Chapter 4

RANDOM NUMBERS

4.1 Random Numbers

The earlier chapters have very clearly illustrated that random numbers are a necessary basic ingredient in the application of Monte Carlo method or simulation of situations involving randomness. There are a large number of systems, where chance plays a part. These systems are called *stochastic* systems. Even for the solution of problems, which are deterministic, random numbers are required for simulation.

What are random numbers? These numbers are samples drawn from a uniformly distributed random variable between some specified intervals, and they have equal probability of occurrence.

Properties of Random Numbers :

A sequence of random numbers has two important statistical properties.

- uniformity, and
- independence.

Each random number is an independent sample drawn from a continuous uniform distribution between an interval 0 to 1. The probability density function (pdf) is shown in Fig. 4.1 and is given by,

$$f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & otherwise \end{cases}$$

The expected value of each random number R_i is given by

$$E(R) = \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

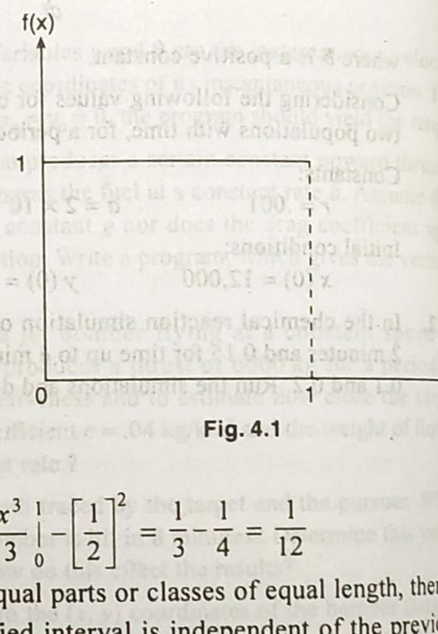

Fig. 4.1

and variance is given by $V(R) = \int_0^1 x^2 dx - [E(R)]^2 = \frac{x^3}{3}\Big|_0^1 - \left[\frac{1}{2}\right]^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

If the interval between 0 and 1 is divided into n equal parts or classes of equal length, then,

- the probability of observing a value in a specified interval is independent of the previous values drawn.
- if a total of m observations are taken, then the expected number of observations in each interval is m/n, for uniform distribution.

4.2 Random Number Table

Let us conduct a simple experiment to demonstrate the generation of random numbers. Take ten identical chips of paper and write down the digits 0, 1, 2, 3, ..., 9 on them. Put them in a box, mix them well, and take out one chip. It is a random number between 0 and 9 both inclusive. Repeat this experiment, each time returning the chip to the box and mixing them well. Instead of 10 pieces of paper we can have say 50 with digits 0, 1, 2, 3, ..., 9 repeated 5 times. Each time, draw 5 pieces and note down their numbers. Thus each time 5 random digits are obtained. These can be listed in the form of a table similar to Appendix Table A-1. Such a table is called a random number table. The most comprehensive of all published tables of random numbers is due to RAND Corporation, which contains one million random digits. These numbers were generated by using a special roulette, which incorporated electric devices. A simple roulette wheel, shown in Fig. 4.2, comprises of a disc divided into 10 equal sectors numbered from 0 to 9. The rotating disc is abruptly stopped and the number against the pointer is noted down as a random digit.

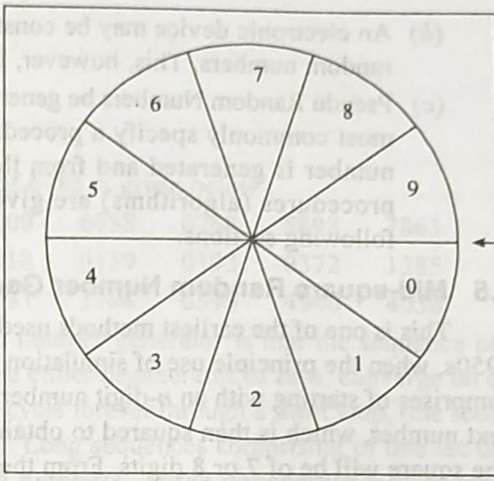

Fig. 4.2

4.3 Pseudo Random Numbers

The 'pseudo' means false. But here word 'pseudo' implies that the random numbers are generated by using some known arithmetic operation. Since, the arithmetic operation is known and the sequence of random numbers can be repeatedly obtained, the numbers cannot be called truly random. However, the pseudo random numbers generated by many computer routines, very closely fulfill the requirement of desired randomness.

If the method of random number generation that is the random number generator is defective, the generated pseudo random numbers may have following departures from ideal randomness.

- The generated numbers may not be uniformly distributed.
- The generated numbers may not be continuous.
- The mean of the generated numbers may be too high or too low.
- The variance may be too high or too low.
- There may be cyclic patterns in the generated numbers, like;
  - (*a*) Auto correction between numbers.
  - (*b*) A group of numbers continuously above the mean, followed by a group continuously below the mean.

Thus, before employing a pseudo random number generator, it should be properly validated, by testing the generated random numbers for randomness. The random number tests commonly used are explained in Sections 4.10 to 4.14.

4.4 Generation of Random Numbers

In the examples discussed so far, random number table was used to obtain the random observations. In computer simulation, where a very large number of random numbers is generally required, the random numbers can be obtained by the following methods.

(*a*) Random numbers may be drawn from the random number tables stored in the memory of the computer. This process, however, is neither practicable nor economical. It is a very slow process, and the numbers occupy considerable space of computer memory. Above all, in real system simulation, a single run may require many times more random numbers than available in tables.

(*b*) An electronic device may be constructed as part of the digital computer to generate truly random numbers. This, however, is considered very expensive.

(*c*) Pseudo Random Numbers be generated by using some arithmetic operation. These methods most commonly specify a procedure, where starting with an initial number, the second number is generated and from that a third number and so on. A number of recursive procedures (algorithms) are given in literature, some of which are described in the following sections.

4.5 Mid-square Random Number Generator

This is one of the earliest methods used for generating pseudo random numbers. It was used in 1950s, when the principle use of simulation was in designing thermonuclear weapons. The method comprises of starting with an *n*-digit number, squaring it and taking the *n* digits in the middle as the next number, which is then squared to obtain the next number. Say, we start with a 4-digit number. The square will be of 7 or 8 digits. From the squared number chop off the two low order digits and one or two high order digits, to obtain a 4-digit number in the middle.

Let the *seed* number be 5673, when squared we get 32182929. After removing two low order digits and two high order digits, we get the next random number 1829. Its square is 3345241. After removing two low order and one high order digit, we get 3452 as random number. Some random numbers obtained from the seed 5673 are;

5673	1829	3452	9163	9605	2560	5536
6472	8867	6236	8876	7833	3558	6593
4676	8649	8052	8347	6724	2121	4986
8601	9772	4919	1965	8612	1665	7722
6292	5892	7156	2083	3388	4785	8962

A computer program in C language for generating random numbers by the mid-square algorithm is given below :

```
#include<stdio.h>
#include<stdlib.h>
#include<math.h>
main()
  {
     /* Mid-Square Method of Generating 4-digit Random Numbers*/
     /* seed is the starting 4-digit random number,
        n is the number of random numbers to be generated
        x is the random number */

     long int i,s,x,y,z,nd,seed;
     int n;
     seed= 6785;
     printf("\n Number of random numbers to be generated n=");
     scanf("%d",&n);

     for(i=1;i<=n;++i) {
     y= seed*seed / 100.;
     z= y/10000.;
```

```
x=int( (y/10000.-z)*10000.);
seed=x;
printf(" %41d ",x);
}
}
```

A sequence of 30 numbers generated with a seed of 6785 is given below :

0361	1303	6978	6924	9416	6609	6788	0768	5898	7863
8267	3431	7716	5366	7938	0118	0139	0193	0372	1385
9126	2837	484	2342	4848	5031	3108	6595	4940	4036

The problem with the mid-square pseudo random number generator is that the sequence of numbers is limited. With very few exceptions mid-square either degenerates to zero, converge on a constant (the seed 2500 never departs from that value) or cycle forever through a short loop, (the seed 7777 ends up in the cycle 2100, 4100, 8100, 6100, ...). Long sequences comprising of one lac or more numbers can be obtained by using longer seed numbers. Three degenerate mid-square sequences are given below.

Seed 2061 gives;

2061	2477	1355	8360	8896	1388	9265	8402
5936	2360	5696	4444	7491	1150	3225	4006
0480	2304	3084	5110	1121	2566	5843	1406
9768	4138	1230	5129	3066	4003	3240	3576
3317	0024	0005	0000	0000	0000	0000	0000

Seed 1357 gives;

1357	8414	7953	2502	2600	7600	7600	7600	....

Seed 1379 gives;

1379	9016	2882	3059	3574	7734	8147	
3736	9576	6997	9580	7764	2796	8176	
8469	7236	4031	2489	1951	8064	0280	
0784	6146	7733	7992	8720	0384	1474	
1726	9790	8441	2504	2700	2900	4100	
8100	6100	2100	4100	8100	6100	2100	....

4.6 Congruence Method or Residue Method

The most commonly employed pseudo random number generators use the congruence method, also called the method of power residues. This algorithm is described by the expression,

$$r_{i+1} = (ar_i + b) \text{ modulo } m$$

Where a, b and m are constants, r_i and r_{i+1} are ith and $(i + 1)$th random numbers. The expression implies multiplication of a by r_i addition of b and then dividing by m. The r_{i+1} is the remainder or residue. To begin the process of random number generation, in addition to a, b and m, the value of r_0 is also required. It may be any random number and is called seed.

The congruential random number generator may be of the additive, multiplicative or mixed type. The expression given above with $a > 1$ and $b > 0$ is of the mixed type.

If $a = 1$, the expression reduces to the *additive* type.

$$r_{i+1} = (r_i + b) \text{ modulo } m$$

If $b = 0$, the expression reduces to the multiplicative congruential method.

$$r_{i+1} = ar_i \text{ modulo } m.$$

The multiplicative methods are considered better than the additive methods and are as good as the mixed methods.

The selection of values for the constants a, b and m is very important, because on them depends the length of the sequence of random numbers, after which the sequence repeats. It is not possible to generate a non-repeating sequence of numbers with these methods. However, a sufficiently long sequence can be obtained by making a suitable selection of the constants. Since the number can be predicted, rather computed from r_i, and the whole string is reproducible, the numbers obtained are not truly random. They are called pseudo random numbers and hence the method is termed as pseudo random number generator.

Most of the computer languages have a standard function for generating random numbers. In the modern scientific calculators, a random number key is provided. While pressed a random number between 0.000 and 0.999 is generated.

Example 4.1. The pseudo random number generation by the congruential methods can be illustrated by taking some values for a, b and m in the recursive equation.

$$r_{i+1} = (ar_i + b) \text{ mod } m$$

It is better to start with a prime number as modulus m, and prime multiplier a; b can be taken any, say 1. The seed r_0 may be any.

(a) Mixed Multiplicative Congruential (MMC) Generator :

Taking $a = 13$, $b = 1$ and $m = 19$

And let $r_0 = 1$

$$\begin{aligned}
r_1 &= (1 \times 13 + 1) \text{ mod } 19 = 14 \text{ mod } 19 = 0 \text{ residue } 14 = 14\\
r_2 &= (14 \times 13 + 1) \text{ mod } 19 = 183 \text{ mod } 19 = 9 \text{ residue } 12 = 12\\
r_3 &= (12 \times 13 + 1) \text{ mod } 19 = 157 \text{ mod } 19 = 8 \text{ residue } 5 = 5\\
r_4 &= (5 \times 13 + 1) \text{ mod } 19 = 66 \text{ mod } 19 = 3 \text{ residue } 9 = 9\\
r_5 &= (9 \times 13 + 1) \text{ mod } 19 = 118 \text{ mod } 19 = 6 \text{ residue } 4 = 4\\
r_6 &= (4 \times 13 + 1) \text{ mod } 19 = 53 \text{ mod } 19 = 2 \text{ residue } 15 = 15\\
r_7 &= (15 \times 13 + 1) \text{ mod } 19 = 196 \text{ mod } 19 = 10 \text{ residue } 6 = 6\\
r_8 &= (6 \times 13 + 1) \text{ mod } 19 = 79 \text{ mod } 19 = 4 \text{ residue } 3 = 3\\
r_9 &= (3 \times 13 + 1) \text{ mod } 19 = 40 \text{ mod } 19 = 2 \text{ residue } 2 = 2\\
r_{10} &= (2 \times 13 + 1) \text{ mod } 19 = 27 \text{ mod } 19 = 1 \text{ residue } 8 = 8\\
r_{11} &= (8 \times 13 + 1) \text{ mod } 19 = 105 \text{ mod } 19 = 5 \text{ residue } 10 = 10\\
r_{12} &= (10 \times 13 + 1) \text{ mod } 19 = 131 \text{ mod } 19 = 6 \text{ residue } 17 = 17\\
r_{13} &= (17 \times 13 + 1) \text{ mod } 19 = 222 \text{ mod } 19 = 11 \text{ residue } 13 = 13\\
r_{14} &= (13 \times 13 + 1) \text{ mod } 19 = 170 \text{ mod } 19 = 8 \text{ residue } 18 = 18\\
r_{15} &= (18 \times 13 + 1) \text{ mod } 19 = 235 \text{ mod } 19 = 12 \text{ residue } 7 = 7\\
r_{16} &= (7 \times 13 + 1) \text{ mod } 19 = 92 \text{ mod } 19 = 4 \text{ residue } 16 = 16\\
r_{17} &= (16 \times 13 + 1) \text{ mod } 19 = 209 \text{ mod } 19 = 11 \text{ residue } 0 = 00\\
r_{18} &= (0 \times 13 + 1) \text{ mod } 19 = 01 \text{ mod } 19 = 0 \text{ residue } 1 = 01
\end{aligned}$$

This sequence of random numbers between 0 and 18 both inclusive will continue repeating. The number 18, which is one less than modulus is called **Euler function.** This sequence is very short and is no better than the mid-square sequence. However, if the number chosen as the modulus is very large, the pseudo random number sequence will be large and acquires the properties of a true random number sequence.

Random numbers between 0 and 1 can be generated by

$$R_i = \frac{r_i}{m}, \; i = 1, 2, 3, ...$$

which gives the sequence as,

$$R_1 = \frac{14}{19} = 0.7368 \qquad R_2 = \frac{12}{19} = 0.6316 \qquad R_3 = \frac{5}{19} = 0.2632$$

$$R_4 = \frac{9}{19} = 0.4737 \qquad R_5 = \frac{4}{19} = 0.2105 \qquad R_6 = \frac{15}{19} = 0.7895$$

$$R_7 = \frac{6}{19} = 0.3158 \qquad \text{etc .....}$$

(b) Multiplicative Congruential (MC) Generator :

$$r_{i+1} = ar_i \bmod m$$

Again taking $a = 13$, $m = 19$, and seed $r_0 = 1$

$$r_1 = 1 \times 13 \bmod 19 = 0 \quad \text{residue } 13 = 13$$
$$r_2 = 13 \times 13 \bmod 19 = 8 \quad \text{residue } 18 = 18$$
$$r_3 = 13 \times 18 \bmod 19 = 12 \quad \text{residue } 6 = 6$$
$$r_4 = 13 \times 6 \bmod 19 = 4 \quad \text{residue } 2 = 2$$
$$r_5 = 13 \times 2 \bmod 19 = 1 \quad \text{residue } 7 = 7$$
$$r_6 = 13 \times 7 \bmod 19 = 4 \quad \text{residue } 15 = 15$$
$$r_7 = 13 \times 15 \bmod 19 = 10 \quad \text{residue } 5 = 5$$
$$r_8 = 13 \times 5 \bmod 19 = 3 \quad \text{residue } 8 = 8$$
$$r_9 = 13 \times 8 \bmod 19 = 5 \quad \text{residue } 9 = 9$$
$$r_{10} = 13 \times 9 \bmod 19 = 6 \quad \text{residue } 3 = 3$$
$$r_{11} = 13 \times 3 \bmod 19 = 2 \quad \text{residue } 1 = 1$$

The sequence of numbers obtained is 1, 13, 18, 6, 2, 7, 15, 5, 8, 9, 3, 1, which will repeat forever.

(c) Additive Congruential Generator :

$$r_{i+1} = (r_i + b) \bmod m$$

Again taking $m = 19$ and $b = 11$

Taking seed $r_0 = 1$

$$r_1 = (1 + 11) \bmod 19 = 12$$
$$r_2 = (12 + 11) \bmod 19 = 4$$
$$r_3 = (4 + 11) \bmod 19 = 15$$
$$r_4 = (15 + 11) \bmod 19 = 7$$
$$r_5 = (7 + 11) \bmod 19 = 18$$
$$r_6 = (18 + 11) \bmod 19 = 10$$
$$r_7 = (10 + 11) \bmod 19 = 2$$
$$r_8 = (2 + 11) \bmod 19 = 13$$
$$r_9 = (13 + 11) \bmod 19 = 5$$
$$r_{10} = (5 + 11) \bmod 19 = 16$$
$$r_{11} = (16 + 11) \bmod 19 = 8$$
$$r_{12} = (8 + 11) \bmod 19 = 1$$

A computer program in C language for generating the random numbers by the mixed congruential method is given below:

```
#include<stdio.h>
#include<stdlib.h>
#include<math.h>
main()
  {   /*Mixed congruential method of generating random numbers
    a,band m are constants,the values of which are to be suitablly
    selected and entered.
    nn is the number of random numbers to be generated.
    seed is the starting random number, which also is to be
    entered. */

    int a,b,k,i,j,m,nn,seed,r[50];
    printf("\n Enter the INTEGER values of a,b,m");
    scanf("%d %d %d",&a,&b,&m);
    printf("\n Enter the INTEGER value of seed ");
    scanf("%d",&seed);
    printf("\n Enter the number of random numbers to be generated");
    scanf("%d",&nn);
    /*a=21; b=53; m=1000; nn=30; */
    r[0]= seed;
    for(i=1;i<=nn;++i) {
    r[i]=(a*r[i-1]+b) % m;
    printf("%4d",r[i]);
    }

  }
```

The output of the above program is given below :

145	98	111	384	117	510	763	76	649
682	375	928	541	414	747	740	593	506
679	312	605	758	971	444	377	970	423
936	709	942	835	588	407	474	007	200
253	366	739	572	065	418	831	504	637
430	83	796	769	202	295	53	281	282

4.7 Arithmetic Congruential Generator

Another kind of pseudo random number generator is the arithmetic congruential algorithm, which is given as,

$$r_{i+1} = (r_{i-1} + r_i) \bmod m$$

The process starts with two random numbers, which are added and divided by m with the residue giving the third number. Then 2nd and 3rd numbers result into 4th number, and so on.

For example,

If $r_1 = 9$, $r_2 = 13$ and $m = 17$

$$r_3 = (9 + 13) \bmod 17 = 5$$
$$r_4 = (13 + 5) \bmod 17 = 1$$
$$r_5 = (5 + 1) \bmod 17 = 6$$
$$r_6 = (1 + 6) \bmod 17 = 7$$
$$r_7 = (6 + 7) \bmod 17 = 13$$
$$r_8 = (7 + 13) \bmod 17 = 3$$
$$r_9 = (13 + 3) \bmod 17 = 16$$
$$r_{10} = (3 + 16) \bmod 17 = 2$$

and so on as 1, 3, 4, 7, 11, 13, 8, 4; 12, 16, 11, 10, 4, 14, 1, 15, 16, 14, 13, 10, 6, ...

This results into quite a long sequence.

Remarks: Some studies reported in literature indicated that the multiplicative congruence method is superior to the additive method, and that mixed congruence methods are not noticeably better than simple multiplicative methods. In each method, the quality and length of the random number sequence obtained depends upon the values of constants, the selection of which is a complex problem. To be considered random the sequence of numbers produced must meet various tests to ensure that they are uniformly distributed and that there is no significant correlation between the digits of individual numbers and between the sequential numbers. The length of the sequence before it starts repeating should be sufficiently long.

According to the available guidelines modulus m should be the largest prime number that can be filled to the computer word size, and the multiplier a should be a positive primitive root of m. A positive primitive root is defined as a prime factor or any positive integral power thereof.

$a = 5^5 = 3125$ and $m = 2^{35} - 31 = 3435973837$ is one combination suitable for a 36 word size computer.

$a = 7^5 = 16807$ and $m = 2^{31} - 1 = 2147483647$ is one combination suitable for a 32 bit word size computer.

These values ensure a sequence of over two billion random numbers. Further, specify a seed $r_0 = 123457$. The first few numbers generated using the above values for a, m and r_0 are as under.

$$r_1 = 7^5\,(123457) \bmod (2^{31} \quad 1) = 2{,}074{,}941{,}799$$

$$R_1 = \frac{r_1}{2^{31}} = 0.9662$$

$$r_2 = 7^5\,(2{,}074{,}941{,}799) \bmod (2^{31} - 1) = 559{,}872{,}160$$

$$R_2 = \frac{r_2}{2^{31}} = 0.2607$$

$$r_3 = 7^5\,(559{,}872{,}160) \bmod (2^{31} - 1) = 1{,}645{,}535{,}613$$

$$R_3 = \frac{r_3}{2^{31}} = 0.7662$$

In this case while determining random number between 0 and 1, r_i has been divided not by m but by $m + 1$. This does not make much difference, as the value of m is very large.

4.8 Combined Congruential Generators

The random number generators discussed so far, may not be adequate for some complex applications, where hundreds and thousands of elementary events must be simulated before a significant event occur. The simulation of complex computer networks, in which thousands of users are executing hundreds of programs, require substantially longer periods. One method of meeting such a demand is to combine two or more multiplicative congruential generators in such a way that the combined generator has good statistical properties and a longer period.

If $r_{i,1}, r_{i,2}, \ldots, r_{i,k}$ are the ith output from k different multiplicative congruential generators, where the jth generator has prime modulus m_j and the multiplier a_i chosen so that the period is $m_j - 1$, then the combined generator will give,

$$r_i = \left(\sum_{j=1}^{k} (-1)^{j-1} r_{i,j} \right) \bmod m_1 - 1$$

with

$$R_i = \begin{cases} \dfrac{r_i}{m_i}, & r_i > 0 \\ \dfrac{m_i - 1}{m_i} & r_i = 0 \end{cases}$$

The maximum possible period for such a generator is

$$P = \frac{(m_1 - 1)(m_2 - 1) \ldots\ldots (m_k - 1)}{2^k - 1}$$

One such a generator suggested by Le Ecuyer is for $k = 2$. Values of m_1 and m_2 are 2147483563 and 2147483399, while a_1 and a_2 are 40014 and 40692. The value of seed $r_{1,0}$ is selected between 1 and 2147483562 and that of $r_{2,0}$ is selected between 1 and 2147483398.

4.9 Qualities of an Efficient Random Number Generator

- It should have a sufficiently long cycle, that is, it should generate a sufficiently long sequence of random numbers, before beginning to repeat the sequence.
- The random numbers generated should be replicable, that is by specifying the starting conditions, it should be possible to obtain the same set of random numbers, as and when desired. Many times common random numbers are required for the comparison of two systems.
- The generated random numbers should fulfill the requirements of uniformity and independence.
- The random number generator should be fast and cost-effective.
- It should be portable to different computers and ideally to different programming languages.

4.10 Testing Numbers for Randomness

A sequence of random numbers is considered to be random, if;

(*i*) The numbers are uniformly distributed, that is every number has an equal chance of occurrence.

(*ii*) The numbers are not serially autocorrelated. This means that there is no correlation between adjacent pairs or numbers, or that the appearance of one number does not influence the appearance of next number. The sequence 1, 3, 5, 7, 9, ... or 1, 4, 2, 6, 3, 1, 4, 7, 5, ... or 1, 3, 5, 2, 4, 6, 3, 5, 9, ... are serially correlated.

There are a number of tests, which are used to ensure that the random numbers are uniformity distributed and are not serially autocorrelated, some of which are discussed below.

4.11 Uniformity Test

The test of uniformity or frequency test is a basic test that should always be performed to validate a random number generator. Two frequency tests are available. They are, Kolmogorov-Smirnov test and the Chi-Squared test. Both of these tests compare the generated random numbers with the theoretical uniform distribution.

The Kolmogorov-Smirnov Test

This test compares the continuous cdf, $F(x)$ of the uniform distribution to the empirical cdf, $S_N(x)$, of the sample of N random numbers. The largest absolute deviation between $F(x)$ and $S_N(x)$ is determined and is compared with the critical value, which is available as function of N in Appendix Table A-6, for various levels of significance. The procedure of employing the Kolmogorov-Smirnov uniformity test is clearly illustrated in the next example.

Example 4.2. The Kolmogorov-Smirnov test is to be performed to test the uniformity of following random numbers with a level of significance of $\alpha = 0.05$.

.24, .89, .11, .61, .23, .86, .41, .64, .50, .65

The calculations of the test are given in Table 4.1. The top row of the table lists the given random numbers R_i $(i = 1, N)$ in the ascending order. Here $N = 10$. In the second row, the numbers are computed from the empirical distribution, i.e., i/N values are listed. In the third row deviation, $\frac{i}{N} - R_i$ is computed maximum of which gives D^+, while in the fourth row, the deviation $R_i - \frac{(i-1)}{N}$ is computed, the maximum of which given D^-.

The largest deviation, $D = \max(D^+, D^-)$.

From the table, $D^+ = 0.15$, $D^- = 0.13$ giving the largest deviation $D = 0.15$.

The critical value of D obtained from Appendix Table A-6 for $\alpha = 0.05$ and $N = 10$ is 0.410. Since the computed value 0.15 is less than the critical value, the given random numbers are uniform at 95% level of significance. At $\alpha = 0.01$, critical values is 0.368, which again is more than 0.15, hence, the given random numbers are uniform even at 99% level of significance.

Table 4.1

R_i	.11	.23.	.24	.41	.50	.61	.64	.65	.86	.89
i/N	.10	.20	.30	.40	.50	.60	.70	.80	.90	1.0
$i/N - R_i$	—	—	.06	—	00	—	.06	.15	.04	.11
$R_i - (i-1)/N$	.11	.13	.04	.11	.10	.11	.04	—	.06	—

4.12 Chi-Squared Test

The Chi-Squared test uses the sample statistic

$$\chi_0^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$$

where O_i is the observed number in the ith class, E_i is the expected number in the ith class and n is the number of classes. For the uniform distribution, E_i, the expected number in each class is given by

$$E_i = \frac{N}{n}$$

for equally spaced classes, where N is the total number of observations. It can be shown that the sampling distribution of χ_0^2 is approximately the chi-square distribution with $n - 1$ degrees of freedom.

This test involves the classification of 500 random numbers between 0 and 1 into 10 equal intervals, that is numbers less than or equal to 0.1 less than or equal to 0.2, 0.3, ..., 1.0. A bar chart or histogram can than be plotted to illustrate the uniformity of distribution. If the 500 numbers were distributed among the 10 classes, with perfect uniformity, all bars will be of equal length of 50 numbers.

Chi-square is a characteristic of the distribution, which is a measure of its randomness. The statistic Chi-square is computed by subtracting the number of random numbers in each class from the expected number (that is 50), squaring the difference, adding the squares for the ten classes, and dividing the sum by the expectation (50).

For example, if 47, 49, 55, 54, 46, 53, 51, 43, 46 and 54 are the numbers in 10 classes.

Then their differences from 50 are, 3, 1, 5, 4, 4, 3, 1, 7, 4, and 4; and sum of the squares of differences is $9 + 1 + 25 + 16 + 16 + 9 + 1 + 49 + 16 + 16 = 158$

$$\text{Chi-square} = \frac{158}{50} = 3.16$$

There are tables of Chi-square that will tell us about the goodness of the results. The acceptable value of Chi-square will depend upon the degrees of freedom which are one less than the number of classes, and the level of confidence we wish to place in our results. In the present case, since there are 10 classes in which we have divided the random numbers, there are 9 degrees of freedom. At 95% confidence level the acceptable value of Chi-square for 9 degrees of freedom is 16.919 (Appendix Table A-2). This means that when the value of Chi-square is less than or equal to 16.9, there are only 5 chances out of 100 that our results are wrong.

Chi-Squared Test can be employed to compare different sets of random numbers or different random number generators. For each set of random numbers the statistics Chi-square is computed. The one with smaller value of Chi-square is more uniformly distributed.

Example 4.3. The two-digit random numbers generated by a multiplicative congruential method are given below. Determine Chi-square. Is it acceptable at 95% confidence level?

36,	91,	51,	02,	54,	06,	58,	06,	58,	02,	54,	01,	48,	97,	43,
22,	83,	25,	79,	95,	42,	87,	73,	17,	02,	42,	95,	38,	79,	29,
65,	09,	55,	97,	39,	83,	31,	77,	17,	62,	03,	49,	90,	37,	13,
17,	58,	11,	51,	92,	33,	78,	21,	66,	09,	54,	49,	90,	35,	84,
26,	74,	22,	62,	12,	90,	36,	83,	32,	75,	31,	94,	34,	87,	40,
07,	58,	05,	56,	22,	58,	77,	71,	10,	73,	23,	57,	13,	36,	89,
22,	68,	02,	44,	99,	27,	81,	26,	85.						

Solution: The given 100 random numbers can be divided into 10 classes as given below:

Class	*Count*	*Frequency*	*Diff*	*(Diff)*^{2}
$0 < r \le 10$	*************	13	3	9
$10 < r \le 20$	********	8	2	4
$20 < r \le 30$	*********	9	1	1
$30 < r \le 40$	*************	13	3	9
$40 < r \le 50$	*******	7	3	9
$50 < r \le 60$	*************	13	3	9
$60 < r \le 70$	*****	5	5	25
$70 < r \le 80$	************	12	2	4
$80 < r \le 90$	************	12	2	4
$90 < r \le 100$	********	8	2	4
				78

$$\text{Chi-square} = \frac{78}{10} = 7.8$$

For 9 (10 – 1) degrees of freedom, at 95% confidence level, the acceptable value of Chi-square is 16.9, which is more than 7.8. Hence, the given set of random numbers is acceptable, so far as its uniformity in distribution is concerned.

4.13 Testing for Autocorrelation

The uniformity test of random numbers is only a necessary test for randomness, not a sufficient one. A sequence of numbers may be perfectly uniform, and still not random. For example, the sequence .1, .2, .3., .4., .5, .6, .7, .8, .9, .1, .2, .3, ..., would give a perfectly uniform distribution with the chi-square value as zero. But the sequence can by no means be regarded as random. The numbers are not independent, as the occurrence of one number say .3 decides the next, which is to be .4 etc. This defect is called 'serial autocorrelation' of adjacent pairs of numbers.

The Chi-Squared test for serial autocorrelation makes use of a 10 × 10 matrix (checker board). The 10 classes described in the uniformity test are represented both along the rows and columns. If the classes are to be represented on a bar chart, 100 bars, one for each cell of the matrix, will be required. To reduce the number of groups, instead of 10, random numbers are divided into smaller number of classes as 3, or 4. Three classes will be: less than or equal to 0.33, less than or equal to 0.67 and less than or equal to 1.0. With three classes in rows and three in columns, there will be 9 cells.

Let us consider the following random numbers :

49	95	82	19	41	31	12	53	62	40	87	83	26	01	91
55	38	75	90	35	71	57	27	85	52	08	35	57	88	38
77	86	29	18	09	96	58	22	08	93	85	45	79	68	20
11	78	93	21	13	06	32	63	79	54	67	35	18	81	40
62	13	76	74	76	45	29	36	80	78	95	25	52.		

These 73 random numbers giving 72 pairs, are grouped into 9 classes with expectation of 8 in each group.

Class	*Count*	*Frequency*	*Diff*	*(Diff)*^{2}
$R_1 \le .33$ & $R_2 \le 0.33$	* * * * * * * * *	9	1	1
$R_1 \le .67$ & $R_2 \le 0.33$	* * * * * * *	7	1	1
$R_1 \le 1.0$ & $R_2 \le 0.33$	* * * * * *	6	2	4
$R_1 \le .33$ & $R_2 \le 0.67$	* * * * * *	.6	2	4
$R_1 \le .67$ & $R_2 \le 0.67$	* * * * * * * *	8	0	0
$R_1 \le 1.0$ & $R_2 \le 0.67$	* * * * * * * * *	9	1	1
$R_1 \le .33$ & $R_2 \le 1.0$	* * * * * * *	7	1	1
$R_1 \le .67$ & $R_2 \le 1.0$	* * * * * * * * *	9	1	1
$R_1 \le 1.0$ & $R_2 \le 1.0$	* * * * * * * * * * *	11	3	9
		72		24

$$\text{Chi-square} = \frac{24}{8} = 3.0$$

The counts in different classes have been determined by taking the pairs of random numbers. Pair .49 and .95 falls in class $R_1 \le .67$ and $R_2 \le 1.0$. Then the next pair is .95 and .82 which falls in class $R_1 \le 1.0$ and $R_2 \le 1.0$, Pair .82 and .19 falls in class $R_1 \le 1.0$ and $R_2 \le 0.33$ and so on. Since the total number of pairs in 72, one less than the number of random numbers, the expectation is 8, that is 8 pairs in each class. Then squares of differences (frequency – expectation) are determined and their sum is obtained, which on division by expectation gives the value of Chi-square as 3.0.

In this case there are two variables R_1 and R_2 and hence the degrees of freedom are nine minus two that is seven. The criterion value of χ^2 (chi-square) for seven degrees of freedom at 95% confidence level is 14.067. The value of Chi-square obtained for the given set of random numbers is well within the acceptable limit, and hence, they are not serially autocorrelated.

Example 4.4. Given below is a sequence of random numbers. Perform the Chi-Squared tests to check the numbers for uniform distribution and serial autocorrelation.

07	05	96	14	10	90	21	15	84	28	20	78	35	25
72	42	30	66	49	35	60	56	40	54	63	45	48	70
50	42	77	55	36	84	60	30	91	65	24	98	70	18
07	75	12	14	80	06	21	85	96	28	90	90	35	95
84	42	05	78	49	10	72	56	15	66	63	20	60	70
25	54	77	30	48	84	35	42	91	40	36	98	45	30
07	50	24	14	55	18	21							

4.13.1 Uniformity Test

For checking the random numbers for the uniformity, we will divide these in 10 class and take 90 random numbers to have the expected value in each class as 9.

There are ten classes and one variable, giving 9 degrees of freedom. For 9 degrees of freedom at 95% confidence level, the acceptable value of χ^2 is up to 16.9. Our value of χ^2 is well within the acceptable limit and hence, the random numbers in the given sequence are uniformly distributed.

Class	Count	Frequency	(Diff)²
$0 \le R \le 10$	* * * * * * * * *	9	0
$11 \le R \le 20$	* * * * * * * * *	9	0
$21 \le R \le 30$	* * * * * * * * * * * *	12	9
$31 \le R \le 40$	* * * * * * * *	8	1
$41 \le R \le 50$	* * * * * * * * * * * *	12	9
$51 \le R \le 60$	* * * * * * * * *	9	0
$61 \le R \le 70$	* * * * * * * *	8	1
$71 \le R \le 80$	* * * * * * * *	8	1
$81 \le R \le 90$	* * * * * * * *	8	1
$91 \le R \le 100$	* * * * * * *	7	4
		90	26

Chi-square, $\chi^2 = \frac{26}{9} = 2.9$

4.13.2 Chi-Squared Test for Autocorrelation

For this test, the overlapping pairs of random numbers are to be taken. Taking three classes 33; 34 to 67 and 68 to 100 for each random number in the pair, the number of pairs in each s is obtained as follows. Since there are 9 classes, the expectation value is 10.

R_1	R_2	Count	Frequency	$(Diff)^2$
$\le 33,$	≤ 33	* * * * * * * * * *	10	0
$\le 67,$	≤ 33	* * * * * * * * * *	10	0
$\le 100,$	≤ 33	* * * * * * * * *	9	1
$\le 33,$	≤ 67	* * * * * * *	7	9
$\le 67,$	≤ 67	* * * * * * * * * * * * * * * * * * * *	20	100
$\le 100,$	≤ 67	* * * * * * * * * * *	11	1
$\le 33,$	≤ 100	* * * * * * * * * *	13	9
$\le 67,$	≤ 100	* * * * * *	6	16
$\le 100,$	≤ 100	* * * *	4	36
			90	172

$$\text{Chi-square } (\chi^2) = \frac{172}{10} = 17.2$$

The criterion value of χ^2 for 7 degrees of freedom at 95% confidence level is 14.1, which is less than the value we have obtained for the given sequence. Thus, the given sequence of random numbers is serially autocorrelated at 95% confidence level.

4.14 Poker Test

This test gets its name from a game of cards called poker. This test not only tests the randomness of the sequence of numbers, but also the digits comprising of each number. Every random number of five digits or every sequence of five digits is treated as a poker hand.

71549 are five different digits
55137 would be a pair
33669 would be two pairs
55513 would be three of a kind
44477 would be a full house
77774 would be four of a kind
88888 would be five of a kind

The occurrence of five of a kind is rare. The order of the 'cards' within a 'hand' is unimportant, the straights, flushes and royals of the poker game are disregarded in Poker Test.

In 10,000 random and independent numbers of five digits each, you may expect the following distribution of various combinations.

Five different digits	3024	or	30.24%
Pairs	5040		50.40%
Two-pairs	1080		10.80%
Three of a kinds	720		7.20%
Full houses	90		0.90%
Four of a kinds	45		0.45%
Five of a kinds	1		0.01%

Example 4.5 : Poker Test

A sequence of 10,000 five-digit random numbers has been generated, and an analysis of numbers indicate that there are 3075 numbers having five different digits, 4935 having a pair, 1135 having two pairs, 695 having three of a kind, 105 having full house (three of a kind and a pair) 54 having four of a kind and one having all five of a kind. Use Poker Test to determine if these random numbers are independent, at $\alpha = 0.01$.

The calculations of χ^2 are given in Table 4.2.

Table 4.2

Combination Distribution i	*Observed Distribution* O_i	*Expected* E_i	$\frac{(O_i - E_i)^2}{E_i}$
Five different digits	3075	3024	0.8601
Pairs	4935	5040	2.1875
Two-pairs	1135	1080	1.8750
Three of a kind	695	720	0.868
Full houses	105	90	2.5
Four of a kind	54	45	1.8
Five of a kind	01	01	0.0
	10,000	10,000	10.0907

The appropriate degrees of freedom in this case are 6, one less than the number of combinations. The critical value of χ^2 for six degrees of freedom at $\alpha = 0.01$ is 16.8. The value of χ^2 obtained from calculations is 10.097, which is less than the critical value. Hence, the hypothesis of independence cannot be rejected on the basis of Poker Test.

Example 4.6. Use the mixed congruential method to generate the following sequence of random numbers.

(*a*) A sequence of ten two-digit numbers, such that $r_{n+1} = (21r_n + 53)$ modulo 100. Take $r_0 = 52$.

(*b*) A sequence of ten random numbers between 0 and 31 such that,

$r_{n+1} = (13r_n + 15)$ modulo m. Table $r_0 = 11$.

Solution:

(*a*) $r_{n+1} = (21r_n + 53) \bmod 100$

Given $r_0 = 52$

$$
\begin{aligned}
r_1 &= (21 \times 52 + 53) \bmod 100 = 1145 \bmod 100 = 45\\
r_2 &= (21 \times 45 + 53) \bmod 100 = 998 \bmod 100 = 98\\
r_3 &= (21 \times 98 + 53) \bmod 100 = 2111 \bmod 100 = 11\\
r_4 &= (21 \times 11 + 53) \bmod 100 = 284 \bmod 100 = 84\\
r_5 &= (21 \times 84 + 53) \bmod 100 = 1817 \bmod 100 = 17\\
r_6 &= (21 \times 17 + 53) \bmod 100 = 410 \bmod 100 = 10\\
r_7 &= (21 \times 10 + 53) \bmod 100 = 263 \bmod 100 = 63\\
r_8 &= (21 \times 63 + 53) \bmod 100 = 1376 \bmod 100 = 76\\
r_9 &= (21 \times 76 + 53) \bmod 100 = 1649 \bmod 100 = 49\\
r_{10} &= (21 \times 49 + 53) \bmod 100 = 1082 \bmod 100 = 82
\end{aligned}
$$

The required sequence of random numbers is 52, 45, 98, 11, 84, 17, 10, 63, 76, 49, 82.

(*b*) $r_{n+1} = (13r_n + 15) \bmod m$

Since the random numbers required have to be between 0 and 31, the value of m will be taken as 32.

Given $r_0 = 11$

$$
\begin{aligned}
r_1 &= (13 \times 11 + 15) \bmod 32 = 158 \bmod 32 = 30\\
r_2 &= (13 \times 30 + 15) \bmod 32 = 405 \bmod 32 = 21\\
r_3 &= (13 \times 21 + 15) \bmod 32 = 288 \bmod 32 = 00\\
r_4 &= (13 \times 0 + 15) \bmod 32 = 15 \bmod 32 = 15\\
r_5 &= (13 \times 15 + 15) \bmod 32 = 210 \bmod 32 = 18\\
r_6 &= (13 \times 18 + 15) \bmod 32 = 249 \bmod 32 = 25\\
r_7 &= (13 \times 25 + 15) \bmod 32 = 340 \bmod 32 = 20\\
r_8 &= (13 \times 20 + 15) \bmod 32 = 275 \bmod 32 = 19\\
r_9 &= (13 \times 19 + 15) \bmod 32 = 262 \bmod 32 = 06
\end{aligned}
$$

The required sequence of 10 numbers is 11, 30, 21, 00, 15, 18, 25, 20, 19, 06.

Example 4.7. Generate a sequence of five random numbers such that

(*a*) $r_{i+1} = ar_1 \bmod m$, taking $a = 16$, $m = 23$ and $r_0 = 15$

(*b*) $r_{i+1} = (r_1 + b) \bmod m$, taking $b = 11$, $m = 17$ and $r_0 = 1$

Solution:

(*a*) $r_{i+1} = ar_1 \bmod m = 16r_i \bmod 23$

Given $r_0 = 15$

$$
\begin{aligned}
r_1 &= 16 \times 15 \bmod 23 = 240 \bmod 23 = 10\\
r_2 &= 16 \times 10 \bmod 23 = 160 \bmod 23 = 22\\
r_3 &= 16 \times 22 \bmod 23 = 352 \bmod 23 = 7\\
r_4 &= 16 \times 7 \bmod 23 = 112 \bmod 23 = 20\\
r_5 &= 16 \times 20 \bmod 23 = 320 \bmod 23 = 21\\
r_6 &= 16 \times 21 \bmod 23 = 336 \bmod 23 = 14\\
r_7 &= 16 \times 14 \bmod 23 = 224 \bmod 23 = 17\\
r_8 &= 16 \times 17 \bmod 23 = 272 \bmod 23 = 19\\
r_9 &= 16 \times 19 \bmod 23 = 304 \bmod 23 = 5\\
r_{10} &= 16 \times 5 \bmod 23 = 80 \bmod 23 = 11\\
r_{11} &= 16 \times 11 \bmod 23 = 176 \bmod 23 = 15
\end{aligned}
$$

The required sequence of random numbers is 15, 10, 22, 07, 20, 21, 14, 17, 19, 05.

(*b*) $r_{i+1} = (r_1 + b) \bmod m = (r_i + 11) \bmod 17$

Given $r_0 = 1$

$$
\begin{aligned}
r_1 &= (1 + 11) \bmod 17 = 12\\
r_2 &= (12 + 11) \bmod 17 = 06\\
r_3 &= (6 + 11) \bmod 17 = 00\\
r_4 &= (0 + 11) \bmod 17 = 11\\
r_5 &= (11 + 11) \bmod 17 = 05\\
r_6 &= (5 + 11) \bmod 17 = 16\\
r_7 &= (16 + 11) \bmod 17 = 10\\
r_8 &= (10 + 11) \bmod 17 = 04\\
r_9 &= (4 + 11) \bmod 17 = 15\\
r_{10} &= (15 + 11) \bmod 17 = 09
\end{aligned}
$$

The required sequence of random numbers is 01, 12, 06, 00, 11, 05, 16, 10, 04, 15, 09.

Example 4.8. Use the mid of square method to generate 10 four-digit numbers, taking the seed as 9876.

Solution: After squaring the given four-digit number, we will chop off the last two digits and then take the next four as the mid of the square. This will give the next four-digit random number.

r_i	r_i^2	r_{i+1}
9876	97,5353,76	5353
5353	28,6546,09	6546
6546	42,8501,16	8501
8501	72,2670,01	2670
2670	7,1289,00	1289
1289	1,6615,21	6615
6615	43,7582,25	7582
7582	57,4867,24	4867
4867	23,6876,89	6876
6876		

Thus, 9876, 5353, 6546, 8501, 2670, 1289, 6615, 7582, 4867, 6876 is the required sequence of random numbers.

Example 4.9. Employ the arithmetic congruential generator, to generate a sequence of 10 random numbers given $r_1 = 987$ $r_2 = 535$ and modulo $m = 1000$

Solution:

$$r_{i+2} = (r_i + r_{i+1}) \text{ modulo } m$$
$$r_3 = (r_1 + r_2) \text{ modulo } 1000$$
$$= (987 + 535) \text{ mod } 1000 = 1522 \text{ mod } 1000 = 522$$
$$r_4 = (535 + 522) \text{ mod } 1000 = 1057 \text{ mod } 1000 = 057$$
$$r_5 = (522 + 057) \text{ mod } 1000 = 579 \text{ mod } 1000 = 579$$
$$r_6 = (057 + 579) \text{ mod } 1000 = 636 \text{ mod } 1000 = 636$$
$$r_7 = (579 + 636) \text{ mod } 1000 = 1215 \text{ mod } 1000 = 215$$
$$r_8 = (636 + 215) \text{ mod } 1000 = 851 \text{ mod } 1000 = 851$$
$$r_9 = (215 + 851) \text{ mod } 1000 = 1066 \text{ mod } 1000 = 066$$
$$r_{10} = (851 + 66) \text{ mod } 1000 = 917 \text{ mod } 1000 = 917$$

987, 535, 522, 057, 579, 636, 215, 851, 066, 917 is the required sequence of random numbers.

Example 4.10. A sequence of 10,000 random numbers, each of four digits has been generated. This sequence is to be tested for independence using Poker Test. The analysis of the numbers reveals that in 5120 numbers all four digits are different, in 4230, there is one pair in each number, in 560, there are two pairs, while in 75, there are three digits of a kind and in 15 cases all the four digits are same. Determine, if the random numbers are independence at $\alpha = 0.05$.

In this example, each random number is of four digits. The following combination are possible:

(*a*) Four different digits

(*b*) One pair in the digits

(*c*) Two pairs

(*d*) Three digits of one kind

(*e*) All four digits of one kind.

The probability of occurrence of combination (*a*), that is four different digits is,

$$P(a) = .9 \times .8 \times .7 = 0.504$$

The probability having one pair and the other two different digits is,

$$P(b) = \binom{4}{2} \times .1 \times .9 \times .8 = 0.432$$

The probability of having two pairs of like digits is,

$$P(c) = \binom{4}{2} \times .1 \times .1 \times .9 = 0.054$$

The probability of having three digits of a kind is,

$$P(d) = .1 \times .1 \times .9 = 0.009$$

The probability of having all four digits of one kind is

$$P(e) = .1 \times .1 \times .1 = 0.001$$

The test is summarized below in Table 4.3.

Table 4.3

Combination Distribution i	*Observed Distribution* O_i	*Expected* E_i	$\frac{(O_i - E_i)^2}{E_i}$
Four different digits	5120	5040	1.2698
Pairs	4230	4320	1.8750
Two pairs	560	540	0.7407
Three of a kind	75	90	2.5000
Four of a kind	15	10	2.5000
	10000	10000	8.8856

Since, there are five combinations, number of degrees of freedom is four. The critical value $\chi^2_{0.05,\,4} = 9.49$, which is more than the χ^2 obtained above. Hence, the hypothesis that the random numbers are independent cannot be rejected on the basis of Poker Test.

Example 4.11. A sequence of 1000 three-digit random numbers has been generated and their analysis indicates that 680 have three different digits, 289 contain exactly one pair of like digits and 31 contain three like digits. Using Poker Test, these numbers are to be tested for independence at $\alpha = 0.05$.

In case of three-digit number, the possible combinations are

(*a*) All three of different kind

(*b*) One pair

(*c*) All three of same kind.

The probabilities of occurrence of these three combinations are,

$$p(a) = .9 \times .8 = 0.72$$

$$p(b) = \binom{3}{2} \times .1 \times .9 = 0.27$$

$$p(c) = .1 \times .1 = 0.01$$

The calculations of χ^2 are given in Table 4.4.

Table 4.4

Combination Distribution i	Observed Distribution O_i	Expected E_i	$\frac{(O_i - E_i)^2}{E_i}$
Three different digits	680	720	2.2222
One Pair	289	270	1.3370
Three of a kind	31	10	44.1000
	1000	1000	47.6592

The number of degrees of freedom in this case is 2, one less than the number of combinations. Since $\chi^2_{0.05, 2} = 5.99 < 47.66$, the hypothesis of independence of the numbers is rejected.

4.15 Exercises

1. Describe a procedure to physically generate random numbers on the interval [0, 1] with two-digit accuracy.
2. Why do the random numbers generated by computer are called pseudo random numbers. Demonstrate the Mid-square random number generation method, taking the following numbers as seeds. Generate 40 random numbers in each case.

   (*a*) (*i*) 2061 (*ii*) 1357 (*iii*) 1379 (*iv*) 3452

   What is wrong with each sequence of random numbers ?

   (*b*) (*i*) 4789 (*ii*) 7583 (*iii*) 3789
3. Use the mixed congruential method to generate the following sequences of random numbers.

   (*a*) A sequence of 10 two-digit random numbers such that

   $$r_{n+1} = (21r_n + 53) \text{ modulo } 100.$$

   Take $r_0 = 46$.

   (*b*) A sequence of 10 random numbers between 0 and 40, such that

   $$r_{n+1} = (9r_n + 15) \text{ modulo } m.$$

   Take $r_0 = 12$.
4. Repeat problem 3, when the mixed congruential method is reduced to

   (*a*) multiplicative congruential method

   (*b*) additive congruential method
5. Generate a sequence of 10 random numbers employing the Arithmetic congruential generator, when $r_1 = 89$ and $r_2 = 38$ modulus $m = 23$.
6. Use the multiplicative congruential method to generate a sequence of four three-digit random numbers. Let $r_0 = 117$, $a = 3$, $m = 1000$. **[PTU, B. Tech (Prod.) May 2006]**
7. The following sequence of random numbers have been generated

   0.37, 0.55, 0.71, 0.97, 0.65, 0.29, 0.84, 0.78, 0.23

   Use the Kolmogorov-Smirnov Test with $\alpha = 0.05$ to determine, if these numbers are uniformly distributed over the interval 0 to 1.
8. Take 100 two-digit random numbers from the random number table and test their uniformity employing the Chi-Squared Test. Are the numbers uniformly distributed at (*a*) 95% confidence level, (*b*) 99% confidence level?

9. Test the random numbers taken in Problem 8 for serial autocorrelation, by employing the Chi-Squared Test.

10. A sequence of 100 random numbers is given below. Use Chi-Squared Test with $\alpha = 0.05$ to test whether these numbers are uniformly distributed.

21	81	92	23	96	20	68	57	79	84
82	62	12	08	92	83	74	85	60	49
48	37	65	74	22	11	28	10	55	82
72	95	08	85	79	95	86	11	16	52
70	55	50	87	67	51	72	38	29	62
71	12	07	75	56	34	40	67	24	86
18	82	41	29	63	06	84	01	20	06
06	33	14	79	25	65	57	47	74	68
54	35	81	07	88	96	70	85	29	13
12	91	26	57	30	22	90	03	13	31

11. Test the numbers given in Exercise 10, for independence, by employing the Chi-Squared Test. Take $\alpha = 0.05$.

12. Take 100 five-digit random numbers from the random number table and employ Poker Test to check the uniformity of their distribution.

13. A sequence of 1000 four-digit numbers has been generated and an analysis indicates the following combinations and frequencies. Use Poker Test to determine, whether these numbers are independent. Use $\alpha = 0.05$.

Combination	*Observed frequency*
Four different digits	567
One pair	390
Two pairs	19
Three digits of a kind	23
Four digits of a kind	1
	1000

14. Write a computer program that will generate four-digit random numbers using the multiplicative congruential method. Allow the user to input values of r_0, a, b and m.

15. Test the following sequence of numbers for uniformity and independence using procedures you learned in this chapter:

.883, .748, .969, .302, .866, .053, .214, .111, .554, .611,
.964, .033, .444, .459, .000, .725, .724, .405, .160, .736

16. (*a*) What do you mean by system simulation?

(*b*) Why the random numbers generated by computer are called pseudo random numbers? Discuss the congruence method of generating random numbers.