Linear Programming

Objectives of business < Maximizing Profits / Minimizing Costs

Linear programming is a model that consists of linear relationships representing a firm's decision(s), given an objective and resource constraints.

Model formulation

decision variables are mathematical symbols that represent level of activity.

objective function is a linear relationship that reflects objective of an operation.

A constraint is a linear relationship that represents a restriction on decision-making.

Parameters are numerical values that are included in objective function and constraints

Maximization Problem

A company makes two products; Bowls and Mugs. These two products have the following resource requirements for production and profit per item produced (i.e. model parameters):

Product	Resource Requirements: Labour (Hr/unit)	Clay (LB/unit)	Profit (Rs/unit)
Bowl	1	4	40
Mug	2	3	50

There are 40 hours of labour and 120 pounds of clay available each day for production.

Model Formulation

Decision Variables

X_1 = no. of Bowls to Produce

X_2 = no of Mugs to produce

Objective function

$$Z_{Max} = 40X_1 + 50X_2$$

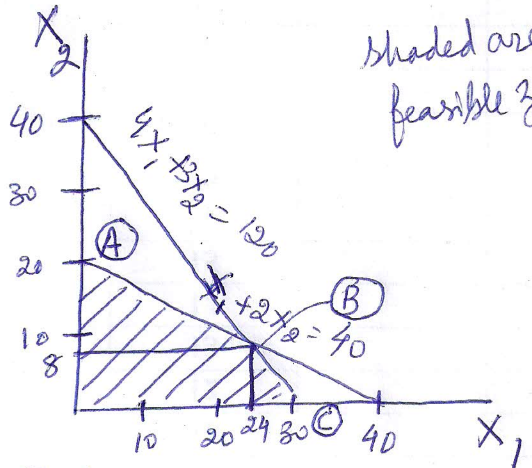
Z = Total Profit

Model Constraints

$X_1 + 2X_2 \leq 40$ hrs

$4X_1 + 3X_2 \leq 120$ LB

$X_1, X_2 \geq 0$



Shaded area is feasible zone.

At A,

$Z = 20 \times 50 = 1000$

At B,

$Z = (40 \times 24) + (8 \times 50) = 1360$

At C,

$Z = (40 \times 30) = 1200$

Profits are highest at point B

$X_1 = 24$ (no. of Bowls)

$X_2 = 8$ (no. of Mugs)

Alternatively.

$X_1 + 2X_2 = 40$

$X_1 = 40 - 2X_2$ —— (1)

And $4X_1 + 3X_2 = 120$

$4X_1 = 120 - 3X_2$

$X_1 = 30 - (3X_2/4)$ — (2)

$40 - 2X_2 = 30 - (3X_2/4)$

$40 - 30 = 2X_2 - \left(\frac{3X_2}{4}\right)$

$10 = \frac{5X_2}{4}$

$40 = 5X_2$

$X_2 = 8$

Putting it into eq (1)

$X_1 = 40 - 2 \times 8 = 24$

If objective function is changed as under

$Z = 70X_1 + 20X_2$

$20X_2 = Z - 70X_1$

$$\boxed{X_2 = \frac{Z}{20} - \frac{7}{2}X_1}$$

Then optimal solution point is C

i.e. $X_1 = 30$ & $X_2 = 0$

$Z = 30 \times 70 = 2100$ ✓

At B, $Z = (24 \times 70) + (8 \times 20)$

$= 1680 + 160 = 1840$

At A $Z = 20 \times 20 = 400$

Comparing

$X_2 = \frac{Z}{20} - \frac{7}{2}X_1$

$(y) = a + (-bx)$ with

$X_2 = \frac{Z}{50} - \frac{4}{5}X_1$

We find that with change in coefficients of objective function, optimal solution has changed.

similarly, if constraint coefficients are changed, optimal solution will also change.

Slack Variables

Slack variable is added to inequality to convert it into equation

$x_1 + 2x_2 \leq 40 \longrightarrow x_1 + 2x_2 + S_1 = 40$ hrs

$4x_1 + 3x_2 \leq 120 \longrightarrow 4x_1 + 3x_2 + S_2 = 120$ LB

A slack variable represents unused resources

Objective function is revised as

$\text{Max } Z = 40x_1 + 50x_2 + 0S_1 + 0S_2$

Minimization Problem

A person needs a fertilizer. There are two brands of fertilizer to chose from, Super-gro and Crop-quick. Each brand yields a specific amount of nitrogen and phosphate per bag, as follows:

	Chemical Contribution	
Brand	nitrogen (LB/Bag)	Phosphate (LB/Bag)
Super gro	2	4
Crop quick	4	3

The person requires at least 16 pounds of nitrogen and 24 pounds of phosphate. Super-gro costs Rs 6/- bag and Crop-quick costs Rs 3/- bag. How many bags of each brand should he purchase to minimize cost?

Decision Variables

x_1 = no. of bags of Super gro

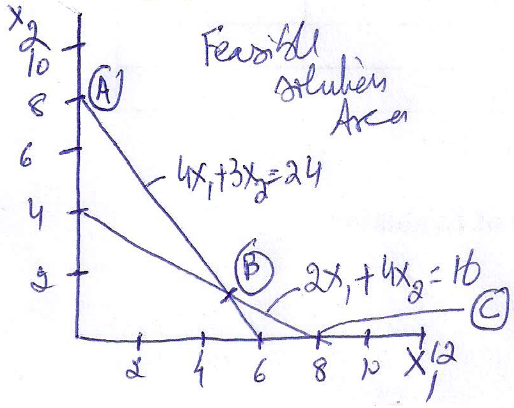
x_2 = no. of bags of Crop quick

Model Constraints

$2x_1 + 4x_2 \geq 16$ LB

$4x_1 + 3x_2 \geq 24$ LB

$x_1, x_2 \geq 0$



$\text{min } Z = 6x_1 + 3x_2$

At A, $Z = 8 \times 3 = 24$ ✓

At B, $Z = (4.8 \times 6) + (1.6 \times 3)$
$= 28.8 + 4.8 = 33.6$

At C, $Z = 8 \times 6 = 48$

$x_1 = 0, \; x_2 = 8, \; Z = 24$

Surplus variables are subtracted from constraint inequalities to convert these into equations as follows.

$2x_1 + 4x_2 - S_1 = 16$ LB

$4x_1 + 3x_2 - S_2 = 24$ LB

Revised objective function

$\text{Min } Z = 6x_1 + 3x_2 + 0S_1 + 0S_2$