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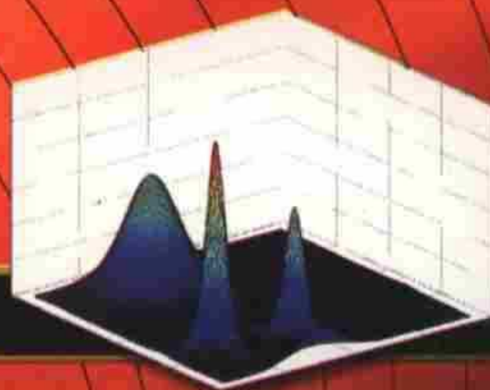
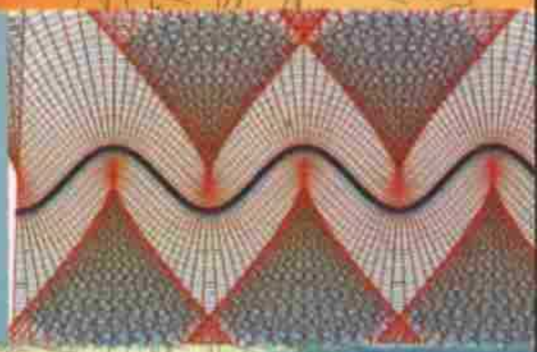


Higher Engineering Mathematics

Dr. B.S. Grewal



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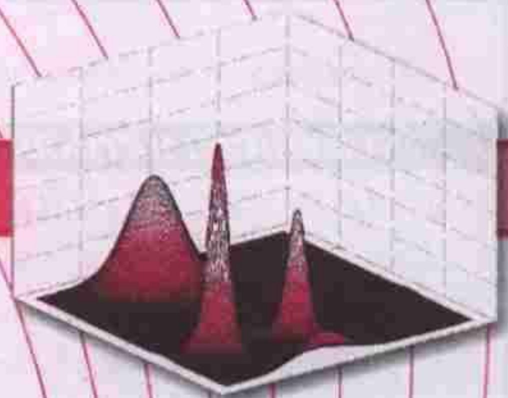


Higher Engineering Mathematics

42nd Edition

Higher Engineering Mathematics

B.S. GREWAL, *Ph.D.*
Professor of Applied Mathematics
Principal Scientific Officer (Ex.)
Defence Research & Development Organisation,
New Delhi
Formerly of
College of Military Engineering, Poona,
Delhi College of Engineering, Delhi

J.S. GREWAL, *M.I.E., I. Engr. (U.K.), M.I. Mar. Engg.*

KHANNA PUBLISHERS

4575/15, Onkar House, Opp. Happy School
Daryaganj, New Delhi-110002
Phone : 011-2324 30 42, 9811541460 ; *Fax* : 011-2324 30 43
e-mail : khannapublishers@yahoo.in
Website : www.khannapublishers.in

Published by :
R.C. Khanna
&
Vineet Khanna
for KHANNA PUBLISHERS
Nai Sarak, Delhi-110006 (India).

ISBN No. : 978-81-7409-195-5

Forty Second Edition : ***2012, June***

Price : ***₹ 450.00***

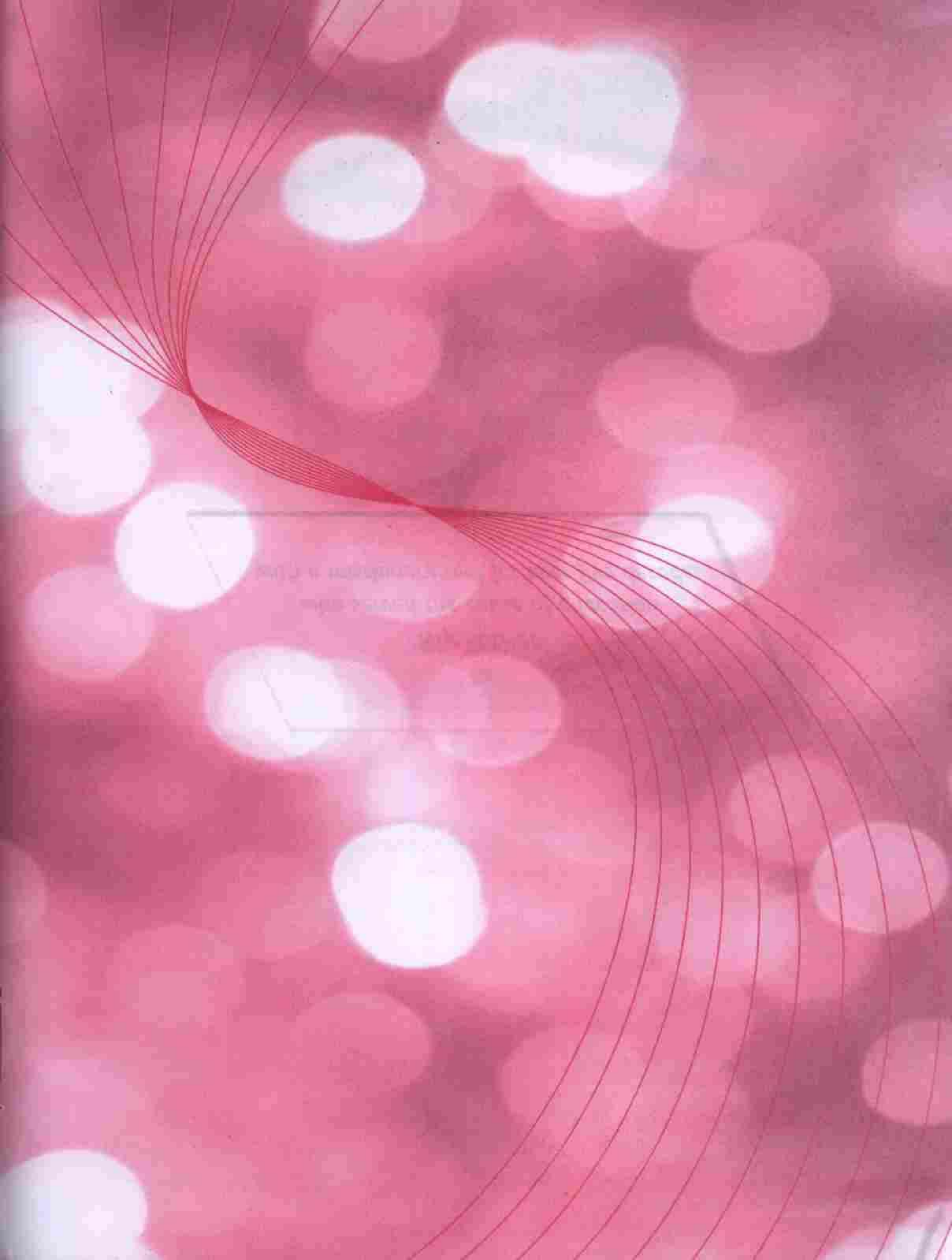
Typesetted at : Goswami Printers, Delhi.

Printed at : Print India, Dilshad Garden, Delhi.

To

My Father

who served the cause of education
with a missionary zeal for over five decades



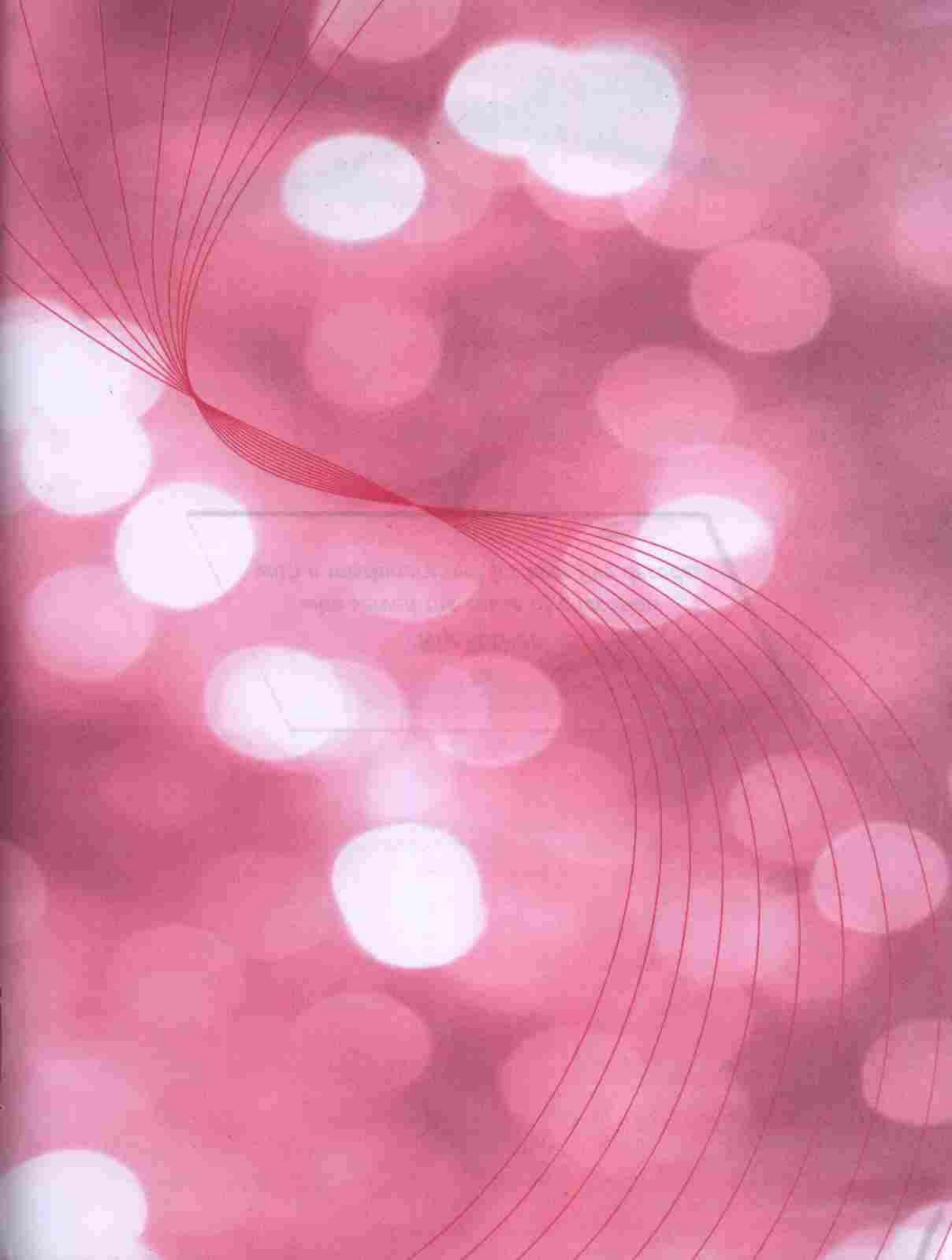
Preface to the 42nd Edition

The book has now been recast in an attractive new format, retaining its main features which have made it so popular. The text has been carefully revised, the number of illustrative examples has been increased and problems from the latest university question papers have been added. The 'Objective Type of Questions' have been updated and given at the end of each chapter. It is hoped that the book in its new form will enjoy its ever increasing popularity.

The author takes this opportunity to thank the numerous readers in India and abroad for their letters of appreciation and fellow professors for their suggestions and patronage of the book. In particular, he is grateful to Prof. Jeevargi Phakirappa, V.N. Engg. College, Bellary (Kar.); Prof. P. Annapurna, N.B.K.R. Inst. of Technology, Vidyanagar (A.P.); Dr. A.P. Burnwal, R.I.T., Koderma (Jh. Kh.); Prof. M. Vasudeva Reddy, Vaishnavi Inst. of Technology, Tarapalli (Tirupati); Dr. K.P. Ghadle, B.A.M. University, Aurangabad (Mah.); Prof. B.K. Yadav, Chauksey Engg. College, Bilaspur (C.G.); Prof. D. Ravi Kumar, Vignan University, Guntur (A.P.); Dr. J.C. Prajapati, Charotara University of Sc. & Technology, Changa (Guj.); Prof. Ramesh Chandra, S.R. Technology Institute, Nalgonda (A.P.); Dr. Latika Bhandari, R.V.S. College of Engg. & Technology, Bhillai; Prof. R. Saraswathi, Sri Padmavati Engg. College, Kavalli (A.P.) and Prof. Vikas Goyal, J.M. Inst. of Technology, Radur (Haryana).

Suggestions for improvement of the text and intimation of misprints will be thankfully acknowledged.

New Delhi **B.S. GREWAL**



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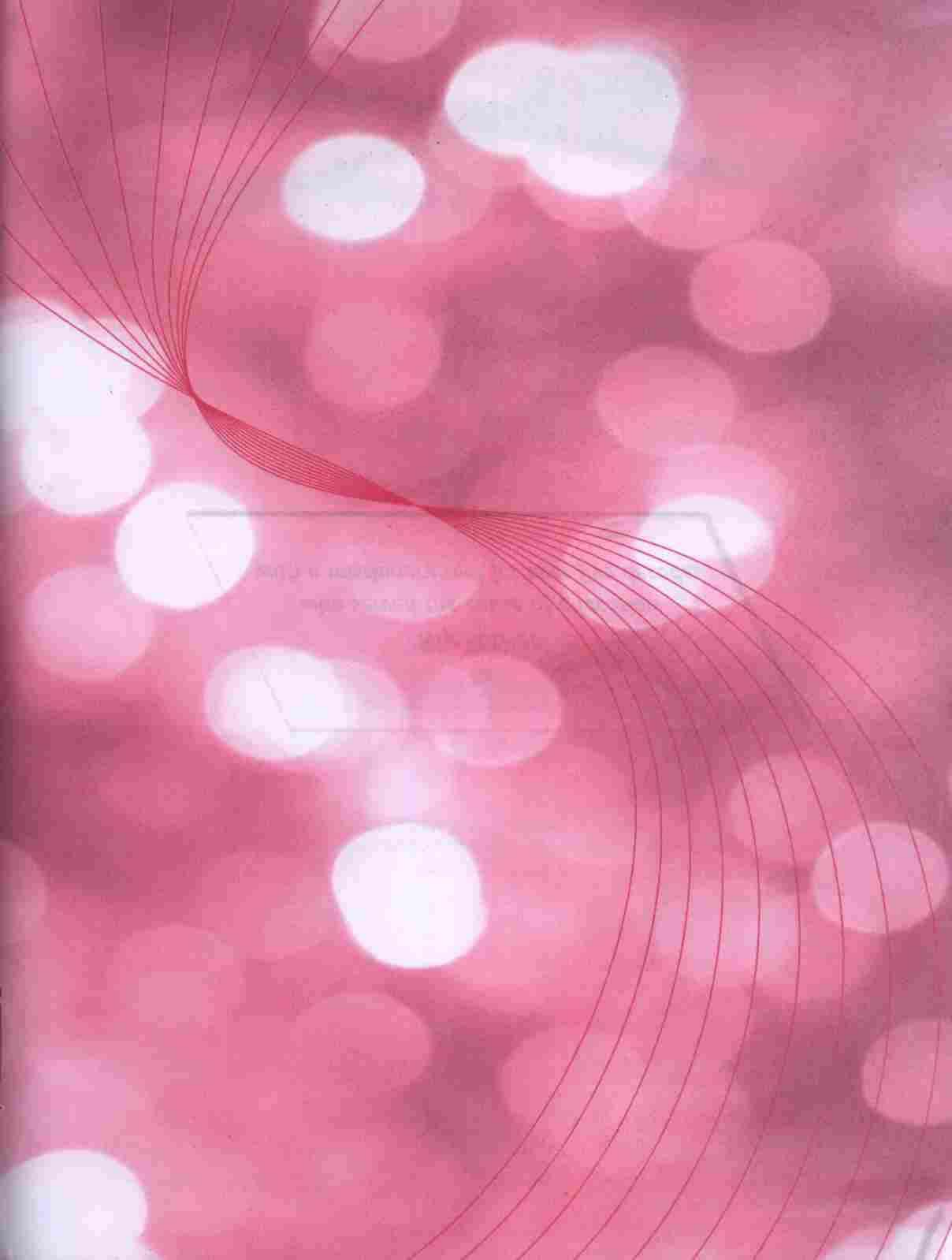
Unit VIII : SPECIAL TOPICS

Note : *The references given alongside the problems pertain to the* **Degree Engineering Examinations** *of the various universities and professional bodies. The abbreviations used for some of these are given below :*

Agra	*stands for*	Dr. B.R. Ambedkar University, Agra
Andhra	"	Andhra University, Waltair
Anna	"	Anna University, Chennai
Bhopal	"	Rajiv Gandhi Technical University, Bhopal
B.P.T.U.	"	Biju Patnaik Technical University, Rourkela
Coimbatore	"	Bharathiyar University, Coimbatore
CUSAT	"	Cochin University of Science and Technology, Kochi
Calicut	"	Calicut University, Cochin
Hazaribag	"	Vinoba Bhave University, Hazaribag
Hissar	"	Guru Jambeshwar University, Hissar
I.E.T.E.	"	Graduateship Examination of the Institute of Electronics and Telecommunication Engineers (India)
I.I.T.	"	Degree Engineering Examination of Indian Institute of Technology
I.S.M.	"	Indian School of Mines, Dhanbad
Kottayam	"	Mahatama Gandhi Memorial University, Kottayam
Kurukshetra	"	National Institute of Technology, Kurukshetra
Madurai	"	Madurai Kamaraj University, Madurai
Marathwada	"	B.A.M. University, Aurangabad
Nagarjuna	"	Acharya Nagarjuna University
P.T.U.	"	Punjab Technical University, Jalandhar
Raipur	"	Pt. Ravi Shankar Shukla University, Raipur
R.T.U.	"	Rajasthan Technical University, Kota
Rohtak	"	Maharishi Dayanand University, Rohtak
S. Patel	"	Sardar Patel University, Vallabh Vidyanagar
S.V.T.U.	"	Swami Vivekanand Technical University, Chhatisgarh
Tirupati	"	Sri Venkateswara University, Tirupati
Tiruchirapalli	"	Bharathidasan University, Tiruchirapalli
U.P.T.U.	"	UP Technical University, Lucknow
U.K.T.U.	"	Uttarakhand Technical University, Dehradun
V.T.U.	"	Visveswaraiah Technological University, Belgaum
Warangal	"	Warangal University of Technology
W.B.T.U.	"	West Bengal University of Technology, Kolkata



CHAPTER 1

Solution of Equations

1. Introduction. 2. General properties. 3. Transformation of equations. 4. Reciprocal equations. 5. Solution of cubic equations–Cardan's method. 6. Solution of biquadratic equations—Ferrari's method ; Descarte's method. 7. Graphical solution of equations. 8. Objective Type Questions.

1.1 INTRODUCTION

The expression $f(x) = a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n$
where a's are constants ($a_0 \neq 0$) and n is a positive integer, is called a *polynomial in x* of degree n. The polynomial $f(x) = 0$ is called an *algebraic equation of degree n*. If $f(x)$ contains some other functions such as trigonometric, logarithmic, exponential etc. ; then $f(x) = 0$ is called a *transcendental equation*.

The value of x which satisfies $f(x) = 0$, ...(1)
is called its root. Geometrically, a root of (1) is that value of x where the graph of $y = f(x)$ crosses the x-axis. The process of finding the roots of an equation is known as *solution* of that equation. This is a problem of basic importance in applied mathematics. We often come across problems in deflection of beams, electrical circuits and mechanical vibrations which depend upon the solution of equations. As such, a brief account of solution of equations is given in this chapter.

1.2 GENERAL PROPERTIES

I. *If α is a root of the equation $f(x) = 0$, then the polynomial $f(x)$ is exactly divisible by $x - \alpha$ and conversely.*
For instance, 3 is a root of the equation $x^4 - 6x^2 - 8x - 3 = 0$, because $x = 3$ satisfies this equation.
$\therefore$ $x - 3$ divides $x^4 - 6x^2 - 8x - 3$ completely, *i.e.*, $x - 3$ is its factor.

II. *Every equation of the nth degree has n roots (real or imaginary).*
Conversely if $\alpha_1, \alpha_2, ..., \alpha_n$ be the roots of the nth degree equation $f(x) = 0$, then
$$f(x) = A(x - \alpha_1)(x - \alpha_2) ... (x - \alpha_n) \text{ where } A \text{ is a constant.}$$

Obs. If a polynomial of degree n vanishes for more than n value of x, it must be identically zero.

Example 1.1. *Solve the equation $2x^3 + x^2 - 13x + 6 = 0$.*

Solution. By inspection, we find $x = 2$ satisfies the given equation.
$\therefore$ 2 is its root, *i.e.* $x - 2$ is a factor of $2x^3 + x^2 - 13x + 6$. Dividing this polynomial by $x - 2$, we get the quotient $2x^2 + 5x - 3$ and remainder 0.

Equating the quotient to zero, we get $2x^2 + 5x - 3 = 0$.

Solving this quadratic, we get $x = \dfrac{-5 \pm \sqrt{[5^2 - 4.(2).(-3)]}}{2 \times 2} = \dfrac{-5 \pm 7}{4} = -3, \dfrac{1}{2}$.

Hence, the roots of the given equation are 2, – 3, 1/2.

Note. *The labour of dividing the polynomial by $x - 2$ can be saved considerably by the following simple device called* **synthetic division.**

2	1	–13	6	2
	4	10	– 6	
2	5	– 3	0	

[*Explanation* : (*i*) Write down the coefficient of the powers of x in order (supplying the missing powers of x by zero coefficients and write 2 on extreme right.

(*ii*) Put 2 as the first term of 3rd row and multiply it by 2, write 4 under 1 and add, giving 5.

(*iii*) Multiply 5 by 2, write 10 under – 13 and add, giving – 3.

(*iv*) Multiply – 3 by 2, write – 6 under 6 and add given zero].

Thus the quotient is $2x^2 + 5x - 3$ and remainder is zero.

Obs. To divide a polynomial by $x + h$, we write $- h$ on the extreme right.

III. Intermediate value property. If $f(a)$ and $f(b)$ *have different signs, then the equation* $f(x) = 0$ *has atleast one root between* $x = a$ *and* $x = b$.

The polynomial $f(x)$ is a continuous function of x (Fig. 1.1). So while x changes from a to b, $f(x)$ must pass through all the values from $f(a)$ to $f(b)$. But since one of these quantities $f(a)$ or $f(b)$ is positive and the other negative, it follows that at least for one value of x(say α) lying between a and b, $f(x)$ must be zero. Then α is the required root.

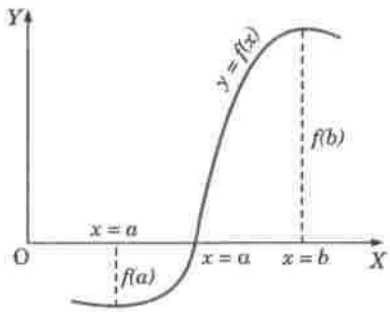


Fig. 1.1

IV. *In an equation with real coefficients, imaginary roots occur in conjugate pairs, i.e.*, if $\alpha + i\beta$ is a root of the equation $f(x) = 0$, then $\alpha - i\beta$ must also be its root. (See p. 534)

Similarly if $a + \sqrt{b}$ is an irrational root of an equation, then $a - \sqrt{b}$ must also be its root.

Obs. *Every equation of the odd degree has at least one real root.*

This follows from the fact that imaginary roots occur in conjugate pairs.

Example 1.2. *Solve the equation* $3x^3 - 4x^2 + x + 88 = 0$, *one root being* $2 + \sqrt{7}i$.

Solution. Since one root is $2 + \sqrt{7}i$, the other root must be $2 - \sqrt{7}i$.

$\therefore$ The factors corresponding to these roots are

$$(x - 2 - \sqrt{7}i) \text{ and } (x - 2 + \sqrt{7}i)$$

or
$$(x - 2 - \sqrt{7}i)(x - 2 + \sqrt{7}i) = (x - 2)^2 + 7 = x^2 - 4x + 11,$$

which is a divisor of $3x^3 - 4x^2 + x + 88$ ...(*i*)

$\therefore$ Division of (*i*) by $x^2 - 4x + 11$ gives $3x + 8$ as the quotient.

Thus the depressed equation is $3x + 8 = 0$. Its root is $- 8/3$. Hence the roots of the given equation are $2 \pm \sqrt{7}i, - 8/3$.

V. Descarte's rule of signs. **The equation* $f(x) = 0$ *cannot have more positive roots than the changes of signs in* $f(x)$; *and more negative roots than the changes of signs in* $f(-x)$.

For instance, consider the equation $f(x) = 2x^7 - x^5 + 4x^3 - 5 = 0$ ...(1)

Sign of $f(x)$ are + – + –

Clearly, $f(x)$ has 3 changes of signs (from + to – or – to +).

Thus (*i*) cannot have more than 3 positive roots.

---

*After the French mathematician and philosopher *Rene Descartes* (1596–1650), who invented Analytic geometry in 1637.

Also $\quad f(-x) = 2(-x)^7 - (-x)^5 + 4(-x)^3 - 5$

$$= -2x^7 + x^5 - 4x^3 - 5$$

This shows that $f(x)$ has 2 changes of signs. Thus (i) cannot have more than 2 negative roots.

Obs. Existence of imaginary roots. *If an equation of the nth degree has at the most p positive roots and at the most q negative roots, then it follows that the equation has at least n – (p + q) imaginary roots.*

Evidently (1) above is an equation of the 7th degree and has at the most 3 positive roots and 2 negative roots. Thus (1) has at least 2 imaginary roots.

VI. Relations between roots and coefficients, *If* $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ *be the roots of the equation*

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0 \qquad \ldots(1)$$

then $\quad \Sigma\alpha_1 = -\dfrac{a_1}{a_0}, \quad \Sigma\alpha_1\alpha_2 = \dfrac{a_2}{a_0}, \quad \Sigma\alpha_1\alpha_2\alpha_3 = -\dfrac{a_3}{a_0}$

....

$$\alpha_1\alpha_2\alpha_3 \ldots\ldots \alpha_n = (-1)^n \frac{a_n}{a_0}.$$

Example 1.3. *Solve the equation $x^3 - 7x^2 + 36 = 0$, given that one root is double of another.*

Solution. Let the roots be α, β, γ such that $\beta = 2\alpha$.

Also $\quad \alpha + \beta + \gamma = 7, \ \alpha\beta + \beta\gamma + \gamma\alpha = 0, \ \alpha\beta\gamma = -36$

$\therefore \quad 3\alpha + \gamma = 7 \qquad \ldots(i)$

$2\alpha^2 + 3\alpha\gamma = 0 \qquad \ldots(ii)$

$2\alpha^2\gamma = -36 \qquad \ldots(iii)$

Solving (i) and (ii), we get $\alpha = 3, \gamma = -2$.

[The values $\alpha = 0, \gamma = 7$ are inadmissible, as they do not satisfy (iii)].

Hence the required roots are 3, 6 and – 2.

Example 1.4. *Solve the equation $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$, given that the sum of two of its roots is zero.* *(Cochin, 2005 ; Madras, 2003)*

Solution. Let the roots be $\alpha, \beta, \gamma, \delta$ such that $\alpha + \beta = 0$.

Also $\quad \alpha + \beta + \gamma + \delta = 2 \quad \therefore \quad \gamma + \delta = 2$

Thus the quadratic factor corresponding to α, β is of the form $x^2 - 0x + p$, and that corresponding to γ, δ is of the form of $x^2 - 2x + q$.

$\therefore \quad x^4 - 2x^3 + 4x^2 + 6x - 21 = (x^2 + p)(x^2 - 2x + q) \qquad \ldots(i)$

Equating the coefficients of x^2 and x from both sides of (i), we get

$$4 = p + q, \quad 6 = -2p.$$

$\therefore \quad p = -3, \quad q = 7.$

Hence the given equation is equivalent to $(x^2 - 3)(x^2 - 2x + 7) = 0$

$\therefore$ The roots are $x = \pm\sqrt{3}, 1 \pm i\sqrt{6}$.

Example 1.5. *Find the condition that the cubic $x^3 - lx^2 + mx - n = 0$ should have its roots in*

(a) arithmetical progression. *(Madras, 2000 S)*

(b) geometrical progression.

Solution. (a) Let the roots be $a - d, a, a + d$ so that the sum of the roots $= 3a = l$ *i.e.*, $a = l/3$.

Since a is the root of the given equation

$\therefore \quad a^3 - la^2 + ma - n = 0 \qquad \ldots(i)$

Substituting $a = l/3$, we get $(l/3)^3 - l(l/3)^2 + m(l/3) - n = 0$.

or $\quad 2l^3 - 9lm + 27n = 0, \quad$ which is the required condition.

(*b*) Let the roots be a/r, a, ar, so that the product of the roots $= a^3 = n$.

Putting $a = (n)^{1/3}$, in (*i*), we get $n - ln^{2/3} + mn^{1/3} - n = 0$ or $m = ln^{1/3}$

Cubing both sides, we get $m^3 = l^3 n$, which is the required condition.

Example 1.6. *Solve the equation $x^4 - 2x^3 - 21x^2 + 22x + 40 = 0$ whose roots are in A.P.*

Solution. Let the roots be $a - 3d$, $a - d$, $a + d$, $a + 3d$, so that the sum of the roots $= 4a = 2$.

$\therefore \qquad a = 1/2$

Also product of the roots $= (a^2 - 9d^2)(a^2 - d^2) = 40$

or $$\left(\frac{1}{4} - 9d^2\right)\left(\frac{1}{4} - d^2\right) = 40 \quad \text{or} \quad 144d^4 - 40d^2 - 639 = 0$$

$\therefore \qquad d^2 = 9/4 \quad \text{or} \quad -7/36$

Thus, $d = \pm 3/2$, the other value is not admissible.

Hence the required roots are $-4, -1, 2, 5$.

Example 1.7. *Solve the equation $2x^4 - 15x^3 + 35x^2 - 30x + 8 = 0$, whose roots are in G.P.*

Solution. Let the roots be a/r^3, a/r, ar, ar^3, so that product of the roots $= a^4 = 4$.

Also the product of a/r^3, ar^3 and a/r, ar are each $= a^2 = 2$.

$\therefore$ The factors corresponding to a/r^3, ar^3 and a/r, ar are $x^2 + px + 2$, $x^2 + qx + 2$.

Thus, $\quad 2x^4 - 15x^3 + 35x^2 - 30x + 8 = 2(x^2 + px + 2)(x^2 + qx + 2)$

Equating the coefficients of x^3 and x^2

$$-15 = 2p + 2q \quad \text{and} \quad 35 = 8 + 2pq$$

$\therefore \qquad p = -9/2, q = -3.$

Thus the given equation is $2\left(x^2 - \frac{9}{2}x + 2\right)(x^2 - 3x + 2) = 0$

Hence the required roots are 1/2, 4 and 1, 2 *i.e.*, $\frac{1}{2}$, 1, 2, 4.

Example 1.8. *If α, β, γ be the roots of the equation $x^3 + px + q = 0$, find the value of*
(a) $\Sigma\alpha^2\beta$, *(b)* $\Sigma\alpha^4$ *(c)* $\Sigma\alpha^3\beta$.

Solution. We have $\quad \alpha + \beta + \gamma = 0 \qquad ...(i)$

$$\alpha\beta + \beta\gamma + \gamma\alpha = p \qquad ...(ii)$$

$$\alpha\beta\gamma = -q \qquad ...(iii)$$

(*a*) Multiplying (*i*) and (*ii*), we get

$\alpha^2\beta + \alpha^2\gamma + \beta^2\gamma + \beta^2\alpha + \gamma^2\alpha + \gamma^2\beta + 3\alpha\beta\gamma = 0$

or $$\Sigma\alpha^2\beta = -3\alpha\beta\gamma = 3q \qquad \text{[By (iii)]}$$

(*b*) Multiplying the given equation by x, we get $x^4 + px^2 + qx = 0$

Putting $x = \alpha, \beta, \gamma$ successively and adding, we get $\Sigma\alpha^4 + p\Sigma\alpha^2 + q\Sigma\alpha = 0$

or $$\Sigma\alpha^4 = -p\Sigma\alpha^2 - q(0) \qquad ...(iv)$$

Now squaring (*i*), we get $\alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha) = 0$

or $$\Sigma\alpha^2 = -2p \qquad \text{[By (ii)]}$$

Hence, substituting the value of $\Sigma\alpha^2$ in (*iv*), we obtain

$$\Sigma\alpha^4 = -p(-2p) = 2p^2.$$

(*c*) $\Sigma\alpha^3\beta = \Sigma\alpha^2\,\Sigma\alpha\beta - \alpha\beta\gamma\Sigma\alpha = -2p(p) - (-q)(0) = -2p^2.$

PROBLEMS 1.1

1. Form the equation of the fourth degree whose roots are $3 + i$ and $\sqrt{7}$. (*Madras, 2000 S*)
2. Solve the equation (*i*) $x^3 + 6x + 20 = 0$, one root being $1 + 3i$.
   (*ii*) $x^4 - 2x^3 - 22x^2 + 62x - 15 = 0$, given that $2 + \sqrt{3}$ is a root.
3. Show that $x^7 - 3x^4 + 2x^3 - 1 = 0$ has at least four imaginary roots. (*Cochin, 2005*)
4. Show that the equation $x^4 + 15x^2 + 7x - 11 = 0$ has one positive, one negative and two imaginary roots.
5. Find the number and position of real roots of $x^4 + 4x^3 - 4x - 13 = 0$.
6. Solve the equation $3x^3 - 11x^2 + 8x + 4 = 0$, given that two of its roots are equal.
7. If r_1, r_2, r_3 are the roots of the equation $2x^3 - 3x^2 + kx - 1 = 0$, find constant k if sum of two roots is 1. (*S.V.T.U., 2009*)
8. The equation $x^4 - 4x^3 + ax^2 + 4x + b = 0$ has two pairs of equal roots. Find the values of a and b.

Solve the following equations 9–14 :

9. $x^3 - 9x^2 + 14x + 24 = 0$, given that two of its roots are in the ratio 3 : 2.
10. $x^3 - 4x^2 - 20x + 48 = 0$ given that the roots α and β are connected by the relation $\alpha + 2\beta = 0$. (*S.V.T.U., 2007*)
11. $x^4 - 6x^3 + 13x^2 - 12x + 4 = 0$, given that it has two parts of equal roots. (*Madras, 2003*)
12. $x^4 - 8x^3 + 21x^2 - 20x + 5 = 0$ given that the sum of two of the roots is equal to the sum of the other two.
13. $x^3 - 12x^2 + 39x - 28 = 0$, roots being in arithmetical progression. (*Madras, 2001 S*)
14. $8x^3 - 14x^2 + 7x - 1 = 0$, roots being in geometrical progression. (*Osmania, 1999*)
15. O, A, B, C are the four points on a straight line such that the distances of A, B, C from O are the roots of equation $ax^3 + 3bx^2 + 3cx + d = 0$. If B is the middle point of AC, show that $a^2d - 3abc + 2b^3 = 0$. (*S.V.T.U., 2006*)
16. Solve the equations (*i*) $x^4 + 2x^3 - 21x^2 - 22x + 40 = 0$ whose roots are in A.P.
    (*ii*) $x^4 + 5x^3 - 30x^2 + 40x + 64 = 0$ whose roots are in G.P.
17. If α, β, γ be the roots of the equation $x^3 - lx^2 + mx - n = 0$, find the value of
    (*i*) $\Sigma\alpha^2\beta^2$, (*ii*) $(\beta + \gamma)(\gamma + \alpha)(\alpha + \beta)$
18. Find the sum of the cubes of the roots of the equation $x^3 - 6x^2 + 11x - 6 = 0$.
19. If α, β, γ are the roots of $x^3 + 4x - 3 = 0$, find the value of (*i*) $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$ (*ii*) $\alpha^{-2} + \beta^{-2} + \gamma^{-2}$.
20. If α, β, γ be the roots of $x^3 + px + q = 0$, show that
    (*i*) $\alpha^5 + \beta^5 + \gamma^5 = 5\alpha\beta\gamma(\beta\gamma + \gamma\alpha + \alpha\beta)$, (*ii*) $3\Sigma\alpha^2\,\Sigma\alpha^5 = 5\Sigma\alpha^3\,\Sigma\alpha^4$.

1.3 TRANSFORMATION OF EQUATIONS

(1) To find an equation whose roots are *m* times the roots of the given equation, *multiply the second term by m, third term by m^2 and so on* (all missing terms supplied with zero coefficients).

For instance, let the given equation be

$$3x^4 + 6x^3 + 4x^2 - 8x + 11 = 0 \quad ...(i)$$

To multiply its roots by m, put $y = mx$ (or $x = y/m$) in (*i*).

Then $$3(y/m)^4 + 6(y/m)^3 + 4(y/m)^2 + 8(y/m) + 11 = 0$$

Multiplying by m^4, we get $$3y^4 + m(6y^3) + m^2(4y^2) - m^3(8y) + m^4(11) = 0$$

This is same as multiplying the second term by m, third term by m^2 and so on in (*i*).

Cor. To find an equation whose roots are with opposite signs to those of the given equation, *change the signs of the every alternative term of the given equation beginning with the second.*

Changing the signs of the roots of (*i*) is same as multiplying its roots by – 1.

∴ The required equation will be

$$3x^4 + (-1)6x^3 + (-1)^2\,4x^2 - (-1)^3\,8x + (-1)^4\,11 = 0$$

or $$3x^4 - 6x^3 + 4x^3 + 8x + 11 = 0$$

which is (*i*) with signs of every alternate term changed beginning with the second.

(2) To find an equation whose roots are reciprocal of the root of the given equation, *change x to 1/x.*

Example 1.9. *Solve $6x^3 - 11x^2 - 3x + 2 = 0$, given that its roots are in harmonic progression.*

Solution. Since the roots of the given equation are in H.P., the roots of the equation having reciprocal roots will be in A.P.

The equation with reciprocal roots is $6(1/x)^3 - 11(1/x)^2 - 3(1/x) + 2 = 0$

or $$2x^3 - 3x^2 - 11x + 6 = 0 \qquad ...(i)$$

Since the roots of the given equation are in H.P., therefore, the roots of (i) are in A.P. Let the root be $a - d$, a, $a + d$. Then

$$3a = 3/2 \text{ and } a(a^2 - d^2) = -3.$$

Solving these equations, we get $a = 1/2$, $d = 5/2$.

Thus the roots of (i) are -2, $1/2$, 3.

Hence the required roots of the given equation are $-1/2$, 2, $1/3$.

Example 1.10. *If α, β, γ be the roots of the cubic equation $x^3 - px^2 + qx - r = 0$, form the equation whose roots are $\beta\gamma + 1/\alpha$, $\gamma\alpha + 1/\beta$, $\alpha\beta + 1/\gamma$.*

Hence evaluate $\Sigma(\alpha\beta + 1/\gamma)(\beta\gamma + 1/\alpha)$. (S.V.T.U., 2008)

Solution. If x is a root of the given equation and y a root of the required equation, then

$$y = \beta\gamma + 1/\alpha = \frac{\alpha\beta\gamma + 1}{\alpha} = \frac{r+1}{\alpha} \qquad [\because \ \alpha\beta\gamma = r]$$

or $$y = \frac{r+1}{x} \Rightarrow x = \frac{r+1}{y}$$

Thus substituting $x = (r + 1)/y$ in the given equation, we get

$$\left(\frac{r+1}{y}\right)^3 - p\left(\frac{r+1}{y}\right)^2 + q\left(\frac{r+1}{y}\right) - r = 0$$

or $ry^3 - q(r + 1)\,y^2 + p(r + 1)^2\,y - (r + 1)^3 = 0$, which is the required equation.

Hence $\Sigma\,(\alpha\beta + 1/\gamma)\,(\beta\gamma + 1/\alpha) = p(r + 1)^2/r$.

Example 1.11. *Form an equation whose roots are cubes of the roots of $x^3 - 3x^2 + 1 = 0$.* ...(i)

Solution. If y be a root of the required equation, then $y = x^3$ ...(ii)

Now we have to eliminate x from (i) and (ii)

$\therefore$ Rewriting (i) as $x^3 + 1 = 3x^2$

Cubing both sides, $x^9 + 3x^6 + 3x^3 + 1 = 27x^6$

Substituting $x^3 = y$, we get $y^3 - 24y^2 + 3y + 1 = 0$, which is the required equation.

(3) To diminish the roots of an equation $f(x) = 0$ by h, *divide $f(x)$ by $x - h$ successively. Then the successive remainders determine the coefficients of the required equation.*

Let the given equation be

$$a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0 \qquad ...(i)$$

To diminish its roots by h, put $y = x - h$ (or $x = y + h$) in (i) so that

$$a_0(y + h)^n + a_1(y + h)^{n-1} + \ldots + a_n = 0 \qquad ...(ii)$$

On simplification, it takes the form

$$A_0\,y^n + A_1\,y^{n-1} + \ldots + A_n = 0 \qquad ...(iii)$$

Its coefficient A_0, A_1, ... A_n can easily be found with the help of *synthetic division* (p. 2). For this, we put $y = x - h$ in (iii) so that

$$A_0\,(x - h)^n + A_1(x - h)^{n-1} + \ldots + A_n = 0 \qquad ...(iv)$$

Clearly, (i) and (iv) are identical. If we divide L.H.S. of (iv) by $x - h$, the remainder is A_n and the quotient $Q = A_0(x - h)^{n-1} + A_1(x - h)^{n-2} + \ldots + A_{n-1}$. Similarly, if we divide Q by $x - h$, the remainder is A_{n-1} and the quotient is Q_1(say). Again dividing Q_1 by $x - h$, A_{n-2} will be obtained as remainder and so on.

Obs. To increase the roots by h, we take h negative.

Example 1.12. *Transform the equation $x^3 - 6x^2 + 5x + 8 = 0$ into another in which the second term is missing. Hence find the equation of its squared differences.* (Cochin, 2005)

Solution. Sum of the roots of the given equation = 6.

In order that the second term in the transformed equation is missing, the sum of the roots is to be zero.

Since the equation has 3 roots, if we decrease each root by 2, the sum of the roots of the new equation will become zero.

$\therefore$ Dividing $x^3 - 6x^2 + 5x + 8$ by $x - 2$ successively, we have

$$
\begin{array}{ccccc}
1 & -6 & 5 & 8 & (2 \\
 & 2 & -8 & -6 & \\
\hline
 & -4 & -3 & \mathbf{2} & \\
 & 2 & -4 & & \\
\hline
 & -2 & \mathbf{-7} & & \\
 & 2 & & & \\
\hline
\mathbf{1} & \mathbf{0} & & &
\end{array}
$$

Thus the transformed equation is $x^3 - 7x + 2 = 0$. ...(*i*)

If α, β, γ be the roots of the given equation, then the roots of (*i*) are $\alpha - 2, \beta - 2, \gamma - 2$.

Let these roots be denoted by a, b, c.

Then $b - c = \beta - \gamma$. Also $a + b + c = 0$, $abc = -2$.

Now $(b - c)^2 = (b + c)^2 - 2bc = (a + b + c - a)^2 - \dfrac{2abc}{a} = a^2 + 4/a$

$\therefore$ The equation of squared differences of (*i*) is given by the transformation $y = x^2 + 4/x$

or $$x^3 - xy + 4 = 0 \qquad ...(ii)$$

Subtracting (*ii*) from (*i*), we get $-7x + xy - 2 = 0$ or $x = 2/(y - 7)$

Substituting for x in (*i*), the equation becomes

$$[2/(y - 7)]^3 - 7[2/(y - 7)] + 2 = 0 \quad \text{or} \quad y^3 - 28y^2 + 245y - 682 = 0 \qquad ...(iii)$$

Roots of this equation are $(b - c)^2, (c - a)^2, (a - b)^2$ *i.e.*, $(\beta - \gamma)^2, (\gamma - \alpha)^2, (\alpha - \beta)^2$.

Hence (*iii*) is the required equation.

1.4 RECIPROCAL EQUATIONS

If an equation remains unaltered on changing x to $1/x$, it is called a **reciprocal equation.**

Such equations are of the following standard types :

I. *A reciprocal equation of an odd degree having coefficients of terms equidistant from the beginning and end equal.* It has a root $= -1$.

II. *A reciprocal equation of an odd degree having coefficients of terms equidistant from the beginning and end equal but opposite in sign.* It has root = 1.

III. *A reciprocal equation of an even degree having coefficients of terms equidistant from the beginning and end equal but opposite in sign.* Such an equation has two roots = 1 and -1.

The substitution $x + 1/x = y$ reduces the degree of the equation of half its former degree.

Example 1.13. *Solve $6x^5 - 41x^4 + 97x^3 - 97x^2 + 41x - 6 = 0$.* (Coimbatore, 2001 S)

Solution. This is a reciprocal equation of odd degree with opposite signs. $\therefore$ $x = 1$ is a root.

Dividing L.H.S. by $x - 1$, the given equation reduces to

$$6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0$$

Dividing by x^2, we have

$$6(x^2 + 1/x^2) - 35(x + 1/x) + 62 = 0$$

Putting $x + 1/x = y$ and $x^2 + 1/x^2 = y^2 - 2$, we get

$$6(y^2 - 2) - 35y + 62 = 0 \quad \text{or} \quad 6y^2 - 35y + 50 = 0 \quad \text{or} \quad (3y - 1)(2y - 5) = 0$$

$\therefore$ $$x + 1/x = y = 1/3 \quad \text{or} \quad 5/2$$

i.e., $3x^2 - 10x + 3 = 0$ or $2x^2 - 5x + 2 = 0$

i.e., $(3x - 1)(x - 3) = 0$ or $(2x - 1)(x - 2) = 0$

$\therefore$ $x = 1/3, 3$ or $1/2, 2$

Hence the required roots are 1, 1/3, 3, 1/2, 2.

Example 1.14. *Solve* $6x^6 - 25x^5 + 31x^4 - 31x^2 + 25x - 6 = 0$. *(Madras, 2003)*

Solution. This is a reciprocal equation of even degree with opposite signs. $\therefore$ $x = 1, -1$ are its roots.

Dividing L.H.S. by $x - 1$ and $x + 1$, the given equation reduces to

$$6x^4 - 25x^3 + 37x^2 - 25x + 6 = 0$$

Dividing by x^2, we get

$$6(x^2 + 1/x^2) - 25(x + 1/x) + 37 = 0.$$

Putting $x + 1/x = y$ and $x^2 + 1/x^2 = y^2 - 2$, it becomes

$$6(y^2 - 2) - 25y + 37 = 0 \quad \text{or} \quad 6y^2 - 25y + 25 = 0$$

or $(2y - 5)(3y - 5) = 0$

$\therefore$ $x + 1/x = y = 5/2$ or $5/3$.

i.e., $2x^2 - 5x + 2 = 0$ or $3x^2 - 5x + 3 = 0$

$\therefore$ $x = 2, 1/2$ or $x = \dfrac{5 \pm i\sqrt{11}}{6}$

Hence the required roots of the given equation are $1, -1, 2, 1/2, \dfrac{5 \pm i\sqrt{11}}{6}$.

PROBLEMS 1.2

1. Find the equation whose roots are 3 times the roots of $x^3 + 2x^2 - 4x + 1 = 0$.
2. Form the equation whose roots are the reciprocals of the roots of $2x^5 + 4x^3 - 13x^2 + 7x - 6 = 0$. *(S.V.T.U., 2009)*
3. Find the equation whose roots are the negative reciprocals of the roots of
$$x^4 + 7x^3 + 8x^2 - 9x + 10 = 0.$$
4. Solve the equation $6x^3 - 11x^2 - 3x + 2 = 0$, given that its roots are in H.P.
5. Find the equation whose roots are the roots of
   (*i*) $x^3 - 6x^2 + 11x - 6 = 0$ each increased by 1. *(S.V.T.U., 2009)*
   (*ii*) $x^4 + x^3 - 3x^2 - x + 2 = 0$ each diminished by 3.
   (*iii*) $x^5 - 5x^4 + 10x^3 - 10x^2 + 5x + 6 = 0$ each diminished by 1.
6. Find the equation whose roots are the squares of the roots of $x^3 - x^2 + 8x - 6 = 0$.
7. Find the equation whose roots are the cubes of the roots of $x^3 + px^2 + q = 0$.
8. If α, β, γ are the roots of the equation $2x^3 + 3x^2 - x - 1 = 0$, form the equation whose roots are $(1 - \alpha)^{-1}$, $(1 - \beta)^{-1}$ and $(1 - \gamma)^{-1}$.
9. If a, b, c are the roots of the equation $x^3 + px^2 + qx + r = 0$, find the equation whose roots are ab, bc and ca. *(Madras, 2003)*
10. If α, β, γ be the roots of $x^3 + mx + n = 0$, form the equation whose roots are
   (*a*) $\alpha + \beta - \gamma, \beta + \gamma - \alpha, \gamma + \alpha - \beta$, (*b*) $\beta\gamma/\alpha, \gamma\alpha/\beta, \alpha\beta/\gamma$ (*c*) $\dfrac{1}{\beta} + \dfrac{1}{\gamma}, \dfrac{1}{\gamma} + \dfrac{1}{\alpha}, \dfrac{1}{\alpha} + \dfrac{1}{\beta}$.
11. Find the equation of squared differences of the roots of the cubic $x^3 + 6x^2 + 7x + 2 = 0$.
12. Solve the equations :
   (*i*) $6x^4 + 5x^3 - 38x^2 + 5x + 6 = 0$ (*ii*) $4x^4 - 20x^3 + 33x^2 - 20x + 4 = 0$. *(Madras, 2003)*
   (*iii*) $8x^5 - 22x^4 - 55x^3 + 55x^2 + 22x - 8 = 0$. (*iv*) $6x^5 + x^4 - 43x^3 - 43x^2 + x + 6 = 0$ *(S.V.T.U., 2006)*
   (*v*) $3x^6 + x^5 - 27x^4 + 27x^2 - x - 3 = 0$.
13. Show that the equation $x^4 - 10x^3 + 23x^2 - 6x - 15 = 0$ can be transformed into reciprocal equation by diminishing the roots by 2. Hence solve the equation.
14. By suitable transformation, reduce the equation $x^4 + 16x^3 + 83x^2 + 152x + 84 = 0$ to an equation in which term in x^3 is absent and hence solve it. *(Madras, 2002)*

1.5 SOLUTION OF CUBIC EQUATIONS–CARDAN'S METHOD*

Consider the equation $ax^3 + bx^2 + cx + d = 0$ ...(1)

Dividing by a, we get an equation of the form $x^3 + lx^2 + mx + n = 0$.

To remove the x^2 term, put $y = x - (-l/3)$ or $x = y - l/3$ so that the resulting equation is of the form

$$y^3 + py + q = 0 \qquad ...(2)$$

To solve (2), put $y = u + v$

so that $y^3 = u^3 + v^3 + 3uv(u + v) = u^3 + v^3 + 3uvy$

or $y^3 - 3uvy - (u^3 + v^3) = 0$ ...(3)

Comparing (2) and (3), we get

$$uv = -p/3,\ u^3 + v^3 = -q \text{ or } u^3 + v^3 = -q \text{ and } u^3v^3 = -p^3/27$$

$\therefore$ u^3, v^3 are the roots of the equation $t^2 + qt - p^3/27 = 0$

which gives $u^3 = \frac{1}{2}(-q + \sqrt{q^2 + 4p^3/27}) = \lambda^3$ (say)

and $v^3 = \frac{1}{2}(-q - \sqrt{q^2 + 4p^3/27})$

$\therefore$ The three values of u are $\lambda, \lambda\omega, \lambda\omega^2$, where ω is one of the imaginary cube roots of unity.

From $uv = -p/3$, we have $v = -p/3u$

$\therefore$ When $u = \lambda, \lambda\omega$ and $\lambda\omega^2$,

$$v = -\frac{p}{3\lambda}, -\frac{p\omega^2}{3\lambda} \text{ and } -\frac{p\omega}{3\lambda}. \qquad [\because \omega^3 = 1]$$

Hence the three roots of (2) are $\lambda - \frac{p}{3\lambda}, \lambda\omega - \frac{p\omega^2}{3\lambda}, \lambda\omega^2 - \frac{p\omega}{3\lambda}$ (Being $= u + v$)

Having known y, the corresponding values of x can be found from the relation $x = y - l/3$.

Obs. 1. If one value of u is found to be a rational number, find the corresponding value of v giving one root $y = u + v$. Then find the corresponding root $x = \alpha$ (say). Finally, divide the left hand side of (1) by $x - \alpha$, giving the remaining quadratic equation from which the other two roots can be found readily.

Obs. 2. If u^3 and v^3 turn out to be conjugate complex numbers, the roots of the given cubic can be obtained in neat forms by employing De Moivre's theorem. (§ 19.5)

Example 1.15. *Solve by Cardan's method* $x^3 - 3x^2 + 12x + 16 = 0$. *(U.P.T.U., 2008)*

Solution. Given equation is $x^3 - 3x^2 + 12x + 16 = 0$ ...(*i*)

To remove the second term from (i), diminish each root of (*i*) by 3/3 = 1, *i.e.*, put $y = x - 1$ or $x = y + 1$ [$\therefore$ Sum of roots = 3]. Then (*i*) becomes

$$(y + 1)^3 - 3(y + 1) + 12(y + 1) + 16 = 0 \text{ or } y^3 + 9y^2 + 26 = 0 \qquad ...(ii)$$

To solve (*ii*), put $y = u + v$ so that $y^3 - 3uvy - (u^3 + v^3) = 0$ ...(*iii*)

Comparing (*ii*) and (*iii*), we get $uv = -3$ and $u^3 + v^3 = -26$

$\therefore$ u^3, v^3 are the roots of the equation $t^2 + 26t - 27 = 0$

or $(t + 27)(t - 1) = 0$ whence $t = -27, t = 1$.

or $u^3 = -27$ *i.e.*, $u = -3$ and $v^3 = 1$ *i.e.*, $v = 1$

$\therefore$ $y = u + v = -3 + 1 = -2$ and $x = y + 1 = -1$

Dividing L.H.S. of (*i*) by $x + 1$, we obtain $x^2 - 4x + 16 = 0$

or $$x = \frac{4 \pm \sqrt{(16 - 64)}}{2} = 2 \pm i\ 2\sqrt{3}$$

Hence the required roots of the given equation are $-1, 2 \pm i\ 2\sqrt{3}$.

*Named after an Italian mathematician *Girolamo Cardan* (1501–1576) who was the first to use complex number as roots of an equation.

Example 1.16. *Solve the cubic equation $28x^3 - 9x^2 + 1 = 0$ by Cardan's method.*

Solution. Since the term in x is missing, let us put $x = 1/y$ in the given equation so that the transformed equation is $y^3 - 9y + 28 = 0$ ...(*i*)

To solve (*i*), put $y = u + v$ so that $y^3 - 3uvy - (u^3 + v^3) = 0$ ...(*ii*)

Comparing (*ii*) and (*iii*), we get $uv = 3$ and $u^3 + v^3 = -28$.

$\therefore$ u^3, v^3 are the roots of $t^2 + 28t + 27 = 0$

or $(t + 1)(t + 27) = 0$ or $t = -1, -27$ or $u = -1, v = -3$

$\therefore$ $y = u + v = -4$. Dividing L.H.S. of (*i*) by $y + 4$, we obtain $y^2 - 4y + 7 = 0$ whence $y = 2 \pm i\sqrt{3}$.

$\therefore$ Roots of (*i*) are $-4, 2 \pm i\sqrt{3}$.

Hence the roots of the given cubic equation are $-\dfrac{1}{4}, \dfrac{1}{2 \pm i\sqrt{3}}$ or $-\dfrac{1}{4}, (2 - i\sqrt{3})/7, (2 + i\sqrt{3})/7$.

Example 1.17. *Solve the equation $x^3 + x^2 - 16x + 20 = 0$.*

Solution. Instead of diminishing the roots of the given equation by $-1/3$, we first multiply its roots by 3, so that the equation becomes

$$x^3 + 3x^2 - 144x + 540 = 0 \quad ...(i)$$

To remove the x^2 term, put $y = x - (-3/3)$ or $x = y - 1$ in (*i*)

so that $$(y - 1)^3 + 3(y - 1)^2 - 144(y - 1) + 540 = 0$$

or $$y^3 - 147y + 686 = 0 \quad ...(ii)$$

To solve (iii), let $y = u + v$, so that

$$y^3 - 3uvy - (u^3 + v^3) = 0 \quad ...(iii)$$

Comparing (*ii*) and (*iii*), we get

$$uv = 49,\ u^3 + v^3 = -686, \text{ so that } u^3 v^3 = (343)^2.$$

$\therefore$ u^3, v^3 are the roots of the quadratic

$$t^2 + 686t + (343)^2 = 0 \quad \text{or} \quad (t + 343)^2 = 0$$

$\therefore$ $t = -343$ *i.e.*, $u^3 = v^3 = -343$ or $u = v = -7$.

Thus $y = u + v = -14$ and $x = y - 1 = -15$.

Dividing L.H.S. of (*i*) by $x + 15$, we get

$$(x - 6)^2 = 0 \quad \text{or} \quad x = 6, 6.$$

$\therefore$ The root of (*i*) are $-15, 6, 6$.

Hence the roots of the given equation are $-5, 2, 2$.

Example 1.18. *Solve $x^3 - 3x^2 + 3 = 0$.* *(S.V.T.U., 2006)*

Solution. Given equation is $x^3 - 3x^2 + 3 = 0$ ...(*i*)

To remove the x^2 term, put $y = x - 3/3$ or $x = y + 1$,

so that (*i*) becomes $(y + 1)^3 - 3(y + 1)^2 + 3 = 0$

or $$y^3 - 3y + 1 = 0 \quad ...(ii)$$

To solve it, put $y = u + v$

so that $$y^3 - 3uvy - (u^3 + v^3) = 0 \quad ...(iii)$$

Comparing (*ii*) and (*iii*), we get $uv = 1, u^3 + v^3 = -1$

$\therefore$ u^3, v^3 are the roots of the equation $t^2 + t + 1 = 0$

Hence $u^3 = \dfrac{-1 + i\sqrt{3}}{2}$ and $v^3 = \dfrac{-1 - i\sqrt{3}}{2}$

$\therefore$ $u = \left(\dfrac{-1 + i\sqrt{3}}{2}\right)^{1/3}$ put $-\dfrac{1}{2} = r\cos\theta$ and $\sqrt{3}/2 = r\sin\theta$

$= [r(\cos\theta + i\sin\theta)]^{1/3}$ so that $r = 1, \theta = 2\pi/3$

$= [\cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi)]^{1/3}$,

where n is any integer or zero. Using De Moivre's theorem (p. 647).

$$u = \cos\left(\frac{\theta + 2n\pi}{3}\right) + i \sin\left(\frac{\theta + 2n\pi}{3}\right)$$

Giving n the value 0, 1, 2 successively we get the three values of u to be

$$\cos\frac{\theta}{3} + i\sin\frac{\theta}{3}, \cos\frac{\theta + 2\pi}{3} + i\sin\frac{\theta + 2\pi}{3}, \cos\frac{\theta + 4\pi}{3} + i\sin\frac{\theta + 4\pi}{3}$$

i.e., $$\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}, \cos\frac{8\pi}{9} + i\sin\frac{8\pi}{9}, \cos\frac{14\pi}{9} + i\sin\frac{14\pi}{9}.$$

The corresponding values of v are

$$\cos\frac{2\pi}{9} - i\sin\frac{2\pi}{9}, \cos\frac{8\pi}{9} - i\sin\frac{8\pi}{9}, \cos\frac{14\pi}{9} - i\sin\frac{14\pi}{9}.$$

$\therefore$ The three values of $y = u + v$ are $2\cos 2\pi/9$, $2\cos 8\pi/9$, $2\cos 14\pi/9$.

Hence the roots of (i) are found from $x = 1 + y$ to be

$$1 + 2\cos 2\pi/9,\ 1 + 2\cos 8\pi/9,\ 1 + 2\cos 14\pi/9.$$

PROBLEMS 1.3

Solve the following equations by Cardan's method :

1. $x^3 - 27x + 54 = 0$. (*U.P.T.U., 2003*)
2. $x^3 - 18x + 35 = 0$ (*Osmania, 2003*)
3. $x^3 - 15x = 126$ (*S.V.T.U., 2009*)
4. $2x^3 + 5x^2 + x - 2 = 0$ (*U.P.T.U., 2003*)
5. $9x^3 + 6x^2 - 1 = 0$ (*S.V.T.U., 2008*)
6. $x^3 - 6x^2 + 6x - 5 = 0$
7. $x^3 - 3x + 1 = 0$
8. $27x^3 + 54x^2 + 198x - 73 = 0$

1.6 SOLUTION OF BIQUADRATIC EQUATIONS

(1) Ferrari's method

This method of solving a biquadratic equation is illustrated by the following examples :

Example 1.19. *Solve the equation $x^4 - 12x^3 + 41x^2 - 18x - 72 = 0$ by Ferrari's method.* (*S.V.T.U., 2007*)

Solution. Combining x^4 and x^3 terms into a perfect square, the given equation can be written as

$$(x^2 - 6x + \lambda)^2 + (5 - 2\lambda)x^2 + (12\lambda - 18)x - (\lambda^2 + 72) = 0$$

or $$(x^2 - 6x + \lambda)^2 = \{(2\lambda - 5)x^2 + (18 - 12\lambda)x + (\lambda^2 + 72)\} \quad ...(i)$$

This equation can be factorised if R.H.S. is a perfect square

i.e., if $$(18 - 12\lambda)^2 = 4(2\lambda - 5)(\lambda^2 + 72) \quad [b^2 = 4ac]$$

i.e., if $$2\lambda^3 - 41\lambda^2 + 252\lambda - 441 = 0 \quad \text{which gives} \quad \lambda = 3.$$

$\therefore$ (i) reduces to $(x^2 - 6x + 3)^2 = (x - 9)^2$

i.e., $$(x^2 - 5x - 6)(x^2 - 7x + 12) = 0.$$

Hence the roots of the given equation are -1, 3, 4 and 6.

Example 1.20. *Solve the equation $x^4 - 2x^3 - 5x^2 + 10x - 3 = 0$ by Ferrari's method.*

Solution. Combining x^4 and x^3 terms into a perfect square, the given equation can be written as $(x^2 - x + \lambda)^2 = (2\lambda + 6)x^2 - (2\lambda + 10)x + (\lambda^2 + 3)$. This equation can be factorised, if R.H.S. is a perfect square *i.e.,* if $(2\lambda + 10)^2 = 4(2\lambda + 6)(\lambda^2 + 3)$ $[b^2 = 4ac]$

or $$2\lambda^3 + 5\lambda^2 - 4\lambda - 7 = 0, \text{ which gives } \lambda = -1.$$

$\therefore$ (i) reduces to $(x^2 - x - 1)^2 = 4x^2 - 8x + 4$

or $$(x^2 - x - 1)^2 - (2x - 2)^2 = 0 \quad \text{or} \quad (x^2 + x - 3)(x^2 - 3x + 1) = 0$$

$\therefore$ $$x = \frac{-1 \pm \sqrt{1 + 12}}{2} \quad \text{or} \quad \frac{3 \pm \sqrt{9 - 4}}{2}$$

Hence the roots are $\dfrac{-1 \pm \sqrt{13}}{2}, \dfrac{3 \pm \sqrt{5}}{2}$.

(2) Descarte's method

This method of solving a biquadratic equations consists in removing the term in x^3 and then expressing the new equation as product of two quadratics. It has been best illustrated by the following examples :

Example 1.21. *Solve the equation $x^4 - 8x^2 - 24x + 7 = 0$ by Descarte's method.* (U.P.T.U., 2001)

Solution. In the given equation, the term in x^3 is already absent so we assume that

$$x^4 - 8x^2 - 24x + 7 = (x^2 + px + q)(x^2 - px + q') \qquad ...(i)$$

Equating coefficients of the like powers of x in (*i*), we get

$$-8 = q + q' - p^2, -24 = p(q' - q) ; 7 = qq'$$

$\therefore \quad q + q' = p^2 - 8, q - q' = 24/p$

$\therefore \quad (p^2 - 8)^2 - (24/p)^2 = 4 \times 7$

$$p^2 - 16p^4 + 36p^2 - 576 = 0 \quad \text{or} \quad t^3 - 16t^2 + 36t - 576 = 0 \text{ where } t = p^2$$

Now $t = 16$ satisfies this cubic so that $p = 4$.

$\therefore \quad q + q' = 8, q - q' = 6 \quad \therefore \quad q = 7, q' = 1$

Thus (*i*) takes the form $(x^2 + 4x + 7)(x^2 - 4x + 1) = 0$

whence $$x = \frac{-4 \pm \sqrt{(16 - 28)}}{2} \text{ or } x = \frac{4 \pm \sqrt{(16 - 4)}}{2}$$

Hence $x = -2 \pm \sqrt{3}i, 2 \pm \sqrt{3}$.

Example 1.22. *Solve the equation $x^4 - 6x^3 - 3x^2 + 22x - 6 = 0$ by Descarte's method.*

Solution. Here sum of roots = 6 and number of roots = 4

$\therefore$ To remove the second term, we have to diminish the roots by 6/4 (= 3/2) which will be a problem. Therefore, we first multiply the roots by 2. $\therefore \quad y^4 - 12y^3 + 12y^2 + 176y - 96 = 0$ where $y = 2x$. Now diminishing the roots by 3, we obtain $z^4 - 42z^2 + 32z + 297 = 0$ where $z = y - 3$.

Assuming that $$z^4 - 42z^2 + 32z + 297 = (z^2 + pz + q)(z^2 - pz + q') \qquad ...(i)$$

and comparing coefficients, we get

$$-42 = q + q' - p^2 ; 32 = p(q' - q) ; 297 = q q'$$

$\therefore \quad q + q' = p^2 - 42 ; q - q' = -32/p, q q' = 297$

$\therefore \quad (p^2 - 42)^2 - (-32/p)^2 = 4 \times 297$

or $\quad t^3 - 84t^2 + 576t - 1024 = 0 \quad$ where $t = p^2$

Now $t = 4$ satisfies this cubic so that $p = 2$.

$\therefore \quad q + q' = -38, q - q' = -16, \quad \therefore \quad q = -27, q' = -11.$

Thus (*i*) takes the form $(z^2 + 2z - 27)(z^2 - 2z - 11) = 0$

Whence $$z = \frac{-2 \pm \sqrt{(4 + 108)}}{2} \quad \text{or} \quad z = \frac{2 \pm \sqrt{(4 + 44)}}{2}$$

or $$x = \frac{1}{2}y = \frac{1}{2}(z + 3) = \frac{1}{2}(2 \pm \sqrt{28}) = \frac{1}{2}(4 \pm \sqrt{12})$$

Hence $\quad x = 1 \pm \sqrt{7}, 2 \pm \sqrt{3}$.

PROBLEMS 1.4

Solve by Ferrari's method, the equations :

1. $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$ (U.P.T.U., 2003)
2. $x^4 + 2x^3 - 7x^2 - 8x + 12 = 0$ (U.P.T.U., 2002)
3. $x^4 - 10x^2 - 20x - 16 = 0$
4. $x^4 - 8x^3 - 12x^2 + 60x + 63 = 0$ (U.P.T.U., 2005)

Solve the following equations by Descartes method :

5. $x^4 - 6x^3 + 3x^2 + 22x - 6 = 0$
6. $x^4 + 12x - 5 = 0$
7. $x^4 - 8x^3 - 24x + 7 = 0$ (U.P.T.U., 2001)
8. $x^4 - 10x^3 + 44x^2 - 104x + 96 = 0$

Obs. We have obtained algebraic solutions of cubic and biquadratic equations. But the need often arises to solve higher degree or transcendental equations for which no algebraic methods are available in general. Such equations can be best solved by *graphical method* (explained below) or by *numerical methods* (§28.2).

1.7 GRAPHICAL SOLUTION OF EQUATIONS

Let the equation be $f(x) = 0$.

(i) *Find the interval (a, b) in which a root of $f(x) = 0$ lies.*

[At least one root of $f(x) = 0$ lies in (a, b) if $f(a)$ and $f(b)$ are of opposite signs—§1.2(III) p. 2].

(ii) *Write the equation $f(x) = 0$ as $\phi(x) = \psi(x)$ where $\psi(x)$ contains only terms in x* and the constants.

(iii) *Draw the graphs of $y = \phi(x)$ and $y = \psi(x)$ on the same scale and with respect to the same axes.*

(iv) *Read the abscissae of the points of intersection of the curves $y = \phi(x)$ and $y = \psi(x)$. These are required real roots of $f(x) = 0$.*

Sometimes it may not be convenient to write the given equation $f(x) = 0$ in the form $\phi(x) = \psi(x)$. In such cases, we proceed as follows :

(i) Form a table for the value of x and $y = f(x)$ directly.

(ii) Plot these points and pass a smooth curve through them.

(iii) Read the abscissae of the points where this curve cuts the x-axis. These are the required roots of $f(x) = 0$.

Obs. The roots, thus located graphically are approximate and to improve their accuracy, the curves are replotted on the larger scale in the immediate vicinity of each point of intersection. This gives a better approximation to the value of desired root. The above graphical operation may be repeated until the root is obtained correct upto required number of decimal places. But this method of repeatedly drawing graphs is very tedious. It is, therefore, advisable to improve upon the accuracy of an approximate root by numerical method of §28.2.

Example 1.23. *Find graphically an approximate value of the root of the equation*

$$3 - x = e^{x-1}.$$

Solution. Let $\quad f(x) = e^{x-1} + x - 3 = 0 \quad$ ...(i)

$f(1) = 1 + 1 - 3 = -\text{ve}$

and $\quad f(2) = e + 2 - 3 = 2.718 - 1 = +\text{ve}$

$\therefore$ A root of (i), lies between $x = 1$ and $x = 2$.

Let us rewrite (i) as $\quad e^{x-1} = 3 - x.$

The abscissa of the point of intersection of the curves

$$y = e^{x-1} \quad \text{...(ii)}$$

and $$y = 3 - x \quad \text{...(iii)}$$

will give the required root.

To plot (ii), we form the following table of values :

$x =$	$y = e^{x-1}$
1.1	1.11
1.2	1.22
1.3	1.35
1.4	1.49
1.5	1.65
1.6	1.82
1.7	2.01
1.8	2.23
1.9	2.46
2.0	2.72

Taking the origin at (1, 1) and 1 small unit along either axis = 0.02, we plot these points and pass a smooth curve through them as shown in Fig. 1.2.

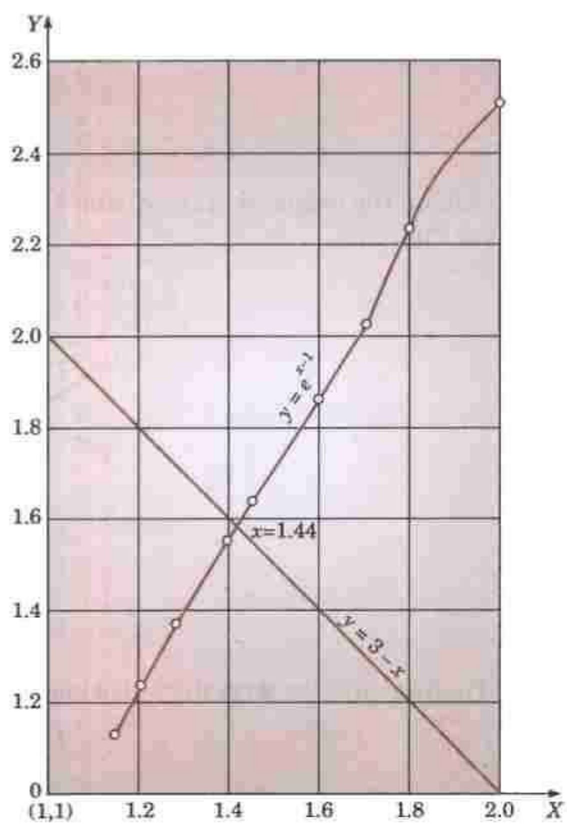


Fig. 1.2

To draw the line (iii), we join the points (1, 2) and (2, 1) on the same scale and with the same axes.
From the figure, we get the required root to be $x = 1.44$ nearly.

Example 1.24. *Obtain graphically an approximate value of the root of $x = \sin x + \pi/2$.*

Solution. Let us write the given equation as $\sin x = x - \pi/2$

The abscissa of the point of intersection of the curve $y = \sin x$ and the line $y = x - \pi/2$ will give a rough estimate of the root.

To draw a curve $y = \sin x$, we form the following table :

x	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
y	0	0.71	1	0.71	0

Taking 1 unit along either axis $= \pi/4 = 0.8$ nearly, we plot the curve as shown in Fig. 1.3.

Also we draw the line $y = x - \pi/2$ to the same scale and with the same axis.

From the graph, we get $x = 2.3$ radians approximately.

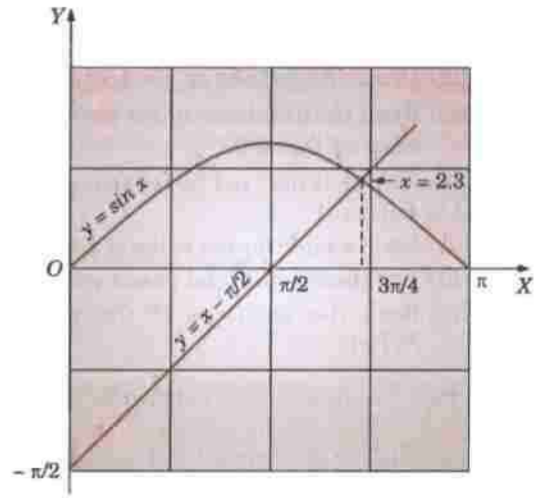


Fig. 1.3

Example 1.25. *Obtain graphically the lowest root of $\cos x \cosh x = -1$.*

Solution. Let $\quad f(x) = \cos x \cosh x + 1 = 0$ ...(*i*)

$\because \quad f(0) = +\text{ ve}, f(\pi/2) = +\text{ ve and } f(\pi) = -\text{ ve}.$

$\therefore$ The lowest root of (*i*) lies between $x = \pi/2$ and $x = \pi$.

Let us write (*i*) as $\cos x = -\operatorname{sech} x$.

The abscissa of the point of intersection of the curves

$y = \cos x$ ...(*ii*) and $y = -\operatorname{sech} x$ ...(*iii*)

will give the required root. To draw (*ii*), we form the following table of values :

$x =$	$\pi/2 = 1.57$	$3\pi/4 = 2.36$	$\pi = 3.14$
$y = \cos x$	0	-0.71	-1

Taking the origin at (1.57, 0) and 1 unit along either axes $= \pi/8 = 0.4$ nearly, we plot the cosine curve as shown in Fig. 1.4.

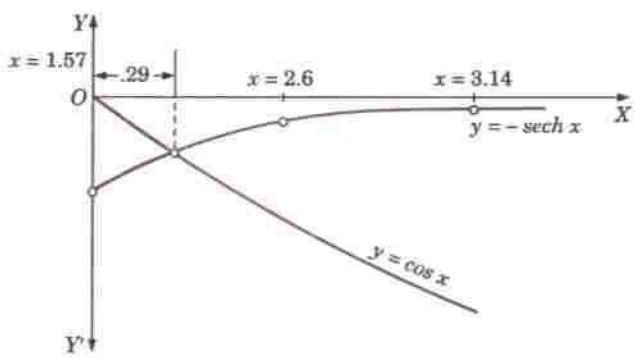


Fig. 1.4

To draw (iii), we form the following table :

$x =$	1.57	2.36	3.14
$\cosh x =$	2.58	5.56	11.12
$y = -\operatorname{sech} x$	-0.39	-0.18	-0.09

Then we plot the curve (*iii*) to the same scale with the same axes.

From the figure we get the lowest root to be approximately $x = 1.57 + 0.29 = 1.86$.

PROBLEMS 1.5

Solve the following equations graphically :

1. $x^3 - x - 1 = 0$ (*Madras, 2000 S*)
2. $x^3 - 3x - 5 = 0$
3. $x^3 - 6x^2 + 9x - 3 = 0$.
4. $\tan x = 1.2\, x$
5. $x = 3 \cos (x - \pi/4)$
6. $e^x = 5x$ which is near $x = 0.2$.

1.8 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 1.6

Choose the correct answer or fill up the blanks in the following problems :

1. If for the equation $x^3 - 3x^2 + kx + 3 = 0$, one root is the negative of another, then the value of k is
   (*a*) 3 (*b*) -3 (*c*) 1 (*d*) -1.
2. If the roots of the equation $x^n - 1 = 0$ are $1, \alpha_1, \alpha_2, \ldots\ldots, \alpha_{n-1}$, then $(1 - \alpha_1)(1 - \alpha_2) \ldots\ldots (1 - \alpha_{n-1})$ is equal to
   (*a*) 0 (*b*) 1 (*c*) n (*d*) $n + 1$.
3. If α, β, γ are the roots of $2x^3 - 3x^2 + 6x + 1 = 0$, then $\alpha^2 + \beta^2 + \gamma^2$ is
   (*a*) 15/4 (*b*) -3 (*c*) $-15/4$ (*d*) 33/4.
4. $x + 2$ is a factor of
   (*a*) $x^4 + 2$ (*b*) $x^4 - x^2 + 12$
   (*c*) $x^4 - 2x^3 - x + 2$ (*d*) $x^4 + 2x^3 - x - 2$
5. If $\alpha + \beta + \gamma = 5$; $\alpha\beta + \beta\gamma + \gamma\alpha = 7$; $\alpha\beta\gamma = 3$, then the equation whose roots are α, β and γ is
   (*a*) $x^3 - 7 = 0$ (*b*) $x^3 - 7x^2 + 3 = 0$
   (*c*) $x^3 - 5x^2 + 7x - 3 = 0$ (*d*) $x^3 + 7x^2 - 3 = 0$.
6. If one of the roots of the equation $x^3 - 6x^2 + 11x - 6 = 0$ is 2, then the other two roots are
   (*a*) 1 and 3 (*b*) 0 and 4
   (*c*) -1 and 5 (*d*) -2 and 6.
7. The equation whose roots are the reciprocals of the roots of $x^3 + px^2 + r = 0$ is
   (*a*) $x^3 + 1/p.x^2 + 1/r = 0$ (*b*) $1/r \,.\, x^3 + 1/p.x + 1 = 0$
   (*c*) $rx^3 + px^2 + 1 = 0$ (*d*) $rx^3 + px + 1 = 0$.
8. If 1 and 2 are two roots of the equation $x^4 - x^3 - 19x^2 + 49x - 30 = 0$, then the remaining two roots are
   (*a*) -3 and 5 (*b*) 3 and -5
   (*c*) -6 and 5 (*d*) 6 and -5.
9. If the roots of $x^3 - 3x^2 + px + 1 = 0$, are in arithmetic progression, then the sum of squares of the largest and the smallest roots is
   (*a*) 3 (*b*) 5 (*c*) 6 (*d*) 10.
10. A root of $x^3 - 8x^2 + px + q = 0$ where p and q are real numbers is $3 + i\sqrt{3}$. The real root is
    (*a*) 2 (*b*) 6 (*c*) 9 (*d*) 12.
11. One of the roots of the equation $f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 = 0$ where $a_0, a_1, \ldots a_{n-1}$ are real, is given to be $2 - 3i$. Of the remaining, the next $n - 2$ roots are given to be $1, 2, 3, \ldots, n - 2$. The nth root is
    (*a*) n (*b*) $n - 1$ (*c*) $2 + 3i$ (*d*) $-2 + 3i$.
12. If a real root of $f(x) = 0$ lies in $[a, b]$, then the sign of $f(a) . f(b)$ is ......
13. Descartes rule of signs states that ......
14. If α, β, γ are the roots of the equation $x^3 - px + q = 0$, then $\Sigma\, 1/\alpha =$ ......
15. If α, β, γ are the roots of $x^3 = 7$, then $\Sigma\alpha^3$ is ......
16. One real root of the equation $x^3 + 2x^2 + 5 = 0$ lies between ....

17. In an equation with real coefficients, imaginary roots must occur in ......
18. If $f(\alpha)$ and $f(\beta)$ are of opposite signs, then $f(x) = 0$ has at least one root between α and β provided ......
19. If α, β, γ are the roots of the equation $x^3 + 2x + 3 = 0$, then $\alpha + 3, \beta + 3, \gamma + 3$ are the roots of the equation ......
20. If one root is double of another in $x^3 - 7x^2 + 36 = 0$, then its roots are ......
21. The equation whose roots are 10 times those $x^3 - 2x - 7 = 0$, is ......
22. If α, β, γ are the roots of $x^3 + px^2 + qx + r = 0$, then $\Sigma (1/\alpha\beta) =$ ......
23. $\sqrt{3}$ and $-1 + i$ are the roots of the biquadratic equation ......
24. If α, β, γ are the roots of $x^3 - 3x + 2 = 0$, then the value of $\alpha^2 + \beta^2 + \gamma^2$ is ......
25. If there is a root of $f(x) = 0$ in the interval $[a, b]$, then sign of $f(a)/f(b)$ is ......
26. If α, β, γ are the roots of $x^3 + px^2 + qx + r = 0$, then the condition for $\alpha + \beta = 0$ is ......
27. The three roots of $x^3 = 1$ are ......
28. One real root of the equation $x^3 + x - 5 = 0$ lies in the interval
    (*i*) (2, 3), (*ii*) (3, 4), (*iii*) (1, 2), (*iv*) (– 3, – 2)
29. If two roots of $x^3 - 3x^2 + 2 = 0$ are equal, then its roots are ......
30. The cubic equation whose two roots are 5 and $1 - i$ is ......
31. The sum and product of the roots of the equation $x^5 = 2$ are ...... and ......
32. If the roots of the equation $x^4 + 2x^3 - ax^2 - 22x + 40 = 0$ are $-5, -2$, 1 and 4, then $a =$ ......
33. A root of $x^3 - 3x^2 + 2.5 = 0$ lies between 1.1 and 1.2. (True or False)
34. The equation $x^6 - x^5 - 10x + 7 = 0$ has four imaginary roots. (True or False)

CHAPTER 2

Linear Algebra : Determinants, Matrices

2.1 INTRODUCTION

Linear algebra comprises of the theory and applications of linear system of equation, linear transformations and eigen value problems. In linear algebra, we make a systematic use of matrices and to a lesser extent determinants and their properties.

Determinants were first introduced for solving linear systems and have important engineering applications in systems of differential equations, electrical networks, eigen-value problems and so on. Many complicated expressions occurring in electrical and mechanical systems can be elegently simplified by expressing them in the form of determinants.

Cayley* discovered matrices in the year 1860. But it was not until the twentieth century was well-advanced that engineers heard of them. These days, however, matrices have been found to be of great utility in many branches of applied mathematics such as algebraic and differential equations, mechanics theory of electrical circuits, nuclear physics, aerodynamics and astronomy. With the advent of computers, the usage of matrix methods has been greatly facilitated.

2.2 DETERMINANTS

(1) **Definition.** The expression $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is called a *determinant of the second order* and stands for '$a_1b_2 - a_2b_1$'. It contains 4 numbers a_1, b_1, a_2, b_2 (called *elements*) which are arranged along two horizontal lines (called *rows*) and two vertical lines (called *columns*).

Similarly, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is called a *determinant of the third order*. It consists of 9 *elements* which are arranged in 3 *rows* and 3 *columns*.

**Arthur Cayley* (1821–1895) was a professor at Cambridge and is known for his important contributions to algebra, matrices and differential equations.

In general, *a determinant of the nth order* is denoted by $\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \ldots l_1 \\ a_2 & b_2 & c_2 & d_2 \ldots l_2 \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_n & b_n & c_n & d_n \ldots l_n \end{vmatrix}$

which is a block of n^2 *elements* arranged in the form of a square along *n-rows* and *n-columns*. The diagonal through the left hand top corner which contains the elements $a_1, b_2, c_3, \ldots l_n$ is called the *leading or principal diagonal*.

(2) Cofactors

The cofactor *of any element in a determinant is obtained by deleting the row and column which intersect in that element with the proper sign. The sign of an element in the ith row and jth column is* $(-1)^{i+j}$. The cofactor of an element is usually denoted by the corresponding capital letter.

For instance, in $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, the cofactor of b_3 *i.e.*, $B_3 = (-1)^{3+2} \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$ and $C_2 = -\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$.

(3) Laplace's expansion.* *A determinant can be expanded in terms of any row (or column) as follows :*

Multiply each element of the row (or column) in terms of which we intend expanding the determinant, by its cofactor and then add up all these terms.

$\therefore$ Expanding by R_1 (*i.e.*, 1st row),

$$\Delta = a_1A_1 + b_1B_1 + c_1C_1 = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

Similarly, expanding by C_2 (*i.e.*, 2nd column)

$$\Delta = b_1B_1 + b_2B_2 + b_3B_3 = -b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

$$= -b_1(a_2c_3 - a_3c_2) + b_2(a_1c_3 - a_3c_1) - b_3(a_1c_2 - a_2c_1)$$

and expanding by R_3 (*i.e.*, 3rd row), $\Delta = a_3A_3 + b_3B_3 + c_3C_3$.

Thus Δ is the sum of the products of the elements of any row (or column) by the corresponding cofactors.

If, however, the sum of the products of the elements of any row (or column) by the cofactors of another row (or column) be taken, the result is zero.

e.g., in Δ, $$a_3A_2 + b_3B_2 + c_3C_2 = -a_3 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_3 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - c_3 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$$

$$= -a_3(b_1c_3 - b_3c_1) + b_3(a_1c_3 - a_3c_1) - c_3(a_1b_3 - a_3b_1) = 0$$

In general, $a_iA_j + b_iB_j + c_iC_j = \Delta$ when $i = j$
$= 0$ when $i \neq j$

Example 2.1. *Expand* $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$.

Solution. Expanding by R_1, $\Delta = a \begin{vmatrix} b & f \\ f & c \end{vmatrix} - h \begin{vmatrix} h & f \\ g & c \end{vmatrix} + g \begin{vmatrix} h & b \\ g & f \end{vmatrix}$

$$= a(bc - f^2) - h(hc - gf) + g(hf - gb) = abc + 2fgh - af^2 - bg^2 - ch^2.$$

*Named after a great French mathematician *Pierre Simon Marquis De Laplace* (1749–1827). He made important contributions to probability theory, special functions, potential theory and astronomy. While a professor in Paris, he taught Napolean Bonapart for a year.

Example 2.2. *Find the value of* $\Delta = \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{vmatrix}$.

Solution. Since there are two zeros in the second row, therefore, expanding by R_2, we get

$$\Delta = -\begin{vmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix} + 0 - 3\begin{vmatrix} 0 & 1 & 3 \\ 2 & 3 & 1 \\ 3 & 0 & 2 \end{vmatrix} + 0$$

(Expand by C_1) (Expand by R_1)

$$= -[1(0 \times 2 - 1 \times 1) - 3(2 \times 2 - 1 \times 3) + 0] - 3[0 - (2 \times 2 - 3 \times 1) + 3(2 \times 0 - 3 \times 3)]$$

$$= -(-1-3) - 3(-1-27) = 4 + 84 = 88.$$

2.3 PROPERTIES OF DETERMINANTS

The following properties, are proved for determinants of the third order, but these hold good for determinants of any order. These properties enable us to simplify a given determinant and evaluate it without expanding the given determinant.

I. *A determinant remains unaltered by changing its rows into columns and columns into rows.*

Let $$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{[Expand by } R_1]$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

Then $$\Delta' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad \text{[Expand by } R_1]$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) = \Delta.$$

Obs. 1. Any theorem concerning the rows of a determinant, therefore, applies equally to its columns and *vice-versa*.

2. When a *row* or a *column* is referred to in a general manner, it is called a *line*.

II. *If two parallel lines of a determinant are interchanged, the determinant retains its numerical value but changes in sign.*

Let $$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{[Expand by } R_1]$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

Interchanging C_2 and C_3, we have

$$\Delta' = \begin{vmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{vmatrix} \quad \text{[Expand by } R_1]$$

$$= a_1(c_2b_3 - c_3b_2) - c_1(a_2b_3 - a_3b_2) + b_1(a_2c_3 - a_3c_2)$$

$$= -[a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)] = -\Delta.$$

Cor. *If a line of* Δ *be passed over two parallel lines, i.e., if the resulting determinant is like*

$$\Delta' = \begin{vmatrix} b_1 & c_1 & a_1 \\ b_2 & c_2 & a_2 \\ b_3 & c_3 & a_3 \end{vmatrix}, \quad \text{then } \Delta' = (-1)^2\,\Delta.$$

In general, if any line of a determinant be passed over m parallel lines, the resulting determinant

$$\Delta' = (-1)^m\,\Delta.$$

III. *A determinant vanishes if two parallel lines are identical.*

Consider a determinant Δ in which two parallel lines are identical.

Interchange of the identical lines leaves the determinant unaltered yet by the previous property, the interchanges of two parallel lines changes the sign of the determinant.

Hence $\Delta = \Delta' = -\Delta$ or $2\Delta = 0$, or $\Delta = 0$.

IV. *If each element of a line be multiplied by the same factor, the whole determinant is multiplied by that factor.*

i.e.,
$$\begin{vmatrix} a_1 & pb_1 & c_1 \\ a_2 & pb_2 & c_2 \\ a_3 & pb_3 & c_3 \end{vmatrix} = p\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

For on expanding by C_2,

$$\text{L.H.S.} = -pb_1(a_2c_3 - a_3c_2) + pb_2(a_1c_3 - a_3c_1) - pb_3(a_1c_2 - a_2c_1)$$
$$= p\{-b_1B_1 + b_2B_2 - b_3B_3\} = \text{R.H.S.}$$

Similarly,
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ ka_1 & kb_2 & kc_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Cor. *If two parallel lines be such that the elements of one are equi-multiples of the elements of the other, the determinant vanishes.*

i.e.,
$$\begin{vmatrix} a_1 & b_1 & pb_1 \\ a_2 & b_2 & pb_2 \\ a_3 & b_3 & pb_3 \end{vmatrix} = p\begin{vmatrix} a_1 & b_1 & b_1 \\ a_2 & b_2 & b_2 \\ a_3 & b_3 & b_3 \end{vmatrix} = p(0) = 0$$

V. *If each element of a line consists of m terms, the determinant can be expressed as the sum of m determinants.*

Consider the determinant $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 + d_1 - e_1 \\ a_2 & b_2 & c_2 + d_2 - e_2 \\ a_3 & b_3 & c_3 + d_3 - e_3 \end{vmatrix}$

end of whose third column elements consists of three terms.

Expanding Δ by C_3, we have

$$\Delta = (c_1 + d_1 - e_1)(a_2b_3 - a_3b_2) - (c_2 + d_2 - e_2)(a_1b_3 - a_3b_1) + (c_3 + d_3 - e_3)(a_1b_2 - a_2b_1)$$
$$= [c_1(a_2b_3 - a_3b_2) - c_2(a_1b_3 - a_3b_1) + c_3(a_1b_2 - a_2b_1)] + [d_1(a_2b_3 - a_3b_2) - d_2(a_1b_3 - a_3b_1)$$
$$+ d_3(a_1b_2 - a_2b_1)] - [e_1(a_2b_3 - a_3b_2) - e_2(a_1b_3 - a_3b_1) + e_3(a_1b_2 - a_2b_1)]$$
$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} - \begin{vmatrix} a_1 & b_1 & e_1 \\ a_2 & b_2 & e_2 \\ a_3 & b_3 & e_3 \end{vmatrix}$$

Further, *if the elements of three parallel lines consist of m, n and p terms respectively, the determinants can be expressed as the sum of m × n × p determinants.*

Example 2.3. *If* $\begin{vmatrix} a & a^2 & a^3 - 1 \\ b & b^2 & b^3 - 1 \\ c & c^2 & c^3 - 1 \end{vmatrix} = 0$ *in which a, b, c are different, show that abc = 1.*

Solution. As each term of C_3 in the given determinant consists of two terms, we express it as a sum of two determinants.

$$\begin{vmatrix} a & a^2 & a^3 - 1 \\ b & b^2 & b^3 - 1 \\ c & c^2 & c^3 - 1 \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} + \begin{vmatrix} a & a^2 & -1 \\ b & b^2 & -1 \\ c & c^2 & -1 \end{vmatrix} = abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

[Taking common a, b, c from R_1, R_2, R_3 respectively of the first determinant and -1 from C_3 of the second determinant.]

$$= abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

[Passing C_3 over C_2 and C_1 in the second determinant]

$$\therefore \quad \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} (abc - 1) = 0. \text{ Hence } abc = 1, \text{ since } \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \neq 0 \text{ as } a, b, c \text{ are all different.}$$

VI. *If to each elements of a line be added equi-multiples of the corresponding elements of one or more parallel lines, the determinants remains unaltered.*

Let
$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Then
$$\Delta' = \begin{vmatrix} a_1 + pb_1 - qc_1 & b_1 & c_1 \\ a_2 + pb_2 - qc_2 & b_2 & c_2 \\ a_3 + pb_3 - qc_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} pb_1 & b_1 & c_1 \\ pb_2 & b_2 & c_2 \\ pb_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} -qc_1 & b_1 & c_1 \\ -qc_2 & b_2 & c_2 \\ -qc_3 & b_3 & c_3 \end{vmatrix}$$

$$= \Delta + 0 + 0 = \Delta. \qquad \text{[by } IV\text{-}Cor.\text{]}$$

Obs. *This property is very useful for simplifying determinants.* To add equi-multiples of parallel lines, we shall employ the following **notation :**

Suppose to the elements of the second row, we add p times the elements of the first row and q times the element of the third row ; then we say :

Operate $R_2 + pR_1 + qR_3$.

Similarly Operate '$C_3 + mC_1 - nC_2$'

means that to the elements of the third column add m times the elements of the first column and $-n$ times the elements of the second column.

Example 2.4. *Evaluate* $\begin{vmatrix} 21 & 17 & 7 & 10 \\ 24 & 22 & 6 & 10 \\ 6 & 8 & 2 & 3 \\ 6 & 7 & 1 & 2 \end{vmatrix}$.

Solution. Operating $R_1 - R_2 - R_4$, $R_2 - 3R_3$, $R_3 - 2R_4$, the given determinant becomes

$$\Delta = \begin{vmatrix} -8 & -12 & 0 & -2 \\ 6 & -2 & 0 & 1 \\ -4 & -6 & 0 & -1 \\ 5 & 7 & 1 & 2 \end{vmatrix} \qquad \text{[Expand by } C_1\text{]}$$

$$= -\begin{vmatrix} -8 & -12 & -2 \\ 6 & -2 & 1 \\ -4 & -6 & -1 \end{vmatrix} = 0 \qquad [\because \quad R_1 = 2R_2]$$

Example 2.5. *Solve the equation* $\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 3x+5 & 5x+8 & 10x+17 \end{vmatrix} = 0.$

Solution. Operating $R_3 - (R_1 + R_2)$, we get

$$\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 0 & 1 & 3x+8 \end{vmatrix} = 0 \qquad \text{(Operate } R_2 - R_1 \text{ and } R_1 + R_3\text{)}$$

or
$$\begin{vmatrix} x+2 & 2x+4 & 6x+12 \\ x+1 & x+1 & x+1 \\ 0 & 1 & 3x+8 \end{vmatrix} = 0 \quad \text{or} \quad (x+1)(x+2) \begin{vmatrix} 1 & 2 & 6 \\ 1 & 1 & 1 \\ 0 & 1 & 3x+8 \end{vmatrix} = 0$$

To bring one more zero in C_1, operate $R_1 - R_2$.

$$\therefore \qquad (x+1)(x+2)\begin{vmatrix} 0 & 1 & 5 \\ 1 & 1 & 1 \\ 0 & 1 & 3x+8 \end{vmatrix} = 0$$

Now expand by C_1. $\therefore \quad -(x+1)(x+2)(3x+8-5) = 0$ or $-3(x+1)(x+2)(x+1) = 0$

Thus, $\qquad x = -1, -1, -2.$

Example 2.6. *Prove that* $\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix} = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$.

Solution. Let Δ be the given determinant. Taking a, b, c, d common from R_1, R_2, R_3, R_4 respectively, we get

$$\Delta = abcd\begin{vmatrix} a^{-1}+1 & a^{-1} & a^{-1} & a^{-1} \\ b^{-1} & b^{-1}+1 & b^{-1} & b^{-1} \\ c^{-1} & c^{-1} & c^{-1}+1 & c^{-1} \\ d^{-1} & d^{-1} & d^{-1} & d^{-1}+1 \end{vmatrix}$$

[Operate $R_1 + (R_2 + R_3 + R_4)$ and take out the common factor from R_1]

$$= abcd\,(1 + a^{-1} + b^{-1} + c^{-1} + d^{-1})\begin{vmatrix} 1 & 1 & 1 & 1 \\ b^{-1} & b^{-1}+1 & b^{-1} & b^{-1} \\ c^{-1} & c^{-1} & c^{-1}+1 & c^{-1} \\ d^{-1} & d^{-1} & d^{-1} & d^{-1}+1 \end{vmatrix}$$

[Operate $C_2 - C_1, C_3 - C_1, C_4 - C_1$]

$$= abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\begin{vmatrix} 1 & 0 & 0 & 0 \\ b^{-1} & 1 & 0 & 0 \\ c^{-1} & 0 & 1 & 0 \\ d^{-1} & 0 & 0 & 1 \end{vmatrix} = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

Obs. If all elements on one side of the leading diagonal are zero, then the determinant is equal to the product of leading diagonal elements and such a determinants is called a *triangular determinant.*

VII. Factor Theorem. *If the elements of a determinant Δ are functions of x and two parallel lines become identical when $x = a$, then $x - a$ is a factor of Δ.*

Let $\qquad \Delta = f(x)$

Since $\qquad \Delta = 0$ when $x = a$, $\qquad \therefore \; f(a) = 0.$

i.e., $(x - a)$ is a factor of $f(x)$.

Hence $x - a$ is a factor of Δ.

Obs. *If k parallel lines of a determinant Δ become identical when $x = a$, then $(x - a)^{k-1}$ is a factor of Δ.*

Example 2.7. *Factorize* $\Delta = \begin{vmatrix} a^3 & a^2 & a & 1 \\ b^3 & b^2 & b & 1 \\ c^3 & c^2 & c & 1 \\ d^3 & d^2 & d & 1 \end{vmatrix}$.

Solution. Putting $a = b$, $R_1 \equiv R_2$ and hence $\Delta = 0$. $\quad \therefore \quad a - b$ is a factor of Δ.

Similarly, $a - c$ and $a - d$ are also factors of Δ.

Again putting $b = c$, $R_2 \equiv R_3$ and hence $\Delta = 0$. $\quad \therefore \quad b - c$ is a factor of Δ.

Similarly $b - d$ and $c - d$ are also factors of Δ.

Also Δ is of the sixth degree in a, b, c, d and therefore, there cannot be any other algebraic factor of Δ.

$\therefore \quad$ Suppose $\Delta = k(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$, where k is a numerical constant.

The leading term in $\Delta = a^3b^2c$. The corresponding term on R.H.S. $= ka^3b^2c$.

$\therefore \qquad k = 1.$

Hence, $\qquad \Delta = (a-b)(a-c)(a-d)(b-c)(b-d)(c-d).$

Example 2.8. *Prove that* $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc\,(a+b+c)^3.$ *(J.N.T.U., 1998)*

Solution. Let the given determinant be Δ. If we put $a = 0$,

$$\Delta = \begin{vmatrix} (b+c)^2 & 0 & 0 \\ 0 & c^2 & b^2 \\ c^2 & c^2 & b^2 \end{vmatrix} = 0$$

$\therefore$ a is a factor of Δ. Similarly b and c are its factors.

Again if we put $a + b + c = 0$.

$$\Delta = \begin{vmatrix} (-a)^2 & a^2 & a^2 \\ b^2 & (-b)^2 & b^2 \\ c^2 & c^2 & (-c)^2 \end{vmatrix} = 0$$

In this, three columns being identical, $(a + b + c)^2$ is a factor of Δ.

As Δ is of the sixth degree and is symmetrical in a, b, c the remaining factor must therefore, be of the first degree and of the form $k\,(a + b + c)$.

Thus $\Delta = kabc\,(a + b + c)^3$

To determine k, put $a = b = c = 1$, then

$$\begin{vmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{vmatrix} = 27k \quad \text{or} \quad 54 = 27k \quad i.e., k = 2$$

Hence $\Delta = 2abc\,(a + b + c)^3$.

Otherwise : Operating $C_1 - C_3$ and $C_2 - C_3$, we have

$$\Delta = \begin{vmatrix} (b+c)^2 - a^2 & 0 & a^2 \\ 0 & (c+a)^2 - b^2 & b^2 \\ c^2 - (a+b)^2 & c^2 - (a+b)^2 & (a+b)^2 \end{vmatrix} \quad [\text{Take } (a+b+c) \text{ common from } C_1 \text{ and } C_2]$$

$$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ c-a-b & c-a-b & (a+b)^2 \end{vmatrix} \quad [\text{Operate } R_3 - R_1 - R_2]$$

$$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix} \quad \left[\text{Operate } C_1 + \frac{1}{a} C_3, C_2 + \frac{1}{b} C_3\right]$$

$$= (a+b+c)^2 \begin{vmatrix} b+c & a^2/b & a^2 \\ b^2/a & c+a & b^2 \\ 0 & 0 & 2ab \end{vmatrix} \quad [\text{Expand by } R_3]$$

$$= 2ab(a+b+c)^2\,[(b+c)(c+a) - ab] = 2abc\,(a+b+c)^3.$$

VIII. Multiplication of Determinants. *The product of two determinants of the same order is itself a determinant of that order.*

Let $\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$

then their product is defined as

$$\Delta_1 \Delta_2 = \begin{vmatrix} a_1 l_1 + b_1 m_1 + c_1 n_1, & a_1 l_2 + b_1 m_2 + c_1 n_2, & a_1 l_3 + b_1 m_3 + c_1 n_3 \\ a_2 l_1 + b_2 m_1 + c_2 n_1, & a_2 l_2 + b_2 m_2 + c_2 n_2, & a_2 l_3 + b_2 m_3 + c_2 n_3 \\ a_3 l_1 + b_3 m_1 + c_3 n_1, & a_3 l_2 + b_3 m_2 + c_3 n_2, & a_3 l_3 + b_3 m_3 + c_3 n_3 \end{vmatrix}$$

Similarly, the product of two determinants of the nth order is a determinant of the nth order.

Example 2.9. *Evaluate* $\begin{vmatrix} a^2+\lambda^2 & ab+c\lambda & ca-b\lambda \\ ab-c\lambda & b^2+\lambda^2 & bc+a\lambda \\ ca+b\lambda & bc-a\lambda & c^2+\lambda^2 \end{vmatrix} \times \begin{vmatrix} \lambda & c & -b \\ -c & \lambda & a \\ b & -a & \lambda \end{vmatrix}$

Solution. By the rule of multiplication of determinants, the resulting determinant

$$\Delta = \begin{vmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{vmatrix}$$

where $d_{11} = (a^2+\lambda^2)\lambda + (ab+c\lambda)c + (ca-b\lambda)(-b) = \lambda(a^2+b^2+c^2+\lambda^2)$

$d_{12} = (a^2+\lambda^2)(-c) + (ab+c\lambda)\lambda + (ca-b\lambda)a = 0$

$d_{13} = 0,$

$d_{21} = 0, d_{22} = \lambda(a^2+b^2+c^2+\lambda^2), d_{23} = 0.$

$d_{31} = 0, d_{32} = 0, d_{33} = \lambda(a^2+b^2+c^2+\lambda^2).$

Hence $$\Delta = \begin{vmatrix} \lambda(a^2+b^2+c^2+\lambda^2) & 0 & 0 \\ 0 & \lambda(a^2+b^2+c^2+\lambda^2) & 0 \\ 0 & 0 & \lambda(a^2+b^2+c^2+\lambda^2) \end{vmatrix}$$

$$= \lambda^3(a^2+b^2+c^2+\lambda^2)^3.$$

Example 2.10. *Show that* $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$ *where A, B etc. are the co-factors of a, b, etc. respectively in the determinant* $(a_1b_2c_3)$.

Solution. Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta' = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$

Then $$\Delta\Delta' = \begin{vmatrix} a_1A_1+b_1B_1+c_1C_1, & a_1A_2+b_1B_2+c_1C_2, & a_1A_3+b_1B_3+c_1C_3 \\ a_2A_1+b_2B_1+c_2C_1, & a_2A_2+b_2B_2+c_2C_2, & a_2A_3+b_2B_3+c_2C_3 \\ a_3A_1+b_3B_1+c_3C_1, & a_3A_2+b_3B_2+c_3C_2, & a_3A_3+b_3B_3+c_3C_3, \end{vmatrix} = \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix} = \Delta^3$$

Hence $\Delta' = \Delta^2$.

Obs. Δ' is called the **reciprocal** *or* **adjugate determinant** *of* Δ.

Example 2.11. *Express* $\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}$ *as the square of a determinant, and hence find its value.*

Solution. Given determinant

$$= \begin{vmatrix} a.(-a)+b.c+c.b, & a.(-b)+b.a+c.c, & a.(-c)+b.b+c.a \\ b.(-a)+c.c+a.b, & b.(-b)+c.a+a.c, & b.(-c)+c.b+a.a \\ c.(-a)+a.c+b.b, & c.(-b)+a.a+b.c, & c.(-c)+a.b+b.a \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix}$$

[Taking out (-1) common from C_1 and interchange C_2, C_3.]

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times (-1)^2 \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \Delta^2$$

where $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3+b^3+c^3-3abc)$

Hence the given determinant $= \Delta^2 = (a^3+b^3+c^3-3abc)^2$.

PROBLEMS 2.1

1. Prove, without expanding, that $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix}$ vanishes.

2. If $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$, then prove, without expansion, that $xyz = -1$ where x, y, z are unequal.

(*Andhra, 1999 ; Assam, 1999*)

3. Show that (*i*) $\begin{vmatrix} x & l & m & 1 \\ \alpha & x & n & 1 \\ \alpha & \beta & x & 1 \\ \alpha & \beta & \gamma & 1 \end{vmatrix} = (x-\alpha)(x-\beta)(x-\gamma)$.

(*ii*) $\begin{vmatrix} a & b & c \\ b+c & c+a & a+b \\ a^2 & b^2 & c^2 \end{vmatrix} = -(a-b)(b-c)(c-a)(a+b+c)$.

4. If a, b, c are all different and $\begin{vmatrix} a & a^3 & a^4-1 \\ b & b^3 & b^4-1 \\ c & c^3 & c^4-1 \end{vmatrix} = 0$, then show that $abc\,(bc+ca+ab) = a+b+c$.

5. Evaluate (*i*) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 3 & 4 \\ 1 & 2 & 4 & 4 \\ 1 & 2 & 3 & 5 \end{vmatrix}$ (*ii*) $\begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix}$

Prove the following results : (6 to 12)

6. $\begin{vmatrix} a+b & b+c & c+a \\ l+m & m+n & n+l \\ p+q & q+r & r+p \end{vmatrix} \div \begin{vmatrix} a & b & c \\ l & m & n \\ p & q & r \end{vmatrix} = 2$

7. $\begin{vmatrix} a-b-c & 2b & 2c \\ 2a & b-c-a & 2c \\ 2a & 2b & c-a-b \end{vmatrix} = (a+b+c)^3$

8. $\begin{vmatrix} 1+a^2-b^2 & 2b & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a^2 & 1-a^2-b^2 \end{vmatrix}$ is a perfect cube.

9. $\begin{vmatrix} 1 & \cos A & \sin A \\ 1 & \cos B & \sin B \\ 1 & \cos C & \sin C \end{vmatrix} = 4 \sin \dfrac{B-C}{2} \sin \dfrac{C-A}{2} \sin \dfrac{A-B}{2}$.

10. $\begin{vmatrix} 4 & 5 & 6 & x \\ 5 & 6 & 7 & y \\ 6 & 7 & 8 & z \\ x & y & z & 0 \end{vmatrix}$ is a perfect square.

11. $\begin{vmatrix} 1 & a & a^2 & a^3+bcd \\ 1 & b & b^2 & b^3+cda \\ 1 & c & c^2 & c^3+dab \\ 1 & d & d^2 & d^3+abc \end{vmatrix}$ vanishes.

12. $\begin{vmatrix} a^2+\lambda & ab & ac & ad \\ ab & b^2+\lambda & bc & bd \\ ac & bc & c^2+\lambda & cd \\ ad & bd & cd & d^2+\lambda \end{vmatrix} = \lambda^3(a^2+b^2+c^2+d^2+\lambda)$

Factorize each of the following determinants : (13 to 15)

13. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$ (*Andhra, 1998*)

14. $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$.

15. $\begin{vmatrix} b^2c^2+a^2 & bc+a & 1 \\ c^2a^2+b^2 & ca+b & 1 \\ a^2b^2+c^2 & ab+c & 1 \end{vmatrix}$

16. $\begin{vmatrix} a^2 & b^2 & c^2 & d^2 \\ a & b & c & d \\ 1 & 1 & 1 & 1 \\ bcd & cda & dab & abc \end{vmatrix}$

17. If $a+b+c=0$, solve $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$ (*Andhra, 1999*)

18. Solve the equation $\begin{vmatrix} x+1 & 2x+1 & 3x+1 \\ 2x & 4x+3 & 6x+3 \\ 4x+1 & 6x+4 & 8x+4 \end{vmatrix} = 0.$

19. Show that $\begin{vmatrix} b^2+c^2 & ab & ac \\ ba & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix} = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = 4a^2b^2c^2.$

2.4 MATRICES

(1) Definition. *A system of mn numbers arranged in a rectangular formation along m rows and n columns and bounded by the brackets [ ] is called an m by n* **matrix** *; which is written as m × n matrix.* A matrix is also denoted by a single capital letter.

Thus
$$A = \begin{bmatrix} a_{11} & a_{12} & ...a_{1j} & ...a_{1n} \\ a_{21} & a_{22} & ...a_{2j} & ...a_{2n} \\ ... & ... & ... & ... \\ a_{i1} & a_{i2} & ...a_{ij} & ...a_{in} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & a_{mj} & ...a_{mn} \end{bmatrix}$$

is a matrix of *order mn*. It has *m rows* and *n columns*. Each of the *mn* numbers is called an element of the matrix.

To locate any particular element of a matrix, the elements are denoted by a letter followed by two suffixes which respectively specify the rows and columns. Thus a_{ij} is the element in the i-th row and j-th column of A. In this notation, the matrix A is denoted by $[a_{ij}]$.

A matrix should be treated as a single entity with a number of components, rather than a collection of numbers. For example, the coordinates of a point in solid geometry, are given by a set of three numbers which can be represented by the matrix $[x, y, z]$. Unlike a determinant, a matrix cannot reduce to a single number and the question of finding the value of a matrix never arises. The difference between a determinant and a matrix is brought out by the fact that an interchange of rows and columns does not alter the determinant but gives an entirely different matrix.

(2) Special matrices

Row and column matrices. *A matrix having a single row is called a row matrix, e.g.,* $\begin{bmatrix} 1 & 3 & 5 & 7 \end{bmatrix}$.

A matrix having a single column is called a column matrix, e.g., $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$

Row and column matrices are sometimes called *row vectors* and *column vectors*.

Square matrix. *A matrix having n rows and n columns is called a square matrix of order n.*

The determinant having the same elements as the square matrix A is called the *determinant of the matrix* and is denoted by the symbol $|A|$. For example, if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}, \quad \text{then } |A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$$

The diagonal of this matrix containing the elements 1, 3, 5 is called the *leading or principal diagonal. The sum of the diagonal elements of a square matrix A is called the* **trace** *of A.*

A square matrix is said to be **singular** *if its determinant is zero otherwise* **non-singular.**

Diagonal matrix. *A square matrix all of whose elements except those in the leading diagonal, are zero is called a diagonal matrix.*

A diagonal matrix whose all the leading diagonal elements are equal is called a *scalar matrix.* For example,

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 6 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

are the diagonal and scalar matrices respectively.

Unit matrix. *A diagonal matrix of order n which has unity for all its diagonal elements, is called a unit matrix or an identity matrix of order n* and is denoted by I_n. For example, unit matrix of order 3 is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Null matrix. *If all the elements of a matrix are zero, it is called a null or zero matrix* and is denoted by '**0**' ; *e.g.*,

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ is a null matrix.}$$

Symmetric and skew-symmetric matrices. *A square matrix* $A = [a_{ij}]$ *is said to be symmetric when* $a_{ij} = a_{ji}$ *for all i and j.*

If $a_{ij} = -a_{ji}$ *for all i and j so that all the leading diagonal elements are zero, then the matrix is called a skew-symmetric matrix.* Examples of symmetric and skew-symmetric matrices are

$$\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & h & -g \\ -h & 0 & f \\ g & -f & 0 \end{bmatrix} \text{ respectively.}$$

Triangular matrix. A square matrix all of whose elements below the leading diagonal are zero, is called an *upper triangular matrix.* A square matrix all of whose elements above the leading diagonal are zero, is called a *lower triangular matrix.* Thus

$$\begin{bmatrix} a & h & g \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & -5 & 4 \end{bmatrix}$$

are upper and lower triangular matrices respectively.

2.5 MATRICES OPERATIONS

(1) Equality of Matrices

Two matrices A and B are said to equal if and only if

(i) they are of the same order

and (ii) each element of A is equal to the corresponding element of B.

(2) Addition and subtraction of matrices. *If A, B be two matrices of the same order, then their sum A + B is defined as the matrix each element of which is the sum of the corresponding elements of A and B.*

Thus, $$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} + \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \\ c_3 & d_3 \end{bmatrix} = \begin{bmatrix} a_1 + c_1 & b_1 + d_1 \\ a_2 + c_2 & b_2 + d_2 \\ a_3 + c_3 & b_3 + d_3 \end{bmatrix}$$

Similarly, $A - B$ *is defined as a matrix whose elements are obtained by subtracting the elements of B from the corresponding elements of A.*

Thus, $$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} - \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 - c_1 & b_1 - d_1 \\ a_2 - c_2 & b_2 - d_2 \end{bmatrix}$$

Obs. 1. Only matrices of the same order can be added or subtracted.

2. Addition of matrices is *commutative*,

i.e., $A + B = B + A.$

3. Addition and subtraction of matrices is *associative*.

i.e. $(A + B) - C = A + (B - C) = B + (A - C)$.

(3) Multiplication of matrix by a scalar. *The product of a matrix A by a scalar k is a matrix whose each element is k times the corresponding elements of A.*

Thus, $$k\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} = \begin{bmatrix} ka_1 & kb_1 & kc_1 \\ ka_2 & kb_2 & kc_2 \end{bmatrix}$$

The distributive law holds for such products, *i.e.*, $k(A + B) = kA + kB$.

Obs. All the laws of ordinary algebra hold for the addition or subtraction of matrices and their multiplication by scalars.

Example 2.12. *Find x, y, z and w given that*

$$3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 5 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 6 & x+y \\ z+w & 5 \end{bmatrix}$$

Solution. We have $\begin{bmatrix} 3x & 3y \\ 3z & 3w \end{bmatrix} = \begin{bmatrix} x+6 & 5+x+y \\ -1+z+w & 2w+5 \end{bmatrix}$

Equating the corresponding elements, we get

$$3x = x + 6,\ 3y = 5 + x + y,\ 3z = -1 + z + w,\ 3w = 2w + 5.$$

or $$2x = 6,\ 2y = 5 + x,\ 2z = w - 1,\ w = 5$$

Hence $x = 3, y = 4, z = 2, w = 5$.

Example 2.13. *Express* $\begin{bmatrix} 3 & 5 & -7 \\ -8 & 11 & 4 \\ 13 & -14 & 6 \end{bmatrix}$ *as the sum of a lower triangular matrix with zero leading diagonal and an upper triangular matrix.*

Solution. Let $L = \begin{bmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{bmatrix}$ be the lower triangular matrix with zero leading diagonal.

and $U = \begin{bmatrix} l & m & n \\ 0 & p & q \\ 0 & 0 & r \end{bmatrix}$ be the upper triangular matrix.

Then $$\begin{bmatrix} 3 & 5 & -7 \\ -8 & 11 & 4 \\ 13 & -14 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{bmatrix} + \begin{bmatrix} l & m & n \\ 0 & p & q \\ 0 & 0 & r \end{bmatrix}$$

Equating corresponding elements from both sides, we obtain $3 = l, 5 = m, -7 = n, -8 = a, 11 = p, 4 = q, 13 = b, -14 = c, 6 = r$.

Hence $$L = \begin{bmatrix} 0 & 0 & 0 \\ -8 & 0 & 0 \\ 13 & -14 & 0 \end{bmatrix} \text{ and } U = \begin{bmatrix} 3 & 5 & -7 \\ 0 & 11 & 4 \\ 0 & 0 & 6 \end{bmatrix}$$

(4) Multiplication of matrices. *Two matrices can be multiplied only when the number of columns in the first is equal to the number of rows in the second. Such matrices are said to be* **conformable.**

For instance, the product $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{bmatrix} \times \begin{bmatrix} l_1 & l_2 \\ m_1 & m_2 \\ n_1 & n_2 \end{bmatrix}$

is defined as the matrix $\begin{bmatrix} a_1l_1 + b_1m_1 + c_1n_1 & a_1l_2 + b_1m_2 + c_1n_2 \\ a_2l_1 + b_2m_1 + c_2n_1 & a_2l_2 + b_2m_2 + c_2n_2 \\ a_3l_1 + b_3m_1 + c_3n_1 & a_3l_2 + b_3m_2 + c_3n_2 \\ a_4l_1 + b_4m_1 + c_4n_1 & a_4l_2 + b_4m_2 + c_4n_2 \end{bmatrix}$

In general, if $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix}$

be two $m \times n$ and $n \times p$ conformable matrices, then their product is defined as the $m \times p$ matrix

$$AB = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ c_{m1} & c_{m2} & \cdots & c_{mp} \end{bmatrix}$$

where $c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + a_{i3} b_{3j} + \ldots + a_{in} b_{nj}$, *i.e.*, the element in the ith row and the jth column of the matrix AB is obtained by weaving the ith row of A with jth column of B. The expression for c_{ij} is known as the *inner product* of the ith row with the jth column.

Post-multiplication and Pre-multiplication. In the product AB, the matrix A is said to be *post-multiplied* by the matrix B. Whereas in BA, the matrix A is said to be *pre-multiplied* by B. In one case the product may exist and in the other case it may not. Also the product in both cases may exist yet may or may not be equal.

Obs. 1. *Multiplication of matrices is associative. i.e., $(AB)C = A(BC)$*
provided A, B are conformable for the product AB and B, C are conformable for the product BC. (Ex. 2.16).

Obs. 2. *Multiplication of matrices is distributive. i.e., $A(B + C) = AB + AC$.*
provided A, B are conformable for the product AB and A, C are conformable for the product AC.

Obs. 3. Power of a matrix. If A be a square matrix, then the product AA is defined as A^2. Similarly, we define higher powers of A. *i.e.*, $A \cdot A^2 = A^3$, $A^2 \cdot A^2 = A^4$ etc.

If $A^2 = A$, then the matrix A is called idempotent.

Example 2.14. *If* $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix}$, *form the product of AB. Is BA defined?*

Solution. Since the number of columns of A = the number of rows of B (each being = 3).

$\therefore$ The product AB is defined and

$$= \begin{bmatrix} 0.1 + 1.-1 + 2.2, & 0.-2 + 1.0 + 2.-1 \\ 1.1 + 2.-1 + 3.2, & 1.-2 + 2.0 + 3.-1 \\ 2.1 + 3.-1 + 4.2, & 2.-2 + 3.0 + 4.-1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 5 & -5 \\ 7 & -8 \end{bmatrix}$$

Again since the number of columns of $B \neq$ the number of rows of A.

$\therefore$ The product BA is not possible.

Example 2.15. *If* $A = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$, *compute AB and BA and show that $AB \neq BA$.*

Solution. Considering rows of A and columns of B, we have

$$AB = \begin{bmatrix} 1.2 + 3.1 + 0.-1, & 1.3 + 3.2 + 0.1, & 1.4 + 3.3 + 0.2 \\ -1.2 + 2.1 + 1.-1, & -1.3 + 2.1 + 1.1, & -1.4 + 2.3 + 1.2 \\ 0.2 + 0.1 + 2.-1, & 0.3 + 0.2 + 2.1, & 0.4 + 0.3 + 2.2 \end{bmatrix} = \begin{bmatrix} 5 & 9 & 13 \\ -1 & 2 & 4 \\ -2 & 2 & 4 \end{bmatrix}$$

Again considering the rows of B and columns of A, we have

$$BA = \begin{bmatrix} 2.1 + 3.-1 + 4.0, & 2.3 + 3.2 + 4.0 & 2.0 + 3.1 + 4.2 \\ 1.1 + 2.-1 + 3.0, & 1.3 + 2.2 + 3.0 & 1.0 + 2.1 + 3.2 \\ -1.1 + 1.-1 + 2.0, & -1.3 + 1.2 + 2.0 & -1.0 + 1.1 + 2.2 \end{bmatrix} = \begin{bmatrix} -1 & 12 & 11 \\ -1 & 7 & 8 \\ -2 & -1 & 5 \end{bmatrix}$$

Evidently $AB \neq BA$.

Example 2.16. *If* $A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 3 & 1 \\ 5 & 3 & 4 \end{bmatrix}$, *find the matrix B such that* $AB = \begin{bmatrix} 3 & 4 & 2 \\ 1 & 6 & 1 \\ 5 & 6 & 4 \end{bmatrix}$. *(Mumbai, 2005)*

Solution. Let $AB = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 3 & 1 \\ 5 & 3 & 4 \end{bmatrix} \begin{bmatrix} l & m & n \\ p & q & r \\ u & v & w \end{bmatrix}$

$$= \begin{bmatrix} 3l+2p+2u & 3m+2q+2v & 3n+2r+2w \\ l+3p+u & m+3q+v & n+3r+w \\ 5l+3p+4u & 5m+3q+4v & 5n+3r+4w \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 4 & 2 \\ 1 & 6 & 1 \\ 5 & 6 & 4 \end{bmatrix} \quad \text{(given)}$$

Equating corresponding elements, we get

$$3l + 2p + 2u = 3, \quad l + 3p + u = 1, \quad 5l + 3p + 4u = 5 \qquad ...(i)$$

$$3m + 2q + 2v = 4, \quad m + 3q + v = 6, \quad 5m + 3q + 4v = 6 \qquad ...(ii)$$

$$3n + 2r + 2w = 2, \quad n + 3r + w = 1, \quad 5n + 3r + 4w = 4 \qquad ...(iii)$$

Solving the equations (*i*), we get $l = 1, p = 0, u = 0$

Similarly equations (*ii*) give $m = 0, q = 2, v = 0$

and equations (*iii*) give $n = 0, r = 0, w = 1$

Thus, $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Example 2.17. *Prove that* $A^3 - 4A^2 - 3A + 11I = 0$, *where* $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$.

Solution. $A^2 = A \times A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1+6+2 & 3+0+4 & 2-3+6 \\ 2+0-1 & 6+0-2 & 4+0-3 \\ 1+4+3 & 3+0+6 & 2-2+9 \end{bmatrix} = \begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix}$

$$A^3 = A^2 \times A = \begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix} \times \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 9+14+5 & 27+0+10 & 18-7+15 \\ 1+8+1 & 3+0+2 & 2-4+3 \\ 8+18+9 & 24+0+18 & 16-9+27 \end{bmatrix} = \begin{bmatrix} 28 & 37 & 26 \\ 10 & 5 & 1 \\ 35 & 42 & 34 \end{bmatrix}$$

$A^3 - 4A^2 - 3A + 11I$

$$= \begin{bmatrix} 28 & 37 & 26 \\ 10 & 5 & 1 \\ 35 & 42 & 34 \end{bmatrix} - 4 \begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix} - 3 \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} + 11 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 28-36-3+11 & 37-28-9+0 & 26-20-6+0 \\ 10-4-6-0 & 5-16+0+11 & 1-4+3+0 \\ 35-32-3+0 & 42-36-6+0 & 34-36-9+11 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

Example 2.18. *By mathematical induction, prove that if*

$$A = \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix}, \text{ then } A^n = \begin{bmatrix} 1+10n & -25n \\ 4n & 1-10n \end{bmatrix}.$$

Solution. When $n = 1$, A^n gives $A^1 = \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix}$ ...(*i*)

Let us assume that the result is true for any positive integer k, so that

$$A^k = \begin{bmatrix} 1+10k & -25k \\ 4k & 1-10k \end{bmatrix}$$

$\therefore$ $$A^{k+1} = A^k \cdot A^1 = \begin{bmatrix} 1+10k & -25k \\ 4k & 1-10k \end{bmatrix}\begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} 11(1+10k)-100k & -25(1+10k)+225k \\ 44k+4(1-10k) & -100k-9(1-10k) \end{bmatrix}$$

$$= \begin{bmatrix} 1+10(k+1) & -25(k+1) \\ 4(k+1) & 1-10(k+1) \end{bmatrix}$$

This is true for $n = k + 1$ ...(*ii*)

We have seen in (*i*) that the result is true for $n = 1$.

$\therefore$ It is true for $n = 1 + 1 = 2$ [by (*ii*)]

Similarly, it is true for $n = 2 + 1 = 3$ and so on.

Hence by mathematical induction, the result is true for all positive integers n.

Example 2.19. *Prove that (AB)C = A(BC), where A, B, C are matrices conformable for the products.* (*J.N.T.U., 2002 S*)

Solution. Let $A = [a_{ij}]$ be of order $m \times n$, $B = [b_{ij}]$ be of order $n \times p$ and $C = [c_{ij}]$ be of order of $p \times q$.

Then $$AB = [a_{ik}]\,[b_{kj}] = \sum_{k=1}^{n} a_{ik}\, b_{kj}$$

$\therefore$ $$(AB)C = \left[\sum_{k=1}^{n} a_{ik}\, b_{kl}\right] \cdot [c_{lj}] = \left[\sum_{l=1}^{p}\left(\sum_{k=1}^{n} a_{ik}\, b_{kl}\right) c_{lj}\right] = \left[\sum_{k=1}^{n}\sum_{l=1}^{p} a_{ik}\, b_{kl}\, c_{lj}\right]$$

Similarly, $$BC = [b_{kl}] \cdot [c_{lj}] = \sum_{l=1}^{p} b_{kl}\, c_{lj}$$

$\therefore$ $$A(BC) = [a_{ik}]\left[\sum_{l=1}^{p} b_{kl}\, c_{lj}\right] = \left[\sum_{k=1}^{n} a_{ik}\left(\sum_{l=1}^{p} b_{kl}\, c_{lj}\right)\right] = \left[\sum_{k=1}^{n}\left(\sum_{l=1}^{p} a_{ik}\, b_{kl}\, c_{lj}\right)\right]$$

Hence $(AB)C = A(BC)$.

PROBLEMS 2.2

1. For what values of x, the matrix $\begin{bmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -1-x \end{bmatrix}$ is singular ?
2. Find the values of x, y, z and a which satisfy the matrix equation $\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4a-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix}$.
3. Matrix A has x rows and $x + 5$ columns. Matrix B has y rows and $11 - y$ columns. Both AB and BA exist. Find x and y.
4. If $A + B = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix}$ and $A - B = \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}$, calculate the product AB.
5. If $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, find AB or BA, whichever exists.
6. If $A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix}$, verify that $(AB)C = A(BC)$ and $A(B + C) = AB + AC$.
7. Evaluate (*i*) $[x, y, z]\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$; (*ii*) $\begin{bmatrix} 2 & 1 & -1 \\ 4 & -5 & 6 \\ -3 & 7 & 3 \end{bmatrix} \times \begin{bmatrix} 3 & 1 \\ -6 & 4 \\ -2 & 5 \end{bmatrix} \times \begin{bmatrix} 5 & 3 \\ -2 & 1 \end{bmatrix}$; (*iii*) $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \times [4 \;\; 5 \;\; 2] \times \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} \times [3 \;\; 2]$

8. Prove that the product of two matrices

$$\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \text{ and } \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$$

is a null matrix when θ and ϕ differ by an odd multiple of $\pi/2$.

9. If $A = \begin{bmatrix} 0 & -\tan\alpha/2 \\ \tan\alpha/2 & 0 \end{bmatrix}$, show that $I + A = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$.

10. If $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$, find the value of $A^2 - 6A + 8I$, where I is a unit matrix of second order. *(B.P.T.U., 2006)*

11. If $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$, and I is the unit matrix of order 3, evaluate $A^2 - 3A + 9I$.

12. If $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 0 & 2 \\ 4 & -3 & 2 \end{bmatrix}$, verify the result $(A + B)^2 = A^2 + BA + AB + B^2$.

13. If $E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $F = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$,

calculate the products EF and FE and show that $E^2F + FE^2 = E$.

14. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, show that $A^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$, when n is a positive integer.

15. Factorize the matrix $A = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$ into the form LU, where L is lower triangular and U is upper triangular matrix.

2.6 RELATED MATRICES

(1) Transpose of a matrix. *The matrix obtained from any given matrix A, by interchanging rows and columns is called the* **transpose** *of A and is denoted by A′.*

Thus the transposed matrix of $A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix}$ is $A' = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix}$

Clearly, the transpose of an $m \times n$ matrix is an $n \times m$ matrix. Also the transpose of the transpose of a matrix coincides with itself, *i.e.*, $(A')' = A$.

For a symmetric matrix, $A' = A$ *and for a skew-symmetric matrix,* $A' = -A$.

Obs. 1. *The transpose of the product of the two matrices is the product of their transposes taken in the reverse order i.e.,* $(\mathbf{AB})' = \mathbf{B'A'}$.

For, the element in the ith row and jth col. of $(AB)'$

= element in the jth row and ith col. of AB = inner product of jth row of A with ith col. of B

= inner product of jth col. of A' with ith row of B' = element in the ith row and jth col. of $B'A'$

Hence $(AB)' = B'A'$.

Obs. 2. *Every square matrix can be uniquely expressed as a sum of a symmetric and a skew-symmetric matrix.* *(J.N.T.U., 2001)*

Let A be the given square matrix, then $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$.

Let $B = \frac{1}{2}(A + A')$ and $C = \frac{1}{2}(A - A')$

$\therefore$ $B' = \left[\frac{1}{2}(A + A')\right]' = \frac{1}{2}[A' + (A')'] = \frac{1}{2}(A' + A) = B$, *i.e.*, $B = \frac{1}{2}(A + A')$ is a symmetric matrix.

Again, $C' = \left[\frac{1}{2}(A - A')\right]' = \frac{1}{2}[A' - (A')'] = \frac{1}{2}(A' - A) = -C$, *i.e.*, $C = \frac{1}{2}(A - A')$ is a skew-symmetric matrix.

Hence A can be expressed as the sum of a symmetric and a skew-symmetric matrix.

To prove the uniqueness, assume that P is a symmetric matrix and Q is a skew-symmetric matrix such that $A = P + Q$.

Then $A' = (P + Q)' = P' + Q' = P - Q$

Thus, $P = \frac{1}{2}(A + A')$ and $Q = \frac{1}{2}(A - A')$

which shows that there is one and only one way of expressing A as the sum of a symmetric and skew-symmetric matrix.

Example 2.20. *Express the matrix A as the sum of a symmetric and a skew-symmetric matrix where*

$$A = \begin{bmatrix} 4 & 2 & -3 \\ 1 & 3 & -6 \\ -5 & 0 & -7 \end{bmatrix}$$

Solution. We have $A' = \begin{bmatrix} 4 & 1 & -5 \\ 2 & 3 & 0 \\ -3 & -6 & -7 \end{bmatrix}$

Then $A + A' = \begin{bmatrix} 8 & 3 & -8 \\ 3 & 6 & -6 \\ -8 & -6 & -14 \end{bmatrix}$ and $A - A' = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -6 \\ -2 & 6 & 0 \end{bmatrix}$

$$\therefore \quad A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A') = \begin{bmatrix} 4 & 1.5 & -4 \\ 1.5 & 3 & -3 \\ -4 & -3 & -7 \end{bmatrix} + \begin{bmatrix} 0 & 0.5 & 1 \\ -0.5 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}.$$

(2) Adjoint of a square matrix. The determinant of the square matrix

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \text{ is } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

The matrix formed by the cofactors of the elements in Δ is

$\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$. Then the transpose of this matrix, *i.e.*, $\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$

is called the *adjoint of the matrix A* and is written as *Adj. A*.

Thus the **adjoint** *of A is the transposed matrix of cofactors of A.*

(3) Inverse of a matrix. *If A be any matrix, then a matrix B if it exists, such that AB = BA = I, is called the* **Inverse** *of A* which is denoted by A^{-1} so that $AA^{-1} = I$.

Also $$\mathbf{A^{-1} = \frac{Adj.A}{|A|}}$$

For $$A\,(Adj\,A) = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} = |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or $A \cdot \dfrac{Adj.A}{|A|} = I \quad [\because \ |A| \neq 0]$ or $\dfrac{Adj.A}{|A|}$ is the inverse of A.

Obs. 1. *Inverse of a matrix, is unique.*

If possible, let the two inverses of the matrix A be B and C,

then $AB = BA = I$ and $AC = CA = I$

$\therefore$ $CAB = (CA)B = IB = B$ and $CAB = C(AB) = CI = C$

Thus, $B = C$.

Obs. 2. *The reciprocal of the product of two matrices is the product of their reciprocals taken in the reverse order i.e.,* $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ *(Assam, 1999)*

If A, B be two matrices, then the reciprocal of their product is $(AB)^{-1}$.

Clearly, $(AB).(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$ [by Associative law]

$$= AIA^{-1} = AA^{-1} = I.$$

Similarly, $(B^{-1}A^{-1}).(AB) = I$

Hence $B^{-1}A^{-1}$ is the reciprocal of AB.

Obs. 3. Multiplication by an inverse matrix plays the same role in matrix algebra that division plays in ordinary algebra.

i.e., if $[A][B] = [C][D]$, then $[A]^{-1}[A][B] = [A^{-1}][C][D]$

or $B = A^{-1}[C][D]$, *i.e.*, $\dfrac{[C][D]}{[A]} = A^{-1}[C][D]$.

Example 2.21. *Find the inverse of* $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

Solution. The determinant of the given matrix A is

$$\Delta = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \text{ (say)}$$

If $A_1, A_2, \ldots\ldots$ be the cofactors of $a_1, a_2 \ldots$ in Δ, then $A_1 = -24, A_2 = -8, A_3 = -12$; $B_1 = 10, B_2 = 2, B_3 = 6$; $C_1 = 2, C_2 = 2, C_3 = 2$.

Thus $\Delta = a_1A_1 + a_2A_2 + a_3A_3 = -8.$

and $$adj\,A = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix}.$$

Hence the inverse of the given matrix A

$$= \frac{adj\,A}{\Delta} = \frac{1}{-8}\begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

Note. For other methods see Examples 2.25 ; 2.28 and 2.46.

Example 2.22. *Find the matrix A if* $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix}$ *(Mumbai, 2008)*

Solution. If $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = B, \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = C$ and $\begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix} = D$, then

$$BAC = D \quad \text{or} \quad AC = B^{-1}D$$

$$\therefore \quad A = B^{-1}DC^{-1}$$

Now, $$B^{-1} = \frac{adj\,B}{|B|} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

Similarly, $$C^{-1} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

Hence, $$A = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 14 & 8 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 24 & 13 \\ -34 & -18 \end{bmatrix}.$$

PROBLEMS 2.3

1. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, verify that $AA' = I = A'A$, where I is the unit matrix.

2. Express each of the following matrices as the sum of a symmetric and a skew-symmetric matrix :

(i) $\begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} a & a & b \\ c & b & b \\ c & a & c \end{bmatrix}$

3. If A is a non-singular matrix of order n, prove that $A \; adj\, A = |A| I$. (*Mumbai, 2006*)

Verify that $A\,(adj\, A) = (adj\, A)\,A = |A|\, I$, where $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$.

4. Find the inverse of the matrix (i) $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ (*Mumbai, 2009*) (ii) $\begin{bmatrix} 5 & -2 & 4 \\ -2 & 1 & 1 \\ 4 & 1 & 0 \end{bmatrix}$ (*B.P.T.U., 2005*)

5. If $A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 3 & 1 \\ 5 & 3 & 4 \end{bmatrix}$, compute $adj\, A$ and A^{-1}. Also find B such that $AB = \begin{bmatrix} 3 & 4 & 2 \\ 1 & 6 & 1 \\ 5 & 6 & 4 \end{bmatrix}$. (*Mumbai, 2008*)

6. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, (i) find A^{-1} ; (ii) show that $A^3 = A^{-1}$.

7. Find the inverse of the matrix

$$S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \text{ and if } A = \frac{1}{2}\begin{bmatrix} 4 & -1 & 1 \\ -2 & 3 & -1 \\ 2 & 1 & 5 \end{bmatrix},$$

show that SAS^{-1} is a diagonal matrix dig (2, 3, 1). (*Mumbai, 2007*)

8. If $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$, prove that $A^{-1} = A'$.

9. Show that $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & -\tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix}\begin{bmatrix} 1 & \tan\theta/2 \\ -\tan\theta/2 & 1 \end{bmatrix}^{-1}$.

10. If $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$, verify that $(AB)' = B'A'$, where A' is the transpose of A.

11. $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 9 & 3 \\ 1 & 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$, verify that $(AB)^{-1} = B^{-1}A^{-1}$.

12. If A is a square matrix, show that (i) $A + A'$ is symmetric, and (ii) $A - A'$ is skew-symmetric. (*P.T.U., 1999*)

13. If $D = \text{diag}\,[d_1, d_2, d_3]$, $d_1, d_2, d_3 \neq 0$, prove that $D^{-1} = \text{diag}\,[d_1^{-1}, d_2^{-1}, d_3^{-1}]$.

14. If A and B are square matrices of the same order and A is symmetrical, show that $B'AB$ is also symmetrical. [**Hint.** Show that $(B'AB)' = B'AB$]

15. If a non-singular matrix A is symmetric, show that A^{-1} is also symmetric.

2.7 (1) RANK OF A MATRIX

If we select any r rows and r columns from any matrix A, deleting all the other rows and columns, then the determinant formed by these $r \times r$ elements is called the *minor of A of order r*. Clearly, there will be a number of different minors of the same order, got by deleting different rows and columns from the same matrix.

Def. *A matrix is said to be of rank r when*

(i) *it has at least one non-zero minor of order r,*

and (ii) *every minor of order higher than r vanishes.*

Briefly, the rank of a matrix is the largest order of any non-vanishing minor of the matrix.

If a matrix has a non-zero minor of order r, its rank is $\geq r$.

If all minors of a matrix of order $r + 1$ are zero, its rank is $\leq r$.

The rank of a matrix A shall be denoted by $\rho(A)$.

(2) Elementary transformation of a matrix. The following operations, three of which refer to rows and three to columns are known as *elementary transformations :*

I. The interchange of any two rows (columns).

II. The multiplication of any row (column) by a non-zero number.

III. The addition of a constant multiple of the elements of any row (column) to the corresponding elements of any other row (column).

Notation. The *elementary row transformations* will be denoted by the following symbols :

(*i*) R_{ij} for the interchange of the ith and jth rows.

(*ii*) kR_i for multiplication of the ith row by k.

(*iii*) $R_i + pR_j$ for addition to the ith row, p times the jth row.

The corresponding column transformation will be denoted by writing C in place of R.

Elementary transformations do not change either the order or rank of a matrix. While the value of the minors may get changed by the transformation I and II, their zero or non-zero character remains unaffected.

(3) Equivalent matrix. *Two matrices A and B are said to be equivalent if one can be obtained from the other by a sequence of elementary transformations.* Two equivalent matrices have the same order and *the same rank.* The symbol ~ is used for equivalence.

Example 2.23. *Determine the rank of the following matrices :*

(*i*) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ (*V.T.U., 2011*)

Solution. (*i*) Operate $R_2 - R_1$ and $R_3 - 2R_1$ so that the given matrix

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix} = A \text{ (say)}$$

Obviously, the 3rd order minor of A vanishes. Also its 2nd order minors formed by its 2nd and 3rd rows are all zero. But another 2nd order minor is $\begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} = -1 \neq 0$.

$\therefore$ $\rho(A) = 2$. Hence the rank of the given matrix is 2.

(*ii*) Given matrix

$$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 3 & 1 & -3 & -1 \\ 1 & 1 & -3 & -1 \end{bmatrix} \qquad \sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

[Operating $C_3 - C_1, C_4 - C_1$] [Operating $R_3 - R_1, R_4 - R_1$]

$$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = A \text{ (say)}$$

[Operating $R_3 - 3R_2, R_4 - R_2$] [Operating $C_3 + 3C_2, C_4 + C_2$]

Obviously, the 4th order minor of A is zero. Also every 3rd order minor of A is zero. But, of all the 2nd order minors, only $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1 \neq 0$. $\therefore$ $\rho(A) = 2$.

Hence the rank of the given matrix is 2.

(4) Elementary matrices. *An elementary matrix is that, which is obtained from a unit matrix, by subjecting it to any of the elementary transformations.*

Examples of elementary matrices obtained from

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ are } R_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = C_{23}\,;\, kR_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}\,;\, R_1 + pR_2 = \begin{bmatrix} 1 & p & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(5) Theorem. *Elementary row (column) transformations of a matrix A can be obtained by pre-multiplying (post-multiplying) A by the corresponding elementary matrices.*

Consider the matrix $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

Then $$R_{23} \times A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \end{bmatrix}$$

So a pre-multiplication by R_{23} has interchanged the 2nd and 3rd rows of A. Similarly, pre-multiplication by kR_2 will multiply the 2nd row of A by k and pre-multiplication by $R_1 + pR_2$ will result in the addition of p times the 2nd row of A to its 1st row.

Thus the pre-multiplication of A by elementary matrices results in the corresponding elementary row transformation of A. It can easily be seen that post mutiplication will perform the elementary column transformations.

(6) Gauss-Jordan method of finding the inverse*. *Those elementary row transformations which reduce a given square matrix A to the unit matrix, when applied to unit matrix I give the inverse of A.*

Let the successive row transformations which reduce A to I result from pre-multiplication by the elementary matrices $R_1, R_2, \ldots R_i$ so that

$$R_i R_{i-1} \ldots R_2 R_1 A = I$$

$\therefore$ $$R_i R_{i-1} \ldots R_2 R_1 A A^{-1} = I A^{-1}$$

or $$R_i R_{i-1} \ldots R_2 R_1 I = A^{-1} \qquad [\because \; AA^{-1} = I]$$

Hence the result.

Working rule to evaluate A^{-1}. *Write the two matrices A and I side by side. Then perform the same row transformations on both. As soon as A is reduced to I, the other matrix represents A^{-1}.*

Example 2.24. *Using the Gauss-Jordan method, find the inverse of the matrix*

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}.$$ (Kurukshetra, 2006)

Solution. Writing the same matrix side by side with the unit matrix of order 3, we have

$$\left[\begin{array}{rrr:rrr} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right] \qquad \text{(Operate } R_2 - R_1 \text{ and } R_3 + 2R_1)$$

$$\sim \left[\begin{array}{rrr:rrr} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right] \qquad \text{(Operate } \tfrac{1}{2}R_2 \text{ and } \tfrac{1}{2}R_3)$$

$$\sim \left[\begin{array}{rrr:rrr} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & 1 & 1 & 0 & \frac{1}{2} \end{array}\right] \qquad \text{(Operate } R_1 - R_2 \text{ and } R_3 + R_2)$$

*Named after the great German mathematician *Carl Friedrich Gauss* (1777–1855) who made his first great discovery as a student at Gottingen. His important contributions are to algebra, number theory, mechanics, complex analysis, differential equations, differential geometry, non-Euclidean geometry, numerical analysis, astronomy and electromagnetism. He became director of the observatory at Gottingen in 1807.

Name after another German mathematician and geodesist *Wilhelm Jordan* (1842–1899).

$$\sim\begin{bmatrix} 1 & 0 & 6: & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -3: & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -2: & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad \left[\text{Operate } R_1 + 3R_3,\ R_2 - \frac{3}{2}R_3 \text{ and } \left(-\frac{1}{2}\right)R_2\right]$$

$$\sim\begin{bmatrix} 1 & 0 & 0: & 3 & 1 & \frac{3}{2} \\ 0 & 1 & 0: & -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ 0 & 0 & 1: & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

Hence the inverse of the given matrix is $\begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$ [cf. Example 2.21]

(7) Normal form of a matrix. *Every non-zero matrix A of rank r, can be reduced by a sequence of elementary transformations, to the form*

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \text{ called the } \textbf{normal form} \text{ of } A. \qquad ...(i)$$

Cor. 1. *The rank of a matrix A is r if and only if it can be reduced to the normal form (i).*

Cor. 2. *Since each elementary transformation can be affected by pre-multiplication or post-multiplication with a suitable elementary matrix and each elementary matrix is non-singular, therefore, we have the following result :*

Corresponding to every matrix A of rank r, there exist non-singular matrices P and Q such that PAQ equals (i).

If A be a $m \times n$ matrix, then P and Q are square matrices of orders m and n respectively.

Example 2.25. *Reduce the following matrix into its normal form and hence find its rank.*

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}. \qquad \textit{(U.P.T.U., 2005)}$$

Solution.

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix} \qquad [\text{By } R_{12}]$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix} \qquad [\text{By } R_2 - 2R_1, R_3 - 3R_1, R_4 - 6R_1]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix} \qquad [\text{By } C_2 + C_1, C_3 + 2C_1, C_4 + 4C_1]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad [\text{By } R_4 - R_2 - R_3]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad [\text{By } R_2 - R_3]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad [\text{By } R_3 - 4R_2]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad [\text{By } C_3 + 6C_2,\ C_4 + 3C_2]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \left[\text{By } \frac{1}{33} C_3\right]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad [\text{By } C_4 - 22C_3]$$

$$\sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence $\rho(A) = 3$.

Example 2.26. *For the matrix* $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$,
find non-singular matrices P and Q such that PAQ is in the normal form. Hence find the rank of A.
(Kurukshetra, 2005)

Solution. We write $A = IAI$, *i.e.*, $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

We shall affect every elementary row (column) transformation of the product by subjecting the pre-factor (post-factor) of A to the same.

Operate $C_2 - C_1$, $C_3 - 2C_1$, $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Operate $R_2 - R_1$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Operate $C_3 - C_2$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

Operate $R_3 + R_2$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

which is of the normal form $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$

Hence, $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ and $\rho(A) = 2$.

PROBLEMS 2.4

Determine the rank of the following matrices (1–4) :

1. $\begin{bmatrix} 1 & 4 & 5 \\ 2 & 6 & 8 \\ 3 & 7 & 22 \end{bmatrix}$ (*P.T.U., 2005*)

2. $\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}$ (*W.B.T.U., 2005*)

3. $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ (*Kottayam, 2005*)

4. $\begin{bmatrix} 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \\ 11 & 12 & 13 & 14 \\ 16 & 17 & 18 & 19 \end{bmatrix}$ (*Rohtak, 2004*)

5. $\begin{bmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{bmatrix}$ (*Bhopal, 2008*)

6. Determine the values of p such that the rank of $A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 4 & 4 & -3 & 1 \\ p & 2 & 2 & 2 \\ 9 & 9 & p & 3 \end{bmatrix}$ is 3. (*Mumbai, 2007*)

7. Use Gauss-Jordan method to find the inverse of the following matrices :

(*i*) $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ (*Mumbai, 2008*)

(*iii*) $\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ (*B.P.T.U., 2006*) (*iv*) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ (*Kurukshetra, 2006*)

8. Find the non-singular matrices P and Q such that $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$ is reduced to normal form. Also find its rank. (*S.V.T.U., 2009 ; Mumbai, 2007*)

9. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, find A^{-1}. Also find two non-signular matrices P and Q such that $PAQ = I$, where I is the unit matrix and verify that $A^{-1} = QP$.

10. Find non-singular matrices P and Q such that PAQ is in the normal form for the matrices :

(*i*) $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ (*Rohtak, 2004*) (*ii*) $A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$

11. Reduce each of the following matrices to normal form and hence find their ranks :

(*i*) $\begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$ (*Kurukshetra, 2005*) (*ii*) $A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ (*Bhopal 2009*)

(*iii*) $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$ (*Mumbai, 2008*) (*iv*) $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix}$ (*U.T.U., 2010*)

2.8 PARTITION METHOD OF FINDING THE INVERSE

According to this method of finding the inverse, if the inverse of a matrix A_n of order n is known, then the inverse of the matrix A_{n+1} can easily be obtained by adding $(n + 1)$th row and $(n + 1)$th column to A_n.

Let $$A = \begin{bmatrix} A_1 & : & A_2 \\ \cdots & : & \cdots \\ A_3' & : & \alpha \end{bmatrix} \text{ and } A^{-1} = \begin{bmatrix} X_1 & : & X_2 \\ \cdots & : & \cdots \\ X_3' & : & x \end{bmatrix}$$

where A_2, X_2 are column vectors and A_3', X_3' are row vectors (being transposes of column vectors A_3, X_3) and α, x are ordinary numbers. We also assume that A_1^{-1} is known.

Then, $AA^{-1} = I_{n+1}$, *i.e.*, $$\begin{bmatrix} A_1 & : & A_2 \\ \cdots & : & \cdots \\ A_3' & : & \alpha \end{bmatrix} \begin{bmatrix} X_1 & : & X_2 \\ \cdots & : & \cdots \\ X_3' & : & x \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & 1 \end{bmatrix}$$

gives
$$A_1X_1 + A_2X_3' = I_n \qquad \ldots(i)$$
$$A_1X_2 + A_2x = 0 \qquad \ldots(ii)$$
$$A_3'X_1 + \alpha X_3' = 0 \qquad \ldots(iii)$$
$$A_3'X_2 + \alpha x = 1 \qquad \ldots(iv)$$

From (*ii*), $X_2 = -A_1^{-1} A_2 x$ and using this, (*iv*) gives $x = (\alpha - A_3' A_1^{-1} A_2)^{-1}$

Hence x and then X_2 are given.

Also from (*i*), $X_1 = A_1^{-1} (I_n - A_2X_3')$

and using this, (*iii*) gives $X_3' = -A_3' A_1^{-1} (\alpha - A_3'A_1^{-1} A_2)^{-1} = -A_3'A_1^{-1} x$

Then X_1 is determined and hence A^{-1} is computed.

Obs. This is also known as the *'Escalator method'*. For evaluation of A^{-1} we only need to determine two inverse matrices A_1^{-1} and $(\alpha - A_3'A_1^{-1}A_2)^{-1}$.

Example 2.27. *Using the partition method, find the inverse of* $\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$.

Solution. Let $$A = \begin{bmatrix} 1 & 1 & : & 1 \\ 4 & 3 & : & -1 \\ \cdots & \cdots & : & \cdots \\ 3 & 5 & : & 3 \end{bmatrix} = \begin{bmatrix} A_1 & : & A_2 \\ \cdots & : & \cdots \\ A_3' & : & \alpha \end{bmatrix}$$

so that $$A_1^{-1} = \begin{bmatrix} 1 & 1 \\ 4 & 3 \end{bmatrix}^{-1} = -\begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix}$$

Let $$A^{-1} = \begin{bmatrix} X_1 & : & X_2 \\ \cdots & : & \cdots \\ X_3' & : & x \end{bmatrix} \text{ so that } AA^{-1} = I.$$

$$\alpha - A_3' A_1^{-1} A_2 = 3 + [3 \quad 5] = \begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -10$$

$\therefore$ $$x = (\alpha - A_3' A_1^{-1} A_2)^{-1} = -\frac{1}{10}$$

Also, $$X_2 = -A_1^{-1} A_2 x = \begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \left(-\frac{1}{10}\right) = -\frac{1}{10} \begin{bmatrix} 4 \\ -5 \end{bmatrix}$$

Then $$X_3' = -A_3' A_1^{-1} x = [3 \quad 5] \begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix} \left(-\frac{1}{10}\right) = -\frac{1}{10} [-11 \quad 2]$$

Finally, $$X_1 = A_1^{-1}(I - A_2 X_3') = -\begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix} - \frac{1}{10} \begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} [-11 \quad 2]$$

$$= \begin{bmatrix} -3 & 1 \\ 4 & -1 \end{bmatrix} - \frac{1}{10}\begin{bmatrix} -44 & 8 \\ 55 & -10 \end{bmatrix} = \begin{bmatrix} 1.4 & 0.2 \\ -1.5 & 0 \end{bmatrix}$$

Hence $$A^{-1} = \begin{bmatrix} 1.4 & 0.2 & -0.4 \\ -1.5 & 0 & 0.5 \\ 1.1 & -0.2 & -0.1 \end{bmatrix}.$$

Example 2.28. *If A and C are non-singular matrices,then show that* $\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & 0 \\ -C^{-1}BA^{-1} & C^{-1} \end{bmatrix}$

Hence find inverse of $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 0 & 4 & 0 \\ 0 & 1 & 0 & 3 \end{bmatrix}$. *(Mumbai, 2005)*

Solution. Let the given matrix be $M = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$ and its inverse be $M^{-1} = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$ both in the partioned form where A, B, C, P, Q, R, S are all matrices.

$\therefore$ $$MM^{-1} = \begin{bmatrix} A & O \\ B & C \end{bmatrix}\begin{bmatrix} P & Q \\ R & S \end{bmatrix} = I$$

or $$\begin{bmatrix} AP + OR & AQ + OS \\ BP + CR & BQ + CS \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix}$$

$\therefore$ *Equating corresponding elements, we have*

$$AP + OR = I, AQ + OS = 0, BP + CR = 0, BQ + CS = I.$$

Second relation gives $AQ = 0$, *i.e.,* $Q = 0$ as A is non-singular.

First relation gives $AP = I$, *i.e.,* $P = A^{-1}$.

From third equation, $BP + CR = 0$, *i.e.,* $CR = -BP = -BA^{-1}$

$\therefore$ $$C^{-1}CR = -C^{-1}BA^{-1} \quad \text{or} \quad IR = -C^{-1}BA^{-1} \quad \text{or} \quad R = -C^{-1}BA^{-1}$$

From fourth equation, $BQ + CS = I$, or $CS = I$ or $S = C^{-1}$

Hence $$M^{-1} = \begin{bmatrix} A^{-1} & 0 \\ -C^{-1}BA^{-1} & C^{-1} \end{bmatrix}.$$

(*ii*) Let $$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 0 & 4 & 0 \\ 0 & 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} A & O \\ B & C \end{bmatrix}$$

Whence $$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$$

$\therefore$ $$A^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, C^{-1} = \frac{1}{12}\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$$

$\therefore$ $$-C^{-1}(BA^{-1}) = -\frac{1}{12}\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}\left\{\frac{1}{2}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

$$= -\frac{1}{24}\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix} = -\frac{1}{24}\begin{bmatrix} 18 & 0 \\ 0 & 4 \end{bmatrix}$$

Hence, $$M^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ -3/4 & 0 & 1/4 & 0 \\ 0 & -1/6 & 0 & 1/3 \end{bmatrix}.$$

PROBLEMS 2.5

Find the inverse of each of the following matrices using the partition method :

1. $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ (*Nagpur, 1997*)

2. $\begin{bmatrix} 3 & 2 & 4 \\ 2 & 1 & 1 \\ 1 & 3 & 5 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 3 & 2 \\ 2 & 4 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 3 & -1 & 10 & 2 \\ 5 & 1 & 20 & 3 \\ 9 & 7 & 39 & 4 \\ 1 & -2 & 2 & 1 \end{bmatrix}$.

2.9 SOLUTION OF LINEAR SYSTEM OF EQUATIONS

(1) Method of determinants—Cramer's* rule

Consider the equations $\left\{\begin{matrix} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{matrix}\right\}$ ...(*i*)

If the determinant of coefficient be $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

then $$x\Delta = \begin{vmatrix} xa_1 & b_1 & c_1 \\ xa_2 & b_2 & c_2 \\ xa_3 & b_3 & c_3 \end{vmatrix} \qquad [\text{Operate } C_1 + yC_2 + zC_3]$$

$$= \begin{vmatrix} a_1x + b_1y + c_1z & b_1 & c_1 \\ a_2x + b_2y + c_2z & b_2 & c_2 \\ a_3x + b_3y + c_3z & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \qquad [\text{By } (i)]$$

Thus $$x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ provided } \Delta \neq 0. \qquad ...(ii)$$

Similarly, $$y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad ...(iii)$$

and $$z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad ...(iv)$$

Equation (*ii*), (*iii*) and (*iv*) *giving the values of x, y, z constitute the* **Cramer's rule,** which reduces the solution of the linear equations (*i*) to a problem in evaluation of determinants.

(2) Matrix inversion method

If $$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

then the equations (*i*) are equivalent to the matrix equation $AX = D$ ...(*v*)

where A is the *coefficient matrix.*

Multiplying both sides of (*v*) by the reciprocal matrix A^{-1}, we get

$$A^{-1}AX = A^{-1}D \quad \text{or} \quad IX = A^{-1}D \qquad [\because A^{-1}A = I]$$

**Gabriel Cramer* (1704–1752), a Swiss mathematician.

or $$X = A^{-1}D \quad i.e., \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} \times \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \quad ...(vi)$$

where A_1, B_1 etc. are the cofactors of a_1, b_1 etc. in the determinant $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ $(\Delta \neq 0)$

Hence equating the values of x, y, z to the corresponding elements in the product on the right side of (vi), we get the desired solutions.

Obs. *When A is a singular matrix, i.e., $\Delta = 0$, the above methods fail. These also fail when the number of equations and the number of unknowns are unequal. Matrices can, however, be usefully applied to deal with such equations as will be seen in § 2.10(2).*

Example 2.29. *Solve the equations $3x + y + 2z = 3$, $2x - 3y - z = -3$, $x + 2y + z = 4$ by (i) determinants (ii) matrices.*

Solution. (*i*) *By determinants :*

Here $$\Delta = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 3(-3+2) - 2(1-4) + (-1+6) = 8 \quad \text{[Expanding by } C_1]$$

$\therefore$ $$x = \frac{1}{\Delta}\begin{vmatrix} 3 & 1 & 2 \\ -3 & -3 & -1 \\ 4 & 2 & 1 \end{vmatrix} \quad \text{[Expand by } C_1]$$

$$= \frac{1}{8}[3(-3+2) + 3(1-4) + 4(-1+6)] = 1$$

Similarly, $$y = \frac{1}{\Delta}\begin{vmatrix} 3 & 3 & 2 \\ 2 & -3 & -1 \\ 1 & 4 & 1 \end{vmatrix} = 2 \quad \text{and} \quad z = \frac{1}{\Delta}\begin{vmatrix} 3 & 1 & 3 \\ 2 & -3 & -3 \\ 1 & 2 & 4 \end{vmatrix} = -1$$

Hence $x = 1, y = 2, z = -1$.

Note. The use of Cramer's rule involves a lot of labour when the number of equations exceeds four. In such and other cases, the numerical methods given in § 28.4 to 28.6 are perferable.

(*ii*) *By matrices :*

Here $$\Delta = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{(say).}$$

Then $A_1 = -1, A_2 = 3, A_3 = 5$; $B_1 = -3, B_2 = 1, B_3 = 7$; $C_1 = 7, C_2 = -5, C_3 = -11$.

Also $\Delta = a_1A_1 + a_2A_2 + a_3A_3 = 8$.

$\therefore$ $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{\Delta}\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \times \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix} \times \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -3-9+20 \\ -9-3+28 \\ 21+15-44 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Hence $x = 1, y = 2, z = -1$.

Example 2.30. *Solve the equations $x_1 - x_2 + x_3 + x_4 = 2$; $x_1 + x_2 - x_3 + x_4 = -4$; $x_1 + x_2 + x_3 - x_4 = 4$; $x_1 + x_2 + x_3 + x_4 = 0$, by finding the inverse by elementary row operations.*

Solution. Given system can be written as $AX = B$, where

$$A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, B = \begin{bmatrix} 2 \\ -4 \\ 4 \\ 0 \end{bmatrix}$$

To find A^{-1}, we write

$$[A : I] = \begin{bmatrix} 1 & -1 & 1 & 1 & : & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 1 & : & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 & : & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & : & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} R_2 - R_1 \\ R_3 + R_1 \\ R_4 + R_1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 1 & 1 & : & 1 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 & : & -1 & 1 & 0 & 0 \\ 2 & 0 & 2 & 0 & : & 1 & 0 & 1 & 0 \\ 2 & 0 & 2 & 2 & : & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} R_2 \\ \frac{1}{2} R_3 \\ \frac{1}{2} R_4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 1 & 1 & : & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & : & -1/2 & 1/2 & 0 & 0 \\ 1 & 0 & 1 & 0 & : & 1/2 & 0 & 1/2 & 0 \\ 1 & 0 & 1 & 1 & : & 1/2 & 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} R_3 - R_2 \\ R_4 - R_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 1 & : & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & -1 & 0 & : & -1/2 & 1/2 & 0 & 0 \\ 1 & 0 & 1 & 0 & : & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 & : & 0 & 0 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} R_1 - R_4 \\ R_2 + R_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & : & 1/2 & 1/2 & +1/2 & -1/2 \\ 1 & 1 & 0 & 0 & : & 0 & 1/2 & 1/2 & 0 \\ 1 & 0 & 1 & 0 & : & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 & : & 0 & 0 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} R_2 - R_1 \\ R_3 - R_1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & : & 1/2 & 1/2 & 1/2 & -1/2 \\ 0 & 1 & 0 & 0 & : & -1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & : & 0 & -1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & : & 0 & 0 & -1/2 & 1/2 \end{bmatrix}$$

Thus, $$A^{-1} = \begin{bmatrix} 1/2 & 1/2 & 1/2 & -1/2 \\ -1/2 & 0 & 0 & 1/2 \\ 0 & -1/2 & 0 & 1/2 \\ 0 & 0 & -1/2 & 1/2 \end{bmatrix}$$

Hence, $$X = A^{-1}B = \begin{bmatrix} 1/2 & 1/2 & 1/2 & -1/2 \\ -1/2 & 0 & 0 & 1/2 \\ 0 & -1/2 & 0 & 1/2 \\ 0 & 0 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix}$$

i.e., $x_1 = 1, x_2 = -1, x_3 = 2, x_4 = -2.$

PROBLEMS 2.6

Solve the following equations with the help of determinants (1 to 4) :

1. $x + y + z = 4 \,;\, x - y + z = 0 \,;\, 2x + y + z = 5.$ *(Osmania, 2003)*
2. $x + 3y + 6z = 2 \,;\, 3x - y + 4z = 9 \,;\, x - 4y + 2z = 7.$
3. $x + y + z = 6.6 \,;\, x - y + z = 2.2 \,;\, x + 2y + 3z = 15.2.$
4. $x^2 z^3/y = e^8 \,;\, y^2 z/x = e^4 \,;\, x^3 y/z^4 = 1.$
5. $2vw - wu + uv = 3uvw \,;\, 3vw + 2wu + 4uv = 19uvw \,;\, 6vw + 7wu - uv = 17uvw.$

Solve the following system of equations by matrix method (6 to 8) :

6. $x_1 + x_2 + x_3 = 1, x_1 + 2x_2 + 3x_3 = 6, x_1 + 3x_2 + 4x_3 = 6.$ *(P.T.U., 2006)*
7. $x + y + z = 3 \,;\, x + 2y + 3z = 4 \,;\, x + 4y + 9z = 6.$ *(Bhopal, 2003)*
8. $2x - 3y + 4z = -4, x + z = 0, -y + 4z = 2.$ *(W.B.T.U., 2005)*
9. $2x - y + 3z = 8 \,;\, x - 2y - z = -4 \,;\, 3x + y - 4z = 0.$ *(Mumbai, 2005)*
10. $2x_1 + x_2 + 2x_3 + x_4 = 6, 4x_1 + 3x_2 + 3x_3 - 3x_4 = -1, 6x_1 - 6x_2 + 6x_3 + 12x_4 = 36, 2x_1 + 2x_2 - x_3 + x_4 = 10.$ *(U.P.T.U., 2001)*

11. By finding A^{-1}, solve the linear equation $AX = B$, where $A = \begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 0 \\ 5 & 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix}$.

12. In a given electrical network, the equations for the currents i_1, i_2, i_3 are
$3i_1 + i_2 + i_3 = 8$; $2i_1 - 3i_2 - 2i_3 = -5$; $7i_1 + 2i_2 - 5i_3 = 0$.
Calculate i_1 and i_3 by Cramer's rule.

13. Using the loop current method on a circuit, the follwoing equations are obtained :
$7i_1 - 4i_2 = 12, -4i_1 + 12i_2 - 6i_3 = 0, -6i_2 + 14i_3 = 0.$
By matrix method, solve for i_1, i_2 and i_3.

14. Solve the following equations by calculating the inverse by elementary row operations :
$2x_1 + 2x_2 + 2x_3 - 3x_4 = 2$; $3x_1 + 6x_2 - 2x_3 + x_4 = 8$; $x_1 + x_2 - 3x_3 - 4x_4 = -1$; $2x_1 + x_2 + 5x_3 + x_4 = 5$.

2.10 (1) CONSISTENCY OF LINEAR SYSTEM OF EQUATIONS

Consider the system of m linear equations

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= k_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= k_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= k_m \end{aligned}\right\} \qquad \dots(i)$$

containing the n unknowns $x_1, x_2, \dots, x_n$. To determine whether the equations (i) are consistent (*i.e.*, possess a solution) or not, we consider the ranks of the matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & k_1 \\ a_{21} & a_{22} & \dots & a_{2n} & k_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & k_m \end{bmatrix}$$

A is the *co-efficient matrix* and K is called the *augmented matrix* of the equations (i).

(2) Rouche's theorem. *The system of equations (i) is consistent if and only if the coefficient matrix A and the augmented matrix K are of the same rank otherwise the system is inconsistent.*

Proof. We consider the following two possible cases :

I. Rank of A = rank of K = r ($r \leq$ the smaller of the numbers m and n). The equations (i) can, be suitable row operations, be reduced to

$$\left.\begin{aligned} b_{11}x_1 + b_{12}x_2 + \dots + b_{1n}x_n &= l_1 \\ 0.x_1 + b_{22}x_2 + \dots + b_{2n}x_n &= l_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ 0.x_1 + 0.x_2 + \dots + b_{rn}x_n &= l_r \end{aligned}\right\} \qquad \dots(ii)$$

and the remaining $m - r$ equations being all of the form $0.x_1 + 0.x_2 + \dots\dots + 0.x_n = 0$.

The equations (ii) will have a solution, though $n - r$ of the unknowns may be chosen arbitrarily. The solution, will be unique only when $r = n$. Hence the equations (i) are consistent.

II. Rank of A (i.e., r) < rank of K. In particular, let the rank of K be $r + 1$. In this case, the equations (i) will reduce, by suitable row operations, to

$$\begin{aligned} b_{11}x_1 + b_{12}x_2 + \dots + b_{1n}x_n &= l_1, \\ 0.x_1 + b_{22}x_2 + \dots + b_{2n}x_n &= l_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots&, \\ 0.x_1 + 0.x_2 + \dots + b_{rn}x_n &= l_r, \\ 0.x_1 + 0.x_2 + \dots + 0.x_n &= l_{r+1}, \end{aligned}$$

and the remaining $m - (r + 1)$ equations are of the form $0.x_1 + 0.x_2 + \dots + 0.x_n = 0$.

Clearly, the $(r + 1)$th equation cannot be satisfied by any set of values for the unknowns. Hence the equations (i) are inconsistent.

[Procedure to test the consistency of a system of equations in n unknowns :

Find the ranks of the coefficient matrix A and the augmented matrix K, by reducing A to the triangular form by elementary row operations. Let the rank of A be r and that of K be r'.

(*i*) *If* $r \neq r'$, *the equations are inconsistent, i.e., there is no solution.*

(*ii*) *If* $r = r' = n$, *the equations are consitent and there is a unique solution.*

(*iii*) *If* $r = r' < n$, *the equations are consistent and there are infinite number of solutions.* Giving arbitrary values to $n - r$ of the unknowns, we may express the other r unknowns in terms of these.]

Example 2.31. *Test for consistency and solve*

$$5x + 3y + 7z = 4,\ 3x + 26y + 2z = 9,\ 7x + 2y + 10z = 5.$$

(*Bhopal, 2008 ; J.N.T.U., 2005 ; P.T.U., 2005*)

Solution. We have
$$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$

Operate $3R_1, 5R_2$,
$$\begin{bmatrix} 15 & 9 & 21 \\ 15 & 130 & 10 \\ 7 & 2 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 45 \\ 5 \end{bmatrix}$$

Operate $R_2 - R_1$,
$$\begin{bmatrix} 15 & 9 & 21 \\ 0 & 121 & -11 \\ 7 & 2 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 33 \\ 5 \end{bmatrix}$$

Operate $\frac{7}{3}R_1, 5R_3, \frac{1}{11}R_2$,
$$\begin{bmatrix} 35 & 21 & 49 \\ 0 & 11 & -1 \\ 35 & 10 & 50 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 28 \\ 3 \\ 25 \end{bmatrix}$$

Operate $R_3 - R_1 + R_2, \frac{1}{7}R_1$,
$$\begin{bmatrix} 5 & 3 & 7 \\ 0 & 11 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$$

The ranks of coefficient matrix and augmented matrix for the last set of equations, are both 2. Hence the equations are consistent. Also the given system is equivalent to

$$5x + 3y + 7z = 4,\ 11y - z = 3, \quad \therefore \quad y = \frac{3}{11} + \frac{z}{11} \quad \text{and} \quad x = \frac{7}{11} - \frac{16}{11}z$$

where z is a parameter.

Hence $x = \frac{7}{11}, y = \frac{3}{11}$ and $z = 0$, is a particular solution.

Obs. In the above solution, the coefficient matrix is reduced to an upper triangular matrix by row-transformations.

Example 2.32. *Investigate the values of* λ *and* μ *so that the equations*

$$2x + 3y + 5z = 9,\ 7x + 3y - 2z = 8,\ 2x + 3y + \lambda z = \mu,$$

have (i) no solution, (ii) a unique solution and (iii) an infinite number of solutions.

(*Mumbai, 2007 ; V.T.U., 2007*)

Solution. We have
$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

The system admits of unique solution if, and only if, the coefficient matrix is of rank 3. This requires that

$$\begin{vmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{vmatrix} = 15(5 - \lambda) \neq 0$$

Thus for a unique solution $\lambda \neq 5$ and μ may have any value. If $\lambda = 5$, the system will have no solution for those values of μ for which the matrices

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & 5 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & 5 & \mu \end{bmatrix}$$

are not of the same rank. But A is of rank 2 and K is not of rank 2 unless $\mu = 9$. Thus if $\lambda = 5$ and $\mu \neq 9$, the system will have no solution.

If $\lambda = 5$ and $\mu = 9$, the system will have an infinite number of solutions.

Example 2.33. *Test for consistency the following equations and solve them if consistent :* $x - 2y + 3t = 2$, $2x + y + z + t = -4$; $4x - 3y + z + 7t = 8$. *(Mumbai, 2008)*

Solution. Given equation can be written as

$$\begin{bmatrix} 1 & -2 & 0 & 3 \\ 2 & 1 & 1 & 1 \\ 4 & -3 & 1 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 8 \end{bmatrix}$$

Operate $R_2 - 2R_1, R_3 - 4R_1$,

$$\begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 5 & 1 & -5 \\ 0 & 5 & 1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Operate $R_3 - R_2$,

$$\begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 5 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Clearly, rank of the coefficient matrix is 2 and the rank of augmented matrix is also 2. Hence the given equations are consistent. But the rank $2 < 4$, the number of unknowns.

$\therefore$ The number of parameters is $4 - 2 = 2$

Thus the equations have doubly infinite solutions. Now putting $t = k_1$ and $z = k_2$ in

$$x - 2y + 3t = 2 \quad \text{and} \quad 5y + z - 5t = 0,$$

we get $x - 2y + 3k_1 = 2$ and $5y + k_2 - 5k_1 = 0$

Hence $y = k_1 - k_2/5$

and
$$x = 2 + 2y - 3k_1$$
$$= 2 + 2(k_1 - k_2/5) - 3k_1$$
$$= 2 - k_1 - \frac{2}{5} k_2$$

(3) System of linear homogeneous equations. Consider the homogeneous linear equations

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = 0 \end{array}\right\} \quad \ldots(iii)$$

Find the rank r of the coefficient matrix A by reducing it to the triangular form by elementary row operations.

I. *If $r = n$, the equations (iii) have only a trivial zero solution*

$$x_1 = x_2 = \ldots = x_n = 0$$

If $r < n$, the equation (iii) have $(n - r)$ linearly independent solutions.

The number of linearly independent solutions is $(n - r)$ means, if arbitrary values are assigned to $(n - r)$ of the variables, the values of the remaining variables can be uniquely found.

Thus the equations (*iii*) will have an infinite number of solutions.

II. *When $m < n$* (*i.e.*, the number of equations is less than the number of variables), *the solution is always other than $x_1 = x_2 = \ldots = x_n = 0$.* The number of solutions is infinite.

III. *When $m = n$* (*i.e.*, the number of equations = the number of variables), *the necessary and sufficient condition for solutions other than $x_1 = x_2 = \ldots = x_n = 0$, is that the determinant of the coefficient matrix is zero.* In this case the equations are said to be *consistent* and such a solution is called *non-trivial solution. The determinant is called the* **eliminant** *of the equations.*

Example 2.34. *Solve the equations*
(i) $x + 2y + 3z = 0,\ 3x + 4y + 4z = 0,\ 7x + 10y + 12z = 0$
(ii) $4x + 2y + z + 3w = 0,\ 6x + 3y + 4z + 7w = 0,\ 2x + y + w = 0.$

Solution. (*i*) Rank of the coefficient matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 7 & 10 & 12 \end{bmatrix} \qquad \text{[Operating } R_3 - 3R_1]$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{[Operating } R_3 - 7R_1 - 2R_2]$$

is 3 which = the number of variables (*i.e.*, $r = n$)

$\therefore$ The equations have only a trivial solution : $x = y = z = 0$.

(*ii*) Rank of the coefficient matrix

$$\begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 5/2 & 5/2 \\ 0 & 0 & -1/2 & -1/2 \end{bmatrix} \qquad \text{[Operating } R_2 - \frac{3}{2} R_1, R_3 - \frac{1}{2} R_1$$

$$\sim \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 5/2 & 5/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{[Operating } R_3 + \frac{1}{5} R_2$$

is 2 which < the number of variable (*i.e.*, $r < n$)

$\therefore$ Number of independent solutions $= 4 - 2 = 2$. Given system is equivalent to

$$4x + 2y + z + 3w = 0,\ z + w = 0.$$

$\therefore$ We have $z = -w$ and $y = -2x - w$

which give an infinite number of non-trivial solutions, x and w being the parameters.

Example 2.35. *Find the values of k for which the system of equations* $(3k - 8)x + 3y + 3z = 0,\ 3x + (3k - 8)y + 3z = 0,\ 3x + 3y + (3k - 8)z = 0$ *has a non-trivial solution.* (*U.P.T.U., 2006*)

Solution. For the given system of equations to have a non-trivial solution, the determinant of the coefficient matrix should be zero.

i.e., $$\begin{vmatrix} 3k-8 & 3 & 3 \\ 3 & 3k-8 & 3 \\ 3 & 3 & 3k-8 \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 3k-2 & 3 & 3 \\ 3k-2 & 3k-8 & 3 \\ 3k-2 & 3 & 3k-8 \end{vmatrix} = 0 \qquad \text{[Operating } C_1 + (C_2 + C_3)]$$

or $$(3k-2)\begin{vmatrix} 1 & 3 & 3 \\ 1 & 3k-8 & 3 \\ 1 & 3 & 3k-8 \end{vmatrix} = 0 \quad \text{or} \quad (3k-2)\begin{vmatrix} 1 & 3 & 3 \\ 0 & 3k-11 & 0 \\ 0 & 0 & 3k-11 \end{vmatrix} = 0 \qquad \text{[Operating } R_2 - R_1, R_3 - R_1]$$

or $$(3k - 2)(3k - 11)^2 = 0 \text{ whence } k = 2/3,\ 11/3,\ 11/3.$$

Example 2.36. *If the following system has non-trivial solution, prove that* $a + b + c = 0$ *or* $a = b = c$: $ax + by + cz = 0,\ bx + cy + az = 0,\ cx + ay + bz = 0.$ (*Mumbai, 2006*)

Solution. For the given system of equations to have non-trivial solution, the determinant of the coefficient matrix is zero.

i.e., $$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{vmatrix} = 0 \qquad \text{[Operating } R_1 + R_2 + R_3]$$

or $$(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix} = 0 \quad \text{or} \quad (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ b & c-b & a-b \\ c & a-c & b-c \end{vmatrix} = 0 \qquad \text{[Operating } C_2 - C_1, C_3 - C_1]$$

or $\quad (a+b+c)\,[(c-b)(b-c)-(a-c)(a-b)]=0$

or $\quad (a+b+c)(-a^2-b^2-c^2+ab+bc+ca)=0$

i.e., $\quad a+b+c=0 \quad \text{or} \quad a^2+b^2+c^2-ab-bc-ca=0$

or $\quad a+b+c=0 \quad \text{or} \quad \frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]=0$

or $\quad a+b+c=0\,;\ a=b,\ b=c,\ c=a.$

Hence the given system has a non-trivial solution if $a+b+c=0$ or $a=b=c$.

Example 2.37. *Find the values of λ for which the equations*

$$(\lambda-1)x+(3\lambda+1)y+2\lambda z=0$$
$$(\lambda-1)x+(4\lambda-2)y+(\lambda+3)z=0$$
$$2x+(3\lambda+1)y+3(\lambda-1)z=0$$

are consistent, and find the ratios of $x:y:z$ when λ has the smallest of these values. What happens when λ has the greatest of these values. (*Kurukshetra, 2006 ; Delhi, 2002*)

Solution. The given equations will be consistent, if

$$\begin{vmatrix} \lambda-1 & 3\lambda+1 & 2\lambda \\ \lambda-1 & 4\lambda-2 & \lambda+3 \\ 2 & 3\lambda+1 & 3(\lambda-1) \end{vmatrix}=0 \qquad [\text{Operate } R_2-R_1]$$

or if,
$$\begin{vmatrix} \lambda-1 & 3\lambda+1 & 2\lambda \\ 0 & \lambda-3 & 3-\lambda \\ 2 & 3\lambda+1 & 3(\lambda-1) \end{vmatrix}=0 \qquad [\text{Operate } C_3+C_2]$$

or if,
$$\begin{vmatrix} \lambda-1 & 3\lambda+1 & 5\lambda+1 \\ 0 & \lambda-3 & 0 \\ 2 & 3\lambda+1 & 6\lambda-2 \end{vmatrix}=0 \qquad [\text{Expand by } R_2]$$

or if, $\quad (\lambda-3)\begin{vmatrix} \lambda-1 & 5\lambda+1 \\ 2 & 2(3\lambda+1) \end{vmatrix}=0$ or if, $2(\lambda-3)\,[(\lambda-1)(3\lambda-1)-(5\lambda+1)]=0$

or if, $\quad 6\lambda(\lambda-3)^2=0$ or if, $\lambda=0$ or 3.

(*a*) *When* $\lambda=0$, the equations become $-x+y=0$ ...(*i*)

$$-x-2y+3z=0 \qquad ...(ii)$$
$$2x+y-3z=0 \qquad ...(iii)$$

Solving (*ii*) and (*iii*), we get $\dfrac{x}{6-3}=\dfrac{y}{6-3}=\dfrac{z}{-1+4}$. Hence $x=y=z$.

(*b*) *When* $\lambda=3$, equations becomes identical.

PROBLEMS 2.7

1. Investigate for consistency of the following equations and if possible find the solutions :
$$4x-2y+6z=8,\ x+y-3z=-1,\ 15x-3y+9z=21.$$
2. For what values of k the equations $x+y+z=1,\ 2x+y+4z=k,\ 4x+y+10z=k^2$ have a solution and solve them completely in each case. (*Bhopal, 2008 ; Mumbai, 2008 ; V.T.U., 2006*)
3. Investigate for what values of λ and μ the simultaneous equations
$$x+y+z=6,\ x+2y+3z=10,\ x+2y+\lambda z=\mu,$$
have (*i*) no solution, (*ii*) a unique solution, (*iii*) an infinite number of solutions. (*Mumbai, 2007 ; U.P.T.U., 2006 ; Rohtak, 2004*)
4. Test for consistency and solve,
   (*i*) $2x-3y+7z=5,\ 3x+y-3z=13,\ 2x+19y-47z=32.$ (*Bhopal, 2009 ; Kurukshetra, 2005 ; Raipur, 2005*)
   (*ii*) $x+2y+z=3,\ 2x+3y+2z=5,\ 3x-5y+5z=2,\ 3x+9y-z=4.$ (*Bhilai, 2005 ; Madras, 2002*)
   (*iii*) $2x+6y+11=0,\ 6x+20y-6z+3=0,\ 6y-18z+1=0.$ (*Rajasthan, 2005*)
   (*iv*) $3x+3y+2z=1,\ x+2y=4,\ 10y+3z=-2,\ 2x-3y-z=5.$ (*U.T.U., 2010 ; Nagarjuna, 2008*)

5. Find the values of a and b for which the equations
$$x + ay + z = 3, x + 2y + 2z = b, x + 5y + 3z = 9$$
are consistent. When will these equations have a unique solution ? *(Kurukshetra, 2005 ; Madras, 2003)*

6. Show that if $\lambda \neq -5$, the system of equations
$$3x - y + 4z = 3, x + 2y - 3z = -2, 6x + 5y + \lambda z = -3,$$
have a unique solution. If $\lambda = -5$, show that the equations are consistent. Determine the solutions in each case.

7. Show that the equations
$$3x + 4y + 5z = a, 4x + 5y + 6z = b, 5x + 6y + 7z = c$$
do not have a solution unless $a + c = 2b$. *(Raipur, 2004 ; Nagpur, 2001)*

8. Prove that the equations $5x + 3y + 2z = 12$, $2x + 4y + 5z = 2$, $39x + 43y + 45z = c$ are incompatible unless $c = 74$; and in that case the equations are satisfied by $x = 2 + t$, $y = 2 - 3t$, $z = -2 + 2t$, where t is any arbitrary quantity.

9. Find the values of λ for which the equations $(2 - \lambda)x + 2y + 3 = 0$, $2x + (4 - \lambda)y + 7 = 0$, $2x + 5y + (6 - \lambda) = 0$ are consistent and find the values of x and y corresponding to each of these values of λ.

10. Show that there are three real values of λ for which the equations $(a - \lambda)x + by + cz = 0$, $bx + (c - \lambda)y + az = 0$, $cx + ay + (b - \lambda)z = 0$ are simultaneously true and that the product of these values of λ is
$$D = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}.$$

11. Determine the values of k for which the following system of equations has non-trivial solutions and find them :
$$(k-1)x + (4k-2)y + (k+3)z = 0, (k-1)x + (3k+1)y + 2kz = 0, 2x + (3k+1)y + 3(k-1)z = 0.$$
(Mumbai, 2005)

12. Show that the system of equations $2x_1 - 2x_2 + x_3 = \lambda x_1$, $2x_1 - 3x_2 + 2x_3 = \lambda x_2$, $-x_1 + 2x_2 = \lambda x_3$ can possess a non-trivial solution only if $\lambda = 1$, $\lambda = -3$. Obtain the general solution in each case.

13. Determine the values of λ for which the following set of equations may possess non-trivial solution :
$$3x_1 + x_2 - \lambda x_3 = 0, 4x_1 - 2x_2 - 3x_3 = 0, 2\lambda x_1 + 4x_2 + \lambda x_3 = 0.$$
For each permissible value of λ, determine the general solution. *(Kurukshetra, 2006)*

14. Solve completely the system of equations

(*i*) $x + y - 2z + 3w = 0$; $x - 2y + z - w = 0$; $4x + y - 5z + 8w = 0$; $5x - 7y + 2z - w = 0$.

(*ii*) $3x + 4y - z - 6w = 0$; $2x + 3y + 2z - 3w = 0$; $2x + y - 14z - 9w = 0$; $x + 3y + 13z + 3w = 0$. *(J.N.T.U., 2002 S)*

2.11 (1) LINEAR TRANSFORMATIONS

Let (x, y) be the co-ordinates of a point P referred to set of rectangular axes OX, OY. Then its co-ordinates (x', y') referred to OX', OY', obtained by rotating the former axes through an angle θ given by
$$\left.\begin{aligned} x' &= x\cos\theta + y\sin\theta, \\ y' &= -x\sin\theta + y\cos\theta \end{aligned}\right\} \quad \ldots(i)$$

A more general transformation than (*i*) is
$$\left.\begin{aligned} x' &= a_1 x + b_1 y \\ y' &= a_2 x + b_2 y \end{aligned}\right\} \quad \ldots(ii)$$

which in matrix notation is $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Such transformations as (*i*) and (*ii*), are called *linear transformations* in two dimensions.

Similarly, the relations of the type
$$\left.\begin{aligned} x' &= l_1 x + m_1 y + n_1 z \\ y' &= l_2 x + m_2 y + n_2 z \\ z' &= l_3 x + m_3 y + n_3 z \end{aligned}\right\} \quad \ldots(iii)$$

give a *linear transformation* from (x, y, z) to (x', y', z') in three dimensional problems.

In general, the relation $Y = AX$ where $Y = \begin{bmatrix} y_1 \\ y_2 \\ \ldots \\ y_n \end{bmatrix}$, $A = \begin{bmatrix} a_1 & b_1 & c_1 & \ldots & k_1 \\ a_2 & b_2 & c_2 & \ldots & k_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_n & b_n & c_n & \ldots & k_n \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix}$ $\quad \ldots(iv)$

give *linear transformation* from n variables $x_1, x_2, ..., x_n$ to the variables $y_1, y_2, ..., y_n$ *i.e.*, the transformation of the vector X to the vector Y.

This transformation is called linear because the linear relations $A(X_1 + X_2) = AX_1 + AX_2$ and $A(bX) = bAX$, hold for this transformation.

If the transformation matrix A is singular, the transformation also is said to be singular otherwise non-singular. For a non-singular transformation $Y = AX$, we can also write the inverse transformation $X = A^{-1} Y$. A non-singular transformation is also called a *regular* transformation.

Cor. If a transformation from (x_1, x_2, x_3) to (y_1, y_2, y_3) is given by $Y = AX$ and another transformation of (y_1, y_2, y_3) to (z_1, z_2, z_3) is given by $Z = BY$, then the transformation from (x_1, x_2, x_3) to (z_1, z_2, z_3) is given by

$$Z = BY = B(AX) = (BA)X.$$

(2) Orthogonal transformation. *The linear transformation (iv), i.e., $Y = AX$, is said to be* ***orthogonal*** *if, it transforms*

$$y_1^2 + y_2^2 + ... + y_n^2 \text{ into } x_1^2 + x_2^2 + ... + x_n^2$$

The matrix of an orthogonal transformation is called an **orthogonal matrix.**

We have $$X'X = [x_1 x_2 ......x_n] \times \begin{bmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{bmatrix} = x_1^2 + x_2^2 + ... + x_n^2$$

and similarly, $$Y'Y = y_1^2 + y_2^2 + ... + y_n^2.$$

$\therefore$ If $Y = AX$ is an orthogonal transformation, then

$$X'X = Y'Y = (AX)'(AX) = X'A'AX \text{ which is possible only if } A'A = I.$$

But $A^{-1}A = I$, therefore, $A' = A^{-1}$ for an orthogonal transformation.

Hence a *square matrix A is said to be orthogonal if $AA' = A'A = I$.*

Obs. 1. *If A is orthogonal, A' and A^{-1} are also orthogonal.*

Since A is orthogonal, $A' = A^{-1}$.

$\therefore$ $(A')' = (A^{-1})' = (A')^{-1}$, *i.e.*, $B' = B^{-1}$ where $B = A'$

Hence B (*i.e.*, A') is orthogonal. As $A' = A^{-1}$, A^{-1} is also orthogonal.

Obs. 2. *If A is orthogonal, then $|A| = \pm 1$.*

Since $AA' = A'A = I$ $\therefore$ $|A|\,|A'| = |I|$ *(Mumbai, 2006)*

But $|A'| = |A|$, $\therefore$ $|A|\,|A| = |1|$

or $|A|^2 = 1$ *i.e.*, $|A| = \pm 1$.

Example 2.38. *Show that the transformation*

$$y_1 = 2x_1 + x_2 + x_3,\ y_2 = x_1 + x_2 + 2x_3,\ y_3 = x_1 - 2x_3$$

is regular. Write down the inverse transformation.

Solution. The given transformation may be written as

$$Y = AX$$

where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix}$

Now $$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{vmatrix} = -1$$

Thus the matrix A is non-singular and hence the transformatiuon is regular.

$\therefore$ The inverse transformation is given by

$$X = A^{-1} Y$$

where $A^{-1} = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix}$

Thus $x_1 = 2y_1 - 2y_2 - y_3$; $x_2 = -4y_1 + 5y_2 + 3y_3$; $x_3 = y_1 - y_2 - y_3$ is the inverse transformation.

Example 2.39. *Prove that the following matrix is orthogonal :*

$$\begin{bmatrix} -2/3 & 1/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix}$$ *(Kurukshetra, 2005)*

Solution. We have $AA' = \begin{bmatrix} -2/3 & 1/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix} \times \begin{bmatrix} -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \end{bmatrix}$

$$= \begin{bmatrix} 4/9 + 1/9 + 4/9 & -4/9 + 2/9 + 2/9 & -2/9 - 2/9 + 4/9 \\ -4/9 + 2/9 + 2/9 & 4/9 + 4/9 + 1/9 & 2/9 - 4/9 + 2/9 \\ -2/9 - 2/9 + 4/9 & 2/9 - 4/9 + 2/9 & 1/9 + 4/9 + 4/9 \end{bmatrix} = I.$$

Hence the matrix is orthogonal.

Example 2.40. *If* $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & b \\ 2 & -2 & c \end{bmatrix}$ *is orthogonal, find a, b, c and* A^{-1}. *(Mumbai, 2006)*

Solution. As A is orthogonal, $AA' = I$

$$\therefore \qquad \frac{1}{3}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & b \\ 2 & -2 & c \end{bmatrix} \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & b & c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or
$$\begin{bmatrix} 1 + 4 + a^2 & 2 + 2 + ab & 2 - 4 + ac \\ 2 + 2 + ab & 4 + 1 + b^2 & 4 - 2 + bc \\ 2 - 4 + ac & 4 - 2 + bc & 4 + 4 + c^2 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$\therefore \qquad 5 + a^2 = 9,\ 5 + b^2 = 9,\ 8 + c^2 = 9, \quad i.e.,\ a^2 = 4,\ b^2 = 4,\ c^2 = 1$$

Thus $a = 2, b = 2, c = 1$.

Since A is orthogonal, $A^{-1} = A' = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$.

2.12 (1) VECTORS

Any quantity having n-components is called a *vector of order n*. Therefore, the coefficients in a linear equation or the elements in a row or column matrix will form a vector. Thus any n numbers $x_1, x_2, \ldots, x_n$ written in a particular order, constitute a vector $\mathbf{x}$.

(2) Linear dependence. *The vectors* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ *are said to be* **linearly dependent,** *if there exist numbers* $\lambda_1, \lambda_2, \ldots, \lambda_r$ *not all zero, such that*

$$\lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \ldots + \lambda_r\mathbf{x}_r = \mathbf{0}. \qquad \ldots(i)$$

If no such numbers, other than zero, exist, the vectors are said to be **linearly independent.** If $\lambda_1 \neq 0$, transposing $\lambda_1\mathbf{x}_1$ to the other side and dividing by $-\lambda_1$, we write (i) in the form

$\mathbf{x}_1 = \mu_2\mathbf{x}_2 + \mu_3\mathbf{x}_3 + \ldots + \mu_r\mathbf{x}_r$.

Then the vector $\mathbf{x}_1$ is said to be a linear combination of the vectors $\mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_r$.

Example 2.41. *Are the vectors* $\mathbf{x}_1 = (1, 3, 4, 2)$, $\mathbf{x}_2 = (3, -5, 2, 2)$ *and* $\mathbf{x}_3 = (2, -1, 3, 2)$ *linearly dependent ? If so express one of these as a linear combination of the others.*

Solution. The relation $\qquad \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \lambda_3\mathbf{x}_3 = \mathbf{0}.$

i.e., $\qquad \lambda_1(1, 3, 4, 2) + \lambda_2(3, -5, 2, 2) + \lambda_3(2, -1, 3, 2) = \mathbf{0}$

is equivalent to $\lambda_1 + 3\lambda_2 + 2\lambda_3 = 0,\ 3\lambda_1 - 5\lambda_2 - \lambda_3 = 0,$

$$4\lambda_1 + 2\lambda_2 + 3\lambda_3 = 0,\ 2\lambda_1 + 2\lambda_2 + 2\lambda_3 = 0$$

As these are satisfied by the values $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = -2$ which are not zero, the given vectors are linearly dependent. Also we have the relation,

$$\mathbf{x}_1 + \mathbf{x}_2 - 2\mathbf{x}_3 = 0$$

by means of which any of the given vectors can be expressed as a linear combination of the others.

Obs. Applying elementary row operations to the vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$, we see that the matrices

$$A = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_1 + \mathbf{x}_2 - 2\mathbf{x}_3 \end{bmatrix}$$

have the same rank. The rank of B being 2, the rank of A is also 2. Moreover $\mathbf{x}_1, \mathbf{x}_2$ are linearly independent and $\mathbf{x}_3$ can be expressed as a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$ $\left[\because\ \mathbf{x}_3 = \frac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2)\right]$. Similar results will hold for column operations and for any matrix. In general, we have the following results :

If a given matrix has r linearly independent vectors (rows or columns) and the remaining vectors are linear combinations of these r vectors, then rank of the matrix is r. Conversely, if a matrix is of rank r, it contains r linearly independent vectors are remaining vectors (if any) can be expressed as a linear combination of these vectors.

PROBLEMS 2.8

1. Represent each of the transformations
$$x_1 = 3y_1 + 2y_2, y_1 = z_1 + 2z_2 \text{ and } x_2 = -y_1 + 4y_2, y_2 = 3z_1$$
by the use of matrices and find the composite transformation which express x_1, x_2 in terms of z_1, z_2.
2. If $\xi = x \cos \alpha - y \sin \alpha$, $\eta = x \sin \alpha + y \cos \alpha$, write the matrix A of transformation and prove that $A^{-1} = A'$. Hence write the inverse transformation.
3. A transformation from the variables x_1, x_2, x_3 to y_1, y_2, y_3 is given by $Y = AX$, and another transformation from y_1, y_2, y_3 to z_1, z_2, z_3 is given by $Z = BY$, where
$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -2 \\ -1 & 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix}.$$ Obtain the transformation from x_1, x_2, x_3 to z_1, z_2, z_3.
4. Find the inverse transformation of $y_1 = x_1 + 2x_2 + 5x_3$; $y_2 = 2x_1 + 4x_2 + 11x_3$; $y_3 = -x_2 + 2x_3$.
5. Verify that the following matrix is orthogonal :

(*i*) $\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$ (*Hissar, 2005 S ; P.T.U., 2003*) (*ii*) $\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$ (*Kurukshetra, 2005*)

6. Find the values of a, b, c if $A = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$ is orthogonal ? (*Mumbai, 2005 S*)
7. Prove that $\begin{bmatrix} l & m & n & 0 \\ 0 & 0 & 0 & -1 \\ n & l & -m & 0 \\ -m & n & -l & 0 \end{bmatrix}$ is orthogonal when $l = 2/7, m = 3/7, n = 6/7$.
8. If A and B are orthogonal matrices, prove that AB is also orthogonal. (*Anna, 2005*)
9. Are the following vectors linearly dependent. If so, find the relation between them :
(*i*) (2, 1, 1), (2, 0, – 1), (4, 2, 1). (*Mumbai, 2009*)
(*ii*) (1, 1, 1, 3), (1, 2, 3, 4), (2, 3, 4, 9).
(*iii*) $\mathbf{x}_1 = (1, 2, 4), \mathbf{x}_2 = (2, -1, 3), \mathbf{x}_3 = (0, 1, 2), \mathbf{x}_4 = (-3, 7, 2)$. (*U.P.T.U., 2003 ; Nagpur, 2001*)

2.13 (1) EIGEN VALUES

If A is any square matrix of order n, we can form the matrix $A - \lambda I$, where I is the nth order unit matrix. The determinant of this matrix equated to zero,

i.e.,
$$|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix} = 0$$

is called the *characteristic equation of A*. On expanding the determinant, the characteristic equation takes the form

$$(-1)^n \lambda^n + k_1\lambda^{n-1} + k_2\lambda^{n-2} + \dots + k_n = 0,$$

where k's are expressible in terms of the elements a_{ij}. The roots of this equation are called the *eigenvalues* or *latent roots* or *characteristic roots* of the matrix A.

(2) Eigen vectors

If $X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$ and $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$, then the linear transformation $Y = AX$ ...(*i*)

carries the column vector X into the column vector Y by means of the square-matrix A. In practice, it is often required to find such vectors which transform into themselves or to a scalar multiple of themselves.

Let X be such a vector which transforms into λX by means of the transformation (*i*).

Then $\lambda X = AX$ or $AX - \lambda IX = 0$ or $[A - \lambda I]X = 0$ ...(*ii*)

This matrix equation represents n homogeneous linear equations

$$\left.\begin{array}{l} (a_{11}-\lambda)x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + (a_{22}-\lambda)x_2 + \dots + a_{2n}x_n = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + (a_{nn}-\lambda)x_n = 0 \end{array}\right\} \quad \dots(iii)$$

which will have a non-trivial solution only if the coefficient matrix is singular, *i.e.*, if $|A - \lambda I| = 0$.

This is called the characteristic equation of the transformation and is same as the characteristic equation of the matrix A. It has n roots and corresponding to each root, the equation (*ii*) [or (*iii*)] will have a non-zero solution.

$X = [x_1, x_2, \dots\dots, x_n]'$, which is known as the *eigen vector* or *latent vector*.

Obs. 1. Corresponding to n distinct eigen values, we get n independent eigen vectors. But when two or more eigen values are equal, it may or may not be possible to get linearly independent eigen vectors corresponding to the repeated roots.

Obs. 2. If X_i is a solution for a eigen value λ_i then it follows from (*ii*) that cX_i is also a solution, where c is arbitrary constant. Thus *the eigen vector corresponding to a eigen value is not unique* but may be any one of the vectors cX_i.

Example 2.42. *Find the eigen values and eigen vectors of the matrix* $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$. (*Bhopal, 2008*)

Solution. The characteristic equation is $|A - \lambda I| = 0$

i.e., $\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$ or $\lambda^2 - 7\lambda + 6 = 0$

or $(\lambda - 6)(\lambda - 1) = 0$ $\therefore$ $\lambda = 6, 1.$

Thus the eigen values are 6 and 1.

If x, y be the components of an eigen vector corresponding to the eigen value λ, then

$$[A - \lambda I]X = \begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

Corresponding to $\lambda = 6$, we have $\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$

which gives only one independent equation $-x + 4y = 0$

$\therefore$ $\dfrac{x}{4} = \dfrac{y}{1}$ giving the eigen vector (4, 1).

Corresponding to $\lambda = 1$, we have $\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

which gives only one independent equation $x + y = 0$.

$\therefore \quad \frac{x}{1} = \frac{y}{-1}$ giving the eigen vector $(1, -1)$.

Example 2.43. *Find the eigen values and eigen vectors of the matrix* $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

(*Bhopal, 2009 ; Raipur, 2005*)

Solution. The characteristic equation is $| A - \lambda I | = \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix}$, *i.e.*, $\lambda^3 - 7\lambda^2 + 36 = 0$

Since $\lambda = -2$ satifies it, we can write this equation as

$$(\lambda + 2)(\lambda^2 - 9\lambda + 18) = 0 \quad \text{or} \quad (\lambda + 2)(\lambda - 3)(\lambda - 6) = 0.$$

Thus the eigen values of A are $\lambda = -2, 3, 6$.

If $x, y\ z$ be the components of an eigen vector corresponding to the eigen value λ, we have

$$[A - \lambda I]X = \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \qquad \text{...}(i)$$

Putting $\lambda = -2$, we have $3x + y + 3z = 0$, $x + 7y + z = 0$, $3x + y + 3z = 0$.

The first and third equations being the same, we have from the first two

$$\frac{x}{-20} = \frac{y}{0} = \frac{z}{20} \quad \text{or} \quad \frac{x}{-1} = \frac{y}{0} = \frac{z}{1}$$

Hence the eigen vector is $(-1, 0, 1)$. Also every non-zero multiple of this vector is an eigen vector corresponding to $\lambda = -2$.

Similarly, the eigen vectors corresponding to $\lambda = 3$ and $\lambda = 6$ are the arbitrary non-zero multiples of the vectors $(1, -1, 1)$ and $(1, 2, 1)$ which are obtained from (i).

Hence the three eigen vectors may be taken as $(-1, 0, 1)$, $(1, -1, 1)$, $(1, 2, 1)$.

Example 2.44. *Find the eigen values and eigen vectors of the matrix* $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ (*U.P.T.U., 2005*)

Solution. The characteristic equation is

$$[A - \lambda I] = 0, \quad i.e., \quad \begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix} = 0$$

or $$(3 - \lambda)(2 - \lambda)(5 - \lambda) = 0$$

Thus the eigen values of A are 2, 3, 5.

If x, y, z be the components of an eigen vector corresponding to the eigen value λ, we have

$$[A - \lambda I]X = \begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Putting $\lambda = 2$, we have $x + y + 4z = 0$, $6z = 0$, $3z = 0$, *i.e.*, $x + y = 0$ and $z = 0$.

$$\therefore \qquad \frac{x}{1} = \frac{y}{-1} = \frac{z}{0} = k_1 \text{ (say)}$$

Hence the eigen vector corresponding to $\lambda = 2$ is $k_1(1, -1, 0)$.

Putting $\lambda = 3$, we have $y + 4z = 0$, $-y + 6z = 0$, $2z = 0$, *i.e.*, $y = 0$, $z = 0$.

$$\therefore \qquad \frac{x}{1} = \frac{y}{0} = \frac{z}{0} = k_2$$

Hence the eigen vector corresponding to $\lambda = 3$ is k_2 (1, 0, 0).

Similarly, the eigen vector corresponding to $\lambda = 5$ is k_3 (3, 2, 1).

2.14 PROPERTIES OF EIGEN VALUES

I. Any square matrix A and its transpose A′ have the same eigen values.

We have $$(A - \lambda I)' = A' - \lambda I' = A' - \lambda I$$
$$|(A - \lambda I)'| = |A' - \lambda I|$$
$$|A - \lambda I| = |A' - \lambda I| \qquad [\because |B'| = |B|$$
$\therefore$ $|A - \lambda I| = 0$ if and only if $|A' - \lambda I| = 0$

i.e., λ is an eigen value of A if and only if it is an eigen value of A'.

II. The eigen values of a triangular matrix are just the diagonal elements of the matrix.

Let $$A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix}$$ be a triangular matrix of order n.

Then $|A - \lambda I| = (a_{11} - \lambda)(a_{22} - \lambda) \ldots\ldots (a_{nn} - \lambda)$.

$\therefore$ Roots of $|A - \lambda I| = 0$ are $\lambda = a_{11}, a_{22}, \ldots, a_{nn}$.

Hence the eigen values of A are the diagonal elements of A, *i.e.*, $a_{11}, a_{22}, \ldots, a_{nn}$.

Cor. *The eigen values of a diagonal matrix are just the diagonal elements of the matrix.*

III. The eigen values of an idempotent matrix are either zero or unity.

Let A be an idempotent matrix so that $A^2 = A$. If λ be an eigen value of A, then there exists a non-zero vector X such that

$$AX = \lambda X \qquad \ldots(1)$$

$\therefore$ $A(AX) = A(\lambda X)$, *i.e.*, $A^2X = \lambda(AX)$

i.e. $$AX = \lambda(\lambda X) \qquad [\because A^2 = A \text{ and } AX = \lambda X$$

$\therefore$ $$AX = \lambda^2 X \qquad \ldots(2)$$

From (1) and (2), we get $\lambda^2 X = \lambda X$ or $(\lambda^2 - \lambda) X = 0$

or $$\lambda^2 - \lambda = 0 \text{ whence } \lambda = 0 \text{ or } 1.$$

Hence the result.

IV. The sum of the eigen values of a matrix is the sum of the elements of the principal diagonal.

[This property will be proved for a matrix of order 3, but the method will be capable of easy extension to matrices of any order.]

Consider the square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \ldots(i)$$

so that $$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} \qquad \text{(On expanding)}$$
$$= -\lambda^3 + \lambda^2(a_{11} + a_{22} + a_{33}) - \ldots \qquad \ldots(ii)$$

If $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of A, then $|A - \lambda I| = (-1)^3 (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$
$$= -\lambda^3 + \lambda^2(\lambda_1 + \lambda_2 + \lambda_3) - \ldots \qquad \ldots(iii)$$

Equating the right hand sides of (*ii*) and (*iii*) and comparing coefficients of λ^2, we get

$$\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}. \text{ Hence the result.}$$

V. The product of the eigen values of a matrix A is equal to its determinant.

Putting $\lambda = 0$ in (*iii*), we get the result.

VI. If λ is an eigen value of a matrix A, then $1/\lambda$ is the eigen value of A^{-1}.

If X be the eigen vector corresponding to λ, then $AX = \lambda X$ $\qquad \ldots(i)$

Premultiplying both sides by A^{-1}, we get $A^{-1}AX = A^{-1}\lambda X$

i.e., $$IX = \lambda A^{-1}X \quad \text{or} \quad X = \lambda(A^{-1}X), \quad i.e., \quad A^{-1}X = (1/\lambda)X$$

This being of the same form as (*i*), shows that $1/\lambda$ is an eigen value of the inverse matrix A^{-1}.

VII. If λ is an eigen value of an orthogonal matrix, then $1/\lambda$ is also its eigen value.

We know that if λ is an eigen value of a matrix A, then $1/\lambda$ is an eigen value of A^{-1}. [Property V]. Since A is an orthogonal matrix, A^{-1} is same as its transpose A'.

$\therefore$ $1/\lambda$ is an eigen value of A'.

But the matrices A and A' have the same eigen values, since the determinants $|A - \lambda I|$ and $|A' - \lambda I|$ are the same.

Hence $1/\lambda$ is also an eigen value of A

VIII. If $\lambda_1, \lambda_2 \ldots, \lambda_n$ are the eigen values of a matrix A, then A^m has the eigen values $\lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m$ (m being a positive integer). (*Mumbai, 2006*)

Let A_i be the eigen value of A and X_i the corresponding eigen vector. Then

$$AX_i = \lambda_i X_i \quad \ldots(i)$$

We have $$A^2X_i = A(AX_i) = A(\lambda_i X_i) = \lambda_i(AX_i) = \lambda(\lambda_i X_i) = \lambda_i^2 X_i$$

Similarly, $A^3X_i = \lambda_i^3 X_i$. In general, $A^mX_i = \lambda_i^m X_i$ which is of the same form as (*i*).

Hence λ_i^m is an eigen value of A^m.

The corresponding eigen vector is the same X_i.

2.15 CAYLEY-HAMILTON THEOREM*

Every square matrix satisfies its own characteristic equation; *i.e.*, if the characteristic equation for the nth order square matrix A is

$$|A - \lambda I| = (-1)^n \lambda^n + k_1\lambda^{n-1} + \ldots + k_n = 0 \quad \ldots(i)$$

then $$(-1)^n A^n + k_1A^{n-1} + \ldots + k_n = 0.$$

Let the adjoint of the matrix $A - \lambda I$ be P. Clearly, the elements of P will be polynomials of the $(n-1)$th degree in λ, for the cofactors of the elements in $|A - \lambda I|$ will be such polynomials.

$\therefore$ P can be split up into a number of matrices, containing terms with the same powers of λ, such that

$$P = P_1\lambda^{n-1} + P_2\lambda^{n-2} + \ldots + P_{n-1}\lambda + P_n \quad \ldots(ii)$$

where $P_1, P_2, \ldots, P_n$ are all the square matrices of order n whose elements are functions of the elements of A.

Since the product of a matrix by its adjoint = determinant of the matrix × unit matrix.

$\therefore$ $$[A - \lambda I]P = |A - \lambda I| \times I$$

$\therefore$ by (*i*) and (*ii*), $$[A - \lambda I][P_1\lambda^{n-1} + P_2\lambda^{n-2} + \ldots + P_{n-1}\lambda + P_n]$$
$$= [(-1)^n\lambda^n + k_1\lambda^{n-1} + \ldots + k_{n-1}\lambda + k_n]I.$$

Equating the coefficients of various powers of λ, we get

$$-P_1 = (-1)^n I \qquad [\because IP_1 = P_1]$$
$$AP_1 - P_2 = k_1 I,$$
$$AP_2 - P_3 = k_2 I,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$AP_{n-1} - P_n = k_{n-1}I,$$
$$AP_n = k_n I.$$

Now pre-multiplying the equations by $A^n, A^{n-1}, \ldots, A, I$ respectively and adding, we get

$$(-1)^n A^n + k_1A^{n-1} + \ldots + k_{n-1}A + k_n I = 0, \quad \ldots(iii)$$

for the terms on the left cancel in pairs. This proves the theorem.

Cor. Another method of finding the inverse.

Multiplying (*iii*) by A^{-1}, we get

$$(-1)^n A^{n-1} + k_1A^{n-2} + \ldots + k_{n-1}I + k_nA^{-1} = 0$$

whence $$A^{-1} = -\frac{1}{k_n}[(-1)^nA^{n-1} + k_1A^{n-2} + \ldots + k_{n-1}I].$$

*See footnote on p.17. *William Rowan Hamilton* (1805–1865) an Irish mathematician who is known for his work in dynamics.

This result gives the inverse of A in terms of $\overline{n-1}$ *powers of A and is considered as a practical method for the computation of the inverse of the large matrices.* As a by-product of the computation, the characteristic equation and the determinant of the matrix are also obtained.

Example 2.45. *Verify Cayley-Hamilton theorem for the matrix* $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ *and find its inverse. Also express* $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ *as a linear polynomial in A.* *(Bhopal, 2009)*

Solution. The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda^2 - 4\lambda - 5 = 0 \qquad ...(i)$$

By Cayley-Hamilton theorem, A must satisfy its characteristic equation (i), so that

$$A^2 - 4A - 5I = 0 \qquad ...(ii)$$

Now $$A^2 - 4A - 5I = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

This verifies the theorem.

Multiplying (ii) by A^{-1}, we get $A - 4I - 5A^{-1} = 0$

or $$A^{-1} = \frac{1}{5}(A - 4I) = \frac{1}{5}\left\{\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\} = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$$

Now dividing the polynomial $\lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10I$ by the polynomial $\lambda^2 - 4\lambda - 5$, we obtain

$$\lambda^5 - 4\lambda^4 - 7\lambda^3 - \lambda - 10I = (\lambda^2 - 4\lambda - 5)(\lambda^3 - 2\lambda + 3) + \lambda + 5$$

$$= \lambda + 5 \qquad \text{[By } (i)\text{]}$$

Hence $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I = A + 5$, which is a linear polynomial in A.

Example 2.46. *Find the characteristic equation of the matrix* $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ *and hence find its inverse.*

Solution. The characteristic equation is $\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 3-\lambda & -3 \\ -2 & -4 & -4-\lambda \end{vmatrix} = 0$, *i.e.*, $\lambda^3 - 20\lambda + 8 = 0$.

By Cayley-Hamilton theorem, $A^3 - 20A + 8I = 0$, whence $A^{-1} = \frac{5}{2}I - \frac{1}{8}A^2$,

$$= \frac{5}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{8}\begin{bmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix} \qquad \text{[cf. Ex. 2.21}$$

Example 2.47. *Find the characteristic equation of the matrix,* $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ *and hence compute* A^{-1}. *(U.T.U., 2010)*

Also find the matrix represented by

$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$. *(Anna, 2009 ; Rajasthan, 2005 ; U.P.T.U., 2003)*

Solution. The characteristic equation of the matrix A is

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0 \quad \text{or} \quad [\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0]$$

According to Cayley-Hamilton theorem, we have $A^3 - 5A^2 + 7A - 3I = 0$ ...(*i*)

Multiplying (*i*) by A^{-1}, we get

$$A^2 - 5A + 7I - 3A^{-1} = 0 \quad \text{or} \quad A^{-1} = \frac{1}{3}[A^2 - 5A + 7I] \qquad ...(ii)$$

But $$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4+0+1 & 2+1+1 & 2+0+2 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ 2+0+2 & 1+1+2 & 1+0+4 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$\therefore \quad A^2 - 5A + 7I = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} - 5\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

Hence from (*ii*), $A^{-1} = \dfrac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$

Now $$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$$
$$= A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 7A - 3I) + A^2 + A + I$$
$$= A^2 + A + I \qquad [\because \; A^3 - 5A^2 + 7A - 3I = 0]$$
$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}.$$

PROBLEMS 2.9

1. Find the sum and product of the eigen values of $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$. (*Madras, 2006*)

2. Find the eigen values and eigen vectors of the matrices :
   (*a*) $\begin{bmatrix} 4 & 3 \\ 2 & 9 \end{bmatrix}$ (*W.B.T.U., 2005*) (*b*) $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$ (*Bhopal, 2002 S*)

3. Find the latent roots and the latent vectors of the matrices :
   (*a*) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ (*Bhopal, 2008 ; Nagarjuna, 2008 ; S.V.T.U., 2008 ; J.N.T.U., 2006*)
   (*b*) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ (*J.N.T.U., 2005 ; Kurukshetra, 2005*)
   (*c*) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ (*Mumbai, 2006 ; B.P.T.U., 2006 ; U.P.T.U., 2006*)
   (*d*) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ (*Kurukshetra, 2006*) (*e*) $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$ (*Madras, 2006*)

4. If λ be an eigen value of a non-singular matrix A, show that $|A|/\lambda$ is an eigen value of the matrix *adj A*. (*U.P.T.U., 2001*)

5. Find the eigen values of adj A and of $A^2 - 2A + I$, where $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ (*Mumbai, 2006*)

6. Two eigen values of the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are = 1 each. Find the eigen values of A^{-1}.

7. Show that if $\lambda_1, \lambda_2, ..., \lambda_n$ are the latent roots of a matrix A, then A^2 has the latent roots $\lambda_1^2, \lambda_2^2, ... \lambda_n^2$. (*P.T.U., 2005*)

8. For a symmetrical square matrix, show that the eigen vectors corresponding to two unequal eigen values are orthogonal.
9. Using Cayley-Hamilton theorem, find the inverse of

(i) $\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ *(Osmania, 2000 S)*

(iii) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ 2 & -4 & -4 \end{bmatrix}$ *(Bhopal, 2002 S)* (iv) $\begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ *(U.P.T.U., 2006)*

10. Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$. Show that the equation is satisfied by A and hence obtain the inverse of the given matrix. *(Bhopal, 2008 ; Anna, 2005 ; Kerala, 2005)*
11. Verify Cayley-Hamilton theorem for the matrix A and find its inverse.

(i) $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ *(Anna, 2009 ; S.V.T.U., 2008 ; Madras, 2006)*

(ii) $\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$ *(Coimbatore, 2001)* (iii) $\begin{bmatrix} 3 & 2 & 4 \\ 4 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix}$ *(P.T.U., 2006)*

12. Using Cayley-Hamilton theorem, find A^8, if $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$. *(Anna, 2003)*
13. If $A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, find A^4. *(Madras, 2006)*
14. Using Cayley-Hamilton theorem, find A^{-2}, where $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. *(Bhopal, 2008)*
15. If $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$, evaluate A^{-1}, A^{-2} and A^{-3}.
16. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, show that $A^n = A^{n-2} + A^2 - 1$. Hence find A^{50}. *(Mumbai, 2006)*

2.16 (1) REDUCTION TO DIAGONAL FORM

If a square matrix A of order n has n linearly independent eigen vectors, then a matrix P can be found such that $P^{-1}AP$ is a diagonal matrix.

[This result will be proved for a square matrix of order 3 but the method will be capable of easy extension to matrices of any order.]

Let A be a square matrix of order 3. Let $\lambda_1, \lambda_2, \lambda_3$ be its eigen values and

$$X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \text{ and } X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} \text{ be the corresponding eigen vectors.}$$

Denoting the square matrix $[X_1 X_2 X_3] = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$ by P, we have

$$AP = A[X_1 X_2 X_3] = [AX_1, AX_2, AX_3] = [\lambda_1 X_1, \lambda_2 X_2, \lambda_3 X_3]$$

$$= \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \lambda_3 x_3 \\ \lambda_1 y_1 & \lambda_2 y_2 & \lambda_3 y_3 \\ \lambda_1 z_1 & \lambda_2 z_2 & \lambda_3 z_3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \times \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = PD, \text{ where } D \text{ is the diagonal matrix.}$$

$\therefore$ $P^{-1} AP = P^{-1} PD = D$, which proves the theorem.

Obs. 1. *The matrix P which diagonalises A is called the* ***modal matrix*** *of A and the resulting diagonal matrix D is known as the* ***spectral matrix*** *of A.*

2. *The diagonal matrix has the eigen values of A as its diagonal elements.*

3. *The matrix P, which diagonalise A, constitutes the eign vectors of A.*

(2) Similarity of matrices. *A square matrix* $\hat{A}$ *of order n is called* **similar** *to a square matix A of order n if*

$$\hat{A} = P^{-1}AP \text{ for some non-singular } n \times n \text{ matrix } P.$$

This transformation of a matrix A by a non-singular matrix P to $\hat{A}$ is called a **similarity transformation.**

Obs. *If the matrix* $\hat{A}$ *is similar to the matrix A, then* $\hat{A}$ *has the same eigen values as A.*

If $\mathbf{x}$ *is an eigen vector of A, then* $y = P^{-1}\mathbf{x}$ *is an eigen vector of* $\hat{A}$ *corresponding to the same eigen value.*

(3) Powers of a matrix. *Diagonalisation of a matrix is quite useful for obtaining powers of a matrix.*

Let A be the square matrix. Then a non-singular matrix P can be found such that

$$D = P^{-1} AP$$

$\therefore$ $$D^2 = (P^{-1}AP)(P^{-1}AP) = P^{-1}A^2P \qquad [\because PP^{-1} = I]$$

Similarly, $$D^3 = P^{-1}A^3P \text{ and in general, } D^n = P^{-1}A^nP \qquad ...(i)$$

To obtain A^n, premultiply (i) by P and post-multiply by P^{-1}.

Then $$PD^nP^{-1} = PP^{-1}A^n PP^{-1} = A^n \text{ which gives } A^n.$$

Thus, $$A^n = PD^nP^{-1} \text{ where, } D^n = \begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix}$$

Working procedure :

1. Find the eigen values of the square matrix A.
2. Find the corresponding eigen vectors and write the modal matrix P.
3. Find the diagonal matrix D from $D = P^{-1}AP$
4. Obtain A^n from $A^n = PD^nP^{-1}$.

Example 2.48. *Reduce the matrix* $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ *to the diagonal form.*

(V.T.U., 2011 ; U.T.U., 2010 ; Bhopal, 2009 ; U.P.T.U., 2006)

Solution. The characteristic equation of A is

$$\begin{vmatrix} -1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda^3 - \lambda^2 - 5\lambda + 5 = 0.$$

Solving, we get $\lambda_1 = 1, \lambda_2 = \sqrt{5}, \lambda_3 = -\sqrt{5}$ as the eigen values of A.

When $\lambda = 1$, the corresponding eigen vector is given by

$$-2x + 2y - 2z = 0, x + y + z = 0, -x - y - z = 0$$

Solving the first two equations, we get $\dfrac{x}{2} = \dfrac{y}{0} = \dfrac{z}{-2}$ giving the eigen vector $(1, 0, -1)$

When $\lambda = \sqrt{5}$, the corresponding eigen vector is given by

$$(-1-\sqrt{5})x + 2y - 2z = 0, x + (2-\sqrt{5})y + z = 0, -x - y - \sqrt{5}z = 0.$$

Solving 2nd and 3rd equations, we get

$$\frac{x}{6-2\sqrt{5}}=\frac{y}{-1+\sqrt{5}}=\frac{z}{1-\sqrt{5}} \quad \text{or} \quad \frac{x}{\sqrt{5}-1}=\frac{y}{1}=\frac{z}{-1}$$

giving the eigen vector $(\sqrt{5}-1, 1, -1)$.

Similarly the eigen vector corresponding to $\lambda = -\sqrt{5}$, is $(\sqrt{5}+1, -1, 1)$.

Writing the three eigen vectors as the three columns, we get the transformation (*modal*) matrix as

$$P=\begin{bmatrix}1 & \sqrt{5}-1 & \sqrt{5}+1\\ 0 & 1 & -1\\ 1 & -1 & 1\end{bmatrix}.$$

Hence the diagonal matrix is

$$D=\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & \sqrt{5} & 0\\ 0 & 0 & -\sqrt{5}\end{bmatrix}.$$

Example 2.49. *Find the matrix P which transforms the matrix* $A=\begin{bmatrix}1 & 1 & 3\\ 1 & 5 & 1\\ 3 & 1 & 1\end{bmatrix}$ *to the diagonal form. Hence calculate* A^4.

Solution. The eigen values of A (found in Ex. 2.43) are $-2, 3, 6$ and the eigen vectors are $(-1, 0, 1)$, $(1, -1, 1)$, $(1, 2, 1)$. Writing these eigen vectors as the three columns, the required transformation matrix (*modal matrix*) is

$$P=\begin{bmatrix}-1 & 1 & 1\\ 0 & -1 & 2\\ 1 & 1 & 1\end{bmatrix}$$

To find P^{-1},

$$|P|=\begin{vmatrix}-1 & 1 & 1\\ 0 & -1 & 2\\ 1 & 1 & 1\end{vmatrix}=\begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix} \quad \text{(say)}$$

$$A_1=-3, B_1=2, C_1=1, A_2=0, B_2=-2, C_2=2, A_3=3, B_3=2, C_3=1$$

Also $\quad |P| = a_1A_1 + b_1B_1 + c_1C_1 = 6$

$$\therefore \quad P^{-1}=\frac{1}{|P|}\begin{bmatrix}A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\\ C_1 & C_2 & C_3\end{bmatrix}=\frac{1}{6}\begin{bmatrix}-3 & 0 & 3\\ 2 & -2 & 2\\ 1 & 2 & 1\end{bmatrix}$$

Thus

$$D=P^{-1}AP=\begin{bmatrix}-2 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 6\end{bmatrix}$$

$$\therefore \quad D^4=\begin{bmatrix}(-2)^4 & 0 & 0\\ 0 & 3^4 & 0\\ 0 & 0 & 6^4\end{bmatrix}=\begin{bmatrix}16 & 0 & 0\\ 0 & 81 & 0\\ 0 & 0 & 1296\end{bmatrix}$$

Hence

$$A^4=PD^4P^{-1}=\frac{1}{6}\begin{bmatrix}-1 & 1 & 1\\ 0 & -1 & 2\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}16 & 0 & 0\\ 0 & 81 & 0\\ 0 & 0 & 1296\end{bmatrix}\begin{bmatrix}-3 & 0 & 3\\ 2 & -2 & 2\\ 1 & 2 & 1\end{bmatrix}$$

$$=\begin{bmatrix}-1 & 1 & 1\\ 0 & -1 & 2\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}-8 & 0 & 8\\ 27 & -27 & 27\\ 216 & 512 & 216\end{bmatrix}=\begin{bmatrix}251 & 485 & 235\\ 485 & 1051 & 485\\ 235 & 485 & 251\end{bmatrix}$$

Example 2.50. *Find e^A and 4^A if* $A = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix}$. *(Mumbai, 2006)*

Solution. The characteristic equation of A is

$$\begin{vmatrix} 3/2-\lambda & 1/2 \\ 1/2 & 3/2-\lambda \end{vmatrix} = 0, \quad i.e., (3/2-\lambda)^2 - 1/4 = 0.$$

$$\therefore \qquad \lambda^2 - 3\lambda + 2 = 0 \quad \text{whence } \lambda = 1, 2.$$

When $\lambda = 1$, $[A - \lambda I]\, X = 0$, gives

$$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad [\text{By } 2R_1, 2R_2]$$

or
$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad [\text{By } R_2 - R_1]$$

$\therefore$ $x_1 + x_2 = 0$. If $x_2 = -1, x_1 = 1$, *i.e.*, the eigen vector is $[1, -1]'$.

When $\lambda = 2$, $[A - \lambda I]\, X = 0$, gives $\begin{bmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

or
$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} \text{By } 2R_1 \\ 2R_2 \end{bmatrix}$$

or
$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad [\text{By } R_2 - R_1]$$

$$\therefore \qquad -x_1 + x_2 = 0, \quad i.e., \quad x_1 = x_2$$

If $x_2 = 1, x_1 = 1$, *i.e.*, the eigen vector is $[1, 1]'$

Now $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$$\therefore \qquad P^{-1} = \frac{adj\, P}{|P|} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

If
$$f(A) = e^A, f(D) = e^D = \begin{bmatrix} e^1 & 0 \\ 0 & e^2 \end{bmatrix}$$

$$\therefore \qquad e^A = P\, f(D)\, P^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} e^1 & 0 \\ 0 & e^2 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} e & e^2 \\ -e & e^2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e+e^2 & -e+e^2 \\ -e+e^2 & e+e^2 \end{bmatrix}$$

Replacing e by 4, we get

$$4^A = \frac{1}{2}\begin{bmatrix} 20 & 12 \\ 12 & 20 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}.$$

2.17 REDUCTION OF QUADRATIC FORM TO CANONICAL FORM

A homogeneous expression of the second degree in any number of variables is called a *quadratic form.*

For instance, if $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $X' = [x \quad y \quad z]$, then

$$X'AX = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy \qquad ...(i)$$

which is a *quadratic form.*

Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of the matrix A and

$$X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}, X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$$

be its corresponding eigen vectors in the normalized form (*i.e.*, each element is divided by square root of sum of the squares of all the three elements in the eigen vector).

Then by § 2.16(1), $P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ where $P = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$

Hence the *quadratic form (i) is reduced to a* **canonical form (or sum of squares form** or **Principal axes form).**

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2$$

and P is the **matrix of transformation** which is an orthogonal matrix.

Note. Congruent (or orthogonal) transformation. The diagonal matrix D and the matrix A are called *congruent matrices* and *the above method of reduction is called* **congruent (or orthogonal) transformation.**

Remember that the matrix A corresponding to the quadratic form

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$

is $$\begin{bmatrix} \text{coeff. of } x^2 & \frac{1}{2}\text{coeff. of } yz & \frac{1}{2}\text{coeff. of } zx \\ \frac{1}{2}\text{coeff. of } yz & \text{coeff. of } y^2 & \frac{1}{2}\text{coeff. of } xy \\ \frac{1}{2}\text{coeff. of } zx & \frac{1}{2}\text{coeff. of } xy & \text{coeff. of } z^2 \end{bmatrix}, \quad i.e., \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}.$$

Example 2.51. *Reduce the quadratic form $3x^2 + 5y^2 + 3z^2 - 2yz + 2zx - 2xy$ to the canonical form and specify the matrix of transformation.* *(Bhopal, 2009; Kurukshetra, 2006)*

Solution. The matrix of the given quadratic form is $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Its characteristic equation is $|A - \lambda I| = 0$, *i.e.*, $\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$

which gives $\lambda = 2, 3, 6$ as its eigen values. Hence the given quadratic form reduces to the canonical form

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2, \qquad i.e., \qquad 2x^2 + 3y^2 + 6z^2.$$

To find the matrix of transformation

From $[A - \lambda I]\, X = 0$, we obtain the equations

$$(3-\lambda)\,x - y + z = 0\,;\; -x + (5-\lambda)\,y - z = 0\,;\; x - y + (3-\lambda)\,z = 0.$$

Now corresponding to $\lambda = 2$, we get $x - y + z = 0$, $-x + 3y - z = 0$, and $x - y + z = 0$,

whence $$\frac{x}{1} = \frac{y}{0} = \frac{z}{-1}$$

$\therefore$ The eigen vector is X_1 (1, 0, – 1) and its normalised form is $(1/\sqrt{2}, 0, -1/\sqrt{2})$.

Similarly, corresponding to $\lambda = 3$, the eigen vector is X_2 (1, 1, 1) and its normalised form is $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$.

Finally, corresponding to $\lambda = 6$, the eigen vector is X_3 (1, – 2, 1) and its normalised form is $(1/\sqrt{6}, -2/\sqrt{6}, 1/\sqrt{6})$.

Hence the matrix of transformation is $P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$.

2.18 NATURE OF A QUADRATIC FORM

Let $Q = X' AX$ be a quadratic form in n variables $x_1, x_2, \dots x_n$.

Index. *The number of positive terms in its canonical form is called the index of the quadratic form.*

Signature (S) *of the quadratic form is the difference of positive and negative terms in the canonical form. If the rank of the matrix A is r and the signature of the quadratic form Q is s, then the quadratic form is said to be*

(*i*) *positive definite* if $r = n$ and $s = n$
(*ii*) *negative definite* if $r = n$ and $s = 0$
(*iii*) *positive semidefinite* if $r < n$ and $s = r$
(*iv*) *negative semidefinite* if $r < n$ and $s = 0$
(*v*) *indefinite* in all other cases.

In other words *a real quadratic form X′AX in a variable is said to be*

(*i*) **positive definite** *if all the eigen values of A > 0.*
(*ii*) **negative definite** *if all the eigen values of A < 0.*
(*iii*) **positive semidefinite** *if all the eigen values of A ≥ 0 and at least one eigen value = 0.*
(*iv*) **negative semidefinite** *if all the eigen values of A ≤ 0 and at least one eigen value = 0.*
(*v*) **indefinite** *if some of the eigen values of A are positive and others negative.*

Example 2.52. *Reduce the quadratic form $2x_1x_2 + 2x_1x_3 - 2x_2x_3$ to a canonical form by an orthogonal reduction and discuss its nature.* *(Madras, 2006)*
Also find the modal matrix.

Solution. (*i*) The matrix of the given quadratic form is $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$

Its characteristic equation is $[A - \lambda I] = 0$, *i.e.*, $\begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & -1 \\ 1 & -1 & -\lambda \end{bmatrix} = 0$

which gives $\lambda^3 - 3\lambda + 2 = 0$

Solving, we get $\lambda = 1, 1, -2$ as the eigen values. Hence the given quadratic form reduces to the canonical form

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = 0, \quad i.e., \quad x^2 + y^2 - 2z^2 = 0$$

(*ii*) Since some of the eigen values of A are positive and others are negative, the given quadratic form is *Indefinite.*

(*iii*) *To find the matrix of transformation*

From $[A - \lambda I]\, X = 0$, we get the equations

$$-\lambda x + y + z = 0, x - \lambda y + z = 0, x - y - \lambda z = 0$$

When $\lambda = -2$, we get $\quad 2x + y + z = 0, x + 2y - z = 0, x - y + 2z = 0.$

Solving first and second equations, we get

$$\frac{x}{-1} = \frac{y}{1} = \frac{z}{1}$$

$\therefore$ The corresponding eigen vector $X_1 = (-1, 1, 1)$ and its normalised form is $(-1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$

When $\lambda = 1$, we get $\quad -x + y + z = 0, x - y - z = 0, x - y - z = 0.$

These equations are same. Take $y = 0$ so that $x = z$.

$\therefore$ The corresponding eigen vector $X_2 = (1, 0, 1)$ and its normalised form is $(1/\sqrt{2}, 0, 1/\sqrt{2})$

To find the eigen vector $X_3 = (l, m, n)$ (say)

Since X_3 is orthogonal to X_1, $\therefore\ -l + m + n = 0$

Since X_3 is orthogonal to X_2, $\therefore\ l + n = 0$

These equations give $\dfrac{l}{1} = \dfrac{m}{2} = \dfrac{n}{-1}$.

$\therefore$ The eigen vector $X_3 = (1, 2, -1)$ and normalised form is $(1/\sqrt{6}, 2/\sqrt{6}, -1/\sqrt{6})$.

Hence the modal matrix is

$$P = \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}.$$

PROBLEMS 2.10

1. If $A = \begin{bmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $P = \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, show that $P^{-1}AP$ is a diagonal matrix.

2. Show that the linear transformation

   $H = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, where $\theta = \frac{1}{2}\tan^{-1}\frac{2h}{a-b}$, changes the matrix

   $C = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$ to the diagonal form $D = HCH'$.

3. Reduce the matrix $A = \begin{bmatrix} -19 & 7 \\ -42 & 16 \end{bmatrix}$ to the diagonal form. *(B.P.T.U., 2005)*

4. If $A = \begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}$, find A^n and A^4. *(Mumbai, 2006)*

5. If $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$, calculate A^4. *(Coimbatore, 2001)*

6. If $A = \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}$, then prove that $3\tan A = A\tan 3$. *(Mumbai, 2006)*

7. Find the eigen vectors of the matrix $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ and hence reduce $6x^2 + 3y^2 + 3z^2 - 2yz + 4zx - 4xy$ to a 'sum of squares'. Also write the nature of the matrix. *(Calicut, 2005)*

8. Reduce the quadratic form $2xy + 2yz + 2zx$ into canonical form. *(Anna, 2009 ; Kurukshetra, 2006 ; Mumbai, 2003)*

9. (*a*) Find the eigen values, eigen vectors and the modal of matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$.

   (*b*) Reduce the quadratic form $x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3$ to a canonical form. *(Anna, 2009)*

10. Reduce the following quadratic forms into a 'sum of squares' by an orthogonal transformation and give the matrix of transformation. Also state the nature of each of these.

    (*i*) $3x_1^2 + 3x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3$.

    (*ii*) $8x^2 + 7y^2 + 3z^2 - 12xy - 8yz + 4zx$ *(Anna, 2002 S)*

11. Find the index and signature of the quadratic form $x_1^2 + 2x_2^2 - 3x_3^2$. *(Madras, 2006)*

12. Find the nature of the quadratic form $x^2 + 5y^2 + z^2 + 2xy + 2yz + 6zx$. *(Bhopal, 2009)*

13. Show that the form $5x_1^2 + 26x_2^2 + 10x_3^2 + 4x_2x_3 + 14x_3x_1 + 6x_1x_2$ is a positive semi-definite and find a non-zero set of values of x_1, x_2, x_3 which make the form zero. *(P.T.U., 2003)*

2.19 COMPLEX MATRICES

So far, we have considered matrices whose elements were real numbers. The elements of a matrix can, however, be complex numbers also.

(1) Conjugate of a matrix. *If the elements of a matrix $A = [a_{rs}]$ are complex numbers $\alpha_{rs} + i\,\beta_{rs}$, α_{rs} and β_{rs} being real, then the matrix*

$\overline{A} = [\bar{a}_{rs}] = [\alpha_{rs} - i\beta_{rs}]$ *is called the* **conjugate matrix** *of* A.

The transpose of a conjugate of a matrix A is denoted by A^* or A^θ, *i.e.*, $(\overline{A})' = A^*$.

(2) Hermitian matrix. *A square matrix A such that* $A' = \overline{A}$ *is said to be a* **Hermitian matrix*.** The elements of the leading diagonal of a Hermitian matrix are evidently real, while every other element is the complex conjugate of the element in the transposed position. For instance $A = \begin{bmatrix} 2 & 3+4i \\ 3-4i & -5 \end{bmatrix}$ is a Hermitian matrix, since $A' = \begin{bmatrix} 2 & 3-4i \\ 3+4i & -5 \end{bmatrix} = \overline{A}$

(3) Skew-Hermitian matrix. *A square matrix A such that* $A' = -\overline{A}$ *is said to be a* **skew-Hermitian matrix.** This implies that the leading diagonal elements of a skew-Hermitian matrix are either all zeros or all purely imaginary.

Obs. *A Hermitian matrix is a generalisation of a real symmetric matrix* as every real symmetric matrix is Hermitian. Similarly, *a skew-Hermitian matrix is a generalisation of a real skew-symmetric matrix.*

Properties

I. Any square matrix A can be written as the sum of a Hermitian and skew-Hermitian matrices. (*Mumbai, 2007*)

Take $B = \frac{1}{2}(A + \overline{A}')$ and $C = \frac{1}{2}(A - \overline{A}')$

Then $B' = \frac{1}{2}(A + \overline{A}')' = \frac{1}{2}(A' + \overline{A})$

and $\overline{B} = \frac{1}{2}\overline{(A + \overline{A}')} = \frac{1}{2}(\overline{A} + A') = B'$

i.e., B is a Hermitian matrix.

Again, $C' = \frac{1}{2}(A - \overline{A}')' = \frac{1}{2}(A' - \overline{A})$

and $\overline{C} = \frac{1}{2}\overline{(A - \overline{A}')} = \frac{1}{2}(\overline{A} - A') = -C'$

$\therefore$ $C' = -C$, *i.e.*, C is a skew-Hermitian matrix.

Thus, $A = \frac{1}{2}(A + \overline{A}') + \frac{1}{2}(A - \overline{A}') = B + C$

Hence the result.

II. If A is a Hermitian matrix, then (iA) is a skew-Hermitian matrix. (*Mumbai, 2007*)

We have $(\overline{iA})' = (\bar{i}\,\overline{A})' = (-i\overline{A})' = -i\overline{A}'$

$= -iA$ $\quad [\because \overline{A}' = A]$

Thus (iA) is a skew-Hermitian matrix.

Similarly *if A is a skew-Hermitian matrix then (iA) is a Hermitian matrix.*

III. The eigen values of a Hermitian matrix are real. (see Fig. 2.1)

Let λ be the eigen value and X the corresponding eigen vector of a Hermitian matrix A, so that

$$AX = \lambda X$$

$$\overline{X}'AX = \overline{X}'\lambda X = \lambda \overline{X}'X \quad \text{or} \quad \lambda = \overline{X}'AX / \overline{X}'X$$

Since $\overline{X}'X = \bar{x}_1x_1 + \bar{x}_2x_2 + \ldots + \bar{x}_nx_n = |x_1|^2 + |x_2|^2 + \ldots + |x_n|^2$ is real and non-zero. Also $\overline{X}'AX$ is a Hermitian form which is always real.

$\therefore$ λ, the eigen value of a Hermitian matrix is real.

IV. The eigen values of a skew-Hermitian matrix are purely imaginary or zero.

* Named after the French mathematician *Charles Hermite* (1822–1901), known for his contributions to algebra and number theory.

Let λ be the eigen value and X the corresponding eigen vector of a skew-Hermitian matrix B so that $BX = \lambda X$.

$$\therefore \qquad \overline{X}'BX = \overline{X}'\lambda X = \lambda\overline{X}'X \quad \text{or} \quad \lambda = \overline{X}'BX / \overline{X}'X$$

Since $\overline{X}'X$ is real and non-zero. Also $\overline{X}BX$ is a skew-Hermitian form which is purely imaginary or zero.

$\therefore$ λ, the eigen value of a skew-Hermitian matrix is purely imaginary or zero.

4. **Unitary matrix.** A square matrix U such that $\overline{U'} = U^{-1}$ is called a **unitary matrix.** For a unitary matrix, U, $U \cdot U^* = U^* \cdot U = I$.

This is a generalisation of the orthogonal matrix in the complex field.

Properties

I. Inverse of a unitary matrix is unitary

If U is a unitary matrix, then

$$\overline{U'} = U^{-1}$$

or
$$U' = \overline{U^{-1}}$$

$$\therefore \qquad [(U^{-1})^{-1}]' = \overline{U^{-1}}$$

Writing $U^{-1} = V$, we have

$$[V^{-1}]' = \overline{V} \quad \text{or} \quad V^{-1} = \overline{V'}$$

Thus $V\ (= U^{-1})$ is also unitary.

Cor. *Inverse of an orthogonal matrix is orthogonal.*

II. Transpose of a unitary matrix is unitary

If U is a unitary matrix, $\qquad \overline{U'} = U^{-1}$

or
$$(\overline{U'}) = U^{-1}$$

or
$$[(\overline{U'})]' = [U^{-1}]' = [U']^{-1}$$

Writing $U' = V$, we have $\overline{V'} = V^{-1}$

Thus V (*i.e.*, U') is also unitary.

Cor. *Transpose of an orthogonal matrix is orthogonal.*

III. Product of two unitary matrices is a unitary matrix.

If U and V are unitary matrices then

$$U' = \overline{U}^{-1},\ V' = \overline{V}^{-1}$$

Now,
$$\begin{aligned}(\overline{UV})^{-1} &= (\overline{U}\overline{V})^{-1} = \overline{V}^{-1}\overline{U}^{-1} \\ &= V'U' \\ &= (UV)' \qquad [\because \ U, V \text{ are unitary.}]\end{aligned}$$

Thus, UV is a unitary matrix.

Cor. *Product of two orthogonal matrices is an orthogonal matrix.*

IV. The eigen value of a unitary matrix has absolute value 1. (*U.T.U., 2010*)

If U is a unitary matrix then $\qquad UX = \lambda X \qquad$ ...(1)

Taking conjugate transpose of (1),

$$(\overline{UX})' = (\overline{U}\,\overline{X})' = \overline{X}'\overline{U}' = \overline{X}'\overline{U}^{-1}$$

Also
$$(\overline{UX})' = (\overline{\lambda X})' = \overline{\lambda X'}$$

i.e.,
$$\overline{X}'U^{-1} = \overline{\lambda X'} \qquad ...(2)$$

Post-multiplying (2) by (1), we get

$$(\overline{X}'U^{-1})(UX) = (\overline{\lambda X'}) = (\lambda X)$$

$$\overline{X}'(U^{-1}U)X = (\overline{\lambda}\lambda)(\overline{X}'X) \qquad [\because \ U^{-1}U = I]$$

$$\overline{X}'X = (\lambda\lambda')\,\overline{X}'X$$

Thus
$$\lambda\lambda' = |\lambda|^2 = 1. \qquad [\because \ \overline{X}X \neq 0]$$

Hence the result.

Cor. *The eigen value of an orthogonal matrix has absolute value 1.*

Example 2.53. *If* $A = \begin{bmatrix} 2+i & 3 & -1+3i \\ -5 & i & 4-2i \end{bmatrix}$*, show that* AA^* *is a Hermitian matrix, where* A^* *is the conjugate transpose of A..* *(J.N.T.U., 2005 ; U.P.T.U., 2003)*

Solution. We have $A' = \begin{bmatrix} 2+i & -5 \\ 3 & i \\ -1+3i & 4-2i \end{bmatrix}$

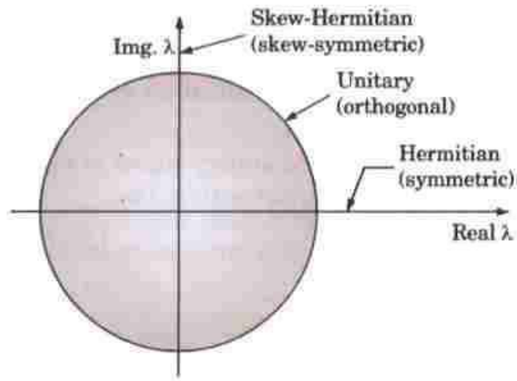


Fig. 2.1. Eigen values of various matrices.

and $$A^* = \begin{bmatrix} 2-i & -5 \\ 3 & -i \\ -1-3i & 4+2i \end{bmatrix}$$

$$\therefore \quad AA^* = \begin{bmatrix} 2-i & 3 & -1+3i \\ -5 & i & 4-2i \end{bmatrix} \begin{bmatrix} 2-i & -5 \\ 3 & -i \\ -1-3i & 4+2i \end{bmatrix}$$

$$= \begin{bmatrix} 4-i^2+9+1-9i^2 & -10-5i-3i-10+10i \\ -10+5i+3i-10-10i & 25-i^2+16-4i^2 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & -20+2i \\ -20-2i & 46 \end{bmatrix}$$, which is a Hermitian matrix.

Example 2.54. *Prove that the matrix* $A = \begin{bmatrix} \frac{1}{2}(1+i) & \frac{1}{2}(-1+i) \\ \frac{1}{2}(1+i) & \frac{1}{2}(1-i) \end{bmatrix}$ *is unitary and find* A^{-1}. *(Mumbai, 2006)*

Solution. Conjugate of A, *i.e.*, $\overline{A} = \begin{bmatrix} \frac{1}{2}(1-i) & \frac{1}{2}(-1-i) \\ \frac{1}{2}(1-i) & \frac{1}{2}(1+i) \end{bmatrix}$

$\therefore$ Transpose of $\overline{A}$, *i.e.*, $$A^\theta = \begin{bmatrix} \frac{1}{2}(1-i) & \frac{1}{2}(1-i) \\ \frac{1}{2}(-1-i) & \frac{1}{2}(1+i) \end{bmatrix}$$

Now $$A^\theta.A = \begin{bmatrix} \frac{1}{2}(1-i) & \frac{1}{2}(1-i) \\ \frac{1}{2}(-1-i) & \frac{1}{2}(1+i) \end{bmatrix} \begin{bmatrix} \frac{1}{2}(1+i) & \frac{1}{2}(-1+i) \\ \frac{1}{2}(1+i) & \frac{1}{2}(1-i) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4}(1+1)+\frac{1}{4}(1+1) & -\frac{1}{4}(1-i)^2+\frac{1}{4}(1-i)^2 \\ -\frac{1}{4}(1+i)^2+\frac{1}{4}(1+i)^2 & \frac{1}{4}(1+1)+\frac{1}{4}(1+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Similarly, $AA^\theta = I$.
Hence A is a unitary matrix.

Also $$A^{-1} = A^\theta = \begin{bmatrix} \frac{1}{2}(1-i) & \frac{1}{2}(1-i) \\ -\frac{1}{2}(1+i) & \frac{1}{2}(1+i) \end{bmatrix}$$

Example 2.55. *Given that* $A = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$, *show that* $(1 - A)(1 + A)^{-1}$ *is a unitary matrix.*

(*Mumbai, 2007*)

Solution. $I + A = \begin{bmatrix} 1 & 1+2i \\ -1+2i & 1 \end{bmatrix}$, $|I + A| = 1 - (-1 - 4) = 6$

$(I + A)^{-1} = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} \div 6$. Also $I - A = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$

$$\therefore \quad (I - A)(I + A)^{-1} = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} \div 6 = \frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix} \quad ...(i)$$

Its conjugate-transpose $= \frac{1}{6}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}$ ...(*ii*)

$\therefore$ Product of (*i*) and (*ii*) $= \frac{1}{36}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix} = \frac{1}{36}\begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix} = I.$

Hence the result.

PROBLEMS 2.11

1. Prove that every Hermitian matrix can be written as $A + iB$, where A is real and symmetric and B is real and skew-symmetric : (*P.T.U., 1999*)
2. Show that every square matrix can be uniquely expressed as $P + iQ$, where P and Q are Hermitian matrices. (*Mumbai, 2008 ; Bhopal, 2002 S*)
3. Show that a Hermitian matrix remains Hermitian when transformed by an orthogonal matrix.
4. Show that the matrix $\begin{bmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{bmatrix}$ is a unitary matrix, if $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$. (*U.P.T.U., 2006*)
5. Show that $\begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$ is a Hermitian matrix.
6. If $A = \begin{bmatrix} -1 & 2+i & 5-3i \\ 2-i & 7 & 5i \\ 5+3i & -5i & 2 \end{bmatrix}$, show that A is a Hermitian matrix and iA is a skew-Hermitian matrix. (*Sambalpur, 2002*)
7. Show that the following matrix is unitary

   (*i*) $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ (*U.P.T.U., 2002*) (*ii*) $\begin{bmatrix} \frac{2+i}{3} & \frac{2i}{3} \\ \frac{2i}{3} & \frac{2-i}{3} \end{bmatrix}$ (*Mumbai, 2008*)
8. Express $A = \begin{bmatrix} 2i & 2+i & 1-i \\ -2+i & -i & 3i \\ -1-i & 3i & 0 \end{bmatrix}$ as $P + iQ$ where P is real and skew-symmetric and Q is real and symmetric. (*Mumbai, 2006*)
9. If $S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix}$, where $a = e^{2i\pi/3}$, prove that $S^{-1} = \frac{1}{3}\bar{S}$. (*Kurukshetra, 2006 ; J.N.T.U., 2001*)

2.20 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 2.12

Choose the correct answer or fill up the blanks in the following problems :

1. To multiply a matrix by scalar k, multiply

 (a) any row by k (b) every element by k (c) any column by k.

2. If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then A^n is

 (a) $\begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$ (b) $\begin{bmatrix} 3^n & (-4)^n \\ 1 & (-1)^n \end{bmatrix}$ (c) $\begin{bmatrix} 1+3n & 1-4n \\ 1+n & 1-n \end{bmatrix}$ (d) $\begin{bmatrix} 1+2n & -4n \\ 1+n & 1-2n \end{bmatrix}$

3. The inverse of the matrix $\begin{bmatrix} -0.5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is

 (a) $\begin{bmatrix} 0.5 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0.5 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} -2 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -0.25 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

4. If $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$, then the determinant AB has the value

 (a) 4 (b) 8 (c) 16 (d) 32

5. The system of equations $x + 2y + z = 9$, $2x + y + 3z = 7$ can be expressed as

 (a) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

 (c) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix}$ (d) none of the above.

6. If $\begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$, then X equals

 (a) $\begin{bmatrix} -3 & -14 \\ 4 & 17 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & -14 \\ 4 & -17 \end{bmatrix}$

7. If $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$, then A (adj A) equals

 (a) $\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 10 \\ 10 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 10 & 1 \\ 1 & 10 \end{bmatrix}$ (d) none of the above.

8. If $3x + 2y + z = 0$, $x + 4y + z = 0$, $2x + y + 4z = 0$, be a system of equations, then

 (a) it is inconsistent

 (b) it has only the trivial solution $x = 0, y = 0, z = 0$.

 (c) it can be reduced to a single equation and so a solution does not exist.

 (d) determinant of the matrix of coefficients is zero.

9. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then

 (a) $C = A\cos\theta - B\sin\theta$ (b) $C = A\sin\theta + B\cos\theta$

 (c) $C = A\sin\theta - B\cos\theta$ (d) $C = A\cos\theta + B\sin\theta$.

10. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \beta & \gamma & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}$, then

(*a*) A is row equivalent to B only when $\alpha = 2$, $\beta = 3$, and $\gamma = 4$

(*b*) A is row equivalent to B only when $\alpha \neq 0$, $\beta \neq 0$, and $\gamma = 0$

(*c*) A is not row equivalent to B

(*d*) A is row equivalent to B for all value of α, β, γ.

11. If $A \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$ where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then A is

(*a*) $\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$ (*b*) $\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$ (*c*) $\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$ (*d*) $\begin{bmatrix} 2 & 1 \\ -1/2 & -1/2 \end{bmatrix}$

12. Matrix has a value. This statement

(*a*) is always true (*b*) depends upon the matrices

(*c*) is false

13. If A is a square matrix such that $AA' = I$, then value of $A'A$ is

(*a*) A^2 (*b*) I (*c*) A^{-1}

14. If every minor of order r of a matrix A is zero, then rank of A is

(*a*) greater than r (*b*) equal to r (*c*) less than or equal to r (*d*) less than r.

15. A square matrix A is called orthogonal if

(*a*) $A = A^2$ (*b*) $A' = A^{-1}$ (*c*) $AA^{-1} = I$

16. The rank of matrix $\begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 4 & -2 & 1 \\ 5 & 2 & 4 & 3 \end{bmatrix}$ is ....

17. The sum of the eigen values of a matrix is the .... of the elements of the principal diagonal.

18. The sum and product of the eigen values of the matrix $\begin{bmatrix} 2 & -3 \\ 4 & -2 \end{bmatrix}$ are .... and .... respectively. (*Anna*, 2009)

19. Inverse of $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ is $\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & k \\ 2 & 2 & 5 \end{bmatrix}$ then k is ....

20. Using Cayley-Hamilton theorem, the value of $A^4 - 4A^3 - 5A^2 - A + 2I$ when $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$ is .... (*Anna*, *2009*)

21. If two eigen values of $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ are 3 and 15, then the third eigen value is ....

22. A quadratic form is positive semi-definite when ....

23. $A_{m \times n}$ and $B_{p \times q}$ are two matrices. When will

(*a*) $A \, . \, B$ exist (*b*) $A + B$ exist ?

24. The product of the eigen values of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ is ....

25. The quadratic form corresponding to the diagonal matrix diag $(\lambda_1, \lambda_2, ..., \lambda_n)$ is

(*a*) $x_1^2 + x_2^2 + ... + x_n^2$ (*b*) $\lambda_1 x_1^2 + \lambda_2 x_2^2 + ... + \lambda_n x_n^2$

(*c*) $\lambda_1^2 x_1^2 + \lambda_2^2 x_2^2 + ... + \lambda_n^2 x_n^2$

26. An example of a 3×3 matrix of rank one is ....

27. The quadratic form corresponding to the symmetric matrix $\begin{bmatrix} 1 & 2 \\ 2 & -4 \end{bmatrix}$ is ....

28. Solving the equations $x + 2y + 3z = 0$, $3x + 4y + 4z = 0$, $7x + 10y + 12z = 0$, $x =$ ...., $y =$ ...., $z =$ ....

29. The eigen values of the matrix $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ are ....

30. A matrix A is *idempotent* if ....

31. The rank of the matrix $\begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$ is ....

32. If A = $\begin{bmatrix} -1 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & -2 \end{bmatrix}$, then the eigen values of A^2 are ....

33. The sum of the eigen values of $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & -1 & -1 \end{bmatrix}$ is

(*i*) – 2 (*ii*) 3 (*iii*) 6 (*iv*) 7 (*S.V.T.U., 2009*)

34. The maximum value of the rank of a 4 × 5 matrix is ....

35. The sum of two eigen values and trace of a 3 × 3 matrix are equal, then the value of $|A|$ is .... (*Anna, 2009*)

36. If the sum of the eigen values of the matrix of the quadratic form is zero, then the nature of the quadratic form is ....

37. The eigen values of matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{bmatrix}$ are ....

38. The eigen values of a triangular matrix are ....

39. If the product of two eigen values of the matrix $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ is 16, then the third eigen value is ....

40. If λ_i, $i = 1, 2, \ldots\ldots\ldots n$ are the eigen values of a square matrix A, then the eigen values of A^T are ....

41. By applying elementary transformations to a matrix, its rank

(*a*) increases (*b*) decreases (*c*) does not change

42. If λ is an eigen value of A, then it is an eigen value of B, only if $B =$ ....

43. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$, then eigen values of A^{-1} are ....

44. The characteristic equation of $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ is ....

45. If $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, then eigen values of A^{-1} are ....

46. Matrix $\begin{bmatrix} x & 2 \\ 1 & x-1 \end{bmatrix}$ is singular for $x =$ ....

47. Every Hermitian matrix can be written as $A + iB$, where A is real and .... and B is real and ....

48. The sum and product of the eigen values of $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are .... and ....

49. If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then $A^3 =$ ....

50. The product of the eigen values of a matrix is equal to ....

51. The eigen values of $A = \begin{bmatrix} 1 & 1 \\ 2 & 5 \end{bmatrix}$ are the roots of the equation ....

52. A system of linear non-homogeneous equations is consistent, if and only if the rank of coefficient matrix is equal to rank of ....
53. The matrix of the quadratic form $q = 4x^2 - 2y^2 + z^2 - 2xy + 6zx$ is ....
54. If $\lambda_1, \lambda_2, \lambda_3$ are the eigen values of a matrix A, then A^3 has the eigen values ....
55. If λ is an eigen value of a non-singular matrix A, then the eigen value of A^{-1} is ....
56. The matrix corresponding to the quadratic form $x^2 + 2y^2 - 7z^2 - 4xy + 8xy + 5yz$ is ....
57. The sum of the squares of the eigen values of $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ is ....
58. If the rank of a matrix A is 2, then the rank of A' is ....
59. The index and signature of the quadratic form $x_1^2 + 2x_2^2 - 3x_3^2$ are respectively .... and ....
60. The equations $x + 2y = 1$, $7x + 14y = 12$ are consistent. (True or False)
61. If rank $(A) = 2$, rank $(B) = 3$, then rank $(AB) = 6$. (True or False)
62. Any set of vectors which includes the zero vector is linearly independent. (True or False)
63. If λ is an eigen value of a symmetric matrix,then λ is real. (True or False)
64. Every square matrix does not satisfy its own characteristic equation. (True or False)
65. If λ is an eigen value of an orthogonal matrix, then $1/\lambda$ is also its eigen value. (True or False)
66. If the rank of a matrix A is 3, then the rank of $3A^T$ is 1. (True or False)
67. The vectors $[1, 1, -1, 1]$, $[1, -1, 2, -1]$, $[3, 1, 0, 1]$ are linearly dependent. (True or False)
68. The eigen values of a skew-symmetric matrix are real. (True or False)
69. Inverse of a unitary matrix is a unitary matrix. (True or False)
70. A is a non-zero column matrix and B is a non-zero row matrix, then rank of AB is one. (True or False)
71. The sum of the eigen values of A equals to the trace of $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$. (True or False)

CHAPTER 3

Vector Algebra & Solid Geometry

1. Vectors. 2. Space coordinates, Resolution of Vectors, Direction cosines. 3. Section formulae. 4–6. Products of two vectors. 7. Physical applications. 8–10. Products of three or more vectors. 11. Equations of a plane. 12. Equations of a straight line. 13. Condition for a line to lie in a plane. 14. Coplanar lines. 15. S.D. between two lines. 16. Intersection of three planes. 17. Equation of a sphere. 18. Tangent plane to a sphere. 19. Cone. 20. Cylinder. 21. Quadric surfaces. 22. Surfaces of Revolution. 23. Objective Type of Questions.

VECTOR ALGEBRA

3.1 (1) VECTORS

A quantity which is completely specified by its magnitude only is called a *scalar*. Length, time, mass, volume, temperature, work, electric charge and numerical data in Statistics are all examples of scalar quantities.

A quantity which is completely specified by its magnitude and direction is called a **vector.** Weight, displacement, velocity, acceleration and electric current density are all vector quantities for each involves magnitude and direction.

A vector is represented by a directed line segment. Thus $\vec{PQ}$ represents a vector whose magnitude is the length PQ and direction is from P (starting point) to Q (end point). We denote a vector by a single letter in capital bold type (or with an arrow on it) and its magnitude (length) by the corresponding small letter in italics type. Thus if **V** is the velocity vector, its magnitude is v, the speed.

A vector of unit magnitude is called a *unit vector*. The idea of unit vector is often used to represent concisely the direction of any vector. Unit vector corresponding to the vector **A** is written as $\hat{\mathbf{A}}$.

A vector of zero magnitude (which can have no direction associated with it) is called a *zero (or null) vector* and is denoted by **0**–a thick zero.

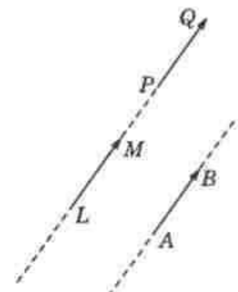


Fig. 3.1

The vector $\vec{QP}$ represents the negative of $\vec{PQ}$, *i.e.*, – **A.**

Two vectors **A** *and* **B** *having the same magnitude and the same* (or parallel) directions are said to be equal and we write **A** = **B.** Clearly the vectors $\vec{AB}$, $\vec{LM}$ and $\vec{PQ}$ are all equal (Fig. 3.1).

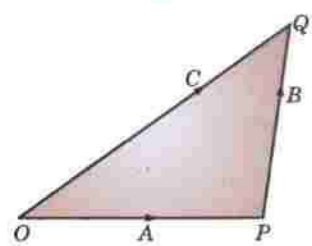


Fig. 3.2

(2) Addition of vectors. Vectors are added according to the *triangle law of addition*, which is a matter of common knowledge. Let **A** and **B** be represented by two vectors $\vec{OP}$ and $\vec{PQ}$ respectively then $\vec{OQ} = \mathbf{C}$ is called the sum or resultant of **A** and **B**. Symbolically, we write,

$$\mathbf{C} = \mathbf{A} + \mathbf{B}$$

(3) Subtraction of vectors. *The subtraction of a vector* **B** *from* **A** *is taken to be the addition of* –**B** *to* **A** *and we write*

$$\mathbf{A} + (-\mathbf{B}) = \mathbf{A} - \mathbf{B}$$

(4) Multiplication of vectors by scalars.

We have just seen that $\quad \mathbf{A} + \mathbf{A} = 2\mathbf{A}$

and $\quad -\mathbf{A} + (-\mathbf{A}) = -2\,\mathbf{A}$

where both 2**A** and – 2**A** denote vectors of magnitude twice that of **A** ; the former having the same direction as **A** and the latter the opposite direction.

In general, *the product m***A** *of a vector* **A** *and a scalar m is a vector whose magnitude is m times that of* **A** *and direction is the same or opposite to* **A** *according as m is positive or negative.*

Thus $\quad \mathbf{A} = a\,\hat{\mathbf{A}}$.

Example 3.1. *If* **A** *and* **B** *are the vectors determined by two adjacent sides of a regular hexagon. What are the vectors represented by the other sides taken in order ?*

Solution. Let *ABCDEF* be the given hexagon, such that

$$\vec{AB} = \mathbf{A} \text{ and } \vec{BC} = \mathbf{B}$$

$\therefore \quad \vec{AC} = \vec{AB} + \vec{BC} = \mathbf{A} + \mathbf{B}$

Also $\quad \vec{AD} = 2\vec{BC} = 2\mathbf{B}$

$\therefore \quad \vec{CD} = \vec{AD} - \vec{AC} = 2\mathbf{B} - (\mathbf{A} + \mathbf{B}) = \mathbf{B} - \mathbf{A}$

Now $\quad \vec{DE} = -\vec{AB} = -\mathbf{A} \quad [\because AB = \text{and} \parallel ED]$

$\quad \vec{EF} = -\vec{BC} = -\mathbf{B} \quad [\because BC = \text{and} \parallel FE]$

and $\quad \vec{FA} = -\vec{CD} = -(\mathbf{B} - \mathbf{A}) = \mathbf{A} - \mathbf{B} \quad [\because CD = \text{and} \parallel AF]$

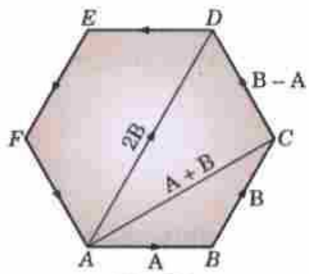


Fig. 3.3

3.2. (1) Space coordinates. Let $X'OX$ and $Y'OY$, $Z'OZ$ be three mutually perpendicular lines which intersect at *O*. Then *O* is called the origin.

X′OX is called the **x-axis,** *Y′OY the* **y-axis,** *Z′OZ the* **z-axis** *and taken together these are called the* **coordinate axes.**

The plane YOZ is called the **yz-plane,** *the plane ZOX the* **zx-plane,** *the plane XOY the* **xy-plane** *and taken together these are called the* **coordinate planes.**

Let *P* be any point in space, Draw *PL*, *PM*, *PN* ⊥s to the *yz*, *zx* and *xy*-planes. Then *LP*, *MP*, *NP* are respectively called the coordinates of *P* (Fig. 3.4). In practice, if $OA = x$, $AN\ y$, $NP = z$, then (x, y, z) are the coordinates of *P* which are positive along *OX*, *OY*, *OZ* respectively and negative along *OX′*, *OY′*, *OZ′*.

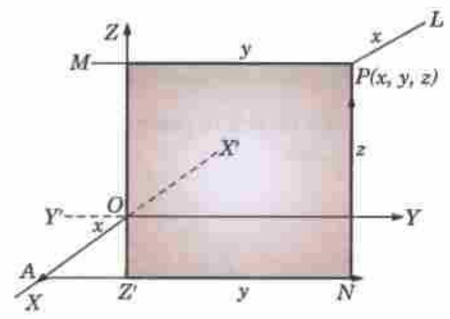


Fig. 3.4

The three coordinate planes divide the space into eight compartments called **octants.** The octant *OXYZ* in which all the coordinates are positive is called the **positive** or **first octant.**

Note. *Three non-coplanar vectors* **A**, **B**, **C** *are said to form a* **right-handed** *(or a* **left-handed***) system according as a right threaded screw rotated through an angle less than* 180° *from* **A** *to* **B** *will advance along (or opposite to)* **C** as shown in Fig. 3.5.

An area of a closed curve described in a given manner is represented by a vector whose magnitude is the given area and direction normal to the plane of the area. Thus the vector **A** representing the area is taken to be positive or negative according as the direction of description of the boundary of the curve and the sense of **A** correspond to a right-handed or a left-handed system.

We have explained the most commonly used system of coordinates namely the *Rectangular Cartesian Coordinates.* The other two systems of coordinates often used to locate a point in space are the *Polar spherical coordinates and Cylindrical coordinates,* which are explained in § 8.21 and 8.20.

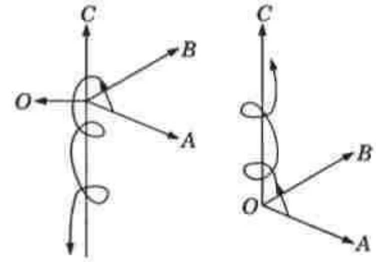


Fig. 3.5

(2) Resolution of vectors. Let **I, J, K** denote unit vectors along OX, OY, OZ respectively. Let $P(x, y, z)$ be a point in space. On OP as diagonal, construct a rectangular parallelopiped with edges OA, OB, OC along the axes so that

$$\vec{OA} = x\mathbf{I},\ \vec{OB} = y\mathbf{J},\ \vec{OC} = z\mathbf{K}$$

Then $$\mathbf{R} = \vec{OP} = \vec{OC'} + \vec{C'P}$$

$$= \vec{OA} = \vec{AC'} + \vec{OC} = \vec{OA} + \vec{OB} + \vec{OC}$$

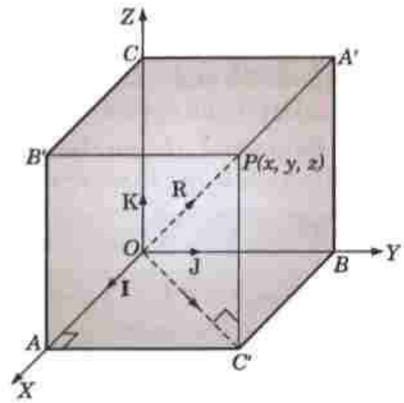


Fig. 3.6

Hence $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$ is called the *position vector* of P relative to origin O and.

$$r = |\mathbf{R}| = \sqrt{(x^2 + y^2 + z^2)}$$

$$[\because\ r^2 = OP^2 = OC'^2 + C'P^2 = OA^2 + AC'^2 + C'P^2]$$

(3) Direction cosines. Let any line L or its parallel OP, make angles α, β, γ with OX, OY, OZ respectively, then *$\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the direction cosines of this line* which are usually denoted by l, m, n.

If l, m, n are direction cosines of a vector $\mathbf{R}$, then

(*i*) $\hat{\mathbf{R}} = l\mathbf{I} + m\mathbf{J} + n\mathbf{K}$, (*ii*) $\mathbf{l^2 + m^2 + n^2 = 1}$

Proof. Let D be the foot of the perpendicular from $P(x, y, z)$ on OY. Then

$$y = OD = r\cos\beta = mr. \text{ Similarly, } z = nr \text{ and } x = lr.$$

$\therefore$ $$\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K} = r(l\mathbf{I} + m\mathbf{J} + n\mathbf{K})$$

or $$\hat{\mathbf{R}} = \frac{\mathbf{R}}{r} = l\mathbf{I} + m\mathbf{J} + n\mathbf{K}$$

which expresses a unit vector in terms of its direction cosines.

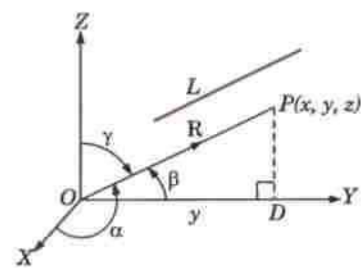


Fig. 3.7

Also $$1 = |\hat{\mathbf{R}}| = \sqrt{(l^2 + m^2 + n^2)} \text{ thus } \mathbf{l^2 + m^2 + n^2 = 1}$$

i.e., $$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$ (*V.T.U., 2010*)

Obs. Directions ratios. If the direction cosines of a line be proportional to a, b, c, then these are called proportional direction cosines or direction ratios of the line.

If the direction cosines be l, m, n, then

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \frac{\sqrt{(l^2 + m^2 + n^2)}}{\sqrt{(a^2 + b^2 + c^2)}} = \frac{1}{\sqrt{(\Sigma a^2)}}$$

$\therefore$ $$l = \frac{a}{\sqrt{(\Sigma a^2)}},\ m = \frac{b}{\sqrt{(\Sigma a^2)}},\ n = \frac{c}{\sqrt{(\Sigma a^2)}}$$

(4) Distance between two points *$P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is*

$$\sqrt{[(\mathbf{x_2 - x_1})^2 + (\mathbf{y_2 - y_1})^2 + (\mathbf{z_2 - z_1})^2]}$$

and **direction ratios** *of* $\vec{PQ}$ *are* $\mathbf{x_2 - x_1, y_2 - y_1, z_2 - z_1}$

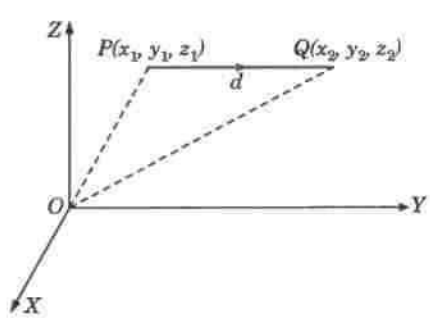


Fig. 3.8

We have

$$\vec{OP} = x_1\mathbf{I} + y_1\mathbf{J} + z_1\mathbf{K}$$

and $$\vec{OQ} = x_2\mathbf{I} + y_2\mathbf{J} + z_2\mathbf{K}$$

$\therefore$ $$\vec{PQ} = \vec{OQ} - \vec{OP}$$

$$= (x_2 - x_1)\mathbf{I} + (y_2 - y_1)\mathbf{J} + (z_2 - z_1)\mathbf{K}$$

Thus, $$d = |\vec{PQ}| = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$$

and direction cosines of $\vec{PQ}$ are proportional to $x_2 - x_1, y_2 - y_1, z_2 - z_1$.

Example 3.2. *Show that the points A(–4, 9, 6), B(–1, 6, 6) and C(0, 7, 10) form a right angled isosceles triangle. Also find the direction cosines of AB.*

Solution. We have $$AB = \sqrt{[(-1+4)^2 + (6-9)^2 + (6-6)^2]} = 3\sqrt{2}$$

$$BC = \sqrt{[(0+1)^2 + (7-6)^2 + (10-6)^2]} = 3\sqrt{2}$$

and $$CA = \sqrt{[(-4-0)^2 + (9-7)^2 + (6-10)^2]} = 6$$

Since $AB^2 + BC^2 = CA^2$ and $AB = BC$, it follows that ΔABC is a right-angled isosceles triangle. The direction ratios of $\vec{AB}$ are $-1+4, 6-9, 6-6$.

$\therefore$ Its direction cosines are $\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0$.

3.3 SECTION FORMULAE

The point **R** (x, y, z) dividing the join of the points $\mathbf{A}(x_1, y_1, z_1)$ *and* $\mathbf{B}(x_2, y_2, z_2)$ *in the ratio* $m_1 : m_2$ *is*

$$\mathbf{R} = \frac{m_1\mathbf{B} + m_2\mathbf{A}}{m_1 + m_2}, \quad i.e., \quad \left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}, \frac{m_1z_2 + m_2z_1}{m_1 + m_2}\right) \quad ...(i)$$

Let $P(\mathbf{A})$ and $Q(\mathbf{B})$ be the given points referred to origin O. Let $R(\mathbf{R})$ be the point dividing the line joining P and Q in the ratio $m_1 : m_2$ so that

$$\frac{PR}{RQ} = \frac{m_1}{m_2}, \quad i.e., \quad m_2 . PR = m_1 . RQ$$

$\therefore$ We have $$m_2 \vec{PR} = m_1 \vec{RQ}$$

or $$m_2(\vec{OR} - \vec{OP}) = m_1(\vec{OQ} - \vec{OR})$$

or $$m_2(\mathbf{R} - \mathbf{A}) = m_1(\mathbf{B} - \mathbf{R})$$

whence $$\mathbf{R} = \frac{m_1\mathbf{B} + m_2\mathbf{A}}{m_1 + m_2}$$

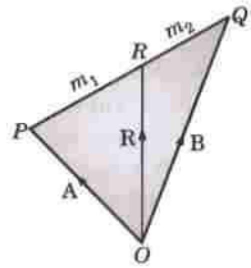


Fig. 3.9

Since $$\mathbf{A} = x_1\mathbf{I} + y_1\mathbf{J} + z_1\mathbf{K}, \ \mathbf{B} = x_2\mathbf{I} + y_2\mathbf{J} + z_2\mathbf{K}$$

and $$\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$$

$$\therefore \quad x\mathbf{I} + y\mathbf{J} + z\mathbf{K} = \frac{m_1(x_2\mathbf{I} + y_2\mathbf{J} + z_2\mathbf{K}) + m_1(x_1\mathbf{I} + y_1\mathbf{J} + z_1\mathbf{K})}{m_1 + m_2}$$

Equating coefficient of **I, J, K,** we get the desired results (*i*).

Cor. 1. *Mid-point of* $P(\mathbf{A})$ *and* $Q(\mathbf{B})$ *is* $\frac{1}{2}(\mathbf{A} + \mathbf{B})$.

2. Point R dividing the join of $P(\mathbf{A})$ and $Q(\mathbf{B})$ in the ratio $m_1 : m_2$ externally is $\mathbf{R} = \frac{m_1\mathbf{B} - m_2\mathbf{A}}{m_1 - m_2}$.

Obs. Rewriting (*i*) as $m_2\mathbf{A} + m_1\mathbf{B} - (m_1 + m_2)\mathbf{R} = 0$, we note that the sum of the coefficients of **A, B** and **R** is zero. Hence it follows that any *three points with position vectors* **A, B** *and* **C** *are collinear if*

$$\lambda\mathbf{A} + \mu\mathbf{B} + \gamma\mathbf{C} = 0, \text{ where } \lambda + \mu + \gamma = 0.$$

Example 3.3. *In a trapezium, prove that the straight line joining the mid-points of the diagonals is parallel to the parallel sides and half their difference.*

Solution. Consider a trapezium $OABC$ with parallel sides OA and BC. Take O as the origin and let the other vertices be $A(\mathbf{A})$, $B(\mathbf{B})$, $C(\mathbf{C})$.

Since CB is parallel to OA, therefore,

$$\mathbf{B} - \mathbf{C} = \vec{CB} = \lambda \vec{OA} = \lambda \mathbf{A}.$$

The mid-points of the diagonals OB and AC are $D(\mathbf{B}/2)$ and $E(\mathbf{A} + \mathbf{C})/2$.

$$\therefore \quad \vec{DE} = \vec{OE} - \vec{OD} = \frac{1}{2}(\mathbf{A} + \mathbf{C}) - \frac{1}{2}\mathbf{B} = \frac{1}{2}[\mathbf{A} - (\mathbf{B} - \mathbf{C})] \qquad ...(i)$$

$$= \frac{1}{2}(1 - \lambda)\mathbf{A} \qquad ...(ii)$$

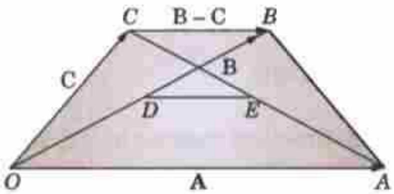


Fig. 3.10

From (*ii*), it is clear that $\vec{DE}$ is parallel to $\vec{OA}$; from (*i*), it follows that $DE = \frac{1}{2}(OA - CB)$.

Hence the result.

Example 3.4. *Show that the line joining one vertex of a parallelogram to the mid-point of an opposite side trisects the diagonal and is itself trisected there at.*

Solution. Consider a parallelogram $OABC$. Take O as the origin and let the other vertices be $A(\mathbf{A})$, $B(\mathbf{B})$ and $C(\mathbf{C})$.

The mid-point D of OA is $\mathbf{A}/2$.

Now since OA is equal to and parallel to CB.

$$\therefore \quad \vec{OA} = \vec{CB}, \text{ i.e., } \mathbf{A} = \mathbf{B} - \mathbf{C}$$

which may be written as $\dfrac{2(\mathbf{A}/2) + 1 \cdot \mathbf{C}}{2 + 1} = \dfrac{\mathbf{B}}{3} = \mathbf{P}$ so that P trisects DC and OB.

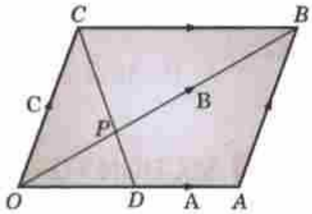


Fig. 3.11

PROBLEMS 3.1

1. Given $\mathbf{R}_1 = 5\mathbf{I} - 2\mathbf{J} + 4\mathbf{K}$ and $\mathbf{R}_2 = \mathbf{I} + 3\mathbf{J} + 7\mathbf{K}$, find the magnitude and direction cosines of the vectors $\mathbf{R}_1 + \mathbf{R}_2$ and $2\mathbf{R}_1 - \mathbf{R}_2$.
2. Show that the points (0, 4, 1) ; (2, 3, − ...), (4, 5, 0) and (2, 6, 2) are the vertices of a square. (*Osmania, 1999 S*)
3. A straight line is inclined to the axes of x and y at angles of 30° and 60°. Find the inclination of the line to the z-axis. (*Madras, 2003*)
4. If a line makes angles α, β, γ with the axes, prove that
   (*i*) $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 2$. (*V.T.U., 2000 ; Osmania, 1999*)
   (*ii*) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -1$.
5. If $\mathbf{A}$ and $\mathbf{B}$ are non-collinear vectors and $\mathbf{P} = (2x + 3y - 2)\mathbf{A} + (3x + 2y + 5)\mathbf{B}$ and $\mathbf{Q} = (-x + 4y - 2)\mathbf{A} + (3x - 4y + 7)\mathbf{B}$, find x, y such that $7\mathbf{P} = 3\mathbf{Q}$.
6. Prove that the line joining the mid-points of the two sides of a triangle is parallel to the third side and half of it.
7. Prove that (*i*) the diagonals of a parallelogram bisect each other ;
   (*ii*) a quadrilateral whose diagonals bisect each other is a parallelogram.
8. In a skew quadrilateral, prove that :
   (*i*) the figure formed by joining the mid-points of the adjacent sides is a parallelogram.
   (*ii*) the joins of the mid-points of opposite sides bisect each other.
9. In a trapezium, prove that the straight line joining the mid-points of the non-parallel sides is parallel to the parallel sides and half their sum.
10. Prove that the vectors $\mathbf{A} = 3\mathbf{I} + \mathbf{J} - 2\mathbf{K}$, $\mathbf{B} = -\mathbf{I} + 3\mathbf{J} + 4\mathbf{K}$, $\mathbf{C} = 4\mathbf{I} - 2\mathbf{J} - 6\mathbf{K}$ can form the sides of a triangle. Also find the length of the median bisecting the vector $\mathbf{C}$. (*J.N.T.U., 1995 S*)
11. Find the ratio in which the line joining (2, 4, 16) and (3, 5, − 4) is divided by the plane $2x - 3y + z + 6 = 0$. (*Mysore, 1995*)
12. Show that the three points $\mathbf{I} - 2\mathbf{J} + 3\mathbf{K}$, $2\mathbf{I} + 3\mathbf{J} - 4\mathbf{K}$, $-7\mathbf{J} + 10\mathbf{K}$ are collinear.
13. If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be the position vectors of the vertices A, B, C of the triangle ABC, show that the three
    (*i*) medians concur at the point $\frac{1}{3}(\mathbf{A} + \mathbf{B} + \mathbf{C})$, called the *centroid*.
    (*ii*) internal bisectors of the angles concur at the point $\dfrac{a\mathbf{A} + b\mathbf{B} + c\mathbf{C}}{a + b + c}$, called the *incentre*.

14. Show that the coordinates of the *centroid of the triangle* whose vertices are $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$ are

$$\left[\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right].$$

15. Show that the coordinates of the *centroid of the tetrahedron* whose vertices are $(x_r, y_r, z_r) : r = 1, 2, 3, 4$ are

$$\left[\frac{1}{4}(x_1+x_2+x_3+x_4), \frac{1}{4}(y_1+y_2+y_3+y_4); \frac{1}{4}(z_1+z_2+z_3+z_4)\right].$$

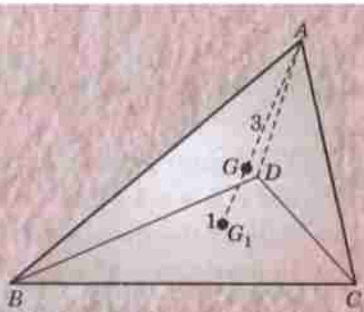


Fig. 3.12

[**Def.** *A* **tetrahedron** *is a solid bounded by four triangular faces.* Thus the tetrahedron $ABCD$ has four faces—the Δs ABC, ACD, ADB, BCD. (Fig. 3.12.)

It has four **vertices** A, B, C, D and three pairs of **opposite edges** AB, CD ; BC, AD ; CA, BD.

The centroid of the tetrahedron divides the join of each vertex to the centroid of the opposite triangular face in the ratio 3 : 1].

16. M and N are the mid-points of the diagonals AC and BD respectively of a quadrilateral $ABCD$. Show that the resultant of the vectors $\vec{AB}, \vec{AD}, \vec{CB}, \vec{CD}$ is $4\,\vec{MN}$. *(Cochin, 1999)*

3.4 PRODUCTS OF TWO VECTORS

Unlike the product of two scalars or that of a vector by a scalar, the product of two vectors is sometimes seen to result in a scalar quantity and sometimes in a vector. As such, we are led to define two types of such products, called the *scalar product* and the *vector product* respectively.

The scalar and vector products of two vectors **A** and **B** are usually written as **A . B** and **A × B** respectively and are read as **A** dot **B** and **A** cross **B**. In view of this notation, the former is sometimes called the *dot product* and the latter the *cross product*.

In vector algebra, the division of a vector by another vector is not defined.

3.5 SCALAR OR DOT PRODUCT

(1) Definition. *The scalar or dot product of two vectors* **A** *and* **B** *is defined as the scalar ab cos θ, where θ is the angle between* **A** *and* **B**.

Thus $\quad \mathbf{A.B} = ab \cos \theta.$

(2) Geometrical interpretation. **A.B** *is the product of the length of one vector and the length of the projection of the other in the direction of the former.*

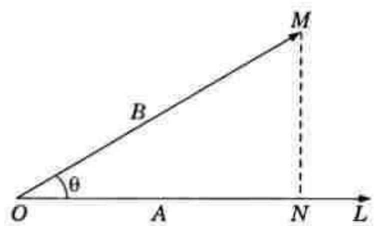


Fig. 3.13

Let $\quad \vec{OL} = \mathbf{A}, \vec{OM} = \mathbf{B} \quad$ then

$\mathbf{A} . \mathbf{B} = ab \cos \theta = a(OM \cos \theta) = a(ON) = |\, A \,|$ Proj. of $|\, \mathbf{B} \,|$ in the direction of **A**.

Similarly, $\mathbf{A} . \mathbf{B} = |\, \mathbf{B} \,|$ Proj. of $|\, \mathbf{A} \,|$ in the direction of **B**.

(3) Properties and other results.

I. Scalar product of two vectors is commutative.

i.e., $\mathbf{A} . \mathbf{B} = \mathbf{B} . \mathbf{A}$ for $\mathbf{A} . \mathbf{B} = ab \cos \theta = ba \cos (-\theta) = \mathbf{B} . \mathbf{A}$

II. The necessary and sufficient condition for two vectors to be perpendicular is that their scalar product should be zero.

When the vectors **A** and **B** are perpendicular, $\mathbf{A} . \mathbf{B} = ab \cos 90° = 0$.

Conversely, when $\mathbf{A} . \mathbf{B} = 0$, $ab \cos \theta = 0$, *i.e.*, $\cos \theta = 0$. $\quad (\because\ a \neq 0, b \neq 0)$, or $\theta = 90°$.)

III. $\mathbf{A} . \mathbf{A} = a^2$ which is written as $\mathbf{A}^2$. Thus *the square of a vector is a scalar which stands for the square of its magnitude.*

IV. For the mutually perpendicular unit vectors, **I, J, K,** we have the relations.

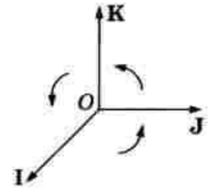


Fig. 3.14

$$\mathbf{I} . \mathbf{J} = \mathbf{J} . \mathbf{K} = \mathbf{K} . \mathbf{I} = 0$$

and
$$\mathbf{I}^2 = \mathbf{J}^2 = \mathbf{K}^2 = 1$$

which are of great utility.

V. Scalar product of two vectors is distributive i.e.,

$$(\mathbf{A}+\mathbf{B})\,.\,\mathbf{C} = \mathbf{A}\,.\,\mathbf{C} + \mathbf{B}\,.\,\mathbf{C}$$

VI. Schwarz inequality :* $|\,\mathbf{A}\,.\,\mathbf{B}\,| \le |\,\mathbf{A}\,|\;|\mathbf{B}|$

$$|\,\mathbf{A}\,.\,\mathbf{B}\,| = |\,\mathbf{A}\,|\;|\,\mathbf{B}\,|\;|\cos\theta\,| \le |\,\mathbf{A}\,|\;|\,\mathbf{B}\,| \qquad [\because \;\; |\cos\theta| \le 1]$$

VII. Scalar product of two vectors is equal to the sum of the products of their corresponding components.

For if $\mathbf{A} = a_1\mathbf{I} + a_2\mathbf{J} + a_3\mathbf{K}$, $\mathbf{B} = b_1\mathbf{I} + b_2\mathbf{J} + b_3\mathbf{K}$

then by the distributive law, $\mathbf{A}\,.\,\mathbf{B} = a_1b_1 + a_2b_2 + a_3b_3$

In particular, $\mathbf{A}^2 = a_1{}^2 + a_2{}^2 + a_3{}^2$.

VIII. **Angle between two lines** *whose direction cosines are l, m, n and l′, m′, n′ is* $\cos^{-1}(\mathbf{ll'} + \mathbf{mm'} + \mathbf{nn'})$.

The unit vectors in the direction of the given lines are $\mathbf{U} = l\mathbf{I} + m\mathbf{J} + n\mathbf{K}$ and $\mathbf{U'} = l'\mathbf{I} + m'\mathbf{J} + n'\mathbf{K}$.

If θ be the angle between the lines, then

$$\mathbf{U}\,.\,\mathbf{U'} = (l\mathbf{I} + m\mathbf{J} + n\mathbf{K})\cdot(l'\mathbf{I} + m'\mathbf{J} + n'\mathbf{K})$$

or $$1\,.\,1\,.\cos\theta = ll' + mm' + nn' \qquad \text{(V.T.U., 2008)}$$

Hence $$\cos\theta = ll' + mm' + nn' \qquad ...(i)$$

Cor. 1.
$$\sin^2\theta = 1 - \cos^2\theta = 1 - (ll' + mm' + nn')^2$$
$$= (l^2 + m^2 + n^2)(l'^2 + m'^2 + n'^2) - (ll' + mm' + nn')^2$$
$$= (mn' - nm')^2 + (nl' - ln')^2 + (lm' - ml')^2$$

$\therefore$ $$\sin\theta = \pm\sqrt{\Sigma(mn' - nm')^2}. \qquad ...(ii)$$

Cor. 2. *The condition that the lines whose direction cosines are l, m, n and l′, m′, n′ should be* **perpendicular** *is*

$$\mathbf{ll'} + \mathbf{mm'} + \mathbf{nn'} = 0 \qquad ...(iii)$$

and **parallel** *is* $$\mathbf{l} = \mathbf{l'},\ \mathbf{m} = \mathbf{m'},\ \mathbf{n} = \mathbf{n'} \qquad ...(iv)$$

These conditions easily follow from (*i*) and (*ii*).

Cor. 3. *The angle θ between two lines whose direction ratios are a, b, c, and a′, b′, c′ is given by*

$$\cos\theta = \frac{aa' + bb' + cc'}{\sqrt{(\Sigma a^2)}\sqrt{(\Sigma a'^2)}}$$

or $$\sin\theta = \frac{\sqrt{[(bc' - cb')^2 + (ca' - ac')^2 + (ab' - ba')^2}}{\sqrt{(\Sigma a^2)}\sqrt{(\Sigma a'^2)}}$$

These lines are (i) perpendicular if $aa' + bb' + cc' = 0$, *(ii) parallel if* $a/a' = b/b' = c/c'$.

IX. **Projection of the line joining two points** (x_1, y_1, z_1) *and* (x_2, y_2, z_2) *on a line whose direction cosines are l, m, n is* $$l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1).$$

Let $$\vec{OP} = x_1\mathbf{I} + y_1\mathbf{J} + z_1\mathbf{K},\ \vec{OQ} = x_2\mathbf{I} + y_2\mathbf{J} + z_2\mathbf{K}$$

$\therefore$ $$\vec{PQ} = (x_2 - x_1)\mathbf{I} + (y_2 - y_1)\mathbf{J} + (z_2 - z_1)\mathbf{K}$$

Also unit vector $\mathbf{U}$ along the given lines is $l\mathbf{I} + m\mathbf{J} + n\mathbf{K}$.

$\therefore$ Projection of PQ on the given line $= \vec{PQ}\,.\,\mathbf{U}$.

$$= l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1).$$

Example 3.5. *Find the sides and angles of the triangle whose vertices are* $\mathbf{I} - 2\mathbf{J} + 2\mathbf{K}$, $2\mathbf{I} + \mathbf{J} - \mathbf{K}$, *and* $3\mathbf{I} - \mathbf{J} + 2\mathbf{K}$.

Solution. Let $\vec{OA} = \mathbf{I} - 2\mathbf{J} + 2\mathbf{K}$, $\vec{OB} = 2\mathbf{I} + \mathbf{J} - \mathbf{K}$, $\vec{OC} = 3\mathbf{I} - \mathbf{J} + 2\mathbf{K}$

Then $$\vec{BC} = \mathbf{I} - 2\mathbf{J} + 3\mathbf{K}$$

$$\vec{CA} = -2\mathbf{I} - \mathbf{J}$$

* Named after the German mathematician *Hermann Amandus Schwarz* (1843—1921) who is known for his work in conformal mapping, calculus of variations and differential geometry. He succeeded *Weierstrass* in Berlin University.

and $\vec{AB} = \mathbf{I} + 3\mathbf{J} - 3\mathbf{K}$

$\therefore$ $BC = \sqrt{14}, CA = \sqrt{5}, AB = \sqrt{19}.$

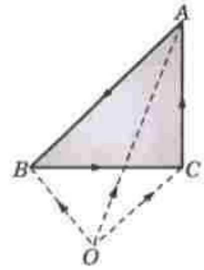


Fig. 3.15

Now d.c.'s of AB and AC being

$$1/\sqrt{19}, 3/\sqrt{19}, -3/\sqrt{19} \text{ and } 2/\sqrt{5}, 1/\sqrt{5}, 0,$$

We have $\cos A = \frac{1}{\sqrt{19}} \cdot \frac{2}{\sqrt{5}} + \frac{3}{\sqrt{19}} \cdot \frac{1}{\sqrt{5}} + \frac{-3}{\sqrt{19}} \cdot 0 = \sqrt{(5/19)}$

i.e., $\angle A = \cos^{-1} \sqrt{(5/19)}$. Again d.c.'s of BC and BA being

$$1/\sqrt{14}, -2/\sqrt{14}, 3/\sqrt{14} \text{ and } -1/\sqrt{19}, -3/\sqrt{19}, 3/\sqrt{19};$$

we have $\cos B = \frac{1}{\sqrt{14}} \cdot \frac{-1}{\sqrt{19}} + \frac{-2}{\sqrt{14}} \cdot \frac{-3}{\sqrt{19}} + \frac{3}{\sqrt{14}} \cdot \frac{3}{\sqrt{19}} = \sqrt{(14/19)}$, *i.e.*, $\angle B = \cos^{-1} \sqrt{(14/19)}$

Finally, d.c.'s of CA and CB being $-2/\sqrt{5}, -1/\sqrt{5}, 0$ and $-1/\sqrt{14}, 2/\sqrt{14}, -3/\sqrt{14}$;

we have $\cos C = \frac{-2}{\sqrt{5}} \cdot \frac{-1}{\sqrt{14}} + \frac{-1}{\sqrt{5}} \cdot \frac{2}{\sqrt{14}} + 0 \cdot \frac{-3}{\sqrt{14}} = 0$, *i.e.*, $\angle C = 90°$

Example 3.6. *Prove that the right -bisectors of the sides of a triangle concur at its circumcentre.*

Solution. Let $A(\mathbf{A})$, $B(\mathbf{B})$, $C(\mathbf{C})$ be the vertices of any triangle ABC. The mid-points of the sides BC, CA and AB are

$$D\left(\frac{\mathbf{B}+\mathbf{C}}{2}\right), E\left(\frac{\mathbf{C}+\mathbf{A}}{2}\right), F\left(\frac{\mathbf{A}+\mathbf{B}}{2}\right)$$

Let the perpendicular at D and E to BC and CA respectively intersect at the point $P(\mathbf{R})$. Then $\vec{DP} \cdot \vec{BC} = 0$

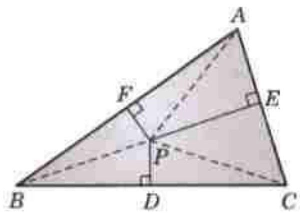


Fig. 3.16

i.e., $\left(\mathbf{R} - \frac{\mathbf{B}+\mathbf{C}}{2}\right) \cdot (\mathbf{C} - \mathbf{B}) = 0$ ...(*i*)

and $\vec{EP} \cdot \vec{CA} = 0$, *i.e.*, $\left(\mathbf{R} - \frac{\mathbf{C}+\mathbf{A}}{2}\right) \cdot (\mathbf{A} - \mathbf{C}) = 0$ ...(*ii*)

Adding (*i*) and (*ii*), we get $\left(\mathbf{R} - \frac{\mathbf{A}+\mathbf{B}}{2}\right) \cdot (\mathbf{A} - \mathbf{B}) = 0$

which shows that FP is perpendicular to AB. Hence the result.

Further $PA = PB$ if $|\mathbf{A} - \mathbf{R}| = |\mathbf{B} - \mathbf{R}|$

or if, $(\mathbf{A} - \mathbf{R})^2 = (\mathbf{B} - \mathbf{R})^2$ or if, $\mathbf{A}^2 - 2\mathbf{A} \cdot \mathbf{R} = \mathbf{B}^2 - 2\mathbf{B} \cdot \mathbf{R}$

of if, $\left(\mathbf{R} - \frac{\mathbf{A}+\mathbf{B}}{2}\right) \cdot (\mathbf{A} - \mathbf{B}) = 0$, which is true.

Example 3.7. *If the distance between two points P and Q is d and the lengths of the projections of PQ on the coordinate planes d_1, d_2, d_3, show that $2d^2 = d_1^2 + d_2^2 + d_3^2$.*

Solution. Let P be (x_1, y_1, z_1) and Q be (x_2, y_2, z_2), then

$$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2.$$

The feet of the perpendiculars drawn from P and Q on the XY-plane are the projections of P and Q on this plane. If these are L and M, then L is $(x_1, y_1, 0)$ and M is $(x_2, y_2, 0)$.

$\therefore$ d_1 = projection of PQ on XY-plane, *i.e.*, LM

or $d_1^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

Similarly, $d_2^2 = (y_1 - y_2)^2 + (z_1 - z_2)^2$ and $d_3^2 = (z_1 - z_2)^2 + (x_1 - x_2)^2$

$\therefore$ $d_1^2 + d_2^2 + d_3^2 = 2[(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2] = 2d^2.$

Example 3.8. *A line makes angles α, β, γ, δ with diagonals of a cube, prove that*

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = 4/3.$$ (V.T.U., 2006 ; Osmania, 2000 S)

Solution. Take O, a corner of the cube as origin and OA, OB, OC the three edges through it, as the axes. Let $OA = OB = OC = a$. Then the coordinates of the corners are as shown in Fig. 3.17.

The four diagonals are OP, AA', BB' and CC'.

Clearly, direction cosines of OP are

$$\frac{a-0}{\sqrt{(\Sigma a^2)}}, \frac{a-0}{\sqrt{(\Sigma a^2)}}, \frac{a-0}{\sqrt{(\Sigma a^2)}} \text{ i.e., } \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}.$$

Similarly, direction cosines of AA' are $-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$;

Similarly, direction cosines of BB' are $\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$;

and Similarly direction cosines of CC' are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$.

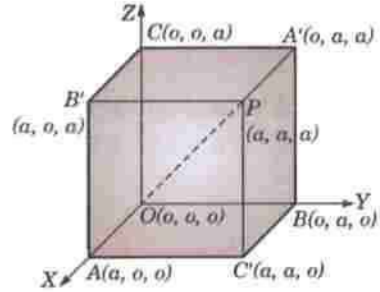


Fig. 3.17

Let l, m, n be the direction cosines of the given line which makes angles α, β, γ, δ with OP, AA', BB', CC' respectively. Then

$$\cos\alpha = \frac{1}{\sqrt{3}}(l+m+n) ; \cos\beta = \frac{1}{\sqrt{3}}(-l+m+n)$$

$$\cos\gamma = \frac{1}{\sqrt{3}}(l-m+n); \cos\delta = \frac{1}{\sqrt{3}}(l+m-n)$$

Squaring and adding, we get

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{1}{3}[(l+m+n)^2 + (-l+m+n)^2 + (l-m+n)^2 + (l+m-n)^2]$$

$$= \frac{1}{3}[4(l^2+m^2+n^2)] = \frac{4}{3}. \qquad [\because l^2+m^2+n^2 = 1]$$

Example 3.9. *If the edges of a rectangular parallelopiped are a, b, c, show that the angle between the four diagonals are* $\cos^{-1}\left(\frac{\pm a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}\right)$.

Solution. Let $OA = a$, $OB = b$, $OC = c$ be the edges of the rectangular parallelopiped. Then the coordinates of the corners are as shown in Fig. 3.18. The four diagonals taken in pairs are (*i*) (OP, AA'), (*ii*) (OP, BB'), (*iii*) (OP, CC'), (*iv*) (AA', BB'), (*v*) (AA', CC') and (*vi*) (BB', CC').

Let the angles between these pairs of diagonals be $\theta_1, \theta_2, \ldots \theta_6$ respectively. Clearly d.r.'s OP are a, b, c ; d.r.'s, of AA' are $-a, b, c$, d.r.'s of BB' are $a, -b, c$ and d.r.'s of CC' are $a, b, -c$.

∴ For the pair (*i*) *i.e.*, (OP, AA') ;

$$\cos\theta_1 = \frac{-a^2+b^2+c^2}{\sqrt{(a^2+b^2+c^2} \sqrt{(a^2+b^2+c^2)}} = \frac{-a^2+b^2+c^2}{a^2+b^2+c^2}$$

Similarly, $\cos\theta_2 = \frac{a^2-b^2+c^2}{a^2+b^2+c^2}$, $\quad \cos\theta_3 = \frac{a^2+b^2-c^2}{a^2+b^2+c^2}$;

$\cos\theta_4 = \frac{-a^2-b^2+c^2}{a^2+b^2+c^2}$; $\quad \cos\theta_5 = \frac{-a^2+b^2-c^2}{a^2+b^2+c^2}$;

$\cos\theta_6 = \frac{a^2-b^2-c^2}{a^2+b^2+c^2}$

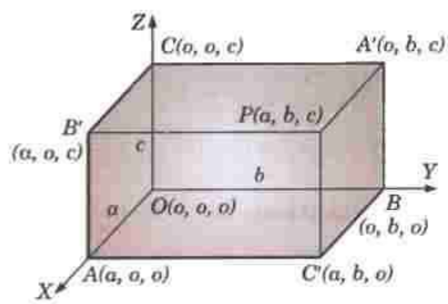


Fig. 3.18

Thus, noting that at least one term in the numerator is negative, we have in general

$$\cos\theta = \frac{\pm a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}.$$

Example 3.10. *Prove that the lines whose direction cosines are given by the relations $al + bm + cn = 0$ and $mn + nl + lm = 0$ are*

(i) Perpendicular if $a^{-1} + b^{-1} + c^{-1} = 0$ (Burdwan, 2003)

(ii) parallel if $\sqrt{a} + \sqrt{b} + \sqrt{c} = 0$.

Solution. Eliminating n from the given relations, we have

$$(m + l)\left(-\frac{al + bm}{c}\right) + lm = 0 \quad \text{or} \quad al^2 + (c - a - b)lm + bm^2 = 0$$

or $$a(l/m)^2 + (c - a - b)(l/m) + b = 0 \qquad ...(1)$$

If l_1, m_1, n_1 ; l_2, m_2, n_2, are the direction cosines of these lines then l_1/m_1, l_2/m_2 are the roots of the quadratic (1).

$$\therefore \quad \frac{l_1}{m_1}\cdot\frac{l_2}{m_2} = \frac{b}{a} \quad \text{or} \quad \frac{l_1 l_2}{1/a} = \frac{m_1 m_2}{1/b} = \frac{n_1 n_2}{1/c} \quad \text{(by symmetry)} = k \text{ (say)}.$$

The lines will be perpendicular if $l_1 l_2 + m_1 m_2 + n_1 n_2 = k\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = 0$

or if, $$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0.$$

The lines will be parallel if $l_1 = l_2$, $m_1 = m_2$, $n_1 = n_2$.

i.e., if, $$l_1/m_1 = l_2/m_2; \quad \text{i.e. if,} \quad (c - a - b)^2 = 4ab$$

or if, $$c - a - b = \pm 2\sqrt{(ab)} \quad \text{or if,} \quad c = a + b \pm 2\sqrt{(ab)} = (\sqrt{a} \pm \sqrt{b})^2$$

or if, $$\pm\sqrt{c} = \sqrt{a} \pm \sqrt{b} \quad \text{or if,} \quad \sqrt{a} + \sqrt{b} + \sqrt{c} = 0$$ [Taking necessary signs]

Example 3.11. *Find the angle between the lines whose direction cosines are given by the equation $l + 3m + 5n = 0$ and $5lm - 2mn + 6nl = 0$.*

Solution. Let us eliminate l from the given relations, by substituting $l = -3m - 5n$ in the second relation

$$5m(-3m - 5n) - 2mn + 6n(-3m - 5n) = 0$$

i.e., $$15m^2 + 45mn + 30n^2 = 0 \quad \text{or} \quad m^2 + 3mn + 2n^2 = 0$$

or $$(m + n)(m + 2n) = 0, \quad \text{i.e.,} \quad m + n = 0 \text{ or } m + 2n = 0$$

Now let us first solve the equations $l + 3m + 5n = 0$ and $m + n = 0$

These give $m = -n$ and $l = -2n$, *i.e.*, $= \frac{l}{-2} = \frac{m}{-1} = \frac{n}{1}$ ...(*i*)

Similarly, solving the equations $l + 3m + 5n = 0$ and $m + 2n = 0$,

We get $$\frac{l}{1} = \frac{m}{-2} = \frac{n}{1} \qquad ...(ii)$$

(*i*) and (*ii*) give the direction ratios of the two lines.

If θ be the angle between these two lines, then

$$\cos\theta = \frac{(-2)\times 1 + (-1)\times(-2) + 1\times 1}{\sqrt{(2^2 + 1^2 + 1^2)}\sqrt{(1^2 + 2^2 + 1^2)}} = \frac{1}{6}, \quad \text{i.e.,} \quad \theta = \cos^{-1}\left(\frac{1}{6}\right).$$

PROBLEMS 3.2

1. If $\mathbf{A} = \mathbf{I} + 2\mathbf{J} + 3\mathbf{K}$, $\mathbf{B} = -\mathbf{I} + 2\mathbf{J} + \mathbf{K}$ and $\mathbf{C} = 3\mathbf{I} + \mathbf{J}$, find t such that $\mathbf{A} + t\mathbf{B}$ is perpendicular to $\mathbf{C}$.
2. (*i*) Show that $\left(\frac{\mathbf{A}}{a^2} - \frac{\mathbf{B}}{b^2}\right)^2 = \left(\frac{\mathbf{A} - \mathbf{B}}{ab}\right)^2$.

   (*ii*) Interpret geometrically $(\mathbf{C} - \mathbf{A}) \cdot (\mathbf{B} - \mathbf{C}) = 0$.
3. If $|\mathbf{A} + \mathbf{B}| = |\mathbf{A} - \mathbf{B}|$, show that $\mathbf{A}$ and $\mathbf{B}$ are mutually perpendicular.

4. If $\mathbf{A} = \mathbf{I} + 2\mathbf{J} - 3\mathbf{K}$ and $\mathbf{B} = 3\mathbf{I} - \mathbf{J} + 2\mathbf{K}$, show that $\mathbf{A} + \mathbf{B}$ is perpendicular to $\mathbf{A} - \mathbf{B}$. Also calculate the angle between $2\mathbf{A} + \mathbf{B}$ and $\mathbf{A} + 2\mathbf{B}$.
5. Show that the three concurrent lines with direction cosines (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3) are coplanar if
$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0.$$
6. Find the projection of the vector $\mathbf{I} - 2\mathbf{J} + \mathbf{K}$ on $4\mathbf{I} - 4\mathbf{J} + 7\mathbf{K}$.
7. The projection of a line on the coordinate axes are 12, 4, 3. Find the length and direction cosines of the line. *(Rajasthan, 2006)*
8. Show (by vector methods) that the mid-point of the hypotenuse of a right-angled triangle is equidistant from its vertices.
9. Prove (by vector methods) that the angle in a semi-circle is a right angle.
10. Show (by vector methods) that the diagonals of a rhombus intersect at right angles.
11. Show that the altitudes of a triangle meet in a point (called the *orthocentre*).
12. *ABCD* is a tetrahedron having the edges *BC* and *AC* at right angles to opposite edges *AD* and *BD* respectively. Show that the third pair of opposite edges *AB* and *CD* are also at right angles.
13. Find the angle between the lines whose direction cosines are given by the equations $l + m + n = 0$, $l^2 + m^2 + n^2 = 0$. *(Rajasthan, 2005)*
14. Show that the lines whose direction cosines are given by the equations $4lm - 3mn - nl = 0$, and $3l + m + 2n = 0$ are perpendicular. *(Anna, 2005)*
15. Show that the lines whose direction cosines are given by the equations $l + m + n = 0$, $al^2 + bm^2 + cn^2 = 0$ are
(*i*) perpendicular, if $a + b + c = 0$, (*ii*) parallel, if $a^{-1} + b^{-1} + c^{-1} = 0$.
16. Show that the straight lines whose direction cosines are given by the equations
$al + bm + cn = 0$, $fmn + gnl + hlm = 0$ are (*i*) perpendicular if $\frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0$ *(Osmania, 2003)*
(*ii*) parallel if $\sqrt{af} \pm \sqrt{bg} \pm \sqrt{ch} = 0$.
17. Show that the angle between any two diagonals of a cube is $\cos^{-1} 1/3$. *(V.T.U., 2009 ; Assam, 1999)*
18. (l_1, m_1, n_1), (l_2, m_2, n_2) and (l_3, m_3, n_3) are the direction cosines of three mutually perpendicular lines. Prove that the line whose d.c.'s are proportional to $l_1 + l_2 + l_3$, $m_1 + m_2 + m_3$, $n_1 + n_2 + n_3$ makes equal angles with the axes. *(V.T.U. 2003)*
19. *AB*, *BC* are the diagonals of adjacent faces of a rectangular box with its centre at the origin *O*, its edges are parallel to the axes. If the angles *BOC*, *COA* and *AOB* are equal to θ, ϕ, ψ respectively, prove that
$$\cos\theta + \cos\phi + \cos\psi = -1.$$

3.6 VECTOR, OR CROSS PRODUCT

(1) Definition. *The vector, or cross product of two vectors* **A** *and* **B** *is defined as a vector such that*

(*i*) *its magnitude is ab* $\sin\theta$, θ *being the angle between* **A** *and* **B**,

(*ii*) *its direction is perpendicular to the plane of* **A** *and* **B**,

and (*iii*) *it forms with* **A** *and* **B** *a right-handed system.*

If **N** be a unit vector normal to the plane of **A** and **B** (**A**, **B**, **N** forming a right-handed system), then
$$\mathbf{A} \times \mathbf{B} = ab \sin\theta\, \mathbf{N}.$$

(2) Geometrical interpretation. $\mathbf{A} \times \mathbf{B}$ *represents twice the vector area of the triangle having the vectors* **A** *and* **B** *as its adjacent sides.*

If **N** be a unit vector normal to the plane of the triangle *OAB*, then
$$\mathbf{A} \times \mathbf{B} = ab \sin\theta\, \mathbf{N}$$
$$= 2\left(\frac{1}{2} ab \sin\theta\right)\mathbf{N} = 2\Delta\, OAB\, \mathbf{N} = 2\Delta\, \overrightarrow{OA}\, B.$$

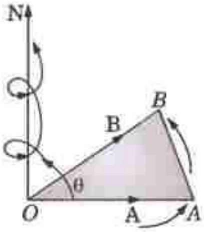


Fig. 3.19

(3) Properties and other results

I. Vector product of two vectors is not commutative,
$$\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A}. \text{ In fact, } \mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}.$$

for $\quad \mathbf{A} \times \mathbf{B} = ab \sin\theta\, \mathbf{N}$ or $2\Delta\, \overrightarrow{OAB}$.

and $\quad \mathbf{B} \times \mathbf{A} = ab \sin(-\theta)\, \mathbf{N} = -ab \sin\theta\, \mathbf{N}$ or $2\Delta \overrightarrow{OBA}$.

II. The necessary and sufficient condition for two non-zero vectors to be parallel is that their vector product should be zero.

When the vectors **A** and **B** are parallel, the angle θ between them is 0 and 180° so that $\sin\theta = 0$, and as such $\mathbf{A} \times \mathbf{B} = \mathbf{0}$.

Conversely, when $\quad \mathbf{A} \times \mathbf{B} = \mathbf{0}$; $ab \sin\theta = 0$

i.e., $\quad \sin\theta = 0 \qquad (\because\ a \neq 0, b \neq 0)$

or $\quad \theta = 0$ or 180°. In particular, $\mathbf{A} \times \mathbf{A} = 0$.

III. For the orthonormal vector trial **I, J, K,** we have the relations :

$$\mathbf{I} \times \mathbf{I} = \mathbf{J} \times \mathbf{J} = \mathbf{K} \times \mathbf{K} = \mathbf{0}$$
$$\mathbf{I} \times \mathbf{J} = \mathbf{K}, \quad \mathbf{J} \times \mathbf{I} = -\mathbf{K}$$
$$\mathbf{J} \times \mathbf{K} = \mathbf{I}, \quad \mathbf{K} \times \mathbf{J} = -\mathbf{I}$$
$$\mathbf{K} \times \mathbf{I} = \mathbf{J}, \quad \mathbf{I} \times \mathbf{K} = -\mathbf{J}.$$

IV. Relation between scalar and vector products.

We have $\quad (\mathbf{A} \,.\, \mathbf{B})^2 = a^2b^2 \cos\theta = a^2b^2 - a^2b^2 \sin^2\theta = a^2b^2 - (\mathbf{A} \times \mathbf{B}) \,.\, (\mathbf{A} \times \mathbf{B})$

$$(\mathbf{A} \times \mathbf{B})^2 = \mathbf{A}^2\mathbf{B}^2 - (\mathbf{A} \,.\, \mathbf{B})^2.$$

V. Vector product of two vectors is distributive

i.e., $\quad (\mathbf{A} + \mathbf{B}) \times \mathbf{C} = \mathbf{A} \times \mathbf{C} + \mathbf{B} \times \mathbf{C}.$

VI. Analytical expression for the vector product.

If $\quad \mathbf{A} = a_1\mathbf{I} + a_2\mathbf{J} + a_3\mathbf{K}, \mathbf{B} = b_1\mathbf{I} + b_2\mathbf{J} + b_3\mathbf{K}$ *then* $\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

For we get

$$\mathbf{A} \times \mathbf{B} = (a_2b_3 - a_3b_2)\mathbf{I} + (a_3b_1 - a_1b_3)\mathbf{J} + (a_1b_2 - a_2b_1)\mathbf{K}$$

whence follows the required result.

Example 3.12. *If* $\mathbf{A} = 4\mathbf{I} + 3\mathbf{J} + \mathbf{K}$, $\mathbf{B} = 2\mathbf{I} - \mathbf{J} + 2\mathbf{K}$, *find a unit vector* **N** *perpendicular to vectors* **A** *and* **B** *such that* **A, B, N** *form a right handed system. Also find the angle between the vectors A and B.*

Solution. Since $\quad \mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ 4 & 3 & 1 \\ 2 & -1 & 2 \end{vmatrix} = 7\mathbf{I} - 6\mathbf{J} - 10\mathbf{K}$

and $\quad |\, \mathbf{A} \times \mathbf{B}\, | = \sqrt{[(7)^2 + (-6)^2 + (-10)^2]} = \sqrt{185}$

$\therefore$ Unit vector $\mathbf{N} \perp$ to **A** and **B** $= \dfrac{\mathbf{A} \times \mathbf{B}}{|\,\mathbf{A} \times \mathbf{B}\,|} = (7\mathbf{I} - 6\mathbf{J} - 10\mathbf{K})/\sqrt{185}$

Also $\quad a = \sqrt{4^2 + 3^2 + 1^2} = \sqrt{26}$ and $b = 3$.

If θ be the angle between **A** and **B**, then $|\, \mathbf{A} \times \mathbf{B}\, | = ab \sin\theta$, *i.e.*, $\sin\theta = |\, \mathbf{A} \times \mathbf{B}\, |/ab$

Thus $\quad \sin\theta = \sqrt{185}/3\sqrt{26}$ whence $\theta = 62°\ 40'$.

Example 3.13. (*i*) *Prove that the area of the triangle whose vertices are* **A, B, C** *is*

$$\frac{1}{2}\ |\mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A} + \mathbf{A} \times \mathbf{B}|$$

(*ii*) *Calculate the area of the triangle whose vertices are A*(1, 0 – 1), *B*(2, 1, 5) *and C*(0, 1, 2).

Solution. (*i*) Let $A(\mathbf{A})$, $B(\mathbf{B})$, $C(\mathbf{C})$ be the vertices of the triangle ABC (Fig. 3.20) and O, the origin so that

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \mathbf{C} - \mathbf{B}$$

and $\quad \vec{BA} = \vec{OA} - \vec{OB} = \mathbf{A} - \mathbf{B}$

$\therefore$ Vector area of ΔABC

$$= \frac{1}{2}[\vec{BC} \times \vec{BA}] = \frac{1}{2}[(\mathbf{C} - \mathbf{B}) \times (\mathbf{A} - \mathbf{B})]$$

$$= \frac{1}{2}[\mathbf{C} \times \mathbf{A} - \mathbf{C} \times \mathbf{B} - \mathbf{B} \times \mathbf{A} + \mathbf{B} \times \mathbf{B}]$$

$$= \frac{1}{2}[\mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A} + \mathbf{A} \times \mathbf{B}] \quad [\because \; \mathbf{B} \times \mathbf{B} = 0]$$

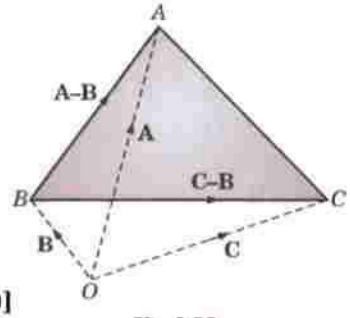


Fig. 3.20

Thus area of $\quad \Delta ABC = \frac{1}{2}\,|\mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A} + \mathbf{A} \times \mathbf{B}|.$

(*ii*) Let O be the origin so that

$$\vec{OA} = \mathbf{I} - \mathbf{K},\; \vec{OB} = 2\mathbf{I} + \mathbf{J} + 5\mathbf{K} \text{ and } \vec{OC} = \mathbf{J} + 2\mathbf{K}$$

Then $\quad \vec{BC} = \vec{OC} - \vec{OB} = -2\mathbf{I} - 3\mathbf{K}$

and $\quad \vec{BA} = \vec{OA} - \vec{OB} = -\mathbf{I} - \mathbf{J} - 6\mathbf{K}$

$\therefore$ Vector area of $\quad \Delta ABC = \frac{1}{2}(\vec{BC} \times \vec{BA}) = \frac{1}{2}\begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ -2 & 0 & -3 \\ -1 & -1 & -6 \end{vmatrix}$

Thus area of $\quad \Delta ABC = \frac{1}{2}\,|-3\mathbf{I} - 9\mathbf{J} + 2\mathbf{K}| = \frac{1}{2}\sqrt{94}.$

Example 3.14. *In a triangle ABC; D, E, F are the mid-points of the sides BC, CA, AB; prove that*

$$\Delta DEF = \Delta FCE = \frac{1}{4}\Delta ABC.$$

Solution. Take B as the origin and let the position vectors of C and A be $\mathbf{C}$ and $\mathbf{A}$ (Fig 3.21); so that the position vectors of D, E, F are

$$\mathbf{C}/2,\; (\mathbf{C} + \mathbf{A})/2,\; \mathbf{A}/2.$$

$$\therefore \quad \Delta \vec{DEF} = \frac{1}{2}(\vec{DE} \times \vec{DF}) = \frac{1}{2}\left(\frac{\mathbf{C}+\mathbf{A}}{2} - \frac{\mathbf{C}}{2}\right)\left(\frac{\mathbf{A}}{2} - \frac{\mathbf{C}}{2}\right)$$

$$= \frac{1}{8}[\mathbf{A} \times (\mathbf{A} - \mathbf{C})] = \frac{1}{8}\,\mathbf{C} \times \mathbf{A} = \frac{1}{4}\Delta \vec{ABC}$$

$$\Delta \vec{FCE} = \frac{1}{2}(\vec{FC} \times \vec{FE}) = \frac{1}{2}[\mathbf{C} - \mathbf{A}/2) \times \mathbf{C}/2]$$

$$= \frac{1}{8}\,\mathbf{C} \times \mathbf{A} = \frac{1}{4}\Delta \vec{ABC}. \text{ Hence the result.}$$

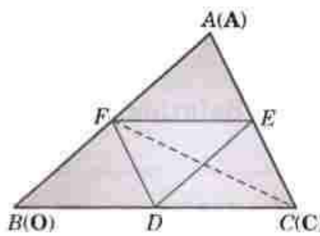


Fig. 3.21

Example 3.15. *Prove that*

(*i*) *sin (A + B) = sin A cos B + cos A sin B.*

(*ii*) *cos (A + B) = cos A cos B – sin A sin B.*

Solution. Let $\mathbf{I}, \mathbf{J}$ denote unit vectors along two perpendicular lines OX, OY so that

$$\mathbf{I}^2 = \mathbf{J}^2 = 1,\; \mathbf{I}.\mathbf{J} = 0$$

and $\quad \mathbf{I} \times \mathbf{I} = \mathbf{J} \times \mathbf{J} = \mathbf{0}$

Let $\quad \angle POX = A$ and $\angle XOQ = B,$

so that $\quad \angle POQ = A + B.$

If $OP = p$ and $OQ = q$, then the coordinates of P are $(p \cos A, -p \sin A)$ and those of Q are $(q \cos B, q \sin B)$ so that

$$\vec{OP} = (p \cos A)\mathbf{I} - (p \sin A)\mathbf{J}$$

$$\vec{OQ} = (q \cos B)\mathbf{I} + (q \sin B)\mathbf{J}$$

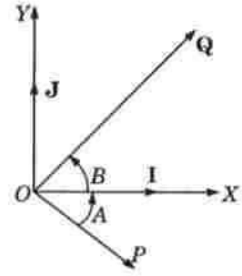


Fig. 3.22

Then $|\vec{OP} \times \vec{OQ}| = |[(p \cos A)\mathbf{I} - (p \sin A)\mathbf{J}] \times [(q \cos B)\mathbf{I} + (q \sin B)\mathbf{J}|$

$= pq\ |\cos A \sin B\ (\mathbf{I} \times \mathbf{J}) - \sin A \cos B\ (\mathbf{J} \times \mathbf{I})|$

$= pq\ (\cos A \sin B + \sin A \cos B)$ for $|\ \mathbf{I} \times \mathbf{J}\ | = 1$

Also $|\vec{OP} \times \vec{OQ}| = pq \sin (A + B)$. Equating the two expressions, we get (*i*).

Similarly, (*ii*) follows from $\vec{OP} . \vec{OQ} = pq \cos (A + B)$.

Example 3.16. *In any triangle ABC, prove that*

(*i*) $a/\sin A = b/\sin B = c/\sin C.$ *(Sine formula)*

(*ii*) $a = b \cos C + c \cos B.$ *(Projection formula)*

(*iii*) $a^2 = b^2 + c^2 - 2bc \cos A.$ *(Cosine formula)*

Solution. From ΔABC, we have $\vec{BC} + \vec{CA} + \vec{AB} = 0$

or $$\vec{CA} + \vec{AB} = -\vec{BC} \qquad ...(\lambda)$$

(*i*) Multiplying (λ) vectorially by $\vec{AB}$, we get

$$\vec{CA} \times \vec{AB} = -\vec{BC} \times \vec{AB}$$

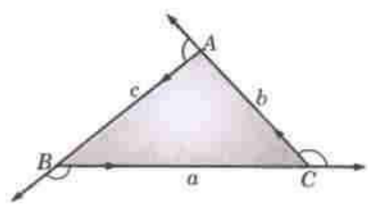


Fig. 3.23

or $$|\vec{CA} \times \vec{AB}| = |\vec{BC} \times \vec{AB}|$$

$\therefore$ $$bc \sin (\pi - A) = ac \sin (\pi - B)$$

or $$a/\sin A = b/\sin B.$$

Similarly, multiplying (λ) vectorially by $\vec{CA}$, we get

$a/\sin A = c/\sin C$, whence follows the result.

(*ii*) Multiplying (λ) scalarly by $\vec{BC}$, we get $\vec{CA} . \vec{BC} + \vec{AB} . \vec{BC} = -(\vec{BC})^2$

$\therefore\quad ba \cos (\pi - C) + ca \cos (\pi - B) = -a^2 \quad$ or $\quad a = b \cos C + c \cos B.$

(*iii*) Squaring (λ), we get

$$(\vec{CA})^2 + (\vec{AB})^2 + 2\vec{CA} . \vec{AB} = (\vec{BC})^2$$

i.e., $$b^2 + c^2\ 2bc \cos (\pi - A) = a^2 \text{ or } a^2 = b^2 + c^2 - 2bc \cos A.$$

PROBLEMS 3.3

1. Given $\mathbf{A} = 2\mathbf{I} + 2\mathbf{J} - \mathbf{K}$, $\mathbf{B} = 6\mathbf{I} - 3\mathbf{J} + 2\mathbf{K}$, find $\mathbf{A} \times \mathbf{B}$ and the unit vector perpendicular to both **A** and **B**. Also determine the sine of the angle between **A** and **B**.
2. If **A** and **B** are unit vectors and θ is the angle between them, show that $\sin \dfrac{\theta}{2} = \dfrac{1}{2}\,|\mathbf{A} - \mathbf{B}|$.
3. Find a unit vector normal to the plane of $\mathbf{A} = 3\mathbf{I} - 2\mathbf{J} + 4\mathbf{K}$ and $\mathbf{B} = \mathbf{I} + \mathbf{j} - 2\mathbf{K}$.
4. For any vector **A**, show that $|\mathbf{A} \times \mathbf{I}|^2 + |\mathbf{A} \times \mathbf{J}|^2 + |\mathbf{A} \times \mathbf{K}|^2 = 2\,|\mathbf{A}|^2$.
5. By vector method, find the area of the triangle whose vertices are $(3, -1, 2)$, $(1, -1, -3)$ and $(4, -3, 1)$.
6. (*a*) Prove that the vector area of the quadrilateral *ABCD* is $\dfrac{1}{2}\ \vec{AC} \times \vec{BD}$.

   (*b*) If $3\mathbf{I} + \mathbf{J} - 2\mathbf{K}$ and $\mathbf{I} - 3\mathbf{J} - 4\mathbf{K}$ are the diagonals of a parallelogram. Find its area.

7. Given vectors $\mathbf{A} = \mathbf{I} - 3\mathbf{J} + 2\mathbf{K}$ and $\mathbf{B} = 2\mathbf{I} + \mathbf{J} - \mathbf{K}$. Find the projection of $\mathbf{A} \times \mathbf{B}$ parallel to $5\mathbf{I} - \mathbf{K}$.
8. If $\mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{0}$, prove that $\mathbf{A} \times \mathbf{B} = \mathbf{B} \times \mathbf{C} = \mathbf{C} \times \mathbf{A}$, and interpret it geometrically.
9. Show that the perpendicular distance of the point **C** from the line joining **A** and **B** is $|\mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A} + \mathbf{A} \times \mathbf{B}| \div |\mathbf{B} - \mathbf{A}|$
10. In AC, diagonal of the parallelogram $ABCD$, a point **P** is taken. Prove that $\Delta BAP = \Delta DAP$.
11. Prove by vector methods, that
(i) $\sin (A - B) = \sin A \cos B - \cos A \sin B$; (ii) $\cos (A - B) = \cos A \cos B + \sin A \sin B$. (Cochin, 1999)
12. In any triangle ABC, prove by vector methods, that
(i) $b = c \cos A + a \cos C$; (ii) $c^2 = a^2 + b^2 - 2ab \cos C$.

3.7 PHYSICAL APPLICATIONS

(1) Work done as a scalar product. If constant force **F** acting on a particle displaces it from the position A to position B, then

Work done = (resolved part of F in the direction of AB). AB

$$= F \cos \theta \,.\, AB = \mathbf{F}.\,\vec{AB}$$

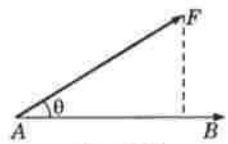


Fig. 3.24

Thus, the work done by a constant force is the scalar (or dot) product of the vectors representing the force and the displacement.

Example 3.17. *Constant forces* $\mathbf{P} = 2\mathbf{I} - 5\mathbf{J} + 6\mathbf{K}$ *and* $\mathbf{Q} = -\mathbf{I} + 2\mathbf{J} - \mathbf{K}$ *act on a particle. Determine the work done when the particle is displaced from A to B the position vectors of A and B being* $4\mathbf{I} - 3\mathbf{J} - 2\mathbf{K}$ *and* $6\mathbf{I} + \mathbf{J} - 3\mathbf{K}$ *respectively.*

Solution. Resultant force $\qquad \mathbf{F} = \mathbf{P} + \mathbf{Q} = \mathbf{I} - 3\mathbf{J} + 5\mathbf{K}$

and $\qquad \vec{AB} = \vec{OB} - \vec{OA} = (6\mathbf{I} + \mathbf{J} - 3\mathbf{K}) - (4\mathbf{I} - 3\mathbf{J} - 2\mathbf{K}) = 2\mathbf{I} + 4\mathbf{J} - \mathbf{K}$

$\therefore$ *Work done* $\qquad = \mathbf{F}.\,\vec{AB} = (\mathbf{I} - 3\mathbf{J} + 5\mathbf{K}) \,.\, (2\mathbf{I} + 4\mathbf{J} - \mathbf{K})$

$\qquad = 1 \,.\, 2 - 3 \,.\, 4 + 5 \,.\, (-1) = -15$ units.

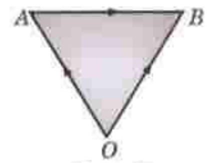


Fig. 3.25

(2) Normal flux. Consider the flow of a liquid through an element of area δs with a velocity **V** inclined at an angle θ to the out ward unit normal **N** to the surface δs (Fig. 3.26).

$\therefore$ Normal flux of the liquid through δs in unit time

$$V \cos \theta \,.\, \delta s = \mathbf{V} \,.\, \mathbf{N}\delta s.$$

Thus, *the rate of normal flux per unit area* = **V** . **N**

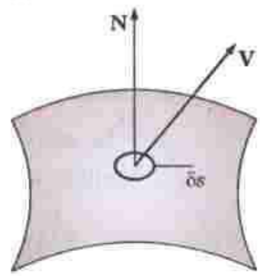


Fig. 3.26

Obs. We can also apply this result to the case of electric or magnetic flux.

(3) Moment of a force about a point. Suppose the moment of the force **F** acting at the point P about the point A is required.

Draw $AM \perp$ the line of action of **F** (Fig. 3.27). If θ be the angle between $\vec{AP}$ and **F** and **N** be a unit vector $\perp$ to their plane, then $\vec{AP} \times \mathbf{F} = (AP.\, F \sin \theta)\, \mathbf{N} = F(AP \sin \theta)\, \mathbf{N} = (F \,.\, AM)\, \mathbf{N}$

Clearly, (i) the magnitude of $\vec{AP} \times \mathbf{F} = F \,.\, AM$ which is the numerical measure of the moment of **F** about A.

and (ii) the direction of $\vec{AP} \times \mathbf{F}$ is the direction of the moment of **F** about A.

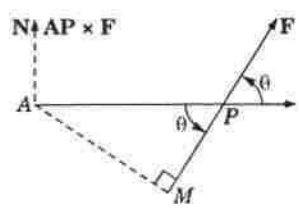


Fig. 3.27

Hence the moment (or torque) of **F** about A is $\vec{AP} \times \mathbf{F}$.

Example 3.18. *Find the torque about the point* $2\mathbf{I} + \mathbf{J} - \mathbf{K}$ *of a force represented by* $4\mathbf{I} + \mathbf{K}$ *acting through the point* $\mathbf{I} - \mathbf{J} + 2\mathbf{K}$.

Solution. Let O be the origin and P be the point, moment about which of the force $\vec{AB}$ through A, is required (Fig. 3.28).

$\therefore$ $\qquad \overrightarrow{OP} = 2\mathbf{I} + \mathbf{J} - \mathbf{K},$

$\qquad \overrightarrow{OA} = \mathbf{I} - \mathbf{J} + 2\mathbf{K}$, and $\overrightarrow{AB} = 4\mathbf{I} + \mathbf{K}$.

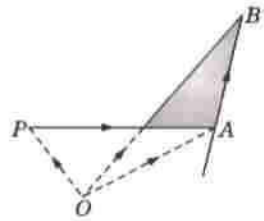


Fig. 3.28

Then, $\qquad \overrightarrow{PA} = \overrightarrow{OA} - \overrightarrow{OP} = -\mathbf{I} - 2\mathbf{J} + 3\mathbf{K}$

$\therefore$ Moment of the force $\overrightarrow{AB}$ about P

$$= \overrightarrow{PA} \times \overrightarrow{AB} = (-\mathbf{I} - 2\mathbf{J} + 3\mathbf{K}) \times (4\mathbf{I} + \mathbf{K})$$

$$= \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ -1 & -2 & 3 \\ 4 & 0 & 1 \end{vmatrix} = -2\mathbf{I} + 13\mathbf{J} + 8\mathbf{K}$$

$\therefore$ Magnitude of the moment $= \sqrt{(4 + 169 + 64)} = 15.4$

(4) Moment of a force about a line.

Def. *The moment of a force* **F** *about a line* **D** *is the resolved part along* **D** *of the moment of* **F** *about any point on* **D**.

Example 3.19. *Find the moment about a line through the origin having direction of* $2\mathbf{I} + 2\mathbf{J} + \mathbf{K}$, *due to a 30 kg force acting at a point* $(-4, 2, 5)$ *in the direction of* $12\mathbf{I} - 4\mathbf{J} - 3\mathbf{K}$.

Solution. Let **D** be the given line through the origin O and **F** the force through **A**$(-4, 2, 5)$.

Clearly, $\qquad \overrightarrow{OA} = -4\mathbf{I} + 2\mathbf{J} + 5\mathbf{K}$

and the force $\qquad \mathbf{F} = 30\left(\dfrac{12\mathbf{I} - 4\mathbf{J} - 3\mathbf{K}}{13}\right)$

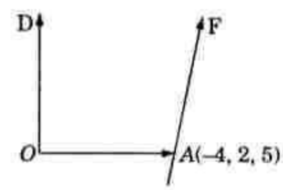


Fig. 3.29

$\therefore$ Moment of **F** about $O = \overrightarrow{OA} \times \mathbf{F}$

$$= \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ -4 & 2 & 5 \\ \dfrac{360}{13} & \dfrac{-120}{13} & \dfrac{-90}{13} \end{vmatrix} = \frac{60}{13}(7\mathbf{I} + 24\mathbf{J} - 4\mathbf{K})$$

Thus the moment of **F** *about the line* **D**

= resolved part of the moment of **F** about O along **D**,

i.e., $$\frac{60}{13}(7\mathbf{I} + 24\mathbf{J} - 4\mathbf{K}) \cdot \hat{\mathbf{D}}$$

$$= \frac{60}{13}(7\mathbf{I} + 24\mathbf{J} - 4\mathbf{K}) \cdot \frac{2\mathbf{I} + 2\mathbf{J} + \mathbf{K}}{\sqrt{(4 + 4 + 1)}} = \frac{20}{13}(7 \times 2 + 24 \times 2 - 4 \times 1) = 89.23.$$

(5) Angular velocity of a rigid body

Let a rigid body be rotating about the axis OM with angular velocity ω radians per second (Fig. 3.30). Let P be a point of the body such that $\overrightarrow{OP} = \mathbf{R}$ and $\angle MOP = \theta$. Draw $PM \perp OM$.

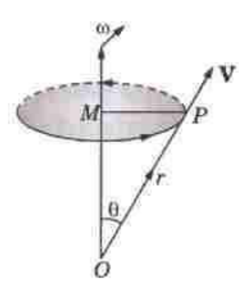


Fig. 3.30

Now if **N** be a unit vector $\perp \vec{\omega}$ **R** then

$$\vec{\omega} \times \mathbf{R} = \omega r \sin\theta \,.\, \mathbf{N} = \omega MP \,.\, \mathbf{N}$$

= (speed of **P**) **N**

= velocity **V** of **P** in a direction $\perp$ to the plane MOP.

Hence $\qquad \mathbf{V} = \vec{\omega} \times \mathbf{R}$.

Example 3.20. *A rigid body is spinning with angular velocity 27 radians per second about an axis parallel to* $2\mathbf{I} + \mathbf{J} - 2\mathbf{K}$ *passing through the point* $\mathbf{I} + 3\mathbf{J} - \mathbf{K}$. *Find the velocity of the point of the body whose position vector is* $4\mathbf{I} + 8\mathbf{J} + \mathbf{K}$.

Solution. Unit vector along the direction of $\vec{\omega} = \dfrac{2\mathbf{I} + \mathbf{J} - 2\mathbf{K}}{\sqrt{(4+1+4)}} = \dfrac{1}{3}(2\mathbf{I} + \mathbf{J} - 2\mathbf{K})$

$\therefore$ Angular velocity $\vec{\omega} = \dfrac{27}{3}(2\mathbf{I} + \mathbf{J} - 2\mathbf{K}) = 9(2\mathbf{I} + \mathbf{J} - 2\mathbf{K})$

Let A be the point $\mathbf{I} + 3\mathbf{J} - \mathbf{K}$ and the point P of the body be $(4\mathbf{I} + 8\mathbf{J} - \mathbf{K})$ so that

$$\vec{AP} = (4\mathbf{I} + 8\mathbf{J} + \mathbf{K}) - (\mathbf{I} + 3\mathbf{J} - \mathbf{K}) = 3\mathbf{I} + 5\mathbf{J} + 2\mathbf{K}$$

$\therefore$ Velocity vector of $P = \mathbf{V} = \vec{\omega} \times \vec{AP} = 9(2\mathbf{I} + \mathbf{J} - 2\mathbf{K}) \times (3\mathbf{I} + 5\mathbf{J} + 2\mathbf{K})$

$$= 9\begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ 2 & 1 & -2 \\ 3 & 5 & 2 \end{vmatrix} = 9(12\mathbf{I} - 10\mathbf{J} + 7\mathbf{K})$$

and its magnitude $9\sqrt{(144+100+49)} = 9\sqrt{293}$.

PROBLEMS 3.4

1. A particle acted on by constant forces $4\mathbf{I} + \mathbf{J} - 3\mathbf{K}$ and $3\mathbf{I} + \mathbf{J} - \mathbf{K}$ is displace from the point $\mathbf{I} + 2\mathbf{J} + 3\mathbf{K}$ to the point $5\mathbf{I} + 4\mathbf{J} + \mathbf{K}$. Find the total work done by the forces.
2. Forces $2\mathbf{I} - 5\mathbf{J} + 6\mathbf{K}$, $-\mathbf{I} + 2\mathbf{J} - \mathbf{K}$ and $2\mathbf{I} + 7\mathbf{J}$ act on a particle P whose position vector is $4\mathbf{I} - 3\mathbf{J} - 2\mathbf{K}$. Determine the work done by the forces in a displacement of the particle to the point $Q\,(6, 1, -3)$.
   Also find the vector moment of the resultant of three forces acting at P about the point Q.
3. Forces of magnitudes 5, 3 1 units act in the directions $6\mathbf{I} + 2\mathbf{J} + 3\mathbf{K}$, $3\mathbf{I} - 2\mathbf{J} + 6\mathbf{K}$, $2\mathbf{I} - 3\mathbf{J} - 6\mathbf{K}$ respectively on a particle which is displaced from the point $(2, 1, -3)$ to $(5, -1, 1)$. Find the work done by the forces.
4. The point of application of the force $(-2, 4, 7)$ is displaced from the point $(3, -5, 1)$ to the point $(5, 9, 7)$. But the force is suddenly halved when the point of application moves half the distance. Find the work done.
5. A force $\mathbf{F} = 3\mathbf{I} + 2\mathbf{J} - 4\mathbf{K}$ is applied at the point $(1, -1, 2)$. Find the moment of the force about the point $(2, -1, 3)$. *(Assam, 1999)*
6. A force with components $(5, -4, 2)$ acts at a point P which is at a distance 3 units from the origin. If the moment of the force about origin has components $(12, 8, -14)$, find the co-ordinates of P.
7. Find the moment of the force $\mathbf{F} = 2\mathbf{I} + 2\mathbf{J} - \mathbf{K}$ acting at the point $(1, -2, 1)$ about z-axis.
8. A force of 10 kg acts in a direction equally inclined to the co-ordinate axes through the point $(3, -2, 5)$. Find the magnitude of the moment of the force about a line through the origin and whose direction ratios are $(2, -3, 6)$.
9. A rigid body is rotating at 2.5 radians per second about an axis OR, where R is the point $2\mathbf{I} - 2\mathbf{J} + \mathbf{K}$ relative to O. Find the velocity of the particle of the body at the point $4\mathbf{I} + \mathbf{J} + \mathbf{K}$. (All lengths are in cm).

3.8 PRODUCTS OF THREE OR MORE VECTORS

With any three vectors **A**, **B**, **C**, we can form the products (**A** . **B**) **C**, (**A** × **B**) . **C** and (**A** × **B**) × **C**. The first being the product of a scalar **A** . **B** and a vector **C**, represents a vector in the direction of **C**. The second being the scalar product of vectors **A** × **B** and **C**, represents a scalar and is usually called the *scalar product of three vectors*. The third being the vector product of the vectors **A** × **B** and **C**, represents a vector and is usually known as the *vector product of three vectors*.

The reader must, however, note that the products of the form **A** . (**B** . **C**), **A** × (**B** . **C**) and **A**(**B** × **C**) are meaningless.

In practical applications, we seldom come across products of more than three vectors. Such products if they occur can, in general, be reduced by using successively the expansion formula for vector triple products. As an illustration, we shall consider two products (**A** × **B**). (**C** × **D**) and (**A** × **B**) × (**C** × **D**) of any four vectors, the former being a scalar and a latter a vector.

3.9 SCALAR PRODUCT OF THREE VECTORS

(1) Definition. *If* **A, B, C** *be any three vectors then the scalar or dot product of* **A × B** *with* **C** *is called the scalar product of the three vectors* **A, B, C** *and is written as* (**A × B**) . **C** *or* [**ABC**].

No ambiguity can arise by omitting the brackets in $(A \times B). C$ as $A \times (B . C)$ would be meaningless .

(2) Geometrical interpretation. *The Product* **A × B . C** *represents numerically the volume of a parallelopiped having* **A, B, C** *as coterminous edges.*

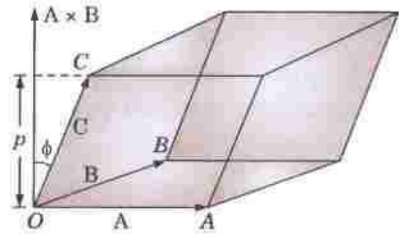


Fig. 3.31

Consider a parallelopiped with $\vec{OA} = \mathbf{A}$, $\vec{OB} = \mathbf{B}$, $\vec{OC} = \mathbf{C}$ as coterminous edges (Fig. 3.31).

Let V be its volume, α the area of each of the two faces parallel to the vectors **A** and **B** and p the perpendicular distance between these faces.

Then $|\mathbf{A} \times \mathbf{B}| = \alpha$ and $|\mathbf{C}| \cos \phi = p$ or $-p$ according as **A, B, C** form a right-handed or left-handed triad.

$$\therefore \quad \mathbf{A} \times \mathbf{B} . \mathbf{C} = |\mathbf{A} \times \mathbf{B}| . |\mathbf{C}| \cos \phi = \pm \alpha p = \pm V.$$

Thus [**ABC**] = V *or* $-V$ *according as* **A, B, C** *form a right-handed or left-handed triad.*

(*Kerala, 1990; J.N.T.U., 1988*)

In particular, for an orthonormal right-handed vector triad **I, J, K**,

$$[\mathbf{IJK}] = \mathbf{I} \times \mathbf{J} . \mathbf{K} = \mathbf{K} . \mathbf{K} = I.$$

(3) Properties and other results.

I. The condition for three vectors to be coplanar is that their scalar triple product should vanish.

If three vectors **A, B, C** anti coplanar, then the volume of the parallelopiped so formed is zero, *i.e.,* [**ABC**] = 0.

II. If any two vectors of a scalar triple product are equal, the product vanishes, i.e., [**ABC**] = 0 when either **A** = **B** or **B** = **C**, or **C** = **A**, for in this case the parallelopiped has zero volume.

III. **Two important rules** (*for evaluating a scalar triple product*). *Every scalar triple product*

(*i*) *is independent of the position of the dot or cross.*

and (*ii*) *depends upon the cyclic order of the vectors.*

It is easy to note that if **A, B, C** is a right-handed triad so are **B, C, A** and **C, A, B.**

Moreover a parallelopiped having **A, B, C** as coterminous edges is the same as that having **B, C, A** or **C, A, B** as coterminous edges.

Thus, if V be the volume of this parallelopiped,

$$\mathbf{A} \times \mathbf{B} . \mathbf{C} = V, \mathbf{B} \times \mathbf{C} . \mathbf{A} = V, \mathbf{C} \times \mathbf{A} . \mathbf{B} = V$$

Also, since **A . B = B . A**, we have

$$\mathbf{C} . \mathbf{A} \times \mathbf{B} = \mathbf{A} \times \mathbf{B} . \mathbf{C} = V$$
$$\mathbf{A} . \mathbf{B} \times \mathbf{C} = \mathbf{B} \times \mathbf{C} . \mathbf{A} = V$$
$$\mathbf{B} . \mathbf{C} \times \mathbf{A} = \mathbf{C} \times \mathbf{A} . \mathbf{B} = V$$

Thus
$$\left.\begin{matrix} \mathbf{AA} \times \mathbf{B} . \mathbf{C} = \mathbf{A} . \mathbf{B} \times \mathbf{C} \\ \mathbf{B} \times \mathbf{C} . \mathbf{A} = \mathbf{B} . \mathbf{C} \times \mathbf{A} \\ \mathbf{C} \times \mathbf{A} . \mathbf{B} = \mathbf{C} . \mathbf{A} \times \mathbf{B} \end{matrix}\right\} = V \quad ...(\alpha)$$

Further a right-handed triad becomes left-handed when the cyclic order of the vectors is changed. Therefore **A, C, B; B, A, C; C, B, A** being left-handed triads, it follows that

$$\mathbf{A} \times \mathbf{C} . \mathbf{B} = -V, \mathbf{B} \times \mathbf{A} . \mathbf{C} = -V, \mathbf{C} \times \mathbf{B} . \mathbf{A} = -V.$$

Thus
$$\left.\begin{matrix} \mathbf{AA} \times \mathbf{C} . \mathbf{B} = \mathbf{A} . \mathbf{C} \times \mathbf{B} \\ \mathbf{B} \times \mathbf{A} . \mathbf{C} = \mathbf{B} . \mathbf{A} \times \mathbf{C} \\ \mathbf{C} \times \mathbf{B} . \mathbf{A} = \mathbf{C} . \mathbf{B} \times \mathbf{A} \end{matrix}\right\} = -V \quad ...(\beta)$$

Obs. In support of the above rules, our notation [**ABC**] indicates the cyclic order of the factors and has nothing to do with position of the dot or the cross.

$\therefore$ The relations (α) and (β) can be compactly written as

$$[\mathbf{ABC}] = [\mathbf{BCA}] = [\mathbf{CAB}] = V \quad \text{and} \quad [\mathbf{ACB}] = [\mathbf{BAC}] = [\mathbf{CBA}] = -V.$$

IV. Scalar triple product is distributive

i.e., $$[\mathbf{A}, \mathbf{B}+\mathbf{C}, \mathbf{D}-\mathbf{E}] = [\mathbf{ABD}] - [\mathbf{ABE}] + [\mathbf{ACD}] - [\mathbf{ACE}]$$

V. If $\mathbf{A} = a_1\mathbf{I} + a_2\mathbf{J} + a_3\mathbf{K}$, $\mathbf{B} = b_1\mathbf{I} + b_2\mathbf{J} + b_3\mathbf{K}$, $\mathbf{C} = c_1\mathbf{I} + c_2\mathbf{J} + c_3\mathbf{K}$

then $$[\mathbf{ABC}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

As $$\mathbf{A}\times\mathbf{B} = (a_2b_3 - a_3b_2)\,\mathbf{I} + (a_3b_1 - a_1b_3)\,\mathbf{J} + (a_1b_2 - a_2b_1)\mathbf{K}$$

$\therefore$ $$[\mathbf{ABC}] = [(a_2b_3 - a_3b_2)\,\mathbf{I} + (a_3b_1 - a_1b_3)\,\mathbf{J} + (a_1b_2 - a_2b_1)\,\mathbf{K}]\,.\,(c_1\mathbf{I} + c_2\mathbf{J} + c_3\mathbf{K})$$
$$= c_1(a_2b_3 - a_3b_2) + c_2(a_3b_1 - a_1b_3) + c_3(a_1b_2 - a_2b_1)$$ which is the required result.

Obs. Linear dependence of vectors. Any three vectors **A, B, C** are said to be *linearly dependent* if one of these can be expressed as a linear combination of other two *i.e.,*

$$\mathbf{A} = m\mathbf{B} + n\mathbf{C}$$

where m, n are constants. This means that **A** lies in the plane of **B**, **C** *i.e.,* $[\mathbf{ABC}] = 0$. Thus *three vectors are linearly dependent if their scalar triple product is zero. Otherwise these vectors are linearly independent.*

Example 3.21. *Show that the points* $-6\mathbf{I} + 3\mathbf{J} + 2\mathbf{K}$, $3\mathbf{I} - 2\mathbf{J} + 4\mathbf{K}$, $5\mathbf{I} + 7\mathbf{J} + 3\mathbf{K}$ *and* $-13\mathbf{I} + 17\mathbf{J} - \mathbf{K}$ *are coplanar.*

Solution. Let $\vec{OA} = -6\mathbf{I} + 3\mathbf{J} + 2\mathbf{K}$, $\vec{OB} = 3\mathbf{I} - 2\mathbf{J} + 4\mathbf{K}$, $\vec{OC} = 5\mathbf{I} + 7\mathbf{J} + 3\mathbf{K}$

and $\vec{OD} = -13\mathbf{I} + 17\mathbf{J} - \mathbf{K}$. Then $\vec{AB} = \vec{OB} - \vec{OA} = 9\mathbf{I} - 5\mathbf{J} + 2\mathbf{K}$

Similarly, $\vec{AC} = 11\mathbf{I} + 4\mathbf{J} + \mathbf{K}$, and $\vec{AD} = -7\mathbf{I} + 14\mathbf{J} - 3\mathbf{K}$.

The given points will be coplanar if $\vec{AB}, \vec{AC}, \vec{AD}$ are coplanar, *i.e.,* if their scalar triple product is zero. Now

$$[\vec{AB}, \vec{AC}, \vec{AD}] = \begin{vmatrix} 9 & -5 & 2 \\ 11 & 4 & 1 \\ -7 & 14 & -3 \end{vmatrix} = 9(-12-14) + 5(-33+7) + 2(154+28) = 0$$

Hence the points A, B, C, D are coplanar.

Example 3.22. *Show that the volume of the tetrahedron ABCD is* $\frac{1}{6}[\vec{AB}, \vec{AC}, \vec{AD}]$.

Hence find the volume of the tetrahedron formed by the points (1, 1, 1), (2, 1, 3), (3, 2, 2) and (3, 3, 4).

Solution. (*i*) Volume of the tetrahedron $ABCD$

$$= \frac{1}{3}(\text{area of } \Delta ABC) \times (\text{height } h \text{ of } D \text{ above the plane } ABC)$$
$$= \frac{1}{6}(2 \text{ area of } \Delta ABC)h$$
$$= \frac{1}{6}(\text{volume of the parallelopiped with } AB, AC, AD \text{ as coterminus edges})$$
$$= \frac{1}{6}[\vec{AB}, \vec{AC}, \vec{AD}].$$

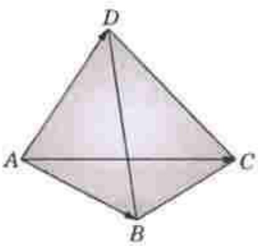


Fig. 3.32

(*ii*) Let $\vec{OA} = \mathbf{I} + \mathbf{J} + \mathbf{K}$, $\vec{OB} = 2\mathbf{I} + \mathbf{J} + 3\mathbf{K}$, $\vec{OC} = 3\mathbf{I} + 2\mathbf{J} + 2\mathbf{K}$ and $\vec{OD} = 3\mathbf{I} + 3\mathbf{J} + 4\mathbf{K}$.

Then $\vec{AB} = \vec{OB} - \vec{OA} = \mathbf{I} + 2\mathbf{K}$

Similarly, $AC = 2\mathbf{I} + \mathbf{J} + \mathbf{K}$ and $\vec{AD} = 2\mathbf{I} + 2\mathbf{J} + 3\mathbf{K}$

$\therefore$ Volume of the tetrahedron $ABCD = \frac{1}{6}[\vec{AB}, \vec{AC}, \vec{AD}] = \frac{1}{6}\begin{vmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 2 & 2 & 3 \end{vmatrix} = \frac{5}{6}.$

3.10 VECTOR PRODUCT OF THREE VECTORS

(1) Definition. *If* **A, B, C** *be any three vectors, then the vector or cross product of* **A × B** *with* **C** *is called the vector product of three vectors* **A, B, C** *and is written as* **(A × B) × C.**

Here the brackets are essential as $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$, expressing the fact that vector triple product is not associative.

(2) Expansion formula. *If* **A, B, C** *be any three vectors,* $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{C} \cdot \mathbf{A})\mathbf{B} - (\mathbf{C} \cdot \mathbf{B})\mathbf{A}$

In words (extreme × adjacent) × outer = (outer . extreme) adjacent – (outer . adjacent) extreme.

The vector $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ is perpendicular to the vector $\mathbf{A} \times \mathbf{B}$ and the latter is perpendicular to the plane containing **A** and **B**. Hence $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ lies in the plane of **A** and **B**. As such we can write

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = l\mathbf{A} + m\mathbf{B} \qquad \text{...(1)}$$

where l and m are some scalars.

Multiply both sides scalarly by **C**, then $\mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = l\mathbf{C} \cdot A + m\mathbf{C} \cdot \mathbf{B}$

The scalar triple product on the left-hand side is zero, since two of its vectors are equal.

$\therefore \qquad l(\mathbf{C} \cdot \mathbf{A}) + m(\mathbf{C} \cdot \mathbf{B}) = 0$

or
$$\frac{l}{\mathbf{C} \cdot \mathbf{B}} = \frac{m}{-\mathbf{C} \cdot \mathbf{A}} = n, \text{ say.}$$

Substituting the values of l and m in (1), we get

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = n(\mathbf{C} \cdot \mathbf{B})\mathbf{A} - n(\mathbf{C} \cdot \mathbf{A})\mathbf{B} \qquad \text{...(2)}$$

Evidently n is some numerical constant. To find it, take the special case $\mathbf{A} = \mathbf{I}, \mathbf{B} = \mathbf{C} = \mathbf{J}$. Then (2) gives

$$(\mathbf{I} \times \mathbf{J}) \times \mathbf{J} = n(\mathbf{J} \cdot \mathbf{J})\mathbf{I} - n(\mathbf{J} \cdot \mathbf{I})\mathbf{J}$$

i.e.,
$$\mathbf{K} \times \mathbf{J} = n\mathbf{I} \text{ or } -\mathbf{I} = n\mathbf{I}.$$

This gives $n = -1$. Hence (2) reduces to the required result.

Similarly, it can be shown that $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$

Cor. $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = \mathbf{0}.$

For L.H.S. $= (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} + (\mathbf{B} \cdot \mathbf{A})\mathbf{C} - (\mathbf{B} \cdot \mathbf{C})\mathbf{A} + (\mathbf{C} \cdot \mathbf{B})\mathbf{A} - (\mathbf{C} \cdot \mathbf{A})\mathbf{B}$ which vanishes identically.

Example 3.23. *If A, B, C, D be any four vectors, prove that*

(i) $(A \times B) \cdot (C \times D) = \begin{vmatrix} A \cdot C & B \cdot C \\ A \cdot D & B \cdot D \end{vmatrix}$ *(ii)* $(A \times B) \times (C \times D) = [ACD]\,B - [BCD]A$

Solution. (*i*) $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = [\mathbf{A} \times \mathbf{B}) \times \mathbf{C}] \cdot \mathbf{D}$ (interchanging the dot and cross)

$= [(\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{B} \cdot \mathbf{C})\mathbf{A}] \cdot \mathbf{D}$

$= (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})$ whence follows the result.

In particular, we have $(\mathbf{A} \times \mathbf{B})^2 = \mathbf{A}^2\mathbf{B}^2 - (\mathbf{A} \cdot \mathbf{B})^2$ which has already been proved in § 3.6 (3) – IV.

(*ii*) $(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \times \mathbf{B}) \times \mathbf{P}$, where $\mathbf{P} = \mathbf{C} \times \mathbf{D}$

$= (\mathbf{A} \cdot \mathbf{P})\mathbf{B} - (\mathbf{B} \cdot \mathbf{P})\mathbf{A} = (\mathbf{A} \cdot \mathbf{C} \times \mathbf{D})\mathbf{B} - (\mathbf{B} \cdot \mathbf{C} \times \mathbf{D})\mathbf{A}$

$= [\mathbf{ACD}]\mathbf{B} - [\mathbf{BCD}]\mathbf{A}.$

Example 3.24. *Show that the components of a vector* **B** *along and perpendicular to a vector* **A**, *in the plane of* **A** *and* **B**, *are*

$$\frac{\mathbf{A} \cdot \mathbf{B}}{\mathbf{A}^2} \text{ and } \frac{(\mathbf{A} \times \mathbf{B}) \times \mathbf{A}}{\mathbf{A}^2}.$$

Solution. Let $\vec{\mathbf{OA}} = \mathbf{A}$, $\vec{\mathbf{OB}} = \mathbf{B}$ and **OM** be the projection of **B** on **A** (Fig. 3.33)

$\therefore$ Component of **B** along **A** = OM (unit vector along **A**)

$$= (\mathbf{B} \cdot \hat{\mathbf{A}})\hat{\mathbf{A}} = \left(\frac{\mathbf{B} \cdot \mathbf{A}}{a}\right)\frac{\mathbf{A}}{a} \qquad [\because \mathbf{A} = a\,\hat{\mathbf{A}}]$$

$$= \frac{\mathbf{B} \cdot \mathbf{A}}{\mathbf{A}^2}\mathbf{A} \qquad [\because a^2 = \mathbf{A}^2]$$

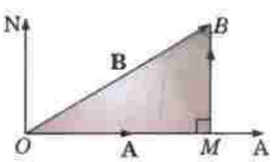


Fig. 3.33

Also component of $\mathbf{B} \perp \mathbf{A} = \overrightarrow{\mathbf{MB}}$

$$= \overrightarrow{OB} - \overrightarrow{OM} = \mathbf{B} - \frac{\mathbf{B}\cdot\mathbf{A}}{\mathbf{A}^2}\,\mathbf{A} = \frac{(\mathbf{A}\cdot\mathbf{A})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{A}}{\mathbf{A}^2} = \frac{(\mathbf{A}\times\mathbf{B})\times\mathbf{A}}{\mathbf{A}^2}.$$

Example 3.25. *Prove the formula*

$$(\mathbf{B}\times\mathbf{C})\cdot(\mathbf{A}\times\mathbf{D}) + (\mathbf{C}\times\mathbf{A})\cdot(\mathbf{B}\times\mathbf{D}) + (\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = 0.$$

and hence show that $\sin(\theta+\phi)\sin(\theta-\phi) = \sin^2\theta - \sin^2\phi$.

Solution. We know that

$$(\mathbf{B}\times\mathbf{C})\cdot(\mathbf{A}\times\mathbf{D}) = (\mathbf{B}\cdot\mathbf{A})(\mathbf{C}\cdot\mathbf{D}) - (\mathbf{B}\cdot\mathbf{D})(\mathbf{C}\cdot\mathbf{A})$$
$$(\mathbf{C}\times\mathbf{A})\cdot(\mathbf{B}\times\mathbf{D}) = (\mathbf{C}\cdot\mathbf{B})(\mathbf{A}\cdot\mathbf{D}) - (\mathbf{C}\cdot\mathbf{D})(\mathbf{A}\cdot\mathbf{B})$$
$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$$

Adding, we get

$$(\mathbf{B}\times\mathbf{C})\cdot(\mathbf{A}\times\mathbf{D}) + (\mathbf{C}\times\mathbf{A})\cdot(\mathbf{B}\times\mathbf{D}) + (\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = 0 \qquad ...(i)$$

Let the vectors **A, B, C, D** be acting along coplanar lines *OA, OB, OC, OD* respectively (Fig. 3.34).

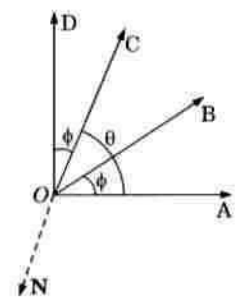


Fig. 3.34

Take $\angle AOC = \theta$ and $\angle AOB = \angle COD = \phi$,

so that $\angle AOD = \theta + \phi$ and $\angle BOC = \theta - \phi$

If **N** be a unit vector normal to the plane of these lines, then

$$(\mathbf{B}\times\mathbf{C})\cdot(\mathbf{A}\times\mathbf{D}) = [bc\sin(\theta-\phi)\mathbf{N}]\cdot[ad\sin(\theta+\phi)\mathbf{N}]$$
$$= abcd\sin(\theta+\phi)\sin(\theta-\phi) \qquad ...(ii)$$
$$(\mathbf{C}\times\mathbf{A})\cdot(\mathbf{B}\times\mathbf{D}) = [ca\sin(-\theta)\mathbf{N}]\cdot[bd\sin\theta\mathbf{N}]$$
$$= -abcd\sin^2\theta \qquad ...(iii)$$

and
$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = [ab\sin\phi\mathbf{N}]\cdot[cd\sin\phi\mathbf{N}]$$
$$= abcd\sin^2\phi \qquad ...(iv)$$

Substituting the values from (*ii*), (*iii*), (*iv*) in (*i*), we get

$abcd\sin(\theta+\phi)\sin(\theta-\phi) - abcd\sin^2\theta + abcd\sin^2\phi = 0$ whence follows the required result.

Example 3.26. *Prove that*

(*i*) $[\mathbf{B}\times\mathbf{C}, \mathbf{C}\times\mathbf{A}, \mathbf{A}\times\mathbf{B}] = [\mathbf{ABC}]^2$. (*Nagpur, 2009*)

(*ii*) $\mathbf{A}\times\{\mathbf{B}\times(\mathbf{C}\times\mathbf{D})\} = \mathbf{B}\cdot\mathbf{D}(\mathbf{A}\times\mathbf{C}) - \mathbf{B}\cdot\mathbf{C}(\mathbf{A}\times\mathbf{D})$.

Solution. (*i*) $[\mathbf{B}\times\mathbf{C}, \mathbf{C}\times\mathbf{A}, \mathbf{A}\times\mathbf{B}] = (\mathbf{B}\times\mathbf{C})\cdot(\mathbf{C}\times\mathbf{A})\times(\mathbf{A}\times\mathbf{B})$

$$= (\mathbf{B}\times\mathbf{C})\cdot\{[(\mathbf{C}\times\mathbf{A})\cdot\mathbf{B}]\mathbf{A} - [(\mathbf{C}\times\mathbf{A})\cdot\mathbf{A}]\mathbf{B}\}$$
$$= (\mathbf{B}\times\mathbf{C})\cdot\{[\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})]\mathbf{A}\} \qquad [\because\ [(\mathbf{C}\times\mathbf{A})\cdot\mathbf{A}] = 0$$
$$= [\mathbf{B}\times\mathbf{C})\cdot\mathbf{A}]\,[(\mathbf{B}\times\mathbf{C})\cdot\mathbf{A}] = [\mathbf{BCA}]^2 = [\mathbf{ABC}]^2 \qquad [\because\ [\mathbf{BCA}] = [\mathbf{ABC}]$$

(*ii*)
$$\mathbf{A}\times\{\mathbf{B}\times(\mathbf{C}\times\mathbf{D})\} = \mathbf{A}\times\{(\mathbf{B}\cdot\mathbf{D})\mathbf{C} - (\mathbf{B}\cdot\mathbf{C})\mathbf{D}\}$$
$$= (\mathbf{A}\times\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\times\mathbf{D})(\mathbf{B}\cdot\mathbf{C}) = (\mathbf{B}\cdot\mathbf{D})(\mathbf{A}\times\mathbf{C}) - \mathbf{B}\cdot\mathbf{C}(\mathbf{A}\times\mathbf{D}).$$

PROBLEMS 3.5

1. Find the volume of the parallelopied whose edges are represented by the vectors $\mathbf{A} = 2\mathbf{I} - 3\mathbf{J} + 4\mathbf{K}$, $\mathbf{B} = \mathbf{I} + 2\mathbf{J} - \mathbf{K}$, $\mathbf{C} = 3\mathbf{I} - \mathbf{J} + 2\mathbf{K}$.
2. Find a such that the vectors $2\mathbf{I} - \mathbf{J} + \mathbf{K}$, $\mathbf{I} + 2\mathbf{J} - 3\mathbf{K}$ and $3\mathbf{I} + a\mathbf{J} + 5\mathbf{K}$ are coplanar.
3. (*i*) Prove that the vectors $\mathbf{I} - 2\mathbf{J} + 3\mathbf{K}$, $-2\mathbf{I} + 3\mathbf{J} - 4\mathbf{K}$ and $\mathbf{I} - 3\mathbf{J} + 5\mathbf{K}$ are coplanar.
   (*ii*) Do the points (4, – 2, 1), (5, 1, 6), (2, 2 – 5) and (3, 5, 0) lie in a plane.
4. (*a*) Test the linear dependency of the vectors (1, 2, 1), (2, 1, 4), (4, 5, 6) and (1, 8, – 5).
   (*b*) Verify whether the following set of vectors are linearly independent (4, 2, 9), (3, 2, 1), (– 4, 6, 9). (*B.P.T.U., 2005*)
5. Find the volume of the tetrahedron, three of whose coterminus edges are $3\mathbf{I} - \mathbf{J} + 2\mathbf{K}$, $2\mathbf{I} + \mathbf{J} - \mathbf{K}$ and $\mathbf{I} - 2\mathbf{J} + 2\mathbf{K}$.

6. Find the volume of the tetrahedron formed by the points
   (*i*) (1, 3, 6), (3, 7, 12), (8, 8, 9) and (2, 2, 8). (*B.P.T.U., 2005*)
   (*ii*) (2, 1, 1), (1, – 1, 2), (0, 1, – 1) and (1, – 2, 1).
7. If $\mathbf{A}.\mathbf{N} = 0$, $\mathbf{B}.\mathbf{N} = 0$, $\mathbf{C}.\mathbf{N} = 0$, prove that $[\mathbf{ABC}] = 0$. Interpret this result geometrically.
8. (*a*) Prove that $[\mathbf{A} + \mathbf{B}, \mathbf{B} + \mathbf{C}, \mathbf{C} + \mathbf{A}] = 2[\mathbf{ABC}]$.
   (*b*) Show that volume of the tetrahedron having $\mathbf{A} + \mathbf{B}$, $\mathbf{B} + \mathbf{C}$ and $\mathbf{C} + \mathbf{A}$ as concurrent edges is twice the volume of the tetrahedron having $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ as concurrent edges.
9. If $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$, show that $(\mathbf{A} \times \mathbf{C}) \times \mathbf{B} = 0$.
10. Show that $\mathbf{I} \times (\mathbf{R} \times \mathbf{I}) + \mathbf{J} \times (\mathbf{R} \times \mathbf{J}) + \mathbf{K} \times (\mathbf{R} \times \mathbf{K}) = 2\mathbf{R}$. (*Assam, 1999*)
11. If $\mathbf{A} = \mathbf{I} - 2\mathbf{J} - 3\mathbf{K}$, $\mathbf{B} = 2\mathbf{I} + \mathbf{J} - \mathbf{K}$, $\mathbf{C} = \mathbf{I} + 3\mathbf{J} - \mathbf{K}$, find
    (*i*) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ (*ii*) $(\mathbf{A} \times \mathbf{B}) \times (\mathbf{B} \times \mathbf{C})$.
12. (*a*) Given $\mathbf{A} = 2\mathbf{I} - \mathbf{J} + 3\mathbf{K}$, $\mathbf{B} = -\mathbf{I} + 3\mathbf{J} + 3\mathbf{K}$, $\mathbf{C} = \mathbf{I} + \mathbf{J} - 2\mathbf{K}$, find the reciprocal triad $(\mathbf{A}', \mathbf{B}', \mathbf{C}')$ and verify that $[\mathbf{ABC}]\,[\mathbf{A'B'C'}] = 1$.
    (*b*) Prove that $\mathbf{A} \times \mathbf{A}' + \mathbf{B} \times \mathbf{B}' + \mathbf{C} \times \mathbf{C}' = 0$
13. Prove that (*i*) $[\mathbf{A} \times \mathbf{B}, \mathbf{C} \times \mathbf{D}, \mathbf{E} \times \mathbf{F}] = [\mathbf{ABD}][\mathbf{CEF}] - [\mathbf{ABC}][\mathbf{DEF}]$
    (*ii*) $\{(\mathbf{A} + \mathbf{B} + \mathbf{C}) \times (\mathbf{B} + \mathbf{C})\}\,.\,\mathbf{C} = [\mathbf{ABC}]$.
14. Show that
    (*i*) $(\mathbf{B} \times \mathbf{C}) \times (\mathbf{A} \times \mathbf{D}) + (\mathbf{C} \times \mathbf{A}) \times (\mathbf{B} \times \mathbf{D}) + (\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = -2[\mathbf{ABC}]\mathbf{D}$. (*Mumbai, 2007*)
    (*ii*) $\mathbf{A} \times [\mathbf{F} \times \mathbf{B}) \times (\mathbf{G} \times \mathbf{C})] + \mathbf{B} \times [\mathbf{F} \times \mathbf{C}) \times (\mathbf{G} \times \mathbf{A}) + \mathbf{C} \times [(\mathbf{F} \times \mathbf{A}) \times (\mathbf{G} \times \mathbf{B})] = 0$.
15. (*a*) Prove that $[\mathbf{LMN}][\mathbf{ABC}] = \begin{vmatrix} \mathbf{L}.\mathbf{A} & \mathbf{L}.\mathbf{B} & \mathbf{L}.\mathbf{C} \\ \mathbf{M}.\mathbf{A} & \mathbf{M}.\mathbf{B} & \mathbf{M}.\mathbf{C} \\ \mathbf{N}.\mathbf{A} & \mathbf{N}.\mathbf{B} & \mathbf{N}.\mathbf{C} \end{vmatrix}$
    (*b*) The length of the edges *OA*, *OB*, *OC* of the tetrahedron *OABC* are *a*, *b*, *c* and the angles *BOC*, *COA*, *AOB* are θ, ϕ, ψ, find its volume.

SOLID GEOMETRY

3.11 (1) EQUATION OF A PLANE

Let $P(x, y, z)$ be any point on the plane through $A(x_1, y_1\, z_1)$ which is normal to the vector $\mathbf{N} = a\mathbf{I} + b\mathbf{J} + c\mathbf{K}$.

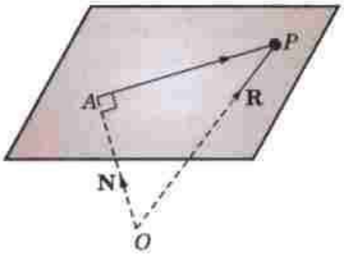


Fig. 3.35

Then $\vec{OP} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$ and $\vec{OA} = x_1\mathbf{I} + y_1\mathbf{J} + z_1\mathbf{K}$

Clearly the vectors $\vec{AP} = (x - x_1)\,\mathbf{I} + (y - y_1)\,\mathbf{J} + (z - z_1)\,\mathbf{K}$ and $\mathbf{N}$ are perpendicular to each other.

$$\therefore \qquad \vec{AP}\,.\,\mathbf{N} = 0 \qquad ...(i)$$

or $$[x - x_1)\,\mathbf{I} + (y - y_1)\,\mathbf{J} + (z - z_1)\,\mathbf{K}]\,.\,(a\mathbf{I} + b\mathbf{J} + c\mathbf{K}) = 0$$

or $$\mathbf{a(x - x_1) + b(y - y_1) + c(z - z_1) = 0} \qquad ...(ii)$$

which is *the equation of any plane through the point* (x_1, y_1, z_1).

Obs. Equation (*ii*) written as $ax + by + cz + d = 0$ *is the general equation of a plane.*

Conversely, *every linear equation in x, y, z represents a plane and the coefficients of x, y, z are the direction ratios of the normal to the plane.*

Cor. IF *l*, *m*, *n* be the direction cosines of the normal to the plane, then

$$\mathbf{lx + my + nz = p} \qquad ...(iii)$$

which is called the **normal form** *of the equation of the plane* where *p* is the perpendicular distance from the origin.

(2) Angle between two planes. Def. *The angle between two planes is equal to the angle between their normals.*

Let the two planes be

$$ax + by + cz + d = 0 \quad \text{and} \quad a'x + b'y + c'z + d' = 0.$$

Now the direction ratio of their normals are *a*, *b*, *c* and *a*′, *b*′, *c*′.

Hence the angle θ between the planes is given by $\cos\theta = \dfrac{aa' + bb' + cc'}{\sqrt{(a^2 + b^2 + c^2)}\sqrt{a'^2 + b'^2 + c'^2}}$

The planes will be perpendicular (if their normal are parallel), *i.e., if* $aa' + bb' + cc' = 0$

The planes will be parallel (if their normals are parallel), *i.e., if* $\dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{c}{c'}$.

Cor. *Any plane parallel to the plane* $ax + by + cz + d = 0$

is $$\mathbf{ax + by + cz + k = 0} \qquad (k \text{ being any constant})$$

for the direction-ratios of their normals are the same.

(3) Perpendicular distance of the point (x_1, y_1, z_1) *from the plane*

$$ax + by + cz + d = 0 \qquad \text{...}(i)$$

is $$\frac{\mathbf{ax_1 + by_1 + cz_1 + d}}{\sqrt{(\mathbf{a^2 + b^2 + c^2})}}$$

Let PL be the perpendicular distance of $P(x_1, y_1, z_1)$ from the plane (i) so that the direction cosines of $\vec{LP}$ are

$$\frac{a}{\sqrt{(\Sigma a^2)}}, \frac{b}{\sqrt{(\Sigma a^2)}}, \frac{c}{\sqrt{(\Sigma a^2)}}.$$

If $Q\,(f, g, h)$ be a point on (i) then

$$af + bg + ch + d = 0 \qquad \text{...}(ii)$$

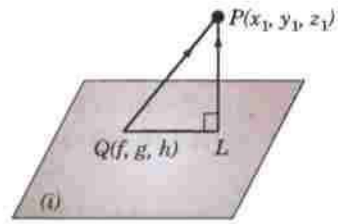


Fig. 3.36

$$\therefore \quad PL = \text{projection of } \vec{QP} \text{ on } \vec{LP} = \vec{QP} \cdot \vec{LP}$$

$$= \frac{(x_1 - f)a + (y_1 - g)b + (z_1 - h)c}{\sqrt{(a^2 + b^2 + c^2)}} \qquad [\text{By } IX \text{ p. 82}]$$

$$= \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{(a^2 + b^2 + c^2)}} \text{ by virtue of } (ii) \qquad \text{...}(iii)$$

The sign of the radical in (iii) is taken to be positive or negative according as d is positive or negative.

Obs. The perpendicular to a plane from two points are taken to be of the same sign if the points lie on the same side and of different signs if they lie on the opposite sides of the plane.

∴ *The two points* (x_1, y_1, z_1) *and* (x_2, y_2, z_2) *lie on the same side or on opposite sides of the plane* $ax + by + cz + d = 0$, according as $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_2 + d$ are of the same sign or of opposite signs.

Cor. Planes bisecting the angles between two planes.

Let $$ax + by + cz + d = 0 \qquad \text{...}(i)$$

and $$a'x + b'y + c'z + d' = 0 \qquad \text{...}(ii)$$

be the given planes.

Let $P(x, y, z)$ be any point on either of the planes bisecting the angles between the planes (i) and (ii).

Then ⊥ distance of P from (i) = ⊥ distance of P from (ii),

$$\therefore \quad \frac{ax + by + cz + d}{\sqrt{(a^2 + b^2 + c^2)}} = \pm \frac{a'x + b'y + c'z + d'}{\sqrt{(a'^2 + b'^2 + c'^2)}}$$

which are the required equations of the bisecting planes.

Example 3.27. *Find the equation of the plane which*

(i) cuts off intercepts a, b, c from the axes.

(ii) passes through the points A(0, 1, 1), B(1, 1, 2) and C(–1, 2, –2).

Solution. (i) **Intercept form** *of the equation of the plane.* Let the required equation of the plane be

$$\alpha x + \beta y + \gamma z + \delta = 0 \qquad \text{...}(1)$$

The plane cuts the axes at A, B, C such that $OA = a$, $OB = b$, $OC = c$,

i.e., it passes through the points $A(a, 0, 0)$, $B(0, b, 0)$, $C(0, 0, c)$.

$\therefore$ $\alpha a + \delta = 0, \beta b + \delta = 0, \gamma c + \delta = 0$

whence $\alpha = -\delta/a, \beta = -\delta/b, \gamma = -\delta/c$

Substituting these values of α, β, γ in (1), $-\frac{\delta}{a}x - \frac{\delta}{b}y - \frac{\delta}{c}z + \delta = 0$ or $\frac{\mathbf{x}}{\mathbf{a}} + \frac{\mathbf{y}}{\mathbf{b}} + \frac{\mathbf{z}}{\mathbf{c}} = 1.$

(*ii*) **Three points form** *of the equation of the plane.*

Any plane through (0, 1, 1) is $a(x-0) + b(y-1) + c(z-1) = 0$ ...(2)

It will pass through (1, 1, 2) and (–1, 2, – 2), if $a + c = 0$ and $-a + b - 3c = 0$.

By cross-multiplication, $\frac{a}{-1} = \frac{b}{2} = \frac{c}{1}$.

Substituting these values in (2), we get $-1 \cdot x + 2(y-1) + 1(z-1) = 0$

or $x - 2y - z + 3 = 0$, which is the required equation of the plane.

Example 3.28. *Find the equation of the plane which passes through the point (3, – 3, 1) and is*
(i) parallel to the plane $2x + 3y + 5z + 6 = 0$.
(ii) normal to the line joining the points (3, 2, – 1) and (2, – 1, 5). (V.T.V., 2006)
(iii) Perpendicular to the planes $7x + y + 2z = 6$ and $3x + 5y - 6z = 8$. (Cochin, 2005 ; V.T.U., 2005)

Solution. (*i*) Any plane parallel to the given plane is

$2x + 3y + 5z + k = 0$ which goes through (3, – 3, 1), if $k = -2$

Thus the required plane is $2x + 3y + 5z - 2 = 0$

(*ii*) Any plane through (3, – 3, 1) is $a(x-3) + b(y+3) + c(z-1) = 0$

The direction cosines of the line joining the points (3, 2, – 1) and (2, – 1, 5) are proportional to 1, 3, – 6.

This line is normal to the plane (1). $\therefore$ a, b, c are proportional to 1, 3, – 6.

Substituting these values in (1), the required equation is

$$1(x-3) + 3(y+3) - 6(z-1) = 0 \quad \text{or} \quad x + 3y - 6z + 12 = 0.$$

(*iii*) Any plane through (3, – 3, 1) is

$a(x-3) + b(y+3) + c(z-1) = 0$ which will be $\perp$ to the planes

$$7x + y + 2z = 6 \text{ and } 3x + 5y - 6z = 8$$

if $$7a + b + 2c = 0 \text{ and } 3a + 5b - 6c = 0.$$

Solving these by cross-multiplication, we get $\frac{a}{1} = \frac{b}{-3} = \frac{c}{-2}$.

Hence the required equation is $1(x-3) - 3(y+3) - 2(z-1) = 0$ or $x - 3y - 2z - 10 = 0$.

Example 3.29. *The plane $4x + 5y - z = 7$ is rotated through a right angle about its line of intersection with the plane $2x + 3y - 3z = 5$. Find the equation of this plane in its new position.*

Solution. Any plane through the line of intersection of

$$4x + 5y - z = 7 \quad ...(i)$$

and $$2x + 3y - 3z = 5 \quad ...(ii)$$

is $$4x + 5y - z - 7 + k(2x + 3y - 3z - 5) = 0$$

i.e., $$(4 + 2k)x + (5 + 3k)y - (1 + 3k)z - (7 + 5k) = 0 \quad ...(iii)$$

Then new position of (*i*) when rotated through a right angle, is such that (*i*) and (*iii*) are perpendicular. This requires that

$$4(4 + 2k) + 5(5 + 3k) + (1 + 3k) = 0$$

i.e., $$26k + 42 = 0 \quad \text{or} \quad k = -21/13$$

Substituting $k = -21/13$ in (*iii*), we get $10x + 2y + 50z + 14 = 0$.

or $5x + y + 25x + 7 = 0$, which is the required plane.

Example 3.30. *Find the distance between the parallel planes $2x - 2y + z + 3 = 0$ and $4x - 4y + 2z + 9 = 0$. Find also the equation of the parallel plane that lies mid-way between the given planes.* (Madras, 2003)

Solution. The distance between the given planes is the perpendicular distance of any point on one of the planes from the other.

A point on the first plane is $(0, 0, -3)$.

$\therefore$ Required distance = $\perp$ distance of $(0, 0, -3)$ from $4x - 4y + 2z + 9 = 0$

$$= \frac{-6+9}{\sqrt{(16+16+4)}} = \frac{3}{6} = \frac{1}{2}$$

Let the equation of the parallel plane that lies mid-way between the given planes be

$$2x - 2y + z + k = 0 \quad \text{...(i)}$$

Now distance of any point $(0, 0, -3)$ on the first plane from (*i*) should be 1/4.

$$\therefore \qquad \pm \frac{-3+k}{\sqrt{(4+4+1)}} = 1/4 \quad i.e., \quad k = 15/4 \text{ or } 9/4.$$

Thus the required plane is $2x - 2y + z + 15/4 = 0$.

Assume that $k = 15/4$ and verify that the distance of a point on this plane $4x - 4y + 2z + 9 = 0$ is also 1/4.

A point on this plane is $(0, 0, -9/4)$. Its distance from the plane (*i*) = $\dfrac{-9/2+15/4}{3} = \dfrac{1}{4}$ (in magnitude)

Thus $k = 9/4$ is not admissible.

$\therefore$ The required plane is $2x - 2y + z + 15/4 = 0$.

Example 3.31. *A variable plane is at a constant distance p from the origin and meets the axes at A, B, C. Find the locus of the centroid of the tetrahedron OABC.*

Solution. As the given plane is at a $\perp$ distance p from the origin, therefore its equation is of the form

$$lx + my + nz = p \quad \text{...(i)} \quad \text{where } l, m, n \text{ are the d.c's of the } \perp \text{ from the origin.}$$

(*i*) may be rewritten as $\dfrac{x}{(p/l)} + \dfrac{y}{(p/m)} + \dfrac{z}{(p/n)} = 1$

so that $OA = p/l$, $OB = p/m$, $OC = p/n$.

$$\therefore \qquad A = (p/l, 0, 0), B = (0, p/m, 0), C = (0, 0, p/n).$$

Thus the coordinates of the centroid G of the tetrahedron $OABC$ are

$$(x_1, y_1, z_1) = (p/4l, p/4m, p/4n)$$ [See p. 81]

$$\therefore \qquad \frac{1}{x_1^2} + \frac{1}{y_1^2} + \frac{1}{z_1^2} = \frac{16}{p^2}(l^2 + m^2 + n^2) = \frac{16}{p^2}$$

Thus the locus of G is $x^{-2} + y^{-2} + z^{-2} = 16p^{-2}$.

Example 3.32. *A variable plane at a constant distance p from the origin meets the axes in A, B, C. Planes are drawn through A, B, C parallel to the coordinate planes, Show that the locus of their point of intersection is given by $x^{-2} + y^{-2} + z^{-2} = p^{-2}$.*

Solution. Let the variable plane be $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$.

Its distance from origin = $\dfrac{1}{\sqrt{(a^{-2} + b^{-2} + c^{-2})}} = p$ (given)

$$i.e., \qquad a^{-2} + b^{-2} + c^{-2} = p^{-2} \quad \text{...(i)}$$

Since $OA = a$, $OB = b$ and $OC = c$, therefore equations of the planes through A, B, C parallel to yz, zx and xy-planes are $x = a$, $y = b$, $z = c$

Let the point of intersection of these three planes be (x_1, y_1, z_1).

Then $$x_1 = a, y_1 = b, z_1 = c \quad \text{...(ii)}$$

Substituting (*ii*) in (*i*), we get $x_1^{-2} + y_1^{-2} + z_1^{-2} = p^{-2}$

Thus the locus of (x_1, y_1, z_1) is $x^{-2} + y^{-2} + z^{-2} = p^{-2}$.

Example 3.33. *A variable plane passes through the fixed point (a, b, c) and meets the coordinate axes in A, B, C. Show that the locus of the point common to the planes through A, B, C parallel to the coordinate planes is*

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=1.$$

Solution. Let ABC be any plane through the fixed point $H(a, b, c)$ such that $OA = x_1$, $OB = y_1$, $OC = z_1$. Then its equation is

$$\frac{x}{x_1}+\frac{y}{y_1}+\frac{z}{z_1}=1 \qquad \text{[See Ex. 3.27 (}i\text{)]}$$

Since H lies on it.

$$\therefore \qquad \frac{a}{x_1}+\frac{b}{y_1}+\frac{c}{z_1}=1. \qquad ...(1)$$

The planes through A, B, C parallel to the coordinate planes are $x = x_1$, $y = y_1$, $z = z_1$, which meet in $P(x_1, y_1, z_1)$.

Thus changing x_1 to x, y_1, to y and z_1 to z in the locus of the P is

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=1.$$

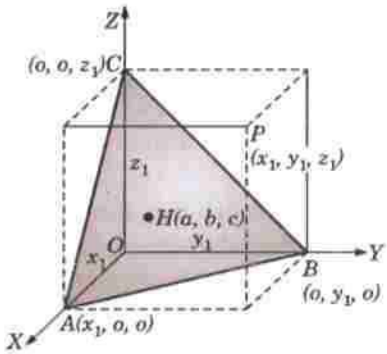


Fig. 3.37

Example 3.34. *Find the equations to the two planes which bisect the angles between the planes $3x - 4y + 5z = 3$, $5x + 3y - 4z = 9$.*

Also point out which of the planes bisects the acute angle. *(V.T.U., 2007)*

Solution. The equations of the planes bisecting the angles between the given planes are

$$\frac{3x-4y+5z-3}{\sqrt{[3^2+(-4)^2+5^2]}}=\pm\frac{5x+3y-4z-9}{\sqrt{[5^2+3^2+(-4)^2]}}$$

or $\qquad 2x + 7y - 9z - 6 = 0 \qquad ...(i)$

and $\qquad 8x - y + z - 12 = 0 \qquad ...(ii)$

which are the required planes.

Let θ be the angle between (*i*) and either of the given planes, say:

$$5x + 3y - 4z = 9.$$

Then,
$$\cos\theta=\frac{2\times5+7\times3\,(-9)\times(-4)}{\sqrt{[2^2+7^2+(-9)^2]}\sqrt{[5^2+3^2+(-4)^2]}}=\frac{67}{5\sqrt{(268)}}$$

$$\therefore \qquad \tan\theta=\frac{\sqrt{2211}}{67} \text{ which is less than 1.}$$

i.e., $\qquad \theta < 45°.$

Now θ is half the angle between the given planes, so that (*i*) bisects that angle between the planes which is $2\theta < 90°$.

Hence the plane $2x + 7y - 9z = 6$, bisects the acute angle.

PROBLEMS 3.6

1. Find the equation of the plane passing through the point (1, 2, 3) and having the vector $\mathbf{N} = 2\mathbf{I} - \mathbf{J} + 3\mathbf{K}$ normal to it.
2. Find the equation of the plane through the points (3, – 1, 1), (1, 2, – 1) and (1, 1, 1).
3. Find a unit vector normal to the plane through the points (– 1, 2, 3), (1, 1, 1) and (2, – 1, 3).
4. Find the distance of the point (1, 4, 5) from the plane passing through the points (2, – 1, 5), (0, – 4, 1) and (2, – 6, 0). *(Rajasthan, 2006)*
5. Show that the four points (0, – 1, 0), (2, 1, – 1), (1, 1, 1) and (3, 3, 0) are coplanar. Find the equation of the plane through them. *(V.T.U., 2008)*

6. Show that the point $(-1/2, 2, 0)$ is the circumcentre of the triangle formed by the points $(1, 1, 0)$, $(1, 2, 1)$, $(-2, 2, -1)$.
   [**Hint.** Show that the point $(-1/2, 2, 0)$ lies in the plane of the triangle and is equidistant from its vertices.]
7. Find the equation of the plane through the point $(2, 1, 0)$ and perpendicular to the planes $2x - y - z = 5$ and $x + 2y - 3z = 5$.
8. Find the equations of the plane through $(0, 0, 0)$ parallel to the plane $x + 2y = 3z + 4$. *(Madras, 2006)*
9. Find the equation of the plane which bisects the join of the points (x_1, y_1, z_1) and (x_2, y_2, z_2) at right angles.
10. Find the equation of the plane through the points $(-1, 2, 1)$, $(-3, 2, -3)$ and parallel to y-axis *(V.T.U., 2010)*
11. Find the equation of the plane through the points $(2, 2, 1)$ and $(9, 3, 6)$ and perpendicular to the plane $2x + 6y + 6z = 9$. *(V.T.U., 2004 ; Osmania, 1999)*
12. A plane contains the points $A(-4, 9, -9)$ and $B(5, -9, 6)$ and is perpendicular to the line which joins B and $C(4, -6, k)$. Evaluate k and find the equation of the plane.
13. Find the distance between the parallel planes
    $$2x - 3y + 6z + 12 = 0 \text{ and } 6x - 9y + 18z - 6 = 0.$$
    Also find the equation of the parallel plane that lies mid-way between the given planes.
14. Find the angle between the plane $x + y + z = 8$ and $2x + y - z = 3$. *(B.P.TU., 2006)*
15. Two planes are given by $x + 2y - 3z - 2 = 0$ and $2x + y + z + 3 = 0$, find
    (*i*) direction cosines of their line of intersection,
    (*ii*) acute angle between the planes, and
    (*iii*) equation of the plane perpendicular to both of them through the point $(2, 2, 1)$.
16. The plane $lx + my = 0$ is rotated about its line of intersection with the plane $z = 0$, through an angle α.
    Prove that the equation of the plane is $lx + my + z\sqrt{(l^2 + m^2)}\tan\alpha = 0$. *(Anna, 2005 S)*
17. Find the equations of the two planes through the points $(0, 4, -3)$, $(6, -4, 3)$ other than the plane through the origin which cut off from the axes intercepts whose sum is zero.
18. A plane meets the coordinates axes at A, B, C, such that the centroid of the triangle ABC is the point (a, b, c), show that the equation of the plane is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 3$. *(Assam, 1999)*
19. A plane passes through a fixed point (a, b, c), show that the locus of the foot of the perpendicular from the origin on the plane is a sphere. *(P.T.U., 2005)*
20. A variable plane is at a constant distance p from the origin and meets the axes at A, B, C. Find the locus of the centroid of the triangle ABC. *(Rajasthan, 2005)*
21. A variable plane makes with the coordinate axes a tetrahedron of constant volume $64\,k^3$. Find the locus of the centroid of the tetrahedron. *(Rajasthan, 2006 ; Osmania, 2003)*
22. Find equations of the planes bisecting the angle between the planes
    $$x + 2y + 2z = 9,\ 4x - 3y + 12z + 12 = 0$$
    and specify the one which bisects the acute angle.

3.12 EQUATIONS OF A STRAIGHT LINE

(1) General form. Two linear equations in x, y, z

i.e., $$ax + by + cz + d = 0 \quad \ldots(i)$$

and $$a'x + b'y + c'z + d' = 0 \quad \ldots(ii)$$

taken together represent a straight line which is the line of intersection of the planes (*i*) and (*ii*). (Fig. 3.38).

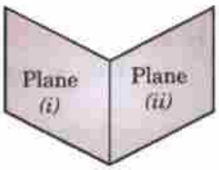


Fig. 3.38

(2) Symmetrical form. *Equations of the line through the point $A(x_1, y_1, z_1)$ and having direction cosines l, m, n are*

$$\frac{\mathbf{x - x_1}}{\mathbf{l}} = \frac{\mathbf{y - y_1}}{\mathbf{m}} = \frac{\mathbf{z - z_1}}{\mathbf{n}}$$

Let $P(x, y, z)$ be any point on the given line through $A(x_1, y_1, z_1)$ and parallel to the unit vector $\mathbf{U} = l\mathbf{I} + m\mathbf{J} + n\mathbf{K}$.

Since $\vec{AP}$ is parallel to $\mathbf{U}$, we can write $\vec{AP} = t\mathbf{U}$, where t is a parameter. $\quad \ldots(i)$

or $$(x - x_1)\,\mathbf{I} + (y - y_1)\,\mathbf{J} + (z - z_1)\,\mathbf{K} = t\,(l\mathbf{I} + m\mathbf{J} + n\mathbf{k})$$

$\therefore$ $$x - x_1 = tl,\ y - y_1 = tm,\ z - z_1 = tn \quad ...(ii)$$

Every point P on the line is given by (*ii*) for some value of t. Thus these are the parametric equations of the given line. Eliminating t, we get

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \quad ...(iii)$$

which are the *symmetrical form of the equations of the line.*

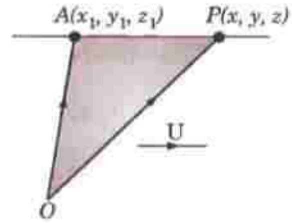


Fig. 3.39

Obs. Any point on the line (*iii*) is $(x_1 + lt, y_1 + mt, z_1 + nt)$.

Cor. *The equations of the line joining the points* (x_1, y_1, z_1) *and* (x_2, y_2, z_2) *are*

$$\frac{\mathbf{x} - \mathbf{x_1}}{\mathbf{x_2} - \mathbf{x_1}} = \frac{\mathbf{y} - \mathbf{y_1}}{\mathbf{y_2} - \mathbf{y_1}} = \frac{\mathbf{z} - \mathbf{z_1}}{\mathbf{z_2} - \mathbf{z_1}}$$

for the direction-ratios of the line joining the given points are

$$x_2 - x_1, y_2 - y_1, z_2 - z_1.$$

[**To reduce the general equation of a line of the symmetrical form:**

(*i*) *find a point on the line,* by putting $z = 0$ in the the given equations and solving the resulting equations for x and y.

(*ii*) *find the direction cosines of the line,* from the fact that it is perpendicular to the normals to the given planes.

(*iii*) *write the equations of the line in the symmetrical form.*]

Example 3.35. *Find in symmetrical form, the equations of the line*

$$x + y + z + 1 = 0,\ 4x + y - 2z + 2 = 0.$$ (*Osmania, 1999*)

Solution. (*i*) To find a point on the line.

Putting $z = 0$ in the given equations, we have

$$x + y + 1 = 0;\ 4x + y + 2 = 0$$

Solving, $\dfrac{x}{1} = \dfrac{y}{2} = \dfrac{1}{-3}$ $\quad\therefore$ A point on the line is $(-1/3, -2/3, 0)$.

(*ii*) *To find the direction cosines l, m, n of the line.*

Since the line lies on both the given planes.

$\therefore$ It is perpendicular to their normals whose direction cosines are proportional to 1, 1, 1 and 4, 1, – 2.

i.e., $$l + m + n = 0;\ 4l + m - 2n = 0.$$

Solving, $$\frac{l}{-1} = \frac{m}{2} = \frac{n}{-1}$$

$\therefore$ The direction cosines of the given line are proportional to – 1, 2, – 1.

(*iii*) Thus the equations of the line in the symmetrical form are

$$\frac{x + 1/3}{-1} = \frac{y + 2/3}{2} = \frac{z}{-1}.$$

Example 3.36. *Find the distance of the point (1, – 2, 3) from the plane $x - y + z = 5$ measured parallel to the line*

$$\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$$ (*Calicut, 1999*)

Solution. The line through $P(1, -2, 3)$ having direction ratios $(2, 3, -6)$ is

$$\frac{x - 1}{2} = \frac{y + 2}{3} = \frac{z - 3}{-6} = r.$$

Any point on this line is $(2r + 1, 3r - 2, 3 - 6r)$.

This point will lie on the plane $x - y + z = 5$

if $$2r + 1 - (3r - 2) + 3 - 6r = 5 \quad \text{or} \quad r = 1/7.$$

$\therefore$ The point of intersection is $Q(9/7, -11/7, 15/7)$

Thus the required distance $= PQ = \sqrt{\left(\frac{4}{49}+\frac{9}{49}+\frac{36}{49}\right)} = 1$

$x + 2y + 2z = 9$, $4x - 3y + 12z + 12 = 0$ and specify the one which bisects the acute angle.

Example 3.37. (a) *Find the image (reflection) of the point (p, q, r) in the plane $2x + y + z = 6$.*

(b) *Find the image (reflection) of the line $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-3}{4}$ in the same plane.* (Delhi, 2002)

[If two points P, P′ be such that the line PP′ is bisected perpendicularly by a plane then either of the points is the image (or reflection) of the other in the plane.]

Solution. (a) Let $P'(p', q', r')$ be the image of $P(p, q, r)$. Then the mid-point of PP' must lie on the given plane.

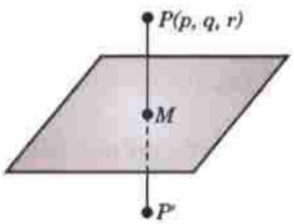


Fig. 3.40(a)

$$\therefore \quad \frac{p+p'}{2}+\frac{q+q'}{2}+\frac{r+r'}{2}=6 \quad ...(i)$$

Also the line PP' must be perpendicular to the plane. The direction ratios of PP' being $p - p', q - q', r - r'$, we therefore, have

$$\frac{p-p'}{2}=\frac{q-q'}{1}=\frac{r-r'}{1}=k \text{ (say)}$$

whence $p' = p - 2k$, $q' = q - k$, $r' = r - k$.

Substituting these in (*i*) and solving, we get

$$k = \frac{1}{3}(2p+q+r-6).$$

Hence P' is

$$\left[\frac{1}{3}(12-p-2q-2r), \frac{1}{3}(6-2p+2q-r), \frac{1}{3}(6-2p-q+2r)\right] \quad ...(ii)$$

(*b*) Any two points on the given line are evidently $P(1, 2, 3)$ and (on putting $z = 7$) $Q(3, 3, 7)$. Their images are [by using (*ii*)] $P'\left(\frac{1}{3},\frac{5}{3},\frac{8}{3}\right)$ and $Q'\left(-\frac{11}{3},-\frac{1}{3},\frac{11}{3}\right)$. The line joining P' and Q' is, therefore

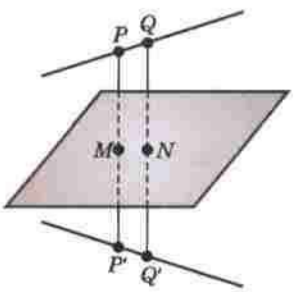


Fig. 3.40(b)

$$\frac{x-\frac{1}{3}}{-\frac{11}{3}-\frac{1}{3}}=\frac{y-\frac{5}{3}}{-\frac{1}{3}-\frac{5}{3}}=\frac{z-\frac{8}{3}}{\frac{11}{3}-\frac{8}{3}}, \text{ i.e., } \frac{3x-1}{4}=\frac{3y-5}{2}=\frac{3z-8}{-1}$$

which is the required image of the given line PQ [Fig. 3.40(*b*)].

Example 3.38. *Find the angle between the line*

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

and the plane $ax + by + cz + d = 0$.

Solution. If θ be the angle between the line and the plane, then $90° - \theta$ is the angle between the line and the normal to the plane (Fig. 3.41).

Now the direction ratios of the line are l, m, n and the direction ratios of the normal to the plane are a, b, c.

$$\therefore \quad \cos(90° - \theta) = \frac{la+mb+nc}{\sqrt{(l^2+m^2+n^2)}\sqrt{(a^2+b^2+c^2)}}$$

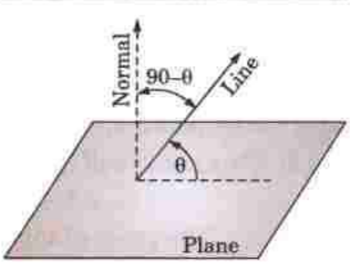


Fig. 3.41

or $$\sin\theta = \frac{la + mb + nc}{\sqrt{(\Sigma l^2)}\sqrt{(\Sigma a^2)}}$$

Hence the required angle $\theta = \sin^{-1}\left(\frac{al + bm + cn}{\sqrt{(\Sigma l^2)}\sqrt{(\Sigma a^2)}}\right)$

Cor. *If the line is parallel to the plane, $\sin\theta = 0$*

$\therefore$ $$\mathbf{al + bm + cn = 0}$$

If the line is perpendicular to the plane, it will be parallel to its normal.

$\therefore$ $$l/a = m/b = n/c.$$

Example 3.39. *Find the equations of the two straight lines through the origin, each of which intersects the straight line $\frac{1}{2}(x-3) = y - 3 = z$ and is inclined at an angle of 60° to it.*

Solution. Let AB be the given line so that any point A on it is $(2r + 3, r + 3, r)$. (Fig. 3.42)

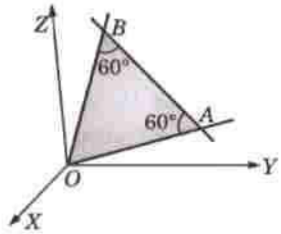


Fig. 3.42

$\therefore$ Direction ratios of OA are $2r + 3 - 0, r + 3 - 0, r - 0$.

Angle between AO and AB has to be 60°,

$\therefore$ $$\cos 60° = \frac{2(2r+3) + 1(r+3) + 1.(r)}{\sqrt{(2^2+1^2+1^2)}\sqrt{[(2r+3)^2+(r+3)^2+r^2]}}$$

or $$\frac{1}{2} = \frac{6r+9}{\sqrt{[6(6r^2+18r+18)]}} \text{ or } r^2 + 3r + 2 = 0 \text{ i.e., } r = -1, -2$$

$\therefore$ Coordinates of A and B are $(1, 2, -1)$ and $(-1, 1, -2)$.

Hence the equations of the required lines OA and OB are $\frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$ and $\frac{x}{-1} = \frac{y}{1} = \frac{z}{-2}$

PROBLEMS 3.7

1. Prove that the points (3, 2, 4), (4, 5, 2) and (5, 8, 0) are collinear. Find the equations of the line through them.
2. Find the angle between the line of intersection of the planes
$2x + 2y - z + 15 = 0$, $4y + z + 29 = 0$ and the line $\frac{x+4}{4} = \frac{y-3}{-3} = \frac{z+2}{1}$. (*V.T.U., 2003 S*)
3. Find the angle between the line of intersection of the planes $3x + 2y + z = 5$ and $x + y - 2z = 3$ and the line of intersection of the plane $2x = y + z$ and $7x + 10y = 8z$.
4. Find the equation of the line through the point $(-2, 3, 4)$ and parallel to the planes $2x + 3y + 4z = 5$ and $4x + 3y + 5z = 6$.
5. Show that the line $\frac{x-1}{3} = \frac{y+2}{-2} = \frac{z-1}{2}$ is parallel to the plane $2x + 2y - z = 6$, and find the distance between them.
6. Find the equation of the line through $(1, 2, -1)$ perpendicular to each of the lines
$$\frac{x}{1} = \frac{y}{0} = \frac{z}{-1} \text{ and } \frac{x}{3} = \frac{y}{4} = \frac{z}{5}.$$
7. Find the equation of the lines bisecting the angle between the lines
$$\frac{x-1}{2} = \frac{y+2}{-2} = \frac{z-3}{1}; \frac{x-1}{12} = \frac{y+2}{4} = \frac{z-3}{-3}.$$
8. Find the foot of the perpendicular from (1, 1, 1) to the line joining the points (1, 4, 6) and (5, 4, 4). (*V.T.U., 2010*)
9. Find the perpendicular distance of the point (1, 1, 1) from the line
$$\frac{x-2}{2} = \frac{y+3}{2} = \frac{z}{-1}.$$

10. Find the distance of the point $(3, -4, 5)$ from the plane $2x + 5y - 6z = 16$, measured parallel to the line $x/2 = y/1 = z/-2$. (V.T.U., 2002)
11. Find the reflection (image) of the point
  (i) $(1, 2, 3)$ in the plane $x + y + z = 9$. (V. T.U., 2010)
  (ii) $(2, -1, 3)$ in the plane $3x - 2y - z - 9 = 0$.
12. Find the angle between the line $\frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6}$ and the plane $3x + y + z = 7$.
13. Find the equation of the plane through the points $(1, 0, -1), (3, 2, 2)$ and parallel to the line
$$x - 1 = \frac{1}{2}(1 - y) = \frac{1}{3}(z - 2).$$ (V.T.U., 2000)
14. Find the equations of the straight line which passes through the point $(2, -1, -1)$, is parallel to the plane $4x + y + z + 2 = 0$ and is perpendicular to the line $2x + y = 0 = x - z + 5$.

3.13 CONDITIONS FOR A LINE TO LIE IN A PLANE

To find the conditions that the line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ ...(1)

may lie in the plane $ax + by + cz + d = 0$ ...(2)

Any point on the line (1) is $(lr + x_1, mr + y_1, nr + z_1)$ which will lie on the plane (2), if
$$a(lr + x_1) + b(mr + y_1) + c(nr + z_1) + d = 0.$$
or if
$$(al + bm + cn)\, r + (ax_1 + by_1 + cz_1 + d) = 0 \quad ...(3)$$
The line (1) will lie in the plane (2), if every point of the line lies in the plane so that (3) is satisfied by all values of r.

$\therefore$ The coefficient of $r = 0$ and the constant term $= 0$.

i.e.,
$$\mathbf{al + bm + cn = 0} \quad ...(4)$$
and
$$\mathbf{ax_1 + by_1 + cz_1 + d = 0} \quad ...(5)$$
These are the required conditions which state that

(i) *the line should be parallel to the plane,* (ii) *a point of line should lie in the plane.*

Thus the equation of any plane through the line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$

is
$$\mathbf{a(x - x_1) + b(y - y_1) + c(z - z_1) = 0} \textit{ where } \mathbf{al + bm + cn = 0.}$$

Obs. The equation of any plane through the line of intersection of the planes
$$ax + by + cz + d = 0 \quad ...(i)$$
and
$$a'x + b'y + c'z + d' = 0. \quad ...(ii)$$
is
$$\mathbf{ax + by + cz + d + k(a'x + b'y + c'z + d') = 0.}$$
For (i) is an equation of the first degree in x, y, z representing a plane and (ii) it is satisfied by the coordinates of the points which satisfy both the given planes, *i.e.,* it contains all the points common to these planes.

Example 3.40. *Obtain the equation of a plane passing through the line of intersection of the planes $7x - 4y + 7z + 16 = 0$ and $4x + 3y - 2z + 13 = 0$ and perpendicular to the plane $x - y - 2z + 5 = 0$.* (V.T.U., 2009)

Solution. The equation of any plane through the line of intersection of the two given planes is
$$7x - 4y + 7z + 16 + k(4x + 3y - 2z + 13) = 0$$
or
$$(7 + 4k)x + (-4 + 3k)y + (7 - 2k)z + (16 + 13k) = 0 \quad ...(i)$$
The plane (i) will be perpendicular to the plane

$x - y - 2z + 5 = 0$ if their normals are perpendicular,

i.e., if $(7 + 4k) . 1 + (-4 + 3k) . (-1) + (7 - 2k) . (-2) = 0$ or if, $k = 3/5$.

Substituting this value of k in (i), we get
$$(7 + 12/5)x + (-4 + 9/5)y + (7 - 6/5)x + (16 + 39/5) = 0$$
or
$$47x - 11y + 29z + 119 = 0 \text{ which is the required equation.}$$

Example 3.41. *Find the equation in the symmetrical form of the projection of the line* $\frac{x-1}{2} = -(y+1) = \frac{z-3}{4}$ *on the plane* $x + 2y + z = 12$.

Solution. Any plane through the given line is

$$A(x-1) + B(y+1) + C(z-3) = 1 \quad ...(i)$$

where $$2A - B + 4C = 0 \quad ...(ii)$$

The plane (*i*) will be perpendicular to the given plane, if

$$A + 2B + C = 0 \quad ...(iii)$$

Solving (*ii*) and (*iii*), we get $\frac{A}{-9} = \frac{B}{2} = \frac{C}{5}$.

Substituting these values in (*i*), we get $9x - 2y - 5z + 4 = 0$ ...(*iv*)

which cuts the given plane $x + 2y + z = 12$ ...(*v*)

along the required line of projection.

One point on this line is got by putting $z = 0$ in (*iv*) and (*v*) and solving, it is (4/5, 28/5, 0).

The direction ratios of the line are found, by solving

$$l + 2m + n = 0 \quad \text{and} \quad 9l - 2m - 5n = 0$$

to be 4, – 7, 10.

Hence the required equations of the line of projection are

$$\frac{x - 4/5}{4} = \frac{y - 28/5}{-7} = \frac{z}{10}$$

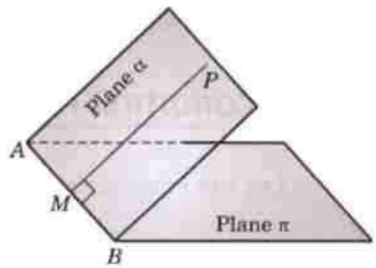


Fig. 3.43

[**The line of greatest slope** in a plane is a line which lies in the plane and is perpendicular to the line of intersection of the plane with the horizontal plane.

In Fig. 3.43, *AB* is the line of intersection of the given plane α with the horizontal plane π. Then *PM* drawn perpendicular to *AB*, is the line of greatest slope on the plane α through the point *P*.]

Example 3.42. *Assuming the line* $x/4 = y/-3 = z/7$ *as vertical, find the equations of the line of greatest slope in the plane* $2x + y - 5z = 12$ *and passing through the point* $(2, 3, -1)$.

Solution. The equation of the horizontal plane through the origin is $4x - 3y + 7z = 0$ ...(*i*)

[The direction ratios of the normal are those of the given vertical line.]

If l, m, n be the direction ratios of the line of intersection of the plane (*i*) and

$$2x + y - 5z = 12 \quad ...(ii)$$

then solving, $4l - 3m + 7n = 0$ and $2l + m - 5n = 0$, we have $l/4 = m/17 = n/5$ ...(*iii*)

Let l', m', n' be the direction ratios of the line of greatest slope which lies in the plane (*ii*).

$$\therefore \quad 2l' + m' - 5n' = 0 \quad ...(iv)$$

Also the line of greatest slope is perpendicular to the line of intersection of the planes (*i*) and (*ii*).

$$\therefore \quad 4l' + 17m' + 5n' = 0 \quad ...(v)$$

Solving (*iv*) and (*v*), $\frac{l'}{3} = \frac{m'}{-1} = \frac{n'}{1}$.

Hence the equations of the line of greatest slope through (2, 3, – 1) and having direction ratios 3, – 1, 1 are

$$\frac{x-2}{3} = \frac{y-3}{-1} = \frac{z+1}{1}.$$

PROBLEMS 3.8

1. Find the equation of the plane which contains the line $\frac{x-1}{2} = y + 1 = \frac{z-3}{4}$ and is perpendicular to the plane $x + 2y + z = 12$. (V.T.U., 2006)

2. Find the equation of the plane through the line $\frac{x-1}{3}=\frac{y-4}{2}=\frac{z-4}{-2}$ and parallel to the line $\frac{x+1}{3}=\frac{y-1}{-4}=\frac{z+2}{1}$.
3. Find the equation of the plane passing through the line of intersection of the planes $x+y+z=1$ and $2x+3y-z+4=0$ and perpendicular to the plane $2y-3z=4$.
4. Find the equation of the plane which contains the line of intersection of the planes $x+y+z=3$ and $2x-y+3z=4$ and is parallel to the line joining the points (2, 1, 1) and (3, 2, 4). *(Madras, 2006)*
5. Find in symmetric form the equations of the line which lies in the plane $2x-y-3z=4$ and is perpendicular to the line
$$\frac{x+1}{3}=\frac{y-1}{3}=\frac{z+4}{2}$$
at the point where the line pierces the plane.
6. A plane is drawn through the line $x+y=1, z=0$ to make an angle $\sin^{-1}(1/3)$ with the plane $x+y+z=0$. Prove that two such planes can be drawn and find their equations. Prove also that the angle between the planes is $\cos^{-1}(7/9)$.
7. Find the equations of the projection of the line $3x-y+2z-1=x+2y-z-2=0$ on the plane $3x+2y+z=0$ in the symmetrical form.
8. Assuming the plane $4x-3y+7z=0$ to be horizontal, find the equations of the line of greatest slope through the point (2, 1, 1) in the plane $2x+y-5z=0$. *(Roorkee, 2000)*

3.14 CONDITION FOR THE TWO LINES TO INTERSECT (OR TO BE COPLANAR)

Let the equations of the lines be $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ ...(1)

$$\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2} \quad ...(2)$$

The equation of any plane through the line (1) is $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$ ...(3)

where $al_1+bm_1+cn_1=0$ ...(4)

The line (2) will lie in the plane (3), if it is parallel to the plane and its point (x_2, y_2, z_2) lies on this plane.

$\therefore$ $al_2+bm_2+cn_2=0$ ...(5)

and $a(x_2-x_1)+b(y_2-y_1)+c(z_2-z_1)=0$ ...(6)

Eliminating a, b, c from (6), (4) and (5), we get

$$\begin{vmatrix} \mathbf{x_2-x_1} & \mathbf{y_2-y_1} & \mathbf{z_2-z_1} \\ \mathbf{l_1} & \mathbf{m_1} & \mathbf{n_1} \\ \mathbf{l_2} & \mathbf{m_2} & \mathbf{n_2} \end{vmatrix} = 0 \textit{ which is the required condition.}$$

Also eliminating a, b, c from (3), (4) and (5), we get $\begin{vmatrix} \mathbf{x-x_1} & \mathbf{y-y_1} & \mathbf{z-z_1} \\ \mathbf{l_1} & \mathbf{m_1} & \mathbf{n_1} \\ \mathbf{l_2} & \mathbf{m_2} & \mathbf{n_2} \end{vmatrix} = \mathbf{0}$

which is the equation of the plane containing the lines (1) and (2).

Example 3.43. *Show that the lines* $\frac{x-5}{4}=\frac{y-7}{4}=\frac{z+3}{-5}$; $\frac{x-8}{7}=\frac{y-4}{1}=\frac{z-5}{3}$ *are coplanar; find their common point and the equation of the plane in which they lie.* *(Madurai, 2002)*

Solution. Any point on the first line is $(5+4r, 7+4r, -3-5r)$ ...(*i*)

which lies on the second line if $\frac{-3+4r}{7}=\frac{3+4r}{7}=\frac{-8-5r}{3}$ ...(*ii*)

$\therefore$ From $\frac{-3+4r}{7}=3+4r$, we have $r=-1$.

This value clearly satisfies the equation $\frac{3+4r}{7}=\frac{-8-5r}{3}$

Hence the lines intersect, (*i.e.*, are coplanar) and from (*i*) their point of intersection is (1, 3, 2).

The equation of the plane in which they lie, is $\begin{vmatrix} x-5 & y-7 & z+3 \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix} = 0$

i.e., $\quad 17x - 47y - 24z + 172 = 0.$

Example 3.44. *Show that the lines*

$$\frac{x+4}{3} = \frac{y+6}{5} = \frac{z-1}{-2} \text{ and } 3x - 2y + z + 5 = 0 = 2x + 3y + 4z - 4$$

are coplanar. Find their point of intersection and the plane in which they lie.

Solution. Any point on the first line is $P(3r - 4, 5r - 6, -2r + 1)$, which lie in the plane

$$3x - 2y + z + 5 = 0$$

if $\quad 3(3r - 4) - 2(5r - 6) + (-2r + 1) + 5 = 0 \quad \text{or} \quad r = 2,$

The point P will also lie in the plane $2x + 3y + 4z - 4 = 0$

if $\quad 2(3r - 4) + 3(5r - 6) + 4(-2r + 1) - 4 = 0 \quad \text{or} \quad r = 2.$

Since the two values of r are equal, the given lines intersect, *i.e.*, are coplanar.

Putting $r = 2$ in the coordinates of P, we get $(2, 4, -3)$ as their point of intersection.

The equation of a plane containing the second line is

$$3x - 2y + z + 5 + k(2x + 3y + 4z - 4) = 0$$

which will contain the first line if its point $(-4, -6, 1)$ lies on it.

$\therefore \quad -12 + 12 + 1 + 5 + k(-8 - 18 + 4 - 4) = 0$

i.e., $\quad k = 3/13$

Substituting this value of k, (i) becomes $45x - 17y + 25z + 53 = 0$, which is the required plane.

Example 3.45. *Find the equations of the line drawn through the point $(1, 0, -1)$ and intersecting the lines*

$$x = 2y = 2z \quad \text{and} \quad 3x + 4y = 1,\ 4x + 5z = 2.$$

(V.T.U., 2007)

Solution. The required line will comprise of

(a) the plane containing the first line and the point $(1, 0, -1)$.

(b) the plane containing the second line and the point $(1, 0, -1)$.

The equation of any plane containing the first line

i.e., $\quad \dfrac{x}{2} = \dfrac{y}{1} = \dfrac{z}{1}$

is $\quad a(x - 0) + b(y - 0) + c(z - 0) = 0 \quad \text{...}(i)$

where $\quad 2a + b + c = 0 \quad \text{...}(ii)$

Also $(1, 0, -1)$ lies on (i) $\therefore \quad a - c = 0 \quad \text{...}(iii)$

Solving (ii) and (iii), we have $\dfrac{a}{1} = \dfrac{b}{-3} = \dfrac{c}{1}$.

Substituting these values in (i), we get $x - 3y + z = 0 \quad \text{...}(iv)$

Again, the equation of any plane containing the second line is

$3x + 4y - 1 + k(4x + 5z - 2) = 0$. Also $(1, 0, -1)$ lies on it. $\quad \text{...}(v)$

$\therefore \quad 3 + 0 - 1 + k(4 - 5 - 2) = 0, \quad$ *i.e.,* $\quad k = \dfrac{2}{3}$.

Substituting $k = 2/3$ in (v), we get $17x + 12y + 10z - 7 = 0 \quad \text{...}(iv)$

Hence (iv) and (vi) constitute the required line.

PROBLEMS 3.9

1. Prove that the lines

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4} \text{ and } \frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$$

are coplanar and find the equation of the plane containing them.

2. Prove that the lines $\frac{x-4}{1}=\frac{y+3}{-4}=\frac{1+z}{7}$ and $\frac{x-1}{2}=\frac{y+1}{-3}=\frac{z+10}{8}$ intersect and find the coordinates of their point of intersection. (*V.T.U., 2000 S ; Andhra, 1999*)

3. Find the condition that the lines $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ and $ax+by+cz+d=0=a'x+b'y+c'z+d'$ are coplanar.

4. Show that the lines $\frac{x+1}{1}=\frac{y+1}{2}=\frac{z+1}{3}$ and $x+2y+3z-8=0=2x+3y+4z-11$ intersect. Find their point of intersection and the equation of the plane containing them. (*V.T.U., 2009*)

5. Show that the lines $x-3y+2z-4=0=2x+y+4z+1$ and

$3x+2y+5z-1=0=2y-z$, are coplanar. (*Andhra, 2000*)

6. Prove that the lines $x=ay+b=cz+d$ and $x=\alpha y+\beta=\gamma z+\delta$ are coplanar if

$(\gamma-c)(a\beta-b\alpha)-(\alpha-a)(c\delta-d\gamma)=0$ (*Rajasthan, 2006*)

7. Obtain the equations of the straight line lying in the plane.

$$x-2y+4z-51=0$$

and interesecting the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-6}{7}$ at right angles.

8. Find the equation of the straight line perpendicular to both the lines

$$\frac{x-1}{1}=\frac{y-1}{2}=\frac{z+2}{3} \quad \text{and} \quad \frac{x+2}{2}=\frac{y-5}{-1}=\frac{z+3}{2}$$

and passing through their point of intersection.

9. A line with direction cosines proportional to 2, 7, − 5 is drawn to intersect the lines

$$\frac{x-8}{3}=\frac{y-6}{-1}=\frac{z+1}{1} \quad \text{and} \quad \frac{x+3}{-3}=\frac{y-3}{2}=\frac{z-6}{4}.$$

Find the coordinates of the point of intersection and the length intercepted.

3.15 SHORTEST DISTANCE BETWEEN TWO LINES

Two straight lines which do not lie in one plane are called *skew lines*. Such lines possess a common perpendicular which is the *shortest distance* between them.

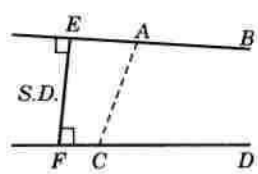


Fig. 3.44

Let the given skew lines AB and CD be

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1} \quad \text{and} \quad \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$$

so that $A \equiv (x_1, y_1, z_1)$ and $C \equiv (x_2, y_2, z_2)$.

Let l, m, n be the direction cosines of the shortest distance EF.

Since $EF \perp$ to both AB and CD.

$\therefore$ $ll_1+mm_1+nn_1=0$ and $ll_2+mm_2+nn_2=0$.

Solving,
$$\frac{l}{m_1n_2-m_2n_1}=\frac{m}{n_1l_2-n_2l_1}=\frac{n}{l_1m_2-l_2m_1}$$

$$=\frac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{[\Sigma(m_1n_2-m_2n_1)^2]}}=\frac{1}{\sin\theta} \qquad ...(1)$$

where θ is the angle between the lines AB and CD.

$\therefore$ *Length of S.D.* (EF) = projection of AC on EF

$= l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)$ where l, m, n have the values as given by (1).

To find the equations of the line of shortest distance, we observe that it is coplanar with both AB and CD.

Plane containing the lines AB and EF is, $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0$ ...(2)

Plane containing the lines CD and EF is $\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix} = 0$ ...(3)

Hence (2) and (3) are the equations of the line of shortest distance.

Obs. The condition for the given lines to be coplanar is also obtained by equating the shortest distance (EF) to zero.

Example 3.46. *Find the magnitude and the equations of the shortest distance between the lines*

$$\frac{x}{2} = \frac{y}{-3} = \frac{z}{1} \quad \text{and} \quad \frac{x-2}{3} = \frac{y-1}{-5} = \frac{z+2}{2}.$$

(V.T.U., 2009 ; Cochin, 2005)

Solution. Let l, m, n be the direction cosines of the shortest distance EF.

$\because$ $EF \perp$ to both AB and CD,

$\therefore$ $2l - 3m + n = 0,\ 3l - 5m + 2n = 0.$

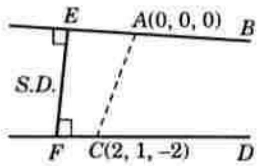


Fig. 3.45

Solving $\quad \dfrac{l}{1} = \dfrac{m}{1} = \dfrac{n}{1} = \dfrac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{(1+1+1)}} = \dfrac{1}{\sqrt{3}}.$

$\therefore$ *Length of S.D.* (EF) = projection of AC on EF

$$= (2-0)\frac{1}{\sqrt{3}} + (1-0)\frac{1}{\sqrt{3}} + (-2-0)\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

The equations of the line of shortest distance (EF) are

$$\begin{vmatrix} x & y & z \\ 2 & -3 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0 \text{ and } \begin{vmatrix} x-2 & y-1 & z+2 \\ 3 & -5 & 2 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

i.e., $\quad 4x + y - 5z = 0$ and $7x + y - 8z = 31.$

Example 3.47. *Find the points on the lines*

$$\frac{x-6}{3} = \frac{y-7}{-1} = \frac{z-4}{1} \qquad ...(i)$$

$$\frac{x}{-3} = \frac{y+9}{2} = \frac{z-2}{4} \qquad ...(ii)$$

which are nearest to each other. Hence find the shortest distance between the lines and its equations.

(V.T.U., 2004 ; Burdwan, 2003 ; Osmania, 2003)

Solution. Any point on the line (*i*) is $E(6+3r, 7-r, 4+r)$ ...(*iii*)

and any point on the line (*ii*) is $F(-3r', -9+2r', 2+4r')$ ...(*iv*)

Then the direction cosines of EF are proportional to $6+3r+3r', 16-r-2r', 2+r-4r'$

Since $EF \perp$ both the lines (*i*) and (*ii*), $\therefore$ $3(6+3r+3r') - (16-r-2r') + (2+r-4r') = 0$

and $\quad -3(6+3r+3r') + 2(16-r-2r') + 4(2+r-4r') = 0$

or $\quad 11r + 7r' + 4 = 0,\ 7r + 29r' - 22 = 0$, whence $r = -1, r' = 1.$

Substituting $r = -1$ in (*iii*) and $r' = 1$ in (*iv*), we get $E = (3, 8, 3)$ and $F = (-3, -7, 6)$ which are the points on (*i*) and (*ii*) nearest to each other.

$\therefore$ Length of the shortest distance (EF) = $\sqrt{[(3+3)^2 + (8+7)^2 + (3-6)^2]} = 3\sqrt{30}$

The equations of the shortest distance (EF) is $\dfrac{x-3}{2} = \dfrac{y-8}{5} = \dfrac{z-3}{-1}.$

Obs. This method is sometimes very convenient and is especially useful when the points of intersection of the line of shortest distance with the given lines are required.

Example 3.48. *Two control cables in the form of straight lines AB and CD are laid such that the coordinates of A, B, C and D are respectively (1, 2, 3), (2, 1, 1), (– 1, 1, 2) and (2, – 1, –3). Determine the amount of clearance between the cables.*

Solution. The direction ratios of AB are 1, – 1, – 2 and those of CD are 3, – 2, – 5.

The amount of clearance between AB and CD is nothing but the shortest distance PQ between the cables. If the direction cosines of PQ be l, m, n then

$$l - m - 2n = 0 \text{ and } 3l - 2m - 5n = 0$$

$$\therefore \quad \frac{l}{1} = \frac{m}{-1} = \frac{n}{1} \qquad [\because \; PQ \perp \text{ to both } AB + CD].$$

Thus the clearance between the cables

= shortest distance between AB and CD

= projection of AC (or BD) on PQ

$$= \frac{1(-1-1)-1(1-2)+1(2-3)}{\sqrt{(1+1+1)}} = \frac{2}{\sqrt{3}} \text{ (in magnitude)}$$

Example 3.49. *Find the equation of the plane through the line*

$$\frac{x-1}{3} = \frac{y-4}{2} = \frac{z-4}{-2} \qquad ...(i)$$

and parallel to the line $\dfrac{x+1}{2} = \dfrac{y-1}{-4} = \dfrac{z+2}{1}$ ...(*ii*)

Hence find the shortest distance between them (*Hazaribagh, 2009*)

Solution. The equation of the plane containing the line (*i*) and parallel to (*ii*) is

$$\begin{vmatrix} x-1 & y-4 & z-4 \\ 3 & 2 & -2 \\ 2 & -4 & 1 \end{vmatrix} = 0$$

or
$$6x + 7y + 16z = 98 \qquad ...(iii)$$

Now the shortest distance between the lines (*i*) and (*ii*)

= Length of the perpendicular drawn from the point (– 1, 1, – 2) of (*ii*) on the plane (*iii*)

$$= \frac{-6+7-32-98}{\sqrt{(6^2+7^2+16^2)}} = \frac{120}{\sqrt{(341)}}, \text{ numerically.}$$

Example 3.50. *Show that the shortest distance between z-axis and the line* $ax + by + cz + d = 0 = a'x + b'y + c'z + d'$ *is* $\dfrac{dc' - d'c}{\sqrt{[(ac' - a'c)^2 + (bc' - b'c)^2]}}$.

Solution. The plane containing the given line is

$$(ax + by + cz + d) + k(a'x + b'y + c'z + d') = 0 \qquad ...(i)$$

or
$$(a + ka')x + (b + kb')y + (c + kc')z + (d + kd') = 0$$

This plane is parallel to the z-axis (d. c's, 0, 0, 1) if $c + kc' = 0$ or $k = -c/c'$. Then (*i*) becomes

$$(ac' - a'c)x + (bc' - b'c)y + (dc' - d'c) = 0 \qquad ...(ii)$$

A point on the z-axis is the origin.

$\therefore \quad \perp$ distance of the origin from the plane (*ii*)

$$= \frac{dc' - d'c}{\sqrt{[(ac' - a'c)^2 + (bc' - b'c)^2]}} \text{ which is the required S.D.}$$

Example 3.51. *A square ABCD of diagonal 2a is folded along the diagonal AC, so that the planes DAC and BAC are at right angles. Find the shortest distance between DC and AB.*

Solution. Let the diagonals AC and BD intersect at O the folded position of the square. Let OB, OC and OD be the axes. Then equations of DC are

$$\frac{x-0}{0-0}=\frac{y-a}{a-0}=\frac{z-0}{0-a} \quad \text{or} \quad \frac{x}{0}=\frac{y-a}{a}=\frac{z}{-a}$$

and those of AB are $\frac{x-a}{a}=\frac{y}{a}=\frac{z}{0}$

The equation of the plane through DC and parallel to AB is

$$\begin{vmatrix} x & y-a & z \\ 0 & a & -a \\ a & a & 0 \end{vmatrix}=0 \quad \text{or} \quad x-y-z+a=0 \qquad ...(i)$$

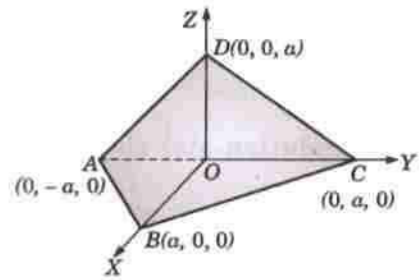


Fig. 3.46

A point on the line AB is $(a, 0, 0)$.

Hence required S.D. = $\perp$ distance of $(a, 0, 0)$ from the plane (i)

$$=\frac{a+a}{\sqrt{1+1+1}}=\frac{2a}{\sqrt{3}}.$$

PROBLEMS 3.10

1. Find the magnitude and the equations of the shortest distance between the lines.

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4} \text{ and } \frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}.$$

(V.T.U., 2008 ; Rajasthan, 2005 ; Madras, 2003)

2. Find the magnitude and equations of the shortest distance between the lines

$$\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1} \text{ and } \frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$$ *(Anna, 2005 S ; Osmania, 2000 S)*

Find also the points where it intersects the lines.

3. Find the shortest distance and the equation of the line of shortest distance between the line $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and the y-axis. *(V.T.U., 2010)*

4. Show that the shortest distance between the lines $y-mx=0=z-c$ and $y+mx=0=z+c$ is c units.

5. If the shortest distance between the lines $\frac{y}{b}+\frac{z}{c}=1, x=0$ and $\frac{x}{a}-\frac{z}{c}=1, y=0$ be $2d$, then show that $d^{-2}=a^{-2}+b^{-2}+c^{-2}$.

6. Show that the shortest distance between x-axis and the line $ax+by+cz+d=0=a'x+b'y+c'z+d'$ is

$$\frac{da'-d'a}{\sqrt{[(ba'-b'a)^2+(ca'-c'a)^2]}}.$$

7. Show that the shortest distance between a diagonal of a rectangular parallelopiped whose edges are a, b, c and the edges not meeting it, are
$bc/(b^2+c^2)^{1/2}, ca/(c^2+a^2)^{1/2}, ab/(a^2+b^2)^{1/2}$.

8. Show that the shortest distance between two opposite edges of the tetrahedron formed by the planes $x+y=0$, $y+z=0, z+x=0$ and $x+y+z=a$ is $2a/\sqrt{6}$.

3.16 INTERSECTION OF THREE PLANES

Any three planes (no two of which are parallel) intersect in one of the following ways :

(1) *The planes may meet in a point,* if the line of section of two of them is not parallel to the third.

(2) *The planes may have a common line of section,* if the line of section of two of them lies on the third (Fig. 3.47).

(3) *The planes may form a triangular prism,* if the line of section of two of them is parallel to the third but does not lie on it. (See Fig. 3.48)

Example 3.52. *Prove that the planes*
(i) $12x - 15y + 16z - 28 = 0$, *(ii)* $6x + 6y - 7z - 8 = 0$. *and (iii)* $2x + 35y - 39z + 92 = 0$,
have a common line of intersection. Prove that the point in which the line $\frac{x-1}{3} = \frac{y}{-2} = \frac{z-3}{1}$ *meets the third plane is equidistant from other two planes.*

Solution. Any plane through the line of intersection of the planes (*i*) and (*ii*) is

$$12x - 15y + 16z - 28 + \lambda(6x + 6y - 7z - 8) = 0$$

or
$$(12 + 6\lambda)x + (-15 + 6\lambda)y + (16 - 7\lambda)z - (28 + 8\lambda) = 0 \quad ...(iv)$$

Three planes will intersect in a common line if the planes (iii) and (iv) represent the same plane.

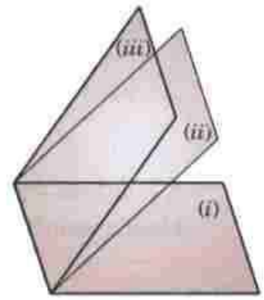


Fig. 3.47

$$\therefore \quad \frac{12 + 6\lambda}{2} = \frac{-15 + 6\lambda}{35} = \frac{16 - 7\lambda}{-39} = \frac{-28 - 8\lambda}{12} \quad ...(v)$$

From $\frac{12 + 6\lambda}{2} = \frac{-15 + 6\lambda}{35}$, we have $\lambda = \frac{-25}{11}$ which satisfies all the equations (*v*).

Hence the given planes intersect in a line.

Any point on the line $\frac{x-1}{3} = \frac{y}{-2} = \frac{z-3}{1} = r$ (say) ...(*vi*)

is $(3r + 1, -2r, r + 3)$ which lies in the plane (*iii*)

if $2(3r + 1) + 35(-2r) - 39(r + 3) + 12 = 0$, *i.e.* if $r = -1$.

$\therefore$ The coordinates of the point P in which (*vi*) meets (*iii*) are $(-2, 2, 2)$.

$$\text{Distance of } P \text{ from plane } (i) = \frac{12(-2) - 15(2) + 16(2) - 28}{\sqrt{144 + 225 + 256}} = \frac{-50}{\sqrt{(625)}} = 2 \text{ (in magnitude)}$$

$$\text{Distance of } P \text{ from plane } (ii) = \frac{6(-2) + 6(2) - 7(2) - 8}{\sqrt{36 + 36 + 49}} = 2 \text{ (in magnitude)}$$

Hence the point P is equidistant from the planes (*i*) and (*ii*).

Example 3.53. *Prove that the three planes*
(i) $2x + y + z = 3$, *(ii)* $x - y + 2z = 4$, *(iii)* $x + z = 2$,
form a triangular prism and find the area of the normal section of the prism.

Solution. Let l, m, n be the direction cosines of the line of intersection of the planes (*ii*) and (*iii*) so that $l - m + 2n = 0, l + n = 0$,

whence
$$\frac{l}{1} = \frac{m}{-1} = \frac{n}{-1}.$$

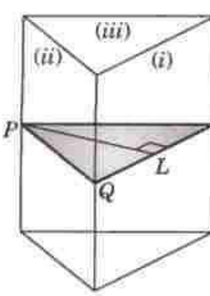


Fig. 3.48

To find a point P on this line, put $x = 0$ in (*ii*) and (*iii*), $-y + 2z = 4$ and $z = 2$. Thus the point P is $(0, 0, 2)$.

Now the line of intersection of (*ii*) and (*iii*) is parallel to the plane (*i*).

$$[\because \quad 2 \times 1 + 1 \times (-1) + 1 \times (-1) = 0]$$

Also the point P does not lie on the plane (*i*).

Hence the given planes form a triangular prism.

Let ΔPQR be its normal section through P.

The equation of the plane through P perpendicular to the line of intersection of the planes (*i*) and (*iii*) is,

$$1(x - 0) - 1(y - 0) - 1(z - 2) = 0$$

or
$$x - y - z + 2 = 0 \quad ...(iv)$$

Solving the equations (*i*), (*ii*) and (*iv*), we get

$$Q \equiv \left(\frac{1}{3}, \frac{1}{3}, 2\right).$$

Solving the equation (*i*), (*iii*) and (*iv*), we get

$$R \equiv \left(\frac{1}{3}, \frac{2}{3}, \frac{5}{3}\right).$$

$$\therefore \qquad QR = \sqrt{\left\{\left(\frac{1}{3}-\frac{1}{3}\right)^2 + \left(\frac{1}{3}-\frac{2}{3}\right)^2 + \left(2-\frac{5}{3}\right)^2\right\}} = \sqrt{\left(\frac{2}{9}\right)}$$

Also $PL \perp$ from P on the plane (*i*) $= \dfrac{3-2}{\sqrt{(4+1+1)}} = \dfrac{1}{\sqrt{6}}$.

Hence the area of $\Delta PQR = \dfrac{1}{2} QR \times PL = \dfrac{1}{2} \cdot \dfrac{\sqrt{2}}{3} \cdot \dfrac{1}{\sqrt{6}} = \dfrac{1}{6\sqrt{3}}$.

PROBLEMS 3.11

1. Prove that the three planes $2x - 3y - 7z = 0$, $3x - 14y - 13z = 0$, $8x - 31y - 33z = 0$ pass through one line.
2. Prove that the planes $x = cy + bz$, $y = az + cx$, $z = bx + ay$ intersect in a line if $a^2 + b^2 + c^2 + 2abc = 1$ and show that the equations of this line are
$$\frac{x}{\sqrt{1-a^2}} = \frac{y}{\sqrt{1-b^2}} = \frac{z}{\sqrt{1-c^2}}$$
(*Rajasthan, 2005*)
3. Show that the planes $x + 2y - 3 = 0$, $3x - 4y + z - 4 = 0$ and $4x + 3y - 2z - 24 = 0$ from a triangular prism.
4. Prove that the planes $2x + 3y + 4z = 6$, $3x + 4y + 5z = 20$, $x + 2y + 3z = 0$ form a prism : obtain the equation of one of its edges in the symmetrical form.

3.17 SPHERE

(1) Def. *A **sphere** is the locus of a point which remains at a constant distance from a fixed point.*
The fixed point is called the *centre* and the constant distance the *radius* of the sphere

(2) The equation of the sphere whose centre is (a, b, c) and radius r, is

$$(\mathbf{x} - \mathbf{a})^2 + (\mathbf{y} - \mathbf{b})^2 + (\mathbf{z} - \mathbf{c})^2 = \mathbf{r}^2.$$

For the distance of any point $P(x, y, z)$ on the sphere from the centre $C(a, b, c)$ = the radius r.

In particular the *equation of the sphere whose centre is the origin and radius a, is*

$$\mathbf{x^2 + y^2 + z^2 = a^2}$$

(3) The equation $\mathbf{x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0}$ represents a sphere whose centre is $(-\mathbf{u}, -\mathbf{v}, -\mathbf{w})$ and radius

$$= \sqrt{\mathbf{u^2 + v^2 + w^2 - d}}.$$

For on writing it as $(x^2 + 2ux) + (y^2 + 2vy) + (z^2 + 2wz) = -d$

or as $\qquad (x + u)^2 + (y + v)^2 + (z + w)^2 = u^2 + v^2 + w^2 - d$

and comparing with $\qquad (x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$,

it clearly represents a sphere whose centre is

$$(a, b, c) = (-u, -v, -w) \text{ and radius } r = \sqrt{(u^2 + v^2 + w^2 - d)}$$

Thus *the general equation of a sphere is such that*

(*i*) *it is the second degree in x, y, z,*

(*ii*) *the coefficient of x^2, y^2, z^2 are equal,*

and (*iii*) *there are no terms containing yz, zx or xy*

(4) Section of a sphere by a plane *is a circle and the section of a sphere by a plane through its centre is called a* **great circle.**

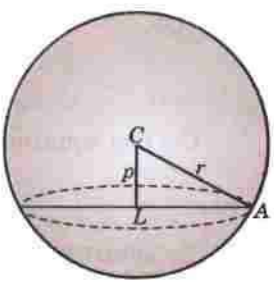


Fig. 3.49

Thus the equations $\qquad x^2 + y^2 + z^2 + 2ux\ 2vy + 2wz + d = 0 \quad$ [Sphere]

and $\qquad Ax + By + Cz + D = 0 \quad$ [Plane]

taken together represent a circle (Fig. 3.49) having centre L and radius $LA = \sqrt{(r^2 - p^2)}$.

(5) The equation of any sphere through the circle *of intersection of*

the sphere $S = 0$

and the plane $U = 0$

is $S + kU = 0$

For the equation $S + kU = 0$

represents a sphere and the points of intersection of the sphere $S = 0$ and the plane $U = 0$ satisfy it.

Example 3.54. *Find the equation of the sphere through the points (0, 0, 0), (0, 1, – 1), (– 1, 2, 0) and (1, 2, 3). Locate its centre and find the radius.*

Solution. Let the required equation of the sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad ...(i)$$

It passes through (0, 0, 0), (0, 1, – 1), (– 1, 2, 0) and (1, 2, 3).

$$\therefore \qquad d = 0,$$

$$1 + 1 + 2v - 2w + d = 0 \quad \text{or} \quad v - w + 1 = 0 \qquad ...(ii)$$

$$1 + 4 - 2u + 4v + d = 0 \quad \text{or} \quad -2u + 4v + 5 = 0 \qquad ...(iii)$$

$$1 + 4 + 9 + 2u + 4v + 6w + d = 0 \quad \text{or} \quad u + 2v + 3w + 7 = 0 \qquad ...(iv)$$

Multiplying (*ii*) by (*iii*) and adding to (*iv*), we get

$$u + 5v + 10 = 0 \qquad ...(v)$$

Solving (*iii*) and (*v*), we get $\quad u = -\dfrac{15}{14}, v = -\dfrac{25}{14}$

From (*ii*), $\quad w = v + 1 = \dfrac{-25}{14} + 1 = \dfrac{-11}{14}$

Substituting these values of u, v, w, d in (*i*), we get

$$x^2 + y^2 + z^2 - \frac{15}{7}x - \frac{25}{7}y - \frac{11}{7}z = 0 \qquad ...(vi)$$

which is the required equaton of the sphere.

Its centre is (15/14, 25/14, 11/14) $\qquad [(-u, -v, -w)]$

and the radius = $[(-15/14)^2] + (-25/14)^2 + (-11/14)^2 - 0] = \sqrt{971/14}$.

Example 3.55. (*a*) *Find the equation of the sphere which has* (x_1, y_1, z_1) *and* (x_2, y_2, z_2) *as the extremities of a diameter.*

(*b*) *Deduce the equation of the sphere described on the line joining the points (2, – 1, 4) and (– 2, 2, – 2) as diameter. Find the area of the circle in which the sphere is intersected by the plane $2x + y - z = 3$.*

(Anna, 2009 ; Hazaribagh, 2009)

Solution. (*a*) Let $P(x, y, z)$ be any point on the sphere having $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ as ends of diameter (Fig. 3.50. (*a*)). Then *AP* and *BP* are at right angles.

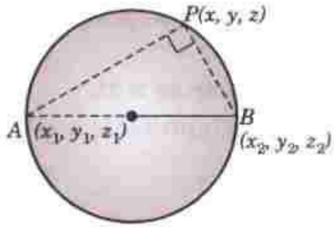


Fig. 3.50 (*a*)

Now direction ratio of *AP* are $x - x_1, y - y_1, z - z_1$ and those of *BP* are $x - x_2, y - y_2, z - z_2$.

Hence

$$(\mathbf{x} - \mathbf{x_1})(\mathbf{x} - \mathbf{x_2}) + (\mathbf{y} - \mathbf{y_1})(\mathbf{y} - \mathbf{y_2}) + (\mathbf{z} - \mathbf{z_1})(\mathbf{z} - \mathbf{z_2}) = 0$$

which is the required equation.

(*b*) The equation of the required sphere is

$$(x - 2)(x + 2) + (y + 1)(y - 2) + (z - 4)(z + 2) = 0$$

or

$$x^2 + y^2 + z^2 - y - 2z - 14 = 0 \qquad ...(i)$$

Its centre is $C(0, 1/2, 1)$

and radius $\quad (r) = \sqrt{(0, 1/4 + 1 + 14)} = \sqrt{(61/4)}$.

Let the given plane $2x + y - z - 3 = 0 \qquad ...(ii)$

cut the sphere (1) in the circle PP' having centre L.

$\therefore$ p = perpendicular CL from C on the plane (2)

$$= \frac{1/2 - 1 - 3}{\sqrt{4+1+1}} = \frac{7}{2\sqrt{6}} \text{ (in magnitude)}$$

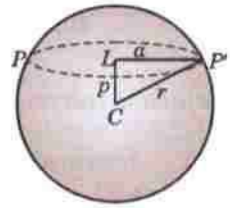


Fig. 3.50 (b)

If a be the radius of the circle PP', then

$$a^2 = r^2 - p^2 = \frac{61}{4} - \frac{49}{24} = \frac{317}{24}$$

Hence the area of circle $PP' = \pi a^2 = \dfrac{317}{24}\pi.$

Example 3.56. *A plane passes through a fixed point (a, b, c) and cuts the axes in A, B, C. Show that the locus of the centre of the sphere OABC is*

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2. \qquad \textit{(P.T.U., 2010)}$$

Solution. Let the centre of the sphere $OABC$ be $P(f, g, h)$ so that its radius $OP = \sqrt{(f^2 + g^2 + h^2)}$.

$\therefore$ The equation of the sphere is

$$(x - f)^2 + (y - g)^2 + (z - h)^2 = f^2 + g^2 + h^2$$

or $$x^2 + y^2 + z^2 - 2fx - 2gy - 2hz = 0 \qquad ...(i)$$

To find OA, putting $y = 0, z = 0$ in (i), we have

$$x^2 - 2fx = 0, \text{ i.e., } OA = x = 2f. \text{ Similarly, } OB = 2g, OC = 2h.$$

Thus the equation of the plane ABC is $\dfrac{x}{2f} + \dfrac{y}{2g} + \dfrac{z}{2h} = 1$

Since the plane passes through (a, b, c) $\therefore$ $\dfrac{a}{2f} + \dfrac{b}{2g} + \dfrac{c}{2h} = 1.$

Hence the locus of the centre (f, g, h) of the sphere is,

$$\frac{a}{2x} + \frac{b}{2y} + \frac{c}{2z} = 1 \quad \text{or} \quad \frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2.$$

Example 3.57. *Find the equation of the sphere having the circle*

$$x^2 + y^2 + z^2 + 10y - 4z - 8 = 0, x + y + z = 3$$

as a great circle. *(Anna, 2009 ; Madras, 2001 S)*

Solution. The equation of any sphere through the given circle is

$$x^2 + y^2 + z^2 + 10y - 4z - 8 + k(x + y + z - 3) = 0$$

i.e., $$x^2 + y^2 + z^2 + kx + (10 + k)y - (4 - k)z - (8 + 3k) = 0 \qquad ...(i)$$

In order that (i) may have the given circle as its great circle, its centre $[-k/2, -(10 + k)/2, (4 - k)/2]$ must lie on the plane $x + y + z = 3$

$$\therefore \qquad -\frac{k}{2} - \frac{10+k}{2} + \frac{4-k}{2} = 3, \text{ i.e., } k = -4$$

whence (i) becomes, $x^2 + y^2 + z^2 - 4x + 6y - 8z + 4 = 0$ which is the required equation.

Example 3.58. *Find the equation of the smallest sphere which contains the circle $x^2 + y^2 + z^2 + 2x + 6y + 4z - 11 = 0$ and $2x + 2y + z + 1 = 0$.*

Solution. Equation of any sphere containing the given circle is

$$x^2 + y^2 + z^2 + 2x + 6y + 4z - 11 + \lambda(2x + 2y + z + 1) = 0$$

or $$x^2 + y^2 + z^2 + (2 + 2\lambda)x + (6 + 2\lambda)y + (4 + \lambda)z - 11 + \lambda = 0 \qquad ...(i)$$

Its radius r is given by

$$r^2 = (1 + \lambda)^2 + (3 + \lambda)^2 + (2 + \tfrac{1}{2}\lambda)^2 - (\lambda - 11) = \frac{9}{4}\left[\lambda^2 + 4\lambda + \frac{100}{9}\right] = \frac{9}{4}\left[(\lambda + 2)^2 + \frac{64}{9}\right]$$

Now r^2 has the least value when $\lambda = -2$.

$\therefore$ Substituting $\lambda = -2$ in (*i*), we get

$$x^2 + y^2 + z^2 - 2x + 2y + 2z - 13 = 0$$

which is the required smallest sphere.

Example 3.59. *Prove that the circles $x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 = 0$, $5y + 6z + 1 = 0$ and $x^2 + y^2 + z^2 - 3x - 4y + 5z - 6 = 0$, $x + 2y - 7z = 0$ lie on the same sphere and find its equation.*

Solution. Equation of any sphere containing the first circle is

$$x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 + \lambda(5y + 6z + 1) = 0$$

or
$$x^2 + y^2 + z^2 - 2x + (3 + 5\lambda)y + (4 + 6\lambda)z - 5 + \lambda = 0 \qquad ...(i)$$

Similarly equation of any sphere containing the second given circle is

$$x^2 + y^2 + z^2 - 3x - 4y + 5z - 6 + \lambda'(x + 2y - 7z) = 0$$

or
$$x^2 + y^2 + z^2 + (-3 + \lambda')x + (-4 + 2\lambda')y + (5 - 7\lambda')z - 6 = 0 \qquad ...(ii)$$

(*i*) and (*ii*) will represent the same sphere when

$$-2 = -3 + \lambda' \qquad ...(iii); \qquad 3 + 5\lambda = -4 + 2\lambda' \qquad ...(iv)$$

$$4 + 6\lambda = 5 - 7\lambda' \qquad ...(v); \qquad -5 + \lambda = -6 \qquad ...(vi)$$

Now (*iii*) gives $\lambda' = 1$ and (*vi*) gives $\lambda = -1$.

Clearly $\lambda = -1$ and $\lambda' = 1$ also satisfy (*iv*) and (*v*). This shows that the given circles lie on the same sphere.

Substituting $\lambda = -1$ in (*i*) or $\lambda' = 1$ in (*ii*), we get

$$x^2 + y^2 + z^2 - 2x - 2y - 2z - 6 = 0$$

which is the desired sphere.

PROBLEMS 3.12

1. Find the equation of the sphere through the points (2, 0, 1), (1, – 5, – 1), (0, –2, 3) and (4, – 1, 2). Also find its centre and radius.
2. Find the equation of the sphere whose diameter is the line joining the origin to the point (2, – 2, 4). Also find its centre and radius.
3. Obtain the equation of the sphere which passes through the points (1, 0, 0), (0, 1, 0), (0, 0, 1) and
   (*a*) has its centre on the plane $x + y + z = 6$.
   (*b*) has its radius as small as possible.
4. A sphere of constant radius k passes through the origin and meets the axes in A, B, C. Prove that the centroid of the
   (*i*) triangle ABC lies on the sphere $9(x^2 + y^2 + z^2) = 4k^2$. (*Assam, 1999*)
   (*ii*) tetrahedron $OABC$ lies on the sphere $x^2 + y^2 + z^2 = k^2/4$.
5. A plane passes through a fixed points (a, b, c), show that the locus of the foot of the perpendicular from the origin on the plane is the sphere $x^2 + y^2 + z^2 - ax - by - cz = 0$.
6. A sphere of constant radius r passes through the origin O and cuts the axes in A, B, C. Prove that the locus of the foot of the perpendicular from O on the plane ABC is given by
   $$(x^2 + y^2 + z^2)^2 (x^{-2} + y^{-2} + z^{-2}) = 4r^2.$$
7. A plane cuts the coordinate axes at A, B, C. If $OA = a$, $OB = b$, $OC = c$, find the equation of the
   (*i*) circumsphere of the tetrahedron $OABC$, (*Assam, 1999*)
   (*ii*) circum-circle of the triangle ABC. Also obtain the coordinates of its centre.
8. Find the centre and radius of the circle $x^2 + y^2 + z^2 - 2y - 4z = 11$, $x + 2y + 2z = 15$. (*P.T.U., 2009 S ; Burdwan, 2003 ; Cochin, 2001*)
9. Show that the points (2, – 6, 0), (4, – 9, 6), (5, 0, 2), (7, – 3, 8) are concyclic.
10. Find the equation of the sphere for which the circle $x^2 + y^2 + z^2 - 3x + 4y - 2z - 5 = 0$, $5x - 2y + 4z + 7 = 0$ is a great circle.
11. Find the equation of the sphere having its centre on the plane $4x - 5y - z = 3$ and passing through the circle $x^2 + y^2 + z^2 - 2x - 3y + 4z + 8 = 0$, $x - 2y + z = 8$. (*Delhi, 2001*)
12. Prove that the plane $x + 2y - z = 4$ cuts the sphere $x^2 + y^2 + z^2 - x - z - 2 = 0$ in a circle of radius unity. Find also the equation of the sphere which has this circle as one of its great circles. (*Nagpur, 2009*)
13. Find the equation of the sphere passing through the circle $x^2 + y^2 + z^2 + 2x + 3y + 6 = 0$, $x - 2y + 4z = 9$ and the centre of the sphere $x^2 + y^2 + z^2 - 2x + 4y - 6z + 5 = 0$. (*Anna, 2009*)

3.18 EQUATION OF THE TANGENT PLANE

The equation of the tangent plane at any point (x_1, y_1, z_1) of the sphere

$$x^2 + y^2 + z^2 = a^2 \textit{ is } \mathbf{xx_1 + yy_1 + zz_1 = a^2}.$$

If $P(x, y, z)$ be any point on the tangent plane at $P_1(x_1, y_1, z_1)$ to the given sphere, the direction ratios of P_1P are $x - x_1, y - y_1, z - z_1$. Also the direction ratios of radius OP_1 are $x_1 - 0, y_1 - 0, z_1 - 0$.

Since OP_1 is normal to the tangent plane at P_1, $OP_1 \perp P_1P$.

$\therefore$ $$x_1(x - x_1) + y_1(y - y_1) + z_1(z - z_1) = 0$$

or $$xx_1 + yy_1 + zz_1 = x_1^2 + y_1^2 + z_1^2 = a^2 \qquad [\because P_1(x_1, y_1, z_1) \text{ lies on the sphere.}$$

This is the desired equation of the tangent plane.

Similarly, *the tangent plane at (x_1, y_1, z_1) to the sphere*

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

is $$\mathbf{xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0}$$

Thus to write the equation of the tangent plane at (x_1, y_1, z_1) to a sphere, change x^2 to xx_1, y^2 to yy_1, z^2 to zz_1, $2x$ to $x + x_1$, $2y$ to $y + y_1$, $2z$ to $z + z_1$.

Obs. *The condition for a plane (or a line) to touch a sphere is that the perpendicular distance of the centre from the plane (or the line) = the radius.*

Example 3.60. *Find the equations of the spheres passing through the circle $x^2 + y^2 + z^2 - 6x - 2z + 5 = 0$, $y = 0$ and touching the plane $3y + 4z + 5 = 0$.*

Solution. The equation of any sphere through the given circle is

$$x^2 + y^2 + z^2 - 6x - 2z + 5 + ky = 0$$

or $$x^2 + y^2 + z^2 - 6x + ky - 2z + 5 = 0 \qquad ...(i)$$

$\therefore$ Its centre = $(3, -k/2, 1)$ and radius = $\sqrt{[9 + (k^2/4) + 1 - 5]} = \sqrt{(5 + k^2/4)}$.

The sphere (i) will touch the plane $3y + 4z + 5 = 0$, if $\perp$ distance of the centre $(3, -k/2, 1)$ from the plane = radius.

i.e., $$\frac{3(-k/2) + 4 + 5}{\sqrt{(9 + 16)}} = \sqrt{\left(5 + \frac{k^2}{4}\right)} \quad \text{or} \quad \text{if,} \quad 4k^2 + 27k + 44 = 0$$

$\therefore$ $$k = \frac{-27 \pm \sqrt{[(27)^2 - 704]}}{8} = -\frac{11}{4} \text{ or } -4$$

Substituting the value of k in (1), we get

$$x^2 + y^2 + z^2 - 6x - \frac{11}{4}y + 2z + 5 = 0 \text{ and } x^2 + y^2 + z^2 - 6x - 4y - 2z + 5 = 0$$

as the two required spheres.

Example 3.61. *Find the equation of the sphere which touches the plane $x - 2y - 2z = 7$ at the point $L(3, -1, -1)$ and passes through the point $M(1, 1, -3)$.*

Solution. If C is the centre of the sphere, then CL is perpendicular to the given plane $x - 2y - 2z = 7$.

$\therefore$ The direction ratios of CL being $1, -2, -2$, the equation of CL is

$$\frac{x-3}{1} = \frac{y+1}{-2} = \frac{z+1}{-2} = k \text{ (say)}$$

Any point on CL is $(k + 3, -2k - 1, -2k - 1)$ which will represent C for some value of k.

Since M lies on the sphere, therefore its radius $CL = CM$ or $(CL)^2 = (CM)^2$

i.e. $$(k + 3 - 3)^2 + (-2k - 1 + 1)^2 + (-2k - 1 + 1)^2$$
$$= (k + 3 - 1)^2 + (-2k - 1 - 1)^2 + (-2k - 1 + 3)^2$$

or $$4k = -12 \quad \text{or} \quad k = -3.$$

$\therefore$ The centre C is $(0, 5, 5)$ and radius $CL = \sqrt{(9 + 36 + 36)} = 9$.

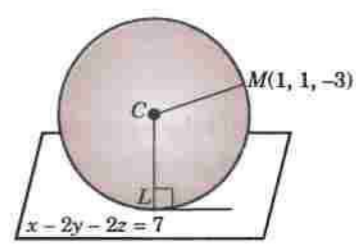


Fig. 3.51

Hence the required equation of the sphere is

$$(x-0)^2+(y-5)^2+(2-5)^2=(9)^2$$

or
$$x^2+y^2+z^2-10y-10z-31=0$$

[**Orthogonal spheres.** Two spheres are said to cut orthogonally if the tangent planes at a point of intersection are at right angles (Fig. 3.52).

The radii of such spheres through their point of intersection P, being $\perp$ to the tangent planes at P are also at right angles. *Thus two spheres cut orthogonally, if the square of the distance between their centre = sum of the squares of their radii*].

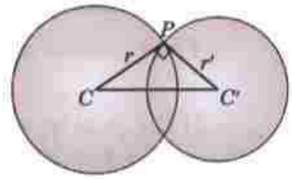


Fig. 3.52

Example 3.62. *Show that the condition for spheres*

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

and
$$x^2+y^2+z^2+2u'x+2v'y+2w'z+d'=0$$

to cut orthogonally is $\mathbf{2uu'+2vv'+2ww'=d+d'}$ *(Anna, 2002 S)*

Solution. The centres of the spheres are

$C(-u,-v,-w)$, $C'(-u',-v',-w')$ and their radii are

$$r=\sqrt{(u^2+v^2+w^2-d)},$$

$$r'=\sqrt{(u'^2+v'^2+w'^2-d')}.$$

Now these spheres will cut orthogonally, if $(CC')^2=r^2+r'^2$

i.e.,
$$(u-u')^2+(v-v')^2+(w-w')^2$$
$$=u^2+v^2+w^2-d+u'^2+v'^2+w'^2-d'$$

or $\quad 2uu'+2vv'+2ww'=d+d'$ which is the required condition.

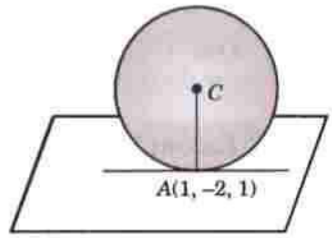


Fig. 3.53

Example 3.63. *Find the equation of the sphere which touches the plane $3x+2y-z+2=0$ at the point $(1,-2,1)$ and cuts the sphere $R^2-2(2I-3J)\,.\,R+4=0$ orthogonally.* *(Roorkee, 2000)*

Solution. The given plane $3x+2y-z+2=0$ ...(*i*)

will touch the required sphere at $A\,(1,-2,1)$ if its centre lies on the normal to (*i*) at A (Fig. 3.53). The equations of the normal to (*i*) at A are $\dfrac{x-1}{3}=\dfrac{y+2}{2}=\dfrac{z-1}{-1}$

Any point on this line is $C\,(3r+1,\,2r-2,\,\pi r+1)$

Also radius (AC) of the required sphere.

$$=\sqrt{[(3r)^2+(2r)^2+(-r)^2]}\;=r\sqrt{14}.$$

Since the required sphere cuts the given sphere

$$x^2+y^2+z^2-4x+6y+4=0 \qquad [\text{Centre}=(2,-3,0) \text{ and + radius}=3$$

orthogonally, therefore (distance between their centres)$^2=\Sigma$ of squares of their radii

i.e.,
$$(3r+1-2)^2+(2r-2+3)^2+(-r+1)^2=14r^2+9 \text{ or } r=-3/2.$$

Thus centre C is $(-7/2,-5,5/2)$ and radius $=\dfrac{3\sqrt{14}}{2}$.

Hence the required sphere is

$$(x+7/2)^2+(y+5)^2+(z-5/2)^2=(3\sqrt{14/2})^2$$

or
$$x^2+y^2+z^2+7x+10y-5z+12=0.$$

PROBLEMS 3.13

1. Find the equations of the tangent planes to the sphere
   (*i*) $x^2+y^2+z^2-4x+2y-6z+11=0$ which are parallel to the plane $x=0$. *(Anna, 2009)*
   (*ii*) $x^2+y^2+z^2=9$ which pass through the line $x+y=6$, $x-2z=3$. *(Madras, 2006)*

2. Find the equations of the spheres which pass through the circle
$x^2 + y^2 + z^2 = 5x + 2y + 3z = 3$, and touch the plane $4x + 3y = 15$. (Anna, 2009)
3. Find the equation of the sphere which is tangential to the plane $x - 2y - 2z = 7$ at $(3, -1, -1)$ and passes through $(1, 1, -3)$.
4. (i) Prove that the equation of the sphere which lies in the first octant and touches the coordinates planes is of the form $(x^2 + y^2 + z^2) - 2\lambda(x + y + z) + 2\lambda^2 = 0$.
(ii) Find the equation of the sphere passing through (1, 4, 9) and touching the coordinate planes.
5. Tangent plane at any point of the sphere $x^2 + y^2 + z^2 = r^2$ meets the coordinates axes at A, B, C. Show that the locus of the point of intersection of the planes drawn parallel to the coordinate planes through A, B, C is the surface $x^{-2} + y^{-2} + z^{-2} = r^{-2}$. (Rajasthan, 2006)
6. Find the equation of the tangent line to the circle $x^2 + y^2 + z^2 = 3$, $3x - 2y + 4z + 3 = 0$ at the point $(1, 1, -1)$.
7. Show that the sphere $x^2 + y^2 + z^2 - 2x + 6y + 14z + 3 = 0$ divides the line joining the points $(2, -1, -4)$ and $(5, 5, 5)$ internally and externally in the ratio $1 : 2$.
8. Find the shortest and the longest distance from the point $(1, 2, -1)$ to the sphere $x^2 + y^2 + z^2 = 24$.
9. Show that the spheres $x^2 + y^2 + z^2 + 6y + 14z + 8 = 0$ and $x^2 + y^2 + z^2 + 6x + 8y + 4z + 20 = 0$, intersect at right angles. Find their plane of intersection.
10. Show that the spheres $x^2 + y^2 + z^2 = 25$ and $x^2 + y^2 + z^2 - 24x - 40y - 18z + 225 = 0$ touch externally and find their point of contact.

3.19 (1) CONE

Def. *A cone is a surface generated by a straight line which passes through a fixed point and satisfies one more condition e.g.*, it may intersect a given curve (called the *guiding curve*).

The fixed point is called the **vertex** *and the straight line in any position is called a* **generator.**

The degree of the equation of a cone depends upon the nature of its guiding curve. In case the guiding curve is a conic, the equation of the cone shall be of the second degree. Such cones are called *Quadric cones*. In what follows, we shall be conncerned only with quadric cones.

Example 3.64. *Find the equation of the cone whose vertex is (3, 1, 2) and base the circle*
$$2x^2 + 3y^2 = 1, z = 1.$$

Solution. Any line through (3, 1, 2) is

$$\frac{x-3}{l} = \frac{y-1}{m} = \frac{z-2}{n} \quad ...(i)$$

It meets $z = 1$, where $\frac{x-3}{l} = \frac{y-1}{m} = \frac{-1}{n}$

whence $x = 3 - l/n$, $y = 1 - m/n$.

Substituting these values of x and y in $2x^2 + 3y^2 = 1$,

$$2(3 - l/n)^2 + 3(1 - m/n)^2 = 1 \quad ...(ii)$$

Eliminating l, m, n from (i) and (ii), the locus of the line (i) is

$$2\left(3 - \frac{x-3}{z-2}\right)^2 + 3\left(1 - \frac{y-1}{z-2}\right)^2 = 1$$

or $\quad 2x^2 + 3y^2 + 20z^2 - 6yz - 12xz + 12x + 6y - 38z + 17 = 0$ which is the required equation.

Example 3.65. *Find the equation of the cone whose vertex is at the origin and guiding curve is* $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{1} = 1, x + y + z = 1.$

Solution. Any line through (0, 0, 0) is $x/l = y/m = z/n$ ...(i)

Any point on it is $P(lr, mr, nr)$.

If (i) intersects the given curve, the coordinates of P should satisfy its equations.

$\therefore$ $$\frac{l^2r^2}{4}+\frac{m^2r^2}{9}+\frac{n^2r^2}{1}=1 \text{ and } lr+mr+nr=1.$$

Eliminating r, $$\left(\frac{l^2}{4}+\frac{m^2}{9}+n^2\right)\Big/(l+m+n)^2=1.$$

Simplifying, $27l^2+32m^2+72(lm+mn+nl)=0$ ...(*ii*)

Eliminating l, m, n from (*i*) and (*ii*), the locus of the line (*i*) is

$$27x^2+32y^2+72(xy+yz+zx)=0 \text{ which is the required equation.}$$

Obs. *The equation of a cone with vertex at the origin is a homogeneous equation of the second degree in x, y, z (i.e., all* terms are of the same degree). The reason is that every generator will have the equation of the form (*i*) above. So the point (lr, mr, nr) will satisfy the equation of the cone for every value of r. This is possible only if the equation is homogeneous.

Example 3.66. *A variable plane parallel to the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$ meets the coordinate axes in A, B, C. Find the equation of the cone whose vertex is the origin and guiding curve the circle ABC.*

Solution. Let the plane $$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=k \quad ...(i)$$

meet the axes at A, B, C, so that $A=(ka, 0, 0)$, $B=(0, kb, 0)$ and $C=(0, 0, kc)$.

$\therefore$ The equation of the sphere through $O\,(0, 0, 0)$ and A, B, C is

$$x^2+y^2+z^2-k(ax+by+cz)=0 \quad ...(ii)$$

Since the equation of the cone with vertex at O is a homogeneous equation of the second degree, therefore, it must be satisfied by points lying on the circle ABC, *i.e.*, on (*i*) and (*ii*) both.

$\therefore$ Making (*ii*) homogeneous with the help of (*i*), we have

$$x^2+y^2+z^2-(ax+by+cz)\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)=0$$

or $$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0 \text{ which is the required equation.}$$

Example 3.67. *Show that the general equation of the cone of the second degree which passes through the axes is of the form $fyz+gzx+hxy=0$.*

Solution. Any cone which passes through the axes will have origin V as its vertex. The general equation of a cone of the second degree having vertex at the origin is of the form

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \quad ...(i)$$

Since it passes through x-axis

$\therefore$ The direction cosines of x-axis (*i.e.*, 1, 0, 0) must satisfy (*i*). This gives $a=0$.

As the cone passes through y-axis, $b=0$.

Similarly, as the cone passes through z-axis, $c=0$.

Hence (*i*) reduces to $fyz+gzx+hxy=0$.

(2) **Right circular cone. Def.** *A right circular cone is a surface generated by a straight line which passes through a fixed point (vertex) and makes a constant angle with a fixed line* (Fig. 3.54).

The constant angle ($\angle AVC$) is called its **semi-vertical angle** *and the fixed line (VC) is called the* **axis.** The section of a right circular cone by a plane perpendicular to its axis is a circle.

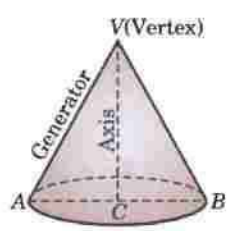


Fig. 3.54

Example 3.68. *Find the equation of the right circular cone whose vertex is the origin, whose axis is the line $x/1=y/2=z/3$ and which has semi-vertical angle of 30°.* (Anna, 2009)

Solution. Let $P(x, y, z)$ be any point on the cone with vertex O and axis (OC)

$$\frac{x}{1}=\frac{y}{2}=\frac{z}{3}, \text{ so that } \angle POC=30°. \qquad \text{(Fig. 3.55)}$$

Now the direction ratios of OP are x, y, z and those of OC are 1, 2, 3.

$$\therefore \qquad \cos 30° = \frac{x(1) + y(2) + z(3)}{\sqrt{(x^2 + y^2 + z^2)} \cdot \sqrt{(1 + 4 + 9)}}$$

or
$$\frac{\sqrt{3}}{2} = \frac{x + 2y + 3z}{\sqrt{[14\,(x^2 + y^2 + z^2)]}}$$

Squaring $\quad 3 \times 14\,(x^2 + y^2 + z^2) = 4(x + 2y + 3z)^2$

or $\quad 19x^2 + 13y^2 + 3z^2 - 8xy - 24yz - 12zx = 0$

which is the required equation of the cone.

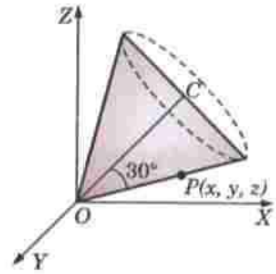


Fig. 3.55

Example 3.69. *Find the equation of the right circular cone generated when the straight line $2y + 3z = 6$, $x = 0$ revolves about z-axis.* *(Hazaribagh, 2009)*

Solution. The vertex is the point of intersection of the line $2y + 3z = 6$, $x = 0$ and the z-axis, *i.e.*, $x = 0$, $y = 0$ (Fig. 3.56).

$\therefore$ Vertex is $A(0, 0, 2)$. A generator of the cone is

$$\frac{x}{0} = \frac{y}{3} = \frac{z-2}{-2}$$

$\therefore$ Direction ratios of the generator are $0, 3, -2$ and the axis (z-axis) are $0, 0, 1$. The semi-vertical angle α is, therefore, given by

$$\cos \alpha = \frac{0 \cdot 0 + 3 \cdot 0 + (-2) \cdot 1}{\sqrt{13}} = \frac{-2}{\sqrt{13}}$$

Let $P(x, y, z)$ be any point on the cone so that the direction ratios of AP are $x, y, z - 2$. Since AP makes an angle α with AZ, we have

$$\cos \alpha = \frac{x \cdot 0 + y \cdot 0 + (z-2) \cdot 1}{\sqrt{[x^2 + y^2 + (z-2)^2]}}$$

Thus
$$\frac{(z-2)^2}{x^2 + y^2 + (z-2)^2} = \cos^2 \alpha = \frac{4}{13}$$

or $\quad 4x^2 + 4y^2 - 9z^2 + 36z - 36 = 0$

which is the required equation of the cone.

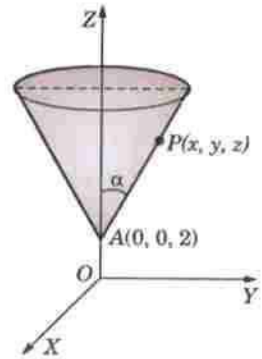


Fig. 3.56

Example 3.70. *Find the equations to the lines in which the plane $2x + y - z = 0$ cuts the cone $4x^2 - y^2 + 3z^2 = 0$.*

Solution. Let $x/l = y/m = z/n$ be one of the two lines in which the given plane $2x + y - z = 0$ ...(*i*)

cuts the given cone $\quad 4x^2 - y^2 + 3z^2 = 0$ ...(*ii*)

$\because$ This line lies on (*i*), $\therefore \quad 2l + m - n = 0$ ...(*iii*)

and it lies on (*ii*), $\therefore \quad 4l^2 - m^2 + 3n^2 = 0$ ...(*iv*)

To eliminate n from (*iii*) and (*iv*), put $n = 2l + m$ in (*iv*).

$$4l^2 - m^2 + 3(2l + m)^2 = 0 \quad \text{or} \quad (4l + m)(2l + m) = 0$$

$\therefore$	Either $4l + m = 0$	or	$2l + m = 0$
From (*iii*)	$2l + m - n = 0$	and	$2l + m - n = 0$
$\therefore$	$\frac{l}{-1} = \frac{m}{4} = \frac{n}{2}$	$\therefore$	$\frac{l}{-1} = \frac{m}{2} = \frac{n}{0}$

Hence the required lines are

$$\frac{x}{-1} = \frac{y}{4} = \frac{z}{2} \text{ and } \frac{x}{-1} = \frac{y}{2} = \frac{z}{0}.$$

Example 3.71. *Find the equation of the enveloping cone of the sphere $x^2 + y^2 + z^2 = a^2$ with vertex at the point (x_1, y_1, z_1).*

Solution. The equation of any generator through $V(x_1, y_1, z_1)$ having direction ratios l, m, n is

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r \text{ (say)} \quad ...(i)$$

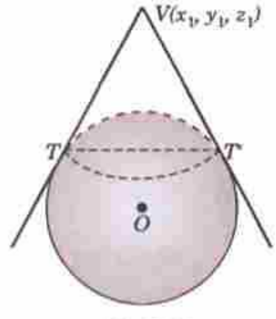


Fig. 3.57

Any point on (i) is $P(x_1 + lr, y_1 + mr, z_1 + nr)$.

It lies on the given sphere if

$(x_1 + lr)^2 + (y_1 + mr)^2 + (z_1 + nr)^2 = a^2$

or $$(l^2 + m^2 + n^2) r^2 + 2(lx_1 + my_1 + nz_1) r + x_1^2 + y_1^2 + z_1^2 - a^2 = 0 \quad ...(ii)$$

The line (i) will touch the given sphere if (ii) has equal roots.

$$\therefore \quad (lx_1 + my_1 + nz_1)^2 = (l^2 + m^2 + n^2)(x_1^2 + y_1^2 + z_1^2 - a^2) \quad ...(iii)$$

The locus of all such lines is the enveloping cone of the given sphere which is obtained by eliminating l, m, n from (i) and (iii).

Thus $[(x - x_1)x_1 + (y - y_1)y_1 + (z - z_1)z_1]^2 = [(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2](x_1^2 + y_1^2 + z_1^2 - a^2)$

which is the equation of the enveloping cone. (Fig. 3.57)

Obs. It can be reduced to the form $SS_1 = T^2$

where $$S = x^2 + y^2 + z^2 - a^2, S_1 = x_1^2 + y_1^2 + z_1^2 - a^2, T = xx_1 + yy_1 + zz_1 - a^2.$$

Thus the enveloping cone of the surface $S = 0$ with vertex (x_1, y_1, z_1) is $SS_1 = T^2$

PROBLEMS 3.14

1. Find the equation of the cone with vertex (α, β, γ) and base $y^2 - 4ax = 0, z = 0$.
2. Find the equation of the cone whose vertex is (3, 4, 5) and base is the conic $3y^2 + 4z^2 = 16, z + 2x = 0$.
3. Find the equation of the cone whose vertex is (1, 2, 3) and whose guiding curve is the circle $x^2 + y^2 + z^2 = 4$, $x + y + z = 1$. *(P.T.U., 2010)*
4. The generators of a cone pass through the point (1, 1, 1) and their direction cosines l, m, n satisfy the relation $l^2 + m^2 = 3n^2$. Obtain the equation of the cone.
5. Find the equation of the right circular cone whose vertex is at the origin and semi-vertical angle is α and having axis of z as its axis. *(V.T.U., 2006 ; Rajasthan, 2005)*
6. Find the equation of the cone whose vertical angle is $\pi/2$, which has its vertex at the origin and its axis along the line $x = -2y = z$. *(V.T.U., 2005)*

   Also show that the plane $z = 0$ cuts the cone in two straight lines inclined at an angle $\cos^{-1} 4/5$.
7. Find the equation of the circular cone which passes through the point (1, 1, 2) and has its vertex at the origin and axis the line $x/2 = -y/4 = z/3$. *(Cochin, 2005 ; Rajasthan, 2005 ; V.T.U., 2004)*
8. Find the equation of the right circular cone generated by revoling the line $x = 0, y - z = 0$ about the axis $x = 0, x = 2$. *(Anna, 2009)*
9. Find the equation of the right circular cone passing through the coordinate axes having vertex at the origin. Obtain the semi-vertical angle and the equation of the axis.
10. Find the semi-vertical angle and the equation of the right circular cone having its vertex at the origin and passing through the circle $y^2 + z^2 = 25, x = 4$. *(Anna, 2009)*
11. Find the equation of the right circular cone which has its vertex at (0, 0, 10) whose intersection with the XY-plane is a circle of radius 5. *(Nagpur, 2009)*
12. Find the equations to the lines in which the plane $3x + y + 5z = 0$ cuts the cone $6yz - 2zx + 5xy = 0$.
13. Prove that the plane $ax + by + cz = 0$ meets the cone $yz + zx + xy = 0$ in perpendicular lines if $a^{-1} + b^{-1} + c^{-1} = 0$.
14. Find the equation of the enveloping cone of the sphere $x^2 + y^2 + z^2 + 2x - 4y + 2z - 1 = 0$ with vertex at (1, 1, 1).

3.20 (1) CYLINDER

Def. *A cylinder is a surface generated by a straight line which is parallel to a fixed line and satisfies one more condition e.g.,* it may intersect a given curve (called the *guiding curve*).

The straight line in any position is called the *generator* and the fixed line the *axis* of the cylinder.

Example 3.72. *Find the equation of a cylinder whose generating lines have the direction cosines l, m, n and which pass through the circumference of the fixed circle $x^2 + z^2 = a^2$ in the ZOX plane.*

Solution. Let $P(x_1, y_1, z_1)$ be any point of the cylinder so that the equation of the generator through P is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \qquad ...(i)$$

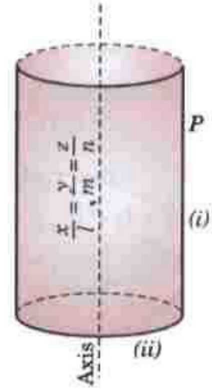


Fig. 3.58

Given guiding circle is $x^2 + z^2 = a^2, y = 0$ ...(ii)

The generator (i) cuts the plane $y = 0$, where

$$\frac{x - x_1}{l} = \frac{- y_1}{m} = \frac{z - z_1}{n}$$

i.e., where $x = x_1 - \dfrac{ly_1}{m}$ and $z = z_1 - \dfrac{ny_1}{m}$

But these values of x and z satisfy $x^2 + z^2 = a^2$

$$\therefore \quad \left(x_1 - \frac{ly_1}{m}\right)^2 + \left(z_1 - \frac{ny_1}{m}\right)^2 = a^2$$

Hence the locus of (x_1, y_1, z_1) is

$(mx - ly)^2 + (mz - ny)^2 = a^2 m^2$, which is the required equation of the cylinder.

(2) Right circular cylinder. Def. *A right circular cylinder is a surface generated by a straight line which is parallel to a fixed line and is at a constant distance from it.*

The constant distance is called the *radius of the cylinder.*

Example 3.73. *The radius of a normal section of a right circular cylinder is 2 units ; the axis lies along the straight line*

$$\frac{x - 1}{2} = \frac{y + 3}{-1} = \frac{z - 2}{5}, \text{ find its equation.} \qquad (P.T.U., 2005)$$

Solution. A point on the axis of the cylinder is $A(1, -3, 2)$ and its direction ratios are $2, -1, 5$.

$\therefore$ Its actual direction cosines are $\dfrac{2}{\sqrt{30}}, \dfrac{-1}{\sqrt{30}}, \dfrac{5}{\sqrt{30}}$.

Let $P(x, y, z)$ be any point on the cylinder. Draw $PM \perp$ to the axis AM. Then $MP = 2$. Now AM = Projection of AP on AM (axis)

$$= (x - 1)\frac{2}{\sqrt{30}} + (y + 3)\frac{-1}{\sqrt{30}} + (z - 2)\frac{5}{\sqrt{30}}$$

$$= \frac{2x - y + 5z - 15}{\sqrt{30}}$$

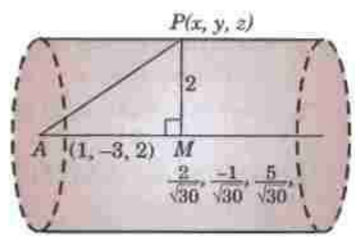


Fig. 3.59

Also $AP = \sqrt{(x - 1)^2 + (y + 3)^2 + (z - 2)^2}$

$\therefore$ From the rt. $\angle d \ \Delta AMP$, $(AM)^2 + (MP)^2 = (AP)^2$

or $$\frac{1}{30}(2x - y + 5z - 15)^2 + 4 = (x - 1)^2 + (y + 3)^2 + (z - 2)^2$$

or $$26x^2 + 29y^2 + 5z^2 + 4xy + 10yz - 20zx + 150y + 30z + 75 = 0.$$

This is the required equation of the right circular cylinder. (Fig. 3.59)

Example 3.74. *Find the equation of the circular cylinder having for its base the circle $x^2 + y^2 + z^2 = 9$, $x - y + z = 3$.* (P.T.U., 2006 ; Cochin, 2005)

Solution. The axis of the cylinder is the line through the centre L of the given circle (or through O (0, 0, 0) the centre of the sphere) (Fig. 3.60) and perpendicular to the plane of the circle.

i.e. $$x - y + z = 3 \qquad ...(i)$$

$\therefore$ Axis of the cylinder is $\quad \frac{x}{1} = \frac{y}{-1} = \frac{z}{1}$

Also $OL \perp$ from O (0, 0, 0) on (*i*)

$$= \frac{3}{\sqrt{(1+1+1)}} = \sqrt{3}.$$

$\therefore$ r, radius of the circle $\quad = \sqrt{(OA^2 - OL^2)} = \sqrt{(9-3)} = \sqrt{6}$

Thus radius of the cylinder $(= r) = \sqrt{6}$

If $P(x, y, z)$ be any point on the cylinder, then

$$OP^2 = OM^2 + MP^2$$

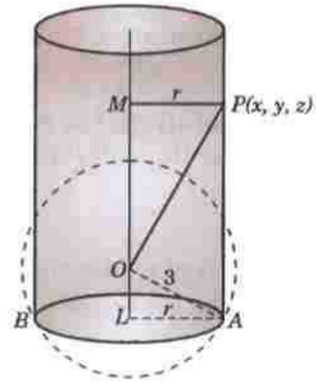


Fig. 3.60

i.e., $$x^2 + y^2 + z^2 = \left[\frac{1}{\sqrt{3}}(x-0) - \frac{1}{\sqrt{3}}(y-0) + \frac{1}{\sqrt{3}}(z-0)\right]^2 + 6$$

i.e., $$x^2 + y^2 + z^2 + xy + yz - zx - 9 = 0$$ which is the required equation.

Example 3.75. *Find the equation of the enveloping cylinder of the sphere $x^2 + y^2 + z^2 = 9$ having generator parallel to the line $x/3 = y/2 = z/1$.*

Solution. If $P(x_1, y_1, z_1)$ be a point on the enveloping cylinder, then the equation of the generator is

$$\frac{x - x_1}{3} = \frac{y - y_1}{2} = \frac{z - z_1}{1} = r(\text{say}). \quad ...(i)$$

Any point on (*i*) is $(x_1 + 3r, y_1 + 2r, z_1 + r)$. It lies on the sphere $x^2 + y^2 + z^2 = 9$. ...(*ii*)

Then $$(x_1 + 3r)^2 + (y_1 + 2r)^2 + (z_1 + r)^2 = 9$$

or $$14r^2 + 2(3x_1 + 2y_1 + z_1)\, r + x_1^2 + y_1^2 + z_1^2 - 9 = 0 \quad ...(iii)$$

In order that (*i*) touches (*ii*), the equation (*iii*) must have equal roots for which

$$4(3x_1 + 2y_1 + z_1)^2 = 4 \times 14\,(x_1^2 + y_1^2 + z_1^2 - 9) \quad [\because\ b^2 = 4ac]$$

or $$5x_1^2 + 10y_1^2 + 13z_1^2 + 12x_1y_1 + 4y_1z_1 + 6z_1x_1 = 126$$

$\therefore$ The locus of (x_1, y_1, z_1) is

$$5x^2 + 10y^2 + 13z^2 + 12xy + 4yz + 6zx = 126$$

which is the required equation of the enveloping cylinder.

PROBLEMS 3.15

1. Find the equation of the right circular cylinder whose axis is the line $x = 2y = -z$ and radius 4. *(Anna, 2009)*
2. Find the equation of the cylinder whose generators are parallel to the line $x = -y/2 = z/3$ and whose guiding curve is the ellipse $x^2 + 2y^2 = 1, z = 3$. *(Rajasthan, 2005 ; Roorkee, 2000)*
3. Find the equation of the right circular cylinder of radius 2 whose axis passes through (1, 2, 3) and has direction ratios (2, – 3, 6). *(V.T.U., 2006 ; Anna, 2005 S)*
4. Find the equation of the right circular cylinder describe on the circle through the points $(a, 0, 0), (0, a, 0), (0, 0, a)$ guiding curve.
5. Find the equation of the cylinder whose directing curve is $x^2 + z^2 - 4x - 2z + 4 = 0, y = 0$ and whose axis contains the point (0, 3, 0). Find also the area of the section of the cylinder by a plane parallel to xz-plane.
6. Find the equation of the enveloping cylinder of the sphere $x^2 + y^2 + z^2 - 2x - 4y - 6z - 2 = 0$ whose generators are perpendicular to the lines $\frac{x}{3} = \frac{y}{-1} = \frac{z}{0}$ and $\frac{x}{1} = \frac{y}{2} = \frac{z}{0}$.
7. Find the equation to the cylinder whose generators intersect the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, z = 0$ and are parallel to the line $x/l = y/m = z/n$.

3.21 QUADRIC SURFACES

The surface represented by general equation of the second degree in x, y, z is called a **quadric surface** *or a* **conicoid.**

Thus the general equation of a *quadric surface* is of the form

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$$

which can be reduced to any of the following standard forms so useful in engineering problems. We now, proceed to study their shapes.

(1) Ellipsoid : $\mathbf{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1}$.

(*i*) It is symmetrical about each of the coordinate planes for only even powers of x, y, z occur in its equation.

(*ii*) It meets the x-axis at $A\ (a, 0, 0), A'\ (-a, 0, 0)$;
the y-axis at $B\ (0, b, 0), B'\ (0, -b, 0)$;
and the z-axis at $C(0, 0, c), C'\ (0, 0, -c)$.

(*iii*) Its sections by the coordinate planes are ellipses. For the section by the yz-plane $(x = 0)$ is the ellipse.

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \text{ etc.}$$

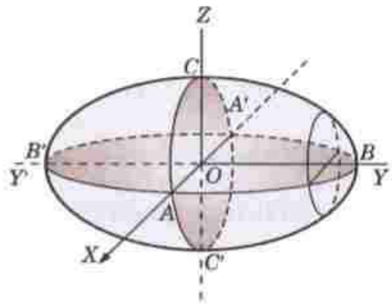


Fig. 3.61

(*iv*) The surface is generated by a variable ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{k^2}{c^2}, z = k\ ;$$

(as k varies from $-c$ to c) and is limited in every direction.

Hence its shape is as shown in Fig. 3.61 which is like that of an *egg*.

(2) Hyperboloid of one sheet : $\mathbf{\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1}$.

(*i*) It is symmetrical about each of the coordinate planes for only even powers of x, y, z occur in its equation.

(*ii*) It meets the x-axis at $A(a, 0, 0), A'\ (-a, 0, 0)$; the y-axis at $B\ (0, b, 0), B'\ (0, -b, 0)$; and the z-axis in imaginary points.

(*iii*) Its section by the yz-plane $(x = 0)$ is the hyperbola $\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, (*i.e.*, $DE, D'E'$)

Its section by the zx-plane $(y = 0)$ is the hyperbola $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$. (*i.e.*, $FG, F'G'$)

Its section by the xy-plane $(z = 0)$ is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(*iv*) The surface is generated by a variable ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{k^2}{c^2}, z = k$ (as k varies from $-\infty$ to ∞) and extends to infinity on both sides of the xy-plane.

Hence its shape is as shown in Fig. 3.62 which is like that of *juggler's dabru*.

(3) Hyperboloid of two sheets : $\mathbf{\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1}$.

(*i*) It is symmetrical about each of the coordinate planes for only even powers of x, y, z occur in its equation.

(*ii*) It meets the z-axis at $C(0, 0, c), C'(0, 0, -c)$ and the x and y-axes in imaginary points.

(*iii*) Its section by the yz-plane $(x = 0)$ is the hyperbola $\frac{z^2}{c^2} - \frac{y^2}{b^2} = 1$. (*i.e.*, $ACB, A'C'B'$)

Its section by the zx-plane $(y = 0)$ is the hyperbola $\frac{z^2}{c^2} - \frac{x^2}{a^2} = 1$. (*i.e.*, $DCE, D'C'E'$)

Its section by the xy-plane ($z = 0$), is the imaginary ellipse $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$.

Its section by the xy-plane ($z = 0$) is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(*iv*) The surface is generated by a variable ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{k^2}{c^2} - 1, z = k$,

(as k varies from $-\infty$ to $-c$ and c to $+\infty$) and extends to infinity on both sides of the xy-plane.

Hence its shape is as shown in Fig. 3.63.

(4) Cone : $\mathbf{\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0}$.

(*i*) It is symmetrical about each of the coordinate planes.

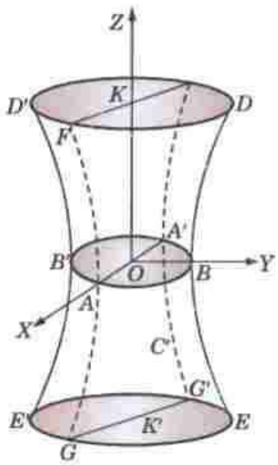


Fig. 3.62

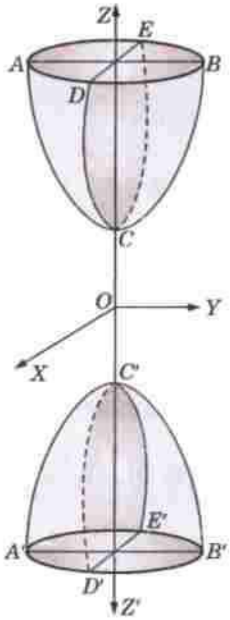


Fig. 3.63

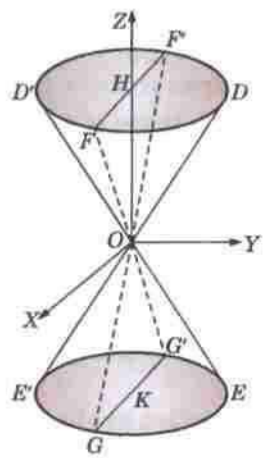


Fig. 3.64

(*ii*) It meets the axes only at the origin.

(*iii*) Its section by the yz-plane ($x = 0$) is the pair of straight lines

$$y = \pm \frac{b}{c} z \qquad (i.e., DOE' \text{ and } D'OE).$$

Its section by the zx-plane ($y = 0$) is the pair of straight lines

$$x = \pm \frac{a}{c} z \qquad (i.e., FOG' \text{ and } F'OG).$$

Its section by the zx-plane ($z = 0$) is the point ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$.

(*iv*) The surface is generated by a variable ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{k^2}{c^2}$, $z = k$ (k varies)

and extends to infinity on both sides of the xy-plane. Hence its shape is as shown in Fig. 3.64.

(5) Elliptic paraboloid : $\mathbf{\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2z}{c}}$

(*i*) It is symmetrical about yz- and zx-planes for only even powers of x and y occur in its equation

(*ii*) It meets the axes at the origin only and touches the xy-plane threat.

(*iii*) Its section by the yz-plane ($x = 0$) is the parabola $y^2 = \frac{2b^2}{c} z$, (*i.e.*, DOD').

Its section by the zx-plane ($y = 0$) is the parabola $x^2 = \frac{2a^2}{c} z$ (*i.e.*, EOE').

Its section by the xy-plane ($z = 0$) is the point ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$

(*iv*) The surface is generated by a variable ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2k}{c}$, $z = k$ (as k varies from 0 to ∞) and it extends to infinity above the xy-plane.

Hence its shape is as shown in Fig. 3.65 and is like that of *tabla*.

(6) Hyperbolic paraboloid : $\frac{\mathbf{x}^2}{\mathbf{a}^2} - \frac{\mathbf{y}^2}{\mathbf{b}^2} = \frac{\mathbf{2z}}{\mathbf{c}}$.

(*i*) It is symmetrical about the yz and zx-planes for only even powers of x and y occur in its equation.

(*ii*) It meets the axes only at the origin and touches the xy-plane threat.

(*iii*) Its section by the yz-plane ($x = 0$) is the parabola $y^2 = -\frac{2b^2}{c} z$. (*i.e.*, DOD')

Its section by the zx-plane ($y = 0$) is the parabola $x^2 = \frac{2a^2}{c} z$ (*i.e.*, EOE').

Its section by the xy-plane ($z = 0$) is the part of lines $y = \pm \frac{b}{a} x$ (not shown in Fig. 3.66.)

(*iv*) The surface is generated by a variable hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{2k}{c}$, $z = k$

and it extends to infinity on both sides of xy-plane. Hence its shape is as shown in Fig. 3.66.

(7)· Cylinder. An equation of the form $f(x, y) = 0$ represents a cylinder generated by a straight line which is parallel to the z-axis and its section by the xy-plane is the curve $f(x, y) = 0$ (Fig. 3.67).

In particular (*i*) $y^2 = 4ax$ represents a *parabolic cylinder*,

(*ii*) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ represents an *elliptic cylinder*, (*iii*) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ represents a *hyperbolic cylinder*.

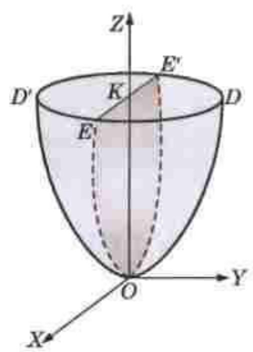


Fig. 3.65

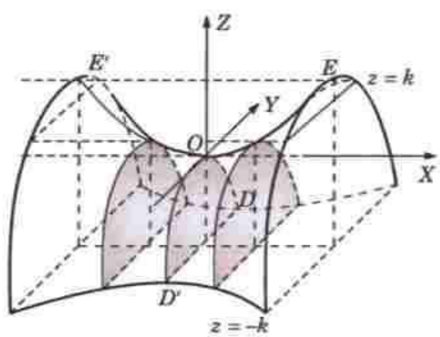


Fig. 3.66

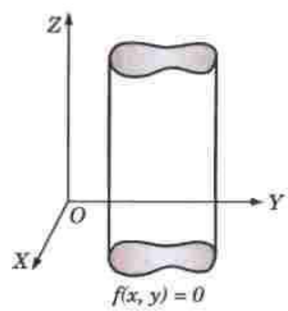


Fig. 3.67

3.22 SURFACES OF REVOLUTION

Let $P(x, y)$ be any point on the curve $y = f(x)$ in the xy-plane. Draw $PM \perp$ to x-axis so that $OM = x$ and $MP = y$. Thus the equation of this curve can be written as

$$MP = f(OM) \quad \text{...(1)}$$

As this curve revolves about the x-axis, the point P describe a circle with centre M and radius MP. Let $Q(x, y, z)$ be any other position of P. Draw $QN \perp$ to zx-plane and join MN so that $OM = x$, $MN = z$, $NQ = y$

and $\angle MNQ = 90°$. $\therefore$ $MP^2 = MQ^2 = MN^2 + NQ^2 = z^2 + y^2$.

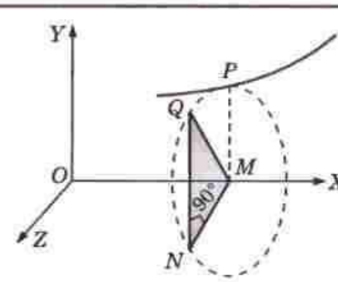


Fig. 3.68

Now substituting the values of MP and MO in (1), we have

$$\sqrt{(y^2 + z^2)} = f(x) \quad \text{or} \quad \mathbf{y^2 + z^2 = [f(x)]^2}$$

which is *the equation of the surface generated by the revolution of the curve $y = f(x)$ about the x-axis* (Fig. 3.68)

Similarly, *the surface generated by the revolution of the curve*

(*i*) *$x = f(y)$ about y-axis is* $\mathbf{z^2 + x^2 = [f(y)]^2}$, (*ii*) $x = f(z)$ about z-axis is $\mathbf{x^2 + y^2 = [f(z)]^2}$

The given revolving curve is called the *generating curve.*

Some standard surfaces of revolution :

Let the generating curve be $y = f(x)$ in the xy-plane and the axis of rotation be the x-axis ; then the surface generated is $y^2 + z^2 = [f(x)]^2$.

(1) Right-circular cylinder. *When $f(x) = a$, the generating curve is a straight line ($y = a$) parallel to the x-axis.*

$\therefore$ The surface generated is $\mathbf{y^2 + z^2 = a^2}$

which represents a *right-circular cylinder of radius a and axis as x-axis* (Fig. 3.69).

(2) Right-circular cone. *When $f(x) = mx$*, the generating curve is a straight line ($y = mx$) passing through the origin.

$\therefore$ The surface generated is $y^2 + z^2 = m^2x^2$ or $\mathbf{y^2 + z^2 = x^2 \tan^2 \alpha}$

which represents *a right-circular cone of semi-vertical angle α and axis as the x-axis* (Fig. 3.70).

(3) Sphere. *When $f(x) = \sqrt{(a^2 - x^2)}$*, the generating curve is a circle ($x^2 + y^2 = a^2$).

$\therefore$ The surface generated is

$$y^2 + z^2 = a^2 - x^2 \quad i.e., \quad \mathbf{x^2 + y^2 + z^2 = a^2},$$

which is *a sphere of radius a and centre (0, 0, 0).*

(4) Ellipsoid of revolution. *When $f(x) = b\sqrt{(1 - x^2/a^2)}$*, the generating curve is an ellipse

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right). \qquad \therefore \text{ The surface generated is } y^2 + z^2 = b^2(1 - x^2/a^2)$$

or $$\mathbf{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1,}$$ which is called an *ellipsoid of revolution.*

If $a^2 > b^2$, the major axis of the generating ellipse is along the x-axis—the axis of revolution and the surface generated, in this case, is called a **prolate spheroid** (Fig. 3.71).

If $a^2 < b^2$, the minor axis of the ellipse lies along the x-axis—the axis of revolution and the surface thus generated is called an **oblate spheroid** (Fig. 3.72).

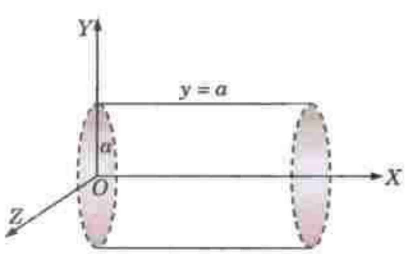


Fig. 3.69

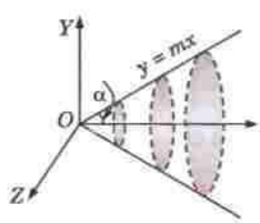
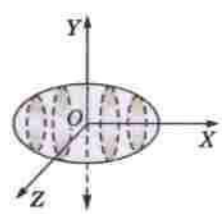
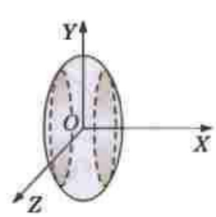


Fig. 3.70



Prolate spheroid

Fig. 3.71



Oblate spheroid

Fig. 3.72

(5) Hyperboloids of revolution

(*i*) *When $f(x) = b\sqrt{(1 + x^2/a^2)}$*, the generating curve is $\frac{y^2}{b^2} - \frac{z^2}{a^2} = 1$ which represents a hyperbola having conjugate axis along the x-axis.

$\therefore$ The surface generated is $y^2 + z^2 = b^2(1 + x^2/a^2)$

or $$\mathbf{\frac{y^2}{b^2} + \frac{z^2}{b^2} - \frac{x^2}{a^2} = 1}$$ which is called a *hyperboloid of revolution of one sheet* (Fig. 3.73).

(*ii*) When $f(x) = b\sqrt{(x^2/a^2 - 1)}$, the generating curve is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ which represents a hyperbola having transverse axis along the x-axis.

$\therefore$ The surface generated is $y^2 + z^2 = b^2(x^2/a^2 - 1)$

or $$\frac{\mathbf{x}^2}{\mathbf{a}^2} - \frac{\mathbf{y}^2}{\mathbf{b}^2} - \frac{\mathbf{z}^2}{\mathbf{b}^2} = \mathbf{1},$$ which is called a *hyperboloid of revolution of two sheets* (Fig. 3.74).

(6) Paraboloid of revolution. When $f(x) = \sqrt{(ax)}$, the generating curve is a parabola ($y^2 = ax$). The surface generated is $\mathbf{y^2 + z^2 = ax}$. which is called a *paraboloid of revolution* (Fig. 3.75).

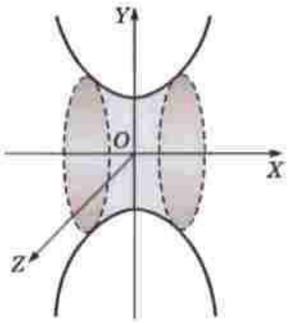


Fig. 3.73

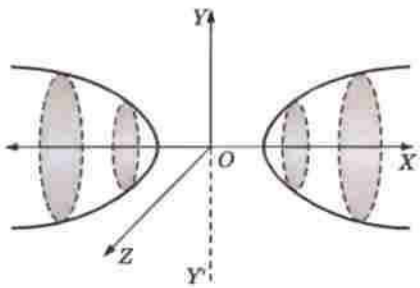


Fig. 3.74

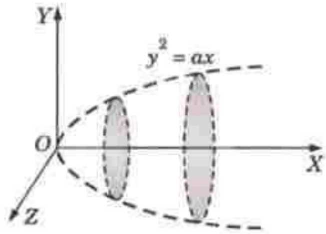


Fig. 3.75

PROBLEMS 3.16

1. What surface is represented by $4x^2 + 9y^2 + 16z^2 = 144$? Trace it roughly. Find the area of the plane curve in which $y = 2$ cuts it.
2. Sketch (roughly) the surface $5(x^2 + z^2) - y^2 = 6$.
   In what curve does the plane $z = 2$ intersect it ? Find the area of the curve of intersection ? What surfaces are represented by the following equations ? Draw diagrams to show their shapes.
3. $x^2 + y^2 = 16$.
4. $x^2/2 - y^2/3 = z$.
5. $z^2 = 4(1 + x^2 + y^2)$.
6. $y^2 = 4z - 8$ (*Andhra, 2000*)
7. $x^2 + y^2 = 5 - 2y$.
8. $x^2 + y^2 = 9z^2$.
9. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{2z}{c}$. (*P.T.U., 2009*)
10. $4x^2 - y^2 - 16z^2 = 36$.

Note. For the equations of the tangent plane and the normal line to a surface refer to § 5.8 (2).

3.23 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 3.17

Select the correct answer or fill up the blanks in each of the following questions :

1. The line $x = ay + b, z = cy + d$ and $x = a'y + b', z = c'y + d'$ are perpendicular if
   (*a*) $aa' + cc' = 1$ (*b*) $aa' + cc' = -1$ (*c*) $bb' + dd' = 1$ (*d*) $bb' + dd' = -1$.
2. The coordinates of the point of intersection of the line $\frac{x+1}{1} = \frac{y+3}{3} = \frac{z+2}{-2}$ with the plane $3x + 4y + 5z = 5$ is
   (*a*) (5, 15, – 14) (*b*) (3, 4, 5) (*c*) (1, 3, – 2) (*d*) (3, 12, – 10).
3. The equation of a right circular cylinder, whose axis is the z-axis and radius a is
   (*a*) $x^2 + y^2 + z^2 = a^2$ (*b*) $z^2 + y^2 = a^2$ (*c*) $x^2 + y^2 = a^2$ (*d*) $z^2 + x^2 = a^2$.
4. The equation $\sqrt{fx} + \sqrt{gy} + \sqrt{hz} = 0$ represent a
   (*a*) sphere (*b*) cylinder (*c*) cone (*d*) pair of planes.

5. The plane $ax + by + cz = 0$ cuts the cone $xy + yz + zx = 0$ in perpendicular lines if
(a) $a + b + c = 0$ (b) $1/a + 1/b + 1/c = 0$
(c) $a^2 + b^2 + c^2 = 0$ (d) $abc = 0$.

6. The equation of the cylinder which intersects the curve $x^2 + y^2 + z^2 = 1$, $x + y + z = 1$ and whose generators are parallel to the axis of z, is
(a) $x^2 + y^2 + xy - x - y = 0$ (b) $x^2 + y^2 + xy + x + y = 0$
(c) $x^2 + y^2 - xy - x - y = 0$ (d) $x^2 + y^2 - xy + x + y = 0$.

7. The equation $x^2 + y^2 + z^2 + xy + yz - zx = 9$ represents
(a) a sphere with $x^2 + y^2 + z^2 = 9$, $x - y + z = 3$ as a great circle
(b) a cone with $x^2 + y^2 + z^2 = 9$, $x - y + z = 3$ as a guiding circle
(c) a cylinder with $x^2 + y^2 + z^2 = 9$, $x - y + z = 3$ as a guiding circle
(d) none of the above.

8. The sum of the direction cosines of a straight line is
(a) zero (b) one (c) constant (d) n one of the above.

9. The angle between the planes $x - y + 2z - 9 = 0$ and $2x + y + z = 7$ is
(a) 30° (b) 90° (c) 60° (d) 120° (V.T.U., 2010 S)

10. The equation of the right circular cone whose axis is $x = y = z$, vertex is the origin and the semi-vertical angle is 45° is given as
(a) $x^2 + y^2 + z^2 = 0$ (b) $2(x^2 + y^2 + z^2) = 3(x + y + z)^2$
(c) $3(x^2 + y^2 + z^2) = 2(x + y + z)^2$ (d) $x^2 + y^2 + z^2 + xy + yz + zx = 0$.

11. The equation of a straight line parallel to the x-axis is given by
(a) $\frac{x-a}{1} = \frac{y-b}{1} = \frac{z-c}{1}$ (b) $\frac{x-a}{0} = \frac{y-b}{1} = \frac{z-c}{1}$
(c) $\frac{x-a}{0} = \frac{y-b}{0} = \frac{z-c}{1}$ (d) $\frac{x-a}{1} = \frac{y-b}{0} = \frac{z-c}{0}$.

12. The equation of the plane passing through $(4, -2, 1)$ and perpendicular to the line with direction ratios $7, 2, -3$ is
(a) $x + 3y - 4z - 8 = 0$ (b) $2x + 7y - 3z - 24 = 0$
(c) $7x + 2y - 3z - 21 = 0$. (d) $7x + 3y - 2z + 21 = 0$. (V.T.U., 2009 S)

13. The equation to the axis of the right circular cylinder whose guiding circle is $x^2 + y^2 + z^2 = 9$, $x - y + z = 3$ is given by
(a) $x = y = z$ (b) $x = -y = z$ (c) $x = y = -z$ (d) $x = -y = -z$.

14. In three dimensions, the equation $x^2 - y^2 = a^2$ represents
(a) a pair of straight lines (b) a hyperbola
(c) a cylinder (d) a cone.

15. Section of a sphere by a plane is
(a) parabola (b) ellipse (c) circle.

16. A line makes angles α, β, γ with the coordinate axes, then $\cos 2\alpha + \cos 2\beta + \cos 2\gamma$ is equal to
(a) 1 (b) 2 (c) -1 (d) -2. (V.T.U., 2010 S)

17. Three lines are coplanar if
(a) they are concurrent
(b) a line is perpendicular to each of them
(c) they are concurrent and a line is perpendicular to each of them.

18. The distance between the planes $2x + 2y + z - 6 = 0$ and $4x + 4y + 2z - 7 = 0$ is
(a) 1/3 (b) 5/6 (c) 13/3.

19. The line $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$ and the plane $x + y - z = 0$ are
(a) parallel (b) perpendicular (c) such that the line lies in the plane.

20. The radius of a great circle of a sphere is
(a) greater than the radius of the sphere (b) less than the radius of the sphere
(c) equal to the radius of the sphere.

21. Which of the following lines are generators of the cone $yz + 4zx + 3xy = 0$?
(a) $x = y = z$ (b) $x = -y = z$ (c) $x = 2y = -z$.

22. The semi-vertical angle of the cone generated by revolving the line $x + y = 0$, $z = 0$ about the x-axis is
(a) $90°$ (b) $45°$ (c) $30°$.

23. All cones passing through the coordinate axes are given by the equation
(a) $x^2 + y^2 + z^2 - yz - zx - xy = 0$ (b) $ax^2 + by^2 + cz^2 - yz - zx - xy = 0$
(c) $ayz + bzx + cxy = 0$.

24. The line $\frac{x+1}{3} = \frac{y-2}{6} = \frac{z-3}{9}$ is perpendicular to the plane $ax + by + cz + d = 0$, if
(a) $a = 2b, b = 3c$ (b) $2a = b, b = 3c$ (c) $2a = b, 3b = 2c$ (d) $a = 3b, 2b = c$.

25. The equation $2(x^2 + y^2 + z^2) - 2xy + 2yz + 2zx = 3a^2$ represents a
(a) cone (b) right-circular cylinder
(c) sphere (d) pair of planes.

26. The equation of the plane through the point $(2, -3, 1)$ and parallel to the plane $3x - 4y + 2z = 5$ is
(a) $3x - 4y + 2z - 20 = 0$ (b) $3x + 4y - 2z + 20 = 0$
(c) $3x - 4y - 2z + 20 = 0$ (d) $3x + 4y + 2z - 20 = 0$.

27. The direction cosines of a line which is equally inclined to the coordinate axes are ....
28. The direction cosines of the line $x = 0 = y$ are ....
29. The equation of the axis of the cylinder $x^2 + y^2 = 25$ is ......
30. The image of the point $(3, 2, -1)$ in the YOZ plane is ......
31. The plane $x - 2y - 2z = k$ touches the sphere $x^2 + y^2 + z^2 - 2x + 4y - 6z + 5 = 0$ for k = ... (P.T.U., 2010)
32. The condition for the three concurrent lines to be coplanar is ......
33. The equation of the cone whose vertex is at the origin and base the circle $x = a$, $y^2 + z^2 = b^2$ is given by ....
34. The plane through points $(2, 2, 1)$, $(9, 3, 6)$ and perpendicular to the plane $2x + 6y + 6z = 9$ is ......
35. Volume of the sphere $x^2 + y^2 + z^2 + 2x - 4y + 8z - 2 = 0$ is ......
36. Angle between the planes $x - y + z = 1$ and $2x - 3y + z = 7$ is ......
37. The equation of the cone whose vertex is the origin and ginding curve is $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{1} = 1$, $x + y + z = 1$, is ...... (Anna, 2009)
38. Any two points on the line $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ other than $(1, 2, 3)$ are ......
39. The equation of the line joining the points $(1, 2, 3)$ and $(2, 1, -3)$ is ......
40. The equation of the sphere on the line joining $(1, 5, 6)$ and $(-2, 1, 1)$ as diameter is ......
41. The conditions for the line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ to lie on the plane $ax + by + cz + d = 0$ are ......
42. The distance between the planes $4x + 3y + z + 4 = 0$ and $8x + 6y + 2z + 12 = 0$ is ....
43. The centre and radius of the sphere $2x^2 + 2y^2 + 2z^2 - 6x + 8y - 8z - 1 = 0$ are ......
44. The radius of the circle $x^2 + y^2 + z^2 - 2x - 4y - 11 = 0$, $x + 2y + 2z = 15$ is ......
45. The symmetric form of the line $x + y + z + 1 = 0 = 4x + y - 2z + 2$ is ......
46. The equation $y^2 = 4z - 8$ represents a ....
47. The equation $x^2 + y^2 = \frac{1}{4}z^2 - 1$ represents a ......
48. Angle between the lines whose d.r.s. are $1, 2, 3$ and $-1, 1, 2$ is ......
49. The intercepts of the plane $2x - 3y + z = 12$ on the coordinate axes are ....
50. The radius of the sphere whose centre is $(4, 4, -2)$ and which passes through the origin is ......
51. The points $(0, 4, 1)$, $(2, 3, -1)$, $(4, 5, 0)$ and $(2, 6, 2)$ are the vertices of a square. (True or False)
52. The points $(3, -1, 1)$, $(5, -4, 2)$ and $(11, -13, 5)$ are collinear. (True or False)
53. The plane $5x + 6y + 7z = 110$, $2x + 3y - 4z = 29$ are perpendicular to each other. (True or False)
54. In three dimensional space, $9x^2 + 16y^2 = 144$ represents ....
55. Equation of the right circular cone with vertex at origin and passing through the curve $x^2 + y^2 + z^2 = 9$, $x + y + z = 1$ is ....
56. A unit vector perpendicular to the vectors $-2\mathbf{I} + 3\mathbf{J} + \mathbf{K}$ and $4\mathbf{I} + 2\mathbf{J}$ is ....

CHAPTER 4

Differential Calculus & Its Applications

4.1 (1) SUCCESSIVE DIFFERENTIATION

The reader is already familiar with the process of differentiating a function $y = f(x)$. For ready reference, a list of derivatives of some standard functions is given in the beginning.

The derivative dy/dx is, in general, another function of x which can be differentiated. The derivative of dy/dx is called the *second derivative* of y and is denoted by d^2y/dx^2. Similarly, the derivative of d^2y/dx^2 is called the *third derivative* of y and is denoted by d^3y/dx^3. In general, the nth derivative of y is denoted by d^ny/dx^n.

Alternative notations for the successive derivatives of $y = f(x)$ are

$$Dy, D^2y, D^3y, \ldots, D^ny\ ;$$

or $$y_1, y_2, y_3, \ldots, y_n\ ;$$

or $$f'(x), f''(x), f'''(x) \ldots, f^n(x).$$

The nth derivative of $y = f(x)$ at $x = a$ is denoted by $(d^ny/dx^n)_a$, $(y_n)_a$ or $f^n(a)$.

Example 4.1. *If $y = e^{ax} \sin bx$, prove that $y_2 - 2ay_1 + (a^2 + b^2)y = 0$.* (Cochin, 2005)

Solution. We have $\quad y = e^{ax} \sin bx \qquad \ldots(i)$

$\therefore \quad y_1 = e^{ax} (\cos bx \,.\, b) + \sin bx\, (e^{ax} \,.\, a) = be^{ax} \cos bx + ay \qquad$ [By (i)]

or $\quad y_1 - ay = be^{ax} \cos bx \qquad \ldots(ii)$

Again differentiating both sides,

$$y_2 - ay_1 = be^{ax} (-\sin bx \,.\, b) + b \cos bx\, (e^{ax} \,.\, a) = -b^2y + a(y_1 - ay)$$

or $$y_2 - 2ay_1 + (a^2 + b^2)y = 0.$$

Example 4.2. *If $x = a (\cos t + t \sin t)$, $y = a (\sin t - t \cos t)$, find d^2y/dx^2.*

Solution. We have $\quad \dfrac{dx}{dt} = a(-\sin t + t \cos t + \sin t) = at \cos t$

and $\quad \dfrac{dy}{dt} = a(\cos t + t \sin t - \cos t) = at \sin t$

$$\therefore \qquad \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{at\sin t}{at\cos t} = \tan t$$

$$\therefore \qquad \frac{d^2y}{dx^2} = \frac{d}{dt}(\tan t)\cdot\frac{dt}{dx} = \sec^2 t\cdot\frac{1}{at\cos t} = 1/at\cos^3 t.$$

Example 4.3. *Given $y^2 = f(x)$, a polynomial of third degree, then evaluate* $\dfrac{d}{dx}\left(y^3\dfrac{d^2y}{dx^2}\right)$.

Solution. Differentiating $y^2 = f(x)$ w.r.t. x, we get

$$2y\frac{dy}{dx} = f'(x) \qquad ...(i)$$

Differentiating (i) w.r.t. x again, we obtain

$$2\left(\frac{dy}{dx}\cdot\frac{dy}{dx} + y\frac{d^2y}{dx^2}\right) = f''(x) \quad \text{or} \quad 2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = f''(x)$$

Again differentiating, we get

$$4\cdot\frac{dy}{dx}\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\frac{d^2y}{dx^2} + 2y\frac{d^3y}{dx^3} = f'''(x)$$

or
$$3y^2\frac{dy}{dx}\frac{d^2y}{dx^2} + y^3\frac{d^3y}{dx^3} = \frac{1}{2}y^2 f'''(x) \qquad \text{[Multiplying by } y^2]$$

Hence
$$\frac{d}{dx}\left(y^3\frac{d^2y}{dx^2}\right) = \frac{1}{2}f(x)f'''(x). \qquad [\because\ y^2 = f(x)]$$

Example 4.4. *If $ax^2 + 2hxy + by^2 = 1$, prove that* $\dfrac{d^2y}{dx^2} = \dfrac{h^2 - ab}{(hx + by)^3}$.

Solution. Differentiating the given equation w.r.t. x,

$$2ax + 2h\left(x\frac{dy}{dx} + y\right) + 2by\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{ax + hy}{hx + by} \qquad ...(i)$$

Differentiating both sides of (i) w.r.t. x,

$$\frac{d^2y}{dx^2} = -\frac{(hx + by)(a + h\,dy/dx) - (ax + hy)(h + b\,dy/dx)}{(hx + by)^2}$$

[Substituting the value of dy/dx from (i)]

$$= -\frac{(hx + by)\left(a - h\cdot\dfrac{ax + hy}{hx + by}\right) - (ax + hy)\left(h - b\cdot\dfrac{ax + hy}{hx + by}\right)}{(hx + by)^2}$$

$$= \frac{(h^2 - ab)(ax^2 + 2hxy + by^2)}{(hx + by)^3}$$

$$= (h^2 - ab)/(hx + by)^3 \qquad [\because\ ax^2 + 2hxy + by^2 = 1]$$

PROBLEMS 4.1

1. If $y = (ax + b)/(cx + d)$, show that $2y_1y_3 = 3y_2^2$.
2. If $y = \sin(\sin x)$, prove that $\dfrac{d^2y}{dx^2} + \tan x\dfrac{dy}{dx} + y\cos^2 x = 0$.
3. If $y = e^{-kt}\cos(lt + c)$, show that $\dfrac{d^2y}{dx^2} + 2k\dfrac{dy}{dx} + n^2y = 0$, where $n^2 = k^2 + l^2$.

4. If $y = \sinh [m \log \{x + \sqrt{(x^2+1)}\}]$, show that $(x^2 + 1) \dfrac{d^2y}{dx^2} + x \dfrac{dy}{dx} = m^2 y$.

5. If $y = \sin^{-1} x$, show that $(1 - x^2) y_5 - 7xy_4 - 9y_3 = 0$. (*Madras, 2000 S*)

6. If $x = \dfrac{1}{2}\left(t + \dfrac{1}{t}\right), y = \dfrac{1}{2}\left(t - \dfrac{1}{t}\right)$, find $\dfrac{d^2y}{dx^2}$. (*Cochin, 2005*)

7. If $x = 2 \cos t - \cos 2t$, $y = 2 \sin t - \sin 2t$, find the value of d^2y/dx^2 when $t = \pi/2$.

8. If $x = a (\cos t + \log \tan t/2)$, $y = a \sin t$, find d^2y/dx^2.

9. If $x = \sin t$, $y = \sin pt$, prove that $(1 - x^2) \dfrac{d^2y}{dx^2} - x \dfrac{dy}{dx} + p^2 y = 0$.

10. If $x^3 + y^3 = 3axy$, prove that $\dfrac{d^2y}{dx^2} = -\dfrac{2a^2xy}{(y^2 - ax)^3}$.

(2) Standard Results

We have (*1*) $\mathbf{D^n (ax + b)^m = m(m - 1) (m - 2) \ldots (m - n + 1) a^n (ax + b)^{m-n}}$

(*2*) $\mathbf{D^n \left(\dfrac{1}{ax + b}\right) = \dfrac{(-1)^n (n!) a^n}{(ax + b)^{n+1}}}$ (*3*) $\mathbf{D^n \log (ax + b) = \dfrac{(-1)^{n-1} (n - 1)! a^n}{(ax + b)^n}}$

(*4*) $\mathbf{D^n (a^{mx}) = m^n (\log a)^n . a^{mx}}$ (*5*) $\mathbf{D^n (e^{mx}) = m^n e^{mx}}$

(*6*) $\mathbf{D^n \sin (ax + b) = a^n \sin (ax + b + n\pi/2)}$ (*7*) $\mathbf{D^n \cos (ax + b) = a^n \cos (ax + b + n\pi/2)}$

(*8*) $\mathbf{D^n [e^{ax} \sin (bx + c)] = (a^2 + b^2)^{n/2} e^{ax} \sin (bx + c + n \tan^{-1} b/a)}$

(*9*) $\mathbf{D^n [e^{ax} \cos (bx + c)] = (a^2 + b^2)^{n/2} e^{ax} \cos (bx + c + n \tan^{-1} b/a)}$

To prove (1), let $y = (ax + b)^m$

$$y_1 = m . a(ax + b)^{m-1}$$
$$y_2 = m(m - 1)a^2 (ax + b)^{m-2}$$
$$y_3 = m(m - 1) (m - 2) a^3 (ax + b)^{m-3}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

Hence $y_n = m(m - 1) (m - 2) \ldots (m - n + 1) a^n (ax + b)^{m-n}$

In particular, $\mathbf{D^n (x^n) = n!}$.

(2) follows from (1) by taking $m = -1$. The proof of (3) is left as an exercise for the student.

To prove (4), let $y = a^{mx}$

$$y_1 = m \log a . a^{mx}, y_2 = (m \log a)^2 a^{mx}, \text{ etc.}$$

In general $y_n = (m \log a)^n a^{mx}$.

(5) follows from (4) by taking $a = e$.

To prove (6), let $y = \sin (ax + b)$

$\therefore$
$$y_1 = a \cos (ax + b) = a \sin (ax + b + \pi/2)$$
$$y_2 = a^2 \cos (ax + b + \pi/2) = a^2 \sin (ax + b + 2\pi/2)$$
$$y_3 = a^3 \cos (ax + b + 2\pi/2) = a^3 \sin (ax + b + 3\pi/2)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

In general, $y_n = a^n \sin (ax + b + n\pi/2)$.

The proof of (7), is left as an exercise for the reader.

To prove (8), let $y = e^{ax} \sin (bx + c)$

$\therefore$
$$y_1 = e^{ax} \cos (bx + c) . b + ae^{ax} \sin (bx + c)$$
$$= e^{ax} [a \sin (bx + c) + b \cos (bx + c)]$$

Put $a = r \cos \alpha$, $b = r \sin \alpha$ so that $r = \sqrt{(a^2 + b^2)}$, $\alpha = \tan^{-1} b/a$

$\therefore$
$$y_1 = re^{ax} [\sin (bx + c) \cos \alpha + \cos (bx + c) \sin \alpha]$$
$$= re^{ax} \sin (bx + c + \alpha)$$

Similarly,
$$y_2 = r^2 e^{ax} \sin (bx + c + 2\alpha)$$
$$y_3 = r^3 e^{ax} \sin (bx + c + 3\alpha)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

In general, $y_n = r^n e^{ax} \sin(bx + c + n\alpha)$ (V.T.U., 2000)

where $r = \sqrt{(a^2 + b^2)}$ and $\alpha = \tan^{-1} b/a$.

Proceeding as in (8), the student should prove (9) himself.

(3) **Preliminary transformations.** Quite often preliminary simplification reduces the given function to one of the above standard forms and then the nth derivative can be written easily.

To find the nth derivative of the powers of sines or cosines or their products, we first express each of these as a series of sines or cosines of multiple angles and then use the above formulae (6) and (7).

Example 4.5. *If* $y = x \log \dfrac{x-1}{x+1}$, *show that* $y_n = (-1)^{n-2}(n-2)!\left[\dfrac{x-n}{(x-1)^n} - \dfrac{x+n}{(x+1)^n}\right]$. (U.P.T.U., 2003)

Solution. Differentiating y w.r.t. x, we have

$$y_1 = \log \frac{x-1}{x+1} + x\left[\frac{1}{x-1} - \frac{1}{x+1}\right]$$

$$= \log(x-1) - \log(x+1) + \frac{1}{x-1} + \frac{1}{x+1} \quad ...(i)$$

Now differentiating (i) $(n-1)$ times w.r.t. x,

$$y_n = \frac{(-1)^{n-2}(n-2)!}{(x-1)^{n-1}} - \frac{(-1)^{n-2}(n-2)!}{(x+1)^{n-1}} + \frac{(-1)^{n-1}(n-1)!}{(x-1)^n} + \frac{(-1)^{n-1}(n-1)!}{(x+1)^n}$$

$$= (-1)^{n-2}(n-2)!\left\{\frac{x-1}{(x-1)^n} - \frac{x+1}{(x+1)^n} + \frac{-(n-1)}{(x-1)^n} + \frac{-(n-1)}{(x+1)^n}\right\}$$

$$= (-1)^{n-2}(n-2)!\left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right].$$

Example 4.6. *Find the nth derivative of* (i) $\cos x \cos 2x \cos 3x$ (S.V.T.U., 2009)
(ii) $e^{2x} \cos^2 x \sin x$.

Solution. (i) $y = \cos x \cos 2x \cos 3x = \frac{1}{2} \cos x (\cos 5x + \cos x)$

$$= \tfrac{1}{4}(2\cos x \cos 5x + 2\cos^2 x) = \tfrac{1}{4}[(\cos 6x + \cos 4x) + (1 + \cos 2x)]$$

$$= \tfrac{1}{4}(1 + \cos 2x + \cos 4x + \cos 6x)$$

$$\therefore \quad y_n = \tfrac{1}{4}[2^n \cos(2x + n\pi/2) + 4^n \cos(4x + n\pi/2 + 6^n \cos(6x + n\pi/2)]$$

(ii) $\cos^2 x \sin x = \cos x(\sin x \cos x) = \cos x \cdot \frac{1}{2} \sin 2x$

$$= \tfrac{1}{4}(2 \sin 2x \cos x) = \tfrac{1}{4}(\sin 3x + \sin x)$$

$$\therefore \quad D^n(e^{2x}\cos^2 x \sin x) = \tfrac{1}{4}[D^n(e^{2x}\sin 3x) + D^n(e^{2x}\sin x)]$$

$$= \tfrac{1}{4}[(2^2 + 3^2)^{n/2}\sin(3x + n\tan^{-1} 3/2) + (2^2 + 1^2)^{n/2}\sin(x + n\tan^{-1}\tfrac{1}{2})]$$

$$= \tfrac{1}{4}[(13)^{n/2}\sin(3x + n\tan^{-1} 3/2) + (5)^{n/2}\sin(x + n\tan^{-1}\tfrac{1}{2})].$$

(4) Use of partial fractions. To find the nth derivative of any rational algebraic fraction, we first split it up into partial fractions. Even when the denominator cannot be resolved into real factors, the method of partial fractions can still be used after breaking the denominator into complex linear factors. Then to put the result back in a real form, we apply De Moivre's theorem (p. 647).

Example 4.7. *Find the nth derivative of* $\dfrac{x}{(x-1)(2x+3)}$.

Solution.
$$\frac{x}{(x-1)(2x+3)} = \frac{1}{(x-1)(2.1+3)} + \frac{-3/2}{(-3/2-1)(2x+3)}$$
$$= \frac{1}{5}\cdot\frac{1}{x-1} + \frac{3}{5}\cdot\frac{1}{2x+3}$$

Hence
$$D^n\left[\frac{x}{(x-1)(2x+3)}\right] = \frac{1}{5}\cdot\frac{(-1)^n\, n!}{(x-1)^{n+1}} + \frac{3}{5}\cdot\frac{(-1)^n\,(n!)2^n}{(2x+3)^{n+1}}$$
$$= \frac{(-1)^n\, n!}{5}\left\{\frac{1}{(x-1)^{n+1}} + \frac{3.2^n}{(2x+3)^{n+1}}\right\}.$$

Example 4.8. *Find the nth derivative of* $\dfrac{1}{x^2+a^2}$.

Solution. We have
$$y = \frac{1}{x^2+a^2} = \frac{1}{(x+ia)(x-ia)} = \frac{1}{2ia}\left(\frac{1}{x-ia} - \frac{1}{x+ia}\right)$$

$\therefore$
$$y_n = \frac{1}{2ia}\left\{\frac{(-1)^n\, n!}{(x-ia)^{n+1}} - \frac{(-1)^n\, n!}{(x+ia)^{n+1}}\right\}$$

[Put $x = r\cos\theta$, $a = r\sin\theta$ so that $r = \sqrt{(x^2+a^2)}$, $\theta = \tan^{-1}(a/x)$]

$$= \frac{(-1)^n\, n!}{2ia}\left\{\frac{1}{r^{n+1}(\cos\theta - i\sin\theta)^{n+1}} - \frac{1}{r^{n+1}(\cos\theta + i\sin\theta)^{n+1}}\right\}$$
$$= \frac{(-1)^n\, n!}{2iar^{n+1}}\{(\cos\theta - i\sin\theta)^{-(n+1)} - (\cos\theta + i\sin\theta)^{-(n+1)}\}$$
$$= \frac{(-1)^n\, n!}{2iar^{n+1}}[\cos(n+1)\theta + i\sin(n+1)\theta - [\cos(n+1)\theta - i\sin(n+1)\theta]]$$

[By De Moivre's theorem]

$$= \frac{(-1)^n\, n!}{2iar^{n+1}}\,.\,2i\sin(n+1)\theta \qquad \left[\text{Put } \frac{1}{r} = \frac{\sin\theta}{a}\right]$$
$$= \frac{(-1)^n\, n!}{a^{n+2}}\sin(n+1)\theta\,\sin^{n+1}\theta.$$

PROBLEMS 4.2

Find the nth derivative of (1 to 11) :

1. $\log(4x^2-1)$ *(V.T.U., 2010)*
2. $\dfrac{x+2}{x+1} + \log\dfrac{x+2}{x+1}$
3. $\sin^3 x\cos^2 x$ *(V.T.U., 2006)*
4. $\cos^9 x$ *(Mumbai, 2008)*
5. $\sinh 2x \sin 4x$ *(V.T.U., 2010 S)*
6. $e^{5x}\cos x\cos 3x$ *(Mumbai, 2007)*
7. $\dfrac{x+3}{(x-1)(x+2)}$ *(V.T.U., 2009)*
8. $\dfrac{x^2}{2x^2+7x+6}$ *(V.T.U., 2005)*
9. $\dfrac{1}{1+x+x^2+x^3}$ *(Mumbai, 2009)*
10. $\dfrac{x}{x^2+a^2}$ *(Mumbai, 2007)*
11. Find the nth derivative of $\tan^{-1}\dfrac{2x}{1-x^2}$ in terms of r and θ. *(U.P.T.U., 2002)*

4.2 LEIBNITZ'S THEOREM for the *n*th Derivative of the product of two functions*

If u, v be two function of x possessing derivatives of the nth order, then

$$(\mathbf{uv})_n = \mathbf{u}_n\, \mathbf{v} + {}^n\mathbf{C}_1\, \mathbf{u}_{n-1}\, \mathbf{v}_1 + {}^n\mathbf{C}_2\, \mathbf{u}_{n-2}\, \mathbf{v}_2 + \dots + {}^n\mathbf{C}_r \mathbf{u}_{n-r}\, \mathbf{v}_r + \dots + {}^n\mathbf{C}_n\, \mathbf{uv}_n$$

We shall prove this theorem by mathematical induction.

Step I. By actual differentiation,

$$(uv)_1 = u_1 v + uv_1$$

$$(uv)_2 = (u_2 v + u_1 v_1) + (u_1 v_1 + uv_2)$$

$$= u_2 v + {}^2C_1\, u_1 v_1 + {}^2C_2\, uv_2 \qquad [\because\ 2 = {}^2C_1,\ 1 = {}^2C_2]$$

Thus we see that the theorem is true for $n = 1, 2$.

Step II. Assume the theorem to be true for $n = m$ (say) so that

$$(uv)_m = u_m v + {}^mC_1\, u_{m-1} v_1 + {}^mC_2\, u_{m-2} v_2 + \dots + {}^mC_{r-1} u_{m-r+1} v_{r-1} + {}^mC_r\, u_{m-r}\, v_r + \dots + {}^mC_m\, uv_m$$

Differentiating both sides,

$$(uv)_{m+1} = (u_{m+1} v + u_m v_1) + {}^mC_1\, (u_m v_1 + u_{m-1} v_2) + {}^mC_2\, (u_{m-1} v_2 + u_{m-2} v_3) + \dots$$
$$+ {}^mC_{r-1}\, u_{m-r+2}\, v_{r-1} + u_{m-r+1}\, v_r) + {}^mC_r\, (u_{m-r+1}\, v_r + u_{m-r} v_{r+1}) + \dots$$
$$+ {}^mC_m\, (u_1 v_m + uv_{m+1})$$
$$= u_{m+1} v + (1 + {}^mC_1)\, u_m v_1 + ({}^mC_1 + {}^mC_2)\, u_{m-1} v_2 + \dots$$
$$+ ({}^mC_{r-1} + {}^mC_r)\, u_{m-r+1}\, v_r + \dots + {}^mC_m\, uv_{m+1}$$

But $\quad 1 + {}^mC_1 = {}^mC_0 + {}^mC_1 = {}^{m+1}C_1,\ {}^mC_1 + {}^mC_2 = {}^{m+1}C_2 \dots$

${}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r, \dots$ and ${}^mC_m = 1 = {}^{m+1}C_{m+1}$

$$\therefore \quad (uv)_{m+1} = u_{m+1} v + {}^{m+1}C_1\, u_m v_1 + {}^{m+1}C_2\, u_{m-1} v_2 + \dots + {}^{m+1}C_r\, u_{m+1-r} v_r + \dots + {}^{m+1}C_{m+1} uv_{m+1}$$

which is of exactly the same form as the given formula with n replaced by $m + 1$. Hence if the theorem is true for $n = m$, it is also true for $n = m + 1$.

Step III. In step I, the theorem has been seen to be true for $n = 2$, and by step II, it must be true for $n = 2 + 1$ *i.e.*, 3 and so for $n = 3 + 1$ *i.e.*, 4 and so on.

Hence the theorem is true for all positive integral values of n.

Example 4.9. *Find the nth derivative of* $e^x (2x + 3)^3$.

Solution. Take $u = e^x$ and $v = (2x + 3)^3$, so that $u_n = e^x$ for all integral values of n, and $v_1 = 6\,(2x + 3)^2$, $v_2 = 24(2x + 3)$, $v_3 = 48$, v_4, v_5 etc. are all zero.

$\therefore$ By Leibnitz's theorem,

$$(uv)_n = u_n\, v + {}^nC_1\, u_{n-1}\, v_1 + {}^nC_2\, u_{n-2}\, v_2 + {}^nC_3\, u_{n-3}\, v_3$$

i.e.,

$$[e^x (2x + 3)^3]_n = e^x (2x + 3)^3 + ne^x[6\,(2x + 3)^2]$$
$$+ \frac{n(n-1)}{1\,.\,2} e^x\, [24(2x + 3)] + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\, e^x\, [48]$$
$$= e^x\, \{(2x + 3)^3 + 6n(2x + 3)^2 + 12n\,(n-1)\,(2x + 3) + 8n(n-1)\,(n-2)\}.$$

Example 4.10. *If* $y = (\sin^{-1} x)^2$, *show that* $(1 - x^2)\, y_{n+2} - (2n + 1)\, xy_{n+1} - n^2\, y_n = 0$. *Hence find* $(y_n)_0$.
(*U.P.T.U.*, *2005*)

Solution. We have $\qquad y = (\sin^{-1} x)^2$

Differentiating, $\qquad y_1 = \dfrac{2 \sin^{-1} x}{\sqrt{(1 - x^2)}}$ or $(1 - x^2)\, y_1^{\,2} = 4\,(\sin^{-1} x)^2 = 4y \qquad \dots(i)$

Again differentiating, $\qquad (1 - x^2)\, 2y_1 y_2 + (-2x)\, y_1^{\,2} = 4y_1 \qquad \dots(ii)$

Dividing by $2y_1$, $(1 - x^2)\, y_2 - xy_1 - 2 = 0$

Differentiating it n times by Leibnitz's theorem,

*Named after the German mathematician and philosopher *Gottfried Wilhelm Leibnitz* (1646–1716) who invented the differential and integral calculus independent of Sir Issac Newton.

$$(1 - x^2)\, y_{n+2} + n(-2x)\, y_{n+1} + \frac{n(n-1)}{2}(-2)y_n - [xy_{n+1} + n(1)\, y_n] = 0$$

or $$(1 - x^2)\, y_{n+2} - (2n+1)\, xy_{n+1} - n^2 y_n = 0$$

which is the required result.

Putting $x = 0$, $(y_{n+2})_0 = n^2 (y_n)_0$ ...(*iii*)

From (*i*), $(y_1)_0 = 0$. From (*ii*), $(y_2)_0 = 2$.

Putting $n = 1, 3, 5, 7, \ldots$ in (*iii*), $0 = y_1 = y_3 = y_5 = y_7 = \ldots$

i.e., if n is odd, $(y_n)_0 = 0$

Again putting $n = 2, 4, 6, \ldots$ in (*iii*)

$$(y_4)_0 = 2^2\, (y_2)_0 = 2\,.\, 2^2$$
$$(y_6)_0 = 4^2\, (y_6)_0 = 2.\, 2^2\,.\, 4^2$$
$$(y_8)_0 = 6^2\, (y_6)_0 = 2.\, 2^2\,.\, 4^2\,.\, 6^2$$

In general, if n is even, $(y_n)_0 = 2\,.\, 2^2\,.\, 4^2\,.\, 6^2 \ldots (n-2)^2, (n \neq 2)$.

Example 4.11. *If $y = e^{a \sin^{-1} x}$, prove that $(1 - x^2)\, y_{n+2} - (2n+1)\, x\, y_{n+1} - (n^2 + a^2) y_n = 0$. Hence find the value of y_n when $x = 0$.* (V.T.U., 2003)

Solution. We have $y = e^{a \sin^{-1} x}$ ...(*i*)

Differentiating, $$y_1 = e^{a \sin^{-1} x} \frac{a}{\sqrt{(1 - x^2)}} = \frac{ay}{\sqrt{(1 - x^2)}}$$ ...(*ii*)

or $$(1 - x^2)\, y_1^2 = a^2 y^2.$$

Again differentiating, $(1 - x^2)\, 2y_1 y_2 + (-2x) y_1^2 = 2a^2\, yy_1$.

Dividing by $2y_1$, $(1 - x^2) y_2 - xy_1 - a^2 y = 0$ ...(*iii*)

Differentiating it n times by Leibnitz's theorem,

$$(1 - x^2)\, y_{n+2} + n\,.\,(-2x)\, y_{n+1} + \frac{n(n-1)}{2}\,.\,(-2)\, y_n - [xy_{n+1} + n\,.\,1\,.\, y_n] - a^2 y_n = 0$$

or $$(1 - x^2) y_{n+2} - (2n+1)\, xy_{n+1} - (n^2 + a^2)\, y_n = 0$$

which is the required result.

Putting $x = 0$, $(y_{n+2})_0 = (n^2 + a^2)\, (y_n)_0$ ...(*iv*)

From (*i*), (*ii*), (*iii*) : $(y)_0 = 1, (y_1)_0 = a, (y_2)_0 = a^2$

Putting $n = 1, 2, 3, 4 \ldots$ in (*iv*),

$$(y_3)_0 = (1^2 + a^2)\, (y_1)_0 = a(1^2 + a^2)$$
$$(y_4)_0 = (2^2 + a^2)\, (y_2)_0 = a^2(2^2 + a^2)$$
$$(y_5)_0 = (3^2 + a^2)\, (y_3)_0 = a(1^2 + a^2)\, (3^2 + a^2)$$
$$(y_6)_0 = (4^2 + a^2)\, (y_4)_0 = a^2(2^2 + a^2)\, (4^2 + a^2).$$

Hence in general, $(y_n)_0 = a(1^2 + a^2)\, (3^2 + a^2) \ldots [(n-2)^2 + a^2]$, when n is odd.

$= a^2(2^2 + a^2)\, (4^2 + a^2) \ldots\ldots [(n-2)^2 + a^2]$, when n is even.

Example 4.12. *If $y^{1/m} + y^{-1/m} = 2x$, prove that*

$$(x^2 - 1) y_{n+2} + (2n+1)\, xy_{n+1} + (n^2 - m^2)\, y_n = 0.$$

(V.T.U., 2008 S ; Mumbai, 2007 ; S.V.T.U., 2007)

Solution. We have $$y^{1/m} + \frac{1}{y^{1/m}} = 2x$$

or $$(y^{1/m})^2 - 2x(y^{1/m}) + 1 = 0$$

$\therefore$ $$y^{1/m} = \frac{2x \pm \sqrt{(4x^2 - 4)}}{2} = x \pm \sqrt{(x^2 - 1)}$$

Hence $$y = [x \pm \sqrt{(x^2 - 1)}\,]^m$$

Taking logarithm, $$\log y = m \log [x \pm \sqrt{(x^2 - 1)}\,]$$

Differentiating both sides w.r.t. x,

$$\frac{1}{y}y_1 = m \cdot \frac{1}{x \pm \sqrt{(x^2-1)}} \cdot \left\{1 \pm \frac{x}{\sqrt{(x^2-1)}}\right\} = \pm \frac{m}{\sqrt{(x^2-1)}}$$

Squaring, $\quad y_1^2 (x^2 - 1) = m^2 y^2$

Again differentiating, $\quad (x^2 - 1)\, 2y_1 y_2 + y_1^2 (2x) = m^2 \cdot 2y \cdot y_1$

Dividing by $2y_1$, $\quad (x^2 - 1) y_2 + xy_1 - m^2 y = 0$

Differentiating it n times by Leibnitz's theorem,

$$(x^2 - 1) y_{n+2} + ny_{n+1}(2x) + \frac{n(n-1)}{2} y_n (2) + xy_{n+1} + n \cdot y_n (1) - m^2 y_n = 0$$

or $$(x^2 - 1) y_{n+2} + (2n + 1) xy_{n+1} + (n^2 - m^2) y_n = 0.$$

PROBLEMS 4.3

1. Find the nth derivative of (*i*) $x^2 \log 3x$. (*ii*) $2^x \cos^9 x$. (*Mumbai, 2009*)
2. If $y = a \cos(\log x) + b \sin(\log x)$, show that $x^2 y_2 + xy_1 + y = 0$ and $x^2 y_{n+2} + (2n+1) xy_{n+1} + (n^2 + 1) y_n = 0$. (*U.P.T.U., 2004 ; Madras, 2000*)
3. If $y = \sin^{-1} x$, prove that $(1 - x^2) y_{n+2} - (2n + 1) xy_{n+1} - n^2 y_n = 0$. Also find (y_n). (*S.V.T.U., 2009*)
4. If $\cos^{-1}(y/b) = \log (x/n)^n$, prove that $x^2 y_{n+2} + (2n + 1) xy_{n+1} + 2n^2 y_n = 0$. (*U.P.T.U., 2006*)
5. If $y = \tan^{-1} x$, prove that $(1 + x^2) y_{n+1} + 2nxy_n + n(n-1) y_{n-1} = 0$. Find $y_{n(0)}$.
6. If $y = \cos(m \sin^{-1} x)$, prove that $(1 - x^2) y_{n+2} - (2n + 1) xy_{n+1} + (m^2 - n^2) y_n = 0$. (*Mumbai, 2008 S*)
7. If $y = \sin(m \sin^{-1} x)$, prove that $(1 - x^2) y_2 - xy_1 + m^2 y = 0$
   and $\quad (1 - x^2) y_{n+2} - 2(n + 1) xy_{n+1} + (m^2 - n^2) y_n = 0$. (*V.T.U., 2009 ; Cochin, 2005*)
   Also find $(y_n)_0$. (*U.P.T.U., 2005*)
8. If $y = e^{m \cos^{-1} x}$, prove that (*i*) $(1 - x^2) y_2 - xy_1 = m^2 y$
   (*ii*) $(1 - x^2) y_{n+2} - (2n + 1) xy_{n+1} - (n^2 + m^2) y_n = 0$. Also find $(y_n)_0$.
9. If $y = (x^2 - 1)^n$, prove that $(x^2 - 1) y_{n+2} + 2xy_{n+1} - n(n + 1) y_n = 0$. (*U.T.U., 2010*)
10. If $\sin^{-1} y = 2 \log (x + 1)$, prove that $(x + 1)^2 y_{n+2} + (2n + 1)(x + 1) y_{n+1} + (x^2 + 4) y_n = 0$. (*V.T.U., 2003*)
11. If $y = x^n \log x$, prove that $y_{n+1} = n!/x$. (*Mumbai, 2008*)
12. If $V_n = \dfrac{d^n}{dx^n}(x^n \log x)$, show that $V_n = nV_{n-1} + (n - 1)!$

    Hence, show that $V_n = n!\left(\log x + 1 + \dfrac{1}{2} + \dfrac{1}{3} + \dots + \dfrac{1}{n}\right)$. (*V.T.U., 2001*)
13. Show that $\dfrac{d^n}{dx^n}\left(\dfrac{\log x}{x}\right) = \dfrac{(-1)^n n!}{x^{n+1}}\left\{\log x - 1 - \dfrac{1}{2} - \dfrac{1}{3} - \dots - \dfrac{1}{n}\right\}$. (*V.T.U., 2006*)
14. If $y = x \log\left(\dfrac{x-1}{x+1}\right)$, show that $y_n = (-1)^{n-2}(n - 2)!\left[\dfrac{x-n}{(x-1)^n} - \dfrac{x+n}{(x+1)^n}\right]$. (*U.P.T.U., 2003*)
15. If $x = \sin t$, $y = \cos pt$, show that $(1 - x^2) y_2 - xy_1 + p^2 y = 0$. Hence prove that
    $$(1 - x^2) y_{n+2} - (2n + 1) xy_{n+1} - (n^2 - p^2) y_n = 0.$$ (*Raipur, 2005 ; V.T.U., 2005*)
16. If $y = \log [x + \sqrt{(1 + x^2)}]^2$, prove that $(1 + x^2) y_{n+2} + (2n + 1) xy_{n+1} + n^2 y_n = 0$. (*V.T.U., 2007 ; Bhillai, 2005*)
    Hence show that $(y_{2k})_0 = (-1)^{k-1} \cdot 2^k \cdot [(k - 1)!]^2$, where k is positive integer.
17. If $y = [x + \sqrt{(x^2 + 1)}]^m$, prove that (*i*) $(x^2 + 1) y_2 + xy_1 - m^2 y = 0$, (*ii*) $y_{n+2} + (n^2 - m^2) y_n = 0$ at $x = 0$. (*V.T.U., 2009 S*)
    Hence find $y_n(0)$. (*Madras, 2000*)
18. If $y = \sin \log (x^2 + 2x + 1)$, prove that (*i*) $(x + 1)^2 y_2 + (x + 1) y_1 + 4y = 0$
    (*ii*) $(x + 1)^2 y_{n+2} + (2n + 1)(x + 1) y_{n+1} + (n^2 + 4) y_n = 0$. (*U.P.T.U., 2006*)

19. If $y = \dfrac{\sinh^{-1} x}{\sqrt{(1+x^2)}}$, show that $(1 + x^2) y_{n+2} + (2n + 3) xy_{n+1} + (n + 1)^2 y_n = 0$. (V.T.U., 2010)

20. If $y = \sinh [m \log (x + \sqrt{x^2 + 1})]$, prove that $(x^2 + 1) y_{n+2} + (2n + 1) xy_{n+1} + (n^2 - m^2) y_n = 0$. (V.T.U., 2010 S)

4.3 FUNDAMENTAL THEOREMS

(1) Rolle's Theorem

If (i) $f(x)$ is continuous in the closed interval $[a, b]$, (ii) $f'(x)$ exists for every value of x in the open interval (a, b) and (iii) $f(a) = f(b)$, then there is at least one value c of x in (a, b) such that $f'(c) = 0$.

Consider the portion AB of the curve $y = f(x)$, lying between $x = a$ and $x = b$, such that

(i) it goes continuously from A to B,

(ii) it has a tangent at every point between A and B, and

(iii) ordinate of A = ordinate of B.

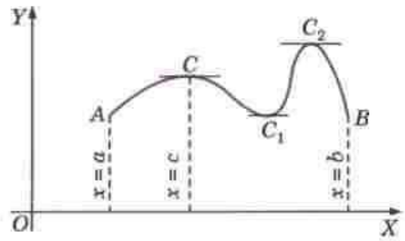


Fig. 4.1

From the Fig. 4.1, it is self-evident that there is at least one point C (may be more) of the curve at which the tangent is parallel, to the x-axis.

i.e., slope of the tangent at C $(x = c) = 0$

But the slope of the tangent at C is the value of the differential coefficient of $f(x)$ w.r.t. x thereat, therefore $f'(c) = 0$.

Hence the theorem is proved.

Example 4.13. *Verify Rolle's theorem for (i) $\sin x / e^x$ in $(0, \pi)$.* (J.N.T.U., 2003)
(ii) $(x - a)^m (x - b)^n$ where m, n are positive integers in $[a, b]$. (V.T.U., 2010 ; Nagarjuna, 2008)

Solution. (i) Let $f(x) = \sin x/e^x$.

$f(x)$ is derivable in $(0, \pi)$.

Also $f(0) = f(\pi) = 0$.

Hence the conditions of Rolle's theorem are satisfied.

$$\therefore \quad f'(x) = \frac{e^x \cos x - e^x \sin x}{e^{2x}} \quad \text{vanishes where } e^x (\cos x - \sin x) = 0$$

or $\tan x = 1$ *i.e.,* $x = \pi/4$.

The value $x = \pi/4$ lies in $(0, \pi)$, so that Rolle's theorem is verified.

(ii) Let $f(x) = (x - a)^m (x - b)^n$.

Since every polynomial is continuous for all values, $f(x)$ is also continuous in $[a, b]$.

$$f'(x) = m(x-a)^{m-1} (x-b)^n + (x-a)^m \cdot n(x-b)^{n-1}$$
$$= (x-a)^{m-1} (x-b)^{n-1} [(m+n) x - (mb + na)]$$

which exists, *i.e.,* $f(x)$ is derivable in (a, b).

Also $f(a) = 0 = f(b)$.

Thus all the conditions of Rolle's theorem are satisfied and there exists c in (a, b) such that $f'(c) = 0$.

$$\therefore \quad (c-a)^{m-1} (c-b)^{n-1} [(m+n) c - (mb + na)] = 0 \quad \text{or} \quad c = (mb + na)/(m + n).$$

Hence, Rolle's theorem is verified.

(2) Lagrange's Mean-Value Theorem*

First form. *If (i) $f(x)$ is continuous in the closed interval $[a, b]$, and*
(ii) $f'(x)$ exists in the open interval (a, b), then there is at least one value c of x in (a, b), such that

$$\frac{\mathbf{f(b) - f(a)}}{\mathbf{b - a}} = \mathbf{f'(c)}.$$

---

*Named after the great French mathematician *Joseph Louis Lagrange* (1736–1813) who became professor at Military Academy, Turin when he was just 19 and director of Berlin Academy in 1766. His important contribution are to algebra, number theory, differential equations, mechanics, approximation theory and calculus of variations.

Consider the function $\phi(x) = f(x) - \dfrac{f(b) - f(a)}{b - a}\, x$

Since $f(x)$ is continuous in $[a, b]$; $\therefore$ $\phi(x)$ is also continuous in $[a, b]$.

Since $f'(x)$ exists in (a, b) ;

$\therefore$ $\phi'(x)$ also exists in (a, b) and $= f'(x) - \dfrac{f(b) - f(a)}{b - a}$ ...(*i*)

Clearly, $$\phi(a) = \frac{b\,f(a) - a\,f(b)}{b - a} = \phi(b).$$

Thus $\phi(x)$ satisfies all the conditions of Rolle's theorem.

$\therefore$ There is at least one value c of x between a and b such that $\phi'(c) = 0$. Substituting $x = c$ in (1), we get

$$f'(c) = \frac{f(b) - f(a)}{b - a} \qquad \text{...(2)}$$

which proves the theorem.

Second form. If we write $b = a + h$, then since $a < c < b$,

$$c = a + \theta h \text{ where } 0 < \theta < 1.$$

Thus the mean value theorem may be stated as follows :

If (i) f(x) is continuous in the closed interval [a, a + h] and (ii) f′ (x) exists in the open interval (a, a + h), then there is at least one number θ(0 < θ < 1) such that

$$\mathbf{f(a + h) = f(a) + hf'(a + \theta h)}$$

Geometrical Interpretation. Let A, B be the points on the curve $y = f(x)$ corresponding to $x = a$ and $x = b$ so that $A = [a, f(a)]$ and $B = [b, f(b)]$. (Fig. 4.2)

$\therefore$ Slope of chord $AB = \dfrac{f(b) - f(a)}{b - a}$

By (2), the slope of the chord $AB = f'(c)$, the slope of the tangent of the curve at $C(x = c)$.

Hence the Lagrange's mean value theorem asserts that *if a curve AB has a tangent at each of its points, then there exists at least one point C on this curve, the tangent at which is parallel to the chord AB.*

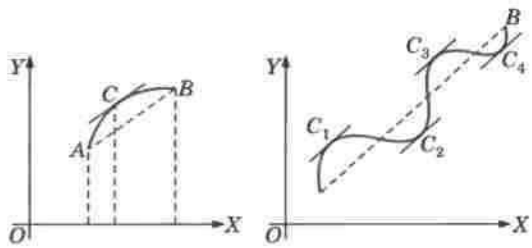


Fig. 4.2

Cor. *If f′(x) = 0 in the interval (a, b) then f (x) is constant in [a, b].* For, if x_1, x_2 be any two values of x in (a, b), then by (2),

$$f(x_2) - f(x_1) = (x_2 - x_1)\, f'(c) = 0 \ (x_1 < c < x_2)$$

Thus, $f(x_1) = f(x_2)$ *i.e.*, $f(x)$ has the same value for every value of x in (a, b).

Example 4.14. *In the Mean value theorem* $f(b) - f(a) = (b - a)\, f'(c)$, ...(*i*)
determine c lying between a and b, if $f(x) = x(x - 1)(x - 2)$, $a = 0$ *and* $b = 1/2$. (*Gorakhpur, 1999*)

Solution. $f(a) = 0,$ $\qquad f(b) = \dfrac{1}{2}\left(-\dfrac{1}{2}\right)\left(-\dfrac{3}{2}\right) = \dfrac{3}{8}$

$f'(x) = 3x^2 - 6x + 2,$ $\qquad f'(c) = 3c^2 - 6c + 2$

Substituting in (*i*), $$\frac{3}{8} - 0 = \left(\frac{1}{2} - 0\right)(3c^2 - 6c + 2)$$

or $$12c^2 - 24c + 5 = 0$$

whence $$c = \frac{24 \pm \sqrt{(24)^2 - 12 \times 5 \times 4}}{24} = 1 \pm 0.764 = 1.764 \ ; 0.236.$$

Hence $c = 0.236$, since it only lies between 0 and 1/2.

Example 4.15. *Prove that (if* $0 < a < b < 1$*),* $\dfrac{b - a}{1 + b^2} < \tan^{-1} b - \tan^{-1} a < \dfrac{b - a}{1 + a^2}$.

Hence show that $\dfrac{\pi}{4} + \dfrac{3}{25} < \tan^{-1}\dfrac{4}{3} < \dfrac{\pi}{4} + \dfrac{1}{6}$. (*Mumbai, 2009 ; V.T.U., 2006*)

Solution. Let $f(x) = \tan^{-1} x$, so that $f'(x) = \dfrac{1}{1+x^2}$.

By Mean value theorem, $\dfrac{\tan^{-1} b - \tan^{-1} a}{b-a} = \dfrac{1}{1+c^2}, a < c < b$ ...(*i*)

Now $a < c < b$, $\therefore$ $1 + a^2 < 1 + c^2 < 1 + b^2$.

$$\therefore \quad \frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2} \quad i.e., \quad \frac{1}{1+b^2} < \frac{1}{1+c^2} < \frac{1}{1+a^2}$$

i.e.,
$$\frac{1}{1+b^2} < \frac{\tan^{-1} b - \tan^{-1} a}{b-a} < \frac{1}{1+a^2} \qquad \text{[By } (i)]$$

Hence
$$\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}$$

Now let $a = 1$, $b = 4/3$.

Then
$$\frac{1/3}{1+16/9} < \tan^{-1}\frac{4}{3} - \frac{\pi}{4} < \frac{1/3}{1+1}$$

i.e.,
$$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}.$$

Example 4.16. *Prove that* $\log(1+x) = x/(1+\theta x)$, *where* $0 < \theta < 1$ *and hence deduce that*

$\dfrac{x}{1+x} < \log(1+x) < x, x > a$ *(Mumbai, 2008)*

Solution. Let $f(x) = \log(1+x)$, then by second form of Lagrange's mean value theorem

$$f(a+h) = f(a) + h f'(a+\theta h), \qquad (0 < \theta < 1)$$

we have
$$f(x) = f(0) + x f'(\theta x) \qquad \text{[Taking } a = 0, h = x]$$

or
$$\log(1+x) = \log(1) + x \cdot 1/(1+\theta x) \qquad [\because \; f'(x) = 1/1+x)$$

Hence
$$\log(1+x) = x/(1+\theta x) \qquad \text{...}(i) \; [\because \; \log(1) = 0$$

Since $0 < \theta < 1$, $\therefore$ $0 < \theta x < x$ for $x > 0$.

or
$$1 < 1 + \theta x < 1 + x \quad \text{or} \quad 1 > \frac{1}{1+\theta x} > \frac{1}{1+x}$$

or
$$x > \frac{x}{1+\theta x} > \frac{x}{1+x}$$

or
$$\frac{x}{1+x} < \log(1+x) < x, x > 0. \qquad \text{[By } (i)]$$

(3) Cauchy's Mean-value Theorem*

If (i) $f(x)$ and $g(x)$ *be continuous in* $[a, b]$

(*ii*) $f'(x)$ and $g'(x)$ exist in (a, b)

and (*iii*) $g'(x) \neq 0$ *for any value of x in* (a, b),

then there is at least one value c of x in (a, b), such that $\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{f'(c)}{f'(c)}$

Consider the function $\phi(x) = f(x) - \dfrac{f(b) - f(a)}{g(b) - g(a)} g(x)$

Since $f(x)$ and $g(x)$ are continuous in $[a, b]$

$\therefore$ $\phi(x)$ is also continuous in $[a, b]$.

Again since $f'(x)$ and $g'(x)$ exist in (a, b).

*Named after the great French mathematician *Augustin-Louis Cauchy* (1789–1857) who is considered as the father of modern analysis and creator of complex analysis. He published nearly 800 research papers of basic importance. Cauchy is also well known for his contributions to differential equations, infinite series, optics and elasticity.

$\therefore$ $\phi'(x)$ also exists in (a, b) and $= f'(x) - \dfrac{f(b)-f(a)}{g(b)-g(a)} g'(x)$

Clearly, $\phi(a) = \phi(b)$.

Thus, $\phi(x)$ satisfies all the conditions of Rolle's theorem. There is therefore, at least one value c of x between a and b, such that $\phi'(c) = 0$

i.e., $$0 = f'(c) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(c) \text{ whence follows the result.}$$

(*P.T.U., 2007 S ; V.T.U., 2006*)

Obs. Cauchy's mean value theorem is a generalisation of Lagrange's mean value theorem, where $g(x) = x$.

Example 4.17. *Verify Cauchy's Mean-value theorem for the functions e^x and e^{-x} in the interval (a, b).*

Solution. $f(x) = e^x$ and $g(x) = e^{-x}$ are both continuous in $[a, b]$ and both functions are differentiable in (a, b).

$\therefore$ $$f'(x) = e^x, g'(x) = -e^{-x}$$

By Cauchy's mean value theorem.

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

$\therefore$ $$\frac{e^b - e^a}{e^{-b} - e^{-a}} = \frac{e^c}{-e^{-c}} \quad i.e., c = \frac{1}{2}(a+b)$$

Thus c lies in (a, b) which verifies the Cauchy's Mean value theorem.

(4) Taylor's Theorem* (Generalised mean value theorem)

If (i) $f(x)$ and its first $(n-1)$ derivatives be continuous in $[a, a+h]$, and (ii) $f^n(x)$ exists for every value of x in $(a, a+h)$, then there is at least one number θ $(0 < \theta < 1)$, such that

$$\mathbf{f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^n}{n!} f^n(a+\theta h)} \quad \dots(1)$$

which is called Taylor's theorem with Lagrange's form remainder, the remainder R_n being $\dfrac{h^n}{n!} f^n(a+\theta h)$.

Proof. Consider the function

$$\phi(x) = f(x) + (a+h-x) f'(x) + \frac{(a+h-x)^2}{2!} f''(x) + \dots + \frac{(a+h-x)^n}{n!} K$$

where K is defined by

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^n}{n!} K \quad \dots(2)$$

(*i*) Since $f(x), f'(x), \dots, f^{n-1}(x)$ are continuous in $[a, a+h]$, therefore $\phi(x)$ is also continuous in $[a, a+h]$,

(*ii*) $\phi'(x)$ exists and $= \dfrac{(a+h-x)^{n-1}}{(n-1)!} [f^n(x) - K]$

(*iii*) Also $\phi(a) = \phi(a+h)$. [By (2)]

Hence $\phi(x)$ satisfies all the conditions of Rolle's theorem, and therefore, there exists at least one number $\theta (0 < \theta < 1)$, such that $\phi'(a+\theta h) = 0$ *i.e.*, $K = f^n(a+\theta h)\ (0 < \theta < 1)$

Substituting this value of K in (2), we get (1).

Cor. 1. Taking $n = 1$ in (1), *Taylor's theorem reduces to Lagrange's Mean-value theorem.*

Cor. 2. Putting $a = 0$ and $h = x$ in (1), we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^n}{n!} f^n(\theta x). \quad \dots(3)$$

which is known as *Maclaurin's theorem with Lagrange's form of remainder.*

*Named after an English mathematician, *Brooke Taylor* (1685–1731).

Example 4.18. *Find the Maclaurin's theorem with Lagrange's form of remainder for $f(x) = \cos x$.* (*J.N.T.U., 2003*)

Solution. $f^n(x) = \frac{d^n}{dx^n}(\cos x) = \cos\left(\frac{n\pi}{2} + x\right)$ so that $f^n_{(0)} = \cos(n\pi/2)$

Thus $f(0) = 1$,

$$f^{2n}(0) = \cos(2n\pi/2) = (-1)^n$$

$$f^{2n+1}(0) = \cos[(2n+1)\pi/2] = 0$$

Substituting these values in the Maclaurin's theorem with Lagrange's form of remainder *i.e.*,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + ... + \frac{x^{2n}}{(2n)!}f^{2n}(0) + \frac{x^{2n+1}}{(2n+1)!}f^{2n+1}(\theta x)$$

We get
$$\cos x = 1 + 0 + \frac{x^2}{2!}(-1) + 0 + ... + \frac{x^{2n}}{(2n)!}(-1)^n + \frac{x^{2n+1}}{(2n+1)!}(-1)^n(-1)\cos(\theta x)$$

i.e.,
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ... + (-1)^n\frac{x^{2n}}{(2n)!} + \frac{(-1)^{n+1}x^{2n+1}}{(2n+1)!}\cos(\theta x)$$

Example 4.19. *If $f(x) = \log(1 + x)$, $x > 0$, using Maclaurin's theorem, show that for $0 < \theta < 1$,*

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3(1 + \theta x)^3}.$$

Deduce that $\log(1 + x) < x - \frac{x^2}{2} + \frac{x^3}{3}$ for $x > 0$. (*J.N.T.U., 2005*)

Solution. By Maclaurin's theorem with remainder R_3, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(\theta x) \qquad ...(i)$$

Here $f(x) = \log(1 + x)$, $\qquad f(0) = 0$

$\therefore$ $f'(x) = \frac{1}{1+x}$, $\qquad f'(0) = 1$

$f''(x) = \frac{-1}{(1+x)^2}$, $\qquad f''(0) = -1$

and $f'''(x) = \frac{2}{(1+x)^3}$, $\qquad f'''(\theta x) = \frac{2}{(1+\theta x)^3}$

Substituting in (*i*), we get $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3}$ ...(*ii*)

Since $x > 0$ and $\theta > 0$, $\theta x > 0$

or $(1 + \theta x)^3 > 1$ *i.e.*, $\frac{1}{(1+\theta x)^3} < 1$

$$\therefore \quad x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3} < x - \frac{x^2}{2} + \frac{x^3}{3}$$

Hence $\log(1 + x) < x - \frac{x^2}{2} + \frac{x^3}{3}$ [By (*ii*)]

PROBLEMS 4.4

1. Verify Rolle's theorem for (*i*) $f(x) = (x + 2)^3(x - 3)^4$ in $(-2, 3)$.
(*ii*) $y = e^x(\sin x - \cos x)$ in $(\pi/4, 5\pi/4)$. (*iii*) $f(x) = x(x + 3)e^{-1/2x}$ in $(-3, 0)$.
(*iv*) $f(x) = \log\left\{\frac{x^2 + ab}{x(a+b)}\right\}$ in (a, b). (*V.T.U., 2005*)

2. Using Rolle's theorem for $f(x) = x^{2m-1}(a-x)^{2n}$, find the value of x between o and a where $f'(x) = 0$.
3. Verify Lagrange's Mean value theorem for the following functions and find the appropriate value of c in each case :
   (i) $f(x) = (x-1)(x-2)(x-3)$ in $(0, 4)$ (V.T.U., 2009)
   (ii) $f(x) = \sin x$ in $[0, \pi]$ (Nagpur, 2008)
   (iii) $f(x) = \log_e x$ in $[1, e]$. (Burdwan, 2003)
   (iv) $f(x) = e^x$ in $[0, 1]$. (V.T.U., 2007)
4. By applying Mean value theorem to

$$f(x) = \log 2 \, . \sin \frac{\pi x}{2} + \log x, \text{ prove that } \frac{\pi}{2} \log 2 \, . \cos \frac{\pi x}{2} + \frac{1}{x} = 0 \text{ for some } x \text{ between 1 and 2.}$$

5. In the Mean value theorem : $f(x+h) = f(x) + h f'(x + \theta h)$,
   show that $\theta = 1/2$ for $f(x) = ax^2 + bx + c$ in $(0, 1)$.
6. If $f(h) = f(0) + h f'(0) + \dfrac{h^2}{2!} f''(\theta h)$, $0 < \theta < 1$, find θ when $h = 1$ and $f(x) = (1-x)^{5/2}$.
7. If x is positive, show that $x > \log(1+x) > x - \dfrac{1}{2}x^2$. (V.T.U., 2000)
8. If $f(x) = \sin^{-1} x$, $0 < a < b < 1$, use Mean value theorem to prove that

$$\frac{b-a}{\sqrt{(1-a^2)}} < \sin^{-1} b - \sin^{-1} a < \frac{b-a}{\sqrt{(1-b^2)}}$$

9. Prove that $\dfrac{b-a}{b} < \log\left(\dfrac{b}{a}\right) < \dfrac{b-a}{b}$ for $0 < a < b$.

   Hence show that $\dfrac{1}{4} < \log \dfrac{4}{3} < \dfrac{1}{3}$. (Mumbai, 2008)
10. Verify the result of Cauchy's mean value theorem for the functions
    (i) $\sin x$ and $\cos x$ in the interval $[a, b]$. (J.N.T.U., 2006 S)
    (ii) $\log_e x$ and $1/x$ in the interval $[1, e]$.
11. If $f(x)$ and $g(x)$ are respectively e^x and e^{-x}, prove that 'c' of Cauchy's mean value theorem is the arithmetic mean between a and b. (Mumbai, 2008)
12. Verify Maclaurin's theorem $f(x) = (1-x)^{5/2}$ with Lagrange's form of remainder upto 3 terms where $x = 1$.
13. Using Taylor's theorem, prove that

$$x - \frac{x^3}{6} < \sin x < x - \frac{x^3}{6} + \frac{x^5}{120}, \quad \text{for } x > 0.$$

4.4 EXPANSIONS OF FUNCTIONS

(1) Maclaurin's series. *If $f(x)$ can be expanded as an infinite series, then*

$$\mathbf{f(x) = f(0) + x\, f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots \infty} \qquad ...(1)$$

If $f(x)$ possess derivatives of all orders and the remainder R_n in (3) on page 145 tends to zero as $n \to \infty$, then the Maclaurin's theorem becomes the Maclaurin's series (1).

Example 4.20. *Using Maclaurin's series, expand tan x upto the term containing x^5.* (V.T.U., 2006)

Solution. Let $f(x) = \tan x$ $\qquad f(0) = 0$

$\therefore$ $f'(x) = \sec^2 x = 1 + \tan^2 x$ $\qquad f'(0) = 1$

$$f''(x) = 2\tan x \sec^2 x = 2 \tan x (1 + \tan^2 x)$$
$$= 2\tan x + 2\tan^3 x \qquad f''(0) = 0$$
$$f'''(0) = 2\sec^2 x + 6 \tan^2 x \sec^2 x$$
$$= 2(1 + \tan^2 x) + 6 \tan^2 x (1 + \tan^2 x)$$
$$= 2 + 8\tan^2 x + 6 \tan^4 x \qquad f'''(0) = 2$$
$$f^{iv}(0) = 16 \tan x \sec^2 x + 24 \tan^3 x \sec^2 x$$

$$= 16 \tan x (1 + \tan^2 x) + 24 \tan^3 x (1 + \tan^2 x)$$

$$= 16 \tan x + 40 \tan^3 x + 24 \tan^5 x \qquad f^{iv}(0) = 0$$

$$f^{v}(0) = 16 \sec^2 x + 120 \tan^2 x \sec^2 x + 120 \tan^4 x \sec^2 x. \qquad f^{v}(0) = 16$$

and so on.

Substituting the values of $f(0)$, $f'(0)$, etc. in the Maclaurin's series, we get

$$\tan x = 0 + x \cdot 1 + \frac{x^2}{2!} + \frac{x^3}{3!} \cdot 2 + \frac{x^4}{4!} \cdot 0 + \frac{x^5}{5!} \cdot 16 + \ldots = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$$

(2) Expansion by use of known series. When the expansion of a function is required only upto first few terms, it is often convenient to employ the following well-known series :

1. $\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots$
2. $\sinh \theta = \theta + \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \frac{\theta^7}{7!} + \ldots$
3. $\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots$
4. $\cosh \theta = 1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \frac{\theta^6}{6!} + \ldots$
5. $\tan \theta = \theta + \frac{\theta^3}{3} + \frac{2}{15}\theta^5 + \ldots$
6. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$
7. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$
8. $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$
9. $\log(1 - x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots\right)$
10. $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$

Example 4.21. *Expand $e^{\sin x}$ by Maclaurin's series or otherwise upto the term containing x^4.*
(Bhopal, 2009; V.T.U., 2011)

Solution. We have $e^{\sin x} = 1 + \sin x + \frac{(\sin x)^2}{2!} + \frac{(\sin x)^3}{3!} + \frac{(\sin x)^4}{4!} + \ldots$

$$= 1 + \left(x - \frac{x^3}{3!} + \ldots\right) + \frac{1}{2!}\left(x - \frac{x^3}{3!} + \ldots\right)^2 + \frac{1}{3!}\left(x - \frac{x^3}{3!} + \ldots\right)^3 + \frac{1}{4!}(x - \ldots)^4 + \ldots$$

$$= 1 + \left(x - \frac{x^3}{6} + \ldots\right) + \frac{1}{2}\left(x^2 - \frac{x^4}{3} + \ldots\right) + \frac{1}{6}\left(x^3 - \ldots\right) + \frac{1}{24}(x^4 + \ldots) + \ldots$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \ldots$$

Otherwise, let $f(x) = e^{\sin x}$ $\qquad f(0) = 1$

$\therefore$ $f'(x) = e^{\sin x} \cos x = f(x) \cdot \cos x$ $\qquad f'(0) = 1$

$f''(x) = f'(x) \cos x - f(x) \sin x,$ $\qquad f''(0) = 1$

$f'''(x) = f''(x) \cos x - 2f'(x) \sin x - f(x) \cos x,$ $\qquad f'''(0) = 0$

$f^{iv}(x) = f'''(x) \cos x - 3f''(x) \sin x - 3f'(x) \cos x + f(x) \sin x,$ $\qquad f^{iv}(0) = -3$

and so on.

Substituting the values of $f(0)$, $f'(0)$ etc., in the Maclaurin's series, we obtain

$$e^{\sin x} = 1 + x \cdot 1 + \frac{x^2}{2!} \cdot 1 + \frac{x^3}{3!} \cdot 0 + \frac{x^4}{4!} \cdot (-3) + \ldots$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \ldots$$

Example 4.22. *Expand log $(1 + \sin^2 x)$ in powers of x as far as the term in x^6.* (*Hissar, 2005 S*)

Solution. We have $\sin^2 x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)^2 = \left[x - \left(\frac{x^3}{6} - \frac{x^5}{120} + \dots\right)\right]^2$

$$= x^2 - 2x\left(\frac{x^3}{6} - \frac{x^5}{120} + \dots\right) + \left(\frac{x^3}{6} - \frac{x^5}{120} + \dots\right)^2$$

$$= x^2 - \frac{x^4}{3} + \frac{x^6}{60} + \frac{x^6}{36} + \dots = x^2 - \frac{x^4}{3} + \frac{2x^6}{45} + \dots = t, \text{ say.}$$

Now $\log(1 + \sin^2 x) = \log(1 + t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \dots$

Substituting the value of t, we get

$$\log(1 + \sin^2 x) = x^2 - \frac{x^4}{3} + \frac{2x^6}{45} + \dots - \frac{1}{2}\left(x^2 - \frac{x^4}{3} + \dots\right)^2 - \frac{1}{3}(x^2 - \dots)^3 - \dots$$

$$= x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \frac{1}{2}\left(x^4 - \frac{2x^6}{3} + \dots\right) + \frac{1}{3}(x^6 + \dots) + \dots$$

$$= x^2 - \frac{5}{6}x^4 + \frac{32}{45}x^6 + \dots$$

Obs. As it is very cumbersome to find the successive derivatives of $\log(1 + \sin^2 x)$, therefore the above method is preferable to Maclaurin's series method.

Example 4.23. *Expand $e^{a\sin^{-1} x}$ in ascending powers of x.*

Solution. Let $y = e^{a\sin^{-1} x}$. In Ex. 4.9, we have shown that

$(y)_0 = 1, (y_1)_0 = a, (y_2)_0 = a^2, (y_3)_0 = a(1 + a^2), (y_4)_0 = a^2(2^2 + a^2)$

and so on.

Substituting these values in the Maclaurin's series

$$y = (y)_0 + \frac{(y_1)_0}{1!}x + \frac{(y_2)_0}{2!}x^2 + \frac{(y_3)_0}{3!}x^3 + \frac{(y_4)_0}{4!}x^4 + \dots$$

we get
$$e^{a\sin^{-1} x} = 1 + ax + \frac{a^2}{2!}x^2 + \frac{a(1^2 + a^2)}{3!}x^3 + \frac{a^2(2^2 + a^2)}{4!}x^4 + \dots$$

(3) Taylor's series. *If $f(x + h)$ can be expanded as an infinite series, then*

$$f(x + h) = f(x) + h f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots \infty \qquad \dots(1)$$

If $f(x)$ possesses derivatives of all orders and the remainder R_n in (1) on page 147, tends to zero as $n \to \infty$, then the Taylor's theorem becomes the *Taylor's series* (1).

Cor. Replacing x by a and h by $(x - a)$ in (1), we get

$$f(x) = f(a) + (x - a) f'(a) + \frac{(x - a)^2}{2!}f''(a) + \frac{(x - a)^3}{3!}f'''(a) + \dots \infty$$

Taking $a = 0$, we get *Maclaurin's series.*

Example 4.24. *Expand $\log_e x$ in powers of $(x - 1)$ and hence evaluate $\log_e 1{\cdot}1$ correct to 4 decimal places.* (*Bhopal, 2007 ; Kurukshetra 2006*)

Solution. Let $f(x) = \log_e x \qquad f(1) = 0$

$\therefore \quad f'(x) = \frac{1}{x}, \qquad f'(1) = 1$

$f''(x) = -\frac{1}{x^2}, \qquad f''(1) = -1$

$$f'''(x) = \frac{2}{x^3}, \qquad f'''(1) = 2$$

$$f^{iv}(x) = -\frac{6}{x^4}, \qquad f^{iv}(0) = -6$$

etc. etc.

Substituting these values in the Taylor's series

$$f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f'''(1) + \ldots,$$

we get
$$\log x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots$$

Now putting $x = 1.1$, so that $x - 1 = 0.1$, we have

$$\log(1.1) = 1.1 - \tfrac{1}{2}(0.1)^2 + \tfrac{1}{3}(0.1)^3 - \tfrac{1}{4}(0.1)^4 + \ldots$$
$$= 0.1 - 0.005 + 0.0003 - 0.00002 + \ldots = 0.0953.$$

Example 4.25. *Use Taylor's series, to prove that*

$$\tan^{-1}(x+h) = \tan^{-1}x + (h\sin z)\cdot\frac{\sin z}{1} - (h\sin z)^2\cdot\frac{\sin 2z}{2} + (h\sin z)^3\cdot\frac{\sin 3z}{3} - \ldots$$

where $z = \cot^{-1}x$. *(Bhillai, 2005)*

Solution. We have $\cot z = x$ ...(*i*)

$\therefore$ $-\operatorname{cosec}^2 z.\ dz/dx = 1$ or $dz/dx = -\sin^2 z$ ...(*ii*)

Now let $f(x+h) = \tan^{-1}(x+h)$, so that $f(x) = \tan^{-1}x$

$$\therefore \quad f'(x) = \frac{1}{1+x^2} = \frac{1}{1+\cot^2 z} = \sin^2 z \qquad \text{[By } (i)]$$

$$f''(x) = 2\sin z\cos z\,\frac{dz}{dx} = \sin 2z\cdot(-\sin^2 z) \qquad \text{[By } (ii)]$$

$$f'''(x) = -[2\cos 2z\cdot\sin^2 z + \sin 2z\cdot 2\sin z\cos z]\,\frac{dz}{dx}$$
$$= -2\sin z\,[\sin z\cos 2z + \sin 2z\cos z]\,(-\sin^2 z) = 2\sin^3 z\sin 3z$$

and so on.

Substituting these values in the Taylor's series

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots,$$

we get the required result.

PROBLEMS 4.5

Using Maclaurin's series, expand the following functions :

1. $\log(1+x)$. Hence deduce that $\log\sqrt{\dfrac{1+x}{1-x}} = x + \dfrac{x^3}{3} + \dfrac{x^5}{5} + \ldots$
2. $\sin x$ *(P.T.U., 2005)*
3. $\sqrt{(1+\sin 2x)}$ *(V.T.U., 2010)*
4. $\sin^{-1}x$ *(Mumbai, 2007)*
5. $\tan^{-1}x$
6. $\log\sec x$ *(Mumbai, 2009 S ; V.T.U., 2009)*

Prove that :

7. $\sec x = 1 + \dfrac{x^2}{2!} + \dfrac{5x^4}{4!} + \ldots$
8. $x\operatorname{cosec} x = 1 + \dfrac{x^2}{6} + \dfrac{7x^4}{360} + \ldots$ *(Mumbai, 2007)*
9. $\sin^{-1}\dfrac{2x}{1+x^2} = 2\left\{x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \ldots\right\}$
10. $\tan^{-1}\dfrac{\sqrt{(1+x^2)}-1}{x} = \dfrac{1}{2}\left(x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \ldots\right)$

11. $\sin^{-1}(3x-4x^3) = 3\left(x+\frac{x^3}{3}+\frac{3x^5}{40}+\dots\right)$

12. $e^x \cos x = 1 + x - \frac{2x^3}{3!} - \frac{2^2 x^4}{4!} \dots$ (Raipur, 2005)

13. $e^{x \sin x} = 1 + x^2 + \frac{x^4}{3} + \frac{x^6}{120} + \dots$ (Kurukshetra, 2009)

14. $e^{\cos^{-1} x} = e^{\pi/2}\left(1-x+\frac{x^2}{3}-\frac{x^3}{3}+\dots\right)$ (Mumbai, 2008)

15. $\log \frac{\sin x}{x} = -\left(\frac{x^2}{6}+\frac{x^4}{180}+\frac{x^6}{2835}+\dots\right)$

16. $\log(1+\sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \dots$ (S.V.T.U. 2009 ; J.N.T.U., 2006 S)

17. $\sqrt{(1+\sin x)} = 1 + \frac{x}{2} - \frac{x^2}{2} - \frac{x^3}{48} + \frac{x^4}{384} + \dots$ (V.T.U., 2006)

18. $\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \dots$ (Bhopal, 2008)

19. $\frac{e^x}{e^x+1} = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \dots$ (Bhopal, 2008 S)

20. $\frac{x}{2}\left(\frac{e^x+1}{e^x-1}\right) = 1 + \frac{1}{6}\cdot\frac{x^2}{2!} - \frac{1}{30}\cdot\frac{x^4}{4!} + \dots$ (Mumbai, 2007)

21. $\sin x \cosh x = x + \frac{x^3}{3} - \frac{x^5}{30} + \dots$

By forming a differential equation, show that

22. $(\sin^{-1} x)^2 = 2\frac{x^2}{2!} + 2\cdot 2^2 \frac{x^4}{4!} + 2\cdot 2^2\cdot 4^2 \cdot \frac{x^6}{6!} + \dots$

23. $\log[1+\sqrt{(1+x^2)}] = x - \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \dots$

24. If $y = \sin(m \sin^{-1} x)$, show that $(1-x^2)y_2 - xy_1 + m^2 y = 0$
Hence expand $\sin m\theta$ in powers of $\sin\theta$. (S.V.T.U., 2008)

25. Using Taylor's theorem, express the polynomial $2x^3 + 7x^2 + x - 6$ in powers of $(x-1)$ (Burdwan, 2003)

26. Expand (i) e^x (Cochin., 2005) (ii) $\tan^{-1} x$, in powers of $(x-1)$ upto four terms.

27. Expand $\sin x$ in powers of $(x-\pi/2)$. Hence find the value of $\sin 91°$ correct to 4 decimal places. (Rohtak, 2003)

28. Prove that $\log \sin x = \log \sin a + (x-a)\cot a - \frac{1}{2}(x-a)^2 \operatorname{cosec}^2 a + \dots$

29. Find the Taylor's series expansion for $\log \cos x$ about the point $\pi/3$.

30. Compute to four decimal places, the value of $\cos 32°$, by the use of Taylor's series. (Kurukshetra, 2006)

31. Calculate approximately (i) $\log_{10} 404$, given $\log 4 = 0.6021$. (Rohtak, 2005 S)
(ii) $(1.04)^{3.01}$ (Mumbai, 2007)

4.5 INDETERMINATE FORMS

In general $\underset{x\to a}{\text{Lt}}[f(x)/\phi(x)] = \underset{x\to a}{\text{Lt}} f(x) / \underset{x\to a}{\text{Lt}} \phi(x)$. But when $\underset{x\to a}{\text{Lt}} f(x)$ and $\underset{x\to a}{\text{Lt}} \phi(x)$ are both zero, then the quotient reduces to the indeterminate form 0/0. This does not imply that $\underset{x\to a}{\text{Lt}}[f(x)/\phi(x)]$ is meaningless or it does not exist. In fact, in many cases, it has a finite value. We shall now, study the methods of evaluating the limits in such and similar other cases :

(1) Form 0/0. *If $f(a) = \phi(a) = 0$, then*

$$\underset{x\to a}{\text{Lt}} \frac{f(x)}{\phi(x)} = \underset{x\to a}{\text{Lt}} \frac{f'(x)}{\phi'(x)}$$

By Taylor's series,

$$\underset{x\to a}{\text{Lt}} \frac{f(x)}{\phi(x)} = \underset{x\to a}{\text{Lt}} \frac{f(a) + (x-a)f'(a) + \frac{1}{2!}(x-a)^2 f''(a) + \dots}{\phi(a) + (x-a)\phi'(a) + \frac{1}{2!}(x-a)^2 \phi''(a) + \dots}$$

$$= \underset{x \to a}{\text{Lt}} \frac{f'(a) + \frac{1}{2}(x-a) f'(a) + \ldots}{\phi'(a) + \frac{1}{2}(x-a)\phi'(a) + \ldots}$$

$$= \frac{f'(a)}{\phi'(a)} = \underset{x \to a}{\text{Lt}} \frac{f'(x)}{\phi'(x)} \quad \ldots(1)$$

This is known as *L'Hospital's rule.*

In general, if

$$f(a) = f'(a) = f''(a) = \ldots = f^{n-1}(a) = 0, \text{ but } f^n(a) \neq 0,$$

and
$$\phi(a) = \phi'(a) = \phi''(a) = \ldots = \phi^{n-1}(a) = 0, \text{ but } \phi^n(a) \neq 0,$$

then from (1),

$$\underset{x \to a}{\textbf{Lt}} \frac{\mathbf{f(x)}}{\boldsymbol{\phi}\mathbf{(x)}} = \frac{\mathbf{f^n(a)}}{\boldsymbol{\phi}^\mathbf{n}\mathbf{(a)}} = \underset{x \to a}{\textbf{Lt}} \frac{\mathbf{f^n(x)}}{\boldsymbol{\phi}^\mathbf{n}\mathbf{(x)}}$$

[Rule to evaluate Lt $[f(x)/\phi(x)]$ in 0/0 form :
Differentiating the numerator and denominator separately as many times as would be necessary to arrive a determinate form].

Example 4.26. *Evaluate (i)* $\underset{x \to 0}{Lt} \dfrac{xe^x - \log(1+x)}{x^2}$. *(V.T.U., 2004 ; Osmania, 2000 S)*

(ii) $\underset{x \to 1}{Lt} \dfrac{x^x - x}{x - 1 - \log x}$

Solution. *(i)*

$$\underset{x \to 0}{\text{Lt}} \frac{xe^x - \log(1+x)}{x^2} \quad \left(\text{form } \frac{0}{0}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{(xe^x + e^x \cdot 1) - 1/(1+x)}{2x} \quad \left(\text{form } \frac{0}{0}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{xe^x + e^x + e^x + 1/(1+x)^2}{2} = \frac{0+1+1+1}{2} = 1\frac{1}{2}.$$

(ii)

$$\underset{x \to 1}{\text{Lt}} \frac{x^x - x}{x - 1 - \log x} \quad \left(\text{form } \frac{0}{0}\right)$$

$$= \underset{x \to 1}{\text{Lt}} \frac{d(x^x)/dx - 1}{1 - 0 - 1/x}$$

$$= \underset{x \to 1}{\text{Lt}} \frac{x^x(1 + \log x) - 1}{1 - 1/x} \quad \left(\text{form } \frac{0}{0}\right)$$

Let $y = x^x$ so that

$\log y = x \log x$

$\therefore \quad \dfrac{1}{y}\dfrac{dy}{dx} = x \cdot \dfrac{1}{x} + 1 \,.\, \log x$

or $\quad \dfrac{d}{dx}(x^x) = x^x(1 + \log x) \quad \ldots(i)$

$$= \underset{x \to 1}{\text{Lt}} \frac{d(x^x)/dx \,.\, (1 + \log x) + x^x(1/x) - 0}{1/x^2}$$

$$= \underset{x \to 1}{\text{Lt}} \frac{x^x(1 + \log x)^2 + x^x(1/x)}{x^{-2}} \quad [\text{By } (i)]$$

$$= \frac{1(1+0)^2 + 1 \cdot 1}{1} = 2.$$

Example 4.27. *Find the values of a and b such that* $\underset{x \to 0}{Lt} \dfrac{x(a + b\cos x) - c\sin x}{x^5} = 1.$ *(Mumbai., 2007)*

Solution. $$\underset{x \to 0}{\text{Lt}} \frac{x(a + b\cos x) - c\sin x}{x^5} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{a + b\cos x - bx\sin x - c\cos x}{5x^4} \qquad ...(i)$$

As the denominator is 0 for $x = 0$, (i) will tend to a finite limit if and only if the numerator also becomes 0 for $x = 0$. This requires $a + b - c = 0$ ...(ii)

With this condition, (i) assumes the form 0/0.

$$\therefore \qquad (i) = \underset{x \to 0}{\text{Lt}} \frac{-b\sin x - b(\sin x + x\cos x) + c\sin x}{20x^3}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{(c - 2b)\sin x - bx\cos x}{20x^3} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{(c - 2b)\cos x - b(\cos x - x\sin x)}{60x^2} \qquad ...(iii)$$

$$= \frac{c - 2b - b}{0} = \frac{c - 3b}{0} = 1 \qquad \text{(Given)}$$

$$\therefore \qquad c - 3b = 0 \quad i.e., \quad c = 3b.$$

Now (iii) $$= \underset{x \to 0}{\text{Lt}} \frac{b\cos x - b\cos x + bx\sin x}{60x^2}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{b\sin x}{60x} = \frac{b}{60} \underset{x \to 0}{\text{Lt}} \left(\frac{\sin x}{x}\right) = \frac{b}{60} = 1.$$

i.e., $b = 60$, and $\therefore \quad c = 180$.

Form (ii), $a = 120$.

(2) Form ∞/∞. It can be shown that L'Hospital's rule can also be applied to this case by differentiating the numerator and denominator separately as many times as would be necessary.

Example 4.28. *Evaluate* $\underset{x \to 0}{Lt} \dfrac{\log x}{\cot x}$.

Solution. $$\underset{x \to 0}{\text{Lt}} \frac{\log x}{\cot x} = \underset{x \to 0}{\text{Lt}} \frac{1/x}{-\operatorname{cosec}^2 x} = -\underset{x \to 0}{\text{Lt}} \frac{\sin^2 x}{x} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= -\underset{x \to 0}{\text{Lt}} \frac{2\sin x\cos x}{1} = 0$$

Obs. Use of known series and standard limits. In many cases, it would be found more convenient to use expansions of known functions and standard limits for evaluating the indeterminate forms. For this purpose, remember the series of § 4.4 (2) and the following limits :

$$\underset{x \to 0}{\text{Lt}} \frac{\sin x}{x} = 1, \qquad \underset{x \to 0}{\text{Lt}} (1 + x)^{1/x} = e$$

Example 4.29. *Evaluate* $\underset{x \to 0}{Lt} \dfrac{e^x \sin x - x - x^2}{x^2 + x\log(1 - x)}$.

Solution. Using the expansions of e^x, $\sin x$ and $\log(1 - x)$, we get

$$\underset{x \to 0}{\text{Lt}} \frac{e^x \sin x - x - x^2}{x^2 + x\log(1 - x)}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + ...\right)\left(x - \frac{1}{3!}x^3 + ...\right) - x - x^2}{x^2 + x\left(-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - ...\right)}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\left(x + x^2 + \frac{1}{3}x^3 - 0 \cdot x^4 + \ldots\right) - x - x^2}{x^2 - \left(x^2 + \frac{1}{2}x^3 + \frac{1}{3}x^4 + \ldots\right)} = \underset{x \to 0}{\text{Lt}} \frac{\frac{1}{3}x^3 - 0 \cdot x^4 + \ldots}{-\frac{1}{2}x^3 - \frac{1}{3}x^4 - \ldots} = \underset{x \to 0}{\text{Lt}} \frac{\frac{1}{3} + \ldots}{-\frac{1}{2} - \frac{1}{3}x - \ldots} = -\frac{2}{3}.$$

Example 4.30. *Evaluate* $\underset{x \to 0}{Lt} \dfrac{(1+x)^{1/x} - e}{x}$.

Solution. Let $y = (1 + x)^{1/x}$

$\therefore$ $$\log y = \frac{1}{x} \log (1 + x) = \frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\right) = 1 - \frac{x}{2} + \frac{x^2}{3} - \ldots$$

or $$y = e^{1 - \frac{1}{2}x + \frac{1}{3}x^2 - \ldots} = e \cdot e^{-\frac{1}{2}x + \frac{1}{3}x^2 - \ldots}$$

$$= e\left[1 + \left(-\frac{1}{2}x + \frac{1}{3}x^2 - \ldots\right) + \frac{1}{2!}\left(-\frac{1}{2}x + \frac{1}{3}x^2 - \ldots\right)^2 + \ldots\right] = e\left(1 - \frac{x}{2} + \frac{11}{24}x^2 + \ldots\right)$$

$\therefore$ $$\underset{x \to 0}{\text{Lt}} \frac{(1+x)^{1/x} - e}{x} = \underset{x \to 0}{\text{Lt}} \frac{e\left(1 - \frac{x}{2} + \frac{11}{24}x^2 + \ldots\right) - e}{x}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{e\left(-\frac{1}{2}x + \frac{11}{24}x^2 + \ldots\right)}{x} = \underset{x \to 0}{\text{Lt}} \left(\frac{-e}{2} + \frac{11}{24}ex + \ldots\right) = -\frac{e}{2}.$$

PROBLEMS 4.6

Evaluate the following limits :

1. $\underset{x \to 0}{\text{Lt}} \dfrac{a^x - b^x}{x}$ *(V.T.U., 2008)*
2. $\underset{x \to 0}{\text{Lt}} \dfrac{x \cos x - \sin x}{x^2 \sin x}$ *(J.N.T.U., 2006 S)*
3. $\underset{\theta \to 0}{\text{Lt}} \dfrac{\theta - \sin \theta}{\sin \theta (1 - \cos \theta)}$
4. $\underset{x \to \pi/2}{\text{Lt}} \dfrac{a^{\sin x} - a}{\log_e \sin x}$
5. $\underset{x \to 0}{\text{Lt}} \dfrac{2 \sin x - \sin 2x}{x^3}$
6. $\underset{x \to 0}{\text{Lt}} \dfrac{\sin x \sin^{-1} x - x^2}{x^6}$
7. $\underset{x \to 0}{\text{Lt}} \dfrac{x \cos x - \log (1 + x)}{x^2}$
8. $\underset{x \to 0}{\text{Lt}} \dfrac{\log \sec x - \frac{1}{2} x^2}{x^4}$
9. $\underset{x \to 0}{\text{Lt}} \dfrac{x e^x - \log (1 + x)}{\cosh x - \cos x}$
10. $\underset{x \to 0}{\text{Lt}} \dfrac{\cos x - \log (1 + x) - 1 + x}{\sin^2 x}$
11. $\underset{x \to 0}{\text{Lt}} \dfrac{e^x + 2 \sin x - e^{-x} - 4x}{x^5}$
12. $\underset{x \to 0}{\text{Lt}} \dfrac{\log (x - a)}{\log (e^x - e^a)}$
13. $\underset{x \to 0}{\text{Lt}} \dfrac{\tan x - x}{x^2 \tan x}$
14. $\underset{x \to 0}{\text{Lt}} \dfrac{e^x - e^{-x} - 2x}{x - \sin x}$
15. $\underset{x \to 0}{\text{Lt}} \dfrac{e^x - e^{\sin x}}{x - \sin x}$
16. $\underset{x \to 0}{\text{Lt}} \dfrac{\sin \{(\log (1 + x)\}}{\log (1 + \sin x)}$
17. $\underset{x \to 0}{\text{Lt}} \dfrac{e^x + \sin x - 1}{\log (1 + x)}$
18. $\underset{x \to 0}{\text{Lt}} \dfrac{(1 + x)^{1/x} - e + \frac{1}{2} ex}{x^2}$
19. If $\underset{x \to 0}{\text{Lt}} \dfrac{\sin 2x + a \sin x}{x^3}$ is finite, find the value of a and the limit. *(Nagpur, 2009)*
20. Find a, b if $\underset{x \to 0}{\text{Lt}} \dfrac{a \sinh x + b \sin x}{x^3} = \dfrac{5}{3}$. *(Mumbai, 2009)*
21. Find a, b, c so that $\underset{x \to 0}{\text{Lt}} \dfrac{ae^x - b \cos x + ce^{-x}}{x \sin x} = 2$. *(Mumbai, 2008)*

(3) Forms reducible to 0/0 form. Each of the following indeterminate forms can be easily reduced to the form 0/0 (or ∞/∞) by suitable transformation and then the limits can be found as usual.

I. Form $0 \times \infty$. If $\underset{x \to 0}{Lt} f(x) = 0$ and $\underset{x \to \infty}{Lt} \phi(x) = \infty$, then

$$\underset{x \to a}{\text{Lt}} [f(x) \cdot \phi(x)] \text{ assumes the form } 0 \times \infty.$$

To evaluate this limit, we write

$$f(x) \cdot \phi(x) = f(x)/[1/\phi(x)] \text{ to take the form } 0/0.$$
$$= \phi(x)/[1/f(x)] \text{ to take the form } \infty/\infty.$$

Example 4.31. *Evaluate* $\underset{x \to 0}{Lt} (\tan x \log x)$ *(V.T.U., 2009)*

Solution.
$$\underset{x \to 0}{\text{Lt}} (\tan x \log x) = \underset{x \to 0}{\text{Lt}} \left(\frac{\log x}{\cot x}\right) \qquad \left(\text{form } \frac{\infty}{\infty}\right)$$
$$= \underset{x \to 0}{\text{Lt}} \left(\frac{1/x}{-\operatorname{cosec}^2 x}\right) = -\underset{x \to 0}{\text{Lt}} \left(\frac{\sin^2 x}{x}\right) \qquad \left(\text{form } \frac{0}{0}\right)$$
$$= \underset{x \to 0}{\text{Lt}} \frac{2 \sin x \cos x}{1} = 0.$$

II. Form $\infty - \infty$. If $\underset{x \to a}{Lt} f(x) = \infty = \underset{x \to a}{Lt} \phi(x)$, then $\underset{x \to a}{\text{Lt}} [f(x) - \phi(x)]$ assumes the form $\infty - \infty$.

It can be reduced to the from 0/0 by writing

$$f(x) - \phi(x) = \left[\frac{1}{\phi(x)} - \frac{1}{f(x)}\right] \Big/ \frac{1}{f(x)\,\phi(x)}$$

Example 4.32. *Evaluate* $\underset{x \to 0}{Lt} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$.

Solution.
$$\underset{x \to 0}{\text{Lt}} \left(\frac{1}{\sin x} - \frac{1}{x}\right) = \underset{x \to 0}{\text{Lt}} \frac{x - \sin x}{x \sin x} = \underset{x \to 0}{\text{Lt}} \frac{1 - \cos x}{x \cos x + \sin x} \qquad \left(\text{form } \frac{0}{0}\right)$$
$$= \underset{x \to 0}{\text{Lt}} \frac{\sin x}{x(-\sin x) + \cos x + \cos x} = \frac{0}{0 + 1 + 1} = 0.$$

III. Forms 0^0, 1^∞, ∞^0. If $y = \underset{x \to a}{\text{Lt}} [f(x)]^{\phi(x)}$ assumes one of these forms, then $\log y = \underset{x \to a}{\text{Lt}} \phi(x) \log f(x)$ takes the form $0 \times \infty$, which can be evaluated by the method given in I above. If $\log y = l$, then $y = e^l$.

Example 4.33. *Evaluate (i)* $\underset{x \to \pi/2}{Lt} (\sin x)^{\tan x}$ *(ii)* $\underset{x \to 0}{Lt} \left(\frac{a^x + b^x + c^x}{3}\right)^{1/x}$ *(V.T.U., 2011)*

(iii) $\underset{x \to 0}{Lt} \left(\frac{\tan x}{3}\right)^{1/x^2}$

Solution. (*i*) Let
$$y = \underset{x \to \pi/2}{\text{Lt}} (\sin x)^{\tan x}.$$

$\therefore$
$$\log y = \underset{x \to \pi/2}{\text{Lt}} \tan x \log \sin x = \underset{x \to \pi/2}{\text{Lt}} \frac{\log \sin x}{\cot x} \qquad \left(\text{form } \frac{0}{0}\right)$$
$$= \underset{x \to \pi/2}{\text{Lt}} \frac{(1/\sin x) \cos x}{-\operatorname{cosec}^2 x} = -\underset{x \to \pi/2}{\text{Lt}} (\sin x \cos x) = 0$$

Hence
$$y = e^0 = 1.$$

(*ii*) Let $$y = \underset{x \to 0}{\text{Lt}} \left(\frac{a^x + b^x + c^x}{3} \right)^{1/x}$$

so that $$\log y = \underset{x \to 0}{\text{Lt}} \frac{\log (a^x + b^x + c^x) - \log 3}{x} \qquad \text{form} \left(\frac{0}{0}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{(a^x + b^x + c^x)^{-1} (a^x \log a + b^x \log b + c^x \log c)}{1}$$

$$= (1 + 1 + 1)^{-1} (\log a + \log b + \log c) = \frac{1}{3} \log (abc) = \log (abc)^{1/3}.$$

$\therefore$ $$y = (abc)^{1/3}$$

(*iii*) $$\underset{x \to 0}{\text{Lt}} \left(\frac{\tan x}{x} \right)^{1/x^2} = \underset{x \to 0}{\text{Lt}} \left(\frac{x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots}{x} \right)^{1/x^2}$$

$$= \underset{x \to 0}{\text{Lt}} \left(1 + \frac{x^2}{3} + \frac{2}{15}x^4 + \ldots \right)^{1/x^2}$$

$$= \underset{x \to 0}{\text{Lt}} (1 + tx^2)^{1/x^2} \qquad \text{where } t = \frac{1}{3} + \frac{2}{15}x^2 + \ldots$$

$$= \underset{x \to 0}{\text{Lt}} [(1 + tx^2)^{1/tx^2}]^t = \underset{x \to 0}{\text{Lt}} e^t = e^{1/3}. \qquad \left[\because \underset{z \to 0}{\text{Lt}} (1 + z)^{1/z} = e\right]$$

PROBLEMS 4.7

Evaluate the following limits :

1. $\underset{x \to 0}{\text{Lt}} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x} \right)$

2. $\underset{x \to 0}{\text{Lt}} \left(\frac{1}{x} - \frac{1}{e^x - 1} \right)$ (*Burdwan, 2003*)

3. $\underset{x \to 1}{\text{Lt}} (2x \tan x - \pi \sec x)$ (*V.T.U., 2008*)

4. $\underset{x \to 0}{\text{Lt}} \left(\frac{\cot x - 1/x}{x} \right)$

5. $\underset{x \to 0}{\text{Lt}} \left(\frac{1}{x^2} - \cot^2 x \right)$

6. $\underset{x \to 1}{\text{Lt}} (x)^{1/(1-x)}$

7. $\underset{x \to 0}{\text{Lt}} (a^x + x)^{1/x}$ (*V.T.U., 2007*)

8. $\underset{x \to \pi/2}{\text{Lt}} (\sec x)^{\cot x}$

9. $\underset{x \to 0}{\text{Lt}} (1 + \sin x)^{\cot x}$

10. $\underset{x \to 0}{\text{Lt}} (\cos x)^{1/x^2}$

11. $\underset{x \to \pi/2}{\text{Lt}} (\tan x)^{\tan 2x}$ (*V.T.U., 2004*)

12. $\underset{x \to 0}{\text{Lt}} (\cot x)^{1/\log x}$

13. $\underset{x \to \pi/2}{\text{Lt}} (\cos x)^{\frac{\pi}{2} - x}$

14. $\underset{x \to 0}{\text{Lt}} \left(\frac{\sin x}{x} \right)^{1/x}$

15. $\underset{x \to 0}{\text{Lt}} \left(\frac{\sin x}{x} \right)^{1/x^2}$ (*V.T.U., 2001*)

16. $\underset{x \to 1}{\text{Lt}} (1 - x^2)^{1/\log (1-x)}$

17. $\underset{x \to a}{\text{Lt}} \left(2 - \frac{x}{a} \right)^{\tan (\pi x/2a)}$ (*V.T.U., 2010 ; Nagpur, 2009*)

18. $\underset{x \to 0}{\text{Lt}} \left\{ \frac{2 (\cosh x - 1)^{1/x^2}}{x^2} \right\}$

19. $\underset{x \to 2}{\text{Lt}} \left\{ \frac{1}{x - 2} - \frac{1}{\log (x - 1)} \right\}$ (*Osmania, 2000 S*)

20. $\underset{x \to 0}{\text{Lt}} \left(\frac{1^x + 2^x + 3^x}{3} \right)^{1/x}.$ (*V.T.U., 2008*)

4.6 TANGENTS AND NORMALS – CARTESIAN CURVES

(1) Equation of the tangent *at the point* (x, y) *of the curve* $y = f(x)$ *is*

$$\mathbf{Y - y = \frac{dy}{dx}(X - x).}$$

The equation of any line through $P(x, y)$ is

$$Y - y = m(X - x)$$

where X, Y are the current coordinates of any point on the line (Fig. 4.3).

If this line is the tangent PT, then

$$m = \tan\psi = dy/dx$$

Hence the equation of the tangent at (x, y) is

$$Y - y = \frac{dy}{dx}(X - x) \qquad ...(2)$$

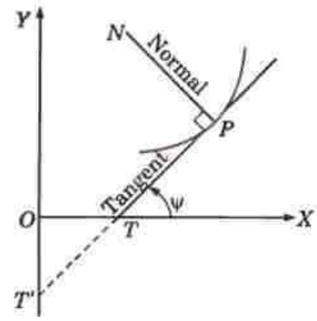


Fig. 4.3

Cor. Intercepts. Putting $Y = 0$ in (2)

$$-y = \frac{dy}{dx}(X - x) \quad \text{or} \quad X = x - y/\frac{dy}{dx}$$

$\therefore$ *Intercept which the tangent cuts off from x-axis* $(= OT) = \mathbf{x - y\, dx/dy}$

Similarly putting $X = 0$ *in (2), we see that*

the intercept which the tangent cuts off from the y-axis

$$(= OT') = \mathbf{y - x\frac{dy}{dx}}$$

(2) Equation of the normal *at the point* (x, y) *of the curve* $y = f(x)$ is

$$\mathbf{Y - y = -\frac{dx}{dy}(X - x)}$$

A normal *to the curve* $y = f(x)$ *at* $P(x, y)$ *is a line through P perpendicular to the tangent there at.*

$\therefore$ Its equation is $Y - y = m'(X - x)$

where $$m' \cdot dy/dx = -1 \quad \text{or} \quad m' = -1/\frac{dy}{dx} = -dx/dy$$

Hence the equation of the normal at (x, y) is $Y - y = -\dfrac{dx}{dy}(X - x)$.

Example 4.34. *Find the equation of the tangent at any point (x, y) to the curve $x^{2/3} + y^{2/3} = a^{2/3}$. Show that the portion of the tangent intercepted between the axes is of constant length.*

Solution. Equation of the curve is $x^{2/3} + y^{2/3} = a^{2/3}$. ...(i)

Differentiating (i) w.r.t. x,

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

$\therefore$ Slope of the tangent at $(x, y) = \dfrac{dy}{dx} = -\left(\dfrac{y}{x}\right)^{1/3}$

$\therefore$ Equation of the tangent at (x, y) is

$$Y - y = -(y/x)^{1/3}(X - x) \qquad ...(ii)$$

Put $Y = 0$ in (ii). Then $\quad X = x + x^{1/3} . y^{2/3}$

i.e., Intercept on x-axis $\quad = (x^{2/3} + y^{2/3})x^{1/3} = a^{2/3} . x^{1/3}$ [By (i)]

Put $X = 0$ in (ii). Then $\quad Y = y + y^{1/3} . x^{2/3}$

i.e., Intercept on y-axis $\quad = (x^{2/3} + y^{2/3})\, y^{1/3} = a^{2/3} . y^{1/3}$ [By (i)]

Thus the portion of the tangent intercepted between the axes

$$= \sqrt{[(\text{Intercept on } x\text{-axis})^2 + (\text{Intercept on } y\text{-axis})^2]}$$

$$= \sqrt{[(a^{2/3} \cdot x^{1/3})^2 + (a^{2/3} \cdot y^{1/3})^2]}$$

$$= \sqrt{[a^{4/3}(x^{2/3} + y^{2/3})]} = a^{2/3}\sqrt{(a)^{2/3}} \quad \text{[By } (i)]$$

$= a$, which is a constant length.

Example 4.35. *Show that the conditions for the line $x \cos \alpha + y \sin \alpha = p$ to touch the curve $(x/a)^m + (y/b)^m = 1$ is $(a \cos \alpha)^{m/(m-1)} + (b \sin \alpha)^{m/(m-1)} = p^{m/(m-1)}$.*

Solution. Equation of the curve is $\dfrac{x^m}{a^m} + \dfrac{y^m}{b^m} = 1$ ...(i)

Differentiating (*i*) w.r.t. x, $\dfrac{mx^{m-1}}{a^m} + \dfrac{my^{m-1}}{b^m}\dfrac{dy}{dx} = 0$

$\therefore$ Slope of the tangent at $(x, y) = \dfrac{dy}{dx} = -\left(\dfrac{b}{a}\right)^m \left(\dfrac{x}{y}\right)^{m-1}$

$\therefore$ Equation of the tangent at (x, y) is

$$Y - y = -\left(\frac{b}{a}\right)^m \left(\frac{x}{y}\right)^{m-1} (X - x)$$

or $$\frac{x^{m-1}X}{a^m} + \frac{y^{m-1}Y}{b^m} = \frac{x^m}{a^m} + \frac{y^m}{b^m} = 1 \quad \text{...(ii) [By } (i)]$$

If the given line touches (*i*) at (x, y) then (*ii*) must be same as $X \cos \alpha + Y \sin \alpha = p$ ...(iii)

Comparing coefficients in (*ii*) and (*iii*),

$$\frac{x^{m-1}}{a^m}\bigg/\cos \alpha = \frac{y^{m-1}}{b^m}\bigg/\sin \alpha = \frac{1}{p}$$

or $$\left(\frac{x}{a}\right)^{m-1} = \frac{a \cos \alpha}{p}, \left(\frac{y}{b}\right)^{m-1} = \frac{b \sin \alpha}{p}$$

or $$\left(\frac{a \cos \alpha}{p}\right)^{\frac{m}{m-1}} + \left(\frac{b \sin \alpha}{p}\right)^{\frac{m}{m-1}} = \left(\frac{x}{a}\right)^m + \left(\frac{y}{b}\right)^m = 1 \quad \text{[By } (i)]$$

whence follows the required condition.

Example 4.36. *Find the equation of the normal at any point θ to the curve $x = a (\cos \theta + \theta \sin \theta)$, $y = a (\sin \theta - \theta \cos \theta)$. Verify that these normals touch a circle with its centre at the origin and whose radius is constant.*

Solution. We have $\dfrac{dx}{d\theta} = a(-\sin \theta + \sin \theta + \theta \cos \theta) = a\theta \cos \theta$

$$\frac{dy}{d\theta} = a(\cos \theta - \cos \theta + \theta \sin \theta) = a\theta \sin \theta$$

$\therefore$ $$\frac{dy}{dx} = \frac{dy}{d\theta}\bigg/\frac{dx}{d\theta} = \frac{\sin \theta}{\cos \theta}$$

$\therefore$ Slope of the normal at $\theta = -\dfrac{\cos \theta}{\sin \theta}$

Hence the equation of the normal at θ

$$y - a(\sin \theta - \theta \cos \theta) = -\frac{\cos \theta}{\sin \theta}[x - a(\cos \theta + \theta \sin \theta)]$$

i.e., $$y \sin \theta - a \sin^2 \theta + a\,\theta \sin \theta \cos \theta = -x \cos \theta + a \cos^2 \theta + a\,\theta \sin \theta \cos \theta$$

i.e., $$x \cos \theta + y \sin \theta = a(\cos^2 \theta + \sin^2 \theta) = a.$$

Now the perpendicular distance of this normal from $(0, 0) = a$, which is a constant. Hence it touches a circle of radius a having its centre at $(0, 0)$.

(3) Angle of intersection of two curves *is the angle between the tangents to the curves at their point of intersection.*

To find this angle θ, proceed as follows :

(i) Find P, the point of intersection of the curves by solving their equations simultaneously.

(ii) Find the values of dy/dx at P for the two curves (say : m_1, m_2).

(iii) Find $\angle\theta$, using the $\tan\theta = \dfrac{m_1 - m_2}{1 + m_1 m_2}$.

When $m_1 m_2 = -1$, $\theta = 90°$ *i.e.*, the curves cut orthogonally.

Example 4.37. *Find the angle of intersection of the curves* $x^2 = 4y$ ...(i)

and $y^2 = 4x$. ...(ii)

Solution. We have $x^4 = 16y^2 = 16.4\,x = 64x$

or $x(x^3 - 64) = 0$ whence $x = 0$ and 4.

Substituting these values in (i), $y = 0$ and 4.

∴ The curves intersect at (0, 0) and (4, 4).

For the curve (i), $dy/dx = x/2$. For the curve (ii), $dy/dx = 2/y$

At (0, 0), slope of tangent to (i) $(= m_1) = 0/2 = 0$ and slope of tangent to (ii) $(= m_2) = 2/0 = \infty$.

Evidently the curves intersect at right angles.

At (4, 4), slope of tangent to (i) $(= m_1) = 4/2 = 2$ and slope of tangent to (ii) $(= m_2) = 2/4 = \frac{1}{2}$

∴ Angle of intersection of the curves

$$= \tan^{-1}\frac{m_1 - m_2}{1 + m_1 m_2} = \tan^{-1}\frac{2 - \frac{1}{2}}{1 + 2\cdot\frac{1}{2}} = \tan^{-1}\frac{3}{4}.$$

Example 4.38. *Show that the condition that the curves* $ax^2 + by^2 = 1$ *and* $a'x^2 + b'y^2 = 1$ *should intersect orthogonally is that*

$$\frac{1}{a} - \frac{1}{b} = \frac{1}{a'} - \frac{1}{b'}.$$

Solution. Given curves are $ax^2 + by^2 = 1$ ...(i) and $a'x^2 + b'y^2 = 1$ ...(ii)

Let $P(h, k)$ be a point of intersection of (i) and (ii) so that

$$ah^2 + bk^2 = 1 \quad \text{and} \quad a'h^2 + b'k^2 = 1$$

$$\therefore \quad \frac{h^2}{-b + b'} = \frac{k^2}{-a' + a} = \frac{1}{ab' - a'b}$$

or $$h^2 = (b' - b)/(ab' - a'b),\ k^2 = (a - a')/(ab' - a'b) \quad \text{...(iii)}$$

Differentiating (i) w.r.t. x,

$$2ax + 2by\,dy/dx = 0 \quad \text{or} \quad dy/dx = -ax/by.$$

Similarly for (ii), $dy/dx = -a'x/b'y$

∴ m_1 = slope of tangent to (i) at $P = -ah/bk$; m_2 = slope of tangent to (ii) at $P = -a'h/b'k$

For orthogonal intersection, we should have $m_1 m_2 = -1$.

i.e., $$\frac{-ah}{bk} \times \frac{-a'h}{b'k} = 1 \text{ i.e., } aa'h^2 + bb'k^2 = 0$$

Substituting the values of h^2 and k^2 from (iii),

$$\frac{aa'(b' - b)}{ab' - a'b} + \frac{bb'(a - a')}{ab' - a'b} = 0 \quad \text{or} \quad \frac{b' - b}{bb'} + \frac{a - a'}{aa'} = 0$$

i.e., $\dfrac{1}{b} - \dfrac{1}{b'} = \dfrac{1}{a} - \dfrac{1}{a'}$ which leads to the required condition.

(4) Lengths of tangent, normal, subtangent and subnormal. Let the tangent and the normal at any point $P(x, y)$ of the curve meet the x-axis at T and N respectively. (Fig. 4.4). Draw the ordinate PM. Then PT *and* PN *are called the lengths of the tangent and the normal respectively. Also* TM *and* MN *are called the subtangent and subnormal respectively.*

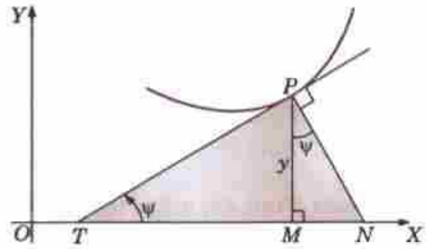


Fig. 4.4

Let $\angle MTP = \psi$ so that $\tan \psi = dy/dx$.

Clearly, $\angle MPN = \psi$.

(1) Tangent $= TP = MP \operatorname{cosec} \psi = y\sqrt{(1 + \cot^2 \psi)} = \mathbf{y\sqrt{[1 + (dx/dy)^2]}}$

(2) Normal $= NP = MP \sec \psi = y\sqrt{(1 + \tan^2 \psi)} = \mathbf{y\sqrt{[1 + (dy/dx)^2]}}$

(3) Subtangent $= TM = y \cot \psi = \mathbf{y\ dx/dy}$

(4) Subnormal $= MN = y \tan \psi = \mathbf{y\ dy/dx}$.

Example 4.39. *For the curve* $x = a(\cos t + \log \tan t/2)$, $y = a \sin t$, *prove that the portion of the tangent between the curve and x-axis is constant.*

Also find its subtangent.

Solution. Differentiating with respect to t,

$$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{\tan t/2} \cdot \sec^2 \frac{t}{2} \cdot \frac{1}{2}\right) = a\left(-\sin t + \frac{\cos t/2}{2 \sin t/2} \cdot \frac{1}{\cos^2 t/2}\right)$$

$$= a\left(-\sin t + \frac{1}{\sin t}\right) = \frac{a(1 - \sin^2 t)}{\sin t} = a \cos^2 t/\sin t\ ; \quad \frac{dy}{dt} = a \cos t\ .$$

$$\therefore \qquad \frac{dy}{dx} = \frac{dy}{dt}\bigg/\frac{dx}{dt} = a \cos t \cdot \frac{\sin t}{a \cos^2 t} = \tan t.$$

Thus *length of the tangent between the curve and x-axis*

$$= y\sqrt{[1 + (dx/dy)^2]} = a \sin t \cdot \sqrt{(1 + \cot^2 t)} = a \sin t\ .\ \operatorname{cosec} t = a \text{ which is a constant.}$$

Also *subtangent* $= y\dfrac{dx}{dy} = a \sin t \cdot \cot t = a \cos t$.

PROBLEMS 4.8

1. Find the equation of the tangent and the normal to the curve $y(x - 2)(x - 3) - x + 7 = 0$ at the point where it cuts the x-axis.
2. The straight line $x/a + y/b = 2$ touches the curve $(x/a)^n + (y/b)^n = 2$ for all values of n. Find the point of contact. *(Bhopal, 2008)*
3. Prove that $\dfrac{x}{a} + \dfrac{y}{b} = 1$ touches the curve $y = be^{-x/a}$ at the point where the curve crosses the axis of y. *(Bhopal, 2009)*
4. If $p = x \cos \alpha + y \sin \alpha$, touches the curve $(x/a)^{n/(n-1)} + (y/b)^{n/(n-1)} = 1$, prove that
$$p^n = (a \cos \alpha)^n + (b \sin \alpha)^n.$$
5. Prove that the condition for the line $x \cos \alpha + y \sin \alpha = p$ to touch the curve $x^m y^n = a^{m+n}$, is
$$p^{m+n} \cdot m^m \cdot n^n = (m + n)^{m+n} a^{m+n} \cos^m \alpha \sin^n \alpha.$$
6. Show that the sum of the intercepts on the axes of any tangent to the curve $\sqrt{x} + \sqrt{y} = a$ is a constant.
7. If x, y be the parts of the axes of x and y intercepted by the tangent at any point (x, y) on the curve $(x/a)^{2/3} + (x/b)^{2/3} = 1$, then show that $(x_1/a)^2 + (y_1/b)^2 = 1$. *(Bhopal, 2008)*
8. If the tangent at (x_1, y_1) to the curve $x^3 + y^3 = a^3$ meets the curve again in (x_2, y_2), show that
$$\frac{x_2}{x_1} + \frac{y_2}{y_1} = -1.$$

9. If the normal to the curve $x^{2/3} + y^{2/3} = a^{2/3}$ makes an angle ϕ with the axis of x, show that its equation is $y \cos \phi - x \sin \phi = a \cos 2\phi$.

10. Find the angle of intersection of the curves $x^2 - y^2 = a^2$ are $x^2 + y^2 = a^2\sqrt{2}$.

11. Show that the parabolas $y^2 = 4ax$ and $2x^2 = ay$ intersect at an angle $\tan^{-1}(3/5)$.

12. Prove that the curves $\frac{x^2}{a} + \frac{y^2}{b} = 1$ and $\frac{x^2}{a'} + \frac{y^2}{b'} = 1$ will cut orthogonally if $a - b = a' - b'$.

13. Show that in the exponential curve $y = be^{x/a}$, the subtangent is of constant length and that the subnormal varies as the square of the ordinate. *(Madras, 2000 S)*

14. Find the lengths of the tangent, normal, subtangent and subnormal for the cycloid:

$$x = a(t + \sin t),\ y = a(1 - \cos t),$$

15. For the curve $x = a \cos^3 \theta$, $y = a \sin^3 \theta$, show that the portion of the tangent intercepted between the point of contact and the x-axis is $y \operatorname{cosec} \theta$. Also find the length of the subnormal.

4.7 POLAR CURVES

(1) Angle between radius vector and tangent. *If ϕ be the angle between the radius vector and the tangent at any point of the curve $r = f(\theta)$,* $\mathbf{\tan \theta = r \dfrac{d\theta}{dr}}$.

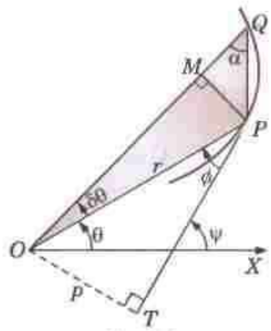


Fig. 4.5

Let $P(r, \theta)$ and $Q(r + \delta r, \theta + \delta\theta)$ be two neighbouring points on the curve (Fig. 4.5). Join PQ and draw $PM \perp OQ$. Then from the rt. angled ΔOMP, $MP = r \sin \delta\theta$, $OM = r \cos \delta\theta$.

$$\therefore \quad MQ = OQ - OM = r + \delta r - r \cos \delta\theta$$
$$= \delta r + r(1 - \cos \delta\theta) = \delta r + 2r \sin^2 \delta\theta/2.$$

If $\angle MQP = \alpha$, then $\qquad \tan \alpha = \dfrac{MP}{MQ} = \dfrac{r \sin \delta\theta}{\delta r + 2r \sin^2 \delta\theta/2}$

In the limit as $Q \to P$ (*i.e.*, $\delta\theta \to 0$), the chord PQ turns about P and becomes the tangent at P and $\alpha \to \phi$.

$$\therefore \quad \tan \phi = \underset{Q \to P}{\text{Lt}} (\tan \alpha) = \underset{\delta\theta \to 0}{\text{Lt}} \frac{r \sin \delta\theta}{\delta r + 2r \sin^2 \delta\theta/2}$$

$$= \underset{\delta\theta \to 0}{\text{Lt}} \frac{r(\sin \delta\theta/\delta\theta)}{(\delta r/\delta\theta) + r \sin \delta\theta/2 \cdot (\sin \delta\theta/2 \div \delta\theta/2)}$$

$$= \frac{r \cdot 1}{(dr/d\theta) + r \cdot 0 \cdot 1} = r\frac{d\theta}{dr}$$

Cor. Angle of intersection of two curves. *If ϕ_1, ϕ_2 be the angles between the common radius vector and the tangents to the two curves at their point of intersection, then the angle of intersection of these curves is $\phi_1 \sim \phi_2$.*

(2) Length of the perpendicular from pole on the tangent. *If p be the perpendicular from the pole on the tangent, then*

(*i*) $\mathbf{p = r \sin \phi}$ $\qquad$ (*ii*) $\dfrac{\mathbf{1}}{\mathbf{p^2}} = \dfrac{\mathbf{1}}{\mathbf{r^2}} + \dfrac{\mathbf{1}}{\mathbf{r^4}}\left(\dfrac{\mathbf{dr}}{\mathbf{d\theta}}\right)^2$

From the rt. $\angle$ed ΔOTP, $p = r \sin \phi$

$$\therefore \quad \frac{1}{p^2} = \frac{1}{r^2} \operatorname{cosec}^2 \phi = \frac{1}{r^2}(1 + \cot^2 \phi)$$

$$= \frac{1}{r^2}\left[1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right] = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \qquad \text{[By (1)]}$$

(3) Polar subtangent and subnormal. Let the tangent and the normal at any point $P(r, \theta)$ of a curve meet the line through the pole perpendicular to the radius vector OP in T and N respectively (Fig. 4.6). Then OT is called the *polar subtangent* and ON the *polar subnormal.*

Let $\angle OTP = \phi$ so that $\tan \phi = r d\theta/dr$

Clearly, $\angle PNO = \phi$.

$\therefore$ (*i*) **Polar subtangent**

$$= OT = r \tan \phi = r \cdot r d\theta/dr = \mathbf{r^2 \frac{d\theta}{dr}}$$

(*ii*) **Polar subnormal**

$$= ON = r \cot \phi = r \cdot \frac{1}{r}\frac{dr}{d\theta} = \mathbf{\frac{dr}{d\theta}}$$

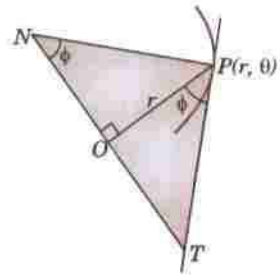


Fig. 4.6

Example 4.40. *For the cardioid $r = a(1 - \cos \theta)$, prove that*

(i) $\phi = \theta/2$ *(ii)* $p = 2a \sin^3 \theta/2$

(iii) polar subtangent $= 2a \sin^2 \frac{\theta}{2} \tan \frac{\theta}{2}$.

Solution. We have $\dfrac{dr}{d\theta} = a \sin \theta$

$\therefore$ $\tan \phi = r\dfrac{d\theta}{dr} = a(1 - \cos \theta) \cdot \dfrac{1}{a \sin \theta}$

$= 2 \sin^2 \theta/2 \div 2 \sin \theta/2 \cos \theta/2 = \tan \theta/2$. Thus $\phi = \theta/2$ ...(*i*)

Also $p = r \sin \phi = a(1 - \cos \theta) \cdot \sin \theta/2 = a \cdot 2 \sin^2 \theta/2 \cdot \sin \theta/2$

$= 2a \sin^3 \theta/2$ ...(*ii*)

Polar subtangent $= r^2 \, d\theta/dr = [a(1 - \cos \theta)]^2 \div a \sin \theta$

$= 4a \sin^4 \theta/2 \div 2 \sin \theta/2 \cos \theta/2 = 2a \sin^2 \theta/2 \tan \theta/2$. ...(*iii*)

Example 4.41. *Find the angle of intersection of the curves $r = \sin \theta + \cos \theta$, $r = 2 \sin \theta$.*

Solution. To find the point of intersection of the curves $r = \sin \theta + \cos \theta$

and $r = 2 \sin \theta$, ...(*ii*), we eliminate r.

Then $2 \sin \theta = \sin \theta + \cos \theta$ or $\tan \theta = 1$ *i.e.*, $\theta = \pi/4$.

For (*i*), $\dfrac{dr}{d\theta} = \cos \theta - \sin \theta$

$\therefore$ $\tan \phi = r\dfrac{d\theta}{dr} = \dfrac{\sin \theta + \cos \theta}{\cos \theta - \sin \theta}$ which $\to \infty$ at $\theta = \pi/4$. Thus $\phi = \pi/2$.

For (*ii*), $dr/d\theta = 2 \cos \theta$ $\therefore$ $\tan \phi' = r\dfrac{d\theta}{dr} = \dfrac{2 \sin \theta}{2 \cos \theta} = 1$ at $\theta = \pi/4$. Thus $\phi' = \pi/4$

Hence the angle of intersection of (*i*) and (*ii*) $= \phi - \phi' = \pi/4$.

PROBLEMS 4.9

1. For a curve in Cartesian form, show that $\tan \phi = \dfrac{xy' - y}{x + yy'}$.
2. Show that in the equiangular spiral $r = ae^{\theta \cot \alpha}$, the tangent is inclined at a constant angle to the radius vector.
3. Show that the tangent to the cardioid $r = a(1 + \cos \theta)$ at the points $\theta = \pi/3$ and $\theta = 2\pi/3$ are respectively parallel and perpendicular to the initial line. (*V.T.U., 2006*)
4. Prove that, in the parabola $2a/r = 1 - \cos \theta$,

   (*i*) $\phi = \pi - \theta/2$ (*ii*) $\pi = a$ cosec $\theta/2$, and (*iii*) polar subtangent $= 2a$ cosec θ.
5. Show that the angle between the tangent at any point P and the line joining P to the origin is the same at all points of the curve

$$\log(x^2 + y^2) = k \tan^{-1}(y/x).$$

6. Show that in the curve $r = a\theta$, the polar subnormal is constant and in the curve $r\,\theta = a$ the polar subtangent is constant.
7. Find the angle of intersection of the curves
   (i) $r = 2 \sin \theta$, and $r = 2 \cos \theta$ *(Bhopal, 1991)*
   (ii) $r = a/(1 + \cos \theta)$ and $r = b/(1 - \cos \theta)$. *(V.T.U., 2008 S)*
8. Prove that the curves $r = a(1 + \cos \theta)$ and $r = b(1 - \cos \theta)$ intersect at right angles. *(V.T.U., 2011 S)*
9. Show that the curves $r^n = a^n \cos n\theta$ and $r^n = b^n \sin n\theta$ cut each other orthogonally.
10. Show that the angle of intersection of the curves $r = a \log \theta$ and $r = a/\log \theta$ is $\tan^{-1} [2e/(1 - e^2)]$. *(V.T.U., 2005)*

4.8 PEDAL EQUATION

If r be the radius vector of any point on the curve and p, the length of the perpendicular from the pole on the tangent at that point, them *the relation between* p *and* r *is called pedal equation of the curve.*

Given the cartesian or polar equation of a curve, we can derive its pedal equation. The method is explained through the following examples.

Example 4.42. *Find the pedal equation of the ellipse* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$ ...(i)

Solution. Equation of the tangent at (x, y) is $\dfrac{Xx}{a^2} + \dfrac{Yy}{b^2} = 1$ ...(ii)

p, length of $\perp$ from $(0, 0)$ on (ii) $= \dfrac{-1}{\sqrt{[(x/a^2)^2 + (y/b^2)^2]}}$

or $$\frac{1}{p^2} = \frac{x^2}{a^4} + \frac{y^2}{b^4} \quad ...(iii)$$

Also $$r^2 = x^2 + y^2 \quad ...(iv)$$

Substituting the value of y^2 from (iv) in (i),

$$\frac{x^2}{a^2} = \frac{r^2 - b^2}{a^2 - b^2}.$$

Then from (i), $$\frac{y^2}{b^2} = \frac{a^2 - r^2}{a^2 - b^2}$$

Now substituting these values of x^2/a^2 and y^2/b^2 in (iii),

$$\frac{1}{p^2} = \frac{1}{a^2}\left(\frac{r^2 - b^2}{a^2 - b^2}\right) + \frac{1}{b^2}\left(\frac{a^2 - r^2}{a^2 - b^2}\right)$$

or $$\frac{a^2 b^2}{p^2} = \frac{r^2 b^2 - b^4 + a^4 - a^2 r^2}{a^2 - b^2} = a^2 + b^2 - r^2$$

Hei e required pedal equation.

Example 4.43. of the curves
(i) $2a/r = $ $r^n = a^n \cos n\,\theta$ *(V.T.U., 2010)*

Solution.
Takin

log $2a$ – lo

Differentiating both sides with respect to θ, we get

$$-\frac{1}{r}\frac{dr}{d\theta}=\frac{1}{1-\cos\theta}\cdot\sin\theta=\cot\frac{\theta}{2}$$

∴ $$\tan\phi=r\frac{d\theta}{dr}=-\tan\theta/2=\tan(\pi-\theta/2)\text{ i.e., }\phi=\pi-\theta/2$$

Also $$p=r\sin\phi=r\sin(\pi-\theta/2)\text{ i.e., }p=r\sin\theta/2$$

or $$p^2=r^2\sin^2\theta/2=r^2\left(\frac{1-\cos\theta}{2}\right)=r^2.\,a/r \qquad \text{[By (i)]}$$

Hence $p^2 = ar$, which is the required pedal equation.

(*ii*) From the given equation, $nr^{n-1}\dfrac{dr}{d\theta}=-na^n\sin n\theta$

so that $$\tan\phi=r\,d\theta/dr=r\frac{nr^{n-1}}{-na^n\sin n\theta}=-\cot n\theta=\tan\left(\frac{\pi}{2}+n\theta\right)$$

i.e., $$\phi=\pi/2+n\theta$$

∴ $$p=r\sin\phi=r\sin\left(\frac{\pi}{2}+n\theta\right)=r\cos n\theta=r\,.\,(r^n/a^n)=r^{n+1}/a^n.$$

Hence $p\,a^n = r^{n+1}$, which is the required pedal equation.

4.9 DERIVATIVE OF ARC

(1) *For the curve $y = f(x)$, we have*

$$\mathbf{\frac{ds}{dx}=\sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}}$$

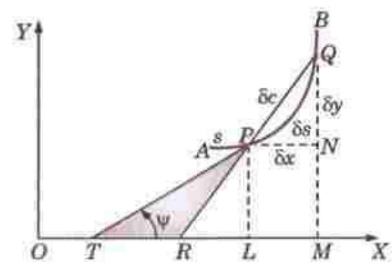


Fig. 4.7

Let $P(x, y)$, $Q(x + \delta x, y + \delta y)$ be two neighbouring points on the curve AB (Fig. 4.7). Let arc $AP = s$, arc $PQ = \delta s$ and chord $PQ = \delta c$.

Draw PL, QM ⊥s on the x-axis and $PN \perp QM$.

∴ From the rt. ∠ed ΔPNQ,

$$PQ^2 = PN^2 + NQ^2$$

i.e., $$\delta c^2=\delta x^2+\delta y^2$$

or $$\left(\frac{\delta c}{\delta x}\right)^2=1+\left(\frac{\delta y}{\delta x}\right)^2$$

∴ $$\left(\frac{\delta s}{\delta x}\right)^2=\left(\frac{\delta s}{\delta c}\cdot\frac{\delta c}{\delta x}\right)^2$$

$$=\left(\frac{\delta s}{\delta c}\right)^2\cdot\left[1+\left(\frac{\delta y}{\delta x}\right)^2\right]$$

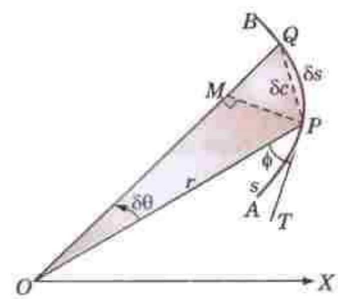


Fig. 4.8

Taking limits as $Q \to P$ (*i.e.*, $\delta c \to 0$),

$$\left(\frac{ds}{dx}\right)^2=1\cdot\left[1+\left(\frac{dy}{dx}\right)^2\right] \qquad \left.\frac{\delta s}{\delta c}=1\right]$$

If s increases with x as in Fig. 4.7, dy/dx is positive.

Thus $$\frac{ds}{dx}=\sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]},\text{ taking positive sign before the radical.} \qquad ...(1)$$

Cor. 1. *If the equation of the curve is $x = f(y)$, then*

$$\frac{ds}{dy}=\frac{ds}{dx}\cdot\frac{dx}{dy}=\sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\cdot\frac{dx}{dy}$$

$\therefore$ $$\mathbf{\frac{ds}{dy}=\sqrt{\left[1+\left(\frac{dx}{dy}\right)^2\right]}} \quad ...(2)$$

Cor. 2. *If the equation of the curve is in parametric form $x = f(t)$, $y = \phi(t)$, then*

$$\frac{ds}{dt}=\frac{ds}{dx}\cdot\frac{dx}{dt}=\sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\cdot\frac{dx}{dt}$$

$$=\sqrt{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dx}\cdot\frac{dx}{dt}\right)^2\right]}$$

$\therefore$ $$\mathbf{\frac{ds}{dt}=\sqrt{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]}} \quad ...(3)$$

Cor. 3. We have $$\frac{ds}{dx}=\sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}=\sqrt{(1+\tan^2\psi)}=\sec\psi$$

$\therefore$ $$\mathbf{\cos\psi=\frac{dx}{ds}}. \quad ...(4)$$

Also $$\sin\psi=\tan\psi\cos\psi=\frac{dy}{dx}\cdot\frac{dx}{ds}$$

$\therefore$ $$\mathbf{\sin\psi=\frac{dy}{ds}} \quad ...(5)$$

(2) *For the curve $r = f(\theta)$, we have* $\mathbf{\dfrac{ds}{d\theta}=\sqrt{\left[r^2+\left(\dfrac{dr}{d\theta}\right)^2\right]}}$.

Let $P(r, \theta)$, $Q(r + \delta r, \theta + \delta\theta)$ be two neighbouring points on the curve AB (Fig. 4.8). Let arc $AP = s$, arc $PQ = \delta s$ and chord $PQ = \delta c$.

Draw $PM \perp OQ$, then

$$PM = r\sin\delta\theta \text{ and } MQ = OQ - OM = r + \delta r - r\cos\delta\theta = \delta r + 2r\sin^2\delta\theta/2$$

From the rt. $\angle$ed ΔPMQ,

$$PQ^2 = PM^2 + MQ^2$$

or $$\delta c^2 = (r\sin\delta\theta)^2 + (\delta r + 2r\sin^2\delta\theta/2)^2$$

$\therefore$ $$\left(\frac{\delta s}{\delta\theta}\right)^2=\left(\frac{\delta s}{\delta c}\cdot\frac{\delta c}{\delta\theta}\right)^2=\left(\frac{\delta s}{\delta c}\right)^2\left[\left(\frac{r\sin\delta\theta}{\delta\theta}\right)^2+\left(\frac{\delta r}{\delta\theta}+\frac{2r\sin^2\delta\theta/2}{\delta\theta}\right)^2\right]$$

$$=\left(\frac{\delta s}{\delta c}\right)^2\left[r^2\left(\frac{\sin\delta\theta}{\delta\theta}\right)^2+\left(\frac{\delta r}{\delta\theta}+r\sin\frac{\delta\theta}{2}\cdot\frac{\sin\delta\theta/2}{\delta\theta/2}\right)^2\right]$$

Taking limits as $Q \to P$

$$\left(\frac{ds}{d\theta}\right)^2=1^2\cdot\left[r^2\cdot 1^2+\left(\frac{dr}{d\theta}+r\cdot 0\cdot 1\right)^2\right]=r^2+\left(\frac{dr}{d\theta}\right)^2$$

As s increases with the increase of θ, $ds/d\theta$ is positive. Thus

$$\frac{ds}{d\theta}=\sqrt{\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]} \quad ...(1)$$

Cor. 1. If the equation of the curve is $\theta = f(r)$, then

$$\frac{ds}{dr}=\frac{ds}{d\theta}\cdot\frac{d\theta}{dr}=\sqrt{\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]}\cdot\frac{d\theta}{dr}$$

$$\therefore \qquad \frac{ds}{dr} = \sqrt{\left[1 + r^2\left(\frac{d\theta}{dr}\right)^2\right]} \qquad ...(2)$$

Cor. 2. We have $\dfrac{ds}{dr} = \sqrt{\left[1 + \left(r\dfrac{d\theta}{dr}\right)^2\right]} = \sqrt{[1 + \tan^2\phi]}$ $\dfrac{ds}{dr} = \sqrt{\left[1 + \left(r\dfrac{d\theta}{dr}\right)^2\right]} = \sqrt{[1 + \tan^2\phi]} = \sec\phi$

$$\therefore \qquad \cos\phi = \frac{dr}{ds} \qquad ...(3)$$

Also $\sin\phi = \tan\phi \,.\, \cos\phi = r\dfrac{d\theta}{dr}\cdot\dfrac{dr}{ds}$

$$\therefore \qquad \sin\phi = r\frac{d\theta}{ds} \qquad ...(4)$$

PROBLEMS 4.10

Prove that the pedal equation of :

1. the parabola $y^2 = 4a(x + a)$ is $p^2 = ar$.
2. the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is $a^2 b^2/p^2 = r^2 - a^2 + b^2$.
3. the astroid $x = a\cos^3 t, y = a\sin^3 t$ is $r^2 = a^2 - 3p^2$.

Find the pedal equations of the following curves :

4. $r = a(1 + \cos\theta)$ (V.T.U., 2009)
5. $r^2 = a^2\sin^2\theta$
6. $r^m\cos m\theta = a^m$. (V.T.U., 2004)
7. $r^m = a^m(\cos m\theta + \sin m\theta)$ (V.T.U., 2010)
8. $r = ae^{m\theta}$. (V.T.U., 2007)
9. Calculate ds/dx for the following curves :
   (i) $ay^2 = x^3$. (ii) $y = c\cosh x/c$.
10. Find $ds/d\theta$ for the curve $x = a(\cos\theta + \theta\sin\theta), y = a(\sin\theta - \theta\cos\theta)$ (V.T.U., 2007)
11. Find $ds/d\theta$ for the following curves :
    (i) $r = a(1 - \cos\theta)$ (V.T.U., 2004) (ii) $r^2 = a^2\cos^2 2\theta$
    (iii) $r = \frac{1}{2}\sec^2\theta$ (V.T.U., 2007)
12. For the curves $\theta = \cos^{-1}(r/k) - \sqrt{(k^2 - r^2)}/r$, prove that $r\dfrac{ds}{dr}$ = constant. (V.T.U., 2005)
13. With the usual meanings for r, s, θ and ϕ for the polar curve $r = f(\theta)$, show that $\dfrac{d\phi}{d\theta} + r\,\text{cosec}^2\theta\,\dfrac{d^2r}{ds^2} = 0$. (V.T.U., 2000)

4.10 CURVATURE

Let P be any point on a given curve and Q a neighbouring point. Let arc $AP = s$ and arc $PQ = \delta s$. Let the tangents at P and Q make angle ψ and $\psi + \delta\psi$ with the x-axis, so that the angle between the tangents at P and $Q = \delta\psi$ (Fig. 4.9).

In moving from P to Q through a distance δs, the tangent has turned through the angle $\delta\psi$. This is called the *total bending or total curvature* of the arc PQ.

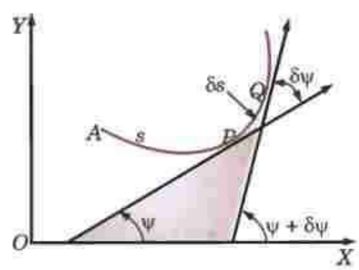


Fig. 4.9

$\therefore$ The *average curvature* of arc $PQ = \dfrac{\delta\psi}{\delta s}$

The limiting value of average curvature when Q approaches P (i.e., $\delta s \to 0$) is defined as the curvature of the curve at P.

Thus **curvature** K (at P) $= \dfrac{d\psi}{ds}$

Obs. Since $\delta\psi$ is measured in radians, the *unit of curvature is radians per unit length e.g.*, radians per centimetre.

(2) Radius of curvature. *The reciprocal of the curvature of a curve at any point P is called the* **radius of curvature** *at P and is denoted by* ρ, so the $\rho = \mathbf{ds/d\psi}$.

(3) Centre of curvature. *A point C on the normal at any point P of a curve distant* ρ *from it, is called the* **centre of curvature** *at P.*

(4) Circle of curvature. *A circle with centre C (centre of curvature at P) and radius* ρ *is called the* **circle of curvature** *at P.*

4.11 (1) RADIUS OF CURVATURE FOR CARTESIAN CURVE y = f (x), is given by

$$\rho = \frac{(1 + \mathbf{y_1}^2)^{3/2}}{\mathbf{y_2}}$$

We know that $\tan\psi = dy/dx = y_1$ or $\psi = \tan^{-1}(y_1)$

Differentiating both sides w.r.t. x,

$$\frac{d\psi}{dx} = \frac{1}{1 + y_1^2} \cdot \frac{d(y_1)}{dx} = \frac{y_2}{1 + y_1^2}$$

$$\therefore \qquad \rho = \frac{ds}{d\psi} = \frac{ds}{dx} \cdot \frac{dx}{d\psi} = \sqrt{(1 + y_1^2)} \cdot \frac{1 + y_1^2}{y_2} = \frac{(1 + y_1^2)^{3/2}}{y_2} \qquad \text{...(1)}$$

(2) Radius of curvature for parametric equations

$$x = f(t), \qquad y = \phi(t).$$

Denoting differentiations with respect to t by dashes,

$$y_1 = \frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = y'/x'.$$

$$y_2 = \frac{d}{dx}(y_1) = \frac{d}{dt}\left(\frac{y'}{x'}\right) \cdot \frac{dt}{dx} = \frac{x'y'' - y'x''x}{(x')^2} \cdot \frac{1}{x'}$$

Substituting the values of y_1 and y_2 in (1)

$$\rho = \left[1 + \left(\frac{y'}{x'}\right)^2\right]^{3/2} \Big/ \left[\frac{x'y'' - y'x''}{(x')^3}\right] = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''} \qquad \textit{(Rajasthan, 2005)}$$

(3) Radius of curvature at the origin. Newton's formulae*

(*i*) *If x-axis is tangent to a curve at the origin, then*

$$\rho \text{ at } (0, 0) = \mathop{\mathbf{Lt}}_{\mathbf{x \to 0}} \left(\frac{\mathbf{x^2}}{\mathbf{2y}}\right)$$

Since x-axis is a tangent at (0, 0), $(dy/dx)_0$ or $(y_1)_0 = 0$

$$\text{Also} \qquad \mathop{\mathrm{Lt}}_{x \to 0} \left(\frac{x^2}{2y}\right) = \mathop{\mathrm{Lt}}_{x \to 0} \left(\frac{2x}{2dy/dx}\right) = \mathop{\mathrm{Lt}}_{x \to 0} \frac{1}{d^2y/dx^2} = \frac{1}{(y_2)_0} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$\therefore \qquad \rho \text{ at } (0, 0) = \frac{\{1 + (y_1^2)_0\}^{3/2}}{(y_2)_0} = \frac{1}{(y_2)_0} = \mathop{\mathrm{Lt}}_{x \to 0} \frac{x^2}{2y} \qquad \text{[From (1)]}$$

(*ii*) Similarly, *if y-axis is tangent to a curve at the origin, then*

$$\rho \text{ at } (0, 0) = \mathop{\mathrm{Lt}}_{x \to 0} \left(\frac{\mathbf{y^2}}{\mathbf{2x}}\right)$$

---

* Named after the great English mathematician and physicist *Sir Issac Newton* (1642–1727) whose contributions are of utmost importance. He discovered many physical laws, invented Calculus alongwith *Leibnitz* (see footnote p. 139) and created analytical methods of investigating physical problems. He became professor at Cambridge in 1699, but his '*Mathematical Principles of Natural Philosophy*' containing development of classical mechanics had been completed in 1687.

(*iii*) In case the curve passes through the origin but neither x-axis nor y-axis is tangent at the origin, we write the equation of the curve as

$$y = f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots \qquad \text{[By Maclaurin's series]}$$

$$= px + qx^2/2 + \ldots \qquad [\because \; f(0) = 0]$$

where $p = f'(0)$ and $q = f''(0)$

Substituting this in the equation $y = f(x)$, we find the values of p and q by equating coefficients of like powers of x. Then $\rho\,(0, 0) = (1 + p^2)^{3/2}/q$.

Obs. *Tangents at the origin to a curve are found by equating to zero the lowest degree terms in its equation.*

Example 4.44. *Find the radius of curvature at the point (i) $(3a/2, 3a/2)$ of the Folium $x^3 + y^3 = 3axy$.* (*Anna, 2009 ; Kurukshetra, 2009 S ; V.T.U., 2008*)

(ii) $(a, 0)$ on the curve $xy^2 = a^3 - x^3$. (*Anna, 2009 ; Kerala, 2005*)

Solution. (*i*) Differentiating with respect to x, we get

$$3x^2 + 3y^2 \frac{dy}{dx} = 3a\left(y + x\frac{dy}{dx}\right)$$

or $$(y^2 - ax)\frac{dy}{dx} = ay - x^2 \qquad \ldots(i) \qquad \therefore \quad \frac{dy}{dx} \text{ at } (3a/2, 3a/2) = -1$$

Differentiating (*i*),

$$\left(2y\frac{dy}{dx} - a\right)\frac{dy}{dx} + (y^2 - ax)\frac{d^2y}{dx^2} = a\frac{dy}{dx} - 2x \quad \therefore \quad \frac{d^2y}{dx^2} \text{ at } (3a/2, 3a/2) = -32/3a$$

Hence ρ at $(3a/2, 3a/2)$ $$= \frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} = \frac{[1 + (-1)^2]^{3/2}}{-32/3a} = \frac{3a}{8\sqrt{2}} \quad \text{(in magnitude)}.$$

(*ii*) We have $$y^2 = a^3 x^{-1} - x^2$$

$\therefore$ $$2y\, dy/dx = -a^3x^{-2} - 2x \quad \text{or} \quad dy/dx = -a^3/(2x^2y) - x/y$$

At $(a, 0)$, $dy/dx \to \infty$, so we find dx/dy from $xy^2 = a^3 - x^3$

$\therefore$ $$x \cdot 2y + y^2\frac{dx}{dy} = -3x^2\frac{dx}{dy}$$

or $$\frac{dx}{dy} = \frac{-2xy}{3x^2 + y^2} \quad \text{or} \quad \frac{dx}{dy} \text{ at } (a, 0) = 0.$$

$\therefore$ $$\frac{d^2x}{dy^2} = \frac{(3x^2 + y^2)\left(-2y\frac{dx}{dy} - 2x\right) - (-2xy)\left(6x\frac{dx}{dy} + 2y\right)}{(3x^2 + y^2)^2}$$

or $$\frac{d^2x}{dy^2} \text{ at } (a, 0) = \frac{(3a^2 + 0)(0 - 2a) - 0}{(3a^2 + 0)^2} = \frac{-2}{3a}$$

Hence $$\rho \text{ at } (a, 0) = \frac{\left[1 + \left(\frac{dx}{dy}\right)_{(a,0)}\right]^{3/2}}{\left(\frac{d^2x}{dy^2}\right)_{(a,0)}} = \frac{(1 + 0)^{3/2}}{(-2/3a)} = -\frac{3a}{2}.$$

Example 4.45. *Show that the radius of curvature at any point of the cycloid $x = a\,(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ is $4a\cos\theta/2$.* (*V.T.U., 2011 ; P.T.U., 2006*)

Solution. We have $$\frac{dx}{d\theta} = a\,(1 + \cos\theta), \; \frac{dy}{d\theta} = a\sin\theta.$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{a \sin\theta}{a(1+\cos\theta)} = \frac{2\sin\theta/2 \cos\theta/2}{2\cos^2\theta/2} = \tan\theta/2$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right) \cdot \frac{d\theta}{dx} = \frac{1}{2}\sec^2\frac{\theta}{2} \cdot \frac{1}{a(1+\cos\theta)}$$

$$= \frac{1}{2}\sec^2\frac{\theta}{2} \cdot \frac{1}{2a\cos^2\theta/2} = \frac{1}{4a}\sec^4\frac{\theta}{2}.$$

$$\therefore \quad \rho = \frac{[1+(dy/dx)^2]^{3/2}}{d^2y/dx^2} = \frac{4a(1+\tan^2\theta/2)^{3/2}}{\sec^4\theta/2}$$

$$= 4a \cdot (\sec^2\theta/2)^{3/2} \cdot \cos^4\theta/2 = 4a\cos\theta/2.$$

Example 4.46. *Prove that the radius of curvature at any point of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$, is three times the length of the perpendicular from the origin to the tangent at that point.*

(J.N.T.U., 2005 ; Bhopal, 2002 S)

Solution. The parametric equation of the curve is

$$x = a\cos^3 t,\ y = a\sin^3 t.$$

$$\therefore \quad x' (= dx/dt) = -3a\cos^2 t \sin t,\ y' = 3a\sin^2 t\cos t.$$

$$x'' = -3a(\cos^3 t - 2\cos t\sin^2 t) = 3a\cos t(2\sin^2 t - \cos^2 t)$$

$$y'' = 3a(2\sin t\cos^2 t - \sin^3 t) = 3a\sin t(2\cos^2 t - \sin^2 t)$$

$$x'^2 + y'^2 = 9a^2(\cos^4 t\sin^2 t + \sin^4 t\cos^2 t) = 9a^2\sin^2 t\cos^2 t$$

$$x'y'' - y'x'' = -9a^2\cos^2 t\sin^2 t(2\cos^2 t - \sin^2 t)$$
$$- 9a^2\cos^2 t\sin^2 t(2\sin^2 t - \cos^2 t) = -9a^2\sin^2 t\cos^2 t$$

$$\therefore \quad \rho = \frac{(x'^2+y'^2)^{3/2}}{x'y'' - y'x''} = \frac{27a^3\sin^3 t\cos^3 t}{-9a^2\sin^2 t\cos^2 t} = -3a\sin t\cos t.$$

Since $\quad dy/dx = y'/x' = -\tan t,$

$\therefore$ Equation of the tangent at $(a\cos^3 t, a\sin^3 t)$ is $y - a\sin^3 t = -\tan t(x - a\cos^3 t)$

i.e., $\quad x\tan t + y - a\sin t = 0 \quad \ldots(i)$

p, length of $\perp$ from $(0, 0)$ on $(i) = \dfrac{0+0-a\sin t}{\sqrt{(\tan^2 t+1)}} = -a\sin t\cos t$. Thus $\rho = 3p$.

EExample 4.47. *If ρ_1 and ρ_2 be the radii of curvature at the ends of a focal chord of the parabola $y^2 = 4ax$, then show that $\rho_1^{-2/3} + \rho_2^{-2/3} = (2a)^{-2/3}$.* *(Rohtak, 2006 S ; Kurukshetra, 2005)*

Solution. Given parabola is $y^2 = 4ax$ or $x = at^2$, $y = 2at$. If dashes denote differentiation w.r.t. t, then

$$x' = 2at,\ y' = 2a\ ;\ x'' = 2a,\ y'' = 0.$$

$$\therefore \quad \rho \text{ at } (at^2, 2at) = \frac{(x'^2+y'^2)^{3/2}}{x'y'' - x''y'} = \frac{(4a^2t^2+4a^2)^{3/2}}{0-4a^2} = 2a(1+t^2)^{3/2} \quad \text{(Numerically)}$$

If $P(t_1)$ and $Q(t_2)$ be the extremities of the focal chord of the parabola, then

$$t_1 t_2 = -1 \quad i.e., \quad t_2 = -1/t_1 \quad \ldots(i)$$

$$\therefore \quad \rho_1 \text{ at } P(t_1) = 2a(1+t_1^2)^{3/2};\ \rho_2 \text{ at } Q(t_2) = 2a(1+t_2^2)^{3/2}$$

Thus $\quad \rho_1^{-2/3} + \rho_2^{-2/3} = (2a)^{-2/3} = [(1+t_1^2)^{-1} + (1+t_2^2)^{-1}]$

$$= (2a)^{-2/3}\left[\frac{1}{1+t_1^2} + \frac{t_1^2}{1+t_1^2}\right] \quad \text{[By } (i)]$$

$$= (2a)^{-2/3}$$

Example 4.48. *Show that the radius of curvature of P on an ellipse $x^2/a^2 + y^2/b^2 = 1$ is CD^3/ab where CD is the semi-diameter conjugate to CP.* *(J.N.T.U., 2002)*

Solution. Two diameters of an ellipse are said to be conjugate if each bisects chords parallel to the other. If CP and CD are two semi-conjugate diameters and P is $(a \cos \theta, b \sin \theta)$ then D is $a \cos\left(\theta + \frac{\pi}{2}\right), b \sin\left(\theta + \frac{\pi}{2}\right)$ *i.e.*, $(-a \sin \theta, b \cos \theta)$.

Also $C(0, 0)$ is the centre of the ellipse.

$$\therefore \quad CD = \sqrt{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)}$$

At P, we have $x = a \cos \theta, y = b \sin \theta$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{b \cos \theta}{-a \sin \theta} = \frac{-b}{a} \cot \theta ; \frac{d^2y}{dx^2} = \frac{b}{a} \operatorname{cosec}^2 \theta \cdot \frac{d\theta}{dx} = \frac{-b}{a^2} \operatorname{cosec}^3 \theta .$$

$$\therefore \quad \rho = \frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} = \frac{\left(1 + \frac{b^2}{a^2} \cot^2 \theta\right)^{3/2}}{-\frac{b}{a^2} \operatorname{cosec}^3 \theta}$$

$$= \frac{a^2}{b \operatorname{cosec}^3 \theta} \frac{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{3/2}}{a^3 \sin^3 \theta} \qquad \text{(Numerically)}$$

$$= \frac{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{3/2}}{ab} = \frac{CD^3}{ab}.$$

Example 4.49. *Find ρ at the origin for the curves*

(i) $y^4 + x^3 + a(x^2 + y^2) - a^2 y = 0$ *(ii)* $y - x = x^2 + 2xy + y^2$

Solution. (*i*) Equating to zero the lowest degree terms, we get $y = 0$.

$\therefore$ x-axis is the tangent at the origin. Dividing throughout by y, we have

$$y^3 + x \cdot \frac{x^2}{y} + a\left(\frac{x^2}{y} + y\right) - a^2 = 0$$

Let $x \to 0$, so that $\qquad \underset{x \to \infty}{\text{Lt}} (x^2/2y) = \rho$.

$$\therefore \quad 0 + 0.2\rho + a(2\rho + 0) - a^2 = 0 \quad \text{or} \quad \rho = a/2.$$

(*ii*) Equating to zero the lowest degree terms, we get $y = x$, as the tangent at the origin, which is neither of the coordinates axes.

$\therefore$ Putting $y = px + qx^2/2 + \ldots$ in the given equation, we get

$$px + qx^2/2 + \ldots - x = x^2 + 2x\,(px + qx^2/2 + \ldots) + (px + qx^2/2 + \ldots)^2$$

Equating coefficients of x and x^2,

$$p - 1 = 0, q/2 = 1 + 2p + p^2 \quad i.e., \quad p = 1 \text{ and } q = 2 + 4 \cdot 1 + 2 \cdot 1^2 = 8.$$

$$\therefore \quad \rho\,(0, 0) = (1 + p^2)^{3/2}/q = (1 + 1)^{3/2}/8 = 1/2\sqrt{2}.$$

(4) Radius of curvature for polar curve $r = f(\theta)$ is given by

$$\rho = \frac{(\mathbf{r}^2 + \mathbf{r}_1^2)^{3/2}}{\mathbf{r}^2 + 2\mathbf{r}_1^2 - \mathbf{r}\mathbf{r}_2}$$

With the usual notations, we have from Fig. 4.10.

$$\psi = \theta + \phi$$

Differentiating w.r.t. s,

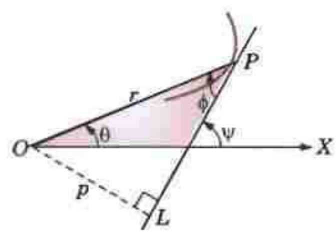


Fig. 4.10

$$\frac{1}{\rho} = \frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{d\theta} \cdot \frac{d\theta}{ds}$$

$$= \frac{d\theta}{ds}\left(1 + \frac{d\phi}{d\theta}\right) \qquad \ldots(1)$$

Also we know that

$$\tan\phi = r\frac{d\theta}{dr} = \frac{r}{r_1} \quad \text{or} \quad \phi = \tan^{-1}\left(\frac{r}{r_1}\right) \qquad \text{where } r_1 = \frac{dr}{d\theta}$$

Differentiating w.r.t. θ,

$$\frac{d\phi}{d\theta} = \frac{1}{1+(r/r_1)^2}\cdot\frac{r_1\cdot r_1 - rr_2}{r_1^2} = \frac{r_1^2 - rr_2}{r^2 + r_1^2} \qquad ...(2)$$

Also, $$\frac{ds}{d\theta} = \sqrt{(r^2 + r_1^2)} \qquad ...(3)$$

Substituting the value from (2) and (3) in (1),

$$\frac{1}{\rho} = \frac{1}{\sqrt{r^2 + r_1^2}}\cdot\left(1 + \frac{r_1^2 - rr_2}{r^2 + r_1^2}\right)$$

Hence $$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$$

(5) Radius of curvature for pedal curve $p = f(r)$ *is given by*

$$\boldsymbol{\rho = r\frac{dr}{dp}}$$

With the usual notation (Fig. 4.10), we have $\psi = \theta + \phi$

Differentiating w.r.t. s,

$$\frac{1}{\rho} = \frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds} \qquad ...(1)$$

Also we know that $p = r\sin\phi$

$$\therefore \quad \frac{dp}{dr} = \sin\phi + r\cos\phi\,\frac{d\phi}{ds}$$

$$= r\frac{d\theta}{ds} + r\frac{dr}{ds}\cdot\frac{d\phi}{dr} \qquad \text{[By (3) and (4) of § 4.9 (2)]}$$

$$= r\left(\frac{d\theta}{ds} + \frac{d\phi}{ds}\right) = \frac{r}{\rho} \qquad \text{[By (1)]}$$

Hence $$\rho = r\frac{dr}{dp}.$$

Example 4.50. *Show that the radius of curvature at any point of the cardioid $r = a\,(1 - \cos\theta)$ varies as $\sqrt{r}$.* *(V.T.U., 2003)*

Solution. Differentiating w.r.t. θ, we get

$$r_1 = a\sin\theta,\ r_2 = a\cos\theta$$

$$\therefore \quad (r^2 + r_1^2)^{3/2} = [a^2(1-\cos\theta)^2 + a^2\sin^2\theta]^{3/2} = a^3[2\,(1-\cos\theta)]^{3/2}$$

$$r^2 - rr_2 + 2r_1^2 = a^2(1-\cos\theta)^2 - a^2\,(1-\cos\theta)\cos\theta + 2a^2\sin^2\theta = 3a^2\,(1-\cos\theta)$$

Thus $$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 - rr_2 + 2r_1^2} = \frac{a^3\,2\sqrt{2}(1-\cos\theta)^{3/2}}{3a^2(1-\cos\theta)}$$

$$= \frac{2\sqrt{2}}{3}a\,(1-\cos\theta)^{1/2} = \frac{2\sqrt{2}a}{3}\left(\frac{r}{a}\right)^{1/2} \propto \sqrt{r}.$$

Otherwise. The pedal equation of this cardioid is $2ap^2 = r^3$ ...(*i*)

Differentiating w.r.t. p, we get

that $$4ap = 3r^2\frac{dr}{dp} \text{ whence } \rho = r\frac{dr}{dp} = \frac{4ap}{3r} = \frac{4ar^{3/2}}{3r\cdot\sqrt{(2a)}} \propto \sqrt{r}.$$

$$[\because \quad p = r^{3/2}/\sqrt{(2a)} \text{ from } (i)]$$

PROBLEMS 4.11

1. Find the radius of curvature at any point
   (i) $(at^2, 2at)$ of the parabola $y^2 = 4ax$.
   (ii) (o, c) of the catenary $y = c \cosh x/c$.
   (iii) $(a, 0)$ of the curve $y = x^3 (x - a)$. *(V.T.U., 2010)*
2. Show that for (i) the rectangular hyperbola $xy = c^2$, $\rho = \dfrac{(x^2+y^2)^{3/2}}{2c^2}$. *(Rohtak, 2005 ; Madras, 2000)*
   (ii) the curve $y = ae^{x/a}$, $\rho = a \sec^2 \theta \operatorname{cosec} \theta$ where $\theta = \tan^{-1}(y/a)$. *(Rajasthan, 2006)*
3. Show that the radius of curvature at
   (i) $(a, 0)$ on the curve $y^2 = a^2 (a - x)/x$ is $a/2$. *(V.T.U., 2000 S)*
   (ii) $(a/4, a/4)$ on the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ is $a/\sqrt{2}$. *(J.N.T.U., 2006 S)*
   (iii) $x = \pi/2$ of the curve $y = 4 \sin x - \sin 2x$ is $5\sqrt{5}/4$. *(V.T.U., 2009 S)*
4. For the curve $y = \dfrac{ax}{a+x}$, show that $\left(\dfrac{2\rho}{a}\right)^{2/3} = \left(\dfrac{x}{y}\right)^2 + \left(\dfrac{y}{x}\right)^2$. *(V.T.U., 2008)*
5. Find the radius of curvature at any point on the
   (i) *ellipse* : $x = a \cos \theta$, $y = b \sin \theta$. *(V.T.U., 2003)*
   (ii) *cycloid* : $x = a (\theta - \sin \theta)$, $y = a (1 - \cos \theta)$.
   (iii) *curve* : $x = a (\cos t + t \sin t)$, $y = a (\sin t - t \cos t)$.
6. Show that the radius of curvature (i) at the point $(a \cos^3 \theta, a \sin^3 \theta)$ on the curve
   $x^{2/3} + y^{2/3} = a^{2/3}$ is $3a \sin \theta \cos \theta$. *(Anna, 2009)*
   (ii) at the point t on the curve $x = e^t \cos t$, $y = e^t \sin t$ is $\sqrt{2}e^t$. *(Calicut, 2005)*
7. If ρ be the radius of curvature at any point P on the parabola, $y^2 = 4ax$ and S be its focus, then show that ρ^2 varies as $(SP)^3$. *(Kurukshetra, 2006)*
8. Prove that for the ellipse in pedal form $\dfrac{1}{p^2} = \dfrac{1}{a^2} + \dfrac{1}{b^2} - \dfrac{1}{a^2b^2}$, the radius of curvature at the point (p, r) is $\rho = a^2 b^2/p^3$. *(V.T.U., 2010 S)*
9. Show that the radius of curvature at an end of the major axis of the ellipse $x^2/a^2 + y^2/b^2 = 1$ is equal to the semi-latus rectum. *(Osmania, 2000 S)*
10. Show that the radius of curvature at each point of the curve $x = a (\cos t + \log \tan t/2)$, $y = a \sin t$, is inversely proportional to the length of the normal intercepted between the point on the curve and the x-axis. *(J.N.T.U., 2003)*
11. Find the radius of curvature at the origin for
    (i) $x^3 + y^3 - 2x^2 + 6y = 0$ *(Burdwan, 2003)*
    (ii) $2x^4 + 3y^4 + 4x^2y + xy - y^2 + 2x = 0$
    (iii) $y^2 = x^2 (a + x)/(a - x)$.
12. Find the radius of the curvature at the point (r, θ) on each of the curves :
    (i) $r = a (1 - \cos \theta)$ *(Kurukshetra, 2005)*
    (ii) $r^n = a^n \cos n\theta$. *(P.T.U., 2010; J.N.T.U., 2006)*
13. For the cardioid $r = a(1 + \cos \theta)$, show that ρ^2/r is constant. *(P.T.U., 2005)*
14. Find the radius of curvature for the parabola $2a/r = 1 + \cos \theta$. *(Kurukshetra, 2006)*
15. If ρ_1, ρ_2 be the radii of curvature at the extremities of any chord of the cardioid $r = a (1 + \cos \theta)$ which passes through the pole, show that $\rho_1{}^2 + \rho_2{}^2 = 16 a^2/9$.
16. For any curve $r = f(\theta)$, prove that $\dfrac{r}{\rho} = \sin \phi \left(1 + \dfrac{d\phi}{d\theta}\right)$.

4.12 (1) CENTRE OF CURVATURE at any point P (x, y) on the curve y = f (x) is given by

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}, \quad \bar{y} = y + \frac{1+y_1^2}{y_2}.$$

Let $C(\bar{x}, \bar{y})$ be the centre of curvature and ρ the radius of curvature of the curve at $P(x, y)$ (Fig. 4.11). Draw PL and CM ⊥s to OX and $PN \perp CM$. Let the tangent at P make an $\angle\psi$ with the x-axis. Then $\angle NCP = 90° - \angle NPC = \angle NPT = \psi$

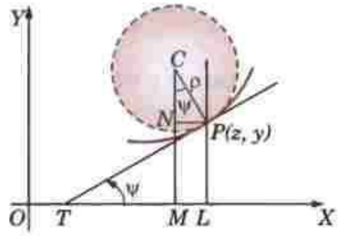


Fig. 4.11

$$\therefore \quad \bar{x} = OM = OL - ML = OL - NP$$

$$= x - \rho \sin\psi = x - \frac{(1+y_1^2)^{3/2}}{y_2} \cdot \frac{y_1}{\sqrt{(1+y_1^2)}}$$

$$\left[\because \tan\psi = y_1, \quad \therefore \sin\psi = \frac{y_1}{\sqrt{(1+y_1^2)}}\right.$$

$$= x - \frac{y_1(1+y_1^2)}{y_2}$$

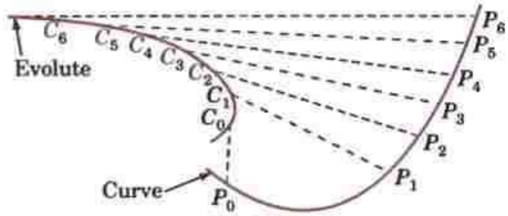


Fig. 4.12

and $$\bar{y} = MC = MN + NC = LP + \rho\cos\psi$$

$$\left[\because \sec\psi = \sqrt{(1+\tan^2\psi)} = \sqrt{(1+y_1^2)}\right.$$

$$= y + \frac{(1+y_1^2)^{3/2}}{y_2} \cdot \frac{1}{\sqrt{(1+y_1^2)}} = y + \frac{1+y_1^2}{y_2}$$

Cor. Equation of the circle of curvature at P is $(x - \bar{x})^2 + (y - \bar{y})^2 = \rho^2$.

(2) Evolute. *The locus of the centre of curvature for a curve is called its* **evolute** *and the curve is called an* **involute** *of its evolute.* (*Fig. 4.12*)

Example 4.51. *Find the coordinates of the centre of curvature at any point of the parabola $y^2 = 4ax$. Hence show that its evolute is*

$$27ay^2 = 4(x - 2a)^3. \qquad \text{(V.T.U., 2000)}$$

Solution. We have $2yy_1 = 4a$ *i.e.*, $y_1 = 2a/y$

and $$y_2 = -\frac{2a}{y^2} \cdot y_1 = -\frac{4a^2}{y^3}$$

If $(\bar{x}, \bar{y})$ be the centre of curvature, then

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} = x - \frac{2a/y(1+4a^2/y^2)}{-4a^2/y^3}$$

$$= x + \frac{y^2+4a^2}{2a} = x + \frac{4ax+4a^2}{2a} = 3x + 2a \qquad [\because y^2 = 4ax] \qquad ...(i)$$

and $$\bar{y} = y + \frac{1+y_1^2}{y_2} = y + \frac{1+4a^2/y^2}{-4a^2/y^2}$$

$$= y - \frac{y(y^2+4a^2)}{4a^2} = \frac{-y^3}{4a^2} = -\frac{2x^{3/2}}{\sqrt{a}} \qquad ...(ii)$$

To find the evolute, we have to eliminate x from (*i*) and (*ii*)

$$\therefore \quad (\bar{y})^2 = \frac{4x^3}{a} = \frac{4}{a}\left(\frac{\bar{x}-2a}{3}\right)^3 \quad \text{or} \quad 27a(\bar{y})^2 = 4(\bar{x} - 2a)^3.$$

Thus the locus of $(\bar{x}, \bar{y})$ *i.e.*, evolute, is $27ay^2 = 4(x - 2a)^3$.

Example 4.52. *Show that the evolute of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ is another equal cycloid.* (*Madras, 2006*)

Solution. We have $y_1 = \dfrac{dy}{d\theta} \div \dfrac{dx}{d\theta} = \dfrac{a\sin\theta}{a(1-\cos\theta)} = \cot\dfrac{\theta}{2}$.

$$y^2 = \frac{d}{dx}(y_1) = \frac{d}{d\theta}\left(\cot\frac{\theta}{2}\right)\cdot\frac{d\theta}{dx}$$

$$= -\operatorname{cosec}^2\frac{\theta}{2}\cdot\frac{1}{2}\cdot\frac{1}{a(1-\cos\theta)} = -\frac{1}{4a\sin^4\theta/2}$$

If $(\bar{x}, \bar{y})$ be the centre of curvature, then

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} = a(\theta - \sin\theta) + \cot\frac{\theta}{2}\left(-4a\sin^4\frac{\theta}{2}\right)\left(1+\cot^2\frac{\theta}{2}\right)$$

$$= a(\theta - \sin\theta) + \frac{\cos\theta/2}{\sin\theta/2}\cdot 4a\sin^4\frac{\theta}{2}\cdot\operatorname{cosec}^2\frac{\theta}{2}$$

$$= a(\theta - \sin\theta) + 4a\sin\theta/2\cos\theta/2 = a(\theta - \sin\theta) + 2a\sin\theta = a(\theta + \sin\theta)$$

$$\bar{y} = y + \frac{1+y_1^2}{y_2} = a(1-\cos\theta) + \left(1+\cot^2\frac{\theta}{2}\right)\left(-4a\sin^4\frac{\theta}{2}\right)$$

$$= a(1-\cos\theta) - 4a\sin^4\theta/2\,.\operatorname{cosec}^2\theta/2$$

$$= a(1-\cos\theta) - 4a\sin^2\theta/2$$

$$= a(1-\cos\theta) - 2a(1-\cos\theta) = -a(1-\cos\theta)$$

Hence the locus of $(\bar{x}, \bar{y})$ *i.e.*, the evolute, is given by

$x = a(\theta + \sin\theta)$, $y = -a(1-\cos\theta)$ which is another equal cycloid.

(3) Chord or curvature *at a given point of a curve*

(*i*) parallel to x-axis $= 2\rho\sin\psi$

(*ii*) parallel to y-axis $= 2\rho\cos\psi$

Consider the circle of curvature at a given point P on a curve. Let C be the centre and ρ the radius of curvature at P so that $PQ = 2\rho$. (Fig. 4.13)

Let PL, PM be the chords of curvature parallel to the axes of x and y respectively. Let the tangent PT make an $\angle\psi$ with the x-axis so that $\angle LQP = \angle QPM = \psi$.

Then from the rt. $\angle$ed ΔPLQ,

$$PL = 2\rho\sin\psi\sigma$$

and
$$PM = 2\rho\cos\psi.$$

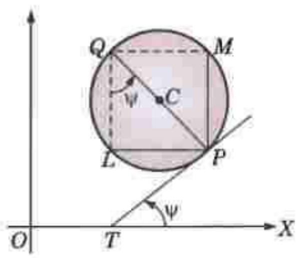


Fig. 4.13

4.13 (1) ENVELOPE

The equation $x\cos\alpha + y\sin\alpha = 1$ ...(1)

represents a straight line for a given value of α. If different values are given to α, we get different straight lines. All these straight lines thus obtained are said to constitute a family of straight lines.

In general, the *curves corresponding to the equation $f(x, y, \alpha) = 0$ for different values of α, constitute a* **family of curves** *and α is called the* **parameter** *of the family.*

The envelope of a family of curves is the curve which touches each member of the family. For example, we know that all the straight lines of the family (1) touch the circle

$$x^2 + y^2 = 1 \quad ...(2)$$

i.e., the envelope of the family of lines (1) is the circle (2)—Fig. 4.14, which may also be seen as the locus of the ultimate points of intersection of the consecutive members of the family of lines (1). This leads to the following :

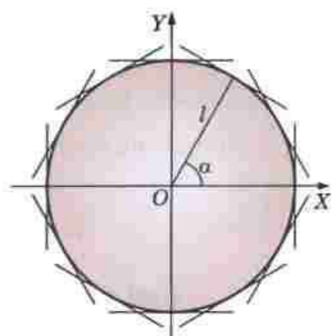


Fig. 4.14

Def. *If $f(x, y, \alpha) = 0$ and $f(x, y, \alpha + \delta\alpha) = 0$ be two consecutive members of a family of curves, then the locus of their ultimate points of intersection is called the* **envelope** *of that family.*

(2) Rule to find the envelope of the family of curves $f(x, y, \alpha) = 0$:

Eliminate α from $f(x, y, \alpha) = 0$ and $\dfrac{\partial f(x, y, \alpha)}{\partial \alpha} = 0$.

Example 4.53. *Find the envelope of the family of lines $y = mx + \sqrt{(1 + m^2)}$, m being the parameter.*

Solution. We have $(y - mx)^2 = 1 + m^2$ ...(*i*)

Differentiating (*i*) partially with respect to m,

$$2(y - mx)(-x) = 2m \quad \text{or} \quad m = xy/(x^2 - 1) \qquad ...(ii)$$

Now eliminating m from (*i*) and (*ii*)

Substituting the value of m in (*i*), we get

$$\left(y - \frac{x^2 y}{x^2 - 1}\right)^2 = 1 + \left(\frac{xy}{x^2 - 1}\right)^2 \quad \text{or} \quad y^2 = (x^2 - 1)^2 + x^2y^2$$

or $\qquad x^2 + y^2 = 1 \qquad$ which is the required equation of the envelope.

Obs. Sometimes the equation to the family of curves contains two parameters which are connected by a relation. In such cases, we eliminate one of the parameters by means of the given relation, then proceed to find the envelope.

Example 4.54. *Find the envelope of a system of concentric and coaxial ellipses of constant area.*

Solution. Taking the common axes of the system of ellipses as the coordinate axes, the equation to an ellipse of the family is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ where } a \text{ and } b \text{ are the parameters.} \qquad ...(i)$$

The area of the ellipse = πab which is given to be constant, say = πc^2.

$$\therefore \qquad ab = c^2 \quad \text{or} \quad b = c^2/a. \qquad ...(ii)$$

Substituting in (*i*), $\dfrac{x^2}{a^2} + \dfrac{y^2}{(c^2/a^2)} = 1 \quad$ or $\quad x^2 a^{-2} + (y^2/c^4)\, a^2 = 0$ ...(*iii*)

which is the given family of ellipses with a as the only parameter.

Differentiating partially (*iii*) with respect to a,

$$-2x^2 a^{-3} + 2(y^2/c^4)\, a = 0 \quad \text{or} \quad a^2 = c^2\, x/y \qquad ...(iv)$$

Eliminate a from (*iii*) and (*iv*).

Substituting the value of a^2 in (*iii*), we get

$$x^2(y/c^2x) + (y^2/c^4)(c^2x/y) = 1 \text{ or } 2xy = c^2$$

which is the required equation of the envelope. P

(3) Evolute *of a curve is the envelope of the normals to that curve* (*Fig. 4.12*)

Example 4.55. *Find the evolute of the parabola $y^2 = 4ax$.* (*Madras, 2003*)

Solution. Any normal to the parabola is $y = mx - 2\,am - am^3$ ...(*i*)

Differentiating it with respect to m partially,

$$0 = x - 2a - 3am^2 \text{ or } m = [(x - 2a)/3a]^{1/2}$$

Substituting this value of m in (*i*),

$$y = \left(\frac{x - 2a}{3a}\right)^{1/2} \left[x - 2a - a \cdot \frac{x - 2a}{3a}\right]$$

Squaring both sides, we have

$$27ay^2 = 4(x - 2a)^3$$

which is the evolute of the parabola. (*cf.* Example 4.51).

PROBLEMS 4.12

1. Find the coordinates of the centre of curvature at $(at^2, 2at)$ on the parabola $y^2 = 4ax$. *(V.T.U., 2000 S)*
2. If the centre of curvature of the ellipse $x^2/a^2 + y^2/b^2 = 1$ at one end of the minor axis lies at the other end, then show that the eccentricity of the ellipse is $1/\sqrt{2}$. *(Anna, 2005 S ; Madras, 2003)*
3. Show that the equation of the evolute of the
   (*i*) *parabola* $x^2 = 4ay$ is $4(y - 2a)^3 = 27ax^2$. *(Anna, 2009)*
   (*ii*) *ellipse* $x = a \cos \theta, y = b \sin \theta$ (*i.e.*, $x^2/a^2 + y^2/b^2 = 1$) is
   $$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}.$$
   (*iii*) *rectangular hyperbola* $xy = c^2$, (*i.e.*, $x = ct, y = c/t$) is
   $$(x + y)^{2/3} - (x - y)^{2/3} = (4c)^{2/3}.$$ *(Anna, 2003)*
4. Find the evolute of (*i*) cycloid $x = a(t + \sin t), y = a(1 - \cos t)$
   (*ii*) the curve $x = a(\cos \theta + \theta \sin \theta), y = a(\sin \theta - \theta \cos \theta)$. *(Anna, 2009 S)*
5. Find the evolute of the curve $x = a \cos^3 \theta, y = a \sin^3 \theta$ *i.e.*, $x^{2/3} + y^{2/3} = a^{2/3}$. *(Osmania, 2002)*
6. Show that the evolute of the curve $x = a(\cos t + \log \tan t/2), y = a \sin t$ is $y = a \cosh x/a$. *(Anna, 2005 S)*
7. Find the circle of curvature at the point (*i*) $(a/4, a/4)$ of the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$.
   (*ii*) $(3/2, 3/2)$ of the curve $x^3 + y^3 = 3xy$ *(Anna, 2009 ; Madras, 2006 ; Calicut, 2005)*
8. Show that the circle of curvature at the origin for the curve $x + y = ax^2 + by^2 + cx^3$ is
   $$(a + b)(x^2 + y^2) = 2(x + y).$$ *(Nagpur, 2009)*
9. If C_x, C_y be the chords of curvature parallel to the axes at any point on the curve $y = ae^{x/a}$, prove that
   $$\frac{1}{C_x^2} + \frac{1}{C_y^2} = \frac{1}{2aC_x}.$$
10. In the curve $y = a \cosh x/a$, prove that the chord of curvature parallel to y-axis is the double the ordinate.

Find the envelope of the following family of lines :

11. $y = mx + a/m$, m being the parameter. *(Madras, 2006)*
12. $\dfrac{x}{a} \cos \alpha + \dfrac{y}{b} \sin \alpha = 1$, α being the parameter.
13. $y = mx - 2am - am^3$.
14. $y = mx + \sqrt{(a^2m^2 + b^2)}$, m being the parameter. *(Anna, 2009)*
15. Find the envelope of the family of parabolas $y = x \tan \alpha - \dfrac{gx^2}{2u^2 \cos \alpha}$, α being the parameter.
16. Find the envelope of the straight line $x/a + y/b = 1$, where the parameters a and b are connected by the relation :
    (*i*) $a + b = c$. (*ii*) $ab = c^2$
    (*iii*) $a^2 + b^2 = c^2$.
17. Find the envelope of the family of ellipses $x^2/a^2 + y^2/b^2 = 1$ for which $a + b = c$. *(Madras, 2006)*

Prove that the evolute of the

18. *ellipse* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$. *(J.N.T.U., 2006 ; Anna, 2005)*
19. *hyperbola* $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is $(ax)^{2/3} - (by)^{2/3} = (a^2 + b^2)^{2/3}$. *(Anna, 2009)*
20. *parabola* $x^2 = 4by$ is $27bx^2 = 4(y - 2b)^3$.

4.14 (1) INCREASING AND DECREASING FUNCTIONS

In the function $y = f(x)$, *if y increases as x increases* (as at A), *it is called an* **increasing function of** x. On the contrary, *if y decreases as x increases* (as at C), *it is called a* **decreasing function** *of* x.

Let the tangent at any point on the graph of the function make an $\angle\psi$ with the x-axis (Fig. 4.15) so that

$$dy/dx = \tan\psi$$

At any point such as A, where the function is increasing $\angle\psi$ is acute *i.e.*, dy/dx is positive. At a point such as C, where the function is decreasing $\angle\psi$ is obtuse *i.e.*, dy/dx is negative.

Hence the *derivative of an increasing function is + ve*, and *the derivative of a decreasing function is – ve.*

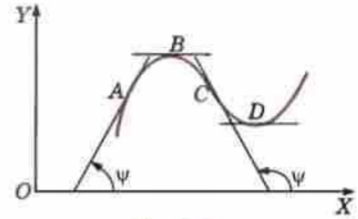


Fig. 4.15

Obs. *If the derivative is zero* (as at B or D), *then y is neither increasing nor decreasing.* In such cases, we say *that the function is* **stationary.**

(2) Concavity, Convexity and Point of Inflexion

(*i*) *If a portion of the curve on both sides of a point, however small it may be, lies above the tangent* (*as at D*), *then the curve is said to be* **concave upwards** at D where d^2y/dx^2 is positive.

(*ii*) *If a portion of the curve on both sides of a point lies below the tangent* (as at B), *then the curve is said to be* **Convex upwards** at B where d^2y/dx^2 is negative.

(*iii*) *If the two portions of the curve lie on different sides of the tangent thereat* (*i.e.*, the curve crosses the tangent (as at C), *then the point C* is said to be a **point of inflexion** of the curve.

At a point of inflexion $\dfrac{d^2y}{dx^2} = 0$ and $\dfrac{d^3y}{dx^3} \neq 0.$

4.15 (1) MAXIMA AND MINIMA

Consider the graph of the continuous function $y = f(x)$ in the interval (x_1, x_2) (Fig. 4.16). Clearly the point P_1 is the highest in its own immediate neighbourhood. So also is P_3. At each of these points P_1, P_3 the function is said to have a *maximum* value.

On the other hand, the point P_2 is the lowest in its own immediate neighbourhood. So also is P_4. At each of these points P_2, P_4 the function is said to have a *minimum* value.

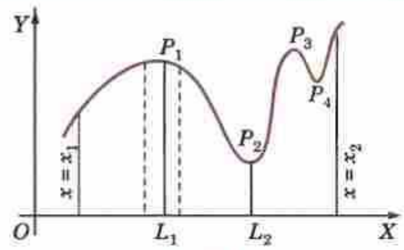


Fig. 4.16

Thus, we have

Def. *A function f(x) is said to have a* **maximum** *value at x = a, if there exists a small number h, however small, such that f(a) > both f(a – h) and f(a + h).*

A function f(x) is said to have a **minimum** *value at x = a, if there exists a small number h, however small, such that f(a) < both f(a – h) and f(a + h).*

Obs. 1. *The maximum and minimum values of a function taken together are called its* **extreme** *values and the points at which the function attains the extreme values are called the* **turning points** *of the function.*

Obs. 2. *A maximum or minimum value of a function is not necessarily the greatest or least value of the function in any finite interval.* The maximum value is simply the greatest value in the immediate neighbourhood of the maxima point or the minimum value is the least value in the immediate neighbourhood of the minima point. In fact, *there may be several maximum and minimum values of a function in an interval and a minimum value may be even greater than a maximum value.*

Obs. 3. It is seen from the Fig. 4.16 that *maxima and minima values occur alternately.*

(2) Conditions for maxima and minima. At each point of extreme value, it is seen from Fig. 4.16 that the tangent to the curve is parallel to the x-axis, *i.e.*, its slope ($= dy/dx$) is zero. Thus if *the function is maximum or minimum at $x = a$, then $(dy/dx)_a = 0$.*

Around a maximum point say, P_1 $(x = a)$, the curve is increasing in a small interval $(a - h, a)$ before L_1 and decreasing in $(a, a + h)$ after L_1 where h is positive and small.

i.e., in $(a - h, a)$, $dy/dx \geq 0$; at $x = a$, $dy/dx = 0$ and in $(a, a + h)$, $dy/dx \leq 0$.

Thus dy/dx (which is a function of x) changes sign from positive to negative in passing through P_1, *i.e.*, it is a decreasing function in the interval $(a - h, a + h)$ and therefore, its derivative d^2y/dx^2 is negative at $P_1(x = a)$.

Similarly, around a minimum point say P_2, dy/dx changes sign from negative to positive in passing through P_2, *i.e.*, it is an increasing function in the small interval around L_2 and therefore its derivative d^2y/dx^2 is positive at P_2.

Hence (*i*) *$f(x)$ is maximum at $x = a$ if $f'(a) = 0$ and $f''(a)$ is – ve [i.e., $f'(a)$ changes sign from + ve to – ve]*
(*ii*) *$f(x)$ is minimum at $x = a$, if $f'(a) = 0$ and $f''(a)$ is + ve [i.e., $f'(a)$ changes sign from – ve to + ve]*

Obs. A maximum or a minimum value is a stationary value but a stationary value may neither be a maximum nor a minimum value.

(3) Procedure for finding maxima and minima

(*i*) *Put the given function = $f(x)$*

(*ii*) *Find $f'(x)$ and equate it to zero. Solve this equation and let its roots be a, b, c, ...*

(*iii*) *Find $f''(x)$ and substitute in it by turns $x = a, b, c, ...$*

If $f''(a)$ is – ve, $f(x)$ is maximum at $x = a$.

If $f''(a)$ is + ve, $f''(x)$ is minima at $x = a$.

(*iv*) Sometimes $f''(x)$ may be difficult to find out or $f''(x)$ may be zero at $x = a$. In such cases, see if $f'(x)$ *changes sign from + ve to – ve as x passes through a, then $f(x)$ is maximum at $x = a$.*

If $f'(x)$ changes sign from – ve to + ve as x passes through a, $f(x)$ is minimum at $x = a$.

If $f'(x)$ does not change sign while passing through $x = a$, $f(x)$ is neither maximum nor minimum at $x = a$.

Example 4.56. *Find the maximum and minimum values of $3x^4 - 2x^3 - 6x^2 + 6x + 1$ in the interval (0, 2).*

Solution. Let $f(x) = 3x^4 - 2x^3 - 6x^2 + 6x + 1$

Then $f'(x) = 12x^3 - 6x^2 - 12x + 6 = 6(x^2 - 1)(2x - 1)$

$\therefore$ $f'(x) = 0$ when $x = \pm 1, \frac{1}{2}$.

So in the interval (0, 2) $f(x)$ can have maximum or minimum at $x = \frac{1}{2}$ or 1.

Now $f''(x) = 36x^2 - 12x - 12 = 12(3x^2 - x - 1)$ so that $f''\left(\frac{1}{2}\right) = -9$ and $f''(1) = 12$.

$\therefore$ $f(x)$ has a maximum at $x = \frac{1}{2}$ and a minimum at $x = 1$.

Thus the maximum value $= f\left(\frac{1}{2}\right) = 3\left(\frac{1}{2}\right)^4 - 2\left(\frac{1}{2}\right)^3 - 6\left(\frac{1}{2}\right)^2 + 6\left(\frac{1}{2}\right) + 1 = 2\frac{7}{16}$

and the minimum value $= f(1) = 3(1)^4 - 2(1)^3 - 6(1)^2 + 6(1) + 1 = 2$.

Example 4.57. *Show that $\sin x\,(1 + \cos x)$ is a maximum when $x = \pi/3$.*

(Bhopal, 2009 ; Rajasthan, 2005)

Solution. Let $f(x) = \sin x\,(1 + \cos x)$

Then
$$f'(x) = \cos x\,(1 + \cos x) + \sin x\,(-\sin x)$$
$$= \cos x\,(1 + \cos x) - (1 - \cos^2 x) = (1 + \cos x)(2\cos x - 1)$$

$\therefore$ $f'(x) = 0$ when $\cos x = \frac{1}{2}$ or -1 *i.e.*, when $x = \pi/3$ or π.

Now $f''(x) = -\sin x\,(2\cos x - 1) + (1 + \cos x)(-2\sin x) = -\sin x\,(4\cos x + 1)$

so that $f''(\pi/3) = -3\sqrt{2}/2$ and $f''(\pi) = 0$.

Thus $f(x)$ has a maximum at $x = \pi/3$.

Since $f''(\pi)$ is 0, let us see whether $f'(x)$ changes sign or not.

When x is slightly $< \pi$, $f'(x)$ is – ve, then when x is slightly $> \pi$, $f'(x)$ is again – ve *i.e.*, $f'(x)$ does not change sign as x passes through π. So $f(x)$ is neither maximum nor minimum at $x = \pi$.

(4) Practical Problems

In many problems, the function (whose maximum or minimum value is required) is not directly given. It has to be formed from the given data. If the function contains two variables, one of them has to be eliminated with the help of the other conditions of the problem. A number of problems deal with triangles, rectangles, circles, spheres, cones, cylinders etc. The student is therefore, advised to remember the formulae for areas, volumes, surfaces etc. of such figures.

Example 4.58. *A window has the form of a rectangle surmounted by a semi-circle. If the perimeter is 40 ft., find its dimensions so that the greatest amount of light may be admitted.* *(Madras, 2000 S)*

Solution. The greatest amount of light may be admitted means that the area of the window may be maximum.

Let x ft. be the radius of the semi-circle so that one side of the rectangle is $2x$ ft. (Fig. 4.17). Let the other side of the rectangle y ft. Then the perimeter of the whole figure

$$= \pi x + 2x + 2y = 40 \text{ (given) and the area } A = \tfrac{1}{2}\pi x^2 + 2xy. \qquad ...(i)$$

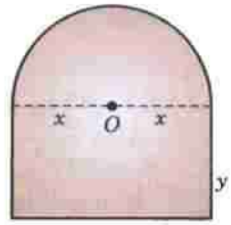


Fig. 4.17

Here A is a function of two variables x and y. To express A in terms of one variable x (say), we substitute the value of y from (i) in it.

$$\therefore \qquad A = \tfrac{1}{2}\pi x^2 + x[40 - (\pi + 2)x] = 40\,x - \left(\frac{\pi}{2} + 2\right)x^2$$

Then $$\frac{dA}{dx} = 40 - (\pi + 4)x$$

For A to be maximum or minimum, we must have $dA/dx = 0$ *i.e.*, $40 - (\pi + 4)x = 0$

or $$x = 40/(\pi + 4)$$

$\therefore$ From (i), $$y = \tfrac{1}{2}[40 - (\pi + 2)x] = \tfrac{1}{2}[40 - (\pi + 2)\,40/(\pi + 4)] = 40/(\pi + 4) \text{ i.e., } x = y$$

Also $$\frac{d^2A}{dx^2} = -(\pi + 4), \text{ which is negative.}$$

Thus the area of the window is maximum when the radius of the semi-circle is equal to the height of the rectangle.

Example 4.59. *A rectangular sheet of metal of length 6 metres and width 2 metres is given. Four equal squares are removed from the corners. The sides of this sheet are now turned up to form an open rectangular box. Find approximately, the height of the box, such that the volume of the box is maximum.*

Solution. Let the side of each of the squares cut off be x m so that the height of the box is x m and the sides of the base are $6 - 2x$, $2 - 2x$ m (Fig. 4.18).

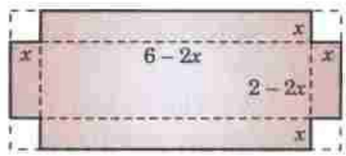


Fig. 4.18

$\therefore$ Volume V of the box

$$= x(6 - 2x)(2 - 2x) = 4(x^3 - 4x^2 + 3x)$$

Then $$\frac{dV}{dx} = 4(3x^2 - 8x + 3)$$

For V to be maximum or minimum, we must have

$$dV/dx = 0 \text{ i.e., } 3x^2 - 8x + 3 = 0$$

$$\therefore \qquad x = \frac{8 \pm \sqrt{[64 - 4 \times 3 \times 3]}}{6} = 2.2 \text{ or } 0.45 \text{ m.}$$

The value $x = 2.2$ m is inadmissible, as no box is possible for this value.

Also $\dfrac{d^2V}{dx^2} = 4(6x - 8)$, which is – ve for $x = 0.45$ m.

Hence the volume of the box is maximum when its height is 45 cm.

Example 4.60. *Show that the right circular cylinder of given surface (including the ends) and maximum volume is such that its height is equal to the diameter of the base.*

Solution. Let r be the radius of the base and h, the height of the cylinder.

Then given surface $\quad S = 2\pi rh + 2\pi r^2 \quad ...(i) \quad$ and the volume $V = \pi r^2 h \quad ...(ii)$

Hence V is a function of two variables r and h. To express V in terms of one variable only (say r), we substitute the value of h from (i) in (ii).

Then $$V = \pi r^2\left(\frac{S - 2\pi r^2}{2\pi r}\right) = \frac{1}{2}Sr - \pi r^3 \quad \therefore \quad \frac{dV}{dr} = \frac{1}{2}S - 3\pi r^2.$$

For V to be maximum or minimum, we must have $dV/dr = 0$,

i.e., $$\tfrac{1}{2}S - 3\pi r^2 = 0 \quad \text{or} \quad r = \sqrt{(S/6\pi)}.$$

Also $\dfrac{d^2V}{dr^2} = -6\pi r$, which is negative for $r = \sqrt{(S/6\pi)}$.

Hence V is maximum for $r = \sqrt{(S/6\pi)}$.

i.e., for $6\pi r^2 = S = 2\pi rh + 2\pi r^2$ *i.e.,* for $h = 2r$, which proves the required result. [By (*i*)]

Example 4.61. *Show that the diameter of the right circular cylinder of greatest curved surface which can be inscribed in a given cone is equal to the radius of the cone.*

Solution. Let r be the radius OA of the base and α the semi-vertical angle of the given cone (Fig. 4.19). Inscribe a cylinder in it with base-radius $OL = x$.

Then the height of the cylinder LP

$$= LA \cot\alpha = (r - x)\cot\alpha$$

$\therefore$ The curved surface S of the cylinder

$$= 2\pi x.\, LP = 2\pi x(r - x)\cot\alpha$$
$$= 2\pi\cot\alpha\,(rx - x^2)$$

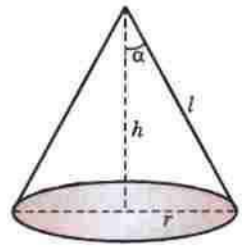


Fig. 4.19

$\therefore$ $$\frac{dS}{dx} = 2\pi\cot\alpha\,(r - 2x) = 0 \text{ for } x = r/2.$$

and $$\frac{d^2S}{dx^2} = -4\pi\cot\alpha$$

Hence S is maximum when $x = r/2$.

Example 4.62. *Find the altitude and the semi-vertical angle of a cone of least volume which can be circumscribed to a sphere of radius a.*

Solution. Let h be the height and α the semi-vertical angle of the cone so that its radius $BD = h\tan\alpha$ (Fig. 4.20).

$\therefore$ The volume V of the cone is given by

$$V = \frac{1}{3}\pi(h\tan\alpha)^2 h = \frac{1}{3}\pi h^3\tan^2\alpha.$$

Now we must express $\tan\alpha$ in terms of h.

In the rt. $\angle d\ \Delta AEO$,

$$EA = \sqrt{(OA^2 - a^2)} = \sqrt{[(h-a)^2 - a^2]} = \sqrt{(h^2 - 2ha)}$$

$\therefore$ $$\tan\alpha = \frac{EO}{EA} = \frac{a}{\sqrt{(h^2 - 2ha)}}$$

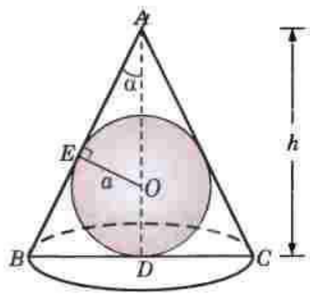


Fig. 4.20

Thus $$V = \frac{1}{3}\pi h^3 \cdot \frac{a^2}{h^2 - 2ha} = \frac{1}{3}\pi a^3 \cdot \frac{h^2}{h - 2a}$$

$\therefore$ $$\frac{dV}{dh} = \frac{1}{3}\pi a^2 \cdot \frac{(h-2a)2h - h^2\cdot 1}{(h-2a)^2} = \frac{1}{3}\pi a^2 \cdot \frac{h(h-4a)}{(h-2a)^2}$$

Thus $\dfrac{dV}{dh} = 0$ for $h = 4a$, the other value ($h = 0$) being not possible.

Also dV/dh is – ve when h is slightly $< 4a$, and it is + ve when h is slightly $> 4a$.

Hence V is minimum (*i.e.* least) when $h = 4a$

and $$\alpha = \sin^{-1}\left(\frac{a}{OA}\right) = \sin^{-1}\left(\frac{a}{3a}\right) = \sin^{-1}\frac{1}{3}.$$

Example 4.63. *Find the volume of the largest possible right-circular cylinder that can be inscribed in a sphere of radius a.*

Solution. Let O be the centre of the sphere of radius a. Construct a cylinder as shown in Fig. 4.21. Let $OA = r$.

Then $$AB = \sqrt{(OB^2 - OA^2)} = \sqrt{(a^2 - r^2)}$$

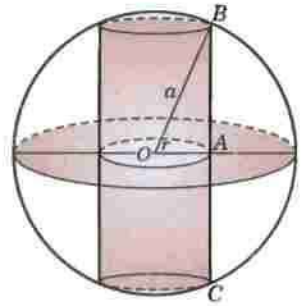


Fig. 4.21

$\therefore$ Height h of the cylinder $= 2 \cdot AB = 2\sqrt{(a^2 - r^2)}$.

Thus volume V of the cylinder

$$= \pi r^2 h = 2\pi r^2 \sqrt{(a^2 - r^2)}$$

$$\therefore \quad \frac{dV}{dr} = 2\pi \{2r\sqrt{(a^2 - r^2)} + r^2 \cdot \frac{1}{2}(a^2 - r^2)^{-1/2}(-2r)\}$$

$$= \frac{2\pi r(2a^2 - 3r^2)}{\sqrt{(a^2 - r^2)}}$$

The $dV/dr = 0$ when $r^2 = 2a^2/3$, the other value ($r = 0$) being not admissible.

Now $$\frac{d^2V}{dr^2} = 2\pi \frac{\sqrt{(a^2 - r^2)}(2a^2 - 9r^2) - r(2a^2 - 3r^2) \times \frac{1}{2}(a^2 - r^2)^{-1/2} . (-2r)}{(a^2 - r^2)}$$

$$= 2\pi \frac{(a^2 - r^2)(2a^2 - 9r^2) + r^2(2a^2 - 3r^2)}{(a^2 - r^2)^{3/2}}$$ which is – ve for $r^2 = 2a^2/3$.

Hence V is maximum for $r^2 = 2a^2/3$ and maximum volume

$$= 2\pi r^2 \sqrt{(a^2 - r^2)} = 4\pi a^3/3\sqrt{3}.$$

Example 4.64. *Assuming that the petrol burnt (per hour) in driving a motor boat varies as the cube of its velocity, show that the most economical speed when going against a current of c miles per hour is $\frac{3}{2}c$ miles per hour.*

Solution. Let v m.p.h. be the velocity of the boat so that its velocity relative to water (when going against the current) is $(v - c)$ m.p.h.

$\therefore$ Time required to cover a distance of s miles $= \dfrac{s}{v - c}$ hours.

Since the petrol burnt per hour $= kv^3$, k being a constant.

$\therefore$ The total petrol burnt, y, is given by

$$y = k\frac{v^3 s}{v - c} = ks\frac{v^3}{v - c} \quad \therefore \quad \frac{dy}{dv} = ks \cdot \frac{(v - c)3v^2 - v^3 \cdot 1}{(v - c)^2}$$

$$= ks \cdot \frac{v^2(2v - 3c)}{(v - c)^2}$$

Thus $dy/dv = 0$ for $v = 3c/2$, the other value ($v = 0$) is inadmissible.

Also dy/dv is $-ve$, when v is slightly $< 3c/2$ and it is $+ve$, when v is slightly $> 3c/2$.

Hence y is minimum for $v = 3c/2$.

PROBLEMS 4.13

1. (*i*) Test the curve $y = x^4$ for points of inflexion ? (*Burdwan, 2003*)

   (*ii*) Show that the points of inflexion of the curve $y^2 = (x - a)^2 (x - b)$ lie on the straight line $3x + a = 4b$. (*Rajasthan, 2005*)

2. The function $f(x)$ defined by $f(x) = a/x + bx$, $f(2) = 1$, has an extremum at $x = 2$. Determine a and b. Is this point (2, 1), a point of maximum or minimum on the graph of $f(x)$?
3. Show that $\sin^p\theta \cos^q\theta$ attains a maximum when $\theta = \tan^{-1}(p/q)$. (*Rajasthan, 2006*)
4. If a beam of weight w per unit length is built-in horizontally at one end A and rests on a support O at the other end, the deflection y at a distance x from O is given by

$$EIy = \frac{w}{48}(2x^4 - 3lx^3 + l^3x),$$

where l is the distance between the ends. Find x for y to be maximum.

5. The horse-power developed by an aircraft travelling horizontally with velocity v feet per second is given by

$$H = \frac{aw^2}{v} + bv,$$

where a, b and w are constants. Find for what value of v the horse-power is maximum.

6. The velocity of waves of wave-length λ on deep water is proportional to $\sqrt{(\lambda/a + a/\lambda)}$, where a is a certain constant, prove that the velocity is minimum when $\lambda = a$.
7. In a submarine telegraph cable, the speed of signalling varies as $x^2 \log_e(1/x)$, where x is the ratio of the radius of the core to that of the covering. Show that the greatest speed is attained when this ratio is $1/\sqrt{e}$.
8. The efficiency e of a screw-jack is given by $e = \tan\theta/\tan(\theta + \alpha)$, where α is a constant. Find θ if this efficiency is to be maximum. Also find the maximum efficiency.
9. Show that of all rectangles of given area, the square has the least parameter.
10. Find the rectangle of greatest perimeter that can be inscribed in a circle of radius a.
11. A gutter of rectangular section (open at the top) is to be made by bending into shape of a rectangular strip of metal. Show that the capacity of the gutter will be greatest if its width is twice its depth.
12. Show that the triangle of maximum area that can be inscribed in a given circle is an equilateral triangle.
13. An open box is to be made from a rectangular piece of sheet metal 12 cms × 18 cms, by cutting out equal squares from each corner and folding up the sides. Find the dimensions of the box of largest volume that can be made in this manner.
14. An open tank is to be constructed with a square base and vertical sides to hold a given quantity of water. Find the ratio of its depth to the width so that the cost of lining the tank with lead is least.
15. A corridor of width b runs perpendicular to a passageway of width a. Find the longest beam which can be moved in a horizontal plane along the passageway into the corridor?
16. One corner of a rectangular sheet of paper of width a is folded so as to reach the opposite edge of the sheet. Find the minimum length of the crease.
17. Show that the height of closed cylinder of given volume and least surface is equal to its diameter.
18. Prove that a conical vessel of a given storage capacity requires the least material when its height is $\sqrt{2}$ times the radius of the base. (*Warangal, 1996*)
19. Show that the semi-vertical angle of a cone of maximum volume and given slant height is $\tan^{-1}\sqrt{2}$.
20. The shape of a hole bored by a drill is cone surmounting a cylinder. If the cylinder be of height h and radius r and the semi-vertical angle of the cone be α where $\tan\alpha = h/r$, show that for a total fixed depth H of the hole, the volume removed is maximum if $h = \frac{H}{6}(1 + \sqrt{7})$. (*Raipur, 2005*)
21. A cylinder is inscribed in a cone of height h. If the volume of the cylinder is maximum, show that its height is $h/3$.
22. Show that the volume of the biggest right circular cone that can be inscribed in a sphere of given radius is 8/27 times that of the sphere.
23. A given quantity of metal is to be cast into a half-cylinder with a rectangular base and semi-circular ends. Show that in order that the total surface area may be a minimum, the ratio of the length of the cylinder to the diameter of its semi-circular ends is $\pi/(\pi + 2)$.
24. A person being in a boat a miles from the nearest point of the beach, wishes to reach as quickly as possible a point b miles from that point along the shore. The ratio of his rate of walking to his rate of rowing is $\sec\alpha$. Prove that he should land at a distance $b - a\cot\alpha$ from the place to be reached.
25. The cost per hour of propelling a steamer is proportional to the cube of her speed through water. Find the relative speed at which the steamer should be run against a current of 5 km per hour to make a given trip at the least cost.

4.16 ASYMPTOTES

(1) Def. *An asymptote of a curve is a straight line at a finite distance from the origin, to which a tangent to the curve tends as the point of contact recedes to infinity.*

In other words, an asymptote is a straight line which cuts a curve on two points, at an infinite distance from the origin and yet is not itself wholly at infinity.

(2) Asymptotes parallel to axes. Let the equation of the curve arranged according to powers of x be

$$a_0x^n + (a_1y + b_1)x^{n-1} + (a_2y^2 + b_2y + c_2)x^{n-2} + \dots 0 \qquad \dots(1)$$

If $a_0 = 0$ and y be so chosen that $a_1y + b_1 = 0$, then the coefficients of two highest powers of x in (1) vanish and therefore, two of its roots are infinite. Hence $a_1y + b_1 = 0$ is an asymptote of (1) which is parallel to x-axis.

Again if a_0, a_1, b_1 are all zero and if y be so chosen that $a_2y^2 + b_2y + c_2 = 0$, then three roots of (1) become infinite. Therefore, the two lines represented by $a_2y^2 + b_2y + c_2 = 0$ are the asymptotes of (1) which are parallel to x-axis, and so on.

Similarly, for asymptotes parallel to y-axis.

Thus we have the following **rules :**

I. *To find the asymptotes parallel to x-axis, equate to zero the coefficient of the highest power of x in the equation,* provided this is not merely a constant.

II. *To find the asymptotes parallel to y-axis, equate to zero the coefficient of the highest power of y in the equation,* provided this is not merely a constant.

Example 4.65. *Find the asymptotes of the curve*

$$x^2y^2 - x^2y - xy^2 + x + y + 1 = 0.$$

Solution. The highest power of x is x^2 and its coefficient is $y^2 - y$.

$\therefore$ The asymptotes parallel to the x-axis are given by

$$y(y-1) = 0 \text{ i.e., by } y = 0 \text{ and } y = 1.$$

The highest power of y is y^2 and its coefficient is $x^2 - x$.

$\therefore$ The asymptotes parallel to the y-axis are given by

$$x(x-1) = 0 \text{ i.e., by } x = 0 \text{ and } x = 1.$$

Hence the asymptotes are $x = 0, x = 1, y = 0$ and $y = 1$.

(3) Inclined asymptotes. Let the equation of the curve be of the form

$$x^n\phi_n(y/x) + x^{n-1}\phi_{n-1}(y/x) + x^{n-2}\phi_{n-2}(y/x) + \dots = 0 \qquad \dots(1)$$

where $\phi_r(y/x)$ is an expression of degree r is y/x.

To find where this curve is cut by the line $y = mx + c$, $\qquad \dots(2)$

put $y/x = m + c/x$ in (1). The resulting equation is

$$x^n\phi_n(m + c/x) + x^{n-1}\phi_{n-1}(m + c/x) + x^{n-2}\phi_{n-2}(m + c/x) + \dots = 0$$

which gives the abscissae of the points of intersection.

Expanding each of the ϕ-functions by Taylor's series,

$$x^n\left\{\phi_n(m) + \frac{c}{x}\phi_n'(m) + \frac{c^2}{2!\,x^2}\phi_n''(m) + \dots\right\} + x^{n-1}\left\{\phi_{n-1}(m) + \frac{c}{x}\phi'_{n-1}(m) + \dots\right\}$$
$$+ x^{n-2}\{\phi_{n-2}(m) + \dots\} = 0$$

or

$$x^n\phi_n(m) + x^{n-1}\{c\phi'_n(m) + \phi_{n-1}(m)\}$$
$$+ x^{n-2}\left\{\frac{c^2}{2!}\phi_n''(m) + c\phi'_{n-1}(m) + \phi_{n-2}(m)\right\} + \dots = 0 \qquad \dots(3)$$

If the line (2) is an asymptote to the curve, it cuts the curve in two points at infinity *i.e.*, the equation (3) has two infinite roots for which the coefficients of two highest terms should be zero.

i.e., $\qquad \phi_n(m) = 0 \qquad \dots(4) \qquad$ and $\qquad c\,\phi'_n(m) + \phi_{n-1}(m) = 0 \qquad \dots(5)$

If the roots of (4) be $m_1, m_2, \dots m_n$, then the corresponding values of c (*i.e.* $c_1, c_2, \dots\dots c_n$) are given by (5). Hence the asymptotes are

$$y = m_1x + c_1, y = m_2x + c_2, \dots\dots, y = m_nx + c_n.$$

Obs. If (4) gives two equal values of m, then the corresponding values of c cannot be found from (5). Then c is determined by equating to zero the coefficient of x^{n-2} *i.e.*, from

$$\frac{c^2}{2!}\phi''_n(m) + c\,\phi'_{n-1}(m) + \phi_{n-2}(m) = 0 \qquad ...(6)$$

In this case, *there will be two parallel asymptotes.*

Working rule :

1. Put $x = 1, y = m$ in the highest degree terms, thus getting $\phi_n(m)$. Equate it to zero and solve for m. Let its roots be $m_1, m_2, \ldots\ldots$
2. Form $\phi_{n-1}(m)$ by putting $x = 1$ and $y = m$ in the $(n-1)$th degree terms.
3. Find the values of c (*i.e.* $c_1, c_2, \ldots\ldots$) by substituting $m = m_1, m_2, \ldots\ldots$ in turn in the formula
$$c = -\phi_{n-1}(m)/\phi'_n(m)$$
[Sometimes it take (0/0) form, then find c from (6).]
4. Substitute the values of m and c in $y = mx + c$ in turn.

Example 4.66. *Find the asymptotes of the curve*

(*i*) $y^3 - 2xy^2 - x^2y + 2x^3 + 3y^2 - 7xy + 2y^2 + 2y + 2x + 1 = 0.$

(*ii*) $x^3 + 3x^2y - 4y^3 - x + y + 3 = 0.$

(*iii*) $(x + y)^2 (x + y + 2) = x + 9y - 2.$ (*Rohtak, 2005*)

Solution. (*i*) Putting $x = 1$ and $y = m$ in the third degree terms,

$$\phi_3(m) = m^3 - 2m^2 - m + 2, \quad \therefore \quad \phi_3(m) = 0 \text{ gives } m^3 - 2m^2 - m + 2 = 0$$

or $$(m^2 - 1)(m - 2) = 0 \text{ whence } m = 1, -1, 2.$$

Also putting $x = 1$ and $y = m$ in the 2nd degree terms, $\phi_2(m) = 3m^2 - 7m + 2$

$$\therefore \qquad c = -\frac{\phi_2(m)}{\phi'_3(m)} = -\frac{3m^2 - 7m + 2}{3m^2 - 4m - 1}$$

$$= -1 \text{ when } m = 1, = -2 \text{ when } m = -1, = 0 \text{ when } m = 2.$$

Hence the asymptotes are $y = x - 1$, $y = -x - 2$ and $y = 2x$.

(*ii*) Putting $x = 1$ and $y = m$ in the third degree terms,

$$\phi_3(m) = 1 + 3m - 4m^3$$

$$\therefore \qquad \phi_3(m) = 0 \text{ gives } 4m^3 - 3m - 1 = 0, \quad \text{or} \quad (m-1)(2m+1)^2 = 0$$

whence $$m = 1, -1/2, -1/2.$$

Similarly, $$\phi_2(m) = 0$$

$$\therefore \qquad c = -\frac{\phi_2(m)}{\phi'_3(m)} = -\frac{0}{3 - 12m^2}$$

$$= 0 \text{ when } m = 1, = \frac{0}{0} \text{ form when } m = -\frac{1}{2}.$$

Thus $\left(\text{when } m = -\frac{1}{2}\right)$ c is to be obtained from

$$\frac{c^2}{2!}\phi''_3(m) + c\,\phi'_2(m) + \phi_1(m) = 0$$

or $$\frac{c^2}{2}(-24m) + c\,.\,0 + (-1 + m) = 0$$

Putting $m = -1/2$, $6c^2 - 3/2 = 0$ whence $c = \pm 1/2$.

Hence the asymptotes are $y = x$, $y = -\frac{1}{2}x + \frac{1}{2}$, $y = -\frac{1}{2}x - \frac{1}{2}$.

(*iii*) Putting $x = 1$ and $y = m$ in the third degree terms, $\phi_3(m) = (1 + m)^3$.

$$\therefore \qquad \phi_3(m) = 0 \text{ gives } (m + 1)^3 = 0 \text{ whence } m = -1, -1, -1.$$

Similarly, $$\phi_2(m) = 2(1 + m)^2, \phi_1(m) = -1 - 9m, \phi_0(m) = 2.$$

For these three equal values of $m = -1$, values of c are obtained from

$$\frac{c^3}{3!}\phi_3'''(m) + \frac{c^2}{2!}\phi_2''(m) + c\,\phi_1'(m) + \phi_0(m) = 0$$

or $$\frac{c^3}{6}(6) + \frac{c^2}{2}(4) + c(-9) + 2 = 0 \quad \text{or} \quad c^3 + 2c^2 - 9c + 2 = 0.$$

Solving for c, we have $c = 2, -2 \pm \sqrt{5}$.

Hence the three asymptotes are

$$y = -x + 2,\ y = -x - 2 + \sqrt{5},\ y = -x - 2 - \sqrt{5}.$$

4. Asymptotes of polar curves. It can be shown that an *asymptote of the curve* $1/r = f(\theta)$ *is*

$$r \sin(\theta - \alpha) = 1/f'(\alpha),$$

where α *is a root of the equation* $f(\theta) = 0$ *and* $f'(\alpha)$ *is the derivative of* $f(\theta)$ *w.r.t.* θ *at* $\theta = \alpha$.

Example 4.67. *Find the asymptote of the spiral* $r = a/\theta$.

Equation of the curve can be written as $1/r = \theta/a = f(\theta)$, say.

$$f(\theta) = 0, \text{ if } \theta = 0\ (= \alpha). \text{ Also } f'(\theta) = 1/a \quad \therefore \quad f'(\alpha) = 1/a.$$

$\therefore$ The asymptote is $r \sin(\theta - 0) = 1/f'(0)$ or $r \sin\theta = a$.

PROBLEMS 4.14

Find the asymptotes of

1. $x^3 + y^3 = 3axy$ (*Agra, 2002*)
2. $(x^2 - a^2)(y^2 - b^2) = a^2 b^2$ (*Osmania, 2002*)
3. $(a/x)^2 + (b/y)^2 = 1$ (*Burdwan, 2003*)
4. $x^2y + xy^2 + xy + y^2 + 3x = 0.$
5. $4x^3 + 2x^2 - 3xy^2 - y^3 - 1 - xy - y^2 = 0.$ (*U.P.T.U., 2001*)
6. $x^2(x - y)^2 - a^2(x^2 + y^2) = 0$ (*Kurukshetra, 2006*)
7. $(x + y)^2(x + 2y + 2) = (x + 9y - 2)$ (*Rajasthan, 2006*)
8. Show that the asymptotes of the curve $x^2y^2 = a^2(x^2 + y^2)$ form a square of side $2a$.
9. Find the asymptotes of the curve $x^2y - xy^2 + xy + y^2 + x - y = 0$ and show that they cut the curve again in three points which lie on the line $x + y = 0$. (*Kurukshetra, 2006*)

Find the asymptotes of the following curves :

10. $r = a\tan\theta$. (*Rohtak, 2006 S*)
11. $r = a(\sec\theta + \tan\theta)$
12. $r\sin\theta = 2\cos 2\theta$. (*Kurukshetra, 2009 S*)
13. $r\sin n\theta = a$.

4.17 (1) CURVE TRACING

In many practical applications, a knowledge about the shapes of given equations is desirable. On drawing a sketch of the given equation, we can easily study the behaviour of the curve as regards its symmetry asymptotes, the number of branches passing through a point etc.

A point through which two branches of a curve pass is called a **double point.** At such a point P, the curve has two tangents, one for each branch.

If the tangents are real and distinct, the double point is called a **node** [Fig. 4.22 (*a*)].

If the tangents are real and coincident, the double point is called a **cusp** [Fig. 4.22 (*b*)].

If the tangents are imaginary, the double point is called a *conjugate point* (or an *isolated point*). Such a point cannot be shown in the figure.

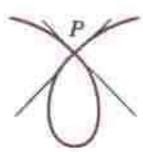


Fig. 4.22 (*a*)

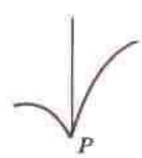


Fig. 4.22 (*b*)

(2) Procedure for tracing cartesian curves.

1. Symmetry. *See if the curve is symmetrical about any line.*

(*i*) *A curve is symmetrical about the x-axis, if only even powers of y occur in its equation.* (*e.g.*, $y^2 = 4ax$ is symmetrical about x-axis).

(*ii*) *A curve is symmetrical about the y-axis, if only even powers of x occur in its equation.* (*e.g.*, $x^2 = 4ay$ is symmetrical about y-axis).

(*iii*) *A curve is symmetrical about the line* $y = x$, *if on interchanging x* and *y its equation remains unchanged,* (*e.g.*, $x^3 + y^3 = 3axy$ *is symmetrical about the line* $y = x$).

2. Origin. (*i*) *See if the curve passes through the origin.*

(A curve passes through the origin if there is no constant term in its equation).

(*ii*) *If it does, find the equation of the tangents thereat,* by equating to zero the lowest degree terms.

(*iii*) *If the origin is a double point, find whether the origin is a node, cusp or conjugate point.*

3. Asymptotes. (*i*) *See if the curve has any asymptote parallel to the axes* (p. 183).

(*ii*) *Then find the inclined asymptotes,* if need be. (p. 183).

4. Points. (*i*) *Find the points where the curve crosses the axes and the asymptotes.*

(*ii*) *Find the points where the tangent is parallel or perpendicular to the x-axis,* (*i.e.* the points where $dy/dx = 0$ or ∞).

(*iii*) *Find the region (or regions) in which no portion of the curve exists.*

Example 4.68. *Trace the curve* $y^2 (2a - x) = x^3$.
(*P.T.U., 2010, V.T.U., 2008 ; Rajasthan, 2006 ; U.P.T.U., 2005*)

Solution. (*i*) *Symmetry*: The curve is symmetrical about the x-axis.
[$\because$ only even powers of y occur in the equation.

(*ii*) *Origin :* The curve passes through the origin
[$\because$ there is no constant term in its equation.

The tangents at the origin are $y = 0, y = 0$ [Equating to zero the lowest degree terms.]

$\therefore$ Origin is a *cusp*

(*iii*) *Asymptotes :* The curve has an asymptote $x = 2a$.
[$\because$ co-eff. of y^3 is absent, co-eff. of y^2 is an asymptote.

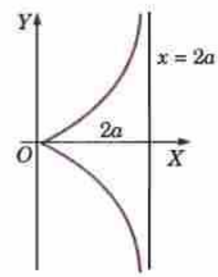


Fig. 4.23

(*iv*) *Points* : (*a*) curve meets the axes at (0, 0) only. (*b*) $y^2 = x^3/(2a - x)$

When x is –ve, y^2 is –ve (*i.e.* y is imaginary) so that no portion of the cuve lies to the left of the y-axis. Also when $x > 2a$, y^2 is again –ve, so that no portion of the curve lies to the right of the line $3x = 2a$.

Hence, the shape of the curve is as shown in Fig. 4.23. This curve is known as *Cissoid.*

Example 4.69. *Trace the curve* $y^2 (a - x) = x^2(a + x)$. (*V.T.U., 2010 ; B.P.T.U., 2005*)

Solution. (*i*) *Symmetry* : The curve is symmetrical about the x-axis.

(*ii*) *Origin* : The curve passes through the origin and the tangents at the origin are $y^2 = x^2$,

i.e. $y = x$ and $y = -x$. $\therefore$ Origin is a *node.*

(*iii*) *Asymptotes* : The curve has an asymptote $x = a$

(*iv*) *Points* : (*a*) When $x = 0, y = 0$; when $y = 0, x = 0$ or $-a$.

$\therefore$ The curve crosses the axes at (0, 0) and $(-a, 0)$.

We have $y = \pm x \sqrt{\left(\dfrac{a + x}{a - x}\right)}$

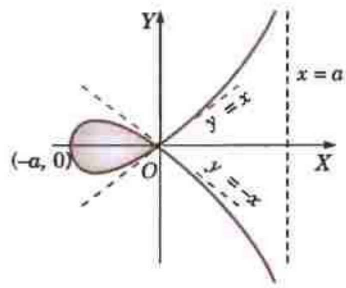


Fig. 4.24

When $x > a$ or $< -a$, y is imaginary.

$\therefore$ No portion of the curve lies to the right of the line $x = a$ or to the left of the line $x = -a$.

Hence the shape of the curve is as shown in Fig. 4.24. This curve is known as *Strophoid.*

Example 4.70. *Trace the curve* $y = x^2/(1 - x^2)$.

Solution. (*i*) *Symmetry* : The curve is symmetrical about y-axis.

(*ii*) *Origin* : It passes through the origin and the tangent at the origin is $y = 0$ (*i.e.*, x-axis).

(iii) *Asymptotes* : The asymptotes are given by $1 - x^2 = 0$ or $x = \pm 1$ and $y = -1$.

(iv) *Points* : (a) The curve crosses the axes at the origin only. (b) When $x \to 1$ from left, $y \to \infty$

When $x \to 1$ from right $y \to -\infty$

When $x > 1$, y is $-$ve

Hence the curve is as shown in Fig. 4.25.

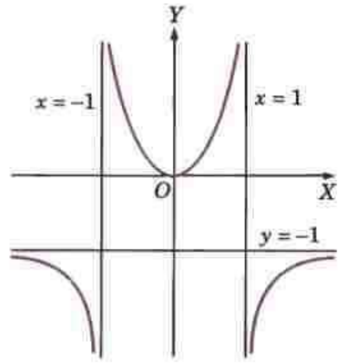


Fig. 4.25

Example 4.71. *Trace the curve $a^2y^2 = x^2(a^2 - x^2)$.* (P.T.U., 2009 ; V.T.U., 2008 S)

Solution. (i) *Symmetry*. The curve is symmetrical about x-axis, y-axis and origin.

(ii) *Origin*. The curve passes through the origin and the tangents at the origin are $a^2y^2 = a^2x^2$ *i.e.*, $y = \pm x$.

(iii) *Asymptotes*. The curve has no asymptote.

(iv) *Points*. (a) The curve cuts x-axis $(y = 0)$ at $x = 0, \pm a$. and cuts y-axis $(x = 0)$ at $y = 0$ *i.e.*, (0, 0) only.

(b) $\dfrac{dy}{dx} = \dfrac{x(a^2 - 2x^2)}{a^2 y} \to \infty$ at $(a, 0)$

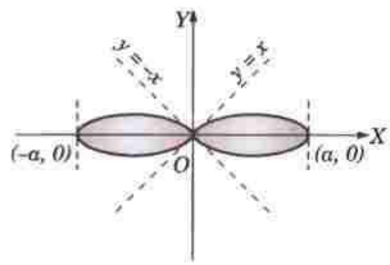


Fig. 4.26

i.e., tangent to the curve at $(a, 0)$ is parallel to y-axis. Similarly the tangent at $(-a, 0)$ is parallel to y-axis.

(c) We have $y = \dfrac{x}{a}\sqrt{a^2 - x^2}$ which is real for $x^2 < a^2$ *i.e.*, $-a < x < a$.

$\therefore$ The curve lies between $x = a$ and $x = -a$

Hence the shape of the curve is as shown is Fig. 4.26.

Example 4.72. *Trace the curve $y = x^3 - 12x - 16$.* (P.T.U., 2008)

Solution. (i) *Symmetry*. The curve has no symmetry.

(ii) *Origin*. It doesn't pass through the origin.

(iii) *Asymptotes* : The curve has no asymptote.

(iv) *Points*. (a) The curve cuts x-axis $(y = 0)$ at $(-2, 0)$, $(4, 0)$ and cuts y-axis $(x = 0)$ at $(0, -16)$.

(b) $\dfrac{dy}{dx} = 3x^2 - 12$

At (– 2, 0), $\dfrac{dy}{dx} = 0$ *i.e.*, tangent is parallel to x-axis at $(-2, 0)$.

At (4, 0), $\dfrac{dy}{dx} = 36$ *i.e.*, $\tan\theta = 36$ *i.e.*, tangent makes an acute angle $\tan^{-1} 36$ with x-axis at (4, 0).

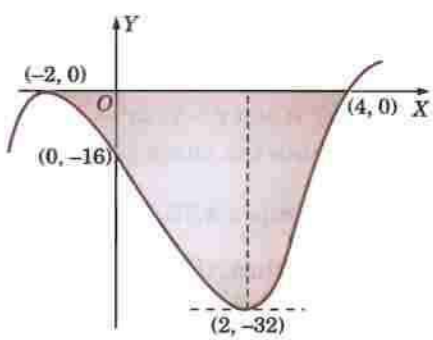


Fig. 4.27

Also $\dfrac{dy}{dx} = 0$ at $3x^2 - 12 = 0$ or $x = \pm 2$ *i.e.*, tangent is also parallel to x-axis at $(2, -32)$.

(c) $y \to \infty$ as $x \to \infty$ and $y \to -\infty$ as $x \to -\infty$; y is + *ve* for $x > 4$ and y is – *ve* for $x < 4$.

Hence the shape of the curve is as shown in Fig. 4.27.

Example 4.73. *Trace the curve $9ay^2 = (x - 2a)(x - 5a)^2$* (J.N.T.U., 2008)

Solution. (i) *Symmetry*. The curve is symmetrical about the x-axis.

(ii) *Origin*. The curve does'nt pass through the origin.

(iii) *Asymptotes*. It has no asymptotes.

(iv) *Points.* (a) The curve cuts the x-axis ($y = 0$) at $x = 2a$, and $x = 5a$. *i.e.*, at A $(2a, 0)$ and $B(5a, 0)$.

It cuts the y-axis ($x = 0$) at $y^2 = -50\,a^2/9$, *i.e.*, y is imaginary.

So the curve doesn't cut the y-axis.

(b) $y = \dfrac{(x-5a)\sqrt{(x-2a)}}{3\sqrt{a}}$ *i.e.*, y is imaginary for $x < 2a$. So the curve exists only for $x \geq 2a$.

(c) $\dfrac{dy}{dx} = \pm \dfrac{x-3a}{2\sqrt{a}\,\sqrt{(x-2a)}}$

At A $(2a, 0)$, $\dfrac{dy}{dx} \to \infty$ *i.e.*, tangent is parallel to y-axis.

At B $(5a, 0)$, $\dfrac{dy}{dx} = \pm \dfrac{1}{\sqrt{3}}$ *i.e.*, there are two distinct tangents.

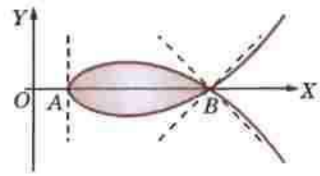


Fig. 4.28

So there is a node at B $(5a, 0)$.

Hence the shape of the curve is as shown in Fig. 4.28.

Example 4.74. *Trace the curve* $x^3 + y^3 = 3axy$ *(Kurukshetra, 2005 ; U.P.T.U., 2003)*

or $$r = \frac{3a \sin\theta\cos\theta}{\sin^3\theta + \cos^3\theta}.$$

Solution. (i) *Symmetry* : The curve is symmetrical about the line $y = x$.

(ii) *Origin* : It passes through the origin and tangents at the origin are $xy = 0$, *i.e.*, $x = 0$, $y = 0$.

$\therefore$ Origin is a *node*.

(iii) *Asymptotes* : (a) It has no asymptote parallel to the axes.

(b) Putting $y = m$ and $x = 1$ in the third degree terms,

$$\phi_3(m) = 1 + m^3,\ \phi_3(m) = 0 \text{ gives } m = -1.$$

$$\therefore \qquad c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\left(\frac{-3am}{3m^2}\right) = \frac{a}{m}.$$
$$= -a, \text{ when } m = -1.$$

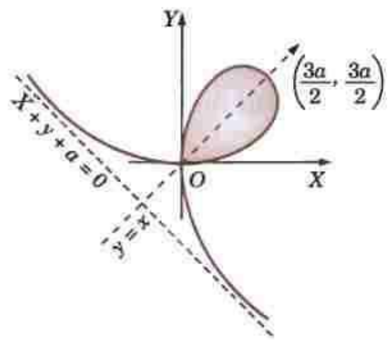


Fig. 4.29

Hence $y = -x - a$ $\left(i.e., \ \dfrac{x}{-a} + \dfrac{y}{-a} = 1\right)$ is an asymoptote.

(iv) *Points* : (a) It meets the axes at the origin only.

(b) When $y = x$, $2x^3 = 3ax^2$, *i.e.* $x = 0$ or $3a/2$. *i.e.*, the curve crosses the line $y = x$ at $(3a/2, 3a/2)$.

Hence the shape of the curve is as shown in Fig. 4.29. This curve is known as *Folium of Descartes*.

Example 4.75. *Trace the curve* $x^3 + y^3 = 3ax^2$.

Solution. (i) *Symmetry* : The curve has no symmetry.

(ii) *Origin* : The curve passes through the origin and the tangents at the origin are $x = 0$ and $x = 0$.

$\therefore$ The origin is a *cusp*.

(iii) *Asymptotes* : (a) The curve has no asymptote parallel to the axes.

(b) Putting $x = 1$, $y = m$ in the third degree terms, we get

$$\phi_3(m) = m^3 + 1\ ; \quad \therefore \quad \phi_3(m) = 0, \text{ gives } m = -1.$$

$$\therefore \qquad c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{-3a}{3m^2} = a \text{ for } m = 1.$$

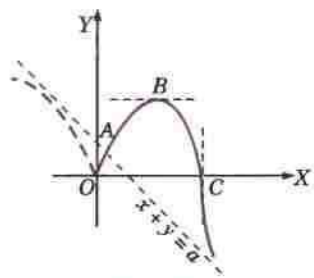


Fig. 4.30

Thus $x + y = a$ is the only asymptote.

The curve lies above the asymptote when x is positive and large and it lies below the asymptote when x is negative.

(*iv*) *Points.* (*a*) The curve crosses the axes at O (0, 0) and $C(3a, 0)$. It crosses the asymptote at $A(a/3, 2a/3)$.

(*b*) Since $y^2\, dy/dx = x(2a - x)$. $\therefore$ $dy/dx = 0$ for $x = 2a$.

(*c*) Now $y = [x^2 (3a - x)]^{1/3}$.

When $0 < x < 3a$, y is positive. As x increases from 0, y also increases till $x = 2a$ where the tangent is parallel to the x-axis. As x increases from $2a$ to $3a$, y constantly decreases to zero.

When $x > 3a$, y is negative.

When $x < 0$, y is positive and constantly increases as x varies from 0 to $-\infty$.

Combining all these facts we see that the shape of the curve is as shown in Fig. 4.30.

Example 4.76. *Trace the curve* $y^2 (x - a) = x^2(x + a)$.

Solution. (*i*) *Symmetry* : The curve is symmetrical about the x-axis.

(*ii*) *Origin* : The curve passes through the origin and the tangents at the origin are $y^2 = -x^2$ *i.e.*, $y = \pm ix$, which are imaginary lines. $\therefore$ The origin is an *isolated point.*

(*iii*) *Asymptotes* : (*a*) $x = a$ is the only asymptote parallel to the y-axis.

(*b*) Putting $x = 1$ and $y = m$ in the third degree terms, we get

$$\phi_3(m) = m^2 - 1.$$

$\therefore$ $$\phi_3(m) = 0 \text{ gives } m = \pm 1$$

$\therefore$ $$c = \frac{\phi_2(m)}{\phi_3'(m)} = -\frac{-a(m^2+1)}{2m} = \pm a \text{ for } m = \pm 1.$$

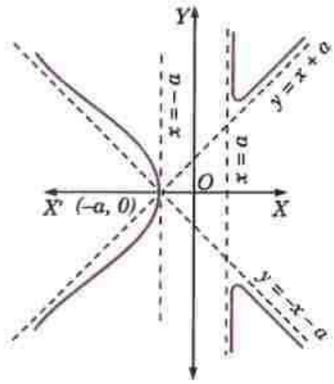


Fig. 4.31

Thus the other two asymptotes are $y = x + a$; $y = -x - a$.

(*iv*) *Points* : (*a*) The curve crosses the axes at $(-a, 0)$ and $(0, 0)$.

It crosses the asymptotes $y = x + a$ and $y = -x - a$ at $(-a, 0)$.

(*b*) $y = \pm x\sqrt{\left(\dfrac{x+a}{x-a}\right)}$

When $x < a$ and $x > -a$, y is imaginary.

$\therefore$ no portion of the curve lies between the lines $x = a$ and $x = -a$. Thus the vertical asymptote must be approached from the right.

(*c*) $\dfrac{dy}{dx} = \pm\dfrac{x^2 - ax + a^2}{(x-a)^{3/2}(x+a)^{1/2}}$

$\therefore$ $$dy/dx = 0, \text{ when } x = \tfrac{1}{2}(1 + \sqrt{5})a = 1.6a \text{ approx.}$$

[rejecting the value $\frac{1}{2}(1 - \sqrt{5})a$ which lies between $-a$ and a]

and $$dy/dx \to \infty, \text{ when } x = \pm a.$$

Thus the tangent is parallel to the x-axis at $x = 1.6a$ and perpendicular to the x-axis at $x = \pm a$.

Hence the shape of the curve is as shown in Fig. 4.31.

4.17 (3) PROCEDURE FOR TRACING CURVES IN PARAMETRIC FORM : $x = f(t)$ and $y = \phi(t)$

1. Symmetry. *See if the curve has any symmetry.*

(*i*) *A curve is symmetrical about the x-axis, if on replacing t by – t, f(t) remains unchanged and φ(t) changes to – φ(t).*

(*ii*) *A curve is symmetrical about the y-axis if on replacing t by – t, f(t) changes to – f(t) and φ(t) remains unchanged.*

(*iii*) *A curve is symmetrical in the opposite quadrants, if on replacing t by – t, both f(t) and φ(t) remains unchanged.*

2. **Limits.** *Find the greatest and least values of x and y so as to determine the strips, parallel to the axes, within or outside which the curve lies.*
3. **Points.** (*a*) *Determine the points where the curve crosses the axes.*

The points of intersection of the curve with the x-axis given by the roots of $\phi(t) = 0$, while those with the y-axis are given by the roots of $f(t) = 0$.

(*b*) *Giving t a series of value, plot the corresponding values of x and y, noting whether x and y increase or decrease for the intermediates values of t.* For this purpose, we consider the sign of dx/dt and dy/dt for the different values of t.

(*c*) *Determine the points where the tangent is parallel or perpendicular to the x-axis, (i.e.,* where $dy/dx = 0$ or $\to \infty$).

(*d*) *When x and y are periodic functions of t with a common period, we need to study the curve only for one period,* because the other values of t will repeat the same curve over and over again.

Obs. Sometimes it is convenient to eliminate t between the given equations and use the resulting cartesian equation to trace the curve.

Example 4.77. *Trace the curve* $x = a\cos^3 t$, $y = a\sin^3 t$ *or* $x^{2/3} + y^{2/3} = a^{2/3}$.
(*P.T.U., 2009 S ; U.P.T.U., 2005 ; V.T.U., 2003*)

Solution. (*i*) *Symmetry.* The curve is symmetrical about the x-axis.

[$\because$ On changing t to $-t$, x remains unchanged but y changed to $-y$]

(*ii*) *Limits.* $\because$ $|x| \le a$ and $|y| \le a$.

$\therefore$ The curve lies entirely within the square bounded by the lines $x = \pm a$, $y = \pm a$.

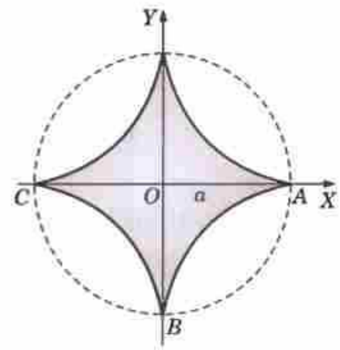


Fig. 4.32

(*iii*) *Points* : We have
$$\frac{dx}{dt} = -3a\cos^2 t \sin t,$$
$$\frac{dy}{dt} = 3a\sin^2 t\cos t,\ \frac{dy}{dx} = -\tan t.$$
$\therefore$ $dy/dx = 0$ when $t = 0$ or π

and $dy/dx \to \infty$, when $t = \pi/2$.

The following table gives the corresponding values of t, x, y and dy/dx.

As t increases	*x*	*y*	*dy/dx varies*	*Portion traced*
from 0 to $\pi/2$	+ve and decreases from a to 0	+ve and increases from 0 to a	from 0 to ∞	*A* to *B*
from $\pi/2$ to π	+ ve and increases numerically from 0 to $-a$	+ve and decreases from a to 0	from ∞ to 0	*B* to *C*

As t increases from π to 2π, we get the reflection of the curve ABC in the x-axis. The values of $t > 2\pi$ give no new points.

Hence the shape of the curve is as shown in Fig. 4.32 and is known as **Astroid.**

Example 4.78. *Trace the curve* $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$. (*J.N.T.U., 2009 S*)

Solution. (*i*) *Symmetry.* The curve is symmetrical about the y-axis.

[$\because$ On changing θ to $-\theta$, x changes to $-x$ and y remains unchanged]

Thus we may consider the curve only for positive value of x, *i.e.,* for $\theta > 0$.

(*ii*) *Limits.* The greatest value of y is $2a$ and the least value is zero.

Hence the curve lies entirely between the lines $y = 2a$ and $y = 0$.

(*iii*) *Points.* We have
$$\frac{dx}{d\theta} = a(1 + \cos\theta),\ \frac{dy}{d\theta} = a\sin\theta \text{ and } \frac{dy}{dx} = -\tan\theta/2.$$

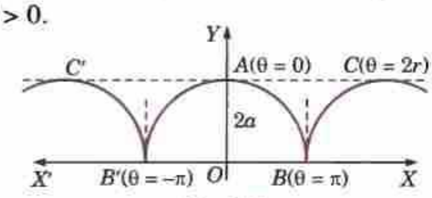


Fig. 4.33

$\therefore$ $dy/dx = 0$ when $\theta = 0$ or 2π and $dy/dx \to \infty$ when $\theta = \pi$.

The following table gives the corresponding values of θ, x, y and $\dfrac{dy}{dx}$:

As θ increases	*x*	*y*	*dy/dx varies*	*Portion traced*
from 0 to π	increases from 0 to $a\pi$	decreases from $2a$ to 0	from 0 to ∞	A to B
from π to 2π	increases from $a\pi$ to $2a\pi$	increases from 0 to $2a$	from ∞ to 0	B to C

As θ decreases from 0 to -2π, we get the reflection of the curve ABC in the y-axis.

The curve consists of congruent arches extending to infinity in both the directions of the x-axis in the intervals ... $(-3\pi, -\pi)$ $(-\pi, \pi)$ $(\pi, 3\pi)$, ...

Hence the shape of the curve is as shown in Fig. 4.33 and is known as **Cycloid.**

Obs. 1. Cycloid is the curve described by a point on the circumference of a circle which rolls without sliding on a fixed straight line. This fixed line (x-axis) is called the *base* and the farthest point (A) from it the *vertex* of the cycloid.

The complete cycloid consists of the arch $B'AB$ and its endless repetitions on both sides.

2. Inverted cycloid: $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$.

The complete inverted cycloid consists of the arch BOA and an endless repetitions of the same on both sides. Here AB is the base and O the *vertex* of this cycloid. (Fig. 4.34).

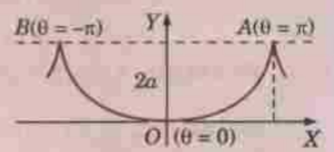


Fig. 4.34

4.17 (4) PROCEDURE FOR TRACING POLAR CURVES

1. **Symmetry.** *See if the curve is symmetrical about any line.*

(*i*) *A curve is symmetrical about the initial line OX, if only* $\cos\theta$ *(or* $\sec\theta$*) occur in its equation.* (*i.e.*, it remains unchanged when θ is changed to $-\theta$) *e.g.*, $r = a(1 + \cos\theta)$ is symmetrical about the initial line.

(*ii*) *A curve is symmetrical about the line through the pole* $\perp$ *to the initial line (i.e., OY), if only* $\sin\theta$ *(or* $\operatorname{cosec}\theta$*) occur in its equation.* (*i.e.*, it remains unchanged when θ is changed to $\pi - \theta$) *e.g.*, $r = a\sin 3\theta$ is symmetrical about OY.

(*iii*) *A curve is symmetrical about the pole, if only even powers of r occur in the equation* (*i.e.*, it remains unchanged when r is changed to $-r$) *e.g.*, $r^2 = a^2\cos 2\theta$ is symmetrical about the pole.

2. **Limits.** See if r and θ are confined between certain limits.

(*i*) *Determine the numerically greatest value of r, so as to notice whether the curve lies within a circle or not e.g.*, $r = a\sin 3\theta$ lies wholly within the circle $r = a$.

(*ii*) *Determine the region in which no portion of the curve lies by finding those values of* θ *for which r is imaginary e.g.*, $r^2 = a^2\cos 2\theta$ does not lie between the lines $\theta = \pi/4$ and $\theta = 3\pi/4$.

3. **Asymptotes.** *If the curve possesses an infinite branch, find the asymptotes (p. 183).*

4. **Points.** (*i*) *Giving successive values to* θ*, find the corresponding values of r.*

(*ii*) *Determine the points where the tangent coincides with the radius vector or is perpendicular to it* (*i.e.*, the points where $\tan\phi = r\, d\theta/dr = 0$ or ∞).

Example 4.79. *Trace the curve* $r = a\sin 3\theta$. (*U.P.T.U., 2002*)

Solution. (*i*) *Symmetry.* The curve is symmetrical about the line through the pole $\perp$ to the initial line.

(*ii*) *Limits.* The curve wholly lies within the curve $r = a$. ($\because$ r is never $> a$)

(*iii*) *Asymptotes.* It has no asymptotes.

(*iv*) *Points.* (*a*) $\tan\phi = r\dfrac{d\theta}{dr} = \dfrac{a\sin 3\theta}{3a\cos 3\theta} = \dfrac{1}{3}\tan 3\theta$

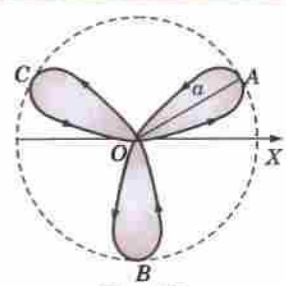


Fig. 4.35

$\therefore$ $\phi = 0$, when $\theta = 0, \pi/3, ...$

$\phi = \pi/2$, when $\theta = \pi/6, \pi/2, ...$

Hence the curve of the curve

(*b*) The following table gives the variations of r, θ and ϕ :

As θ varies from	*r varies from*	*ϕ varies from*	*Portion traced from*
0 to $\pi/6$	0 to a	0 to $\pi/2$	O to A
$\pi/6$ to $\pi/3$	a to 0	$\pi/2$ to 0	A to O
$\pi/3$ to $\pi/2$	0 to $-a$	0 to $\pi/2$	O to B

As θ increases from $\pi/2$ to π, portions of the curve from B to O, O to C and C to O are traced by symmetry about the line $\theta = \pi/2$.

Hence the curve consists of three loops as shown in Fig. 4.35 and is known as *three-leaved rose.*

Obs. *The curves of the form $r = a \sin n\theta$ or $r = a \cos n\theta$ are called **Roses** having*

(*i*) n leaves (loops) when n is odd,

(*ii*) $2n$ leaves (loops) when n is even.

Example 4.80. *Trace the curve $r = a \sin 2\theta$.(Four Leaved Rose)* (*V.T.U., 2009*)

Solution. (*i*) *Symmetry.* The curve is symmetrical about the line through the pole,$\perp$ to the initial line.

(*ii*) *Limits* : The curve lies wholly within the circle $r = a$ ($\because$ r is never $> a$)

(*iii*) *Points* : (*a*)

As θ increases from	*r varies from*	*Loop*	
0 to $\frac{\pi}{4}$	0 to a		
		no : 1,	
$\frac{\pi}{4}$ to $\frac{\pi}{2}$	a to 0		
$\frac{\pi}{2}$ to $\frac{3\pi}{4}$	0 to $-a$		
		no : 2,	
$\frac{3\pi}{4}$ to $\frac{\pi}{2}$	$-a$ to 0		etc. etc.

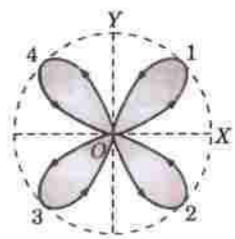


Fig. 4.36

(*b*) $$\tan \phi = r\frac{d\theta}{dr} = \frac{1}{2} \tan 2\theta \; ;$$

$\therefore$ $$\phi = 0, \text{ when } \theta = 0, \frac{\pi}{2}, \pi, 3\frac{\pi}{2}, 2\pi \, ...$$

$$\phi = \frac{\pi}{2}, \text{ when } \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \, ...$$

Hence, the shape of the curve is as shown in Fig. 4.36.

Example 4.81. *Trace the curve $r^2 = a^2 \cos 2\theta$.* (*V.T.U., 2007 ; Kurukshetra, 2006 ; B.P.T.U., 2005*)

Solution. (*i*) *Symmetry.* The curve is symmetrical about the pole.

(*ii*) *Limits* : (*a*) The curve lies wholly within the circle $r = a$.

(*b*) No portion of the curve lies between the lines $\theta = \pi/4$ and $\theta = 3\pi/4$.

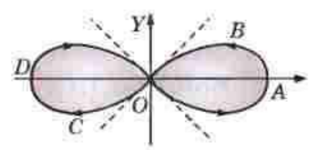


Fig. 4.37

(*iii*) *Points* : (*a*) $\tan \phi = r \dfrac{d\theta}{dr} = -\cot 2\theta = \tan\left(\dfrac{\pi}{2} + 2\theta\right)$

i.e., $$\phi = \frac{\pi}{2} + 2\theta \quad \therefore \quad \phi = 0, \text{ when } \theta = -\pi/4 \; ; \; \phi = \pi/2 \text{ when } \theta = 0.$$

Thus, the tangent at O is $\theta = -\pi/4$ and the tangent at A is $\perp$ to the intial line.

(*b*) The variations of r and θ are given below :

As θ varies from	*r varies from*	*Portion traced*
0 to $\pi/4$	a to 0	*ABO*
$3\pi/4$ to π	0 to a	*OCD*

As θ increase from π to 2π, we get the reflection of the arc *ABOCD* in the initial line. Hence the shape of the curve is as shown in Fig. 4.37. This curve is known as *Lemniscate of Bernoulli.*

Example 4.82. *Trace the curve* $r = a + b\cos\theta$ *(Limacon)*

Solution. (*i*) *Symmetry.* It is symmetrical about the initial line.

(*ii*) *Limits* : The curve wholly lies within the circle $r = a + b$

($\because$ r is never $> a + b$)

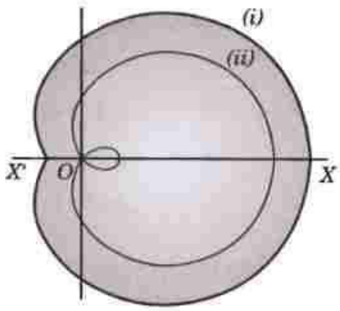


Fig. 4.38

(*iii*) *Points* : (α) when $a > b$.

As θ increases from 0 to $\pi/2$; r decreases from $a + b$ to a

As θ increases from $\pi/2$ to π ; r decreases from a to $a - b$

The shape of the curve is as shown in Fig. 4.38 (*i*).

(β) when $a < b$.

As θ increases from 0 to $\pi/2$; r decreases from $a + b$ to a

As θ increases from $\pi/2$ to α ; r decreases from a to 0

As θ increases from α to π ; r decreases from 0 to $a - b$

when $\alpha = \cos^{-1}\left(-\dfrac{a}{b}\right)$

In this case, the curve consists of two parts, one of which forms a loop within the other and the shape is as shown in Fig. 4.38 (*ii*).

Example 4.83. *Trace the curve* $r\theta = a$. *(Spiral)*

Solution. (*i*) *Symmetry.* There is no symmetry.

(*ii*) *Limits* : There are no limits to the values of r.

The curve does not pass through the pole for r does not become zero for any real value of θ.

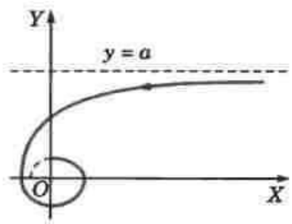


Fig. 4.39

(*iii*) *Asymptotes* : $\dfrac{1}{r} = \dfrac{\theta}{a} = f(\theta)$

$$f(\theta) = 0 \text{ for } \theta = 0 \;;\; f'(\theta) = 1/a,\; f'(0) = 1/a.$$

$\therefore$ Asymptote is $r \sin(\theta - 0) = 1/f'(0)$

i.e., $y = r\sin\theta = a$ is an asymptote.

(*iv*) *Points* : As θ increases from 0 to ∞, r to positive and decreases from ∞ to 0.

Hence the space of the curve is as shown in Fig. 4.39.

Example 4.84. *Trace the curve* $x^5 + y^5 = 5ax^2y^2$.

Solution. (*i*) *Symmetry.* The curve is symmetrical about the line $y = x$.

$\because$ On interchanging x and y, it remains unchanged.]

(*ii*) *Origin* : It passes through the origin and the tangents at the origin are given by

$$x^2 y^2 = 0,\ i.e.,\ x = 0, x = 0 \;;\; y = 0, y = 0.$$

Hence the curve has both *node* and the *cusp* at the origin.

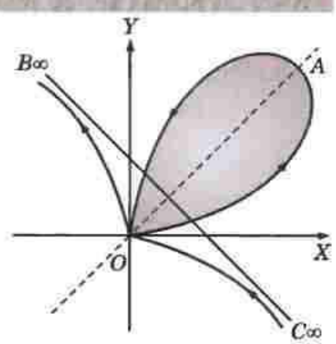


Fig. 4.40

(*iii*) *Asymptotes* : (*a*) It has no asymptotes parallel to the axes.

(*b*) Putting $x = 1, y = m$ in the fifth degree terms, we get

$$\phi_5(m) = 1 + m^5. \quad \therefore \quad \phi_5(m) = 0 \text{ gives } m = -1.$$

$$\therefore \qquad c = -\frac{\phi_4(m)}{\phi_5'(m)} = -\frac{-5am^2}{5m^4} = a \text{ for } m = -1.$$

Hence $y = -x + a$ or $x + y = a$ is an asymptote.

(*iv*) *Points* : Since it is not convenient to express y as a function of x or *vice versa*, hence we change the equation into polar coordinates by putting, $x = r\cos\theta$ and $y = r\sin\theta$. The equation of the curve becomes :

$$r = \frac{5a\sin^2\theta\cos^2\theta}{\cos^5\theta + \sin^5\theta} = \frac{5a}{4}\cdot\frac{\sin^5 2\theta}{\cos^5\theta + \sin^5\theta}$$

As θ increases from	*r*	*Portion traced from*
0 to $\pi/4$	is +ve and increases from 0 to $\frac{5\sqrt{2}}{2}a$	0 to *A*
$\pi/4$ to $\pi/2$	is +ve and decreases from $\frac{5\sqrt{2}}{2}a$ to 0	*A* to 0
$\pi/2$ to $3\pi/4$	is +ve and increases from 0 to ∞	0 to B_∞
$3\pi/4$ to π	is –ve and decreases from ∞ to 0	C_∞ to 0

As θ increases from π to 2π, the curve will retraced.

Hence the shape of the curve is as shown in Fig. 4.40.

PROBLEMS 4.15

Trace the following curves :

1. $y^2(a + x) = x^2(a - x)$. (*S.V.T.U., 2008 ; U.P.T.U., 2006 ; Rajasthan, 2005*)
2. $y^2(a^2 + x^2) = x^2(a^2 - x^2)$ (*V.T.U., 2010*)
3. $y = (x^2 + 1)/(x^2 - 1)$
4. $ay^2 = x^2(a - x)$ (*Kuruksehtra, 2009 S ; V.T.U., 2004*)
5. $x^2y^2 = a^2(y^2 - x^2)$
6. $x = a\cos^3\theta, y = b\sin^3\theta$
7. $x = a(\theta - \sin\theta), y = a(1 - \cos\theta)\ (0 < \theta < 2\pi)$
8. $x = (a\cos t + \log\tan t/2), y = a\sin t$.
9. $r = a\cos 2\theta$
10. $r = a\cos 3\theta$
11. $r = a(1 - \cos\theta)$
12. $r = 2 + 3\cos\theta$
13. $r^2\cos 2\theta = a^2$. (*S.V.T.U., 2009*)

[**Hint.** Changing to Cartesian form $x^2 - y^2 = a^2$. This is a rectangular hyperbola with asymptotes $x + y = 0$ and $x - y = 0$]

4.18 OBJECTIVE TYPES OF QUESTIONS

PROBLEMS 4.16

Select the correct answer or fill up the blanks in each of the following questions :

1. The radius of curvature of the catenary $y = c\cosh x/c$ at the point where it crosses the y-axis is ......
2. The envelope of the family of straight lines $y = mx + am^2$, (m being the parameter) is ......
3. The curvature of the circle $x^2 + y^2 = 25$ at the point (3, 4) is ......
4. The value of $\underset{x\to\pi/2}{\text{Lt}} \dfrac{\log\sin x}{(\pi/2 - x)^2}$ is

   (*a*) zero (*b*) 1/2 (*c*) – 1/2 (*d*) – 2. (*V.T.U., 2010*)
5. Taylor's expansion of the function $f(x) = \dfrac{1}{1+x^2}$ is

(a) $\sum_{n=0}^{\infty} (-1)^n x^{2n}$ for $-1 < x < 1$ (b) $\sum_{n=0}^{\infty} x^{2n}$ for $-1 < x < 1$

(c) $\sum_{n=0}^{\infty} (-1)^n x^{2n}$ for any real x (d) $\sum_{n=0}^{\infty} (-1)^n x^n$ for $-1 < x \le 1$.

6. A triangle of maximum area inscribed in a circle of radius r
   (a) is a right angled triangle with hypotenuse measuring $2r$
   (b) is an equilateral triangle
   (c) is an isosceles triangle of height r
   (d) does *not* exist.
7. The extreme value of $(x)^{1/x}$ is
   (a) e (b) $(1/e)^e$ (c) $(e)^{1/e}$ (d) 1.
8. The percentage error in computing the area of an ellipse when an error of 1 per cent is made in measuring the major and minor axes is
   (a) 0.2% (b) 2% (c) 0.02%.
9. The length of subtangent of the rectangular hyperbola $x^2 - y^2 = a^2$ at the point $(a, \sqrt{2}a)$ is
   (a) $\sqrt{2}\,a$ (b) $2a$ (c) $\frac{1}{2a}$ (d) $\frac{a^{3/2}}{\sqrt{2}}$.
10. The length of subnormal to the curve $y = x^3$ at (2, 8) is
    (a) 2/3 (b) 32 (c) 96 (d) 64.
11. If the normal to the curve $y^2 = 5x - 1$ at the point $(1, -2)$ is of the form $ax - 5y + b = 0$, then a and b are
    (a) 4, 14 (b) 4, $-$14 (c) $-$4, 14 (d) $-$4, $-$14.
12. The radius of curvature of the curve $y = e^x$ at the point where it crosses the y-axis is
    (a) 2 (b) $\sqrt{2}$ (c) $2\sqrt{2}$ (d) $\frac{1}{2}\sqrt{2}$.
13. The equation of the asymptotes of $x^3 + y^3 = 3axy$, is
    (a) $x + y - a = 0$ (b) $x - y + a = 0$ (c) $x + y + a = 0$ (d) $x - y - a = 0$.
14. If ϕ be the angle between the tangent and radius vector at any point on the curve $r = f(\theta)$, then $\sin\phi$ equals to
    (a) $\frac{dr}{ds}$ (b) $r\frac{d\theta}{ds}$ (c) $r\frac{d\theta}{dr}$.
15. Envelope of the family of lines $x = my + 1/m$ is ...
16. The chord of curvature parallel to y-axis for the curve $y = a \log \sec x/a$ is ....
17. $\sinh x = \ldots x + \ldots x^3 + \ldots x^5 + \ldots$
18. The nth derivative of $(\cos x \cos 2x \cos 3x)$ = ...
19. If $x^3 + y^3 - 3axy = 0$, then d^2y/dx^2 at $(3a/2, 3a/2)$ = ...
20. When the tangent at a point on a curve is parallel to x-axis, then the curvature at that point is same as the second derivative at that point. *(True or False)*
21. If $x = at^2, y = 2at$, t being the parameter, then $xy\, d^2y/dx^2$ = ...
22. The radius of curvature for the parabola $x = a, y = 2at$ at any point t = ...
23. If (a, b) are the coordinates of the centre of curvature whose curvature is k, then the equation of the circle of curvature is ....
24. Evolute is defined as the .... of the normals for a given curve.
25. Envelope of the family of lines $\frac{x}{t} + yt = 2c$ (where t is the parameter) is ....
26. The angle between the radius vector and tangent for the curve $r = ae^{\theta \cot \alpha}$ is ....
27. The subnormal of the parabola $y^2 = 4ax$ is ....
28. The fourth derivative of $(e^{-x} x^3)$ is ....

29. If $y^2 = P(x)$, a polynomial of degree 3, then $\frac{2d}{dx}\left(y^3 \frac{d^2y}{dx^2}\right)$ equals ....

(*a*) $P'''(x) + P'(x)$ (*b*) $P''(x) + P'''(x)$ (*c*) $P(x)\,P'''(x)$.

30. The envelope of the family of straight line $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$.

31. Curvature of a straight line is

(*A*) ∞ (*B*) zero (*C*) Both (*A*) and (*B*) (*D*) None of these.

32. The value of '*c*' of the Cauchy's Mean value theorem for $f(x) = e^x$ and $g(x) = e^{-x}$ in $[2, 3]$ is ....

33. If the equation of a curve remains unchanged when x and y are interchanged, then the curve is symmetrical about ....

34. For the curve $y^2(1 + x) = x^2(1 - x)$, the origin is a .... (node/cusp/conjugate point).

35. The number of loops of $r = a \sin 2\theta$ are .... and these of $r = a \cos 3\theta$ are ....

36. Tangents at the origin for the curve $y^2(x^2 + y^2) + a^2(x^2 - y^2) = 0$ are ....

37. The asymptote to the curve $y^2(4 - x) = x^3$ is ....

38. The points of inflexion of the curve $y^2 = (x - a)^2(x - b)$ lie on the line $3x + a =$ ....

39. The curve $r = a/(1 + \cos\theta)$ intersects orthogonally with the curve

(*A*) $r = b/(1 - \cos\theta)$ (*B*) $r = b/(1 + \sin\theta)$ (*C*) $r = b/(1 + \sin^2\theta)$ (*D*) $r = b/(1 + \cos^2\theta)$. (*V.T.U., 2010*)

40. The region where the curve $r = a \sin\theta$ does not lie is ....

41. If $f(x)$ is continuous in the closed interval $[a, b]$, differentiable in (a, b) and $f(a) = f(b)$, then there exists at least one value c of x in (a, b) such that $f'(c)$ is equal to

(*A*) 1 (*B*) -1 (*C*) 2 (*D*) 0. (*V.T.U., 2009*)

42. If two curves intersect orthogonally in cartesian form, then the angle between the same two curves in polar form is

(*A*) $\pi/4$ (*B*) Zero (*C*) 1 radian (*D*) None of these.

43. If the angle between the radius vector and the tangent is constant, then the curve is,

(*A*) $r = a\cos\theta$ (*B*) $r^2 = a^2\cos^2\theta$ (*C*) $r = ae^{b\theta}$. (*V.T.U., 2009*)

CHAPTER

5

Partial Differentiation and Its Applications

5.1 (1) FUNCTIONS OF TWO OR MORE VARIABLES

We often come across quantities which depend on two or more variables. For example, the area of a rectangle of length x and breadth y is given by $A = xy$. For a given pair of values of x and y, A has a definite value. Similarly, the volume of a parallelopiped ($= xyh$) depends on the three variables x(= length), y(= breadth) and h(=height).

Def. *A symbol z which has a definite value for every pair of values of x and y is called a function of two independent variables x and y and we write $z = f(x, y)$ or $\phi(x, y)$.*

We may interpret (x, y) as the coordinates of a point in the XY-plane and z as the height of the surface $z = f(x, y)$. We have come across several examples of such surfaces in Chapter 4.

The set R of points (x, y) such that any two points P_1 and P_2 of R can be so joined that any arc P_1P_2 wholly lies in R, is called as *region* in the XY-plane. A region is said to be a *closed region* if it includes all the points of its boundary, otherwise it is called an *open region.*

A set of points lying within a circle having centre at (a, b) and radius $\delta > 0$, is said to be *neighbourhood* of (a, b) in the circular region $R : (x - a)^2 + (y - b)^2 < \delta^2$.

When z is a function of three or more variables $x, y, t, \ldots$, we represent the relation by writing $z = f(x, y, t, \ldots)$. For such functions, no geometrical representation is possible. However, the concepts of a region and neighbourhood can easily be extended to functions of three or more variables.

(2) Limits. *The function $f(x, y)$ is said to tend to the limit l as $x \to a$ and $y \to b$ if and only if the limit l is independent of the path followed by the point (x, y) as $x \to a$ and $y \to b$ and we write*

$$\underset{\substack{x \to a \\ y \to b}}{Lt} f(x, y) = l$$

In terms of a circular neighbourhood, we have the following *definition of the limit :*

The function $f(x, y)$ defined in a region R, is said to tend to the limit l as $x \to a$ and $y \to b$ if and only if corresponding to a positive number ε, there exists another positive number δ such that $| f(x, y) - l | < \varepsilon$ for $0 < (x - a)^2 + (y - b)^2 < \delta^2$ for every point (x, y) in R.

(3) Continuity. *A function $f(x, y)$ is said to be continuous at the point (a, b) if*

$$\underset{\substack{x \to a \\ y \to b}}{Lt} f(x, y) \text{ exists and } = f(a, b)$$

If a function is continuous at all points of a region, then it is said to be *continuous in that region*. A function which is not continuous at a point is said to be *discontinuous* at that point.

Obs. *Usually, the limit is the same irrespective of the path along which the point (x, y) approaches (a, b) and*

$$\underset{x\to a}{Lt}\left\{\underset{y\to b}{Lt} f(x, y)\right\} = \underset{y\to b}{Lt}\left\{\underset{x\to a}{Lt} f(x, y)\right\}$$

But it is not always so, as the following examples show :

$$\text{Lt}\left(\frac{x-y}{x+y}\right) \textit{ as } (x, y) \to (0, 0) \text{ along the line } y = mx$$

$$= \underset{x\to 0}{\text{Lt}} \frac{x-mx}{x+mx} = \frac{1-m}{1+m} \text{ which is different for lines with different slopes.}$$

Also $\underset{x\to 0}{\text{Lt}}\left[\underset{y\to 0}{\text{Lt}}\left(\frac{x-y}{x+y}\right)\right] = \underset{x\to 0}{\text{Lt}}\left(\frac{x}{x}\right) = 1$, whereas $\underset{y\to 0}{\text{Lt}}\left[\underset{x\to 0}{\text{Lt}}\left(\frac{x-y}{x+y}\right)\right] = \underset{y\to 0}{\text{Lt}}\left(\frac{-y}{y}\right) = 1.$

$\therefore$ As (x, y) is made to approach $(0, 0)$ along different paths, $f(x, y)$ approaches different limits. Hence the two repeated limits are not equal and $f(x, y)$ is discontinuous at the origin.

Also the function is not defined at $(0, 0)$ since $f(x, y) = 0/0$ for $x = 0, y = 0$.

(4) *As in the case of functions of one variable, the following results hold :*

I. *If* $\underset{\substack{x\to a\\ y\to b}}{Lt} f(x, y) = l$ and $\underset{\substack{x\to a\\ y\to b}}{Lt} g(x, y) = m$,

then (*i*) *If* $\underset{\substack{x\to a\\ y\to b}}{Lt} [f(x, y) \pm g(x, y)] = l \pm m$ (*ii*) $\underset{\substack{x\to a\\ y\to b}}{Lt} [f(x,y) \,.\, g(x, y)] = l \,.\, m$

(*iii*) $\underset{\substack{x\to a\\ y\to b}}{Lt} [f(x,y)/ g(x, y)] = l/m$ $(m \neq 0)$

II. *If* $f(x, y), g(x, y)$ *are continuous at (a, b) then so also are the functions*

$f(x, y) \pm g(x, y), f(x, y) \,.\, g(x, y)$ and $f(x, y)/g(x, y)$

provided $g(x, y) \neq 0$ in the last case.

PROBLEMS 5.1

Evaluate the following limits :

1. $\underset{\substack{x\to 1\\ y\to 2}}{\text{Lt}} \dfrac{2x^2y}{x^2+y^2+1}$ 2. $\underset{\substack{x\to 0\\ y\to 0}}{\text{Lt}} \dfrac{xy}{x^2+y^2}$ 3. $\underset{\substack{x\to \infty\\ y\to 2}}{\text{Lt}} \dfrac{xy+1}{x^2+2y^2}$ 4. $\underset{\substack{x\to 1\\ y\to 1}}{\text{Lt}} \dfrac{x(y-1)}{y(x-1)}$

5. If $f(x, y) = \dfrac{x-y}{2x+y}$, show that $\underset{x\to 0}{\text{Lt}}\left[\underset{y\to 0}{\text{Lt}} f(x, y)\right] \neq \underset{y\to 0}{\text{Lt}}\left[\underset{x\to 0}{\text{Lt}} f(x, y)\right]$

   Also show that the function is discountinuous at the origin.

6. Show that the function $f(x, y) = x^2 + 2y$, $(x, y) \neq (1, 2)$

   $3(x, y) = (1, 2)$ $= 0$

   is discontinuous at $(1, 2)$.

7. Investigate the continuity of the function

   $f(x, y) = xy/(x^2 + y^2)$, $(x, y) \neq (0, 0)$

   $= 0$ $(x, y) = (0, 0)$

   at the origin.

   Note. In whatever follows, all the functions considered are continuous and their partial derivatives (as defined below) exist.

5.2 PARTIAL DERIVATIVES

Let $z = f(x, y)$ be a function of two variables x and y.

If we keep y as constant and vary x alone, then z is a function of x only. The derivative of z with respect to x, treating y as constant, is called the *partial derivative of z with respect to x* and is denoted by one of the symbols

$$\frac{\partial z}{\partial x}, \frac{\partial f}{\partial x}, f_x(x, y), D_x f. \qquad \text{Thus} \quad \frac{\partial z}{\partial x} = \underset{\delta x \to 0}{\text{Lt}} \frac{f(x + \delta x, y) - f(x, y)}{\delta x}$$

Similarly, the derivative of z with respect to y, keeping x as constant, is called the *partial derivative of z with respect to y* and is denoted by one of the symbols.

$$\frac{\partial z}{\partial y}, \frac{\partial f}{\partial y}, f_y(x, y), D_y f. \qquad \text{Thus} \quad \frac{\partial z}{\partial y} = \underset{\partial y \to 0}{\text{Lt}} \frac{f(x, y + \delta y) - f(x, y)}{\delta y}$$

Similarly, if z is a function of three or more variables $x_1, x_2, x_3, \ldots$ *the partial derivative of z with respect to x_1, is obtained by differentiating z with respect to x_1, keeping all other variables constant* and is written as $\partial z/\partial x_1$.

In general f_x and f_y are also functions of x and y and so these can be differentiated further partially with respect to x and y.

Thus $\quad \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right) = \dfrac{\partial^2 z}{\partial x^2}$ or $\dfrac{\partial^2 z}{\partial x^2}$ or f_{xx}, $\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{\partial^2 z}{\partial x \partial y}$ or $\dfrac{\partial^2 f}{\partial x \partial y}$ or f_{yx}*

$\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial x}\right) = \dfrac{\partial^2 z}{\partial y \partial x}$ or $\dfrac{\partial^2 f}{\partial y \partial x}$ or f_{xy}, and $\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{\partial^2 z}{\partial y^2}$ or $\dfrac{\partial^2 f}{\partial y^2}$ or f_{yy}.

It can easily be verified that, in all ordinary cases,

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}.$$

Sometimes we use the following notation

$$\frac{\partial z}{\partial x} = p, \ \frac{\partial z}{\partial y} = q, \ \frac{\partial^2 z}{\partial x^2} = r, \ \frac{\partial^2 z}{\partial x \partial y} = s, \ \frac{\partial^2 z}{\partial y^2} = t.$$

Example 5.1. *Find the first and second partial derivatives of $z = x^3 + y^3 - 3axy$.*

Solution. We have $\quad z = x^3 + y^3 - 3axy$.

$$\therefore \quad \frac{\partial z}{\partial x} = 3x^2 + 0 - 3ay(1) = 3x^2 - 3ay, \text{ and } \frac{\partial z}{\partial y} = 0 + 3y^2 - 3ax(1) = 3y^2 - 3ax$$

Also $\quad \dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial}{\partial x}(3x^2 - 3ay) = 6x, \ \dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial}{\partial y}(3x^2 - 3ay) = -3a$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(3y^2 - 3ax) = 6y, \text{ and } \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(3y^2 - 3ax) = -3a.$$

We observe that $\quad \dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y}$.

Example 5.2. *If $u = x^2 \tan^{-1} \dfrac{y}{x} - y^2 \tan^{-1} \dfrac{x}{y}$,*

show that $\quad \dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{x^2 - y^2}{x^2 + y^2}$. *(Mumbai, 2008 S)*

and $\quad \dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$ *(Madras, 2000)*

Solution. We have $\quad \dfrac{\partial u}{\partial y} = x^2 \cdot \dfrac{1}{1 + (y/x)^2} \cdot \dfrac{1}{x} - \left\{2y \cdot \tan^{-1} \dfrac{x}{y} + y^2 \cdot \dfrac{1}{1 + (x/y)^2} \cdot \left(-\dfrac{x}{y}\right)\right\}$

$$= \frac{x^3}{x^2 + y^2} - 2y \tan^{-1} \frac{x}{y} + \frac{xy^2}{x^2 + y^2} = x - 2y \tan^{-1} \frac{x}{y}.$$

*It is important to note that in the subscript notation the subscripts are written in the same order in which we differentiate whereas in the '∂' notation the order is opposite.

$\therefore \quad \dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left\{x - 2y \tan^{-1} \dfrac{x}{y}\right\} = 1 - 2y \cdot \dfrac{1}{1+(x/y)^2} \cdot \dfrac{1}{y} = 1 - \dfrac{2y^2}{x^2+y^2} = \dfrac{x^2-y^2}{x^2+y^2}.$

Similarly, $\quad \dfrac{\partial u}{\partial x} = 2x \tan^{-1} y/x - y$

and $\quad \dfrac{\partial^2 u}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left\{2x \tan^{-1} \dfrac{y}{x} - y\right\} = \dfrac{x^2-y^2}{x^2+y^2}$. Hence the result.

Example 5.3. *If* $z = f(x+ct) + \phi(x-ct)$, *prove that*

$$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$$ (J.N.T.U., 2006 ; V.T.U., 2003 S)

Solution. We have $\dfrac{\partial z}{\partial x} = f'(x+ct) \cdot \dfrac{\partial}{\partial x}(x+ct) + \phi'(x-ct)\dfrac{\partial}{\partial x}(x-ct) = f'(x+ct) + \phi'(x-ct)$

and $\quad \dfrac{\partial^2 z}{\partial x^2} = f''(x+ct) + \phi''(x-ct)$ ...(*i*)

Again $\quad \dfrac{\partial z}{\partial t} = f'(x+ct)\dfrac{\partial}{\partial t}(x+ct) + \phi'(x-ct)\dfrac{\partial}{\partial t}(x-ct) = cf'(x-ct) - c\phi'(x-ct)$

and $\quad \dfrac{\partial^2 z}{\partial t^2} = c^2 f''(x+ct) + c^2\phi''(x-ct) = c^2[f''(x+ct) + \phi''(x-ct)]$ ...(*ii*)

From (*i*) and (*ii*), it follows that $\dfrac{\partial^2 z}{\partial t^2} = c^2 \dfrac{\partial^2 z}{\partial x^2}$.

Obs. This is an important partial differential equation, known as *wave equation* (§ 18.4).

Example 5.4. *If* $\theta = t^n e^{-r^2/4t}$, *what value of n will make* $\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial \theta}{\partial r}\right) = \dfrac{\partial \theta}{\partial t}$?

(Nagpur, 2009 ; Kurukshetra, 2006 ; U.P.T.U., 2006)

Solution. We have $\dfrac{\partial \theta}{\partial r} = t^n \cdot e^{-r^2/4t} \cdot \left(\dfrac{-2r}{4t}\right) = -\dfrac{r}{2}t^{n-1}e^{-r^2/4t}$

$\therefore \quad r^2 \dfrac{\partial \theta}{\partial r} = -\dfrac{r^3}{2}t^{n-1} \cdot e^{-r^2/4t}$

and $\quad \dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial \theta}{\partial r}\right) = -\dfrac{3r^2}{2}t^{n-1}e^{-r^2/4t} - \dfrac{r^3}{2}t^{n-1} \cdot e^{-r^2/4t}\left(-\dfrac{2r}{4t}\right)$

$\therefore \quad \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial \theta}{\partial r}\right) = \left(-\dfrac{3}{2}t^{n-1} + \dfrac{r^2}{4}t^{n-2}\right)e^{-r^2/4t}$

Also $\quad \dfrac{\partial \theta}{\partial t} = nt^{n-1} \cdot e^{-r^2/4t} + t^n \cdot e^{-r^2/4t} \cdot \dfrac{r^2}{4t^2} = \left(nt^{n-1} + \dfrac{1}{4}r^2t^{n-2}\right)e^{-r^2/4t}$

Since $\quad \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial \theta}{\partial r}\right) = \dfrac{\partial \theta}{\partial t}$,

$\therefore \quad \left(-\dfrac{3}{2}t^{n-1} + \dfrac{1}{4}r^2t^{n-2}\right)e^{-r^2/4t} = \left(nt^{n-1} + \dfrac{1}{4}r^2t^{n-2}\right)e^{-r^2/4t}$ or $\left(n + \dfrac{3}{2}\right)t^{n-1}e^{-r^2/4t} = 0.$

Hence $\quad n = -3/2.$

Example 5.5. *If* $v = (x^2+y^2+z^2)^{-1/2}$, *prove that*

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0. \text{ (Laplace equation)*}$$ (V.T.U., 2006 ; Osmania, 2003 S)

*See footnote p. 18.

Solution. We have $\frac{\partial v}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2x = -x(x^2+y^2+z^2)^{-3/2}$

and
$$\frac{\partial^2 v}{\partial x^2} = -1[1 \cdot (x^2+y^2+z^2)^{-3/2} + x(-3/2)(x^2+y^2+z^2)^{-5/2} \cdot 2x]$$
$$= -(x^2+y^2+z^2)^{-5/2}[x^2+y^2+z^2-3x^2] = (x^2+y^2+z^2)^{-5/2}(2x^2-y^2-z^2)$$

Similarly,
$$\frac{\partial^2 v}{\partial y^2} = (x^2+y^2+z^2)^{-5/2}(-x^2+2y^2-z^2) \text{ and } \frac{\partial^2 v}{\partial z^2} = (x^2+y^2+z^2)^{-5/2}(-x^2-y^2+2z^2)$$

Hence
$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = (x^2+y^2+z^2)^{-5/2} \cdot (0) = 0.$$

Obs. *A function v satisfying the Laplace equation is said to be a* **harmonic function.**

Example 5.6. *If* $u = \log(x^3+y^3+z^3-3xyz)$, *show that* $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$.

(P.T.U., 2010 ; Anna, 2009 ; Bhopal, 2008 ; U.P.T.U., 2006)

Solution. We have $\frac{\partial u}{\partial x} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$, $\frac{\partial u}{\partial y} = \frac{3y^2-3zx}{x^3+y^3+z^3-3xyz}$, $\frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$

$\therefore$
$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz}$$
$$= \frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} = \frac{3}{x+y+z}$$ *(V.T.U., 2009)*

Now
$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$
$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) = \left(\frac{3}{x+y+z}\right)$$
$$= -\frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} = -\frac{9}{(x+y+z)^2}.$$

Example 5.7. *If* $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$, *prove that*
$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right)$$ *(U.P.T.U., 2003)*

Solution. We have $x^2(a^2+u)^{-1} + y^2(b^2+u)^{-1} + z^2(c^2+u)^{-1} = 1$ ...(*i*)

Differentiating (*i*) partially w.r.t. x, we get
$$2x(a^2+u)^{-1} - x^2(a^2+u)^{-2}\frac{\partial u}{\partial x} - y^2(b^2+u)^{-2}\frac{\partial u}{\partial y} - z^2(c^2+u)^{-2}\frac{\partial u}{\partial z} = 0$$

or
$$\frac{2x}{a^2+u} = \left\{\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{y^2}{(c^2+u)^2}\right\}\frac{\partial u}{\partial x}$$

or
$$\frac{\partial u}{\partial x} = \frac{2x}{(a^2+u)v} \text{ where } v = \frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{y^2}{(c^2+u)^2}$$

Similarly differentiating (*i*) partially w.r.t. y, we get
$$\frac{2y}{b^2+u} = \left\{\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right\}\frac{\partial u}{\partial y} \text{ or } \frac{\partial u}{\partial y} = \frac{2y}{(b^2+u)v}$$

Similarly, differentiating (i) partially w.r.t. z, we get

$$\frac{2z}{(b^2+u)} = \left\{\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right\}\frac{\partial u}{\partial z} \text{ or } \frac{\partial u}{\partial z} = \frac{2z}{(c^2+u)v}$$

$$\therefore \quad \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = \frac{4}{v^2}\left\{\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{y^2}{(c^2+u)^2}\right\} = \frac{4}{v} \quad ...(ii)$$

Also $$2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right) = 2\left\{\frac{2x^2}{(a^2+u)v} + \frac{2y^2}{(b^2+u)v} + \frac{2z^2}{(c^2+u)v}\right\}$$

$$= \frac{4}{v}\left\{\frac{x^2}{(a^2+u)} + \frac{y^2}{(b^2+u)} + \frac{z^2}{(c^2+u)}\right\} = \frac{4}{v} \quad \text{[By } (i)\text{] } ...(iii)$$

Hence the equality of (ii) and (iii) proves the result.

Example 5.8. *If $u = x^y$, show that $\dfrac{\partial^3 u}{\partial x^2 \partial y} = \dfrac{\partial^3 u}{\partial x \partial y \partial x}$.* (Anna, 2009)

Solution. We have $\dfrac{\partial u}{\partial y} = x^y \log_e x$ and $\dfrac{\partial^2 u}{\partial x \partial y} = yx^{y-1} \cdot \log x + x^y \cdot \dfrac{1}{x} = x^{y-1}(y \log x + 1)$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2 \partial y} = \frac{\partial}{\partial x}[x^{y-1}(y \log x + 1)] \quad ...(i)$$

Again $$\frac{\partial u}{\partial x} = yx^{y-1} \text{ and } \frac{\partial^2 u}{\partial y \partial x} = 1 \cdot x^{y-1} + y\left(\frac{1}{x}x^y \log x\right) = x^{y-1}(1 + y \log x)$$

$$\therefore \quad \frac{\partial^3 u}{\partial x \partial y \partial x} = \frac{\partial}{\partial x}[x^{y-1}(y \log x + 1)] \quad ...(ii)$$

From (i) and (ii) follows the required result.

PROBLEMS 5.2

1. Evaluate $\partial z/\partial x$ and $\partial z/\partial y$, if
   (i) $z = x^2 y - x \sin xy$; (ii) $z = \log(x^2 + y^2)$;
   (iii) $z = \tan^{-1}\{(x^2 + y^2)/(x + y)\}$; (iv) $x + y + z = \log z$.
2. If $z(x + y) = x^2 + y^2$, show that $\left(\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)^2 = 4\left(1 - \dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)$. (V.T.U., 2003)
3. If $z = e^{ax+by} f(ax - by)$, prove that $b\dfrac{\partial z}{\partial x} + a\dfrac{\partial z}{\partial y} = 2abz$. (V.T.U., 2010)
4. Given $u = e^{r\cos\theta}\cos(r\sin\theta)$, $v = e^{r\cos\theta}\sin(r\sin\theta)$; prove that $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$ and $\dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$.
5. If $z = \tan(y + ax) - (y - ax)^{3/2}$, show that $\dfrac{\partial^2 z}{\partial x^2} = a^2\dfrac{\partial^2 z}{\partial y^2}$. (Mumbai, 2009)
6. Verify that $f_{xy} = f_{yx}$, when f is equal to (i) $\sin^{-1}(y/x)$; (ii) $\log x \tan^{-1}(x^2 + y^2)$.
7. If $f(x, y) = (1 - 2xy + y^2)^{-1/2}$, show that $\dfrac{\partial}{\partial x}\left[(1 - x^2)\dfrac{\partial f}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[y^2\dfrac{\partial f}{\partial y}\right] = 0$. (Rohtak, 2006 S)
8. Prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ if (i) $u = \tan^{-1}\left[\dfrac{2xy}{x^2 - y^2}\right]$, (ii) $u = \log(x^2 + y^2) + \tan^{-1}(y/x)$. (Anna, 2009)
9. If $v = \dfrac{1}{\sqrt{t}}e^{-x^2/4a^2t}$, prove that $\dfrac{\partial v}{\partial t} = a^2\dfrac{\partial^2 v}{\partial x^2}$.

10. The equation $\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2}$ refers to the conduction of heat along a bar without radiation, show that if $u = Ae^{-gx} \sin (nt - gx)$, where A, g, n are positive constants then $g = \sqrt{(n/2\mu)}$.
11. Find the value of n so that the equation $V = r^n (3 \cos^2 \theta - 1)$ satisfies the relation
$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial V}{\partial \theta}\right) = 0.$$
12. If $z = \log (e^x + e^y)$, show that $rt - s^2 = 0$ where $r = \partial^2 z/\partial x^2$, $s = \partial^2 z/\partial x \partial y$, $t = \partial^2 z/\partial y^2$.
13. If $u = \frac{y}{z} + \frac{z}{x}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.
14. Let $r^2 = x^2 + y^2 + z^2$ and $V = r^m$, prove that $V_{xx} + V_{yy} + V_{zz} = m(m+1)r^{m-2}$. *(Raipur, 2005)*
15. If $v = \log (x^2 + y^2 + z^2)$, prove that $(x^2 + y^2 + z^2)\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = 2$.
16. If $v = x^y \cdot y^x$, prove that $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = v(x + y + \log v)$. *(Anna, 2005)*
17. If $x^x y^y z^z = c$, show that at $x = y = z$, $\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}$. *(Bhopal, 2008)*
18. If $u = e^{xyz}$, find the value of $\frac{\partial^3 u}{\partial x \partial y \partial z}$. *(Rajasthan, 2005 ; Osmania, 2003 S)*

5.3 WHICH VARIABLE IS TO BE TREATED AS CONSTANT

(1) Consider the equation $x = r \cos \theta$, $y = r \sin \theta$ ...(1)

To find $\partial r/\partial x$, we need a relation between r and x. Such a relation will contain one more variable θ or y, for we can eliminate only one variable out of four from the relations (1). Thus the two possible relations are

$$r = x \sec \theta \quad ...(2) \qquad \text{and} \qquad r^2 = x^2 + y^2 \quad ...(3)$$

Now we can find $\partial r/\partial x$ either from (2) by treating θ as constant or from (3) by regarding y as constant. And there is no reason to suppose that the two values of $\partial r/\partial x$ so found, are equal. *To avoid confusion as to which variable is regarded constant, we introduce the following :*

Notation : $(\partial r/\partial x)_\theta$ *means the partial derivative of r with respect to x keeping* θ *constant in* a relation expressing r as a function of x and θ.

Thus from (2), $(\partial r/\partial x)_\theta = \sec \theta$.

When no indication is given regarding the variable to be kept constant, then according to convention $(\partial/\partial x)$ always means $(\partial/\partial x)_y$ and $\partial/\partial y$ means $(\partial/\partial y)_x$. Similarly, $\partial/\partial r$ means $(\partial/\partial r)_\theta$ and $\partial/\partial \theta$ means $(\partial/\partial \theta)_r$.

(2) In thermodynamics, we come across ten variables such as p (pressure), v (volume), T (temperature), W (work), ϕ (entropy) etc. Any one of these can be expressed as a function of other two variables *e.g.*, $T = f(p, v)$, $T = g(p, \phi)$

As we shall see, these respectively give rise to the following results :

$$dT = \frac{\partial T}{\partial p} dp + \frac{\partial T}{\partial v} dv \quad ...(i)$$

$$dT = \frac{\partial T}{\partial p} dp + \frac{\partial T}{\partial \phi} d\phi \quad ...(ii)$$

Now, $\partial T/\partial p$ appearing in (*i*), has been obtained from T as function of p and v, treating v as constant, we write it as $(\partial T/\partial p)_v$.

Similarly, $\partial T/\partial p$ occurring in (*ii*), is written as $(\partial T/\partial p)_\phi$.

Example 5.9. *If $u = f(r)$ and $x = r \cos \theta$, $y = r \sin \theta$, prove that*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r).$$ *(S.V.T.U., 2008 ; Rajasthan, 2006 ; U.P.T.U., 2005)*

Solution. We have $\dfrac{\partial u}{\partial x} = f'(r) \cdot \dfrac{\partial r}{\partial x}$ and $\dfrac{\partial^2 u}{\partial x^2} = f''(r) \cdot \left(\dfrac{\partial r}{\partial x}\right)^2 + f'(r) \cdot \dfrac{\partial^2 r}{\partial x^2}$

Similarly, $$\frac{\partial^2 u}{\partial y^2} = f''(r) \cdot \left(\frac{\partial r}{\partial y}\right)^2 + f'(r) \cdot \frac{\partial^2 r}{\partial y^2}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) \cdot \left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right] + f'(r)\left[\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2}\right]$$

Now to find $\dfrac{\partial r}{\partial x}, \dfrac{\partial r}{\partial y}$ etc., we write $r = (x^2 + y^2)^{1/2}$

$$\therefore \quad \frac{\partial r}{\partial x} = \frac{1}{2}(x^2 + y^2)^{-1/2} \cdot 2x = \frac{x}{r} \quad \text{and} \quad \frac{\partial^2 r}{\partial x^2} = \frac{r \cdot 1 - x \cdot \partial r/\partial x}{r^2} = \frac{r - x^2/r}{r^2} = \frac{y^2}{r^3}.$$

Similarly, $$\frac{\partial r}{\partial y} = \frac{y}{x} \quad \text{and} \quad \frac{\partial^2 r}{\partial y^2} = \frac{x^2}{r^3}.$$

Substituting the values of $\partial r/\partial x$ etc. in (*i*), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r)\left[\frac{x^2}{r^2} + \frac{y^2}{r^2}\right] + f'(r)\left[\frac{y^2}{r^3} + \frac{x^2}{r^3}\right] = f''(r) + \frac{1}{r}f'(r).$$

Example 5.10. *If $x = e^{r\cos\theta}\cos(r\sin\theta)$ and $y = e^{r\cos\theta}\sin(r\sin\theta)$, prove that $\dfrac{\partial x}{\partial \theta} = -r\dfrac{\partial y}{\partial r}, \dfrac{\partial y}{\partial \theta} = r\dfrac{\partial x}{\partial r}$.*

Hence show that $\dfrac{\partial^2 x}{\partial \theta^2} + r\dfrac{\partial x}{\partial r} + r^2\dfrac{\partial^2 x}{\partial r^2} = 0$.

Solution. We have $x = e^{r\cos\theta}\cos(r\sin\theta)$

$$\therefore \quad \frac{\partial x}{\partial \theta} = e^{r\cos\theta}(-r\sin\theta) \cdot \cos(r\sin\theta) + e^{r\cos\theta}[-\sin(r\sin\theta)] \cdot r\cos\theta$$

$$= -re^{r\cos\theta}[\sin\theta\cos(r\sin\theta) + \cos\theta\sin(r\sin\theta)]$$

$$= -re^{r\cos\theta}\sin(\theta + r\sin\theta) \qquad \text{...}(i)$$

and $$\frac{\partial x}{\partial r} = e^{r\cos\theta} \cdot \cos\theta \cdot \cos(r\sin\theta) - e^{r\cos\theta}\sin\theta\,(r\sin\theta)\sin\theta$$

$$= e^{r\cos\theta}\cos(\theta + r\sin\theta) \qquad \text{...}(ii)$$

Similarly, $y = e^{r\cos\theta}\sin(r\sin\theta)$ gives

$$\frac{\partial y}{\partial \theta} = re^{r\cos\theta}\cos(\theta + r\sin\theta) \qquad \text{...}(iii)$$

and $$\frac{\partial y}{\partial r} = e^{r\cos\theta}\sin(\theta + r\sin\theta) \qquad \text{...}(iv)$$

From (*i*) and (*iv*), $\dfrac{\partial x}{\partial \theta} = -r\dfrac{\partial y}{\partial r}$ ...(*v*)

From (*ii*) and (*iii*), $\dfrac{\partial y}{\partial \theta} = r\dfrac{\partial x}{\partial r}$ ...(*vi*)

From (*v*), $$\frac{\partial^2 x}{\partial \theta^2} = -r\frac{\partial^2 y}{\partial \theta \partial r} = -r\frac{\partial^2 y}{\partial r \partial \theta}$$

From (*vi*), $\dfrac{\partial x}{\partial r} = \dfrac{1}{r}\dfrac{\partial y}{\partial \theta}$ which gives $\dfrac{\partial^2 x}{\partial r^2} = -\dfrac{1}{r^2}\dfrac{\partial y}{\partial \theta} + \dfrac{1}{r}\dfrac{\partial^2 y}{\partial r \partial \theta}$

$$\therefore \quad \frac{\partial^2 x}{\partial \theta^2} + r\frac{\partial x}{\partial r} + r^2\frac{\partial^2 x}{\partial r^2} = -r\frac{\partial^2 y}{\partial r \partial \theta} + \frac{\partial y}{\partial \theta} - \frac{\partial y}{\partial \theta} + r\frac{\partial^2 y}{\partial r \partial \theta} = 0.$$

PROBLEMS 5.3

1. If $x = r\cos\theta$, $y = r\sin\theta$, show that (i) $\dfrac{\partial r}{\partial x} = \dfrac{\partial x}{\partial r}$ (ii) $\dfrac{1}{r}\dfrac{\partial x}{\partial \theta} = r\dfrac{\partial \theta}{\partial x}$, (iii) $\left(\dfrac{\partial r}{\partial x}\right)^2 + \left(\dfrac{\partial r}{\partial y}\right)^2 = 1.$ (Burdwan, 2003)

2. If $x^2 = au + bv$, $y^2 = au - bv$, prove that $\left(\dfrac{\partial u}{\partial x}\right)_y \cdot \left(\dfrac{\partial x}{\partial u}\right)_v = \dfrac{1}{2} = \left(\dfrac{\partial v}{\partial y}\right)_x \cdot \left(\dfrac{\partial y}{\partial v}\right)_u$.

3. If $u = lx + my$, $v = mx - ly$, show that $\left(\dfrac{\partial u}{\partial x}\right)_y \cdot \left(\dfrac{\partial x}{\partial u}\right)_v = \dfrac{l^2}{l^2 + m^2}$, $\left(\dfrac{\partial y}{\partial v}\right)_x \left(\dfrac{\partial v}{\partial y}\right)_u = \dfrac{l^2 + m^2}{l^2}$.

4. If $x = r\cos\theta$, $y = r\sin\theta$, prove that

(i) $\dfrac{\partial^2 r}{\partial x^2} + \dfrac{\partial^2 r}{\partial y^2} = \dfrac{1}{r}\left[\left(\dfrac{\partial r}{\partial x}\right)^2 + \left(\dfrac{\partial r}{\partial y}\right)^2\right]$ (ii) $\dfrac{\partial^2 \theta}{\partial x^2} + \dfrac{\partial^2 \theta}{\partial y^2} = 0$ $(x \neq 0, y \neq 0)$.

5. If $z = x\log(x + r) - r$ where $r^2 = x^2 + y^2$, prove that $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = \dfrac{1}{x + y}$, $\dfrac{\partial^3 z}{\partial x^3} = -\dfrac{x}{r^3}$. (Mumbai, 2008)

6. If $u = f(r)$ where $r = \sqrt{(x^2 + y^2 + z^2)}$, show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = f''(r) + \dfrac{2}{r}f'(r)$.

5.4 (1) HOMOGENEOUS FUNCTIONS

An expression of the form $a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \ldots + a_ny^n$ in which every term is of the nth degree, is called a homogeneous function of degree n. This can be rewritten as

$$x^n[a_0 + a_1(y/x) + a_2(y/x)^2 + \ldots + a_n(y/x)^n].$$

Thus *any function $f(x, y)$ which can be expressed in the form $x^n\phi(y/x)$, is called a* **homogeneous function** of degree n in x and y.

For instance, $x^3\cos(y/x)$ is a homogeneous function of degree 3, in x and y.

In general, a function $f(x, y, z, t, \ldots)$ is said to be a homogeneous function of degree n in $x, y, z, t, \ldots$, if it can be expressed in the form $x^n\phi(y/x, z/x, t/x, \ldots)$.

(2) Euler's theorem on homogeneous functions*. *If u be a homogeneous function of degree n in x and y, then*

$$\mathbf{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.}$$

Since u is a homogeneous function of degree n in x and y, therefore,

$$u = x^n f(y/x)$$

$$\therefore \quad \frac{\partial u}{\partial x} = nx^{n-1}f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right)\cdot y\left(-\frac{1}{x^2}\right) = nx^{n-1}f\left(\frac{y}{x}\right) - yx^{n-2}f'\left(\frac{y}{x}\right)$$

and $$\frac{\partial u}{\partial y} = x^n f'\left(\frac{y}{x}\right)\cdot\frac{1}{x} = x^{n-1}f'\left(\frac{y}{x}\right). \text{ Hence } x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n f\left(\frac{y}{x}\right) = nu.$$

In general, if u be a homogeneous function of degree n in $x, y, z, t, \ldots$, then,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + t\frac{\partial u}{\partial t} \ldots = nu.$$

Example 5.11. *Show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2u\log u$ where $\log u = (x^3 + y^3)/(3x + 4y)$.*

Solution. Since $$z = \log u = \frac{x^3 + y^3}{3x + 4y} = x^2\cdot\frac{1 + (y/x)^3}{3 + 4(y/x)},$$

* After an enormously creative Swiss mathematician *Leonhard Euler* (1707–1783). He studied under John *Bernoulli* and became a professor of mathematics in St. Petersburg, Russia. Even after becoming totally blind in 1771, he contributed to almost all branches of mathematics.

$\therefore$ z is a homogeneous function of degree 2 in x and y.

By Euler's theorem, we get

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z \qquad ...(i)$$

But $$\frac{\partial z}{\partial x} = \frac{1}{u}\frac{\partial u}{\partial x} \text{ and } \frac{\partial z}{\partial y} = \frac{1}{u}\frac{\partial u}{\partial y}$$

Hence (i) becomes

$$x \cdot \frac{1}{u}\frac{\partial u}{\partial x} + y \cdot \frac{1}{u}\frac{\partial u}{\partial y} = 2\log u \quad \text{or} \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u \log u.$$

Example 5.12. *If* $u = \sin^{-1}\frac{x+2y+3z}{x^8+y^8+z^8}$, *find the value of* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$. *(U.P.T.U., 2004)*

Solution. Here u is not a homogeneous function. We therefore, write

$$\omega = \sin u = \frac{x+2y+3z}{x^8+y^8+z^8} = x^{-7} \cdot \frac{1+2(y/x)+3(z/x)}{1+(y/x)^8+(z/x)^8}$$

Thus ω is a homogeneous function of degree -7 in x, y, z. Hence by Euler's theorem

$$x\frac{\partial \omega}{\partial x} + y\frac{\partial \omega}{\partial y} + z\frac{\partial \omega}{\partial z} = (-7)\,\omega \qquad ...(i)$$

But $$\frac{\partial \omega}{\partial x} = \cos u\,\frac{\partial u}{\partial x}, \frac{\partial \omega}{\partial y} = \cos u\,\frac{\partial u}{\partial y}, \frac{\partial \omega}{\partial z} = \cos u\,\frac{\partial u}{\partial z}$$

$\therefore$ (i) becomes $x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} + z\cos u\frac{\partial u}{\partial z} = -7\sin u$ or $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -7\tan u.$

Example 5.13. *If* $u = \frac{x^3y^3z^3}{x^3+y^3+z^3} + \log\left(\frac{xy+yz+zx}{x^2+y^2+z^2}\right)$, *find the value of* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$.

(Mumbai, 2009)

Solution. Let $v = \frac{x^3y^3z^3}{x^3+y^3+z^3}$ and $w = \log\left(\frac{xy+yz+zx}{x^2+y^2+z^2}\right)$ ...(i)

so that $u = v + w$

Since $v = x^6\frac{(y/x)^3(z/x)^3}{1+(y/x)^3+(z/x)^3}$, therefore v is a homogeneous function of degree 6 in x, y, z.

Hence by Euler's theorem $$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = 6v \qquad ...(ii)$$

Since $w = \log\left\{\frac{\frac{y}{x}+\frac{y}{x}\cdot\frac{z}{x}+\frac{z}{x}}{1+\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2}\right\}$ therefore w is a homogeneous function of degree zero in x, y, z.

Hence by Euler's theorem $$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = 0 \qquad ...(iii)$$

Addint (ii) and (iii), we obtain

$$x\left(\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\right) + y\left(\frac{\partial v}{\partial y}+\frac{\partial w}{\partial y}\right) + z\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial z}\right) = 6v$$

or $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 6\,\frac{x^3y^3z^3}{x^3+y^3+z^3} \qquad \text{[By }(i)\text{]}$$

Example 5.14. *If z is a homogeneous function of degree n in x and y, show that*

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial z^2}=n(n-1)\,z.$$

(Anna, 2009 ; V.T.U., 2007 ; U.P.T.U., 2006)

Solution. By Euler's theorem, $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz$ ...(*i*)

Differentiating (*i*) partially w.r.t. x, we get $x\frac{\partial^2 z}{\partial x^2}+\frac{\partial z}{\partial x}+y\frac{\partial^2 z}{\partial x\partial y}=n\frac{\partial z}{\partial x}$

i.e., $$x\frac{\partial^2 z}{\partial x^2}+y\frac{\partial^2 z}{\partial x\partial y}=(n-1)\frac{\partial z}{\partial x}$$ ...(*ii*)

Again differentiating (*i*) partially w.r.t. y, we get $x\frac{\partial^2 z}{\partial y\partial x}+\frac{\partial z}{\partial y}+y\frac{\partial^2 z}{\partial y^2}=n\frac{\partial z}{\partial y}$

i.e., $$x\frac{\partial^2 z}{\partial x\partial y}+y\frac{\partial^2 z}{\partial y^2}=(n-1)\frac{\partial z}{\partial y}$$ ...(*iii*)

Multiplying (*ii*) by x and (*iii*) by y and adding, we get

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=(n-1)\left(x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right)=n(n-1)\,z.$$ [By (*i*)]

Example 5.15. *If* $u=\sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, *prove that* $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\tan u.$

(Rajasthan, 2006 ; Calicut, 2005)

and $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-\frac{\sin u\cos 2u}{4\cos^3 u}.$$ *(P.T.U., 2006)*

Solution. Here u is not a homogeneous function but $z=\sin u=\frac{x+y}{\sqrt{x}+\sqrt{y}}$ is a homogeneous function of degree 1/2 in x and y.

$\therefore$ By Euler's theorem, $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\frac{1}{2}z$

or $$x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}=\frac{1}{2}\sin u$$

Thus $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\tan u$$ ...(*i*)

Differentiating (*i*) w.r.t. x partially, we get

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=\frac{1}{2}\sec^2 u\frac{\partial u}{\partial x}\quad\text{or}\quad x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=\left(\frac{1}{2}\sec^2 u-1\right)\frac{\partial u}{\partial x}$$ ...(*ii*)

Again differentiating (*i*) w.r.t. y partially, we get

$$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=\frac{1}{2}\sec^2 u\frac{\partial u}{\partial y}\quad\text{or}\quad x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}=\left(\frac{1}{2}\sec^2 u-1\right)\frac{\partial u}{\partial x}$$ ...(*iii*)

Multiplying (*ii*) by x and (*iii*) by y and adding, we obtain

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\left(\frac{1}{2}\sec^2 u-1\right)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

or $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\left(\frac{1}{2}\sec^2 u-1\right)\left(\frac{1}{2}\tan u\right)$$ [By (*i*)]

$$=\frac{1}{4}\frac{\sin u}{\cos^3 u}-\frac{1}{2}\frac{\sin u}{\cos u}=-\frac{\sin u\,(2\cos^2 u-1)}{4\cos^3 u}$$

Hence $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-\frac{\sin u\cos 2u}{4\cos^3 u}.$$

PROBLEMS 5.4

1. Verify Euler's theorem, when (i) $f(x, y) = ax^2 + 2hxy + by^2$
   (ii) $f(x, y) = x^2(x^2 - y^2)^3/(x^2 + y^2)^3$.
   (iii) $f(x, y) = 3x^2yz + 5xy^2z + 4z^4$ (J.N.T.U., 1999)
2. If $u = \sin^{-1}\dfrac{x}{y} + \tan^{-1}\dfrac{y}{x}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$. (Hazaribagh, 2009 ; Osmania, 2003 S)
3. If $u = \sin^{-1}\dfrac{x^2 + y^2}{x + y}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u$ (Bhopal, 2009 ; V.T.U., 2003)
4. If $\sin u = \dfrac{x^2y^2}{x + y}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3\tan u$. (Kottayam, 2005 ; V.T.U., 2003 S)
5. If $u = \cos^{-1}\dfrac{x + y}{\sqrt{x} + \sqrt{y}}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = -\dfrac{1}{2}\cot u$. (V.T.U., 2004)
6. Show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2u\log u$, where $u = e^{x^2 + y^2}$ (P.T.U., 2010)
7. If $z = f(y/x) + \sqrt{(x^2 + y^2)}$, show that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = \sqrt{x^2 + y^2}$. (Mumbai, 2008)
8. If $u = \dfrac{x}{y + z} + \dfrac{y}{z + x} + \dfrac{z}{x + y}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0$. (V.T.U., 2000 S)
9. If $\sin u = \dfrac{x + 2y + 3z}{\sqrt{(x^8 + y^8 + z^8)}}$, show that $xu_x + yu_y + zu_z + 3\tan u = 0$. (S.V.T.U., 2009 ; U.T.U., 2009)
10. If $z = x\phi\left(\dfrac{y}{x}\right) + \psi\left(\dfrac{y}{x}\right)$, prove that $x^2\dfrac{\partial^2 z}{\partial x^2} + 2xy\dfrac{\partial^2 z}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 0$. (S.V.T.U., 2009 ; U.P.T.U., 2006)
11. If $u = \tan^{-1}\dfrac{x^3 + y^3}{x + y}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sin 2u$. (P.T.U., 2009 S)
    and $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 2\cos 3u\sin u$. (Mumbai, 2009 ; Bhopal, 2008 ; S.V.T.U., 2007)
12. Given $z = x^n f_1(y/x) + y^{-n} f_2(x/y)$, prove that $x^2\dfrac{\partial^2 z}{\partial x^2} + 2xy\dfrac{\partial^2 z}{\partial x\partial y} + y^2\dfrac{\partial^2 z}{\partial y^2} + x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = n^2 z$.
    (Kurukshetra, 2009 S ; Rohtak, 2003)
13. If $u = x^2\tan^{-1}(y/x) - y^2\tan^{-1}(x/y)$, evaluate $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2}$. (U.T.U., 2009 ; Hissar, 2005 S)
14. If $u = \tan^{-1}(y^2/x)$, prove that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = -\sin^2 u \,.\, \sin 2u$. (Bhillai, 2005 ; P.T.U., 2005)
15. If $u = \operatorname{cosec}^{-1}\left(\dfrac{x^{1/2} + y^{1/2}}{x^{1/3} + y^{1/3}}\right)^{1/2}$, prove that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = \dfrac{\tan u}{12}\left(\dfrac{13}{12} + \dfrac{\tan^2 u}{12}\right)$
    (Mumbai, 2008 ; Rohtak, 2006 S)

5.5 (1) TOTAL DERIVATIVE

If $u = f(x, y)$, where $x = \phi(t)$ and $y = \psi(t)$, then we can express u as a function of t alone by substituting the values of x and y in $f(x, y)$. Thus we can find the ordinary derivative du/dt which is called the *total derivative* of u to distinguish it from the partial derivatives $\partial u/\partial x$ and $\partial u/\partial y$.

Now to find du/dt without actually substituting the values of x and y in $f(x, y)$, we establish the following **Chain rule :**

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} \qquad ...(i)$$

Proof. We have $u = f(x, y)$

Giving increment δt to t, let the corresponding increments of x, y and u be δx, δy and δu respectively.

Then $\quad u + \delta u = f(x + \delta x, y + \delta y)$

Subtracting, $\quad \delta u = f(x + \delta x, y + \delta y) - f(x, y)$

$$= \{f(x + \delta x, y + \delta y) - f(x, y + \delta y)\} + \{f(x, y + \delta y) - f(x, y)\}$$

$$\therefore \quad \frac{\delta u}{\delta t} = \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x} \cdot \frac{\delta x}{\delta t} + \frac{f(x, y + \delta y) - f(x, y)}{\delta y} \cdot \frac{\delta y}{\delta t}$$

Taking limits as $\delta t \to 0$, δx and δy also $\to 0$, we have

$$\frac{du}{dt} = \operatorname*{Lt}_{\delta y \to 0}\left[\operatorname*{Lt}_{\delta y \to 0}\left\{\frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x}\right\}\right]\frac{dx}{dt} + \operatorname*{Lt}_{\delta y \to 0}\left\{\frac{f(x, y + \delta y) - f(x, y)}{\delta y}\right\}\frac{dy}{dt}$$

$$= \operatorname*{Lt}_{\delta y \to 0}\left\{\frac{\partial f(x, y + \delta y)}{\delta y}\right\} \cdot \frac{dx}{dt} + \frac{\partial f(x, y)}{\partial y} \cdot \frac{dy}{dt}$$

[Supposing $\partial f(x, y)/\partial x$ to be a continuous function of y]

$$= \frac{\partial f(x, y)}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f(x, y)}{\partial y} \cdot \frac{dy}{dt} \text{ which is the desired formula.}$$

Cor. Taking $t = x$, (*i*) becomes, $\dfrac{du}{dx} = \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} \cdot \dfrac{dy}{dx}$ ...(*ii*)

Obs. If $u = f(x, y, z)$, where x, y, z are all functions of a variable t, then **Chain rule** is

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt} \quad \text{...(iii)}$$

(2) Differentiation of implicit functions. If $f(x, y) = c$ be an implicit relation between x and y which defines as a differentiable function of x, then (*ii*) becomes

$$0 = \frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

This gives the *important formula* $\mathbf{\dfrac{dy}{dx} = -\dfrac{\partial f}{\partial x} \Big/ \dfrac{\partial f}{\partial y}}$ $\quad \left[\dfrac{\partial f}{\partial y} \neq 0\right]$

for the first differential coefficient of an implicit function.

Example 5.16. *Given $u = \sin(x/y)$, $x = e^t$ and $y = t^2$, find du/dt as a function of t. Verify your result by direct substitution.*

Solution. We have $\dfrac{du}{dt} = \dfrac{\partial u}{\partial x} \cdot \dfrac{dx}{dt} + \dfrac{\partial u}{\partial y} \cdot \dfrac{dy}{dt} = \left(\cos\dfrac{x}{y}\right)\dfrac{1}{y} \cdot e^t + \left(\cos\dfrac{x}{y}\right)\left(-\dfrac{x}{y^2}\right)2t$

$$= \cos(e^t/t^2) \cdot e^t/t^2 - 2\cos(e^t/t^2) \cdot e^t/t^3 = \{(t-2)/t^3\}e^t \cos(e^t/t^2)$$

Also $u = \sin(x/y) = \sin(e^t/t^2)$

$$\therefore \quad \frac{du}{dt} = \cos\left(\frac{e^t}{t^2}\right) \cdot \frac{t^2 e^t - e^t \cdot 2t}{t^4} = \frac{t-2}{t^3} e^t \cos\left(\frac{e^t}{t^2}\right) \text{ as before.}$$

Example 5.17. *If x increases at the rate of 2 cm/sec at the instant when $x = 3$ cm. and $y = 1$ cm., at what rate must y be changing in order that the function $2xy - 3x^2y$ shall be neither increasing nor decreasing ?*

Solution. Let $\quad u = 2xy - 3x^2y$, so that

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} = (2y - 6xy)\frac{dx}{dt} + (2x - 3x^2)\frac{dy}{dt} \quad \text{...(i)}$$

when $x = 3$ and $y = 1$, $dx/dt = 2$, and u is neither increasing nor decreasing, *i.e.*, $du/dt = 0$.

$\therefore$ (*i*) becomes $\quad 0 = (2 - 6 \times 3)\, 2 + (2 \times 3 - 3 \times 9)\dfrac{dy}{dt}$

or $\quad \dfrac{dy}{dt} = -\dfrac{32}{21}$ cm/sec. Thus y is decreasing at the rate of 32/21 cm/sec.

Example 5.18. *If $u = x \log xy$ where $x^3 + y^3 + 3xy = 1$, find du/dx.* (V.T.U., 2009)

Solution. From $f(x, y) = x^3 + y^3 + 3xy - 1$, we have

$$\frac{dy}{dx} = -\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y} = -\frac{3x^2 + 3y}{3y^2 + 3x} = -\frac{x^2 + y}{y^2 + x} \quad ...(i)$$

Also $$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} = (1 \cdot \log xy + x \cdot 1/x) + (x/y)\cdot dy/dx.$$

Hence $$du/dx = 1 + \log xy - x(x^2 + y)/y(y^2 + x) \quad \text{[By (i)]}$$

Example 5.19. *If $u = u\left(\frac{y - x}{xy}, \frac{z - x}{xz}\right)$, show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$.* (U.P.T.U., 2005)

Solution. Let $$v = \frac{y - x}{xy} = \frac{1}{x} - \frac{1}{y} \quad \text{and} \quad w = \frac{z - x}{xz} = \frac{1}{x} - \frac{1}{z} \quad ...(i)$$

so that $$u = u(v, w)$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial x} = \frac{\partial u}{\partial v}\left(-\frac{1}{x^2}\right) + \frac{\partial u}{\partial w}\left(-\frac{1}{x^2}\right) \quad \text{[Using (i)]}$$

or $$x^2\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w} \quad ...(ii)$$

Also $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial y} = \frac{\partial u}{\partial v}\left(\frac{1}{y^2}\right) + \frac{\partial u}{\partial w}(0) \quad \text{[Using (i)]}$$

or $$y^2\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v} \quad ...(iii)$$

Similarly $$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial v}\,\frac{\partial v}{\partial z} + \frac{\partial u}{\partial w}\,\frac{\partial w}{\partial z} = \frac{\partial u}{\partial v}(0) + \frac{\partial u}{\partial w}\left(\frac{1}{z^2}\right) \quad \text{[Using (i)]}$$

or $$z^2\frac{\partial u}{\partial z} = \frac{\partial u}{\partial w} \quad ...(iv)$$

Adding (*ii*), (*iii*) and (*iv*), we have

$$x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0.$$

Example 5.20. *Formula for the second differential coefficient of an implicit function.*
If $f(x, y) = 0$, show that

$$\frac{d^2y}{dx^2} = -\frac{q^2r - 2pqs + p^2t}{q^3}$$ (Kurukshetra, 2006)

Solution. We have $$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{p}{q} \quad ...(i)$$

$$\therefore \quad \frac{d^2y}{dx^2} = -\frac{d}{dx}\left(\frac{dy}{dx}\right) = -\frac{d}{dx}\left(\frac{p}{q}\right) = -\frac{q(dp/dx) - p(dq/dx)}{q^2} \quad ...(ii)$$

Using the notations : $r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial p}{\partial x}$, $s = \frac{\partial^2 f}{\partial x\partial y} = \frac{\partial q}{\partial x}$, $t = \frac{\partial^2 f}{\partial y^2} = \frac{\partial q}{\partial y}$,

we have $$\frac{dp}{dx} = \frac{\partial p}{\partial x} + \frac{\partial p}{\partial y}\cdot\frac{dy}{dx} = r + s(-p/q) = \frac{qr - ps}{q}$$

and $$\frac{dq}{dx} = \frac{\partial q}{\partial x} + \frac{\partial q}{\partial y}\cdot\frac{dy}{dx} = s + t(-p/q) = \frac{qs - pt}{q}$$

Substituting the values of dp/dx and dq/dx in (*ii*), we get

$$\frac{d^2y}{dx^2} = -\frac{1}{q^2}\left[q\left(\frac{qr-ps}{q}\right) - p\left(\frac{qs-pt}{q}\right)\right] = -\frac{q^2r - 2pqs + p^2t}{q^3}.$$

PROBLEMS 5.5

1. If $z = u^2 + v^2$ and $u = at^2$, $v = 2at$, find dz/dt. (*P.T.U., 2005*)
2. If $u = \tan^{-1}(y/x)$ where $x = e^t - e^{-t}$, and $y = e^t + e^{-t}$, find du/dt. (*V.T.U., 2003*)
3. Find the value of $\frac{du}{dt}$ given $u = y^2 - 4ax$, $x = at^2$, $y = 2at$. (*Anna, 2009*)
4. At a given instant the sides of a rectangle are 4 ft. and 3 ft. respectively and they are increasing at the rate of 1.5 ft./sec. and 0.5 ft./sec. respectively, find the rate at which the area is increasing at that instant.
5. If $z = 2xy^2 - 3x^2y$ and if x increases at the rate of 2 cm. per second and it passes through the value $x = 3$ cm., show that if y is passing through the value $y = 1$ cm., y must be decreasing at the rate of $2\frac{2}{15}$ cm. per second, in order that z shall remain constant.
6. If $u = x^2 + y^2 + z^2$ and $x = e^{2t}$, $y = e^{2t}\cos 3t$, $z = e^{2t}\sin 3t$. Find $\frac{du}{dt}$ as a total derivative and verify the result by direct substitution.
7. If $\phi\,(cx - az, cy - bz) = 0$, show that $\frac{a\partial z}{\partial x} - \frac{b\partial z}{\partial y} = c$.
8. If $f(x, y) = 0$, $\phi\,(y, z) = 0$, show that $\frac{\partial f}{\partial y}\cdot\frac{\partial \phi}{\partial z}\cdot\frac{dz}{dx} = \frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial y}$.
9. If the curves $f(x, y) = 0$ and $\phi(y, z) = 0$ touch, show that at the point of contact, $\frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial y} = \frac{\partial f}{\partial y}\cdot\frac{\partial \varphi}{\partial \gamma}$.
10. If $f(x, y) = 0$, show that $\left(\frac{\partial f}{\partial y}\right)^3 \frac{d^2y}{dx^2} = 2\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial^2 f}{\partial x\partial y}\right) - \left(\frac{\partial f}{dy}\right)^2 \frac{\partial^2 f}{\partial x^2} - \left(\frac{\partial f}{\partial x}\right)^2\left(\frac{\partial^2 f}{\partial y^2}\right)$.

5.6 CHANGE OF VARIABLES

If $\qquad u = f(x, y)$ ...(1)

where $\qquad x = \phi(s, t)$ and $y = \Psi(s, t)$ ...(2)

it is often necessary to change expressions involving $u, x, y, \partial u/\partial x, \partial u/\partial y$ etc. to expressions involving $u, s, t, \partial u/\partial s, \partial u/\partial t$ etc.

The necessary formulae for the change of variables are easily obtained. If t is regarded as a constant, then x, y, u will be functions of s alone. Therefore, by (*i*) of page 208, we have

$$\frac{\partial \mathbf{u}}{\partial \mathbf{s}} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}}\cdot\frac{\partial \mathbf{x}}{\partial \mathbf{s}} + \frac{\partial \mathbf{u}}{\partial \mathbf{y}}\cdot\frac{\partial \mathbf{y}}{\partial \mathbf{s}} \quad ...(3)$$

where the ordinary derivatives have been replaced by the partial derivatives because x, y are functions of two variables s and t.

$\because$ Similarly, regarding s as constant, we obtain $\frac{\partial \mathbf{u}}{\partial \mathbf{t}} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}}\cdot\frac{\partial \mathbf{x}}{\partial \mathbf{t}} + \frac{\partial \mathbf{u}}{\partial \mathbf{y}}\cdot\frac{\partial \mathbf{y}}{\partial \mathbf{t}}$ ...(4)

On solving (3) and (4) as simultaneous equations in $\partial u/\partial x$ and $\partial u/\partial y$, we get their values in terms of $\partial u/\partial s$, $\partial u/\partial t$, u, s, t.

If instead of the equations (2), s and t are given in terms of x and y, say: $s = \xi(x, y)$ and $t = \eta(x, y)$, ...(5)

then it is easier to use the formulae $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$ ...(6)

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y} \quad ...(7)$$

The higher derivatives of u can be found by repeated application of formulae (3) and (4) or of (6) and (7).

Example 5.21. *If $u = F(x-y, y-z, z-x)$, prove that*

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$$

(V.T.U., 2010 ; U.T.U. 2009 ; U.P.T.U., 2003)

Solution. Put $x-y=r,\ y-z=s$ and $z-x=t$, so that $u=f(r,s,t)$.

$$\therefore \quad \frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$=\frac{\partial u}{\partial r}.(1)+\frac{\partial x}{\partial s}.(0)+\frac{\partial u}{\partial t}.(-1)=\frac{\partial u}{\partial r}-\frac{\partial u}{\partial t} \quad ...(i)$$

Similarly,
$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}=-\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s} \quad ...(ii)$$

and
$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}=-\frac{\partial u}{\partial s}+\frac{\partial u}{\partial t} \quad ...(iii)$$

Adding (i), (ii) and (iii), we get the required result.

Example 5.22. *If $z=f(x,y)$ and $x=e^u\cos v$, $y=e^u\sin v$, prove that $x\frac{\partial z}{\partial v}+y\frac{\partial z}{\partial u}=e^{2u}\frac{\partial z}{\partial y}$*

and
$$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial u}{\partial v}\right)^2\right]$$

(Mumbai, 2009)

Solution. We have $\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$

$$=\frac{\partial z}{\partial x}(e^u\cos v)+\frac{\partial z}{\partial y}(e^u\sin v) \quad ...(i)$$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$=\frac{\partial z}{\partial x}(-e^u\sin v)+\frac{\partial x}{\partial y}(e^u\cos v) \quad ...(ii)$$

$$\therefore \quad x\frac{\partial x}{\partial v}+y\frac{\partial z}{\partial u}=(e^u\cos v)\left[-e^u\sin v\frac{\partial z}{\partial x}+e^u\cos v\frac{\partial z}{\partial y}\right]+(e^u\sin v)\left[e^u\cos v\frac{\partial z}{\partial x}+e^u\sin v\frac{\partial z}{\partial y}\right]$$

$$=(e^{2u}\cos^2 v+e^{2u}\sin^2 v)\frac{\partial z}{\partial y}=e^{2u}\frac{\partial z}{\partial y}$$

Now squaring (i) and (ii) and adding, we get

$$\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2=e^{2u}\left(\cos v\frac{\partial z}{\partial x}+\sin v\frac{\partial z}{\partial y}\right)^2+e^{2u}\left(-\sin v\frac{\partial z}{\partial x}+\cos v\frac{\partial z}{\partial y}\right)^2$$

or
$$e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2\right]=\cos^2 v\left(\frac{\partial z}{\partial x}\right)^2+\sin^2 v\left(\frac{\partial z}{\partial y}\right)^2+2\sin v\cos v\frac{\delta z}{\delta z}\frac{\delta z}{\delta y}$$

$$+\sin^2 v\left(\frac{\partial z}{\partial x}\right)^2+\cos^2 v\left(\frac{\partial z}{\partial y}\right)^2-2\sin v\cos v\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}$$

$$=(\cos^2 v+\sin^2 v)\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2$$

Hence $$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right].$$

Example 5.23. *If $x + y = 2e^{\theta}\cos\phi$ and $x - y = 2ie^{\theta}\sin\phi$, show that*

$$\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial \phi^2} = 4xy\frac{\partial^2 u}{\partial x \partial y}$$ *(Nagpur, 2009 ; U.P.T.U., 2002)*

Solution. We have $x = e^{\theta}(\cos\phi + i\sin\phi) = e^{\theta}\,.\,e^{i\phi}$ [See p. 205]

and $y = e^{\theta}(\cos\phi - i\sin\phi) = e^{\theta}\,.\,e^{-i\phi}$

Here u is a composite function of θ and ϕ.

$$\therefore \quad \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$= \frac{\partial u}{\partial x}\cdot(e^{\theta}\,.\,e^{i\phi}) + \frac{\partial u}{\partial y}(e^{\theta}\,.\,e^{-i\phi}) = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

or $$\frac{\partial}{\partial \theta} = x\frac{\partial}{\partial u} + y\frac{\partial}{\partial y} \quad ...(i)$$

Also $$\frac{\partial u}{\partial \phi} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \phi} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \phi} = \frac{\partial u}{\partial x}\cdot(e^{\theta}\,.\,ie^{i\phi}) + \frac{\partial u}{\partial y}(e^{\theta}\,.\,-ie^{-i\phi}) = ix\frac{\partial u}{\partial x} - iy\frac{\partial u}{\partial y}$$

$$\frac{\partial}{\partial \phi} = ix\frac{\partial}{\partial x} - iy\frac{\partial}{\partial y} \quad ...(ii)$$

Using the operator (i), we have

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right) = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$= x\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) + x\frac{\partial}{\partial x}\left(y\frac{\partial u}{\partial y}\right) + y\frac{\partial}{\partial y}\left(x\frac{\partial u}{\partial x}\right) + y\frac{\partial}{\partial y}\left(y\frac{\partial u}{\partial y}\right)$$

$$= x\left(x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\right) + xy\frac{\partial^2 u}{\partial x \partial y} + yx\frac{\partial^2 u}{\partial y \partial x} + y\left(y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y}\right)$$

$$= x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \quad ...(iii)$$

Similarly using (ii), $$\frac{\partial^2 u}{\partial \phi^2} = \frac{\partial}{\partial \phi}\left(\frac{\partial u}{\partial \phi}\right) = \left(ix\frac{\partial}{\partial x} - iy\frac{\partial}{\partial y}\right)\left(ix\frac{\partial u}{\partial x} - iy\frac{\partial u}{\partial y}\right)$$

$$= -x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} - y^2\frac{\partial^2 u}{\partial y^2} - x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} \quad ...(iv)$$

Adding (iii) and (iv), we get $$\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial \phi^2} = 4xy\frac{\partial^2 u}{\partial x \partial y}$$

Example 5.24. *Transform the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ into polar coordinates.* *(P.T.U., 2010)*

Solution. We have $x = r\cos\theta$, $y = r\sin\theta$ and $r = \sqrt{(x^2 + y^2)}$, $\theta = \tan^{-1}(y/x)$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{(x^2 + y^2)}} = \cos\theta \text{ and } \frac{\partial \theta}{\partial x} = -\frac{y}{x^2 + y^2} = -\frac{\sin\theta}{r}$$

Thus, $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} = \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}$$

i.e., $$\frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \qquad \text{Similarly, } \frac{\partial}{\partial y} = \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}.$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}\right)$$

$$= \cos^2\theta\frac{\partial^2 u}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta} \quad ...(i)$$

and $$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}\right)$$

$$= \sin^2\theta\frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial r} \quad ...(ii)$$

Adding (*i*) and (*ii*), we get $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial\theta^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r}$.

Hence the transformed equation is $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial\theta^2} = 0$.

PROBLEMS 5.6

1. If $z = f(x, y)$ and $x = e^u + e^{-v}$, $y = e^{-u} - e^v$, prove that $\dfrac{\partial z}{\partial u} - \dfrac{\partial z}{\partial v} = x\dfrac{\partial z}{\partial x} - y\dfrac{\partial z}{\partial y}$. (*V.T.U., 2006*)
2. If $u = f(r, s)$, $r = x + at$, $s = y + bt$ and x, y, t are independent variables, show that $\dfrac{\partial u}{\partial t} = a\dfrac{\partial u}{\partial x} + b\dfrac{\partial u}{\partial y}$.
3. If $\phi(z/x^3, y/x) = 0$, prove that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 3z$. (*Mumbai, 2007*)
4. If $u = f(x, y)$ and $x = r\cos\theta$, $y = r\sin\theta$, prove that $\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial u}{\partial y}\right)^2 = \left(\dfrac{\partial u}{\partial r}\right)^2 + \dfrac{1}{r^2}\left(\dfrac{\partial u}{\partial\theta}\right)^2$.
(*V.T.U., 2010 ; Madras 2006 ; Rohtak, 2005*)
5. If $u = f(2x - 3y, 3y - 4z, 4z - 2x)$, prove that $\dfrac{1}{2}\dfrac{\partial u}{\partial x} + \dfrac{1}{3}\dfrac{\partial u}{\partial y} + \dfrac{1}{4}\dfrac{\partial u}{\partial z} = 0$. (*U.P.T.U., 2006 ; Raipur, 2005*)
6. If $u = f(e^{y-z}, e^{z-x}, e^{x-y})$, prove that $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$ (*Mumbai, 2008 S*)
7. If $u = f(r, s, t)$ and $r = x/y$, $s = y/z$, $t = z/x$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0$. (*Anna, 2009 ; Kurukshetra, 2006*)
8. If $x = u + v + w$, $y = vw + wu + uv$, $z = uvw$ and F is a function of x, y, z, show that
$$u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} + w\frac{\partial F}{\partial w} = x\frac{\partial F}{\partial x} + 2y\frac{\partial F}{\partial y} + 3z\frac{\partial F}{\partial z}.$$
9. Given that $u(x, y, z) = f(x^2 + y^2 + z^2)$ where $x = r\cos\theta\cos\phi$, $y = r\cos\theta\sin\phi$ and $z = r\sin\theta$, find $\dfrac{\partial u}{\partial\theta}$ and $\dfrac{\partial u}{\partial\phi}$.
10. If the three thermodynamic variables P, V, T are connected by a relation $f(P, V, T) = 0$, show that
$$\left(\frac{\partial P}{\partial T}\right)_V\left(\frac{\partial T}{\partial V}\right)_P\left(\frac{\partial V}{\partial P}\right)_T = -1.$$
11. If by the substitution $u = x^2 - y^2$, $v = 2xy$, $f(x, y) = \theta(u, v)$, show that
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4(x^2 + y^2)\left(\frac{\partial^2\theta}{\partial u^2} + \frac{\partial^2\theta}{\partial v^2}\right).$$ (*Anna, 2003*)
12. Transform $\dfrac{\partial^2 z}{\partial x^2} + 2xy^2\dfrac{\partial z}{\partial x} + 2(y - y^3)\dfrac{\partial z}{\partial y} + x^2y^2z = 0$ by the substitution $x = uv$, $y = 1/v$. Hence show that z is the same function of u and v as of x and y.

5.7 (1) JACOBIANS

If u and v are functions of two independent variables x and y, then the determinant

$\begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}$ is called the *Jacobian** of u, v with respect to x, y

and is written as $\dfrac{\partial(u, v)}{\partial(x, y)}$ or $J\left(\dfrac{u, v}{x, y}\right)$.

Similarly the Jacobian of u, v, w with respect to x, y, z is

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{vmatrix}$$

Likewise, we can define Jacobians of four or more variables. An important application of Jacobians is in connection with the change of variables in multiple integrals (§ 7.7).

(2) Properties of Jacobians. We give below two of the important properties of Jacobians. For simplicity, the properties are stated in terms of two variables only, but these are evidently true in general.

I. If $J = \partial(u, v)/\partial(x, y)$ and $J' = \partial(x, y)/\partial(u, v)$ then $JJ' = 1$.

Let $u = f(x, y)$ and $v = g(x, y)$.

Suppose, on solving for x and y, we get $x = \phi(u, v)$ and $y = \psi(u, v)$.

Then
$$\left.\begin{aligned} \frac{\partial u}{\partial u} &= 1 = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial u}, \\ \frac{\partial u}{\partial v} &= 0 = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial v}, \\ \frac{\partial v}{\partial u} &= 0 = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial u}, \\ \text{and} \quad \frac{\partial v}{\partial v} &= 1 = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial v}, \end{aligned}\right\} \quad ...(i)$$

$$\therefore \quad JJ' = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} \times \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial y/\partial x & \partial v/\partial y \end{vmatrix} \times \begin{vmatrix} \partial x/\partial u & \partial y/\partial u \\ \partial x/\partial v & \partial y/\partial v \end{vmatrix}$$

(Interchanging rows and columns of the 2nd determinant).

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1. \qquad \text{[By virtue of } (i)]$$

II. **Chain rule for Jacobians.** *If u, v are functions of r, s and r, s are functions of x, y, then*

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(r, s)}\cdot\frac{\partial(r, s)}{\partial(x, y)}.$$

$$\frac{\partial(u, v)}{\partial(r, s)}\cdot\frac{\partial(r, s)}{\partial(x, y)} = \begin{vmatrix} \partial u/\partial r & \partial u/\partial s \\ \partial v/\partial r & \partial v/\partial s \end{vmatrix} \times \begin{vmatrix} \partial r/\partial x & \partial s/\partial x \\ \partial r/\partial y & \partial s/\partial y \end{vmatrix}$$

[Interchanging rows and columns of the 2nd det.]

$$= \begin{vmatrix} \dfrac{\partial u}{\partial r}\cdot\dfrac{\partial r}{\partial x} + \dfrac{\partial u}{\partial s}\cdot\dfrac{\partial s}{\partial x} & \dfrac{\partial u}{\partial r}\cdot\dfrac{\partial r}{\partial y} + \dfrac{\partial u}{\partial s}\cdot\dfrac{\partial s}{\partial y} \\ \dfrac{\partial v}{\partial r}\cdot\dfrac{\partial r}{\partial x} + \dfrac{\partial v}{\partial s}\cdot\dfrac{\partial s}{\partial x} & \dfrac{\partial v}{\partial r}\cdot\dfrac{\partial r}{\partial y} + \dfrac{\partial v}{\partial s}\cdot\dfrac{\partial s}{\partial y} \end{vmatrix} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \frac{\partial(u, v)}{\partial(x, y)}.$$

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* Called after the German mathematician *Carl Gustav Jacob Jacobi* (1804–1851), who made significant contributions to mechanics, partial differential equations, astronomy, elliptic functions and the calculus of variations.

Example 5.25. *(i) In polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, show that*

$$\frac{\partial(x, y)}{\partial(r, \theta)} = r.$$ *(U.P.T.U., 2006 ; V.T.U., 2004 ; Andhra, 2000)*

(ii) In cylindrical coordinates (Fig. 8.28), $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$, show that

$$\frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \rho.$$

(iii) In spherical polar coordidnates (Fig. 8.29), $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, show that

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2\sin\theta.$$ *(Anna, 2009 ; Hazaribagh, 2009 ; Rohtak, 2003)*

Solution. *(i)* We have $\frac{\partial x}{\partial r} = \cos\theta,\ \frac{\partial x}{\partial \theta} = -r\sin\theta$ and $\frac{\partial y}{\partial r} = \sin\theta,\ \frac{\partial y}{\partial \theta} = -r\cos\theta$

$$\therefore \quad \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

(ii) We have
$$\frac{\partial x}{\partial \rho} = \cos\phi,\ \frac{\partial x}{\partial \phi} = -\rho\sin\phi,\ \frac{\partial x}{\partial z} = 0,$$
$$\frac{\partial y}{\partial \rho} = \sin\phi,\ \frac{\partial y}{\partial \phi} = \rho\cos\phi,\ \frac{\partial y}{\partial z} = 0 \text{ and } \frac{\partial z}{\partial \rho} = 0,\ \frac{\partial z}{\partial \phi} = 0,\ \frac{\partial z}{\partial z} = 1$$

$$\therefore \quad \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \begin{vmatrix} \cos\phi & -\rho\sin\theta & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho.$$

(iii) We have
$$\frac{\partial x}{\partial r} = \sin\theta\cos\theta,\ \frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi,\ \frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi,$$
$$\frac{\partial y}{\partial r} = \sin\theta\sin\phi,\ \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi,\ \frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi,$$
and
$$\frac{\partial z}{\partial r} = \cos\theta,\ \frac{\partial z}{\partial \theta} = -r\sin\theta,\ \frac{\partial z}{\partial \phi} = 0.$$

$$\therefore \quad \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta.$$

Example 5.26. *If $y_1 = \frac{x_2x_3}{x_1}$, $y_2 = \frac{x_3x_1}{x_2}$, $y_3 = \frac{x_1x_2}{x_3}$, show that the Jacobian of y_1, y_2, y_3 with respect to x_1, x_2, x_3 is 4.* *(U.P.T.U., 2006)*

Solution. We have $\frac{\partial y_1}{\partial x_1} = -\frac{x_2x_3}{x_1^2},\ \frac{\partial y_1}{\partial x_2} = \frac{x_3}{x_1},\ \frac{\partial y_1}{\partial x_3} = \frac{x_2}{x_1}$

$$\frac{\partial y_2}{\partial x_1} = \frac{x_3}{x_2},\ \frac{\partial y_2}{\partial x_2} = -\frac{x_3x_1}{x_2^2},\ \frac{\partial y_2}{\partial x_3} = \frac{x_1}{x_2}$$

and
$$\frac{\partial y_3}{\partial x_1} = \frac{x_2}{x_3},\ \frac{\partial y_3}{\partial x_2} = \frac{x_1}{x_3},\ \frac{\partial y_3}{\partial x_3} = -\frac{x_1x_2}{x_3^2}$$

$$\therefore \quad \frac{\partial(y_1y_2y_3)}{\partial(x_1x_2x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix} \begin{vmatrix} \frac{-x_2x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & \frac{-x_3x_1}{x_2^2} & \frac{x_1}{x_1} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & \frac{-x_1x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{1}{x_1^2x_2^2x_3^2}\begin{vmatrix} -x_2x_3 & x_3x_1 & x_1x_2 \\ x_2x_3 & -x_3x_1 & x_1x_2 \\ x_2x_3 & x_3x_1 & -x_1x_2 \end{vmatrix} = -\frac{x_1^2x_2^2x_3^2}{x_1^2x_2^2x_3^2}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= -1(1-1) - 1(-1-1) + 1(1+1) = 0 + 2 + 2 = 4.$$

Example 5.27. *In $u = x + 3y^2 - z^3$, $v = 4x^2yz$, $w = 2z^2 - xy$, evaluate $\partial(u, v, w)/\partial(x, y, z)$ at $(1, -1, 0)$.* (V.T.U., 2006)

Solution. $$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 6y & -3z^2 \\ 8xyz & 4x^2z & 4x^2y \\ -y & -x & 4z \end{vmatrix}$$

$\therefore$ At the point $(1, -1, 0)$ $$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} 1 & -6 & 0 \\ 0 & 0 & -4 \\ 1 & -1 & 0 \end{vmatrix} = 4(-1+6) = 20.$$

Example 5.28. *If $u = x^2 - y^2$, $v = 2xy$ and $x = r\cos\theta$, $y = r\sin\theta$, find $\dfrac{\partial(u, v)}{\partial(r, \theta)}$.* (V.T.U., 2009 ; Madras, 2006)

Solution. We have $$\frac{\partial(u, v)}{\partial(r, \theta)} = \frac{\partial(u, v)}{\partial(x, y)} \times \frac{\partial(x, y)}{\partial(r, \theta)}$$

Since $u = x^2 - y^2$, $u = 2xy$

$$\therefore \quad \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2 + y^2) \quad \text{...(ii)}$$

Since $x = r\cos\theta$, $y = r\sin\theta$,

$$\therefore \quad \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \quad \text{...(iii)}$$

Hence, $$\frac{\partial(u, v)}{\partial(r, \theta)} = 4(x^2 + y^2).\,r = 4(r^2\cos^2\theta + r^2\sin^2\theta).\,r = 4r^3 \quad \text{[Using (ii) \& (iii)]}$$

(3) Jacobian of Implicit functions. *If u_1, u_2, u_3 instead of being given explicitly in terms x_1, x_2, x_3, be connected with them equations such as*

$$f_1(u_1, u_2, u_3, x_1, x_2, x_3) = 0,\ f_2(u_1, u_2, u_3, x_1, x_2, x_3) = 0,\ f_3(u_1, u_2, u_3, x_1, x_2, x_3) = 0, \text{ then}$$

$$\frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} = (-1)^3 \frac{\partial(f_1, f_2, f_3)}{\partial(x_1, x_2, x_3)} \div \frac{\partial(f_1, f_2, f_3)}{\partial(u_1, u_2, u_3)}$$

Obs. This result can be easily generalised. It bears analogy to the result $\dfrac{dy}{dx} = -\dfrac{\partial f}{\partial x}\Big/\dfrac{\partial f}{\partial y}$, where x, y are connected by the relation $f(x, y) = 0$.

Example 5.29. *If $u = x, y, z$, $v = x^2 + y^2 + z^2$, $w = x + y + z$, find $\partial(x, y, z)/\partial(u, v, w)$.* (U.P.T.U., 2003)

Solution. Let $f_1 = u - x\,y\,z$, $f_2 = v - x^2 - y^2 - z^2$, $f_3 = w - x - y - z$.

We have $$\frac{\partial(x, y, z)}{\partial(u, v, w)} = (-1)^3 \frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} \div \frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)} \quad ...(i)$$

Now, $$\frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)} = \begin{vmatrix} \partial f_1/\partial x & \partial f_1/\partial y & \partial f_1/\partial z \\ \partial f_2/\partial x & \partial f_2/\partial y & \partial f_2/\partial z \\ \partial f_3/\partial x & \partial f_3/\partial y & \partial f_3/\partial z \end{vmatrix} = \begin{vmatrix} -yz & -xz & -xy \\ -2x & -2y & -2z \\ -1 & -1 & -1 \end{vmatrix}$$

$$= -2(x-y)(y-z)(z-x) \quad ...(ii)$$

and $$\frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \quad ...(iii)$$

Substituting values from (*ii*) and (*iii*) in (*i*), we get

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = (-1) \times 1/[-2(x-y)(y-z)(z-x)] = 1/2(x-y)(y-z)(z-x).$$

(4) Functional relationship. *If u_1, u_2, u_3 be functions of x_1, x_2, x_3 then the necessary and sufficient condition for the existence of a functional relationship of the form $f(u_1, u_2, u_3) = 0$, is*

$$J\left(\frac{u_1, u_2, u_3}{x_1, x_2, x_3}\right) = 0.$$

Example 5.30. *If $u = x\sqrt{(1-y^2)} + y\sqrt{(1-x^2)}$, $v = \sin^{-1} x + \sin^{-1} y$, show that u, v are functionally related and find the relationship.* (*Kurukshetra, 2006*)

Solution. We have $$\frac{\partial u}{\partial x} = \sqrt{(1-y^2)} - \frac{xy}{\sqrt{(1-x^2)}}, \frac{\partial u}{\partial y} = \frac{-xy}{\sqrt{(1-y^2)}} + \sqrt{(1-x^2)}$$

and $$\frac{\partial v}{\partial x} = \frac{1}{\sqrt{(1-x^2)}}, \frac{\partial v}{\partial y} = \frac{1}{\sqrt{(1-y^2)}}$$

$\therefore$ $$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \sqrt{(1-y^2)} - \frac{xy}{\sqrt{(1-x^2)}}, & \sqrt{(1-x^2)} - \frac{xy}{\sqrt{(1-y^2)}} \\ \frac{1}{\sqrt{(1-x^2)}}, & \frac{1}{\sqrt{(1-y^2)}} \end{vmatrix}$$

$$= 1 - \frac{xy}{\sqrt{[(1-x^2)(1-y^2)]}} - 1 + \frac{xy}{\sqrt{[(1-x^2)(1-y^2)]}} = 0$$

Hence u and v are functionally related *i.e.*, they are not independent.

We have $$v = \sin^{-1} x + \sin^{-1} y = \sin^{-1} [x\sqrt{(1-y^2)} + y\sqrt{(1-x^2)}]$$

i.e., $$u = \sin v$$

which is the required relationship between u and v.

PROBLEMS 5.7

1. If $x = r\cos\theta$, $y = r\sin\theta$, evaluate $\partial(r, \theta)/\partial(x, y)$, $\partial(x, y)/\partial(r, \theta)$ and prove that $[\partial(r, \theta)\,\partial(x, y)]$. $[\partial(x, y)/\partial(r, \theta)] = 1$. (*V.T.U., 2010*)
2. If $x = u(1-v)$, $y = uv$, prove that $JJ' = 1$. (*V.T.U., 2000 S*)
3. If $x = a\cosh\xi\cos\eta$, $y = a\sinh\xi\sin\eta$, show that $\partial(x, y)/\partial(\xi, \eta) = \frac{1}{2}a^2(\cosh 2\xi - \cos 2\eta)$ (*S.V.T.U., 2007*)
4. If $x = e^u \sec v$, $y = e^u \tan v$, find $J = \partial(u, v)/\partial(x, y)$, $J' = \partial(x, y)/\partial(u, v)$. Hence show $JJ' = 1$. (*V.T.U., 2007 S*)
5. If $u = x^2 - 2y^2$, $v = 2x^2 - y^2$ where $x = r\cos\theta$, $y = r\sin\theta$, show that $\frac{\partial(u, v)}{\partial(r, \theta)} = 6r^3 \sin 2\theta$.
6. If $u = x^2 + y^2 + z^2$, $v = xy + yz + zx$, $w = x + y + z$, find $\frac{\partial(u, v, w)}{\partial(x, y, z)}$. (*U.T.U., 2009 ; V.T.U., 2008*)

7. If $F = xu + v - y$, $G = u^2 + vy + w$, $H = zu - v + vw$, compute $\partial(F, G, H)/\partial(u, v, w)$.

8. If $u = x + y + z$, $uv = y + z$, $uvw = z$, show that $\partial(x, y, z)/\partial(u, v, w) = u^2 v$.
(*Kurukshetra, 2009; P.T.U., 2009 S; V.T.U., 2003*)

9. If $u^3 + v^3 = x + y$ and $u^2 + v^2 = x^3 + y^3$, show that $\dfrac{\partial(u, v)}{\partial(x, y)} = \dfrac{1}{2}\dfrac{y^2 - x^2}{uv(u - v)}$ (*U.P.T.U., 2006 MCA*)

10. If $u = \dfrac{x + y}{1 - xy}$ and $v = \tan^{-1} x + \tan^{-1} y$, find $\dfrac{\partial(u, v)}{\partial(x, y)}$. Are u and v functionally related. If so, find this relationship.
(*Nagpur, 2008*)

11. If $u = 3x + 2y - z$, $v = x - 2y + z$ and $w = x(x + 2y - z)$, show that they are functionally related, and find the relation.
(*Nagpur, 2009*)

5.8 (1) GEOMETRICAL INTERPRETATION

If $P(x, y, z)$ be the coordinates of a point referred to axes OX, OY, OZ then the equation $z = f(x, y)$ represents a surface. (Fig. 5.1)

Let a plane $y = b$ parallel to the XZ-plane pass through P cutting the surface along the curve APB given by

$$z = f(x, b).$$

As y remains equal to b and x varies then P moves along the curve APB and $\partial z/\partial x$ is the ordinary derivative of $f(x, b)$ w.r.t. x.

Hence $\partial z/\partial x$ *at* P *is the tangent of the angle which the tangent at* P *to the section of the surface* $z = f(x, y)$ *by a plane through* P *parallel to the plane XOZ, makes with a line parallel to the x-axis.*

Similarly, $\partial z/\partial y$ *at* P *is the tangent of the angle which the tangent at* P *to the curve of intersection of the surface* $z = f(x, y)$ *and the plane* $x = a$, *makes with a line parallel to the y-axis.*

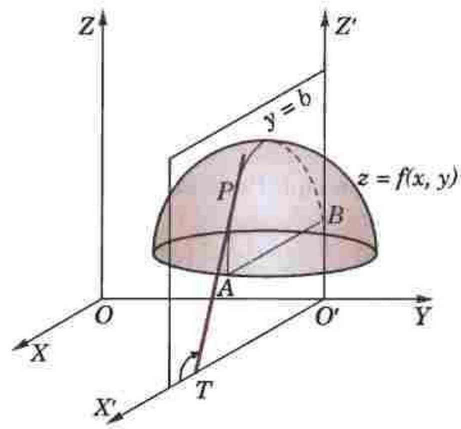

Fig. 5.1

(2) Tangent plane and Normal to a surface. Let $P(x, y, z)$ and $Q(x + \delta x, y + \delta y, z + \delta z)$ be two neighbouring points on the surface $F(x, y, z) = 0$. (Fig. 5.2) ...(*i*)

Let the arc PQ be δs and the chord PQ be δc, so that (as for plane curves)

$$\underset{Q \to P}{\text{Lt}} (\delta s/\delta c) = 1.$$

The direction cosines of PQ are $\dfrac{\delta x}{\delta c}, \dfrac{\delta y}{\delta c}, \dfrac{\delta z}{\delta c}$ *i.e.*, $\dfrac{\delta x}{\delta s}\cdot\dfrac{\delta s}{\delta c}, \dfrac{\delta y}{\delta s}\cdot\dfrac{\delta s}{\delta c}, \dfrac{\delta z}{\delta s}\cdot\dfrac{\delta s}{\delta c}$

When $\delta s \to 0$, $Q \to P$ and PQ tends to tangent line PT. Then noting that the coordinates of any point on arc PQ are functions of s only, the direction cosines of PT are

$$\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds} \qquad ...(ii)$$

Differentiating (*i*) with respect to s, we obtain $\dfrac{\partial F}{\partial x}\cdot\dfrac{dx}{ds} + \dfrac{\partial F}{\partial y}\cdot\dfrac{dy}{ds} + \dfrac{\partial F}{\partial z}\cdot\dfrac{dz}{ds} = 0.$

This shows that the tangent line whose direction cosines are given by (*ii*), is perpendicular to the line having direction ratios

$$\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \qquad ...(iii)$$

Since we can take different curves joining Q to P, we get a number of tangent lines at P and the line having direction ratios (*iii*) will be perpendicular to *all* these tangent lines at P. Thus all the tangent lines at P lie in a plane through P perpendicular to line (*iii*).

Hence *the equation of the tangent plane to (i) at the point P is*

$$\frac{\partial \mathbf{F}}{\partial \mathbf{x}}(\mathbf{X} - \mathbf{x}) + \frac{\partial \mathbf{F}}{\partial \mathbf{y}}(\mathbf{Y} - \mathbf{y}) + \frac{\partial \mathbf{F}}{\partial \mathbf{z}}(\mathbf{Z} - \mathbf{z}) = \mathbf{0}$$

where (X, Y, Z) are the current coordinates of any point on this tangent plane.

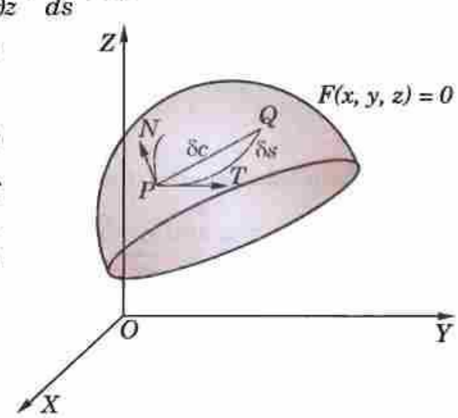

Fig. 5.2

Also *the equation of the normal to the surface at P (i.e., the line through P,* perpendicular to the tangent plane at *P*) is

$$\frac{\mathbf{X}-\mathbf{x}}{\partial \mathbf{F}/\partial \mathbf{x}} = \frac{\mathbf{Y}-\mathbf{y}}{\partial \mathbf{F}/\partial \mathbf{y}} = \frac{\mathbf{Z}-\mathbf{z}}{\partial \mathbf{F}/\partial \mathbf{z}}.$$

Example 5.31. *Find the equations of the tangent plane and the normal to the surface $z^2 = 4(1 + x^2 + y^2)$ at (2, 2, 6).*

Solution. We have $F(x, y, z) = 4x^2 + 4y^2 - z^2 + 4.$

$\therefore$ $\partial F/\partial x = 8x$, $\partial F/\partial y = 8y$, $\partial F/\partial z = -2z$, and at the point (2, 2, 6)

$\partial F/\partial x = 16$, $\partial F/\partial y = 16$, $\partial F/\partial z = -12$

Hence the equation of the tangent plane at (2, 2, 6) is $16(X-2) + 16(Y-2) - 12(Z-6) = 0$

i.e., $4X + 4Y - 3Z + 2 = 0$ ...(*i*)

Also the equation of the normal at (2, 2, 6) [being perpendicular to (*i*)] is

$$\frac{X-2}{4} = \frac{Y-2}{4} = \frac{Z-6}{-3}.$$

PROBLEMS 5.8

Find the equations of the tangent plane and normal to each of the following surfaces at the given points :

1. $2x^2 + y^2 = 3 - 2z$ at $(2, 1, -3)$ (*Assam, 1998*)
2. $x^3 + y^3 + 3xyz = 3$ at $(1, 2, -1)$ (*Osmania, 2003 S*)
3. $xyz = a^2$ at (x_1, y_1, z_1).
4. $2xz^2 - 3xy - 4x = 7$ at $(1, -1, 2)$.
5. Show the plane $3x + 12y - 6z - 17 = 0$ touches the conicoid $3x^2 - 6y^2 + 9z^2 + 17 = 0$. Find also the point of contact.
6. Show that the plane $ax + by + cz + d = 0$ touches the surface $px^2 + qy^2 + 2z = 0$, if $\frac{a^2}{p} + \frac{b^2}{q} + 2cd = 0$.
7. Find the equation of the normal to the surface $x^2 + y^2 + z^2 = a^2$. (*P.T.U., 2009 S*)

5.9 TAYLOR'S THEOREM FOR FUNCTIONS OF TWO VARIABLES

Considering $f(x + h, y + k)$ as a function of a single variable x, we have by Taylor's theorem*

$$f(x+h, y+k) = f(x, y+k) + h\frac{\partial f(x, y+k)}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 f(x, y+k)}{\partial x^2} + \ldots \quad \ldots(i)$$

Now expanding $f(x, y + k)$ as a function of y only,

$$f(x, y+k) = f(x, y) + k\frac{\partial f(x, y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2} + \ldots$$

$\therefore$ (*i*) takes the form $f(x+h, y+k) = f(x, y) + k\dfrac{\partial f(x, y)}{\partial y} + \dfrac{k^2}{2!}\dfrac{\partial^2 f(x, y)}{\partial y^2} + \ldots$

$$+ h\frac{\partial}{\partial x}\left\{f(x, y) + k\frac{\partial f(x, y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2} + \ldots\right\} + \frac{h^2}{2!}\frac{\partial^2}{\partial x^2}\left\{f(x, y) + k\frac{\partial (x, y)}{\partial y} + \ldots\right\}$$

Hence, $$f(x+h, y+k) = f(x, y) + h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + \ldots \quad \ldots(1)$$

In symbols we write it as $f(x+h, y+k) = f(x, y) + \left(h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y}\right)f + \dfrac{1}{2!}\left(h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y}\right)^2 f + \ldots$

Taking $x = a$ and $y = b$, (1) becomes

$$f(a+h, b+k) = f(a, b) + [hf_x(a, b) + kf_y(a, b)] + \frac{1}{2!}[h^2 f_{xx}(a, b) + 2hkf_{xy}(a, b) + k^2 f_{yy}(a, b)] + \ldots$$

*See footnote on page 145.

Putting $a + h = x$ and $b + k = y$ so that $h = x - a$, $k = y - b$, we get

$$\mathbf{f(x, y) = f(a, b) + [(x - a)\, f_x\, (a, b) + (y - b)\, f_y\, (a, b)]}$$

$$\mathbf{+ \frac{1}{2!}\, [(x - a)^2\, f_{xx}\, (a, b) + 2(x - a)(y - b)\, f_{xy}\, (a, b) + (y - b)^2\, f_{yy}(a, b)] + ...} \quad ...(2)$$

This is Taylor's expansion of f(x, y) in powers of (x – a) and (y – b). It is used to expand f(x, y) in the neighbourhood of (a, b).

Cor. Putting $a = 0$, $b = 0$, in (2), we get

$$\mathbf{f(x, y) = f(0, 0) + [xf_x(0, 0) + yf_y\, (0, 0)] + \frac{1}{2!}\, [x^2\, f_{xx}\, (0, 0) + 2xy\, f_{xy}\, (0, 0) + y^2\, f_{yy}\, (0, 0)] + ...} \quad ...(3)$$

This is Maclaurin's expansion of $f(x, y)$.

Example 5.32. *Expand $e^x \log (1 + y)$ in powers of x and y upto terms of third degree.*

(V.T.U., 2010 ; P.T.U., 2009 ; J.N.T.U., 2006)

Solution. Here

$f(x, y) = e^x \log (1 + y)$	$\therefore$	$f(0, 0) = 0$
$f_x(x, y) = e^x \log (1 + y)$		$f_x(0, 0) = 0$
$f_y(x, y) = e^x \dfrac{1}{1+y}$		$f_y(0, 0) = 1$
$f_{xx}(x, y) = e^x \log (1 + y)$		$f_{xx}(0, 0) = 0$
$f_{xy}(x, y) = e^x \dfrac{1}{1+y}$		$f_{xy}(0, 0) = 1$
$f_{yy}(x, y) = - e^x (1 + y)^{-2}$		$f_{yy}(0, 0) = - 1$
$f_{xxx}(x, y) = e^x \log (1 + y)$		$f_{xxx}(0, 0) = 0$
$f_{xxy}(x, y) = e^x \dfrac{1}{1+y}$		$f_{xxy}(0, 0) = 1$
$f_{xyy}(x, y) = - e^x (1 + y)^{-2}$		$f_{xyy}(0, 0) = - 1$
$f_{yyy}(x, y) = 2e^x (1 + y)^{-3}$		$f_{yyy}(0, 0) = 2$

Now Maclaurin's expansion of $f(x, y)$ gives

$$f(x, y) = f(0, 0) + xf_x(0, 0) + yf_y(0, 0) + \frac{1}{2!}\, \{x^2 f_{xx}(0, 0) + 2xy\, f_{xy}(0, 0) + y^2 f_{yy}(0, 0)\}$$

$$+ \frac{1}{3!}\, \{x^3 f_{xxx}(0, 0) + 3x^2 y f_{xxy}(0, 0) + 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)\} + ...$$

$$\therefore \quad e^x \log (1 + y) = 0 + x(0) + y(1) + \frac{1}{2!}\, \{x^2 (0) + 2xy(1) + y^2 (-1)\}$$

$$+ \frac{1}{3!}\, \{x^3(0) + 3x^2y(1) + 3xy^2(-1) + y^3(2)\} + ...$$

$$= y + xy - \frac{1}{2} y^2 + \frac{1}{2} (x^2y - xy^2) + \frac{1}{3} y^3 + ...$$

Example 5.33. *Expand $x^2y + 3y - 2$ in powers of (x – 1) and (y + 2) using Taylor's theorem.*

(P.T.U., 2010 ; V.T.U., 2008 ; U.P.T.U., 2006 ; Anna, 2005)

Solution. Taylor's expansion of $f(x, y)$ in powers of $(x - a)$ and $(x - b)$ is given by

$$f(x, y) = f(a, b) + [(x - a)\, f_x(a, b) + (y - b)\, f_y(a, b)] + \frac{1}{2!}\, [(x - a)^2 f_{xx}(a, b)$$

$$+ 2\, (x - a)(y - b)\, f_{xy}(a, b) + (y - b)^2 f_{yy}(a, b)] + \frac{1}{3!}\, [(x - a)^3 f_{xxx}(a, b)$$

$$+ 3\, (x - a)^2\, (y - b)\, f_{xxy}(a, b) + 3\, (x - a)\, (y - b)^2 f_{xyy}\, (a, b)$$

$$+ (y - b)^3 f_{yyy}\, (a, b)] + ... \quad ...(i)$$

Hence $a = 1$, $b = - 2$ and $f(x, y) = x^2y + 3y - 2$

$\therefore$ $f(1,-2)=-10, f_x = 2xy, f_x(1,-2)=-4; f_y = x^2+3, f_y(1,-2)=4; f_{xx}=2y,$
$f_{xx}(1,-2)=-4; f_{xy}=2x, f_{xy}(1,-2)=2; f_{yy}=0, f_{yy}(1,-2)=0; f_{xxx}=0, f_{xxx}=(1,-2)=0;$
$f_{xxy}(1,-2)=2, f_{xyy}(1,-2)=0, f_{yyy}(1,-2)=0$

All partial derivatives of higher order vanish.

Substituting these in (i), we get

$$x^2y+3y-2=-10+[(x-1)(-4)+(y+2)4]+\frac{1}{2}[(x-1)^2(-4)+2(x-1)(y+2)(2)$$
$$+(y+2)^2(0)]+\frac{1}{6}[(x-1)^3(0)+3(x-1)^2(y+2)(2)+3(x-1)(y+2)^2(0)+(y+2)^3(0)]$$
$$=-10-4(x-1)+4(y+2)-2(x-1)^2+2(x-1)(y+2)+(x-1)^2(y+2).$$

Example 5.34. *Expand $f(x,y)=\tan^{-1}(y/x)$ in powers of $(x-1)$ and $(y-1)$ upto third-degree terms. Hence compute $f(1.1, 0.9)$ approximately.* *(V.T.U., 2010 ; J.N.T.U., 2006 ; U.P.T.U., 2006)*

Solution. Here $a=1$, $b=1$ and $f(1,1)=\tan^{-1}(1)=\pi/4$.

$$f_x=\frac{-y}{x^2+y^2} \qquad f_x(1,1)=-\frac{1}{2}; \qquad f_y=\frac{x}{x^2+y^2}, \qquad f_y(1,1)=\frac{1}{2}$$

$$f_{xx}=\frac{2xy}{(x^2+y^2)^2}, \qquad f_{xx}(1,1)=\frac{1}{2}; \qquad f_{xy}=\frac{y^2-x^2}{(x^2+y^2)^2}, \qquad f_{xy}(1,1)=0$$

$$f_{yy}=\frac{-2xy}{(x^2+y^2)^2}, \qquad f_{yy}(1,1)=-\frac{1}{2};$$

$$f_{xxx}=\frac{2y^3-6x^2y}{(x^2+y^2)^3}, \qquad f_{xxx}(1,1)=-\frac{1}{2}; \qquad f_{xxy}=\frac{2x^3-6xy^2}{(x^3+y^2)^3}, \qquad f_{xxy}(1,1)=-\frac{1}{2}$$

$$f_{xyy}=\frac{6x^2y-2y^3}{(x^2+y^2)^3}, \qquad f_{xyy}(1,1)=\frac{1}{2}; \qquad f_{yyy}=\frac{6xy^2-2x^3}{(x^2+y^2)^3}, \qquad f_{yyy}(1,1)=\frac{1}{2}$$

Taylor's expansion of $f(x,y)$ in powers of $(x-1)$ and $(y-1)$ is given by

$$f(x,y)=f(1,1)+\frac{1}{1!}\{(x-1)f_x(1,1)+(y-1)f_y(1,1)\}+\frac{1}{2!}\{(x-1)^2 f_{xx}(1,1)+2(x-1)(y-1)$$
$$f_{xy}(1,1)+(y-1)^2 f_{yy}(1,1)+\frac{1}{3!}\{(x-1)^3 f_{xxx}(1,1)+3(x-1)^2(y-1)f_{xxy}(1,1)$$
$$+3(x-1)(y-1)^2 f_{xyy}(1,1)+(y-1)^3 f_{yyy}(1,1)\}+\ldots$$

$$\therefore \quad \tan^{-1}\left(\frac{y}{x}\right)=\frac{\pi}{4}+\left\{(x-1)\left(-\frac{1}{2}\right)+(y-1)\frac{1}{2}\right\}+\frac{1}{2!}\left\{(x-1)^2\frac{1}{2}+2(x-1)(y-1)(0)+(y-1)^2\left(-\frac{1}{2}\right)\right\}$$
$$+\frac{1}{3!}\left\{(x-1)^3\left(-\frac{1}{2}\right)+3(x-1)^2(y-1)\left(-\frac{1}{2}\right)+3(x-1)(y-1)^2\frac{1}{2}+(y-1)^3\frac{1}{2}\right\}+\ldots$$
$$=\frac{\pi}{4}-\frac{1}{2}\{(x-1)-(y-1)\}+\frac{1}{4}\{(x-1)^2-(y-1)^2\}-\frac{1}{12}\{(x-1)^3+3(x-1)^2(y-1)$$
$$-3(x-1)(y-1)^2-(y-1)^3\}+\ldots$$

Putting $x=1.1$ and $y=0.9$, we get

$$f(1.1, 0.9)=\frac{\pi}{4}-\frac{1}{2}(0.2)+\frac{1}{4}(0)-\frac{1}{12}\{(0.1)^3-3(0.1)^3-3(0.1)^3-(-0.1)^3\}$$
$$=0.7854-0.1000+0.0003=0.6857.$$

5.10 (1) ERRORS AND APPROXIMATIONS

Let $f(x,y)$ be a continuous function of x and y. If δx and δy be the increments of x and y, then the new value of $f(x,y)$ will be $f(x+\delta x, y+\delta y)$. Hence

$$\delta f = f(x + \delta x, y + \delta y) - f(x, y).$$

Expanding $f(x + \delta x, y + \delta y)$ by Taylor's theorem and supposing δx, δy to be so small that their products, squares and higher powers can be neglected, we get

$$\boldsymbol{\delta f = \frac{\partial f}{\partial x}\,\delta x + \frac{\partial f}{\partial y}\,\delta y}, \text{ approximately.}$$

Similarly if f be a function of several variables $x, y, z, t, \ldots$, then

$$\delta f = \frac{\partial f}{\partial x}\,\delta x + \frac{\partial f}{\partial y}\,\delta y + \frac{\partial f}{\partial z}\,\delta z + \frac{\partial f}{\partial t}\,\delta t + \ldots \text{ approximately.}$$

These formulae are very useful in correcting the effect of small errors in measured quantities.

(2) Total Differential

If u is a function of two variables x and y, the *total differential* of u is defined as

$$\mathbf{du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy} \qquad \ldots(1)$$

The differentials dx and dy are respectively the increments δx and δy in x and y. If x and y are not independent variables but functions of another variable t even then the formula (1) holds and we write $dx = \frac{dx}{dt}\,dt$ and $dy = \frac{dy}{dt}\,dt$. Similar definition can be given for a function of three or more variables.

Example 5.35. *The diameter and altitude of a can in the shape of a right circular cylinder are measured as 4 cm and 6 cm respectively. The possible error in each measurement is 0.1 cm. Find approximately the maximum possible error in the values computed for the volume and the lateral surface.*

Solution. Let x be the diameter and y the height of the can. Then its volume $V = \frac{\pi}{4}\,x^2y$

$$\therefore \qquad \delta V = \frac{\partial V}{\partial x}\,\delta x + \frac{\partial V}{\partial y}\,\delta y = \frac{\pi}{4}\,(2xy\delta x + x^2\,\delta y)$$

When $x = 4$ cm., $y = 6$ cm. and $\delta x = \delta y = 0.1$ cm.

$$\therefore \qquad \delta V = \frac{\pi}{4}(2 \times 4 \times 6 \times 0.1 + 4^2 \times 0.1) = 1.6\pi \text{ cm}^3$$

Also its lateral surface $S = \pi xy$

$$\therefore \qquad \delta S = \pi(y\,\delta x + x\,\delta y)$$

When $x = 4$ cm., $y = 6$ cm. and $\delta x = \delta y = 0.1$ cm., we have $\delta S = \pi(6 \times 0.1 + 4 \times 0.1) = \pi$ cm^2.

Example 5.36. *The period of a simple pendulum is $T = 2\pi\sqrt{(l/g)}$, find the maximum error in T due to the possible error upto 1% in l and 2.5% in g.* (U.P.T.U., 2004)

Solution. We have $T = 2\pi\sqrt{(l/g)}$

or
$$\log T = \log 2\pi + \frac{1}{2}\log l - \frac{1}{2}\log g$$

$$\therefore \qquad \frac{1}{T}\,\delta T = 0 + \frac{1}{2}\,\frac{1}{l}\,\delta l - \frac{1}{2}\,\frac{1}{g}\,\delta g$$

$$\frac{\delta T}{T}\,100 = \frac{1}{2}\left(\frac{\delta l}{l}\,100 - \frac{\delta g}{g}\,100\right) = \frac{1}{2}(1 \pm 2.5) = 1.75 \text{ or } -0.75$$

Thus the maximum error in $T = 1.75\%$

Example 5.37. *A balloon is in the form of right circular cylinder of radius 1.5 m and length 4 m and is surmounted by hemispherical ends. If the radius is increased by 0.01 m and length by 0.05 m, find the percentage change in the volume of balloon.* (U.P.T.U., 2005)

Solution. Let the volume of the balloon (Fig. 5.3) be V, so that

$$V = \pi r^2 h + \frac{2}{3}\pi r^3 + \frac{2}{3}\pi r^3 = \pi r^2 h + \frac{4}{3}\pi r^3$$

$\therefore$ $$\delta V = 2\pi r\delta r h + \pi r^2 \delta h + \frac{4}{3}\pi 3r^2 \delta r$$

or $$\frac{\delta V}{V} = \frac{\pi r[2h\delta r + r\delta h + 4r\delta r]}{\pi r^2 h + \frac{4}{3}\pi r^3}$$

$$= \frac{2(h+2r)\,\delta r + r\delta h}{rh + \frac{4}{3}r^2} = \frac{2(4+3)(.01) + 1.5(.05)}{1.5 \times 4 + \frac{4}{3}(1.5)^2}$$

$$= \frac{0.14 + 0.075}{6+3} = \frac{0.215}{9}$$

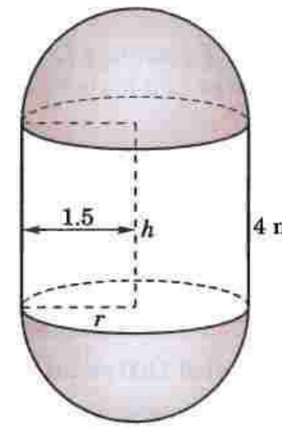

Fig. 5.3

Hence, the percentage change in $V = 100\,\dfrac{\delta V}{V} = \dfrac{21.5}{9} = 2.39\%$

Example 5.38. *In estimating the cost of a pile of bricks measured as 2 m × 15 m × 1.2 m, the tape is stretched 1% beyond the standard length. If the count is 450 bricks to 1 cu. m. and bricks cost ₹ 530 per 1000, find the approximate error in the cost.* (V.T.U., 2001)

Solution. Let x, y and z m be the length, breadth and height of the pile so that its volume $V = xyz$

or $$\log V = \log x + \log y + \log z \;\therefore\; \frac{\delta V}{V} = \frac{\delta x}{x} + \frac{\delta y}{y} + \frac{\delta z}{z}$$

Since $$V = 2 \times 15 \times 1.2 = 36 \text{ m}^3, \text{ and } \frac{\delta x}{x} = \frac{\delta y}{y} = \frac{\delta z}{z} = \frac{1}{100}$$

$\therefore$ $$\delta V = 36\left(\frac{3}{100}\right) = 1.08 \text{ m}^3.$$

Number of bricks in $\delta V = 1.08 \times 450 = 486$

Thus error in the cost $= 486 \times \dfrac{530}{1000} =$ ₹ 257.58 which is a loss to the brick seller.

Example 5.39. *The height h and semi-vertical angle α of a cone are measured and from them A, the total area of the surface of the cone including the base is calculated. If h and α are in error by small quantities δh and δα respectively, find the corresponding error in the area. Show further that if α = π/6, an error of + 1% in h will be approximately compensated by an error of – 0.33 degrees in α.*

Solution. If r be the base radius and l the slant height of the cone, (Fig. 5.4), then total area

$$A = \text{area of base} + \text{area of curved surface}$$
$$= \pi r^2 + \pi r l = \pi r(r + l)$$
$$= \pi h \tan\alpha\,(h\tan\alpha + h\sec\alpha)$$
$$= \pi h^2(\tan^2\alpha + \tan\alpha\sec\alpha)$$

$\therefore$ $$\delta A = \frac{\delta A}{\delta h}\delta h + \frac{\delta A}{\delta \alpha}\delta\alpha$$
$$= 2\pi h(\tan^2\alpha + \tan\alpha\sec\alpha)\delta h$$
$$+ \pi h^2(2\tan\alpha\sec^2\alpha + \sec^3\alpha + \tan\alpha\sec\alpha\tan\alpha)\delta\alpha$$

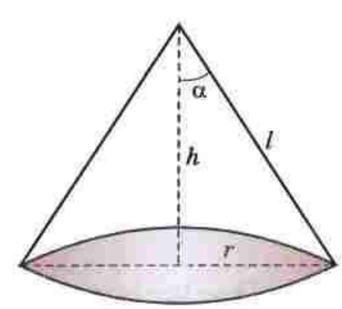

Fig. 5.4

which gives the error in the area A.

Putting $\delta h = h/100$ and $\alpha = \pi/6$, we get

$$\delta A = 2\pi h\left[\left(\frac{1}{\sqrt{3}}\right)^2 + \frac{1}{\sqrt{3}}\cdot\frac{2}{\sqrt{3}}\right]\frac{h}{100} + \pi h^2\left[2.\frac{1}{\sqrt{3}}\cdot\frac{4}{3} + \frac{8}{3\sqrt{3}} + \frac{1}{\sqrt{3}}\cdot\frac{2}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}}\right]\delta\alpha$$

$$= \frac{2\pi h^2}{100} + 2\sqrt{3}\pi h^2 \,\delta\alpha$$

The error in h will be compensated by the error in α, when

$$\delta A = 0 \text{ i.e., } \frac{2\pi h^2}{100} + 2\sqrt{3}\pi h^2 \,\delta\alpha = 0$$

or $$\delta\alpha = -\frac{1}{100\sqrt{3}} \text{ radians} = -\frac{.01}{1.732} \times 57.3° = -0.33°.$$

Example 5.40. *Show that the approximate change in the angle A of a triangle ABC due to small changes δa, δb, δc in the sides a, b, c respectively, is given by*

$$\delta A = \frac{a}{2\Delta} (\delta a - \delta b \cos C - \delta c \cos B)$$

where Δ is the area of the triangle. Verify that $\delta A + \delta B + \delta C = 0$.

Solution. We know that $a^2 = b^2 + c^2 - 2bc \cos A$

so that $$2a\delta a = 2b\delta b + 2c\delta c - 2(c\delta b \cos A - b\delta c \cos A + bc \sin A\, \delta A)$$

$\therefore$ $$bc \sin A\, \delta A = a\delta a - (b - c \cos A)\, \delta b - (c - b \cos A)\, \delta c$$

or $$2\,\Delta\delta A = a\delta a - (c \cos A + a \cos C - c \cos A)\, \delta b - (a \cos B + b \cos A - b \cos A)\, \delta c$$

[$\because$ $b = c \cos A + a \cos C$ etc. ...(*i*)]

$$= a\delta a - a \cos C\, \delta b - a \cos B\, \delta c$$

or $$\delta A = \frac{a}{2\Delta} (\delta a - \delta b \cos C - \delta c \cos B)$$

By symmetry, we have

$$\delta B = \frac{b}{2\Delta} (\delta b - \delta c \cos A - \delta a \cos C)$$

$$\delta C = \frac{c}{2\Delta} (\delta c - \delta a \cos B - \delta b \cos A)$$

$\therefore$ $$\delta A + \delta B + \delta C = \frac{1}{2\Delta} (a - b \cos C - c \cos B)\, \delta a + (b - c \cos A - a \cos C)\, \delta b$$
$$+ (c - a \cos B - b \cos A)]$$

$$= \frac{1}{2\Delta} [(a - a)\, \delta a + (b - b)\, \delta b + (c - c)\, \delta c] = 0 \qquad \text{[By (i)]}$$

Example 5.41. *If the sides of a plane triangle ABC vary in such a way that its circumradius remains constant, prove that* $\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0.$

Solution. The circumradius R of ΔABC is given by

$$R = \frac{a}{2 \sin A} = \frac{b}{2 \sin B} = \frac{c}{2 \sin C}$$

$\therefore$ $$a = 2R \sin A$$ [$\because$ R is constant

Taking differentials, $da = 2R \cos A\, dA$ or $\frac{da}{\cos A} = 2RdA$

Similarly, $$\frac{db}{\cos B} = 2RdB, \quad \frac{dc}{\cos C} = 2RdC$$

$\therefore$ $$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 2R\,(dA + dB + dC)$$

Now $A + B + C = \pi$, gives $dA + dB + dC = 0$ ...(*i*)

Thus $$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0 \qquad \text{[By (i)]}$$

PROBLEMS 5.9

1. Expand the following functions as far as terms of third degree :
   (*i*) $\sin x \cos y$ (*V.T.U., 2009*) (*ii*) $e^x \sin y$ at $(-1, \pi/4)$ (*Anna, 2009*)
   (*iii*) $xy^2 + \cos xy$ about $(1, \pi/2)$. (*Hissar, 2005 S ; V.T.U., 2003*)
2. Expand $f(x, y) = x^y$ in powers of $(x - 1)$ and $(y - 1)$. (*U.T.U., 2009*)
3. If $f(x, y) = \tan^{-1} xy$, compute $f(0.9, -1.2)$ approximately.
4. If the kinetic energy $k = wv^2/2g$, find approximately the change in the kinetic energy as w changes from 49 to 49.5 and v changes from 1600 to 1590. (*V.T.U., 2006*)
5. Find the possible percentage error in computing the resistance r from the formula $1/r = 1/r_1 + 1/r_2$, if r_1, r_2 are both in error by 2%.
6. The voltage V across a resistor is measured with an error h, and the resistance R is measured with an error k. Show that the error in calculating the power $W(V, R) = V^2/R$ generated in the resistor, is $VR^{-2}(2Rh - Vk)$. (*V.T.U., 2009*)
7. Find the percentage error in the area of an ellipse if one per cent error is made in measuring the major and minor axes. (*V.T.U., 2011*)
8. The time of oscillation of a simple pendulum is given by the equation $T = 2\pi\sqrt{(l/g)}$. In an experiment carried out to find the value of g, errors of 1.5% and 0.5% are possible in the values of l and T respectively. Show that the error in the calculated value of g is 0.5%. (*Cochin, 2005*)
9. If $pv^2 = k$ and the relative errors in p and v are respectively 0.05 and 0.025, show that the error in k is 10%. (*Mysore, 1999*)
10. If the H.P. required to propel a steamer varies as the cube of the velocity and square of the length. Prove that a 3% increase in velocity and 4% increase in length will require an increase of about 17% in H.P.
11. The range R of a projectile which starts with a velocity v at an elevation α is given by $R = (v^2 \sin 2\alpha)/g$. Find the percentage error in R due to an error of 1% in v and an error of $\frac{1}{2}$% in α. (*Kurukshetra, 2009*)
12. In estimating the cost of a pile of bricks measured as 6 m × 50 m × 4 m, the tape is stretched 1% beyond the standard length. If the count is 12 bricks in 1 m³ and bricks cost ₹ 100 per 1000, find the approximate error in the cost. (*U.T.U., 2010 ; U.P.T.U., 2005*)
13. The deflection at the centre of a rod of length l and diameter d supported at its ends, loaded at the centre with a weight w varies at wl^3d^{-4}. What is the increase in the deflection corresponding to p% increase in w, q% decrease in l and r% increase in d ?
14. The work that must be done to propel a ship of displacement D for a distance s in time t is proportional to $(s^2D^{2/3}/t^2)$. Find approximately the increase of work necessary when the displacement is increased by 1%, the time is diminished by 1% and the distance diminished by 2%.
15. The indicated horse power I of an engine is calculated from the formula $I = PLAN/33{,}000$, where $A = \pi d^2/4$. Assuming that error of r per cent may have been made in measuring P, L, N and d, find the greatest possible error in I.
16. The torsional rigidity of a length of wire is obtained from the formula $N = 8\pi\, Il/t^2r^4$. If l is decreased by 2%, r is increased by 2%, t is increased by 1.5%, show that the value of N is diminished by 13% approximately. (*V.T.U., 2003*)
17. If $x^2 + y^2 + z^2 - 2xyz = 1$, show that $\dfrac{dx}{\sqrt{(1-x^2)}} + \dfrac{dy}{\sqrt{(1-y^2)}} + \dfrac{dz}{\sqrt{(1-z^2)}} = 0$.
   [**Hint.** $2(x - yz)\,dx + 2(y - zx)\,dy + 2(z - xy)\,dz = 0$. Also $(x - yz)^2 = (1 - y^2)(1 - z^2)$, ...]

5.11 (1) MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES

Def. *A function $f(x, y)$ is said to have a* **maximum** *or* **minimum** *at $x = a, y = b$, according as*

$$f(a + h, b + k) < \text{or} > f(a, b),$$

for all positive or negative small values of h and k.

In other words, if $\Delta = f(a + h, b + k) - f(a, b)$, *is of the same sign for all small values of h, k, and if this sign is negative, then $f(a, b)$ is a maximum. If this sign is positive, $f(a, b)$ is a minimum.*

Considering $z = f(x, y)$ as a surface, maximum value of z occurs at the top of an elevation (*e.g.*, a dome) from which the surface descends in every direction and a minimum value occurs at the bottom of a depression (*e.g.*, a bowl) from which the surface ascends in every direction. Sometimes the maximum or minimum value may form a *ridge* such that the surface descends or ascends in all directions except that of the ridge. Besides these, we have such a point of the surface, where the tangent plane is horizontal and the surface looks like leather seat on the horse's back [Fig. 5.5 (*c*)] which falls for displacement in certain directions and rises for displacements in other directions. Such a point is called a **saddle point.**

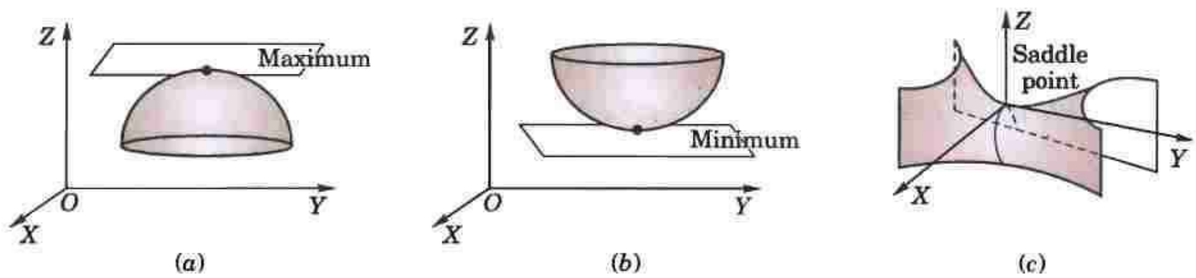

Fig. 5.5

Note. *A maximum or minimum value of a function is called its* **extreme value.**

(2) Conditions for f(x, y) to be maximum or minimum

Using Taylor's theorem page 235, we have $\Delta = f(a + h, b + k) - f(a, b)$

$$= \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)_{a,b} + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + \ldots \qquad \ldots(i)$$

For small values of h amd k, the second and higher order terms are still smaller and hence may be neglected. Thus

$$\text{sign of } \Delta = \text{sign of } [hf_x(a, b) + kf_y(a, b)].$$

Taking $h = 0$ we see that the right hand side changes sign when k changes sign. Hence $f(x, y)$ cannot have a maximum or a minimum at (a, b) unless $f_y(a, b) = 0$.

Similarly taking $k = 0$, we find that $f(x, y)$ cannot have a maximum or minimum at (a, b) unless $f_x(a, b) = 0$.

Hence *the necessary conditions for $f(x, y)$ to have a maximum or minimum at (a, b) are that*

$$\mathbf{f_x(a, b) = 0,\ f_y(a, b) = 0.}$$

If these conditions are satisfied, then for small value of h and k, (i) gives

$$\text{sign of } \Delta = \text{sign of } \left[\frac{1}{2!}(h^2 r + 2hks + k^2 t)\right] \text{ where } r = f_{xx}(a, b),\ s = f_{xy}(a, b) \text{ and } t = f_{yy}(a, b).$$

Now $\quad h^2r + 2hks + k^2t = \frac{1}{r}\left[(h^2r^2 + 2hkrs + k^2rt)\right] = \frac{1}{r}\left[(hr + ks)^2 + k^2(rt - s^2)\right]$

Thus sign of Δ = sign of $\frac{1}{2r}\left\{(hr + ks)^2 + k^2(rt - s^2)\right\}$ $\qquad \ldots(ii)$

In (ii), $(hr + ks)^2$ is always positive and $k^2(rt - s^2)$ will be positive if $rt - s^2 > 0$. In this case, Δ will have the same sign as that of r for all values of h and k.

Hence if $rt - s^2 > 0$, then *$f(x, y)$ has a maximum or a minimum at (a, b) according as $r <$ or > 0.*

If $rt - s^2 < 0$, then Δ will change with h and k and hence there is no maximum or minimum at (a, b) *i.e.*, it is a *saddle point*.

If $rt - s^2 = 0$, further investigation is required to find whether there is a maximum or minimum at (a, b) or not.

Note. Stationary value. *$f(a, b)$ is said to be a stationary value of $f(x, y)$, if $f_x(a, b) = 0$ and $f_y(a, b) = 0$ i.e.* the function is stationary at (a, b).

Thus every extreme value is a stationary value but the converse may not be true.

(3) Working rule to find the maximum and minimum values of f(x, y)

1. Find $\partial f/\partial x$ and $\partial f/\partial y$ *and equate each to zero. Solve these as simultaneous equations in x and y.* Let (a, b), *(c, d), ... be the pairs of values.*
2. *Calculate the value of $r = \partial^2 f/\partial x^2$, $s = \partial^2 f/\partial x\partial y$, $t = \partial^2 f/\partial y^2$ for each pair of values.*

3. *(i) If $rt - s^2 > 0$ and $r < 0$ at (a, b), $f(a, b)$ is a max. value.*
(ii) If $rt - s^2 > 0$ and $r > 0$ at (a, b), $f(a, b)$ is a min. value.
(iii) If $rt - s^2 < 0$ at (a, b), $f(a, b)$ is not an extreme value, i.e., (a, b) is a saddle point.
(iv) If $rt - s^2 = 0$ at (a, b), the case is doubtful and needs further investigation.

Similarly examine the other pairs of values one by one.

Example 5.42. *Examine the following function for extreme values :*

$$f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2. \quad \textit{(J.N.T.U., 2003)}$$

Solution. We have $f_x = 4x^3 - 4x + 4y$; $f_y = 4y^3 + 4x - 4y$

and $r = f_{xx} = 12x^2 - 4,\ s = f_{xy} = 4,\ t = f_{yy} = 12y^2 - 4$ ...(*i*)

Now $f_x = 0, f_y = 0$ give $x^3 - x + y = 0$, ...(*i*) $y^3 + x - y = 0$ ...(*ii*)

Adding these, we get $4(x^3 + y^3) = 0$ or $y = -x$.

Putting $y = -x$ in (*i*), we obtain $x^3 - 2x = 0$, *i.e.* $x = \sqrt{2}, -\sqrt{2}, 0$.

$\therefore$ Corresponding values of y are $-\sqrt{2}, \sqrt{2}, 0$.

At $(\sqrt{2}, -\sqrt{2})$, $rt - s^2 = 20 \times 20 - 4^2 = +$ ve and r is also + ve. Hence $f(\sqrt{2}, -\sqrt{2})$ is a minimum value.

At $(-\sqrt{2}, \sqrt{2})$ also both $rt - s^2$ and r are + ve.

Hence $f(-\sqrt{2}, \sqrt{2})$, is also a minimum value.

At $(0, 0)$, $rt - s^2 = 0$ and, therefore, further investigation is needed.

Now $f(0, 0) = 0$ and for points along the x-axis, where $y = 0$, $f(x, y) = x^4 - 2x^2 = x^2(x^2 - 2)$, which is negative for points in the neighbourhood of the origin.

Again for points along the line $y = x$, $f(x, y) = 2x^4$ which is positive.

Thus in the neighbourhood of $(0, 0)$ there are points where $f(x, y) < f(0, 0)$ and there are points where $f(x, y) > f(0, 0)$.

Hence $f(0, 0)$ is not an extreme value *i.e.*, it is a *saddle point*.

Example 5.43. *Discuss the maxima and minima of $f(x, y) = x^3y^2(1 - x - y)$.*

(Anna, 2009 ; J.N.T.U., 2006 ; Bhopal, 2002)

Solution. We have $f_x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$; $f_y = 2x^3y - 2x^4y - 3x^3y^2$

and $r = f_{xx} = 6xy^2 - 12x^2y^2 - 6xy^3$; $s = f_{xy} = 6x^2y - 8x^3y - 9x^2y^2$; $t = f_{yy} = 2x^3 - 2x^4 - 6x^3y$.

When $f_x = 0, f_y = 0$, we have $x^2y^2(3 - 4x - 3y) = 0$, $x^3y(2 - 2x - 3y) = 0$

Solving these, the stationary points are (1/2, 1/3), (0, 0).

Now $rt - s^2 = x^4y^2[12(1 - 2x - y)(1 - x - 3y) - (6 - 8x - 9y)^2]$

At $(1/2, 1/3)$, $rt - s^2 = \dfrac{1}{16} \cdot \dfrac{1}{9}\left[12\left(1 - 1 - \dfrac{1}{3}\right)\left(1 - \dfrac{1}{2} - 1\right) - (6 - 4 - 3)^2\right] = \dfrac{1}{14} > 0$

Also $r = 6\left(\dfrac{1}{2} \cdot \dfrac{1}{9} - \dfrac{2}{4} \cdot \dfrac{1}{9} - \dfrac{1}{2} \cdot \dfrac{1}{27}\right) = -\dfrac{1}{9} < 0$

Hence $f(x, y)$ has a maximum at (1/2, 1/3) and maximum value $= \dfrac{1}{8} \cdot \dfrac{1}{9}\left(1 - \dfrac{1}{2} - \dfrac{1}{3}\right) = \dfrac{1}{432}$.

At $(0, 0)$, $rt - s^2 = 0$ and therefore further investigation is needed.

For points along the line $y = x$, $f(x, y) = x^5(1 - 2x)$ which is positive for $x = 0.1$ and negative for $x = -0.1$ *i.e.*, in the neighbourhood of $(0, 0)$ there are points where $f(x, y) > f(0, 0)$ and there are points where $f(x, y) < f(0, 0)$. Hence $f(0, 0)$ is not an extreme value.

Example 5.44. *In a plane triangle, find the maximum value of cos A cos B cos C.*

(V.T.U., 2010 ; Nagpur, 2009 ; Anna, 2005 S)

Solution. We have $A + B + C = \pi$ so that $C = \pi - (A + B)$.

$$\cos A \cos B \cos C = \cos A \cos B \cos [\pi - (A + B)]$$
$$= -\cos A \cos B \cos (A + B) = f(A, B), \text{ say.}$$

We get $$\frac{\partial f}{\partial A} = \cos B\,[\sin A \cos (A + B) + \cos A \sin (A + B)]$$
$$= \cos B \sin (2A + B)$$

and $$\frac{\partial f}{\partial B} = \cos A \sin (A + 2B)$$

$$\frac{\partial f}{\partial A} = 0,\ \frac{\partial f}{\partial B} = 0 \quad \text{only when } A = B = \pi/3.$$

Also $$r = \frac{\partial^2 f}{\partial A^2} = 2 \cos B \cos (2A + B),\ t = \frac{\partial^2 f}{\partial B^2} = 2 \cos A \cos (A + 2B)$$

$$s = \frac{\partial^2 f}{\partial A \partial B} = -\sin B \sin (2A + B) + \cos B \cos (2A + B) = \cos (2A + 2B)$$

When $A = B = \pi/3$, $r = -1$, $s = -1/2$, $t = -1$ so that $rt - s^2 = 3/4$.

These show that $f(A, B)$ is maximum for $A = B = \pi/3$.

Then $C = \pi - (A + B) = \pi/3$.

Hence $\cos A \cos B \cos C$ is maximum when each of the angles is $\pi/3$ *i.e.*, triangle is equilateral and its maximum value = 1/8.

5.12 LAGRANGE'S METHOD OF UNDERTERMINED MULTIPLIERS

Sometimes it is required to find the stationary values of a function of several variables which are not all independent but are connected by some given relations. Ordinarily, we try to convert the given function to the one, having least number of independent variables with the help of given relations. Then solve it by the above method. When such a procedure becomes impracticable, Lagrange's method* proves very convenient. Now we explain this method.

Let $$u = f(x, y, z) \quad \text{...(1)}$$

be a function of three variables x, y, z which are connected by the relation.

$$\phi(x, y, z) = 0 \quad \text{...(2)}$$

For u to have stationary values, it is necessary that

$$\partial u/\partial x = 0,\ \partial u/\partial y = 0,\ \partial u/\partial z = 0.$$

$$\therefore \quad \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = du = 0 \quad \text{...(3)}$$

Also differentiating (2), we get $$\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = d\phi = 0 \quad \text{...(4)}$$

Multiply (4) by a parameter λ and add to (3). Then

$$\left(\frac{\partial u}{\partial x} + \lambda \frac{\partial \phi}{\partial x}\right) dx + \left(\frac{\partial u}{\partial y} + \lambda \frac{\partial \phi}{\partial y}\right) dy + \left(\frac{\partial u}{\partial z} + \lambda \frac{\partial \phi}{\partial z}\right) dz = 0$$

This equation will be satisfied if $\frac{\partial u}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0,\ \frac{\partial u}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0,\ \frac{\partial u}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0.$

These three equations together with (2) will determine the values of x, y, z and λ for which u is stationary.

Working rule : 1. *Write* $F = f(x, y, z) + \lambda\phi(x, y, z)$

2. *Obtain the equations* $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0.$

3. *Solve the above equations together with* $\phi(x, y, z) = 0$.

The values of x, y, z so obtained will give the stationary value of $f(x, y, z)$.

Obs. Although the Lagrange's method is often very useful in application yet *the drawback is that we cannot determine the nature of the stationary point*. This can sometimes, be determined from physical considerations of the problem.

*See footnote page 142.

Example 5.45. *A rectangular box open at the top is to have volume of 32 cubic ft. Find the dimensions of the box requiring least material for its construction.* (Kurukshetra, 2006 ; P.T.U., 2006 ; U.P.T.U., 2005)

Solution. Let x, y and z ft. be the edges of the box and S be its surface.

Then $$S = xy + 2yz + 2zx \quad ...(i)$$

and $$xyz = 32 \quad ...(ii)$$

Eliminating z from (i) with the help of (ii), we get $S = xy + 2(y + x)\dfrac{32}{xy} = xy + 64\left(\dfrac{1}{x} + \dfrac{1}{y}\right)$

$\therefore$ $$\partial S/\partial x = y - 64/x^2 = 0 \quad \text{and} \quad \partial S/\partial y = x - 64/y^2 = 0.$$

Solving these, we get $x = y = 4$.

Now $r = \partial^2 S/\partial x^2 = 128/x^3$, $s = \partial^2 S/\partial x \partial y = 1$, $t = \partial^2 S/\partial y^2 = 128/y^3$.

At $x = y = 4$, $rt - s^2 = 2 \times 2 - 1 = +$ ve and r is also $+$ ve.

Hence S is minimum for $x = y = 4$. Then from (ii), $z = 2$.

Otherwise (*by Lagrange's method*) :

Write $$F = xy + 2yz + 2zx + \lambda(xyz - 32)$$

Then $$\frac{\partial F}{\partial x} = y + 2z + \lambda yz = 0 \quad ...(iii)$$

$$\frac{\partial F}{\partial y} = x + 2z + \lambda zx = 0 \quad ...(iv)$$

$$\frac{\partial F}{\partial z} = 2y + 2x + \lambda xy = 0 \quad ...(v)$$

Multiplying (iii) by x and (iv) by y and subtracting, we get $2zx - 2zy = 0$ or $x = y$.

[The value $z = 0$ is neglected, as it will not satisfy (ii)]

Again multiplying (iv) by y and (v) by z and subtracting, we get $y = 2z$.

Hence the dimensions of the box are $x = y = 2z = 4$ ...(vi)

Now let us see what happens as z increases from a small value to a large one. When z is small, the box is flat with a large base showing that S is large. As z increases, the base of the box decreases rapidly and S also decreases. After a certain stage, S again starts increasing as z increases. Thus S must be a minimum at some intermediate stage which is given by (vi). Hence S is minimum when $x = y = 4$ ft and $z = 2$ ft.

Example 5.46. *Given $x + y + z = a$, find the maximum value of $x^m y^n z^p$.* (Anna, 2009)

Solution. Let $f(x, y, z) = x^m y^n z^p$ and $\phi(x, y, z) = x + y + z - a$.

Then $$F(x, y, z) = f(x, y, z) + \lambda\, \phi(x, y, z)$$
$$= x^m y^n z^p + \lambda(x + y + z - a).$$

For stationary values of F, $\dfrac{\partial F}{\partial x} = 0, \dfrac{\partial F}{\partial y} = 0, \dfrac{\partial F}{\partial z} = 0$

$\therefore$ $$mx^{m-1} y^n z^p + \lambda = 0,\ nx^m y^{n-1} z^p + \lambda = 0,\ px^m y^n z^{p-1} + \lambda = 0$$

or $$-\lambda = mx^{n-1} y^n z^p = nx^m y^{n-1} z^p = px^m y^n z^{p-1}$$

i.e. $$\frac{m}{x} = \frac{n}{y} = \frac{p}{z} = \frac{m + n + p}{x + y + z} = \frac{m + n + p}{a} \quad [\because \ x + y + z = a$$

$\therefore$ The maximum value of f occurs when

$$x = am/(m + n + p),\ y = an/(m + n + p),\ z = ap/(m + n + p)$$

Hence the maximum value of $f(x, y, z) = \dfrac{a^{m+n+p} \cdot m^m \cdot n^n \cdot p^p}{(m + n + p)^{m+n+p}}$.

Example 5.47. *Find the maximum and minimum distances of the point (3, 4, 12) from the sphere $x^2 + y^2 + z^2 = 4$.* (U.T.U., 2010)

Solution. Let $P(x, y, z)$ be any point on the sphere and $A(3, 4, 12)$ the given point so that

$$AP^2 = (x - 3)^2 + (y - 4)^2 + (z - 12)^2 = f(x, y, z), \text{ say} \quad ...(i)$$

We have to find the maximum and minimum values of $f(x, y, z)$ subject to the condition

$$\phi(x, y, z) = x^2 + y^2 + z^2 - 4 = 0 \quad ...(ii)$$

Let $F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z)$

$$= (x-3)^2 + (y-4)^2 + (z-12)^2 + \lambda(x^2 + y^2 + z^2 - 4)$$

Then $\dfrac{\partial F}{\partial x} = 2(x-3) + 2\lambda x,\ \dfrac{\partial F}{\partial y} = 2(y-4) + 2\lambda y,\ \dfrac{\partial F}{\partial z} = 2(z-12) + 2\lambda z$

$\therefore$ $\dfrac{\partial F}{\partial x} = 0, \dfrac{\partial F}{\partial y} = 0$ and $\dfrac{\partial F}{\partial z} = 0$ give

$$x - 3 + \lambda x = 0,\ y - 4 + \lambda y = 0,\ z - 12 + \lambda z = 0 \quad ...(iii)$$

which give $\lambda = -\dfrac{x-3}{x} = -\dfrac{y-4}{y} = -\dfrac{z-12}{z}$

$$= \pm\frac{\sqrt{[(x-3)^2 + (y-4)^2 + (z-12)^2]}}{\sqrt{(x^2 + y^2 + z^2)}} = \pm\frac{\sqrt{f}}{1}$$

Substituting for λ in (iii), we get

$$x = \frac{3}{1+\lambda} = \frac{3}{1 \pm \sqrt{f}},\ y = \frac{4}{1 \pm \sqrt{f}},\ z = \frac{12}{1 \pm \sqrt{f}}$$

$\therefore$ $x^2 + y^2 + z^2 = \dfrac{9 + 16 + 144}{(1 \pm \sqrt{f})^2} = \dfrac{169}{(1 \pm \sqrt{f})^2}$

Using (ii), $1 = \dfrac{169}{(1 \pm \sqrt{f})^2}$ or $1 \pm \sqrt{f} = \pm 13$, $\sqrt{f} = 12, 14.$

[We have left out the negative values of $\sqrt{f}$, because $\sqrt{f} = AP$ is + ve by (i)]

Hence maximum $AP = 14$ and minimum $AP = 12$.

Example 5.48. *Show that the rectangular solid of maximum volume that can be inscribed in a sphere is a cube.* *(Kurukshetra, 2006 ; U.P.T.U., 2004)*

Solution. Let $2x$, $2y$, $2z$ be the length, breadth and height of the rectangular solid so that its volume

$$V = 8xyz \quad ...(i)$$

Let R be the radius of the sphere so that $x^2 + y^2 + z^2 = R^2$ ...(ii)

Then $F(x, y, z) = 8xyz + \lambda(x^2 + y^2 + z^2 - R^2)$

and $\partial F/\partial x = 0, \partial F/\partial y = 0$ and $\partial F/\partial z = 0$ give

$$8yz + 2x\lambda = 0,\ 8zx + 2y\lambda = 0,\ 8xy + 2z\lambda = 0$$

or $2x^2\lambda = -8xyz = 2y^2\lambda = 2z^2\lambda$

Thus for a maximum volume $x = y = z$.

i.e., the rectangular solid is a cube.

Example 5.49. *A tent on a square base of side x, has its sides vertical of height y and the top is a regular pyramid of height h. Find x and y in terms of h, if the canvas required for its construction is to be minimum for the tent to have a given capacity.*

Solution. Let V be the volume enclosed by the tent and S be its surface area (Fig. 5.6).

Then V = cuboid $(ABCD, A'B'C'D')$ + pyramid $(K, A'B'C'D')$

$$= x^2y + \frac{1}{3}x^2h = x^2(y + h/3)$$

$$S = 4(ABGF) + 4\Delta\, KGH = 4xy + 4\frac{1}{2}(x \cdot KM)$$

$$= 4xy + x\sqrt{(x^2 + 4h^2)} \qquad [\because\ KM = \sqrt{(KL^2 + LM^2)} = \sqrt{[h^2 + (x/2)^2]}]$$

For constant V, we have

$$\delta V = 2x(y + h/3)\,\delta x + x^2(\delta y) + \frac{x^2}{3}\,\delta h = 0$$

For minimum S, we have

$$\delta S = [4y + \sqrt{(x^2+4h^2)} + x\cdot\frac{1}{2}(x^2+4h^2)^{-1/2}\cdot 2x]\,\delta x$$
$$+ 4x\delta y + x\cdot\frac{1}{2}(x^2+4h^2)^{-1/2}\cdot 8h\delta h = 0$$

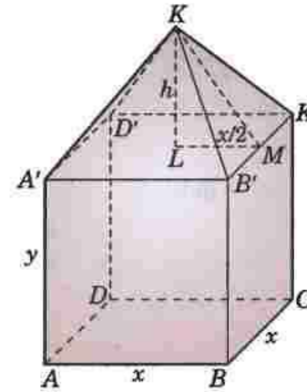

Fig. 5.6

By Lagrange's method,

$$[4y + \sqrt{(x^2+4h^2)} + x^2(x^2+4h^2)^{-1/2}] + \lambda\cdot 2x(y+h/3) = 0 \quad ...(i)$$
$$4x + \lambda\cdot x^2 = 0 \quad ...(ii)$$
$$4hx(x^2+4h^2)^{-1/2} + \lambda\cdot x^2/3 = 0 \quad ...(iii)$$

(ii) gives $\lambda = -4/x$. Then (iii) becomes

$$4hx(x^2+4h^2)^{-1/2} - 4x/3 = 0 \quad \text{or} \quad x = \sqrt{5}\,h$$

Now putting $x = \sqrt{5}\,h$, $\lambda = -4/x$ in (i), we get

$$4y + 3h + \frac{5}{3}h - \frac{4}{x}\cdot 2x(y+h/3) = 0 \quad \text{or} \quad 4y + \frac{14}{3}h - 8y - \frac{8h}{3} = 0, \quad i.e., \quad y = h/2.$$

Example 5.50. *If $u = a^3x^2 + b^3y^2 + c^3z^2$ where $x^{-1} + y^{-1} + z^{-1} = 1$, show that the stationary value of u is given by $x = \Sigma a/a$, $y = \Sigma a/b$, $z = \Sigma a/c$.* *(Kerala, 2005)*

Solution. Let $u = f(x,y,z) = a^3x^2 + b^3y^2 + c^3z^2$

and $\phi(x,y,z) = x^{-1} + y^{-1} + z^{-1} - 1$

Let $F(x,y,z) = f(x,y,z) + \lambda\,\phi(x,y,z)$
$= a^3x^2 + b^3y^2 + c^3z^2 + \lambda(x^{-1} + y^{-1} + z^{-1} - 1)$

Then $\dfrac{\partial F}{\partial x} = 0, \dfrac{\partial F}{\partial y} = 0$ and $\dfrac{\partial F}{\partial z} = 0$ give

$$2a^3x^2 - \lambda/x^2 = 0, \quad 2b^3y - \lambda/y^2 = 0, \quad 2c^3z - \lambda/z^2 = 0$$

or
$$2a^3x^3 = \lambda, \; 2b^3y^3 = \lambda, \; 2c^3z^3 = \lambda$$

which give $ax = by = cz = k$ (say) *i.e.*, $x = k/a$, $y = k/b$, $z = k/c$.

Substituting these in $x^{-1} + y^{-1} + z^{-1} = 1$, we get $k = a + b + c$

Hence the stationary value of u is given by

$$x = \Sigma a/a, \quad y = \Sigma a/b \quad \text{and} \quad z = \Sigma a/c.$$

Example 5.51. *Find the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$ *(U.T.U., 2010 ; Anna, 2009 ; Madras, 2006)*

Solution. Let the edges of the parallelopiped be $2x$, $2y$ and $2z$ which are parallel to the axes. Then its volume $V = 8xyz$.

Now we have to find the maximum value of V subject to the condition that

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \quad ...(i)$$

Write
$$F = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

Then
$$\frac{\partial F}{\partial x} = 8yz + \lambda\left(\frac{2x}{a^2}\right) = 0 \quad ...(ii)$$

$$\frac{\partial F}{\partial y} = 8zx + \lambda\left(\frac{2y}{b^2}\right) = 0 \quad ...(iii) \qquad \frac{\partial F}{\partial z} = 8xy + \lambda\left(\frac{2z}{c^2}\right) = 0 \quad ...(iv)$$

Equating the values of λ from (*ii*) and (*iii*), we get $x^2/a^2 = y^2/b^2$

Similarly from (*iii*) and (*iv*), we obtain $y^2/b^2 = z^2/c^2 \quad \therefore \quad x^2/a^2 = y^2/b^2 = z^2/c^2$

Substituting these in (*i*), we get $x^2/a^2 = \frac{1}{3}$ *i.e.* $\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} = \frac{1}{3}$

These give $x = a/\sqrt{3}$, $y = b/\sqrt{3}$, $z = c/\sqrt{3}$ ...(*v*)

When $x = 0$, the parallelopiped is just a rectangular sheet and as such its volume $V = 0$.

As x increases, V also increases continuously.

Thus V must be greatest at the stage given by (*v*).

Hence the greatest volume $= \frac{8abc}{3\sqrt{3}}$.

PROBLEMS 5.10

1. Find the maximum and minimum values of
   (*i*) $x^3 + y^3 - 3axy$ (*U.P.T.U., 2005*) (*ii*) $xy + a^3/x + a^3/y$.
   (*iii*) $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$ (*Mumbai, 2007*) (*iv*) $2(x^2 - y^2) - x^4 + y^4$ (*Osmania, 2003*)
   (*v*) $\sin x \sin y \sin (x + y)$.
2. If $xyz = 8$, find the values of x, y for which $u = 5xyz/(x + 2y + 4z)$ is a maximum. (*S.V.T.U., 2007 ; Kurukshetra, 2005*)
3. Find the minimum value of $x^2 + y^2 + z^2$, given that
   (*i*) $xyz = a^3$ (*P.T.U., 2009 ; Osmania, 2003*) (*ii*) $ax + by + cz = p$. (*V.T.U., 2010 ; U.P.T.U., 2006*)
   (*iii*) $xy + yz + zx = 3a^2$ (*Anna, 2009*)
4. Find the dimensions of the rectangular box, open at the top, of maximum capacity whose surface is 432 sq. cm. (*Madras, 2000 S*)
5. The sum of three numbers is constant. Prove that their product is maximum when they are equal.
6. Find the points on the surface $z^2 = xy + 1$ nearest to the origin. (*Burdwan, 2003 ; Andhra, 2000*)
7. Show that, if the perimeter of a triangle is constant, the triangle has maximum area when it is equilateral.
8. Find the maximum and minimum distances from the origin to the curve $5x^2 + 6xy + 5y^2 - 8 = 0$.
9. The temperature T at any point (x, y, z) in space is $T = 400\,xyz^2$. Find the highest temperature on the surface of the unit sphere $x^2 + y^2 + z^2 = 1$. (*V.T.U., 2009 ; Hissar, 2005 S*)
10. Divide 24 into three parts such that the continued product of the first, square of the second and the cube of the third may be maximum. (*Bhillai, 2005*)
11. Find the stationary values of $u = x^2 + y^2 + z^2$ subject to $ax^2 + by^2 + cz^2 = 1$ and $lx + my + nz = 0$. (*S.V.T.U., 2008*)

5.13 DIFFERENTIATION UNDER THE INTEGRAL SIGN

If a function $f(x, \alpha)$ of two variables x and α (called a parameter), be integrated with respect to x between the limits a and b, then $\int_a^b f(x, \alpha)\, dx$ is a function of $\alpha : F(\alpha)$, say. To find the derivative of $F(\alpha)$, when it exists, it is not always possible to first evaluate this integral and then to find the derivative. Such problems are solved by the following rules :

(1) Leibnitz's rule*

If $f(x, \alpha)$ and $\frac{\partial f(x, \alpha)}{\partial \alpha}$ *be continuous functions of x and* α, *then*

$$\frac{d}{d\alpha}\left[\int_a^b f(x, \alpha)\, dx\right] = \int_a^b \frac{\partial f(x, \alpha)}{\partial \alpha}\, dx$$ *where, a, b are constants independent of* α.

*See foot note on p. 139.

Let $\displaystyle\int_a^b f(x, \alpha)dx = F(\alpha),$

then $\displaystyle F(\alpha + \delta\alpha) - F(\alpha) = \int_a^b f(x, \alpha + \delta\alpha)dx - \int_a^b f(x, \alpha)dx = \int_a^b [f(x, \alpha + \delta\alpha) - f(x, \alpha)]dx$

$$= \delta\alpha \int_a^b \frac{\partial f(x, \alpha + \theta\delta\alpha)}{\partial \alpha} dx, \ (0 < \theta < 1) \qquad \left\{\begin{array}{l} \because \ f(x, \alpha + h) - f(x, \alpha) = hf'(x, \alpha + \theta h) \\ \text{where } 0 < \theta < 1, \text{ by Mean Value Theorem} \end{array}\right.$$

Proceeding to limits as $\delta\alpha \to 0$, $\displaystyle\underset{\delta\alpha \to 0}{\text{Lt}} \frac{F(\alpha + \delta\alpha) - F(\alpha)}{\delta\alpha} = \int_a^b \frac{\partial f(x, \alpha + \theta \cdot 0)}{\partial \alpha} dx$

or $\displaystyle\frac{dF}{d\alpha} = \int_a^b \frac{\partial f(x, \alpha)}{\partial \alpha} dx$ which is the desired result.

Obs. 1. Leibnitz's rule enables us to derive from the value of a simple definite integral, the value of another definite integral which it may otherwise be difficult, or even impossible, to evaluate.

Obs. 2. The rule for differentiation under the integral sign of an infinite integral is the same as for a definite integral.

Example 5.52. *Evaluate* $\displaystyle\int_0^1 \frac{x^\alpha - 1}{\log x} dx, \ \alpha \geq 0.$ *(V.T.U., 2010)*

Solution. Let $\displaystyle F(\alpha) = \int_0^1 \frac{x^\alpha - 1}{\log x} dx \qquad ...(i)$

then $$F(\alpha) = \int_0^1 \frac{\partial}{\partial \alpha}\left(\frac{x^\alpha - 1}{\log x}\right) dx = \int_0^1 \frac{x^\alpha \log x}{\log x} dx$$

$$= \int_0^1 x^\alpha \, dx = \left| \frac{x^{\alpha+1}}{\alpha + 1} \right|_0^1 = \frac{1}{1 + \alpha} \qquad \left[\because \ \frac{d}{dt}(n^t) = n^t \log n\right]$$

Now integrating both sides w.r.t. α, $F(\alpha) = \log(1 + \alpha) + c \qquad ...(ii)$

From (*i*), when $\alpha = 0$, $F(0) = 0$

$\therefore$ From (*ii*), $F(0) = \log(1) + c$, *i.e.*, $c = 0$. Hence (*ii*) gives, $F(\alpha) = \log(1 + \alpha)$.

Example 5.53. *Given* $\displaystyle\int_0^\pi \frac{dx}{a + b\cos x} = \frac{\pi}{\sqrt{(a^2 - b^2)}} \ (a > b),$

evaluate $\displaystyle\int_0^\pi \frac{dx}{(a + b\cos x)^2}$ *and* $\displaystyle\int_0^\pi \frac{\cos x}{(a + b\cos x)^2} dx$ *(Madras, 2006)*

Solution. We have $\displaystyle\int_0^\pi \frac{dx}{a + b\cos x} = \frac{\pi}{\sqrt{(a^2 - b^2)}} \qquad ...(i)$

Differentiating both sides of (*i*) w.r.t. a,

$$\int_0^\pi \frac{\partial}{\partial a}\left(\frac{1}{a + b\cos x}\right) dx = \frac{\partial}{\partial a}\left\{\frac{\pi}{\sqrt{(a^2 - b^2)}}\right\}$$

i.e. $$\int_0^\pi \frac{-dx}{(a + b\cos x)^2} = \pi \cdot \left(-\frac{1}{2}\right)(a^2 - b^2)^{-3/2} \cdot 2a$$

$\therefore$ $$\int_0^\pi \frac{dx}{(a + b\cos x)^2} = \frac{\pi a}{(a^2 - b^2)^{3/2}}$$

Now differentiating both sides of (*i*) w.r.t. b,

$$\int_0^\pi -(a + b\cos x)^{-2} \cdot \cos x \, dx = \pi\left(-\frac{1}{2}\right)(a^2 - b^2)^{-3/2} \cdot (-2b)$$

$$\therefore \qquad \int_0^{\pi} \frac{\cos x}{(a+b\cos x)^2}dx = \frac{\pi b}{(a^2-b^2)^{3/2}}.$$

(2) Leibnitz's rule *for variable limits of integration*

If $f(x, \alpha)$, $\dfrac{\partial f(x,\alpha)}{\partial \alpha}$ *be continuous functions of x and α, then*

$$\frac{d}{d\alpha}\left\{\int_{\phi(\alpha)}^{\psi(\alpha)} f(x,\alpha)\,dx\right\} = \int_{\phi(\alpha)}^{\psi(\alpha)} \frac{\partial f(x,\alpha)}{\partial \alpha}dx + \frac{d\psi}{d\alpha} f[\psi(\alpha),\alpha] - \frac{d\phi}{d\alpha} f[\phi(\alpha),\alpha]$$

provided $\phi(\alpha)$ *and* $\psi(\alpha)$ *possesses continuous first order derivatives w.r.t.* α.

Its proof is beyond the scope of this book.

Example 5.54. *Evaluate* $\int_0^{\alpha} \frac{\log(1+\alpha x)}{1+x^2}dx$ *and hence show that*

$$\int_0^1 \frac{\log(1+x)}{1+x^2}dx = \frac{\pi}{8}\log_e 2$$ (Hissar, 2005 S)

Solution. Let $$F(\alpha) = \int_0^{\alpha} \frac{\log(1+\alpha x)}{1+x^2}dx \qquad ...(i)$$

Then by the above rule, $$F'(\alpha) = \int_0^{\alpha} \frac{\partial}{\partial\alpha}\left(\frac{\log(1+\alpha x)}{1+x^2}\right)dx + \frac{d(\alpha)}{d\alpha}\cdot\frac{\log(1+\alpha^2)}{1+\alpha^2} - 0$$

$$= \int_0^{\alpha} \frac{x}{(1+\alpha x)(1+x^2)}dx + \frac{\log(1+\alpha^2)}{1+\alpha^2} \qquad ...(ii)$$

Breaking the integrand into partial fractions,

$$\int_0^{\alpha} \frac{x\,dx}{(1+\alpha x)(1+x^2)} = -\frac{\alpha}{1+\alpha^2}\int_0^{\alpha}\frac{dx}{1+\alpha x} + \frac{1}{2(1+\alpha^2)}\int_0^{\alpha}\frac{2x}{1+x^2}dx + \frac{\alpha}{1+\alpha^2}\int_0^{\alpha}\frac{dx}{1+x^2}$$

$$= -\frac{1}{1+\alpha^2}\Big|\log(1+\alpha x)\Big|_0^{\alpha} + \frac{1}{2(1+\alpha^2)}\times\Big|\log(1+x^2)\Big|_0^{\alpha} + \frac{\alpha}{1+\alpha^2}\Big|\tan^{-1}x\Big|_0^{\alpha}$$

$$= -\frac{\log(1+\alpha^2)}{1+\alpha^2} + \frac{\log(1+\alpha^2)}{2(1+\alpha^2)} + \frac{\alpha\tan^{-1}\alpha}{1+\alpha^2}$$

Substituting this value in (*ii*), $F'(\alpha) = \dfrac{\log(1+\alpha^2)}{2(1+\alpha^2)} + \dfrac{\alpha\tan^{-1}\alpha}{1+\alpha^2}$

Now integrating both sides w.r.t. α,

$$F(\alpha) = \frac{1}{2}\int \log(1+\alpha^2)\cdot\frac{1}{1+\alpha^2}d\alpha + \int\frac{\alpha\tan^{-1}\alpha}{1+\alpha^2}d\alpha \qquad \text{[Integrating by parts]}$$

$$= \frac{1}{2}\left[\log(1+\alpha^2)\cdot\tan^{-1}\alpha - \int\frac{2\alpha}{1+\alpha^2}\cdot\tan^{-1}\alpha\,d\alpha\right] + \int\frac{\alpha\tan^{-1}\alpha}{1+\alpha^2}d\alpha + c$$

$$= \frac{1}{2}\log(1+\alpha^2)\cdot\tan^{-1}\alpha + c \qquad ...(iii)$$

But from (*i*), when $\alpha = 0$, $F(0) = 0$.

$\therefore$ From (*iii*), $F(0) = 0 + c$, *i.e.*, $c = 0$. Hence (*iii*) gives, $F(\alpha) = \dfrac{1}{2}\log(1+\alpha^2)\tan^{-1}\alpha$

Putting $\alpha = 1$, we get $\displaystyle\int_0^1 \frac{\log(1+x)}{1+x^2}dx = F(1) = \frac{\pi}{8}\log_e 2.$

PROBLEMS 5.11

1. Differentiating $\int_0^x \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$ under the integral sign, find the value of $\int_0^x \frac{dx}{(x^2+a^2)^2}$.

2. By successive differentiation of $\int_0^1 x^m\, dx = \frac{1}{m+1}$ w.r.t. m, evaluate $\int_0^1 x^m (\log x)^n\, dx$.

3. Evaluate $\int_0^\pi \log(1 + a\cos x)\, dx$, using the method of differentiation under the sign of integration.

4. Given that $\int_0^\pi \frac{dx}{a-\cos x} = \frac{\pi}{\sqrt{(a^2-1)}}$, evaluate $\int_0^\pi \frac{dx}{(a-\cos x)^2}$. (V.T.U., 2009)

Prove that :

5. $\int_0^\infty e^{-x}\cdot\frac{\sin ax}{x}dx = \tan^{-1} a.$ $\left[\textbf{Hint.} \text{ Use } \int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\right]$

6. $\int_0^\infty e^{-ax}\frac{\sin x}{x}dx = \tan^{-1}\frac{1}{a}$. Hence show that $\int_0^\infty \frac{\sin x}{x} = \frac{\pi}{2}$. (Rohtak, 2003)

7. $\int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)}dx = \frac{\pi}{2}\log(1+a)$ where $a \geq 0$. (V.T.U., 2010 ; S.V.T.U., 2009 ; Rohtak, 2006 S ; Anna, 2005 S)

8. $\int_0^\infty \frac{e^{-x}}{x}(1-e^{-ax})\, dx = \log(1+a), (a > -1).$

9. $\int_0^{\pi/2} \log(\alpha^2\cos^2\theta + \beta^2\sin^2\theta)\, d\theta = \pi\log\frac{\alpha+\beta}{2}$ (S.V.T.U., 2008)

10. $\int_0^{\pi/2} \frac{\log(1+y\sin^2 x)}{\sin^2 x}dx = \pi[\sqrt{(1+y)}-1]$ (S.V.T.U., 2008)

11. $\int_0^\pi \frac{\log(1+\alpha\cos x)}{\cos x}dx = \pi\sin^{-1}\alpha.$ (V.T.U., 2007)

12. $\int_0^\infty e^{-x^2}\cos\alpha x\, dx = \frac{\sqrt{\pi}}{2}e^{-\alpha^2/4}$ (Mumbai, 2009 S)

13. $\frac{d}{da}\int_0^{a^2} \tan^{-1}\frac{x}{a}dx = 2a\tan^{-1}a - \frac{1}{2}\log(a^2+1)$. Verify your result by direct integration.

14. $\int_{\pi/2-\alpha}^{\pi/2} \sin\theta\cos^{-1}(\cos\alpha\cos\theta)\, d\theta = \frac{\pi}{2}(1-\cos\alpha).$ (Burdwan, 2003)

15. If $y = \int_0^x f(t)\sin[k(x-t)]dt$, prove that y satisfies the differential equation $\frac{d^2y}{dx^2} + k^2 y = k f(x)$.

5.14 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 5.12

Select the correct answer or fill up the blanks in each of the following problems :

1. If $u = e^x(x\cos y - y\sin y)$, then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \ldots$

2. If $x = uv$, $y = u/v$, then $\frac{\partial(x, y)}{\partial(u, v)}$ is

   (a) $-2u/v$ (b) $-2v/u$ (c) 0 (d) 1. (V.T.U., 2010)

3. If $J_1 = \dfrac{\partial(u, v)}{\partial(x, y)}$ and $J_2 = \dfrac{\partial(x, y)}{\partial(u, v)}$, then $J_1J_2 = ...$

4. If $u = f(y/x)$, then

(a) $x\dfrac{\partial u}{\partial x} - y\dfrac{\partial u}{\partial y} = 0$ (b) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$ (c) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2u$ (d) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 1$.

5. If $u = x^y$, then $\partial u/\partial x$ is

(a) 0 (b) yx^{y-1} (c) $x^y \log x$.

6. If $x = r\cos\theta$, $y = r\sin\theta$, then

(a) $\dfrac{\partial x}{\partial r} = 1/\dfrac{\partial r}{\partial x}$ (b) $\dfrac{\partial x}{\partial r} = \dfrac{\partial r}{\partial x}$ (c) $\dfrac{\partial x}{\partial r} = 0$.

7. If $u = x^y$, then $\partial u/\partial y$ is

(a) yx^{y-1} (b) 0 (c) $x^y \log x$.

8. If $u = x^3 + y^3$, then $\dfrac{\partial^2 u}{\partial x\partial y}$ is equal to

(a) -3 (b) 3 (c) 0 (d) $3x + 3y$ (V.T.U., 2010 S)

9. If $u = x^2 + 2xy + y^2 + x + y$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$ is equal to

(a) $2u$ (b) u (c) 0 (d) none of these.

10. If $u = \log\dfrac{x^2}{y}$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$ is equal to

(a) $2u$ (b) $3u$ (c) u (d) 1. (V.T.U., 2010 S)

11. If $x = r\cos\theta$, $y = r\sin\theta$, then $\dfrac{\partial(x, y)}{\partial(r, \theta)}$ is equal to

(a) 1 (b) r (c) $1/r$ (d) 0. (V.T.U., 2010 S)

12. If $A = f_{xx}(a, b)$, $B = f_{xy}(a, b)$, $C = f_{yy}(a, b)$, then $f(x, y)$ will have a maximum at (a, b) if

(a) $f_x = 0, f_y = 0,\ AC < B^2$ and $A < 0$ (b) $f_x = 0, f_y = 0,\ AC = B^2$ and $A > 0$

(c) $f_x = 0, f_y = 0,\ AC > B^2$ and $A > 0$ (d) $f_x = 0, f_y = 0,\ AC > B^2$ and $A < 0$.

13. If $z = \sin^{-1}\dfrac{\sqrt{x^2 + y^2}}{x + y}$, then $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y}$ is

(a) 0 (b) 1/2 (c) 1 (d) 2. (Bhopal, 2008)

14. If $u = \sin^{-1}(x/y) + \tan^{-1}(y/x)$, then $x\,\partial u/\partial x + y\,\partial u/\partial y$ equals

(a) $\sin^{-1}(x/y) + \tan^{-1}(y/x)$ (b) $2[\sin^{-1}(x/y) + \tan^{-1}(y/x)]$

(c) $3[\sin^{-1}(x/y) + \tan^{-1}(y/x)]$ (d) zero.

15. If an error of 1% is made in measuring its length and breadth, the percentage error in the area of a rectangle is

(a) 0.2% (b) 0.02% (c) 2% (d) 1%. (V.T.U., 2010)

16. $\dfrac{\partial(u, v)}{\partial(x, y)} \times \dfrac{\partial(x, y)}{\partial(u, v)}$ equals

(a) -1 (b) 1 (c) zero (d) none of these.

17. $\dfrac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}$ is a homogeneous function of degree ......

18. If $z = \log(x^3 + y^3 - x^2y - xy^2)$, then $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y}$ is equal to ......

19. If $r = \partial^2 f/\partial x^2$, $s = \partial^2 f/\partial x\partial y$ and $t = \partial^2 f/\partial y^2$, then the condition for the saddle point is ......

20. If $f(x, y) = \dfrac{1}{x^3} + \dfrac{1}{y^3} + \dfrac{1}{x^3 + y^3}$, then $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y}$ is

(a) 0 (b) $3f$ (c) 9 (d) $-3f$. (V.T.U., 2009 S)

21. If $u = x^4 + y^4 + 3x^2y^2$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = ...$

22. If u and v are functions of r, s where r, s are functions of x, y, then $\frac{\partial(u, v)}{\partial(r, s)} \cdot \frac{\partial(r, s)}{\partial(x, y)} = ...$

23. The necessary conditions for a function $f(x, y)$ to have an extreme at (a, b) are ......

24. If $u = (x-y)^4 + (y-z)^4 + (z-x)^4$, then $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$ is

(a) 1 (b) u (c) $4u$ (d) 0. (V.T.U., 2010)

25. If u is a composite function of t, defined by the relations $u = f(x, y)$; $x = \phi(t)$, $y = \psi(t)$, then total derivative $\frac{du}{dt} = ...$

26. If $u = \cos^{-1}(x/y) + \tan^{-1}(y/x)$, then $x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy}$ is

(a) u (b) $2u$ (c) 0 (d) 1. (V.T.U., 2010)

27. If $f(x, y, z) = 0$, then $\frac{\partial x}{\partial y} \cdot \frac{\partial y}{\partial z} \cdot \frac{\partial z}{\partial x} = ...$

28. If $u = f(x + ay) + g(x - ay)$, then $\frac{\partial^2 u}{\partial y^2}$ equals

(a) $\frac{\partial^2 u}{\partial x^2}$ (b) $a \frac{\partial^2 u}{\partial x^2}$ (c) $a^2 \frac{\partial^2 u}{\partial x^2}$ (d) $\frac{\partial^2 u}{\partial x \partial y}$. (V.T.U., 2010)

29. If sum of three numbers is constant, then their product is a maximum when the numbers are ......

30. $y = \cosh(\lambda x) \cosh(-\lambda at)$ is a solution of $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$. (*True* or *False*)

CHAPTER 6

Integral Calculus and Its Applications

1. Reduction formulae. 2. Reduction formulae for $\int \sin^n x\, dx$, $\int \cos^n x\, dx$ and evaluation of $\int_0^{\pi/2} \sin^n x\, dx$, $\int_0^{\pi/2} \cos^n x\, dx$. 3. Reduction formula for $\int \sin^m x \cos^n x\, dx$ and evaluation of $\int_0^{\pi/2} \sin^m x \cos^n x\, dx$. 4. Reduction formulae for $\int \tan^n x\, dx$ $\int \cot^n x\, dx$. 5. Reduction formulae for $\int \sec^n x\, dx$, $\int \operatorname{cosec}^n x\, dx$. 6. Reduction formulae for $\int x^n e^{ax}\, dx$, $\int x^m (\log x)^n\, dx$. 7. Reduction formulae for $\int x^n \sin mx\, dx$, $\int x^n \cos nx\, dx$ and $\int \cos^m x \sin nx\, dx$. 8. Definite integrals. 9. Integral as the limit of a sum. 10. Areas of curves. 11. Lengths of curves. 12. Volumes of revolution. 13. Surface areas of revolution. 14. Objective Type of Questions.

6.1 REDUCTION FORMULAE

The reader is already familiar with some standard methods of integrating functions of a single variable. However, there are some integrals which cannot be evaluated by the afore-said methods. In such cases, the method of reduction formulae proves useful. A reduction formula connects an integral with another of the same type but of lower order. The successive application of the reduction formula enables us to evaluate the given integral. Now we shall derive some standard reduction formulae.

6.2 (1) REDUCTION FORMULAE for

(a) $\int \sin^n x\, dx$ **(b)** $\int \cos^n x\, dx.$

(a) $\int \sin^n x\, dx = \int \sin^{n-1} x \cdot \sin x\, dx$ [Integrated by parts]

$$= \sin^{n-1} x \cdot (-\cos x) - \int (n-1) \sin^{n-2} x \cos x\, (-\cos x)\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x\, (1 - \sin^2 x)\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x\, dx - (n-1) \int \sin^n x\, dx$$

Transposing

$$n \int \sin^n x\, dx = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x\, dx$$

or

$$\int \sin^n x\, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x\, dx \qquad ...(i)$$

(*b*) Similarly, $$\int \cos^n x\,dx = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

Thus we have the required reduction formulae.

Obs. To integrate $\int \sin^n x\,dx$ or $\int \cos^n x\,dx$,

(a) when the index of sin x is odd put cos x = t

when the index of cos x is odd, put sin x = t

(b) when the index is an even positive integer, express the integrand as a series of cosines of multiple angles and integrate term by term if n is small, *otherwise use the method of reduction formulae.*

(2) To show that $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$

$$= \frac{(n-1)(n-3)(n-5)\ldots}{n(n-2)(n-4)\ldots} \times \left(\frac{\pi}{2}, \text{only if } n \text{ is even}\right)$$

From (*i*), we have

$$I_n = \int_0^{\pi/2} \sin^n x\,dx = -\left|\frac{\sin^{n-1} x \cos x}{n}\right|_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

i.e. $$I_n = \frac{n-1}{n} I_{n-2}$$

Case I. *When n is odd*

Similarly $$I_{n-2} = \frac{n-3}{n-2} I_{n-4},\quad I_{n-4} = \frac{n-5}{n-4} I_{n-6}$$

....

$$I_5 = \frac{4}{5} I_3,\ I_3 = \frac{2}{3} I_1 = \frac{2}{3}\int_0^{\pi/2} \sin x\,dx = \frac{2}{3}\left| -\cos x\right|_0^{\pi/2} = \frac{2}{3}.$$

Form these, we get $I_n = \dfrac{(n-1)(n-3)(n-5)\ldots 2}{n(n-2)(n-4)\ldots 3}$ ...(*ii*)

Case II. *When n is even*

We have $$I_{n-2} = \frac{n-3}{n-2} I_{n-4},\ I_{n-4} = \frac{n-5}{n-4} I_{n-6}$$

....

$$I_4 = \frac{3}{4} I_2,\ I_2 = \frac{1}{2} I_0 = \int_0^{\pi/2} \sin^0 x\,dx = \frac{1}{2}\int_0^{\pi/2} dx = \frac{1}{2}\cdot\frac{\pi}{2}.$$

Form these, we obtain $I_n = \dfrac{(n-1)(n-3)(n-5)\ldots 3\cdot 1}{n(n-2)(n-4)\ldots 4\cdot 2}\cdot\dfrac{\pi}{2}$ ...(*iii*)

Combining (*ii*) and (*iii*), we get the required result for $\int_0^{\pi/2} \sin^n x\,dx$.

Proceeding exactly as above, we get the result for $\int_0^{\pi/2} \cos^n x\,dx$.

Example 6.1. *Integrate* (*i*) $\int \sin^4 x\,dx$ (*ii*) $\int_0^{\pi/2} \cos^6 x\,dx$.

Solution. (*i*) We have the reduction formula

$$\int \sin^n x\,dx = \frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

Putting $n = 4, 2$ successively,

$$\int \sin^4 x\,dx = -\frac{\sin^3 x \cos x}{4} + \frac{3}{4}\int \sin^2 x\,dx \qquad \ldots(\alpha)$$

$$\int \sin^2 x\,dx = -\frac{\sin x \cos x}{2} + \frac{1}{2}\int (\sin x)^0\,dx$$

But $\int (\sin x)^0\,dx = \int dx = x. \quad \therefore \quad \int \sin^2 x\,dx = -\frac{\sin x \cos x}{2} + \frac{x}{2}$

Substituting this in (α), we get

$$\int \sin^4 x\,dx = -\frac{\sin^3 x \cos x}{4} + \frac{3}{4}\left(-\frac{\sin x \cos x}{2} + \frac{x}{2}\right)$$

(*ii*) We know that $\int_0^{\pi/2} \cos^n x\,dx = \frac{(n-1)(n-3)(n-5)\ldots}{n(n-2)(n-4)\ldots}\left(\frac{\pi}{2} \text{ if } n \textit{ is even}\right)$

Putting $n = 6$, we get

$$\int_0^{\pi/2} \cos^6 x\,dx = \frac{5.3.1}{6.4.2}\frac{\pi}{2} = \frac{5\pi}{16}.$$

Example 6.2. *Evaluate*

(*i*) $\int_0^a \frac{x^7\,dx}{\sqrt{(a^2-x^2)}}$ (*V.T.U., 2006*) (*ii*) $\int_0^\pi \frac{\sqrt{(1-\cos x)}}{1+\cos x}\sin^2 x\,dx$ (*iii*) $\int_0^\infty \frac{dx}{(a^2+x^2)^n}$.

Solution. (*i*) $\int_0^a \frac{x^7}{\sqrt{(a^2-x^2)}}\,dx$ | Put $x = a\sin\theta$, so that $dx = a\cos\theta\,d\theta$
Also when $x = 0, \theta = 0$, when $x = a, \theta = \pi/2$

$$= \int_0^{\pi/2} \frac{a^7 \sin^7\theta}{a\cos\theta}\cdot a\cos\theta\,d\theta = a^7 \int_0^{\pi/2} \sin^7\theta\,d\theta = a^7 \cdot \frac{6.4.2}{7.5.3.1} = \frac{16}{35}a^7$$

(*ii*) Putting $x = 2\theta$, we get

$$\int_0^\pi \frac{\sqrt{(1-\cos x)}}{1+\cos x}\sin^2 x\,dx = \int_0^{\pi/2} \frac{\sqrt{(1-\cos 2\theta)}}{1+\cos 2\theta}\sin^2 2\theta \cdot 2d\theta$$

$$= 2\int_0^{\pi/2} \frac{\sqrt{2}\sin\theta}{2\cos^2\theta}\cdot(2\sin\theta\cos\theta)^2\,d\theta = 4\sqrt{2}\int_0^{\pi/2}\sin^3\theta\,d\theta = 4\sqrt{2}\cdot\frac{2}{3} = \frac{8\sqrt{2}}{3}.$$

(*iii*) $\int_0^\infty \frac{dx}{(a^2+x^2)^n}$ | Put $x = a\tan\theta$, so that $dx = a\sec^2\theta\,d\theta$
Also when $x = 0, \theta = 0$, when $x = \infty, \theta = \pi/2$

$$= \int_0^{\pi/2} \frac{a\sec^2\theta\,d\theta}{a^{2n}\sec^{2n}\theta} = \frac{1}{a^{2n-1}}\int_0^{\pi/2}\cos^{2n-2}\theta\,d\theta = \frac{1}{a^{2n-1}}\cdot\frac{(2n-3)(2n-5)\ldots3.1}{(2n-2)(2n-4)\ldots4.2}\cdot\frac{\pi}{2}.$$

Example 6.3. *Evaluate* $\int_0^a \frac{x^n}{\sqrt{(a^2-x^2)}}\,dx$. *Hence find the value of* $\int_0^1 x^n \sin^{-1} x\,dx$.

Solution. Putting $x = a\sin\theta$, we get

$$\int_0^a \frac{x^n}{\sqrt{(a^2-x^2)}}\,dx = \int_0^{\pi/2}\frac{(a\sin\theta)^n}{a\cos\theta}(a\cos\theta)\,d\theta = a^n\int_0^{\pi/2}\sin^n\theta\,d\theta$$

$$\left.\begin{aligned} &= \frac{(n-1)(n-3)\ldots.2}{n(n-2)\ldots.3}a^n, \text{ if } n \text{ is odd} \\ &= \frac{(n-1)(n-3)\ldots1}{n(n-2)\ldots2}\cdot\frac{\pi}{2}a^n, \text{ if } n \text{ is even}\end{aligned}\right\} \quad \ldots(i)$$

Now integrating by parts, we have

$$\int_0^1 x^n\sin^{-1}x\,dx = \left|(\sin^{-1}x)\cdot\frac{x^{n+1}}{n+1}\right|_0^1 - \int_0^1\frac{x^{n+1}}{n+1}\frac{1}{\sqrt{(1-x^2)}}\,dx$$

$$= \frac{1}{(n+1)}\left[\frac{\pi}{2} - \int_0^1 \frac{x^{n+1}}{(1-x^2)}\right] \qquad \text{[Using } (i) \text{ p. 241]}$$

$$= \frac{1}{n+1}\left\{\frac{\pi}{2} - \frac{n(n-2)(n-4)\ldots 1}{(n+1)(n-1)(n-3)\ldots 2}\frac{\pi}{2}\right\} \qquad \text{when } n \text{ is odd}$$

$$= \frac{1}{n+1}\left\{\frac{\pi}{2} - \frac{n(n-2)(n-4)\ldots 2}{(n+1)(n-1)(n-3)\ldots 3}\right\} \qquad \text{when } n \text{ is even}$$

Evaluate 6.4. *Evaluate $I_n = \int_0^a (a^2 - x^2)^n\, dx$ where n is a positive integer. Hence show that*

$$I_n = \frac{2n}{2n+1} a^2 I_{n-1}.$$

Solution. Putting $n = a \sin\theta$, we get

$$I_n = \int_0^a (a^2 - x^2)^n\, dx = \int_0^{\pi/2} (a^2 - a^2 \sin^2\theta)^n\, a\cos\theta\, d\theta = a^{2n+1}\int_0^{\pi/2} \cos^{2n+1}\theta\, d\theta$$

$$= a^{2n+1}\cdot \frac{(2n)(2n-2)(2n-4)\ldots 4.2}{(2n+1)(2n-1)(2n-3)\ldots 5.3} \qquad [\because \ (2n+1) \text{ is always odd}]$$

Now replacing n by $n-1$, we get

$$I_{n-1} = a^{2n-1}\frac{(2n-2)(2n-4)\ldots 4.2}{(2n-1)(2n-3)\ldots 5.3} \quad \therefore \quad \frac{I_n}{I_{n-1}} = a^2\cdot\frac{2n}{2n+1} \quad \text{or} \quad I_n = \frac{2n}{2n+1}a^2 I_{n-1}.$$

which is the second desired result.

6.3 (1) REDUCTION FORMULAE for $\int \sin^m x \cos^n x\, dx$

$$\int \sin^m x \cos^n x\, dx = \int \sin^{m-1} x \cdot \cos^n x \cdot \sin x\, dx \qquad \text{[Integrate by parts]}$$

$$= \sin^{m-1} x \cdot \left(\frac{-\cos^{n+1} x}{n+1}\right) - \int (m-1)\sin^{m-2} x \cos x \cdot \left(-\frac{\cos^{n+1} x}{n+1}\right) dx.$$

$$= -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{m+1}\int \sin^{m-2} x (1 - \sin^2 x)\cos^n x\, dx$$

$$= -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1}\int \sin^{m-2} x \cos^n x\, dx - \frac{m-1}{n+1}\int \sin^m x \cos^n x\, dx$$

Transposing the last term to the left and dividing by $1 + (m-1)/(n+1)$, *i.e.*, $(m+n)/(n+1)$, we obtain the reduction formula

$$\int \sin^m x \cos^n x\, dx = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2} x \cos^n x\, dx \qquad \ldots(1)$$

Obs. To integrate $\int \sin^m x \cos^n x\, dx$,

(*a*) *when m is odd,* put $\cos x = t$

when n is odd, put $\sin x = t$

(*b*) *when m and n both are even integers,* express the integrand as a series of cosines of multiple angles and integrate term by term if m and n are small, *otherwise* use the method of reduction formulae.

(2) To show that

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{(m-1)(m-3)\ldots\times(n-1)(n-3)\ldots}{(m+n)(m+n-2)(m+n-4)\ldots} \times \left(\frac{\pi}{2}, \textit{ only if both m and n are even}\right)$$

From (*i*), we have

$$I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x \, dx = \left| -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} \right|_0^{\pi/2} + \frac{m-1}{m+n} \int_0^{\pi/2} \sin^{m-2} x \cos^n x \, dx$$

i.e.,
$$I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}.$$

Case I. *When m is odd*

Similarly, $I_{m-2,n} = \dfrac{m-3}{m+n-2} I_{m-4,n}, \; I_{m-4,n} = \dfrac{m-5}{m+n-4} I_{m-6,n}$

....

$$I_{5,n} = \frac{4}{n+5} I_{3,n}$$

Finally $I_{3,n} = \dfrac{2}{n+3} I_{1,n} = \dfrac{2}{n+3} \displaystyle\int_0^{\pi/2} \sin x \cos^n x \, dx$

$$= \frac{2}{n+3} \left| -\frac{\cos^{n+1} x}{n+1} \right|_0^{\pi/2} = \frac{2}{(n+3)(n+1)} \quad ...(ii)$$

From these, we obtain

$$I_{m,n} = \frac{(m-1)(m-3)(m-5) \dots 4.2}{(m+n)(m+n-2)(m+n-4) \dots (n+3)(n+1)}$$

Case II. *When m is even*

We have, $I_{m-2,n} = \dfrac{m-3}{m+n-2} I_{m-4,n}, \quad I_{m-4,n} = \dfrac{m-5}{m+n-4} I_{m-6,n}$

....

$$I_{4,n} = \frac{3}{n+4} I_{2,n}, I_{2,n} = \frac{1}{n+2} I_{0,n} = \frac{1}{n+2} \int_0^{\pi/2} \cos^n x \, dx$$

From these, we have $I_{m,n} = \dfrac{(m-1)(m-3)(m-5) \dots 1}{(m+n)(m+n-2)(m+n-4) \dots (n+2)} \displaystyle\int_0^{\pi/2} \cos^n x \, dx$

$$= \frac{(m-1)(m-3) \dots 1}{(m+n)(m+n-2) \dots (n+2)} \cdot \frac{(n-1)(n-3) \dots}{n(n-2) \dots} \times (\pi/2 \text{ only if } n \text{ is even}) \quad ...(iii)$$

Combining (*ii*) and (*iii*), we get the desired result.

Example 6.5. *Integrate* (*i*) $\int \sin^4 x \cos^2 x \, dx$ *(Raipur, 2005)*

(*ii*) $\displaystyle\int_0^{\infty} \frac{t^6}{(1+t^2)^7} dt$ (*iii*) $\displaystyle\int_0^{\infty} \frac{x^2}{(1+x^2)^{7/2}} dx$ *(V.T.U., 2010 S)*

Solution. (*i*) Taking $n = 2$, in (*i*) of page 241, we have the reduction formula :

$$\int \sin^m x \cos^2 x \, dx = \frac{\sin^{m-1} x \cos^3 x}{m+2} + \frac{m-1}{m+2} \int \sin^{m-2} x \cos^2 x \, dx$$

Putting $m = 4, 2$ successively,

$$\int \sin^4 x \cos^2 x \, dx = -\frac{\sin^3 x \cos^3 x}{6} + \frac{3}{6} \int \sin^2 x \cos^2 x \, dx \quad ...(1)$$

$$\int \sin^2 x \cos^2 x \, dx = -\frac{\sin x \cos^3 x}{4} + \frac{1}{4} \int \cos^2 x \, dx$$

But $\displaystyle\int \cos^2 x \, dx = \frac{1}{2} \int (1 + \cos 2x) \, dx = \frac{1}{2}\left(x + \frac{1}{2} \sin 2x\right)$

$$\therefore \qquad \int \sin^2 x \cos^2 x\, dx = -\frac{\sin x \cos^3 x}{4} + \frac{1}{16}(2x + \sin 2x)$$

Substituting this in (1), we get

$$\int \sin^4 x \cos^2 x\, dx = -\frac{\sin^3 x \cos^3 x}{6} + \frac{1}{2}\left\{-\frac{\sin x \cos^3 x}{4} + \frac{1}{16}(2x + \sin 2x)\right\}$$

(*ii*) Putting $t = \tan\theta$, so that

$$\int_0^\infty \frac{t^6}{(1+t^2)^7}\, dt = \int_0^{\pi/2} \frac{\tan^6\theta}{\sec^{14}\theta} \sec^2\theta\, d\theta = \int_0^{\pi/2} \sin^6\theta \cos^6\theta\, d\theta = \frac{5\cdot 3\cdot 1 \times 5\cdot 3\cdot 1}{12\cdot 10\cdot 8\cdot 6\cdot 4\cdot 2} \times \frac{\pi}{2} = \frac{5\pi}{2048}.$$

(*iii*) Putting $x = \tan\theta$, so that

$$\int_0^\infty \frac{x^2}{(1+x^2)^{7/2}}\, dx = \int_0^{\pi/2} \frac{\tan^2\theta}{\sec^7\theta} \sec^2\theta\, d\theta = \int_0^{\pi/2} \sin^2\theta \cos^3\theta\, d\theta = \frac{1.2}{53.1} = \frac{2}{15}.$$

Example 6.6. *Evaluate : (i)* $\int_0^{\pi/6} \cos^4 3\theta \sin^3 6\theta\, d\theta$ *(V.T.U., 2003 S)*

(ii) $\int_0^1 x^4(1-x^2)^{3/2}\, dx$ *(iii)* $\int_0^{2a} x^2\sqrt{(2ax - x^2)}\, dx.$ *(V.T.U., 2010)*

Solution. (*i*) $\int_0^{\pi/6} \cos^4 3\theta \sin^3 6\theta\, d\theta = \int_0^{\pi/6} \cos^4 3\theta\,(2\sin 3\theta \cos 3\theta)^3\, d\theta$

$$= 8\int_0^{\pi/6} \sin^3 3\theta \cos^7 3\theta\, d\theta$$

Put $3\theta = x$ so that $3d\theta = dx$
Also when $\theta = 0, x = 0$; when $\theta = \pi/6, x = \pi/2$.

$$= \frac{8}{3}\int_0^{\pi/2} \sin^3 x \cos^7 x\, dx$$

$$= \frac{8}{3}\cdot\frac{2\times 6\cdot 4\cdot 2}{10\cdot 8\cdot 6\cdot 4\cdot 2} = \frac{1}{15}.$$

(*ii*) $\int_0^1 x^4(1-x^2)^{3/2}\, dx$

Put $x = \sin t$ so that $dx = \cos t\, dt$
When $x = 0, t = 0$; when $x = 1, t = \pi/2$

$$= \int_0^{\pi/2} \sin^4 t\,(\cos^2 t)^{3/2}\cdot\cos t\, dt = \int_0^{\pi/2} \sin^4 t \cos^4 t\, dt$$

$$= \frac{3\cdot 1\times 3\cdot 1}{8\cdot 6\cdot 4\cdot 2}\cdot\frac{\pi}{2} = \frac{3\pi}{256}.$$

(*iii*) $\int_0^{2a} x^2\sqrt{(2ax - x^2)}\, dx$

$$= \int_0^{\pi/2} x^{5/2}\sqrt{(2a - x)}\, dx$$

Put $x = 2a\sin^2\theta$
$\therefore\ dx = 4a\sin\theta\cos\theta\, d\theta$

$$= \int_0^{\pi/2} (2a\sin^2\theta)^{5/2}\sqrt{(2a)}\cos\theta\cdot 4a\sin\theta\cos\theta\, d\theta$$

$$= 2^5 a^4 \int_0^{\pi/2} \sin^6\theta\cos^2\theta\, d\theta = 32\, a^4\cdot\frac{5\cdot 3\cdot 1\times 1}{8\cdot 6\cdot 4\cdot 2}\cdot\frac{\pi}{2} = \frac{5\pi a^4}{8}.$$

PROBLEMS 6.1

Evaluate :

1. (*i*) $\int_0^{\frac{\pi}{2}} \cos^9 x\, dx$ (*ii*) $\int_0^{\pi/6} \sin^5 3\theta\, d\theta$
2. (*i*) $\int_0^1 \frac{x^9}{\sqrt{(1-x^2)}}\, dx$ (*ii*) $\int_0^1 x^5 \sin^{-1} x\, dx$
3. (*i*) $\int_0^\infty \frac{dx}{(1+x^2)^n}\ (n > 1)$ (*V.T.U., 2008 S*) (*ii*) $\int_0^{\pi/4} \sin^2 x \cos^4 x\, dx.$ (*J.N.T.U., 2003*)

4. If $I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x \, dx$ $(m > 0, n > 0)$, show that $I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}$.

Hence evaluate $\int_0^{\pi/2} \sin^4 x \cos^8 x \, dx$

Evaluate :

5. (i) $\int_0^{\pi/2} \sin^4 x \cos^6 x \, dx$ *(Cochin, 2005)* (ii) $\int_0^{\pi/2} \sin^{15} x \cos^3 x \, dx$

6. (i) $\int_0^1 x^6 \sqrt{(1-x^2)} \, dx$ (ii) $\int_0^{\pi/2} \cos^4 3\theta \sin^3 6\theta \, d\theta$

7. (i) $\int_0^{2a} x^{7/2} (2a-x)^{-1/2} \, dx$ (ii) $\int_0^{2a} \frac{x^3 \, dx}{\sqrt{(2ax-x^2)}}$ *(Madras, 2000 S)*

8. (i) $\int_0^2 x^{5/2} \sqrt{(2-x)} \, dx$ (ii) $\int_0^4 x^3 \sqrt{(4x-x^2)} \, dx$ *(V.T.U., 2004)*

9. If $I_n = \int x^n \sqrt{(a-x)} \, dx$, prove that $(2n+3) I_n = 2an \, I_{n-1} - 2x^n (a-x)^{3/2}$ *(Marathwada, 2008)*

10. If n is a positive integer, show that $\int_0^{2a} x^n \sqrt{(2ax-x^2)} \, dx = \frac{2n+1}{(n+2)\, n!} \cdot \frac{a^{n+2}}{2n} \pi$ *(V.T.U., 2007)*

6.4 REDUCTION FORMULAE for (a) $\int \tan^n x \, dx$ (b) $\int \cot^n x \, dx$

(a) Let
$$I_n = \int \tan^n x \, dx = \int \tan^{n-2} x \cdot \tan^2 x \, dx = \int \tan^{n-2} x \cdot (\sec^2 x - 1) \, dx$$
$$= \int \tan^{n-2} x \sec^2 x \, dx - \int \tan^{n-2} x \, dx$$

Thus, $I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$ which is the required reduction formula.

(b) Let
$$I_n = \int \cot^n x \, dx = \int \cot^{n-2} x \cot^2 x \, dx = \int \cot^{n-2} x \, (\operatorname{cosec}^2 x - 1) \, dx$$
$$= \int \cot^{n-2} x \operatorname{cosec}^2 x \, dx - \int \cot^{n-2} x \, dx$$

Thus
$$I_n = -\frac{\cot^{n-1} x}{n-1} - I_{n-2}$$
which is the required reduction formula.

Example 6.7. *Evaluate (i) $\tan^5 x \, dx$ (ii) $\int \cot^6 x \, dx$.*

Solution. (i) Putting $n = 5, 3$ successively in the reduction formula for $\int \tan^n x \, dx$, we get

$$I_5 = \frac{1}{4} \tan^4 x - I_3 \, ; \quad I_3 = \frac{1}{2} \tan^2 x - I_1$$

Thus
$$I_5 = \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x + I_1$$

i.e.,
$$\int \tan^5 x \, dx = \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x + \int \tan x \, dx = \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x - \log \cos x.$$

(ii) Putting $n = 6, 4, 2$ successively in the reduction formula for $\int \cot^n x \, dx$, we get

$$I_6 = -\frac{1}{5} \cot^5 x - I_4 \, ; I_4 = -\frac{1}{3} \cot^3 x - I_2 \, ; I_2 = -\cot x - I_0$$

Thus
$$I_6 = \frac{-1}{5} \cot^5 x + \frac{1}{3} \cot^3 x - \cot x - \int dx$$

i.e., $$\int \cot^6 x\,dx = -\frac{1}{5}\cot^5 x + \frac{1}{3}\cot^3 x - \cot x - x.$$

Example 6.8. *If* $I_n = \int_0^{\pi/4} \tan^n \theta\,d\theta$, *prove that* $n\,(I_{n-1} + I_{n+1}) = 1.$ *(V.T.U., 2003)*

Solution. The reduction formula for $\int_0^{\pi/4} \tan^n \theta\,d\theta$ is

$$I_n = \frac{1}{n-1}\Big|\tan^n x\Big|_0^{\pi/4} - I_{n-2} = \frac{1}{n-1} - I_{n-2} \quad \text{or} \quad I_n + I_{n-2} = \frac{1}{n-1}$$

Changing n to $n+1$, we obtain

$$I_{n+1} + I_{n-1} = \frac{1}{(n+1)} \quad \text{or} \quad (n+1)(I_{n+1} + I_{n-1}) = 1.$$

6.5 REDUCTION FORMULAE for (a) $\int \sec^n x\,dx$ (b) $\int \operatorname{cosec}^n x\,dx$

(*a*) Let $$I_n = \int \sec^n x\,dx = \int \sec^{n-2} x\,.\,\sec^2 x\,dx$$

Integrating by parts, we have

$$I_n = \sec^{n-2}x.\,\tan x - \int \{(n-2)\sec^{n-3}x \cdot \sec x \tan x\}\tan x\,dx$$
$$= \sec^{n-2}x \tan x - (n-2)\int \sec^{n-2}x \cdot \tan^2 x\,dx$$
$$= \sec^{n-2}x \tan x - (n-2)\int \sec^{n-2}x \cdot (\sec^2 x - 1)\,dx$$
$$= \sec^{n-2}x \tan x - (n-2)\,I_n + (n-2)\,I_{n-2}$$

Transposing, we have

$$(n-1)I_n = \sec^{n-2}x \tan x + (n-2)I_{n-2}$$

Thus $$I_n = \frac{\sec^{n-2}x \tan x}{n-1} + \frac{n-2}{n-1}I_{n-2}$$ which is the desired reduction formula.

(*b*) Let $$I_n = \int \operatorname{cosec}^n x\,dx = \int \operatorname{cosec}^{n-2} x \cdot \operatorname{cosec}^2 x\,dx$$

Integrating by parts, we have

$$I_n = \operatorname{cosec}^{n-2}x.\,(-\cot x) - \int [(n-2)\operatorname{cosec}^{n-3}x \cdot (-\operatorname{cosec} x \cot x) \cdot (-\cot x)\,dx$$
$$= -\cot x \operatorname{cosec}^{n-2}x - (n-2)\int \operatorname{cosec}^{n-2}x\,(\operatorname{cosec}^2 x - 1)\,dx$$
$$= -\cot x \operatorname{cosec}^{n-2}x - (n-2)I_n + (n-2)I_{n-2}$$

or $$[1 + (n-2)]I_n = -\cot x \operatorname{cosec}^{n-2}x + (n-2)I_{n-2}$$

Thus $$I_n = -\frac{\cot x \operatorname{cosec}^{n-2}x}{n-1} + \frac{n-2}{n-1}I_{n-2}$$

which is the required reduction formula.

Example 6.9. *Evaluate* (*i*) $\int_0^{\pi/4} \sec^4 x\,dx$ (*ii*) $\int_{\pi/3}^{\pi/2} \operatorname{cosec}^3 \theta\,d\theta.$ *(V.T.U., 2008)*

Solution. (*i*) Putting $n = 4$ in the reduction formula for $\int \sec^n x\,dx$, we get $I_4 = \dfrac{\sec^2 x \tan x}{3} + \dfrac{2}{3}I_2$

$$\therefore \quad \int_0^{\pi/4} \sec^4 x\,dx = \left|\frac{\sec^2 x \tan x}{3}\right|_0^{\pi/4} + \frac{2}{3}\int_0^{\pi/4} \sec^2 x\,dx$$

$$= \frac{2}{3} + \frac{2}{3}\Big|\tan x\Big|_0^{\pi/4} = \frac{2}{3}(1+1) = 4/3.$$

(*ii*) Putting $n = 3$ in the reduction formula for $\int \operatorname{cosec}^n x\, dx$, we get

$$I_3 = -\frac{1}{2}\cot x \operatorname{cosec} x + \frac{1}{2} I_1$$

$$\therefore \quad \int_{\pi/3}^{\pi/2} \operatorname{cosec}^3 x\, dx = -\frac{1}{2}\Big| \cot x \operatorname{cosec} x \Big|_{\pi/3}^{\pi/2} + \frac{1}{2}\int_{\pi/3}^{\pi/2} \operatorname{cosec} x\, dx$$

$$= -\frac{1}{2}\left(0 - \frac{2}{3}\right) + \frac{1}{2}\Big| \log(\operatorname{cosec} x - \cot x)\Big|_{\pi/3}^{\pi/2}$$

$$= \frac{1}{3} + \frac{1}{2}\left[\log 1 - \log\left(\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right)\right] = \frac{1}{3} + \frac{1}{4}\log 3.$$

PROBLEMS 6.2

1. Evaluate (*i*) $\int \tan^6 x\, dx$ (*V.T.U., 2007*) (*ii*) $\int \cot^5 x\, dx$.
2. Show that $\int_0^{\pi/4} \tan^7 x\, dx = \frac{1}{12}(5 - 6\log 2)$
3. If $I_n = \int_0^{\pi/4} \tan^n x\, dx$, prove that $(n-1)(I_n + I_{n-2}) = 1$. (*V.T.U., 2009*)
   Hence evaluate I_5. (*Madras, 2000*)
4. If $I_n = \int_{\pi/4}^{\pi/2} \cot^n \theta\, d\theta\ (n > 2)$, prove that $I_n = \frac{1}{n-1} - I_{n-1}$. Hence evaluate I_4. (*Marathwada, 2008*)
5. Obtain the reduction formula for $\int_0^{\pi/4} \sec^n \theta\, d\theta$. (*V.T.U., 2010 S*)
6. Evaluate (*i*) $\int \sec^6 \theta\, d\theta$ (*ii*) $\int_{\pi/6}^{\pi/2} \operatorname{cosec}^5 d\theta$.
7. Evaluate $\int_0^a (a^2 + x^2)^{5/2}\, dx$.
8. If $I_n = \int \frac{t^n}{1+t^2}\, dt$, show that $I_{n+2} = \frac{t^{n+1}}{n+1} - I_n$. Hence evaluate I_6.

6.6 REDUCTION FORMULAE for

(*a*) $\int \mathbf{x^n\, e^{ax}\, dx}$ **(*b*)** $\int \mathbf{x^m (\log x)^n\, dx}$.

(*a*) Let $\quad I_n = \int x^n e^{ax}\, dx$

Integrating by parts, we have

$$I_n = x^n \cdot \frac{e^{ax}}{a} - \int n x^{n-1} \cdot \frac{e^{ax}}{a}\, dx$$

or $\quad I_n = \dfrac{x^n e^{ax}}{a} - \dfrac{n}{a} I_{n-1}$ which is the required reduction formula. (*Madras, 2006*)

(*b*) Let $\quad I_{m,n} = \int x^m (\log x)^n\, dx = \int (\log x)^n \,.\, x^m\, dx$

Integrating by parts, we have

$$I_{m,n} = (\log x)^n \cdot \frac{x^{m+1}}{m+1} - \int n(\log x)^{n-1} \cdot \frac{1}{x} \cdot \frac{x^{m+1}}{m+1}\, dx$$

$$= \frac{x^{m+1}}{m+1}(\log x)^n - \frac{n}{m+1}\int x^m (\log x)^{n-1}\, dx \quad \text{or} \quad I_{m,n} = \frac{x^{m+1}}{m+1}(\log x)^n - \frac{n}{m+1} I_{m,n-1}$$

which is the desired reduction formula.

6.7 REDUCTION FORMULAE for

(*a*) $\int \mathbf{x^n \sin mx\, dx}$ (*b*) $\int \mathbf{x^n \cos mx\, dx}$ (*c*) $\int \mathbf{\cos^m x \sin nx\, dx}$

(*a*) Let $I_n = \int x^n \sin mx\, dx$

Integrating by parts, we get

$$I_n = x^n\left(\frac{-\cos mx}{m}\right) - \int n x^{n-1}\left(\frac{-\cos mx}{m}\right) dx$$

$$= -\frac{x^n \cos mx}{m} + \frac{n}{m}\int x^{n-1} \cos mx\, dx \qquad \text{[Again integrate by parts]}$$

$$= -\frac{x^n \cos mx}{m} + \frac{n}{m}\left\{x^{n-1}\cdot\frac{\sin mx}{m} - \left\{\int (n-1)x^{n-2}\cdot\frac{\sin mx}{m}\, dx\right\}\right\}$$

or $$I_n = -\frac{x^n \cos mx}{m} + \frac{n}{m^2} x^{n-1}\sin mx - \frac{n(n-1)}{m^2} I_{n-2}$$

which is the desired reduction formula. (*Madras, 2003*)

(*b*) Let $I_n = \int x^n \cos mx\, dx$

Integrating twice by parts as above, we get

$$I_n = \frac{x^n \sin mx}{m} + \frac{n}{m^2} x^{n-1}\cos mx - \frac{n(n-1)}{m^2} I_{n-2}$$

(*c*) Let $I_{m,n} = \int \cos^m x \sin nx\, dx$

Integrating by parts,

$$I_{m,n} = -\cos^m x\cdot\frac{\cos nx}{n} - \int m\cos^{m-1} x(-\sin x)\cdot\left(\frac{-\cos nx}{n}\right)dx$$

$$= -\frac{1}{n}\cos^m x\cos nx - \frac{m}{n}\int \cos^{m-1} x\,.\cos nx \sin x\, dx$$

$$\left[\begin{array}{l}\because \ \sin(n-1)x = \sin nx\cos x - \cos nx\sin x\\ \text{or } \cos nx\sin x = \sin nx\cos x - \sin(n-1)x\end{array}\right.$$

$$= -\frac{1}{n}\cos^m x\cos nx - \frac{m}{n}\int \cos^{m-1} x\,\{\sin nx\cos x - \sin(n-1)x\}\, dx$$

$$= -\frac{1}{n}\cos^m x\cos nx - \frac{m}{n}(I_{m,n} - I_{m-1,n-1})$$

Transposing, we get

$$\left(1+\frac{m}{n}\right) I_{m,n} = -\frac{1}{n}\cos^m x\cos nx + \frac{m}{n} I_{m-1,n-1}$$

or $$I_{m,n} = -\frac{\cos^m x\cos nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$$

which is the desired reduction formula.

Example 6.10. *Show that* $\int_0^{\pi/2} \cos^m x\cos nx\, dx = \frac{m}{m+n}\int_0^{\pi/2}\cos^{m-1} x\cos(n-1)x\, dx$

Hence deduce that $\int_0^{\pi/2}\cos^n x\cos nx\, dx = \frac{\pi}{2^{n+1}}$. (*S.V.T.U., 2008*)

Solution. Let $I_{m,n} = \int_0^{\pi/2}\cos^m x\cos nx\, dx$

Integrating by parts

$$I_{m,n} = \left|\cos^m x\cdot\frac{\sin nx}{n}\right|_0^{\pi/2} - \int_0^{\pi/2} m\cos^{m-1} x(-\sin x)\times\frac{\sin nx}{n}\, dx$$

$$= \frac{m}{n}\int_0^{\pi/2} \cos^{m-1} x \sin nx \sin x \, dx \qquad \left[\because \cos(n-1)x = \cos nx \cos x + \sin nx \sin x \text{ or } \sin nx \sin x = \cos(n-1)x - \cos nx \cos x\right.$$

$$= \frac{m}{n}\int_0^{\pi/2} \cos^{m-1} x\,[\cos(n-1)x - \cos nx \cos x]\, dx = \frac{m}{n}(I_{m-1,\,n-1} - I_{m,\,n})$$

Transposing and dividing by $(1 + m/n)$, we get

$$I_{m,\,n} = \frac{m}{m+n} I_{m-1,\,n-1}$$

which is the required result.

Putting $\quad m = n,\ I_n \left(= \int_0^{\pi/2} \cos^n x \cos nx \, dx\right) = \frac{1}{2} I_{n-1}$

Changing n to $n - 1$,

$$I_{n-1} = \frac{1}{2} I_{n-2}$$

$$\therefore \quad I_n = \frac{1}{2}\left(\frac{1}{2} I_{n-2}\right) = \frac{1}{2^2} I_{n-2} = \frac{1}{2^3} I_{n-3} \ldots\ldots = \frac{1}{2^n} I_{n-n} = \frac{1}{2^n}\cdot\int_0^{\pi/2} (\cos x)^0 \, dx$$

Hence $\quad I_n = \frac{1}{2^n}\cdot\frac{\pi}{2} = \frac{\pi}{2^{n+1}}.$

Example 6.11. *Find a reduction formula for* $\int e^{ax} \sin x \, dx$. *Hence evaluate* $\int e^x \sin^3 x \, dx$.

Solution. Let $I_n = \int e^{ax} \sin^n x \, dx = \int \underset{I}{\sin^n x} \cdot \underset{II}{e^{ax}} \, dx$

Integrating by parts,

$$I_n = \sin^n x \cdot \frac{e^{ax}}{a} - \int (n \sin^{n-1} x \cos x)\cdot\frac{e^{ax}}{a}\, dx$$

$$= \frac{e^{ax}\sin^n x}{a} - \frac{n}{a}\int (\sin^{n-1} x \cos x)\,.\, e^{ax} \, dx \qquad \text{[Again integrating by parts]}$$

$$= \frac{e^{ax}\sin^n x}{a} - \frac{n}{a}\left[\sin^{n-1} x \cos x \cdot \frac{e^{ax}}{a} - \int \{(n-1)\sin^{n-2} x \times \cos x \cdot \cos x + \sin^{n-1} x(-\sin x)\}\frac{e^{ax}}{a}\, dx\right]$$

$$= \frac{e^{ax}\sin^{n-1} x}{a^2}(a \sin x - n \cos x) + \frac{n}{a^2}\int [(n-1)\sin^{n-2} x \times (1 - \sin^2 x) - \sin^n x]\, e^{ax}\, dx$$

$$= \frac{e^{ax}\sin^{n-1} x}{a}(a \sin x - n \cos x) + \frac{n(n-1)}{a^2} I_{n-2} - \frac{n^2}{a^2} I_n$$

Transposing and dividing by $(1 + n^2/a^2)$, we get

$$I_n = \frac{e^{ax}\sin^{n-1} x\,(a \sin x - n \cos x)}{a^2 + n^2} + \frac{n(n-1)}{a^2 + n^2} I_{n-2}$$

which is the required reduction formula.

Putting $a = 1$ and $n = 3$, we get

$$I_3 = \frac{e^x \sin^2 x\,(\sin x - 3\cos x)}{1^2 + 9} + \frac{3\,.\,2}{1^2 + 9} I_1$$

But $\quad I_1 = \int e^x \sin x \, dx = \frac{e^{ax}}{\sqrt{2}} \sin(x - \tan^{-1} 1).$

$$\therefore \quad I_3 = \frac{e^x \sin^2 x\,(\sin x - 3\cos x)}{10} + \frac{3}{5}\cdot\frac{e^{ax}}{\sqrt{2}} \sin(x - \pi/4).$$

PROBLEMS 6.3

1. If $I_n = \int x^n e^x \, dx$, show that $I_n + n I_{n-1} = x^n e^x$. Hence find I_4. *(Madras, 2000)*

2. If $u_n = \int_0^a x^n e^{-x} \, dx$, prove that $u_n - (n+a) u_{n-1} + a(n-1) u_{n-2} = 0$. *(Madras, 2003)*

3. Obtain a reduction formula for $\int x^m (\log x)^n \, dx$. Hence evaluate $\int_0^1 x^5 (\log x)^3 \, dx$. *(S.V.T.U., 2009 ; Bhillai, 2005)*

4. If n is a positive integer, show that $\int_0^1 x^m (\log x)^n \, dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$, $m > -1$.

5. If $I_n = \int_0^{\pi/2} x \sin^n x \, dx$ $(n > 1)$, prove that $n^2 I_n = n(n-1) I_{n-2} + 1$. Hence evaluate I_5.

6. If $I_n = \int_0^{\pi/2} x \cos^n x \, dx$ $(n > 1)$, prove that $I_n = \frac{n-1}{n} I_{n-2} - \frac{1}{n^2}$. Hence evaluate I_4.

7. If $u_n = \int_0^{\pi/2} x^n \sin x \, dx$, $(n > 1)$, prove that $u_n + n(n-1) u_{n-2} = n(\pi/2)^{n-1}$. Hence evaluate u_2. *(Madras, 2000 S)*

8. If $I_n = \int x^n \sin ax \, dx$, show that $a^2 I_n = -ax^n \cos ax + nx^{n-1} \sin ax - n(n-1) I_{n-2}$. *(Marathwada, 2008)*

9. Prove that $\int_0^{\pi/2} \cos^{n-2} x \sin nx \, dx = \frac{1}{n-1}$, $n > 1$.

10. If $I_{m,n} = \int_0^{\pi/2} \cos^m x \cos nx \, dx$, prove that $I_{m,n} = \frac{m(m-1)}{m^2 - n^2} I_{m-2,n}$

11. Find a reduction formula for $\int e^{ax} \cos^n x \, dx$. Hence evaluate $\int_0^{\pi/2} e^{2x} \cos^3 x \, dx$.

12. Obtain a reduction formula for $I_m = \int_0^\infty e^{-x} \sin^m x \, dx$ where $m \geq 2$ in the form
$(1 + m^2) I_m = m(m-1) I_{m-2}$. Hence evaluate I_4. *(Gorakhpur, 1999)*

6.8 DEFINITE INTEGRALS

Property I. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

(*i.e., the value of a definite integral depends on the limits and not on the variable of integration*).

Let $\int f(x)\,dx = \phi(x)$; $\therefore \int_a^b f(x)\,dx = \phi(b) - \phi(a)$.

Then $\int f(t)\,dt = \phi(t)$; $\therefore \int_a^b f(t)\,dt = \phi(b) - \phi(a)$.

Hence the result.

Property II. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(*i.e., the interchange of limits changes the sign of the integral*).

Let $\int f(x)\,dx = \phi(x)$; $\therefore \int_a^b f(x)\,dx = \phi(b) - \phi(a)$

and $-\int_b^a f(x)\,dx = -\left| \phi(x) \right|_b^a = -[\phi(a) - \phi(b)] = \phi(b) - \phi(a)$.

Hence the result.

Property III. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Let $\int f(x)\,dx = \phi(x)$, so that $\int_a^b f(x)\,dx = \phi(b) - \phi(a)$ ...(1)

Also
$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = |\,\phi(x)\,|_a^c + |\,\phi(x)\,|_c^b$$
$$= [\phi(c) - \phi(a)] + [\phi(b) - \phi(c)] = \phi(b) - \phi(a) \qquad ...(2)$$

Hence the result follows from (1) and (2).

Property IV. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Put $x = a - t$, so that $dx = -dt$. Also when $x = 0, t = a$; when $x = a, t = 0$.

$$\therefore \quad \int_0^a f(x)\,dx = -\int_a^0 f(a-t)\,dt = \int_0^a f(a-t)\,dt = \int_0^a f(a-x)\,dx \qquad \text{[Prop. II]}$$

Example 6.12. *Evaluate* $\int_0^{\pi/2} \frac{\sqrt{(\sin x)}}{\sqrt{(\sin x)} + \sqrt{(\cos x)}}\,dx$.

Solution. Let
$$I = \int_0^{\pi/2} \frac{\sqrt{(\sin x)}}{\sqrt{(\sin x)} + \sqrt{(\cos x)}}\,dx$$

Then
$$I = \int_0^{\pi/2} \frac{\sqrt{\left[\sin\left(\frac{1}{2}\pi - x\right)\right]}}{\sqrt{\left[\sin\left(\frac{1}{2}\pi - x\right)\right]} + \sqrt{\left[\cos\left(\frac{1}{2}\pi - x\right)\right]}}\,dx \qquad \text{[Prop. IV]}$$
$$= \int_0^{\pi/2} \frac{\sqrt{(\cos x)}}{\sqrt{(\cos x)} + \sqrt{(\sin x)}}\,dx$$

Adding
$$2I = \int_0^{\pi/2} \frac{\sqrt{(\sin x)} + \sqrt{(\cos x)}}{\sqrt{(\sin x)} + \sqrt{(\cos x)}}\,dx = \int_0^{\pi/2} dx = |\,x\,|_0^{\pi/2} = \frac{\pi}{2}.$$

Hence $I = \pi/4$.

Example 6.13. *Evaluate* $\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx$. *(Cochin, 2005)*

Solution. Let
$$I = \int_0^1 \frac{\log(1+x)}{1+x^2}\,dx \qquad \left[\begin{array}{l}\text{Put } x = \tan\theta \text{ so that } dx = \sec^2\theta\,d\theta \\ \text{When } x = 0, \theta = 0 \text{ ; when } x = 1, \theta = \pi/4\end{array}\right.$$
$$= \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{1+\tan^2\theta}\,.\sec^2\theta\,d\theta = \int_0^{\pi/4} \log(1+\tan\theta)\,d\theta$$
$$= \int_0^{\pi/4} \log\left[1 + \tan\left(\frac{\pi}{4} - \theta\right)\right]d\theta = \int_0^{\pi/4} \log\left(1 + \frac{1-\tan\theta}{1+\tan\theta}\right)d\theta \qquad \text{[Prop. IV]}$$
$$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right)d\theta = \log 2\int_0^{\pi/4} d\theta - I$$

Transposing, $2I = \log 2 \cdot |\,\theta\,|_0^{\pi/4} = \frac{\pi}{4}\log 2$. Hence $I = \frac{\pi}{8}\log 2$.

Example 6.14. *Evaluate* $\int_0^{\pi} \frac{x\sin^3 x}{1+\cos^2 x}\,dx$. *(Madras, 2006)*

Solution. Let
$$I = \int_0^{\pi} \frac{x\sin^3 x}{1+\cos^2 x}\,dx$$

Then $$I = \int_0^\pi \frac{(\pi - x)\sin^3(\pi - x)}{1 + \cos^2(\pi - x)}\,dx$$ [Prop. IV]

$$= \int_0^\pi \frac{(\pi - x)\sin^3 x}{1 + \cos^2 x}\,dx = \pi \int_0^\pi \frac{\sin^3 x}{1 + \cos^2 x}\,dx - I$$

Transposing, $$2I = \pi \int_0^\pi \frac{\sin^3 x}{1 + \cos^2 x}\,dx$$

$$= -\pi \int_1^{-1} (1 - t^2)\frac{dt}{1 + t^2}$$ | Put $\cos x = t$ so that $-\sin x\,dx = dt$ When $x = 0, t = 1$; When $x = \pi, t = -1$;

$$= \pi \int_1^{-1} \frac{-2 + (1 + t^2)}{1 + t^2}\,dt = -2\pi \int_1^{-1} \frac{dt}{1 + t^2} + \pi \int_1^{-1} dt$$

$$= -2\pi \left| \tan^{-1} t \right|_1^{-1} + \pi \left| t \right|_1^{-1} = -2\pi\left(-\frac{\pi}{4} - \frac{\pi}{4}\right) - 2\pi.$$ Hence, $I = \pi^2/2 - \pi$.

Property V. $\int_{-a}^{a} \mathbf{f(x)\,dx = 2\int_0^a f(x)\,dx}$**, if f(x) is an even function,**

= 0 if f(x) is an odd function. *(Bhopal, 2008)*

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx \quad ...(1) \quad \text{[Prop. I]}$$

In $\int_{-a}^{0} f(x)\,dx$, put $x = -t$, so that $dx = -dt$

$$\therefore \quad \int_{-a}^{0} f(x)\,dx = -\int_a^0 f(-t)\,dt = \int_0^a f(-t)\,dt = \int_0^a f(-x)\,dx \quad \text{[Prop. II]}$$

Substituting in (1), we get

$$\int_{-a}^{a} f(x)\,dx = \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx \quad ...(2)$$

(i) If $f(x)$ is an even function, $f(-x) = f(x)$.

$\therefore$ from (2), $\int_{-a}^{a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx$

(ii) If $f(x)$ is an odd function, $f(-x) = -f(x)$.

$\therefore$ from (2), $\int_{-a}^{a} f(x)\,dx = -\int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 0.$

Property VI. $\int_0^{2a} \mathbf{f(x)\,dx = 2\int_0^a f(x)\,dx}$**, if f (2a – x) = f(x)**

= 0, if f(2a – x) = – f(x)

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx \quad ...(1) \quad \text{[Prop. III]}$$

In $\int_0^{2a} f(x)\,dx$, put $x = 2a - t$, so that $dx = -dt$

Also when $x = a, t = a$; when $x = 2a, t = 0$.

$$\therefore \quad \int_0^{2a} f(x)\,dx = -\int_a^0 f(2a - t)\,dt = \int_0^a f(2a - t)\,dt = \int_0^a f(2a - x)\,dx \quad \text{[Prop. II]}$$

Subsituting in (1), we get

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a - x)\,dx \quad ...(2)$$

(i) If $f(2a - x) = f(x)$, then from (2)

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx$$

(*ii*) If $f(2a-x) = -f(x)$, then from (2)

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx - \int_0^a f(x)\,dx = 0.$$

Cor. 1. *If n is even,* $\int_0^{\pi} \sin^m x \cos^n x\,dx = 2\int_0^{\pi/2} \sin^m x \cos^n x\,dx$ *and if n is odd,* $\int_0^{\pi} \sin^m x \cos^n x\,dx = 0.$

Cor. 2. *If m is odd,* $\int_0^{2\pi} \sin^m x \cos^n x\,dx = 0$

and *if m is even,* $\int_0^{2\pi} \sin^m x \cos^n x\,dx = 2\int_0^{\pi} \sin^m x \cos^n x\,dx$

$$= 4\int_0^{\pi/2} \sin^m x \cos^n x\,dx,\ \text{if } n \text{ is even} = 0,\ \text{if } n \text{ is odd.}$$

Example 6.15. *Evaluate* $\int_0^{\pi} \theta \sin^2\theta \cos^4\theta\,d\theta.$ *(V.T.U., 2009 S)*

Solution. Let $\quad I = \int_0^{\pi} \theta \sin^2\theta \cos^4\theta\,d\theta$

Then $\quad I = \int_0^{\pi} (\pi-\theta)\sin^2(\pi-\theta)\cos^4(\pi-\theta)\,d\theta = \pi\int_0^{\pi} \sin^2\theta\cos^4\theta\,d\theta - I$ [Prop. IV]

or $\quad 2I = \pi\int_0^{\pi} \sin^2\theta\cos^4\theta\,d\theta = 2\pi\int_0^{\pi/2} \sin^2\theta\cos^4\theta\,d\theta$ [Prop. VI Cor. 2]

$$= 2\pi \cdot \frac{1.3.1}{6.4.2} = \frac{\pi}{2} = \frac{\pi^2}{16}$$

Hence $\quad I = \dfrac{\pi^2}{32}$

Example 6.16. *Evaluate* $\int_0^{\pi/2} \log \sin x\,dx.$ *(Anna, 2005 S)*

Solution. Let $\quad I = \int_0^{\pi/2} \log \sin x\,dx$ ...(*i*)

then $\quad I = \int_0^{\pi/2} \log\sin(\pi/2 - x)\,dx = \int_0^{\pi/2} \log\cos x\,dx$ ...(*ii*)

Adding (*i*) and (*ii*)

$$2I = \int_0^{\pi/2} (\log\sin x + \log\cos x)\,dx$$

$$= \int_0^{\pi/2} \log(\sin x + \cos x)\,dx = \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right)dx$$

$$= \int_0^{\pi/2} \log\sin 2x\,dx - \int_0^{\pi/2} \log 2\,dx = \int_0^{\pi/2} \log\sin 2x\,dx - \log 2\int_0^{\pi/2} dx$$

$$= \int_0^{\pi/2} \log\sin 2x\,dx - \log 2\,|x|_0^{\pi/2} = I' - \frac{\pi}{2}\log 2 \quad ...(iii)$$

where $\quad I' = \int_0^{\pi/2} \log\sin 2x\,dx$ $\quad$ [Put, $2x = t$, so that $2dx = dt$; When $x = 0, t = 0$; when $x = \pi/2, t = \pi$]

$$= \frac{1}{2}\int_0^{\pi} \log\sin t\,dt = \frac{1}{2}\int_0^{\pi} \log\sin x\,dx$$ [$\because$ $\log\sin(\pi - x) = \log\sin x$, Prop. IV]

$$= \frac{1}{2}\cdot 2\int_0^{\pi/2} \log\sin x\,dx = I.$$

Thus from (*iii*), $2I = I - (\pi/2)\log 2$, *i.e.*, $I = -(\pi/2)\log 2.$

Obs. The following are its immediate deductions :

$$\int_0^{\pi/2} \log \sin x\, dx = \int_0^{\pi/2} \log \cos x\, dx = -\frac{\pi}{2}\log 2$$

and
$$\int_0^{\pi} \log \sin x\, dx = -\pi \log 2.$$

Example 6.17. *Evaluate* $\int_0^1 \frac{\sin^{-1} x}{x}\, dx$.

Solution. Put $\sin^{-1} x = \theta$ or $x = \sin\theta$ so that $dx = \cos\theta\, d\theta$

Also when $x = 0, \theta = 0$; when $x = 1, \theta = \pi/2$.

$$\therefore \quad \int_0^1 \frac{\sin^{-1} x}{x}\, dx = \int_0^{\pi/2} \theta \cdot \frac{\cos\theta}{\sin\theta}\, d\theta \qquad \text{[Integrate by parts]}$$

$$= [\theta \cdot \log \sin\theta]_0^{\pi/2} - \int_0^{\pi/2} 1 \cdot \log \sin\theta\, d\theta$$

$$= -\int_0^{\pi/2} \log\sin\theta\, d\theta = -\left(-\frac{\pi}{2}\log 2\right) = \frac{\pi}{2}\log 2 \qquad \left[\because \underset{x\to 0}{\text{Lt}}\,(x \log x) = 0\right]$$

PROBLEMS 6.4

Prove that :

1. (i) $\int_0^{\pi/2} \log \tan x\, dx = 0$ (ii) $\int_0^{\pi/2} \sin 2x \log \tan x\, dx = 0$

2. (i) $\int_0^{\infty} \frac{x^7(1-x^{12})}{(1+x)^{28}}\, dx = 0$ (ii) $\int_0^{\pi/4} \log(1+\tan\theta)\, d\theta = \frac{\pi}{8}\log_e 2$ *(Madras, 2000)*

3. (i) $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx = \frac{\pi}{4}$ (ii) $\int_0^{a} \frac{dx}{x + \sqrt{(a^2 + x^2)}} = \frac{\pi}{4}$

4. (i) $\int_0^{\pi/2} \frac{dx}{1+\sqrt{\cot x}} = \frac{\pi}{4}$. (ii) $\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\, dx = \frac{\pi}{4}$.

5. (i) $\int_0^{\pi/2} \frac{x \tan x}{\sec x + \cos x}\, dx = \frac{\pi^2}{4}$. (ii) $\int_0^{\pi} \frac{x}{1+\sin x}\, dx = \pi$ *(Anna, 2002 S)*

6. (i) $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} = \frac{1}{2}\pi(\pi - 2)$. (ii) $\int_0^{\pi/2} \frac{x\, dx}{\sin x + \cos x} = \frac{\pi}{2\sqrt{2}}\log(\sqrt{2}+1)$.

Evaluate :

7. (i) $\int_0^{\pi} \sin^4 x\, dx$ (ii) $\int_0^{2\pi} \cos^6 x\, dx$

   (iii) $\int_0^{\pi} \sin^6 x \cos^4 x\, dx$ *(V.T.U., 2001)* (iv) $\int_0^{2\pi} \sin^4 x \cos^6 x\, dx$

8. (i) $\int_0^{\pi} x \sin^7 x\, dx$ *(V.T.U., 2009)* (ii) $\int_0^{\pi} x \cos^4 x \sin^5 x\, dx$ *(Marathwada, 2008)*

Prove that :

9. (i) $\int_0^{\pi} \frac{x\, dx}{a^2\cos^2 x + b^2 \sin^2 x} = \frac{\pi^2}{2ab}$ (ii) $\int_0^{\pi/2} \frac{x\, dx}{2\sin^2 x + \cos^2 x} = \frac{\pi^2}{2\sqrt{2}}$

10. (i) $\int_0^{\pi} \frac{x\, dx}{a^2 - \cos^2 x} = \frac{\pi^2}{2a\sqrt{(a^2-1)}}$ $(a > 1)$ (ii) $\int_0^{\pi} \frac{x\, dx}{1+\sin^2 x} = \frac{\pi^2}{2\sqrt{2}}$.

11. $\int_0^{\pi} \log(1+\cos\theta)\, d\theta = -\pi \log_e 2$ (Madras, 2003)

12. (i) $\int_0^{\infty} \frac{\log(1+x^2)}{1+x^2} dx = \pi \log_e 2$ (ii) $\int_0^{\infty} \frac{\log(x+1/x)}{1+x^2} dx = \pi \log_e 2.$

6.9 (1) INTEGRAL AS THE LIMIT OF A SUM

We have so far considered integration as inverse of differentiation. We shall now define the definite integral as the limit of a sum :

Def. *If $f(x)$ is continuous and single valued in the interval $[a, b]$, then the definite integral of $f(x)$ between the limits a and b is defined by the equation*

$$\int_a^b f(x)\, dx = \underset{h \to 0}{\text{Lt}}\, h\,[f(a) + f(a+h) + f(a+2h) + \ldots + f(a + \overline{n-1}\, h)],$$

where $nh = b - a$. ...(1)

(2) Evaluation of limits of series

The summation definition of a definite integral enables us to express the limits of sums of certain types of series as definite integrals which can be easily evaluated. We rewrite (1) as follows :

$$\int_a^b f(x)\, dx = \underset{n \to \infty}{\text{Lt}}\, h \sum_{r=0}^{n-1} f(a + rh), \text{ where } nh = b - a.$$

Putting $a = 0$ and $b = 1$, so that $h = 1/n$, we get

$$\underset{n \to \infty}{\text{Lt}} \frac{1}{n} \sum_{r=0}^{n-1} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\, dx$$

Thus to express a given series as definite integral:

(i) Write the general term (T_r or T_{r+1} whichever involves r)

i.e., $f(r/n) \cdot 1/n$

(ii) Replace r/n by x and $1/n$ by dx,

(iii) Integrate the resulting expression, taking

the lower limit = $\underset{n \to \infty}{\text{Lt}} (r/n)$ where r is as in the first term,

and the upper limit = $\underset{n \to \infty}{\text{Lt}} (r/n)$ where r is as in the last term.

Example 6.18. *Find the limit, when $n \to \infty$, of the series*

$$\frac{n}{n^2} + \frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \ldots + \frac{n}{n^2+(n-1)^2}$$

Solution. Here the general term $(= T_{r+1}) = \frac{n}{n^2+r^2} = \frac{n}{1+(r/n)^2} \cdot \frac{1}{n}$

$$= \frac{1}{1+x^2}\, dx \qquad \text{[Putting } r/n = x \text{ and } 1/n = dx]$$

Now for the first term $r = 0$ and for the last term $r = n - 1$

$\therefore$ the lower limit of integration $= \underset{n \to \infty}{\text{Lt}} \left(\frac{0}{n}\right) = 0$

and the upper limit of integration $= \underset{n \to \infty}{\text{Lt}} \left(\frac{n-1}{n}\right) = \underset{n \to \infty}{\text{Lt}} \left(1 - \frac{1}{n}\right) = 1.$

Hence, the required limit $= \int_0^1 \frac{dx}{1+x^2} = \left| \tan^{-1} x \right|_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \pi/4.$

[**To find limit of a product by integration :**

Let $P = \underset{n \to \infty}{\text{Lt}}$ (given product)

Take logs of both sides, so that

$$\log P = \underset{n \to \infty}{\text{Lt}} \text{ (a series) } = k \text{ (say). Then } P = e^k.]$$

Example 6.19. *Evaluate* $\underset{n \to \infty}{Lt} \left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\dots\left(1+\frac{n}{n}\right)\right\}^{1/n}$. (*Bhopal, 2008*)

Solution. Let $$P = \underset{n \to \infty}{\text{Lt}} \left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\dots\left(1+\frac{n}{n}\right)\right\}^{1/n}.$$

Taking logs of both sides,

$$\log P = \underset{n \to \infty}{\text{Lt}} \frac{1}{n}\left\{\log\left(1+\frac{1}{n}\right) + \log\left(1+\frac{2}{n}\right) + \dots + \log\left(1+\frac{n}{n}\right)\right\}$$

Its general term $= \log\left(1+\frac{r}{n}\right) \cdot \frac{1}{n} = \log(1+x) \cdot dx$ [Putting $r/n = x$ and $1/n = dx$]

Also for first term $r = 1$ and for the last term $r = n$.

$\therefore$ The lower limit of integration $= \underset{n \to \infty}{\text{Lt}} (1/n) = 0$ and the upper limit $= \underset{n \to \infty}{\text{Lt}} (n/n) = 1$

Hence $$\log P = \int_0^1 \log(1+x)\, dx = \int_0^1 \log(1+x) \cdot 1\, dx \quad \text{[Integrate by parts]}$$

$$= \left| \log(1+x) \cdot x \right|_0^1 - \int_0^1 \frac{1}{1+x} \cdot x\, dx$$

$$= \log 2 - \int_0^1 \frac{1+x-1}{1+x}\, dx = \log 2 - \int_0^1 dx + \int_0^1 \frac{dx}{1+x}$$

$$= \log 2 - \left| x \right|_0^1 + \left| \log(1+x) \right|_0^1 = \log 2 - 1 + \log 2$$

$$= \log 2^2 - \log_e e = \log(4/e). \text{ Hence, } P = 4/e.$$

PROBLEMS 6.5

Find the limit, as $n \to \infty$, of the series :

1. $\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{2n}$. (*Bhopal, 2009*)
2. $\frac{1}{n^3+1} + \frac{4}{n^3+8} + \frac{9}{n^3+27} + \dots + \frac{r^2}{n^3+r^3} + \dots + \frac{1}{2n}$.
3. $\underset{n \to \infty}{\text{Lt}} \frac{\sqrt{n}}{\sqrt{n^3}} + \frac{\sqrt{n}}{\sqrt{(n+3)^3}} + \frac{\sqrt{n}}{\sqrt{(n+6)^3}} + \dots + \frac{\sqrt{n}}{\sqrt{[n+3(n-1)]^3}}$.

Evaluate :

4. $\underset{n \to \infty}{\text{Lt}} \sum_{r=1}^{n-1} \frac{1}{\sqrt{(n^2-r^2)}}$ (*Bhopal, 2008*)
5. $\underset{n \to \infty}{\text{Lt}} \frac{[(n+1)(n+2)\dots(n+n)]^{1/n}}{n}$.
6. $\underset{n \to \infty}{\text{Lt}} \left\{\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n^2}\right)\dots\left(1+\frac{n^2}{n^2}\right)\right\}^{1/n}$ (*Bhopal, 2008*)

6.10 AREAS OF CARTESIAN CURVES

(1) *Area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$, $x = b$ is* $\int_a^b \mathbf{y\ dx}$.

Let AB be the curve $y = f(x)$ between the ordinates LA $(x = a)$ and MB $(x = b)$. (Fig. 6.1)

Let $P(x, y)$, $P'(x + \delta x, y + \delta y)$ be two neighbouring points on the curve and NP, $N'P'$ be their respective ordinates.

Let the area $ALNP$ be A, which depends on the position of P whose abscissa is x. Then the area $PNN'P' = \delta A$.

Complete the rectangles PN' and $P'N'$.

Then the area $PNN'P'$ lies between the areas of the rectangles PN' and $P'N$.

i.e., δA lies between $y\delta x$ and $(y + \delta y)\,\delta x$

$\therefore$ $\frac{\delta A}{\delta x}$ lies between y and $y + \delta y$.

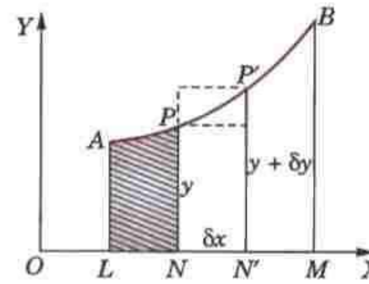

Fig. 6.1

Now taking limits as $P' \to P$ *i.e.*, $\delta x \to 0$ (and $\therefore$ $\delta y \to 0$),

$$dA/dx = y$$

Integrating both sides between the limits $x = a$ and $x = b$, we have

$$\left| A \right|_a^b = \int_a^b y\,dx$$

or (value of A for $x = b$) – (value of A for $x = a$) = $\int_a^b y\,dx$

Thus area $ALMB = \int_a^b y\,dx.$

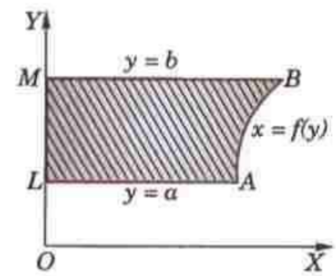

Fig. 6.2

(2) Interchanging x and y in the above formula, we see that *the area bounded by the curve $x = f(y)$, the y-axis and the abscissae $y = a$, $y = b$ is* $\int_a^b$ **x dy.** (Fig. 6.2)

Obs. 1. The area bounded by a curve, the x-axis and two ordinates is called the **area under the curve.** The process of finding the area of plane curves is often called **quadrature.**

Obs. 2. **Sign of an area.** *An area whose boundary is described in the anti-clockwise direction is considered positive and an area whose boundary is described in the clockwise direction is taken as negative.*

In Fig. 6.3, the area $ALMB \left(= \int_a^b y\,dx\right)$ which is described in the anti-clockwise direction and lies above the x-axis, will give a positive result.

In Fig. 6.4, the area $ALMB \left(= \int_a^b y\,dx\right)$ which is described in the clockwise direction and lies below the x-axis, will give a negative result.

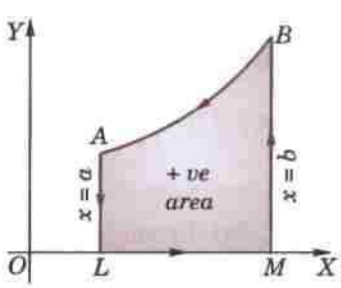

Fig. 6.3

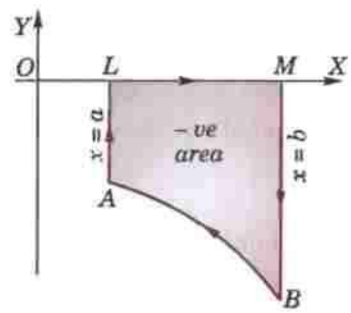

Fig. 6.4

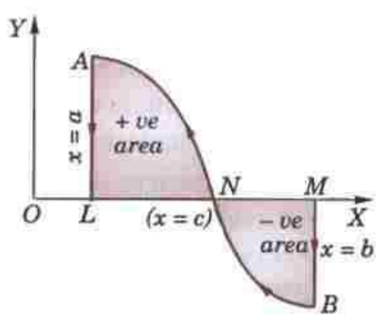

Fig. 6.5

In Fig. 6.5, the area $ALMB \left(= \int_a^b y\,dx\right)$ will not consist of the sum of the area $ALN \left(= \int_a^c y\,dx\right)$ and the area NMB $\left(= \int_c^b y\,dx\right)$, but their difference.

Thus to find the total area in such cases the numerical value of the area of each portion must be evaluated separately and their results added afterwards.

Example 6.20. *Find the area of the loop of the curve $ay^2 = x^2(a - x)$.* (*S.V.T.U., 2009 ; Osmania, 2000*)

Solution. Let us trace the curve roughly to get the limits of integration.

(*i*) The curve is symmetrical about x-axis.

(*ii*) It passes through the origin. The tangents at the origin are $ay^2 = ax^2$ or $y = \pm x$. $\therefore$ Origin is a node.

(*iii*) The curve has no asymptotes.

(*iv*) The curve meets the x-axis at (0, 0) and (a, 0). It meets the y-axis at (0, 0) only.

From the equation of the curve, we have $y = \dfrac{x}{\sqrt{a}}\sqrt{(a-x)}$

For $x > a$, y is imaginary. Thus no portion of the curve lies to the right of the line $x = a$. Also $x \to -\infty, y \to \infty$.

Thus the curve is as shown in Fig. 6.6.

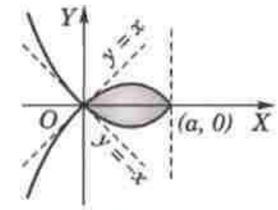

Fig. 6.6

$\therefore$ Area of the loop = 2 (area of upper half of the loop)

$$= 2\int_0^a y\,dx = 2\int_0^a x\sqrt{\left(\frac{a-x}{a}\right)}\,dx = \frac{2}{\sqrt{a}}\int_0^a [a-(a-x)]\sqrt{(a-x)}\,dx$$

$$= \frac{2}{\sqrt{a}}\int_0^a [a(a-x)^{1/2} - (a-x)^{3/2}]\,dx = 2\sqrt{a}\left|\frac{(a-x)^{3/2}}{-3/2}\right|_0^a - \frac{2}{\sqrt{a}}\left|\frac{(a-x)^{5/2}}{-5/2}\right|_0^a$$

$$= -\frac{4}{3}\sqrt{a}\,(0-a^{3/2}) + \frac{4}{5\sqrt{a}}(0-a^{5/2}) = \frac{4}{3}a^2 - \frac{4}{5}a^2 = \frac{8}{15}a^2.$$

Example 6.21. *Find the area included between the curve $y^2(2a-x) = x^3$ and its asymptote.* (V.T.U., 2003)

Solution. The curve is as shown in Fig. 4.23.

Area between the curve and the asymptote

$$= 2\int_0^{2a} y\,dx = 2\int_0^{2a}\sqrt{\left(\frac{x^3}{2a-x}\right)}\,dx$$

Put $x = 2a\sin^2\theta$ so that $dx = 4a\sin\theta\cos\theta\,d\theta$

$$= 2\int_0^{\pi/2}\sqrt{\left\{\frac{(2a\sin^2\theta)^3}{2a\cos^2\theta}\right\}}\cdot 4a\sin\theta\cos\theta\,d\theta$$

$$= 16a^2\int_0^{\pi/2}\sin^4\theta\,d\theta = 16a^2\cdot\frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2} = 3\pi a^2.$$

Example 6.22. *Find the area enclosed by the curve $a^2x^2 = y^3(2a-y)$.*

Solution. Let us first find the limits of integration.

(*i*) The curve is symmetrical about y-axis.

(*ii*) It passes through the origin and the tangents at the origin are $x^2 = 0$ or $x = 0, x = 0$.

$\therefore$ There is a cusp at the origin.

(*iii*) The curve has no asymptote.

(*iv*) The curve meets the x-axis at the origin only and meets the y-axis at (0, $2a$). From the equation of the curve, we have

$$x = \frac{y}{a}\sqrt{\{y(2a-y)\}}$$

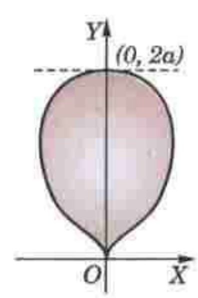

Fig. 6.7

For $y < 0$ or $y > 2a$, x is imaginary. Thus the curve entirely lies between $y = 0$ (x-axis) and $y = 2a$, which is shown in Fig. 6.7.

$$\therefore \text{ Area of the curve } = 2\int_0^{2a} x\,dy = \frac{2}{a}\int_0^{2a} y\sqrt{[y(2a-y)]}\,dy$$

Put $y = 2a\sin^2\theta$ $\therefore$ $dy = 4a\sin\theta\cos\theta\,d\theta$

$$= \frac{2}{a}\int_0^{\pi/2} 2a\sin^2\theta\sqrt{[2a\sin^2\theta\,(2a-2a\sin^2\theta)]}\times 4a\sin\theta\cos\theta\,d\theta$$

$$= 32a^2\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta = 32a^2\,\frac{3\cdot 1\times 1}{6\cdot 4\cdot 2}\cdot\frac{\pi}{2} = \pi a^2.$$

Example 6.23. *Find the area enclosed between one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$; and its base.* (V.T.U., 2000)

Solution. To describe its first arch, θ varies from 0 to 2π *i.e.*, x varies from 0 to $2a\pi$ (Fig. 6.8).

$\therefore$ Required area $= \int_{x=0}^{2\pi a} y\, dx$

where $y = a(1 - \cos\theta)$, $dx = a(1 - \cos\theta)\, d\theta$.

$$= \int_{\theta=0}^{\pi/2} a(1 - \cos\theta) \cdot a(1 - \cos\theta)\, d\theta$$

$$= 2a^2 \int_0^{\pi} (1 - \cos\theta)^2\, d\theta = 8a^2 \int_0^{\pi} \sin^4 \frac{\theta}{2}\, d\theta$$

$$= 16a^2 \int_0^{\pi/2} \sin^4 \phi\, d\phi, \text{ putting } \theta/2 = \phi \text{ so that } d\theta = 2d\phi.$$

$$= 16a^2 \cdot \frac{3 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2} = 3\pi a^2.$$

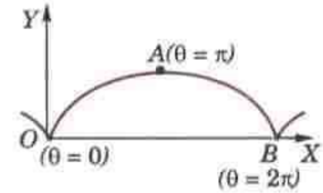

Fig. 6.8

Example 6.24. *Find the area of the tangent cut off from the parabola $x^2 = 8y$ by the line $x - 2y + 8 = 0$.*

Solution. Given parabola is $x^2 = 8y$ ...(*i*)

and the straight line is $x - 2y + 8 = 0$ ...(*ii*)

Substituting the value of y from (*ii*) in (*i*), we get

$$x^2 = 4(x + 8) \text{ or } x^2 - 4x - 32 = 0$$

or $\quad (x - 8)(x + 4) = 0 \quad \therefore \quad x = 8, -4.$

Thus (*i*) and (*ii*) intersect at P and Q where $x = 8$ and $x = -4$. (Fig. 6.9)

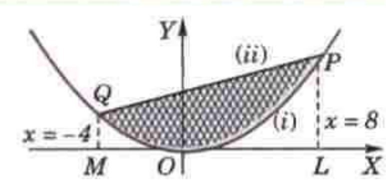

Fig. 6.9

$\therefore$ Required area POQ (*i.e.*, dotted area) = area bounded by straight line (*ii*) and x-axis from $x = -4$ to $x = 8$ – area bounded by parabola (*i*) and x-axis from $x = -4$ to $x = 8$.

$$= \int_{-4}^{8} y\, dx, \text{ from } (ii) - \int_{-4}^{8} y\, dx, \text{ from } (i)$$

$$= \int_{-4}^{8} \frac{x+8}{2}\, dx - \int_{-4}^{8} \frac{x^2}{8}\, dx = \frac{1}{2}\left| \frac{x^2}{2} + 8x \right|_{-4}^{8} - \frac{1}{8}\left| \frac{x^3}{3} \right|_{-4}^{8}$$

$$= \frac{1}{2}\{(32 + 64) - (-24)\} - \frac{1}{24}(512 + 64) = 36.$$

Example 6.25. *Find the area common to the parabola $y^2 = ax$ and the circle $x^2 + y^2 = 4ax$.*

Solution. Given parabola is $\quad y^2 = ax$ ...(*i*)

and the circle is $\quad x^2 + y^2 = 4ax.$ ...(*ii*)

Both these curves are symmetrical about x-axis. Solving (*i*) and (*ii*) for x, we have

$$x^2 + ax = 4ax \text{ or } x(x - 3a) = 0$$

or $\quad x = 0, 3a.$

Thus the two curves intersect at the points where $x = 0$ and $x = 3a$. (Fig. 6.10).

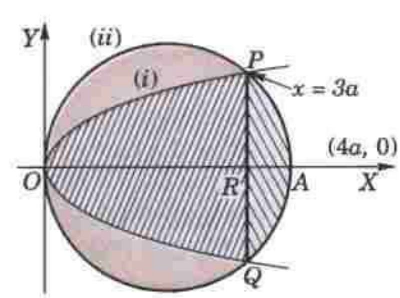

Fig. 6.10

Also (*ii*) meets the x-axis at $A(4a, 0)$.

Area common to (*i*) and (*ii*) *i.e.*, the shaded area

$$= 2[\text{Area } ORP + \text{Area } PRA] \qquad \text{(By symmetry)}$$

$$= 2\left[\int_0^{3a} y\, dx, \text{ from } (i) + \int_{3a}^{4a} y\, dx, \text{ from } (ii)\right]$$

$$= 2\left[\int_0^{3a} \sqrt{(ax)}\, dx + \int_{3a}^{4a} \sqrt{(4ax - x^2)}\, dx\right]$$

$$= 2\sqrt{a} \left| \frac{x^{3/2}}{3/2} \right|_0^{3a} + 2 \int_{3a}^{4a} \sqrt{[4a^2 - (x-2a)^2]}\, dx$$

$$= \frac{4\sqrt{a}}{3}(3a)^{3/2} + 2\left[\frac{1}{2}(x-2a)\sqrt{[4a^2-(x-2a)^2]} + \frac{4a^2}{2}\sin^{-1}\frac{x-2a}{2a}\right]_{3a}^{4a}$$

$$= 4\sqrt{3}\,a^2 + 2[\{0 - \frac{1}{2}a\,\sqrt{3}a\} + 2a^2\{\pi/2 - \pi/6\}]$$

$$= 4\sqrt{3}\,a^2 - \sqrt{3}a^2 + \frac{4}{3}\pi a^2 = \left(3\sqrt{3} + \frac{4}{3}\pi\right)a^2.$$

PROBLEMS 6.6

1. (*i*) Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. (*Kerala, 2005 ; V.T.U., 2003 S*)

   (*ii*) Find the area bounded by the parabola $y^2 = 4ax$ and its latus-rectum.
2. Find the area bounded by the curve $y = x(x-3)(x-5)$ and the x-axis.
3. Find the area included between the curve $ay^2 = x^3$, the x-axis and the ordinates $x = a$.
4. Find the area of the loop of the curve :

   (*i*) $3ay^2 = x(x-a)^2$ (*Rajasthan, 2005*) (*ii*) $x(x^2+y^2) = a(x^2-y^2)$ (*P.T.U., 2010*)
5. Find the whole area of the curve :

   (*i*) $a^2x^2 = y^3(2a-y)$ (*Nagpur, 2009*) (*ii*) $8a^2y^2 = x^2(a^2-x^2)$ (*V.T.U., 2006*)
6. Find the area included between the curve and its asymptotes in each case :

   (*i*) $xy^2 = a^2(a-x)$. (*V.T.U., 2003*) (*ii*) $x^2y^2 = a^2(y^2-x^2)$. (*V.T.U., 2007*)
7. Show that the area of the loop of the curve $y^2(a+x) = x^2(3a-x)$ is equal to the area between the curve and its asymptote.
8. Find the whole area of the *astroid* $x^{2/3} + y^{2/3} = a^{2/3}$ or $x = a\cos^3\theta, y = a\sin^3\theta$. (*V.T.U., 2005*)
9. Find the area bounded by the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ and the coordinate axes.
10. Find the area included between the *cycloid* $x = a(\theta + \sin\theta), y = a(1-\cos\theta)$ *and its base*. Also find the area between the curve and the x-axis. (*Gorakhpur, 1999*)
11. Find the area common to the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 4x$.
12. Prove that the area common to the parabolas $x^2 = 4ay$ and $y^2 = 4ax$ is $16a^2/3$. (*S.V.T.U., 2008 ; Kurukshetra, 2005*)
13. Find the area included between the circle $x^2 + y^2 = 2ax$ and the parabola $y^2 = ax$.
14. Find the area bounded by the parabola $y^2 = 4ax$ and the line $x + y = 3a$.
15. Find the area of the segment cut off from the parabola $y = 4x - x^2$ by the straight line $y = x$. (*V.T.U., 2010 ; S.V.T.U., 2008*)

(2) Areas of polar curves. *Area bounded by the curve $r = f(\theta)$ and the radii vectors*

$$\theta = \alpha, \theta = \beta \textit{ is } \frac{1}{2}\int_\alpha^\beta \mathbf{r^2}\, \mathbf{d\theta}.$$

Let AB be the curve $r = f(\theta)$ between the radii vectors $OA\,(\theta = \alpha)$ and $OB\,(\theta = \beta)$. Let $P(r, \theta)$, $P'(r + \delta r, \theta + \delta\theta)$ be any two neighbouring points on the curve. (Fig. 6.11)

Let the area $OAP = A$ which is a function of θ. Then the area $OPP' = \delta A$. Mark circular arcs PQ and $P'Q'$ with centre O and radii OP and OP'.

Evidently area OPP' lies between the sectors OPQ and $OP'Q'$ *i.e.*, δA lies between $\frac{1}{2}r^2\,\delta\theta$ and $\frac{1}{2}(r + \delta r)^2\,\delta\theta$.

$$\therefore \quad \frac{\delta A}{\delta\theta} \text{ lies between } \tfrac{1}{2}r^2 \text{ and } \tfrac{1}{2}(r+\delta r)^2.$$

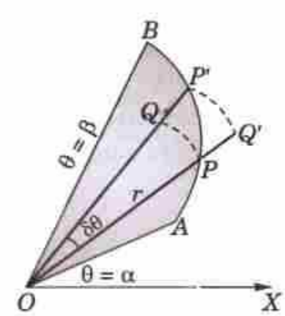

Fig. 6.11

Now taking limits as $\delta\theta \to 0$ ($\therefore \quad \delta r \to 0$), $\dfrac{dA}{d\theta} = \dfrac{1}{2} r^2$

Integrating both sides from $\theta = \alpha$ to $\theta = \beta$, we get $\left| A \right|_\alpha^\beta = \int_\alpha^\beta \frac{1}{2} r^2 \, d\theta$

or (value of A for $\theta = \beta$) – (value of A for $\theta = \alpha$) $= \dfrac{1}{2} \displaystyle\int_\alpha^\beta r^2 \, d\theta$

Hence the required area $OAB = \dfrac{1}{2} \displaystyle\int_\alpha^\beta r^2 \, d\theta$.

Example 6.26. *Find the area of the cardioid $r = a(1 - \cos\theta)$.* (V.T.U., 2004)

Solution. The curve is as shown in Fig. 6.12. Its upper half is traced from $\theta = 0$ to $\theta = \pi$.

$\therefore$ Area of the curve

$$= 2 \cdot \frac{1}{2} \int_0^\pi r^2 \, d\theta = a^2 \int_0^\pi (1 - \cos\theta)^2 \, d\theta$$

$$= a^2 \int_0^\pi (2\sin^2\theta/2)^2 \, d\theta = 4a^2 \int_0^\pi \sin^4\theta/2 \cdot d\theta$$

$$= 8a^2 \int_0^{\pi/2} \sin^4\phi \, d\phi, \text{ putting } \theta/2 = \phi \text{ and } d\theta = 2d\phi.$$

$$= 8a^2 \cdot \frac{3 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2} = \frac{3\pi a^2}{2}.$$

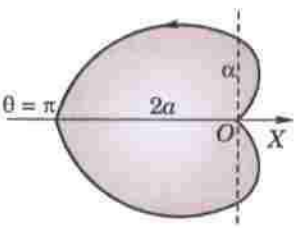

Fig. 6.12

Example 6.27. *Find the area of a loop of the curve $r = a \sin 3\theta$.*

Solution. The curve is as shown in Fig. 4.35. It consists of three loops.
Putting $r = 0$, $\sin 3\theta = 0 \quad \therefore \quad 3\theta = 0$ or π *i.e.*, $\theta = 0$ or $\pi/3$ which are the limits for the first loop.

$\therefore$ Area of a loop

$$= \frac{1}{2} \int_0^{\pi/3} r^2 \, d\theta = \frac{1}{2} a^2 \int_0^{\pi/3} \sin^2 3\theta \, d\theta = \frac{a^2}{4} \int_0^{\pi/3} (1 - \cos 6\theta) \, d\theta$$

$$= \frac{a^2}{4} \left| \theta - \frac{\sin 6\theta}{6} \right|_0^{\pi/3} = \frac{a^2}{4}\left(\frac{\pi}{3} - 0\right) = \frac{\pi a^2}{12}.$$

Obs. The limits of integration for a loop of $r = a \sin n\theta$ or $r = a \cos n\theta$ are the two consecutive values of θ when $r = 0$.

Example 6.28. *Prove that the area of a loop of the curve $x^3 + y^3 = 3axy$ is $3a^2/2$.*

Solution. Changing to polar form (by putting $x = r\cos\theta$, $y = r\sin\theta$), $r = \dfrac{3a \sin\theta \cos\theta}{\cos^3\theta + \sin^3\theta}$

Putting $r = 0$, $\sin\theta\cos\theta = 0$.

$\therefore \quad \theta = 0, \pi/2$, which are the limits of integration for its loop.

$\therefore$ Area of the loop

$$= \frac{1}{2} \int_0^{\pi/2} r^2 \, d\theta = \frac{1}{2} \int_0^{\pi/2} \frac{9a^2 \sin^2\theta \cos^2\theta}{(\cos^3\theta + \sin^3\theta)^2} \, d\theta$$

$$= \frac{9a^2}{2} \int_0^{\pi/2} \frac{\tan^2\theta \sec^2\theta}{(1 + \tan^3\theta)^2} \, d\theta \qquad \text{[Dividing num. and denom. by } \cos^6\theta]$$

$$= \frac{3a^2}{2} \int_1^\infty \frac{dt}{t^2}, \qquad \text{putting } 1 + \tan^3\theta = t \text{ and } 3\tan^2\theta \sec^2\theta \, d\theta = dt.$$

$$= \frac{3a^2}{2} \left| \frac{t^{-1}}{-1} \right|_1^\infty = \frac{3a^2}{2}(-0 + 1) = \frac{3a^2}{2}.$$

Example 6.29. *Find the area common to the circles*

$$r = a\sqrt{2} \text{ and } r = 2a\cos\theta$$

Solution. The equations of the circles are $r = a\sqrt{2}$ ...(*i*) and $r = 2a \cos \theta$ ...(*ii*)

(*i*) represents a circle with centre at (0, 0) and radius $a\sqrt{2}$. (*ii*) represents a circle symmetrical about *OX*, with centre at $(a, 0)$ and radius a.

The circles are shown in Fig. 6.13. At their point of intersection *P*, eliminating *r* from (*i*) and (*ii*),

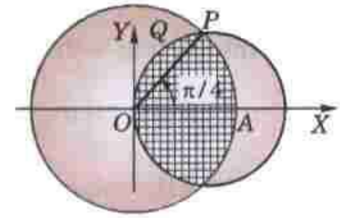

Fig. 6.13

$$a\sqrt{2} = 2a \cos \theta \text{ i.e., } \cos \theta = 1 \sqrt{2}$$

or $$\theta = \pi/4$$

$\therefore$ Required area = 2 × area *OAPQ* (By symmetry)

$$= 2(\text{area } OAP + \text{area } OPQ)$$

$$= 2\left[\frac{1}{2}\int_0^{\pi/4} r^2\, d\theta, \text{ for } (i) + \frac{1}{2}\int_{\pi/4}^{\pi/2} r^2\, d\theta, \text{ for } (ii)\right]$$

$$= \int_0^{\pi/4} (a\sqrt{2})^2\, d\theta + \int_{\pi/4}^{\pi/2} (2a\cos\theta)^2\, d\theta = 2a^2 \,|\, \theta \,|_0^{\pi/4} + 4a^2 \int_{\pi/4}^{\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= 2a^2(\pi/4 - 0) + 2a^2 \left| \theta + \frac{\sin 2\theta}{2} \right|_{\pi/4}^{\pi/2} = \frac{\pi a^2}{2} + 2a^2\left(\frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{2}\right) = a^2(\pi - 1).$$

Example 6.30. *Find the area common to the cardioids $r = a(1 + \cos \theta)$ and $r = a(1 - \cos \theta)$.*

(*Kurukshetra, 2006 ; V.T.U., 2006*)

Solution. The cardioid $r = a(1 + \cos \theta)$ is *ABCOB′A* and the cardioid $r = a(1 - \cos \theta)$ is *OC′BA′B′O*. Both the cardioids are symmetrical about the initial line *OX* and intersect at *B* and *B′* (Fig. 6.14)

$\therefore$ Required area (shaded) = 2 area *OC′BCO*

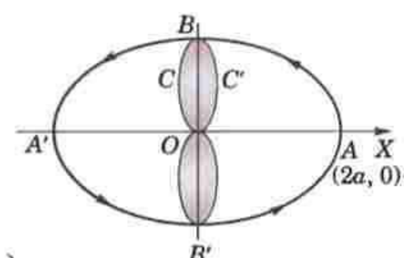

Fig. 6.14

$$= 2\,[\text{area } OC'BO + \text{area } OBCO]$$

$$= 2\left[\left\{\int_0^{\pi/2} \frac{1}{2} r^2\, d\theta\right\}_{r = a(1-\cos\theta)} + \left\{\int_{\pi/2}^{\pi} \frac{1}{2} r^2\, d\theta\right\}_{r = a(1+\cos\theta)}\right]$$

$$= a^2 \int_0^{\pi/2} (1 - \cos\theta)^2\, d\theta + a^2 \int_{\pi/2}^{\pi} (1 + \cos\theta)^2\, d\theta$$

$$= a^2 \left\{\int_0^{\pi/2} (1 - 2\cos\theta + \cos^2\theta)\, d\theta + \int_{\pi/2}^{\pi} [1 + 2\cos\theta + \cos^2\theta]\, d\theta\right\}$$

$$= a^2 \left\{\int_0^{\pi} (1 + \cos^2\theta)\, d\theta - 2\int_0^{\pi/2} \cos\theta\, d\theta + 2\int_{\pi/2}^{\pi} \cos\theta\, d\theta\right\}$$

$$= a^2 \left\{\int_0^{\pi} \left(1 + \frac{1+\cos 2\theta}{2}\right) d\theta - 2\,|\sin\theta\,|_0^{\pi/2} + 2\,|\sin\theta\,|_{\pi/2}^{\pi}\right\}$$

$$= a^2 \left\{\left|\frac{3}{2}\theta + \frac{\sin 2\theta}{4}\right|_0^{\pi} - 2(1-0) + 2(0-1)\right\} = \left(\frac{3\pi}{2} - 4\right)a^2.$$

PROBLEMS 6.7

1. Find the whole area of
   (*i*) the *cardioid* $r = a(1 + \cos\theta)$ (*V.T.U., 2008*) (*ii*) the *lemniscate* $r^2 = a^2 \cos 2\theta$; (*V.T.U., 2006*)
2. Find the area of one loop of the curve
   (*i*) $r = a \sin 2\theta$. (*ii*) $r = a \cos 3\theta$.
3. Show that the area included between the folium $x^3 + y^3 = 3axy$ and its asymptote is equal to the area of loop.
4. Prove that the area of the loop of the curve $x^3 + y^3 = 3axy$ is three times the area of the loop of the curve $r^2 = a^2 \cos 2\theta$.
5. Find the area inside the circle $r = a \sin\theta$ and lying outside the *cardioid* $r = a(1 - \cos\theta)$. (*Anna, 2009*)
6. Find the area outside the *circle* $r = 2a\cos\theta$ and inside the *cardioid* $r = a(1 + \cos\theta)$. (*Kurukshetra, 2006*)

6.11 LENGTHS OF CURVES

(1) *The length of the arc of the curve $y = f(x)$ between the points where $x = a$ and $x = b$ is*

$$\int_a^b \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\, dx$$

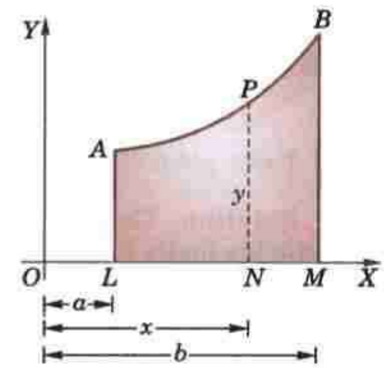

Fig. 6.15

Let AB be the curve $y = f(x)$ between the points A and B where $x = a$ and $x = b$ (Fig. 6.15)

Let $P(x, y)$ be any point on the curve and arc $AP = x$ so that it is a function of x.

Then $$\frac{ds}{dx} = \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]} \qquad \text{[(1) of p. 164]}$$

$\therefore$ $$\int_a^b \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\, dx = \int_a^b \frac{ds}{dx}\cdot dx = |\, s\,|_{x=a}^{x=b}$$

$$= (\text{value of } s \text{ for } x = b) - (\text{value of } s \text{ for } x = a) = \text{arc } AB - 0$$

Hence, the arc $AB = \int_a^b \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\, dx.$

(2) *The length of the arc of the curve $x = f(y)$ between the points where $y = a$ and $y = b$, is*

$$\int_a^b \sqrt{\left[1+\left(\frac{dx}{dy}\right)^2\right]}\, dy \qquad \text{[Use (2) of p. 165]}$$

(3) *The length of the arc of the curve $x = f(t)$, $y = \phi(t)$ between the points where $t = a$ and $t = b$, is*

$$\int_a^b \sqrt{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]}\, dt. \qquad \text{[Use (3) p. 165]}$$

(4) *The length of the arc of the curve $r = f(\theta)$ between the points where $\theta = \alpha$ and $\theta = \beta$, is*

$$\int_\alpha^\beta \sqrt{\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]}\, d\theta. \qquad \text{[Use (1) of p. 165]}$$

(5) *The length of the arc of the curve $\theta = f(r)$ between the points where $r = a$ and $r = b$, is*

$$\int_a^b \sqrt{\left[1+\left(r\frac{d\theta}{dr}\right)^2\right]}\, dr \qquad \text{[Use (2) of p. 166]}$$

Example 6.31. *Find the length of the arc of the parabola $x^2 = 4ay$ measured from the vertex to one extremity of the latus-rectum.* (Delhi, 2002)

Solution. Let A be the vertex and L an extremity of the latus-rectum so that at A, $x = 0$ and at L, $x = 2a$. (Fig. 6.16).

Now $$y = x^2/4a \text{ so that } \frac{dy}{dx} = \frac{1}{4a}\cdot 2x = \frac{x}{2a}$$

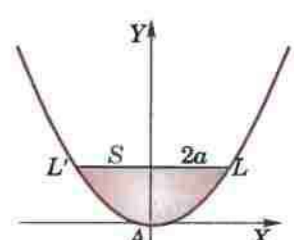

Fig. 6.16

$\therefore$ $$\text{arc } AL = \int_0^{2a} \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\, dx$$

$$= \int_0^{2a} \sqrt{\left[1+\left(\frac{x}{2a}\right)^2\right]}\, dx = \frac{1}{2a}\int_0^{2a} \sqrt{[(2a)^2+x^2]}\, dx$$

$$= \frac{1}{2a}\left[\frac{x\sqrt{[(2a)^2 + x^2]}}{2} + \frac{(2a)^2}{2}\sinh^{-1}\frac{x}{2a}\right]_0^{2a} = \frac{1}{2a}\left[\frac{2a\sqrt{(8a)^2}}{2} + 2a^2\sinh^{-1}1\right]$$

$$= a[\sqrt{2} + \sinh^{-1}1] = a[\sqrt{2} + \log(1+\sqrt{2})] \qquad [\because \sinh^{-1}x = \log[x + \sqrt{(1+x^2)}]]$$

Example 6.32. *Find the perimeter of the loop of the curve $3ay^2 = x(x-a)^2$.*

Solution. The curve is symmetrical about the x-axis and the loop lies between the limits $x = 0$ and $x = a$. (Fig. 6.17).

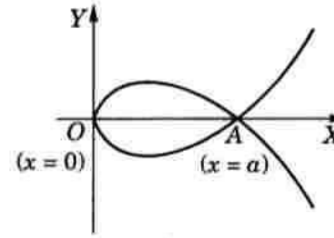

Fig. 6.17

We have $$y = \frac{\sqrt{x}(x-a)}{\sqrt{(3a)}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{(3a)}}\left[\frac{3}{2}x^{1/2} - \frac{a}{2}\cdot x^{-1/2}\right] = \frac{1}{2\sqrt{(3a)}}\frac{3x-a}{\sqrt{x}}$$

$$\therefore \quad \text{Perimeter of the loop} = 2\int_0^a \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}\,dx \qquad \text{(By symmetry)}$$

$$= 2\int_0^a \sqrt{\left[1 + \frac{(3x-a)^2}{12ax}\right]}\,dx = 2\int_0^a \frac{\sqrt{(9x^2 + 6ax + a^2)}}{\sqrt{(12ax)}}\,dx$$

$$= \frac{1}{\sqrt{(3a)}}\int_0^a \frac{3x+a}{\sqrt{x}}\,dx = \frac{1}{\sqrt{(3a)}}\int_0^a (3x^{1/2} + ax^{-1/2})\,dx$$

$$= \frac{1}{\sqrt{(3a)}}\left|\frac{3x^{3/2}}{3/2} + a\frac{x^{1/2}}{1/2}\right|_0^a = \frac{1}{\sqrt{(3a)}}(4a^{3/2}) = \frac{4a}{\sqrt{3}}.$$

Example 6.33. *Find the length of one arch of the cycloid*
$x = a(t - \sin t)$, $y = a(1 - \cos t)$. *(P.T.U., 2009 ; V.T.U., 2004)*

Solution. As a point moves from one end O to the other end of its first arch, the parameter t increases from 0 to 2π. [see Fig. 6.8]

Also $$\frac{dx}{dt} = a(1-\cos t), \quad \frac{dy}{dt} = a\sin t.$$

$$\therefore \quad \text{Length of an arch} = \int_0^{2\pi}\sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]}\,dx$$

$$= \int_0^{2\pi}\sqrt{\{[a(1-\cos t)]^2 + (a\sin t)^2\}}\,dt = a\int_0^{2\pi}\sqrt{[2(1-\cos t)]}\,dt$$

$$= 2a\int_0^{2\pi}\sin t/2\,dt = 2a\left|-\frac{\cos t/2}{1/2}\right|_0^{2\pi} = 4a[(-\cos\pi) - (-\cos 0)] = 8a.$$

Example 6.34. *Find the entire length of the cardioid $r = a(1 + \cos\theta)$.*
(P.T.U., 2010 ; Bhopal, 2008 ; Kurukshetra, 2005)
Also show that the upper half is bisected by $\theta = \pi/3$. *(Bhillai, 2005)*

Solution. The cardioid is symmetrical about the initial line and for its upper half, θ increases from 0 to π (Fig. 6.18)

Also $$\frac{dr}{d\theta} = -a\sin\theta.$$

$$\therefore \quad \text{Length of the curve} = 2\int_0^{\pi}\sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}\,d\theta$$

$$= 2\int_0^{\pi} \sqrt{\{[a(1+\cos\theta)]^2 + (-a\sin\theta)^2\}}\, d\theta = 2a\int_0^{\pi}\sqrt{[2(1+\cos\theta)]}\, d\theta$$

$$= 4a\int_0^{\pi}\cos\theta/2\, d\theta = 4a\left|\frac{\sin\theta/2}{1/2}\right|_0^{\pi} = 8a(\sin\pi/2 - \sin 0) = 8a.$$

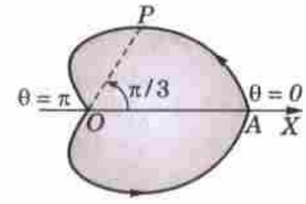

Fig. 6.18

$\therefore$ Length of upper half of the curve is $4a$. Also length of the arc AP from 0 to $\pi/3$.

$$= a\int_0^{\pi/3}\sqrt{[2(1+\cos\theta)]}\, d\theta = 2a\int_0^{\pi/3}\cos\theta/2\cdot d\theta$$

$$= 4a\left|\sin\theta/2\right|_0^{\pi/3} = 2a = \text{half the length of upper half of the cardioid.}$$

PROBLEMS 6.8

1. Find the length of the arc of the *semi-cubical parabola* $ay^2 = x^3$ from the vertex to the ordinate $x = 5a$.
2. Find the length of the curve (*i*) $y = \log\sec x$ from $x = 0$ to $x = \pi/3$. (*V.T.U., 2010 S ; P.T.U., 2007*)
   (*ii*) $y = \log[(e^x - 1)/(e^x + 1)]$ from $x = 1$ to $x = 2$.
3. Find the length of the arc of the parabola $y^2 = 4ax$ (*i*) from the vertex to one end of the latus-rectum.
   (*ii*) cut off by the line $3y = 8x$. (*V.T.U., 2008 S ; Mumbai, 2006*)
4. Find the perimeter of the loop of the following curves :
   (*i*) $ay^2 = x^2(a - x)$ (*ii*) $9y^2 = (x-2)(x-5)^2$.
5. Find the length of the curve $y^2 = (2x-1)^2$ cut off by the line $x = 4$. (*V.T.U., 2000 S*)
6. Show that the whole length of the curve $x^2(a^2 - x^2) = 8a^2y^2$ is $\pi a\sqrt{2}$.
7. (*a*) Find the length of an arch of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$.
   (*b*) By finding the length of the curve show that the curve $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, is divided in the ratio 1 : 3 at $\theta = 2\pi/3$. (*S.V.T.U., 2009*)
8. Find the whole length of the curve $x = a\cos^3 t$, $y = a\sin^3 t$ *i.e.*, $x^{2/3} + y^{2/3} = a^{2/3}$.
   (*V.T.U., 2010 ; Marathwada, 2008 ; Rajasthan, 2006*)
   Also show that the line $\theta = \pi/3$ divides the length of this *astroid* in the first quadrant in the ratio 1 : 3.
   (*Mumbai, 2001*)
9. Find the length of the loop of the curve $x = t^2$, $y = t - t^3/3$. (*Mumbai, 2001*)
10. For the curve $r = ae^{\theta}\cot\alpha$, prove that s/r = constant, s being measured from the origin.
11. Find the length of the curve $\theta = \frac{1}{2}\left(r + \frac{1}{r}\right)$ from $r = 1$ to $r = 3$. (*Marathwada, 2008*)
12. Find the perimeter of the *cardioid* $r = a(1 - \cos\theta)$. Also show that the upper half of the curve is bisected by the line $\theta = 2\pi/3$.
13. Find the whole length of the *lemniscate* $r^2 = a^2\cos 2\theta$.
14. Find the length of the parabola $r(1 + \cos\theta) = 2a$ as cut off by the latus-rectum. (*J.N.T.U., 2003*)

6.12 (1) VOLUMES OF REVOLUTION

(*a*) **Revolution about x-axis.** *The volume of the solid generated by the revolution about the x-axis, of the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$, $x = b$ is*

$$\int_a^b \pi \mathbf{y^2\, dx}.$$

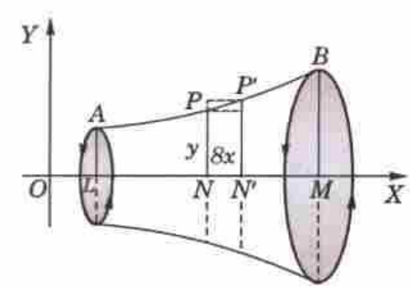

Fig. 6.19

Let AB be the curve $y = f(x)$ between the ordinates $LA(x = a)$ and $MB(x = b)$.

Let $P(x, y)$, $P'(x + \delta x, y + \delta y)$ be two neighbouring points on the curve and NP, $N'P'$ be their respective ordinates (Fig. 6.19).

Let the volume of the solid generated by the revolution about x-axis of the area $ALNP$ be V, which is clearly a function of x. Then the volume of the solid generated by the revolution of the area $PNN'P'$ is δV. Complete the rectangles PN' and $P'N$.

The δV lies between the volumes of the right circular cylinders generated by the revolution of rectangles PN' and $P'N$,

i.e., δV lies between $\pi y^2\,\delta x$ and $\pi(y+\delta y)^2\,\delta x$.

$\therefore$ $\dfrac{\delta V}{\delta x}$ lies between πy^2 and $\pi(y+\delta y)^2$.

Now taking limits as $P'\to P$, *i.e.*, $\delta x\to 0$ (and $\therefore$ $\delta y\to 0$), $\dfrac{dV}{dx}=\pi y^2$

$$\therefore \qquad \int_a^b \frac{dV}{dx}\,dx=\int_a^b \pi y^2\,dx \quad\text{or}\quad |V|_{x=a}^{b}=\int_a^b \pi y^2\,dx$$

or (value of V for $x=b$) – (value of V for $x=a$)

i.e., volume of the solid obtained by the revolution of the area $ALMB=\displaystyle\int_a^b \pi y^2\,dx$.

Example 6.35. *Find the volume of a sphere of radius a.* (S.V.T.U., 2007)

Solution. Let the sphere be generated by the revolution of the semi-circle ABC, of radius a about its diameter CA (Fig. 6.20)

Taking CA as the x-axis and its mid-point O as the origin, the equation of the circle ABC is $x^2+y^2=a^2$.

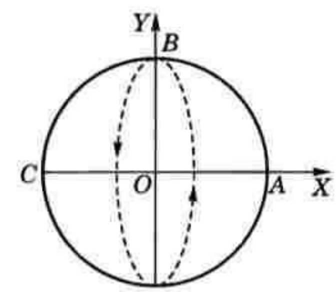

Fig. 6.20

$\therefore$ Volume of the sphere = 2 (volume of the solid generated by the revolution about x-axis of the quadrant OAB)

$$=2\int_0^a \pi y^2\,dx=2\pi\int_0^b (a^2-x^2)\,dx$$

$$=2\pi\left|a^2x-\frac{x^3}{3}\right|_0^a=2\pi\left[a^3-\frac{a^3}{3}-(0-0)\right]=\frac{4}{3}\pi a^3.$$

Example 6.36. *Find the volume formed by the revolution of loop of the curve $y^2(a+x)=x^2(3a-x)$, about the x-axis.* (Marathwada, 2008)

Solution. The curve is symmetrical about the x-axis, and for the upper half of its loop x varies from 0 to $3a$ (Fig. 6.21)

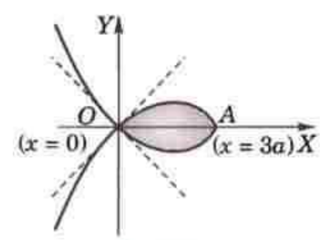

Fig. 6.21

$$\therefore \quad \text{Volume of the loop}=\int_0^{3a}\pi y^2\,dx=\pi\int_0^{3a}\frac{x^2(3a-x)}{a+x}\,dx$$

$$=\pi\int_0^{3a}\frac{-x^3+3ax^2}{x+a}\,dx$$

[Divide the numerator by the denominator]

$$=\pi\int_0^{3a}\left[-x^2+4ax-4a^2+\frac{4a^3}{x+a}\right]dx=\pi\left|-\frac{x^3}{3}+4a\cdot\frac{x^2}{2}-4a^2x+4a^3\log(x+a)\right|_0^{3a}$$

$$=\pi\left[\frac{-27a^3}{3}+2a\cdot 9a^2-4a^2\cdot 3a+4a^3\log 4a-(4a^3\log a)\right]$$

$$=\pi a^3(-3+4\log 4)=\pi a^3(8\log 2-3).$$

Example 6.37. *Prove that the volume of the reel formed by the revolution of the cycloid $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$ about the tangent at the vertex is π^2a^3.* (V.T.U., 2003)

Solution. The arch AOB of the cycloid is symmetrical about the y-axis and the tangent at the vertex is the x-axis. For half the cycloid OA, θ varies from 0 to π. (Fig. 4.31).

Hence the required volume

$$=2\int_{\theta=0}^{\theta=\pi}\pi y^2\,dx=2\pi\int_0^{\pi}a^2(1-\cos\theta)^2\,.\,a\,(1+\cos\theta)\,d\theta$$

$$= 2\pi a^3 \int_0^{\pi} (2\sin^2 \theta/2)^2 \cdot (2\cos^2 \theta/2)\, d\theta$$

$$= 16\pi a^3 \int_0^{\pi} \sin^4 \theta/2 \cdot \cos^2 \theta/2 \cdot d\theta \qquad [\text{Put } \theta/2 = \phi,\ d\theta = 2d\phi]$$

$$= 32\pi a^3 \int_0^{\pi/2} \sin^4 \phi \cos^2 \phi \, d\phi = 32\pi a^3 \cdot \frac{3\,.\,1\times 1}{6\,.\,4\,.\,2} \cdot \frac{\pi}{2} = \pi^2 a^3.$$

Example 6.38. *Find the volume of the solid formed by revolving about x-axis, the area enclosed by the parabola $y^2 = 4ax$, its evolute $27ay^2 = 4(x-2a)^3$ and the x-axis.*

Solution. The curve $27ay^2 = 4(x-2a)^3$ ...(*i*)
is symmetrical about x-axis and is a semi-cubical parabola with vertex at A $(2a, 0)$. The parabola $y^2 = 4ax$ and (*i*) intersect at B and C where $27a(4ax) = 4(x-2a)^3$ or $x^3 - 6ax^2 - 15a^2x - 8a^3 = 0$ which gives $x = -a, -a, 8a$. Since x is not negative, therefore we have $x = 8a$ (Fig. 6.22).

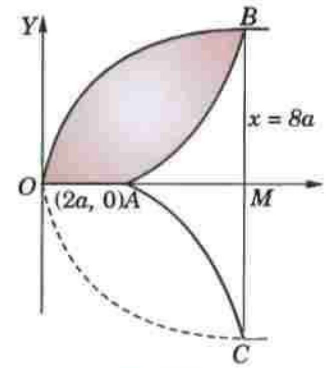

Fig. 6.22

$\therefore$ Required volume = Volume obtained by revolving the shaded area OAB about x-axis = Vol. obtained by revolving area $OMBO$ – Vol. obtained by revolving area $ADBA$

$$= \int_0^{8a} \pi y^2 \,(= 4ax)\, dx - \int_{2a}^{8a} \pi y^2 \,[\text{for } (i)]\, dx$$

$$= 4a\pi \left| \frac{x^2}{2} \right|_0^{8a} - \frac{4\pi}{27a}\int_{2a}^{8a} (x-2a)^3\, dx$$

$$= 128\pi a^3 - \frac{4\pi}{27a} \left| \frac{(x-2a)^4}{4} \right|_{2a}^{8a}$$

$$= 128\pi a^3 - \frac{\pi}{27a}(6a)^4 = 80\pi a^3.$$

(*b*) **Revolution about the y-axis.** Interchanging x and y in the above formula, we see that the *volume of the solid generated by the revolution about y-axis, of the area, bounded by the curve $x = f(y)$, the y-axis and the abscissae $y = a$, $y = b$ is*

$$\int_a^b \pi\, \mathbf{x}^2\, \mathbf{dy}.$$

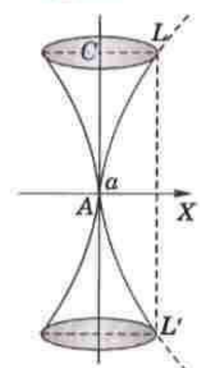

Fig. 6.23

Example 6.39. *Find the volume of the reel-shaped solid formed by the revolution about the y-axis, of the part of the parabola $y^2 = 4ax$ cut off by the latus-rectum.* (Rohtak, 2003)

Solution. Given parabola is $x = y^2/4a$.

Let A be the vertex and L one extremity of the latus-rectum. For the arc AL, y varies from 0 to $2a$ (Fig. 6.23).

$\therefore$ required volume = 2 (volume generated by the revolution about the y-axis of the area ALC)

$$= 2\int_0^{2a} \pi x^2\, dy = 2\pi \int_0^{2a} \frac{y^4}{16a^2} \cdot dy = \frac{\pi}{8a^2} \left| \frac{y^5}{5} \right|_0^{2a} = \frac{\pi}{40a^2}(32a^5 - 0) = \frac{4\pi a^3}{5}.$$

(*c*) **Revolution about any axis.** *The volume of the solid generated by the revolution about any axis LM of the area bounded by the curve AB, the axis LM and the perpendiculars AL, BM on the axis, is*

$$\int_{OL}^{OM} \pi(\mathbf{PN})^2\, \mathbf{d(ON)}$$

where O is a fixed point in LM and PN is perpendicular from any point P of the curve AB on LM.

With O as origin, take OLM as the x-axis and OY, perpendicular to it as the y-axis (Fig. 6.24).

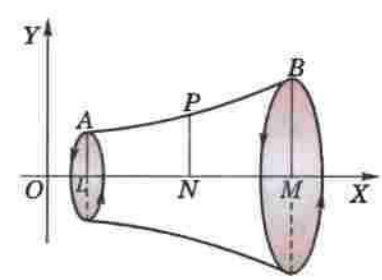

Fig. 6.24

Let the coordinates of P be (x, y) so that $x = ON$, $y = NP$

If $OL = a$, $OM = b$, then required volume $= \int_a^b \pi y^2\, dx = \int_{OL}^{OM} \pi (PN)^2\, d(ON)$.

Example 6.40. *Find the volume of the solid obtained by revolving the cissoid $y^2(2a - x) = x^3$ about its asymptote.* (V.T.U., 2000)

Solution. Given curve is $y = \dfrac{x^3}{2a - x}$ ...(i)

It is symmetrical about x-axis and the asymptote is $x = 2a$. (See Fig. 4.23). If $P(x, y)$ be any point on it and PN is perpendicular on the asymptote AN then $PN = 2a - x$ and

$$AN = y = \frac{x^{3/2}}{\sqrt{(2a - x)}} \qquad \text{[From (i)]}$$

$$\therefore \quad d(AN) = dy = \frac{\sqrt{(2a-x)}\,(3/2)\sqrt{x} - x^{3/2}\cdot \frac{1}{2}(2a-x)^{-1/2}(-1)}{2a - x}\, dx$$

$$= \frac{3\sqrt{x}(2a - x) + x^{3/2}}{2(2a - x)^{3/2}}\, dx = \frac{3ax^{1/2} - x^{3/2}}{(2a - x)^{3/2}}\, dx$$

$$\therefore \quad \text{Required volume} = 2\int_{x=0}^{x=2a} \pi (PN)^2\, d(AN) = 2\pi \int_0^{2a} (2a - x)^2 \cdot \frac{3ax^{1/2} - x^{3/2}}{(2a - x)^{3/2}} \cdot dx$$

$$= 2\pi \int_0^{2a} \sqrt{(2a - x)}\,(3a - x)\sqrt{x}\, dx \qquad \left[\begin{array}{l}\text{Put } x = 2a\sin^2\theta \\ \text{then } dx = 4a \sin\theta\cos\theta\, d\theta\end{array}\right.$$

$$= 2\pi \int_0^{\pi/2} \sqrt{(2a)}\cos\theta\,(3a - 2a\sin^2\theta)\, x\, \sqrt{(2a)}\sin\theta \,.\, 4a\sin\theta\cos\theta\, d\theta$$

$$= 16\pi a^3 \left[3\int_0^{\pi/2} \sin^2\theta\cos^2\theta\, d\theta - 2\int_0^{\pi/2} \sin^4\theta\cos^2\theta\, d\theta\right]$$

$$= 16\pi a^3 \left[3 \cdot \frac{1\times 1}{4\cdot 2}\cdot\frac{\pi}{2} - 2\cdot\frac{3\cdot 1\times 1}{6\cdot 4\cdot 2}\cdot\frac{\pi}{2}\right] = 2\pi^2 a^3.$$

(2) Volumes of revolution (polar curves). *The volume of the solid generated by the revolution of the area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \alpha$, $\theta = \beta$ (Fig. 6.25)*

(a) *about the initial line OX $(\theta = 0)$* $= \int_\alpha^\beta \dfrac{2\pi}{3}\mathbf{r}^3 \sin\theta\, \mathbf{d}\theta$

(b) *about the line OY $(\theta = \pi/2)$* $= \int_\alpha^\beta \dfrac{2\pi}{3}\mathbf{r}^3 \cos\theta\, \mathbf{d}\theta.$

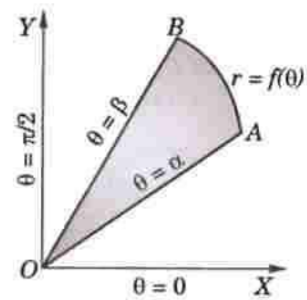

Fig. 6.25

Example 6.41. *Find the volume of the solid generated by the revolution of the cardioid $r = a(1 + \cos\theta)$ about the initial line.* (V.T.U., 2010 ; Kurukshetra, 2009 S)

Solution. The cardioid is symmetrical about the initial line and for its upper half θ varies from 0 to π. [Fig. 6.18].

$$\therefore \quad \text{required volume} = \int_0^\pi \frac{2}{3}\pi r^3 \sin\theta\, d\theta = \frac{2\pi}{3}\int_0^\pi a^3(1 + \cos\theta)^3 \sin\theta\, d\theta$$

$$= -\frac{2\pi a^3}{3}\int_0^\pi (1 + \cos\theta)^3 \cdot (-\sin\theta)\, d\theta = -\frac{2\pi a^3}{3}\left|\frac{(1 + \cos\theta)^4}{4}\right|_0^\pi = -\frac{\pi a^3}{6}[0 - 16] = \frac{8}{3}\pi a^3.$$

Example 6.42. *Find the volume of the solid generated by revolving the lemniscate $r^2 = a^2\cos 2\theta$ about the line $\theta = \pi/2$.* (V.T.U., 2006)

Solution. The curve is symmetrical about the pole. For the upper half of the R.H.S. loop, θ varies from 0 to $\pi/4$. (Fig. 4.34).

$\therefore$ required volume = 2(volume generated by the half loop in the first quadrant)

$$= 2\int_0^{\pi/4} \frac{2}{3}\pi r^3 \cos\theta \, d\theta = \frac{4\pi}{3} \cdot \int_0^{\pi/4} a^3 (\cos 2\theta)^{3/2} \cos\theta \, d\theta \qquad [\because \; r = a(\cos 2\theta)^{1/2}]$$

$$= \frac{4\pi a^3}{3} \int_0^{\pi/4} (1 - 2\sin^2\theta)^{3/2} \cos\theta \, d\theta \qquad \begin{bmatrix} \text{Put } \sqrt{2}\sin\theta = \sin\phi \\ \therefore \; \sqrt{2}\cos\theta\, d\theta = \cos\phi \, d\phi \end{bmatrix}$$

$$= \frac{4\pi a^3}{3} \int_0^{\pi/2} (1 - \sin^2\phi)^{3/2} \cdot \frac{1}{\sqrt{2}} \cos\phi \, d\phi = \frac{4\pi a^3}{3\sqrt{2}} \int_0^{\pi/2} \cos^4\phi \, d\phi = \frac{4\pi}{3\sqrt{2}} a^3 \frac{3 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2} = \frac{\pi a^3}{4\sqrt{2}}.$$

PROBLEMS 6.9

1. Find the volume generated by the revolution of the area bounded by x-axis, the *catenary* $y = c \cosh x/c$ and the ordinates $x = \pm c$, about the axis of x.
2. Find the volume of a spherical segment of height h cut off from a sphere of radius a.
3. Find the volume generated by revolving the portion of the parabola $y^2 = 4ax$ cut off by its latus-rectum about the axis of the parabola. *(V.T.U., 2009)*
4. Find the volume generated by revolving the area bounded by the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$, $x = 0$, $y = 0$ about the x-axis.
5. Find the volume of the solid generated by revolving the ellipse $x^2/a^2 + y^2/b^2 = 1$.
   (*i*) about the major axis. *(Bhopal, 2002 S)* (*ii*) about the minor axis. *(Bhillai, 2005)*
6. Obtain the volume of the frustrum of a right circular cone whose lower base has radius R, upper base is of radius r and altitude is h.
7. Find the volume generated by the revolution of the curve $27ay^2 = 4(x - 2a)^3$ about the x-axis.
8. Find the volume of the solid formed by the revolution, about the x-axis, of the loop of the curve :
   (*i*) $y^2(a - x) = x^2(a + x)$ (*ii*) $2ay^2 = x(x - a)^2$ (*iii*) $y^2 = x(2x - 1)^2$.
9. Find the volume obtained by revolving one arch of the cycloid
   (*i*) $x = a(t - \sin t)$, $y = a(1 - \cos t)$, about its base. *(Kurukshetra, 2006 ; V.T.U., 2005)*
   (*ii*) $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$, about the x-axis.
10. Find the volume of the spindle-shaped solid generated by the revolution of the *astroid* $x^{2/3} + y^{2/3} = a^{2/3}$ about the x-axis. *(P.T.U., 2010 ; S.V.T.U., 2008)*
11. Find the volume of the solid formed by the revolution, about the y-axis, of the area enclosed by the curve $xy^2 = 4a^2(2a - x)$ and its asymptote. *(V.T.U., 2006)*
12. Prove that the volume of the solid formed by the revolution of the curve $(a^2 + x^2) = a^3$, about its asymptote is $\frac{1}{2}\pi^2 a^3$.
13. Find the volume generated by the revolution about the initial line of
    (*i*) $r = 2a\cos\theta$ (*ii*) $r = a(1 - \cos\theta)$. *(P.T.U., 2006)*
14. Determine the volume of the solid obtained by revolving the *lemicon* $r = a + b\cos\theta$ $(a > b)$ about the initial line. *(Gorakhpur, 1999)*
15. Find the volume of the solid formed by revolving a loop of the lemniscate $r^2 = a^2 \cos 2\theta$ about the initial line. *(J.N.T.U., 2003 ; Delhi, 2002)*

6.13 SURFACE AREAS OF REVOLUTION

(*a*) **Revolution about x-axis.** *The surface area of the solid generated by the revolution about x-axis, of the arc of the curve $y = f(x)$ from $x = a$ to $x = b$, is*

$$\int_{x=a}^{x=b} 2\pi y \, ds.$$

Let AB be the curve $y = f(x)$ between the ordinates LA $(x = a)$ and MB $(x = b)$. Let $P(x, y)$, $P'(x + \delta x y + \delta y)$ be two neighbouring points on the curve and NP, $N'P'$ be their respective ordinates (Fig. 6.19).

Let the arc $AP = s$ so that arc $PP' = \delta s$. Let the surface-area generated by the revolution about x-axis of the arc AP be S and that generated by the revolution of the arc PP' be δS.

Since δs is small, the surface area δS may be regarded as lying between the curved surfaces of the right cylinders of radii PN and $P'N'$ and of same thickness δs.

Thus δS lies between $2\pi y\ \delta s$ and $2\pi (y + \delta y)\ \delta s$

$\therefore \quad \dfrac{\delta S}{\delta s}$ lies between $2\pi y$ and $2\pi (y + \delta y)$

Taking limits as $P' \to P$, *i.e.*, $\delta s \to 0$ and $\delta y \to 0$, $dS/dx = 2\pi y$

$$\therefore \quad \int_{x=a}^{x=b} \frac{dS}{ds}\, ds = \int_{x=a}^{x=b} 2\pi y\, ds \quad \text{or} \quad \left| S \right|_{x=a}^{x=b} = \int_{x=a}^{x=b} 2\pi y\, ds$$

or (value of S for $x = b$) – (value of S for $x = a$) = $\displaystyle\int_{x=a}^{x=b} 2\pi y\ dx$

or surface area generated by the revolution of the arc $AB - 0 = \displaystyle\int_{x=a}^{x=b} 2\pi y\ ds.$

Hence, the required surface area = $\displaystyle\int_{x=a}^{x=b} 2\pi y\ ds.$

Obs. Practical forms of the formula $\mathbf{S = \int 2\pi y\ ds.}$

(*i*) *Cartesian form* [*for the curve* $y = f(x)$]

$$S = \int 2\pi y \frac{ds}{dx}\, dx, \text{ where } \frac{ds}{dx} = \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}$$

(*ii*) *Parametric form* [*for the curve* $x = f(t), y = \phi(t)$]

$$S = \int 2\pi y \frac{ds}{dt}\, dt, \text{ where } \frac{ds}{dt} = \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]}$$

(*iii*) *Polar form* [*for the curve* $r = f(\theta)$]

$$S = \int 2\pi y \frac{ds}{d\theta}\, d\theta, \text{ where } y = r \sin\theta,\ \frac{ds}{d\theta} = \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}.$$

Example 6.43. *Find the surface of the solid formed by revolving the cardioid $r = a(1 + \cos\theta)$ about the initial line.* (V.T.U., 2009 ; Rajasthan, 2006 ; J.N.T.U., 2003)

Solution. The cardioid is symmetrical about the initial line and for its upper half, θ varies from 0 to π (Fig. 6.18).

Also
$$\frac{ds}{d\theta} = \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]} = \sqrt{[a^2(1+\cos\theta)^2 + a^2 \sin^2\theta]}$$

$$= a\sqrt{[2(1+\cos\theta)]} = a\sqrt{[2.2\cos^2\theta/2]} = 2a\cos\theta/2$$

$$\therefore \quad \text{required surface} = \int_0^\pi 2\pi y \frac{ds}{d\theta}\, d\theta = 2\pi \int_0^\pi r \sin\theta \cdot 2a\cos\theta/2\ d\theta$$

$$= 4\pi a \int_0^\pi a(1+\cos\theta)\ \sin\theta \cdot \cos\theta/2\ d\theta = 4\pi a^2 \int_0^\pi 2\cos^2\frac{\theta}{2} . 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} . \cos\frac{\theta}{2}\, d\theta$$

$$= 16\pi a^2 \int_0^\pi \cos^4\frac{\theta}{2}\sin\frac{\theta}{2}\, d\theta = 16\pi a^2(-2)\int_0^\pi \cos^4\frac{\theta}{2}\left(-\sin\frac{\theta}{2}\cdot\frac{1}{2}\right) d\theta$$

$$= -32\pi a^2 \left| \frac{\cos^5\theta/2}{5} \right|_0^\pi = \frac{-32\pi a^2}{5}(0-1) = \frac{32\pi a^2}{5}.$$

(*b*) **Revolution about y-axis.** Interchanging x and y in the above formula, we see that *the surface of the solid generated by the revolution about y-axis, of the arc of the curve $x = f(y)$ from $y = a$ to $y = b$ is*

$$\int_{y=a}^{y=b} \mathbf{2\pi x\ ds.}$$

Example 6.44. *Find the surface area of the solid generated by the revolution of the astroid $x = a\cos^3 t$, $y = a\sin^3 t$, about the y-axis.*

Solution. The astroid is symmetrical about the x-axis, and for its portion in the first quadrant t varies from 0 to $\pi/2$. (Fig. 4.29).

Also $$\frac{dx}{dt} = -3a\cos^2 t\sin t,\ \frac{dy}{dt} = 3a\sin^2 t\cos t.$$

$$\frac{ds}{dt} = \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]} = \sqrt{[9a^2\cos^4 t\sin^2 t + 9a^2\sin^4 t\cos^2 t]}$$

$$= 3a\sin t\cos t\sqrt{(\cos^2 t + \sin^2 t)} = 3a\sin t\cos t$$

$\therefore$ $$\text{required surface} = 2\int_0^{\pi/2} 2\pi x\frac{ds}{dt}\cdot dt = 4\pi\int_0^{\pi/2} a\cos^3 t\cdot 3a\sin t\cos t\,dt$$

$$= 12\pi a^2\int_0^{\pi/2}\sin t\cos^4 t\,dt = 12\pi a^2\frac{3.1}{5.3.1} = \frac{12\pi a^2}{5}.$$

PROBLEMS 6.10

1. Find the area of the surface generated by revolving the arc of the catenary $y = c\cosh x/c$ from $x = 0$ to $x = c$ about the x-axis.
2. Find the area of the surface formed by the revolution of $y^2 = 4ax$ about its axis, by the arc from the vertex to one end of the latus-rectum.
3. Find the surface of the solid generated by the revolution of the *ellipse* $x^2/a^2 + y^2/b^2 = 1$ about the x-axis. *(Raipur, 2005 ; Bhopal, 2002 S)*
4. Find the volume and surface of the *right circular cone* formed by the revolution of a right-angled triangle about a side which contains the right angle.
5. Obtain the surface area of a *sphere* of radius a.
6. Show that the surface area of the solid generated by the revolution of the curve $x = a\cos^3 t$, $y = a\sin^3 t$ about the x-axis, is $12\pi^2/5$.
7. The arc of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ in the first quadrant revolves about x-axis. Show that the area of the surface generated is $6\pi a^2/5$.
8. Find the surface area of the solid generated by revolving the *cycloid* $x = a(t - \sin t)$, $y = a(1 - \cos t)$ about the base. *(Marathwada, 2008 ; Cochin, 2005 ; Kurukshetra, 2005)*
9. Find the surface area of the solid got by revolving the arch of the *cycloid* $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$ about the base. *(V.T.U., 2010 S)*
10. Prove that the surface and volume of the solid generated by the revolution about the x-axis, of the loop of the curve $x = t^2$, $y = t - t^3/3$, [or $9y^2 = x(x-3)^2$], are respectively 3π and $3\pi/4$.
11. Prove that the surface of the solid generated by the revolution of the *tractrix* $x = a\cos t + \frac{a}{2}\log\tan^2 t/2$, $y = a\sin t$, about x-axis is $4\pi a^2$.
12. Find the surface area of the solid of revolution of the curve $r = 2a\cos\theta$ about the initial line. *(V.T.U., 2009)*
13. Find the surface of the solid generated by the revolution of the *cardioid* $r = a(1 - \cos\theta)$ about the initial line.
14. Find the surface of the solid generated by the revolution of the *lemniscate* $r^2 = a^2\cos 2\theta$ about the initial line. *(V.T.U., 2005)*
15. The part of *parabola* $y^2 = 4ax$ cut off by the latus-rectum revolves about the tangent at the vertex. Find the curved surface of the reel thus formed.

6.14 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 6.11

Choose the correct answer or fill up the blanks in the following problems :

1. If $f(x) = f(2a - x)$, then $\int_0^{2a} f(x)\,dx$ is equal to

(a) $\int_a^0 f(2a - x)\, dx$ (b) $2\int_0^a f(x)\, dx$ (c) $-2\int_0^a f(x)\, dx$ (d) 0.

2. $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx$ is equal to

(a) 0 (b) 1 (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$.

3. The value of definite integral $\int_{-a}^{a} |x|\, dx$ is equal to

(a) a (b) a^2 (c) 0 (d) $2a$.

4. $\lim_{n \to \infty} \left[\frac{n}{n^2} + \frac{n}{n^2 + 1^2} + \frac{n}{n^2 + 2^2} + \ldots + \frac{n}{n^2 + (n-1)^2}\right]$ is equal to

(a) $-\frac{\pi}{4}$ (b) 0 (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$.

5. $\int_0^{\pi/2} \frac{\cos 2x}{\cos x + \sin x}\, dx$ equals

(a) -1 (b) 0 (c) 1 (d) 2.

6. $\lim_{n \to \infty} \left(\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{3n}\right)$ equals

(a) $\log_e 2$ (b) $2 \log_e 2$ (c) $\log_e 3$ (d) $2 \log_e 3$.

7. $\int_0^{\pi} \sin^5\left(\frac{x}{2}\right)$ is equal to

(a) $\frac{16}{15}$ (b) $\frac{15}{16}\pi$ (c) $\frac{16}{15}\pi^2$ (d) $\frac{15}{16}$.

8. $\int_0^{\pi/2} \sin^{99} x \cos x\, dx$ is equal to

(a) $\frac{1}{99}$ (b) $\frac{\pi}{100}$ (c) $\frac{99}{100}$ (d) None of these. (V.T.U., 2009)

9. The value of $\int_{-\pi/2}^{\pi/2} \cos^7 x\, dx$ is

(a) $\frac{32\pi}{35}$ (b) $\frac{32}{35}$ (c) zero.

10. The length of the arc of the equiangular spiral $r = ae^{\theta \cot \alpha}$ between the points for which the radii vectors are r_1 and r_2 is

(a) $(r_2 - r_1)$ cosec α (b) $(r_2 - r_1) \cos \alpha$ (c) $(r_2 - r_1) \sin \alpha$ (d) $(r_2 - r_1) \sec \alpha$.

11. The area of the region in the first quadrant bounded by the y-axis and the curves $y = \sin x$ and $y = \cos x$ is

(a) $\sqrt{2}$ (b) $\sqrt{2} + 1$ (c) $\sqrt{2} - 1$ (d) $2\sqrt{2} - 1$.

12. The value of $\int_0^1 x^{3/2} (1 - x)^{3/2}\, dx$ is

(a) $\pi/32$ (b) $-\pi/32$ (c) $3\pi/128$ (d) $-3\pi/128$. (V.T.U., 2010)

13. The entire length of the cardioid $r = 5(1 + \cos \theta)$ is

(a) 40 (b) 30 (c) 20 (d) 5. (V.T.U., 2009)

14. The area of the cardioid $r = a(1 - \cos \theta)$ is ....

15. If S_1 and S_2 are surface areas of the solids generated by revolving the ellipses $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ about the y-axis, then

(a) $S_1 > S_2$ (b) $S_1 < S_2$ (c) $S_1 = S_2$ (d) can't predict.

16. The area of the loop of the curve $r = a \sin 3\theta$ is ....

17. If $I_n = \int_0^{\pi/4} \tan^n \theta\, d\theta$, then $n(I_{n-1} + I_{n+1}) =$ ....

18. $\int_0^2 x^3 \sqrt{(2x - x^2)}\, dx =$ ....

19. $\int_0^{\pi/2} \sin 2\theta \log \tan \theta \, d\theta$ is equal to

(a) 1 (b) -1 (c) 0 (d) $\pi/2$.

20. The area of the loop of the folium of Descartes $x^3 + y^3 - 3xy = 0$ is

(a) π (b) $\pi/2$ (c) 1.5 (d) 3.

21. The volume of the frustrum of a right circular cone whose lower base has radius r_1 and upper base has radius r_2 and altitude is h = ...

22. The length of the arc of the curve $y = \log \sec x$ from $x = 0$ to $x = \pi/4$ is

(a) $\log_e 2$ (b) $\log_e 3$ (c) $\log_e (1 + \sqrt{2})$ (d) $\log_e (1 + \sqrt{3})$. *(Bhopal, 2008)*

23. If v_1 = volume of the solid generated by revolving the area included between x-axis and $x^2 + y^2 = a^2$ about x-axis ; v_2 = volume of the solid generated by revolving the entire area of the circle $x^2 + y^2 = a^2$ about x-axis, then

(a) $v_1 = v_2$ (b) $v_2 = 2v_1$ (c) $v_2 = 4v_1$ (d) $v_2 = 16v_1$.

24. If $f(r, \theta) = f(-r, \theta)$, then the curve is symmetrical about the ....

(a) initial line (b) pole (c) origin (d) tangential line. *(V.T.U., 2010)*

25. The volume generated by the revolution of the curve $y = a^3 (a^2 + x^2)^{-1}$ about its asymptote is

(a) $\pi^2 a^3/2$ (b) $\pi a^3/2$ (c) $\pi a^2/2$ (d) $\pi a/2$. *(V.T.U., 2010)*

CHAPTER 7

Multiple Integrals and Beta, Gamma Functions

7.1 DOUBLE INTEGRALS

The definite integral $\int_a^b f(x)\,dx$ is defined as the limit of the sum

$$f(x_1)\,\delta x_1 + f(x_2)\,\delta x_2 + \dots + f(x_n)\,\delta x_n,$$

where $n \to \infty$ and each of the lengths $\delta x_1, \delta x_2, \dots$ tends to zero. A double integral is its counterpart in two dimensions.

Consider a function $f(x, y)$ of the independent variables x, y defined at each point in the finite region R of the xy-plane. Divide R into n elementary areas $\delta A_1, \delta A_2, \dots, \delta A_n$. Let (x_r, y_r) be any point within the rth elementary area δA_r. Consider the sum

$$f(x_1, y_1)\,\delta A_1 + f(x_2, y_2)\,\delta A_2 + \dots + f(x_n, y_n)\,\delta A_n, \text{ i.e., } \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r$$

The limit of this sum, if it exists, as the number of sub-divisions increases indefinitely and area of each sub-division decreases to zero, is defined as the *double integral of f(x, y) over the region R* and is written as $\iint_R f(x, y)\,dA$.

Thus
$$\iint_R f(x, y)\,dA = \underset{\substack{n \to \infty \\ \delta A \to 0}}{\text{Lt}} \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r \qquad \dots(1)$$

The utility of double integrals would be limited if it were required to take limit of sums to evaluate them. However, there is another method of evaluating double integrals by successive single integrations.

For purpose of evaluation, (1) is expressed as the repeated integral $\int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x, y)\,dxdy$.

Its value is found as follows :

(*i*) *When y_1, y_2 are functions of x and x_1, x_2 are constants,* $f(x, y)$ is first integrated w.r.t. y keeping x fixed between limits y_1, y_2 and then resulting expression is integrated w.r.t. x within the limits x_1, x_2 *i.e.,*

$$\boxed{I_1 = \int_{x_1}^{x_2} \boxed{\int_{y_1}^{y_2} f(x, y)\,dy}\,dx}$$

where integration is carried from the inner to the outer rectangle.

Figure 7.1 illustrates this process. Here AB and CD are the two curves whose equations are $y_1 = f_1(x)$ and $y_2 = f_2(x)$. PQ is a vertical strip of width dx.

Then the inner rectangle integral means that the integration is along one edge of the strip PQ from P to Q (x remaining constant), while the outer rectangle integral corresponds to the sliding of the edge from AC to BD.

Thus the whole region of integration is the area $ABDC$.

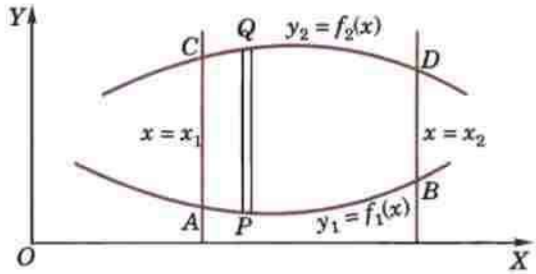

Fig. 7.1

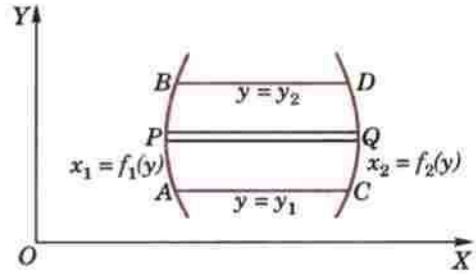

Fig. 7.2

(*ii*) *When x_1, x_2 are functions of y and y_1, y_2 are constants*, $f(x, y)$ is first integrated w.r.t. x keeping y fixed, within the limits x_1, x_2 and the resulting expression is integrated w.r.t. y between the limits y_1, y_2, *i.e.*,

$$I_2 = \boxed{\int_{y_1}^{y_2} \boxed{\int_{x_1}^{x_2} f(x, y)\, dx}\, dy} \quad \text{which is geometrically illustrated by Fig. 7.2.}$$

Here AB and CD are the curves $x_1 = f_1(y)$ and $x_2 = f_2(y)$. PQ is a horizontal strip of width dy.

Then inner rectangle indicates that the integration is along one edge of this strip from P to Q while the outer rectangle corresponds to the sliding of this edge from AC to BD.

Thus the whole region of integration is the area $ABDC$.

(*iii*) *When both pairs of limits are constants*, the region of integration is the rectangle $ABDC$ (Fig. 7.3).

In I_1, we integrate along the vertical strip PQ and then slide it from AC to BD.

In I_2, we integrate along the horizontal strip $P'Q'$ and then slide it from AB to CD.

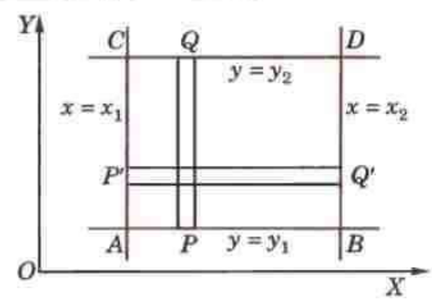

Fig. 7.3

Here obviously $I_1 = I_2$.

Thus *for constant limits, it hardly matters whether we first integrate w.r.t. x and then w.r.t. y or vice versa.*

Example 7.1. *Evaluate* $\int_0^5 \int_0^{x^2} x(x^2 + y^2)\, dxdy$.

Solution. $$I = \int_0^5 dx \int_0^{x^2} (x^3 + xy^3)\, dy = \int_0^5 \left[x^3 y + x \cdot \frac{y^3}{3}\right]_0^{x^2} dx = \int_0^5 \left[x^3 \cdot x^2 + x \cdot \frac{y^6}{3}\right] dx$$

$$= \int_0^5 \left(x^5 + \frac{x^7}{3}\right) dx = \left|\frac{x^6}{6} + \frac{x^8}{24}\right|_0^5 = 5^6\left[\frac{1}{6} + \frac{5^2}{24}\right] = 18880.2 \text{ nearly.}$$

Example 7.2. *Evaluate* $\iint_A xy\, dx\, dy$, *where A is the domain bounded by x-axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$.*

Solution. The line $x = 2a$ and the parabola $x^2 = 4ay$ intersect at $L(2a, a)$. Figure 7.4 shows the domain A which is the area OML.

Integrating first over a vertical strip PQ, *i.e.*, w.r.t. y from $P(y = 0)$ to $Q(y = x^2/4a)$ on the parabola and then w.r.t. x from $x = 0$ to $x = 2a$, we have

$$\iint_A xy\, dx\, dy = \int_0^{2a} dx \int_4^{x^2/4a} xy\, dy = \int_0^{2a} x \left[\frac{y^2}{2}\right]_0^{x^2/4a} dx$$

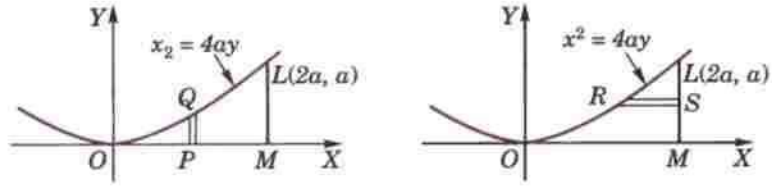

Fig. 7.4

$$= \frac{1}{32a^2} \int_0^{2a} x^5 \, dx = \frac{1}{32a^2} \left| \frac{x^6}{6} \right|_0^{2a} = \frac{a^4}{3}.$$

Otherwise integrating first over a horizontal strip RS, *i.e.*, w.r.t. x from, R ($x = 2\sqrt{ay}$) on the parabola to $S(x = 2a)$ and then w.r.t. y from $y = 0$ to $y = a$, we get

$$\iint_A xy \, dx \, dy = \int_0^a dx \int_{2\sqrt{(ay)}}^{2a} xy \, dx = \int_0^a y \left[\frac{x^2}{2} \right]_{2\sqrt{(ay)}}^{2a} dy$$

$$= 2a \int_0^a (ay - y^2) \, dy = 2a \left[\frac{ay^2}{2} - \frac{y^3}{3} \right]_0^a = \frac{a^4}{3}.$$

Example 7.3. *Evaluate* $\iint_R x^2 \, dx \, dy$ *where R is the region in the first quadrant bounded by the lines* $x = y$, $y = 0$, $x = 8$ *and the curve* $xy = 16$.

Solution. The line AL ($x = 8$) intersects the hyperbola $xy = 16$ at A (8, 2) while the line $y = x$ intersects this hyperbola at B (4, 4). Figure 7.5 shows the region R of integration which is the area $OLAB$. To evaluate the given integral, we divide this area into two parts OMB and $MLAB$.

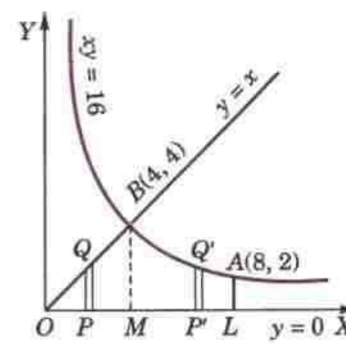

Fig. 7.5

$$\therefore \quad \iint_R x^2 \, dxdy = \int_{x \text{ at } 0}^{x \text{ at } M} \int_{y \text{ at } P}^{y \text{ at } Q} x^2 \, dxdy + \int_{x \text{ at } M}^{x \text{ at } L} \int_{y \text{ at } P'}^{y \text{ at } Q'} x^2 \, dxdy$$

$$= \int_0^4 \int_0^x x^2 \, dxdy + \int_4^8 \int_0^{16/x} x^2 \, dxdy$$

$$= \int_0^4 x^2 \, dx \left| y \right|_0^x + \int_4^8 x^2 \, dx \left| y \right|_0^{16/x}$$

$$= \int_0^4 x^3 \, dx + \int_4^8 16x \, dx = \left| \frac{x^4}{4} \right|_0^4 + 16 \left| \frac{x^2}{2} \right|_4^8 = 448$$

7.2 CHANGE OF ORDER OF INTEGRATION

In a double integral with variable limits, the change of order of integration changes the limit of integration. While doing so, sometimes it is required to split up the region of integration and the given integral is expressed as the sum of a number of double integrals with changed limits. To fix up the new limits, it is always advisable to draw a rough sketch of the region of integration.

The change of order of integration quite often facilitates the evaluation of a double integral. The following examples will make these ideas clear.

Example 7.4. *By changing the order of integration of* $\int_0^\infty \int_0^\infty e^{-xy} \sin px \, dxdy$, *show that*

$$\int_0^\infty \frac{\sin px}{x} \, dx = \frac{\pi}{2}.$$

(U.P.T.U., 2004)

Solution. $\int_0^\infty \int_0^\infty e^{-xy} \sin px \, dxdy = \int_0^\infty \left(\int_0^\infty e^{-xy} \sin px \, dx \right) dy$

$$= \int_0^\infty \left| -\frac{e^{-xy}}{p^2+y^2}(p\cos px + y\sin px)\right|_0^\infty dy$$

$$= \int_0^\infty \frac{p}{p^2+y^2}\,dy = \left|\tan^{-1}\left(\frac{y}{p}\right)\right|_0^\infty = \frac{\pi}{2} \quad ...(i)$$

On changing the order of integration, we have

$$\int_0^\infty \int_0^\infty e^{-xy}\sin px\,dx\,dy = \int_0^\infty \sin px\left\{\int_0^\infty e^{-xy}\,dy\right\}dx$$

$$= \int_0^\infty \sin px\left|\frac{e^{-xy}}{-x}\right|_0^\infty dx = \int_0^\infty \frac{\sin px}{x}\,dx \quad ...(ii)$$

Thus from (*i*) and (*ii*), we have $\int_0^\infty \frac{\sin px}{x}\,dx = \frac{\pi}{2}$.

Example 7.5. *Change the order of integration in the integral*

$$I = \int_{-a}^{a}\int_0^{\sqrt{(a^2-y^2)}} f(x,y)\,dx\,dy.$$

Solution. Here the elementary strip is parallel to x-axis (such as PQ) and extends from $x = 0$ to $x = \sqrt{(a^2-y^2)}$ (*i.e.*, to the circle $x^2+y^2=a^2$) and this strip slides from $y = -a$ to $y = a$. This shaded semi-circular area is, therefore, the region of integration (Fig. 7.6).

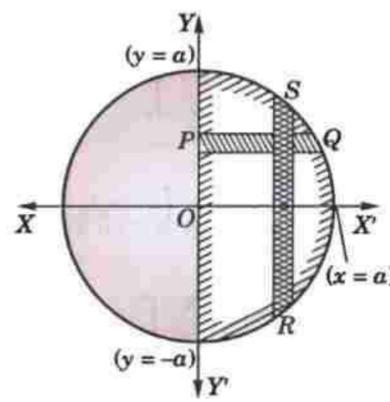

Fig. 7.6

On changing the order of integration, we first integrate w.r.t. y along a vertical strip RS which extends from R [$y = -\sqrt{(a^2-y^2)}$] to S [$y = \sqrt{(a^2-y^2)}$]. To cover the given region, we then integrate w.r.t. x from $x = 0$ to $x = a$.

Thus $$I = \int_0^a dx\int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} f(x,y)\,dy$$

or $$= \int_0^a \int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} f(x,y)\,dydx.$$

Example 7.6. *Evaluate* $\int_0^1\int_{e^x}^{e} dydx/\log y$ *by changing the order of integration.*

Solution. Here the integration is first w.r.t. y from P on $y = e^x$ to Q on the line $y = e$. Then the integration is w.r.t. x from $x = 0$ to $x = 1$, giving the shaded region ABC (Fig. 7.7).

On changing the order of integration, we first integrate w.r.t. x from R on $x = 0$ to S on $x = \log y$ and then w.r.t. y from $y = 1$ to $y = e$.

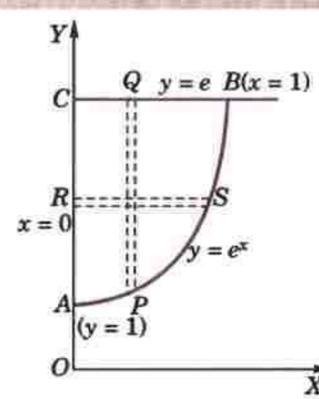

Fig. 7.7

Thus $$\int_0^1\int_{e^x}^{e}\frac{dydx}{\log y} = \int_1^e\int_0^{\log y}\frac{dxdy}{\log y}$$

$$= \int_1^e \frac{dy}{\log y}\left| x\right|_0^{\log y} = \int_1^e dy = \left| y\right|_1^e = e - 1.$$

Example 7.7. *Change the order of integration in* $I = \int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dydx$ *and hence evaluate.*

(*Nagpur, 2009 ; P.T.U., 2009 S*)

Solution. Here integration is first w.r.t. y and P on the parabola $x^2 = 4ay$ to Q on the parabola $y^2 = 4ax$ and then w.r.t. x from $x = 0$ to $x = 4a$ giving the shaded region of integration (Fig. 7.8).

On changing the order of integration, we first integrate w.r.t. x from R to S, then w.r.t. y from $y = 0$ to $y = 4a$

$$\therefore \quad I = \int_0^{4a} \int_{y^2/4a}^{2\sqrt{ay}} dxdy = \int_0^{4a} dy \left| x \right|_{y^2/4a}^{2\sqrt{ay}} = \int_0^{4a} (2\sqrt{ay} - y^2/4a)\, dy$$

$$= \left| 2\sqrt{a}\, \frac{y^{3/2}}{3/2} - \frac{y^3}{12a} \right|_0^{4a} = \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3}.$$

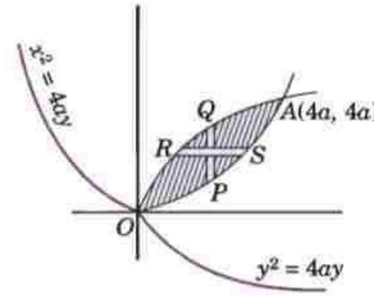

Fig. 7.8

Example 7.8. *Change the order of integration and hence evaluate*

$$I = \int_0^a \int_{\sqrt{ax}}^a \frac{y^2\, dxdy}{\sqrt{(y^4 - a^2x^2)}}$$

(S.V.T.U., 2006 S)

Solution. Here integration is first w.r.t. y from P on the parabola $y^2 = ax$ to Q on the line $y = a$, then w.r.t. x from $x = 0$ to $x = a$, giving the shaded region OAB of integration (Fig. 7.9).

On changing the order of integration, we first integrate w.r.t. x from R to S, then w.r.t. y from $y = 0$ to $y = a$.

$$\therefore \quad I = \int_0^a \int_0^{y^2/a} \frac{y^2\, dy}{\sqrt{(y^4 - a^2x^2)}}\, dx = \frac{1}{a} \int_0^a \int_0^{y^2/a} y^2\, dy \frac{dx}{\sqrt{[(y^2/a)^2 - x^2]}}\, dx$$

$$= \frac{1}{a} \int_0^a y^2 dy \left| \sin^{-1}\left(\frac{xa}{y^2}\right) \right|_0^{y^2/a} = \frac{1}{a} \int_0^a y^2 dy\ [\sin^{-1}(1) - \sin^{-1}(0)]$$

$$= \frac{\pi}{2a} \int_0^a y^2 dy = \frac{\pi}{2a} \left| \frac{y^3}{3} \right|_0^a = \frac{\pi a^2}{6}.$$

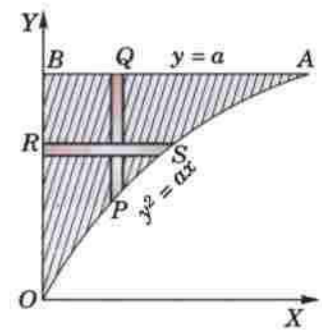

Fig. 7.9

Example 7.9. *Change the order of integration in* $I = \int_0^1 \int_{x^2}^{2-x} xy\, dxdy$ *and hence evaluate the same.*

(Bhopal, 2008 ; V.T.U., 2008 ; S.V.T.U., 2007 ; P.T.U., 2005 ; U.P.T.U., 2005)

Solution. Here the integration is first w.r.t. y along a vertical strip PQ which extends from P on the parabola $y = x^2$ to Q on the line $y = 2 - x$. Such a strip slides from $x = 0$ to $x = 1$, giving the region of integration as the curvilinear triangle OAB (shaded) in Fig. 7.10.

On changing the order of integration, we first integrate w.r.t. x along a horizontal strip $P'Q'$ and that requires the splitting up of the region OAB into two parts by the line AC ($y = 1$), *i.e.*, the curvilinear triangle OAC and the triangle ABC.

For the region OAC, the limits of integration for x are from $x = 0$ to $x = \sqrt{y}$ and those for y are from $y = 0$ to $y = 1$. So the contribution to I from the region OAC is

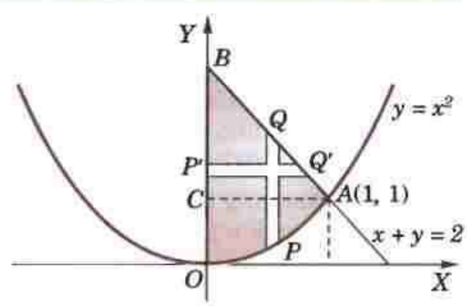

Fig. 7.10

$$I_1 = \int_0^1 dy \int_0^{\sqrt{y}} xy\, dx$$

For the region ABC, the limits of integration for x are from $x = 0$ to $x = 2 - y$ and those for y are from $y = 1$ to $y = 2$. So the contribution to I from the region ABC is

$$I_2 = \int_1^2 dy \int_0^{2-y} xy\, dx.$$

Hence, on reversing the order of integration,

$$I = \int_0^1 dy \int_0^{\sqrt{y}} xy\,dx + \int_1^2 dy \int_0^{2-y} xy\,dx$$

$$= \int_0^1 dy \left| \frac{x^2}{2}.y \right|_0^{\sqrt{y}} + \int_1^2 dy \left| \frac{x^2}{2}.y \right|_0^{2-y} = \frac{1}{2}\int_0^1 y^2 dy + \frac{1}{2}\int_1^2 y(2-y)^2\,dy = \frac{1}{6} + \frac{5}{24} = \frac{3}{8}.$$

Example 7.10. *Change the order of integration in* $I = \int_0^1 \int_x^{\sqrt{(2-x^2)}} \frac{x}{\sqrt{(x^2+y^2)}}\,dx$ *and hence evaluate it.*

(J.N.T.U., 2005 ; Rohtak, 2003)

Solution. Here the integration is first w.r.t. y along PQ which extends from P on the line $y = x$ to Q on the circle $y = \sqrt{(2.x^2)}$. Then PQ slides from $y = 0$ to $y = 1$, giving the region of integration OAB as in Fig. 7.11.

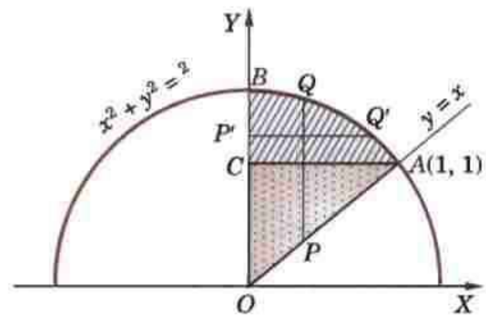

Fig. 7.11

On changing the order of integration, we first integrate w.r.t. x from P' to Q' and that requires splitting the region OAB into two parts OAC and ABC.

For the region OAC, the limits of integration for x are from $x = 0$ to $x = 1$ and those for y are from $y = 0$ to $y = 1$. So the contribution to I from the region OAC is

$$I_1 = \int_0^1 dy \int_0^y \frac{x}{\sqrt{(x^2+y^2)}}\,dx.$$

For the region ABC, the limits of integration for x are 0 to $\sqrt{(2-y^2)}$ and these for y are from 1 to $\sqrt{2}$. So the contribution to I from the region ABC is

$$I_2 = \int_1^{\sqrt{2}} dy \int_0^{\sqrt{(2-y^2)}} \frac{x}{\sqrt{(x^2+y^2)}}\,dx$$

Hence $$I = \int_0^1 \left| (x^2+y^2)^{1/2} \right|_0^y dy + \int_1^{\sqrt{2}} \left| (x^2+y^2)^{1/2} \right|_0^{\sqrt{(2-y^2)}} dy$$

$$= \int_0^1 (\sqrt{2}-1)\,y\,dy + \int_1^{\sqrt{2}} \sqrt{(2-y)}\,dy = \frac{1}{2}(\sqrt{2}-1) + \sqrt{2}\sqrt{(2-1)} - \frac{1}{2} = 1 - 1/\sqrt{2}.$$

7.3 DOUBLE INTEGRALS IN POLAR COORDINATES

To evaluate $\int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r, \theta)\,dr\,d\theta$, we first integrate w.r.t. r between limits $r = r_1$ and $r = r_2$ keeping θ fixed and the resulting expression is integrated w.r.t. θ from θ_1 to θ_2. In this integral, r_1, r_2 are functions of θ and θ_1, θ_2 are constants.

Figure 7.12 illustrates the process geometrically.

Here AB and CD are the curves $r_1 = f_1(\theta)$ and $r_2 = f_2(\theta)$ bounded by the lines $\theta = \theta_1$ and $\theta = \theta_2$. PQ is a wedge of angular thickness $\delta\theta$.

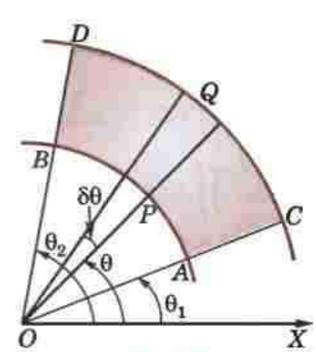

Fig. 7.12

Then $\int_{r_1}^{r_2} f(r, \theta)\,dr$ indicates that the integration is along PQ from P to Q while the integration w.r.t. θ corresponds to the turning of PQ from AC to BD.

Thus the whole region of integration is the area $ACDB$. The order of integration may be changed with appropriate changes in the limits.

Example 7.11. *Evaluate* $\iint r \sin\theta \, dr \, d\theta$ *over the cardioid* $r = a(1 - \cos\theta)$ *above the initial line.* *(Kerala, 2005)*

Solution. To integrate first w.r.t. r, the limits are from 0 ($r = 0$) to P [$r = a(1 - \cos\theta)$] and to cover the region of integration R, θ varies from 0 to π (Fig. 7.13).

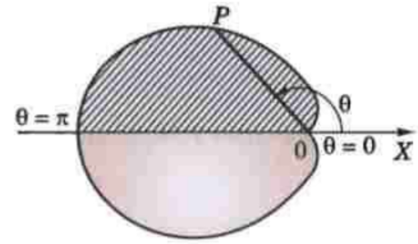

Fig. 7.13

$$\therefore \quad \iint_R r \sin\theta \, dr d\theta = \int_0^{\pi} \sin\theta \left[\int_0^{r = a(1-\cos\theta)} r dr \right] d\theta$$

$$= \int_0^{\pi} \sin\theta \, d\theta \left| \frac{r^2}{2} \right|_0^{a(1-\cos\theta)} = \frac{a^2}{2} \int_0^{\pi} (1 - \cos\theta)^2 . \sin\theta \, d\theta$$

$$= \frac{a^2}{2} \left| \frac{(1-\cos\theta)^3}{3} \right|_0^{\pi} = \frac{a^2}{2} . \frac{8}{3} = \frac{4a^2}{3}.$$

Example 7.12. *Calculate* $\iint r^3 \, dr \, d\theta$ *over the area included between the circles* $r = 2\sin\theta$ *and* $r = 4\sin\theta$.

Solution. Given circles $r = 2\sin\theta$ ...(*i*)

and $r = 4\sin\theta$ ...(*ii*)

are shown in Fig. 7.14. The shaded area between these circles is the region of integration.

If we integrate first w.r.t. r, then its limits are from $P(r = 2\sin\theta)$ to $Q(r = 4\sin\theta)$ and to cover the whole region θ varies from 0 to π. Thus the required integral is

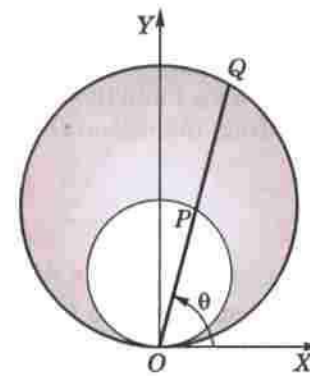

Fig. 7.14

$$I = \int_0^{\pi} d\theta \int_{2\sin\theta}^{4\sin\theta} r^3 \, dr = \int_0^{\pi} d\theta \left[\frac{r^4}{4} \right]_{2\sin\theta}^{4\sin\theta}$$

$$= 60 \int_0^{\pi} \sin^4\theta \, d\theta = 60 \times 2 \int_0^{\pi/2} \sin^4\theta \, d\theta = 120 \times \frac{3.1}{4.2} . \frac{\pi}{2} = 22.5\pi.$$

PROBLEMS 7.1

Evaluate the following integrals (1–7) :

1. $\int_1^2 \int_1^3 xy^2 \, dxdy$.

2. $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2) \, dx \, dy$. *(V.T.U., 2000)*

3. $\int_0^1 \int_0^x e^{x/y} \, dxdy$. *(P.T.U., 2005)*

4. $\int_0^1 \int_0^{\sqrt{(1+x^2)}} \frac{dydx}{1 + x^2 + y^2}$ *(Rajasthan, 2005)*

5. $\iint xy \, dxdy$ over the positive quadrant of the circle $x^2 + y^2 = a^2$. *(Rajasthan, 2006)*

6. $\iint (x + y)^2 dxdy$ over the area bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$. *(Kurukshetra, 2009 S ; U.P.T.U., 2004 S)*

7. $\iint xy(x + y) \, dxdy$ over the area between $y = x^2$ and $y = x$. *(V.T.U., 2010)*

Evaluate the following integrals by changing the order of integration (8–15) :

8. $\int_0^a \int_y^a \frac{xdxdy}{x^2 + y^2}$. *(Bhopal, 2008)*

9. $\int_0^3 \int_1^{\sqrt{(4-y)}} (x + y) \, dxdy$. *(V.T.U., 2005 ; Anna, 2003 S ; Delhi, 2002)*

10. $\int_0^1 \int_x^{\sqrt{(2-x^2)}} \frac{x\, dydx}{\sqrt{(x^2+y^2)}}$. (P.T.U., 2010 ; Marathwada, 2008 ; U.P.T.U., 2006)

11. $\int_0^{a/\sqrt{2}} \int_y^{\sqrt{(a^2-y^2)}} \log (x^2+y^2)\, dxdy\ (a>0)$.

12. $\int_0^1 \int_x^{\sqrt{x}} xy\, dydx$. (V.T.U., 2010)

13. $\int_0^a \int_{a-\sqrt{(a^2-y^2)}}^{a+\sqrt{(a^2-y^2)}} xy\, dx\, dy$. (Anna, 2009)

14. $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dydx$. (Bhopal, 2009 ; S.V.T.U., 2009 ; V.T.U., 2007)

15. $\int_0^\infty \int_0^x xe^{-x^2/y}\, dydx$. (S.V.T.U., 2006 ; U.P.T.U., 2005 ; V.T.U., 2004)

16. Sketch the region of integration of the following integrals and change the order of integrations,

(i) $\int_0^{2a} \int_{\sqrt{(2ax-x^2)}}^{\sqrt{(2ax)}} f(x)\, dxdy$ (Rajasthan, 2006) (ii) $\int_0^{ae^{\pi/4}} \int_{2\log(r/a)}^{\pi/2} f(r, \theta)\, r\, drd\theta$.

17. Show that $\iint_R r^2 \sin\theta\, drd\theta = 2a^2/3$, where R is the semi-circle $r = 2a\cos\theta$ above the initial line.

18. Evaluate $\iint \frac{r\, drd\theta}{\sqrt{a^2+r^2}}$ over one loop of the lemniscate $r^2 = a^2 \cos 2\theta$. (Rohtak, 2006 S ; P.T.U., 2005)

19. Evaluate $\iint r^3\, drd\theta$ over the area bounded between the circles $r = 2\cos\theta$ and $r = 4\cos\theta$.

(Anna, 2009 ; Madras, 2006)

7.4 AREA ENCLOSED BY PLANE CURVES

(1) *Cartesian coordinates.*

Consider the area enclosed by the curves $y = f_1(x)$ and $y = f_2(x)$ and the ordinates $x = x_1$, $x = x_2$ [Fig. 7.15 (*a*)].

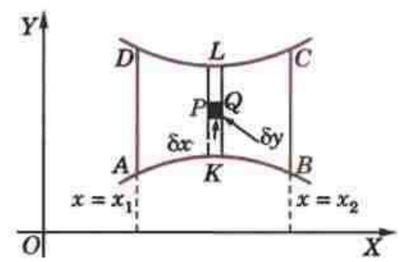

Fig. 7.15(*a*)

Divide this area into vertical strips of width δx. If $P(x, y)$, $Q(x + \delta x, y + \delta y)$ be two neighbouring points, then the area of the small rectangle $PQ = \delta x \delta y$.

$\therefore$ area of strip $KL = \underset{\delta y \to 0}{\text{Lt}} \Sigma\, \delta x\, \delta y$.

Since for all rectangles in this strip δx is the same and y varies from $y = f_1(x)$ to $y = f_2(x)$.

$$\therefore \quad \text{area of the strip } KL = \delta x \underset{\delta y \to 0}{\text{Lt}} \sum_{f_1(x)}^{f_2(x)} dy = \delta x \int_{f_1(x)}^{f_2(x)} dy.$$

Now adding up all such strips from $x = x_1$ to $x = x_2$, we get the area $ABCD$

$$= \underset{\delta x \to 0}{\text{Lt}} \sum_{x_1}^{x_2} \delta x \,.\int_{f_1(x)}^{f_2(x)} dy = \int_{x_1}^{x_2} dx \int_{f_1(x)}^{f_2(x)} dy = \int_{x_1}^{x_2} \int_{f_1(x)}^{f_2(x)} dxdy$$

Similarly, dividing the area $A'B'C'D$ [Fig. 7.15(*b*)] into horizontal strips of width δy, we get the area $A'B'C'D'$.

$$= \int_{y_1}^{y_2} \int_{f_1(y)}^{f_2(y)} dxdy$$

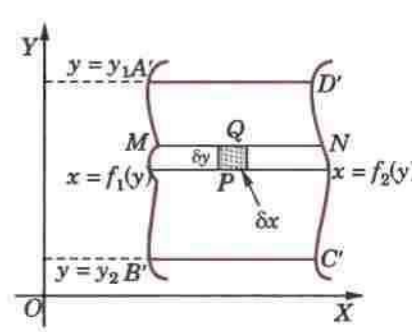

Fig. 7.15 (*b*)

(2) *Polar coordinates.*

Consider an area A enclosed by a curve whose equation is in polar coordinates.

Let $P(r, \theta)$, $Q(r + \delta r, \theta + \delta\theta)$ be two neighbouring points. Mark circular areas of radii r and $r + \delta r$ meeting OQ in R and OP (produced) in S (Fig. 7.16).

Since arc $PR = r\delta\theta$ and $PS = \delta r$.

$\therefore$ area of the curvilinear rectangle $PRQS$ is approximately $= PR \,.\, PS = r\delta\theta \,.\, \delta r$.

If the whole area is divided into such curvilinear rectangles, the sum $\Sigma\Sigma r\delta\theta\delta r$ taken for all these rectangles, gives in the limit the area A.

Hence $$A = \underset{\substack{\delta r \to 0 \\ \delta\theta \to 0}}{\text{Lt}} \Sigma\Sigma r\delta\theta\delta r = \iint r\,d\theta\,dr$$

where the limits are to be so chosen as to cover the entire area.

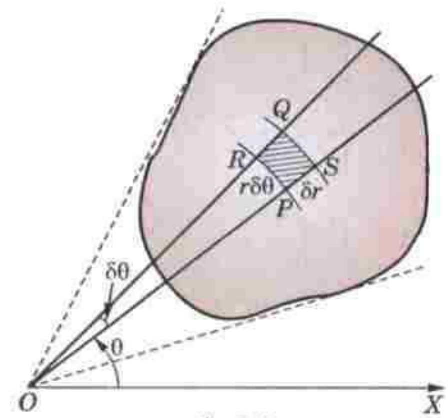

Fig. 7.16

Example 7.13. *Find the area of a plate in the form of a quadrant of the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$ (V.T.U., 2001 ; Osmania, 2000 S)

Solution. Dividing the area into vertical strips of width δx, y varies from $K(y = 0)$ to $L[y = b\sqrt{(1 - x^2/b^2)}]$ and then x varies from 0 to a (Fig. 7.17).

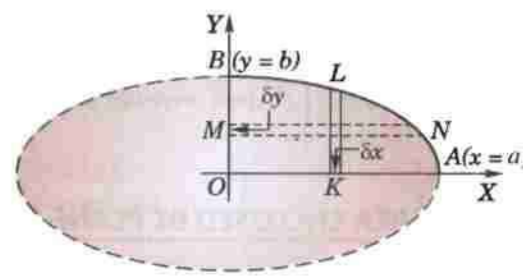

Fig. 7.17

$\therefore$ required area

$$= \int_0^a dx \int_0^{b\sqrt{(1-x^2/a^2)}} dy = \int_0^a dx\,[y]_0^{b\sqrt{(1-x^2/a^2)}}$$

$$= \frac{b}{a}\int_0^a \sqrt{(a^2 - x^2)}\,dx = \pi ab/4.$$

Otherwise, dividing this area into horizontal strips of width δy, x varies from $M(x = 0)$ to $N[x = a\sqrt{(1 - y^2/b^2)}\,]$ and then y varies from 0 to b.

$\therefore$ required area $$= \int_0^b dy \int_0^{a\sqrt{(1-y^2/b^2)}} dx = \int_0^b dy\,[x]_0^{a\sqrt{(1-y^2/b^2)}}$$

$$= \frac{a}{b}\int_0^b \sqrt{(b^2 - y^2)}\,dy = \pi ab/4.$$

Obs. The change of the order of integration does not in any way affect the value of the area.

Example 7.14. *Show that the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$.*

(Kerala, 2005 ; Rohtak, 2003)

Solution. Solving the equations $y^2 = 4ax$ and $x^2 = 4ay$, it is seen that the parabolas intersect at $O\,(0, 0)$ and $A\,(4a, 4a)$. As such for the shaded area between these parabolas (Fig. 7.18) x varies from 0 to $4a$ and y varies from P to Q *i.e.*, from $y = x^2/4a$ to $y = 2\sqrt{(ax)}$. Hence the *required area*

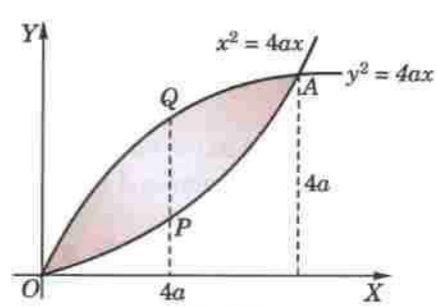

Fig. 7.18

$$= \int_0^{4a}\int_{x^2/4a}^{2\sqrt{(ax)}} dy\,dx = \int_0^{4a} (2\sqrt{(ax)} - x^2/4a)\,dx$$

$$= \left| 2\sqrt{a}\,.\,\frac{2}{3}x^{3/2} - \frac{1}{4a}\,.\,\frac{x^3}{3} \right|_0^{4a} = \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2.$$

Example 7.15. *Calculate the area included between the curve $r = a\ (\sec\theta + \cos\theta)$ and its asymptote.*

Solution. The curve is symmetrical about the initial line and has an asymptote $r = a \sec\theta$ (Fig. 7.19).

Draw any line OP cutting the curve at P and its asymptote at P'. Along this line, θ is constant and r varies from $a \sec\theta$ at P' to $a\ (\sec\theta + \cos\theta)$ at P. Then to get the upper half of the area, θ varies from 0 to $\pi/2$.

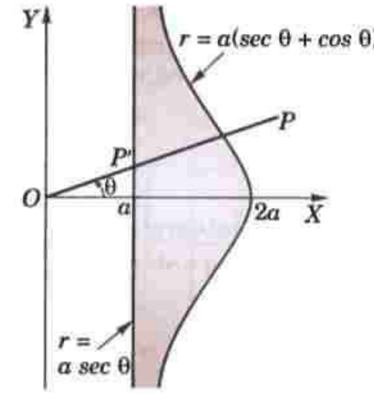

Fig. 7.19

$$\therefore \quad \text{required area} = 2\int_0^{\pi/2}\int_{a\sec\theta}^{a(\sec\theta+\cos\theta)} r\,dr d\theta$$

$$= 2\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_{a\sec\theta}^{a(\sec\theta+\cos\theta)} d\theta$$

$$= a^2\int_0^{\pi/2}(2+\cos^2\theta)\,d\theta = 5\pi a^2/4.$$

Example 7.16. *Find the area lying inside the cardioid $r = a(1 + \cos\theta)$ and outside the circle $r = a$.*

Solution. In Fig. 7.20, $ABODA$ represents the cardioid $r = a(1 + \cos\theta)$ and $CBA'DC$ is the circle $r = a$.

Required area (shaded) = 2 (area $ABCA$)

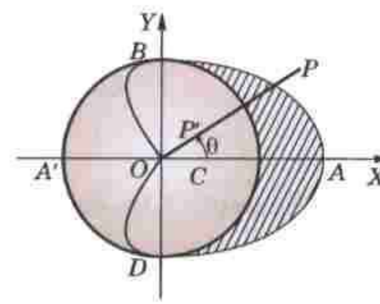

Fig. 7.20

$$= 2\int_0^{\pi/2}\int_{r=OP'}^{r=OP} r\,d\theta\,dr = 2\int_0^{\pi/2}\int_a^{a(1+\cos\theta)}(rdr)\,d\theta$$

$$= 2\int_0^{\pi/2}\left|\frac{r^2}{2}\right|_a^{a(1+\cos\theta)} d\theta = a^2\int_0^{\pi/2}[(1+\cos\theta)^2-1]\,d\theta$$

$$= a^2\int_0^{\pi/2}(\cos^2\theta+2\cos\theta)\,d\theta = a^2\left(\frac{1}{2}\cdot\frac{\pi}{2}+2\right) = \frac{a^2}{4}(\pi+8).$$

PROBLEMS 7.2

1. Find, by double integration, the area lying between the parabola $y = 4x - x^2$ and the line $y = x$.
2. Find the area lying between the parabola $y = x^2$ and the line $x + y - z = 0$. *(Anna, 2009)*
3. By double integration, find the whole area of the curve $a^2x^2 = y^3(2a - y)$. *(U.P.T.U., 2001)*
4. Find, by double integration, the area enclosed by the curves $y = 3x/(x^2 + 2)$ and $4y = x^2$. *(J.N.T.U., 2005)*
5. Find, by double integration, the area of the lemniscate $r^2 = a^2 \cos 2\theta$. *(Madras, 2000 S)*
6. Find, by double integration, the area lying inside the circle $r = a \sin\theta$ and outside the cardioid $r = a(1 - \cos\theta)$. *(Anna 2009 ; Mumbai, 2006)*
7. Find the area lying inside the cardioid $r = 1 + \cos\theta$ and outside the parabola $r(1 + \cos\theta) = 1$.
8. Find the area common to the circles $r = a\cos\theta$, $r = a\sin\theta$ by double integration. *(Mumbai, 2007)*

7.5 TRIPLE INTEGRALS

Consider a function $f(x, y, z)$ defined at every point of the 3-dimensional finite region V. Divide V into n elementary volumes $\delta V_1, \delta V_2, \ldots, \delta V_n$. Let (x_r, y_r, z_r) be any point within the rth sub-division δV_r. Consider the sum

$$\sum_{r=1}^{\infty} f(x_r, y_r, z_r)\delta V_r .$$

The limit of this sum, if it exists, as $n \to \infty$ and $\delta V_r \to 0$ is called the *triple integral of f(x, y, z) over the region V* and is denoted by

$$\iiint f(x, y, z)\,dV.$$

For purposes of evaluation, it can also be expressed as the repeated integral

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x, y, z)\,dxdydz.$$

If x_1, x_2 are constants ; y_1, y_2 are either constants or functions of x and z_1, z_2 are either constants or functions of x and y, then this integral is evaluated as follows :

First $f(x, y, z)$ is integrated w.r.t. z between the limits z_1 and z_2 keeping x and y fixed. The resulting expression is integrated w.r.t. y between the limits y_1 and y_2 keeping x constant. The result just obtained is finally integrated w.r.t. x from x_1 to x_2.

Thus $$I = \boxed{\int_{x_1}^{x_2} \boxed{\int_{y_1(x)}^{y_2(x)} \boxed{\int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)dz}\, dy}\, dx}$$

where the integration is carried out from the innermost rectangle to the outermost rectangle.

The order of integration may be different for different types of limits.

Example 7.17. *Evaluate* $\int_{-1}^{1} \int_{0}^{z} \int_{x-z}^{x+z} (x + y + z)\, dxdydz.$ *(J.N.T.U., 2006 ; Cochin, 2005)*

Solution. Integrating first w.r.t. y keeping x and z constant, we have

$$I = \int_{-1}^{1} \int_{0}^{z} \left| xy + \frac{y^2}{2} + yz \right|_{x-z}^{x+z} dx\, dz = \int_{-1}^{1} \int_{0}^{z} \left[(x+z)(2z) + \frac{1}{2} 4xz \right] dx\, dz$$

$$= 2 \int_{-1}^{1} \left| \frac{x^2 z}{2} + z^2 x + \frac{x^2}{2} z \right|_0^z dz = 2 \int_{-1}^{1} \left(\frac{z^3}{2} + z^3 + \frac{z^3}{2} \right) dz = 4 \left| \frac{z^4}{4} \right|_{-1}^{1} = 0.$$

Example 7.18. *Evaluate* $\int_{0}^{1} \int_{0}^{\sqrt{(1-x^2)}} \int_{0}^{\sqrt{(1-x^2-y^2)}} xyz\, dxdydz.$ *(V.T.U., 2003 S)*

Solution. We have

$$I = \int_0^1 x \left[\int_0^{\sqrt{(1-x^2)}} y \left\{ \int_0^{\sqrt{(1-x^2-y^2)}} z\, dz \right\} dy \right] dx = \int_0^1 x \left\{ \int_0^{\sqrt{(1-x^2)}} y \,.\left| \frac{z^2}{2} \right|_0^{\sqrt{(1-x^2-y^2)}} dy \right\} dx$$

$$= \int_0^1 x \left\{ \int_0^{\sqrt{(1-x^2)}} y \,.\, \frac{1}{2}(1 - x^2 - y^2)\, dy \right\} dx = \frac{1}{2} \int_0^1 x \left| (1-x^2)\frac{y^2}{2} - \frac{y^4}{4} \right|_0^{\sqrt{(1-x^2)}} dx$$

$$= \frac{1}{8} \int_0^1 [(1-x^2)^2 \,.\, 2x - (1-x^2)^4 \,.\, x]\, dx = \frac{1}{8} \int_0^1 (x - 2x^3 + x^5)\, dx$$

$$= \frac{1}{8} \left| \frac{x^2}{2} - \frac{2x^4}{4} + \frac{x^6}{6} \right|_0^1 = \frac{1}{8}\left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6} \right) = \frac{1}{48}.$$

PROBLEMS 7.3

Evaluate the following integrals :

1. $\int_0^a \int_0^b \int_0^c (x^2 + y^2 + z^2)\, dx\, dy\, dz$. *(Anna, 2009)*
2. $\int_{-c}^{c} \int_{-b}^{b} \int_{-a}^{a} (x^2 + y^2 + z^2)\, dx\, dy\, dz$ *(S.V.T.U., 2009 ; V.T.U. 2000)*
3. $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x\, dz\, dx\, dy$ *(Nagpur, 2009)*
4. $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx.$ *(V.T.U., 2010 ; Kurukshetra, 2009 S ; J.N.T.U., 2005)*
5. $\int_0^{\log 2} \int_0^x \int_0^{x+\log y} e^{x+y+z}\, dx\, dy\, dz.$ *(Bhopal, 2008)*
6. $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z\, dz\, dx\, dy.$ *(S.V.T.U., 2008 ; Rohtak, 2005)*
7. $\int_0^{\pi/2} \int_0^{a \sin\theta} \int_0^{\frac{a^2 - r^2}{a}} r dz\, dr\, d\theta.$ *(V.T.U., 2009)*

7.6 VOLUMES OF SOLIDS

(1) Volumes as double integrals. Consider a surface $z = f(x, y)$. Let the orthogonal projection on XY-plane of its portion S' be the area S (Fig. 7.21).

Divide S into elementary rectangles of area $\delta x\, \delta y$ by drawing lines parallel to X and Y-axes. With each of these rectangles as base, erect a prism having its length parallel to OZ.

$\therefore$ volume of this prism between S and the given surface $z = f(x, y)$ is $z\delta x\delta y$.

Hence the volume of the solid cylinder on S as base, bounded by the given surface with generators parallel to the Z-axis.

$$= \underset{\substack{\delta x \to 0 \\ \delta y \to 0}}{\text{Lt}} \Sigma\Sigma\, z\delta x\delta y$$

$$= \iint z\, dx\, dy \quad \text{or} \quad \iint f(x, y)\, dx\, dy$$

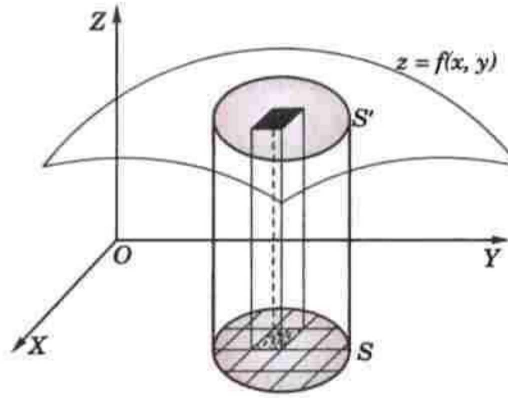

Fig. 7.21

where the integration is carried over the area S.

Obs. While using polar coordinates, divide S into elements of area $r\delta\theta\, \delta r$.

$\therefore$ replacing $dxdy$ by $r\delta\theta\delta r$, we get the required volume $= \iint zr\, d\theta\, dr$.

Example 7.19. *Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.*
(S.V.T.U., 2007 ; Cochin, 2005 ; Madras, 2000 S)

Solution. From Fig. 7.22, it is self-evident that $z = 4 - y$ is to be integrated over the circle $x^2 + y^2 = 4$ in the XY-plane. To cover the shaded half of this circle, x varies from 0 to $\sqrt{(4 - y^2)}$ and y varies from -2 to 2.

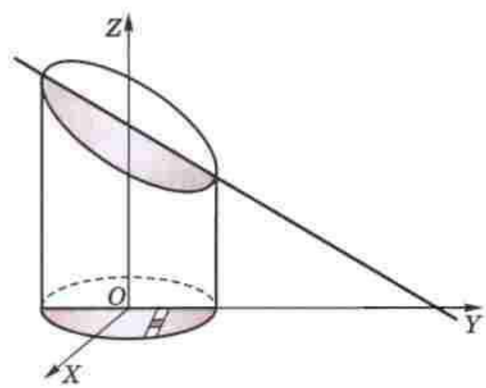

Fig. 7.22

$\therefore$ Required volume

$$= 2\int_{-2}^{2}\int_{0}^{\sqrt{(4-y^2)}} z\, dx\, dy = 2\int_{-2}^{2}\int_{0}^{\sqrt{(4-y^2)}} (4 - y)\, dx\, dy$$

$$= 2\int_{-2}^{2} (4 - y)\,[x]_0^{\sqrt{(4-y^2)}}\, dy = 2\int_{-2}^{2} (4 - y)\sqrt{(4 - y^2)}\, dy$$

$$= 2\int_{-2}^{2} 4\sqrt{(4 - y^2)}\, dy - 2\int_{-2}^{2} y\sqrt{(4 - y^2)}\, dy$$

$$= 8\int_{-2}^{2} \sqrt{(4 - y^2)}\, dy$$ [The second term vanishes as the integrand is an odd function.]

$$= 8\left|\frac{y\sqrt{(4 - y)^2}}{2} + \frac{4}{2}\sin^{-1}\frac{y}{2}\right|_{-2}^{2} = 16\pi.$$

(2) Volume as triple integral

Divide the given solid by planes parallel to the coordinate planes into rectangular parallelopipeds of volume $\delta x\, \delta y\, \delta z$ (Fig. 7.23).

$\therefore$ the total volume $= \underset{\substack{\delta x \to 0 \\ \delta y \to 0 \\ \delta z \to 0}}{\text{Lt}} \Sigma\Sigma\Sigma\, \delta x\, \delta y\, \delta z$

$$= \iiint dx\, dy\, dz$$

with appropriate limits of integration.

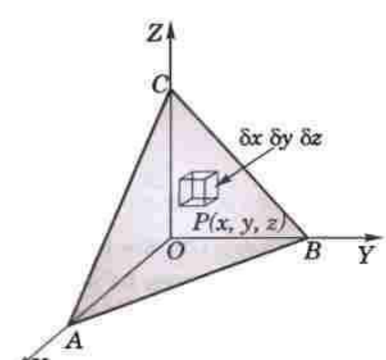

Fig. 7.23

Example 7.20. *Calculate the volume of the solid bounded by the planes $x = 0, y = 0, x + y + z = a$ and $z = 0$.* (P.T.U., 2009)

Solution. Volume required $= \int_0^a \int_0^{a-x} \int_0^{a-x-y} dz\, dy\, dx$

$$= \int_0^a \int_0^{a-x} (a - x - y)\, dy\, dx = \int_0^a \left| (a-x)y - \frac{y^2}{2} \right|_0^{a-x} dx$$

$$= \int_0^a \left\{ (a-x)^2 - \frac{(a-x)^2}{2} \right\} dx = \frac{1}{2} \int_0^a (a-x)^2\, dx = \frac{1}{2} \left| -\frac{(a-x)^3}{3} \right|_0^a = \frac{a^3}{6}.$$

Example 7.21. *Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.* (Anna, 2009 ; P.T.U., 2006 ; Kottayam, 2005)

Solution. Let $OABC$ be the positive octant of the given ellipsoid which is bounded by the planes OAB $(z = 0)$, OBC $(x = 0)$, OCA $(y = 0)$ and the surface ABC, *i.e.*,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Divide this region R into rectangular parallellopipeds of volume $\delta x \delta y \delta z$. Consider such an element at $P(x, y, z)$. (Fig. 7.24)

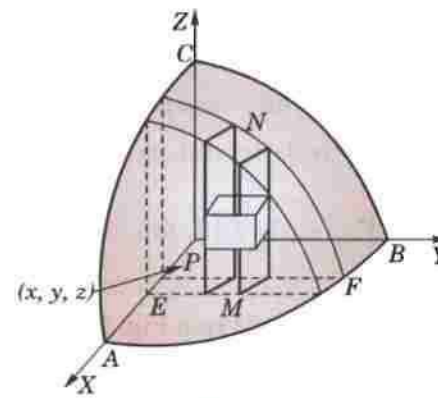

Fig. 7.24

$\therefore$ the required volume $= 8 \iiint_R dx\, dy\, dz$.

In this region R,

(*i*) z varies from 0 to MN where

$$MN = c\sqrt{(1 - x^2/a^2 - y^2/b^2)}.$$

(*ii*) y varies from 0 to EF, where $EF = b\sqrt{(1 - x^2/a^2)}$ from the equation of the ellipse OAB, *i.e.*,

$$x^2/a^2 + y^2/b^2 = 1.$$

(*iii*) x varies from 0 to $OA = a$.

Hence the volume of the whole ellipsoid

$$= 8 \int_0^a \int_0^{b\sqrt{(1-x^2/a^2)}} \int_0^{c\sqrt{(1-x^2/a^2-y^2/b^2)}} dx\, dy\, dz = 8 \int_0^a dx \int_0^{b\sqrt{(1-x^2/a^2)}} dy \left| z \right|_0^{c\sqrt{(1-x^2/a^2-y^2/b^2)}}$$

$$= 8c \int_0^a dx \int_0^{b\sqrt{(1-x^2/a^2)}} \sqrt{(1 - x^2/a^2 - y^2/b^2)}\, dy$$

$$= \frac{8c}{b} \int_0^a dx \int_0^{\rho} \sqrt{(\rho^2 - y^2)}\, dy \quad \text{when } \rho = b\sqrt{1 - x^2/a^2}.$$

$$= \frac{8c}{b} \int_0^a dx \left[\frac{y\sqrt{(\rho^2 - y^2)}}{2} + \frac{\rho^2}{2} \sin^{-1} \frac{y}{\rho} \right]_0^{\rho} = \frac{8c}{b} \int_0^a \frac{b^2}{2} \left(1 - \frac{x^2}{a^2} \right) \frac{\pi}{2}\, dx$$

$$= 2\pi bc \int_0^a \left(1 - \frac{x^2}{a^2} \right) dx = 2\pi bc \left| x - \frac{x^3}{3a^2} \right|_0^a = \frac{4\pi abc}{3}.$$

Otherwise. See Problem 27 page 292.

(3) Volumes of solids of revolution

Consider an elementary area $\delta x \delta y$ at the point $P(x, y)$ of a plane area A. (Fig. 7.25)

As this elementary area revolves about x-axis, we get a ring of volume

$$= \pi[(y + \delta y)^2 - y^2]\, \delta x = 2\pi y\, \delta x\, \delta y,$$

nearly to the first powers of δy.

Hence the total volume of the solid formed by the revolution of the area A about x-axis.

$$= \iint_A 2\pi y \, dxdy.$$

In polar coordinates, the above formula for the volume becomes

$$\iint_A 2\pi r \sin\theta \,.\, r d\theta dr, \text{ i.e. } \iint_A 2\pi r^2 \sin\theta \, d\theta dr$$

Similarly, the volume of the solid formed by the revolution of the area A about y-axis $= \iint_A 2\pi x \, dx \, dy$.

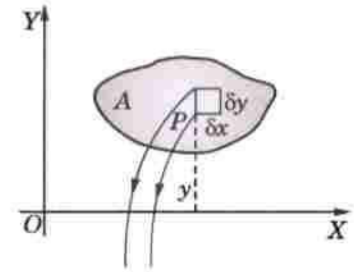

Fig. 7.25

Example 7.22. *Calculate by double integration, the volume generated by the revolution of the cardioid $r = a(1 - \cos\theta)$ about its axis.*

Solution. Required volume

$$= \int_0^{\pi} \int_0^{a(1-\cos\theta)} 2\pi r^2 \sin\theta \, dr \, d\theta$$

$$= 2\pi \int_0^{\pi} \left| \frac{r^3}{3} \right|_0^{a(1-\cos\theta)} \sin\theta \, d\theta$$

$$= \frac{2\pi a^3}{3} \int_0^{\pi} (1-\cos\theta)^3 \,.\, \sin\theta \, d\theta = \frac{2\pi a^3}{3} \left| \frac{(1-\cos\theta)^4}{4} \right|_0^{\pi} = \frac{8\pi a^3}{3}.$$

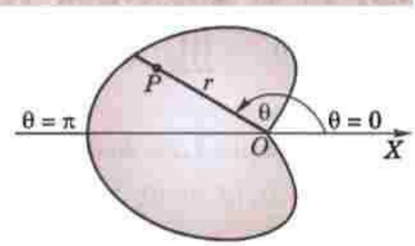

Fig. 7.26

7.7 CHANGE OF VARIABLES

An appropriate choice of co-ordinates quite often facilitates the evaluation of a double or a triple integral. By changing the variables, a given integral can be transformed into a simpler integral involving the new variables.

(1) In a double integral, let the variables x, y be changed to the new variables u, v by the transformation.

$$x = \phi(u, v), y = \psi(u, v)$$

where $\phi(u, v)$ and $\psi(u, v)$ are continuous and have continuous first order derivatives in some region R'_{uv} in the uv-plane which corresponds to the region R_{xy} in the xy-plane. Then

$$\iint_{R_{xy}} f(x, y) \, dxdy = \iint_{R'_{uv}} f[\phi(u, v), \psi(u, v)] \, |J| \, dudv \quad \text{...(1)}$$

where $$J = \frac{\partial(x, y)}{\partial(u, v)} \; (\neq 0)$$

is the *Jacobian of transformation** from (x, y) to (u, v) coordinates.

(2) For triple integrals, the formula corresponding to (1) is

$$\iiint_{R_{xyz}} f(x, y, z) \, dxdydz = \iiint_{R'_{uvw}} f[x(u, v, w), y(u, v, w), z(u, v, w)] \, |J| \, dudvdw$$

where $$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} \; (\neq 0)$$

is the *Jacobian of transformation* from (x, y, z) to (u, v, w) coordinates.

Particular cases :

(i) ***To change cartesian coordinates (x, y) to polar coordinates (r, θ),*** *we have* $x = r\cos\theta$, $y = r\sin\theta$ *and*

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = r \quad \text{[Ex. 5.25, p. 216]}$$

$$\therefore \quad \iint_{R_{xy}} f(x, y) \, dx \, dy = \iint_{R'_{r\theta}} f(r\cos\theta, r\sin\theta) \,.\, r \, dr \, d\theta.$$

* See footnote page 215.

(*ii*) ***To change rectangular coordinates (x, y, z) to cylindrical coordinates*** (ρ, ϕ, z) — *Fig. 8.27, we have*

$$x = \rho \cos \phi,\ y = \rho \sin \phi,\ z = z$$

and $$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \rho$$ [Ex. 5.25]

Then $$\iiint_{R_{xyz}} f(x, y, z)\, dxdydz = \iiint_{R'_{\rho\phi z}} f(\rho \cos \phi, \rho \sin \phi, z)\,.\,\rho d\rho d\phi dz.$$

(*iii*) ***To change rectangular coordinates (x, y, z) to spherical polar coordinates (r,*** θ, ϕ)—*Fig. 8.28, we have*

$$x = r \sin \theta \cos \phi,\ y = r \sin \theta \sin \phi,\ z = r \cos \theta$$

and $$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin \theta$$ [Ex. 5.25]

Then $$\iiint_{R_{xyz}} f(x, y, z)\, dxdydz = \iiint_{R'_{r\theta\phi}} f(r \sin \theta \cos \phi, r \sin \theta, \sin \phi, r \cos \theta)\,.\, r^2 \sin \theta\, dr d\theta d\phi$$

Example 7.23. *Evaluate* $\iint_R (x + y)^2\, dxdy$, *where R is the parallelogram in the xy-plane with vertices (1, 0), (3, 1), (2, 2), (0, 1) using the transformation* $u = x + y$ *and* $v = x - 2y$. *(U.P.T.U., 2004)*

Solution. The region R, *i.e.*, parallelogram $ABCD$ in the xy-plane becomes the region R', *i.e.*, rectangle $A'B'C'D'$ in the uv-plane as shown in Fig. 7.27, by taking

$$u = x + y \quad \text{and} \quad v = x - 2y \qquad ...(i)$$

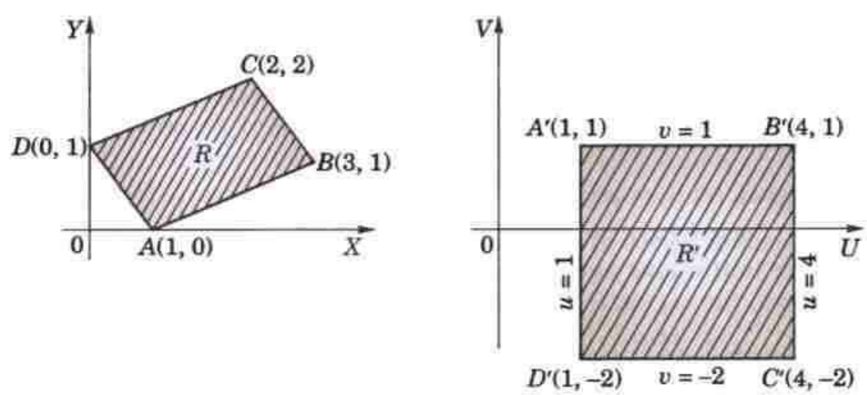

Fig. 7.27

From (*i*), we have $$x = \frac{1}{3}(2u + v),\ y = \frac{1}{3}(u - v)$$

$\therefore$ $$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = -\frac{1}{3}$$

Hence, the given integral

$$= \iint_{R'} u^2\, |J|\, dudv = \int_1^4 \int_{-2}^1 u^2 \,.\, \frac{1}{3}\,.\, dudv = \frac{1}{3} \left| \frac{u^3}{3} \right|_1^4 .\, |\, v\, |_{-2}^1 = 21.$$

Example 7.24. *Evaluate* $\iint_D xy\sqrt{(1 - x - y)}\, dx\, dy$ *where D is the region bounded by* $x = 0$, $y = 0$ *and* $x + y = 1$ *using the transformation* $x + y = u$, $y = uv$. *(Marathwada, 2008)*

Solution. We have $x = u - uv,\ y = uv$

$$\therefore \qquad J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \partial x/\partial u & \partial y/\partial u \\ \partial x/\partial v & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 1-v & v \\ -u & u \end{vmatrix} = u.$$

Also when $x = 0,\ u = 0,\ v = 1$; when $y = 0,\ u = 0,\ v = 0$ and when $x + y = 1,\ u = 1$

$\therefore$ the limits of u are from 0 to 1 and limits of v are from 0 to 1.

Thus
$$\iint_D xy\sqrt{(1-x-y)}\, dxdy = \int_0^1 \int_0^1 u(1-v)\, uv\, (1-u)^{1/2}\, |J|\, dudv$$

$$= \int_0^1 \int_0^1 u^3 (1-u)^{1/2}\, v(1-v)\, du\, dv$$

where $u = \sin^2\theta$
$du = 2\sin\theta\cos\theta\, d\theta$
$u = 0, \theta = 0$
$u = 1, \theta = \pi/2$

$$= \int_0^1 u^3 (1-u)^{1/2}\, du \times \int_0^1 v(1-v)\, dv$$

$$= \int_0^{\pi/2} \sin^6\theta\cos\theta \,.\, 2\sin\theta\cos d\theta \times \left| \frac{v^2}{2} - \frac{v^3}{3} \right|_0^1$$

$$= 2\int_0^{\pi/2} \sin^7\theta\cos^2\theta\, d\theta \left(\frac{1}{6}\right) = \frac{1}{3} \cdot \frac{6\,.\,1}{9\,.\,7\,.\,5\,.\,3} = \frac{2}{945}.$$

Example 7.25. *Evaluate* $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dxdy$ *by changing to polar coordinates.* (Anna, 2003)

Hence show that $\int_0^\infty e^{-x^2}\, dx = \sqrt{\pi/2}$. (Madras, 2003 ; U.P.T.U., 2003 ; J.N.T.U., 2000)

Solution. The region of integration being the first quadrant of the xy-plane, r varies from 0 to ∞ and θ varies from 0 to $\pi/2$. Hence,

$$I = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dxdy = \int_{\theta=0}^{\pi/2} \int_{r=0}^\infty e^{-r^2}\, r dr d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2} \left\{ \int_0^\infty e^{-r^2} (-2r)\, dr \right\} d\theta = -\frac{1}{2}\int_0^{\pi/2} \left| e^{-r^2} \right|_0^\infty d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{\pi}{4}. \qquad ...(i)$$

Also
$$I = \int_0^\infty e^{-x^2}\, dx \times \int_0^\infty e^{-y^2}\, dy = \left\{ \int_0^\infty e^{-x^2}\, dx \right\}^2 \qquad ...(ii)$$

Thus, from (*i*) and (*ii*), we have $\int_0^\infty e^{-x^2} = \sqrt{\pi/2}$. ...(*iii*)

Example 7.26. *Find the volume bounded by the paraboloid* $x^2 + y^2 = az$, *the cylinder* $x^2 + y^2 = 2ay$ *and the plane* $z = 0$.

Solution. The required volume is found by integrating $z = (x^2 + y^2)/a$ over the circle $x^2 + y^2 = 2ay$.

Changing to polar coordinates in the xy-plane, we have $x = r\cos\theta$, $y = r\sin\theta$ so that $z = r^2/a$ and the polar equation of the circle is $r = 2a\sin\theta$.

To cover this circle, r varies from 0 to $2a\sin\theta$ and θ varies from 0 to π. (Fig. 7.28)

Hence the required volume

$$= \int_0^\pi \int_0^{2a\sin\theta} z\,.\,r\, d\theta\, dr = \frac{1}{a}\int_0^\pi d\theta \int_0^{2a\sin\theta} r^3\, dr$$

$$= \frac{1}{a}\int_0^\pi d\theta \left| \frac{r^4}{4} \right|_0^{2a\sin\theta} = 4a^3 \int_0^\pi \sin^4\theta\, d\theta = \frac{3\pi a^3}{2}.$$

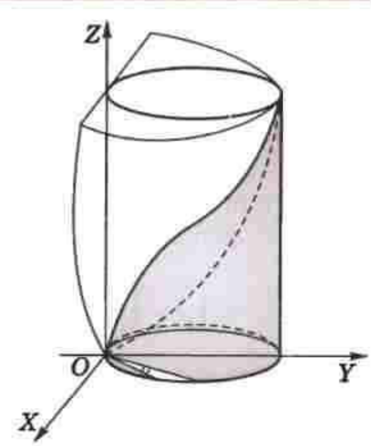

Fig. 7.28

Example 7.27. *Find, by triple integration, the volume of the sphere $x^2 + y^2 + z^2 = a^2$.* (*Bhopal, 2009 ; Madras, 2006 ; V.T.U., 2003 S*)

Solution. Changing to polar spherical coordinates by putting

$$x = r \sin \theta \cos \phi,\ y = r \sin \theta \sin \phi,\ z = r \cos \theta$$

we have $dx\, dy\, dz = r^2 \sin \theta\, dr\, d\theta\, d\phi$.

Also the volume of the sphere is 8 times the volume of its portion in the positive octant for which r varies from 0 to a, θ varies from 0 to $\pi/2$ and ϕ varies from 0 to $\pi/2$.

$\therefore$ volume of the sphere

$$= 8 \int_0^a \int_0^{\pi/2} \int_0^{\pi/2} r^2 \sin \theta\, dr\, d\theta\, d\phi = 8 \int_0^a r^2\, dr \,.\, \int_0^{\pi/2} .\sin \theta\, d\theta \,.\, \int_0^{\pi/2} d\phi$$

$$= 8 \,.\, \left| \frac{r^3}{3} \right|_0^a .\left| -\cos \theta \right|_0^{\pi/2} .\, \frac{\pi}{2} = 4\pi \,.\, \frac{a^3}{3} \,.\, (-0 + 1) = \frac{4}{3} \pi a^3.$$

Example 7.28. *Find the volume of the portion of the sphere $x^2 + y^2 + z^2 = a^2$ lying inside the cylinder $x^2 + y^2 = ay$.* (*Rohtak, 2003*)

Solution. The required volume is easily found by changing to cylindrical coordinates (ρ, ϕ, z). We therefore, have

$$x = \rho \cos \phi,\ y = \rho \sin \phi,\ z = z$$

and $$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \rho.$$

Then the equation of the sphere becomes $\rho^2 + z^2 = a^2$ and that of cylinder becomes $\rho = a \sin \phi$.

The volume inside the cylinder bounded by the sphere is twice the volume shown shaded in the Fig. 7.29 for which z varies from 0 to $\sqrt{(a^2 - \rho^2)}$, ρ varies from 0 to $a \sin \phi$ and ϕ varies from 0 to π.

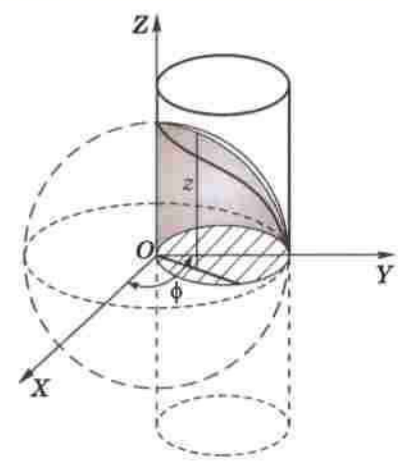

Fig. 7.29

Hence the required volume $= 2 \int_0^{\pi} \int_0^{a \sin \phi} \int_0^{\sqrt{(a^2 - \rho^2)}} \rho dz d\rho d\phi$

$$= 2 \int_0^{\pi} \int_0^{a \sin \phi} \rho \sqrt{(a^2 - \rho^2)}\, d\rho d\phi = 2 \int_0^{\pi} \left| -\frac{1}{3}(a^2 - \rho^2)^{3/2} \right|_0^{a \sin \phi} d\phi$$

$$= \frac{2a^3}{3} \int_0^{\pi} (1 - \cos^3 \phi)\, d\phi = \frac{2a^3}{9}(3\pi - 4).$$

Example 7.29. *Evaluate* $\int_0^1 \int_0^{\sqrt{(1-x^2)}} \int_{\sqrt{(x^2+y^2)}}^1 \frac{dz\, dy\, dx}{\sqrt{(x^2 + y^2 + z^2)}}$. (*V.T.U., 2008*)

Solution. We change to spherical polar coordinates (r, θ, ϕ), so that

$$x = r \sin \theta \cos \phi,\ y = r \sin \theta \sin \phi,\ z = r \cos \theta$$

and $$J = r^2 \sin \theta,\ x^2 + y^2 + z^2 = r^2.$$

The region of integration is common to the cone $z^2 = x^2 + y^2$ and the cylinder $x^2 + y^2 = 1$ bounded by the plane $z = 1$ in the positive octant (Fig. 7.30). Hence θ varies from 0 to $\pi/4$, r varies from 0 to $\sec \theta$ and ϕ varies from 0 to $\pi/2$.

$\therefore$ given integral becomes

$$\int_0^{\pi/2} \int_0^{\pi/4} \int_0^{\sec \theta} \frac{1}{r} . r^2 \sin \theta\, dr\, d\theta\, d\phi = \int_0^{\pi/2} d\phi \int_0^{\pi/4} \left| \frac{r^2}{2} \right|_0^{\sec \theta} \sin \theta\, d\theta$$

$$= \frac{\pi}{2} \int_0^{\pi/4} \frac{\sec^2 \theta}{2} \sin \theta\, d\theta = \frac{\pi}{4} \int_0^{\pi/4} \sec \theta \tan \theta\, d\theta = \frac{\pi}{4} \left| \sec \theta \right|_0^{\pi/4} = \frac{(\sqrt{2} - 1)\, \pi}{4}.$$

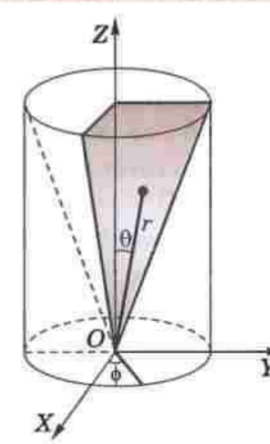

Fig. 7.30

Example 7.30. *Find the volume of the solid surrounded by the surface*

$$(x/a)^{2/3} + (y/b)^{2/3} + (z/c)^{2/3} = 1. \qquad \textit{(Hissar, 2005 S)}$$

Solution. Changing the variables, x, y, z to X, Y, Z where, $(x/a)^{1/3} = X$, $(y/b)^{1/3} = Y$, $(z/c)^{1/3} = Z$

i.e., $x = aX^3, y = bY^3, z = cZ^3$ so that $J = \partial(x, y, z)/\partial(X, Y, Z) = 27\,abc\,X^2Y^2Z^2$.

$$\therefore \quad \text{required volume} = \iiint dx\,dy\,dz = 27\,abc \iiint X^2Y^2Z^2\,dX\,dY\,dZ$$

taken throughout the sphere $X^2 + Y^2 + Z^2 = 1$. ...(*i*)

Now change X, Y, Z to spherical polar coordinates r, θ, ϕ so that $X = r\sin\theta\cos\phi$, $Y = r\sin\theta\sin\phi$, $Z = r\cos\theta$, and $\partial(X, Y, Z)/\partial(r, \theta, \phi) = r^2\sin\theta$. To describe the positive octant of the sphere (*i*), r varies from 0 to 1, θ from 0 to $\pi/2$ and ϕ from 0 to $\pi/2$.

$$\therefore \quad \text{required volume} = 27abc \times 8 \int_0^1 \int_0^{\pi/2} \int_0^{\pi/2} r^2\sin^2\theta\cos^2\phi \times r^2\sin^2\theta\sin^2\phi \,.\, r^2\cos^2\theta \,.\, r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= 216\,abc \int_0^1 r^8\,dr \int_0^{\pi/2} \sin^4\theta\cos^2\theta\,d\theta \int_0^{\pi/2} \sin^2\phi\cos^2\phi\,d\phi = 4\pi\,abc/35.$$

PROBLEMS 7.4

Evaluate the following integrals by changing to polar co-ordinates :

1. $\int_0^1 \int_0^{\sqrt{(1-y^2)}} (x^2 + y^2)\,dy\,dx$. *(P.T.U., 2010)*
2. $\int_0^2 \int_0^{\sqrt{(2x-x^2)}} \dfrac{x\,dx\,dy}{x^2+y^2}$ *(Anna, 2009)*
3. $\int_0^{4a} \int_{y^2/4a}^{y} \dfrac{x^2-y^2}{x^2+y^2}\,dxdy$ *(Mumbai, 2006)*
4. $\iint xy(x^2+y^2)^{n/2}\,dxdy$ over the positive quadrant of $x^2 + y^2 = 4$, supposing $n + 3 > 0$. *(S.V.T.U., 2007)*
5. $\iint \dfrac{dxdy}{(1+x^2+y^2)^2}$ over one loop of the lemniscate $(x^2 + y^2) = x^2 - y^2$. *(Mumbai, 2007)*
6. Transform the following to cartesian form and hence evaluate $\int_0^{\pi} \int_0^{a} r^3 \sin\theta\cos\theta\,drd\theta$. *(P.T.U., 2005)*
7. $\iint y^2\,dx\,dy$ over the area outside $x^2 + y^2 - ax = 0$ and inside $x^2 + y^2 - 2ax = 0$. *(Mumbai, 2006)*
8. By using the transformation $x + y = u$, $y = uv$, show that $\int_0^1 \int_0^{1-x} e^{y/(x+y)}\,dydx = \frac{1}{2}(e-1)$. *(P.T.U., 2003)*
9. Transform $\int_0^{\pi/2} \int_0^{\pi/2} \sqrt{\dfrac{\sin\phi}{\sin\theta}}\,d\phi\,d\theta$ by the substitution $x = \sin\phi\cos\theta$, $y = \sin\phi\sin\theta$ and show that its value is π. *(U.P.T.U., 2001)*

Evaluate the following integrals by changing to spherical coordinates :

10. $\int_0^1 \int_0^{\sqrt{(1-x^2)}} \int_0^{\sqrt{(1-x^2-y^2)}} \dfrac{dx\,dy\,dz}{\sqrt{(1-x^2-y^2-z^2)}}$. *(V.T.U., 2006 ; Kottayam, 2005)*
11. $\iiint_V \dfrac{dx\,dy\,dz}{x^2+y^2+z^2}$ where V is the volume of the sphere $x^2 + y^2 + z^2 = a^2$. *(Anna, 2009)*
12. Evaluate $\iiint \dfrac{dx\,dy\,dz}{(1+x+y+z)^3}$ over the volume of the tetrahedron $x = 0, y = 0, z = 0, x + y + z = 1$. *(Mumbai, 2007)*
13. Show that $\iiint \dfrac{dx\,dy\,dz}{\sqrt{(a^2-x^2-y^2-z^2)}} = \dfrac{\pi^2 a^3}{8}$, the integral being extended for all the values of the variables for which the expression is real. *(U.T.U., 2010)*
14. $\iiint z^2\,dx\,dy\,dz$, taken over the volume bounded by the surfaces $x^2 + y^2 = a^2$, $x^2 + y^2 = z$ and $z = 0$.

15. Find the volume bounded by the xy-plane, the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 3$. *(I.S.M., 2001)*
16. Find the volume bounded by the xy-plane, the paraboloid $2z = x^2 + y^2$ and the cylinder $x^2 + y^2 = 4$. *(Raipur, 2005)*
17. Find the volume cut from the sphere $x^2 + y^2 + z^2 = a^2$ by the cone $x^2 + y^2 = z^2$.
18. Find the volume common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$. *(S.V.T.U., 2006)*
19. Find the volume cut off from the cylinder $x^2 + y^2 = ax$ by the planes $z = 0$ and $z = x$. *(U.P.T.U., 2006)*
20. Find the volume enclosed by the cylinders $x^2 + y^2 = 2ax$ and $z^2 = 2ax$. *(Marathwada, 2008)*
21. Find the volume of the cylinder $x^2 + y^2 - 2ax = 0$, intercepted between the paraboloid $x^2 + y^2 = 2az$ and the xy-plane.
22. Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the hyperboloid $x^2 + y^2 - z^2 = 1$.
23. Find the volume of the region bounded by $z = x^2 + y^2$, $z = 0$, $x = -a$, $x = a$ and $y = -a$, $y = a$.
24. Prove, by using a double integral that the volume generated by the revolution of the cardioid $r = a(1 + \cos\theta)$ about its axis is $8\pi a^3/3$. *(V.T.U., 2000)*
25. Evaluate $\iiint (x + y + z)\, dx\, dy\, dz$ over the tetrahedron bounded by the planes $x = 0, y = 0, z = 0$ and $x + y + z = 1$. [See Fig. 7.34]
26. Find the volume of the tetrahedron bounded by the coordinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. *(Burdwan, 2003)*
27. Work out example 7.21 by changing the variables.

7.8 AREA OF A CURVED SURFACE

Consider a point P of the surface $S : z = f(x, y)$. Let its projection on the xy-plane be the region A. Divide it into area elements by drawing lines parallel to the axes of X and Y. (Fig. 7.31).

On the element $\delta x \delta y$ as base, erect a cylinder having generators parallel to OZ and meeting the surface S in an element of area δS.

As $\delta x \delta y$ is the projection of δS on the xy-plane,

$\therefore$ $\delta x \delta y = \delta S \,.\, \cos\gamma$, where γ is the angle between the xy-plane and the tangent plane to S at P, *i.e.*, it is the angle between the Z-axis and the normal to S at $P(= \angle Z'PN)$.

Now since the direction cosines of the normal to the surface $F(x, y, z) = 0$ proportional to

$$\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}.$$

$\therefore$ the direction cosines of the normal to $S[F = f(x, y) - z]$ are proportional to $-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1$ and those of the z-axis are 0, 0, 1.

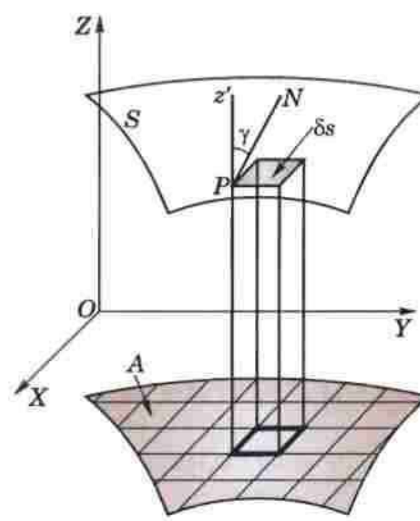

Fig. 7.31

Hence $$\cos\gamma = \frac{1}{\sqrt{\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]}} \quad \therefore \quad \delta S = \frac{\delta x\, \delta y}{\cos\gamma} = \sqrt{\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]}\, \delta x\, \delta y$$

Hence $$S = \underset{\delta S \to 0}{\text{Lt}} \sum \delta S = \iint_A \sqrt{\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]}\, dx dy$$

Similarly, if B and C be the projections of S on the yz-and zx-planes respectively, then

$$S = \iint_B \sqrt{\left[\left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 + 1\right]}\, dy\, dz$$

and $$S = \iint_C \sqrt{\left[\left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial x}\right)^2 + 1\right]}\, dz\, dx.$$

Example 7.31. *Find the area of the portion of the cylinder $x^2 + z^2 = 4$ lying inside the cylinder $x^2 + y^2 = 4$.*

Solution. Figure 7.32 shows one-eighth of the required area. Its projection on the xy-plane is a quadrant circle $x^2 + y^2 = 4$.

For the cylinder $\qquad x^2 + z^2 = 4, \qquad ...(i)$

we have
$$\frac{\partial z}{\partial x} = -\frac{x}{z}, \frac{\partial z}{\partial y} = 0.$$

so that
$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1 = \frac{x^2 + z^2}{z^2} = \frac{4}{4 - x^2}.$$

Hence the required surface area = 8 (surface area of the upper portion of (*i*) lying within the cylinder $x^2 + y^2 = 4$ in the positive octant)

$$= 8 \int_0^2 \int_0^{\sqrt{(4-x^2)}} \frac{2}{\sqrt{(4-x^2)}}\, dxdy = 16 \int_0^2 dx = 32 \text{ sq. units.}$$

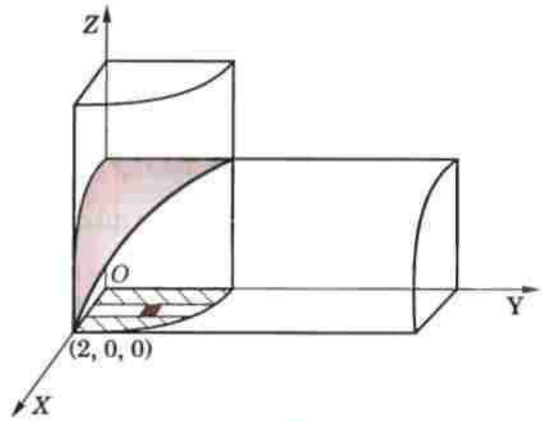

Fig. 7.32

Example 7.32. *Find the area of the portion of the sphere $x^2 + y^2 + z^2 = 9$ lying inside the cylinder $x^2 + y^2 = 3y$.*

Solution. Figure 7.33 shows one-fourth of the required area. Its projection on the xy-plane is the semi-circle $x^2 + y^2 = 3y$ bounded by the Y-axis.

For the sphere

$$x^2 + y^2 + z^2 = 9, \frac{\partial z}{\partial x} = -\frac{x}{z} \text{ and } \frac{\partial z}{\partial y} = -\frac{y}{z}$$

$$\therefore \qquad \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1 = (x^2 + y^2 + z^2)/z^2$$

$$= \frac{9}{9 - x^2 - y^2} = \frac{9}{9 - r^2} \qquad \text{when} \quad x = r\cos\theta, y = r\sin\theta.$$

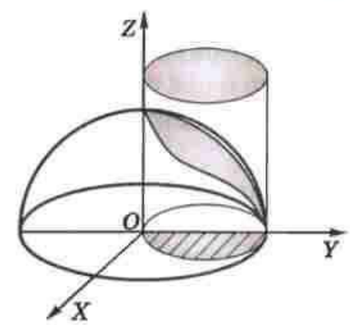

Fig. 7.33

Using polar coordinates, the required area is found by integrating $3/\sqrt{(9-r^2)}$ over the semi-circle $r = 3\sin\theta$, for which r varies from 0 to $3\sin\theta$ and θ varies from 0 to $\pi/2$.

Hence the required surface area

$$= 4 \int_0^{\pi/2} \int_0^{3\sin\theta} \frac{3}{\sqrt{(9-r^2)}}\, r d\theta\, dr = -6 \int_0^{\pi/2} \left| \frac{\sqrt{(9-r^2)}}{1/2} \right|_0^{3\sin\theta} d\theta$$

$$= 36 \int_0^{\pi/2} (1 - \cos\theta)\, d\theta = 36 \left| \theta - \sin\theta \right|_0^{\pi/2} = 18(\pi - 2) \text{ sq. units.}$$

PROBLEMS 7.5

1. Show that the surface area of the sphere $x^2 + y^2 + z^2 = a^2$ is $4\pi a^2$.
2. Find the area of the portion of the cylinder $x^2 + y^2 = 4y$ lying inside the sphere $x^2 + y^2 + z^2 = 16$.
3. Find the area of the portion of the sphere $x^2 + y^2 + z^2 = a^2$ lying inside the cylinder $x^2 + y^2 = ax$.
4. Find the area of the surface of the cone $x^2 + y^2 = z^2$ cut off by the surface of the cylinder $x^2 + y^2 = a^2$ above the xy-plane.
5. Compute the area of that part of the plane $x + y + z = 2a$ which lies in the first octant and is bounded by the cylinder $x^2 + y^2 = a^2$. *(Burdwan, 2003)*

7.9 CALCULATION OF MASS

(a) **For a plane lamina,** if the surface density at the point $P(x, y)$ be $\rho = f(x, y)$ then the elementary mass at $P = \rho\delta x\delta y$.

$$\therefore \quad \textit{total mass of the lamina} = \iint \rho \, dx \, dy \qquad ...(i)$$

with integrals embracing the whole area of the lamina.

In polar coordinates, taking $\rho = \phi(r, \theta)$ at the point $P(r, \theta)$,

$$\textit{total mass of the lamina} = \iint \rho r \, d\theta \, dr \qquad ...(ii)$$

(b) **For a solid,** if the density at the point $P(x, y, z)$ be $\rho = f(x, y, z)$, then

total mass of the solid $= \iiint \rho \, dx \, dy \, dz$ with appropriate limits of integration.

Example 7.33. *Find the mass of the tetrahedron bounded by the coordinates planes and the plane* $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, *the variable density* $\rho = \mu xyz$. *(Rohtak, 2003 ; U.P.T.U., 2003)*

Solution. Elementary mass at $P = \mu xyz \,.\, \delta x \, \delta y \, \delta z$.

$$\therefore \quad \text{the whole mass} = \iiint \mu xyz \, dx \, dy \, dz,$$

the integrals embracing the whole volume $OABC$ (Fig. 7.34). The limits for z are from 0 to $z = c(1 - x/a - y/b)$.

The limits for y are from 0 to $y = b(1 - x/a)$ and limits for x are from 0 to a.

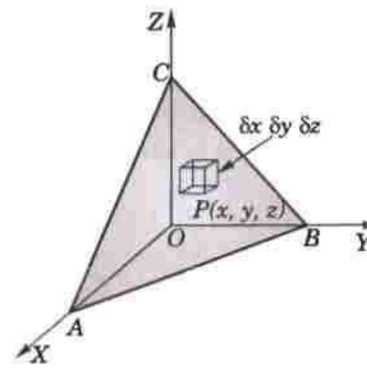

Fig. 7.34

Hence the required mass

$$= \int_0^a \int_0^{b(1-x/a)} \int_0^{c(1-x/a-y/b)} \mu xyz \, dz \, dy \, dx$$

$$= \mu \int_0^a \int_0^{b(1-x/a)} xy \left| z^2/2 \right|_0^{c(1-x/a-y/b)} dydz$$

$$= \mu \int_0^a \int_0^{b(1-x/a)} xy \,.\, \frac{c^2}{2}\left(1 - \frac{x}{a} - \frac{y}{b}\right)^2 dydx$$

$$= \frac{\mu c^2}{2} \int_0^a \int_0^{b(1-x/a)} x \,.\left[\left(1-\frac{x}{a}\right)^2 y - 2\left(1-\frac{x}{a}\right)\frac{y^2}{b} + \frac{y^3}{b^2}\right] dy \, dx$$

$$= \frac{\mu c^2}{2} \int_0^a x \left| \left(1-\frac{x}{a}\right)^2 \frac{y^2}{2} - 2\left(1-\frac{x}{a}\right)\frac{y^3}{3b} + \frac{y^4}{4b^2} \right|_0^{b(1-x/a)} dx$$

$$= \frac{\mu c^2}{2} \int_0^a x \left[\frac{b^2}{2}\left(1-\frac{x}{a}\right)^4 - \frac{2b^2}{3}\left(1-\frac{x}{a}\right)^4 + \frac{b^2}{4}\left(1-\frac{x}{a}\right)^4\right] dx = \frac{\mu b^2 c^2}{24} \int_0^a x\,(1 - x/a)^4 \, dx = \frac{\mu a^2 b^2 c^2}{720}.$$

7.10 CENTRE OF GRAVITY

(a) **To find the C.G. $(\bar{x}, \bar{y})$ of a plane lamina,** take the element of mass $\rho\delta x\delta y$ at the point $P(x, y)$. Then

$$\bar{x} = \frac{\iint x\rho \, dxdy}{\iint \rho \, dxdy}, \quad \bar{y} = \frac{\iint y \, \rho dxdy}{\iint \rho \, dxdy} \quad \text{with integrals embracing the whole lamina.}$$

While using polar coordinates, take the elementary mass as $\rho r\delta\theta\delta r$ at the point $P(r, \theta)$ so that $x = r\cos\theta$, $y = r\sin\theta$.

$$\therefore \quad \bar{x} = \frac{\iint r\cos\theta \, \rho r \, d\theta dr}{\iint \rho \, r d\theta dr}, \quad \bar{y} = \frac{\iint r\sin\theta \, \rho \, r d\theta dr}{\iint \rho \, r d\theta dr}$$

(*b*) **To find the C.G.** $(\bar{x}, \bar{y}, \bar{z})$ **of a solid,** take an element of mass $\rho\delta x\delta y\delta z$ enclosing the point $P(x, y, z)$. *Then*

$$\bar{x} = \frac{\iiint x\rho\, dxdydz}{\iiint \rho\, dxdydz},\ \bar{y} = \frac{\iiint y\rho\, dxdydz}{\iiint \rho\, dxdydz} \text{ and } \bar{z} = \frac{\iiint z\rho\, dxdydz}{\iiint \rho\, dxdydz}.$$

Example 7.34. *Find by double integration, the centre of gravity of the area of the cardioid* $r = a\,(1 + \cos\theta)$.

Solution. The cardioid being symmetrical about the initial line, its C.G. lies on OX, *i.e.*, $\bar{y} = 0$ (Fig. 7.35).

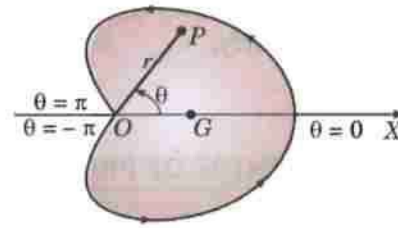

Fig. 7.35

$$\therefore \qquad \bar{x} = \frac{\iint \rho\, r\cos\theta \,.\, rd\theta\, dr}{\iint \rho\, rd\theta\, dr} = \frac{\int_{-\pi}^{\pi}\int_0^{a(1+\cos\theta)} \cos\theta .\, r^2 dr \,.\, d\theta}{\int_{-\pi}^{\pi}\int_0^{a(1+\cos\theta)} rdr\,.\,d\theta}$$

$$= \frac{\int_{-\pi}^{\pi} \cos\theta \left| \frac{r^3}{3} \right|_0^{a(1+\cos\theta)} d\theta}{\int_{-\pi}^{\pi} \left| \frac{r^2}{2} \right|_0^{a(1+\cos\theta)} d\theta} = \frac{2a}{3} . \frac{\int_{-\pi}^{\pi} \cos\theta\,(1+\cos\theta)^3\, d\theta}{\int_{-\pi}^{\pi} (1+\cos\theta)^2\, d\theta}$$

$$= \frac{2a}{3} . \frac{2 . \int_0^{\pi} (3\cos^2\theta + \cos^4\theta)\, d\theta}{2 . \int_0^{\pi} (1+\cos^2\theta)\, d\theta} \qquad \left\{ \because \int_{-\pi}^{\pi} \cos^n\theta\, d\theta = 2\int_0^{\pi} \cos^n\theta\, d\theta \text{ or } 0 \text{ according as } n \text{ in even or odd.} \right\}$$

$$= \frac{2a}{3} . \frac{2 . \int_0^{\pi/2} (3\cos^2\theta + \cos^4\theta)\, d\theta}{2 . \int_0^{\pi/2} (1+\cos^2\theta)\, d\theta} \text{ (as the powers of } \cos\theta \text{ are even)} = \frac{2a}{3} . \frac{3 . \frac{1}{2} . \frac{\pi}{2} + \frac{3 . 1}{4 . 2} . \frac{\pi}{2}}{\frac{\pi}{2} + \frac{1}{2} . \frac{\pi}{2}} = \frac{5a}{6}$$

Hence the C.G. of the cardioid is at $G(5a/6, 0)$.

Example 7.35. *Using double integration, find the C.G. of a lamina in the shape of a quadrant of the curve* $(x/a)^{2/3} + (y/b)^{2/3} = 1$, *the density being* $\rho = kxy$, *where k is a constant.*

Solution. Let $G\ (\bar{x}, \bar{y})$ be the C.G. of the lamina OAB (Fig. 7.36), so that

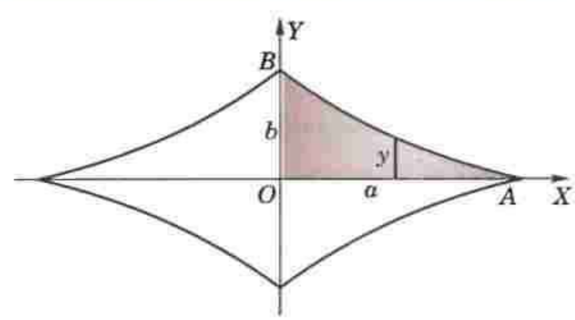

Fig. 7.36

$$\bar{x} = \frac{\iint kxy \,.\, xdxdy}{\iint kxy \,.\, dxdy} = \frac{\iint x^2 y\, dxdy}{\iint xy\, dxdy}$$

where the integrals are taken over the area OAB so that y varies from 0 to y (to be found from the equation of the curve in terms of x) and then x varies from 0 to a.

Thus

$$\bar{x} = \frac{\int_0^a \int_0^y x^2 y\, dydx}{\int_0^a \int_0^y xy\, dydx} = \frac{\int_0^a x^2 . \left| y^2/2 \right|_0^y dx}{\int_0^a x . \left| y^2/2 \right|_0^y dx} = \frac{\int_0^a x^2 y^2\, dx}{\int_0^a xy^2\, dx}$$

For any point on the curve, we have

$$x = a\cos^3\theta,\ y = b\sin^3\theta \text{ so that}$$
$$dx = -3a\cos^2\theta\sin\theta\, d\theta.$$

Also when $\qquad x = 0,\ \theta = \pi/2$; when $x = a,\ \theta = 0$.

Hence $$\bar{x} = \frac{\int_{\pi/2}^{0} a^2\cos^6\theta \,.\, b^2\sin^6\theta \,.\,(-3a\cos^2\theta\sin\theta)\,d\theta}{\int_{\pi/2}^{0} a\cos^3\theta \,.\, b^2\sin^6\theta \,.\,(-3a\cos^2\theta\sin\theta)\,d\theta}$$

$$= a\frac{\int_0^{\pi/2}\sin^7\theta\cos^8\theta\,d\theta}{\int_0^{\pi/2}\sin^7\theta\cos^5\theta\,d\theta} = \frac{128}{429}a$$

Similarly, $$\bar{y} = \frac{\int_0^a\int_0^y kxy \,.\, y\,dxdy}{\int_0^a\int_0^y kxy \,.\, dxdy} = \frac{128}{429}b.$$ Hence the required C.G. is $G\left(\frac{128}{429}a, \frac{128}{429}b\right)$.

7.11 CENTRE OF PRESSURE

Consider plane area A immersed vertically in a homogeneous liquid. Take the line of intersection of the given plane with the free surface of the liquid as the x-axis and any line lying in this plane and perpendicular to it downwards as the y-axis (Fig. 7.37).

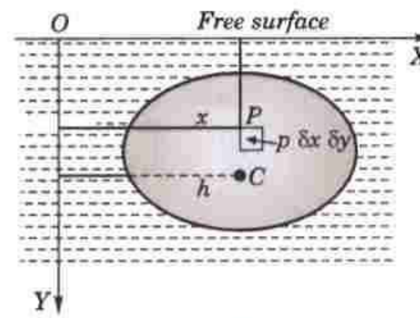

Fig. 7.37

If p be the pressure at the point $P(x, y)$ of the area A, then the pressure on an elementary area $\delta x\delta y$ at P is $p\delta x\delta y$ which is normal to the plane.

$\therefore$ the resultant pressure on $A = \iint p\,dxdy$.

If this resultant pressure acting at $C\,(h, k)$ is equivalent to pressure at various points such as $p\delta x\delta y$ distributed over the whole area A, then C is called the *centre of pressure*.

$\therefore$ taking the moment of the resultant pressure at C and the sum of the moments of the individual pressures such as $p\delta x\delta y$ at $P(x, y)$ about the y-axis, we get

$$h\iint p\,dxdy = \iint x \,.\, p\,dxdy, \quad i.e., \ h = \iint x \,.\, dxdy \Big/ \iint p\,dxdy$$

Similarly, taking moments about x-axis, we have

$$k = \iint y \,.\, p\,dxdy \Big/ \iint p\,dxdy$$ with integrals embracing the whole of the area A.

While using polar coordinates, replace x by $r\cos\theta$, y by $r\sin\theta$ and $dxdy$ by $r\,d\theta dr$ in the above formulae.

Example 7.36. *A horizontal boiler has a flat bottom and its ends are plane and semi-circular. If it is just full of water, show that the depth of the centre of pressure of either end is 0.7 × total depth approximately.*

Solution. Let the semi-circle $x^2 + y^2 = a^2$ represent an end of the given boiler (Fig. 7.38). By symmetry, its centre of pressure lies on OY.

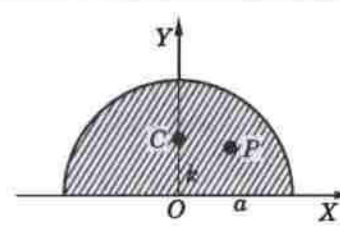

Fig. 7.38

If w be the weight of water per unit volume, then the pressure p at the point $P(x, y) = w(a - y)$.

$\therefore$ the height k of the C.P. above OX, is given by

$$k = \frac{\iint y \cdot p\,dxdy}{\iint p\,dxdy} = \frac{\int_{-a}^{a}\int_0^{\sqrt{(a^2-x^2)}} w(a-y)y\,dy\cdot dx}{\int_{-a}^{a}\int_0^{\sqrt{(a^2-x^2)}} w(a-y)\,dy\cdot dx}$$

$$= \frac{\int_{-a}^{a}\left| ay^2/2 - y^3/3\right|_0^{\sqrt{(a^2-x^2)}}dx}{\int_{-a}^{a}\left| ay - y^2/2\right|_0^{\sqrt{(a^2-x^2)}}dx} = \frac{\int_{-a}^{a}\left[\frac{a}{2}(a^2-x^2) - \frac{1}{3}(a^2-x^2)^{3/2}\right]dx}{\int_{-a}^{a}\left[a(a^2-x^2)^{1/2} - \frac{1}{2}(a^2-x^2)\right]dx}$$

Now put $x = a \sin\theta$, so that $dx = a\cos\theta\, d\theta$.

Also when $x = -a$, $\theta = -\pi/2$, and when $x = a$, $\theta = \pi/2$.

$$\therefore \qquad k = \frac{\int_{-\pi/2}^{\pi/2}\left[\frac{a^3}{2}\cos^2\theta - \frac{a^3}{3}\cos^3\theta\right] a\cos\theta\, d\theta}{\int_{-\pi/2}^{\pi/2}\left[a^2\cos\theta - \frac{a^2}{2}\cos^2\theta\right] a\cos\theta\, d\theta}$$

$$= \frac{a}{3}\cdot\frac{2\int_0^{\pi/2}(3\cos^3\theta - 2\cos^4\theta)\, d\theta}{2\int_0^{\pi/2}(2\cos^2\theta - \cos^3\theta)\, d\theta} = \frac{a}{4}\left(\frac{16-3\pi}{3\pi-4}\right) = 0.3a \text{ nearly.}$$

Hence, the depth of the C.P. $= a - k = 0.7a$ approximately.

PROBLEMS 7.6

1. A lamina is bounded by the curves $y = x^2 - 3x$ and $y = 2x$. If the density at any point is given by λxy, find by double integration, the mass of the lamina.
2. Find the mass of a lamina in the form of cardioid $r = a(1 + \cos\theta)$ whose density at any point varies as the square of its distance from the initial line.
3. Find the mass of a solid in the form of the positive octant of the sphere $x^2 + y^2 + z^2 = 9$, if the density at any point is $2xyz$.
4. Find the centroid of the area enclosed by the parabola $y^2 = 4ax$, the axis of x and its latus-rectum.
5. The density at any point (x, y) of a lamina is $\sigma(x + y)/a$ where σ and a are constants. The lamina is bounded by the lines $x = 0, y = 0, x = a, y = b$. Find the position of its centre of gravity.
6. Find the centroid of a loop of the lemniscate $r^2 = a^2\cos 2\theta$.
7. A plane in the form of a quadrant of the ellipse $(x/a)^2 + (y/b)^2 = 1$ is of small but varying thickness, the thickness at any point being proportional to the product of the distances of that point from the axes ; show that the coordinates of the centroid are $(8a/15, 8b/15)$. (*Nagpur, 2009*)
8. In a semi-circular disc bounded by a diameter OA, the density at any point varies as the distance from O ; find the position of the centre of gravity.
9. Find the centroid of the tetrahedron bounded by the coordinate planes and the plane $x + y + z = 1$, the density at any point varying as its distance from the face $z = 0$.
10. Find $\bar{x}$ where $(\bar{x}, \bar{y}, \bar{z})$ is the centroid of the region R bounded by the parabolic cylinder $z = 4 - x^2$ and the planes $x = 0, y = 0, y = 6, z = 0$. (Assume that the density is constant).
11. If the density at any point of the solid octant of the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$ varies as xyz, find the coordinates of the C.G. of the solid. (*P.T.U., 2005*)
12. A horizontal boiler has a flat bottom and its ends consist of a square 1 metre wide surmounted by an isosceles triangle of height 0.5 metre. Determine the depth of the centre of pressure of either end when the boiler is just full.
13. A quadrant of a circle is just, immersed vertically in a heavy homogeneous liquid with one edge in the surface. Find the centre of pressure.
14. Find the depth of the centre of pressure of a square lamina immersed in the liquid, with one vertex in the surface and the diagonal vertical.
15. Find the centre of pressure of a triangular lamina immersed in a homogeneous liquid with one side in the free surface. (*P.T.U., 2003*)
16. A uniform semi-circular is lamina immersed in a fluid with its plane vertical and its boundary diameter on the free surface. If the density at any point of the fluid varies as the depth of the point below the free surface, find the position of the centre of pressure of the lamina.

7.12 (1) MOMENT OF INERTIA

If a particle of mass m of a body be at a distance r from a given line, then mr^2 is called the *moment of inertia of the particle about the given line* and the sum of similar expressions taken for all the particles of the body, *i.e.*, Σmr^2 is called the *moment of inertia of the body about the given line* (Fig. 7.39).

If M be the total mass of the body and we write its moment of inertia $= Mk^2$, then k is called the *radius of gyration* of the body about the axis.

(2) M.I. of plane lamina. Consider the elementary mass $\rho\delta x\delta y$ at the point $P(x, y)$ of a plane area A so that its M.I. about x-axis $= \rho\delta x\delta y\, y^2$.

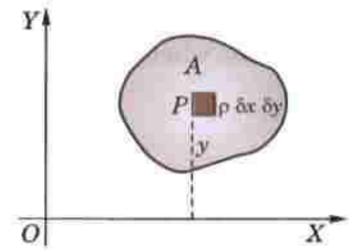

Fig. 7.39

$\therefore$ M.I. of the lamina about x-axis, *i.e.* $I_x = \iint_A \rho\, y^2\, dxdy.$

Similarly, M.I. of the lamina about y-axis' *i.e.*, $I_y = \iint_A \rho\, x^2\, dxdy.$

Also M.I. of the lamina about an axis perpendicular to the xy-plane, *i.e.*,

$$I_z = \iint_A \rho(x^2 + y^2)\, dxdy.$$

(3) M.I. of a solid. Consider an elementary mass $\rho\delta x\delta y\delta z$ enclosing a point $P(x, y, z)$ of a solid of volume V.

Distance of P from the x-axis $= \sqrt{(y^2 + z^2)}$.

$\therefore$ M.I. of this element about the x-axis $= \rho\, \delta x\delta y\delta z\, (y^2 + z^2)$.

Thus *M.I. of this solid about x-axis, i.e.,* $I_x = \iiint_V \rho(y^2 + z^2)\, dxdydz.$

Similarly, *its M.I. about y-axis, i.e.,* $I_y = \iiint_V \rho(z^2 + x^2)\, dxdydz$

and M.I. *about z-axis, i.e.,* $I_z = \iiint_V \rho(x^2 + y^2)\, dxdydz.$

(4) Sometimes we require the moment of inertia of a body about axes other than the principal axes. The following theorems prove useful for this purpose :

I. Theorem of perpendicular axis. *If the moment of inertia of a lamina about two perpendicular axes OX, OY in its plane are* I_x *and* I_y, *then the moment of inertia of the lamina about an axis OZ, perpendicular to it is given by* $I_z = I_x + I_y$.

Its proof follows from the relations giving I_x, I_y and I_z for a plane lamina [(2) above].

II. Steiner's theorem*. *If the moment of inertia of a body of mass M about an axis through its centre of gravity is I, then I′, moment of inertia about a parallel axis at a distance d from the first axis, is given by* $I' = I + Md^2$.

Its proof will be found in any text book on Dynamics of a Rigid Body.

Example 7.37. *Find the M.I. of the area bounded by the curve* $r^2 = a^2 \cos 2\theta$ *about its axis.*

Solution. Given curve is symmetrical about the pole and for half of the loop in the first quadrant θ varies from 0 to $\pi/4$ (Fig. 7.40).

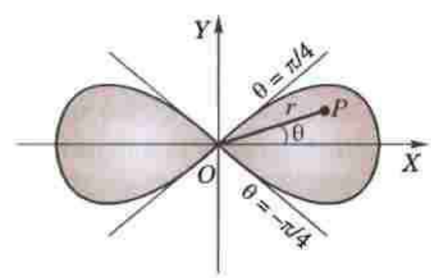

Fig. 7.40

Elementary area at $P(r, \theta) = r\, d\theta dr$.

If ρ be the surface density, then elementary mass

$$= \rho r\, d\theta dr \qquad ...(i)$$

$\therefore$ its total mass $M = 4 \int_0^{\pi/4} \int_0^{a\sqrt{(\cos 2\theta)}} \rho r\, dr d\theta$

$$= 2\rho a^2 \int_0^{\pi/4} \cos 2\theta\, d\theta = \rho a^2 \qquad ...(ii)$$

Now M.I. of the elementary mass (i) about the x-axis.

$$= \rho r\, d\theta dr \cdot y^2 = \rho r d\theta\, dr\, (r \sin\theta)^2 = \rho r^3 \sin^2\theta\, dr d\theta$$

Hence the M.I. of the whole area

$$= 4 \int_0^{\pi/4} \int_0^{a\sqrt{(\cos 2\theta)}} \rho r^3 \sin^2\theta\, dr d\theta = 4\rho \int_0^{\pi/4} \sin^2\theta \cdot \left| r^4/4 \right|_0^{a\sqrt{(\cos 2\theta)}} d\theta$$

$$= \rho a^2 \int_0^{\pi/4} \cos^2 2\theta \cdot \sin^2\theta\, d\theta = \rho a^4 \int_0^{\pi/2} \cos^2\phi \cdot \sin^2\frac{\phi}{2} \cdot \frac{d\phi}{2} \qquad \text{[Put } 2\theta = \phi,\ d\theta = d\phi/2]$$

$$= \frac{\rho a^4}{4} \int_0^{\pi/2} (\cos^2\phi - \cos^3\phi)\, d\phi = \frac{\rho a^4}{48}(3\pi - 8) = \frac{Ma^2}{48}(3\pi - 8). \qquad \text{[By } (ii)]$$

---

*Named after a Swiss geometrer *Jacob Steiner* (1796–1863) who was a professor at Berlin University.

Example 7.38. *Find the moment of inertia of a hollow sphere about a diameter, its external and internal radii being 5 metres and 4 metres.*

Solution. Let ρ be the density of the given hollow sphere. Then the M.I. about the diameter, *i.e.*, x-axis is

$$I_x = \iiint_V \rho(y^2 + z^2)\, dx\, dy\, dz$$

Changing to polar spherical coordinates, we get

$$I_x = \int_0^{2\pi} \int_0^{\pi} \int_4^{5} \rho\{(r \sin\theta \sin\phi)^2 + (r\cos\theta)^2\}\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \rho\left\{\int_0^{2\pi} \sin^2\phi\, d\phi \cdot \int_0^{\pi} \sin^3\theta\, d\theta \left|\frac{r^5}{5}\right|_4^5 + \int_0^{2\pi} d\phi \int_0^{\pi} \cos^2\theta \sin\theta\, d\theta \cdot \left|\frac{r^5}{5}\right|_4^5\right\}$$

$$= \frac{8\pi\rho}{15}(5^5 - 4^5) = 1120.5 \text{ m}.$$

Example 7.39. *A solid body of density ρ is in the shape of the solid formed by revolution of the centroid $r = a(1 + \cos\theta)$ about the initial line. Show that its moment of inertia about a straight line through the pole perpendicular to the initial line is $\frac{352}{105}\pi\rho a^5$.* *(U.P.T.U., 2001)*

Solution. An elementary area $rd\theta dr$, when revolved about OX generates a circular ring of radius $LP = r \sin\theta$ (Fig. 7.41).

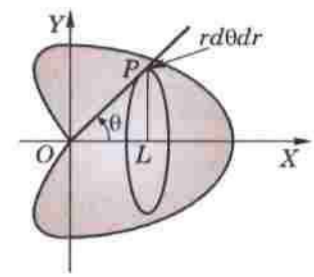

Fig. 7.41

M.I. of this ring about a diameter parallel to OY

$$= (2\pi r \sin\theta)(rd\theta dr)\rho \cdot \frac{(r\sin\theta)^2}{2}.$$

[∵ M.I. of a ring about a diameter $= Ma^2/2$.]

Now using Steiner's theorem, we have M.I. of the ring about OY = M.I. of the ring about a diameter LP parallel to OY + Mass of the ring $(OL)^2$ $(r\cos\theta)^2$

$$= 2\pi\rho r^4 \sin^3\theta\, d\theta dr + 2\pi r \sin\theta\,(rd\theta dr)\,(r\cos\theta)^2$$

Hence M.I. of the solid generated by revolution about OY

$$= \pi\rho \int_0^{\pi} \int_0^{r = a(1+\cos\theta)} (r^4 \sin^3\theta + 2r^4 \sin\theta \cos^2\theta)\, d\theta dr$$

$$= \pi\rho \int_0^{\pi} (\sin^3\theta + 2\sin\theta\cos^2\theta)\, d\theta \int_0^{a(1+\cos\theta)} r^4\, dr$$

$$= \frac{\pi\rho a^5}{5} \int_0^{\pi} \sin\theta\,(1 + \cos^2\theta)(1 + \cos\theta)^5\, d\theta \qquad [\text{Put } \theta = 2\phi]$$

$$= \frac{\pi\rho a^5}{5} \int_0^{\pi/2} \sin 2\phi\,(1 + \cos^2 2\phi)(1 + \cos 2\phi)^5\, 2d\phi$$

$$= \frac{\pi\rho a^5}{5} \int_0^{\pi/2} 2\sin\phi\cos\phi\,\{1 + (2\cos^2\phi - 1)^2\}\,(2\cos^2\phi)^5\, 2d\phi$$

$$= \frac{256\,\pi\rho a^5}{5} \int_0^{\pi/2} (\cos^{11}\phi - 2\cos^{13}\phi + 2\cos^{15}\phi)\sin\phi\, d\phi$$

$$= \frac{256\,\pi\rho a^5}{5}\left| -\frac{\cos^{12}\phi}{12} + \frac{2\cos^{14}\phi}{14} - \frac{2\cos^{16}\phi}{16}\right|_0^{\pi/2} = \frac{352\,\pi\rho a^5}{105}.$$

Example 7.40. *A hemisphere of radius R has a cylindrical hole of radius a drilled through it, the axis of the hole being along the radius normal to the plane face of the hemisphere. Find its radius of gyration about a diameter of this face.*

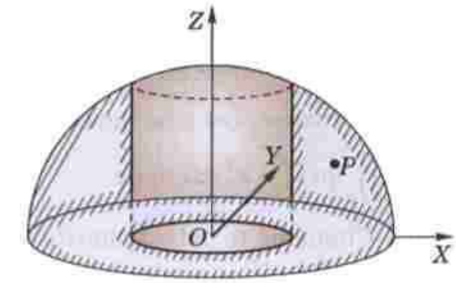

Fig. 7.42

Solution. M.I. of the given solid about x-axis

$$= \iiint \rho(y^2 + z^2)\, dx\, dy\, dz$$

The limits of integration for z are from 0 to $z = \sqrt{(R^2 - x^2 - y^2)}$ found from the equation of the sphere $x^2 + y^2 + z^2 = R^2$. The limits for x and y are to be such as to cover the shaded area A in the xy-plane between the concentric circles of radii a and R (Fig. 7.42).

Thus the required M.I. about x-axis

$$= \rho \iint_A \int_0^{\sqrt{(R^2 - x^2 - y^2)}} (y^2 + z^2)\, dz\, dx\, dy$$

$$= \rho \iint_A \left| y^2 z + z^3/3 \right|_0^{\sqrt{(R^2 - x^2 - y^2)}} dx\, dy = \rho \iint_A \left[y^2 (R^2 - x^2 - y^2)^{1/2} + \frac{1}{3}(R^2 - x^2 - y^2)^{3/2} \right] dx\, dy.$$

Now changing to polar coordinates, we have $x = r\cos\theta$, $y = r\sin\theta$ and $dxdy = rd\theta dr$.

Also to cover the area A, r varies from a to R and θ varies from 0 to 2π.

Hence the required M.I. about x-axis

$$= \rho \int_a^R \int_0^{2\pi} \left[r^2 \sin^2\theta \cdot (R^2 - r^2)^{1/2} + \frac{1}{3}(R^2 - r^2)^{3/2} \right] r d\theta dr$$

$$= \rho \int_a^R \int_0^{2\pi} \left[\frac{1}{2} r^2 (1 - \cos 2\theta) + \frac{1}{3}(R^2 - r^2) \right] d\theta \cdot r(R^2 - r^2)^{1/2}\, dr$$

$$= \rho \int_a^R \left| \frac{r^2}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + \frac{1}{3}(R^2 - r^2)\,\theta \right|_0^{2\pi} \cdot r(R^2 - r^2)^{1/2}\, dr$$

$$= \rho \int_a^R 2\pi \left(\frac{r^2}{2} + \frac{R^2 - r^2}{3} \right) . \, r(R^2 - r^2)^{1/2}\, dr$$

$$= \frac{\pi\rho}{3} \int_a^R (2R^2 + r^2)(R^2 - r^2)^{1/2} \cdot r dr \qquad \text{[Put } r^2 = t \text{ and } rdr = dt/2]$$

$$= \frac{\pi\rho}{6} \int_{a^2}^{R^2} (2R^2 + t)(R^2 - t)^{1/2}\, dt \qquad \text{[Integrate by parts]}$$

$$= \frac{\pi\rho}{9} \left[(2R^2 + a^2)(R^2 - a^2)^{3/2} + \frac{2}{5}(R^2 - a^2)^{5/2} \right] = \frac{2\pi\rho}{3}(R^2 - a^2)^{3/2} \times \frac{1}{10}(4R^2 + a^2)$$

$$\left[\because \ \text{Mass} = \rho \int_0^{2\pi} \int_a^R \int_0^{\sqrt{(R^2 - r^2)}} dz \cdot rdr \cdot d\theta = \frac{2\pi\rho}{3}(R^2 - a^2)^{3/2} \right]$$

Hence, the radius of gyration = $[(4R^2 + a^2)/10]^{1/2}$.

7.13 (1) PRODUCT OF INERTIA

If a particle of mass m of a body be at distances x and y from two given perpendicular lines, then $\sum mxy$ is called the *product of inertia* of the body about the given lines.

Consider an elementary mass $\delta x \delta y \delta z$ enclosing the point $P(x, y, z)$ of solid of volume V. Then the product of inertia (P.I.) of this element about the axes of x and $y = \rho \delta x \delta y \delta z\, xy$.

$\therefore$ P.I. of the solid about x and y-axes, *i.e.*, $P_{xy} = \iiint_V \rho\, xy\, dxdydz$

Similarly, $P_{yz} = \iiint_V \rho\, yz\, dxdydz$ and $P_{zx} = \iiint_V \rho\, zx\, dxdydz$.

In particular, for a plane lamina of surface density ρ and covering a region A in the xy-plane,

$$P_{xy} = \iint_A \rho\, xy\, dxdy \ \text{ whereas } P_{yz} = P_{zx} = 0. \qquad [\because \ z = 0]$$

(2) Principal axes. *The principal axes* of a lamina at a given point are that pair of axes in its plane through the given point, about which the product of inertia of the lamina vanishes.

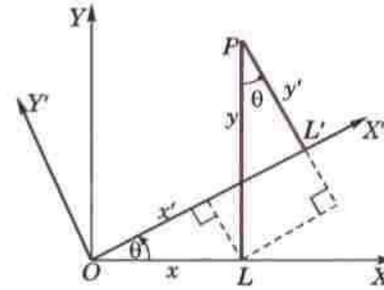

Fig. 7.43

Let $P(x, y)$ be a point of the plane area A referred to rectangular axes OX, OY. Let (x', y') be the coordinates of P referred to another pair of rectangular axes OX', OY' in the same plane and inclined at an angle θ to the first pair (Fig. 7.43).

Then $$x' = x\cos\theta + y\sin\theta$$
$$y' = y\cos\theta - x\sin\theta$$

If I_x, I_y be the moments of inetria of the area A about OX and OY and P_{xy} be its product of inertia about these axes, then

$$I_x = \iint_A \rho y^2\, dA,\ I_y = \iint_A \rho x^2\, dA,\ P_{xy} = \iint \rho xy\, dA.$$

$\therefore$ the product of inertia P'_{xy} about OX' and OY' is given by

$$P'_{xy} = \int_A \rho x'y'\, dA = \iint_A \rho(x\cos\theta + y\sin\theta)(y\cos\theta - x\sin\theta)\, dA$$

$$= \sin\theta\cos\theta \iint_A \rho(y^2 - x^2)\, dA + (\cos^2\theta - \sin^2\theta)\iint_A \rho xy\, dA$$

$$= 1/2 \sin 2\theta \cdot (I_x - I_y) + \cos 2\theta\, P_{xy}.$$

Now OX', OY' will be the principal axes of the area A if P'_{xy} vanishes.

i.e., If $\quad 1/2 \sin 2\theta\, (I_x - I_y) + \cos 2\theta\, P_{xy} = 0$

i.e., If $\quad \mathbf{\tan 2\theta = 2P_{xy}/(I_y - I_x)}$.

This gives two values of θ differing by $\pi/2$.

Example 7.41. *Show that the principal axes at the node of a half-loop of the lemniscate $r^2 = a^2\cos 2\theta$ are inclined to the initial line at angles*

$$\frac{1}{2}\tan^{-1}\frac{1}{2} \text{ and } \frac{\pi}{2} + \frac{1}{2}\tan^{-1}\frac{1}{2}.$$

Solution. Let the element of mass at $P(r, \theta)$ be $\rho r d\theta dr$.

Then $$I_x = \rho\int_0^{\pi/4}\int_0^{a\sqrt{(\cos 2\theta)}} r^2\sin^2\theta \cdot r d\theta dr$$ [See Fig. 7.40]

$$= \frac{\rho a^4}{4}\int_0^{\pi/4}\sin^2\theta\cos^2 2\theta\, d\theta = \frac{\rho a^4}{16}\left(\frac{\pi}{4} - \frac{2}{3}\right)$$

$$I_y = \rho\int_0^{\pi/4}\int_0^{a\sqrt{(\cos 2\theta)}} r^2\cos^2\theta \cdot r\, d\theta dr = \frac{\rho a^4}{16}\left(\frac{\pi}{4} + \frac{2}{3}\right)$$

and $$P_{xy} = \rho\int_0^{\pi/4}\int_0^{a\sqrt{(\cos 2\theta)}} r^2\sin\theta\cos\theta \cdot r d\theta dr = \frac{\rho a^4}{48}.$$

Hence the required direction of the principal axes at O are given by

$$\tan 2\theta = \frac{2P_{xy}}{I_y - I_x} = \frac{\rho a^4/24}{(\rho a^4/16)\times(4/3)} = \frac{1}{2}$$

or by $$\theta = \frac{1}{2}\tan^{-1}\frac{1}{2} \text{ and } \frac{\pi}{2} + \frac{1}{2}\tan^{-1}\frac{1}{2}.$$

PROBLEMS 7.7

1. Using double integrals, find the moment of inertia about the x-axis of the area enclosed by the lines $x = 0, y = 0, (x/a) + (y/b) = 1.$ *(P.T.U., 2005)*
2. Find the moment of inertia of a circular plate about a tangent.
3. Find the moment of inertia of the area $y = \sin x$ from $x = 0$ to $x = 2\pi$ about OX.

4. Find the moment of inertia of a quadrant of the ellipse $(x/a)^2 + (y/b)^2 = 1$ of mass M about the x-axis, if the density at a point is proportional to xy.
5. Find the moment of inertia about the initial line of the cardioid $r = a(1 + \cos\theta)$.
6. Find the moment of inertia of a uniform spherical ball of mass M and radius R about a diameter.
7. Find the moment of inertia of a solid right circular cylinder about (*i*) its axis (*P.T.U., 2006*)
(*ii*) a diameter of the base.
8. Find the M.I. of a solid right circular cone having base-radius r and height h, about (*i*) its axis, (*ii*) an axis through the vertex and perpendicular to its axis, (*iii*) a diameter of its base.
9. Find the moment of inertia of a hollow sphere about a diameter, its external and internal radii being 51 metres and 49 metres.
10. Find the moment of inertia about z-axis of a homogeneous tetrahedron bounded by the planes $x = 0, y = 0, z = x + y$ and $z = 1$.
11. Find the moment of inertia of an octant of the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$, about the x-axis.
12. Find the product of inertia of a quadrant of the ellipse $(x/a)^2 + (y/b)^2 = 1$, about the coordinate axes.
13. Show that the principal axes at the origin of the triangle enclosed by $x = 0, y = 0, (x/a) + (y/b) = 1$ are inclined to the x-axis at angles α and $\alpha + \pi/2$, where $\alpha = \frac{1}{2}\tan^{-1}[ab/(a^2 - b^2)]$ (*U.P.T.U., 2002*)
14. The lengths AB and AD of the sides of a rectangle $ABCD$ are $2a$ and $2b$. Show that the inclination to AB of one of the principal axes at A is $\frac{1}{2}\tan^{-1}\left\{\frac{3ab}{2(a^2 - b^2)}\right\}$.

7.14 BETA FUNCTION

The beta function is defined as

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad \begin{cases} m > 0 \\ n > 0 \end{cases} \qquad ...(1)$$

Putting $x = 1 - y$ in (1), we get $\beta(m, n) = -\int_1^0 (1-y)^{m-1}\, y^{n-1}\,dy$

$$= \int_0^1 y^{n-1}(1-y)^{m-1}\,dy = \beta(n, m)$$

Thus $\beta(m, n) = \beta(n, m)$ ...(2)

Putting $x = \sin^2\theta$ so that $dx = 2\sin\theta\cos\theta\,d\theta$, (1) becomes

$$\beta(m, n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(\cos^2\theta)^{n-1}\,2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta \qquad ...(3)$$

which is another form of $\beta(m, n)$.

This function is also *Euler's integral of the first kind**.

7.15 (1) GAMMA FUNCTION

The gamma function is defined as

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx \ (n > 0) \qquad ...(i)$$

This integral is also known as *Euler's integral of the second kinds.* It defines a function of n for positive values of n.

*After an enormously creative Swiss mathematician *Leonhard Euler (1707–1783).* He studied under *John Bernoulli* and became a professor of mathematics in St. Petersburg, Russia. Even after becoming totally blind in 1771, he contributed to almost all branches of mathematics.

In particular, $\Gamma(1) = \int_0^\infty e^{-x}\, dx = \left| -e^{-x} \right|_0^\infty = 1.$ ...(ii)

(2) Reduction formula for Γ(n).

Since $\Gamma(n+1) = \int_0^\infty e^{-x} x^n\, dx$ [Integrating by parts] $= \left| -x^n e^{-x} \right|_0^\infty + n \int_0^\infty e^{-x} x^{n-1} dx$

$\therefore$ $\Gamma(n+1) = n\Gamma(n)$ ...(iii)

which is the reduction formula for $\Gamma(n)$. From this formula, it is clear that if $\Gamma(n)$ is known throughout a unit interval say : $1 < n \le 2$, then the values of $\Gamma(n)$ throughout the next unit interval $2 < n \le 3$ are found, from which the values of $\Gamma(n)$ for $3 < n \le 4$ are determined and so on. In this way, *the values of $\Gamma(n)$ for all positive values of $n > 1$ may be found by successive application of (iii).*

Also using (*iii*) in the form

$$\Gamma(n) = \frac{\Gamma(n+1)}{n} \quad ...(iv)$$

We can define $\Gamma(n)$ for values of n for which the definition (1) fails. It gives the value of $\Gamma(n)$ for $0 < n \le 1$ in terms of the values of $\Gamma(n)$ for $1 < n \le 2$. Thus we can define $\Gamma(n)$ for all $n < 0$ provided its value for $1 < n \le 2$ is known. Also if $-1 < n < 0$, (4) gives $\Gamma(n)$ in terms of its values for $0 < n < 1$. Then we may find, $\Gamma(n)$ for $-2 < n < -1$ and so on.

Thus (i) and (iv) together give a complete definition of Γ(n) for all values of n except when n is zero or a negative integer and its graph is as shown in Fig. 7.44. The values of *Γ(n)* for *1 < n ≤ 2* are given in (Table I-Appendix 2) from which the values of *Γ(n)* for values of n outside the interval $1 < n \le 2$ ($n \ne 0, -1, -2, -3$ .....) may be found.

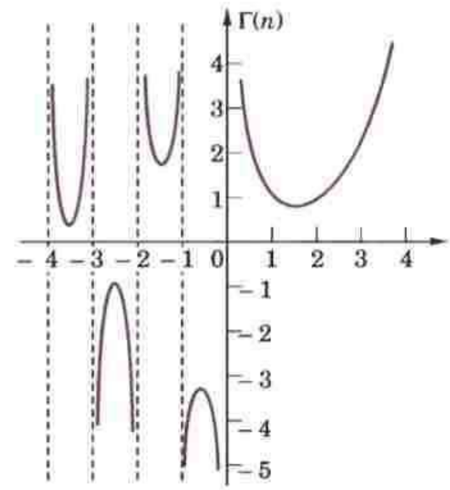

Fig. 7.44

(3) Value of Γ(n) in terms of factorial.

Using $\Gamma(n+1) = n\Gamma(n)$ successively, we get

$\Gamma(2) = 1 \times \Gamma(1) = 1\,!$

$\Gamma(3) = 2 \times \Gamma(2) = 2 \times 1 = 2\,!$

$\Gamma(4) = 3 \times \Gamma(3) = 3 \times 2\,! = 3\,!$

....

In general $\Gamma(n+1) = n\,!$ provided n is a positive integer ...(v)

Taking $n = 0$, it defines $0\,! = \Gamma(1) = 1$.

(4) Value of $\Gamma\left(\frac{1}{2}\right)$**.** We have

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty e^{-x} x^{-1/2}\, dx \qquad \text{[Put } x = y^2 \text{ so that } dx = 2y\, dy]$$

$$= 2\int_0^\infty e^{-y^2}\, dy \text{ which is also } = 2\int_0^\infty e^{-x^2}\, dx$$

$$\therefore \quad \left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy = 4\int_0^{\pi/2} \int_0^\infty e^{-r^2}\, r\, dr\, d\theta$$

$$= 4 \cdot \frac{\pi}{2} \int_0^\infty e^{-r^2}\, r dr = 2\pi \left[\left(-\frac{1}{2}\right) e^{-r^2}\right]_0^\infty = \pi$$

whence $\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi} = 1.772$ ...(vi) (*V.T.U., 2006*)

7.16 RELATION BETWEEN BETA AND GAMMA FUNCTIONS

$$\beta(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

We have $\Gamma(m) = \int_0^\infty e^{-t}\, t^{m-1}$ [Put $t = x^2$ so that $dt = 2x\, dx$

$$= 2\int_0^\infty e^{-x^2} x^{2m-1}\, dx \qquad ...(2)$$

Similarly, $\Gamma(n) = 2\int_0^\infty e^{-y^2} y^{2n-1}\, dy$

$\therefore$ $\Gamma(m)\Gamma(n) = 4\int_0^\infty e^{-x^2} x^{2m-1}\, dx \int_0^\infty e^{-y^2} y^{2n-1}\, dy$

$$= 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1}\, dx\, dy \qquad ...(3)$$ [$\because$ the limits of integration are constant.]

Now change to polar coordinates by writing $x = r\cos\theta$, $y = r\sin\theta$ and $dx\, dy = r d\theta dr$. To cover the region in (3) which is the entire first quadrant, r varies from 0 to ∞ and θ from 0 to $\pi/2$. Thus (3) becomes

$$\Gamma(m)\Gamma(n) = 4\int_0^\infty \int_0^{\pi/2} e^{-r^2} r^{2(m+n)-1} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta dr$$

$$= \left[2\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta\right] \times \left[2\int_0^\infty e^{-r^2} r^{2(m+n)-1}\, dr\right] \qquad ...(4)$$

But by (2), $2\int_0^\infty e^{-r^2} r^{2(m+n)-1}\, dr = \Gamma(m+n)$

and by (3) of § 7.14, $2\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta = \beta(m, n)$.

Thus (4) gives $\Gamma(m)\Gamma(n) = \beta(m, n)\,\Gamma(m+n)$ *(U.T.U., 2010 ; Bhopal, 2009 ; V.T.U., 2008 S)*

whence follows (1) which is extremely useful for evaluating definite integrals in terms of gamma functions.

Cor. *Rule to evaluate* $\int_0^{\pi/2} \sin^p x \cos^q x\, dx$.

$$\int_0^{\pi/2} \sin^p x \cos^q x\, dx = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) \qquad \text{[By (3) of § 7.14]}$$

$$= \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)} \qquad ...(5)$$

In particular, when $q = 0$, and $p = n$, we have

$$\left.\begin{aligned} \int_0^{\pi/2} \sin^n x\, dx &= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \cdot \frac{\sqrt{\pi}}{2} \\ \text{Similarly,} \quad \int_0^{\pi/2} \cos^n x\, dx &= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \cdot \frac{\sqrt{\pi}}{2} \end{aligned}\right\} \qquad ...(6)$$

Example 7.42. *Show that*

(a) $\Gamma(n) = \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy \ (n > 0)$. *(J.N.T.U., 2003 ; Madras, 2003 S)*

(b) $\beta(p, q) = \int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}}\, dy$ *(V.T.U., 2003 ; Gauhati, 1999)*

$= \int_0^1 \frac{x^{p-1} + x^{q-1}}{(1+x)^{p+q}}\, dx.$ *(V.T.U., 2008 ; Osmania, 2003 ; Rohtak, 2003)*

Solution. (a) $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx \ (n > 0)$

$$= \int_1^0 \left(\log \frac{1}{y}\right)^{n-1} y\left(-\frac{1}{y}dy\right) = \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy.$$

Put $y = e^{-x}$
i.e., $x = \log(1/y)$
so that $dx = -(1/y)\,dy$

(*b*) $$\beta(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx$$

$$= \int_\infty^0 \frac{1}{(1+y)^{p-1}}\left(\frac{y}{1+y}\right)^{q-1} \frac{-1}{(1+y)^2}\,dy$$

Put $x = \dfrac{1}{1+y}$ *i.e.*, $y = \dfrac{1}{x} - 1$
so that $dx = \dfrac{-1}{(1+y)^2}\,dy$

$$= \int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}}\,dy = \int_0^1 \frac{y^{q-1}}{(1+y)^{p+q}}\,dy + \int_1^\infty \frac{y^{q-1}}{(1+y)^{p+q}}\,dy$$

Now substituting $y = 1/z$ in the second integral, we get

$$\int_1^\infty \frac{y^{q-1}}{(1+y)^{p+q}}\,dy = \int_1^0 \frac{1}{z^{q-1}} \cdot \frac{1}{(1+1/z)^{p+q}}\left(-\frac{1}{z^2}\right)dz = \int_0^1 \frac{z^{p-1}}{(1+z)^{p+q}}\,dz.$$

Hence, $$\beta(p, q) = \int_0^1 \frac{y^{q-1}}{(1+y)^{p+q}}\,dy + \int_0^1 \frac{z^{p-1}}{(1+z)^{p+q}}\,dz = \int_0^1 \frac{x^{p-1} + x^{q-1}}{(1+x)^{p+q}}\,dx.$$

Example 7.43. *Express the following integrals in terms of gamma functions :*

(*a*) $\int_0^1 \dfrac{dx}{\sqrt{(1-x^4)}}$ (*b*) $\int_0^{\pi/2} \sqrt{(\tan\theta)}\,d\theta.$ (*Madras, 2006*)

(*c*) $\int_0^\infty \dfrac{x^c}{c^x}\,dx$ (*U.P.T.U., 2006*) (*d*) $\int_0^\infty a^{-bx^2}\,dx.$

(*e*) $\int_0^1 x^5\,[\log(1/x)]^3\,dx$ (*Madras, 2000*)

Solution. (*a*) $\int_0^1 \dfrac{dx}{\sqrt{(1-x^4)}}$

Put $x^2 = \sin\theta$, *i.e.*, $x = \sin^{1/2}\theta$
so that $dx = 1/2\sin^{-1/2}\theta\cos\theta\,d\theta$

$$= \int_0^{\pi/2} \frac{1}{2} \cdot \frac{\sin^{-1/2}\theta \cdot \cos\theta\,d\theta}{\sqrt{(1-\sin^2\theta)}} = \frac{1}{2}\int_0^{\pi/2} \sin^{-1/2}\theta\,d\theta = \frac{1}{2} \cdot \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)}{\Gamma\left(\frac{-\frac{1}{2}+2}{2}\right)} = \frac{\sqrt{\pi}}{4}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

(*b*) $$\int_0^{\pi/2} \sqrt{(\tan\theta)}\,d\theta = \int_0^{\pi/2} \sin^{1/2}\theta\cos^{-1/2}\theta\,d\theta$$

$$= \frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)}{2\Gamma\left(\frac{\frac{1}{2}-\frac{1}{2}+2}{2}\right)} = \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{2\Gamma(1)} = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$$

(*c*) $$\int_0^\infty \frac{x^c}{c^x}\,dx = \int_0^\infty \frac{x^c}{e^{x\log c}}\,dx$$ $[\because\ c^x = e^{\log c^x} = e^{x\log c}]$

$$= \int_0^\infty e^{-x\log c}\,x^c\,dx$$ [Put $x\log c = t$ so that $dx = dt/\log c$]

$$= \int_0^\infty e^{-t}\left(\frac{t}{\log c}\right)^c \frac{dt}{\log c} = \frac{1}{(\log c)^{c+1}}\int_0^\infty t^c\, e^{-t}\, dt = \Gamma(c+1)/(\log c)^{c+1}$$

(d) $\int_0^\infty a^{-bx^2}\, dx = \int_0^\infty e^{-bx^2 \log a}\, dx$ $\left[\begin{array}{l}\text{Put } (b \log a)\, x^2 = t \\ \text{so that } dx = dt/2\sqrt{(b \log a)}\end{array}\right.$

$$= \frac{1}{2\sqrt{(b\log a)}}\int_0^\infty e^{-t}\, t^{-1/2}\, dt = \frac{\Gamma\left(\frac{1}{2}\right)}{2\sqrt{(b\log a)}} = \frac{\sqrt{\pi}}{2\sqrt{(b\log a)}}$$

(e) $\int_0^1 x^4 [\log(1/x)]^3\, dx = \frac{1}{625}\int_0^\infty e^{-t}\,.\, t^3\, dt$ $\left[\begin{array}{l}\text{Put } x = e^{-t/5} \text{ so that } \log(1/x) = t/5 \\ dx = -\frac{1}{5}e^{-t/5}\, dt\end{array}\right]$

$$= \frac{\Gamma(4)}{625} = \frac{6}{625}.$$

Example 7.44. *Evaluate* $\int_0^\infty e^{-ax}\, x^{m-1}$ *sin bx dx in terms of Gamma function.* *(U.P.T.U., 2003)*

Solution. We have $\Gamma(m) = \int_0^\infty e^{-x}\, x^{m-1}\, dx$ [Put $x = ay$, $dx = a\,dy$]

$$= \int_0^\infty e^{-ay}\, a^m\, y^{m-1}\, dy \quad \text{or} \quad \int_0^\infty e^{-ay}\, y^{m-1}\, dy = \Gamma(m)/a^m \quad \text{...(i)}$$

Then $I = \int_0^\infty e^{-ax}\, x^{m-1} \sin bx\, dx = \int_0^\infty e^{-ax}\, x^{m-1}$ (Imaginary part of e^{ibx}) dx

$= \text{I.P. of } \int_0^\infty e^{-(a-ib)x}\, x^{m-1}\, dx$

$= \text{I.P. of } \{\Gamma(m)/(a-ib)^m$ [By (i)]

$= \text{I.P. of } \{\Gamma(m)/(r^m(\cos\theta - i\sin\theta)^m]$ where $a = r\cos\theta$, $b = r\sin\theta$

$= \text{I.P. of } \Gamma(m)/(r^m(\cos m\theta - i\sin m\theta)]$ (Using Demoivere's theorem §19.5)

$$= \text{I.P. of } \left\{\frac{\Gamma(m)\,.\,(\cos m\theta + i\sin m\theta)}{r^m(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)}\right\}$$

$= \frac{\Gamma(m)}{r^m}\sin m\theta$ where $r\ \sqrt{(a^2+b^2)}$, $\theta = \tan^{-1} b/a$.

Example 7.45. *Prove that* $\int_0^1 \frac{x^2 dx}{\sqrt{(1-x^4)}} \times \int_0^1 \frac{dx}{\sqrt{(1+x^4)}} = \frac{\pi}{4\sqrt{2}}$

Solution. $\int_0^1 \frac{x^2 dx}{\sqrt{(1-x^4)}} \times \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta}\cdot\frac{\cos\theta}{2\sqrt{(\sin\theta)}}\, d\theta$ $\left[\text{Putting } x^2 = \sin\theta,\ dx = \frac{\cos\theta\, d\theta}{2\sqrt{(\sin\theta)}}\right.$

$$= \frac{1}{2}\int_0^{\pi/2}\sqrt{(\sin\theta)}\, d\theta = \frac{1}{4}\beta\left(\frac{3}{4},\frac{1}{2}\right) = \frac{1}{4}\,\frac{\Gamma(3/4)\,\Gamma(1/2)}{\Gamma(5/4)} = \frac{\Gamma(3/4)\,\Gamma(1/2)}{\Gamma(1/4)}$$

$\int_0^1 \frac{dx}{\sqrt{(1+x^4)}} = \int_0^{\pi/4}\frac{\sec^2\theta\, d\theta}{2\sqrt{(\tan\theta)}\sec\theta}$ $\left[\text{Putting } x^2 = \tan\theta,\ dx = \frac{\sec^2\theta\, d\theta}{2\sqrt{(\tan\theta)}}\right.$

$= \frac{1}{\sqrt{2}}\int_0^{\pi/4}\frac{d\theta}{\sqrt{(\sin 2\theta)}} = \frac{1}{2\sqrt{2}}\int_0^{\pi/2}\sin^{-1/2}\phi\, d\phi$ $\left[\text{Putting } 2\theta = \phi,\ d\theta = \frac{1}{2}d\phi\right.$

$$= \frac{1}{4\sqrt{2}}\beta\left(\frac{1}{4},\frac{1}{2}\right) = \frac{1}{4\sqrt{2}}\,\frac{\Gamma(1/4)\,\Gamma(1/2)}{\Gamma(3/4)}$$

$$\therefore \qquad \int_0^1 \frac{x^2\,dx}{\sqrt{(1-x^4)}} \times \int_0^1 \frac{dx}{\sqrt{(1+x^4)}} = \frac{1}{4\sqrt{2}}\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 = \frac{\pi}{4\sqrt{2}}.$$

Example 7.46. *Prove that (i)* $\beta(m, 1/2) = 2^{2m-1}\beta(m, m)$ *(V.T.U., 2004)*

$$(ii)\ \Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m) \qquad \textit{(Duplication Formula)}$$

(V.T.U., 2010 ; Kerala, M.E., 2005 ; Madras, 2003 S)

Solution. (*i*) We know that $\beta(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$ ...(1)

Putting $n = \frac{1}{2}$, we have $\beta\left(m, \frac{1}{2}\right) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\, d\theta$ ...(2)

Again putting $n = m$ in (*i*), we get $\beta(m, m) = 2\int_0^{\pi/2} (\sin\theta\cos\theta)^{2m-1}\, d\theta$

$$= \frac{1}{2^{2m-2}}\int_0^{\pi/2} \sin^{2m-1} 2\theta\, d\theta$$

$$= \frac{1}{2^{2m-1}}\int_0^{\pi} \sin^{2m-1}\phi\, d\phi, \text{ putting } 2\theta = \phi$$

$$= \frac{1}{2^{2m-1}}\cdot 2\int_0^{\pi/2} \sin^{2m-1}\phi\, d\phi$$

or $$2^{2m-1}\beta(m, m) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\, d\theta = \beta\left(m, \frac{1}{2}\right)$$ [by (2)

(*ii*) Rewriting the above result in terms of Γ functions, we get

$$2^{2m-1}\frac{\Gamma(m)\,\Gamma(m)}{\Gamma(m+m)} = \frac{\Gamma(m)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)} \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

or $$\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma(2m)}{2^{2m-1}}.$$

Example 7.47. *Prove that*

(*a*) $\iint_D x^{l-1} y^{m-1}\, dx\, dy = \dfrac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m+1)} h^{l+m}$ *where D is the domain* $x \geq 0$, $y \geq 0$ *and* $x + y \leq h$.

(U.P.T.U., 2005)

(*b*) $\iiint_V x^{l-1} y^{m-1} z^{n-1}\, dxdydz = \dfrac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)}$

where V is the region $x \geq 0$, $y \geq 0$, $z \geq 0$ *and* $x + y + z \leq 1$. *This important result is known as* **Dirichlet's integral***.

Solution. (*a*) Putting $x/h = X$ and $y/h = Y$, we see that the given integral

$$= \iint_{D'} (hX)^{l-1}(hY)^{m-1} h^2\; dXdY \text{ where } D' \text{ is the domain } X \geq 0, Y \geq 0 \text{ and } X + Y \leq 1.$$

$$= h^{l+m}\int_0^1\int_0^{1-X} X^{l-1}\,Y^{m-1}\, dY\, dX = h^{l+m}\int_0^1 X^{l-1}\left|\frac{Y^m}{m}\right|_0^{1-X} dX$$

$$= \frac{h^{l+m}}{m}\int_0^1 X^{l-1}(1-X)^m\, dX = \frac{h^{l+m}}{m}\beta(l, m+1) = \frac{h^{l+m}}{m}\cdot\frac{\Gamma(l)\,\Gamma(m+1)}{\Gamma(l+m+1)}$$

*Named after a German mathematician *Peter Gustav Lejeune Dirichlet* (1805–1859) who studied under Cauchy and succeeded Gauss at Gottingen. He is known for his contributions to Fourier series and number theory.

$$= h^{l+m} \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m+1)} \qquad ...(i)\ [\because\ \Gamma(m+1)/m = \Gamma(m)]$$

(*b*) Taking $y + z \le 1 - x (= h$: say), the triple integral

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x^{l-1}\, y^{m-1}\, z^{n-1}\, dz dy dx$$

$$= \int_0^1 x^{l-1} \left[\int_0^h \int_0^{h-y} y^{m-1}\, z^{n-1}\, dz dy \right] dx = \int_0^1 x^{l-1} \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n+1)} h^{m+n}\, dx \qquad \text{... [By } (i)]$$

$$= \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n+1)} \int_0^1 x^{l-1} (1-x)^{m+n}\, dx = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n+1)}\, \beta(l, m+n+1)$$

$$= \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n+1)} \cdot \frac{\Gamma(l)\,\Gamma(m+n+1)}{\Gamma(l+m+n+1)} = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)}.$$

Example 7.48. *Evaluate the integral* $\iiint x^{l-1}\, y^{m-1}\, z^{n-1}\, dxdydz$ *where x, y, z are all positive with condition,* $(x/a)^p + (y/b)^q + (z/c)^r \le 1$. *(U.P.T.U., 2005 S)*

Solution. Put $(x/a)^p = u$, *i.e.*, $x = au^{1/p}$ so that $dx = \dfrac{a}{p} u^{1/p-1}\, du$

$(y/b)^q = v$, *i.e.*, $y = bv^{1/q}$ so that $dy = \dfrac{b}{q}\, v^{1/q-1}\, dv$

and $(z/c)^r = w$, *i.e.*, $z = cw^{1/r}$ so that $dz = \dfrac{c}{r} w^{1/r-1}\, dw$

Then $\iiint x^{l-1}\, y^{m-1}\, z^{n-1}\, dxdydz$

$$= \iiint (au^{1/p})^{l-1} (bv^{1/q})^{m-1} (cw^{1/r})^{n-1} \left(\frac{a}{p}\right) u^{1/p-1} \left(\frac{b}{q}\right) v^{1/q-1} \left(\frac{c}{r}\right) w^{1/r-1}\, dudvdw$$

$$= \frac{a^l\, b^m\, c^n}{pqr} \iiint u^{l/p-1}\, v^{m/q-1}\, w^{n/r-1}\, du\, dv\, dw \text{ where } u + v + w \le 1.$$

$$= \frac{a^l\, b^m\, c^n}{pqr} \frac{\Gamma(l/p)\,\Gamma(m/q)\,\Gamma(n/r)}{\Gamma(l/p + m/q + n/r + 1)} \qquad \text{[By Dirichlet's integral]}$$

Example 7.49. *The plane* $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$ *meets the axes in A, B and C. Apply Dirichlet's integral to find the volume of the tetrahedron OABC. Also find its mass if the density at any point is kxyz.* *(U.P.T.U., 2004)*

Solution. Put $x/a = u$, $y/b = v$, $z/c = w$ then the tetrahedron $OABC$ has $u \ge 0$, $v \ge 0$, $w \ge 0$ and $u + v + w \le 1$.

$\therefore$ volume of this tetrahedron $= \iiint_D dx\, dy\, dz$

$$= \iiint_{D'} abc\, du\, dv\, dw \qquad \left[\begin{array}{l} a\ dx = adu,\ dy = bdv,\ dz = cdw \\ \text{for } D' = u \ge 0, v \ge 0, w \ge 0\ \&\ u+v+w \le 1. \end{array} \right.$$

$$= abc \iiint_{D'} u^{1-1}\, v^{1-1}\, w^{1-1}\, du\, dv\, dw$$

$$= abc \frac{\Gamma(1)\,\Gamma(1)\,\Gamma(1)}{\Gamma(1+1+1+1)} = \frac{abc}{6} \qquad \text{[By Dirichlet's integral]}$$

$$\text{Mass} = \iiint kxyz dx\, dy\, dz = \iiint k\,(au)\,(bv)\,(cw)\, abc\, du\, dv\, dw$$

$$= ka^2b^2c^2 \iiint u^{2-1}\, v^{2-1}\, w^{2-1}\, du\, dv\, dw$$

$$= ka^2b^2c^2 \frac{\Gamma(2)\,\Gamma(2)\,\Gamma(2)}{\Gamma(2+2+2+1)} ka^2b^2c^2 \cdot \frac{1}{6!} = \frac{k}{720}\, a^2b^2c^2.$$

PROBLEMS 7.8

1. Compute :
   (*i*) $\Gamma(3.5)$ (*Assam, 1998*) (*ii*) $\Gamma(4.5)$
   (*iii*) $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$ (*S.V.T.U., 2009*) (*iv*) $\beta(2.5, 1.5)$ (*v*) $\beta\left(\frac{9}{2}, \frac{7}{2}\right)$. (*Andhra, 2000*)
2. Express the following integrals in terms of gamma functions :
   (*i*) $\int_0^\infty e^{-x^2}\,dx$ (*ii*) $\int_0^\infty x^{p-1}e^{-kx}\,dx (k > 0)$ (*Delhi, 2002 ; V.T.U., 2000*)
   (*iii*) $\int_0^\infty \sqrt{x}\,e^{-x^3}\,dx$ (*J.N.T.U., 2003*) (*iv*) $\int_0^\infty \frac{dx}{x^{p+1}.(x-1)^q} (-p < q < 1)$
3. Show that :
   (*i*) $\int_0^\infty \frac{x^4}{4^x}\,dx = \frac{\Gamma(5)}{(\log 4)^5}$ (*Marathwada, 2008*)
   (*ii*) $\int_0^{\pi/2} \sqrt{(\cot\theta)}\,d\theta = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$ (*Osmania, 2003 S ; V.T.U., 2001*)
   (*iii*) $\int_0^{\pi/2} [\sqrt{(\tan\theta)} + \sqrt{(\sec\theta)}]\,d\theta = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\left\{\Gamma\left(\frac{3}{4}\right) + \sqrt{\pi}/\Gamma\left(\frac{3}{4}\right)\right\}$ (*J.N.T.U., 2000*)
   (*iv*) $\int_0^{\pi/2} \sqrt{(\sin\theta)}\,d\theta \times \int_0^{\pi/2} \frac{d\theta}{\sqrt{(\sin\theta)}} = \pi$. (*V.T.U., 2007*)
4. Given $\int_0^\infty \frac{x^{n-1}}{1+x}\,dx = \frac{\pi}{\sin n\pi}$, show that $\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$. (*S.V.T.U., 2008*)

   Hence evaluate $\int_0^\infty \frac{dy}{1+y^4}$. (*V.T.U., 2006 ; J.N.T.U., 2005*)
5. Prove that :
   (*i*) $\int_0^1 \frac{x\,dx}{\sqrt{(1-x^5)}} = \frac{1}{5}\beta\left(\frac{2}{5}, \frac{1}{2}\right)$ (*Raipur, 2006*) (*ii*) $\int_0^1 \frac{dx}{\sqrt{(1+x^4)}} = \frac{1}{4\sqrt{2}}\beta\left(\frac{1}{4}, \frac{1}{2}\right)$. (*V.T.U., 2003*)
   (*iii*) $\int_0^1 x^3(1-\sqrt{x})^5\,dx = 2\beta(8, 6)$. (*Marathwada, 2008 ; J.N.T.U., 2006*)
6. Show that (*i*) $\int_a^b (x-a)^{m-1}(b-x)^{n-1}\,dx = (b-a)^{m+n-1}\beta(m, n)$ (*P.T.U., 2010 ; Mumbai, 2005*)
   (*ii*) $\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\,dx = \frac{1}{a^n . b^m}\beta(m, n)$ (*Nagpur, 2009*) (*iii*) $\int_0^\infty \frac{x^{10}-x^{18}}{(1+x)^{30}}\,dx = 0$ (*Mumbai, 2005*)
   (*iv*) $\int_0^1 \frac{(1-x^4)^{3/4}}{(1+x^4)^2}\,dx = \frac{1}{2^{9/2}}\beta\left(\frac{7}{4}, \frac{1}{4}\right)$ (*Mumbai, 2007*)
7. Prove that $\int_0^1 x^m(\log x)^n\,dx = \frac{(-1)^n\,n!}{(m+1)^{n+1}}$, where n is a positive integer and $m > -1$. (*S.V.T.U., 2006*)

   Hence evaluate $\int_0^1 x(\log x)^3\,dx$. (*Nagpur, 2009*)
8. Show that $\int_0^1 y^{q-1}\left(\log\frac{1}{y}\right)^{p-1} dy = \frac{\Gamma(p)}{q^p}$, where $p > 0, q > 0$. (*Rohtak, 2006 S*)
9. Express $\int_0^1 x^m(1-x^n)^p\,dx$ in terms of gamma functions (*Marathwada, 2008*)

   Hence evaluate : (*i*) $\int_0^1 x(1-x^3)^{10}\,dx$. (*Bhopal, 2008*) (*ii*) $\int_0^1 \frac{dx}{\sqrt{(1-x^n)}}$ (*Anna, 2005*)

10. Prove that $\int_0^\infty \frac{dx}{(e^x + e^{-x})^n} = \frac{1}{4}\beta\left(\frac{n}{2}, \frac{n}{2}\right)$ and hence evaluate $\int_0^\infty \operatorname{sech}^8 x \, dx$.

11. Prove that $\beta\left(n + \frac{1}{2}, n + \frac{1}{2}\right) = \frac{\Gamma(n + 1/2)\sqrt{\pi}}{2^{2n}\,\Gamma(n+1)}$. Hence show that $2^n\,\Gamma(n + 1/2) = 1 \cdot 3 \cdot 5 \ldots (2n - 1)\sqrt{\pi}$

(*Mumbai, 2007*)

12. Prove that :

(*i*) $\frac{\beta(m+1, n)}{m} = \frac{\beta(m, n+1)}{n} = \frac{\beta(m, n)}{m+n}$ (*ii*) $\beta(n, n) = \frac{\sqrt{\pi}\,\Gamma(n)}{2^{2n-1}\,\Gamma\left(n + \frac{1}{2}\right)}$

(*iii*) $\Gamma\left(n + \frac{1}{2}\right) = \frac{\Gamma(2n+1)\sqrt{\pi}}{2^{2n}\,.\,\Gamma(n+1)}$ (*iv*) $\beta(m + 1) + \beta(m, n + 1) = \beta(m, n)$

(*Bhopal, 2008 ; J.N.T.U., 2006 ; Madras, 2003*)

13. Show that $\iint x^{m-1}\, y^{n-1}\, dx\, dy$ over the positive quadrant of the ellipse

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \text{ is } \frac{a^m b^n}{2n}\,\beta\left(\frac{m}{2}, \frac{n}{2} + 1\right).$$

14. Show that the area in the first quadrant enclosed by the curve $(x/a)^\alpha + (y/b)^\beta = 1$, $\alpha > 0$, $\beta > 0$, is given by

$$\frac{ab}{\alpha+\beta}\,\frac{\Gamma(1/\alpha)\,\Gamma(1/\beta)}{\Gamma(1/\alpha + 1/\beta)}.$$

15. Find the mass of an octant of the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$, the density at any point being $\rho = kxyz$.

(*U.P.T.U., 2002*)

7.17 (1) ELLIPTIC INTEGRALS

In Applied Mathematics, we often come across integrals of the form $\int_0^1 e^{-x^2}\, dx$ or $\int_0^1 \sin x^2\, dx$ which cannot be evaluated by any of the standard methods of integration. In such cases, we may find the value to any desired degree of accuracy by expanding their integrands as power series. An important class of such integrals is the *elliptic integrals*.

Def. The integral $F(k, \phi) = \int_0^\phi \frac{dx}{\sqrt{(1 - k^2 \sin^2 x)}}\ (k^2 < 1)$ ...(*i*)

which is a function of the two variables k and ϕ, is called the *elliptic integral of the first kind with modulus k* and *amplitude* ϕ.

The integral $E(k, \phi) = \int_0^\phi \sqrt{(1 - k^2 \sin^2 x)}\, dx\ (k^2 < 1)$ ...(*ii*)

is called the *elliptic integral of the second kind* with *modulus k and amplitude* ϕ.

The name *elliptic integral* arose from its original application in finding the length of an elliptic arc (Fig.7.45), For instance, consider the ellipse

$$x = a\cos\phi, \qquad y = b\sin\phi, \qquad (a < b)$$

Then length of its arc

$$AP = \int_0^\phi \sqrt{\left[\left(\frac{dx}{d\phi}\right)^2 + \left(\frac{dy}{d\phi}\right)^2\right]}\, d\phi = \int_0^\phi \sqrt{[(-a\sin\phi)^2 + (b\cos\phi)^2]}\, d\phi$$

$$= \int_0^\phi \sqrt{[(b^2 + (a^2 - b^2)\sin^2\phi]}\, d\phi = b\int_0^\phi \sqrt{\left[1 - \left(1 - \frac{a^2}{b^2}\right)\sin^2\phi\right]}\, d\phi$$

$$= bE(k, \phi) \text{ for } k^2 = 1 - a^2/b^2 \le 1.$$

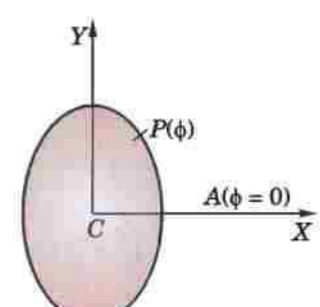

Fig. 7.45

Also the perimeter of the ellipse

$$= 4b \int_0^{\pi/2} \sqrt{(1 - k^2 \sin^2 \phi)}\, d\phi = 4bE(k, \pi/2).$$

This particular integral with upper limit $\phi = \pi/2$ is called the *complete elliptic integral of the second kind and is denoted by* $E(k)$.

Thus $$E(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 \phi)\, d\phi \; (k^2 < 1) \qquad ...(iii)$$

Similarly, the *complete elliptic integral of first kind is*

$$K(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{(1 - k^2 \sin^2 \phi)}} (k^2 < 1) \qquad ...(iv)$$

To evaluate it, we expand the integral in the form

$$(1 - k^2 \sin^2 \phi)^{-1/2} = 1 + \frac{k^2}{2} \sin^2 \phi + \frac{3k^4}{4} \sin^4 \phi + ...$$

This series can be shown to be uniformly convergent for all k, and may, therefore, be integrated term by term [See § 9.19-II]. Then we have

$$K(k) = \int_0^{\pi/2} \left(1 + \frac{k^2}{2} \sin^2 \phi + \frac{3k^4}{8} \sin^4 \phi + \frac{5k^6}{16} \sin^6 \phi + ...\right) d\phi$$

$$= \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1.3}{2.4}\right)^2 k^4 + \left(\frac{1.3.5}{2.4.6.}\right)^2 k^6 + ...\right] \qquad ...(v)$$

This series may be used to compute K for various values of k. In particular, if $k = \sin 10°$; we have

$$K = \frac{\pi}{2} (1 + 0.00754 + 0.00012 + ...) = 1.5828 \qquad ...(vi)$$

In this way tables of the elliptic integrals are constructed. Values of $F(k, \phi)$ and $E(k, \phi)$ are readily available for $0 \le \phi \le \pi/2$, $0 < k < 1$. *(See Peirce's short tables).*

Example 7.50. *Express* $\int_0^{\pi/6} \frac{dx}{\sqrt{(\sin x)}}$ *in terms of elliptic integral.*

Solution. Put $\cos x = \cos^2 \phi$ and $dx = \dfrac{2 \cos \phi \, d\phi}{\sqrt{(1 + \cos^2 \phi)}}$

Then $$I = \int_0^{\pi/2} \frac{2 \cos^2 \phi}{\sqrt{(1 + \cos^2 \phi)}} d\phi = 2 \int_0^{\pi/2} \frac{(1 + \cos^2 \phi) - 1}{\sqrt{(1 + \cos^2 \phi)}} d\phi$$

$$= 2 \left\{ \int_0^{\pi/2} \sqrt{(1 + \cos^2 \phi)}\, d\phi - \int_0^{\pi/2} \frac{d\phi}{\sqrt{(1 + \cos^2 \phi)}} \right\} = 2 \left\{ \int_0^{\pi/2} \sqrt{(2 - \sin^2 \phi)}\, d\phi - \int_0^{\pi/2} \frac{d\phi}{\sqrt{(2 - \sin^2 \phi)}} \right\}$$

$$= 2\sqrt{2} \int_0^{\pi/2} \sqrt{(1 - 1/2 \sin^2 \phi)}\, d\phi - \sqrt{2} \int_0^{\pi/2} \frac{d\phi}{\sqrt{(1 - 1/2 \sin^2 \phi)}} = 2\sqrt{2}\, E\left(\frac{1}{\sqrt{2}}\right) - \sqrt{2}\, K\left(\frac{1}{\sqrt{2}}\right)$$

(2) **Elliptic functions.** By putting $\sin x = t$ and $\sin \phi = z$, (i) becomes

$$u = \int_0^z \frac{dt}{\sqrt{[(1 - t^2)(1 - k^2 t^2)]}} (k^2 < 1) \qquad ...(vii)$$

This is known as *Jacobi's form of the elliptic integral of first kind** whereas (i) is the *Legendre's form†*.

If $k = 0$, (vii) gives $u = \sin^{-1} z$. By analogy, we denote (vii) $sn^{-1} z$ for a fixed non-zero value of k. This leads to the functions $sn\ u = z = \sin \phi$ and $cn\ u = \cos \phi$ which are called the *Jacobi's elliptic functions.*

* See footnote p. 215.

† A French mathematician *Adrien Marie Legendre* (1752–1833) who made important contributions to number theory, special functions, calculus of variations and elliptic integrals.

The elliptic functions $sn\, u$ and $cn\, u$ are periodic with a period depending on k and an amplitude equal to unity. These behave somewhat like $\sin u$ and $\cos u$. For instance

$$sn(0) = 0, cn(1) = 1 \quad \text{and} \quad sn(-u) = -sn(u), cn(-u) = cn(u).$$

Example 7.51. *Show that* $\int_0^{a/2} \frac{dx}{\sqrt{(2ax - x^2)}\sqrt{(a^2 - x^2)}} = \frac{2}{3a} K\left(\frac{1}{3}\right)$.

Solution. Putting $x = \frac{a}{2}(1 - \sin\theta), dx = -\frac{a}{2}\cos\theta\, d\theta,$

$$2ax - x^2 = \frac{a^2}{4}(1 - \sin\theta)(3 + \sin\theta) \text{ and } a^2 - x^2 = \frac{a^2}{4}(1 + \sin\theta)(3 - \sin\theta)$$

Also when $x = 0, \theta = \pi/2$; when $x = a/2, \theta = 0$.

Thus the given integral

$$= \frac{4}{a^2}\int_{\pi/2}^{0} \frac{-(a/2)\cos\theta\, d\theta}{\sqrt{[(1 - \sin^2\theta)(2 - \sin^2\theta)]}} = \frac{2}{3a}\int_0^{\pi/2} \frac{d\theta}{\sqrt{[(1 - (1/3)^2 \sin^2\theta]}} = \frac{2}{3a} K\left(\frac{1}{3}\right).$$

7.18 (1) ERROR FUNCTION OR PROBABILITY INTEGRAL

The error function or the probability integral is defined as

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\, dt.$$

This integral arises in the solution of certain partial differential equations of applied mathematics and occupies an important position in the probability theory.

The complementary error function $erfc(x)$ is defined as $erfc(x) = 1 - erf(x)$.

(2) Properties : (*i*) $erf(-x) = -erf(x)$; (*ii*) $erf(0) = 0$

$$(iii)\ erf(\infty) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2} dt = \frac{2}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2} = 1$$ [By (*iii*), p. 289]

This proves that the total area under the *Normal* or *Gaussian error function curve* is unity – § 26.16.

PROBLEMS 7.9

1. By means of the substitution $k \sin x = \sin z$, show that

(*i*) $\int_0^{\pi} \frac{dx}{\sqrt{(1 - k^2\sin^2 x)}} = \frac{1}{k} F\left(\frac{1}{k}, \phi'\right)$;

(*ii*) $\int_0^{\phi} \sqrt{(1 - k^2\sin^2 x)}\, dx = \left(\frac{1}{k} - k\right) F\left(\frac{1}{k}, \phi'\right) + kE\left(\frac{1}{k}, \phi'\right)$

where $k > 1$ and $\phi' = \sin^{-1}(k \sin\phi)$.

Express the following integrals in terms of elliptic integrals :

2. $\int_0^{\pi/2} \frac{dx}{\sqrt{(1 + 3\sin^2 x)}}$. (*Kerala, M.E., 2005*)
3. $\int_0^{\pi/2} \frac{dx}{\sqrt{(2 - \cos x)}}$.
4. $\int_0^{\pi/2} \sqrt{(\cos x)}\, dx$.
5. Expand $erf(x)$ in ascending powers of x. Hence evaluate $erf(0)$. (*P.T.U., 2009 S*)
6. Compute (*i*) $erf(0.3)$, (*ii*) $erf(0.5)$, correct to three decimal places.
7. Show that (*i*) $erf(x) + erf(-x) = 0$ (*ii*) $erfc(x) + erfc(-x) = 2$
8. Prove that

(*i*) $\frac{d}{dx}[erf(ax)] = \frac{2a}{\sqrt{\pi}} e^{-a^2x^2}$ (*Osmania, 2003*) (*ii*) $\frac{d}{dx}[erfc(ax)] = -\frac{2a}{\sqrt{\pi}} e^{-a^2x^2}$.

9. Prove that $\int_0^\infty e^{-x^2 - 2ax}\, dx = \frac{\sqrt{\pi}}{2} e^{a^2}[1 - erf(0)]$

7.19 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 7.10

Fill up the blanks or choose the correct answer from the following problems :

1. $\int_0^2 \int_0^x (x + y)\, dx\, dy =$ ....
2. $\int_0^1 \int_0^{1-x} dx\, dy$ ....
3. $\int_0^1 e^{-x^2}\, dx =$ ....
4. $\beta\left(\frac{1}{2}, \frac{1}{2}\right) =$ .... (*V.T.U., 2010*)
5. $\Gamma(3.5) =$ ....
6. The surface area of the sphere $x^2 + y^2 + z^2 + 2x - 4y + 8z - 2 = 0$ is ....
7. $\int_0^2 \int_1^3 \int_1^2 xy^2 z\, dz dy dx =$ ....
8. If $u = x + y$ and $v = x - 2y$, then the area-element $dxdy$ is replaced by .... $dudv$.
9. In terms of Beta function $\int_0^{\pi/2} \sin^7 \theta \sqrt{\cos \theta}\, d\theta =$ ....
10. The value of $\beta(2, 1) + \beta(1, 2)$ is ....
11. $\int_0^1 \int_1^2 xy\, dy\, dx =$ ....
12. Volume bounded by $x \geq 0$, $y \geq 0$, $z \geq 0$ and $x^2 + y^2 + z^2 = 1$ as a triple integral integral.
13. Value of $\int_0^1 \int_0^{x^2} xe^y\, dy\, dx$ is equal to

    (*a*) $e/2$ (*b*) $e - 1$ (*c*) $1 - e$ (*d*) $e/2 - 1$. (*Bhopal, 2008*)
14. $\iint x^2 y^3\, dxdy$ over the rectangle $0 \leq x \leq 1$ and $0 \leq y \leq 3$ is ....
15. $\int_0^\pi \int_0^{a \sin \theta} r\, dr\, d\theta =$ ....
16. $\int_{x=0}^{x=3} \int_{y=0}^{y=1/x} ye^{xy}\, dxdy =$ ....
17. $\int_0^{\pi/2} \int_0^\infty \frac{r\, dr\, d\theta}{(r^2 + a^2)} =$ ....
18. $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2)\, dx\, dy =$ ....
19. To change cartesian coordinates (x, y, z) to spherical polar coordinate (r, θ, ϕ); $dx\, dy\, dz$ is replaced by ....
20. $\int_0^2 \int_0^{x^2} e^{y/x}\, dydx =$ ....
21. $\iint (x + y)^2\, dxdy$ over the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is ....
22. $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx\, dy}{1 + x^2 + y^2} =$ ....
23. $\iint xy(x + y)\, dx\, dy$ over the area between $y + x^2$ and $y = x$, is ....
24. Value of $\int_0^1 \int_x^{\sqrt{x}} xy\, dx\, dy$ is

    (*a*) zero (*b*) $-1/24$ (*c*) $1/24$ (*d*) 24. (*V.T.U., 2010*)
25. $\iint dx\, dy$ over the area bounded by $x = 0$, $y = 0$, $x^2 + y^2 = 1$ and $5y = 3$, is ....
26. $\iint_R y\, dx\, dy$ where R is the region bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$, is ....
27. $\iint (x^2 + y^2)\, dx\, dy$ in the positive quadrant for which $x + y \leq 1$, is ....
28. Area between the parabolas $y^2 = 4x$ and $x^2 = 4y$ is ....
29. Changing the order of integration in $\int_{-a}^a \int_0^{\sqrt{a^2-y^2}} f(x, y)\, dx\, dy =$ ....
30. $\Gamma(1/4)\Gamma(3/4) =$ .... (*V.T.U., 2011*)
31. $\beta(5/2, 7/2) =$ ....
32. $\int_0^\infty \int_0^x xe^{-x^2/y}\, dydx =$ ....
33. On changing to polar coordinates $\int_0^{2a} \int_0^{\sqrt{(2ax-x^2)}} dx\, dy$ becomes ....

34. A square lamina is immersed in the liquid with one vertex in the surface and the diagonal of length vertical. Its centre of pressure is at a depth ....
35. The centroid of the area enclosed by the parabola $y^2 = 4x$, x-axis and its latus-rectum is ....
36. The moment of inertia of a uniform spherical ball of mass 10 gm and radius 2 cm about a diameter is ....
37. M.I. of a solid right circular cone (base-radius r and height h) about its axis is ....

38. $erf_c(-x) - erf(x) =$ .... 39. $\int_0^1 \frac{x-1}{\log x}\, dx =$ .... 40. $\Gamma\left(\frac{3}{2}\right) =$ ....

41. Value of $\int_0^a \int_0^b \int_0^c x^2y^2z^2\, dx\, dy\, dz$ is

(a) $\frac{abc}{3}$ (b) $\frac{a^2b^2c^2}{27}$ (c) $\frac{a^3b^3c^3}{27}$ (d) $\frac{a^2b^2c^2}{9}$.

42. The integral $\int_0^1 \int_0^{\sqrt{(1-x^2)}} (x+y)\, dy\, dx$ after changing the order of integration.

(a) $\int_0^2 \int_0^{\sqrt{(1-y^2)}} (x+y)\, dx\, dy$ (b) $\int_0^1 \int_0^{\sqrt{(1-y^2)}} (x+y)\, dx\, dy$

(c) $\int_0^1 \int_0^{\sqrt{(1+y^2)}} (x+y)\, dx\, dy$ (d) $\int_0^{-1} \int_0^{\sqrt{(1-y^2)}} (x+y)\, dx\, dy$. (V.T.U., 2011)