x	y	1st diff.	2nd diff.	3rd diff.	4th diff.
x_0-2h	y_{-2}				
		$\Delta y_{-2}(=\delta y_{-3/2})$			
x_0-h	y_{-1}		$\Delta^2 y_{-2}(=\delta^2 y_{-1})$		
		$\Delta y_{-1}(=\delta y_{-1/2})$		$\Delta^3 y_{-2}(=\delta^3 y_{-1/2})$	
x_0	y_0		$\Delta^2 y_{-1}(=\delta^2 y_0)$		$\Delta^4 y_{-2}(=\delta^4 y_0)$
		$\Delta y_0(=\delta y_{1/2})$		$\Delta^3 y_{-1}(=\delta^3 y_{1/2})$	
x_0+h	y_1		$\Delta^2 y_0(=\delta^2 y_1)$		
		$\Delta y_1(=\delta y_{3/2})$			
x_0+2h	y_2				

7.5. GAUSS'S FORWARD INTERPOLATION FORMULA

The Newton's forward interpolation formula is

$$y_p = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{1.2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{1.2.3}\Delta^3 y_0 + \ldots.. \quad \ldots(1)$$

We have $\Delta^2 y_0 - \Delta^2 y_{-1} = \Delta^3 y_{-1}$

i.e. $$\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1} \quad \ldots(2)$$

Similarly $$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1} \quad \ldots(3)$$

$$\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1} \text{ etc.} \quad \ldots(4)$$

Also $\Delta^3 y_{-1} - \Delta^3 y_{-2} = \Delta^4 y_{-2}$

i.e. $$\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$$

Similarly $$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2} \text{ etc.} \quad \ldots(5)$$

Substituting for $\Delta^2 y_0, \Delta^3 y_0, \Delta^4 y_0$ ..... from (2), (3), (4) ..... in (1), we get

$$y_p = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{1.2}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{p(p-1)(p-2)}{1.2.3}(\Delta^3 y_{-1} + \Delta^4 y_{-1})$$

$$+ \frac{p(p-1)(p-2)(p-3)}{1.2.3.4} \times (\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \ldots..$$

Hence $$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1}$$

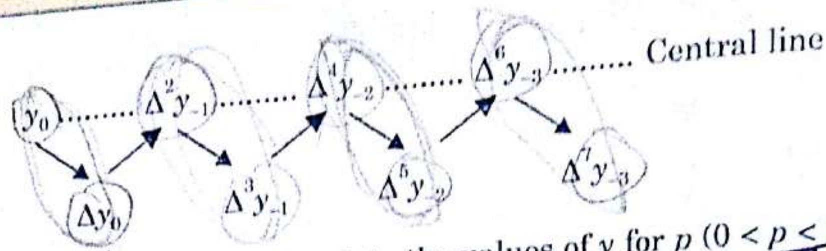
$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \ldots.. \quad [\text{using (5)}$$

which is called *Gauss's forward interpolation formula.*

Cor. *In the central differences notation, this formula will be*

$$y_p = y_0 + p\delta y_{1/2} + \frac{p(p-1)}{2!}\delta^2 y_0 + \frac{(p+1)p(p-1)}{3!}\delta^3 y_{1/2} + \frac{(p+1)p(p-1)(p-2)}{4!}\delta^4 y_0 + \ldots..$$

Obs. 1. *It employs odd differences just below the central line and even difference on the central line* as shown below :



Obs. 2. This formula is used to interpolate the values of y for p $(0 < p < 1)$ measured forwardly from the origin.

7.6. GAUSS'S BACKWARD INTERPOLATION FORMULA

The Newton's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{1.2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{1.2.3}\Delta^3 y_0 + \dots \quad \dots(1)$$

We have $\Delta y_0 - \Delta y_{-1} = \Delta^2 y_{-1}$ ...(2)

i.e. $\Delta y_0 = \Delta y_{-1} + \Delta^2 y_{-1}$ ...(3)

Similarly $\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$ ...(4)

$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$ etc.

Also $\Delta^3 y_{-1} - \Delta^3 y_{-2} = \Delta^4 y_{-2}$ ...(5)

i.e. $\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$ ...(6)

Similarly $\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$ etc.

Substituting for $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \dots$ from (2), (3), (4) in (1), we get

$$y_p = y_0 + p(\Delta y_{-1} + \Delta^2 y_{-1}) + \frac{p(p-1)}{1.2}(\Delta^2 y_{-1} + \Delta^3 y_{-1})$$
$$+ \frac{p(p-1)(p-2)}{1.2.3}(\Delta^3 y_{-1} + \Delta^4 y_{-1}) + \frac{p(p-1)(p-2)(p-3)}{1.2.3.4} \times (\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \dots$$
$$= y_0 + p\Delta y_{-1} + \frac{(p+1)p}{1.2}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{1.2.3}\Delta^3 y_{-1}$$
$$+ \frac{(p+1)p(p-1)(p-2)}{1.2.3.4}\Delta^4 y_{-1} + \frac{p(p-1)(p-2)(p-3)}{1.2.3.4}\Delta^5 y_{-1} + \dots$$
$$= y_0 + p\Delta y_{-1} + \frac{(p+1)p}{1.2}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{1.2.3}(\Delta^3 y_{-2} + \Delta^4 y_{-2})$$
$$+ \frac{(p+1)p(p-1)(p-2)}{1.2.3.4}(\Delta^4 y_{-2} + \Delta^5 y_{-2}) + \dots$$

[using (5) and (6)]

Hence
$$y_p = y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2} + \dots$$

which is called *Gauss's backward interpolation formula.*

Cor. *In the central differences notation, this formula will be*

$$y_p = y_0 + p\delta y_{-1/2} + \frac{(p+1)p}{2!}\delta^2 y_0 + \frac{(p+1)p(p-1)}{3!}\delta^3 y_{-1/2} + \frac{(p+2)(p+1)p(p-1)}{4!}\delta^4 y_0 + \ldots.$$

Obs. 1. This formula contains odd differences above the central line and even differences on the central line as shown below ;

$$\begin{array}{ccccccccccc} & & \Delta y_{-1} & & & \Delta^3 y_{-2} & & & \Delta^5 y_{-3} & & \\ & \nearrow & & \searrow & \nearrow & & \searrow & \nearrow & & \searrow & \\ y_0 & \ldots\ldots & & \Delta^2 y_{-1} & \ldots\ldots & & \Delta^4 y_{-2} & \ldots\ldots & & \Delta^6 y_{-3} & \ldots\ldots \text{ Central line} \end{array}$$

Obs. 2. It is used to interpolate the values of y for a negative value of p lying between -1 and 0.

Obs. 3. Gauss's forward and backward formulae are not of much practical use. However, these serve as intermediate steps for obtaining the important formulae of the following sections.

7.7. STIRLING'S FORMULA

Gauss's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \ldots \quad \ldots(1)$$

Gauss's backward interpolation formula is

$$y_p = y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2} + \ldots.. \quad \ldots(2)$$

Taking the mean of (1) and (2), we obtain

$$y_p = y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!} \times \left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \ldots.. \quad \ldots(3)$$

which is called *Stirling's formula*.

Cor. *In the central differences notation, (3) takes the form*

$$y_p = y_0 + p\,\mu\delta y_0 + \frac{p^2}{2!}\delta^2 y_0 + \frac{p(p^2-1^2)}{3!}\mu\delta^3 y_0 + \frac{p^2(p^2-1^2)}{4!}\delta^4 y_0 + \ldots \quad \ldots(4)$$

for

$$\tfrac{1}{2}(\Delta y_0 + \Delta y_{-1}) = \tfrac{1}{2}(\delta y_{1/2} + \delta y_{-1/2}) = \mu\,\delta y_0,$$

$$\tfrac{1}{2}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) = \tfrac{1}{2}(\delta^3 y_{1/2} + \delta^3 y_{-1/2}) = \mu\,\delta^3 y_0 \text{ etc.}$$

Obs. *This formula involves means of the odd differences just above and below the central line and even differences on this line as shown below :*

$$\ldots\ldots y_0 \ldots \begin{pmatrix}\Delta y_{-1} \\ \Delta y_0\end{pmatrix} \ldots \Delta^2 y_{-1} \ldots \begin{pmatrix}\Delta^3 y_{-2} \\ \Delta^3 y_{-1}\end{pmatrix} \ldots \Delta^4 y_{-2} \ldots \begin{pmatrix}\Delta^5 y_{-3} \\ \Delta^5 y_{-2}\end{pmatrix} \ldots \Delta^6 y_{-3} \ldots\ldots \text{ Central line.}$$

7.8. BESSEL'S FORMULA

Gauss's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \dots \quad \dots(1)$$

We have $\Delta^2 y_0 - \Delta^2 y_{-1} = \Delta^3 y_{-1}$

i.e. $\Delta^2 y_{-1} = \Delta^2 y_0 - \Delta^3 y_{-1}$ ...(2)

Similarly $\Delta^4 y_{-2} = \Delta^4 y_{-1} - \Delta^5 y_{-2}$ etc. ...(3)

Now (1) can be written as

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\left(\frac{1}{2}\Delta^2 y_{-1} + \frac{1}{2}\Delta^2 y_{-1}\right) + \frac{p(p^2-1)}{3!}\Delta^3 y_{-1} + \frac{p(p^2-1)(p-2)}{4!}\left(\frac{1}{2}\Delta^4 y_{-2} + \frac{1}{2}\Delta^4 y_{-2}\right) + \dots$$

$$= y_0 + p\Delta y_0 + \frac{1}{2}\frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{1}{2}\frac{p(p-1)}{2!}(\Delta^2 y_0 - \Delta^3 y_{-1}) + \frac{p(p^2-1)}{3!}\Delta^3 y_{-1} + \frac{1}{2}\frac{p(p^2-1)(p-2)}{4!}\Delta^4 y_{-2} + \frac{1}{2}\frac{p(p^2-1)(p-2)}{4!} \times (\Delta^4 y_{-1} - \Delta^5 y_{-2}) + \dots \quad \text{[using (2), (3) etc.}$$

$$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\cdot\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{p(p-1)}{2!} \times \left(\frac{p+1}{3} - \frac{1}{2}\right)\Delta^3 y_{-1} + \frac{p(p^2-1)(p-2)}{4!}\cdot\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \dots$$

Hence

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\cdot\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{(p-\frac{1}{2})p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\cdot\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \dots \quad \dots(4)$$

which is known as the *Bessel's formula*.

Cor. *In the central differences notation, (4) becomes*

$$y_p = y_0 + p\delta y_{1/2} + \frac{p(p-1)}{2!}\mu\delta^2 y_{1/2} + \frac{(p-\frac{1}{2})p(p-1)}{3!}\delta^3 y_{1/2} + \frac{(p+1)p(p-1)(p-2)}{4!}\mu\delta^4 y_{1/2} + \dots \quad \dots(5)$$

for $\frac{1}{2}(\Delta^2 y_{-1} + \Delta^2 y_0) = \mu\delta^2 y_{1/2}, \frac{1}{2}(\Delta^4 y_{-2} + \Delta^4 y_{-1}) = \mu\delta^4 y_{1/2}$ etc.

Obs. *This is a very useful formula for practical purposes. It involves odd differences below the central line and means of even differences of and below his line* as shown below :

$$y_0 \dots\dots \begin{Bmatrix}\Delta^2 y_{-1} \\ \\ \Delta^2 y_0\end{Bmatrix} \dots\dots \begin{Bmatrix}\Delta^4 y_{-2} \\ \\ \Delta^4 y_{-1}\end{Bmatrix} \dots\dots \begin{Bmatrix}\Delta^6 y_{-3} \\ \\ \Delta^6 y_{-2}\end{Bmatrix} \dots\dots \text{Central line.}$$
$$\Delta y_0 \qquad\qquad \Delta^3 y_{-1} \qquad\qquad \Delta^5 y_{-2}$$

7.9. LAPLACE-EVERETT'S FORMULA

Gauss's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1}$$
$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \frac{(p+2)(p+1)p(p-1)(p-2)}{5!} \times \Delta^5 y_{-2} + \ldots \quad \ldots(1)$$

We eliminate the odd differences in (1) by using the relations
$\Delta y_0 = y_1 - y_0$, $\Delta^3 y_{-1} = \Delta^2 y_0 - \Delta^2 y_{-1}$, $\Delta^5 y_{-2} = \Delta^4 y_{-1} - \Delta^4 y_{-2}$ etc.
Then (1) becomes

$$y_p = y_0 + p(y_1 - y_0) + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}(\Delta^2 y_0 - \Delta^2 y_{-1})$$
$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \frac{(p+2)(p+1)p(p-1)(p-2)}{5!} \times (\Delta^4 y_{-1} - \Delta^4 y_{-2}) + \ldots$$
$$= (1-p)y_0 + py_1 - \frac{p(p-1)(p-2)}{3!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^2 y_0$$
$$- \frac{(p+1)p(p-1)(p-2)(p-3)}{5!}\Delta^4 y_{-2} + \frac{(p+2)(p+1)p(p-1)(p-2)}{5!} \times \Delta^4 y_{-1} - \ldots$$

To change the terms with negative sign, putting $p = 1 - q$, we obtain

$$y_p = qy_0 + \frac{q(q^2-1^2)}{3!}\Delta^2 y_{-1} + \frac{q(q^2-1^2)(q^2-2^2)}{5!}\Delta^4 y_{-2} + \ldots$$
$$+ py_1 + \frac{p(p^2-1^2)}{3!}\Delta^2 y_0 + \frac{p(p^2-1^2)(p^2-2^2)}{5!}\Delta^4 y_{-1} + \ldots$$

This is known as *Laplace-Everett's formula.*

Obs. 1. *This formula is extensively used and involves only even differences on and below the central line* as shown below :

$$\begin{array}{cccccccc} y_0 & \ldots\ldots & \Delta^2 y_{-1} & \ldots\ldots & \Delta^4 y_{-2} & \ldots\ldots & \Delta^6 y_{-3} \ldots\ldots & \text{Central line} \\ — & & — & & — & & — & \\ y_1 & & \Delta^2 y_0 & & \Delta^4 y_{-1} & & \Delta^6 y_{-2} & \end{array}$$

Obs. 2. *There is a close relationship between Bessel's formula and Everett's formula and one can be deduced from the other by suitable rearrangements. It is also interesting to observe that Bessel's formula truncated after third differences is Everett's formula truncated after second differences.*

7.10. CHOICE OF AN INTERPOLATION FORMULA

So far we have derived several interpolation formulae for calculating y_p from equispaced values. Now we have to see which formula yields most accurate results in a particular problem.

The coefficients in the central difference formulae are smaller and converge faster than those in Newton's formulae. After a few terms, the coefficients in the Stirling's formula decrease more rapidly than those of the Bessel's formula and the coefficients of Bessel's formula decrease more rapidly than those of Newton's formula. As such, whenever possible, central difference formulae should be used in preference to Newton's formulae.

The right choice of an interpolation formula however, depends on the position of the interpolated value in the given data.

The following rules will be found useful :

1. To find a tabulated value near the beginning of the table, use Newton's forward formula.

2. To find a value near the end of the table, use Newton's backward formula.

3 To find an interpolated value near the centre of the table, use either Stirling's or Bessel's or Everett's formula.

If interpolation is required for p lying between $-\frac{1}{4}$ and $\frac{1}{4}$, prefer Stirling's formula.

If interpolation is desired for p lying between $\frac{1}{4}$ and $\frac{3}{4}$, use Bessel's or Everett's formula.

Example 7.7. *Find $f(22)$ from the Gauss forward formula :*

x :	20	25	30	35	40	45
$f(x)$:	354	332	291	260	231	204

(J.N.T.U., B. Tech., 2007)

Sol. Taking $x_0 = 25$, $h = 5$, we have to find the value of $f(x)$ for $x = 22$.

i.e., for $$p = \frac{x - x_0}{h} = \frac{22 - 25}{5} = -0.6.$$

The difference table is as follows :

x	p	y_p	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$	$\Delta^4 y_p$	$\Delta^5 y_p$
20	−1	354 $(= y_{-1})$					
			−22 (Δy_{-1})				
25	0	332 $(= y_0)$		−19 $(\Delta^2 y_{-1})$			
			−41 (Δy_0)		29 $(\Delta^3 y_{-1})$		
30	1	291 $(= y_1)$		10 $(\Delta^2 y_0)$		−37	
			−31 (Δy_1)		−8 $(\Delta^3 y_0)$		45
35	2	260 $(= y_2)$		2 $(\Delta^2 y_1)$		8	
			−29 (Δy_2)		0		
40	3	231 $(= y_3)$		2 $(\Delta^2 y_2)$			
			−27 (Δy_3)				
45	4	204 $(= y_4)$					

Gauss forward formula is

$$y_p = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)\,p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)\,p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \frac{(p+1)(p-1)(p-2)(p+2)}{5!}\Delta^5 y_{-2}$$

$$\therefore \quad f(22) = 332 + (0.6)(-41) + \frac{(-0.6)(-0.6-1)}{2!}(-19) + \frac{(-0.6+1)(-0.6)(-0.6-1)}{3!}(-8)$$
$$+ \frac{(-0.6+1)(-0.6)(-0.6-1)(-0.6-2)}{4!}(-37)$$
$$+ \frac{(-0.6+1)(-0.6)(-0.6-1)(-0.6-2)(-0.6+2)}{5!}(45)$$
$$= 332 + 24.6 - 9.12 - 0.512 + 1.5392 - 0.5241$$

Hence $f(22) = 347.983$.

Example 7.8. *Use Gauss's forward formula to evaluate y_{30}, given that $y_{21} = 18.4708$, $y_{25} = 17.8144$, $y_{29} = 17.1070$, $y_{33} = 16.3432$ and $y_{37} = 15.5154$.*

Sol. Taking $x_0 = 29$, $h = 4$, we require the value of y for $x = 30$

i.e. for $$p = \frac{x - x_0}{h} = \frac{30 - 29}{4} = 0.25.$$

The difference table is given below :

x	p	y_p	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$	$\Delta^4 y_p$
21	– 2	18.4708				
			– 0.6564			
25	– 1	17.8144		– 0.0510		
			– 0.7074		– 0.0074	
29	0	17.1070		– 0.0564		– 0.0022
			– 0.7638		– 0.0076	
33	1	16.3432		– 0.0640		
			– 0.8278			
37	2	15.5154				

Gauss's forward formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{1.2}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{1.2.3}\Delta^3 y_{-1}$$
$$+ \frac{(p+1)p(p-1)(p-2)}{1.2.3.4}\Delta^4 y_{-2} + \ldots$$

$$y_{30} = 17.1070 + (0.25)(-0.7638) + \frac{(0.25)(-0.75)}{2}(-0.0564)$$
$$+ \frac{(1.25)(0.25)(-0.75)}{6}(-0.0076) + \frac{(1.25)(0.25)(-0.75)(-1.75)}{24} \times (-0.0022)$$
$$= 17.1070 - 0.19095 + 0.00529 + 0.0003 - 0.00004 = 16.9216 \text{ approx.}$$

Example 7.9. *Using Gauss backward difference formula, find y (8) from the following table :*

x :	0	5	10	15	20	25
y :	7	11	14	18	24	32

(J.N.T.U., B. Tech., 2007)

Sol. Taking $x_0 = 10$, $h = 5$, we have to find y for $x = 8$, *i.e.*, for $p = \frac{x - x_0}{h} = \frac{8 - 10}{5} = -0.4.$

The difference table is as follows :

x	p	y_p	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$	$\Delta^4 y_p$	$\Delta^5 y_5$
0	– 2	7					
			4				
5	– 1	11		– 1			
			3		2		

10	0	14		1		– 1	
			4		1		0
15	1	18		2		– 1	
			6		0		
20	2	24		2			
			8				
25	3	32					

Gauss backward formula is

$$y_p = y_0 + p\Delta y_{-1} + \frac{(p+1)\,p}{2\,!}\,\Delta^2 y_{-1} + \frac{(p+1)\,p\,(p-1)}{3\,!}\,\Delta^3 y_{-2}$$

$$+ \frac{(p+2)\,(p+1)\,p\,(p-1)}{4\,!}\,\Delta^4 y_{-2} + \ldots\ldots$$

$$y_{(8)} = 14 + (-0.4)\,(3) + \frac{(-0.4+1)\,(-0.4)}{2\,!}\,(1) + \frac{(-0.4+1)\,(-0.4)\,(-0.4-1)}{3\,!}\,(2)$$

$$+ \frac{(-0.4+2)\,(-0.4+1)\,(-0.4)\,(-0.4-1)}{4\,!}\,(-1)$$

$$= 14 - 1.2 - 0.12 + 0.112 + 0.034$$

Hence $y_{(8)} = 12.826$

Example 7.10. *Interpolate by means of Gauss's backward formula, the population of a town for the year 1974, given that :*

Year	:	*1939*	*1949*	*1959*	*1969*	*1979*	*1989*
Population (in thousands)	:	*12*	*15*	*20*	*27*	*39*	*52*

(*Kottayam, B. Tech., 2005*)

Sol. Taking $x_0 = 1969$, $h = 10$, the population of the town is to be found for

$$p = \frac{1974 - 1969}{10} = 0.5.$$

The central difference table is

x	p	y_p	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$	$\Delta^4 y_p$	$\Delta^5 y_p$
1939	– 3	12					
			3				
1949	– 2	15		2			
			5		0		
1959	– 1	20		2		3	
			7		3		–10
1969	0	27		5		–7	
			12		– 4		
1979	1	39		1			
			13				
1989	2	52					

Gauss's backward formula is

$$y_p = y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)\,p\,(p-1)}{3!}\Delta^3 y_{-2}$$
$$+ \frac{(p+2)(p+1)\,p\,(p-1)}{4!}\Delta^4 y_{-2} + \frac{(p+2)(p+1)\,p\,(p-1)(p-2)}{5!}\Delta^5 y_3 + \ldots\ldots$$

i.e.
$$y_{0.5} = 27 + (0.5)(7) + \frac{(1.5)(0.5)}{2}(5) + \frac{(1.5)(0.5)(-0.5)}{6}(3)$$
$$+ \frac{(2.5)(1.5)(-0.5)}{24}(-7) + \frac{(2.5)(1.5)(0.5)(-0.5)(1.5)}{120}(-10)$$
$$= 27 + 3.5 + 1.875 - 0.1875 + 0.2743 - 0.1172 = 32.532 \text{ thousands approx.}$$

Example 7.11. *Employ Stirling's formula to compute $y_{12.2}$ from the following table ($y_x = 1 + \log_{10} \sin x$) :*

$x°$:	10	11	12	13	14
$10^5 u_x$:	23,967	28,060	31,788	35,209	38,368

(*V.T.U., B. Tech., 2004*)

Sol. Taking the origin at $x_0 = 12°$, $h = 1$ and $p = x - 12$, we have the following central difference table :

p	y_x	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$	$\Delta^4 y_x$
$-2 = x_{-2}$	$0.23967 = y_{-2}$				
		$0.04093 = \Delta y_{-2}$			
$-1 = x_{-1}$	$0.28060 = y_{-1}$		$-0.00365 = \Delta^2 y_{-2}$		
		$0.03728 = \Delta y_{-1}$		$0.00058 = \Delta^3 y_{-2}$	
$0 = x_0$	$0.31788 = y_0$		$-0.00307 = \Delta^2 y_{-1}$		$-0.00013 = \Delta^4 y_{-2}$
		$0.03421 = \Delta y_0$		$0.00045 = \Delta^3 y_{-1}$	
$1 = x_1$	$0.35209 = y_1$		$-0.00262 = \Delta^2 y_0$		
		$0.03159 = \Delta y_1$			
$2 = x_2$	$0.38368 = y_2$				

At $x = 12.2$, $p = 0.2$. (As p lies between $-\frac{1}{4}$ and $\frac{1}{4}$, the use of Stirling's formula will be quite suitable.)

Stirling's formula is

$$y_p = y_0 + \frac{p}{1}\cdot\frac{\Delta y_{-1} + \Delta y_{-0}}{2} + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p\,(p^2-1)}{3!}\cdot\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}$$
$$+ \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \ldots\ldots$$

When $p = 0.2$, we have

$$\therefore \quad y_{0.2} = 0.31788 + 0.2\left(\frac{0.03728 + 0.03421}{2}\right) + \frac{(0.2)^2}{2}(-0.00307)$$
$$+ \frac{(0.2)\,[(0.2)^2 - 1]}{6}\left(\frac{0.00058 - 0.00045}{2}\right) + \frac{(0.2)^2\,[(0.2)^2 - 1]}{24}(-0.00013)$$
$$= 0.31788 + 0.00715 - 0.00006 - 0.000002 + 0.0000002 = 0.32497.$$

Example 7.12. *Given*

$\theta°$:	0	5	10	15	20	25	30
$\tan\theta$:	0	0.0875	0.1763	0.2679	0.3640	0.4663	0.5774

Using Stirling's formula, estimate the value of tan 16°. (Anna, B. Tech., 2005)

Sol. Taking the origin at $\theta° = 15°$, $h = 5°$ and $p = \dfrac{\theta - 15}{5}$, we have the following central difference table :

p	$y = \tan\theta$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
−3	0.0000					
		0.0875				
−2	0.0875		0.0013			
		0.0888		0.0015		
−1	0.1763		0.0028		0.0002	
		<u>0.0916</u>		<u>0.0017</u>		<u>−0.0002</u>
0	<u>0.2679</u>		<u>0.0045</u>		<u>0.0000</u>	
		<u>0.0961</u>		<u>0.0017</u>		<u>0.0009</u>
1	0.3640		0.0062		0.0009	
		0.1023		0.0026		
2	0.4663		0.0088			
		0.1111				
3	0.5774					

At $\theta = 16°$, $\qquad p = \dfrac{16-15}{5} = 0.2$

Stirling's formula is

$$y_p = y_0 + \frac{p}{1} \cdot \frac{\Delta y_{-1} + \Delta y_0}{2} + \frac{p^2}{2!} \Delta^2 y_{-1} + \frac{p(p^2-1)}{3!} \cdot \frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}$$

$$+ \frac{p^2(p^2-1)}{4!} \Delta^4 y_{-2} + \ldots\ldots$$

$$\therefore \qquad y_{0.2} = 0.2679 + (0.2)\left(\frac{0.0916 + 0.0961}{2}\right) + \frac{(0.2)^2}{2!}(0.0045) + \ldots\ldots$$

$$= 0.2679 + 0.01877 + 0.00009 + \ldots\ldots = 0.28676$$

Hence $\quad \tan 16° = 0.28676.$

Example 7.13. *Apply Bessel's formula to obtain y_{25}, given $y_{20} = 2854$, $y_{24} = 3162$, $y_{28} = 3544$, $y_{32} = 3992$.* (S.V.T.U., B. Tech., 2007)

Sol. Taking the origin at $x_0 = 24$, $h = 4$, we have $p = \frac{1}{4}(x - 24)$.

∴ The central difference table is

p	y	Δy	$\Delta^2 y$	$\Delta^3 y$
– 1	2854			
		308		
0	3162		74	
		382		– 8
1	3544		66	
		448		
2	3992			

At $x = 25$, $p = (25 - 24)/4 = \frac{1}{4}$. (As p lies between $\frac{1}{4}$ and $\frac{3}{4}$, the use of Bessel's formula will yield accurate result.)

Bessel's formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{(p-\frac{1}{2})p(p-1)}{3!}\Delta^3 y_{-1} + \ldots \quad \ldots(1)$$

When $p = 0.25$, we have

$$y_p = 3162 + 0.25 \times 382 + \frac{0.25(-0.75)}{2}\left(\frac{74+66}{2}\right) + \frac{(-0.25)0.25(-0.75)}{2}(-8)$$

$$= 3162 + 95.5 - 6.5625 - 0.0625$$

$$= 3250.875 \text{ approx.}$$

Example 7.14. *Apply Bessel's formula to find the value of f (27.5) from the table :*

x :	25	26	27	28	29	30
$f(x)$:	4.000	3.846	3.704	3.571	3.448	3.333

(U.P.T.U., M.C.A., 2009)

Sol. Taking the origin at $x_0 = 27$, $h = 1$, we have $p = x - 27$

The central difference table is

x	p	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
25	– 2	4.000				
			– 0.154			
26	– 1	3.846		0.012		
			– 0.142		– 0.003	
27	0	3.704		0.009		0.004
			– 0.133		– 0.001	
28	1	3.571		0.010		– 0.001
			– 0.123		– 0.002	
29	2	3.448		0.008		
			– 0.115			
30	3	3.333				

At $x = 27.5$, $p = 0.5$ (As p lies between 1/4 and 3/4, the use of Bessel's formula will yield accurate result)

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Bessel's formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{\left(p-\frac{1}{2}\right)p(p-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \ldots\ldots$$

When $p = 0.5$, we have

$$y_p = 3.704 - \frac{(0.5)(0.5-1)}{2}\left(\frac{0.009+0.010}{2}\right) + 0$$

$$+ \frac{(0.5+1)(0.5)(0.5-1)(0.5-2)}{24}\left(\frac{-0.001+0.004}{2}\right)$$

$$= 3.704 - 0.11875 - 0.00006 = 3.585$$

Hence $f(27.5) = 3.585.$

Example 7.15. *Using Everett's formula, evaluate f(30) if f(20) = 2854, f(28) = 3162, f(36) = 7088, f(44) = 7984.* (U.P.T.U., B. Tech., 2006)

Sol. Taking the origin at $x_0 = 28$, $h = 8$, we have $p = \frac{x-28}{8}$. The central table is

x	p	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	– 1	2854			
			308		
28	0	3162		3618	
			3926		– 6648
36	1	7088		– 3030	
			896		
44	2	7984			

At $x = 30$, $p = \frac{30-28}{8} = 0.25$ and $q = 1 - p = 0.75$

Everett's formula is

$$y_p = qy_0 + \frac{q(q^2-1^2)}{3!}\Delta^2 y_{-1} + \frac{q(q^2-1^2)(q^2-2^2)}{5!}\Delta^4 y_{-2} + \ldots\ldots$$

$$+ py_1 + \frac{p(p^2-1^2)}{3!}\Delta^2 y_0 + \frac{p(p^2-1^2)(p^2-2^2)}{5!}\Delta^4 y_{-1} + \ldots\ldots$$

$$= (0.75)(3162) + \frac{0.75(0.75^2-1)}{6}(3618) + \ldots\ldots$$

$$+ 0.25(7080) + \frac{0.25(0.25^2-1)}{6}(-3030) + \ldots\ldots$$

$$= 2371.5 - 351.75 + 1770 + 94.69 = 3884.4$$

Hence $f(30) = 3884.4$

Example 7.16. *Given the table*

x :	310	320	330	340	350	360
$\log x$:	2.49136	2.50515	2.51851	2.53148	2.54407	2.55630,

find the value of log 337.5 by Everett's formula.

Sol. Taking the origin at $x_0 = 330$ and $h = 10$, we have $p = \dfrac{x - 330}{10}$.

$\therefore$ The central difference table is

p	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
−2	2.49136					
		0.01379				
−1	2.50515		− 0.00043			
		0.01336		0.00004		
0	2.51881		− 0.00039		− 0.00003	
		0.01297		0.00001		0.00004
1	2.53148		− 0.00038		0.00001	
		0.01259		0.00002		
2	2.54407		− 0.00036			
		0.01223				
3	2.55630					

To evaluate log 337.5 *i.e.* for $x = 337.5$, $p = \dfrac{337.5 - 330}{10} = 0.75$

(As $p > 0.5$ and $= 0.75$, Everett's formula will be quite suitable)

Everett's formula is

$$y_p = qy_0 + \frac{q(q^2 - 1^2)}{3!}\Delta^2 y_{-1} + \frac{q(q^2-1^2)(q^2-2^2)}{5!}\Delta^4 y_{-2} + \ldots\ldots$$

$$+ py_1 + \frac{p(p^2-1^2)}{3!}\Delta^2 y_0 + \frac{p(p^2-1^2)(p^2-2^2)}{5!}\Delta^4 y_{-1} + \ldots\ldots$$

$$= 0.25 \times 2.51851 + \frac{0.25(0.0625-1)}{6} \times (-0.00039)$$

$$+ \frac{0.25(0.0625-1)(0.0625-4)}{120} \times (-0.00003)$$

$$+ 0.75 \times 2.53148 + \frac{0.75(0.5625-1)}{6} \times (-0.00038)$$

$$+ \frac{0.75(0.5625-1)(0.5625-4)}{120} \times (0.00001)$$

$$= 0.62963 + 0.00002 - 0.0000002 + 1.89861 + 0.00002 + 0.0000001$$

$$= 2.52828 \text{ nearly.}$$

−0.001 − (−0.007)
−0.00 + 0.003
0.002

PROBLEMS 7.2

1. Find the y (25), given that $y_{20} = 24$, $y_{24} = 32$, $y_{28} = 35$, $y_{32} = 40$, using Gauss forward difference formula. *(J.N.T.U., B. Tech., 2006)*

2. Using Gauss forward formula, find a polynomial of degree four which takes the following values of the function $f(x)$:

x :	1	2	3	4	5
$f(x)$:	1	− 1	1	− 1	1

3. Using Gauss's forward formula, evaluate $f(3.75)$ from the table :

x :	2.5	3.0	3.5	4.0	4.5	5.0
y :	24.145	22.043	20.225	18.644	17.262	16.047

(Bhopal, B. Tech., 2002)

4. From the following table :

x :	1.00	1.05	1.10	1.15	1.20	1.25	1.30
e^x :	2.7183	2.8577	3.0042	3.1582	3.3201	3.4903	3.6693

Find $e^{1.17}$, using Gauss forward formula. *(U.P.T.U., B. Tech., 2006)*

5. Using Gauss's backward formula, estimate the number of persons earning wages between Rs. 60 and Rs. 70 from the following data :

Wages (Rs.) :	Below 40	40—60	60—80	80—100	100—120
No. of persons : (in thousands)	250	120	100	70	50

(Tirchirapalli, B.E., 2001)

6. Apply Gauss's backward formula to find sin 45° from the following table :

$\theta°$:	20	30	40	50	60	70	80
sin θ :	0.34202	0.502	0.64279	0.76604	0.86603	0.93969	0.98481

7. Using Stirling's formula find y_{35}, given $y_{20} = 512$, $y_{30} = 439$, $y_{40} = 346$, $y_{50} = 243$, where y_x represents the number of persons at age x years in a life table. *(Nagarjuna, B.E., 2003 S)*

8. The pressure p of wind corresponding to velocity v is given by the following data. Estimate p when $v = 25$.

v :	10	20	30	40
p :	1.1	2	4.4	7.9

9. Use Stirling's formula to evaluate $f(1.22)$, given

x :	1.0	1.1	1.2	1.3	1.4
$f(x)$:	0.841	0.891	0.932	0.963	0.985

(Tirchirapalli, B.E., 2001)

10. Calculate the value of $f(1.5)$ using Bessels' interpolation formula, from the table :

x :	0	1	2	3
$f(x)$:	3	6	12	15

(U.P.T.U., B. Tech., 2008)

11. Use Bessel's formula to obtain y_{25}, given $y_{20} = 24$, $y_{24} = 32$, $y_{28} = 35$, $y_{32} = 40$. *(Gurukul, M.Sc., 2000)*

12. Employ Bessel's formula to find the value of F at $x = 1.95$, given that

x :	1.7	1.8	1.9	2.0	2.1	2.2	2.3
F :	2.979	3.144	3.283	3.391	3.463	3.997	4.491

Which other interpolation formula can be used here ? Which is more appropriate ? Give reasons.

13. From the following table :

x :	20	25	30	35	40
$f(x)$:	11.4699	12.7834	13.7648	14.4982	15.0463

Find $f(34)$ using Everett's formula. *(Madras, B.E., 2000 S)*

14. Apply Everett's formula to obtain u_{25}, given $u_{20} = 2854$, $u_{24} = 3162$, $u_{28} = 3544$, $u_{32} = 3992$.

15. Given the table :

x :	310	320	330	340	350	360
$\log x$:	2.4914	2.5052	2.5185	2.5315	2.5441	2.5563,

find the value of log 337.5 by Gauss, Stirling, Bessel and Everette formulae.

7.11. INTERPOLATION WITH UNEQUAL INTERVALS

The various interpolation formulae derived so far possess the disadvantage of being applicable only to equally spaced values of the argument. It is, therefore, desirable to develop interpolation formulae for unequally spaced values of x. Now we shall study two such formulae :

(*i*) Lagrange's interpolation formula

(*ii*) Newton's general interpolation formula with divided differences.

7.12. LAGRANGE'S INTERPOLATION FORMULA

If $y = f(x)$ takes the value $y_0, y_1, \ldots\ldots, y_n$ corresponding to $x = x_0, x_1, \ldots\ldots, x_n$, then

$$f(x) = \frac{(x-x_1)(x-x_2)\ldots\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots\ldots(x_0-x_n)}\,y_0 + \frac{(x-x_0)(x-x_2)\ldots\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots\ldots(x_1-x_n)}\,y_1$$
$$+ \ldots\ldots + \frac{(x-x_0)(x-x_1)\ldots\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots\ldots(x_n-x_{n-1})}\,y_n \qquad \ldots(1)$$

This is known as *Lagrange's interpolation formula for unequal intervals.*

Proof. Let $y = f(x)$ be a function which takes the values $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_n, y_n)$. Since there are $n + 1$ pairs of values of x and y, we can represent $f(x)$ by a polynomial in x of degree n. Let this polynomial be of the form

$$y = f(x) = a_0(x - x_1)(x - x_2) \ldots\ldots (x - x_n) + a_1(x - x_0)(x - x_2) \ldots\ldots (x - x_n)$$
$$+ a_2(x - x_0)(x - x_1)(x - x_3) \ldots\ldots (x - x_n) + \ldots + a_n(x - x_0)(x - x_1) \ldots\ldots (x - x_{n-1}) \quad \ldots(2)$$

Putting $x = x_0$, $y = y_0$, in (2), we get

$$y_0 = a_0(x_0 - x_1)(x_0 - x_2) \ldots\ldots (x_0 - x_n)$$
$$a_0 = y_0/[(x_0 - x_1)(x_0 - x_2) \ldots\ldots (x_0 - x_n)]$$

Similarly putting $x = x_1$, $y = y_1$ in (2), we have $a_1 = y_1/[(x_1 - x_0)(x_1 - x_2) \ldots\ldots (x_1 - x_n)]$

Proceeding the same way, we find $a_2, a_3 \ldots\ldots a_n$.

Substituting the values of $a_0, a_1, \ldots\ldots, a_n$ in (2), we get (1).

Obs. Lagranges interpolation formula (1) for n points is a polynomial of degree $(n - 1)$ which is known as *Lagrangian polynomial* and is very simple to implement on a computer.

This formula can also be used to split the given function into partial fractions.

For on dividing both sides of (1) by $(x - x_0)(x - x_1) \ldots\ldots (x - x_n)$, we get

$$\frac{f(x)}{(x-x_0)(x-x_1)\ldots\ldots(x-x_n)} = \frac{y_0}{(x_0-x_1)(x_0-x_2)\ldots\ldots(x_0-x_n)}\cdot\frac{1}{x-x_0}$$
$$+ \frac{y_1}{(x_1-x_0)(x_1-x_2)\ldots\ldots(x_1-x_n)}\cdot\frac{1}{x-x_1} + \ldots\ldots$$
$$+ \frac{y_n}{(x_n-x_0)(x_n-x_1)\ldots\ldots(x_n-x_{n-1})}\cdot\frac{1}{x-x_n}.$$

Example 7.17. *Given the values*

x :	5	7	11	13	17
$f(x)$:	150	392	1452	2366	5202,

evaluate $f(9)$, using Lagrange's formula (V.T.U., B. Tech., 200...)

Sol. (*i*) Here $x_0 = 5,\quad x_1 = 7,\quad x_2 = 11,\quad x_3 = 13,\quad x_4 = 17$

and $y_0 = 150, y_1 = 392, y_2 = 1452, y_3 = 2366, y_4 = 5202.$

Putting $x = 9$ and substituting the above values in Lagrange's formula, we get

$$f(9) = \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)} \times 150 + \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)} \times 392$$

$$+ \frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)} \times 1452$$

$$+ \frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)} \times 2366$$

$$+ \frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)} \times 5202$$

$$= -\frac{50}{3} + \frac{3136}{15} + \frac{3872}{3} - \frac{2366}{3} + \frac{578}{5} = 810$$

Example 7.18. *Find the polynomial $f(x)$ by using Lagrange's formula and hence find $f(3)$ for*

x :	0	1	2	5
$f(x)$:	2	3	12	147

(Anna, B. Tech., 200...)

Sol. Here $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 5$

and $y_0 = 2, y_1 = 3, y_2 = 12, y_3 = 147.$

Lagrange's formula is

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$= \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}(2) + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}(3)$$

$$+ \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}(12) + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}(147)$$

Hence $f(x) = x^3 + x^2 - x + 2$

$\therefore$ $f(3) = 27 + 9 - 3 + 2 = 35.$

Example 7.19. *A curve passes through the points (0, 18), (1, 10), (3, – 18) and (6, 90). Find the slope of the curve at $x = 2$.* (J.N.T.U., B.Tech., 2009)

Sol. Here $x_0 = 0, x_1 = 1, x_2 = 3, x_3 = 6$ and $y_0 = 18, y_1 = 10, y_2 = -18, y_3 = 90$

Since the values of x are unequally spaced, we use the Lagrange's formula :

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$= \frac{(x-1)(x-3)(x-6)}{(0-1)(0-3)(0-6)}(18) + \frac{(x-0)(x-3)(x-6)}{(1-0)(1-3)(1-6)}(10)$$

$$+ \frac{(x-0)(x-1)(x-6)}{(3-0)(3-1)(3-6)}(-18) + \frac{(x-0)(x-1)(x-3)}{(6-0)(6-1)(6-3)}(90)$$

$$= (-x^3 + 10x^2 - 27x + 18) + (x^3 - 9x^2 + 18x) + (x^3 - 7x^2 + 6x) + (x^3 - 4x^2 + 3x)$$

i.e., $$y = 2x^3 - 10x^2 + 18$$

Thus the slope of the curve at $x = 2 = \left(\frac{dy}{dx}\right)_{x=2}$

$$= (6x^2 - 20x)_{x=2} = -16.$$

Example 7.20. *Using Lagrange's formula, express the function* $\frac{3x^2+x+1}{(x-1)(x-2)(x-3)}$ *as a sum of partial fractions.*

Sol. Let us evaluate $y = 3x^2 + x + 1$ for $x = 1$, $x = 2$ and $x = 3$

These values are

x :	$x_0 = 1$	$x_1 = 2$	$x_2 = 3$
y :	$y_0 = 5$	$y_1 = 15$	$y_2 = 31$

Lagrange's formula is

$$y = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

Substituting the above values, we get

$$y = \frac{(x-2)(x-3)}{(1-2)(1-3)}(5) + \frac{(x-1)(x-3)}{(2-1)(2-3)}(15) + \frac{(x-1)(x-2)}{(3-1)(3-2)}(31)$$

$$= 2.5\,(x-2)(x-3) - 15\,(x-1)(x-3) + 15.5\,(x-1)(x-2)$$

Thus $$\frac{3x^2+x+1}{(x-1)(x-2)(x-3)} = \frac{2.5(x-2)(x-3) - 15(x-1)(x-3) + 15.5(x-1)(x-2)}{(x-1)(x-2)(x-3)}$$

$$= \frac{2.5}{x-1} - \frac{15}{x-2} + \frac{15.5}{x-3}$$

Example 7.21. *Find the missing term in the following table using interpolation :*

x :	0	1	2	3	4
y :	1	3	9	...	81

Sol. Since the given data is unevenly spaced, therefore we use Lagrange's interpolation formula :

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

Here we have $x_0 = 0 \quad x_1 = 1 \quad x_2 = 2 \quad x_3 = 4$
$y_0 = 1 \quad y_1 = 3 \quad y_2 = 9 \quad y_3 = 81$

$$\therefore \quad y = \frac{(x-1)(x-2)(x-4)}{(0-1)(0-2)(0-4)}(1) + \frac{(x-0)(x-2)(x-4)}{(1-0)(1-2)(1-4)}(3)$$
$$+ \frac{(x-0)(x-1)(x-4)}{(2-0)(2-1)(2-4)}(9) + \frac{(x-0)(x-1)(x-2)}{(4-0)(4-1)(4-2)}(81)$$

When $x = 3$, then

$$\therefore \quad y = \frac{(3-1)(3-2)(3-4)}{-8} + 3(3-2)(3-4) + \frac{3(3-1)(3-4)}{-4}(9)$$
$$+ \frac{3(3-1)(3-2)}{24}(81) = \frac{1}{4} - 3 + \frac{27}{2} + \frac{81}{4} = 31.$$

Hence the missing term for $x = 3$ is $y = 31$.

Example 7.22. *Find the distance moved by a particle and its acceleration at the end of 4 seconds, if the time verses velocity data is as follows :*

t :	0	1	3	4
v :	21	15	12	10

Sol. Since the values of t are not equispaced, we use Lagrange's formula :

$$v = \frac{(t-t_1)(t-t_2)(t-t_3)}{(t_0-t_1)(t_0-t_2)(t_0-t_3)} v_0 + \frac{(t-t_0)(t-t_2)(t-t_3)}{(t_1-t_0)(t_1-t_2)(t_1-t_3)} v_1$$
$$+ \frac{(t-t_0)(t-t_1)(t-t_3)}{(t_2-t_0)(t_2-t_1)(t_2-t_3)} v_2 + \frac{(t-t_0)(t-t_1)(t-t_2)}{(t_3-t_0)(t_3-t_1)(t_3-t_2)} v_3$$

i.e.,
$$v = \frac{(t-1)(t-3)(t-4)}{(-1)(-2)(-4)}(21) + \frac{t(t-3)(t-4)}{(1)(-2)(-3)}(15)$$
$$+ \frac{t(t-1)(t-4)}{(3)(2)(-1)}(12) + \frac{t(t-1)(t-3)}{(4)(3)(1)}(10)$$

i.e.,
$$v = \frac{1}{12}(-5t^3 + 38t^2 - 105t + 252)$$

$$\therefore \quad \text{Distance moved } s = \int_0^4 v\,dt = \frac{1}{12}\int_0^4 (-5t^3 + 38t^2 - 105t + 252)\,dt \qquad \left[\because v = \frac{ds}{dt}\right]$$

$$= \frac{1}{12}\left(\frac{-5t^4}{4} + \frac{38t^3}{3} - \frac{105t^2}{2} + 252t\right)_0^4$$

$$= \frac{1}{12}\left(-320 + \frac{2432}{3} - 840 + 1008\right) = 54.9$$

Also acceleration $= \frac{dv}{dt} = \frac{1}{2}(-15t^2 + 76t - 105 + 0)$

Hence acceleration at $(t = 4) = \frac{1}{12}(-15(16) + 76(4) - 105) = -3.4.$

PROBLEMS 7.3

1. Use Lagrange's interpolation formula to find the value of y when $x = 10$, if the following values of x and y are given :

x :	5	6	9	11
y :	12	13	14	16.

(J.N.T.U., B. Tech., 2008)

2. The following table gives the viscosity of an oil as a function of temperature. Use Lagrange's formula to find viscosity of oil at a temperature of 140°.

Temp.° :	110	130	160	190
Viscosity :	10.8	8.1	5.5	4.8

3. Given $\log_{10} 654 = 2.8156$, $\log_{10} 658 = 2.8182$, $\log_{10} 659 = 2.8189$, $\log_{10} 661 = 2.8202$, find by using Lagrange's formula, the value of $\log_{10} 656$. *(Hazaribagh, B.E., 2009)*

4. The following are the measurements T made on a curve recorded by oscilograph representing a change of current I due to a change in the conditions of an electric current.

T :	1.2	2.0	2.5	3.0
I :	1.36	0.58	0.34	0.20

Using Lagrange's formula, find I and $T = 1.6$. *(J.N.T.U., B. Tech., 2009)*

5. Using Lagrange's interpolation, calculate the profit in the year 2000 from the following data :

Year :	1997	1999	2001	2002
Profit in Lakhs of Rs :	43	65	159	248

(Anna, B. Tech., 2004)

6. Use Lagrange's formula to find the form of $f(x)$, given

x :	0	2	3	6
$f(x)$:	648	704	729	792.

(Madras, B.E., 2003 S)

7. If $y(1) = -3$, $y(3) = 9$, $y(4) = 30$, $y(6) = 132$, find the Lagrange's interpolation polynomial that takes the same values as y at the given points. *(V.T.U., B. Tech., 2006)*

8. Given $f(0) = -18$, $f(1) = 0$, $f(3) = 0$, $f(5) = -248$, $f(6) = 0$, $f(9) = 13104$, find $f(x)$.

(Nagarjuna, B.E., 2003 S)

9. Find the missing term in the following table using interpolation

x :	1	2	4	5	6
y :	14	15	5	...	9.

10. Using Lagrange's formula, express the function $\dfrac{x^2+x-3}{x^3-2x^2-x+2}$ as sum of partial fractions.

11. Using Lagrange's formula, express the function $\dfrac{x^2+6x-1}{(x^2-1)(x-4)(x-6)}$ as a sum of partial fractions.

[**Hint.** Tabulate the values of $f(x) = x^2 + 6x - 1$ for $x = -1, 1, 4, 6$ and apply Lagrange's formula.]

7.13 DIVIDED DIFFERENCES

The Lagrange's formula has the drawback that if another interpolation value were inserted, then the interpolation coefficients are required to be recalculated. This labour of recomputing the interpolation coefficients is saved by using Newton's general interpolation formula which employs what are called 'divided differences'. Before deriving this formula, we shall first define these differences.

If $(x_0, y_0), (x_1, y_1), (x_2, y_2)$, ... be given points, then the *first divided difference* for the arguments x_0, x_1 is defined by the relation $[x_0, x_1] = \dfrac{y_1 - y_0}{x_1 - x_0}$.

Similarly $[x_1, x_2] = \dfrac{y_2 - y_1}{x_2 - x_1}$ and $[x_2, x_3] = \dfrac{y_3 - y_2}{x_3 - x_2}$ etc.

The *second divided difference* for x_0, x_1, x_2 is defined as $[x_0, x_1, x_2] = \dfrac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$.

The *third divided difference* for x_0, x_1, x_2, x_3 is defined as

$$[x_0, x_1, x_2, x_3] = \frac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_3 - x_0} \text{ and so on.}$$

Obs. 1. *The divided differences are symmetrical in their arguments i.e. independent of the order of the arguments.* For it is easy to write $[x_0, x_1] = \dfrac{y_0}{x_0 - x_1} + \dfrac{y_1}{x_1 - x_0} = [x_1, x_0]$, $[x_0, x_1, x_2]$

$$= \frac{y_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{y_2}{(x_2 - x_0)(x_2 - x_1)}$$

$= [x_1, x_2, x_0]$ or $[x_2, x_0, x_1]$ and so on.

Obs. 2. *The nth divided differences of a polynomial of the nth degree are constant.*

Let the arguments be equally spaced so that $x_1 - x_0 = x_2 - x_1 = \ldots = x_n - x_{n-1} = h$. Then

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h}$$

$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0 - x_1]}{x_2 - x_0} = \frac{1}{2h}\left\{\frac{\Delta y_1}{h} - \frac{\Delta y_0}{h}\right\}$$

$= \frac{1}{2!\,h^2}\,\Delta^2 y_0$ and in general, $[x_0, x_1, x_2, \ldots\ldots, x_n] = \frac{1}{n!\,h^n}\,\Delta^n y_0$.

If the tabulated function is a nth degree polynomial, then $\Delta^n y_0$ will be constant. Hence the nth divided differences will also be constant.

7.14. NEWTON'S DIVIDED DIFFERENCE FORMULA

Let $y_0, y_1, \ldots\ldots, y_n$ be the values of $y = f(x)$ corresponding to the arguments $x_0, x_1, \ldots\ldots, x_n$. Then from the definition of divided differences, we have

$$[x, x_0] = \frac{y - y_0}{x - x_0}$$

so that
$$y = y_0 + (x - x_0)[x, x_0] \qquad \ldots(1)$$

Again
$$[x, x_0, x_1] = \frac{[x, x_0] - [x_0, x_1]}{x - x_1}$$

which gives
$$[x, x_0] = [x_0, x_1] + (x - x_1)[x, x_0, x_1]$$

Substituting this value of $[x, x_0]$ in (1), we get

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x, x_0, x_1] \qquad \ldots(2)$$

Also
$$[x, x_0, x_1, x_2] = \frac{[x, x_0, x_1] - [x_0\ x_1, x_2]}{x - x_2}$$

which gives
$$[x, x_0, x_1] = [x_0, x_1, x_2] + (x - x_2)[x, x_0, x_1, x_2]$$

Substituting this value of $[x, x_0, x_1]$ in (2), we obtain

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)[x, x_0, x_1, x_2]$$

Proceeding in this manner, we get

$$\begin{aligned} y = f(x) = y_0 &+ (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] \\ &+ (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3] + \ldots\ldots \\ &+ (x - x_0)(x - x_1) \ldots\ldots (x - x_n)[x, x_0, x_1, \ldots\ldots x_n] \end{aligned} \qquad \ldots(3)$$

which is called *Newton's general interpolation formula with divided differences.*

7.15 RELATION BETWEEN DIVIDED AND FORWARD DIFFERENCES

If $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots\ldots$ be the given points, then

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}$$

Also
$$\Delta y_0 = y_1 - y_0$$

If $x_0, x_1, x_2, \ldots\ldots$ are equispaced, then $x_1 - x_0 = h$, so that

$$[x_0, x_1] = \frac{\Delta y_0}{h}$$

Similarly
$$[x_1, x_2] = \frac{\Delta y_1}{h}$$

Now
$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$$

$$= \frac{\Delta y_1/h - \Delta y_0/h}{2h} \qquad [\because \; x_2 - x_0 = 2h]$$

$$= \frac{\Delta y_1 - \Delta y_0}{2h^2} = \frac{\Delta^2 y_0}{2h^2}$$

Thus $\qquad [x_0, x_1, x_2] = \dfrac{\Delta^2 y_0}{2!\,h^2}$

Similarly $\qquad [x_1, x_2, x_3] = \dfrac{\Delta^2 y_0}{2!\,h^2}$

$$\therefore \qquad [x_0, x_1, x_2, x_3] = \frac{\Delta^2 y_1/2h^2 - \Delta^2 y_0/2h^2}{x_3 - x_0} = \frac{\Delta^2 y_1 - \Delta^2 y_0}{2h^2(3h)} \qquad [\because \; x_3 - x_0 = 3h]$$

Thus $\qquad [x_0, x_1, x_2, x_3] = \dfrac{\Delta^3 y_0}{3!\,h^3}$

In general, $[x_0, x_1, \ldots\ldots, x_n] = \dfrac{\Delta^n y_0}{n!\,h^n}$.

This is the relation between divided and forward differences.

Example 7.23. *Given the values*

x :	5	7	11	13	17
$f(x)$:	150	392	1452	2366	5202,

evaluate f(9), using Newton's divided difference formula.

Sol. The divided differences table is

x	y	*1st divided differences*	*2nd divided differences*	*3rd divided differences*
5	150			
		$\dfrac{392 - 150}{7 - 5} = 121$		
7	392		$\dfrac{265 - 121}{11 - 5} = 24$	
		$\dfrac{1452 - 392}{11 - 7} = 265$		$\dfrac{32 - 24}{13 - 5} = 1$
11	1452		$\dfrac{457 - 265}{13 - 7} = 32$	
		$\dfrac{2366 - 1452}{13 - 11} = 457$		$\dfrac{42 - 32}{17 - 7} = 1$
13	2366		$\dfrac{709 - 457}{17 - 11} = 42$	
		$\dfrac{5202 - 2366}{17 - 13} = 709$		
17	5202			

Taking $x = 9$ in the Newton's divided difference formula, we obtain

$$f(9) = 150 + (9-5) \times 121 + (9-5)(9-7) \times 24 + (9-5)(9-7)(9-11) \times 1$$
$$= 150 + 484 + 192 - 16 = 810.$$

Example 7.24. *Using Newton's divided differences formula, evaluate f(8) and f(15) given :*

x :	*4*	*5*	*7*	*10*	*11*	*13*
f(x) :	*48*	*100*	*294*	*900*	*1210*	*2028*

(V.T.U., B. Tech., 2008)

Sol. The divided differences table is

x	$f(x)$	*1st divided differences*	*2nd divided differences*	*3rd divided differences*	*4th divided differences*
4	48				
		52			
5	100		15		
		97		1	
7	294		21		0
		202		1	
10	900		27		0
		310		1	
11	1210		33		
		409			
13	2028				

Taking $x = 8$ in the Newton's divided difference formula, we obtain

$$f(8) = 48 + (8-4)\,52 + (8-4)(8-5)\,15 + (8-4)(8-5)(8-7)\,1$$
$$= \mathbf{448.}$$

Similarly $f(15) = 3150$.

Example 7.25. *Determine f(x) as a polynomial in x for the following data :*

x :	*– 4*	*– 1*	*0*	*2*	*5*
f(x) :	*1245*	*33*	*5*	*9*	*1335.*

(V.T.U., B. Tech., 2007)

Sol. The divided differences table is

x	$f(x)$	*1st divided differences*	*2nd divided differences*	*3rd divided differences*	*4th divided differences*
– 4	1245				
		– 404			
– 1	33		94		
		– 28		– 14	
0	5		10		3
		2		13	
2	9		88		
		442			
5	1335				

Applying Newton's divided difference formula

$$f(x) = f(x_0) + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + \ldots$$

$$= 1245 + (x + 4)(-404) + (x + 4)(x + 1)(94)$$

$$+ (x + 4)(x + 1)(x - 0)(-14) + (x + 4)(x + 1)x(x - 2)(3)$$

$$= 3x^4 - 5x^3 + 6x^2 - 14x + 5.$$

Example 7.26. *Using Newton's divided difference formula, find the missing value from the table :*

x :	*1*	*2*	*4*	*5*	*6*
y :	*14*	*15*	*5*	...	*9*

Sol. The divided difference table is

x	y	*1st divided differences*	*2nd divided differences*	*3rd divided differences*
1	14			
		$\frac{15-14}{2-1} = 1$		
2	15		$\frac{-5-1}{4-1} = -2$	
		$\frac{5-15}{4-2} = -5$		$\frac{7/4+2}{6-1} = \frac{3}{4}$
4	5		$\frac{2+5}{6-2} = \frac{7}{4}$	
		$\frac{9-5}{6-4} = 2$		
6	9			

Newton's divided difference formula is

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2]$$
$$+ (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3] + \ldots\ldots$$

$$= 14 + (x - 1)(1) + (x - 1)(x - 2)(-2) + (x - 1)(x - 2)(x - 4)\left(\frac{3}{4}\right)$$

Putting $x = 5$, we get

$$y(5) = 14 + 4 + (4)(3)(-2) + (4)(3)(1)\left(\frac{3}{4}\right) = 3.$$

Hence missing value is 3.

PROBLEMS 7.4

1. Find the third divided difference with arguments 2, 4, 9, 10 of the function $f(x) = x^3 - 2x$. *(U.P.T.U., B. Tech., 2005)*
2. Obtain the Newton's divided difference interpolating polynomial and hence find $f(6)$:

x :	3	7	9	10
$f(x)$:	168	120	72	63

(U.P.T.U., B. Tech., 2007)

3. Using Newton's divided differences interpolation, find $u(3)$, given that $u(1) = -26$, $u(2) = 12$, $u(4) = 256$, $u(6) = 844$. *(Anna, B.E., 2004)*
4. A thermocouple gives the following output for rise in temperature

Temp (°C)	0	10	20	30	40	50
Output (mV)	0.0	0.4	0.8	1.2	1.6	2.0

Find the output of thermocouple for 37°C temperature using Newton's divided difference formula.

5. Using Newton's divided difference interpolation, find the polynomial of the given data :

x :	−1	0	1	3
$f(x)$:	2	1	0	−1

(Anna, B.E., 2007)

6. For the following table, find $f(x)$ as a polynomial in x using Newton's divided difference formula :

x :	5	6	9	11
$f(x)$:	12	13	14	16

7. Use Newton's divided difference formula to find $f(x)$ from the following data :

x :	0	1	2	4	5	6
$f(x)$:	1	14	15	5	6	19

8. The observed values of a function are respectively 168, 120, 72 and 63 at the four positions 3, 7, 9 and 10 of the independent variable. What is the best estimate for the value of the function at the position 6.
9. Find the equation of the cubic curve which passes through the points (4, −43), (7, 83), (9, 327) and (12, 1053).
10. Find the missing term in the following table using Newton's divided difference formula

x :	0	1	2	3	4
y :	1	3	9	...	81

11. Certain corresponding values of x and $\log_{10} x$ are given below :

x :	300	304	305	307
$\log_{10} x$:	2.4771	2.4829	2.4843	2.4871

Find $\log_{10} 310$ by (*i*) Lagrange's formula.
(*ii*) Newton's divided difference formula.

7.16. HERMITE'S INTERPOLATION FORMULA

This formula is similar to the Lagrange's interpolation formula. In Lagrange's method, the interpolating polynomial $P(x)$ agrees with $y(x)$ at the points $x_0, x_1, \ldots\ldots, x_n$, whereas in Hermite's method $P(x)$ and $y(x)$ as well as $P'(x)$ and $y'(x)$ coincide at the $(n + 1)$ points *i.e.*

$$P(x_i) = y(x_i) \text{ and } P'(x_i) = y'(x_i)\ ;\ i = 0, 1, \ldots\ldots, n \qquad \ldots(1)$$

As there are $2(n + 1)$ conditions in (1), $(2n + 2)$ coefficients are to be determined. Therefore $P(x)$ is a polynomial of degree $(2n + 1)$.

We assume that $P(x)$ is expressible in the form

$$P(x) = \sum_{i=0}^{n} U_i(x)\, y(x_i) + \sum_{i=0}^{n} V_i(x)\, y'(x_i) \qquad \ldots(2)$$

$$y = \frac{1}{6}\left[(4-x)^3\left(-\frac{36}{7}\right) + (x-3)^3\left(\frac{30}{7}\right)\right] + (4-x)\left[1 - \frac{1}{6}\left(-\frac{36}{7}\right)\right] + (x-3)\left(0 - \frac{5}{7}\right)$$

i.e., $y = 1.57\,x^3 - 16.71\,x^2 + 57.86x - 64.57 \quad (3 \le x \le 4)$

Taking $i = 3$ in (A), the cubic spline in $(4 \le x \le 5)$ is

$$y = \frac{1}{6}\left[(1-x)^3\left(\frac{30}{7}\right)\right] + (5-x)\left(-\frac{5}{7}\right) + (x-4)\,(1)$$

i.e., $y = -0.71x^3 + 2.14x^2 - 0.43x - 6.86. \quad (4 \le x \le 5)$

PROBLEMS 7.6

1. Find the cubic splines for the following table of values :

x :	1	2	3
y :	-6	-1	16

Hence evaluate $y(1.5)$ and $y'(2)$.

2. The following values of x and y are given :

x :	1	2	3	4
y :	1	5	11	8

Using cubic splines, show that

(*i*) $y(1.5) = 2.575$ (*ii*) $y'(3) = 2.067$.

3. Find the cubic spline corresponding to the interval [2, 3] from the following table :

x :	1	2	3	4	5
y :	30	15	32	18	25

Hence compute (*i*) $y(2.5)$ and (*ii*) $y'(3)$.

7.18. DOUBLE INTERPOLATION

So far, we have derived interpolation formulae to approximate a function of a single variable. In case of functions, of two variables, we interpolate with respect to the first variable keeping the other variable constant. Then interpolate with respect to the second variable.

Similarly, we can extend the said procedure for functions of three variables.

7.19. INVERSE INTERPOLATION

So far, given a set of values of x and y, we have been finding the value of y corresponding to a certain value of x. On the other hand, the process of estimating the value of x for a value of y (which is not in the table) is called *inverse interpolation*. When the values of x are unequally spaced *Lagrange's method* is used and when the values of x are equally spaced, the *Iterative method* should be employed.

7.20. LAGRANGE'S METHOD

This procedure is similar to Lagrange's interpolation formula (p. 207), the only difference being that x is assumed to be expressible as a polynomial in y.

Lagrange's formula is merely a relation between two variables either of which may be taken as the independent variable. Therefore, on inter-changing x and y in the Lagrange's formula, we obtain

$$x = \frac{(y-y_1)(y-y_2)\ldots\ldots(y-y_n)}{(y_0-y_1)(y_0-y_2)\ldots\ldots(y_0-y_n)}\,x_0 + \frac{(y-y_0)(y-y_2)\ldots\ldots(y-y_n)}{(y_1-y_0)(y_1-y_2)\ldots\ldots(y_1-y_n)}\,x_1 + \ldots\ldots + \frac{(y-y_0)(y-y_1)\ldots\ldots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\ldots\ldots(y_n-y_{n-1})} \quad \ldots(1)$$

which is used for inverse interpolation.

Example 7.34. *The following table gives the values of x and y :*

x :	1.2	2.1	2.8	4.1	4.9	6.2
y :	4.2	6.8	9.8	13.4	15.5	19.6

Find the value of x corresponding to $y = 12$, using Lagrange's technique. (V.T.U., B.E., 2009)

Sol. Here $x_0 = 1.2,\ x_1 = 2.1,\ x_2 = 2.8,\ x_3 = 4.1,\ x_4 = 4.9,\ x_5 = 6.2$
and $y_0 = 4.2,\ y_1 = 6.8,\ y_2 = 9.8,\ y_3 = 13.4,\ y_4 = 15.5, y_5 = 19.6.$
Taking $y = 12$, the above formula (1) gives

$$x = \frac{(12-6.8)(12-9.8)(12-13.4)(12-15.5)(12-19.6)}{(4.2-6.8)(4.2-9.8)(4.2-13.4)(4.2-15.5)(4.2-19.6)} \times 1.2$$
$$+ \frac{(12-4.2)(12-9.8)(12-13.4)(12-15.5)(12-19.6)}{(6.8-4.2)(6.8-9.8)(6.8-13.4)(6.8-15.5)(6.8-19.6)} \times 2.1$$
$$+ \frac{(12-4.2)(12-6.8)(12-13.4)(12-15.5)(12-19.6)}{(9.8-4.2)(9.8-6.8)(9.8-13.4)(9.8-15.5)(9.8-19.6)} \times 2.8$$
$$+ \frac{(12-4.2)(12-6.8)(12-9.8)(12-15.5)(12-19.6)}{(13.4-4.2)(13.4-6.8)(13.4-9.8)(13.4-15.5)(13.4-19.6)} \times 4.1$$
$$+ \frac{(12-4.2)(12-6.8)(12-9.8)(12-13.4)(12-19.6)}{(15.5-4.2)(15.5-6.8)(15.5-9.8)(15.5-13.4)(15.5-19.6)} \times 4.9$$
$$+ \frac{(12-4.2)(12-6.8)(12-9.8)(12-13.4)(12-15.5)}{(19.6-4.2)(19.6-6.8)(19.6-9.8)(19.6-13.4)(19.6-15.5)} \times 6.2$$
$$= 0.022 - 0.234 + 1.252 + 3.419 - 0.964 + 0.055 = 3.55.$$

Example 7.35. *Apply Lagrange's formula inversely to obtain a root of the equation $f(x) = 0$, given that $f(30) = -30$, $f(34) = -13$, $f(38) = 3$, and $f'(42) = 18$.* (V.T.U., B. Tech., 2009)

Sol. Here $x_0 = 30, x_1 = 34, x_2 = 38, x_3 = 42$
and $y_0 = -30, y_1 = -13, y_2 = 3, y_3 = 18$

It is required to find x corresponding to $y = f(x) = 0$.

Taking $y = 0$, the Lagrange's formula gives,

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)}\,x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y-y_2)(y_1-y_3)}\,x_1$$
$$+ \frac{(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)}\,x_2 + \frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)}\,x_3$$

$$= \frac{13(-3)(-18)}{(-17)(-33)(-48)} \times 30 + \frac{30(-3)(-18)}{17(-16)(-31)} \times 34$$

$$+ \frac{30(13)(-18)}{33(16)(-15)} \times 38 + \frac{30(13)(-3)}{48(31)(15)} \times 42$$

$$= -0.782 + 6.532 + 33.682 - 2.202 = 37.23.$$

Hence the desired root of $f(x) = 0$ is 37.23.

7.21. ITERATIVE METHOD

Newton's forward interpolation formula (p. 163) is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots\ldots$$

From this, we get

$$p = \frac{1}{\Delta y_0}\left[y_p - y_0 - \frac{p(p-1)}{2!}\Delta^2 y_0 - \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 - \ldots..\right] \qquad \ldots(1)$$

Neglecting the second and higher differences, we obtain the first approximation to p as

$$p_1 = (y_p - y_0)/\Delta y_0 \qquad \ldots(2)$$

To find the second approximation, retaining the term with second differences in (1) and replacing p by p_1, we get

$$p_2 = \frac{1}{\Delta y_0}\left[y_p - y_0 - \frac{p_1(p_1-1)}{2!}\Delta^2 y_0\right] \qquad \ldots(3)$$

To find the third approximation, retaining the term with third differences in (1) and replacing every p by p_2, we have

$$p_3 = \frac{1}{\Delta y_0}\left[y_p - y_0 - \frac{p_2(p_2-1)}{2!}\Delta^2 y_0 - \frac{p_2(p_2-1)(p_2-2)}{3!}\Delta^3 y_0\right]$$

and so on. This process is continued till two successive approximations of p agree with each other.

Obs. This technique can equally well be applied by starting with any other interpolation formula.

This method is a powerful *iterative procedure for finding the roots of an equation to a good degree of accuracy.*

Example 7.36. *The following values of $y = f(x)$ are given*

x :	10	15	20
y :	1754	2648	3564

Find the value of x for $y = 3000$ by iterative method.

Sol. Taking $x_0 = 10$ and $h = 5$, the difference table is

x	y	Δy	$\Delta^2 y$
10	1754		
		894	
15	2648		22
		916	
20	3564		

Here $y_p = 3000, y_0 = 1754, \Delta y_0 = 894$ and $\Delta^2 y_0 = 22$.

$\therefore$ The successive approximations to p are

$$p_1 = \frac{1}{894}(3000 - 1754) = 1.39$$

$$p_2 = \frac{1}{894}\left[3000 - 1754 - \frac{1.39(1.39-1)}{2} \times 22\right] = 1.387$$

$$p_3 = \frac{1}{894}\left[3000 - 1754 - \frac{1.387(1.387-1)}{2} \times 22\right] = 1.3871$$

We, therefore, take $p = 1.387$ correct to three decimal places. Hence the value of x (corresponding to $y = 3000$) $= x_0 + ph = 10 + 1.387 \times 5 = 16.935$.

Example 7.37. *Using inverse interpolation, find the real root of the equation $x^3 + x - 3 = 0$, which is close to 1.2.*

Sol. The difference table is

x	v	$y(= x^3 + x - 3)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	– 0.2	– 1	0.431			
				0.066		
1.1	– 0.1	– 0.569	0.497		0.006	
				0.072		0
1.2	0	– 0.072			0.006	
			0.569			
1.3	0.1	0.497		0.078		
			0.647			
1.4	0.2	1.144				

Clearly the root of the given equation lies between 1.2 and 1.3. Assuming the origin at $x = 1.2$ and using Stirling's formula.

$$y = y_0 + x\,\frac{\Delta y_0 + \Delta y_{-1}}{2} + \frac{x^2}{2}\,\Delta^2 y_{-1} + \frac{x(x^2-1)}{6}\,\frac{(\Delta^3 y_{-1} + \Delta^3 y_{-2})}{2}, \text{ we get}$$

$$0 = -0.072 + x\,.\,\frac{0.569 + 0.467}{2} + \frac{x^2}{2}(0.072) + \frac{x(x^2-1)}{6}\,\frac{0.006 + 0.006}{2} \qquad (\because\ y = 0)$$

or $$0 = -0.072 + 0.532x + 0.036x^2 + 0.001x^3$$

This equation can be written as

$$x = \frac{0.072}{0.532} - \frac{0.036}{0.532}x^2 - \frac{0.001}{0.532}x^3 \qquad \ldots(i)$$

$\therefore$ First approximation $x^{(1)} = \dfrac{0.072}{0.532} = 0.1353$

Putting $x = x^{(1)}$ on R.H.S. of (i), we get

Second approximation $x^{(2)} = 0.1353 - 0.067(0.1353)^2 - 1.8797(0.1353)^3 = 0.134$

Hence the desired root $= 1.2 + 0.1 \times 0.134 = 1.2134$.

PROBLEMS 7.7

1. Apply Lagrange's method to find the value of x when $f(x) = 15$ from the given data :

x :	5	6	9	11
$f(x)$:	12	13	14	16

(Madras, B.E., 2000)

2. Obtain the value of t when $A = 85$ from the following table, using Lagrange's method :

t :	2	5	8	14
A :	94.8	87.9	81.3	68.7

3. Apply Lagrange's formula inversely to obtain the root of the equation $f(x) = 0$, given that $f(30) = -30$, $f(34) = -13$, $f(38) = 3$ and $f(42) = 18$.

(Kerala, B. Tech., 1995 S)

4. From the following data :

x :	1.8	2.0	2.2	2.4	2.6
y :	2.9	3.6	4.4	5.5	6.7,

find x when $y = 5$ using the iterative method.

5. The equation $x^3 - 15x + 4 = 0$ has a root close to 0.3, obtain this root upto 4 decimal places using inverse interpolation.

6. Solve the equation $x = 10 \log x$, by iterative method, given that

x :	1.35	1.36	1.37	1.38
$\log x$:	0.1303	0.1355	0.1367	0.1392.

7.22. OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 7.8

Select the correct answer or fill up the blanks in the following questions :

1. Newton's backward interpolation formula is ....
2. Bessel's formula is most appropriate when p lies between
   (*a*) – 0. 25 and 0.25 (*b*) 0.25 and 0.75 (*c*) 0.75 and 1.00
3. Form the divided difference table for the following data :

x :	5	15	22
y :	7	36	160

4. Interpolation is the technique of estimating the value of a function for any ..... .
5. Bessel's formula for interpolation is ..... .
6. The 4th divided differences for $x_0, x_1, x_2, x_3, x_4 =$ ....
7. Stirling's formula is best suited for p lying between ..... .
8. Newton's divided differences formula is ....

8

NUMERICAL DIFFERENTIATION & INTEGRATION

8.1. NUMERICAL DIFFERENTIATION

It is the process of calculating the value of the derivative of a function at some assigned value of x from the given set of values (x_i, y_i). To compute dy/dx, we first replace the exact relation $y = f(x)$ by the best interpolating polynomial $y = \phi(x)$ and then differentiate the latter as many times as we desire. The choice of the interpolation formula to be used, will depend on the assigned value of x at which dy/dx is desired.

If the values of x are equi-spaced and dy/dx is required near the beginning of the table, we employ Newton's forward formula. If it is required near the end of the table, we use Newton's backward formula. For values near the middle of the table, dy/dx is calculated by means of Stirling's or Bessel's formula.

If the values of x are not equi-spaced, we use Newton's divided difference formula to represent the function.

Hence corresponding to each of the interpolation formulae, we can derive a formula for finding the derivative.

Obs. While using these formulae, it must be observed that the table of values defines the function at these points only and does not completely define the function and the function may not be differentiable at all. As such, *the process of numerical differentiation should be used only if the tabulated values are such that the differences of some order are constants.* Otherwise, errors are bound to creep in which go on increasing as derivatives of higher order are found. This is due to the fact that the difference between $f(x)$ and the approximating polynomial $\phi(x)$ may be small at the data points but $f'(x) - \phi'(x)$ may be large.

8.2. FORMULAE FOR DERIVATIVES

Consider the function $y = f(x)$ which is tabulated for the values $x_i (= x_0 + ih)$, $i = 0, 1, 2, \ldots n$.

(1) Derivatives using forward difference formula. Newton's forward interpolation formula (p. 184) is

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots$$

Differentiating both sides w.r.t. p, we have

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \ldots$$

Since $p = \dfrac{(x-x_0)}{h}$, therefore $\dfrac{dp}{dx} = \dfrac{1}{h}$.

Now
$$\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \frac{4p^3-18p^2+22p-6}{4!}\Delta^4 y_0 + \ldots\right] \quad ...(1)$$

At $x = x_0$, $p = 0$. Hence putting $p = 0$,

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 + \ldots\right] \quad ...(2)$$

Again differentiating (1) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dp}\left(\frac{dy}{dp}\right)\frac{dp}{dx}$$

$$= \frac{1}{h}\left[\frac{2}{2!}\Delta^2 y_0 + \frac{6p-6}{3!}\Delta^3 y_0 + \frac{12p^2-36p+22}{4!}\Delta^4 y_0 + \ldots\right]\frac{1}{h}$$

Putting $p = 0$, we obtain

$$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 + \ldots\right] \quad ...(3)$$

Similarly
$$\left(\frac{d^3y}{dx^3}\right)_{x_0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \frac{5}{4}\right] \quad ...(4)$$

Otherwise : We know that $1 + \Delta = E = e^{hD}$

$\therefore$
$$hD = \log(1+\Delta) = \Delta - \tfrac{1}{2}\Delta^2 + \tfrac{1}{3}\Delta^3 - \tfrac{1}{4}\Delta^4 + \ldots$$

or
$$D = \frac{1}{h}\left[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \ldots\right]$$

and
$$D^2 = \frac{1}{h^2}\left[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \ldots\right]^2 = \frac{1}{h^2}\left[\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 + \ldots\right]$$

and
$$D^3 = \frac{1}{h^2}\left[\Delta^3 - \frac{3}{2}\Delta^4 + \ldots\right]$$

Now applying the above identities to y_0, we get

Dy_0 i.e.
$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 + \ldots\right]$$

$$\left(\frac{d^2 y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 - \ldots\right]$$

and
$$\left(\frac{d^3 y}{dx^3}\right)_{x_0} = \frac{1}{h^3}\left[\Delta^2 y_0 - \frac{3}{2}\Delta^4 y_0 + \ldots\right]$$

which are the same as (2), (3) and (4) respectively.

(2) Derivatives using backward difference formula. Newton's backward interpolation formula (p. 185) is

$$y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \ldots$$

Differentiating both sides w.r.t. p, we get

$$\frac{dy}{dx} = \nabla y_n + \frac{2p+1}{2!}\nabla^2 y_n + \frac{3p^2+6p+2}{3!}\nabla^3 y_n + \ldots$$

Since $\quad p = \dfrac{x - x_n}{h}$, therefore $\dfrac{dp}{dx} = \dfrac{1}{h}$.

Now
$$\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2p+1}{2!}\nabla^2 y_n + \frac{3p^3+6p+2}{3!}\nabla^3 y_n + \ldots\right] \qquad \ldots(5)$$

At $x = x_n$, $p = 0$. Hence putting $p = 0$, we get

$$\left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + \frac{1}{6}\nabla^6 y_n + \ldots\right] \qquad \ldots(6)$$

Again differentiating (5) w.r.t. x, we have

$$\frac{d^2 y}{dx^2} = \frac{d}{dp}\left(\frac{dy}{dx}\right)\frac{dp}{dx} = \frac{1}{h}\left[\nabla^2 y_n + \frac{6p+6}{3!}\nabla^3 y_n + \frac{6p^2+18p+11}{12}\nabla^4 y_n + \ldots\right]$$

Putting $p = 0$, we obtain

$$\left(\frac{d^2 y}{dx^2}\right)_{x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \ldots\right] \qquad \ldots(7)$$

Similarly,
$$\left(\frac{d^3 y}{dx^3}\right)_{x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \ldots\right] \qquad \ldots(8)$$

Otherwise : We know that $\quad 1 - \nabla = E^{-1} = e^{-hD}$

$$\therefore \qquad -hD = \log(1 - \nabla) = -\left[\nabla + \tfrac{1}{2}\nabla^2 + \tfrac{1}{3}\nabla^3 + \tfrac{1}{4}\nabla^4 + \ldots\right]$$

or
$$D = \frac{1}{h}\left[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{1}{4}\nabla^4 + \ldots\right]$$

$$\therefore \qquad D^2 = \frac{1}{h^2}\left[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \ldots\right]^2 = \frac{1}{h^2}\left[\nabla^2 + \nabla^3 + \frac{11}{12}\nabla^4 + \ldots\right]$$

Similarly, $$D^3 = \frac{1}{h^3}\left[\nabla^3 + \frac{3}{2}\nabla^4 + \ldots\right]$$

Applying these identities to y_n, we get

Dy_n i.e. $$\left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + \frac{1}{6}\nabla^6 y_n + \ldots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \ldots\right]$$

and $$\left(\frac{d^3y}{dx^3}\right)_{x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \ldots\right]$$

which are the same as (6), (7) and (8).

(2) Derivatives using central difference formulae. Stirling's formula (p. 195) is

$$y_p = y_0 + \frac{p}{1!}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1^2)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{p^2(p^2-1^2)}{4!}\Delta^4 y_{-2} + \ldots$$

Differentiating both sides w.r.t. p, we get

$$\frac{dy}{dp} = \left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{2p}{2!}\Delta^2 y_{-1} + \frac{3p^2-1}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{4p^3-2p}{4!}\Delta^4 y_{-2} + \ldots\ldots$$

Since $p = \dfrac{x - x_0}{h}$, $\therefore \dfrac{dp}{dx} = \dfrac{1}{h}$.

Now $$\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx} = \frac{1}{h}\left[\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + p\Delta^2 y_{-1} + \frac{3p^2-1}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{2p^3-p}{12}\Delta^4 y_{-2} + \ldots\right]$$

At $x = x_0$, $p = 0$. Hence putting $p = 0$, we get

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} - \frac{1}{6}\,\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} + \frac{1}{30}\,\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2} + \frac{1}{140}\ldots\right] \quad \ldots(9)$$

Similarly $$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3} - \frac{1}{560}\ldots\right] \quad \ldots(10)$$

Obs. We can similarly use any other interpolation formula for computing the derivatives.

Example 8.1. *Given that*

x :	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y :	7.989	8.403	8.781	9.129	9.451	9.750	10.031,

find $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ *at* (a) $x = 1.1$ (V.T.U., B. Tech., 2006)

(b) $x = 1.6$. (Rohtak, B. Tech., 2006)

Sol. (a) The difference table is :

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
1.0	7.989						
		0.414					
1.1	8.403		– 0.036				
		0.378		0.006			
1.2	8.781		– 0.030		– 0.002		
		0.348		0.004		0.001	
1.3	9.129		– 0.026		–0.001		0.002
		0.322		0.003		0.003	
1.4	9.451		– 0.023		+0.002		
		0.299		0.005			
1.5	9.750		– 0.018				
		0.281					
1.6	10.031						

We have

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 + \ldots\right] \quad \ldots(i)$$

and

$$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 - \ldots\right] \quad \ldots(ii)$$

Here $h = 0.1$, $x_0 = 1.1$, $\Delta y_0 = 0.378$, $\Delta^2 y_0 = -0.03$ etc.

Substituting these values in (i) and (ii), we get

$$\left(\frac{dy}{dx}\right)_{1.1} = \frac{1}{0.1}\left[0.378 - \frac{1}{2}(-0.03) + \frac{1}{3}(0.004) - \frac{1}{4}(-0.001) + \frac{1}{5}(0.003)\right] = 3.952$$

$$\left(\frac{d^2y}{dx^2}\right)_{1.1} = \frac{1}{(0.1)^2}\left[-0.03 - (0.004) + \frac{11}{12}(-0.001) - \frac{5}{6}(0.003)\right] = -3.74.$$

(b) We use the above difference table and the backward difference operator ∇ instead of Δ.

$$\left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + \frac{1}{6}\nabla^6 y_n + \ldots\right] \quad \ldots(i)$$

and

$$\left(\frac{d^2y}{dx^2}\right)_{x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \ldots\right] \quad \ldots(ii)$$

Here $h = 0.1$, $x_n = 1.6$, $\nabla y_n = 0.281$, $\nabla^2 y_n = -0.018$ etc.

Putting these values in (*i*) and (*ii*), we get

$$\left(\frac{dy}{dx}\right)_{1.6} = \frac{1}{0.1}\left[0.281 + \frac{1}{2}(-0.018) + \frac{1}{3}(0.005) + \frac{1}{4}(0.002) + \frac{1}{5}(0.003) + \frac{1}{6}(0.002)\right] = 2.75$$

$$\left(\frac{d^2y}{dx^2}\right)_{1.6} = \frac{1}{(0.1)^2}\left[-0.018 + 0.005 + \frac{11}{12}(0.002) + \frac{5}{6}(0.003) + \frac{137}{180}(0.002)\right] = -0.715.$$

Example 8.2. *The following data gives the velocity of a particle for 20 seconds at an interval of 5 seconds. Find the initial acceleration using the entire data :*

Time t (sec) :	*0*	*5*	*10*	*15*	*20*
Velocity v(m/sec) :	*0*	*3*	*14*	*69*	*228*

(Anna, B. Tech., 2004)

Sol. The difference table is :

t	v	Δv	$\Delta^2 v$	$\Delta^3 v$	$\Delta^4 v$
0	0				
		3			
5	3		8		
		11		36	
10	14		44		24
		55		60	
15	69		104		
		159			
20	228				

An initial acceleration $\left(i.e. \frac{dv}{dt}\right)$ at $t = 0$ is required, we use Newton's forward formula :

$$\left(\frac{dv}{dt}\right)_{t=0} = \frac{1}{h}\left(\Delta v_0 - \frac{1}{2}\Delta^2 v_0 + \frac{1}{3}\Delta^3 v_0 - \frac{1}{4}\Delta^4 v_0 + \ldots\ldots\right)$$

$$\therefore \quad \left(\frac{dv}{dt}\right)_{t=0} = \frac{1}{5}\left[3 - \frac{1}{2}(8) + \frac{1}{3}(36) - \frac{1}{4}(24)\right]$$

$$= \frac{1}{5}(3 - 4 + 12 - 6) = 1$$

Hence the initial acceleration is 1 m/sec².

Example 8.3. *Find the value of cos (1.74) from the following table :*

x :	*1.7*	*1.74*	*1.78*	*1.82*	*1.86*
sin x :	*0.9916*	*0.9857*	*0.9781*	*0.9691*	*0.9584*

(J.N.T.U., B.Tech., 2009)

Sol. Let $y = f(x) = \sin x$, so that $f'(x) = \cos x$.

The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.7	0.9916				
		– 0.0059			
1.74	0.9857		– 0.0017		
		– 0.0076		0.0003	
1.78	0.9781		– 0.0014		– 0.0006
		– 0.0090		– 0.0003	
1.82	0.9691		– 0.0017		
		– 0.0107			
1.86	0.9584				

Since we require $f'(1.74)$, we use Newton's forward formula

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\ldots\right] \qquad \ldots(i)$$

Here $\qquad h = 0.04,\ x_0 = 1.7,\ \Delta y_0 = -0.0059,\ \Delta^2 y_0 = -0.0017$ etc.

Substituting these values in (*i*), we get

$$\left(\frac{dy}{dx}\right)_{1.74} = \frac{1}{0.04}\left[0.0059 - \frac{1}{2}(-0.0017) + \frac{1}{3}(0.0003) - \frac{1}{4}(-0.0006)\right] = -0.12.$$

$$= \frac{1}{0.04}(0.007) = 0.175$$

$$\left[-0.0076 - \frac{1}{2}(-0.0014) + \frac{1}{3}(-0.0003)\right]$$

Hence $\quad \cos(1.74) = -0.175$

Example 8.4. *A slider in a machine moves along a fixed straight rod. Its distance x cm. along the rod is given below for various values of the time t seconds. Find the velocity of the slider and its acceleration when t = 0.3 second.*

$t =$	0	0.1	0.2	0.3	0.4	0.5	0.6
$x =$	30.13	31.62	32.87	33.64	33.95	33.81	33.24

(V.T.U., B. Tech., 2009)

Sol. The difference table is :

t	x	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
0	30.13						
		1.49					
0.1	31.62		– 0.24				
		1.25		– 0.24			
0.2	32.87		– 0.48		0.26		
		0.77		0.02		– 0.27	
0.3	33.64		– 0.46		– 0.01		0.29
		0.31		0.01		0.02	

0.4	33.95		– 0.45		0.01
		– 0.14		0.02	
0.5	33.81		– 0.43		
		– 0.57			
0.6	33.24				

As the derivatives are required near the middle of the table, we use Stirling's formulae:

$$\left(\frac{dx}{dt}\right)_{t_0} = \frac{1}{h}\left(\frac{\Delta x_0 + \Delta x_{-1}}{2}\right) - \frac{1}{6}\left(\frac{\Delta^3 x_{-1} + \Delta^3 x_{-2}}{2}\right) + \frac{1}{30}\left(\frac{\Delta^5 x_{-2} + \Delta^5 x_{-3}}{2}\right) + \dots \quad \dots(i)$$

$$\left(\frac{d^2x}{dt^2}\right)_{t_0} = \frac{1}{h^2}\left[\Delta^2 x_{-1} - \frac{1}{12}\Delta^4 x_{-2} + \frac{1}{90}\Delta^6 x_{-3} \dots\right] \quad \dots(ii)$$

Here $h = 0.1$, $t_0 = 0.3$, $\Delta x_0 = 0.31$, $\Delta x_{-1} = 0.77$, $\Delta^2 x_{-1} = -0.46$ etc.

Putting these values in (*i*) and (*ii*), we get

$$\left(\frac{dx}{dt}\right)_{0.3} = \frac{1}{0.1}\left[\frac{0.31+0.77}{2} - \frac{1}{6}\left(\frac{0.01+0.02}{2}\right) + \frac{1}{30}\left(\frac{0.02-0.27}{2}\right) - \dots\dots\right] = 5.33$$

$$\left(\frac{d^2x}{dt^2}\right)_{0.3} = \frac{1}{(0.1)^2}\left[-0.46 - \frac{1}{12}(-0.01) + \frac{1}{90}(0.29) - \dots\dots\right] = -45.6$$

Hence the required velocity is 5.33 cm/sec and acceleration is – 45.6 cm/sec^2.

Example 8.5. *The elevation above a datum line of seven points of a road are given below :*

x :	*0*	*300*	*600*	*900*	*1200*	*1500*	*1800*
y :	*135*	*149*	*157*	*183*	*201*	*205*	*193*

Find the gradient of the road at the middle point.

Sol. Here $h = 300$, $x_0 = 0$, $y_0 = 135$, we require the gradient $\frac{dy}{dx}$ at $x = 900$.

The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	135					
		14				
300	149		– 6			
		8		24		
600	157		18		– 50	
		26		– 26		70
900	183		– 8		20	
		18		– 6		– 16
1200	201		– 14		4	
		4		– 2		
1500	205		– 16			
		– 12				
1800	193					

Using Stirling's formula for the first derivative, we get

$$y'(x_0) = \frac{1}{h}\left[\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) - \frac{1}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{1}{30}\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right)\right]$$

$$= \frac{1}{300}\left[\frac{1}{2}(18+26) - \frac{1}{12}(-6-26) + \frac{1}{60}(-16+70)\right]$$

$$= \frac{1}{300}(22 + 2.666 + 0.9) = 0.085$$

Hence the gradient of the road at the middle point is 0.085.

Example 8.6. *Using Bessel's formula, find f′(7.5) from the following table :*

x :	*7.47*	*7.48*	*7.49*	*7.50*	*7.51*	*7.52*	*7.53*
f(x) :	*0.193*	*0.195*	*0.198*	*0.201*	*0.203*	*0.206*	*0.208*

(J.N.T.U., B. Tech., 2006)

Sol. Taking $x_0 = 7.50$, $h = 0.01$, we have $p = \dfrac{x - x_0}{h} = \dfrac{x - 7.50}{0.01}$

The difference table is

x	p	y_p	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
7.47	−3	0.193						
			0.002					
7.48	−2	0.195		0.001				
			0.003		−0.001			
7.49	−1	0.198		0.000		0.000		
			0.003		−0.001		0.003	
7.50	0	0.201		−0.001		0.003		−0.01
			0.002		0.002		−0.007	
7.51	1	0.203		0.001		−0.004		
			0.003		−0.002			
7.52	2	0.206		−0.001				
			0.002					
7.53	3	0.208						

Bessel's formula (p. 196) is

$$y_p = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\cdot\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{(p-\frac{1}{2})\,p\,(p-1)}{3!}\cdot\Delta^3 y_{-1}$$

$$+ \frac{(p+1)\,p\,(p-1)(p-2)}{4!}\cdot\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \frac{(p-\frac{1}{2})(p+1)\,p\,(p-1)(p-2)}{5!}\cdot\Delta^5 y_{-2}$$

$$+ \frac{(p+2)(p+1)\,p\,(p-1)(p-2)(p-3)}{6!}\cdot\frac{\Delta^6 y_{-3} + \Delta^6 y_{-2}}{2} + \ldots\ldots \qquad \ldots(i)$$

Since $p = \frac{x - x_0}{h}$, $\therefore \frac{dp}{dx} = \frac{1}{h}$ and $\frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx} = \frac{1}{h}\frac{dy}{dp}$

Differentiating (*i*) w.r.t. p and putting $p = 0$, we get

$$\left(\frac{dy}{dx}\right) = \frac{1}{h}\left(\frac{dy}{dp}\right)_{p=0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{4}(\Delta^2 y_{-1} + \Delta^2 y_0) + \frac{1}{12}\Delta^3 y_{-1} + \frac{1}{24}(\Delta^4 y_{-2} + \Delta^4 y_{-1}) - \frac{1}{120}\Delta^5 y_{-2} - \frac{1}{240}(\Delta^6 y_{-3} + \Delta^6 y_{-2})\right]$$

$$\left(\frac{dy}{dx}\right)_{7.5} = \frac{1}{0.01}\left[0.002 - \frac{1}{4}(-0.001 + 0.001) + \frac{1}{12}(0.002) + \frac{1}{24}(-0.004 + 0.003) - \frac{1}{120}(-0.007) - \frac{1}{240}(-0.010 + 0)\right] \quad [\because \Delta^6 y_{-2} = 0]$$

$$= 0.2 + 0 + 0.01666 - 0.0416 + 0.00583 + 0.00416 = 0.223.$$

Example 8.7. *Find $f'(10)$ from the following data :*

x :	3	5	11	27	34
$f(x)$:	– 13	23	899	17315	35606.

Sol. As the values of x are not equi-spaced, we shall use Newton's divided difference formula. The divided difference table is

x	$f(x)$	1st div. diff.	2nd div. diff.	3rd div. diff.	4th div. diff.
3	– 13				
		18			
5	23		16		
		146		0.998	
11	899		39.96		0.0002
		1025		1.003	
27	17315		69.04		
		2613			
34	35606				

Fifth differences being zero, Newton's divided difference formula is

$$f(x) = f(x_0) + (x - x_0)f(x_0, x_1) + (x - x_0)(x - x_1)f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2)f(x_0, x_1, x_2, x_3) + (x - x_0)(x - x_1) \times (x - x_2)(x - x_3)f(x_0, x_1, x_2, x_3, x_4)$$

Differentiating it w.r.t. x, we get

$$f'(x) = f(x_0, x_1) + (2x - x_0 - x_1)f(x_0, x_1, x_2) + [3x^2 - 2x(x_0 + x_1 + x_2) + x_0x_1 + x_1x_2 + x_2x_0] \times f(x_0, x_1, x_2, x_3) + [4x^3 - 3x^2(x_0 + x_1 + x_2 + x_3) + 2x(x_0x_1 + x_1x_2 + x_2x_3 + x_3x_0 + x_1x_3 + x_0x_2) - (x_0x_1x_2 + x_1x_2x_3 + x_2x_3x_0 + x_0x_1x_3)] f(x_0, x_1, x_2, x_3, x_4)$$

Putting $x_0 = 3, x_1 = 5, x_2 = 11, x_3 = 27$ and $x = 10$, we obtain

$$f'(0) = 18 + 12 \times 16 + 23 \times 0.998 - 426 \times 0.0002 = 232.869.$$

8.4. NUMERICAL INTEGRATION

The process of evaluating a definite integral from a set of tabulated values of the integrand $f(x)$ is called *numerical integration*. This process when applied to a function of a single variable, is known as *quadrature*.

The problem of numerical integration, like that of numerical differentiation, is solved by representing $f(x)$ by an interpolation formula and then integrating it between the given limits. In this way, we can derive quadrature formulae for approximate integration of a function defined by a set of numerical values only.

8.5. NEWTON-COTES QUADRATURE FORMULA

Let $$I = \int_a^b f(x)\, dx$$

where $f(x)$ takes the values $y_0, y_1, y_2, \ldots, y_n$ for $x = x_0, x_1, x_2, \ldots, x_n$.

Let us divide the interval (a, b) into n sub-intervals of width h so that $x_0 = a$, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, ... $x_n = x_0 + nh = b$. Then

$$I = \int_{x_0}^{x_0+nh} f(x)\, dx = h \int_0^n f(x_0 + rh)\, dr, \quad \text{Putting } x = x_0 + rh,\ dx = hdr$$

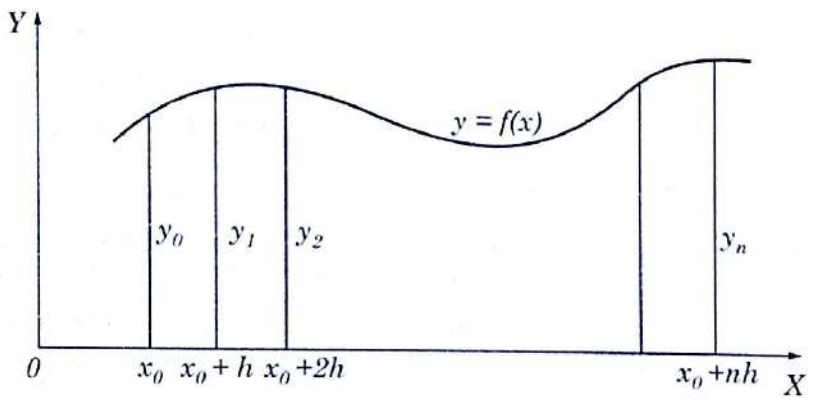

Fig. 8.1

$$= h \int_0^n \Big[y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!} \Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!} \Delta^3 y_0$$
$$+ \frac{r(r-1)(r-2)(r-3)}{4!} \Delta^4 y_0 + \frac{r(r-1)(r-2)(r-3)(r-4)}{5!} \Delta^5 y_0$$
$$+ \frac{r(r-1)(r-2)(r-3)(r-4)(r-5)}{6!} \Delta^6 y_0 + \ldots \Big] dr$$

[by Newton's forward interpolation formula

Integrating term by term, we obtain

$$\int_{x_0}^{x_0+nh} f(x)\, dx = nh \Big[y_0 + \frac{n}{2} \Delta y_0 + \frac{n(2n-3)}{12} \Delta^2 y_0 + \frac{n(n-2)^2}{24} \Delta^3 y_0$$
$$+ \left(\frac{n^4}{5} - \frac{3n^3}{2} + \frac{11n^2}{3} - 3n \right) \frac{\Delta^4 y_0}{4!}$$

$$+\left(\frac{n^5}{6}-2n^4+\frac{35n^3}{4}-\frac{50n^2}{3}+12n\right)\frac{\Delta^5 y_0}{5!}$$

$$\left.+\left(\frac{n^6}{7}-\frac{15n^5}{6}+17n^4-\frac{225n^3}{4}+\frac{274n^2}{3}-60n\right)\frac{\Delta^6 y_0}{6!}+\ldots\right] \quad ...(1)$$

This is known as *Newton-Cotes quadrature formula*. From this general formula, we deduce the following important quadrature rules by taking $n = 1, 2, 3, \ldots\ldots$

I. Trapezoidal rule. Putting $n = 1$ in (1) and taking the curve through (x_0, y_0) and (x_1, y_1) as a straight line *i.e.* a polynomial of first order so that differences of order higher than first become zero, we get

$$\int_{x_0}^{x_0+h} f(x)\,dx = h\left(y_0+\frac{1}{2}\Delta y_0\right)=\frac{h}{2}(y_0+y_1)$$

Similarly $$\int_{x_0+h}^{x_0+2h} f(x)\,dx = h\left(y_1+\frac{1}{2}\Delta y_1\right)=\frac{h}{2}(y_1+y_2)$$

....

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx = \frac{h}{2}(y_{n-1}+y_n)$$

Adding these n integrals, we obtain

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\,[(y_0+y_n)+2(y_1+y_2+\ldots+y_{n-1})] \quad ...(2)$$

This is known as the *trapezoidal rule*.

Obs. The area of each strip (trapezium) is found separately. Then the area under the curve and the ordinates at x_0 and $x_0 + nh$ is approximately equal to the sum of the areas of the n trapeziums.

II. Simpson's one-third rule. Putting $n = 2$ in (1) above and taking the curve through (x_0, y_0), (x_1, y_1) and (x_2, y_2) as a parabola *i.e.* a polynomial of second order so that differences of order higher than second vanish, we get

$$\int_{x_0}^{x_0+2h} f(x)\,dx = 2h(y_0+\Delta y_0+\tfrac{1}{6}\Delta^2 y_0)=\frac{h}{3}(y_0+4y_1+y_2)$$

Similarly $$\int_{x_0+2h}^{x_0+4h} f(x)\,dx = \frac{h}{3}(y_2+4y_3+y_4)$$

....

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)\,dx = \frac{h}{3}(y_{n-2}+4y_{n-1}+y_n),\ n \text{ being even.}$$

Adding all these integrals, we have when n is even

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\,[(y_0+y_n)+4(y_1+y_3+\ldots+y_{n-1}) + 2(y_2+y_4+\ldots+y_{n-2})] \quad ...(3)$$

This is known as the *Simpson's one-third rule* or simply *Simpson's rule* and is most commonly used.

Obs. While applying (3), *the given interval must be divided into even number of equal sub-intervals*, since we find the area of two strips at a time.

III. Simpson's three-eighth rule. Putting $n = 3$ in (1) above and taking the curve through $(x_i, y_i) : i = 0, 1, 2, 3$ as a polynomial of third order so that differences above the third order vanish, we get

$$\int_{x_0}^{x_0+3h} f(x)\,dx = 3h\left(y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right)$$

$$= \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3)$$

Similarly,

$$\int_{x_0+3h}^{x_0+5h} f(x)\,dx = \frac{3h}{8}(y_3 + 3y_4 + 3y_5 + y_6) \text{ and so on.}$$

Adding all such expressions from x_0 to $x_0 + nh$, where n is a multiple of 3, we obtain

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\,[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \ldots + y_{n-1}) + 2(y_3 + y_6 + \ldots + y_{n-3})] \qquad \ldots(4)$$

which is known as *Simpson's three-eighth rule.*

Obs. While applying (4), *the number of sub-intervals should be taken as multiple of 3.*

IV. Boole's rule. Putting $n = 4$ in (1) above and neglecting all differences above the fourth, we obtain

$$\int_{x_0}^{x_0+4h} f(x)\,dx = 4h\left(y_0 + 2\Delta y_0 + \frac{5}{3}\Delta^2 y_0 + \frac{2}{3}\Delta^3 y_0 + \frac{7}{90}\Delta^4 y_0\right)$$

$$= \frac{2h}{45}(7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4)$$

Similarly

$$\int_{x_0+4h}^{x_0+8h} f(x)\,dx = \frac{2h}{45}(7y_4 + 32y_5 + 12y_6 + 32y_7 + 7y_8) \text{ and so on.}$$

Adding all these integrals from x_0 to $x_0 + nh$, where n is a multiple of 4, we get

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{2h}{45}(7y_0 + 32y_1 + 12y_2 + 32y_3 + 14y_4 + 32y_5 + 12y_6 + 32y_7 + 14y_8 + \ldots \qquad \ldots(5)$$

This is known as *Boole's rule.*

Obs. While applying (5), the number of sub-intervals should be taken as a multiple of 4.

V. Weddle's rule. Putting $n = 6$ in (1) above and neglecting all differences above the sixth, we obtain

$$\int_{x_0}^{x_0+6h} f(x)\,dx = 6h\left(y_0 + 3\Delta y_0 + \frac{9}{2}\Delta^2 y_0 + 4\Delta^3 y_0 + \frac{123}{60}\Delta^4 y_0 + \frac{11}{20}\Delta^5 x_0 + \frac{1}{6}\cdot\frac{41}{140}\Delta^6 y_0\right)$$

If we replace $\frac{41}{140}\Delta^6 y_0$ by $\frac{3}{10}\Delta^6 y_0$, the error made will be negligible.

$$\therefore \quad \int_{x_0}^{x_0+6h} f(x)\,dx = \frac{3h}{10}(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6)$$

Similarly

$$\int_{x_0+6h}^{x_0+12h} f(x)\,dx = \frac{3h}{10}(y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}) \text{ and so on.}$$

Adding all these integrals from x_0 to $x_0 + nh$, where n is a multiple of 6, we get

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{10}(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + \ldots) \quad ...(6)$$

This is known as *Weddle's rule*.

Obs. While applying (6), *the number of sub-intervals should be taken as a multiple of 6.*

Weddle's rule is generally more accurate than any of the others. Of the two Simpson rules, the 1/3 rule is better.

Example 8.10. *Evaluate* $\int_0^6 \frac{dx}{1+x^2}$ *by using (i) Trapezoidal rule,*
(ii) Simpson's 1/3 rule, (*Mumbai, B.E., 2005*)
(iii) Simpson's 3/8 rule, (*J.N.T.U., B. Tech., 2008*)
(iv) Weddle's rule and compare the results with its actual value. (*V.T.U., B.E., 2008*)

Sol. Divide the interval (0, 6) into six parts each of width $h = 1$. The values of $f(x) = \frac{1}{1+x^2}$ are given below :

x	0	1	2	3	4	5	6
$f(x)$ $= y$	1 y_0	0.5 y_1	0.2 y_2	0.1 y_3	0.0588 y_4	0.0385 y_5	0.027 y_6

(*i*) By Trapezoidal rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2}[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \tfrac{1}{2}[(1 + 0.027) + 2(0.5 + 0.2 + 0.1 + 0.0588 + 0.0385)] = 1.4108.$$

(*ii*) By Simpson's 1/3 rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{1}{3}[(1 + 0.027) + 4(0.5 + 0.1 + 0.0385) + 2(0.2 + 0.0588)] = 1.3662.$$

(*iii*) By Simpson's 3/8 rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3}{8}[(1 + 0.027) + 3(0.5 + 0.2 + 0.0588 + 0.0385) + 2(0.1)] = 1.3571.$$

(iv) By Weddle's rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6]$$

$$= 0.3[1 + 5(0.5) + 0.2 + 6(0.1) + 0.0588 + 5(0.0385) + 0.027] = 1.3735$$

Also $\int_0^6 \frac{dx}{1+x^2} = \left| \tan^{-1} x \right|_0^6 = \tan^{-1} 6 = 1.4056$

This shows that the value of the integral found by Weddle's rule is the nearest to the actual value followed by its value given by Simpson's 1/3 rule.

Example 8.11. *Evaluate the integral $\int_0^1 \frac{x^2}{1+x^3} dx$ using Simpson's 1/3rd rule. Compare the error with the exact value.* *(Mumbai, B.E., 2004)*

Sol. Let us divide the interval (0, 1) into 4 equal parts so that $h = 0.25$. Taking $y = \frac{x^2}{(1+x^3)}$, we have

x:	0	0.25	0.50	0.75	1.00
y:	0	0.06153	0.22222	0.39560	0.5
	y_0	y_1	y_2	y_3	y_4

By Simpson's 1/3rd rule, we have

$$\int_0^1 \frac{x^2}{1+x^3} dx = \frac{h}{3}[(y_0 + y_4) + 2(y_2) + 4(y_1 + y_3)]$$

$$= \frac{0.25}{3}[(0 + 0.5) + 2(0.22222) + 4(0.06153 + 0.3956)]$$

$$= \frac{0.25}{3}[0.5 + 0.44444 + 1.82852] = 0.2310^{c}$$

Also $\int_0^1 \frac{x^2}{1+x^3} dx = \frac{1}{3} \left| \log(1+x^3) \right|_0^1 = \frac{1}{3}\log_e 2 = 0.23105$

Thus the error $= 0.23108 - 0.23105 = -0.00003.$

Example 8.12. *Use the Trapezoidal rule to estimate the integral $\int_0^2 e^{x^2} dx$ taking the number 10 intervals.* *(U.P.T.U., B. Tech., 2008)*

Sol. Let $y = e^{x^2}$, $h = 0.2$ and $n = 10$.

The values of x and y are as follows :

x:	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
y:	1	1.0408	1.1735	1.4333	1.8964	2.1782	4.2206	7.0993	12.9358	25.5337	54.5981
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

By Trapezoidal rule, we have

$$\int_0^1 e^{x^2} dx = \frac{h}{2} [(y_0 + y_{10}) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9)]$$

$$= \frac{0.2}{2} [(1 + 54.5981) + 2(1.0408 + 1.1735 + 1.4333 + 1.8964 + 2.1782 + 4.2206 + 7.0993 + 12.9358 + 25.5337)]$$

Hence $\int_0^2 e^{x^2} dx = 17.0621$

Example 8.13. *Use Simpson's 1/3rd rule to find* $\int_0^{0.6} e^{-x^2} dx$ *by taking seven ordinates.* *(Bhopal, B. Tech., 2009)*

Sol. Divide the interval (0, 0.6) into six parts each of width $h = 0.1$. The values of $y = f(x) = e^{-x^2}$ are given below :

x	0	0.1	0.2	0.3	0.4	0.5	0.6
x^2	0	0.01	0.04	0.09	0.16	0.25	0.36
y	1	0.9900	0.9608	0.9139	0.8521	0.7788	0.6977
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 1/3rd rule, we have

$$\int_0^{0.6} e^{-x^2} dx = \frac{h}{3} [(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{0.1}{3} [(1 + 0.6977) + 4(0.99 + 0.9139 + 0.7788) + 2(0.9608 + 0.8521)]$$

$$= \frac{0.1}{3} [1.6977 + 10.7308 + 3.6258] = \frac{0.1}{3} (16.0543) = 0.5351.$$

Example 8.14. *Compute the value of* $\int_{0.2}^{1.4} (\sin x - \log x + e^x)\, dx$ *using Simpson's* $\frac{3}{8}$*th rule.* *(Mumbai, B.E., 2005)*

Sol. Let $y = \sin x - \log_e x + e^x$ and $h = 0.2$, $n = 6$.

The values of y are as given below :

x :	0.2	0.4	0.6	0.8	1.0	1.2	1.4
y :	3.0295	2.7975	2.8976	3.1660	3.5597	4.0698	4.4042
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's $\frac{3}{8}$th rule, we have

$$\int_{0.2}^{1.4} y\, dx = \frac{3h}{8} [(y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5)]$$

$$= \frac{3}{8}(0.2)\,[7.7336 + 2\,(3.1660) + 3\,(13.3247)] = 4.053$$

Hence $\int_{0.2}^{1.4} (\sin x - \log_e x + e^x)\, dx = 4.053.$

Obs. Applications of Simpson's rule. If the various ordinates in § 8.5 represent equispaced cross-sectional areas, then Simpson's rule gives the volume of the solid. As such, Simpson's rule is very useful to civil engineers for calculating the amount of earth that must be moved to fill a depression or make a dam. Similarly if the ordinates denote velocities at equal intervals of time, the Simpson's rule gives the distance travelled. The following examples illustrate these applications.

Example 8.15. *The velocity v(km/min) of a moped which starts from rest, is given at fixed intervals of time t (min) as follows :*

t :	2	4	6	8	10	12	14	16	18	20
v :	10	18	25	29	32	20	11	5	2	0

Estimate approximately the distance covered in 20 minutes.

Sol. If s (km) be the distance covered in t (min), then $\frac{ds}{dt} = v$

$$\therefore \qquad \left| s \right|_{t=0}^{20} = \int_0^{20} v\, dt = \frac{h}{3}\,[X + 4.O + 2.E], \text{ by Simpson's rule}$$

Here $h = 2$, $v_0 = 0$, $v_1 = 10$, $v_2 = 18$, $v_3 = 25$ etc.

$$\therefore \qquad X = v_0 + v_{10} = 0 + 0 = 0$$

$$O = v_1 + v_3 + v_5 + v_7 + v_9 = 10 + 25 + 32 + 11 + 2 = 80$$

$$E = v_2 + v_4 + v_6 + v_8 = 18 + 29 + 20 + 5 = 72$$

Hence the required distance $= \left| s \right|_{t=0}^{20} = \frac{2}{3}(0 + 4 \times 80 + 2 \times 72) = 309.33$ km.

Example 8.16. *The velocity v of a particle at distance s from a point on its linear path is given by the following table :*

s (m) :	0	2.5	5.0	7.5	10.0	12.5	15.0	17.5	20.0
v (m/sec) :	16	19	21	22	20	17	13	11	9

Estimate the time taken by the particle to traverse the distance of 20 metres, using Bool's rule. *(U.P.T.U., B. Tech., 2007)*

Sol. If t sec be the time taken to traverse a distance s (m) then $\frac{ds}{dt} = v$

or $$\frac{dt}{ds} = \frac{1}{v} = y \text{ (say)},$$

then $$\left| t \right|_{s=0}^{s=20} = \int_0^{20} y\, ds$$

Here $h = 2.5$ and $n = 8$.

Also $y_0 = \frac{1}{16}, y_1 = \frac{1}{19}, y_2 = \frac{1}{4}, y_3 = \frac{1}{22}, y_4 = \frac{1}{20}, y_5 = \frac{1}{17}, y_6 = \frac{1}{13}\ y_7 = \frac{1}{11}, y_8 = \frac{1}{9}$.

$\therefore$ By Boole's Rules, we have

$$|t|_{s=0}^{s=20} = \int_0^{20} y\,ds = \frac{2h}{45}\,[7y_0 + 32y_1 + 12y_2 + 32y_3 + 14y_4 + 32y_5 + 12y_6 + 32y_7 + 14y_8]$$

$$= \frac{2(2.5)}{45}\left[7\left(\frac{1}{16}\right) + 32\left(\frac{1}{19}\right) + 12\left(\frac{1}{21}\right) + 32\left(\frac{1}{22}\right) + 14\left(\frac{1}{20}\right) + 32\left(\frac{1}{17}\right) + 12\left(\frac{1}{13}\right) + 32\left(\frac{1}{11}\right) + 14\left(\frac{1}{9}\right)\right]$$

$$= \frac{1}{9}\,(12.11776) = 1.35$$

Hence the required time = 1.35 sec.

Example 8.17. *A solid of revolution is formed by rotating about the x-axis, the area between the x-axis, the lines x = 0 and x = 1 and a curve through the points with the following co-ordinates :*

x :	*0.00*	*0.25*	*0.50*	*0.75*	*1.00*
y :	*1.0000*	*0.9896*	*0.9589*	*0.9089*	*0.8415*

Estimate the volume of the solid formed using Simpson's rule. (*Manipal, B.E., 2001*)

Sol. Here $h = 0.25$, $y_0 = 1$, $y_1 = 0.9896$, $y_2 = 0.9589$ etc.

$\therefore$ Required volume of the solid generated

$$= \int_0^1 \pi y^2\,dx = \pi\,.\,\frac{h}{3}\,[(y_0^2 + y_4^2) + 4(y_1^2 + y_3^2) + 2y_2^2]$$

$$= \frac{0.25\pi}{3}\,[\{1 + (0.8415)^2\} + 4\{(0.9896)^2 + (0.9089)^2\} + 2(0.9589)^2]$$

$$= \frac{0.25 \times 3.1416}{3}\,[1.7081 + 7.2216 + 1.839]$$

$$= 0.2618(10.7687) = 2.8192.$$

PROBLEMS 8.2

1. Use trapezoidal rule to evaluate $\int_0^1 x^3\,dx$ considering five sub-intervals.

2. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using (*i*) Trapezoidal rule taking $h = 1/4$.

   (*ii*) Simpson's 1/3 rule taking $h = 1/4$. (*J.N.T.U., B. Tech., 2008*)

   (*iii*) Simpson's 3/8 rule taking $h = 1/6$. (*V.T.U., B. Tech., 2007*)

   (*iv*) Weddle's rule taking $h = 1/6$. (*Bhopal, B.E., 2009*)

   Hence compute an approximate value of π in each case.

3. Evaluate $\int_0^1 \frac{dx}{1+x}$ applying

(*i*) Trapezoidal rule (J.N.T.U., B. Tech., 2009)

(*ii*) Simpson's 1/3 rule

(*iii*) Simpson's 3/8th rule. (Mumbai, B. Tech., 2004)

4. Find an approximate value of $\log_e 5$ by calculating to 4 decimal places, by Simpson's 1/3 rule, $\int_0^5 \frac{dx}{4x+5}$, dividing the range into 10 equal parts. (Anna, B. Tech., 2005)

5. Evaluate $\int_0^4 e^x \, dx$ by Simpson's rule, given that

$$e = 2.72, e^2 = 7.39, e^3 = 20.09, e^4 = 54.6$$

and compare it with the actual value. (Nagarjuna, B. Tech., 2003)

6. Find $\int_0^6 \frac{e^x}{1+x} dx$ using Simpson's $\frac{1}{3}$rd rule, (U.P.T.U., B.Tech., 2006)

7. Evaluate $\int_0^2 e^{-x^2} dx$, using Simpson's rule. (Take $h = 0.25$) (J.N.T.U., B. Tech., 2007)

8. Calculate the value of $\int_0^\pi \sin x \, dx$ by Simpson's $\frac{1}{3}$-rule, using 11 ordinates. Verify your answer by direct integration. (J.N.T.U., B. Tech., 2009)

9. Evaluate $\int_0^{\pi/2} \sqrt{\cos\theta} \, d\theta$ using Simpon's 1/3 rule, taking 9 ordinates. (V.T.U., B.E., 2009)

10. Evaluate $\int_0^{\pi/2} e^{\sin x} dx$ correct to 4 decimal places, By Simpson's $\frac{3}{8}$th rule. (U.P.T.U., B.Tech., 2007)

11. Given that

x :	4.0	4.2	4.4	4.6	4.8	5.0	5.2
$\log x$:	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487,

evaluate $\int_4^{5.2} \log x \, dx$ by

(*a*) Trapezoidal rule, (*b*) Simpson's 1/3-rule, (Kerala, B. Tech., 2003)

(*c*) Simpson's 3/8 rule, (J.N.T.U., B. Tech., 2006)

(*d*) Weddle's rule. (V.T.U., B. Tech., 2008)

Also find the error in each case.

12. The table below shows the temperature $f(t)$ as a function of time :

t :	1	2	3	4	5	6	7
$f(t)$:	81	75	80	83	78	70	60

Using Simpson's $\frac{1}{3}$rd rule to estimate $\int_1^7 f(t)\,dt$ (J.N.T.U., B.Tech., 2007)

13. A curve is drawn to pass through the points given by the following table :

x :	1	1.5	2	2.5	3	3.5	4
y :	2	2.4	2.7	2.8	3	2.6	2.1

Estimate the area bounded by the curve, x-axis and the lines $x = 1$, $x = 4$.

(Bhopal, B.E., 2007)

14. A river is 80 ft wide. The depth d in feet at a distance x ft. from one bank is given by the following table :

x :	0	10	20	30	40	50	60	70	80
y :	0	4	7	9	12	15	14	8	3

Find approximately the area of the cross-section. (Rohtak, B. Tech., 2005)

15. A curve is drawn to pass through the points given by the following table :

x :	1	1.5	2	2.5	3	3.5	4
y :	2	2.4	2.7	2.8	3	2.6	2.1

Using Weddle's rule, estimate the area bounded by the curve, the x-axis and the lines $x = 1$, $x = 4$. (V.T.U., B. Tech., 2005)

16. A curve is given by the table :

x :	0	1	2	3	4	5	6
y :	0	2	2.5	2.3	2	1.7	1.5

The x-coordinate of the C.G. of the area bounded by the curve, the end ordinates and the x-axis is given by $A\bar{x} = \int_0^6 xy\,dx$, where A is the area. Find $\bar{x}$ by using Simpson's rule.

17. A body is in the form of a solid of revolution. The diameter D in cms of its sections at distances x cm. from one end are given below. Estimate the volume of the solid.

x :	0	2.5	5.0	7.5	10.0	12.5	15.0
D :	5	5.5	6.0	6.75	6.25	5.5	4.0.

18. The velocity v of a particle at distance s from a point on its path is given by the table :

s ft :	0	10	20	30	40	50	60
v ft/sec :	47	58	64	65	61	52	38

Estimate the time taken to travel 60 ft by using Simpson's 1/3 rule. Compare the result with Simpson's 3/8 rule. (Madras, B.E., 2003)

19. The following table gives the velocity v of a particle at time t :

t (seconds) :	0	2	4	6	8	10	12
v (m/sec.) :	4	6	16	34	60	94	136

Find the distance moved by the particle in 12 seconds and also the acceleration at $t = 2$ sec.

(S.V.T.U., B. Tech., 2007)

20. A rocket is launched from the ground. Its acceleration is registered during the first 80 seconds and is given in the table below. Using Simpson's $\frac{1}{3}$rd rule, find the velocity of the rocket at $t = 80$ seconds.

t (sec)	0	10	20	30	40	50	60	70	80
f (cm/sec^2)	30	31.63	33.34	35.47	37.75	40.33	43.25	46.69	50.67

(*Mumbai, B. Tech., 2004*)

21. A reservoir discharging water through sluices at a depth h below the water surface has a surface area A for various values of h as given below :

h (ft.)	10	11	12	13	14
A (sq. ft.)	950	1070	1200	1350	1530

If t denotes time in minutes, the rate of fall of the surface is given by $dh/dt = -48\sqrt{h}/A$. Estimate the time taken for the water level to fall from 14 to 10 ft. above the sluices.

8.6. ERRORS IN QUADRATURE FORMULAE

The error in the quadrature formulae is given by

$$E = \int_a^b y\,dx - \int_a^b P(x)\,dx$$

where $P(x)$ is the polynomial representing the function $y = f(x)$, in the interval $[a, b]$.

(1) Error in Trapezoidal rule. Expanding $y = f(x)$ around $x = x_0$ by Taylor's series, we get

$$y = y_0 + (x - x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \ldots \qquad \ldots(1)$$

$$\therefore \int_{x_0}^{x_0+h} y\,dx = \int_{x_0}^{x_0+h} \left[y_0 + (x - x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \ldots\right] dx$$

$$= y_0 h + \frac{h^2}{2!}y_0' + \frac{h^3}{3!}y_0'' + \ldots \qquad \ldots(2)$$

Also A_1 = area of the first trapezium in the interval $[x_0, x_1] = \frac{1}{2}h(y_0 + y_1)$ $\qquad \ldots(3)$

Putting $x = x_0 + h$ and $y = y_1$ in (1), we get $\quad y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \ldots$

Substituting this value of y_1 in (3), we get

$$A_1 = \frac{1}{2}h\left[y_0 + y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \ldots\right]$$

$$= hy_0 + \frac{h^2}{2}y_0' + \frac{h^3}{2.2!}y_0'' + \ldots \qquad \ldots(4)$$

8.11. NUMERICAL DOUBLE INTEGRATION

The double integral $I = \int_c^d \int_a^b f(x, y)\, dx dy$

is evaluated numerically by two successive integrations in x and y directions considering one variable at a time. Repeated application of trapezoidal rule (or Simpson's rule) yields formulae for evaluating I.

(1) Trapezoidal rule. Dividing the interval (a, b) into n equal sub-intervals each of length h and the interval (c, d) into m equal sub-intervals each of length k, we have :

$$x_i = x_0 + ih, \quad x_0 = a, \quad x_n = b.$$
$$y_j = y_0 + jk, \quad y_0 = c, \quad y_m = d.$$

Using trapezoidal rule in both directions, we get

$$I = \frac{h}{2}\int_c^d [f(x_0, y) + f(x_n, y) + 2\{f(x_1, y) + f(x_2, y) + \ldots\ldots + f(x_{n-1}, y)\}]\, dy$$

$$= \frac{hk}{4} [(f_{00} + f_{0m}) + 2(f_{01} + f_{02} + \ldots + f_{0,\, m-1})$$
$$+ (f_{n0} + f_{nm}) + 2(f_{n1} + f_{n2} + \ldots + f_{n,\, m-1})$$
$$+ 2\sum_{i=1}^{n-1} \{(f_{i0} + f_{im}) + 2(f_{i1} + f_{i2} + \ldots + f_{i,\, m-1})\}] \quad \text{where } f_{ij} = f(x_i, y_j).$$

(2) Simpson's rule. We divide the interval (a, b) into $2n$ equal sub-intervals each of length h and the interval (c, d) into $2m$ equal sub-intervals each of length k. Then applying Simpson's rule in both directions, we get

$$\int_{y_{j-1}}^{y_{j+1}} \int_{x_{i-1}}^{x_{i+1}} f(x, y)\, dx dy = \frac{h}{3}\int_{y_{j-1}}^{y_{j+1}} [f(x_{i-1}, y) + 4f(x_i, y) + f(x_{i+1}, y)]\, dy$$

$$= \frac{hk}{9} [(f_{i-1,\, j-1} + 4f_{i-1,\, j} + f_{i-1,\, j+1}) + 4(f_{i,\, j-1} + 4f_{i,\, j} + f_{i,\, j+1}) + (f_{i+1,\, j-1} + 4f_{i+1,\, j} + f_{i+1,\, j+1})]$$

Adding all such intervals, we obtain the value of I.

Example 8.27. *Using trapezoidal rule, evaluate*

$$I = \int_1^2 \int_1^2 \frac{dx dy}{x + y}, \text{ taking four sub-intervals.} \qquad \textit{(Anna, B.E., 2004)}$$

Sol. Taking $h = k = 0.25$ so that $m = n = 4$, we obtain

$$I = \frac{1}{64} [f_{(1,\, 1)} + f_{(1,\, 2)} + 2(f_{(1,\, 1.25)} + f_{(1,\, 1.5)} + f_{(1,\, 1.75)})$$
$$+ f_{(2,\, 1)} + f_{(2,\, 2)} + 2(f_{(2,\, 1.25)} + f_{(2,\, 1.5)} + f_{(2,\, 1.75)})$$
$$+ 2\{f_{(1.25,\, 1)} + f_{(1.25,\, 2)} + 2(f_{(1.25,\, 1.25)} + f_{(1.25,\, 1.5)} + f_{(1.25,\, 1.75)})$$
$$+ f_{(1.5,\, 1)} + f_{(1.5,\, 2)} + 2(f_{(1.5,\, 1.25)} + f_{(1.5,\, 1.5)} + f_{(1.5,\, 1.75)})$$
$$+ f_{(1.75,\, 1)} + f_{(1.75,\, 2)} + 2(f_{(1.75,\, 1.25)} + f_{(1.75,\, 1.5)} + f_{(1.75,\, 1.75)})\}]$$
$$= 0.3407.$$

Example 8.28. *Apply Simpson's rule to evaluate the integral*

$$I = \int_2^{2.6} \int_4^{4.4} \frac{dxdy}{xy}.$$

Sol. Taking $h = 0.2$ and $k = 0.3$ so that $m = n = 2$, we get

$$I = \frac{hk}{9} [f(4, 2) + 4f(4, 2.3) + f(4, 2.6)$$
$$+ 4\{f(4.2, 2) + 4f(4.2, 2.3) + f(4.2, 2.6)\}$$
$$+ f(4.4, 2) + 4f(4.4, 2.3) + f(4.4, 2.6)]$$
$$= \frac{0.06}{9} [0.6559 + 4(0.6246) + 0.5962]$$
$$= \frac{0.02}{3} \times 3.7505 = 0.025.$$

PROBLEMS 8.4

1. Evaluate $\int_0^1 \int_0^1 xe^y \, dxdy$ using Trapezoidal rule ($h = k = 0.5$).

2. Apply Trapezoidal rule to evaluate

(*i*) $\int_1^5 \int_1^5 \frac{dxdy}{\sqrt{(x^2 + y^2)}}$, taking two sub-intervals.

(*ii*) $\int_0^1 \int_1^2 \frac{2xy \, dx \, dy}{(1 + x^2)(1 + y^2)}$, taking $h = k = 0.25$. (*Anna, B.E., 2004*

3. Evaluate $\int_0^2 \int_0^2 f(x, y) \, dx \, dy$ by Trapezoidal rule for the following data :

y/x	0	0.5	1	1.5	2
0	2	3	4	5	5
1	3	4	6	9	11
2	4	6	8	11	14

(*Anna, B.E., 20*

4. Using Trapezoidal and Simpson's rules, evaluate

$\int_0^1 \int_0^1 e^{x+y} \, dxdy$, taking two sub-intervals. (*Madras, B.E., 20*

5. Using Simpson's rule, evaluate

(*i*) $\int_1^{2.8} \int_2^{3.2} \frac{dxdy}{x + y}$ (*Anna, B.E., 2004*) (*ii*) $\int_0^1 \int_0^1 \frac{dxdy}{1 + x + y}$, taking $h = k = 0.5$.