*Solutions to the Exercises in*

# FIRST COURSE IN MATHEMATICAL LOGIC

This is a solutions manual to accompany a
Blaisdell Book in the Pure and Applied Sciences,
*First Course in Mathematical Logic*,
by Patrick Suppes and Shirley Hill.
The consulting editor for the textbook is
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# Solutions to the Exercises in

# FIRST COURSE IN

# MATHEMATICAL

# LOGIC

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# Preface

This book presents solutions to the exercises in *First Course in Mathematical Logic*. The solutions are written to show the teacher how to set up the problem, how to state the premise, and what steps to take in solving the given problem, that is, the steps that lead to the correct conclusion with the specific rule that applies to each step.

It is essential that the teacher work out each problem, not just look at a solution, before presenting the corresponding topic to the class or assigning the problem to the students. Merely reading through solutions does not give a strong enough mastery nor sufficient appreciation of the details or difficulties faced by the students to result in clear and sensitive teaching. Further, in assigning problems, the teacher should have decided whether suggestions are called for to direct the students' efforts in fruitful directions. Finally, when demonstrating a solution in class, the teacher should not have to refer to a key.

There are always many ways of deriving a given conclusion from given premises. In general, the rules simply tell the different things that may be done. Which rule to use and the specific details of its use are not given in the solutions book. This is a matter of strategy determined by experience or ingenuity. One proof may be shorter or may be more elegant than another, but any valid proof is acceptable. The class may enjoy seeing different proofs for the same problem.

The text is to be used as a textbook, not a workbook. Tell the students to copy the exercises on their homework papers and not to write in the text.

*First Course in Mathematical Logic* offers an opportunity to gain experience in a kind of reading that we will call "analytic reading." The students are well acquainted with reading for content, which can be done rapidly as with novels, magazines, newspapers, and so forth. The students have also had some experience with the critical reading of history and the reasoned presentation of ideas. This must be slower reading to gain understanding and prepare one for critical discussion of the ideas which the reading matter presents. Analytic reading proceeds even more slowly. In it the reader seeks to understand in detail how the author did, and how the reader may, obtain certain results. Analytic reading requires that the reader go — at least in his mind and often on paper — through all of the necessary steps to obtain the results. He cannot simply observe the results; he must understand how they are obtained. In analytic reading a person seeks to prepare himself for the action of carrying out the same kind of work. He must acquire understanding *and* technique. The reading of instructions for baking a cake, for assembling a piece of apparatus, or for operating a machine requires this careful "going through it with the author," although in such cases acquiring understanding takes little effort. The effort is almost entirely spent in gaining technique. A person can bake a cake without understanding the chemical or physical principles involved or operate a machine without understanding how it works. This is not true with logic. The student cannot "do logic" without understanding it; he cannot even understand logic without "doing it."

This new kind of reading requires careful instruction. At first, give the students considerable help. Read the text aloud in class carefully, carrying out on the blackboard all of the steps involved. Every time any previous page, rule, definition, or formula is referred to, actually go back to it. Frequently point out to the students that all of these activities are essential in analytic reading. Require that each student do the same writing at his desk. Have him actually write out all the steps involved and the answers to every question you ask. Point out that analytic reading demands that he ask himself such questions as *what*, *how*, and *why*. Then assign additional analytic reading to be done by the student outside of the classroom. Frequently give him help before and after the readings; point out what understandings and skills are to be gained; and require that he put these skills to use immediately.

The teacher may want to give the students the following suggestions as to what to look for when reading a paragraph. A well-written paragraph is built around one idea. There is one and only one idea being presented. This idea is stated in a topic sentence which usually appears at or near the beginning or end of the paragraph. Almost every sentence in a paragraph is one of the following: (1) *topic sentence*, (2) *repetition of the topic sentence*, (3) *build up* preparing for the topic sentence, or (4) *follow up* explaining the topic sentence, giving illustrations or applications. The student needs to recognize which of these each sentence is. It may be of value to study paragraphs together in class, deciding the function of each sentence.

Notice that this procedure demands continuous response from every student. The teacher should elicit student participation at all times. Logic is not something to look at and listen to, but something to be done, that is, a skill to be gained. The lecture method is subject to very rapidly diminishing returns after only a short period of instruction has been given unless active thought and response are elicited. The students must always have pencil and paper on hand and in use. One device the teacher may find useful here is to frequently ask questions having very short answers and to require every student to write out an answer at his desk. This brings about the continuous active participation of students who tend to think only when called on. It is a way of continuous calling on every member of the class.

There are two general groups of things that we prove with logic: (1) that sets of statements (arguments) are valid or invalid or (2) that sets of statements are consistent or inconsistent (that is, possibly all true at once or not). If the set contains only one statement, the second group reduces to whether it is logically false or possibly true. A special case of a statement being possibly true is that in which it is necessarily or logically true, in which case the statement is called a *tautology* or a truth of logic.

Concerning valid arguments, it is well to have in mind that with logic we cannot say simply that we can prove a statement with factual contents. What we prove is that *if* certain other statements (set of premises) are true, then so must be the conclusion. We do not simply prove the conclusion in isolation. In the text, for example, on page 10, problem 14, you are asked,

"Prove: S *assuming* (1) ¬T ∨ R, (2) T, and (3) ¬S → ¬R"

or "Prove S *from* these premises." The 'assuming' or 'from' are understood. This is what is meant by calling the statements 'premises': that is, statements assumed to be, or taken as, true.

However, there are two types of statements that can be asserted without premises. But these statements do not have any factual content; they, indirectly or directly, say something about language. The first type of statements in this category are "definitions." They merely announce a resolution to use some word or expression in a certain way. When we say "a triangle is a 'three-sided figure,'" we are simply saying "Wherever we would use the phrase 'three-sided figure,' we may use instead the word 'triangle.'" The rules to be followed in properly introducing the definitions to be used are the subject of the theory of definition and cannot be given here. (These rules guarantee, for example, that our definitions do not add factual content, or result in inconsistencies in our discourse, or attempt to define entities into existence.) The second type of statements that can be asserted without premises are "truths of logic." For a discussion see pages 236–237 and 250 of the text. On page 250 statements of this type are proved without a dependence on premises, but they do not have any factual content. They simply tell something about the syntax of language, how statements can be constructed.

It might be valuable at some time to discuss with the class distinctions between what it means to say an argument is reasonable and what it means to say an argument is logical. To say that someone or some course of action is reasonable is to say that intelligence and wisdom have been used well. This includes consideration of available evidence, use of verified principles, induction according to the weight of evidence, and the use of logical analysis for valid deduction and the discovery of inconsistencies. It is unreasonable to go against evidence or to attempt simultaneously to accept inconsistencies — that is, to be illogical. Being logical is a necessary part of being reasonable.

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*Solutions to the Exercises in*

# FIRST COURSE IN MATHEMATICAL LOGIC

# Symbolizing Sentences

▶ 1.2 *Sentential Connectives*

**1.2, Exercise 1, pages 3–5**

**A.**
1. A
2. A
3. M or
4. A
5. M and
6. M or
7. M if . . . then . . .
8. M and
9. A
10. M not
11. M if . . . then . . .
12. A
13. M not
14. M if . . . then . . .
15. M not
16. A
17. M if . . . then . . .
18. M if . . . then . . .
19. M or
20. M and
21. M if . . . then . . .
22. A
23. M or
24. M and

**C.**
| | | |
|---|---|---|
| 1. not | 1 | |
| 2. and | 2 | |
| 3. not | 1 | |
| 4. or | 2 | |
| 5. if . . . then . . . | 2 | |
| 6. or | 2 | |
| 7. if . . . then . . . | 2 | |
| 8. and | 2 | |
| 9. or | 2 | |
| 10. if . . . then . . . | 2 | |
| 11. if . . . then . . . | 2 | |
| 12. if . . . then . . . | 2 | |
| 13. or | 2 | |
| 14. if . . . then . . . ; or | 3 | |
| 15. if . . . then . . . ; and | 3 | |
| 16. no connective | 1 | |

▶ 1.3 *The Form of Molecular Sentences*

Treating the form of molecular sentences as having sets of parentheses outside the connectives emphasizes the form and allows two easy transitions: (1) The transition from English to logical symbolism is facilitated since parentheses may be inserted in the English. (2) The transition to formulas containing more than one connective where parentheses are often critical or a help to the student. The parentheses are required up to page 22 (in the textbook) except for negations.

Note that we have included sentences involving more than one connective. It is to be explained (in Section 1.5) that every sentence is one and only one of the following: (1) atomic, (2) negation, (3) conjunction, (4) disjunction, (5) conditional (or later, page 103, biconditional). So what is wanted here is the over-all form. The exercise is thus also a preparation for the idea of dominance (see page 21). However, if some students want to give a complete analysis they may be told of the possibility of parentheses within parentheses. Of course, this has already been implied by the suggestion at the bottom of page 6, that the spaces within the parentheses can be filled with any sentences. Solutions below are given with such complete analysis.

**1.3, Exercise 2, page 9**

**A.**
1. (John is here) and (Mary has left).
2. If ($x+1=10$) then ($x=9$).

1

**1.3, Exercise 2, page 9** (*continued*)

 **A.** 3. Either [not (Mary is here)] or [Jane is gone].

    4. If [either ($x=1$) or ($y=2$)] then [$z=3$].

    5. If [(not ($x=1$)) and ($x+y=2$)] then [$y=2$].

    6. If [either (Smith is at home) or (Jones is in court)] then [Scott is innocent].

    7. ($y=0$) and ($x=0$).

    8. Either [($y=0$) and (not ($x=0$))] or [$z=2$].

    9. It is not the case that (6 = 7).

   10. It is not the case that [if ($x+0=10$) then ($x=5$)].

## ▶ 1.4 *Symbolizing Sentences*

**1.4, Exercise 3, pages 11–12**

 **A.** 2. Let **D** = 'Ducklings do grow into swans'
     Then: Not (**D**)

    3. Let **P** = 'He walked three steps to the right'
      **Q** = 'Then he went two steps forward'
     Then: (**P**) and (**Q**)

    4. Let **P** = 'These problems are easy for me'
     Then: Not (**P**)

    5. Let **B** = 'The buzzer sounds'
      **T** = 'It is time for class to begin'
     Then: If (**B**) then (**T**)

    6. Let **C** = 'Chemistry class has already begun'
      **L** = 'I am late'
     Then: If (**C**) then (**L**)

    7. Let **M** = 'One side of the moon is seen from the earth'
     Then: Not (**M**)

    8. Let **D** = 'Dick will go to the dance'
      **S** = 'He will go to the show'
     Then: Either (**D**) or (**S**)

    9. Let **P** = 'Roses are red'
      **Q** = 'Violets are blue'
     Then: (**P**) and (**Q**)

   10. Let **P** = 'Brazil is in South America'
      **Q** = 'It is in the Southern Hemisphere'
     Then: If (**P**) then (**Q**)

 **B.** The English results should not contain parentheses.

 **C.** 1. and
    John is second. Tim is fourth.

    2. Either . . . or . . .
    Jack is the winner. Jim is the winner.

    3. Not
    Eddie is the winner.

    4. If . . . then . . .
    Jim is the winner. He gets the medal.

**C.**    5. If . . . then . . . ; not

Jim is the winner.    He must have placed second.

6. and

The Alps are young mountains.

The Appalachians are old mountains.

7. not

Spiders are insects.

8. If . . . then . . .

Spiders are insects.    They must have six legs.

9. If . . . then . . .

Material is heated.    It expands.

10. Either . . . or . . .

Most planets are too hot for living things like ourselves.

They are too cold for living things like ourselves.

**D.**    1. If (**P**) then (**Q**)

2. If not (**P**) then (**Q**)

3. If either not (**P**) or not (**Q**) then (**R**)

4. If (**P**) then (**Q**)

5. If (**P**) then not (**Q**)

6. **P** and **Q**

7. (**P**) or (**Q**)

8. If not (**P**) and not (**Q**) then (**R**)

9. If (**P**) and (**Q**) then (**R**)

10. If not (**P**), then not (**Q**) and not (**R**)

▶ 1.5 *The Sentential Connectives and Their Symbols*

The writing of the symbol & requires specific instruction. Although the symbol may be written either toward the bottom or toward the top, in either case the difficulty is that the motion must start backwards, that is, right to left. Start by moving back down ✔ , then up ↺, around ↻ and down forward ⅋. Or, start by moving back up ↖ , over ↻ , back down ↺, and up forward ⅋ : ⅋.

**1.5, Exercise 4, pages 13–14**

**A.**    Example 1

Let **B** = 'Bob lives on our street'

    **J** = 'Janet lives on the next block'

Then: (**B**) & (**J**)

(The remainder of the examples are to be written in the same manner.)

**B.**    1. (**P**) & (**Q**)

2. (**A**) & (**B**)

3. (**H**) & (**K**)

4. (**T**) & (**G**)

5. (**S**) & (**Q**)

**C.**    Example 1

'My favorite color is blue' for (**P**)

'His favorite color is green' for (**Q**)

My favorite color is blue and his favorite color is green.

(The students are to write similar sentences for the remaining examples.)

**D.**    1. $(x=0)$   &   $(y=4)$

2. $(x \neq 0)$   &   $(x+y=2)$

3. $(x-x=0)$   &   $(x+0=x)$

4. $(x+y=y+x)$   &   $(x+(y+z)=(x+y)+z)$

**1.5, Exercise 5, pages 15–16**

**A.**    Example 1

Let **P** = 'The area of triangle $ABC$ is equal to the area of triangle $DEF$'

    **Q** = 'The area of triangle $ABC$ is less than the area of triangle $DEF$'

**1.5, Exercise 5, pages 15-16** (*continued*)

**A.** Then: (P) ∨ (Q)

(The remainder of the examples are done in this same manner.)

**B.** 1. (P) ∨ (Q)  2. (P) ∨ (Q)  3. (R) ∨ (S)  4. (T) ∨ (E)  5. (P) ∨ (N)

**C.** Follow the procedure of Exercise 4C, page 14.

**D.**
1. $(x=0)$ ∨ $(x>0)$
2. $(x\neq0)$ & $(y\neq0)$
3. $(x>1)$ ∨ $(x+y=0)$
4. $(y=x)$ ∨ $(y\neq x)$
5. $(y+x>y+x+z)$ ∨ $(z=0)$
6. $(y+z=z+y)$ & $(0+x=x)$

**E.**
1. $((x+y=0)$ & $(z>0))$ ∨ $(z=0)$
2. $(x=0)$ & $((y+z>x)$ ∨ $(z=0))$
3. $(x\neq0)$ or $((x=0)$ & $(y>0))$
4. $((x=y)$ & $(z=w))$ ∨ $((x<y)$ & $(z=0))$

**1.5, Exercise 6, pages 18-19**

Note that the standard form for a negation is ¬(    ).  Also see the text, page 29.

**A.** Example 1

Let J='In the Southern Hemisphere, July is a summer month'

Then: ¬(J), and so forth.

**B.** 1. ¬(R)  2. ¬(Q)  3. ¬(H)  4. ¬(T)  5. ¬(J)

**C.** 1. (P) & ¬(Q)  2. ¬(R) & ¬(M)  3. (S) ∨ ¬(B)  4. ¬(P) ∨ ¬(Q)  5. (T) & ¬(R)

**D.** 1. (P) ∨ (Q)  2. ¬(P) ∨ (Q)  3. (Q) & ¬(P)  4. ¬(Q) & ¬(P)  5. ¬(P) & (Q)

**E.**
1. negation
2. conjunction
3. negation
4. disjunction
5. conjunction
6. negation
7. disjunction
8. conjunction
9. negation
10. disjunction

**F.**
1. not . . . and . . . not . . .
2. If . . . not . . . then . . .
3. If . . . not . . . then . . .
4. . . . or . . . not . . .
5. If . . . then . . . not . . .
6. . . . or . . .
7. If . . . then . . .
8. If . . . and . . . then . . .
9. If . . . and . . . then . . .
10. Either . . . or . . . not . . .

**1.5, Exercise 7, pages 20-21**

**A.** Example 1

Let C='It is cold enough'.

L='The lake will be frozen over'.

Then: (C) → (L),

and so forth.

**B.**
1. Jane is youngest.
2. Molly is oldest.
3. Janet is youngest.

4. Mike is older.
5. Mike is seventeen.

**C.**
1. Bob is third.
2. He is before John.
3. Carl is fifth.
4. He is after Mark.
5. Mark is first.

**D.** 1. (P) → (R)    2. (S) → (T)    3. (Q) → (P)    4. (P) → ¬(S)    5. ¬(S) → ¬(T)

**E.**   C after sentences: 2, 3, 6, 8, 9

It may be helpful briefly to point out several things that must be true of any formula that adequately symbolizes a sentence.

(1) Each formula is one, but only one of the following: an atomic sentence, a conditional, a conjunction, a disjunction, or a negation.

(2) The connectives can work only with whole sentences.

(3) A formula can begin only with (a) a capital letter, (b) an opening parenthesis, or (c) a negation. It ends only with (a) a capital letter, or (b) a closing parenthesis.

(4) For a conjunction in English: The left member (sometimes called left conjunct) is everything between 'both' and 'and'. The right conjunct begins immediately after 'and' and ends at the end of the whole conjunction that is at the end of the first sentence starting after the 'and' (if no 'both' or 'either' has been left out). If a 'both' or an 'either' is missing then the right conjunct ends with the first punctuation that makes a full sentence after the 'and'.

(5) For a disjunction in English the left disjunct and right disjunct can be recognized in a similar way. And the same is true for the antecedent and consequent of a conditional.

(6) When no 'both' or 'either' is missing the order of dominance is clearly given by the order in which any 'both', 'either', 'if', or 'not' occur; and the place to put &, ∨, →, or ¬ is shown by the place where 'and', 'or', 'then', or 'not' are found.

(7) Inside any pair of parentheses must be a complete sentence.

(8) It is best to consider parentheses to be part of the sentence they enclose. Thus in ¬(P → Q) the negated sentence begins with the opening parenthesis and does not end until its closing parenthesis mate. This is consistent with the discussions on pages 16 and 29 of the text.


▶ 1.6 *Grouping and Parentheses*


## 1.6, Exercise 8, page 25

(He is wrong) and (I am right or I shall be surprised).   (He is wrong and I am right) or (I shall be surprised).


## 1.6, Exercise 9, pages 26–27

**A.**  1. (P ∨ Q) & S
    2. (Q ∨ R) & S
    3. Q & (R ∨ T)
    4. (P ∨ R) & Q
    5. R & (P ∨ T)

**B.**  1. P ∨ (Q & S)
    2. Q ∨ (R & S)
    3. (Q & R) ∨ T
    4. (P & Q) ∨ R
    5. P ∨ (Q & R)

**C.**  1. S ∨ (T & R)
    2. (T ∨ S) & Q
    3. T & (S ∨ R)
    4. P ∨ (Q & T)
    5. (P & Q) ∨ R

**D.**  1. Let R = 'Randy is President'
        J = 'Jim is Treasurer'
        B = 'Bert is Treasurer'
      Then: (R & J) ∨ B

    2. R & (J ∨ B)

3. (N & S) ∨ L
4. N & (S ∨ L)
5. (J ∨ T) & G
6. (N ∨ M) & C

**E.**  1. Let P = '*x* is less than two'
        Q = '*x* is equal to one'
        S = '*x* is equal to 0'
      P → (Q ∨ S)

2. (L & G) → E
3. F & (I → F)
4. (G & L) ∨ ¬E
5. (P & M) → S

**1.6, Exercise 9, pages 26–27** (*continued*)

**F.**   1. $x<2 \rightarrow (x=1 \lor x=0)$        4. $(x>5 \;\&\; x<7) \lor x \neq 6$

       2. $(x<3 \;\&\; x>1) \rightarrow x=2$     5. $(x+3>5 \;\&\; y-4>x) \rightarrow y>6$

       3. $y=4 \;\&\; (x<y \rightarrow x<5)$

**1.6, Exercise 10, pages 28–29**

**A.**   1. $P \rightarrow (R \;\&\; S)$          9. $(P \rightarrow R) \lor Q$       **B.**   1. $J \rightarrow (K \;\&\; C)$

       2. $P \rightarrow (Q \lor R)$        10. $P \rightarrow (R \lor Q)$           2. $(K \lor C) \rightarrow J$

       3. $(P \;\&\; Q) \rightarrow R$       11. $P \;\&\; (Q \rightarrow T)$          3. $(J \rightarrow C) \lor \neg J$

       4. $(R \lor P) \rightarrow Q$       12. $(P \;\&\; Q) \rightarrow T$         4. $K \lor (J \rightarrow C)$

       5. $(P \rightarrow Q) \;\&\; S$       13. $P \lor (T \rightarrow Q)$         5. $(K \rightarrow C) \;\&\; \neg J$

       6. $R \;\&\; (P \rightarrow Q)$       14. $(Q \rightarrow R) \lor \neg S$

       7. $R \lor (Q \rightarrow T)$       15. $Q \rightarrow (R \lor \neg S)$

       8. $(Q \rightarrow P) \lor S$

**1.6, Exercise 11, pages 30–33**

**A.**   1. $\neg(P \lor R)$       **C.**   1. $\neg(P \rightarrow R)$       **E.**   2. $D \rightarrow (R \;\&\; F)$

       2. $\neg(P \rightarrow S)$          2. $\neg P \rightarrow R$             3. $W \lor (O \;\&\; T)$

       3. $\neg(P \;\&\; T)$         3. $\neg P \;\&\; \neg R$           4. $(W \lor O) \;\&\; T$

       4. $\neg(P \rightarrow \neg Q)$       4. $\neg(R \;\&\; T)$          5. $(W \;\&\; O) \lor T$

       5. $\neg(R \lor S)$         5. $P \rightarrow \neg Q$            6. $\neg(J \;\&\; A)$

       6. $\neg(\neg Q \;\&\; \neg S)$  or     6. $\neg(P \rightarrow \neg Q)$      7. $\neg J \;\&\; A$

          $\neg\neg(Q \;\&\; \neg S)$       7. $\neg Q \lor \neg R$          8. $(P \;\&\; D) \rightarrow C$

                            8. $\neg(T \lor S)$          9. $H \lor (\neg H \rightarrow W)$

                            9. $\neg S \;\&\; \neg Q$        10. $\neg H \;\&\; \neg V$

**B.**   1. $\neg S$                  10. $\neg(R \rightarrow S)$      11. $(T \rightarrow S) \;\&\; C$

       2. $\neg(P \lor T)$                          12. $\neg(P \lor D)$

       3. $\neg(S \;\&\; \neg T)$      **D.**   1. $\neg P \lor \neg Q$     13. $\neg R \rightarrow \neg Q$

       4. $\neg(P \rightarrow R)$          2. $\neg(Q \rightarrow P)$      14. $F \rightarrow (D \;\&\; \neg K)$

       5. $\neg(Q \;\&\; R)$          3. $\neg(P \lor Q)$      15. $(F \rightarrow D) \;\&\; \neg K$

       6. $\neg\neg R$                  4. $\neg Q \;\&\; P$

       7. $\neg(T \rightarrow \neg S)$        5. $\neg(P \;\&\; Q)$

       8. $\neg(\neg N \lor M)$        6. $\neg Q \rightarrow \neg P$

       9. $\neg(\neg Q \rightarrow \neg T)$      7. $\neg(P \rightarrow Q)$

      10. $\neg(\neg S \;\&\; P)$        8. $\neg P \lor Q$

      11. $\neg(P \lor \neg S)$        9. $\neg P \;\&\; Q$

      12. $\neg\neg Q$               10. $\neg(Q \;\&\; P)$

**F.**   1. $(x=0 \lor x=1) \rightarrow y=2$      5. $(x=y \lor x=z) \;\&\; y>3$

       2. $x=0 \lor (x \neq 0 \;\&\; y=z)$       6. $(x=y \;\&\; y=z) \rightarrow x=z$

       3. $(x=1 \lor x \neq 1) \;\&\; y \neq 3$      7. $(x>y \;\&\; y>z) \rightarrow x>z$

       4. $x=y \rightarrow (y \neq z \;\&\; y>5)$

▶ 1.7 *Elimination of Some Parentheses*

This section has left the following rule as understood: Unless otherwise indicated by parentheses, the stronger connectives dominate.

Except for Exercise 12 the student should be allowed to retain parentheses if he wishes. In fact, in Chapter Two he may need to be encouraged to do so if he tends, for example, to be confused and think simplification can be applied to $P \rightarrow Q \;\&\; R$. This error is very unlikely to occur when the formula is written $P \rightarrow (Q \;\&\; R)$.

## .7, Exercise 12, page 35

**A.**
1. $P \rightarrow Q \vee R$
2. $P \vee (Q \& R)$
3. $(R \rightarrow S) \& T$
4. $\neg(R \& S)$

5. $P \vee Q \rightarrow \neg R$
6. $\neg(P \rightarrow Q)$
7. $A \& (B \rightarrow C)$

8. $(M \rightarrow N) \vee P$
9. $\neg(P \vee \neg Q)$
10. $(\neg A \vee \neg B) \& \neg C$

**B.**
1. $(x \neq 0 \vee x > y) \& y = z$
2. $x = 0 \rightarrow x > y \& y \neq z$
3. $x = 0 \vee (x \neq 0 \& y = z)$
4. $x > y \& y > z \rightarrow x > z$

5. $x = 0 \vee (x > 0 \rightarrow y = 0)$
6. $x = y \& (y = z \vee x = z)$
7. $x = y \& y = z \rightarrow x = z$
8. $(x = y \vee x = z) \& y \neq z$

**C.**
2. $R \rightarrow H \& U$
3. $W \vee (O \& T)$
4. $(W \vee O) \& T$
5. $(W \& O) \vee T$
6. $\neg(J \& A)$

7. $\neg J \& A$
8. $L \vee S \rightarrow C$
9. $H \vee (\neg H \rightarrow W)$
10. $\neg W \& \neg F$
11. $(T \rightarrow S) \& C$

12. $\neg(P \vee G)$
13. $\neg R \rightarrow \neg Q$
14. $F \rightarrow D \& \neg K$
15. $(F \rightarrow D) \& \neg K$

## 1.7, Exercise 13, pages 36–40

**A.**
1. M and
2. M or
3. A
4. M if . . . then . . .
5. A

6. M not
7. M and
8. A
9. M not
10. M if . . . then . . .

11. A
12. M not
13. M and
14. M not
15. M or

**D.** Example 1
Let P = 'It is after five'
    Q = 'The meeting is over'
Then: $P \rightarrow Q$

2. $W \vee L$
3. $\neg C \rightarrow \neg M$
4. $S \& L$
5. $N \rightarrow \neg P$

**E.**
1. $P \& Q \rightarrow R$
2. $P \vee Q \rightarrow R$
3. $P \& \neg Q \rightarrow \neg R$
4. $R \rightarrow P \vee \neg Q$
5. $R \& (P \& Q)$
6. $\neg R \rightarrow \neg P$
7. $P \vee Q$
8. $\neg Q \vee P \rightarrow R$
9. $R \& (P \vee Q)$

10. $Q \& (P \rightarrow R)$
11. $\neg(P \& Q)$
12. $\neg P \& Q \rightarrow \neg R$

**F.**
1. $P \rightarrow Q$
2. $P \vee Q$
3. $P \vee Q \rightarrow \neg Q$
4. $\neg P \vee \neg Q$
5. $(P \& Q) \vee (R \& S)$

6. $\neg(P \& Q)$
7. $\neg(P \vee Q)$
8. $\neg P \rightarrow \neg Q \& R$
    (See note.)
9. $\neg(P \rightarrow Q)$
10. $\neg(P \& \neg P)$
11. $P \& (Q \vee R)$
12. $(P \& Q) \vee R$
13. $P \& (Q \rightarrow \neg R)$

*Note.* The English here is ambiguous. Could be interpreted: $\neg P \rightarrow \neg(Q \& R)$. However we assume in English also that 'not' is the weakest connective. An unambiguous English form would be: If not P then both not Q and R. The less likely interpretation can be clearly indicated by: If not P then not both Q and R.

**1.7, Exercise 13, pages 36–40** (*continued*)

**G.**  1. c, h                                                5. f
     2. b, e                                                6. d
     3. a                                                   7. g
     4. i, (a, b, c, e, g, h)                               8. j

**H.**  1. Let G = 'x is greater than five'   G          4. ¬(O → D)
     2. ¬O                                                5. S & E → F
     3. E ∨ G                                              6. L ∨ G → ¬E

**I.**   3. $x = 3 \ \lor \ x > 6$
     5. $x + 4 = 7 \ \ \& \ \ y + x = 8 \ \ \rightarrow \ \ y = 5$
     6. $x < 5 \ \lor \ x > 7 \ \ \rightarrow \ \ x \neq 6$

**J.**   1. Let S = 'In Argentina summer begins in June'
        In Argentina summer does not begin in June,
        and so forth.

## Chapter 1, Review Test, pages 40–42

**I.**   a. Let P = 'The book costs more than two dollars'
           Q = 'Jane will buy it'
        Then: P → ¬Q

     b. P ∨ ¬Q                                       d. $x < 3 \ \rightarrow \ x < 4$
     c. P & Q                                         e. $x \neq 5 \ \rightarrow \ x < 5 \ \lor \ x > 5$

**II.**  a. P ∨ Q → R                                  e. ¬R → ¬P & ¬Q
     b. Q → ¬P                                        f. ¬Q ∨ P
     c. R ∨ ¬Q                                        g. ¬Q & ¬P → ¬R
     d. (Q & P) & R

**III.** a. conjunction                                e. conditional
     b. negation                                      f. antecedent
     c. molecular sentence                            g. consequent
     d. atomic                                        h. disjunction

**IV.**  a. (P ∨ Q) & R                                f. (P → Q) ∨ R
     b. ¬(P & Q)                                      g. ¬P → ¬R
     c. ¬P & Q                                        h. P ∨ (Q & R)
     d. P & Q → R                                     i. ¬(P → Q)
     e. ¬(P ∨ ¬Q)                                     j. P & (Q → R)

**V.**   a. not   ¬(P ∨ Q)                             d. and   P & (Q ∨ R)
     b. and   ¬P & ¬Q                                 e. if . . . then . . .   P → Q & R
     c. either . . . or . . .   (P & Q) ∨ ¬R

# Logical Inference

## ▶ 2.2 *Rules of Inference and Proof*

### 2.2, Exercise 1, pages 46–47

**A.**
1. You are on Pacific Standard Time.
2. We will not make it to the plane.
3. It either needs more water or better soil.
4. The office is closed.
5. I do not live in one of the fifty states.

**B.**
1. R
2. ¬R
3. Q
4. Q & R
5. Q ∨ R
6. Q & P

**C.**
1. C
2. C
3. I
4. I
5. C

**D.**
1. $x+y>1$
2. $y+x=z$
3. $x+y$ is a number
4. $x>y$
5. $x=z$

### 2.2, Exercise 2, pages 48–49

**A.**

1. 
   | (1) ¬A → ¬B | P |
   |---|---|
   | (2) ¬A | P |
   | (3) ¬B | PP |

2. 
   | (1) M | P |
   |---|---|
   | (2) M → N | P |
   | (3) N | PP |

3. 
   | (1) R | P |
   |---|---|
   | (2) R → ¬T ∨ Q | P |
   | (3) ¬T ∨ Q | PP |

4. 
   | (1) ¬B → ¬D & A | P |
   |---|---|
   | (2) ¬B | P |
   | (3) ¬D & A | PP |

**B.**

1. Example
   Let W = 'You are on the West Coast'
   T = 'You are on Pacific Standard Time'

   | (1) W → T | P |
   |---|---|
   | (2) W | P |
   | (3) T | PP |

2. 
   | (1) ¬L → ¬M | P |
   |---|---|
   | (2) ¬L | P |
   | (3) ¬M | PP |

3. 
   | (1) ¬G → W ∨ S | P |
   |---|---|
   | (2) ¬G | P |
   | (3) W ∨ S | PP |

4. 
   | (1) F | P |
   |---|---|
   | (2) F → S | P |
   | (3) S | PP |

5. 
   | (1) C → ¬F | P |
   |---|---|
   | (2) C | P |
   | (3) ¬F | PP |

**2.2, Exercise 2, pages 48–49** (*continued*)

**C.**  1. Let P =ʻx =0'

Q =ʻx+y> 1'

(1) ¬P → Q    P
(2) ¬P        P
(3) Q         PP

2. (1) P → Q    P
   (2) P        P
   (3) Q        PP

3. (1) P & Q → R    P
   (2) P & Q        P
   (3) R            PP

4. (1) P & Q → R    P
   (2) P & Q        P
   (3) R            PP

5. (1) P & Q        P
   (2) P & Q → R    P
   (3) R            PP

**2.2, Exercise 3, pages 50–52**

**A.**  1. Prove: ¬T
   (1) R → ¬T    P
   (2) S → R     P
   (3) S         P
   (4) R         PP 2, 3
   (5) ¬T        PP 1, 4

2. Prove: G
   (1) ¬H → ¬J    P
   (2) ¬H         P
   (3) ¬J → G     P
   (4) ¬J         PP 1, 2
   (5) G          PP 3, 4

3. Prove: C
   (1) A → B & D    P
   (2) B & D → C    P
   (3) A            P
   (4) B & D        PP 1, 3
   (5) C            PP 2, 4

4. Prove: M ∨ N
   (1) ¬J → M ∨ N    P
   (2) F ∨ G → ¬J    P
   (3) F ∨ G         P
   (4) ¬J            PP 2, 3
   (5) M ∨ N         PP 1, 4

5. Prove: ¬S
   (1) T            P
   (2) T → ¬Q       P
   (3) ¬Q → ¬S      P
   (4) ¬Q           PP 2, 1
   (5) ¬S           PP 3, 4

**B.**  *1. Let P =ʻ2 is greater than 1'

Q =ʻ3 is greater than 1'

R =ʻ3 is greater than 0'

Prove: R
   (1) P → Q    P
   (2) Q → R    P
   (3) P        P
   (4) Q        PP 1, 3
   (5) R        PP 2, 4

2. Prove: $x = y$
   (1) $x+1 = 2$                   P
   (2) $x+1 = 2$ → $y+1 = 2$       P
   (3) $y+1 = 2$ → $x = y$         P
   (4) $y+1 = 2$                   PP 2, 1
   (5) $x = y$                     PP 3, 4

In those problems having atomic sentences written in mathematical symbols, the logical structure will be clearer to the student and anyone who grades his papers if the student is required to write the atomic mathematical sentences compactly and to space around the logical sentential connectives. (Note in text on page 75

*Note.* It is probably a good idea to put a line under what is to be proved before the proof. This prevents accidental use of the conclusion as though it were a premise.

nd the pages following.)   If the second premise above were spaced in any of the following ways it would be
uite misleading:

$$x \ + \ 1{\rightarrow}y \ + \ 1 \ = \ 2$$
$$x+1{\rightarrow}y+1 \ = \ 2$$
$$x+1 \ \ {\rightarrow}y \ + \ 1=2$$

whole set of premises so poorly spaced would be extremely difficult to follow.  All that a student needs to
ecognize about the atomic sentences is—which ones are the same and which ones are different.  It may help
ome students to place parentheses around either atomic or molecular sentences so as to emphasize the dominant
onnective in each formula.  For example, see problem 4 below.

**B.**    3.  Prove: $x+2=y+2$

| | | |
|---|---|---|
| (1) | $x+0=y \ \rightarrow \ x=y$ | P |
| (2) | $x+0=y$ | P |
| (3) | $x=y \ \rightarrow \ x+2=y+2$ | P |
| (4) | $x=y$ | PP 1, 2 |
| (5) | $x+2=y+2$ | PP 3, 4 |

4.  Prove: $x>10$

| | | |
|---|---|---|
| (1) | $(x>y \ \ \& \ \ y>z) \ \rightarrow$ $(x>z)$ | P |
| (2) | $x>y \ \ \& \ \ y>z$ | P |
| (3) | $(x>z) \ \rightarrow \ (x>10)$ | P |
| (4) | $x>z$ | PP 1, 2 |
| (5) | $x>10$ | PP 3, 4 |

5.  Prove: $z=x$

| | | |
|---|---|---|
| (1) | $x=y \ \ \& \ \ y=z \ \rightarrow \ x=z$ | P |
| (2) | $x=z \ \rightarrow \ z=x$ | P |
| (3) | $x=y \ \ \& \ \ y=z$ | P |
| (4) | $x=z$ | PP 1, 3 |
| (5) | $z=x$ | PP 2, 4 |

6.  Let R = 'The moist air rises'
      L = 'It will cool'
      M = 'Clouds will form'

Prove: M

| | | |
|---|---|---|
| (1) | $R \rightarrow L$ | P |
| (2) | $L \rightarrow M$ | P |
| (3) | R | P |
| (4) | L | PP 1, 3 |
| (5) | M | PP 2, 4 |

**C.**    1.  Prove: $\neg N$

| | | |
|---|---|---|
| (1) | $R \rightarrow \neg S$ | P |
| (2) | R | P |
| (3) | $\neg S \rightarrow Q$ | P |
| (4) | $Q \rightarrow \neg N$ | P |
| (5) | $\neg S$ | PP 1, 2 |
| (6) | Q | PP 3, 5 |
| (7) | $\neg N$ | PP 4, 6 |

2.  Prove: B

| | | |
|---|---|---|
| (1) | $\neg G \rightarrow E$ | P |
| (2) | $E \rightarrow K$ | P |
| (3) | $\neg G$ | P |
| (4) | $K \rightarrow \neg L$ | P |
| (5) | $\neg L \rightarrow M$ | P |
| (6) | $M \rightarrow B$ | P |
| (7) | E | PP 1, 3 |
| (8) | K | PP 2, 7 |
| (9) | $\neg L$ | PP 4, 8 |
| (10) | M | PP 5, 9 |
| (11) | B | PP 6, 10 |

3.  Prove: $R \lor S$

| | | |
|---|---|---|
| (1) | $C \lor D$ | P |
| (2) | $C \lor D \rightarrow \neg F$ | P |
| (3) | $\neg F \rightarrow A \ \& \ \neg B$ | P |
| (4) | $A \ \& \ \neg B \rightarrow R \lor S$ | P |
| (5) | $\neg F$ | PP 2, 1 |
| (6) | $A \ \& \ \neg B$ | PP 3, 5 |
| (7) | $R \lor S$ | PP 4, 6 |

**Double Negation (page 52).**  Notice that the symbolic manipulation is to either:

(1) (a) Surround by parentheses.  This step can be omitted for sentences that are atomic or are negations.
     (b) Prefix two negations, or
(2) Given two negations before parentheses, an atomic letter, or a negation (such as $\neg\neg\neg P$), remove the two
    beginning negations.

## 2.2, Exercise 4, pages 53–54

**A.**  1. It is not the case that all mammals are not warm-blooded animals.
   2. The nucleus of the atom is positively charged.
   3. It is not the case that granite is not a type of igneous rock.
   4. It is not the case that in the United States, presidential elections are not held every four years.
   5. One fifth is equal to twenty per cent.

**B.**  1. not PP
   2. R ∨ S
   3. not PP
   4. ¬P
   5. not PP
   6. Q

**C.**  1. T
   2. F
   3. T
   4. T
   5. F

**D.**  1. Prove: ¬¬T

| | | |
|---|---|---|
| (1) | S → T | P |
| (2) | S | P |
| (3) | T | PP 1, 2 |
| (4) | ¬¬T | DN 3 |

   2. Prove: B

| | | |
|---|---|---|
| (1) | ¬A | P |
| (2) | ¬A → ¬¬B | P |
| (3) | ¬¬B | PP 2, 1 |
| (4) | B | DN 3 |

**D.**  3. Prove: G

| | | |
|---|---|---|
| (1) | H → ¬¬G | P |
| (2) | H | P |
| (3) | ¬¬G | PP 1, 2 |
| (4) | G | DN 3 |

   4. Prove: P ∨ Q

| | | |
|---|---|---|
| (1) | R → ¬¬(P ∨ Q) | P |
| (2) | R | P |
| (3) | ¬¬(P ∨ Q) | PP 1, 2 |
| (4) | P ∨ Q | DN 3 |

   5. Prove: ¬¬N

| | | |
|---|---|---|
| (1) | M → ¬P | P |
| (2) | ¬P → N | P |
| (3) | M | P |
| (4) | ¬P | PP 1, 3 |
| (5) | N | PP 2, 4 |
| (6) | ¬¬N | DN 5 |

   6. Prove: Q

| | | |
|---|---|---|
| (1) | J → K & M | P |
| (2) | J | P |
| (3) | K & M → ¬¬Q | P |
| (4) | K & M | PP 1, 2 |
| (5) | ¬¬Q | PP 3, 4 |
| (6) | Q | DN 5 |

## 2.2, Exercise 5, pages 57–58

**A.**  1. Light is not simply a continuous wave motion.
   2. It is not the case that one angle of this triangle is greater than 90 degrees.   (See note.)
   3. The lease is not held to be valid.
   4. It did not rain last night.
   5. Susan is not my sister.

*Note.*  It is interesting to note that in this case the shorter, "One angle of this triangle is not greater than 90 degrees," would be ambiguous, or even false.

**B.**  1. ¬Q
   2. ¬¬P
   3. ¬R
   4. ¬Q
   5. ¬P
   6. ¬(P ∨ Q)

**C.**  1. Prove: C

| | | |
|---|---|---|
| (1) | ¬B | P |
| (2) | A → B | P |
| (3) | ¬A → C | P |
| (4) | ¬A | TT 2, 1 |
| (5) | C | PP 3, 4 |

**C. 2.** Prove: **F**

|     |                  |         |
| --- | ---------------- | ------- |
| (1) | G → H            | P       |
| (2) | ¬G → ¬¬F         | P       |
| (3) | ¬H               | P       |
| (4) | ¬G               | TT 1, 3 |
| (5) | ¬¬F              | PP 2, 4 |
| (6) | F                | DN 5    |

**3.** Prove: **R & S**

|     |                 |         |
| --- | --------------- | ------- |
| (1) | P → ¬Q          | P       |
| (2) | Q               | P       |
| (3) | ¬P → R & S      | P       |
| (4) | ¬¬Q             | DN 2    |
| (5) | ¬P              | TT 1, 4 |
| (6) | R & S           | PP 3, 5 |

**4.** Prove: **E**

|     |         |         |
| --- | ------- | ------- |
| (1) | F       | P       |
| (2) | ¬E → ¬F | P       |
| (3) | ¬¬F     | DN 1    |
| (4) | ¬¬E     | TT 2, 3 |
| (5) | E       | DN 4    |

**5.** Prove: **¬S**

|     |        |         |
| --- | ------ | ------- |
| (1) | S → ¬R | P       |
| (2) | R      | P       |
| (3) | ¬¬R    | DN 2    |
| (4) | ¬S     | TT 1, 3 |

**More about Negation (page 58).** It must be noted that this special rule about negation is *not* a rule of ference. Several of the rules of inference require that some expression be negated or that one expression be e negation of another. This rule does not justify our negating some formula. It only gives us an alternative ıy of doing the negating which is justified or required by some other rule, or of recognizing one expression to : the negation of another. In terms of symbolic manipulation the rule means we now have two ways of getting e negation of a sentence:

(1) Adding parentheses if needed, we put a negation in front of the whole sentence. Or,
(2) Take away a negation from in front of a whole sentence, if there is one there to be taken away.

hus the negation of ¬G may be written either ¬¬G or G; the negation of ¬(P & Q) may be written as either ⌐(P & Q) or P & Q. But the negation of ¬R ∨ S can be written only ¬(¬R ∨ S) because there is no negation ı front of the whole sentence to be taken away. The negation in ¬R ∨ S is only in front of R not R ∨ S. If ıe negation were for the whole sentence, we would have ¬(R ∨ S).

**.2, Exercise 6, page 60**

**A. 1.** Prove: **¬P**

|     |        |         |
| --- | ------ | ------- |
| (1) | P → ¬Q | P       |
| (2) | Q      | P       |
| (3) | ¬P     | TT 1, 2 |

**2.** Prove: **¬A**

|     |        |         |
| --- | ------ | ------- |
| (1) | A → ¬C | P       |
| (2) | B → C  | P       |
| (3) | B      | P       |
| (4) | C      | PP 2, 3 |
| (5) | ¬A     | TT 1, 4 |

**3.** Prove: **P**

|     |          |         |
| --- | -------- | ------- |
| (1) | ¬P → ¬Q  | P       |
| (2) | Q        | P       |
| (3) | P        | TT 1, 2 |

**4.** Prove: **A**

|     |         |         |
| --- | ------- | ------- |
| (1) | ¬A → ¬B | P       |
| (2) | ¬B → ¬C | P       |
| (3) | C       | P       |
| (4) | ¬¬B     | TT 2, 3 |
|     | or  B   |         |
| (5) | A       | TT 1, 4 |

**5.** Prove: **¬S**

|     |        |         |
| --- | ------ | ------- |
| (1) | P → Q  | P       |
| (2) | Q → R  | P       |
| (3) | S → ¬R | P       |
| (4) | P      | P       |
| (5) | Q      | PP 1, 4 |
| (6) | R      | PP 2, 5 |
| (7) | ¬S     | TT 3, 6 |

**2.2, Exercise 6, page 60** (*continued*)

**A.**   6. Prove: ¬A
       (1) A → B    P
       (2) B → C    P
       (3) C → D    P
       (4) ¬D      P
       (5) ¬C      TT 3, 4
       (6) ¬B      TT 2, 5
       (7) ¬A      TT 1, 6

    With these problems, especially those in mathematical symbols, after copying one, the student should develop a habit of looking back to be sure he has correctly copied it. For example, it is easy to slip down and copy a second or third premise from a problem below.

**B.**   1. Prove: $x = 0$
       (1) $x \neq 0 \rightarrow x+y \neq y$   P
       (2) $x+y = y$   P
       (3) $x = 0$   TT 1, 2

    2. Prove: $x \neq 0$
       (1) $x = 0 \rightarrow x \neq y$   P
       (2) $x = z \rightarrow x = y$   P
       (3) $x = z$   P
       (4) $x = y$   PP 2, 3
       (5) $x \neq 0$   TT 1, 4

    3. Prove: $x = y$
       (1) $x \neq y \rightarrow x \neq z$   P
       (2) $x \neq z \rightarrow x \neq 0$   P
       (3) $x = 0$   P
       (4) $x = z$   TT 2, 3
       or ¬$(x \neq z)$
       or ¬¬$(x = z)$
       (5) $x = y$   TT 1, 4

    4. Prove: $x \neq 0$
       (1) $x = y \rightarrow x = z$   P
       (2) $x = z \rightarrow x = 1$   P
       (3) $x = 0 \rightarrow x \neq 1$   P
       (4) $x = y$   P
       (5) $x = z$   PP 1, 4
       (6) $x = 1$   PP 2, 5
       (7) $x \neq 0$   TT 3, 6

    5. Prove: $x \neq y$
       (1) $x = y \rightarrow y = z$   P
       (2) $y = z \rightarrow y = w$   P
       (3) $y = w \rightarrow y = 1$   P
       (4) $y \neq 1$   P
       (5) $y \neq w$   TT 3, 4
       (6) $y \neq z$   TT 2, 5
       (7) $x \neq y$   TT 1, 6

    6. Prove: $x = 0$
       (1) $x \neq 0 \rightarrow y = 1$   P
       (2) $x = y \rightarrow y = w$   P
       (3) $y = w \rightarrow y \neq 1$   P
       (4) $x = y$   P
       (5) $y = w$   PP 2, 4
       (6) $y \neq 1$   PP 3, 5
       (7) $x = 0$   TT 1, 6

**2.2, Exercise 7, page 63**

**A.**   1. A society is a collection of individuals who pursue a way of life. } either or both are acceptable
       Culture is their way of life.
    2. The atomic number of hydrogen is 1 and the atomic number of helium is 2.
    3. Kofi speaks the Twi language and Ama speaks the Ga language.
    4. Tom likes to ski. } either or both
       There is snow in the mountains.
    5. This inference is valid and that is not valid.

**A.**   6.  Q $\Big\}$ either or both are valid conclusions
       R

    7.  P ∨ Q $\Big\}$ either or both
          S

    8.  (R ∨ S) & Q

    9.  S & T

  10.  (Q & R) & S
           or Q
           or R

**B.**   1. Prove: ¬S

| | | |
|---|---|---|
| (1) | ¬R & T | P |
| (2) | S → R | P |
| (3) | ¬R | S 1 |
| (4) | ¬S | TT 2, 3 |

    2. Prove: A & B

| | | |
|---|---|---|
| (1) | C → A | P |
| (2) | C | P |
| (3) | C → B | P |
| (4) | A | PP 1, 2 |
| (5) | B | PP 3, 2 |
| (6) | A & B | A 4, 5 |

    3. Prove: ¬¬Q

| | | |
|---|---|---|
| (1) | P & Q | P |
| (2) | Q | S 1 |
| (3) | ¬¬Q | DN 2 |

    4. Prove: B & D

| | | |
|---|---|---|
| (1) | B & C | P |
| (2) | B → D | P |
| (3) | B | S 1 |
| (4) | D | PP 2, 3 |
| (5) | B & D | A 3, 4 |

    5. Prove: ¬S & Q

| | | |
|---|---|---|
| (1) | ¬S → Q | P |
| (2) | ¬(T & R) | P |
| (3) | S → T & R | P |
| (4) | ¬S | TT 3, 2 |
| (5) | Q | PP 1, 4 |
| (6) | ¬S & Q | A 4, 5 |

    6. Prove: A & C

| | | |
|---|---|---|
| (1) | A & ¬B | P |
| (2) | ¬C → B | P |
| (3) | ¬B | S 1 |
| (4) | C | TT 2, 3 |
| (5) | A | S 1 |
| (6) | A & C | A 5, 4 |

## 2.2, Exercise 8, pages 67–69

Problem **A.** 2 below involves the question as to what constitutes distinct atomic sentences. The students are already acquainted with the fact that the replacing of a proper name by a pronoun referring to the same person does not make another distinct sentence. Thus in the sentences—'If *Mary has red hair*, then she has blue eyes'. '*She has red hair*'—the two italicized sentences are not distinct (although they are detectably different) and so they are symbolized by the same letter, for example 'R → B', 'R'. Notice the distinction here between 'distinct' and 'different'. We can say that *two sentences carrying precisely the same information are not distinct*. In problem 2, the sentences 'It is on the Atlantic Ocean' and 'It is certainly not on the Atlantic Ocean' carry precisely the same information except for the negation and are therefore not distinct. Another example of nondistinct atomic sentences occurs in problem **A.** 3 on page 85 of the text, the second and third atomic sentences. The rule of usage in translating is that distinct atomic sentences are given different symbols, nondistinct atomic sentences are given the same symbol.

**A.**   1. He is a politician.
      2. The port of New Orleans is on the Gulf of Mexico.
      3. The internal energy of an atom changes only in steps.
      4. He is not going to return it to the library today.
      5. The festival will be held outside.

**2.2, Exercise 8, pages 67–69** (*continued*)

**B.**  1. ¬Q
2. P → Q
3. ¬T
4. P
5. R
6. P & Q
7. R
8. ¬S
9. T
10. U
11. S
12. T
13. R

**C.**  1. Prove: P
(1) P ∨ Q     P
(2) ¬T     P
(3) Q → T     P
(4) ¬Q     TT 3, 2
(5) P     TP 1, 4

2. Prove: B
(1) ¬A ∨ B     P
(2) ¬A → E     P
(3) ¬E     P
(4) A     TT 2, 3
    or  ¬¬A
(5) B     TP 1, 4

3. Prove: M
(1) S & P     P
(2) M ∨ ¬N     P
(3) S → N     P
(4) S     S 1
(5) N     PP 3, 4
(6) M     TP 2, 5

4. Prove: A & B
(1) B     P
(2) B → ¬D     P
(3) A ∨ D     P
(4) ¬D     PP 2, 1
(5) A     TP 3, 4
(6) A & B     A 5, 1

**C.**  5. Prove: H
(1) ¬S     P
(2) S ∨ (H ∨ G)     P
(3) ¬G     P
(4) H ∨ G     TP 2, 1
(5) H     TP 4, 3

6. Prove: P
(1) T → P ∨ Q     P
(2) ¬¬T     P
(3) ¬Q     P
(4) T     DN 2
(5) P ∨ Q     PP 1, 4
(6) P     TP 5, 3

7. Prove: R
(1) ¬Q ∨ S     P
(2) ¬S     P
(3) ¬(R & S) → Q     P
(4) ¬Q     TP 1, 2
(5) R & S     TT 3, 4
(6) R     S 5

**D.**  1. Prove: $x=y$
(1) $x=y$ ∨ $x=z$     P
(2) $x=z$ → $x=6$     P
(3) $x\neq6$     P
(4) $x\neq z$     TT 2, 3
(5) $x=y$     TP 1, 4

2. Prove: $3-2=1$
(1) $1+1=2$ & $2+1=3$     P
(2) $3-2=1$ ∨ $2-1\neq1$     P
(3) $1+1=2$ → $2-1=1$     P
(4) $1+1=2$     S 1
(5) $2-1=1$     PP 3, 4
(6) $3-2=1$     TP 2, 5

3. Prove: $x=0$
(1) $x\neq0$ → $x\neq y$     P
(2) $x=y$ ∨ $x=z$     P
(3) $x\neq z$     P
(4) $x=y$     TP 2, 3
(5) $x=0$     TT 1, 4

4. Prove: $x=0$
(1) $x=0$ ∨ $x=y$     P
(2) $x=y$ → $x=z$     P
(3) $x\neq z$     P
(4) $x\neq y$     TT 2, 3
(5) $x=0$     TP 1, 4

**D.** 5. Prove: $x=w$

| | | |
|---|---|---|
| (1) | $x=y \rightarrow x=z$ | P |
| (2) | $x=z \rightarrow x=w$ | P |
| (3) | $x=y \vee x=0$ | P |
| (4) | $x=0 \rightarrow x+u=1$ | P |
| (5) | $x+u \neq 1$ | P |
| (6) | $x \neq 0$ | TT 4, 5 |
| (7) | $x=y$ | TP 3, 6 |
| (8) | $x=z$ | PP 1, 7 |
| (9) | $x=w$ | PP 2, 8 |

▶ 2.3 *Sentential Derivation*

Strictly, the complete form for a formal proof does not require announcing the conclusion before it is derived. However, it is recommended that the student announce it first and separate it from the proof with a line. The line serves to prevent accidental use of the conclusion as if it were a premise. It helps to have the conclusion before you when giving a formal proof and there is no more convenient place to put it. And when an argument is translated from English the conclusion should be translated when the premises are so that no further attention need be given to the English.

When the students make their own selections of capital letters to represent the various distinct atomic sentences, the resulting nonuniformity makes the grading of complex formal proofs difficult. More uniformity will result and grading simplified if the choice is dictated. But do not assist the student by picking out atomic sentences. Simply list the letters to be used in the order in which they are to appear first in the premises and conclusion. Have it understood that you may give one or two extra letters so as not even to give clues as to how many distinct atomic sentences there are. Thus the last letter or two may not be used. For example, in Exercise 9, page 74, problem **A.** 1, **M, B, D,** and **S** might be assigned. S would remain unused. Remembering which letter represents which atomic sentence is facilitated by choosing the initial letter of some key word. But if the same letter is prominent in distinct atomic sentences, it should not be used for either of them. Consider the sentence, 'If the weather is warm then the water will be wonderful.' Avoid 'W' altogether. Use, for example, 'M' and 'L', the final letters.

**2.3, Exercise 9, pages 73–77**

**A.** 1. Let M = 'This is a matrilinear society'
         B = 'The mother's brother is head of the family'
         D = 'The father does impose discipline'

Prove: ¬D

| | | |
|---|---|---|
| (1) | M → B | P |
| (2) | B → ¬D | P |
| (3) | M | P |
| (4) | B | PP 1, 3 |
| (5) | ¬D | PP 2, 4 |

2. Prove: I

| | | |
|---|---|---|
| (1) | I ∨ S | P |
| (2) | G | P |
| (3) | G → ¬S | P |
| (4) | ¬S | PP 3, 2 |
| (5) | I | TP 1, 4 |

2. Let I = 'This rock is an igneous rock'
         S = 'It is a sedimentary rock'
         G = 'This rock is granite'

3. Let J = 'Jack is taller than Bob'
         M = 'Mary is shorter than Jean'
         S = 'Jack and Bill are the same height'

**2.3, Exercise 9, pages 73–77** (*continued*)

**A.** 3. Prove: ¬S

| | | |
|---|---|---|
| (1) | J → M | P |
| (2) | ¬M | P |
| (3) | S → J | P |
| (4) | ¬J | TT 1, 2 |
| (5) | ¬S | TT 3, 4 |

*Note.* From this point on, the explanation of the symbolization will not be given. You should continue to require this explanation however, until certain the students understand and until the technique is well established.

4. Prove: ¬D

| | | |
|---|---|---|
| (1) | A → B ∨ C | P |
| (2) | B → ¬A | P |
| (3) | D → ¬C | P |
| (4) | A | P |
| (5) | B ∨ C | PP 1, 4 |
| (6) | ¬B | TT 2, 4 |
| (7) | C | TP 5, 6 |
| (8) | ¬D | TT 3, 7 |

5. Prove: B

| | | |
|---|---|---|
| (1) | F → J & S | P |
| (2) | T → ¬S | P |
| (3) | T ∨ B | P |
| (4) | F | P |
| (5) | J & S | PP 1, 4 |
| (6) | S | S 5 |
| (7) | ¬T | TT 2, 6 |
| (8) | B | TP 3, 7 |

**B.** 1. Prove: Q

| | | |
|---|---|---|
| (1) | S → (P ∨ Q) | P |
| (2) | S | P |
| (3) | ¬P | P |
| (4) | P ∨ Q | PP 1, 2 |
| (5) | Q | TP 4, 3 |

2. Prove: R

| | | |
|---|---|---|
| (1) | S → ¬T | P |
| (2) | T | P |
| (3) | ¬S → R | P |
| (4) | ¬S | TT 1, 2 |
| (5) | R | PP 3, 4 |

**B.** 3. Prove: S & T

| | | |
|---|---|---|
| (1) | P & R | P |
| (2) | P → S | P |
| (3) | R → T | P |
| (4) | P | S 1 |
| (5) | S | PP 2, 4 |
| (6) | R | S 1 |
| (7) | T | PP 3, 6 |
| (8) | S & T | A 5, 7 |

4. Prove: ¬S

| | | |
|---|---|---|
| (1) | T → R | P |
| (2) | R → ¬S | P |
| (3) | T | P |
| (4) | R | PP 1, 3 |
| (5) | ¬S | PP 2, 4 |

5. Prove: T

| | | |
|---|---|---|
| (1) | P → S | P |
| (2) | ¬S | P |
| (3) | ¬P → T | P |
| (4) | ¬P | TT 1, 2 |
| (5) | T | PP 3, 4 |

6. Prove: S & T

| | | |
|---|---|---|
| (1) | P → S | P |
| (2) | P → T | P |
| (3) | P | P |
| (4) | S | PP 1, 3 |
| (5) | T | PP 2, 3 |
| (6) | S & T | A 4, 5 |

7. Prove: S

| | | |
|---|---|---|
| (1) | P ∨ Q | P |
| (2) | ¬Q | P |
| (3) | P → S | P |
| (4) | P | TP 1, 2 |
| (5) | S | PP 3, 4 |

**B.**    8. Prove: S

| | | |
|---|---|---|
| (1) | T → R | P |
| (2) | ¬R | P |
| (3) | T ∨ S | P |
| (4) | ¬T | TT 1, 2 |
| (5) | S | TP 3, 4 |

9. Prove: ¬T

| | | |
|---|---|---|
| (1) | P → S | P |
| (2) | P & Q | P |
| (3) | (S & R) → ¬T | P |
| (4) | Q → R | P |
| (5) | P | S 2 |
| (6) | S | PP 1, 5 |
| (7) | Q | S 2 |
| (8) | R | PP 4, 7 |
| (9) | S & R | A 6, 8 |
| (10) | ¬T | PP 3, 9 |

10. Prove: ¬R

| | | |
|---|---|---|
| (1) | S ∨ ¬R | P |
| (2) | T → ¬S | P |
| (3) | T | P |
| (4) | ¬S | PP 2, 3 |
| (5) | ¬R | TP 1, 4 |

11. Prove: S

| | | |
|---|---|---|
| (1) | P → Q & R | P |
| (2) | P | P |
| (3) | T → ¬Q | P |
| (4) | T ∨ S | P |
| (5) | Q & R | PP 1, 2 |
| (6) | Q | S 5 |
| (7) | ¬T | TT 3, 6 |
| (8) | S | TP 4, 7 |

12. Prove: ¬Q

| | | |
|---|---|---|
| (1) | T ∨ ¬S | P |
| (2) | S | P |
| (3) | Q → ¬T | P |
| (4) | T | TP 1, 2 |
| (5) | ¬Q | TT 3, 4 |

13. Prove: Q ∨ R

| | | |
|---|---|---|
| (1) | S → ¬T | P |
| (2) | T | P |
| (3) | ¬S → (Q ∨ R) | P |
| (4) | ¬S | TT 1, 2 |
| (5) | Q ∨ R | PP 3, 4 |

14. Prove: S

| | | |
|---|---|---|
| (1) | ¬T ∨ R | P |
| (2) | T | P |
| (3) | ¬S → ¬R | P |
| (4) | R | TP 1, 2 |
| (5) | S | TT 3, 4 |

15. Prove: ¬R

| | | |
|---|---|---|
| (1) | Q & T | P |
| (2) | Q → ¬R | P |
| (3) | T → ¬R | P |
| (4) | Q | S 1 |
| (5) | ¬R | PP 2, 4 |

**C.**    1. Prove: $y+8<12$

| | | |
|---|---|---|
| (1) | $x+8=12$ ∨ $x \neq 4$ | P |
| (2) | $x=4$ & $y<x$ | P |
| (3) | $x+8=12$ & $y<x$ → $y+8<12$ | P |
| (4) | $x=4$ | S 2 |
| (5) | $x+8=12$ | TP 1, 4 |
| (6) | $y<x$ | S 2 |
| (7) | $x+8=12$ & $y<x$ | A 5, 6 |
| (8) | $y+8<12$ | PP 3, 7 |

**2.3, Exercise 9, pages 73–77** (*continued*)

**C.**    2.  Prove: $x<4$    &    $y<6$

(1) $x+2<6$    →    $x<4$         P
(2) $y<6$    ∨    $x+y\not<10$         P
(3) $x+y<10$    &    $x+2<6$    P
(4) $x+2<6$                         S  3
(5) $x<4$                             PP  1, 4
(6) $x+y<10$                     S  3
(7) $y<6$                             TP  2, 6
(8) $x<4$    &    $y<6$         A  5, 7

4.  Prove: $y>z$

(1) $x=y$    →    $x=z$         P
(2) $x\neq y$    →    $x<z$         P
(3) $x\not<z$    ∨    $y>z$         P
(4) $y\neq z$    &    $x\neq z$     P
(5) $x\neq z$                         S  4
(6) $x\neq y$                         TT  1, 5
(7) $x<z$                             PP  2, 6
(8) $y>z$                             TP  3, 7

3.  Prove: $x=5$    &    $x\neq y$

(1) $x=y$    →    $x\neq y+3$     P
(2) $x=y+3$    ∨    $x+2=y$     P
(3) $x+2\neq y$    &    $x=5$     P
(4) $x+2\neq y$                     S  3
(5) $x=y+3$                         TP  2, 4
(6) $x\neq y$                         TT  1, 5
(7) $x=5$                             S  3
(8) $x=5$    &    $x\neq 4$         A  7, 6

5.  Prove: $x<5$

(1) $x<y$    ∨    $x=y$         P
(2) $x=y$    →    $y\neq 5$         P
(3) $x<y$    &    $y=5$    →
         $x<5$                         P
(4) $y=5$                             P
(5) $x\neq y$                         TT  2, 4
(6) $x<y$                             TP  1, 5
(7) $x<y$    &    $y=5$         A  4, 6
(8) $x<5$                             PP  3, 7

6.  Prove: $\tan\theta\neq 0.577$

(1) $\tan\theta=0.577$    →    $\sin\theta=0.500$    &    $\cos\theta=0.866$    P
(2) $\sin\theta=0.500$    &    $\cos\theta=0.866$    →    $\cot\theta=1.732$    P
(3) $\sec\theta=1.154$    ∨    $\cot\theta\neq 1.732$             P
(4) $\sec\theta\neq 1.154$                                 P
(5) $\cot\theta\neq 1.732$                                 TP  3, 4
(6) $\neg(\sin\theta=0.500$    &    $\cos\theta=0.866)$    TT  2, 3
(7) $\tan\theta\neq 0.577$                                 TT  1, 7

*Note.*  This example is from trigonometry.  For these values, the angle $\theta$ would be 30°.

7.  Prove: $\neg(y>7$    ∨    $x=y)$

(1) $x<6$                                         P
(2) $y>7$    ∨    $x=y$    →    $\neg(y=4$    &    $x<y)$    P
(3) $y\neq 4$    →    $x\not<6$                     P
(4) $x<6$    →    $x<y$                         P
(5) $x<y$                                         PP  4, 1
(6) $y=4$                                         TT  3, 1
(7) $y=4$    &    $x<y$                         A  5, 6
(8) $\neg(y>7$    ∨    $x=y)$                     TT  2, 7

**C.** 8. Prove: $x > 6$

    (1) $x > 5 \rightarrow x = 6 \lor x > 6$    P

    (2) $x \neq 5 \ \& \ x \not< 5 \rightarrow x > 5$    P

    (3) $x < 5 \rightarrow x \neq 3 + 4$    P

    (4) $x = 3 + 4 \ \& \ x \neq 6$    P

    (5) $x = 3 + 4 \rightarrow x \neq 5$    P

    (6) $x = 3 + 4$    S 4

    (7) $x \not< 5$    TT 3, 6

    (8) $x \neq 5$    PP 5, 6

    (9) $x \neq 5 \ \& \ x \not< 5$    A 8, 7

    (10) $x > 5$    PP 2, 9

    (11) $x = 6 \lor x > 6$    PP 1, 10

    (12) $x \neq 6$    S 4

    (13) $x > 6$    TP 11, 12

9. Prove: $x = 4$

    (1) $3x + 2y = 18 \ \&$

        $x + 4y = 16$    P

    (2) $x = 2 \rightarrow 3x + 2y \neq 18$    P

    (3) $x = 2 \lor y = 3$    P

    (4) $x \neq 4 \rightarrow y \neq 3$    P

    (5) $3x + 2y = 18$    S 1

    (6) $x \neq 2$    TT 2, 5

    (7) $y = 3$    TP 3, 6

    (8) $x = 4$    TT 4, 7

10. Prove: $x < 3$

    (1) $x + 2 > 5 \rightarrow x = 4$    P

    (2) $x = 4 \rightarrow x + 4 \not< 7$    P

    (3) $x + 4 < 7$    P

    (4) $x + 2 > 5 \lor$

        $(5 - x > 2 \ \& \ x < 3)$    P

    (5) $x \neq 4$    TT 2, 3

    (6) $x + 2 \not> 5$    TT 1, 5

    (7) $5 - x > 2 \ \& \ x < 3$    TP 4, 6

    (8) $x < 3$    S 7

▶ 2.4 *More about Parentheses*

**2.4, Exercise 10, pages 78–79**

**A.**    $\neg Q \ \& \ R$ asserts that $Q$ is false and $R$ is true.

     $\neg(Q \ \& \ R)$ asserts that $Q$ and $R$ are not both true; that is, at least one (and maybe both) is false, but it does not say which.

**B.**    Same as $\neg Q \ \& \ R$

**C.**    Yes, $\neg P$ by TT.   But without premises, no.

**D.**

| | | |
|---|---|---|
| 1. or | disjunction |
| 2. if . . . then . . . | conditional |
| 3. and | conjunction |
| 4. if . . . then . . . | conditional |
| 5. or | disjunction |
| 6. not | negation |
| 7. or | disjunction |
| 8. and | conjunction |
| 9. if . . . then . . . | conditional |
| 10. not | negation |
| 11. or | disjunction |
| 12. if . . . then . . . | conditional |
| 13. not | negation |
| 14. if . . . then . . . | conditional |
| 15. or | disjunction |

**E.**

  1. $(P \rightarrow R) \ \& \ S$

  2. $P \ \& \ (R \lor S)$

  3. $A \ \& \ B \rightarrow C$   or   $(A \ \& \ B) \rightarrow C$

  4. $(P \ \& \ R) \lor S$

  5. $\neg P \rightarrow R$

  6. $\neg(P \rightarrow R)$

  7. $\neg P \ \& \ \neg R$

  8. $\neg(P \ \& \ R)$

  9. $(A \rightarrow B) \lor C$

  10. $A \rightarrow B \lor C$   or   $A \rightarrow (B \lor C)$

  11. $\neg(P \lor Q)$

  12. $\neg P \lor Q$

  13. $(P \rightarrow Q) \ \& \ (R \rightarrow S)$

  14. $\neg(\neg A \rightarrow \neg B)$

  15. $\neg P \lor \neg Q$

**2.4, Exercise 10, pages 78–79** (*continued*)

**F.**   1. $\dfrac{(P \to Q) \ \& \ R}{R}$

2. $P \to Q \ \& \ R$

$\dfrac{\neg(Q \ \& \ R)}{\neg P}$

3. $\dfrac{Q \ \& \ (P \lor S)}{Q}$

4. $P \to Q \ \& \ S$

$\dfrac{P}{Q \ \& \ S}$

▶ 2.5 *Further Rules of Inference*

**2.5, Exercise 11, pages 81–83**

**A.**   1. It is not the case that some games are not easy to learn.
2. Some games are easy to learn or today is Tuesday.
3. Fire is hot or some games are easy to learn.
4. Any disjunction of which 'some games are easy to learn', or No. 1 above, is the right or left disjunct. (This statement applies to the formation of other sentences as well.)

**B.**   1. valid, law of addition   2. valid, law of addition   3. valid, law of addition and *modus ponendo ponens*   4. valid, law of addition and *modus ponendo ponens*   5. valid, law of addition and *modus ponendo ponens*   6. valid, law of addition and *modus ponendo ponens*   7. valid, law of addition   8. valid, law of addition   9. *not* valid   10. *not* valid

**C.**   1. Prove: $R \lor \neg T$

|   |   |   |
|---|---|---|
| (1) | $Q \lor T$ | P |
| (2) | $Q \to R$ | P |
| (3) | $\neg R$ | P |
| (4) | $\neg Q$ | TT 2, 3 |
| (5) | $T$ | TP 1, 4 |
| (6) | $T \lor S$ | LA 5 |

2. Prove: $R \lor \neg T$

|   |   |   |
|---|---|---|
| (1) | $P$ | P |
| (2) | $\neg R \to \neg P$ | P |
| (3) | $R$ | TT 2, 1 |
| (4) | $R \lor \neg T$ | LA 3 |

3. Prove: $R \lor \neg S$

|   |   |   |
|---|---|---|
| (1) | $S \ \& \ Q$ | P |
| (2) | $T \to \neg Q$ | P |
| (3) | $\neg T \to R$ | P |
| (4) | $Q$ | S 1 |
| (5) | $\neg T$ | TT 2, 4 |
| (6) | $R$ | PP 3, 5 |
| (7) | $R \lor \neg S$ | LA 6 |

4. Prove: $Q$

|   |   |   |
|---|---|---|
| (1) | $\neg S$ | P |
| (2) | $T \to S$ | P |
| (3) | $\neg T \lor R \to Q$ | P |
| (4) | $\neg T$ | TT 1, 2 |
| (5) | $\neg T \lor R$ | LA 4 |
| (6) | $Q$ | PP 3, 5 |

5. Prove: $U$

|   |   |   |
|---|---|---|
| (1) | $P \ \& \ \neg T$ | P |
| (2) | $S \to T$ | P |
| (3) | $S \lor Q$ | P |
| (4) | $Q \lor P \to U$ | P |
| (5) | $P$ | S 1 |
| (6) | $Q \lor P$ | LA 5 |
| (7) | $U$ | PP 4, 6 |

6. Prove: $T \lor Q$

|   |   |   |
|---|---|---|
| (1) | $S \to P \ \& \ Q$ | P |
| (2) | $S$ | P |
| (3) | $P \ \& \ Q \to T$ | P |
| (4) | $P \ \& \ Q$ | PP 1, 2 |
| (5) | $T$ | PP 3, 4 |
| (6) | $T \lor Q$ | LA 5 |

**D.**    1. Prove: $y \nless 4 \;\lor\; x > 2$

     (1) $x > 3 \;\lor\; y \nless 4$      P
     (2) $x > 3 \;\rightarrow\; x > y$      P
     (3) $x \ngtr y$      P
     (4) $x \ngtr 3$      TT 2, 3
     (5) $y \nless 4$      TP 1, 4
     (6) $y \nless 4 \;\lor\; x > 2$      LA 5

2. Prove: $x > y \;\lor\; x \nless 6$

     (1) $x \ngtr y \;\lor\; x > 5$      P
     (2) $x \ngtr 5 \;\lor\; y \nless 6$      P
     (3) $x + y = 1 \;\&\; x > y$      P
     (4) $x > y$      S 3
     (5) $x > 5$      TP 1, 4
     (6) $y \nless 6$      TP 2, 5
     (7) $x > y \;\lor\; y \nless 6$      LA 6
     or as an alternate
     (5) $x > y \;\lor\; y \nless 6$      LA 4

3. Prove: $x \neq 3 \;\lor\; x > 2$

     (1) $x + 2 \neq 5 \;\lor\; 2x = 6$      P
     (2) $x + 2 \neq 5 \;\rightarrow\; x \neq 3$      P
     (3) $2x - 2 = 8 \;\rightarrow\; 2x \neq 6$      P
     (4) $x + 3 = 8 \;\&\; 2x - 2 = 8$      P
     (5) $2x - 2 = 8$      S 4
     (6) $2x \neq 6$      PP 3, 5
     (7) $x + 2 \neq 5$      TP 1, 6
     (8) $x \neq 3$      PP 2, 7
     (9) $x \neq 3 \;\lor\; x \neq 2$      LA 8

4. Prove: $\tan 30° = 0.577 \;\lor\; \cos 60° = 0.5$

     (1) $\sin 30° = 0.5 \;\rightarrow\; \csc 30° = 2.0$      P
     (2) $\sin 30° = 0.5$      P
     (3) $\csc 30° = 2.0 \;\rightarrow\; \tan 30° = 0.577$      P
     (4) $\csc 30° = 2.0$      PP 1, 2
     (5) $\tan 30° = 0.577$      PP 3, 4
     (6) $\tan 30° = 0.577 \;\lor\; \cos 60° = 0.5$      LA 5

5. Prove: $x = 5 \;\lor\; x \neq 4$

     (1) $x = 2 \;\rightarrow\; x < 3$      P
     (2) $x \neq 4 \;\&\; x \nless 3$      P
     (3) $x \neq 2 \;\lor\; x > 4 \;\rightarrow\; x = 5$      P
     (4) $x \nless 3$      S 2
     (5) $x \neq 2$      TT 1, 4
     (6) $x \neq 2 \;\lor\; x > 4$      LA 5
     (7) $x = 5$      PP 3, 6
     (8) $x \neq 4$      S 2
     (9) $x = 5 \;\&\; x \neq 4$      A 7, 8

6. Prove: $x = 2$

     (1) $Dx^3 = 3x^2 \;\&\; D3 = 0$      P
     (2) $Dx^3 = 3x^2 \;\rightarrow\; Dx^2 = 2x$      P
     (3) $Dx^2 = 2x \;\lor\; Dx^3 = 12 \;\rightarrow\; x = 2$      P
     (4) $Dx^3 = 3x^2$      S 1
     (5) $Dx^2 = 2x$      PP 2, 4
     (6) $Dx^2 = 2x \;\lor\; Dx^3 = 12$      LA 5
     (7) $x = 2$      PP 3, 6

**2.5, Exercise 11, pages 81–83** (*continued*)

**D.** 7. Prove: $x=3$

| | | |
|---|---|---|
| (1) $x-2=1$ & $2-x\neq1$ | P | |
| (2) $x=1 \rightarrow 2-x=1$ | P | |
| (3) $x=1 \lor x+2=5$ | P | |
| (4) $x+2=5 \lor x-2=1 \rightarrow x=3$ | P | |
| (5) $2-x\neq1$ | S 1 | |
| (6) $x\neq1$ | TT 2, 5 | or |
| (7) $x+2=5$ | TP 3, 6 | |
| (8) $x+2=5 \lor x-2=1$ | LA 7 | |
| (9) $x=3$ | PP 4, 8 | |

(5) $x-2=1$    S 1
(6) $x+2=5 \lor x-2=1$    LA 5
(7) $x=3$    PP 4, 6

8. Prove: $y=x \lor y>x$

(1) $y<6 \rightarrow y<x$    P
(2) $y\not<6 \lor x=5 \rightarrow y>x$    P
(3) $y\not<x$    P
(4) $y\not<6$    TT 1, 3
(5) $y\not<6 \lor x=5$    LA 4
(6) $y>x$    PP 2, 6
(7) $y=x \lor y>x$    LA 6

9. Prove: $y<3 \lor x>5$

(1) $y<4$ & $x=y+3$    P
(2) $\neg(x\neq y+3) \rightarrow x>2$    P
(3) $y\not>2 \rightarrow x\not>2$    P
(4) $y>2 \lor y=3 \rightarrow x>5$    P
(5) $x=y+3$    S 1
(6) $\neg(x\neq y+3)$    DN 5
(7) $x>2$    PP 2, 6
(8) $y>2$    TT 3, 7
(9) $y>2 \lor y=3$    LA 8
(10) $x>5$    PP 4, 9
(11) $y<3 \lor x>5$    LA 10

10. Prove: $(x=4 \lor y\neq8)$ & $x<3$

(1) $x=y \lor x<y$    P
(2) $y=x+4$    P
(3) $(x<3 \lor x>5)$ & $y=x+4 \rightarrow y\neq8$    P
(4) $x\neq y$    P
(5) $y=6 \lor x<y \rightarrow x<3$    P
(6) $x<y$    TP 1, 4
(7) $y=6 \lor x<y$    LA 6
(8) $x<3$    PP 5, 7
(9) $x<3 \lor x>5$    LA 8
(10) $(x<3 \lor x>5)$ & $y=x+4$    A 9, 2
(11) $y\neq8$    PP 3, 10
(12) $x=4 \lor y\neq8$    LA 11
(13) $(x=4 \lor y\neq8)$ & $x<3$    A 12, 8

**2.5, Exercise 12, pages 85–87**

**A.**  1. If water freezes then water expands.

2. If Tom drives at the rate of 50 miles per hour then he will travel 90 miles farther in the same period than yesterday.

3. If Mr. Lincoln is elected then a civil war will result.

4. If a beam of photons penetrates gas in a cloud chamber then the energy of the light passes into the kinetic energy of the electrons.

5. If representation in the Senate is according to population then New York has more than two Senators.

**B.**  1. Prove: $T \to S$
  (1) $W \to M$    P
  (2) $M \to E$    P
  (3) $W \to E$    HS 1, 2

2. Prove: $T \to S$
  (1) $T \to F$    P
  (2) $F \to S$    P
  (3) $T \to S$    HS 1, 2

3. Prove: $L \to C$
  (1) $L \to S$    P
  (2) $S \to C$    P
  (3) $L \to C$    HS 1, 2

4. Prove: $J \to A$
  (1) $J \to S$    P
  (2) $S \to A$    P
  (3) $J \to A$    HS 1, 2

5. Prove: $R \to M$
  (1) $R \to N$    P
  (2) $N \to M$    P
  (3) $R \to M$    HS 1, 2

**C.**  The atomic sentences are factually untrue although the molecular sentences are true.  Add the sentence:

"New York does not have more than two Senators" or $\neg M$.  Thus by TT we obtain $\neg N$ and $\neg R$.

  (1) $R \to N$    P
  (2) $N \to M$    P
  (3) $\neg M$    P
  (4) $\neg N$    TT 2, 3
  (5) $\neg R$    TT 1, 4

**D.**  1. $Q \to R$
2. $P \to T$
3. $S \vee T \to \neg P$
4. $S \to \neg R$

**E.**  1. Prove: $\neg T$
  (1) $(Q \to R)$ & P    P
  (2) $R \to T$    P
  (3) $(Q \to R) \to \neg T$    P
  (4) $Q \to R$    S 1
  (5) $\neg T$    PP 3, 4

2. Prove: P
  (1) $\neg R$    P
  (2) $\neg P \to Q$    P
  (3) $Q \to R$    P
  (4) $\neg P \to R$    HS 2, 3
  (5) P    TT 4, 1

3. Prove: Q
  (1) $\neg R \to S$    P
  (2) $S \to P$ & Q    P
  (3) $R \to T$    P
  (4) $\neg T$    P
  (5) $\neg R$    TT 3, 4
  (6) $\neg R \to P$ & Q    HS 1, 2
  (7) P & Q    PP 5, 6
  (8) Q    S 7

**F.**  1. Prove: $(2+2)+2=6 \;\to\; 3+3=6$
  (1) $(2+2)+2=6 \;\to\; 3\times2=6$    P
  (2) $3\times2=6 \;\to\; 3+3=6$    P
  (3) $(2+2)+2=6 \;\to\; 3+3=6$    HS 1, 2

**2.5, Exercise 12, pages 85–87** (*continued*)

**F.**    2. Prove: $5x-4=3x+4 \rightarrow x=4$

        (1) $5x-4=3x+4 \rightarrow 5x=3x+8$    P
        (2) $2x=8 \rightarrow x=4$    P
        (3) $5x=3x+8 \rightarrow 2x=8$    P
        (4) $5x-4=3x+4 \rightarrow 2x=8$    HS 1, 3
        (5) $5x-4=3x+4 \rightarrow x=4$    HS 4, 2

3. Prove: $z>6 \lor z<y$

        (1) $x>y \rightarrow x>z$    P
        (2) $\neg(z>6) \rightarrow \neg(x>y \rightarrow z<7)$    P
        (3) $x>z \rightarrow z<7$    P
        (4) $x>y \rightarrow z<7$    HS 1, 3
        (5) $z>6$    TT 2, 4
        (6) $z>6 \lor z<y$    LA 5

4. Prove: $x=6 \lor x>6$

        (1) $x\neq y \rightarrow y<x$    P
        (2) $(x>5 \rightarrow y<x) \rightarrow y=5$    P
        (3) $y\neq 5 \lor x=6$    P
        (4) $x>5 \rightarrow x\neq y$    P
        (5) $x>5 \rightarrow y<x$    HS 4, 1
        (6) $y=5$    PP 2, 5
        (7) $x=6$    TP 3, 6
        (8) $x=6 \lor x>6$    LA 7

5. Prove: $x>y$

        (1) $x\neq y \rightarrow x>y \lor x<y$    P
        (2) $x>y \lor x<y \rightarrow x\neq 4$    P
        (3) $x<y \rightarrow \neg(x\neq y \rightarrow x\neq 4)$    P
        (4) $x\neq y$    P
        (5) $x\neq y \rightarrow x\neq 4$    HS 1, 2
        (6) $x\not<y$    TT 3, 5
        (7) $x>y \lor x<y$    PP 1, 4
        (8) $x>y$    TP 7, 6

6. Prove: $(y\neq 0 \lor x<z) \;\&\; (x<y \rightarrow x=0)$

        (1) $x<z \rightarrow x=0$    P
        (2) $y=0 \rightarrow x\not<y$    P
        (3) $x<y \;\&\; z=3$    P
        (4) $x<y \rightarrow x<z$    P
        (5) $x<y$    S 3
        (6) $y\neq 0$    TT 2, 5
        (7) $y\neq 0 \lor x<z$    LA 6
        (8) $x<y \rightarrow x=0$    HS 4, 1
        (9) $(y\neq 0 \lor x<z) \;\&\; (x<y \rightarrow x=0)$    A 7, 8

**F.** 7. Prove: $\neg(z \neq 5) \ \lor \ z > 5$

| | |
|---|---|
| (1) $x = 3 \ \rightarrow \ x > y$ | P |
| (2) $x \neq 3 \ \rightarrow \ z = 5$ | P |
| (3) $(x = 3 \ \rightarrow \ x < z) \ \rightarrow \ x \not< z$ | P |
| (4) $x > y \ \rightarrow \ x < z$ | P |
| (5) $x = 3 \ \rightarrow \ x < z$ | HS 1, 4 |
| (6) $x \not< z$ | PP 3, 5 |
| (7) $x \not> y$ | TT 4, 6 |
| (8) $x \neq 3$ | TT 1, 7 |
| (9) $z = 5$ | PP 2, 8 |
| (10) $\neg(z \neq 5)$ | DN 9 |
| (11) $\neg(z \neq 5) \ \lor \ z > 5$ | LA 10 |

8. Prove: $x \neq 3 \ \lor \ 4 > x$

| | |
|---|---|
| (1) $5x = 20 \ \rightarrow \ x = 4$ | P |
| (2) $2x = 6 \ \lor \ x \neq 3$ | P |
| (3) $2x = 6 \ \rightarrow \ \neg(5x - 3 = 17 \ \rightarrow \ x = 4)$ | P |
| (4) $5x - 3 = 17 \ \rightarrow \ 5x = 20$ | P |
| (5) $5x - 3 = 17 \ \rightarrow \ x = 4$ | HS 4, 1 |
| (6) $2x \neq 6$ | TT 3, 5 |
| (7) $x \neq 3$ | TP 2, 6 |
| (8) $x \neq 3 \ \lor \ 4 > x$ | LA 7 |

9. Prove: $y + z = 8$

| | |
|---|---|
| (1) $z = 5 \ \rightarrow \ ((y = 3 \ \rightarrow \ y + z = 8) \ \& \ z > y)$ | P |
| (2) $(xy + z = 11 \ \rightarrow \ x = 2) \ \rightarrow \ (y = 3 \ \& \ x = 5)$ | P |
| (3) $xy = 6 \ \rightarrow \ x = 2$ | P |
| (4) $xy + z = 11 \ \rightarrow \ xy = 6$ | P |
| (5) $xy + z = 11 \ \rightarrow \ x = 2$ | HS 4, 3 |
| (6) $y = 3 \ \& \ z = 5$ | PP 2, 5 |
| (7) $z = 5$ | S 6 |
| (8) $(y = 3 \ \rightarrow \ y + z = 8) \ \& \ z > y$ | PP 1, 7 |
| (9) $y = 3 \ \rightarrow \ y + z = 8$ | S 8 |
| (10) $y = 3$ | S 6 |
| (11) $y + z = 8$ | PP 9, 10 |

*Note.* In algebra, when letters are written adjacent to each other, as $xy$, it means they are multiplied, $x \cdot y$ or $x \times y$.

10. Prove: $x + z = 3 \ \rightarrow \ y = 3$

| | |
|---|---|
| (1) $(x + y = 5 \ \rightarrow \ y = 3) \ \lor \ x + z = 3$ | P |
| (2) $z \neq 1 \ \lor \ (x + z = 3 \ \rightarrow \ x + y = 5)$ | P |
| (3) $x + y \neq 5 \ \& \ z = 1$ | P |
| (4) $z = 1$ | S 3 |
| (5) $x + z = 3 \ \rightarrow \ x + y = 5$ | TP 2, 4 |
| (6) $x + y \neq 5$ | S 3 |
| (7) $x + z \neq 3$ | TT 5, 6 |
| (8) $x + y = 5 \ \rightarrow \ y = 3$ | TP 1, 7 |
| (9) $x + z = 3 \ \rightarrow \ y = 3$ | HS 5, 8 |

## 2.5, Exercise 13, pages 89–92

**A.**  1. Either Bob will be treasurer or Dick will be treasurer.
2. This number is either greater than zero or it is less than zero.
3. This rock is either sedimentary or it is igneous.
4. Either it is my camera or Tom is its legal owner.
5. Either the plant makes its own food or it depends upon materials from other plants for its food.

**B.**  1. Prove: T ∨ R

| | | |
|---|---|---|
| (1) | B ∨ D | P |
| (2) | B → T | P |
| (3) | D → R | P |
| (4) | T ∨ R | DS 1, 2, 3 |

2. Prove: G ∨ L

| | | |
|---|---|---|
| (1) | P ∨ N | P |
| (2) | P → G | P |
| (3) | N → L | P |
| (4) | G ∨ L | DS 1, 2, 3 |

3. Prove: S ∨ I

| | | |
|---|---|---|
| (1) | L ∨ G | P |
| (2) | L → S | P |
| (3) | G → I | P |
| (4) | S ∨ I | DS 1, 2, 3 |

4. Prove: T ∨ ¬B

| | | |
|---|---|---|
| (1) | J ∨ M | P |
| (2) | M → T | P |
| (3) | J → ¬B | P |
| (4) | T ∨ ¬B | DS 1, 3, 2 |

5. Prove: M ∨ D

| | | |
|---|---|---|
| (1) | G ∨ N | P |
| (2) | G → M | P |
| (3) | N → D | P |
| (4) | M ∨ D | DS 1, 2, 3 |

**C.**  1. R ∨ ¬S

2. ¬S ∨ ¬T

3. P ∨ Q

4. ¬Q ∨ P

**D.**  1. Prove: R & (P ∨ Q)

| | | |
|---|---|---|
| (1) | P ∨ Q | P |
| (2) | Q → R | P |
| (3) | P → T | P |
| (4) | ¬T | P |
| (5) | R ∨ T | DS 1, 3, 2 |
| (6) | R | TP 5, 4 |
| (7) | R & (P ∨ Q) | A 6, 1 |

**D.**  2. Prove: T

| | | |
|---|---|---|
| (1) | P ∨ ¬R | P |
| (2) | ¬R → S | P |
| (3) | P → T | P |
| (4) | ¬S | P |
| (5) | S ∨ T | DS 1, 3, 2 |
| (6) | T | TP 5, 4 |

3. Prove: ¬Q & S

| | | |
|---|---|---|
| (1) | S & ¬R | P |
| (2) | R ∨ ¬T | P |
| (3) | Q → T | P |
| (4) | ¬R | S 1 |
| (5) | ¬T | TP 2, 4 |
| (6) | ¬Q | TT 3, 5 |
| (7) | S | S 1 |
| (8) | ¬Q & S | A 6, 7 |

4. Prove: S

| | | |
|---|---|---|
| (1) | P → Q | P |
| (2) | Q → ¬R | P |
| (3) | R | P |
| (4) | P ∨ (T & S) | P |
| (5) | P → ¬R | HS 1, 2 |
| (6) | ¬P | TT 5, 3 |
| (7) | T & S | TP 4, 6 |
| (8) | S | S 8 |

5. Prove: ¬T & ¬P

| | | |
|---|---|---|
| (1) | ¬S ∨ ¬R | P |
| (2) | ¬R → ¬T | P |
| (3) | ¬S → P | P |
| (4) | ¬P | P |
| (5) | ¬T ∨ P | DS 1, 3, 2 |
| (6) | ¬T | TP 5, 4 |
| (7) | ¬T & ¬P | A 6, 4 |

**E.**  1. Prove: $x=3$ ∨ $x=2$

| | | |
|---|---|---|
| (1) | $x+y=7 \rightarrow x=2$ | P |
| (2) | $y-x=2 \rightarrow x=3$ | P |
| (3) | $x+y=7$ ∨ $y-x=2$ | P |
| (4) | $x=3$ ∨ $x=2$ | DS 3, 1, 2 |

**E.**    2. Prove: $x>2 \ \lor \ x=2$

    (1) $x<y \ \rightarrow \ x=2$      P

    (2) $x<y \ \lor \ x \not< y$      P

    (3) $x \not< y \ \rightarrow \ x>2$      P

    (4) $x>2 \ \lor \ x=2$      DS 2, 1, 3

3. Prove: $y=1$

    (1) $2x+y=7 \ \rightarrow \ 2x=4$      P

    (2) $2x+y=5 \ \rightarrow \ y=1$      P

    (3) $2x+y=7 \ \lor \ 2x+y=5$      P

    (4) $2x \neq 4$      P

    (5) $2x=4 \ \lor \ y=1$      DS 3, 1, 2

    (6) $y=1$      TP 5, 4

4. Prove: $y=1 \ \lor \ y=9$

    (1) $\neg(x=2 \ \lor \ x=8) \ \rightarrow \ x=6$      P

    (2) $2x+3y=21 \ \ \& \ \ x \neq 6$      P

    (3) $x=2 \ \rightarrow \ y=9$      P

    (4) $x=8 \ \rightarrow \ y=1$      P

    (5) $x \neq 6$      S 2

    (6) $x=2 \ \lor \ x=8$      TT 1, 5

    (7) $y=1 \ \lor \ y=9$      DS 6, 3, 4

5. Prove: $\neg(x \not< z) \ \lor \ \neg(z \neq 6)$

    (1) $x>5 \ \lor \ y \not< 6$      P

    (2) $y \not< 6 \ \rightarrow \ x<z$      P

    (3) $x>5 \ \rightarrow \ y<z$      P

    (4) $y \not< z \ \ \& \ \ z=6$      P

    (5) $y<z \ \lor \ x<z$      DS 1, 3, 2

    (6) $y \not< z$      S 4

    (7) $x<z$      TP 5, 6

    (8) $\neg(x \not< z)$      DN 7

    (9) $\neg(x \not< z) \ \lor \ \neg(z \neq 6)$      LA 8

                 or

    (5) $z=6$      S 4

    (6) $\neg(z \neq 6)$      DN 5

    (7) $\neg(x \not< z) \ \lor \ \neg(z \neq 6)$      LA 6

6. Prove: $x \neq 4 \ \lor \ x>y$

    (1) $y=0 \ \rightarrow \ xy=0$      P

    (2) $y=0 \ \lor \ y \not< 1$      P

    (3) $xy=0 \ \lor \ xy>3 \ \rightarrow \ x \neq 4$      P

    (4) $y \not< 1 \ \rightarrow \ xy>3$      P

    (5) $xy=0 \ \lor \ xy>3$      DS 2, 1, 4

    (6) $x \neq 4$      PP 3, 5

    (7) $x \neq 4 \ \lor \ x>y$      LA 6

7. Prove: $y<12 \ \lor \ x<0$

    (1) $x<y \ \lor \ y<x$      P

    (2) $y<x \ \rightarrow \ x>6$      P

    (3) $x<y \ \rightarrow \ x<7$      P

    (4) $(x>6 \ \lor \ x<7) \ \rightarrow \ y \not> 11$      P

    (5) $y>11 \ \lor \ x<0$      P

    (6) $x>6 \ \lor \ x<7$      DS 1, 3, 2

    (7) $y \not> 11$      PP 4, 6

    (8) $x<0$      TP 5, 7

    (9) $y<12 \ \lor \ x<0$      LA 8

**2.5, Exercise 13, pages 89–92** (*continued*)

**E.**   8.  Prove: $x^2=4 \quad \vee \quad x^2=9$

    (1) $2x^2-10x+12=0 \quad \& \quad x<4$         P

    (2) $x^2-5x+6=0 \quad \rightarrow \quad x=2 \quad \vee \quad x=3$     P

    (3) $x=2 \quad \rightarrow \quad x^2=4$          P

    (4) $x=3 \quad \rightarrow \quad x^2=9$          P

    (5) $2x^2-10x+12=0 \quad \rightarrow \quad x^2-5x+6=0$     P

    (6) $2x^2-10x+12=0$          S 1

    (7) $x^2-5x+6=0$          PP 5, 6

    (8) $x=2 \quad \vee \quad x=3$          PP 2, 7

    (9) $x^2=4 \quad \vee \quad x^2=9$          DS 8, 3, 4

   9.  Prove: $x+1 \not> y \quad \vee \quad x \not> 4$

    (1) $(y=5 \quad \rightarrow \quad x \not> y) \quad \& \quad x>1$     P

    (2) $y>5 \quad \vee \quad y=5$          P

    (3) $x<y \quad \vee \quad y>4 \quad \rightarrow \quad x+1 \not> y \quad \& \quad y<9$     P

    (4) $y>5 \quad \rightarrow \quad y>4$          P

    (5) $y=5 \quad \rightarrow \quad x<y$          S 1

    (6) $x<y \quad \vee \quad y>4$          DS 2, 4, 5

    (7) $x+1 \not> y \quad \& \quad y<9$          PP 3, 6

    (8) $x+1 \not> y$          S 7

    (9) $x+1 \not> y \quad \vee \quad x \not> 4$          LA 8

  10.  Prove: $x=4$

    (1) $x=5 \quad \vee \quad x<y$          P

    (2) $x>3 \quad \vee \quad z<2 \quad \rightarrow \quad z<x \quad \vee \quad =1$     P

    (3) $x<y \quad \rightarrow \quad z<2$          P

    (4) $x=5 \quad \rightarrow \quad x>3$          P

    (5) $z<x \quad \rightarrow \quad x=4$          P

    (6) $y=1 \quad \rightarrow \quad \neg(x>3 \quad \vee \quad z<2)$     P

    (7) $x>3 \quad \vee \quad z<2$          DS 1, 4, 3

    (8) $z<x \quad \vee \quad y=1$          PP 2, 7

    (9) $x=4 \quad \vee \quad \neg(x>3 \quad \vee \quad z<2)$     DS 8, 5, 6

    (10) $x=4$          TP 9, 7

**2.5, Exercise 14, pages 93–95**

**A.**   1. R

    2. ¬S

    3. ¬R

    4. S

**B.**   The argument is valid.   DS and DP are used.

**C.**   1. There will not be a tie vote.

    2. This closed figure is a triangle.

    3. The Knights will be third.

**D.**   1. Prove: ¬T & S
      (1) P → ¬Q          P
      (2) P ∨ R           P
      (3) R → ¬Q          P
      (4) T → Q           P
      (5) S               P
      (6) ¬Q ∨ ¬Q         DS 2, 1, 3
      (7) ¬Q              DP 6
      (8) ¬T              TT 4, 7
      (9) ¬T & S          A 8, 5

   3. Prove: ¬S & R
      (1) S → P           P
      (2) ¬P & ¬T         P
      (3) ¬T → R          P
      (4) ¬P              S 2
      (5) ¬S              TT 1, 4
      (6) ¬T              S 2
      (7) R               PP 3, 6
      (8) ¬S & R          A 5, 7

   2. Prove: **Q**
      (1) Q ∨ S           P
      (2) S → T           P
      (3) ¬T              P
      (4) ¬S              TT 2, 3
      (5) Q               TP 1, 4

**E.**   1. Prove: ¬J
      (1) F ∨ S           P
      (2) F → ¬B          P
      (3) S → ¬B          P
      (4) J → B           P
      (5) ¬B ∨ ¬B         DS 1, 2, 3
      (6) ¬B              DP 5
      (7) ¬J              TT 4, 6

**F.**   1. Prove: $x < 4$
      (1) $x = y \ \lor \ x > y$          P
      (2) $x < 4 \ \lor \ x \not< z$      P
      (3) $x = y \ \rightarrow \ x < z$   P
      (4) $x > y \ \rightarrow \ x < z$   P
      (5) $x < z \ \lor \ x < z$          DS 1, 3, 4
      (6) $x < z$                         DP 5
      (7) $x < 4$                         TP 2, 6

   3. Prove: $x = 2$
      (1) $x < 3 \ \lor \ x > 4$                           P
      (2) $x < 3 \ \rightarrow \ x \neq y$                 P
      (3) $x > 4 \ \rightarrow \ x \neq y$                 P
      (4) $x < y \ \lor \ x \neq y \ \rightarrow \ x \neq 4 \ \& \ x = 2$    P
      (5) $x \neq y$                                       DS 1, 2, 3
      (6) $x < y \ \lor \ x \neq y$                        LA 5
      (7) $x \neq 4 \ \& \ x = 2$                          PP 4, 6
      (8) $x = 2$                                          S 7

   4. Prove: $x^2 = 9$
      (1) $x = (+3) \ \rightarrow \ 2x^2 = 18$    P
      (2) $x = (+3) \ \lor \ x = (-3)$            P
      (3) $x = (-3) \ \rightarrow \ 2x^2 = 18$    P
      (4) $2x^2 = 18 \ \rightarrow \ x^2 = 9$     P
      (5) $2x^2 = 18 \ \lor \ 2x^2 = 18$          DS 2, 1, 3
      (6) $2x^2 = 18$                             DP 5
      (7) $x^2 = 9$                               PP 4, 6

**F.**   2. Prove: $x = 1$
      (1) $2x + y = 5 \ \rightarrow \ 2x = 2$     P
      (2) $2x + y = 5 \ \lor \ y = 3$             P
      (3) $2x = 2 \ \rightarrow \ x = 1$          P
      (4) $y = 3 \ \rightarrow \ 2x = 2$          P
      (5) $2x = 2 \ \lor \ 2x = 2$                DS 2, 1, 4
      (6) $2x = 2$                                DP 5
      (7) $x = 1$                                 PP 3, 6

   5. Prove: $\neg(x \neq 5)$
      (1) $z > x \ \rightarrow \ x \neq 7$    P
      (2) $x < 6 \ \lor \ x = 3$              P
      (3) $x = 3 \ \rightarrow \ z > x$       P
      (4) $x < 6 \ \rightarrow \ z > x$       P
      (5) $x = 7 \ \lor \ x = 5$              P
      (6) $z > x \ \lor \ z > x$              DS 2, 4, 3
      (7) $z > x$                             DP 6
      (8) $x \neq 7$                          PP 1, 7
      (9) $x = 5$                             TP 5, 8
      (10) $\neg(x \neq 5)$                   DN 9

**2.5, Exercise 14, pages 93–95** *(continued)*

**F.**    6. Prove: $y=z \ \lor \ x \neq 5$

         (1) $x=y \ \rightarrow \ x<z$      P

         (2) $x=5 \ \rightarrow \ x \not< z$      P

         (3) $x=3 \ \rightarrow \ x<z$      P

         (4) $x \not< y \ \rightarrow \ x=y$      P

         (5) $x=3 \ \lor \ x \not< y$      P

         (6) $x \not< y \ \rightarrow \ x<z$      HS 4, 1

         (7) $x<z \ \lor \ x<z$      DS 5, 3, 6

         (8) $x<z$      DP 7

         (9) $x \neq 5$      TT 2, 8

        (10) $y=z \ \lor \ x \neq 5$      LA 9

   7. Prove: $x^2=9 \ \lor \ x^2>9$

         (1) $x=3 \ \lor \ x=4$      P

         (2) $x=3 \ \rightarrow \ x^2-7x+12=0$      P

         (3) $x=4 \ \rightarrow \ x^2-7x+12=0$      P

         (4) $x^2-7x+12=0 \ \rightarrow \ x>2$      P

         (5) $x^2<9 \ \rightarrow \ x \not> 2$      P

         (6) $x^2 \not< 9 \ \rightarrow \ x^2=9 \ \lor \ x^2>9$      P

         (7) $x^2-7x+12=0 \ \lor \ x^2-7x+12=0$      DS 1, 2, 3

         (8) $x^2-7x+12=0$      DP 7

         (9) $x>2$      PP 4, 8

        (10) $x^2 \not< 9$      TT 5, 9

        (11) $x^2=9 \ \lor \ x^2>9$      PP 6, 10

   8. Prove: $\cos\theta = \tfrac{1}{2}\sqrt{3} \ \lor \ \csc\theta = 2$

         (1) $\theta = 150° \ \rightarrow \ \sin\theta = \tfrac{1}{2}$      P

         (2) $\theta = 30° \ \lor \ \theta = 150°$      P

         (3) $\sin\theta = \tfrac{1}{2} \ \rightarrow \ \csc\theta = 2$      P

         (4) $\theta = 30° \ \rightarrow \ \sin\theta = \tfrac{1}{2}$      P

         (5) $\sin\theta = \tfrac{1}{2} \ \lor \ \sin\theta = \tfrac{1}{2}$      DS 2, 4, 1

         (6) $\sin\theta = \tfrac{1}{2}$      DP 5

         (7) $\csc\theta = 2$      PP 3, 6

         (8) $\cos\theta = \tfrac{1}{2}\sqrt{3} \ \lor \ \csc\theta = 2$      LA 7

**2.5, Exercise 15, pages 96–98**

**A.**    1. Multiplication is a binary operation and addition is a binary operation.

     2. Either the velocity of a body is not changed or an unbalanced force acts on the body.

     3. Jim's uncle is a representative in the state legislature or he is a senator.

     4. Cathy lives on Sixth Street and Tony lives on Maple Street.

     5. Hydrogen is a gas or it is a liquid.

**B.**    1. $Q \lor \neg P$

     2. $\neg S \ \& \ \neg R$

     3. $S \ \& \ P$

     4. $\neg S \lor \neg T$

     5. $R \lor Q$

**C.** 1. Prove: S & Q

    (1) P ∨ T         P
    (2) ¬T           P
    (3) P → Q & S    P
    (4) P            TP 1, 2
    (5) Q & S       PP 3, 4
    (6) S & Q       CL 5

2. Prove: ¬(R & ¬T)

    (1) (R & ¬T) → ¬S   P
    (2) P → S         P
    (3) P & Q         P
    (4) P            S 3
    (5) S            PP 2, 4
    (6) ¬(R & ¬T)     TT 1, 5

3. Prove: S & R

    (1) (R & S) ∨ P   P
    (2) Q → ¬P      P
    (3) T → ¬P      P
    (4) Q ∨ T        P
    (5) ¬P ∨ ¬P     DS 4, 2, 3
    (6) ¬P          DP 5
    (7) R & S       TP 1, 6
    (8) S & R       CL 7

4. Prove: R ∨ Q

    (1) S → R       P
    (2) S ∨ T       P
    (3) ¬T         P
    (4) S          TP 2, 3
    (5) R          PP 1, 4
    (6) R ∨ Q      LA 5

5. Prove: T

    (1) P → Q      P
    (2) Q → R      P
    (3) (P → R) → ¬S  P
    (4) S ∨ T       P
    (5) P → R      HS 1, 2
    (6) ¬S         PP 3, 5
    (7) T          TP 4, 6

6. Prove: ¬S

    (1) ¬T ∨ ¬S    P
    (2) ¬Q → T     P
    (3) Q → ¬R     P
    (4) R          P
    (5) ¬Q        TT 3, 4
    (6) T          PP 2, 5
    (7) ¬S        TP 1, 6

7. Prove: R

    (1) S → R ∨ T   P
    (2) ¬¬S        P
    (3) ¬T         P
    (4) S          DN 2
    (5) R ∨ T       PP 1, 4
    (6) R          TP 5, 3

8. Prove: ¬Q & P

    (1) T → P & ¬Q   P
    (2) T ∨ ¬R      P
    (3) R          P
    (4) T          TP 2, 3
    (5) P & ¬Q      PP 1, 4
    (6) ¬Q & P     CL 5

9. Prove: T

    (1) P ∨ ¬R     P
    (2) ¬S        P
    (3) P → S      P
    (4) ¬R → T     P
    (5) S ∨ T      DS 1, 3, 4
    (6) T         TP 5, 2

10. Prove: ¬P

    (1) R → T      P
    (2) S → Q      P
    (3) T ∨ Q → ¬P  P
    (4) R ∨ S       P
    (5) T ∨ Q      DS 4, 1, 2
    (6) ¬P        PP 3, 5

11. Prove: $y \not< 4$ & $x < y$

    (1) $x > y$ ∨ $x < 4$     P
    (2) $x < 4$ → $x < y$ & $y \not< 4$   P
    (3) $x > y$ → $x = 4$     P
    (4) $x \neq 4$          P
    (5) $x \not> y$         TT 3, 4
    (6) $x < 4$         TP 1, 5
    (7) $x < y$ & $y \not< 4$   PP 2, 6
    (8) $y \not< 4$ & $x < y$   CL 7

**C.**   12. Prove: $y>3$   &   $y<5$

|   |   |
|---|---|
| (1) $x=3$   $\vee$   $y=3$ | P |
| (2) $x>2$   $\vee$   $x+y\not>5$ | P |
| (3) $y=3$   $\vee$   $x=3$   $\rightarrow$   $x+y>5$ | P |
| (4) $\neg(y<5$   &   $y>3)$   $\rightarrow$   $x\not>2$ | P |
| (5) $y=3$   $\vee$   $x=3$ | CL 1 |
| (6) $x+y>5$ | PP 3, 5 |
| (7) $x>2$ | TP 2, 6 |
| (8) $y<5$   &   $y>3$ | TT 4, 7 |
| (9) $y>3$   &   $y<5$ | CL 8 |

13. Prove: $x<3$   &   $y=7$

|   |   |
|---|---|
| (1) $x<3$   &   $y>6$ | P |
| (2) $y\neq7$   $\rightarrow$   $\neg(x=2$   &   $y>x)$ | P |
| (3) $y>6$   &   $x<3$   $\rightarrow$   $y>x$   &   $x=2$ | P |
| (4) $y>6$   &   $x<3$ | CL 1 |
| (5) $y>x$   &   $x=2$ | PP 3, 4 |
| (6) $x=2$   &   $y>x$ | CL 5 |
| (7) $y=7$ | TT 2, 6 |
| (8) $x<3$ | S 1 |
| (9) $x<3$   &   $y=7$ | A 8, 7 |

14. Prove: $x=1$   &   $(y<1$   $\vee$   $y<2)$

|   |   |
|---|---|
| (1) $x+2y=5$   $\vee$   $3x+4y=11$ | P |
| (2) $x\not<2$   $\vee$   $x>y$   $\rightarrow$   $y<2$   $\vee$   $y<1$ | P |
| (3) $3x+4y=11$   $\rightarrow$   $x=1$ | P |
| (4) $x>y$   $\vee$   $x\not<2$ | P |
| (5) $x+2y=5$   $\rightarrow$   $x=1$ | P |
| (6) $x=1$   $\vee$   $x=1$ | DS 1, 5, 3 |
| (7) $x=1$ | DP 6 |
| (8) $x\not<2$   $\vee$   $x>y$ | CL 4 |
| (9) $y<2$   $\vee$   $y<1$ | PP 2, 8 |
| (10) $y<1$   $\vee$   $y<2$ | CL 9 |
| (11) $x=1$   &   $(y<1$   $\vee$   $y<2)$ | A 7, 10 |

15. Prove: $x<6$

|   |   |
|---|---|
| (1) $x>y$   $\vee$   $x<6$ | P |
| (2) $x>y$   $\rightarrow$   $x>4$ | P |
| (3) $x>y$   $\rightarrow$   $x=5$   &   $x<7$ | P |
| (4) $x<6$   $\rightarrow$   $x=5$   &   $x<7$ | P |
| (5) $x<7$   &   $x=5$   $\rightarrow$   $z>x$   $\vee$   $y<z$ | P |
| (6) $x>y$   $\rightarrow$   $\neg(y<z$   $\vee$   $z>x)$ | P |
| (7) $x>y$   $\rightarrow$   $x=5$   &   $x<7$ | HS 2, 3 |
| (8) $(x=5$   &   $x<7)$   $\vee$   $(x=5$   &   $x<7)$ | DS 1, 3, 4 |
| (9) $x=5$   &   $x<7$ | DP 8 |
| (10) $x<7$   &   $x=5$ | CL 9 |
| (11) $z>x$   $\vee$   $y<z$ | PP 5, 10 |
| (12) $y<z$   $\vee$   $z>x$ | CL 11 |
| (13) $x\not>y$ | TT 6, 12 |
| (14) $x<6$ | TP 1, 13 |

## 2.5, Exercise 16, pages 101–103

**A.**
1. It is not the case that both arachnids are insects and they have eight legs.
2. Air is not a good conductor of heat and water is not a good conductor of heat.
3. Either a number is not greater than zero or it is not negative.
4. It is not the case that either the Mississippi River flows northward or the Nile River flows southward.
5. Bats are not birds and porpoises are not fish.

**B.**

1. $\dfrac{\neg I \lor \neg E}{\neg(I \ \& \ E)}$

2. $\dfrac{\neg(A \lor W)}{\neg A \ \& \ \neg W}$

3. $\dfrac{\neg(G \ \& \ N)}{\neg G \lor \neg N}$

4. $\dfrac{\neg M \ \& \ \neg N}{\neg(M \lor N)}$

5. $\dfrac{\neg(B \lor P)}{\neg B \ \& \ \neg P}$

**C.**
1. $\neg P \lor \neg Q$
2. $\neg(R \ \& \ T)$
3. $R \lor \neg S$
4. $\neg(G \ \& \ H)$
5. $\neg(\neg S \ \& \ T)$
6. $\neg(P \lor Q)$
7. $\neg(\neg P \lor Q)$
8. $\neg C \ \& \ \neg D$
9. $P \lor Q$
10. $\neg(S \ \& \ \neg T)$
11. $\neg(\neg R \lor \neg S)$
12. $A \lor B$

**D.**   1. Prove: $\neg S$

| (1) | $\neg(P \ \& \ Q)$ | P |
| (2) | $\neg Q \to T$ | P |
| (3) | $\neg P \to T$ | P |
| (4) | $S \to \neg T$ | P |
| (5) | $\neg P \lor \neg Q$ | DL 1 |
| (6) | $T \lor T$ | DS 5, 3, 2 |
| (7) | $T$ | DP 6 |
| (8) | $\neg S$ | TT 4, 7 |

**D.**   2. Prove: $\neg(A \lor B)$

| (1) | $C \ \& \ \neg D$ | P |
| (2) | $C \to \neg A$ | P |
| (3) | $D \lor \neg B$ | P |
| (4) | $C$ | S 1 |
| (5) | $\neg A$ | PP 2, 4 |
| (6) | $\neg D$ | S 1 |
| (7) | $\neg B$ | TP 3, 6 |
| (8) | $\neg A \ \& \ \neg B$ | A 5, 7 |
| (9) | $\neg(A \lor B)$ | DL 8 |

3. Prove: $R \ \& \ Q$

| (1) | $\neg S \to \neg(P \lor \neg T)$ | P |
| (2) | $T \to Q \ \& \ R$ | P |
| (3) | $\neg S$ | P |
| (4) | $\neg(P \lor \neg T)$ | PP 1, 3 |
| (5) | $\neg P \ \& \ T$ | DL 4 |
| (6) | $T$ | S 5 |
| (7) | $Q \ \& \ R$ | PP 2, 6 |
| (8) | $R \ \& \ Q$ | CL 7 |

4. Prove: $\neg R$

| (1) | $P \to \neg Q$ | P |
| (2) | $\neg Q \to \neg S$ | P |
| (3) | $(P \to \neg S) \to \neg T$ | P |
| (4) | $R \to T$ | P |
| (5) | $P \to \neg S$ | HS 1, 2 |
| (6) | $\neg T$ | PP 3, 5 |
| (7) | $\neg R$ | TT 4, 6 |

5. Prove: $D$

| (1) | $\neg A \to B$ | P |
| (2) | $C \to B$ | P |
| (3) | $C \lor \neg A$ | P |
| (4) | $\neg B \lor D$ | P |
| (5) | $B \lor B$ | DS 3, 2, 1 |
| (6) | $B$ | DP 5 |
| (7) | $T$ | TP 4, 6 |

**2.5, Exercise 16, pages 101–103** (*continued*)

**D.**  6. Prove: ¬T

(1) T → P & S        P
(2) Q → ¬P           P
(3) R → ¬S           P
(4) R ∨ Q            P
(5) ¬P ∨ ¬S          DS 4, 3, 2
(6) ¬(P & S)         DL 5
(7) ¬T               TT 1, 6

7. Prove: ¬P

(1) R → ¬P           P
(2) (R & S) ∨ T      P
(3) T → (Q ∨ U)      P
(4) ¬Q & ¬U          P
(5) ¬(Q ∨ U)         DL 4
(6) ¬T               TT 3, 5
(7) R & S            TP 2, 6
(8) R                S 7
(9) ¬P               PP 1, 8

8. Prove: R & Q

(1) P ∨ Q            P
(2) S → Q & R        P
(3) P → S            P
(4) Q → S            P
(5) S ∨ S            DS 1, 3, 4
(6) S                DP 5
(7) Q & R            PP 2, 6
(8) R & Q            CL 7

9. Prove: G ∨ ¬H

(1) E ∨ F → ¬H       P
(2) J → E            P
(3) K → F            P
(4) J ∨ K            P
(5) E ∨ F            DS 4, 2, 3
(6) ¬H               PP 1, 5
(7) G ∨ ¬H           LA 6

10. Prove: S & T

(1) ¬(P ∨ ¬R)        P
(2) Q ∨ P            P
(3) R → S            P
(4) (Q & S) → (T & S)  P
(5) ¬P & R           DL 1
(6) ¬P               S 5
(7) Q                TP 2, 6
(8) R                S 5
(9) S                PP 3, 8
(10) Q & S           A 7, 9
(11) T & S           PP 4, 10
(12) S & T           CL 11

**E.**  1. Prove: ¬(x=3  &  x<2)   (See note.)

(1) ¬(x<2  &  x=3)   P
(2) x≮2 ∨ x≠3        DL 1
(3) x≠3 ∨ x≮2        CL 2
(4) ¬(x=3  &  x<2)   DL 3

*Note.* CL cannot be applied to (1) because it applies only to conjunctions and disjunctions, not to negations like (1) irrespective of what it is the negation. Additional examples are the last two lines of Problems 4, 7, where DL must again be applied before CL; and 10, where CL must be applied first.

**E.**  2. Prove: ¬(x=5  &  y=4)

(1) y≠3              P
(2) x+y=8 → y=3      P
(3) x+y=8 ∨ x≠5      P
(4) x+y≠8            TT 1, 2
(5) x≠5              TP 3, 4
(6) x≠5 ∨ y≠4        LA 5
(7) ¬(x=5  &  y=4)   DL 6

3. Prove: x>y

(1) ¬(y>5  &  x≠6)   P
(2) x=6 → x>y        P
(3) y≯5 → x>y        P
(4) y≯5 ∨ x=6        DL 1
(5) x>y ∨ x>y        DS 4, 3, 2
(6) x>y              DP 5

**E.**  4. Prove: $y=2$ & $x>y$

    (1) $x \not< y$                             P

    (2) $x<y$ $\lor$ $\neg(x \not> 3 \lor x+y<5)$      P

    (3) $x>3$ $\rightarrow$ $\neg(x \not> y \lor y \neq 2)$      P

    (4) $\neg(x \not> 3 \lor x+y<5)$             TP 1, 2

    (5) $x>3$ & $x+y \not< 5$            DL 4

    (6) $x>3$                             S 5

    (7) $\neg(x \not> y \lor y \neq 2)$               PP 3, 6

    (8) $x>y$ & $y=2$                 DL 7

    (9) $y=2$ & $x>y$                 CL 8

 

5. Prove: $\neg(x=2 \lor y<5)$

    (1) $\neg(y-x=2 \lor x+y \not> 8)$       P

    (2) $\neg(x>y \lor y<5)$              P

    (3) $x=2$ $\rightarrow$ $x+y \not> 8$          P

    (4) $y-x \neq 2$ & $x+y>8$        DL 1

    (5) $x \not> y$ & $y \not< 5$           DL 2

    (6) $x+y>8$                   S 4

    (7) $x \neq 2$                     TT 3, 6

    (8) $y \not< 5$                    S 5

    (9) $x \neq 2$ & $y \not< 5$          A 7, 8

   (10) $\neg(x=2 \lor y<5)$        DL 9

 

6. Prove: $x<y$ $\lor$ $y \neq 4$

    (1) $x=1$ $\rightarrow$ $x<y$                   P

    (2) $x^2-4x+3=0$ $\rightarrow$ $x=1 \lor x=3$      P

    (3) $\neg(x=y \lor x^2-4x+3 \neq 0)$          P

    (4) $x=3$ $\rightarrow$ $x<y$                   P

    (5) $x \neq y$ & $x^2-4x+3=0$         DL 3

    (6) $x^2-4x+3=0$                S 5

    (7) $x=1 \lor x=3$               PP 2, 6

    (8) $x<y \lor x<y$                DS 7, 1, 4

    (9) $x<y$                       DP 8

   (10) $x<y \lor y \neq 4$           LA 9

 

7. Prove: $2+3 \neq 3 \times 3$ $\lor$ $2 \times 3 \neq 1 \times 4$

    (1) $2 \times 3 = 1 \times 4$ & $2+3=3 \times 3$ $\rightarrow$ $2+3=6$     P

    (2) $2+3 \neq 6$ $\lor$ $2 \times 3 = 5$          P

    (3) $2 \times 3 \neq 5$                   P

    (4) $2+3 \neq 6$                     TP 2, 3

    (5) $\neg(2 \times 3 = 1 \times 4$ & $2+3=3 \times 3)$       TT 1, 4

    (6) $2 \times 3 \neq 1 \times 4$ $\lor$ $2+3 \neq 3 \times 3$      DL 5

    (7) $2+3 \neq 3 \times 3$ $\lor$ $2 \times 3 \neq 1 \times 4$      CL 6

*Note.* Each of the atomic sentences would be true if '$\times$' were the sign for addition and '$+$' were the sign for multiplication.

**2.5, Exercise 16, pages 101-103** (*continued*)

**E.**   8. Prove: $x-y\neq2$

| | | |
|---|---|---|
| (1) | $\neg(x>y$ & $x+y>7)$ | P |
| (2) | $x\not>y \rightarrow x<4$ | P |
| (3) | $x+y\not>7 \rightarrow x<4$ | P |
| (4) | $x-y=2 \rightarrow x\not<4$ | P |
| (5) | $x\not>y \vee x+y\not>7$ | DL 1 |
| (6) | $x<4 \vee x<4$ | DS 5, 2, 3 |
| (7) | $x<4$ | DP 6 |
| (8) | $x-y\neq2$ | TT 4, 7 |

9. Prove: $x=1$

| | | |
|---|---|---|
| (1) | $\neg(z<3 \vee x>y)$ & $y=2$ | P |
| (2) | $x\not<y \vee x=1$ | P |
| (3) | $x>z \rightarrow x>y$ | P |
| (4) | $x\not>z \rightarrow x<y$ | P |
| (5) | $\neg(z<3 \vee x>y)$ | S 1 |
| (6) | $z\not<3$ & $x\not>y$ | DL 5 |
| (7) | $x\not>y$ | S 6 |
| (8) | $x\not>z$ | TT 3, 7 |
| (9) | $x<y$ | PP 4, 9 |
| (10) | $x=1$ | TP 2, 9 |

10. Prove: $\neg(x=y \vee y\not>1)$

| | | |
|---|---|---|
| (1) | $y\neq1$ & $y\not<1$ | P |
| (2) | $y\not>1 \rightarrow y<1 \vee y=1$ | P |
| (3) | $x=3 \vee x>3$ | P |
| (4) | $x>3 \rightarrow x\neq y$ | P |
| (5) | $x=3 \rightarrow x\neq y$ | P |
| (6) | $y\not<1$ & $y\neq1$ | CL 1 |
| (7) | $\neg(y<1 \vee y=1)$ | DL 6 |
| (8) | $y>1$ | TT 2, 7 |
| (9) | $x\neq y \vee x\neq y$ | DS 3, 5, 4 |
| (10) | $x\neq y$ | DP 9 |
| (11) | $x\neq y$ & $y>1$ | A 10, 8 |
| (12) | $\neg(x=y \vee y\not>1)$ | DL 11 |

▶ 2.6 *Biconditional Sentences*

**2.6, Exercise 17, pages 105-107**

**A.**   1. if and only if
   2. if and only if, not
   3. if and only if
   4. if and only if
   5. if and only if

**C.**   1. Prove: P ∨ D

| | | |
|---|---|---|
| (1) | P ↔ M | P |
| (2) | M ∨ G | P |
| (3) | G → D | P |
| (4) | M → P | LB 1 |
| (5) | P ∨ D | DS 2, 4, 3 |

**C.** 2. Prove: S ∨ C

    (1) S ↔ R        P
    (2) R & M       P
    (3) R → S       LB 1
    (4) R             S 2
    (5) S             PP 3, 4
    (6) S ∨ C       LA 5

3. Prove: $3 \times 5 \neq 12$

    (1) $3 \times 5 = 12 \quad \leftrightarrow \quad 5 + 5 + 5 = 12$    P
    (2) $4 \times 4 \neq 13$    P
    (3) $5 + 5 + 5 = 12 \quad \rightarrow \quad 4 \times 4 = 13$    P
    (4) $5 + 5 + 5 \neq 12$    TT 3, 2
    (5) $3 \times 5 = 12 \quad \rightarrow \quad 5 + 5 + 5 = 12$    LB 1
    (6) $3 \times 5 \neq 12$    TT 5, 4

4. Prove: I → T

    (1) C ↔ I       P
    (2) C → T      P
    (3) I → C       LB 1
    (4) I → T       HS 3, 2

5. Prove: A ↔ C

    (1) A ↔ T      P
    (2) T ↔ C      P
    (3) A → T      LB 1
    (4) T → C      LB 2
    (5) A → C      HS 3, 4
    (6) C → T      LB 2
    (7) T → A      LB 1
    (8) C → A      HS 6, 7
    (9) A ↔ C      LB 5, 8

6. Prove: F ↔ L   (See note.)

    (1) ¬(F → L) → W      P
    (2) ¬W               P
    (3) L → D          P
    (4) D → F          P
    (5) ¬¬(F → L)       TT 1, 2
    (6) F → L           DN 5
    (7) L → F           HS 3, 4
    (8) F ↔ L           LB 6, 7

*Note.* The translation of premise (1) proves difficult for many students. Be sure the student's translation is correct before he attempts a derivation.

**D.** 1. Prove: $2 \times 5 = 5 + 5 \quad \rightarrow \quad 2 \times 4 = 4 + 4$

    (1) $2 \times 4 = 4 + 4 \quad \leftrightarrow \quad 2 \times 5 = 5 + 5$    P
    (2) $2 \times 5 = 5 + 5 \quad \rightarrow \quad 2 \times 4 = 4 + 4$    LB 1

2. Prove: $x = 4 \quad \leftrightarrow \quad 3x + 2 = 14$

    (1) $3x + 2 = 14 \quad \leftrightarrow \quad 3x = 12$    P
    (2) $3x = 12 \quad \leftrightarrow \quad x = 4$    P
    (3) $3x + 2 = 14 \quad \rightarrow \quad 3x = 12$    LB 1
    (4) $3x = 12 \quad \rightarrow \quad 3x + 2 = 14$    LB 1
    (5) $3x = 12 \quad \rightarrow \quad x = 4$    LB 2
    (6) $x = 4 \quad \rightarrow \quad 3x = 12$    LB 2
    (7) $x = 4 \quad \rightarrow \quad 3x + 2 = 14$    HS 6, 4
    (8) $3x + 2 = 14 \quad \rightarrow \quad x = 4$    HS 3, 5
    (9) $x = 4 \quad \leftrightarrow \quad 3x + 2 = 14$    LB 7, 8

**D.**   3. Prove: $x+y=5$

(1) $3x+y=11 \leftrightarrow 3x=9$                     P

(2) $3x=9 \rightarrow 3x+y=11 \leftrightarrow y=2$            P

(3) $y \neq 2 \lor x+y=5$                          P

(4) $3x=9 \rightarrow 3x+y=11$                     LB 1

(5) $(3x=9 \rightarrow 3x+y=11) \rightarrow y=2$         LB 2

(6) $y=2$                                     PP 5, 4

(7) $x+y=5$                                   TP 3, 6

4. Prove: $\neg(2x \neq 8 \ \& \ x \neq 3)$

(1) $2x=6 \leftrightarrow x=3$            P

(2) $2x=8 \leftrightarrow x=4$            P

(3) $2x=6 \lor x=4$                   P

(4) $2x=6 \rightarrow x=3$              LB 1

(5) $x=4 \rightarrow 2x=8$              LB 2

(6) $2x=8 \lor x=3$                   DS 3, 4, 5

(7) $\neg(2x \neq 8 \ \& \ x \neq 3)$        DL 6

5. Prove: $\neg(y=2 \lor x+2y \neq 7)$

(1) $5x=15 \leftrightarrow x=3$            P

(2) $5x=15 \ \& \ 4x=12$             P

(3) $x=3 \rightarrow x+2y=7$           P

(4) $5x=15$                          S 2

(5) $5x=15 \rightarrow x=3$            LB 1

(6) $x=3$                            PP 5, 4

(7) $x+2y=7$                         PP 3, 6

(8) $y \neq 2 \lor x+2y=7$             LA 7

(9) $\neg(y=2 \lor x+2y \neq 7)$        DL 8

6. Prove: $x \not< y \ \& \ x \neq y$

(1) $y \not> x \leftrightarrow x=y \lor x<y$   P

(2) $\neg(y<1 \lor y \not> x)$             P

(3) $y \not< 1 \ \& \ y>x$               P

(4) $x=y \lor x<y \rightarrow y \not> x$     LB 1

(5) $y>x$                            S 3

(6) $\neg(x=y \lor x<y)$               TT 4, 5

(7) $x \neq y \ \& \ x \not< y$            DL 6

(8) $x \not< y \ \& \ x \neq y$            CL 7

7. Prove: $x<y \leftrightarrow y>x$

(1) $y>x \leftrightarrow x<y$            P

(2) $y>x \rightarrow x<y$              LB 1

(3) $x<y \rightarrow y>x$              LB 1

(4) $x<y \leftrightarrow y>x$            LB 3, 2

8. Prove: $x<y \ \& \ y=6$

(1) $x<y \leftrightarrow y>4$             P

(2) $y=6 \leftrightarrow x+y=10$          P

(3) $y>4 \ \& \ \neg(x+y \neq 10)$      P

(4) $y>4$                             S 1

(5) $y>4 \rightarrow x<y$              LB 1

(6) $x<y$                             PP 5, 4

(7) $\neg(x+y \neq 10)$                 S 3

(8) $x+y=10$                          DN 7

(9) $x+y=10 \rightarrow y=6$           LB 2

(10) $y=6$                            PP 9, 8

(11) $x<y \ \& \ y=6$                 A 6, 10

9. Prove: $xy \neq 0$

(1) $y>x \leftrightarrow x=0$             P

(2) $xy=0 \leftrightarrow x=0$            P

(3) $y \not> x$                         P

(4) $x=0 \rightarrow y>x$              LB 1

(5) $x \neq 0$                          TT 4, 3

(6) $xy=0 \rightarrow x=0$             LB 2

(7) $xy \neq 0$                         TT 6, 5

10. Prove: $\neg(x < y \;\; \& \;\; x = 1)$

    (1) $x = y \;\rightarrow\; x \nless y$             P

    (2) $y = 0 \;\leftrightarrow\; x \nless y$           P

    (3) $x = 0 \;\lor\; xy = 0 \;\rightarrow\; y = 0$    P

    (4) $(x = y \;\rightarrow\; y = 0) \;\rightarrow\; x = 0$   P

    (5) $x \nless y \;\rightarrow\; y = 0$           LB 2

    (6) $x = y \;\rightarrow\; y = 0$            HS 1, 5

    (7) $x = 0$                       PP 4, 6

    (8) $x = 0 \;\lor\; xy = 0$          LA 7

    (9) $y = 0$                       PP 3, 8

    (10) $y = 0 \;\rightarrow\; x \nless y$         LB 2

    (11) $x \nless y$                   PP 10, 9

    (12) $x \nless y \;\lor\; x \neq 1$         LA 11

    (13) $\neg(x < y \;\; \& \;\; x = 1)$     DL 12

# Truth and Validity

▶ 3.2  *Truth-Value and Truth-Functional Connectives*

**3.2, Exercise 1, page 112**

**A.**  False.  A conjunction whose left conjunct is true and whose right conjunct is false is false.

**B.**  1. T
    2. F
    3. F
    4. T
    5. F

**3.2, Exercise 2, page 113**

**A.**  1. F
    2. T
    3. F
    4. F
    5. T

**3.2, Exercise 3, page 114**

| | | | | | |
|---|---|---|---|---|---|
| **A.** | 1. T | | **B.** | 1. F | |
| | 2. T | | | 2. T | |
| | 3. T | | | 3. T | |
| | 4. F | | | 4. T | |
| | 5. T | | | 5. F | |

**3.2, Exercise 4, pages 116–117**

**A.**  It will be true because a conditional is true if its consequent is true no matter what the truth of its antecedent.

| | | | | | |
|---|---|---|---|---|---|
| **B.** | 1. T | | **C.** | 1. T | |
| | 2. F | | | 2. F | |
| | 3. T | | | 3. T | |
| | 4. T | | | 4. T | |
| | 5. T | | | 5. T | |

**3.2, Exercise 5, page 118**

| | | | | |
|---|---|---|---|---|
| **A.** | 1. F | | 4. T | |
| | 2. T | | 5. T | |
| | 3. F | | | |
| | | | **B.** | Always false. |

## ▶ 3.3 *Diagrams of Truth Value*

Before going to more complex examples as found on page 119, give simple examples like those in Exercises 2, 3, and 4 (especially Exercises 2A, page 113; 3B, page 114; 4C, page 116).

**A & B**

T   F
  F

**¬C ∨ D**

T   T
F
    T

**¬P → Q**

T   F
F
    T

Demonstrate some of these, have the students do others at their desks. Special attention should be given to the negation of molecular sentences such as
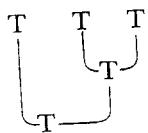
**¬¬S**

T
F
T

**¬(R ∨ ¬S)**

F   T
    F
   F
T

It may be helpful to number off connectives in order of dominance and emphasize that diagraming proceeds in reverse to order of dominance.

Numerous additional exercises may be obtained by using the formulas in Exercise 6 but giving different truth assignments. Similarly the formulas on pages 170 and 174 may be used.
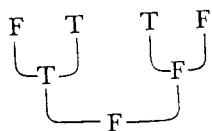
### 3.3, Exercise 6, pages 120–122

**A.**  1.  P → (P → Q)

T   T   T
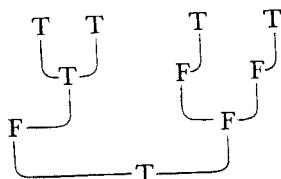      T
  T

2.  (A → P) → (P → A)

F   T   T   F
  T        F
      F

3.  (P → A) → (A → P)

T   F   F   T
  F        T
      T

4.  (P → A) → (¬P → ¬A)

T   F      T   F
  F      F     T
          T
        T

5.  ¬(P & Q) → (¬P ∨ ¬Q)

T   T     T   T
  T      F   F
F          F
      T

6.  ¬(P & B) → (¬P & ¬B)

T   F     T   F
  F      F   T
T         F
      F

## 3.3, Exercise 6, pages 120–122 *(continued)*

**A.**    7. $[(P \ \& \ Q) \to B] \to [P \to (Q \to B)]$

$$
\begin{array}{ccccccc}
T & T & F & & T & T & F \\
& T & & & T & & T \\
& & F & & & F & \\
& & & T & & &
\end{array}
$$

8. $\neg[(P \lor B) \quad \& \quad (B \lor A)]$

$$
\begin{array}{ccc}
T & F & \quad F \quad F \\
T & & F \\
& F & \\
T &
\end{array}
$$

9. $(\neg P \lor B) \quad \lor \quad (\neg B \ \& \ A)$

$$
\begin{array}{cccc}
T & F & F & F \\
F & & T & \\
F & & F & \\
& F &
\end{array}
$$

10. $(P \to Q) \quad \leftrightarrow \quad (Q \to P)$

$$
\begin{array}{cccc}
T & T & T & T \\
T & & T & \\
& T &
\end{array}
$$

**B.**    1. $N \lor C$

$$
\begin{array}{cc}
T & F \\
& T
\end{array}
$$

2. $N \ \& \ C$

$$
\begin{array}{cc}
T & F \\
& F
\end{array}
$$

**B.**    3. $\neg N \ \& \ \neg C$

$$
\begin{array}{cc}
T & F \\
F & T \\
& F
\end{array}
$$

4. $N \leftrightarrow \neg W \lor C$

$$
\begin{array}{ccc}
T & T & F \\
& F & \\
& F & \\
F &
\end{array}
$$

5. $W \lor \neg C \to N$

$$
\begin{array}{ccc}
T & F & T \\
& T & \\
T & & \\
& T &
\end{array}
$$

6. $(W \lor N) \to (W \to \neg C)$

$$
\begin{array}{cccc}
T & T & T & F \\
& T & & T \\
& & & T \\
& T &
\end{array}
$$

7. $(W \leftrightarrow \neg N) \quad \leftrightarrow \quad (N \leftrightarrow C)$

$$
\begin{array}{cccc}
T & T & T & F \\
& F & & F \\
F & & & \\
& & T &
\end{array}
$$

8. $(W \to N) \quad \to \quad [(N \to \neg C) \quad \to \quad (\neg C \to W)]$

$$
\begin{array}{cccccc}
T & T & T & F & F & T \\
& T & & T & T & \\
& & T & & T & \\
& & & T & & \\
& T &
\end{array}
$$

**C.**

**1.** (P & Q) & (R & S) → P ∨ S

```
  T   T      T   F      T   F
   └T─┘       └F─┘        └T─┘
      └───F────┘            │
           └──────T─────────┘
```

**2.** P & Q → R & ¬S

```
  T   T      T   F
   └T─┘       └T─┘
      │          │
      └────T─────┘
          └──T───┘
```

**6.** (P → ¬Q) ↔ (P ∨ R) & S

```
  T    T       T   T   F
   └F─┘         │   │
    └F─┘        └T─┘
       │           └F─┘
       └─────T──────┘
```

**7.** (Q & R) & S → (P ↔ S)

```
  T   T   F      T   F
   └T─┘          └F─┘
      └──F──┘       │
          └───T─────┘
```

**3.** (P → Q) → [(Q → R) → (R → S)]

```
  T   T      T   T      T   F
   └T─┘       └T─┘       └F─┘
      │          └──F──┘   │
      │              └──F──┘
      └───────F──────────┘
```

**8.** (¬P → Q) → (S → R)

```
  T   T       F   T
  F─┘         └T─┘
   └T─┘          │
      └────T─────┘
```

**4.** (P ↔ Q) → (S ↔ R)

```
  T   T      F   T
   └T─┘       └F─┘
      └───F────┘
```

**9.** S → P & Q

```
  F   T   T
      └T─┘
   └T──┘
```

**5.** (P & Q) ∨ S → (P ↔ S)

```
  T   T   F      T   F
   └T─┘   │       └F─┘
      └─T─┘          │
         └───F───────┘
```

**10.** P & Q → S

```
  T   T   F
   └T─┘   │
      └───┘
       └─F─┘
```

**D.** Whenever truth functional analysis (truth diagrams or truth tables) is to be made of mathematical sentences, they should be symbolized with capital letters for atomic sentences, as for English sentences.

Let:  P = 'x = 0'          R = 'y = z'
      Q = 'x = y'          S = 'y = w'

| T | F |
|---|---|
| P | R |
| Q | S |

(Be sure the quotes are present.  P = x = 0 makes no sense at all.)

**3.3, Exercise 6, pages 120–122** (*continued*)

**D.**   1. P & Q → ¬R

```
T   T      F
 └─T─┘    T┘
    └──T─┘
```

2. ¬P ∨ S → R

```
T   F   F
F┘
 └─F─┘
    └───T─┘
```

3. ¬Q ∨ ¬R → S

```
T    F   F
F┘  T┘
 └──T─┘
     └───F─┘
```

4. ¬P ∨ ¬Q → ¬R

```
T    T      F
F┘  F┘     T┘
 └──F─┘
     └───T─┘
```

5. P → ¬Q ∨ ¬S

```
T    T      F
    F┘     T┘
     └──T─┘
 T─┘
```

6. ¬P → R

```
T   F
F┘
 └─T─┘
```

▶ 3.4 *Invalid Conclusions*

Note that the procedure is (1) to write out the argument completely symbolized, (2) assert that the argument is invalid, (3) announce a truth assignment for each distinct atomic sentence, then (4) demonstrate with truth diagrams that these truth assignments do indeed simultaneously make all premises true and the conclusion false.

It needs emphasis with the students that the same truth assignment for any given atomic sentence must be used throughout the premises and conclusion of an argument. To help insure this, it is a good procedure to write out all premises and the conclusion before putting its truth value under any atomic letter. Then one at a time its truth value is put under each distinct letter in all its occurrences.

**3.4, Exercise 7, pages 125–129**

**A.**   1. Prove: ¬M

| (1) E → W |  |
| (2) W ∨ M | *Invalid* |
| (3) E |  |

| T | F |
|---|---|
| M |  |
| E |  |
| W |  |

Premises:

(1) E → W

```
T   T
 └─T─┘
```

(2) W ∨ M

```
T   T
 └─T─┘
```

(3) E

```
T
```

Conclusion:

¬M

```
T
F┘
```

**A.**  2.  Prove: **F**
    (1) C ∨ ¬H
    (2) H
    (3) F → C

*Invalid*

| T | F |
|---|---|
| H | F |
| C | |

Premises:

  (1) C ∨ ¬H      (2) H      (3) F → C      Conclusion:

                                            **F**

      T     T         T            F    T        F
            F
           T                             T

---

3.  Prove: **S**
    (1) G ↔ H
    (2) H
    (3) G ∨ ¬S

*Invalid*

| T | F |
|---|---|
| H | S |
| G | |

Premises:

  (1) G ↔ H      (2) H      (3) G ∨ ¬S      Conclusion:

      T     T         T           T     F        **S**
         T                           T        F
                                T

---

4.  Prove: **W**
    (1) W → J
    (2) W → T
    (3) J & T

*Invalid*

| T | F |
|---|---|
| J | F |
| T | |

Premises:

  (1) W → J      (2) W → T      (3) J & T      Conclusion:

      F     T        F     T        T     T        **W**
          T             T            T        F

---

5.  Prove: **¬E**
    (1) ¬B ∨ W
    (2) B → E
    (3) ¬W

*Invalid*

| T | F |
|---|---|
| E | W |
| | B |

Premises:

  (1) ¬B ∨ W      (2) B → E      (3) ¬W      Conclusion:

      F     F        F     T           F          **¬E**
      T              T            T            T
         T                                  F

**3.4, Exercise 7, pages 125–129** (*continued*)

**A.**    6. Prove: ¬N

       (1) ¬S ∨ M

       (2) S → N

       (3) ¬M

*Invalid*

| T | F |
|---|---|
| N | M |
|   | S |

Premises:                                                Conclusion:

(1) ¬S ∨ M      (2) S → N      (3) ¬M      ¬N



**B.**    1. Prove: ¬S

| | | |
|---|---|---|
| (1) | T & S ↔ R | P |
| (2) | ¬R | P |
| (3) | T | P |
| (4) | T & S → R | LB 1 |
| (5) | ¬(T & S) | TT 2, 4 |
| (6) | ¬T ∨ ¬S | DL 5 |
| (7) | ¬S | TP 6, 3 |

2. Prove: S

| | | |
|---|---|---|
| (1) | Q → R | P |
| (2) | P → Q | P |
| (3) | P ∨ T | P |
| (4) | T → S | P |
| (5) | ¬R | P |
| (6) | Q ∨ S | DS 3, 2, 4 |
| (7) | ¬Q | TT 1, 5 |
| (8) | S | TP 6, 7 |

3. Prove: ¬Q

       (1) T → Q

       (2) ¬T ∨ R

       (3) ¬R

*Invalid*

| T | F |
|---|---|
| Q | T |
|   | R |

Premises:                                                Conclusion:

(1) T → Q      (2) ¬T ∨ R      (3) ¬R      ¬Q



4. Prove: S

| | | |
|---|---|---|
| (1) | R ∨ S | P |
| (2) | ¬P | P |
| (3) | Q ∨ ¬R | P |
| (4) | P ↔ Q | P |
| (5) | Q → P | LB 4 |
| (6) | ¬Q | TT 5, 2 |
| (7) | ¬R | TP 3, 6 |
| (8) | S | TP 1, 7 |

**B.**   5. Prove: T
     (1) ¬(P ∨ Q)
     (2) P ∨ R
     (3) T → R

*Invalid*

| T | F |
|---|---|
| R | P |
|   | Q |
|   | T |

Premises:

Conclusion:

(1) ¬(P ∨ Q)

   F   F
    └─F─┘

    T

(2) P ∨ R

   F   T
   └─T─┘

(3) T → R

   F   T
   └─T─┘

T
F

---

               (a)    or      (b)    or      (c)

6. Prove: ¬R
     (1) P → T
     (2) Q → S
     (3) S ∨ R
     (4) P ∨ ¬Q

*Invalid*

(a)

| T | F |
|---|---|
| R |   |
| P |   |
| T |   |
| Q |   |
| S |   |

(b)

| T |   | F |
|---|---|---|
| R | S | P |
|   | T | Q |

(See note.)

(c)

| T |   | F |
|---|---|---|
| R | S | Q |
| P |   |   |
| T |   |   |

(See note.)

Conclusion:

Premises: (a)

(1) P → T

   T   T
   └─T─┘

(2) Q → S

   T   T
   └─T─┘

(3) S ∨ R

   T   T
   └─T─┘

(4) P ∨ ¬Q

   T   T
       └F─┘
   └──T──┘

¬R

   T
 F┘

*Note.* These could be either true or false but it is necessary that the student have made a particular choice and have been consistent in its application.

---

7. Prove: ¬S
     (1) ¬(P & R)
     (2) Q → R
     (3) Q ∨ ¬S

*Invalid*

| T | F |
|---|---|
| S | P |
| Q |   |
| R |   |

Conclusion:

Premises:

(1) ¬(P & R)

   F   T
   └─F─┘
       T

T

(2) Q → R

   T   T
   └─T─┘

(3) Q ∨ ¬S

   T    T
       └F─┘
   └──T──┘

¬S

   T
 F┘

**3.4, Exercise 7, pages 125–129** (*continued*)

**B.**   8. Prove: ¬P

    (1)  Q → R                        *Invalid*

    (2)  ¬R → S

    (3)  ¬T ∨ ¬P

    (4)  (Q → S) → T

| T | F |
|---|---|
| P | S |
| Q | T |
| R |   |

Premises:                                                                 Conclusion:

(1) Q → R      (2) ¬R → S      (3) ¬T ∨ ¬P      (4) (Q → S) → T      ¬P

Q → R:  T T / T

¬R → S:  T F / F / T

¬T ∨ ¬P:  F T / T F / T

(Q → S) → T:  T F F / F / T

¬P:  T / F

9. Prove: R ∨ ¬Q

    (1)  S & ¬T        P

    (2)  T → P         P

    (3)  S → R         P

    (4)  ¬P → ¬Q       P

    (5)  S             S 1

    (6)  R             PP 3, 5

    (7)  R ∨ ¬Q        LA 6

10. Prove: ¬T

    (1)  ¬P                    *Invalid*

    (2)  ¬Q ∨ ¬R

    (3)  Q ↔ P

    (4)  T → R

| T | F |
|---|---|
| R | P |
| T | Q |

Premises:                                                                 Conclusion:

(1) ¬P      (2) ¬Q ∨ ¬R      (3) Q ↔ P      (4) T → R      ¬T

¬P:  F / T

¬Q ∨ ¬R:  F T / T F / T

Q ↔ P:  F F / T

T → R:  T T / T

¬T:  T / F

**C.**   1. Prove: ¬P ∨ M                        2. Prove: D

    (1)  S ∨ J        P                           (1)  D → N              *Invalid*

    (2)  S → ¬P       P                           (2)  N → S

    (3)  J → M        P                           (3)  S → E

    (4)  ¬P ∨ M       DS 1, 2, 3                  (4)  J → D

                                                  (5)  N

| T | F |
|---|---|
| N | D |
| S | J |
| E |   |

Premises:

(1) D → N      (2) N → S      (3) S ↔ E

D → N:  F T / T

N → S:  T T / T

S ↔ E:  T T / T

## A.  4. Prove: $\neg Q \rightarrow T$

| | | |
|---|---|---|
| (1) | $S \rightarrow R$ | P |
| (2) | $S \lor P$ | P |
| (3) | $P \rightarrow Q$ | P |
| (4) | $R \rightarrow T$ | P |
| (5) | $\neg Q$ | P |
| (6) | $R \lor Q$ | DS 2, 1, 3 |
| (7) | $R$ | TP 6, 5 |
| (8) | $T$ | PP 4, 7 |
| (9) | $\neg Q \rightarrow T$ | CP 5, 8 |

## 5. Prove: $P \rightarrow P \,\&\, Q$

| | | |
|---|---|---|
| (1) | $R \rightarrow T$ | P |
| (2) | $T \rightarrow \neg S$ | P |
| (3) | $(R \rightarrow \neg S) \rightarrow Q$ | P |
| (4) | $P$ | P |
| (5) | $R \rightarrow \neg S$ | HS 1, 2 |
| (6) | $Q$ | PP 3, 5 |
| (7) | $P \,\&\, Q$ | A 4, 6 |
| (8) | $P \rightarrow P \,\&\, Q$ | CP 4, 7 |

## 6. Prove: $S \rightarrow Q$

| | | |
|---|---|---|
| (1) | $R \rightarrow Q$ | P |
| (2) | $T \rightarrow R$ | P |
| (3) | $S \rightarrow T$ | P |
| (4) | $S$ | P |
| (5) | $S \rightarrow R$ | HS 3, 2 |
| (6) | $R$ | PP 5, 4 |
| (7) | $Q$ | PP 1, 6 |
| (8) | $S \rightarrow Q$ | CP 4, 7 |

## 7. Prove: $\neg(R \,\&\, S) \rightarrow T$

| | | |
|---|---|---|
| (1) | $\neg P$ | P |
| (2) | $\neg R \rightarrow T$ | P |
| (3) | $\neg S \rightarrow P$ | P |
| (4) | $\neg(R \,\&\, S)$ | P |
| (5) | $\neg R \lor \neg S$ | DL 4 |
| (6) | $T \lor P$ | DS 5, 2, 3 |
| (7) | $T$ | TP 1, 6 |
| (8) | $\neg(R \,\&\, S) \rightarrow T$ | CP 4, 7 |

## 8. Prove: $T \lor \neg S \rightarrow R$

| | | |
|---|---|---|
| (1) | $\neg R \rightarrow Q$ | P |
| (2) | $T \rightarrow \neg Q$ | P |
| (3) | $\neg S \rightarrow \neg Q$ | P |
| (4) | $T \lor \neg S$ | P |
| (5) | $\neg Q \lor \neg Q$ | DS 4, 3, 2 |
| (6) | $\neg Q$ | DP 5 |
| (7) | $R$ | TT 1, 6 |
| (8) | $T \lor \neg S \rightarrow R$ | CP 4, 7 |

## 9. Prove: $T \rightarrow \neg(P \lor Q)$

| | | |
|---|---|---|
| (1) | $\neg S \lor \neg P$ | P |
| (2) | $Q \rightarrow \neg R$ | P |
| (3) | $T \rightarrow S \,\&\, R$ | P |
| (4) | $T$ | P |
| (5) | $S \,\&\, R$ | PP 3, 4 |
| (6) | $S$ | S 5 |
| (7) | $\neg P$ | TP 1, 6 |
| (8) | $R$ | S 5 |
| (9) | $\neg Q$ | TT 2, 8 |
| (10) | $\neg P \,\&\, \neg Q$ | A 7, 9 |
| (11) | $\neg(P \lor Q)$ | DL 10 |
| (12) | $T \rightarrow \neg(P \lor Q)$ | CP 4, 11 |

## 10. Prove: $\neg Q \rightarrow T \,\&\, S$

| | | |
|---|---|---|
| (1) | $R \rightarrow S$ | P |
| (2) | $S \rightarrow Q$ | P |
| (3) | $R \lor (S \,\&\, T)$ | P |
| (4) | $\neg Q$ | P |
| (5) | $\neg S$ | TT 2, 4 |
| (6) | $\neg R$ | TT 1, 5 |
| (7) | $S \,\&\, T$ | TP 3, 6 |
| (8) | $T \,\&\, S$ | CL 7 |
| (9) | $\neg Q \rightarrow T \,\&\, S$ | CP 4, 8 |

## 11. Prove: $P \,\&\, Q \rightarrow \neg S \,\&\, T$

| | | |
|---|---|---|
| (1) | $\neg R \lor \neg S$ | P |
| (2) | $\neg T \rightarrow \neg P$ | P |
| (3) | $R \,\&\, \neg Q$ | P |
| (4) | $P \,\&\, Q$ | P |
| (5) | $P$ | S 4 |
| (6) | $T$ | TT 2, 5 |
| (7) | $Q$ | S 4 |
| (8) | $R$ | S 3 |
| (9) | $\neg S$ | TP 1, 8 |
| (10) | $\neg S \,\&\, T$ | A 9, 6 |
| (11) | $P \,\&\, Q \rightarrow \neg S \,\&\, T$ | CP 4, 10 |

## 12. Prove: $S \rightarrow P \lor Q$

| | | |
|---|---|---|
| (1) | $S \rightarrow T$ | P |
| (2) | $R \rightarrow P$ | P |
| (3) | $T \rightarrow R$ | P |
| (4) | $S$ | P |
| (5) | $T$ | PP 1, 4 |
| (6) | $R$ | PP 3, 5 |
| (7) | $P$ | PP 2, 6 |
| (8) | $P \lor Q$ | LA 7 |
| (9) | $S \rightarrow P \lor Q$ | CP 4, 8 |

**A.** **13. Prove:** ¬(P ∨ R) → T

(1) Q → P      P
(2) T ∨ S      P
(3) Q ∨ ¬S      P
(4) ¬(P ∨ R)      P
(5) ¬P & ¬R      DL 4
(6) ¬P      S 5
(7) ¬Q      TT 1, 6
(8) ¬S      TP 3, 7
(9) T      TP 2, 8
(10) ¬(P ∨ R) → T      CP 4, 9

**14. Prove:** E → K

(1) E ∨ F → G      P
(2) J → ¬G & ¬H      P
(3) J ∨ K      P
(4) E      P
(5) E ∨ F      LA 4
(6) G      PP 1, 5
(7) G ∨ H      LA 6
(8) ¬(¬G & ¬H)      DL 7
(9) ¬J      TT 2, 8
(10) K      TP 3, 9
(11) E → K      CP 4, 10

**15. Prove:** Q ↔ ¬P

(1) ¬(¬P & ¬Q)      P
(2) S → ¬Q      P
(3) ¬P ∨ S      P
(4) P ∨ Q      DL 1
(5) ¬Q      P
(6) ¬S      TT 2, 5
(7) ¬P      TP 3, 6
(8) Q → ¬P      CP 5, 7
(9) Q      P
(10) ¬P      TP 4, 9
(11) ¬P → Q      CP 9, 10
(12) Q ↔ ¬P      LB 8, 11

**16. Prove:** P → (Q → R)

(1) P & Q → R      P
(2) P      P
(3) Q      P
(4) P & Q      A 2, 3
(5) R      PP 1, 4
(6) Q → R      CP 3, 5
(7) P → (Q → R)      CP 2, 6

**17. Prove:** $x=0 \lor x=1 \rightarrow x^3-3x^2+2x=0$

(1) $x=0 \rightarrow x^2-x=0$      P
(2) $x=1 \rightarrow x^2-x=0$      P
(3) $x=2 \lor x^2-x=0 \rightarrow x^3-3x^2+2x=0$      P
(4) $x=0 \lor x=1$      P
(5) $x^2-x=0 \lor x^2-x=0$      DS 4, 1, 2
(6) $x^2-x=0$      DP 5
(7) $x=2 \lor x^2-x=0$      LA 6
(8) $x^3-3x^2+2x=0$      PP 3, 7
(9) $x=0 \lor x=1 \rightarrow x^3-3x^2+2x=0$      CP 4, 8

**18. Prove:** $y=2 \lor y=4 \rightarrow y<4 \lor y>3$

(1) $(y=4 \rightarrow x>y) \ \& \ x<z$      P
(2) $x>y \lor z>y \rightarrow y<4 \ \& \ y\neq3$      P
(3) $y=2 \rightarrow z>y$      P
(4) $y=4 \rightarrow x>y$      S 1
(5) $y=2 \lor y=4$      P
(6) $x>y \lor z>y$      DS 5, 3, 4
(7) $y<4 \ \& \ y\neq3$      PP 2, 6
(8) $y<4$      S 7
(9) $y<4 \lor y>3$      LA 8
(10) $y=2 \lor y=4 \rightarrow y<4 \lor y>3$      CP 5, 9

## 3.5, Exercise 8, pages 133–138 (continued)

**A.**

19. Prove: $y=2 \rightarrow x=y$

(1) $x \neq y \leftrightarrow x>y \lor y>x$    P
(2) $y \neq 2 \lor x=2$    P
(3) $x>y \lor y>x \rightarrow x \neq 2$    P
(4) $y=2$    P
(5) $x=2$    TP 2, 4
(6) $\neg(x>y \lor y>x)$    TT 3, 5
(7) $x \neq y \rightarrow x>y \lor y>x$    LB 1
(8) $x=y$    TT 7, 6
(9) $y=2 \rightarrow x=y$    CP 4, 8

20. Prove: $x=1 \rightarrow x \neq 2 \ \& \ y \neq 1$

(1) $x=1 \rightarrow xy=2$    P
(2) $x \neq 1 \rightarrow x+y \neq 3$    P
(3) $y=1 \lor x=2 \rightarrow \neg(x+y=3 \ \& \ xy=2)$    P
(4) $x=1$    P
(5) $xy=2$    PP 1, 4
(6) $x+y=3$    TT 2, 4
(7) $x+y=3 \ \& \ xy=2$    A 6, 5
(8) $\neg(y=1 \lor x=2)$    TT 3, 7
(9) $y \neq 1 \ \& \ x \neq 2$    DL 8
(10) $x \neq 2 \ \& \ y \neq 1$    CL 9
(11) $x=1 \rightarrow x \neq 2 \ \& \ y \neq 1$    CP 4, 10

**B.**

1. Prove: $T \rightarrow A$

(1) $\neg T \lor B$    P
(2) $B \rightarrow U$    P
(3) $U \rightarrow K$    P
(4) $T$    P
(5) $B$    TP 1, 4
(6) $B \rightarrow K$    HS 2, 3
(7) $K$    PP 6, 5
(8) $T \rightarrow K$    CP 4, 7

2. Prove: $L \rightarrow \neg M$

(1) $D \lor \neg L$    P
(2) $M \rightarrow \neg D$    P
(3) $L$    P
(4) $D$    TP 1, 3
(5) $\neg M$    TT 2, 4
(6) $L \rightarrow \neg M$    CP 3, 5

3. Prove: $G \rightarrow B$

(1) $P \rightarrow (D \rightarrow B)$    P
(2) $\neg G \lor P$    P
(3) $D$    P
(4) $G$    P
(5) $P$    TP 2, 4
(6) $D \rightarrow B$    PP 1, 5
(7) $B$    PP 3, 6
(8) $G \rightarrow B$    CP 4, 7

4. Prove: $J \rightarrow \neg M$

(1) $J \rightarrow R \lor S$    P
(2) $R \rightarrow \neg J$    P
(3) $M \rightarrow \neg S$    P
(4) $J$    P
(5) $R \lor S$    PP 1, 4
(6) $\neg R$    TT 2, 4
(7) $S$    TP 5, 6
(8) $\neg M$    TT 3, 7
(9) $J \rightarrow \neg M$    CP 4, 8

**B.** 5. Prove: B → W

| | | |
|---|---|---|
| (1) | B → C | P |
| (2) | C → W | P |
| (3) | B | P |
| (4) | B → W | HS 1, 2 |
| (5) | W | PP 3, 4 |
| (6) | B → W | CP 3, 5 |

**C.** 1. Prove: ¬S

| | |
|---|---|
| (1) ¬K → B | *Invalid* |
| (2) ¬B | |
| (3) K ∨ P | |
| (4) S → ¬P | |

| T | F |
|---|---|
| S | B |
| K | P |

Premises:

(1) ¬K → B

(2) ¬B

Conclusion:

(3) K ∨ P

(4) S → ¬P

¬S

3. Prove: S → C ∨ W

| | | |
|---|---|---|
| (1) | S → L | P |
| (2) | ¬L ∨ R | P |
| (3) | R → C | P |
| (3) | S | P |
| (5) | L | PP 1, 4 |
| (6) | R | TP 2, 5 |
| (7) | C | PP 3, 6 |
| (8) | C ∨ W | LA 7 |
| (9) | S → C ∨ W | CP 4, 8 |

2. Prove: ¬C ∨ ¬S

| | | |
|---|---|---|
| (1) | C & S → J | P |
| (2) | ¬J ∨ L | P |
| (3) | ¬L | P |
| (4) | ¬J | TP 2, 3 |
| (5) | ¬(C & S) | TT 1, 4 |
| (6) | ¬C ∨ ¬S | DL 5 |

4. Prove: T ∨ J → S

| | | |
|---|---|---|
| (1) | M | P |
| (2) | T → ¬M | P |
| (3) | J → S | P |
| (4) | T ∨ J | P |
| (5) | ¬M ∨ S | DS 4, 2, 3 |
| (6) | S | TP 5, 1 |
| (7) | T ∨ J → S | CP 4, 6 |

**3.5, Exercise 8, pages 133–138** (*continued*)

**C.**    5. Prove: ¬J

     (1) M ∨ J           *Invalid*

     (2) T → M

     (3) T

| T | F |
|---|---|
| M | |
| J | |
| T | |

Premises:                                           Conclusion:

   (1) M ∨ J      (2) T → M      (3) T        ¬J



**D.**    1. Prove: P

| | | |
|---|---|---|
| (1) ¬T ∨ ¬R | P |
| (2) S → T & R | P |
| (3) Q → S | P |
| (4) Q ∨ P | P |
| (5) ¬(T & R) | DL 1 |
| (6) ¬S | TT 2, 5 |
| (7) ¬Q | TT 3, 6 |
| (8) P | TP 4, 7 |

   2. Prove: P

| | | |
|---|---|---|
| (1) ¬S → ¬T | P |
| (2) T | P |
| (3) S → R & Q | P |
| (4) Q & R → P | P |
| (5) S | TT 1, 2 |
| (6) R & Q | PP 3, 5 |
| (7) Q & R | CL 6 |
| (8) P | PP 4, 7 |

   3. Prove: T → ¬S

| | | |
|---|---|---|
| (1) T → ¬R | P |
| (2) S → Q | P |
| (3) ¬Q ∨ R | P |
| (4)      T | P |
| (5)         ¬R | PP 1, 4 |
| (6)         ¬Q | TP 3, 5 |
| (7)         ¬S | TT 2, 6 |
| (8) T → ¬S | CP 4, 7 |

**E.**    1. Prove: $x = 0 \rightarrow y \neq z$

| | | |
|---|---|---|
| (1) $x = 0 \rightarrow x + y = y$ | P |
| (2) $y = z \rightarrow x + y \neq y$ | P |
| (3)        $x = 0$ | P |
| (4)        $x + y = y$ | PP 1, 3 |
| (5)        $y \neq z$ | TT 2, 4 |
| (6) $x = 0 \rightarrow y \neq z$ | CP 3, 5 |

**E.**    2. Prove: ¬S

     (1) P → ¬Q          *Invalid*

     (2) ¬P ∨ R

     (3) Q

     (4) S → ¬R

| T | F |
|---|---|
| S | P |
| Q | R |

Premises:                                          Conclusion:

   (1) P → ¬Q     (2) ¬P ∨ R     (3) Q     (4) S → ¬R     ¬S

**E.**  3. Prove: $x<y \quad \lor \quad x>y \quad \rightarrow \quad y=z \quad \lor \quad y>z$

| | | |
|---|---|---|
| (1) | $x<y \quad \rightarrow \quad y=z$ | P |
| (2) | $x>y \quad \rightarrow \quad y>z$ | P |
| (3) | $x<y \quad \lor \quad x>y$ | P |
| (4) | $y=z \quad \lor \quad y>z$ | DS 3, 1, 2 |
| (5) | $x<y \quad \lor \quad x>y \quad \rightarrow \quad y=z \quad \lor \quad y>z$ | CP 3, 4 |

4. Prove: $S \rightarrow \neg P$

(1) $P \rightarrow Q$
(2) $Q \rightarrow R$
(3) $\neg R \rightarrow \neg S$

*Invalid*

| T | F |
|---|---|
| S | |
| P | |
| Q | |
| R | |

Premises:

(1) $P \rightarrow Q$

(2) $Q \rightarrow R$

(3) $\neg R \rightarrow \neg S$

Conclusion:

$S \rightarrow \neg P$

5. Prove: $y=w \quad \rightarrow \quad x>y$

| | | |
|---|---|---|
| (1) | $x>y \quad \lor \quad y>x$ | P |
| (2) | $y>x \quad \rightarrow \quad x\neq 0$ | P |
| (3) | $x\neq 0 \quad \rightarrow \quad y\neq w$ | P |
| (4) | $y>x \quad \rightarrow \quad y\neq w$ | HS 2, 3 |
| (5) | $y=w$ | P |
| (6) | $y\not> w$ | TT 4, 5 |
| (7) | $x>y$ | TP 1, 6 |
| (8) | $y=w \quad \rightarrow \quad x>y$ | CP 5, 7 |

6. Prove: $z\neq w \quad \rightarrow \quad x=0 \quad \& \quad x>z$

| | | |
|---|---|---|
| (1) | $y=0 \quad \rightarrow \quad x>z$ | P |
| (2) | $x>z \quad \rightarrow \quad z=w$ | P |
| (3) | $y=0 \quad \lor \quad (x>z \quad \& \quad x=0)$ | P |
| (4) | $y=0 \quad \rightarrow \quad z=w$ | HS 1, 2 |
| (5) | $z\neq w$ | P |
| (6) | $y\neq 0$ | TT 4, 5 |
| (7) | $x>z \quad \& \quad x=0$ | TP 3, 6 |
| (8) | $x=0 \quad \& \quad x>z$ | CL 7 |
| (9) | $z\neq w \quad \rightarrow \quad x=0 \quad \& \quad x>z$ | CP 5, 8 |

▶ 3.6 *Consistency*

**3.6, Exercise 9, page 140**

1. $T \lor \neg T$
2. $J \& \neg J$
3. (1) $\neg(W \lor \neg W)$
   (2) $\neg W \& W$

not logically false
logically false
P    logically false
DL 1

**3.6, Exercise 9, page 140** (*continued*)

4.   (1)  (¬A & M)    &    (M → A)       P      logically false
     (2)  M → A      S 1
     (3)  ¬A & M      S 1
     (4)  M      S 3
     (5)  A      PP 2, 4
     (6)  ¬A      S 3
     (7)  A & ¬A      A 5, 6

5.   (1)  ¬(¬($x<2$ ∨ $x=2$) ∨ ¬($x \not< 2$ & $x \neq 2$))      P      logically false
     (2)  ($x<2$ ∨ $x=2$)    &    ($x \not< 2$ & $x \neq 2$)      DL 1
     (3)  $x \not< 2$    &    $x \neq 2$      S 2
     (4)  ¬($x<2$ ∨ $x=2$)      DL 3
     (5)  $x<2$ ∨ $x=2$      S 2
     (6)  ($x<2$ ∨ $x=2$)    &    ¬($x<2$ ∨ $x=2$)      A 5, 4

In these problems and those to follow, wherever a contradiction is to be derived there will be an endless list of possible contradictions and so an endless variety of acceptable proofs. The solutions here are not necessarily to be referred to others.

**3.6, Exercise 10, pages 142–143**

Notice that here we do not first announce some conclusion to be proved, but simply proceed to derive some contradiction.

**A.**  1.   (1)  ¬Q → R      P
         (2)  ¬R ∨ S      P
         (3)  ¬(P ∨ Q)      P
         (4)  ¬P → ¬S      P
         (5)  ¬P & ¬Q      DL 3
         (6)  ¬P      S 5
         (7)  ¬S      PP 4, 6
         (8)  ¬R      TP 2, 7
         (9)  ¬Q      S 5
       (10)  R      PP 1, 9
       (11)  R & ¬R      A 10, 8

   2.   (1)  T → P      P
         (2)  T & R      P
         (3)  Q → ¬R      P
         (4)  P ∨ S → Q      P
         (5)  T      S 2
         (6)  P      PP 1, 5
         (7)  P ∨ S      LA 6
         (8)  Q      PP 4, 7
         (9)  ¬R      PP 3, 8
       (10)  R      S 2
       (11)  R & ¬R      A 10, 9

   3.   (1)  R → R & Q      P
         (2)  ¬S ∨ R      P
         (3)  ¬T ∨ ¬Q      P
         (4)  S & T      P
         (5)  T      S 4
         (6)  ¬Q      TP 3, 5
         (7)  S      S 4
         (8)  R      TP 2, 7
         (9)  R & Q      PP 1, 8
       (10)  Q      S 9
       (11)  Q & ¬Q      A 10, 6

   4.   (1)  T ∨ ¬R      P
         (2)  ¬(R → S)      P
         (3)  T → S      P
         (4)     R      P
         (5)     T      TP 1, 4
         (6)     S      PP 3, 5
         (7)  R → S      CP 4, 6
         (8)  (R → S) & ¬(R → S)      A 7, 2

**A.**  5.
(1) $Q \rightarrow P$        P
(2) $\neg(P \lor R)$        P
(3) $Q \lor R$        P
(4) $\neg P \ \& \ \neg R$        DL 2
(5) $\neg P$        S 4
(6) $\neg Q$        TT 1, 5
(7) $\neg R$        S 4
(8) $Q$        TP 3, 7
(9) $Q \ \& \ \neg Q$        A 8, 6

6.
(1) $x = 1 \ \rightarrow \ y < x$        P
(2) $y < x \ \rightarrow \ y = 0$        P
(3) $\neg(y = 0 \ \lor \ x \neq 1)$        P
(4) $y \neq 0 \ \& \ x = 1$        DL 3
(5) $x = 1$        S 4
(6) $y < x$        PP 1, 5
(7) $y = 0$        PP 2, 6
(8) $y \neq 0$        S 4
(9) $y = 0 \ \& \ y \neq 0$        A 7, 8

7.
(1) $x = y \ \rightarrow \ x < 4$        P
(2) $x \nless 4 \ \lor \ x < z$        P
(3) $\neg(x < z \ \lor \ x \neq y)$        P
(4) $x \nless z \ \& \ x = y$        DL 3
(5) $x = y$        S 4
(6) $x < 4$        PP 1, 5
(7) $x < z$        TP 2, 6
(8) $x \nless z$        S 4
(9) $x < z \ \& \ x \nless z$        A 7, 8

8.
(1) $2 \times 5 = 5 + 5 \ \leftrightarrow \ 2 \times 6 = 6 + 6$        P
(2) $3 \times 4 = 10 \ \leftrightarrow \ 4 \times 3 = 10$        P
(3) $3 \times 4 = 10 \ \lor \ 2 \times 6 = 6 + 6$        P
(4) $2 \times 5 \neq 5 + 5 \ \& \ 4 \times 3 \neq 10$        P
(5) $2 \times 6 = 6 + 6 \ \rightarrow \ 2 \times 5 = 5 + 5$        LB 1
(6) $3 \times 4 = 10 \ \rightarrow \ 4 \times 3 = 10$        LB 2
(7) $4 \times 3 = 10 \ \lor \ 2 \times 5 = 5 + 5$        DS 3, 6, 5
(8) $2 \times 5 \neq 5 + 5$        S 4
(9) $4 \times 3 = 10$        TP 7, 8
(10) $4 \times 3 \neq 10$        S 4
(11) $4 \times 3 = 10 \ \& \ 4 \times 3 \neq 10$        A 9, 10

9.
(1) $x < y \ \rightarrow \ x \neq y$        P
(2) $y > z \ \rightarrow \ z \nless y$        P
(3) $x = y \ \& \ y > z$        P
(4) $x < y \ \lor \ z > y$        P
(5) $x = y$        S 3
(6) $x \nless y$        TT 1, 5
(7) $z < y$        TP 4, 6
(8) $y \ngtr z$        TT 2, 7
(9) $y > z$        S 3
(10) $y > z \ \& \ y \ngtr z$        A 9, 8

10.
(1) $x = 0 \ \leftrightarrow \ x + y = y$        P
(2) $x > 1 \ \& \ x = 0$        P
(3) $x + y = y \ \rightarrow \ x \ngtr 1$        P
(4) $x = 0 \ \rightarrow \ x + y = y$        LB 1
(5) $x = 0 \ \rightarrow \ x \ngtr 1$        HS 4, 3
(6) $x = 0$        S 2
(7) $x > 1$        S 2
(8) $x \ngtr 1$        PP 5, 6
(9) $x > 1 \ \& \ x \ngtr 1$        A 7, 8

## 3.6, Exercise 11, pages 144–147

**A.**  1.
(1) $Q \ \& \ \neg S$
(2) $\neg(P \lor S)$
(3) $Q \rightarrow T$

*Consistent*

| T | F |
|---|---|
| Q | P |
| T | S |

**3.6, Exercise 11, pages 144–147** (*continued*)

(1) Q & ¬S      (2) ¬(P ∨ S)      (3) Q → T

```
   T   F              F   F              T   T
    └T┘                └F┘                └T┘
   └T┘              T──┘
```

                               (a)    or    (b)    or    (c)

**A.**   2.   (1) P → Q       *Consistent*

| T | F |   | T |   | F |   | T | F |
|---|---|---|---|---|---|---|---|---|
| P |   |   |   | S | P |   | R | P |
| Q |   |   |   |   | Q |   | S | Q |
| R |   |   |   |   | R |   |   |   |
| S |   |   |   |   |   |   |   |   |

(2) Q → R                       (See note.)
(3) ¬R ∨ S

(c)

(1) P → Q      (2) Q → R      (3) ¬R ∨ S

```
   F   F              F   T              T   T
    └T┘                └T┘             F┘    │
                                        └T┘
```

*Note.* This table indicates 'S' may be true or false. Therefore we have a combination of two tables: one with 'S' true, one with 'S' false. What this means in grading is that if the student has chosen to make 'P', 'Q', and 'R' all false, then it does not matter which choice he has made for S so long as he sticks to that particular choice.

3.   (1) T → R       *Consistent*       (a)             (b)
     (2) ¬R → S
     (3) S ∨ T

| T |   | F |   | T |   | F |
|---|---|---|---|---|---|---|
| T | S |   |   | S | R | T |
| R |   |   |   |   |   |   |

(b)

(1) T → R      (2) ¬R → S      (3) S ∨ T

```
   F   F              F   T              T   F
    └T┘             T┘    │              └T┘
                      └ T┘
```

4.   (1) ¬P ∨ ¬R      *Consistent*
     (2) ¬P → S
     (3) ¬S

| T | F |
|---|---|
| P | R |
|   | S |

(1) ¬P ∨ ¬R     (2) ¬P → S     (3) ¬S

   T   F       T   F         F

  F   T       F          T

    T          T

*Consistent*

|  | (a) |  |  | (b) |  |
|---|---|---|---|---|---|
| T |  | F | T |  | F |
|  |  | T | Q |  | P |
|  |  | R |  |  | R |
|  |  | Q |  |  | T |
|  |  | P |  |  |  |

**A.**   5.   (1) R → Q
         (2) P → Q
         (3) Q → ¬T

Table (b) indicates 'Q' true and 'T' false. 'R' and 'P' may be any of four combinations of true or false, say 'R' false and 'P' true:

(1) R → Q     (2) P → Q     (3) Q → ¬T

  F   T       T   T       T   F

    T         T          T

                     T

*Consistent*

| T |  | F |
|---|---|---|
| R | Q | P |

6. Let P = '3 × 5 = 12'
     Q = '6 + 8 = 11'
     R = '13 − 9 = 7'

(1) P → Q
(2) Q → R
(3) ¬P & R

(1) P → Q     (2) Q → R     (3) ¬P & R

  F   F       F   T       F   T

    T         T      T

                   T

*Consistent*

| T | F |
|---|---|
| M | S |
| J | E |

**B.**   1.   (1) M → J
         (2) M & ¬E
         (3) ¬(S ∨ E)

(1) M → J     (2) M & ¬E     (3) ¬(S ∨ E)

  T   T       T   F       F   F

    T         T        F

              T       T

2.   (1) A & ¬C      P
    (2) A → B       P
    (3) ¬C → ¬B    P
    (4) ¬C           S 1
    (5) ¬B           PP 3, 4
    (6) A             S 1
    (7) B             PP 2, 6
    (8) B & ¬B      A 7, 5

*Inconsistent*

3.   (1) J & ¬(T ∨ D)     P
    (2) C → D           P
    (3) K → T           P
    (4) J → C ∨ K       P
    (5) J               S 1
    (6) C ∨ K          PP 4, 5
    (7) ¬(T ∨ D)      S 1
    (8) T ∨ D         DS 6, 2, 3
    (9) (T ∨ D) & ¬(T ∨ D)    A 8, 7

*Inconsistent*

**3.6, Exercise 11, pages 144–147** (*continued*)

**B.**  4.  (1)  J & (S ∨ B)          *Consistent*

| T | F |
|---|---|
| J | B |
| S | C |

(2)  J → ¬B

(3)  S → ¬C

(1)  J & (S ∨ B)        (2)  J → ¬B          (3)  S → ¬C



5.  (1)  B → ¬(S & M)      P
    (2)  B & ¬N          P
    (3)  ¬M → N          P
    (4)  S              P
    (5)  B              S 2
    (6)  ¬(S & M)        PP 1, 5
    (7)  ¬S ∨ ¬M         DL 6
    (8)  ¬M             TP 7, 4
    (9)  N              PP 3, 8
    (10) ¬N             S 2
    (11) N & ¬N          A 9, 10
    *Inconsistent*

6.  (1)  A → C              P
    (2)  ¬(B & C)            P
    (3)  A ∨ (B & C)          P
    (4)  B                P
    (5)  A                TP 2, 3
    (6)  C                PP 1, 5
    (7)  B & C              A 4, 6
    (8)  (B & C) & ¬(B & C)    A 7, 2
    *Inconsistent*

7.  (1)  (B → C) → A          *Consistent*

| T | | F |
|---|---|---|
| A | C | B |
| | | D |

(2)  B → D

(3)  D → C

(4)  ¬A ∨ ¬D

(1)  (B → C) → A      (2)  B → D        (3)  D → C         (4)  ¬A ∨ ¬D



8.  (1)  ¬(C ∨ D)          *Consistent*

| T | F |
|---|---|
| | C |
| | D |
| | B |

(2)  B ↔ C

(3)  C ↔ D

(4)  ¬B

(1)  ¬(C ∨ D)        (2)  B ↔ C          (3)  C ↔ D           (4)  ¬B

**B.** 9. (1) $D \leftrightarrow \neg C$     P
     (2) $B \rightarrow A$     P
     (3) $\neg(A \vee \neg C)$     P
     (4) $D \vee B$     P
     (5) $\neg A \ \& \ \neg\neg C$     DL 3
     (6) $D \rightarrow \neg C$     LB 1
     (7) $A \vee \neg C$     DS 4, 6, 2
     (8) $\neg\neg C$     S 5
     (9) $A$     TP 7, 8
     (10) $\neg A$     S 5
     (11) $A \ \& \ \neg A$     A 9, 10      *Inconsistent*

10. (1) $B \rightarrow A$
     (2) $B \vee D$
     (3) $\neg(A \ \& \ C)$
     (4) $\neg A \leftrightarrow \neg C$

*Consistent*

| T | F |
|---|---|
| D | A |
|   | B |
|   | C |

(1) $B \rightarrow A$



(2) $B \vee D$



(3) $\neg(A \ \& \ C)$



(4) $\neg A \leftrightarrow \neg C$



11. Let $P$ = '$x=y$'    (1) $P \rightarrow \neg P$    *Consistent*
       $Q$ = '$x<y$'    (2) $Q \vee P$
               (3) $\neg Q \rightarrow Q$

| T | F |
|---|---|
| Q | P |

(1) $P \rightarrow \neg P$



(2) $Q \vee P$



(3) $\neg Q \rightarrow Q$



12. (1) $4x-y=1 \quad \& \quad x+y=4$     P
     (2) $x+y=4 \quad \rightarrow \quad x=1$     P
     (3) $4x-y=1 \quad \rightarrow \quad y=3$     P
     (4) $\neg(x=1 \quad \& \quad y=3)$     P
     (5) $x \neq 1 \quad \vee \quad y \neq 3$     DL 4
     (6) $4x-y=1$     S 1
     (7) $x+y=4$     S 1
     (8) $x=1$     PP 2, 7
     (9) $y=3$     PP 3, 6
     (10) $y \neq 3$     TP 5, 8
     (11) $y=3 \quad \& \quad y \neq 3$     A 9, 10     *Inconsistent*

13. Let $P$ = '$x=2$'    (1) $P \vee Q$    *Consistent*
       $Q$ = '$x=3$'    (2) $\neg P \vee \neg Q$

| T | F |
|---|---|
| P | Q |

(or

| T | F |
|---|---|
| Q | P |

)

**3.6, Exercise 11, pages 144–147** (*continued*)

(1) P ∨ Q

$$\begin{array}{cc} T & F \\ \multicolumn{2}{c}{\lfloor_T\rfloor} \end{array}$$

(2) ¬P ∨ ¬Q

$$\begin{array}{cc} T & F \\ F\rfloor & \lfloor T \\ \multicolumn{2}{c}{\lfloor_T\rfloor} \end{array}$$

**B.**   **14.** Let R = '2x+y=4'    (1) R ↔ S     *Consistent*

| | T | F |
|---|---|---|
| R | | |
| S | | |
| P | | |
| Q | | |

     S = 'x+2y=5'    (2) (R → S) → Q

     P = 'y=2'    (3) ¬Q ∨ P

     Q = 'x=1'    (4) P → R

(1) R ↔ S

$$\begin{array}{cc} T & T \\ \multicolumn{2}{c}{\lfloor_T\rfloor} \end{array}$$

(2) (R → S) → Q

$$\begin{array}{ccc} T & T & T \\ \lfloor_T\rfloor & & \\ & \lfloor_T\rfloor & \end{array}$$

(3) ¬Q ∨ P

$$\begin{array}{cc} T & T \\ F\rfloor & \\ \lfloor_T\rfloor & \end{array}$$

(4) P → R

$$\begin{array}{cc} T & T \\ \multicolumn{2}{c}{\lfloor_T\rfloor} \end{array}$$

**15.**   (1) x=y   →   x<z        P

      (2) x≮z   &   (x=y ∨ y<z)     P

      (3) y<z   →   x<z        P

      (4) x=y   ∨   y<z       S 2

      (5) x<z   ∨   x<z       DS 4, 1, 3

      (6) x<z            DP 5

      (7) x≮z           S 2

      (8) x<z   &   x≮z       A 6, 7     *Inconsistent*

**16.** Let D = '3+4=12'    (1) D ↔ C     *Consistent*

| | T | F |
|---|---|---|
| B | B | A |
| | | D |
| | | C |

     C = '3×4=22'    (2) D ∨ B

     B = '3+3=11'    (3) ¬(A ∨ C)

     A = '4+4=10'    (4) A → B

(1) D ↔ C

$$\begin{array}{cc} F & F \\ \multicolumn{2}{c}{\lfloor_T\rfloor} \end{array}$$

(2) D ∨ B

$$\begin{array}{cc} F & T \\ \multicolumn{2}{c}{\lfloor_T\rfloor} \end{array}$$

(3) ¬(A ∨ C)

$$\begin{array}{cc} F & F \\ \multicolumn{2}{c}{\lfloor_F\rfloor} \\ \multicolumn{2}{c}{T\rfloor} \end{array}$$

(4) A → B

$$\begin{array}{cc} F & T \\ \multicolumn{2}{c}{\lfloor_T\rfloor} \end{array}$$

▶ *3.7 Indirect Proof*

In giving an indirect proof, it should be pointed out explicitly that in following the first step specified in the center of page 147, the added premise must be indented, as indicated on page 130 is always done for added premises. This is the clear indication of dependence on the added premise. CP and RAA are the only methods of bringing something from a subordinate proof back out to the main proof, for they are the only methods of eliminating dependence on a used premise.

**3.7, Exercise 12, pages 150–152**

**A.**  1. Prove: ¬P

| | | |
|---|---|---|
| (1) ¬(P & Q) | | P |
| (2) P → R | | P |
| (3) Q ∨ ¬R | | P |
| (4) ¬P ∨ ¬Q | | DL 1 |
| (5) | P | P |
| (6) | ¬Q | TP 4, 5 |
| (7) | ¬R | TP 3, 6 |
| (8) | ¬P | TT 2, 7 |
| (9) | P & ¬P | A 5, 8 |
| (10) ¬P | | RAA 5, 9 |

2. Prove: ¬T

| | | |
|---|---|---|
| (1) T → ¬S | | P |
| (2) F → ¬T | | P |
| (3) S ∨ F | | P |
| (4) | T | P |
| (5) | ¬F | TT 2, 4 |
| (6) | S | TP 3, 5 |
| (7) | ¬S | PP 1, 4 |
| (8) | S & ¬S | A 6, 7 |
| (9) ¬T | | RAA 4, 8 |

3. Prove: R

| | | |
|---|---|---|
| (1) ¬(P & Q) | | P |
| (2) ¬R → Q | | P |
| (3) ¬P → R | | P |
| (4) ¬P ∨ ¬Q | | DL 1 |
| (5) | ¬R | P |
| (6) | Q | PP 2, 5 |
| (7) | ¬¬P | TT 3, 5 |
| (8) | ¬Q | TP 4, 7 |
| (9) | Q & ¬Q | A 6, 8 |
| (10) R | | RAA 5, 9 |

4. Prove: ¬(A & D)

| | | |
|---|---|---|
| (1) A → B ∨ C | | P |
| (2) B → ¬A | | P |
| (3) D → ¬C | | P |
| (4) | (A & D) | P |
| (5) | A | S 4 |
| (6) | B ∨ C | PP 1, 5 |
| (7) | ¬B | TT 2, 5 |
| (8) | C | TP 6, 7 |
| (9) | D | S 4 |
| (10) | ¬C | PP 3, 9 |
| (11) | C & ¬C | A 8, 10 |
| (12) ¬(A & D) | | RAA 4, 11 |

5. Prove: ¬E ∨ M

| | | |
|---|---|---|
| (1) S ∨ O | | P |
| (2) S → ¬E | | P |
| (3) O → M | | P |
| (4) | ¬(¬E ∨ M) | P |
| (5) | ¬E ∨ M | DS 1, 2, 3 |
| (6) | (¬E ∨ M) & ¬(¬E ∨ M) | A 4, 5 |
| (7) ¬E ∨ M | | RAA 4, 6 |

6. Prove: ¬T

| | | |
|---|---|---|
| (1) P ∨ Q | | P |
| (2) T → ¬P | | P |
| (3) ¬(Q ∨ R) | | P |
| (4) ¬Q & ¬R | | DL 3 |
| (5) | T | P |
| (6) | ¬P | PP 2, 5 |
| (7) | Q | TP 1, 6 |
| (8) | ¬Q | S 4 |
| (9) | Q & ¬Q | A 7, 8 |
| (10) ¬T | | RAA 5, 9 |

7. Prove: ¬(T ∨ S)

| | | |
|---|---|---|
| (1) ¬R ∨ ¬B | | P |
| (2) T ∨ S → R | | P |
| (3) B ∨ ¬S | | P |
| (4) ¬T | | P |
| (5) | T ∨ S | P |
| (6) | S | TP 4, 5 |
| (7) | R | PP 2, 5 |
| (8) | ¬B | TP 1, 7 |
| (9) | ¬S | TP 3, 8 |
| (10) | S & ¬S | A 6, 9 |
| (11) ¬(T ∨ S) | | RAA 5, 10 |

**3.7, Exercise 12, pages 150–152** (*continued*)

**A.    8. Prove:** ¬P

| | | |
|---|---|---|
| (1) P → ¬S | | P |
| (2) S ∨ ¬R | | P |
| (3) ¬(T ∨ ¬R) | | P |
| (4) | P | P |
| (5) | ¬S | PP 1, 4 |
| (6) | ¬R | TP 2, 5 |
| (7) | ¬T & ¬¬R | DL 3 |
| (8) | ¬¬R | S 7 |
| (9) | ¬R & ¬¬R | A 6, 8 |
| (10) ¬P | | RAA 4, 9 |

**9. Prove:** ¬S ∨ ¬T

| | | |
|---|---|---|
| (1) ¬P → ¬S | | P |
| (2) ¬P ∨ R | | P |
| (3) R → ¬T | | P |
| (4) | ¬(¬S ∨ ¬T) | P |
| (5) | ¬¬S & ¬¬T | DL 4 |
| (6) | ¬¬S | S 5 |
| (7) | ¬¬P | TT 1, 6 |
| (8) | R | TP 2, 7 |
| (9) | ¬T | PP 3, 8 |
| (10) | ¬¬T | S 5 |
| (11) | ¬T & ¬¬T | A 9, 10 |
| (12) ¬S ∨ ¬T | | RAA 4, 11 |

**10. Prove: R**

| | | |
|---|---|---|
| (1) T & R ↔ ¬S | | P |
| (2) ¬S → T | | P |
| (3) ¬R → ¬S | | P |
| (4) | ¬R | P |
| (5) | ¬S | PP 3, 4 |
| (6) | ¬S → T & R | LB 1 |
| (7) | T & R | PP 5, 6 |
| (8) | R | S 7 (See note.) |
| (9) | R & ¬R | A 8, 4 |
| (10) R | | RAA 4, 9 |

*Note.* Although this is the desired conclusion, the proof is not complete since this appears indented, that is, in a subordinate proof dependent on the added premise.

**11. Prove:** $\neg(y=1 \; \rightarrow \; x^2 \not> xy)$

| | | |
|---|---|---|
| (1) $x=1 \; \lor \; \neg(x+y=y \; \lor \; x \not> y)$ | | P |
| (2) $x>y \; \rightarrow \; x^2>xy \; \& \; y=1$ | | P |
| (3) $x \neq 1$ | | P |
| (4) $\neg(x+y=y \; \lor \; x \not> y)$ | | TP 1, 3 |
| (5) $x+y \neq y \; \& \; x>y$ | | DL 4 |
| (6) $x>y$ | | S 5 |
| (7) $x^2>xy \; \& \; y=1$ | | PP 2, 6 |
| (8) | $y=1 \; \rightarrow \; x^2 \not> xy$ | P |
| (9) | $x^2>xy$ | S 7 |
| (10) | $y \neq 1$ | TT 8, 9 |
| (11) | $y=1$ | S 7 |
| (12) | $y=1 \; \& \; y \neq 1$ | A 11, 10 |
| (13) $\neg(y=1 \; \rightarrow \; x^2 \not> xy)$ | | RAA 8, 12 |

**A.** 12. Prove: $\neg(x=2 \ \leftrightarrow\ x=y)$

    (1) $x<y \ \rightarrow\ xy=x$                         P

    (2) $x\neq y \ \ \& \ \ xy\neq x$                       P

    (3) $x \not< y \ \lor \ y=1 \ \rightarrow \ x=2$          P

    (4) $xy\neq x$                                 S 2

    (5) $x \not< y$                                TT 1, 4

    (6) $x \not< y \ \lor \ y=1$                    LA 5

    (7) $x=2$                              PP 3, 6

    (8)                $x=2 \ \leftrightarrow \ x=y$     P

    (9)                $x=2 \ \rightarrow \ .x=y$     LB 8

  (10)                $x=y$                PP 9, 7

  (11)                $x\neq y$                S 2

  (12)                $x=y \ \ \& \ \ x\neq y$     A 10, 11

  (13) $\neg(x=2 \ \leftrightarrow \ x=y)$          RAA 8, 12

13. Prove: $2x=12 \ \rightarrow \ y=4$

    (1) $2x+3y=24$                                       P

    (2) $(x=6 \ \rightarrow \ y=4) \ \lor \ 2x=12$        P

    (3) $(2x=12 \ \rightarrow \ x=6) \ \lor \ 2x+3y\neq 24$   P

    (4) $x\neq 6$                                       P

    (5)            $\neg(2x=12 \ \rightarrow \ y=4)$        P

    (6)            $2x=12 \ \rightarrow \ x=6$          TP 3, 1

    (7)            $2x\neq 12$                   TT 6, 4

    (8)            $x=6 \ \rightarrow \ y=4$           TP 2, 7

    (9)            $2x=12 \ \rightarrow \ y=4$        HS 6, 8

  (10)           $(2x=12 \ \rightarrow \ y=4) \ \ \& \ \ \neg(2x=12 \ \rightarrow \ y=4)$     A 9, 5

  (11) $2x=12 \ \rightarrow \ y=4$                         RAA 5, 10

14. Prove: $x=0$

    (1) $\neg(y\neq 1 \ \lor \ z\neq -1)$                     P

    (2) $(x<y \ \ \& \ \ x>z) \ \ \& \ \ z=-1 \ \rightarrow \ x=0$    P

    (3) $\neg(y=1 \ \lor \ x=0) \ \lor \ (x<y \ \ \& \ \ x>z)$    P

    (4) $y=1 \ \ \& \ \ z=-1$                        DL 1

    (5)               $x\neq 0$                           P

    (6)               $y=1$                           S 4

    (7)               $y=1 \ \lor \ x=0$              LA 6

    (8)               $x<y \ \ \& \ \ x>z$            TP 3, 7

    (9)               $z=-1$                      S 4

  (10)              $(x<y \ \ \& \ \ x>z) \ \ \& \ \ z=-1$     A 8, 9

  (11)               $x=0$                        PP 2, 10

  (12)               $x=0 \ \ \& \ \ x\neq 0$          A 11, 5

  (13) $x=0$                                   RAA 5, 12

**3.7, Exercise 12, pages 150–152** (*continued*)

**A.**  15.  Prove: $x=0$

(1) $y=1 \;\rightarrow\; x=0 \;\lor\; x>y$       P
(2) $z=-1 \;\rightarrow\; x=0 \;\lor\; x<z$       P
(3) $x \not> y$       P
(4) $x \not< z$       P
(5) $y=1 \;\lor\; z=-1$       P
(6)                    $x \neq 0$       P
(7)                    $x \neq 0 \;\&\; x \not> y$       A 6, 3
(8)                    $\neg(x=0 \;\&\; x>y)$       DL 7
(9)                    $y \neq 1$       TT 1, 8
(10)                    $z=-1$       TP 5, 9
(11)                    $x=0 \;\lor\; x<z$       PP 2, 10
(12)                    $x<z$       TP 11, 6
(13)                    $x<z \;\&\; x \not< z$       A 12, 4
(14) $x=0$       RAA 6, 13

**B.**    Note that several of the proofs below require a rather tricky use of CP, DS, and DP.   The students are unlikely to discover such a proof and should not be expected or required to do so.

1. Prove: $\neg P$

(1) $\neg(P \;\&\; Q)$    P
(2) $P \rightarrow R$    P
(3) $Q \lor \neg R$    P
(4) $\neg P \lor \neg Q$    DL 1
(5)          $\neg R$    P
(6)          $\neg P$    TT 2, 5
(7) $\neg R \rightarrow \neg P$    CP 5, 6
(8)          $Q$    P
(9)          $\neg P$    TP 4, 8
(10) $Q \rightarrow \neg P$    CP 8, 9
(11) $\neg P \lor \neg P$    DS 3, 10,
(12) $\neg P$    DP 12

3. Prove: R

(1) $\neg(P \;\&\; Q)$    P
(2) $\neg R \rightarrow Q$    P
(3) $\neg P \rightarrow R$    P
(4) $\neg P \lor \neg Q$    DL 1
(5)          $\neg Q$    P
(6)          $R$    TT 2, 5
(7) $\neg Q \rightarrow R$    CP 5, 6
(8) $R \lor R$    DS 4, 3, 7
(9) $R$    DP 8

5. Prove: $\neg E \lor M$

(1) $S \lor O$    P
(2) $S \rightarrow \neg E$    P
(3) $O \rightarrow M$    P
(4) $\neg E \lor M$    DS 1, 2, 3

2. Prove: $\neg T$

(1) $T \rightarrow \neg S$    P
(2) $F \rightarrow \neg T$    P
(3) $S \lor F$    P
(4)          $S$    P
(5)          $\neg T$    TT 1, 4
(6) $S \rightarrow \neg T$    CP 4, 5
(7) $\neg T \lor \neg T$    DS 3, 6, 2
(8) $\neg T$    DP 7

6. Prove: $\neg T$

(1) $P \lor Q$    P
(2) $T \rightarrow \neg P$    P
(3) $\neg(Q \lor R)$    P
(4) $\neg Q \;\&\; \neg R$    DL 3
(5) $\neg Q$    S 4
(6) $P$    TP 1, 5
(7) $\neg T$    TT 2, 6

7. Prove: ¬(T ∨ S)

| | | |
|---|---|---|
| (1) | ¬R ∨ ¬B | P |
| (2) | T ∨ S → R | P |
| (3) | B ∨ ¬S | P |
| (4) | ¬T | P |
| (5) | � ¬R | P |
| (6) | ¬(T ∨ S) | TT 2, 5 |
| (7) | ¬T & ¬S | DL 6 |
| (8) | ¬S | S 7 |
| (9) | ¬R → ¬S | CP 5, 8 |
| (10) | ¬B | P |
| (11) | ¬S | TP 3, 10 |
| (12) | ¬B → ¬S | CP 10, 11 |
| (13) | ¬S ∨ ¬S | DS 1, 9, 12 |
| (14) | ¬S | DP 13 |
| (15) | ¬T & ¬S | A 4, 14 |
| (16) | ¬(T ∨ S) | DL 15 |

8. Prove: ¬P

| | | |
|---|---|---|
| (1) | P → ¬S | P |
| (2) | S ∨ ¬R | P |
| (3) | ¬(T ∨ ¬R) | P |
| (4) | ¬T & ¬¬R | DL 3 |
| (5) | ¬¬R | S 4 |
| (6) | S | TP 2, 5 |
| (7) | ¬P | TT 1, 6 |

9. Prove: ¬S ∨ ¬T

| | | |
|---|---|---|
| (1) | ¬P → ¬S | P |
| (2) | ¬P ∨ R | P |
| (3) | R → ¬T | P |
| (4) | ¬S ∨ ¬T | DS 2, 1, 3 |

11. Prove: $\neg(y=1 \rightarrow x^2 \not> xy)$

| | | |
|---|---|---|
| (1) | $x=1 \ \lor \ \neg(x+y=y \ \lor \ x \not> y)$ | P |
| (2) | $x>y \ \rightarrow \ x^2>xy \ \& \ y=1$ | P |
| (3) | $x \neq 1$ | P |
| (4) | $\neg(x+y=y \ \lor \ x \not> y)$ | TP 1, 3 |
| (5) | $x+y \neq y \ \& \ x>y$ | DL 4 |
| (6) | $x>y$ | S 5 |
| (7) | $x^2>xy \ \& \ y=1$ | PP 2, 6 |
| (8) | $y=1 \ \rightarrow \ x^2 \not> xy$ | P |
| (9) | $y=1$ | S 7 |
| (10) | $x^2 \not> xy$ | PP 8, 9 |
| (11) | $(y=1 \ \rightarrow \ x^2 \not> xy) \ \rightarrow \ x^2 \not> xy$ | CP 8, 10 |
| (12) | $x^2>xy$ | S 7 |
| (13) | $\neg(y=1 \ \rightarrow \ x^2 \not> xy)$ | TT 11, 12 |

*Note.* This proof is similar to an indirect proof to the extent that the negation of the conclusion is introduced; but no contradiction is derived here.

13. Prove: $2x=12 \ \rightarrow \ y=4$

| | | |
|---|---|---|
| (1) | $2x+3y=24$ | P |
| (2) | $(x=6 \ \rightarrow \ y=4) \ \lor \ 2x=12$ | P |
| (3) | $(2x=12 \ \rightarrow \ x=6) \ \lor \ 2x+3y \neq 24$ | P |
| (4) | $x \neq 6$ | P |
| (5) | $2x=12 \ \rightarrow \ x=6$ | TP 3, 1 |
| (6) | $2x \neq 12$ | TT 5, 4 |
| (7) | $x=6 \ \rightarrow \ y=4$ | TP 2, 6 |
| (8) | $2x=12 \ \rightarrow \ y=4$ | HS 5, 7 |

**3.7, Exercise 12, pages 150–152** (*continued*)

**B.**   14. Prove: $x=0$

    (1) $\neg(y\neq1 \quad \vee \quad z\neq-1)$           P

    (2) $(x<y \quad \& \quad x>z) \quad \& \quad z=-1 \quad \rightarrow \quad x=0$    P

    (3) $\neg(y=1 \quad \vee \quad x=0) \quad \vee \quad (x<y \quad \& \quad x>z)$    P

    (4) $y=1 \quad \& \quad z=-1$            DL 1

    (5) $z=-1$                S 4

    (6) $y=1$                 S 4

    (7) $y=1 \quad \vee \quad x=0$         LA 6

    (8) $x<y \quad \& \quad x>z$          TP 3, 7

    (9) $(x<y \quad \& \quad x>z) \quad \& \quad z=-1$    A 8, 5

   (10) $x=0$                PP 2, 9

  15. Prove: $x=0$

    (1) $y=1 \quad \rightarrow \quad x=0 \quad \vee \quad x>y$    P

    (2) $z=-1 \quad \rightarrow \quad x=0 \quad \vee \quad x<z$    P

    (3) $x\ngtr y$                P

    (4) $x\nless z$                P

    (5) $y=1 \quad \vee \quad z=-1$         P

    (6) $(x=0 \quad \vee \quad x>y) \quad \vee \quad (x=0 \quad \vee \quad x<z)$    DS 5, 1, 2

    (7)             $x\neq0$           P

    (8)             $x\neq0 \quad \& \quad x\ngtr y$    A 7, 3

    (9)             $\neg(x=0 \quad \vee \quad x>y)$    DL 8

   (10)             $x=0 \quad \vee \quad x<z$    TP 6, 9

   (11)             $x<z$        TP 10, 7

   (12) $x\neq0 \quad \rightarrow \quad x<z$      CP 7, 11

   (13) $x=0$                TT 12, 4

**C.**   1. Prove: $\neg(V \ \& \ R)$

    (1) $V \rightarrow T$         P

    (2) $T \rightarrow S$         P

    (3) $R \rightarrow \neg S$        P

    (4)      $V \ \& \ R$       P

  (4.5)      $V$            S 4 (See note.)

    (5)      $T$            PP 1, 4.5

    (6)      $S$            PP 2, 5

    (7)      $R$            S 4

    (8)      $\neg S$          PP 3, 7

    (9)      $S \ \& \ \neg S$     A 6, 8

   (11) $\neg(V \ \& \ R)$     RAA 4, 9

*Note.* We have numbered this line in this way in order to parallel the numbering of the erroneous proof.

**C.** 2. Prove: ¬T ∨ P

    (1) ¬T ∨ ¬R    P
    (2) ¬R → S    P
    (3) ¬S & ¬P    P
    (5) ¬S    S 3 (See note.)
    (6) ¬¬R    TT 2, 5
    (7) ¬T    TP 1, 6
    (8) ¬P    S 3
    (9) ¬T & ¬P    A 7, 8
    (10) ¬(T ∨ P)    DL 9

## Exercise 13 (Review Exercise), Chapter 3, pages 153–157

**A.** 1.    F & (¬A → ¬F)    *Possibly true*



| T | F |
|---|---|
| F |   |
| A |   |

2.    ¬(T ∨ ¬P) → T    *Possibly true*

(b)



(a)

| T | | | F |
|---|---|---|---|
| T | P | | |

(b)

| T | F |
|---|---|
|   | T |
|   | P |

3. H & ¬H    *Logically false* since this is a contradiction

4.    *Logically false*

    (1) (¬P & Q) & (Q → P)    P
    (2) ¬P & Q    S 1
    (3) Q → P    S 1
    (4) Q    S 2
    (5) P    PP 3, 4
    (6) ¬P    S 2
    (7) P & ¬P    A 5, 6

5.    (P & Q) → (P → Q ∨ R)    *Possibly true*

(b)



(a)

| T | | | F |
|---|---|---|---|
|   | Q | P | |
|   | R | | |

(b)

| T | | | F |
|---|---|---|---|
|   | P | Q | |
|   | R | | |

(c)

| T | | | F |
|---|---|---|---|
|   | P | R | |
|   | Q | | |

**Exercise 13 (Review Exercise), Chapter 3, pages 153–157** (*continued*)

6. (1) ¬[(¬P ∨ Q) ∨ (¬P ∨ ¬Q)]    P       *Logically false*
    (2) ¬(¬P ∨ Q) & ¬(¬P ∨ ¬Q)    DL 1
    (3) ¬(¬P ∨ Q)    S 2
    (4) ¬¬P & ¬Q    DL 3
    (5) ¬(¬P ∨ ¬Q)    S 2
    (6) ¬¬P & ¬¬Q    DL 5
    (7) ¬Q    S 4
    (8) ¬¬Q    S 6
    (9) ¬Q & ¬¬Q    A 7, 8

7. P → ¬P        *Possibly true*

| T | F |
|---|---|
|   | P |

8. (1) ¬(P ∨ ¬P)    P       *Logically false*
    (2) ¬P & ¬¬P    DL 1

9. Let P = 'x = 3'     (P → Q) & ¬P      *Possibly true*
       Q = 'x < 4'

| T | | F |
|---|---|---|
|   | Q | P |

10. Let P = '1 = 2'     ¬((P → Q) → ¬P)      *Possibly true*
       Q = '2 = 1'

| T | F |
|---|---|
| P |   |
| Q |   |

**B.**    1.   (1) W → ¬N       *Consistent*
        (2) W ↔ N
        (3) W & ¬N → ¬W

| T | F |
|---|---|
|   | W |
|   | T |

    (1) W → ¬N        (2) W ↔ N        (3) W & ¬N → ¬W

    2.   (1) A ∨ S       *Consistent*
        (2) A → X
        (3) ¬R
        (4) ¬R → ¬S

| T | F |
|---|---|
| A | S |
| X | R |

(1) A ∨ S     (2) A → X     (3) ¬R     (4) ¬R → ¬S



**B.** 3.   (1) ¬(R ∨ B)    P
        (2) B ∨ ¬S       P
        (3) ¬R → S      P
        (4) ¬R & ¬B     DL 1
        (5) ¬B           S 4
        (6) ¬S           TP 2, 5
        (7) ¬R           S 4
        (8) S            PP 3, 7
        (9) S & ¬S      A 8, 6
       *Inconsistent*

4.   (1) B → P            *Consistent*
     (2) ¬P → M ∨ E
     (3) ¬E
     (4) M & ¬B

| T | | F |
|---|---|---|
| M | P | B |
| | | E |

(1) B → P     (2) ¬P → M ∨ E     (3) ¬E     (4) M & ¬B



5.   (1) F & ¬S → R ∨ C     *Consistent*
     (2) R → H
     (3) C → H
     (4) ¬(F & ¬S)

*Note.* There are a large number of possible combinations of truth assignments that make all four premises true. One is to make all atomic sentences false.

(1) F & ¬S → R ∨ C     (2) R → H     (3) C → H     (4) ¬(F & ¬S)



6.   (1) E → G ∨ H     *Consistent*
     (2) J → ¬H
     (3) G & H
     (4) G → ¬E

| T | F |
|---|---|
| G | E |
| H | J |

**Exercise 13 (Review Exercise), Chapter 3, pages 153–157** (*continued*)

(1) E → G ∨ H    (2) J → ¬H    (3) G & H    (4) G → ¬E

F  T  T
 └─T─┘
└─T──┘

F   T
 └─F─┘
└─T─┘

T  T
 └─T─┘

T    F
 └─T─┘
└─T─┘

**B.**  7.  (1) P & Q                 P
       (2) P → ¬R               P
       (3) ¬R & S → ¬Q          P
       (4) Q → S                P
       (5) P                    S 1
       (6) ¬R                   PP 2, 5
       (7) Q                    S 1
       (8) S                    PP 4, 7
       (9) ¬R & S               A 6, 8
      (10) ¬Q                   PP 3, 9
      (11) Q & ¬Q               A 7, 10
      *Inconsistent*

8.  (1) U → W ∨ (R ∨ S)     P
    (2) W ∨ R → ¬U          P
    (3) U & ¬S              P
    (4) U                   S 3
    (5) W ∨ (R ∨ S)         PP 1, 4
    (6) ¬¬U                 DN 4
    (7) ¬(W ∨ R)            TT 2, 6
    (8) ¬W & ¬R             DL 7
    (9) ¬W                  S 8
   (10) R ∨ S               TP 5, 9
   (11) ¬S                  S 3
   (12) R                   TP 10, 11
   (13) ¬R                  S 8
   (14) R & ¬R              A 12, 13
   *Inconsistent*

9.  (1) P → Q ∨ R       *Consistent*
    (2) Q → ¬P
    (3) R → ¬P

| T | | F |
|---|---|---|
|   | Q | P |
|   | R |   |

(1) P → Q ∨ R       (2) Q → ¬P       (3) R → ¬P

F  T  F
 └─T─┘
└─T──┘

T    F
 └─T─┘
└─T─┘

F   F
 └─T─┘
└─T─┘

10.  (1) $x \neq y$  &  $y \neq z$           P
     (2) $\neg(x = z$  ∨  $x < z)$           P
     (3) $z = 2$  →  $y = z$                 P
     (4) $x < z$  ∨  $z = 2$                 P
     (5) $x \neq z$  &  $x \not< z$          DL 2
     (6) $x \not< z$                         S 5
     (7) $z = 2$                             TP 4, 6
     (8) $y = z$                             PP 3, 7
     (9) $y \neq z$                          S 1
    (10) $y = z$  &  $y \neq z$              A 8, 9
    *Inconsistent*

**C.** 1. Prove: P
    (1) S → D    P
    (2) D → P    P
    (3) S         P
    (4) S → P    HS 1, 2
    (5) P         PP 4, 3

2. Prove: ¬S
    (1) F ∨ R → ¬S        *Invalid*
    (2) I → ¬F
    (3) ¬I

| T | F |
|---|---|
| S | F |
|   | I |
|   | R |

Premises:

(1) F ∨ R → ¬S    (2) I → ¬F    (3) ¬I    Conclusion:
                                                     ¬S

3. Prove: M ∨ A
    (1) D → M             P
    (2) ¬D → A         P
    (3)      ¬(M ∨ A)    P
    (4)      ¬M & ¬A    DL 3
    (5)      ¬M            S 4
    (6)      ¬D            TT 1, 5
    (7)      A             PP 2, 6
    (8)      ¬A            S 4
    (9)      A & ¬A      A 7, 8
    (10) M ∨ A        RAA 3, 9

4. Prove: C ∨ B
    (1) ¬C ∨ M        *Invalid*
    (2) F
    (3) M → (F → B)

| T | F |
|---|---|
| F | C |
|   | B |
|   | M |

Premises:

(1) ¬C ∨ M    (2) F    (3) M → (F → B)    Conclusion:
                                               C ∨ B

5. Prove: P
    (1) P → D        *Invalid*
    (2) F ∨ D
    (3) ¬F

| T | F |
|---|---|
| D | F |
|   | P |

**Exercise 13 (Review Exercise), Chapter 3, pages 153–157** (*continued*)

Premises:                                                  Conclusion:

(1) P → D        (2) F ∨ D        (3) ¬F           P

      F   T            F   T           F            F

        T                T              T

**C.**    6. Prove: M

          (1) H & A                       *Invalid*

          (2) G → R & D

          (3) R & D → O

          (4) O → M

| T | F |
|---|---|
| H | M |
| A | O |
|   | G |
|   | R & D |

*Note.* For R & D to be false, either R is false, D is false, or both are false.

Premises:

(1) H & A        (2) G → R & D

     T   T            F   F   F

       T                 F

                        T

Conclusion:

(3) R & D → O       (4) O → M                 M

     F   F   F         F   F               F

       F               T

          T

7. Prove: ¬B → ¬Q

| (1) | R → N | P |
|---|---|---|
| (2) | K → B ∨ R | P |
| (3) | Q ∨ M → K | P |
| (4) | ¬N | P |
| (5) | ¬R | TT 1, 4 |
| (6) | ¬B | P |
| (7) | ¬B & ¬R | A 6, 5 |
| (8) | ¬(B ∨ R) | DL 7 |
| (9) | ¬K | TT 2, 8 |
| (10) | ¬(Q ∨ M) | TT 3, 9 |
| (11) | ¬Q & ¬M | DL 10 |
| (12) | ¬Q | S 11 |
| (13) | ¬B → ¬Q | CP 6, 12 |

8. Prove: ¬J ∨ C

| (1) | J ∨ S → C & V | P |
|---|---|---|
| (2) | ¬(¬J ∨ C) | P |
| (3) | J & ¬C | DL 2 |
| (4) | J | S 3 |
| (5) | J ∨ S | LA 4 |
| (6) | C & V | PP 1, 5 |
| (7) | C | S 6 |
| (8) | ¬C | S 4 |
| (9) | C & ¬C | A 7, 8 |
| (10) ¬J ∨ C | RAA 2, 9 |

**C.** 9. Prove: P                    *Invalid*

| T | F |
|---|---|
| S | P |
|   | Q |
|   | R |

(1) R ∨ Q → ¬P
(2) S → ¬Q
(3) ¬R & S

Conclusion:

P

F

Premises:

(1) R ∨ Q → ¬P     (2) S → ¬Q     (3) ¬R ∨ S

F  F     F          T     F           F    T
   F        T         T              T
      F        T          T              T
        T                  T                 T

10. Prove: B ∨ C                    *Invalid*

| T | F |
|---|---|
| D | A |
|   | B |
|   | C |

(1) A → B
(2) C → D
(3) A ∨ D

Conclusion:

B ∨ C

Premises:

(1) A → B     (2) C → D     (3) A ∨ D     B ∨ C

F     F        F     T        F    T        F    F
   T              T              T              F

11. Prove: G ∨ J → H ∨ K
(1) G → H                P
(2) J → K                P
(3)     G ∨ J            P
(4)     H ∨ K            DS 3, 1, 2
(5) G ∨ J → H ∨ K        CP 3, 4

12. Prove: C
(1) W → F                P
(2) F & C ↔ W            P
(3) ¬C → W               P
(4)     ¬C               P
(5)     W                PP 3, 4
(6)     W → F & C        LB 2
(7)     F & C            PP 6, 5
(8)     C                S 7
(9)     C & ¬C           A 8, 4
(10) C                   RAA 4, 9

13. Prove: C & ¬D                    *Invalid*

| T | | F |
|---|---|---|
| A | C |   |
| B |   |   |
| D |   |   |

(1) A & C → B
(2) ¬A ∨ (C ∨ D)
(3) A & B

**Exercise 13 (Review Exercise), Chapter 3, pages 153–157** (*continued*)

Premises:                                                                                    Conclusion:

(1) A & C → B        (2) ¬A ∨ (C ∨ D)        (3) A & B        C & ¬D



**C.**  14. Prove: **P & Q**

|      |                 |                   |
|------|-----------------|-------------------|
| (1)  | P ↔ Q           | P                 |
| (2)  | P ∨ Q           | P                 |
| (3)  | ¬(P & Q)        | P (See Note 1.)   |
| (4)  | ¬P ∨ ¬Q         | DL 3              |
| (5)  | P               | P (See Note 2.)   |
| (6)  | ¬Q              | TP 4, 5           |
| (7)  | P → Q           | LB 1              |
| (8)  | Q               | PP 6, 7           |
| (9)  | Q & ¬Q          | A 8, 6            |
| (10) | ¬P              | RAA 5, 9          |
| (11) | Q               | TP 2, 10          |
| (12) | Q → P           | LB 1              |
| (13) | P               | PP 12, 11         |
| (14) | P & ¬P          | A 13, 10          |
| (15) | P & Q           | RAA 3, 15         |

*Note* 1.   When no other method is clear, try RAA.

*Note* 2.   At this point it is necessary to introduce something, anything, for which (1), (2), or (4) can be used.   It could be ¬P, ¬¬P, ¬Q, ¬¬Q; it does not matter which.   The purpose will be an RAA.

A more subtle and elegant indirect proof is the following:

14. Prove: **P & Q**

|      |                 |              |
|------|-----------------|--------------|
| (1)  | P ↔ Q           | P            |
| (2)  | P ∨ Q           | P            |
| (3)  | ¬P              | P            |
| (4)  | Q               | TP 2, 3      |
| (5)  | Q → P           | LB 1         |
| (6)  | P               | PP 5, 4      |
| (7)  | P & ¬P          | A 6, 3       |
| (8)  | P               | RAA 3, 7     |
| (9)  | P → Q           | LB 1         |
| (10) | Q               | PP 8, 9      |
| (11) | P & Q           | A 8, 10      |

The following very elegant proof was written by a fifth-grade student.

**C.**   **14.** Prove: P & Q

    (1) P ↔ Q      P
    (2) P ∨ Q      P
    (3) P → Q      LB 1
    (4) Q → P      LB 1
    (5) P → P      HS 3, 4
    (6) P ∨ P      DS 2, 5, 4
    (7) P          DP 6
    (8) Q          PP 3, 7
    (9) P & Q      A 7, 8

**15.** Prove: ¬S → ¬R

    (1) R → ¬Q      P
    (2) R ∨ Q      P
    (3) R → S      P
    (4)    ¬S      P
    (5)    ¬R      TT 3, 4
    (6) ¬S → ¬R      CP 4, 5

**16.** Prove: M ↔ N

    (1) M ∨ N          *Invalid*
    (2) N ↔ (M → P)
    (3) P ∨ (N & Q)
    (4) Q ↔ (P → N)

| T | | F |
|---|---|---|
| N | P | M |
| Q | | |

*Note.* The parentheses in (2) and (4) are unnecessary.

Premises:

(1) M ∨ N
```
  F   T
  └─T─┘
```

(2) N ↔ M → P
```
  T    F    F
  │    └─T─┘
  └──T──┘
```

(3) P ∨ (N & Q)
```
  F    T   T
  │    └─T─┘
  └─T───┘
```

(4) Q ↔ P → N
```
  T    F   T
  │    └─T─┘
  └─T───┘
```

Conclusion:

M ↔ N
```
  F   T
  └─F─┘
```

**17.** Let:   P = '$x^2 - 5x + 6 = 0$'      Prove: R
           Q = '$x^2 - 7x + 12 = 0$'    (1) P ∨ Q      *Invalid*
           R = '$x = 3$'              (2) Q ↔ R ∨ S
           S = '$x = 4$'              (3) P ↔ R ∨ U
           U = '$x = 2$'

     (a)          (b)          (c)

| T | F | | T | T | F | | T | | F |
|---|---|---|---|---|---|---|---|---|---|
| P | R | | Q | U | R | | P | S | R |
| Q | | | S | | P | | U | | Q |
| S | | | | | | | | | |
| U | | | | | | | | | |

**Exercise 13 (Review Exercise), Chapter 3, pages 153–157** (*continued*)

Premises:     (a)                                                          Conclusion:

(1) P ∨ Q     (2) Q ↔ R ∨ S     (3) P ↔ R ∨ U                              R

<div style="text-align:center">

T  T          T  F  T            T  F  T                                    F

</div>

**C.   18.  Prove:** $z=3$

(1) $x<y$  &  $y<z$  →  $x<z$          P
(2) $(y<z$  →  $x<z)$  →  $z=3$        P
(3) $x<y$                             P
(4)       $y<z$                       P
(5)       $x<y$  &  $y<z$             A 3, 4
(6)       $x<z$                       PP 1, 5
(7) $y<z$  →  $x<z$                   CP 4, 6
(8) $z=3$                            PP 2, 7

19.  Let P = '$x=y$'          Prove: ¬S

Q = '$x=0$'          (1) P → ¬Q         *Invalid*

R = '$z=-1$'         (2) P & R

S = '$x<z$'          (3) S → ¬Q

| T | F |
|---|---|
| S | Q |
| P |   |
| R |   |

Premises:                                                          Conclusion:

(1) P → ¬Q      (2) P & R        (3) S → ¬Q                         ¬S

<div style="text-align:center">

T  F            T  T             T  F                               T

</div>

20.  Prove: ¬($2x+2y=8$  ↔  $y=2$)

(1) $y=2$  →  $4x+y=6$             P
(2) $y=3$  →  $2x+2y=8$            P
(3) ¬($4x+y=6$  ∨  $y≠3$)          P
(4) $4x+y≠6$  &  $y=3$             DL 3
(5) $y=3$                         S 4
(6) $2x+2y=8$                     PP 2, 5
(7)           $2x+2y=8$  ↔  $y=2$  P
(8)           $2x+2y=8$  →  $y=2$  LB 7
(9)           $y=2$               PP 8, 6
(10)          $4x+y≠6$            S 4
(11)          $y≠2$              TT 1, 10
(12)          $y=2$  &  $y≠2$    A 9, 11
(13) ¬($2x+2y=8$  ↔  $y=2$)       RAA 7, 12

**Chapter 3, Review Test, pages 157–160**

I.  a.  (1)  (P ↔ Q)   &   (P & ¬Q)      P               *Logically false*
        (2)  P & ¬Q                      S 1
        (3)  P                           S 2
        (4)  P ↔ Q                       S 1
        (5)  P → Q                       LB 4
        (6)  Q                           PP 5, 3
        (7)  ¬Q                          S 2
        (8)  Q & ¬Q                      A 6, 7

    b.  (1)  P & ¬[(P ∨ Q) ∨ R]      P          *Logically false*
        (2)  P                           S 1
        (3)  ¬[(P ∨ Q) ∨ R]            S 1
        (4)  ¬(P ∨ Q) & ¬R            DL 3
        (5)  ¬(P ∨ Q)                   S 4
        (6)  ¬P & ¬Q                    DL 5
        (7)  ¬P                          S 6
        (8)  P & ¬P                      A 2, 7

    c.  ((P → Q) → Q) → Q                    *Possibly true*



|   | T | F |
|---|---|---|
| P |   |   |
| Q |   |   |

    d.  Let P = '$x < y$'                    *Possibly true*
             Q = '$y = x$'

|   | T | F |
|---|---|---|
| P |   |   |
| Q |   |   |

        (¬P → Q)   &   (Q → P)



    e.  Let P = '$x = 3$'
             Q = '$x = y$'
        (1)  P & ¬(¬Q ∨ P)          P          *Logically false*
        (2)  ¬(¬Q ∨ P)              S 1
        (3)  Q & ¬P                   DL 2
        (4)  ¬P                        S 3
        (5)  P                         S 1
        (6)  P & ¬P                   A 5, 4

**Chapter 3, Review Test, pages 157–160** (*continued*)

**II.**   a.   (1) ¬P ∨ Q             *Consistent*

         (2) ¬R → ¬Q

         (3) P → R

|  (a)  |   |   | (b) |   |   |
|---|---|---|---|---|---|
| T | F |  | T |   | F |
|   | P |  | R | Q | P |
|   | Q |  |   |   |   |
|   | R |  |   |   |   |

(a)

    (1) ¬P ∨ Q        (2) ¬R → ¬Q        (3) P → R



   b.   (1) P → (Q → R)    P          *Inconsistent*

        (2) P & ¬R         P

        (3) Q               P

        (4) P               S 2

        (5) Q → R       PP 1, 4

        (6) R               PP 5, 3

        (7) ¬R            S 2

        (8) R & ¬R       A 6, 7

   c.   (1) P → Q            *Consistent*

        (2) R → S

        (3) P ∨ G

        (4) Q & S

|  (a)  |   |   | (b) |   |   |
|---|---|---|---|---|---|
| T |   |   | T |   | F |
| Q | P |   | Q | G |   |
| S | R |   | S | R |   |
| G |   |   | P |   |   |

    (1) P → Q     (2) R → S     (3) P ∨ G     (4) Q & S



   d.   (1) A → B & C      *Consistent*     Many possibilities.

        (2) D → A ∨ E                For example,

        (3) E → (C → F)

        (4) ¬D ∨ ¬F

| T | F |
|---|---|
|   | A |
|   | D |
|   | E |

    (1) A → B & C    (2) D → A ∨ E    (3) E → (C → F)    (4) ¬D ∨ ¬F

**II.** e.  (1)  P → Q ∨ ¬R     P        *Inconsistent*

    (2)  P → R     P

    (3)  P → ¬Q     P

    (4)  P     P

    (5)  ¬Q     PP 3, 4

    (6)  R     PP 2, 4

    (7)  Q ∨ ¬R     PP 1, 4

    (8)  Q     TP 6, 7

    (9)  Q & ¬Q     A 8, 5

f. Let P = '$x = 1$'     (1) ¬P      *Consistent*

    Q = '$x + y = y$'     (2) ¬Q → R

    R = '$x > 0$'     (3) ¬(R ∨ ¬S)

    S = '$y = 1$'     (4) Q ↔ ¬P

| T | F |
|---|---|
| Q | P |
| S | R |

(1) ¬P    (2) ¬Q → R    (3) ¬(R ∨ ¬S)    (4) Q ↔ ¬P



**III.** a. Prove: **A**

    (1)  M ∨ R     P

    (2)  M → A     P

    (3)  ¬R     P

    (4)  M     TP 1, 3

    (5)  A     PP 2, 4

b. Prove: ¬**A**

    (1)  A → B     P

    (2)  B → T     P

    (3)  R ∨ ¬T     P

    (4)  ¬R     P

    (5)  ¬T     TP 3, 4

    (6)  ¬B     TT 2, 5

    (7)  ¬A     TT 1, 6

c. Prove: **C ∨ ¬S**

    (1)  S → W        *Invalid*

    (2)  ¬W → E

    (3)  E → J

| T | F |
|---|---|
| S | J |
| W | E |

Premises:

Conclusion:

(1) S → W    (2) ¬W → E    (3) E → J    J ∨ ¬S

**Chapter 3, Review Test, pages 157–160** (*continued*)

**III.**  d.  (1) P → Q                    P
          (2) R ∨ ¬Q                    P
          (3) ¬P ∨ ¬R                    P
          (4)      ¬¬P                    P
          (5)      ¬R                    TP 3, 4
          (6)      ¬Q                    TP 2, 5
          (7)      ¬P                    TT 1, 6
          (8)      ¬P & ¬¬P            A 7, 4
          (9) ¬P                        RAA 4, 8

e. Prove: C ∨ ¬A
   (1) A → B                    *Invalid*
   (2) ¬B → C ∨ D
   (3) ¬D

| T | F |
|---|---|
| A | C |
| B | D |

Premises:

(1) A → B          (2) ¬B → C ∨ D          (3) ¬D          Conclusion:

                                                            C ∨ ¬A

f. Prove: ¬E
   (1) E → (G ∨ H)              *Invalid*
   (2) G → (H → K)
   (3) ¬L

(a)
| T | F |
|---|---|
| E | H | L |
| G |   |
| K |   |

(b)
| T | F |
|---|---|
| E | K | L |
| H |   | G |

(c)
| T | F |
|---|---|
| E |   | L |
| G |   | K |
|   |   | H |

Premises: (c)

(1) E → G ∨ H          (2) G → (H → K)          (3) ¬L          Conclusion:

                                                                 ¬E

g.  (1) L ∨ (M & N)                P
    (2) L → N                        P
    (3)      ¬N                        P
    (4)      ¬L                        TT 2, 3
    (5)      M & N                    TP 1, 4
    (6)      N                        S 5
    (7)      N & ¬N                    A 6, 3
    (8) N                            RAA 3, 7

**III.**  **h.**  (1)  A ∨ B → (C ∨ D → E)    P
     (2)      A                        P
     (3)      A ∨ B                    LA 2
     (4)      C ∨ D → E                PP 1, 3
     (5)        C                  P (See note.)
     (6)        C ∨ D              LA 5
     (7)        E                  PP 4, 6
     (8)      C → E                    CP 5, 7
     (9)  A → (C → E)                  CP 2, 8

*Note.* CP is again the obvious step since what is wanted, C → E, is in conditional form.

**i.**  (1)  P → Q                  P
    (2)  P → (Q → R)             P
    (3)  Q → (R → S)             P
    (4)  P → (R → S)             HS 1, 3
    (5)     P                 P
    (6)     R → S             PP 4, 5
    (7)       ¬S          P (See note.)
    (8)       ¬R          TT 6, 7
    (9)       Q → R       PP 2, 5
   (10)       ¬Q          TT 9, 8
   (11)       ¬P          TT 1, 10
   (12)       P & ¬P      A 5, 11
   (13)     S                 RAA 7, 12
   (14)  P → S                   CP 5, 13

*Note.* A premise containing only one atomic sentence is needed here. RAA is the most likely, thus since in this CP proof S is needed, ¬S is introduced. However, a direct proof is possible by first exhausting the possible uses of the P already introduced. See alternative proof.

**i. (Alternative)**
   (1)  P → Q                  P
   (2)  P → (Q → R)             P
   (3)  Q → (R → S)             P
   (4)     P                 P
   (5)     Q → R             PP 2, 4
   (6)     Q                 PP 1, 4
   (7)     R                 PP 5, 6
   (8)     R → S             PP 3, 6
   (9)     S                 PP 8, 7
  (10)  P → S                    CP 4, 9

**j.**  Prove: $x > y \;\rightarrow\; x = 5$
   (1)  $x - y = 2 \;\rightarrow\; (y = 3 \;\rightarrow\; x = 5)$    P
   (2)  $x > y \;\vee\; x - y = 2$            P
   (3)  $y = 3$                          P
   (4)     $x > y$                    P
   (5)     $x - y = 2$                TP 2, 4
   (6)     $y = 3 \;\rightarrow\; x = 5$    PP 1, 5
   (7)     $x = 5$                    PP 6, 3
   (8)  $x > y \;\rightarrow\; x = 5$        CP 4, 7

**Chapter 3, Review Test, pages 157–160** (*continued*)

**III.**    **k.** Let P='$x>y$'     Prove: R → ¬Q

         Q='$y=2$'      (1) ¬P → Q      *Invalid*

         R='$xy=y$'     (2) R ∨ S → U

         S='$x^2=y$'     (3) S → ¬Q

         U='$x=1$'      (4) ¬(P ∨ ¬U)

| T | F |
|---|---|
| R | S |
| Q | P |
| U |   |

Premises:                                                        Conclusion:

(1)   ¬P → Q      (2) R ∨ S → U                    R → ¬Q



(3) S → ¬Q       (4) ¬(P ∨ ¬U)



    **l.** Let P='$x=1$'      Prove: P

          Q='$x=y$'      (1) P ↔ Q      *Invalid*

          R='$x=1$'      (2) Q ∨ ¬R

          S='$x<y$'      (3) ¬R & S

| T | F |
|---|---|
| S | P |
|   | Q |
|   | R |

Premises:                                                       Conclusion:

(1) P ↔ Q      (2) Q ∨ ¬R     (3) ¬R & S      P



**IV.**   **a.**   (1) $x<y$   &   $y≠z$     P
              (2) $y≠z$               S 1

     **b.**   (1) S ↔ R              P
              (2) (S → R)   &   (R → S)    LB 1

     **c.**   (1) ¬R ∨ ¬R     P
              (2) ¬R             DP 1

     **d.**   (1) ¬S ∨ P       P
              (2)       ¬P        P
              (3)       ¬S        TP 1, 2
              (4) ¬P → ¬S     CP 2, 3

     **e.**   (1) ¬A → ¬(B ∨ ¬C)    P
              (2) ¬¬B                 P
              (3) B                   DN 2
              (4) B ∨ ¬C           LA 3
              (5) ¬¬A               TT 1, 4

     **f.**   (1) M → ¬(P ∨ Q)     P
              (2) ¬(P ∨ Q) → N     P
              (3) M → N           HS 1, 2

**IV.** g. (1) $x+y=3$ $\lor$ $(y=2 \to x+y=5)$     P

        (2) $(y=2 \to x+y=5)$ $\lor$ $x+y=3$     CL 1

h. (1) $R \to \lnot Q$          P

    (2) $Q \lor P$          P

    (3)      $\lnot(\lnot R \lor P)$      P

    (4)      $R \, \& \, \lnot P$      DL 3

    (5)      $R$      S 4

    (6)      $\lnot Q$      PP 1, 5

    (7)      $P$      TP 2, 6

    (8)      $\lnot P$      S 4

    (9)      $P \, \& \, \lnot P$      A 7, 8

   (10) $\lnot R \lor P$      RAA 3, 9

i. (1) $(x \neq 3 \ \lor \ y=2) \ \& \ x>y$     P

    (2) $(x=3 \to x=y) \to x \not> y$     P

    (3)     $y=2 \to x=y$     P

    (4)     $x \neq 3 \ \lor \ y=2$     S 1

    (5)       $x=3$     P

    (6)       $y=2$     TP 4, 5

    (7)     $x=3 \to y=2$     CP 5, 6

    (8)     $x=3 \to x=y$     HS 8, 3

    (9)     $x \not> y$     PP 2, 8

   (10)     $x>y$     S 1

   (11)     $x>y \ \& \ x \not> y$     A 10, 9

   (12) $\lnot(y=2 \to x=y)$     RAA 3, 11

j. (1) $\lnot R \lor S$         P

    (2) $\lnot R \to A \, \& \, B$         P

    (3) $S \to C$         P

    (4) $(A \, \& \, B) \lor C$     DS 1, 2, 3

# Truth Tables

▶ 4.1 *Truth Tables*

**4.1, Exercise 1, pages 167–168**

**A.**  1.  (1) B → C
(2) ¬B → ¬C
∴  B ∨ C

| | B* | C* | ¬B | ¬C | (1) B → C | (2) ¬B → ¬C | (con) B ∨ C |
|---|---|---|---|---|---|---|---|
| 1. | T | T | F | F | (T) | (T) | [T] |
| 2. | T | F | F | T | F | T | T |
| 3. | F | T | T | F | T | F | T |
| 4. | F | F | T | T | (T) | (T) | [F] |

*Invalid†*

2.  (1) E → O
(2) ¬E → Y
∴  E ∨ Y
*Valid*

| | E | O | Y | ¬E | (1) E → O | (2) ¬E → Y | (con) E ∨ Y |
|---|---|---|---|---|---|---|---|
| 1. | T | T | T | F | (T) | (T) | [T] |
| 2. | T | T | F | F | (T) | (T) | [T] |
| 3. | T | F | T | F | F | T | T |
| 4. | T | F | F | F | F | T | T |
| 5. | F | T | T | T | (T) | (T) | [T] |
| 6. | F | T | F | T | T | F | F |
| 7. | F | F | T | T | (T) | T | [T] |
| 8. | F | F | F | T | T | F | F |

3.  (1) ¬D ∨ L
∴  D → L
*Valid*

| | D | L | ¬D | (1) ¬D ∨ L | (con) D → L |
|---|---|---|---|---|---|
| 1. | T | T | F | (T) | [T] |
| 2. | T | F | F | F | F |
| 3. | F | T | T | (T) | [T] |
| 4. | F | F | T | (T) | [T] |

* There is a standard form for filling out the T's and F's of the atomic sentences in order to get all possible combinations for them. Require that your students use it so that anyone can easily read work presented by others. (1) Decide how many lines (n) the truth table must have. (See footnote, page 162 of text.) (2) For the column for the last atomic letter alternate T's and F's— T F T F . . . . (3) For the next to last atomic letter alternate groups of two T's and two F's—T T F F T T F F . . . . (4) For the third to last, alternate groups of four T's and four F's. (See B, page 168 of text.) In general there should be twice as many T's or F's in each group in any column as in the next column. In practice it is easier to start at the left with the first atomic letter, but it is easier to give a general explanation starting with the last atomic letter at the right.

Note also that there is a column for every distinct atomic sentence and every distinct molecular sentence. This latter means that there is a column for every occurrence of any connective except repetitions of identical column headings are avoided.

If the T's and F's are placed directly below the dominant connective heading the column it will be in a position corresponding to its position in a truth diagram. This also gives a clearer indication of what kind of sentence it is.

† The fourth line shows the premises both true but the conclusion false.

**A.** 4. (1) P → W

    (2) ¬P

   ∴  ¬W

   *Invalid*

|  |  | P | W | (1)<br>P → W | (2)<br>¬P | (con)<br>¬W |
|---|---|---|---|---|---|---|
| 1. |  | T | T | T | F | F |
| 2. |  | T | F | F | F | T |
| 3. |  | F | T | (T) | (T) | [F] |
| 4. |  | F | F | (T) | (T) | [T] |

5. (1) P & N

   ∴  P ↔ N

   *Valid*

|  |  | P | N | (1)<br>P & N | (con)<br>P ↔ N |
|---|---|---|---|---|---|
| 1. |  | T | T | (T) | [T] |
| 2. |  | T | F | F | F |
| 3. |  | F | T | F | F |
| 4. |  | F | F | F | T |

6. (1) D ↔ ¬R

   ∴  D ∨ R

   *Valid*

|  |  | D | R | ¬R | (1)<br>D ↔ ¬R | (con)<br>D ∨ R |
|---|---|---|---|---|---|---|
| 1. |  | T | T | F | F | T |
| 2. |  | T | F | T | (T) | [T] |
| 3. |  | F | T | F | (T) | [T] |
| 4. |  | F | F | T | F | F |

7. (1) G ∨ ¬E

   ∴  ¬E & G

   *Invalid*

|  |  | G | E | ¬E | (1)<br>G ∨ ¬E | (con)<br>¬E & G |
|---|---|---|---|---|---|---|
| 1. |  | T | T | F | (T) | [F] |
| 2. |  | T | F | T | (T) | [T] |
| 3. |  | F | T | F | F | F |
| 4. |  | F | T | T | (T) | [F] |

8.

|  |  | P | Q | (P ∨ Q) | ¬Q | (con)<br>(P ∨ Q) & ¬Q |
|---|---|---|---|---|---|---|
| 1. |  | T | T | T | F | F |
| 2. |  | T | F | T | T | (T) |
| 3. |  | F | T | T | F | F |
| 4. |  | F | F | F | T | F |

   *Valid*

9.

|  |  | P | Q | ¬Q | ¬P | (1)<br>¬Q → ¬P | (con)<br>P → Q |
|---|---|---|---|---|---|---|---|
| 1. |  | T | T | F | F | (T) | [T] |
| 2. |  | T | F | T | F | F | F |
| 3. |  | F | T | F | T | (T) | [T] |
| 4. |  | F | F | T | T | (T) | [T] |

   *Valid*

**4.1, Exercise 1, pages 167–168** (*continued*)

**A. 10.**

|  | P | Q | (2) ¬Q | (1) P → ¬Q | (con) ¬P |
|---|---|---|---|---|---|
| 1. | T | T | F | F | [F] |
| 2. | T | F | (T) | T | F |
| 3. | F | T | F | T | T |
| 4. | F | F | (T) | T | [T] |

*Invalid*

**B.**

|  | P | Q | R | (1) P → Q | (2) Q → R | (con) P → R |
|---|---|---|---|---|---|---|
| 1. | T | T | T | (T) | (T) | [T] |
| 2. | T | T | F | T | F | F |
| 3. | T | F | T | F | T | T |
| 4. | T | F | F | F | T | F |
| 5. | F | T | T | (T) | (T) | [T] |
| 6. | F | T | F | T | F | T |
| 7. | F | F | T | (T) | (T) | [T] |
| 8. | F | F | F | (T) | (T) | [T] |

**C.**

|  | P | Q | ¬P | ¬Q | P ∨ Q | (1) ¬(P ∨ Q) | (con) ¬P & ¬Q |
|---|---|---|---|---|---|---|---|
| 1. | T | T | F | F | T | F | F |
| 2. | T | F | F | T | T | F | F |
| 3. | F | T | T | F | T | F | F |
| 4. | F | F | T | T | F | (T) | [T] |

*Valid*

**D.**

|  | (1) P | (2) Q | (con) P & Q |
|---|---|---|---|
| 1. | (T) | (T) | [T] |
| 2. | T | F | F |
| 3. | F | T | F |
| 4. | F | F | F |

*Valid*

**E.**

|  | (con) P | Q | (1) P ∨ Q | (2) ¬P |
|---|---|---|---|---|
| 1. | T | T | T | F |
| 2. | T | F | T | F |
| 3. | F | [T] | (T) | (T) |
| 4. | F | F | F | T |

*Valid*

**F. 1.** Let P = '*x* = 3'

         Q = '*y* = 0'

     (1) P

     ∴ Q → P

|  | (1) P | Q | (con) Q → P |
|---|---|---|---|
| 1. | (T) | T | [T] |
| 2. | (T) | F | [T] |
| 3. | F | T | F |
| 4. | F | F | T |

*Valid*

**F.** 2. Let P = '$x=y$'

Q = '$y=1$'

(1) ¬P → P

(2) Q ∨ ¬P

∴ Q

|  |  |  | (con) |  | (1) |  | (2) |
|---|---|---|---|---|---|---|---|
|  | P | Q | ¬P | | ¬P → P | | Q ∨ ¬P |
| 1. | T | [T] | F | | (T) | | T |
| 2. | T | F | F | | T | | F |
| 3. | F | T | T | | F | | T |
| 4. | F | F | T | | F | | T |

*Valid*

3. Let P = '$x<5$'

Q = '$x=y$'

(1) P → ¬Q

(2) ¬Q & P

∴ ¬P & Q

|  |  |  |  |  | (1) | (2) | (con) |
|---|---|---|---|---|---|---|---|
|  | P | Q | ¬P | ¬Q | P → ¬Q | ¬Q & P | ¬P & Q |
| 1. | T | T | F | F | F | F | F |
| 2. | T | F | F | T | (T) | (T) | F |
| 3. | F | T | T | F | T | F | T |
| 4. | F | F | T | T | T | F | F |

*Invalid*

4. Let P = '$x<3$'

(1) P → ¬P

∴ ¬P

|  | (con) |  | (1) |
|---|---|---|---|
|  | P | ¬P | P → ¬P |
| 1. | T | F | F |
| 2. | F | [T] | (T) |

*Valid*

5. Let P = '$x=y$'

Q = '$y=2$'

(1) P → ¬P & Q

∴ ¬P

|  |  |  | (con) |  | (1) |
|---|---|---|---|---|---|
|  | P | Q | ¬P | ¬P & Q | P → ¬P & Q |
| 1. | T | T | F | F | F |
| 2. | T | F | F | F | F |
| 3. | F | T | [T] | T | (T) |
| 4. | F | F | [T] | F | (T) |

*Valid*

6. Let P = '$x=y$'

Q = '$y=2$'

(1) P → P & Q

∴ P

|  | (con) |  |  | (1) |
|---|---|---|---|---|
|  | P | Q | P & Q | P → P & Q |
| 1. | [T] | T | T | (T) |
| 2. | T | F | F | F |
| 3. | [F] | T | F | (T) |
| 4. | [F] | F | F | (T) |

*Invalid*

7. Let P = '$x<z$'

Q = '$x=y$'

(1) P → ¬Q

(2) ¬(¬P & Q)

∴ P ∨ Q

|  |  |  |  |  | (1) |  | (2) | (con) |
|---|---|---|---|---|---|---|---|---|
|  | P | Q | ¬P | ¬Q | P → ¬Q | ¬P & Q | ¬(¬P & Q) | P ∨ Q |
| 1. | T | T | F | F | F | F | T | T |
| 2. | T | F | F | T | (T) | F | (T) | [T] |
| 3. | F | T | T | F | T | T | F | T |
| 4. | F | F | T | T | (T) | F | (T) | [F] |

*Invalid*

## 4.1, Exercise 1, pages 167–168 (*continued*)

**F.**  8.  Let P = '3 < y'

| | | | | | | | | (1) | (2) | (con ) |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | P | Q | ¬P | ¬Q | ¬Q & ¬P | ¬P → Q | Q → ¬Q & ¬P | P ∨ Q |
| Q = 'x > y' | | 1. | T | T | F | F | F | T | F | T |
| (1) ¬P → Q | | 2. | T | F | F | T | F | (T) | (T) | [T] |
| (2) Q → (¬Q & ¬P) | | 3. | F | T | T | F | F | T | F | T |
| ∴ P ∨ Q | | 4. | F | F | T | T | T | F | T | F |

*Valid*

9.  Let P = 'x² = 4'

| | | | | | | | | (1) | (2) | (con) |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | P | Q | ¬P | ¬Q | Q ∨ ¬P | P → Q | ¬(Q ∨ ¬P) | Q & ¬Q |
| Q = 'x = 2' | | 1. | T | T | F | F | T | T | F | F |
| (1) P → Q | | 2. | T | F | F | T | F | F | T | F |
| (2) ¬(Q ∨ ¬P) | | 3. | F | T | T | F | T | T | F | F |
| ∴ Q & ¬Q | | 4. | F | F | T | T | T | T | F | F |

*Valid*   There is no line on which the premises are both true and conclusion false.  This is because the premises are inconsistent and so are never both true at the same time.

10.  Let P = 'x = 2'

| | | | | | | | | (1) | (2) | (3) | (con) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | P | Q | R | ¬P | ¬Q | P ∨ Q | R → ¬P | R → ¬Q | ¬R |
| Q = 'x < 2' | | 1. | T | T | T | F | F | T | F | F | F |
| R = 'x = 3' | | 2. | T | T | F | F | F | (T) | (T) | (T) | [T] |
| (1) P ∨ Q | | 3. | T | F | T | F | T | T | F | T | F |
| (2) R → ¬P | | 4. | T | F | F | F | T | (T) | (T) | (T) | [T] |
| (3) R → ¬Q | | 5. | F | T | T | T | F | T | T | F | F |
| ∴ ¬R | | 6. | F | T | F | T | F | (T) | (T) | (T) | [T] |
| | | 7. | F | F | T | T | T | F | T | T | F |
| | | 8. | F | F | F | T | T | F | T | T | T |

*Valid*

11.  Let P = 'x = y'

| | | | | | (con) | (1) | | (2) |
|---|---|---|---|---|---|---|---|---|
| | | | P | Q | ¬Q | P ↔ ¬Q | P & ¬Q | ¬(P & ¬Q) |
| Q = 'y = 1' | | 1. | T | T | F | F | F | T |
| (1) P ↔ ¬Q | | 2. | T | F | T | T | T | F |
| (2) ¬(P & ¬Q) | | 3. | F | T | [F] | (T) | F | (T) |
| ∴ ¬Q | | 4. | F | F | T | F | F | T |

*Invalid*

**F.** **12.** Let P = '*x* = *y*'
Q = '*x* < 5'
S = '*y* < 6'
(1) ¬P → Q
(2) ¬Q ∨ S
(3) P & ¬Q
∴ P → S

|  | P | Q | S | ¬P | ¬Q | (1) ¬P → Q | (2) ¬Q ∨ S | (3) P & ¬Q | (con) P → S |
|---|---|---|---|---|---|---|---|---|---|
| 1. | T | T | T | F | F | T | T | F | T |
| 2. | T | T | F | F | F | T | F | F | F |
| 3. | T | F | T | F | T | (T) | (T) | (T) | [T] |
| 4. | T | F | F | F | T | (T) | (T) | (T) | [F] |
| 5. | F | T | T | T | F | T | T | F | T |
| 6. | F | T | F | T | F | T | F | F | T |
| 7. | F | F | T | T | T | F | T | F | T |
| 8. | F | F | F | T | T | F | T | F | T |

*Invalid*

## ▶ 4.2 *Tautologies*

### 4.2, Exercise 2, page 170

**A.** These can be done in two long tables. First for 2 and 5 which have only one distinct atomic letter, then for the others which have two. 2, 3, 7, and 10 are tautologies.

|  | P | P ∨ P | 2. P → P ∨ P | P ↔ P | 5. (P ↔ P) ↔ P |
|---|---|---|---|---|---|
| 1. | T | T | T | T | T |
| 2. | F | F | T | T | F |

|  | P | Q | 1. P ↔ Q | P ∨ Q | Q ∨ P | 3. P ∨ Q ↔ Q ∨ P | P → Q | Q → P |
|---|---|---|---|---|---|---|---|---|
| 1. | T | T | T | T | T | T | T | T |
| 2. | T | F | F | T | T | T | F | T |
| 3. | F | T | F | T | T | T | T | F |
| 4. | F | F | T | F | F | T | T | T |

|  | 4. (P → Q) ↔ (Q → P) | 6. (P ∨ Q) → P | P & Q | 7. P & Q ↔ P ∨ Q | ¬P | ¬Q |
|---|---|---|---|---|---|---|
| 1. | T | T | T | T | F | F |
| 2. | F | T | F | T | F | T |
| 3. | F | F | F | T | T | F |
| 4. | T | T | F | T | T | T |

|  | ¬P ∨ ¬Q | 8. ¬P ∨ ¬Q → (P → Q) | P ∨ ¬Q | P → ¬Q |
|---|---|---|---|---|
| 1. | F | T | T | F |
| 2. | T | F | T | T |
| 3. | T | T | F | T |
| 4. | T | T | T | T |

## 4.2, Exercise 2, page 170 (*continued*)

|     | 9. P ∨ ¬Q → (P → ¬Q) | ¬P ∨ Q | 10. ¬P ∨ Q → (P → Q) |
| --- | --- | --- | --- |
| 1. | F | T | T |
| 2. | T | F | T |
| 3. | T | T | T |
| 4. | T | T | T |

**B.** 2.

| P | ¬P | P ∨ ¬P | a tautology |
| --- | --- | --- | --- |
| 1. T | F | T | |
| 2. F | T | T | |

| P | Q | ¬P | 1. P ∨ Q | Q ∨ P | 3. P ∨ Q → Q ∨ P | ¬P → Q | P → (¬P → Q) | 5. P → Q | Q → P |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| 1. T | T | F | T | T | T | T | T | T | T |
| 2. T | F | F | T | T | T | T | T | F | T |
| 3. F | T | T | T | T | T | T | T | T | F |
| 4. F | F | T | F | F | T | F | T | T | T |

|     | 6. (P → Q) → (Q → P) | (P → Q) ↔ Q | 7. [(P → Q) ↔ Q] → P | Q → (Q → P) | 8. P → [Q → (Q → P)] |
| --- | --- | --- | --- | --- | --- |
| 1. | T | T | T | T | T |
| 2. | T | T | T | T | T |
| 3. | F | T | T | F | T |
| 4. | T | F | T | T | T |

These are tautologies: 2, 3, 4, 5, 8, 9, 10

| P | Q | R | 4. (P ∨ Q) ∨ R | P → (P ∨ Q) ∨ R | P & Q | P ∨ R |
| --- | --- | --- | --- | --- | --- | --- |
| 1. T | T | T | T | T | T | T |
| 2. T | T | F | T | T | T | T |
| 3. T | F | T | T | T | F | T |
| 4. T | F | F | T | T | F | T |
| 5. F | T | T | T | T | F | T |
| 6. F | T | F | T | T | F | F |
| 7. F | F | T | T | T | F | T |
| 8. F | F | F | F | T | F | F |

|     | 9. P & Q → P ∨ R | Q ∨ R | P ↔ Q ∨ R | 10. P & Q → (P ↔ Q ∨ R) |
| --- | --- | --- | --- | --- |
| 1. | T | T | T | T |
| 2. | T | T | T | T |
| 3. | T | T | T | T |
| 4. | T | F | F | T |
| 5. | T | T | F | T |
| 6. | T | T | F | T |
| 7. | T | T | F | T |
| 8. | T | F | T | T |

▶ 4.3 *Tautological Implications and Tautological Equivalences*

**4.3, Exercise 3, page 172**

**A.**   1. $[\neg Q \ \& \ (\neg R \rightarrow Q)] \rightarrow R$
       2. $(P \rightarrow Q) \ \rightarrow \ \neg(P \ \& \ \neg Q)$
       3. $[(x=y \ \rightarrow \ x=5) \ \& \ (x=5 \ \rightarrow \ x<z)] \ \rightarrow \ [x=y \ \rightarrow \ x<z]$
       4. $[(C \ \& \ \neg D) \ \& \ (C \rightarrow \neg A) \ \& \ (D \lor \neg B)] \rightarrow \ \neg(A \lor B)$

**B.**   1. Prove: **Q**
         (1) **P**
         (2) $Q \lor \neg P$

       2. Prove: $x \not< 0 \ \lor \ y = x$
         (1) $\neg(x<0 \ \& \ y \neq x)$

       3. Prove: $S \rightarrow Q \ \& \ \neg T$
         (1) $Q \rightarrow T \lor R$
         (2) $\neg S$
         (3) $R \lor T \rightarrow S$

       4. Prove: **S**
         (1) $P \rightarrow Q$
         (2) $P \ \& \ \neg Q$

**4.3, Exercise 4, page 174**

**A.**   1, 2, 3.

       2. Let $P = {}^{\prime}x = 1{}^{\prime}$     (1) $P \lor \neg Q$
          $Q = {}^{\prime}x < 3{}^{\prime}$     (2) $\neg(Q \ \& \ P)$

|   |   |   |   | 1.1 | | |
|   |   |   |   | 2.1 | 1.2 | |
| P | Q | ¬Q | P ∨ ¬Q | Q → P | Q & P |
|---|---|---|---|---|---|
| 1. T | T | F | T | T | T |
| 2. T | F | T | T | T | F |
| 3. F | T | F | F | F | F |
| 4. F | F | T | T | T | F |

|   | 2.2 | | | | 3.1 | 3.2 |
| ¬(Q & P) | ¬P | ¬P & Q | P ∨ ¬Q → ¬P | P → ¬P & Q | | |
|---|---|---|---|---|
| 1. F | F | F | F | F |
| 2. T | F | F | F | F |
| 3. T | T | T | T | T |
| 4. T | T | F | T | T |

| 4. A | B | C | ¬A | ¬B | ¬C | A → ¬B | ¬(A → ¬B) | B & ¬C | ¬(A → ¬B) → C | B & ¬C → ¬A |
|---|---|---|---|---|---|---|---|---|---|---|
|   |   |   |   |   |   |   |   |   | 4.1 | 4.2 |
| 1. T | T | T | F | F | F | F | T | F | T | T |
| 2. T | T | F | F | F | T | F | T | T | F | F |
| 3. T | F | T | F | T | F | T | F | F | T | T |
| 4. T | F | F | F | T | T | T | F | F | T | T |
| 5. F | T | T | T | F | F | T | F | F | T | T |
| 6. F | T | F | T | F | T | T | F | T | T | T |
| 7. F | F | T | T | T | F | T | F | F | T | T |
| 8. F | F | F | T | T | T | T | F | F | T | T |

**4.3, Exercise 4, page 174** (*continued*)

Comparing columns
1. 1.1—1.2   same, therefore equivalent
2. 2.1—2.2   different, therefore not equivalent
3. 3.1—3.2   same, therefore equivalent
4. 4.1—4.2   same, therefore equivalent

**4.3, Exercise 5, page 174**

**A.**  1. $P \vee \neg Q \ \leftrightarrow \ Q \rightarrow P$
2. $x = 1 \ \vee \ x \not< 3 \ \leftrightarrow \ \neg(x < 3 \ \& \ x = 1)$
3. $P \vee \neg Q \rightarrow \neg P \ \leftrightarrow \ P \rightarrow \neg P \ \& \ Q$
4. $\neg(A \rightarrow \neg B) \rightarrow C \ \leftrightarrow \ B \ \& \ \neg C \rightarrow \neg A$

*Note.* Additional parentheses are unnecessary since the $\leftrightarrow$ is the strongest connective.

**4.3, Exercise 6, page 175**

**A.**  Using the tables for Exercise 4, page 174, we need only to add the following columns:

|  | 1. $P \vee \neg Q \ \leftrightarrow \ Q \rightarrow P$ | 2. $P \vee \neg Q \ \leftrightarrow \ \neg(Q \ \& \ P)$ | 3. $P \vee \neg Q \rightarrow \neg P \ \leftrightarrow \ P \rightarrow \neg P \ \& \ Q$ |
|---|---|---|---|
| 1. | T | F | T |
| 2. | T | T | T |
| 3. | T | F | T |
| 4. | T | T | T |

|  | 4. $\neg(A \rightarrow \neg B) \rightarrow C \ \leftrightarrow \ B \ \& \ \neg C \rightarrow \neg A$ |
|---|---|
| 1. | T |
| 2. | T |
| 3. | T |
| 4. | T |
| 5. | T |
| 6. | T |
| 7. | T |
| 8. | T |

1, 3, and 4 are tautologically equivalent.

▶ 4.4 *Summary*

**4.4, Exercise 7, pages 176–178**

**A.**  1. Let **E** = 'Two plus two equals five'
**C** = 'Columbus did discover America'
**L** = 'This is a logic example'

| T | F |
|---|---|
| C | E |
| L |  |

E → ¬C & L

F   T   T

(tree diagram with F, F, T branches resulting in T)

**A.**  2. Let P = 'One is two'

Q = 'Two times three is six'

R = 'Nine minus five is two'

S = 'One is less than two'

| T | F |
|---|---|
| Q | P |
| S | R |

¬P → (¬Q → ¬R & S)

F   T   F   T

(tree diagram: T, F, T branches leading to T, T, and final T)

3. Let M = 'The moon is made of green cheese'

L = 'Cows have four legs'

B = 'Thin china dishes are easily broken'

S = 'Spoons are used for eating food'

| T | F |
|---|---|
| L | M |
| B |   |
| S |   |

¬(M ∨ ¬L) → B ↔ S

F   T   T   T

(tree diagram resulting in T)

¬(M ∨ ¬L)  →  (B ↔ S)

F   T    T   T

(tree diagram resulting in T)

The first solution is preferred, the second assumes a comma before 'thin'.

**B.**  1.

| | | | | | | | | (2) | (1) | (con) |
|---|---|---|---|---|---|---|---|---|---|---|
| A | B | C | ¬A | ¬B | ¬C | A → B | A → C | (A → B) & (A → C) | ¬B ∨ ¬C | |
| 1. T | T | T | F | F | F | T | T | T | F | |
| 2. T | T | F | F | F | T | T | F | F | T | |
| 3. T | F | T | F | T | F | F | T | F | T | |
| 4. T | F | F | F | T | T | F | F | F | T | |
| 5. F | T | T | (T) | F | F | T | T | (T) | F | |
| 6. F | T | F | (T) | F | T | T | T | (T) | T | |
| 7. F | F | T | (T) | T | F | T | T | (T) | T | |
| 8. F | F | F | (T) | T | T | T | T | (T) | T | |

*Invalid*

**4.4, Exercise 7, pages 176–178** (*continued*)

**B.** 2.

|  | A | B | C | ¬A | ¬B | ¬C | A → B | A → C | (con)<br>(1)<br>(A → B) & (A → C) | (2)<br>¬B ∨ ¬C |
|---|---|---|---|---|---|---|---|---|---|---|
| 1. | T | T | T | F | F | F | T | T | T | F |
| 2. | T | T | F | F | F | T | T | F | F | T |
| 3. | T | F | T | F | T | F | F | T | F | T |
| 4. | T | F | F | F | T | T | F | F | F | T |
| 5. | F | T | T | T | F | F | T | T | T | F |
| 6. | F | T | F | [T] | F | T | T | T | (T) | (T) |
| 7. | F | F | T | [T] | T | F | T | T | (T) | (T) |
| 8. | F | F | F | [T] | T | T | T | T | (T) | (T) |

*Valid*

**C.** 1. Let P = '*x = y*'      Conditional

   (1) ¬P → P      (¬P → P) → P

   ∴ P

|  | P | ¬P | ¬P → P | (¬P → P) → P |
|---|---|---|---|---|
| 1. | T | F | T | T |
| 2. | F | T | F | T |

*Valid*

2. ((A → B & A) & (B ∨ ¬A → C)) → (A → C)

|  | A | B | C | ¬A | B & A | A → B & A | B ∨ ¬A | B ∨ ¬A → C | (A → B & A) | & (B ∨ ¬A → C) |
|---|---|---|---|---|---|---|---|---|---|---|
| 1. | T | T | T | F | T | T | T | T | T |
| 2. | T | T | F | F | T | T | T | F | F |
| 3. | T | F | T | F | F | F | F | T | F |
| 4. | T | F | F | F | F | F | F | T | F |
| 5. | F | T | T | T | F | T | T | T | T |
| 6. | F | T | F | T | F | T | T | F | F |
| 7. | F | F | T | T | F | T | T | T | T |
| 8. | F | F | F | T | F | T | T | F | F |

|  | A → C | ((A → B & A) & (B ∨ ¬A → C)) → (A → C) |
|---|---|---|
| 1. | T | T |
| 2. | F | T |
| 3. | T | T |
| 4. | F | T |
| 5. | T | T |
| 6. | T | T |
| 7. | T | T |
| 8. | T | T |

*Valid*

**D.** 1.

| C | D | D ∨ C | ¬(D ∨ C) | C & ¬(D ∨ C) | ¬(C & ¬(D ∨ C)) | |
|---|---|---|---|---|---|---|
| 1. T | T | T | F | F | T | |
| 2. T | F | T | F | F | T | |
| 3. F | T | T | F | F | T | |
| 4. F | F | F | T | F | T | tautology |

2.

| P | Q | P → Q | (P → Q) → P | |
|---|---|---|---|---|
| 1. T | T | T | T | |
| 2. T | F | F | T | |
| 3. F | T | T | F | |
| 4. F | F | T | F | not a tautology |

3.

| A | B | C | A & B | B ∨ C | A ↔ B ∨ C | A & B → (A ↔ B ∨ C) | |
|---|---|---|---|---|---|---|---|
| 1. T | T | T | T | T | T | T | |
| 2. T | T | F | T | T | T | T | |
| 3. T | F | T | F | T | T | T | |
| 4. T | F | F | F | F | F | T | |
| 5. F | T | T | F | T | F | T | |
| 6. F | T | F | F | T | F | T | |
| 7. F | F | T | F | T | F | T | |
| 8. F | F | F | F | F | T | T | tautology |

4. Let P = '$x = 3$'

Q = '$x = y$'     P & (¬Q → ¬P)

| P | Q | ¬Q | ¬P | ¬Q → ¬P | P & (¬Q → ¬P) | |
|---|---|---|---|---|---|---|
| 1. T | T | F | F | T | T | |
| 2. T | F | T | F | F | F | |
| 3. F | T | F | T | T | F | |
| 4. F | F | T | T | T | F | not a tautology |

**E.**

| P | Q | P & Q | 1.<br>P & Q → P | ¬Q | P & ¬Q | 2.<br>P & Q → P & ¬Q | ¬P | ¬P ∨ Q |
|---|---|---|---|---|---|---|---|---|
| 1. T | T | T | T | F | F | F | F | T |
| 2. T | F | F | T | T | T | T | F | F |
| 3. F | T | F | T | F | F | T | T | T |
| 4. F | F | F | T | T | F | T | T | T |

| | 3.<br>P & Q → ¬P ∨ Q | ¬Q → P | 4.<br>P & Q → (¬Q → P) | P ↔ Q | 5.<br>P & Q → (P ↔ Q) |
|---|---|---|---|---|---|
| 1. | T | T | T | T | T |
| 2. | T | T | T | F | T |
| 3. | T | T | T | F | T |
| 4. | T | F | T | T | T |

Thus, **P & Q** tautologically implies: 1, 3, 4, 5.

**4.4, Exercise 7, pages 176–178** (*continued*)

**F.**

| | P | Q | ¬P | ¬P ∨ Q | 1. ¬P ∨ Q → P | Q → P | 2. ¬P ∨ Q → (Q → P) | P → Q |
|----|---|---|----|--------|---------------|-------|---------------------|-------|
| 1. | T | T | F | T | T | T | T | T |
| 2. | T | F | F | F | T | T | T | F |
| 3. | F | T | T | T | F | F | F | T |
| 4. | F | F | T | T | F | T | T | T |

| | 3. ¬P ∨ Q → (P → Q) | ¬Q | ¬Q → ¬P | 4. P ∨ Q → (¬Q → ¬P) | ¬P & Q | 5. ¬P ∨ Q → ¬P & Q |
|----|---------------------|----|---------|----------------------|--------|---------------------|
| 1. | T | F | T | T | F | F |
| 2. | T | T | F | T | F | T |
| 3. | T | F | T | T | T | T |
| 4. | T | T | T | T | F | F |

Thus, ¬P ∨ Q tautologically implies: 3 and 4.

**G.**

| | P | Q | 1. P ∨ Q | ¬P | 2. P ∨ ¬P | 3. ¬P → P | 4. P → ¬P | ¬Q | Q ∨ ¬Q | 5. Q ∨ ¬Q → P | 5. (alternative) P ↔ [Q ∨ ¬Q → P] |
|----|---|---|----------|----|-----------|-----------|-----------|----|--------|----------------|-------------------------------------|
| 1. | T | T | T | F | T | T | F | F | T | T | T |
| 2. | T | F | T | F | T | T | F | T | T | T | T |
| 3. | F | T | T | T | T | F | T | F | T | F | T |
| 4. | F | F | F | T | T | F | T | T | T | F | T |

Comparing the truth table for each of 1.–5. with that for **P** we find the following are identical with **P**: 3 and 5. Thus 3 and 5 are tautologically equivalent to **P**.

The alternative method is illustrated for 5. Since this is an equivalence whose truth table shows it to be a tautology then 5. is tautologically equivalent to **P**.

**H.**

| | P | Q | P & Q | 1.a P & Q → P | 1.b P & Q ↔ P | P | ¬P | ¬¬P | 2.a P → ¬¬P | 2.b P ↔ ¬¬P |
|----|---|---|-------|---------------|---------------|---|----|-----|-------------|-------------|
| 1. | T | T | T | T | T | T | F | T | T | T |
| 2. | T | F | F | T | F | F | T | F | T | T |
| 3. | F | T | F | T | T | | | | | |
| 4. | F | F | F | T | T | | | | | |

| | P | Q | P ∨ Q | 3.a P → P ∨ Q | 3.b P ↔ P ∨ Q | Q ∨ P | 4.a P ∨ Q → Q ∨ P | 4.b P ∨ Q ↔ Q ∨ P |
|----|---|---|-------|---------------|---------------|-------|-------------------|-------------------|
| 1. | T | T | T | T | T | T | T | T |
| 2. | T | F | T | T | T | T | T | T |
| 3. | F | T | T | T | F | T | T | T |
| 4. | F | F | F | T | T | F | T | T |

|  | ¬Q | P & ¬Q | P → Q | ¬(P & ¬Q) | 5.a<br>(P → Q) → ¬(P & ¬Q) | 5.b<br>P → Q ↔ ¬(P & ¬Q) |
|---|---|---|---|---|---|---|
| 1. | F | F | T | T | T | T |
| 2. | T | T | F | F | T | T |
| 3. | F | F | T | T | T | T |
| 4. | T | F | T | T | T | T |

The 'a' columns show that each of the five laws is a tautological implication. In addition 2.b, 4.b, and 5.b show that DN, CL for disjunction, and the new law are also tautological equivalences.

## Chapter 4, Review Test, pages 178–179

I.   a. B → ¬Q          b. L ∨ Q → W          c. ¬L → ¬W ∨ Q



II.  The student has a choice of line-by-line comparison of premises and conclusion, as in a, c, and d, or by constructing the corresponding conditional and determining whether it is a tautological implication, as in b.

a.

|  | A | B | ¬A | ¬B | (1)<br>A → B | (conclusion)<br>¬B → ¬A |
|---|---|---|---|---|---|---|
| 1. | T | T | F | F | (T) | [T] |
| 2. | T | F | F | T | F | F |
| 3. | F | T | T | F | (T) | [T] |
| 4. | F | F | T | T | (T) | [T] |

*Valid*

b. Let P = '*x* < 4'          Conditional
       Q = '*x* = *y*'          P → (Q ∨ P)

|  | P | Q | (1)<br>Q ∨ P | (con)<br>P → Q ∨ P |
|---|---|---|---|---|
| 1. | T | T | T | T |
| 2. | T | F | T | T |
| 3. | F | T | T | T |
| 4. | F | F | F | T |

*Valid*

**Chapter 4, Review Test, pages 178–179** (*continued*)

**II. c.**

| A | B | C | B ∨ C | C & B | (1) A → B ∨ C | (2) C & B → C | (con) A → C |
|---|---|---|-------|-------|-----------|-----------|---------|
| 1. T | T | T | T | T | (T) | (T) | [T] |
| 2. T | T | F | T | F | (T) | (T) | [F] — *Invalid* |
| 3. T | F | T | T | F | (T) | (T) | [T] |
| 4. T | F | F | F | F | F | T | F |
| 5. F | T | T | T | T | (T) | (T) | [T] |
| 6. F | T | F | T | F | (T) | (T) | [T] |
| 7. F | F | T | T | F | (T) | (T) | [T] |
| 8. F | F | F | F | F | (T) | (T) | [T] |

**d.**

| A (con) | B | A ∨ B (1) |
|---------|---|-----------|
| 1. [T] | T | (T) |
| 2. [T] | F | (T) |
| 3. [F] | T | (T)  *Invalid* |
| 4. F | F | F |

**III. a.**

| A | ¬A | A ∨ ¬A |
|---|----|--------|
| 1. T | F | T |
| 2. F | T | T    *tautology* |

**b–e.**

**b.**

| A | B | ¬A | ¬A & B | B → A | ¬A & B ↔ (B → A) | ¬B | ¬A ∨ B | ¬(¬A ∨ B) |
|---|---|----|--------|-------|------------------|----|--------|-----------|
| 1. T | T | F | F | T | F | F | T | F |
| 2. T | F | F | F | T | F | T | F | T |
| 3. F | T | T | T | F | F | F | T | F |
| 4. F | F | T | F | T | F | T | T | F |

| ¬B & A | **c.** ¬(¬A ∨ B) → ¬B & A | ¬A → B | **d.** A → (¬A → B) | A → B |
|--------|-------------------------|--------|--------------------|-------|
| 1. F | T | T | T | T |
| 2. T | T | T | T | F |
| 3. F | T | T | T | T |
| 4. F | T | F | T | T |

**e.**

| (A → B) ↔ B | [(A → B) ↔ B] → A |
|-------------|-------------------|
| 1. T | T |
| 2. T | T |
| 3. T | F |
| 4. F | T |

The tautologies are: a, c, and d.

**IV.** a.

| | P | Q | ¬P | ¬Q | ¬P ∨ ¬Q | ¬P & ¬Q | ¬P ∨ ¬Q → ¬P & ¬Q |
|---|---|---|---|---|---|---|---|
| 1. | T | T | F | F | F | F | T |
| 2. | T | F | F | T | T | F | F |
| 3. | F | T | T | F | T | F | F |
| 4. | F | F | T | T | T | T | T |

not a tautological implication

b.

| | P | Q | R | P→Q | R→P | (P→Q) & (R→P) | R→Q | [(P→Q) & (R→P)] → (R→Q) |
|---|---|---|---|---|---|---|---|---|
| 1. | T | T | T | T | T | T | T | T |
| 2. | T | T | F | T | T | T | T | T |
| 3. | T | F | T | F | T | F | F | T |
| 4. | T | F | F | F | T | F | T | T |
| 5. | F | T | T | T | F | F | T | T |
| 6. | F | T | F | T | T | T | T | T |
| 7. | F | F | T | T | F | F | F | T |
| 8. | F | F | F | T | T | T | T | T |

a tautological implication

**V.**

| | P | Q | P→Q | ¬Q | ¬P | ¬Q→¬P | a. (P→Q) ↔ (¬Q→¬P) | P ∨ Q | b. P↔P ∨ Q |
|---|---|---|---|---|---|---|---|---|---|
| 1. | T | T | T | F | F | T | T | T | T |
| 2. | T | F | F | T | F | F | T | T | T |
| 3. | F | T | T | F | T | T | T | T | F |
| 4. | F | F | T | T | T | T | T | F | T |

c.

| | ¬P ∨ Q | (P→Q) ↔ ¬P ∨ Q |
|---|---|---|
| 1. | T | T |
| 2. | F | T |
| 3. | T | T |
| 4. | T | T |

1. and 3. are tautological equivalences.

# CHAPTER FIVE
# Terms, Predicates, and Universal Quantifiers

▶ 5.1 *Introduction*

**5.1, Exercise 1, pages 181–183**

**A.**

| | | |
|---|---|---|
| 1. valid | 8. invalid | 15. valid |
| 2. valid | 9. valid | 16. invalid |
| 3. invalid | 10. invalid | 17. valid |
| 4. invalid | 11. valid | 18. invalid |
| 5. valid | 12. valid | 19. valid |
| 6. valid | 13. invalid | 20. valid |
| 7. invalid | 14. valid | |

**B.**
1. All the team members received medals.
2. This sentence does not contain connectives.
3. Some mammals are not meat eaters.
4. Mike will attend the meeting.
5. None of the committee members is directly affected by the amendment.

▶ 5.2 *Terms*

**5.2, Exercise 2, pages 184–185**

**A.**
1. This exercise
2. China, the most populous country in the world
3. The game
4. $5+4$, $3+6$
5. seven, three plus three
6. William Shakespeare, the author of Macbeth
7. Two times three, seven times one
8. Mike, the president of our class
9. My mathematics grade
10. $2^3$, 8
11. Elizabeth II, the queen of England
12. Paris, the capital city of France

**B.**

| *name* | *description* |
|---|---|
| 1. The African continent, the North American continent | |
| 2. 5 | the square root of 25 |
| 3. John | The fastest runner on the squad |
| 4. C, XXXII | |

**B.** 5.  
    6. Susan  
    7.  
    8. Larry  
    9. The Andes  
   10.  
   11.  
   12. Canada

4 times 20, 3 times 30  
Today's discussion leader  
That ladder  
The treasurer of the senior class  
The longest mountain chain in the world  
11 + 11, 10 + 12  
That book  
The country to the north of the  
    United States

▶ **5.3** *Predicates*

## 5.3, Exercise 3, pages 186–187

**A.**
| | | |
|---|---|---|
| 1. walks slowly | 6. is walking at the front | 11. listens closely |
| 2. is speaking rapidly | 7. is entering now | 12. rides |
| 3. scores | 8. scores highest | 13. climbs steadily |
| 4. is ending quickly | 9. is running faster | 14. lives nearby |
| 5. is very intelligent | 10. is speaking | 15. sings beautifully |

**B.** For convenience one usually chooses obvious initial letters to refer to the predicates and terms. However, this is unnecessary; and when different predicates or terms have the same initial letter in English it is necessary to use different letters in translating them into logical symbols as done in 14 and 15 below. In such cases it is better to avoid that letter all together.

| | Let '___' be the predicate '_____' | | Let ___ be _____ | | Then: |
|---|---|---|---|---|---|
| 1. | *R* | is refracted | *r* | The light ray | *Rr* |
| 2. | *D* | dispersed quietly | *c* | The crowd | *Dc* |
| 3. | *S* | stirred | *b* | A faint breeze | *Sb* |
| 4. | *G* | grinned | *c* | The Cheshire Cat | *Gc* |
| 5. | *M* | meows | *m* | Mehitabel | *Mm* |
| 6. | *W* | walks gracefully | *s* | Susan | *Ws* |
| 7. | *E* | enters | *j* | Jack | *Ej* |
| 8. | *S* | scowls | *s* | Mr. Smith | *Ss* |
| 9. | *C* | cooks | *b* | Mrs. Brown | *Cb* |
| 10. | *S* | is studying | *c* | Cathy | *Sc* |
| 11. | *L* | is lovely | *p* | That painting | *Lp* |
| 12. | *J* | jumps faultlessly | *h* | John's horse | *Jh* |
| 13. | *W* | works nearby | *w* | Mr. White | *Ww* |
| 14. | *H* | was high overhead | *s* | The sun | *Hs* |
| 15. | *W* | is waiting patiently | *g* | George | *Wg* |

▶ **5.4** *Common Nouns as Predicates*

## 5.4, Exercise 4, pages 188–190

**A.** and **B.**  Students' choices of answers will be varied.

**C.**
| | | |
|---|---|---|
| 1. student | 5. number | 9. salesman |
| 2. colors | 6. nurse | 10. teacher |
| 3. anemone, animal | 7. father, engineer | |
| 4. redwoods, trees | 8. boy | |

**5.4, Exercise 4, pages 188–190** (*continued*)

**D.**   Common nouns occurring outside predicates are underlined.

|   |   |   |
|---|---|---|
| 2. man, books | 8. <u>object</u>, star | 12. <u>package</u> |
| 5. <u>volumes</u>, books | 9. <u>plant</u>, vegetable | 13. <u>phone</u> |
| 6. <u>plan</u> | 10. <u>clock</u> | 14. <u>student</u> |
| 7. planet | 11. <u>package</u>, clothing | 15. pianist |

**E.**

|   |   |   |
|---|---|---|
| 1. Two plus two | 5. 5 + 6 | 9. Three |
| 2. Jack | 6. This bird | 10. John |
| 3. Susan | 7. This rose |   |
| 4. Mary | 8. Jane |   |

**F.**   The students should set forth their choice of symbols explicitly as illustrated in B page 186 of text, but this is omitted below.

|   |   |   |
|---|---|---|
| 1. *Mb* | 6. *Bw* | 11. *Em* |
| 2. *St* | 7. *Sj* | 12. *Ld* |
| 3. *Mc* | 8. *Ca* | 13. *Ps* |
| 4. *Cd* | 9. *Mt* | 14. *Tc* |
| 5. *Rt* | 10. *No* | 15. *Mb* |

**G.**

|   |   |   |
|---|---|---|
| 2. *Lk* $\vee$ *Er* | 6. *Dt* $\vee$ *Ws* | 10. *Hl* $\rightarrow$ *Bm* |
| 3. *Ge* & *Gc* | 7. ¬*Ms* & ¬*Me* | 11. *Wj* $\leftrightarrow$ *Tj* |
| 4. *Et* $\rightarrow$ *Ag* | 8. *Lh* $\vee$ ¬*Pb* | 12. *Ms* $\leftrightarrow$ *Ps* |
| 5. *Sj* $\rightarrow$ ¬*Sb* | 9. *Ib* & *Cb* |   |

▶ 5.5  *Atomic Formulas and Variables*

**5.5, Exercise 5, pages 192–193**

**A.**

|   |   |   |
|---|---|---|
| 1. She | 5. It | 9. It |
| 2. It | 6. She | 10. It |
| 3. He | 7. She |   |
| 4. They | 8. He |   |

**B.**   1. *x* is Terry's friend
2. *y* is an insect,
   and so forth.

**5.5, Exercise 6, pages 193–194**

**A.**

|   |   |   |
|---|---|---|
| 1. yes | 5. yes | 9. no |
| 2. no (a negation) | 6. no | 10. no |
| 3. no | 7. yes | 11. yes |
| 4. yes | 8. yes | 12. yes |

**C.**   1, 2, 4, 5, 8, 12, 16, 17, 19, 20

**5.5, Exercise 7, page 195**

1. Defining:  *Tx* $\leftrightarrow$  *x* is 555 feet tall
               *w* = The Washington Monument

   In symbols:  *Tw*

2. Defining: $Txy \leftrightarrow$ $x$ was the teacher of $y$

   $t =$ Aristotle

   $d =$ Alexander the Great

   In symbols: $Ttd$

3. Defining: $Hx \leftrightarrow$ $x$ has one hundred members

   $s =$ the senate

   In symbols: $Hs$

4. Defining: $Sxy \leftrightarrow$ $x$ made the first solo flight to $y$

   $l =$ Lindbergh

   $p =$ Paris

   In symbols: $Slp$

5. Defining: $Mx \leftrightarrow$ $x$ is a very beautiful mountain

   $f =$ Fujiyama

   In symbols: $Mf$

6. Defining: $Dx \leftrightarrow$ $x$ drops 2,526 feet

   $h =$ the highest falls in Yosemite Park

   In symbols: $Dh$

7. Defining: $Cxy \leftrightarrow$ $x$ gets much coffee from $y$

   $u =$ The United States

   $b =$ Brazil

   In symbols: $Cub$

8. Defining: $Rx \leftrightarrow$ $x$ is red

   $g =$ The Golden Gate Bridge

   In symbols: $Rg$

9. Defining: $Nx \leftrightarrow$ $x$ is a field

   $r =$ the real number system

   In symbols: $Nr$

10. Defining: $Cxy \leftrightarrow$ $x$ is congruent to $y$

    $a =$ triangle $ABC$

    $d =$ triangle $DEF$

    In symbols: $Cad$

## 5.5, Exercise 8, pages 196–197

**A.** Atomic only: 3, 5, 6, 7, 9, 12, 13, 15

Could be molecular: 1, 2, 4, 8, 10, 11, 14

**B.** 1. Defining: $Ex \leftrightarrow$ $x$ is in equilibrium

   $Rx \leftrightarrow$ $x$ is at rest

   $Mx \leftrightarrow$ $x$ is moving at constant speed in a straight line

   $n =$ the Nautilus

   In symbols: $En \rightarrow Rn \lor Mn$

**5.5, Exercise 8, pages 196–197** (*continued*)

**B.**    2. Defining: *Lxy*  ↔  *x* loves *y*
                *m* = Martha
                *d* = Don

   In symbols: *Lmd* & *Ldm*

   3. Defining: *Rxy*  ↔  *x* gives *y* a ride
                *Lx*  ↔  *x* is late to his appointment
                *c* = Mrs. Clark
                *w* = Walter

   In symbols: *Rcw* ∨ *Lw*

   4. Defining: *Ex*  ↔  *x* is exceeded
                *Ox*  ↔  *x* is overcome
                *Rx*  ↔  *x* returns to its original form
                *e* = the elastic limit of the spring
                *m* = the molecular forces of the spring
                *s* = the spring

   In symbols: *Ee* → *Om* & ¬*Rs*

   5. Defining: *Bxy*  ↔  *x* is brother of *y*
                *Sxy*  ↔  *x* is sibling of *y*
                *Fxy*  ↔  *x* is father of *y*
                *Cxy*  ↔  *x* is cousin of *y*
                *j* = Joe
                *m* = Maria
                *l* = Mr. Lopez

   In symbols: ¬*Bjm* ∨ ¬*Smj* → ¬*Flj* & *Cjm*

▶ 5.6 *Universal Quantifiers*

**5.6, Exercise 9, pages 200–201**

   **A.**    1. Byrd is a United States Senator
             2. Mrs. Johnson is a teacher
                and so forth.

   **B.**    1. $(\forall x)(x$ is a United States senator)    False
             2. $(\forall z)(z$ is a teacher)                  False
                and so forth.

**5.6, Exercise 10, pages 202–203**

   **A.**    (1) $(\forall x)(x$ has a name)
             (2) $(\forall y)(y$ is subject to change)
             (3) $(\forall x)(x$ is the value of a variable)
             (4) $(\forall z)(z$ is not absolutely cold)
             (5) $(\forall y)(y$ belongs to a set)
             (6) $(\forall w)(w$ does not change)
             (7) $(\forall y)(y$ is not perfect)
             (8) $(\forall u)(u$ is an atom)
             (9) $(\forall y)(y$ is matter)
            (10) $(\forall y)(y$ is an idea)

   **B.**    (1) $(\forall z)(z > 0)$
             (2) $(\forall x)(x < x + 1)$
             (3) $(\forall x)(x$ is not divisible by zero)
             (4) $(\forall w)(w + 0 = w)$
             (5) $(\forall x)(x \not> x)$

**C.**  1. $(\forall x)(x$ does not welcome disaster$)$

2. $(\forall x)(x$ does not hear those sounds$)$

3. $(\forall x)(x$ does not like to be wrong$)$

4. $(\forall x)(x$ is not perfect$)$

5. $(\forall x)(x$ has certain minimum food requirements$)$

## 5.6, Exercise 11, pages 204–205

**A.**  (A)  1. $(\forall x)(Nx)$

2. $(\forall y)(Sy)$

3. $(\forall x)(Vx)$

4. $(\forall x)(\neg Az)$

5. $(\forall y)(By)$

6. $(\forall w)(\neg Cw)$

7. $(\forall y)(\neg Py)$

8. $(\forall u)(Au)$

9. $(\forall y)(My)$

10. $(\forall y)(Iy)$

(B)  1. $(\forall z)(Gz)$

2. $(\forall x)(Px)$  (See note.)

3. $(\forall x)(\neg Dx)$

4. $(\forall w)(Ew)$  (See note.)

5. $(\forall x)(\neg Gx)$  (See note.)

*Note.* These are one-place predicates in spite of the appearance of the variable more than once in the English or mathematical form.

(C)  1. $(\forall x)(\neg Wxz)$

2. $(\forall x)(\neg Hx)$, where $Hx \leftrightarrow x$ hears those sounds.

or, defining: $Hxy \leftrightarrow x$ hears $y$, $s =$ those sounds, in symbols: $(\forall x)(\neg Hxs)$

3. $(\forall x)(\neg Lx)$

4. $(\forall x)(\neg Px)$

5. $(\forall x)(Rx)$

**B.**  Defining:

In symbols:

1. $Lz \leftrightarrow x$ is living

$(\forall z)(Lz)$

*Note.* Or $(\forall z)Lz$. When the quantifier governs only one predicate, parentheses around that predicate are unnecessary. However, for more than one predicate with binary connectives the parentheses are necessary (see text, page 207, for example). With the single predicate, parentheses may be advisable for the student so that there is minimum change in proceeding to more complex cases.

2. $Wx \leftrightarrow x$ wishes for good luck

Domain restricted to people

$(\forall x)(Wz)$

*Note.* The choice of variables used in defining predicates in no way limits the choice that may be made in symbolizing formulas.

3.

$(\forall x)(x > x - 1)$

*Note.* Arithmetic predicates usually are not replaced by selected letters. However, it could be done. So here we could let '$G$' be the predicate 'is greater than one subtracted from itself'. Then we have: $(\forall x)(Gx)$.

4. $Ry \leftrightarrow y$ is right-handed

Domain restricted to people

$\neg(\forall u)(Ru)$

5.

$(\forall y)(y = y)$

6. $Nz \leftrightarrow z$ is a number

$(\forall z)(Nz)$

7. $Ix \leftrightarrow x$ is impossible

$(\forall x)(\neg Ix)$

8. $Ew \leftrightarrow w$ enjoys defeat

Domain restricted to people

$(\forall u)(\neg Eu)$

**5.6, Exercise 11, pages 204-205** (*continued*)

**B.**  9. $Sy$ ↔ $y$ is absolutely stable                           $(\forall y)(\neg Sy)$

10. $Wx$ ↔ $x$ is worth trying                             $\neg(\forall z)(Wz)$

11. $Kx$ ↔ $x$ is omniscient
    Domain restricted to people                      $(\forall x)(\neg Kx)$

12. $Vz$ ↔ $z$ has value                                 $(\forall y)(Vy)$

13. $Wx$ ↔ $x$ is wise (See note below.)                 $(\forall x)(Wx)$

14. $Fy$ ↔ $y$ is foolish (See note below.)              $(\forall y)(Fy)$

15. $Ex$ ↔ $x$ has two good eyes
    Domain restricted to people                      $\neg(\forall u)(Eu)$

16. $Rz$ ↔ $z$ is relative                               $(\forall z)(Rz)$

17. $Mw$ ↔ $w$ is a man                                  $(\forall w)(Mw)$

*Note.* Domain unrestricted. 13. means, "Everything is wise", not, "Everyone is wise".

18. $Hx$ ↔ $x$ has a history                             $(\forall x)(Hx)$

▶ 5.7 *Two Standard Forms*

**5.7, Exercise 12, pages 208-209**

**A.**  After one example, specific indication of choice of symbols will be omitted. The students, however, should do this for the entire exercise.

1. Defining: $Sx$ ↔ $x$ is a sparrow, $Bx$ ↔ $x$ is a bird.
   In symbols: $(\forall x)(Sx \rightarrow Bx)$.

2. $(\forall x)(Px \rightarrow Gx)$                     7. $(\forall x)(Sx \rightarrow Nx)$ (See note.)

3. $(\forall z)(Cz \rightarrow Bz)$                     8. $(\forall y)(Iy \rightarrow Cy)$

4. $(\forall w)(Fw \rightarrow Ew)$                     9. $(\forall z)(Sz \rightarrow Dz)$

5. $(\forall u)(Mu \rightarrow Wu)$                    10. $(\forall x)(Hx \rightarrow Qx)$

6. $(\forall v)(Fv \rightarrow Dv)$

*Note.* $Sx$ ↔ $x$ is grass; $Nx$ ↔ $x$ is green.

**B.**  1. $(\forall x)(Px \rightarrow \neg Vx)$         **C.**  1. $(\forall z)(Cz \rightarrow \neg Bz)$

2. $(\forall x)(Lx \rightarrow \neg Sx)$                2. $(\forall y)(Ay \rightarrow By)$

3. $(\forall y)(My \rightarrow \neg Iy)$                3. $(\forall x)(Ax \rightarrow \neg Jx)$

4. $(\forall z)(Kz \rightarrow \neg Cz)$                4. $(\forall w)(Cw \rightarrow \neg Hw)$

5. $(\forall u)(Au \rightarrow \neg Mu)$                5. $(\forall v)(Gv \rightarrow Fv)$

6. $(\forall w)(Xw \rightarrow \neg Lw)$                6. $(\forall u)(Du \rightarrow Lu)$

7. $(\forall v)(Fv \rightarrow \neg Iv)$                7. $(\forall t)(Ct \rightarrow \neg Tt)$

8. $(\forall x)(Tx \rightarrow \neg Jx)$                8. $(\forall s)(Cs \rightarrow Ss)$

9. $(\forall x)(Bx \rightarrow \neg Qx)$                9. $(\forall r)(Tr \rightarrow \neg Sr)$

10. $(\forall x)(Fx \rightarrow \neg Hx)$

**D.**  This exercise points up the importance of restating English sentences in if . . . then . . . form. *Only A's are B's* can be restated *All B's are A's* or *For every x, if Bx then Ax* which is symbolized $(\forall x)(Bx \rightarrow Ax)$.

This reverses the order of the predicates in the original. But when 'only' follows 'the' this reversal may not occur. Consider the following example.

(1) The only children allowed on the floor of the Senate are pageboys.

This means

(2) Any child who is allowed on the floor of the Senate is a pageboy.

That is, (3) For any $x$, if both $x$ is a child and $x$ is allowed on the floor of the senate, then $x$ is a pageboy.

(4) $(\forall x)(Cx \quad \& \quad Ax \quad \rightarrow \quad Px)$.

Here the predicates appear in the same order as in the original English.

  This example illustrates another standard form:

(5) All $A$'s who (or which, or that) are $B$'s are $C$'s, which is translated:

(6) $(\forall x)(Ax \quad \& \quad Bx \quad \rightarrow \quad Cx)$.

Sentence (1) might be stated in English.

(7) No children but pageboys are allowed on the floor of the Senate. Or

(8) Any child who is not a pageboy is not allowed on the floor of the Senate.

This exactly fits the form (5) and can be symbolized

(9) $(\forall x)(Cx \quad \& \quad \neg Px \quad \rightarrow \quad \neg Ax)$

which is logically equivalent to (4). Another equivalent form is:

(10) $(\forall x)(Ax \quad \rightarrow \quad \neg Cx \quad \vee \quad Px)$

which would more obviously be translated in English by

(11) Anyone allowed on the floor of the Senate is either not a child or is a pageboy.

  Problems 4, 5, and 6 of this exercise illustrate the fact that the English 'and' need not always be symbolized with &. Problem 4 might be restated:

  4.1 All birds are animals and all fish are animals

and translated:

  4.2 $(\forall x)(Bx \quad \rightarrow \quad Ax) \quad \& \quad (\forall x)(Fx \quad \rightarrow \quad Ax)$.

However, in order to avoid certain difficulties that can arise when formulas using universal quantifiers are used in deductions we will limit ourselves to formulas where all quantifiers are at the beginning. Problem 4 must be restated:

  4.3 Anything that is a bird or a fish is an animal.

  4.4 For all $x$, if $x$ is a bird or $x$ is a fish then $x$ is an animal.

It is then symbolized:

  4.5 $(\forall x)(Bx \quad \vee \quad Fx \quad \rightarrow \quad Ax)$

**D.** 1. $(\forall x)(Lx \quad \rightarrow \quad Px)$

  2. $(\forall x)(Rx \quad \rightarrow \quad Mx)$

  3. $(\forall x)(Fx \quad \rightarrow \quad Ex)$

  4. $(\forall x)(Bx \quad \vee \quad Fx \quad \rightarrow \quad Ax)$

  5. $(\forall x)(Hx \quad \vee \quad Cx \quad \rightarrow \quad Qx)$

  6. $(\forall x)(Mx \quad \vee \quad Ox \quad \rightarrow \quad Dx)$

  7. $\neg(\forall x)(Mx \quad \rightarrow \quad Ix)$

  8. $\neg(\forall x)(Mx \quad \rightarrow \quad Hx)$

  9. $\neg(\forall x)(Sx \quad \rightarrow \quad Nx)$

**E.** 1. $(\forall x)(x>2 \quad \rightarrow \quad x>1)$

  2. $(\forall x)(x+0=x)$

  3. $(\forall x)(x\neq0 \quad \rightarrow \quad x/x=1)$

  4. $(\forall y)(y-y=0)$

  5. $(\forall y)(y-0=y)$

**Chapter 5, Review Test, pages 209–210**

  **I.**  Terms underlined; the rest of the sentence is the predicate. Parts underlined with dashed lines can be considered separate terms or as part of the predicate.

    a.  <u>Jim Taylor</u> is a fullback.

    b.  <u>5s</u>  <u>25</u>.

    c.  <u>The big black bear</u> lumbered slowly toward us.

    d.  <u>Lincoln</u> was <u>the sixteenth President of the United States</u>.

    e.  <u>Two</u> is <u>the cube root of eight</u>.

  **II.**  The student should indicate his choice of symbols.

| | | |
|---|---|---|
| a.  $R(1, 1)$ | e.  $Cb$ | i.  $\neg Ce \rightarrow \neg Cn$ |
| b.  $At$ | f.  $Zn$ | j.  $2+2 \neq 5 \lor 2+3 \neq 6$ |
| c.  $(\forall x)(Gx \rightarrow Ax)$ | g.  $Mcd$ | k.  $Pr \ \& \ Tt$ |
| d.  $Gb$ | h.  $(\forall x)(Sx \rightarrow \neg Ex)$ | l.  $Pg$ |

 **III.**  a.  pronouns        c.  connectives

        b.  variables         (or variables)

  **VI.**  The student should define his symbols.

| | |
|---|---|
| a.  $(\forall y)(y=y \ \& \ \neg(y<y))$ | h.  $(\forall x)(Lx \lor Gx)$ |
|     or | i.  $(\forall x)(Px \leftrightarrow x>0)$ |
|     $(\forall y)(y=y \ \& \ y \not> y)$ | j.  $(\forall x)(Mx \rightarrow \neg Nx)$ |
| b.  $(\forall x)(Sx)$ | k.  $(\forall x)(x>0 \rightarrow Px)$ |
| c.  $(\forall x)(Mx \rightarrow \neg(Lx \ \& \ Rx))$ | l.  $\neg(\forall x)(Nx \rightarrow Px)$ |
| d.  $(\forall z)(Nz \rightarrow \neg(Ez \ \& \ Oz))$ | m.  $(\forall x)\neg(Rx \ \& \ Sx)$ |
| e.  $(\forall y)(Ny \rightarrow Ly)$ |     or |
| f.  $(\forall x)(Lx)$ or $(\forall x)(Lxc)$ |     $(\forall x)(Rx \rightarrow \neg Sx)$ |
| g.  $(\forall w)(Mw \rightarrow \neg(Ww \lor Sw))$ | |

# Universal Specification and
# Laws of Identity

▶ 6.1 *One Quantifier*

**6.1, Exercise 1, pages 213–215**

**A, B.**  1. Prove: $Al$

(1) $(\forall x)(Dx \rightarrow Ax)$    P
(2) $Dl$    P
(3) $Dl \rightarrow Al$    $l/x$ 1
(4) $Al$    PP 3, 2

2. Prove: $\neg Ij$

(1) $(\forall x)(Px \rightarrow \neg Ix)$    P
(2) $Pj$    P
(3) $Pj \rightarrow \neg Ij$    $j/x$ 1
(4) $\neg Ij$    PP 3, 2

3. Prove: $D8$ & $D10$

(1) $(\forall x)(Ex \rightarrow Dx)$    P
(2) $E10$    P
(3) $E8$    P
(4) $E8 \rightarrow D8$    $8/x$ 1
(5) $D8$    PP 4, 3
(6) $E10 \rightarrow D10$    $10/x$ 1
       (See note.)
(7) $D10$    PP 6, 2
(8) $D8$ & $D10$    A 5, 7

*Note.*  A previous universal specification on a line does not prevent a subsequent universal specification on the same line.

4. Prove: $3 \not> 3$

(1) $(\forall x)(Nx \rightarrow x \not> x)$    P
(2) $N3$    P
(3) $N3 \rightarrow 3 \not> 3$    $3/x$ 1
(4) $3 \not> 3$    PP 3, 2

Or alternatives for (6)–(7) could be:
(6) $Pp \rightarrow Vp$    HS 4, 5
(7) $Vp$    PP 6, 3

5. Prove: $4+1>4$

(1) $(\forall x)(Nx \rightarrow x+1>x)$ P
(2) $N4$    P
(3) $N4 \rightarrow 4+1>4$    $4/x$ 1
(4) $4+1>4$    PP 3, 2

6. Prove: $Vp$

(1) $(\forall x)(Px \rightarrow Bx)$    P
(2) $(\forall y)(By \rightarrow Vy)$    P
(3) $Pp$    P
(4) $Pp \rightarrow Bp$    $p/x$ 1
(5) $Bp \rightarrow Vp$    $p/x$ 2
(6) $Bp$    PP 4, 3
(7) $Vp$    PP 5, 6

7. Prove: $\neg F4$

(1) $(\forall x)(Fx \rightarrow \neg Ix)$    P
(2) $I4$    P
(3) $F4 \rightarrow \neg I4$    $4/x$ 1
(4) $\neg F4$    TT 3, 2

8. Prove: $\neg N6$

(1) $(\forall x)(Nx \rightarrow x<0)$    P
(2) $6 \not< 0$    P
(3) $N6 \rightarrow 6<0$    $6/x$ 1
(4) $\neg N6$    TT 3, 2

**6.1, Exercise 1, pages 213–215** (*continued*)

9. Prove: ¬*Pb*

    (1) (∀*x*)(F*x* → E*x*)      P
    (2) (∀*x*)(E*x* → ¬M*x*)      P
    (3) *Mb*      P
    (4) *Eb* → ¬*Mb*      *b*/*x* 2
    (5) ¬*Eb*      TT 4, 3
    (6) *Pb* → *Eb*      *b*/*x* 1
    (7) ¬*Pb*      TT 6, 5

    Or, alternative for (5)–(7):
    (5) *Pb* → *Eb*      *b*/*x* 1
    (6) *Pb* → ¬*Mb*      HS 5, 4
    (7) ¬*Pb*      TT 6, 3

10. Prove: ¬(*O6* ∨ *O8*)

    (1) (∀*x*)(O*x* → ¬D*x*)      P
    (2) *D6*      P
    (3) *D8*      P
    (4) *O6* → ¬*D6*      6/*x* 1
    (5) ¬*O6*      TT 4, 2
    (6) *O8* → ¬*D8*      8/*x* 1
    (7) ¬*O8*      TT 6, 3
    (8) ¬*O6* & ¬*O8*      A 5, 7
    (9) ¬(*O6* ∨ *O8*)      DL 8

11. Prove: *Ca* → *Sa*

    (1) (∀*x*)(C*x* → S*x* ∨ R*x*)      P
    (2) *Wa* & ¬*Ra*      P
    (3) *Ca* → *Sa* ∨ *Ra*      *a*/*x* 1 (See note.)
    (4) ¬*Ra*      S 2 (See note.)
    (5)      *Ca*      P
    (6)      *Sa* ∨ *Ra*      PP 3, 5
    (7)      *Sa*      TP 6, 4
    (8) *Ca* → *Sa*      CP 5, 7

*Note.* Doing these steps before starting the conditional proofs saves making them part of the subordinate proof.

**C.**    1. Prove: *Pb*

    (1) (∀*x*)(P*x* & R*x*)      P
    (2) *Pb* & *Rb*      *b*/*x* 1
    (3) *Pb*      S 2

    2. Prove: *Ft* & *Fr*

    (1) (∀*y*)(G*y* & F*y*)      P
    (2) *Gt* & *Fy*      *t*/*y* 1
    (3) *Ft*      S 2
    (4) *Gr* & *Fr*      *r*/*y* 1
    (5) *Fr*      S 4
    (6) *Ft* & *Fr*      A 3, 5

    3. Prove: *Ga*

    (1) (∀*x*)(H*x* → G*x*)      P
    (2) *Ha*      P
    (3) *Ha* → *Ga*      *a*/*x* 1
    (4) *Ga*      PP 3, 2

    4. Prove: *Fd*

    (1) (∀*y*)(F*y* ↔ H*y*)      P
    (2) *Hd*      P
    (3) *Fd* ↔ *Hd*      *d*/*y* 1
    (4) *Hd* ↔ *Fd*      LB 3
    (5) *Fd*      PP 4, 2

    5. Prove: 2>0

    (1) (∀*x*)(*x*>1 → *x*>0)      P
    (2) 2>1      P
    (3) 2>1 → 2>0      2/*x* 1
    (4) 2>0      PP 3, 2

    6. Prove: 3>0 & 4>0

    (1) 3>0      P
    (2) (∀*y*)(*y*>1 → *y*>0)      P
    (3) 4>1      P
    (4) 3>1 → 3>0      3/*y* 2
    (5) 3>0      PP 4, 1
    (6) 4>1 → 4>0      4/*y* 2
    (7) 4>0      PP 6, 3
    (8) 3>0 & 4>0      A 5, 7

    7. Prove: ¬*Hf*

    (1) (∀*x*)(H*x* → R*x*)      P
    (2) ¬*Rf*      P
    (3) *Hf* → *Rf*      *f*/*x* 1
    (4) ¬*Hf*      TT 3, 2

8. Prove: *Gb*

 (1) $(\forall x)(Gx \lor Jx)$    P
 (2) $\neg Jb$       P
 (3) $Gb \lor Jb$     $b/x$   1
 (4) $Gb$       TP 3, 2

9. Prove: *Jb*

 (1) $\neg Hb$       P
 (2) $(\forall y)(\neg Jy \rightarrow Hy)$   P
 (3) $\neg Jb \rightarrow Hb$    $b/y$   2
 (4) $Jb$       TT 3, 1

10. Prove: *Hx*

 (1) $(\forall x)(Gx \rightarrow Jx \ \& \ Hx)$    P
 (2) $Gx$          P
 (3) $Gx \rightarrow Jx \ \& \ Hx$     $x/x$   1
 (4) $Jx \ \& \ Hx$       PP 3, 2
 (5) $Hx$         S 4

## 6.1, Exercise 2, pages 216–217

**A.** Note that more than one equal sign ($=$) is inserted here to make it clear that the parentheses do make the required term. The students are not required to do this.

1. $(2+6) \times 5 = 8 \times 5 = 40$
2. $2 + (6 \times 5) = 2 + 30 = 32$
3. $-(3^2) = -9$
4. $(-3)^2 = 9$
5. $12 - (3^2) = 12 - 9 = 3$
6. $(12-3)^2 = 9^2 = 81$
7. $(24 \div 3) + 2^2 = 8 + 4 = 12$
8. $24 \div (3 + 2^2) = 24 \div (3+4) = 24 \div 7 = 3\frac{3}{7}$
9. $((24 \div 3) + 2)^2 = (8+2)^2 = 10^2 = 100$
10. $24 \div (3+2)^2 = 24 \div (5)^2 = 24 \div 25 = \frac{24}{25}$

**B.** 1. Prove: $3+7 \neq 2 \times 3$

 (1) $3+7 > 2+5$        P
 (2) $(\forall x)(x > 2+5 \rightarrow x \neq 2 \times 3)$   P
 (3) $3+7 > 2+5 \rightarrow 3+7 \neq 2 \times 3$   $3+7/x$   2
 (4) $3+7 \neq 2 \times 3$       PP 3, 1

 2. Prove: $6 \div 2 > 0$

 (1) $(\forall x)(x \neq 0 \rightarrow x > 0 \lor x < 0)$   P
 (2) $6 \div 2 \neq 0 \ \& \ 6 \div 2 \not< 0$     P
 (3) $6 \div 2 \neq 0 \rightarrow 6 \div 2 > 0 \lor 6 \div 2 < 0$   $6 \div 2/x$   1
 (4) $6 \div 2 \neq 0$         S 2
 (5) $6 \div 2 > 0 \lor 6 \div 2 < 0$     PP 3, 4
 (6) $6 \div 2 \not< 0$        S 2
 (7) $6 \div 2 > 0$        TP 5, 6

**6.1, Exercise 2, pages 216–217** (*continued*)

**B.**    3. Define: $Ex \leftrightarrow x$ is even
$\qquad\qquad Dx \leftrightarrow x$ is divisible by two
      Prove: $\neg D(3 \times 5) \quad \& \quad E(3+5)$

| | | |
|---|---|---|
| (1) | $(\forall x)(Ex \leftrightarrow Dx)$ | P |
| (2) | $\neg E(3 \times 5) \quad \& \quad D(3+5)$ | P |
| (3) | $E(3 \times 5) \leftrightarrow D(3 \times 5)$ | $3 \times 5/x$  1 |
| (4) | $D(3 \times 5) \rightarrow E(3 \times 5)$ | LB 3 |
| (5) | $\neg E(3 \times 5)$ | S 2 |
| (6) | $\neg D(3 \times 5)$ | TT 4, 5 |
| (7) | $E(3+5) \leftrightarrow D(3+5)$ | $3+5/x$  1 |
| (8) | $D(3+5) \rightarrow E(3+5)$ | LB 7 |
| (9) | $D(3+5)$ | S 2 |
| (10) | $E(3+5)$ | PP 8, 9 |
| (11) | $\neg D(3 \times 5) \quad \& \quad E(3+5)$ | A 6, 10 |

4. Define: $Ex \leftrightarrow x$ is even
$\qquad\quad Dx \leftrightarrow x$ is odd
  Prove: $D3 \rightarrow E(1+3)$

| | | |
|---|---|---|
| (1) | $(\forall x)E(x+1) \lor \neg Dx$ | P |
| (2) | $\neg E(1+3) \rightarrow \neg E(3+1)$ | P |
| (3) | $E(3+1) \lor \neg D3$ | $3/x$  1 |
| (4) | $D3$ | P |
| (5) | $E(3+1)$ | TP 3, 4 |
| (6) | $E(1+3)$ | TT 2, 5 |
| (7) | $D3 \rightarrow E(1+3)$ | CP 4, 6 |

5. Define as in 4.
  Prove: $E8$

| | | |
|---|---|---|
| (1) | $(\forall x)(Dx \rightarrow E(x+3))$ | P |
| (2) | $D(2+3)$ | P |
| (3) | $E((2+3)+3) \rightarrow E8$ | P |
| (4) | $D(2+3) \rightarrow E((2+3)+3)$ | $2+3/x$  1 |
| (5) | $E((2+3)+3)$ | PP 4, 2 |
| (6) | $E8$ | PP 3, 5 |

**6.1, Exercise 3, pages 218–220**

**A, B.**    1. Prove: $4+4 < 10$

| | | |
|---|---|---|
| (1) | $(\forall y)(y < 9 \rightarrow y < 10)$ | P |
| (2) | $4+4 < 9$ | P |
| (3) | $4+4 < 9 \rightarrow 4+4 < 10$ | $4+4/y$  1 |
| (4) | $4+4 < 10$ | PP 3, 2 |

2. Prove: $1+1 \not> 4$

| | | |
|---|---|---|
| (1) | $(\forall x)(x > 4 \rightarrow x > 3)$ | P |
| (2) | $1+1 \not> 3$ | P |
| (3) | $1+1 > 4 \rightarrow 1+1 > 3$ | $1+1/x$  1 |
| (4) | $1+1 \not> 4$ | TT 2, 3 |

**A, B.** 3. Prove: $8-4=2+2$

    (1) $(\forall z)(z=3+1 \rightarrow z=2+2)$    P

    (2) $8-4=3+1$    P

    (3) $8-4=3+1 \rightarrow 8-4=2+2$    $8-4/z$  1

    (4) $8-4=2+2$    PP  3, 2

4. Prove: $\neg N2$

    (1) $(\forall x)(Nx \rightarrow x<0)$    P

    (2) $2\not<0$    P

    (3) $N2 \rightarrow 2<0$    $2/x$  1

    (4) $\neg N2$    TT  3, 2

5. Prove: $0<1$

    (1) $(\forall x)(x+1=1 \rightarrow x<1)$    P

    (2) $0+1=1$    P

    (3) $0+1=1 \rightarrow 0<1$    $0/x$  1

    (4) $0<1$    PP  3, 2

6. Prove: $E4$

    (1) $(\forall x)(Dx \rightarrow Ex)$    P

    (2) $O4 \lor D4$    P

    (3) $\neg O4$    P

    (4) $D4 \rightarrow E4$    $4/x$  1

    (5) $D4$    TP  2, 3

    (6) $E4$    PP  4, 5

7. Prove: $E8$ & $E12$

    (1) $(\forall y)(Sy \rightarrow Ey)$    P

    (2) $S8$    P

    (3) $S12$    P

    (4) $S8 \rightarrow E8$    $8/y$  1

    (5) $E8$    PP  4, 2

    (6) $S12 \rightarrow E12$    $12/y$  1

    (7) $E12$    PP  6, 3

    (8) $E8$ & $E12$    A  5, 7

8. Prove: $5+5>8$

    (1) $(\forall x)(x=10 \rightarrow x>8)$    P

    (2) $5+5=10 \lor 5+3=10$    P

    (3) $5+3\neq10$    P

    (4) $5+5=10$    TP  2, 3

    (5) $5+5=10 \rightarrow 5+5>8$    $5+5/x$  1

    (6) $5+5>8$    PP  5, 4

9. Prove: $1+1\not<0$

    (1) $(\forall x)\neg(Px$ & $Nx)$    P

    (2) $(\forall x)(x<0 \rightarrow Nx)$    P

    (3) $P(1+1)$    P

    (4) $1+1<0 \rightarrow N(1+1)$    $1+1/x$  2

    (5) $\neg(P(1+1)$ & $N(1+1))$    $1+1/x$  1

    (6) $\neg P(1+1) \lor \neg N(1+1)$    DL  5

    (7) $\neg N(1+1)$    TP  6, 3

    (8) $1+1\not<0$    TT  4, 7

10. Prove: $2+2>2$

    (1) $(\forall x)(x>2 \rightarrow x+2>2)$    P

    (2) $(\forall x)(x+1>2 \rightarrow x+2>2)$    P

    (3) $2>2 \lor 2+1>2$    P

    (4) $2+1>2 \rightarrow 2+2>2$    $2/x$  2

    (5) $2>2 \rightarrow 2+2>2$    $2/x$  1

    (6) $2+2>2 \lor 2+2>2$    DS  3, 4, 5

    (7) $2+2>2$    DP  6

**6.1, Exercise 3, pages 218–220** (*continued*)

**A, B.** 11. Prove: $Lc$

| | | |
|---|---|---|
| (1) | $(\forall z)(Sz \rightarrow Az)$ | P |
| (2) | $(\forall u)(Au \rightarrow Lu)$ | P |
| (3) | $Sc$ | P |
| (4) | $Sc \rightarrow Ac$ | $c/z$ 1 |
| (5) | $Ac \rightarrow Lc$ | $c/u$ 2 |
| (6) | $Sc \rightarrow Lc$ | HS 4, 5 |
| (7) | $Lc$ | PP 6, 3 |

12. Prove: $\neg Ds$

| | | |
|---|---|---|
| (1) | $(\forall y)(Ty \ \& \ Ay \rightarrow \neg Ey)$ | P (See note.) |
| (2) | $(\forall x)(Dx \rightarrow Tx \ \& \ Ax)$ | P |
| (3) | $Es$ | P |
| (4) | $Ts \ \& \ As \rightarrow \neg Es$ | $s/y$ 1 |
| (5) | $Ds \rightarrow Ts \ \& \ As$ | $s/x$ 2 |
| (6) | $Ds \rightarrow \neg Es$ | HS 5, 4 |
| (7) | $\neg Ds$ | TT 6, 3 |

*Note.* Nowhere in this argument are 'being a pupil' and 'being late' separately employed. Therefore, it is satisfactory to consider 'being a late pupil' as a single predicate, say $Tx$. '$Px \ \& \ Lx$' would be replaced by '$Tx$'.

**C.**   1. Prove: $2+0>1$

| | | |
|---|---|---|
| (1) | $(\forall x)(x=2 \rightarrow x=1+1)$ | P |
| (2) | $(\forall x)(x=1+1 \rightarrow x>1)$ | P |
| (3) | $2+0=2$ | P |
| (4) | $2+0=2 \rightarrow 2+0=1+1$ | $2+0/x$ 1 |
| (5) | $2+0=1+1 \rightarrow 2+0>1$ | $2+0/x$ 2 |
| (6) | $2+0=1+1$ | PP 4, 3 |
| (7) | $2+0>1$ | PP 5, 6 |

2. Prove: $\neg N3$

| | | |
|---|---|---|
| (1) | $(\forall x)(Nx \rightarrow x<0)$ | P |
| (2) | $\neg(3<0)$ | P |
| (3) | $N3 \rightarrow 3<0$ | $3/x$ 1 |
| (4) | $\neg N3$ | TT 3, 2 |

3. Prove: $Fa \rightarrow La$

| | | |
|---|---|---|
| (1) | $(\forall x)(Fx \rightarrow \neg Px)$ | P |
| (2) | $(\forall x)(Px \ \lor \ Lx)$ | P |
| (3) | $Fa \rightarrow \neg Pa$ | $a/x$ 1 |
| (4) | $Pa \ \lor \ La$ | $a/x$ 2 |
| (5) | $Fa$ | P |
| (6) | $\neg Pa$ | PP 3, 5 |
| (7) | $La$ | TP 4, 6 |
| (8) | $Fa \rightarrow La$ | CP 5, 7 |

4. Prove: $\neg N4$

| | | |
|---|---|---|
| (1) | $(\forall x)(x>0 \leftrightarrow Px)$ | P |
| (2) | $(\forall x)(Px \rightarrow \neg Nx)$ | P |
| (3) | $4>0$ | P |
| (4) | $4>0 \leftrightarrow P4$ | $4/x$ 1 |
| (5) | $P4 \rightarrow \neg N4$ | $4/x$ 2 |
| (6) | $4>0 \rightarrow P4$ | LB 4 |
| (7) | $P4$ | PP 6, 3 |
| (8) | $\neg N4$ | PP 5, 7 |

5. Prove: $2\times3\neq0$

| | | |
|---|---|---|
| (1) | $(\forall y)(Py \ \lor \ Ny \rightarrow y\neq0)$ | P |
| (2) | $P(2\times3)$ | P |
| (3) | $P(2\times3) \ \lor \ N(2\times3) \rightarrow 2\times3\neq0$ | $2\times3/y$ 1 |
| (4) | $P(2\times3) \ \lor \ N(2\times3)$ | LA 2 |
| (5) | $2\times3\neq0$ | PP 3, 4 |

**C.**  6. Prove: $5-5=0$

| | | |
|---|---|---|
| (1) | $(\forall x)(\neg Px \rightarrow (\neg Nx \rightarrow x=0))$ | P |
| (2) | $\neg N(5-5)$ | P |
| (3) | $(\forall x)(x>0 \leftrightarrow Px)$ | P |
| (4) | $5-5\not> 0$ | P |
| (5) | $\neg P(5-5) \rightarrow (\neg N(5-5) \rightarrow 5-5=0)$ | $5-5/x$ 1 |
| (6) | $5-5>0 \leftrightarrow P(5-5)$ | $5-5/x$ 3 |
| (7) | $P(5-5) \rightarrow 5-5>0$ | LB 6 |
| (8) | $\neg P(5-5)$ | TT 7, 4 |
| (9) | $\neg N(5-5) \rightarrow 5-5=0$ | PP 5, 8 |
| (10) | $5-5=0$ | PP 9, 2 |

7. Prove: $3<5$

| | | |
|---|---|---|
| (1) | $(\forall x)(x<4 \ \& \ 4<5 \rightarrow x<5)$ | P |
| (2) | $(\forall x)(-4<-z \leftrightarrow z<4)$ | P |
| (3) | $4<5$ | P |
| (4) | $-4<-3$ | P |
| (5) | $3<4 \ \& \ 4<5 \rightarrow 3<5$ | $3/x$ 1 |
| (6) | $-4<-3 \leftrightarrow 3<4$ | $3/z$ 2 |
| (7) | $-4<-3 \rightarrow 3<4$ | LB 6 |
| (8) | $3<4$ | PP 7, 4 |
| (9) | $3<4 \ \& \ 4<5$ | A 7, 3 |
| (10) | $3<5$ | PP 5, 9 |

8. Prove: $Sb \ \& \ Pb \rightarrow \neg Cb$

| | | |
|---|---|---|
| (1) | $(\forall u)(Su \ \& \ Ru \rightarrow \neg Cu)$ | P |
| (2) | $(\forall u)(Pu \rightarrow Ru)$ | P |
| (3) | $Sb \ \& \ Rb \rightarrow \neg Cb$ | $b/u$ 1 |
| (4) | $Pb \rightarrow Rb$ | $b/u$ 2 |
| (5) | $Sb \ \& \ Pb$ | P |
| (6) | $Pb$ | S 5 |
| (7) | $Rb$ | PP 4, 6 |
| (8) | $Sb$ | S 5 |
| (9) | $Sb \ \& \ Rb$ | A 8, 7 |
| (10) | $\neg Cb$ | PP 3, 9 |
| (11) | $Sb \ \& \ Pb \rightarrow \neg Cb$ | CP 5, 10 |

9. Prove: $12=4\times 3$

| | | |
|---|---|---|
| (1) | $(\forall v)(12=v\times 3 \leftrightarrow 3+1=v)$ | P |
| (2) | $(\forall v)(v+1=4 \leftrightarrow 8-v=5)$ | P |
| (3) | $8-3=5$ | P |
| (4) | $12=4\times 3 \leftrightarrow 3+1=4$ | $4/v$ 1 |
| (5) | $3+1=4 \leftrightarrow 8-3=5$ | $3/v$ 2 |
| (6) | $8-3=5 \rightarrow 3+1=4$ | LB 5 |
| (7) | $3+1=4 \rightarrow 12=4\times 3$ | LB 4 |
| (8) | $8-3=5 \rightarrow 12=4\times 3$ | HS 6, 7 |
| (9) | $12=4\times 3$ | PP 8, 3 |

**6.1, Exercise 3, pages 218–220** (*continued*)

**C.**   10. Prove: $3+4<3+7$

    (1) $(\forall x)(x<2+6 \;\;\rightarrow\;\; x<3+7)$      P

    (2) $(\forall x)(x>2+5 \;\;\vee\;\; x<2+6)$      P

    (3) $3+4 \not> 2+5$      P

    (4) $3+4>2+5 \;\;\vee\;\; 3+4<2+6$      $3+4/x$   2

    (5) $3+4<2+6$      TP 4, 3

    (6) $3+4<2+6 \;\;\rightarrow\;\; 3+4<3+7$      $3+4/x$   1

    (7) $3+4<3+7$      PP 6, 5

**6.1, Exercise 4, pages 221–223**

   1. Prove: $5\neq 5+1$

    (1) $(\forall y)(Ey \;\;\leftrightarrow\;\; O(y+1))$      P

    (2) $(\forall x)(x=5+1 \;\;\rightarrow\;\; Ex)$      P

    (3) $\neg O(5+1)$      P

    (4) $E5 \;\;\leftrightarrow\;\; O(5+1)$      $5/x$   1

    (5) $5=5+1 \;\;\rightarrow\;\; E5$      $5/x$   2

    (6) $E5 \;\;\rightarrow\;\; O(5+1)$      LB 4

    (7) $\neg E5$      TT 6, 3

    (8) $5\neq 5+1$      TT 5, 7

   2. Prove: $O(15)$

    (1) $(\forall x)(12=x+4 \;\;\vee\;\; x=5\times 3 \;\;\rightarrow\;\; \neg Ex)$      P

    (2) $(\forall y)(Ey \;\;\vee\;\; Oy)$      P

    (3) $15=5\times 3$      P

    (4) $12=15+4 \;\;\vee\;\; 15=5\times 3 \;\;\rightarrow\;\; \neg E15$      $15/x$   1

    (5) $E(15) \;\;\vee\;\; O(15)$      $15/y$   2

    (6) $12=15+4 \;\;\vee\;\; 15=5\times 3$      LA 3

    (7) $\neg E(15)$      PP 4, 6

    (8) $O(15)$      TP 5, 7

   3. Prove: $P(3+5)$

    (1) $(\forall z)(z>3+4 \;\;\rightarrow\;\; z>0)$      P

    (2) $(\forall y)(Py \;\;\leftrightarrow\;\; y>0)$      P

    (3) $3+5>3+4$      P

    (4) $3+5>3+4 \;\;\rightarrow\;\; 3+5>0$      $3+5/z$   1

    (5) $P(3+5) \;\;\leftrightarrow\;\; 3+5>0$      $3+5/y$   2

    (6) $3+5>0$      PP 4, 3

    (7) $3+5>0 \;\;\rightarrow\;\; P(3+5)$      LB 5

    (8) $P(3+5)$      PP 7, 6

   4. Prove: $\neg He$

    (1) $(\forall x)(Px \;\;\&\;\; Hx \;\;\rightarrow\;\; Ux)$      P

    (2) $Pe \;\;\&\;\; \neg Ue$      P

    (3) $Pe \;\;\&\;\; He \;\;\rightarrow\;\; Ue$      $e/x$   1

    (4) $\neg Ue$      S 2

    (5) $\neg (Pe \;\;\&\;\; He)$      TT 3, 4

    (6) $\neg Pe \;\;\vee\;\; \neg He$      DL 5

    (7) $Pe$      S 2

    (8) $\neg He$      TP 6, 7

5. Be sure the students have this correctly translated before attempting a proof.

Prove: $\neg Cs$

(1) $(\forall u)(Cu \rightarrow Du)$     P
(2) $Hs$     P
(3) $(\forall v)(Mv \rightarrow \neg Dv)$     P  or  $(\forall v)\neg(Mv \ \& \ Dv)$. This requires changes in the lines following.
(4) $(\forall x)(Hx \rightarrow Mx)$     P
(5) $Hs \rightarrow Ms$     $s/x$  4
(6) $Ms$     PP 5, 2
(7) $Ms \rightarrow \neg Ds$     $s/v$  3
(8) $\neg Ds$     PP 7, 6
(9) $Cs \rightarrow Ds$     $s/u$  1
(10) $\neg Cs$     TT 9, 8

6. Prove: $1+2<4$

(1) $(\forall x)(x^2=9 \ \& \ x>2 \rightarrow x=3)$     P
(2) $(\forall y)((y>2 \rightarrow y=3) \rightarrow y<4)$     P
(3) $(1+2)^2=9$     P
(4) $(1+2)^2=9 \ \& \ 1+2>2 \rightarrow 1+2=3$     $1+2/x$  1
(5) $(1+2>2 \rightarrow 1+2=3) \rightarrow 1+2<4$     $1+2/y$  2 (See note.)
(6)         $1+2>2$     P
(7)         $(1+2)^2=9 \ \& \ 1+2>2$     A 3, 6
(8)         $1+2=3$     PP 4, 7
(9) $1+2>2 \rightarrow 1+2=3$     CP 6, 8
(10) $1+2<4$     PP 5, 9

*Note.* We see by line (5) that we could obtain the desired conclusion if we had the conditional '$1+2>2 \rightarrow 1+2=3$'. So we employ the strategy of a conditional proof to obtain it.

7. Prove: $9-1=3+5 \rightarrow E(9-1)$

(1) $(\forall u)(u=3+5 \ \lor \ u=10+2 \rightarrow Du)$     P
(2) $(\forall x)(Dx \ \lor \ Sx \rightarrow Ex)$     P
(3) $9-1=3+5 \ \lor \ 9-1=10+2 \rightarrow D(9-1)$     $9-1/u$  1 (See note.)
(4)         $9-1=3+5$     P
(5)         $9-1=3+5 \ \lor \ 9-1=10+2$     LA 4
(6)         $D(9-1)$     PP 3, 5
(7)         $D(9-1) \ \lor \ S(9-1)$     LA 6
(8)         $D(9-1) \ \lor \ S(9-1) \rightarrow E(9-1)$     $9-1/x$  2
(9)         $E(9-1)$     PP 8, 7
(10) $9-1=3+5 \rightarrow E(9-1)$     CP 4, 9

*Note.* This universal specification is the obvious way to make premise (1) fit the desired conclusion.

**6.1, Exercise 4, pages 221–223** (*continued*)

8. Prove: $\neg V15$

| | | |
|---|---|---|
| (1) | $(\forall x)(Vx \;\rightarrow\; Fx)$ | P |
| (2) | $(\forall y)(Fy \;\rightarrow\; Ey)$ | P |
| (3) | $(\forall z)(Uz \;\lor\; \neg Ez)$ | P |
| (4) | $\neg U15$ | P |
| (5) | $U15 \;\lor\; \neg E15$ | 15/z 3 |
| (6) | $\neg E15$ | TP 5, 4 |
| (7) | $F15 \;\rightarrow\; E15$ | 15/y 2 |
| (8) | $\neg F15$ | TT 7, 6 |
| (9) | $V15 \;\rightarrow\; F15$ | 15/x 1 |
| (10) | $\neg V15$ | TT 9, 8 |

9. Prove: $P4$

| | | |
|---|---|---|
| (1) | $(\forall x)(x<5 \;\lor\; (x>3 \;\&\; Px))$ | P |
| (2) | $(\forall x)(x>0 \;\rightarrow\; (x<5 \;\rightarrow\; Px))$ | P |
| (3) | $4>0$ | P |
| (4) | $4>0 \;\rightarrow\; (4<5 \;\rightarrow\; P4)$ | 4/x 2 |
| (5) | $4<5 \;\rightarrow\; P4$ | PP 4, 3 |
| (6) | $4<5 \;\lor\; (4>3 \;\&\; P4)$ | 4/x 1 |
| (7) | $\neg P4$ | P (See Note 1.) |
| (8) | $4 \not< 5$ | TT 5, 7 |
| (9) | $4>3 \;\&\; P4$ | TP 6, 8 |
| (10) | $P4$ | S 9 (See Note 2.) |
| (11) | $P4 \;\&\; \neg P4$ | A 10, 7 |
| (12) | $P4$ | RAA 7, 11 |

*Note 1.* When other lines fail, try an RAA!

*Note 2.* This is our desired conclusion, but the proof is not complete because it appears indented in a subordinate proof which shows it still depends on the added premise. CP and RAA are the only methods of getting from a subordinate proof back out to the main proof, no longer dependent on the added premise.

An elegant alternative starting line (7) is the following:

| | | |
|---|---|---|
| (7) | $4>8 \;\&\; P4$ | P |
| (8) | $P4$ | S 7 |
| (9) | $4>8 \;\&\; P4 \;\rightarrow\; P4$ | CP 7, 8 |
| (10) | $P4 \;\lor\; P4$ | DS 6, 5, 9 |
| (11) | $P4$ | DP 10 |

10. Prove: $P(4+5) \;\rightarrow\; 3\times(4+5)\neq -6$

| | | |
|---|---|---|
| (1) | $(\forall x)(x>0 \;\lor\; \neg Px)$ | P |
| (2) | $(\forall y)(3xy=-6 \;\rightarrow\; y \not> 0)$ | P |
| (3) | $4+5>0 \;\lor\; \neg P(4+5)$ | 4+5/x 1 |
| (4) | $3\times(4+5)=-6 \;\rightarrow\; 4+5 \not> 0$ | 4+5/y 2 |
| (5) | $P(4+5)$ | P |
| (6) | $4+5>0$ | TP 3, 5 |
| (7) | $3\times(4+5)\neq -6$ | TT 4, 6 |
| (8) | $P(4+5) \;\rightarrow\; 3\times(4+5)\neq -6$ | CP 5, 7 |

## ▶ 6.2 *Two or More Quantifiers*

**6.2, Exercise 5, pages 226–228**

**A.**  1.  $Txy$ ↔ $x$ teases $y$
            $j$ = Jim
            $f$ = Frances
          $Tjf$

2.  $xVy$ ↔ $x$ visited $y$
      $w$ = Mrs. Warren
      $h$ = The Huntington Library
     $wVh$

3.  $xVy$ ↔ $x$ visits $y$
      $l$ = Larry
      $h$ = Harry
     $m$ = Mary
     $s$ = Sherry
     $lVh$ → $mVs$

4.  $Axy$ ↔ $x$ attracts $y$
     $(\forall x)(\forall y)(Axy)$

5.  $Ex$ ↔ $x$ is an eagle
    $Hx$ ↔ $x$ is a hummingbird
    $Lxy$ ↔ $x$ is larger than $y$
    $(\forall x)(\forall y)(Ex \ \& \ Hy \ → \ Lxy)$

6.  $Bxy$ ↔ $x$ is brother of $y$
    $Qxy$ ↔ $x$ sometimes quarrels with $y$
    $(\forall x)(\forall y)(Bxy \ → \ Qxy)$

7.  $Bx$ ↔ $x$ is a boy
    $xWy$ ↔ $x$ wants to play with $y$
    $xPy$ ↔ $x$ must first pump up $y$
     $b$ = the volley ball
    $(\forall x)(Bx \ \& \ xWb \ → \ xPb)$

8.  $xHy$ ↔ $x$ helps $y$
     $e$ = Earl
     $b$ = Betty
     $j$ = Jennifer
    $eHb \ \& \ jHe$

9.  $Bx$ ↔ $x$ is a bird
    $Cx$ ↔ $x$ is a cat
    $Fxy$ ↔ $x$ is afraid of $y$
    $(\forall x)(\forall y)(Bx \ \& \ Cy \ → \ Fxy)$

10.  $Nx$ ↔ $x$ is a nation
     $Pxy$ ↔ $x$ prepares to fight $y$
     $Fxy$ ↔ $x$ fears $y$
     $(\forall x)(\forall y)(Nx \ \& \ Ny \ \& \ Fxy \ → \ Pxy)$

**B.**  1.  Prove: $Gk$

| | | |
|---|---|---|
| (1) | $(\forall x)(Sx \ → \ Px)$ | P |
| (2) | $(\forall x)(\forall y)(Px \ \& \ yFx \ → \ Gy)$ | P |
| (3) | $Sf \ \& \ kFf$ | P |
| (4) | $Pf \ \& \ kFf \ → \ Gk$ | $f/x$   2 |
| (5) | $Sf \ → \ Pf$ | $f/x$   1 |
| (6) | $Sf$ | S 3 |
| (7) | $Pf$ | PP 5, 6 |
| (8) | $kFf$ | S 3 |
| (9) | $Pf \ \& \ kFf$ | A 7, 8 |
| (10) | $Gk$ | PP 4, 9 |

**6.2, Exercise 5, pages 226–228** (*continued*)

**B.**  2. Prove: $Cc \rightarrow \neg Mm$

|     |     |     |
| --- | --- | --- |
| (1) | $(\forall x)(\forall y)(Mx \ \& \ Cy \ \rightarrow \ xKy)$ | P |
| (2) | $\neg(mKc)$ | P |
| (3) | $Mm \ \& \ Cc \ \rightarrow \ mKc$ | $m/x, \ c/y$  1 |
| (4) | $\neg(Mm \ \& \ Cc)$ | TT 3, 2 |
| (5) | $\neg Mm \ \lor \ \neg Cc$ | DL 4 |
| (6) | $\qquad Cc$ | P |
| (7) | $\qquad \neg Mm$ | TP 5, 6 |
| (8) | $Cc \ \rightarrow \ \neg Mm$ | CP 6, 7 |

3. It may be necessary to hint to the students that (2) means, 'For all $y$, if $y$ is a friend of Mike, then Nick is not a friend of $y$'.

Prove: $Fkm \ \rightarrow \ \neg Ljg$

|      |     |     |
| ---- | --- | --- |
| (1) | $(\forall x)(Lxg \ \rightarrow \ Cxn)$ | P |
| (2) | $(\forall y)(Fym \ \rightarrow \ \neg Fny)$ | P |
| (3) | $(\forall z)(Cjz \ \rightarrow \ Fzk)$ | P |
| (4) | $Fkm \ \rightarrow \ \neg Fnk$ | $k/y$  2 (See note.) |
| (5) | $Cjn \ \rightarrow \ Fnk$ | $n/z$  3 (See note.) |
| (6) | $Ljg \ \rightarrow \ Cjn$ | $j/x$  1 (See note.) |
| (7) | $\qquad Fkm$ | P |
| (8) | $\qquad \neg Fnk$ | PP 4, 7 |
| (9) | $\qquad \neg Cjn$ | TT 5, 8 |
| (10) | $\qquad \neg Ljg$ | TT 6, 9 |
| (11) | $Fkm \ \rightarrow \ \neg Ljg$ | CP 7, 10 |

*Note.* These specifications may be introduced after the added premise of line (7), but it is better to do them first if it is foreseen just what specifications will be needed.

4. Prove: $\neg Bn$

|     |     |     |
| --- | --- | --- |
| (1) | $(\forall x)(\forall y)(Bx \ \& \ yFx \ \rightarrow \ Fy)$ | P (See note.) |
| (2) | $cFn \ \& \ \neg Fc$ | P |
| (3) | $Bn \ \& \ cFn \ \rightarrow \ Fy$ | $n/x, \ c/y$  1 |
| (4) | $\neg Fc$ | S 2 |
| (5) | $\neg(Bn \ \& \ cFn)$ | TT 3, 4 |
| (6) | $\neg Bn \ \lor \ \neg(cFn)$ | DL 5 |
| (7) | $cFn$ | S 2 |
| (8) | $\neg Bn$ | TP 6, 7 |

*Note.* 'F' is used here for both predicates to show there is no ambiguity between the two uses, since one is a two-place, and the other a one-place predicate. However, it is usually better to use different letters for clearer

contrast. As practice in translating back to English, ask different pupils to read the problems in English using the given equivalents.

**C.** 1. Prove: $5+3=3+5$

    (1) $(\forall x)(\forall y)(x+y=y+x)$      P

    (2) $5+3=3+5$             $5/x,\ 3/y$    1

2. Prove: $(4+3)+3>6+3$

    (1) $(\forall x)(\forall y)(x>y \;\rightarrow\; x+3>y+3)$      P

    (2) $4+3>6$      P

    (3) $4+3>6 \;\rightarrow\; (4+3)+3>6+3$      $4+3/x,\ 6/y$    1

    (4) $(4+3)+3>6+3$      PP 3, 2

3. Prove: $\frac{3}{4}<1$

    (1) $(\forall w)(\forall z)(w>z \;\rightarrow\; \dfrac{z}{w}<1)$      P

    (2) $4>3$      P

    (3) $4>3 \;\rightarrow\; \frac{3}{4}<1$      $4/w,\ 3/z$

    (4) $\frac{3}{4}<1$      PP 3, 2

4. Prove: $E(3\cdot8)$

    (1) $(\forall x)(\forall y)(Ex \;\rightarrow\; E(x\cdot y))$      P

    (2) $(\forall u)(\forall v)(E(u\cdot v) \;\leftrightarrow\; E(v\cdot u))$      P

    (3) $E8$      P

    (4) $E8 \;\rightarrow\; E(8\cdot3)$      $8/x,\ 3/y$    1

    (5) $E(8\cdot3)$      PP 4, 3

    (6) $E(8\cdot3) \;\leftrightarrow\; E(3\cdot8)$      $8/u,\ 3/v$    2

    (7) $E(8\cdot3) \;\rightarrow\; E(3\cdot8)$      LB 6

    (8) $E(3\cdot8)$      PP 7, 5

5. Prove: $-4\cdot(-4)^2<-4$

    (1) $(\forall x)(\forall y)(x<-1 \;\&\; y>1 \;\rightarrow\; x\cdot y<x)$      P

    (2) $(\forall z)(z<-1 \;\rightarrow\; z^2>1)$      P

    (3) $-4<-1$      P

    (4) $-4<-1 \;\rightarrow\; (-4)^2>1$      $(-4)/z$    2

    (5) $(-4)^2>1$      PP 4, 3

    (6) $-4<1 \;\&\; (-4)^2>1 \;\rightarrow\; -4\cdot(-4)^2<-4$      $-4/x,\ (-4)^2/y$    1

    (7) $-4<-1 \;\&\; (-4)^2>1$      A 3, 5

    (8) $-4\cdot(-4)^2<-4$      PP 6, 7

**6.2, Exercise 5, pages 226–228** (*continued*)

**C.**    6. Prove: $6 \cdot \frac{2}{3} < 6$

| | | |
|---|---|---|
| (1) | $(\forall u)(\forall v)(u - v < 0 \;\; \leftrightarrow \;\; u < v)$ | P |
| (2) | $(\forall w)(\forall z)(w < 1 \;\;\&\;\; w > 0 \;\; \rightarrow \;\; z \cdot w < z)$ | P |
| (3) | $\frac{2}{3} - 1 < 0 \;\;\&\;\; \frac{2}{3} > 0$ | P |
| (4) | $\frac{2}{3} < 1 \;\;\&\;\; \frac{2}{3} > 0 \;\; \rightarrow \;\; 6 \cdot \frac{2}{3} < 6$ | $\frac{2}{3}/w,\; 6/z$   2   (See Note 1.) |
| (5) | $\frac{2}{3} - 1 < 0 \;\; \leftrightarrow \;\; \frac{2}{3} < 1$ | $\frac{2}{3}/u,\; 1/v$   1   (See Note 2.) |
| (6) | $\frac{2}{3} - 1 < 0. \;\; \rightarrow \;\; \frac{2}{3} < 1$ | LB 5 |
| (7) | $\frac{2}{3} - 1 < 0$ | S 3 |
| (8) | $\frac{2}{3} < 1$ | PP 6, 7 |
| (9) | $\frac{2}{3} > 0$ | S 3 |
| (10) | $\frac{2}{3} < 1 \;\;\&\;\; \frac{2}{3} > 0$ | A 7, 8 |
| (11) | $6 \cdot \frac{2}{3} < 6$ | PP 4, 9 |

*Note 1.* The consequent of line (2) is closest in form to the desired conclusion, therefore this specification is chosen to make it exactly so.

*Note 2.* This specification is chosen to make the left side of line (2) to be exactly the left conjunct of line (3).

   7. Prove: $E(5 + 7)$

| | | |
|---|---|---|
| (1) | $(\forall y)(\forall z)(Oy \;\;\&\;\; Oz \;\; \rightarrow \;\; E(y + z))$ | P |
| (2) | $(\forall x)(Ox \;\; \leftrightarrow \;\; \neg Dx)$ | P |
| (3) | $(\forall w)(Ow \;\; \lor \;\; Ew)$ | P |
| (4) | $\neg D7 \;\;\&\;\; \neg E5$ | P |
| (5) | $O5 \;\;\&\;\; O7 \;\; \rightarrow \;\; E(5 + 7)$ | $5/y,\; 7/z$   1   (See Note 1.) |
| (6) | $O7 \;\; \leftrightarrow \;\; \neg D7$ | $7/x$   2   (See Note 2.) |
| (7) | $O5 \;\; \lor \;\; E5$ | $5/w$   3   (See Note 3.) |
| (8) | $\neg D7 \;\; \rightarrow \;\; O7$ | LB 6 |
| (9) | $\neg D7$ | S 4 |
| (10) | $O7$ | PP 8, 9 |
| (11) | $\neg E5$ | S 4 |
| (12) | $O5$ | TP 7, 11 |
| (13) | $O5 \;\;\&\;\; O7$ | A 12, 10 |
| (14) | $E(5 + 7)$ | PP 5, 13 |

*Note 1.* This specification was chosen to make the consequent of line (1) fit the required conclusion, since this is the only item in the premises whose form is the same as that of the conclusion.

*Note 2.* This specification was chosen to make the right side of line (2) fit the first conjunct of the line (4).

*Note 3.* This specification was chosen to make a part of line (3) fit a part of line (4).

   8. Prove: $5 + \frac{1}{4} > 5$

| | | |
|---|---|---|
| (1) | $(\forall x)(\forall y)(Px \;\;\&\;\; Py \;\;\&\;\; x < 1 \;\; \rightarrow \;\; y + x > y)$ | P |
| (2) | $(\forall x)(\forall y)((Py \;\;\&\;\; Px) \;\; \lor \;\; \neg(y + x > x \;\; \lor \;\; y + x > y))$ | P |
| (3) | $\frac{1}{4} < 1$ | P |
| (4) | $\frac{1}{4} + 5 > 5$ | P |
| (5) | $P\frac{1}{4} \;\;\&\;\; P5 \;\;\&\;\; \frac{1}{4} < 1 \;\; \rightarrow \;\; 5 + \frac{1}{4} > 5$ | $\frac{1}{4}/x,\; 5/y$   1   (See Note 1.) |
| (6) | $(P\frac{1}{4} \;\;\&\;\; P5) \;\; \lor \;\; \neg(\frac{1}{4} + 5 > 5 \;\; \lor \;\; \frac{1}{4} + 5 > \frac{1}{4})$ | $5/x,\; \frac{1}{4}/y$   2   (See Note 2.) |
| (7) | $\frac{1}{4} + 5 > 5 \;\; \lor \;\; \frac{1}{4} + 5 > \frac{1}{4}$ | LA 4 |
| (8) | $P\frac{1}{4} \;\;\&\;\; P5$ | TP 6, 7 |
| (9) | $P\frac{1}{4} \;\;\&\;\; P5 \;\;\&\;\; \frac{1}{4} < 1$ | A 3, 8 |
| (10) | $5 + \frac{1}{4} > 5$ | PP 5, 9 |

*Note 1.* The final atomic formulas in both line (1) and line (2) have the same form as the conclusion. So looking further, we note that the third atomic formula of line (1) is the only one in line (1) or line (2) that

has the same form as line (3). This suggests substituting one fourth for $x$ in line (1) and, further, 5 for $y$ in that line, thus making the consequent fit the conclusion.

*Note 2.* Line (4) has the same term immediately before and after the symbol $>$. Since this matches the third atomic formula in line (2), the specification is chosen to make them identical.

**C.**  9. Prove: $P5 \rightarrow (P(-3) \leftrightarrow P(5 \cdot -3)$

| | | |
|---|---|---|
| (1) | $(\forall z)(\forall y)(Pz \quad \& \quad Py \quad \rightarrow \quad P(z \cdot y))$ | P |
| (2) | $(\forall y)(\forall w)(Py \quad \& \quad \neg Pw \quad \rightarrow \quad \neg P(y \cdot w))$ | P |
| (3) | $P5 \quad \& \quad \neg P(-3) \quad \rightarrow \quad \neg P(5 \cdot -3)$ | $5/y,\ -3/w$  2 (See Note 1.) |
| (4) | $P5 \quad \& \quad P(-3) \quad \rightarrow \quad P(5 \cdot -3)$ | $5/z,\ -3/y$  1 (See Note 1.) |
| (5) | $P5$ | P |
| (6) | $P(-3)$ | P (See Note 2.) |
| (7) | $P5 \quad \& \quad P(-3)$ | A 5, 6 |
| (8) | $P(5 \cdot -3)$ | PP 4, 7 |
| (9) | $P-3 \quad \rightarrow \quad P(5 \cdot -3)$ | CP 6, 8 |
| (10) | $P(5 \cdot -3)$ | P |
| (11) | $\neg(P5) \quad \& \quad \neg P(-3))$ | TT 3, 10 |
| (12) | $\neg P5 \quad \vee \quad \neg\neg P(-3)$ | DL 11 |
| (13) | $P(-3)$ | TP 5, 12 |
| (14) | $P(5 \cdot -3) \quad \rightarrow \quad P(-3)$ | CP 10, 13 |
| (15) | $P(-3) \quad \leftrightarrow \quad P(5 \cdot -3)$ | LB 9, 14 |
| (16) | $P5 \quad \rightarrow \quad (P(-3) \quad \leftrightarrow \quad P(5 \cdot -3))$ | CP 5, 15 |

*Note 1.* These specifications are obvious since the conclusion contains $P(5 \cdot -3)$.

*Note 2.* The proof of a biconditional requires the proof of conditionals each way. So a CP for each of them is the obvious approach.

10. Prove: $P7$

| | | |
|---|---|---|
| (1) | $(\forall x)(\forall y)(x > 0 \quad \& \quad y < 0 \quad \rightarrow \quad Nx/y)$ | P |
| (2) | $(\forall u)(\forall v)((u < 0 \quad \rightarrow \quad Nv/u) \quad \rightarrow \quad Pv)$ | P |
| (3) | $7 > 0$ | P |
| (4) | $7 > 0 \quad \& \quad y < 0 \quad \rightarrow \quad N7/y$ | $7/x,\ y/y$  1 (See Note 1.) |
| (5) | $(y < 0 \quad \rightarrow \quad N7/y) \quad \rightarrow \quad P7$ | $y/u,\ 7/v$  2 (See Note 2.) |
| (6) | $y < 0$ | P |
| (7) | $7 > 0 \quad \& \quad y < 0$ | A 3, 6 |
| (8) | $N7/y$ | PP 4, 7 |
| (9) | $y < 0 \quad \rightarrow \quad N7/y$ | CP 6, 8 |
| (10) | $P7$ | PP 5, 9 |

*Note 1.* Specify seven for $x$ to fit line (3); it does not matter what term is specified for $y$.

*Note 2.* Seven for $v$ to fit conclusion; for $u$ specify the same thing specified for $y$ in line (4).

**6.2, Exercise 6, pages 229–230**

**A.**  1. Prove: $hMb \rightarrow mPb$

|      |                                                                                  |                                   |
|------|----------------------------------------------------------------------------------|-----------------------------------|
| (1)  | $(\forall x)(\forall y)(\forall z)(xSy \ \& \ yMz \ \& \ Bz \rightarrow xAz)$     | P (See Note 1.)                   |
| (2)  | $Bb \ \& \ mSh$                                                                   | P                                 |
| (3)  | $(\forall x)(xAb \rightarrow xPb)$                                                | P                                 |
| (4)  | $mSh \ \& \ hMb \ \& \ Bb \rightarrow mAb$                                        | $m/x, \ h/y, \ b/z$   1           |
| (5)  | $mAb \rightarrow mPb$                                                             | $m/x$   3                         |
| (6)  | $hMb$                                                                             | P                                 |
| (7)  | $mSh \ \& \ hMb \ \& \ Bb$                                                        | A 2, 6 (See Note 2.)              |
| (8)  | $mAb$                                                                             | PP 4, 7                           |
| (9)  | $mPb$                                                                             | PP 5, 8                           |
| (10) | $hMb \rightarrow mPb$                                                             | CP 6, 9                           |

*Note 1.* Strictly this and line (4) would require additional parentheses in the antecedent.

*Note 2.* Strictly, to obtain this, line (2) would be separated by simplification and then the results would be adjoined with line (6) in whatever order makes it fit the strict form of the antecedent of line (4).

2. Prove: $bOw$

|      |                                                                                  |                                   |
|------|----------------------------------------------------------------------------------|-----------------------------------|
| (1)  | $(\forall x)(\forall y)(\forall z)(Cx \ \& \ Sy \ \& \ Pz \rightarrow xRy \ \& \ yRz)$ | P                           |
| (2)  | $(\forall x)(\forall y)(xRy \rightarrow yOx)$                                     | P                                 |
| (3)  | $(\forall x)(\forall y)(\forall z)(xRy \ \& \ yRz \rightarrow xRz)$               | P                                 |
| (4)  | $Cw \ \& \ Sh \ \& \ Pb$                                                          | P                                 |
| (5)  | $Cw \ \& \ Sh \ \& \ Pb \rightarrow wRh \ \& \ hRb$                              | $w/x, \ h/y, \ b/z$   1           |
| (6)  | $wRh \ \& \ hRb$                                                                  | PP 5, 4                           |
| (7)  | $wRh \ \& \ hRb \rightarrow wRb$                                                  | $w/x, \ h/y, \ b/z$   3           |
| (8)  | $wRb$                                                                             | PP 7, 6                           |
| (9)  | $wRb \rightarrow bOw$                                                             | $w/x, \ b/y$   2                  |
| (10) | $bOw$                                                                             | PP 9, 8                           |

3. Prove: $1 \not> 1$

|      |                              |                        |
|------|------------------------------|------------------------|
| (1)  | $(\forall x)(\forall y)(x > y \rightarrow y \not> x)$ | P      |
| (2)  | $1 > 1 \rightarrow 1 \not> 1$ | $1/x, \ 1/y$   1      |
| (3)  | $1 > 1$                      | P                      |
| (4)  | $1 \not> 1$                  | PP 2, 3                |
| (5)  | $1 > 1 \ \& \ 1 \not> 1$     | A 3, 4                 |
| (6)  | $1 \not> 1$                  | RAA 3, 5               |

4. Prove: $1 \geq 1$

|      |                              |                        |
|------|------------------------------|------------------------|
| (1)  | $(\forall x)(\forall y)(x \geq y \ \lor \ y \geq x)$ | P       |
| (2)  | $1 \geq 1 \ \lor \ 1 \geq 1$ | $1/x, \ 1/y$   1       |
| (3)  | $1 \geq 1$                   | DP 2                   |

Notice the symbol for 'equal to or greater than' combines the symbols $=$ and $>$ to $\geq$ and omits bottom dash.

**B.**  1. Prove: $4 = 2 + 2$

|      |                                                                    |                              |
|------|--------------------------------------------------------------------|------------------------------|
| (1)  | $(\forall x)(\forall y)(\forall z)(x = y \ \& \ y = z \rightarrow x = z)$ | P                      |
| (2)  | $4 = 2^2$                                                          | P                            |
| (3)  | $2^2 = 2 + 2$                                                      | P                            |
| (4)  | $4 = 2^2 \ \& \ 2^2 = 2 + 2 \rightarrow 4 = 2 + 2$                 | $4/x, \ 2^2/y, \ 2 + 2/z$   1 |
| (5)  | $4 = 2^2 \ \& \ 2^2 = 2 + 2$                                       | A 2, 3                       |
| (6)  | $4 = 2 + 2$                                                        | PP 4, 5                      |

**B.** 2. Prove: $2 \neq 1$

| | | |
|---|---|---|
| (1) $(\forall x)(\forall y)(\forall z)(x=y \quad \& \quad x>z \rightarrow y>z)$ | | P |
| (2) $\neg(1>1)$ | | P |
| (3) $2>1$ | | P |
| (4) $2=1 \quad \& \quad 2>1 \rightarrow 1>1$ | | $2/x, 1/y, 1/z \quad 1$ |
| (5) | $2=1$ | P |
| (6) | $2=1 \quad \& \quad 2>1$ | A 5, 3 |
| (7) | $1>1$ | PP 4, 6 |
| (8) | $1>1 \quad \& \quad \neg(1>1)$ | A 7, 2 |
| (9) $2 \neq 1$ | | RAA 5, 8 |

3. Prove: $2=1+1$

| | | |
|---|---|---|
| (1) $(\forall x)(\forall y)(\forall z)[x=y+1 \quad \lor \quad (x=y \quad \& \quad y=z+1)]$ | | P |
| (2) $2 \neq 1 \quad \lor \quad 1 \neq 0+1$ | | P |
| (3) $2=1+1 \quad \lor \quad (2=1 \quad \& \quad 1=0+1)$ | | $2/x, 1/y, 0/z \quad 1$ |
| (4) | $2 \neq 1+1$ | P |
| (5) | $2=1 \quad \& \quad 1=0+1$ | TP 3, 4 |
| (6) | $2=1$ | S 5 |
| (7) | $1 \neq 0+1$ | TP 2, 6 |
| (8) | $1=0+1$ | S 5 |
| (9) | $1=0+1 \quad \& \quad 1 \neq 0+1$ | A 8, 7 |
| (10) $2=1+1$ | | RAA 4, 9 |

**C.** This can conveniently be done in one continuous proof.

| | | |
|---|---|---|
| (1) $(\forall x)(\forall y)(\forall z)(xQy \quad \& \quad yQz \rightarrow xQz)$ | | P |
| (2) $(\forall x)(\forall y)(xQy \quad \lor \quad yQx)$ | | P |
| (3) $(\forall x)(\forall y)(xIy \quad \leftrightarrow \quad xQy \quad \& \quad yQx)$ | | P |
| (4) $(\forall x)(\forall y)(xPy \quad \leftrightarrow \quad \neg yQx)$ | | P |
| (5) $bQb \quad \lor \quad bQb$ | | $b/x, b/y \quad 2$ |
| (6) $bQb$ | | DP 5 |
| (7) $bQb \quad \& \quad bQb$ | | A 6, 6 |
| (8) $bIb \quad \leftrightarrow \quad bQb \quad \& \quad bQb$ | | $b/x, b/y \quad 3$ |
| (9) $bQb \quad \& \quad bQb \rightarrow bIb$ | | LB 8 |
| (a) (10) $bIb$ | | PP 9, 7 |
| (11) $aQb \quad \lor \quad bQa$ | | $a/x, b/y \quad 2$ |
| (12) $aPb \quad \leftrightarrow \quad \neg bQa$ | | $a/x, b/y \quad 4$ |
| (13) $bPa \quad \leftrightarrow \quad \neg aQb$ | | $b/x, a/y \quad 4$ |
| (14) | $aPb$ | P |
| (15) | $aPb \rightarrow \neg bQa$ | LB 12 |
| (16) | $\neg bQa$ | PP 15, 14 |
| (17) | $aQb$ | TP 11, 16 |
| (18) | $bPa \rightarrow \neg aQb$ | LB 13 |
| (19) | $\neg bPa$ | TT 18, 17 |
| (b) (20) $aPb \rightarrow \neg bPa$ | | CP 14, 19 |

**6.2, Exercise 6, pages 229–230** (*continued*)

|       |        | | | | |        |          |     |
|-------|--------|---|------|---|------|----------|----------|-----|
| (21)  | $aQb$  | & | $bQc$ | → | $aQc$ | $a/x,\ b/y,\ c/z$ | 1 |
| (22)  |        |   | $aPb$ | & | $bQc$ | P        |          |     |
| (23)  |        |   | $aPb$ |   |      | S 22     |          |     |
| (24)  |        |   | $\neg bQa$ |   |  | PP 12, 23 |        |     |
| (25)  |        |   | $aQb$ |   |      | TP 11, 24 |        |     |
| (26)  |        |   | $bQc$ |   |      | S 22     |          |     |
| (27)  |        |   | $aQb$ | &. | $bQc$ | A 25, 26 |        |     |
| (28)  |        |   | $aQc$ |   |      | PP 21, 27 |        |     |
| (c) (29) | $aPb$ | & | $bQc$ | → | $aQc$ | CP 22, 28 |        |     |

▶ **6.3** *Logic of Identity*

**6.3, Exercise 7, page 232**

1. Define: $j$ = July 4th
   $i$ = Independence Day
   In symbols: $j = i$

2. Define: $d$ = Sir Francis Drake
   $Px$ ↔ is a privateer
   In symbols: $Pd$

3. Define: $Cx$ ↔ $x$ is a cat
   $Fx$ ↔ $x$ is a feline
   $(\forall x)(Cx → Fx)$

4. $\frac{1}{5} = 20\%$

5. $f$ = Benjamin Franklin
   $Px$ ↔ $x$ is a printer
   $Pf$

6. $f$ = Franklin
   $a$ = the author of *Poor Richard's Almanac*
   $f = a$

7. $m$ = Monterey
   $c$ = the capital of California
   $m \neq c$

8. $Hx$ ↔ $x$ is a horse
   $Rx$ ↔ $x$ is a good racer
   $\neg(\forall x)(Hx → Rx)$

9. $m$ = Mike
   $Px$ ↔ $x$ is a photographer
   $b$ = the boy I heard about
   $Pm → m = b$

10. $x = 3 → x \neq y$

**6.3, Exercise 8, pages 234–236**

Now that identity, $=$, has been made a logical symbol, it may sometimes be better to space some on each side of it to make this logical symbol stand out as we have done for the connectives.

**A.** The students may have difficulty symbolizing these, particularly 3. and 4. It may be well to be sure they have correctly symbolized them before attempting the deductions.

1. Prove: $2 + 1 > 0$

|      |                        |                  |
|------|------------------------|------------------|
| (1)  | $(\forall x)(Px → x > 0)$ | P              |
| (2)  | $P3$                   | P                |
| (3)  | $3 = 2 + 1$            | P                |
| (4)  | $P(2 + 1)$             | I 2, 3 (See note.) |
| (5)  | $P(2 + 1) → 2 + 1 > 0$ | $2 + 1/x$  1     |
| (6)  | $2 + 1 > 0$            | PP 5, 4          |

*Note.* Very frequently there is a choice as to whether to apply the rule for identities before or after other steps. Note alternative steps:

(4) $P3 \rightarrow 3 > 0$     $3/x$   1
(5) $3 > 0$    PP 4, 2
(6) $2 + 1 > 0$    I 5, 3

**A.**   2. Prove: $Cj$

    (1) $(\forall z)(Mz \rightarrow Cz)$    P
    (2) $Mp$    P
    (3) $j = p$    P
    (4) $Mp \rightarrow Cp$    $p/x$   1
    (5) $Cp$    PP 2, 4
    (6) $Cj$    I 5, 3

3. Prove: $Gw$

    (1) $Se$    P
    (2) $r = w$    P (See note.)
    (3) $Pe \lor Gr$    P
    (4) $(\forall x)(Px \rightarrow \neg Sx)$    P
    (5) $Pe \rightarrow \neg Se$    $e/x$   4
    (6) $\neg Pe$    TT 5, 1
    (7) $Gr$    TP 3, 6
    (8) $Gw$    I 8, 2

*Note.* The conclusion shows us that 'the first defense witness' must be treated as a term. So this must *not* be translated as if 'was the first defense witness' were simply a predicate thus: $Dr$. Here the role the sentence is to play in the argument shows it must be translated as an identity. Translating an identity as simply a predicate weakens what can be done with it in an argument. But note that the verb 'to be' is used in various ways. 'George Sands is a woman', for example, is not an identity since 'George Sands' and 'a woman' do not refer to the same thing.

4. Prove: $Sl \rightarrow Wtl$

    (1) $Cc$    P
    (2) $(\forall x)(\forall y)(Cx \ \& \ Sy \rightarrow \neg Ixy)$    P
    (3) $(\forall x)(\neg Itx \rightarrow Wtx)$    P
    (4) $t = c$    P
    (5) $Ct$    I 1, 4
    (6) $Ct \ \& \ Sl \rightarrow \neg Itl$    $t/x, \ l/y$   2
    (7) $\neg Itl \rightarrow Wtl$    $l/x$   3
    (8) $Ct \ \& \ Sl \rightarrow Wtl$    HS 6, 7
    (9)     $Sl$    P
    (10)     $Ct \ \& \ Sl$    A 5, 9
    (11)     $Wtl$    PP 8, 10
    (12) $Sl \rightarrow Wtl$    CP 9, 11

**B.**   1. Prove: $a \neq b$

    (1) $(\forall x)(Tx \rightarrow Bx)$    P
    (2) $\neg Ba$    P
    (3) $Tb$    P
    (4) $Tb \rightarrow Bb$    $b/x$   1
    (5) $Bb$    PP 4, 3
    (6)     $a = b$    P
    (7)     $Ba$    I 5, 6
    (8)     $Ba \ \& \ \neg Ba$    A 7, 2
    (9) $a \neq b$    RAA 6, 8

**6.3, Exercise 8, pages 234–236** (*continued*)

**B.**     2. Prove: $2^2 + 1 > 2^2$

       (1) $4 = 2^2$                       P

       (2) $4 = 4$                       P

       (3) $(\forall x)(\forall y)(x = y \rightarrow x + 1 > y)$     P

       (4) $4 = 2^2 \rightarrow 4 + 1 > 2^2$        $4/x$, $2^2/y$  3  (See note.)

       (5) $4 + 1 > 2^2$                PP 4, 1

       (6) $2^2 + 1 > 2^2$             I 5, 1

*Note.* Could use $4/x$, $4/y$ or $2^2/x$, $2^2/y$ or $2^2/x$, $4/y$. It would simply alter what applications of line (1) and line (2) would be needed.

   3. Prove: $2 + 3 = 5$

       (1) $(\forall x)(\forall y)(x + y = y + x)$    P

       (2) $3 + 2 = 5$             P

       (3) $3 + 2 = 2 + 3$        $3/x$, $2/y$  1

       (4) $2 + 3 = 5$             I 2, 3

   4. Prove: $3^2 \neq 6$

       (1) $(\forall x)(x < 7 \rightarrow x < 8)$       P

       (2) $\neg(3^2 < 8)$             P

       (3) $6 < 7$                P

       (4) $6 < 7 \rightarrow 6 < 8$        $6/x$  1

       (5) $6 < 8$               PP 3, 4

       (6)          $3^2 = 6$        P

       (7)        $\neg(6 < 8)$       I 2, 6

       (8)      $6 < 8$ &amp; $\neg(6 < 8)$   A 5, 7

       (9) $3^2 \neq 6$             RAA 6, 8

   5. Prove: $\neg(3^2 = 6)$   (*Note.* This is the same as $3^2 \neq 6$.)

       (1) $(\forall x)(x > 7 \rightarrow \neg(x = 6))$    P

       (2) $3^2 = 9$              P

       (3) $9 > 7$               P

       (4) $9 > 7 \rightarrow \neg(9 = 6)$      $9/x$  1

       (5) $\neg(9 = 6)$           PP 4, 3

       (6)        $3^2 = 6$       P

       (7)       $\neg(3^2 = 6)$     I 5, 2

       (8)     $3^2 = 6$ &amp; $\neg(3^2 = 6)$   A 6, 7

       (9) $\neg(3^2 = 6)$         RAA 6, 8

An indirect proof is given here since that is the usual method of proving the negation of an identity. However, a direct proof is possible in one step after line (5).

       (6) $\neg(3^2 = 6)$    I 5, 2

**B.**    6. Prove: $E36$

(1)  $(\forall z)(z^2 = z \cdot z)$      P

(2)  $(\forall x)(\forall y)(Ex \rightarrow E(x \cdot y))$      P

(3)  $E6$      P

(4)  $6^2 = 36$      P

(5)  $6^2 = 6 \cdot 6$      $6/z$   1

(6)  $E6 \rightarrow E6 \cdot 6$      $6/x,\ 6/y$   2

(7)  $E6 \cdot 6$      PP 3, 6

(8)  $E6^2$      I 7, 5

(9)  $E36$      I 8, 4


7. Prove: $4 + 3 \neq 3 \cdot 2$

(1)  $(\forall x)(\forall y)(x + 3 = y + 2 \rightarrow x = 1 + y)$      P

(2)  $4 + 1 \neq 4$      P

(3)  $3 \cdot 2 = 4 + 2$      P

(4)  $4 + 3 = 4 + 2 \rightarrow 4 + 1 = 4$      $4/x,\ 4/y$   1

(5)  $4 + 3 \neq 4 + 2$      TT 2, 4

(6)  $4 + 3 \neq 3 \cdot 2$      I 5, 3


8. Prove: $3 + 2 = 5$

(1)  $(\forall x)(\forall y)(\forall z)(x - y = z \leftrightarrow y + z = x)$      P

(2)  $5 - 3 = 1 + 1$      P

(3)  $1 + 1 = 2$      P

(4)  $5 - 3 = 2 \leftrightarrow 3 + 2 = 5$      $5/x,\ 3/y,\ 2/z$   1

(5)  $5 - 3 = 2 \rightarrow 3 + 2 = 5$      LB 4

(6)  $5 - 3 = 2$      I 2, 3

(7)  $3 + 2 = 5$      PP 5, 6


9. Prove: $O(25)$

(1)  $(\forall u)(\forall v)(\forall w)(u + v = u + w \rightarrow v = w)$      P

(2)  $4 + 5^2 = 29$      P

(3)  $(\forall x)(\forall y)(x^2 = y \rightarrow (Ox \rightarrow Oy))$      P

(4)  $4 + 25 = 29$      P

(5)  $O(5)$      P

(6)  $5^2 = 25 \rightarrow (O(5) \rightarrow O(25))$      $5/x,\ 25/y$   3

(7)  $4 + 5^2 = 4 + 25 \rightarrow 5^2 = 25$      $4/u,\ 5^2/v,\ 25/w$   1

(8)  $4 + 5^2 = 29 \rightarrow 5^2 = 25$      I 4, 7

(9)  $5^2 = 25$      PP 2, 8

(10)  $O(5) \rightarrow O(25)$      PP 6, 9

(11)  $O(25)$      PP 5, 10

**6.3, Exercise 8, pages 234–236** (*continued*)

**B.**  10. Prove: $4 > -4$

    (1) $(\forall x)(\forall y)(\forall z)(x > y \quad \& \quad y > z \quad \rightarrow \quad x > z)$      P

    (2) $4 > 2 + 1$      P

    (3) $(\forall w)(\forall z)(Pw \quad \& \quad Nz \quad \rightarrow \quad w > z)$      P

    (4) $P3 \quad \& \quad N(-4)$      P

    (5) $2 + 1 = 3$      P

    (6) $P3 \quad \& \quad N(-4) \quad \rightarrow \quad 3 > -4$      $3/w, \ -4/z \quad 3$

    (7) $3 > -4$      PP 6, 4

    (8) $4 > 3 \quad \& \quad 3 > -4 \quad \rightarrow \quad 4 > -4$      $4/x, \ 3/y, \ -4/z \quad 1$

    (9) $4 > 3$      I 2, 5

    (10) $4 > 3 \quad \& \quad 3 > -4$      A 9, 7

    (11) $4 > -4$      PP 8, 10

11. Prove: $3 \cdot 7 = 21$

    (1) $(\forall x)(\forall y)(\forall z)(x \cdot (y + z) = (x \cdot y) + (x \cdot z))$      P

    (2) $3 \cdot 5 = 15$      P

    (3) $3 \cdot 2 = 6$      P

    (4) $2 + 5 = 7$      P

    (5) $6 + 15 = 21$      P

    (6) $3 \cdot (2 + 5) = (3 \cdot 2) + (3 \cdot 5)$      $3/x, \ 2/y, \ 5/z \quad 1$

    (7) $3 \cdot 7 = 6 + 15$      I 6; 4, 3, 2

    (8) $3 \cdot 7 = 21$      I 5, 7

▶ 6.4 *Truths of Logic*

**6.4, Exercise 9, pages 237–239**

**A.**  1. Prove: $3 + 1 = (2 + 1) + 1$

    (1) $3 = 2 + 1$      P

    (2) $3 + 1 = 3 + 1$      L (See note.)

    (3) $3 + 1 = (2 + 1) + 1$      I 2, 1

2. Prove: $4 = 2 + 2$

    (1) $2 + 2 = 4$      P

    (2) $4 = 4$      L (See note.)

    (3) $4 = 2 + 2$      I 2, 1

3. Prove: $(2 \cdot 3) \cdot 5 = 30$

    (1) $2 \cdot 3 = 6$      P

    (2) $6 \cdot 5 = 30$      P

    (3) $(2 \cdot 3) \cdot 5 = (2 \cdot 3) \cdot 5$      L (See note.)

    (4) $(2 \cdot 3) \cdot 5 = 6 \cdot 5$      I 3, 1

    (5) $(2 \cdot 3) \cdot 5 = 30$      I 4, 2

**A.**   4. Prove:  $(2 \cdot 7) \cdot 15 = 14 \cdot (3 \cdot 5)$

       (1)  $2 \cdot 7 = 14$             P

       (2)  $3 \cdot 5 = 15$             P

       (3)  $(2 \cdot 7) \cdot 15 = (2 \cdot 7) \cdot 15$     L  (See note.)

       (4)  $(2 \cdot 7) \cdot 15 = 14 \cdot 15$     I  3, 1

       (5)  $(2 \cdot 7) \cdot 15 = 14 \cdot (3 \cdot 5)$   I  4, 2

*Note.* The choice of identity to be introduced has in these four cases been the left side of the desired conclusion set equal to itself. It could just as well be the right side, or some intermediate stage. See, for example, alternatives for Problem 4. Note also that these lines added by rule L are *not* indented into subordinate proofs. They are *not* added premises, assumptions, but statements known, logically, to be true.

    4. (alternatives)

       (1)  $2 \cdot 7 = 14$          P

       (2)  $3 \cdot 5 = 15$          P

       (3)  $14 \cdot 15 = 14 \cdot 15$     L

       (4)  $(2 \cdot 7) \cdot 15 = 14 \cdot 15$    I  1, 3

       (5)  $(2 \cdot 7) \cdot 15 = 14 \cdot (3 \cdot 5)$   I  2, 4

      Or, alternatives for (3) through (5) could be:

       (3)  $(2 \cdot 7) \cdot (3 \cdot 5) = (2 \cdot 7) \cdot (3 \cdot 5)$    L

       (4)  $(2 \cdot 7) \cdot 15 = (2 \cdot 7) \cdot (3 \cdot 5)$    I  3, 2

       (5)  $(2 \cdot 7) \cdot 15 = 14 \cdot (3 \cdot 5)$    I  4, 1

    5. Prove:  $2 \cdot (3 + 4) = (2 \cdot 3) + (2 \cdot 4)$

       (1)  $3 + 4 = 7$           P

       (2)  $2 \cdot 7 = 14$          P

       (3)  $6 + 8 = 14$          P

       (4)  $2 \cdot 3 = 6$           P

       (5)  $2 \cdot 4 = 8$           P

       (6)  $2 \cdot (3 + 4) = 2 \cdot (3 + 4)$     L

       (7)  $2 \cdot (3 + 4) = 2 \cdot 7$     I  6, 1

       (8)  $2 \cdot (3 + 4) = 14$     I  7, 2

       (9)  $2 \cdot (3 + 4) = 6 + 8$     I  8, 3

     (10)  $2 \cdot (3 + 4) = (2 \cdot 3) + 8$    I  9, 4

     (11)  $2 \cdot (3 + 4) = (2 \cdot 3) + (2 \cdot 4)$    I  10, 5

    6. Prove:  $8 + (5 - 2) = (2 \cdot 3) + 5$

       (1)  $(\forall w)(\forall z)(w + z = z + w)$    P

       (2)  $3 + 8 = 11$          P

       (3)  $5 - 2 = 3$           P

       (4)  $2 \cdot 3 = 6$           P

       (5)  $5 + 6 = 11$          P

       (6)  $8 + (5 - 2) = 8 + (5 - 2)$    L

       (7)  $8 + (5 - 2) = 8 + 3$     I  6, 3

       (8)  $8 + 3 = 3 + 8$       $8/w, 3/z$  1

       (9)  $8 + (5 - 2) = 3 + 8$     I  7, 8

     (10)  $8 + (5 - 2) = 11$      I  9, 2

     (11)  $8 + (5 - 2) = 5 + 6$     I  10, 5

     (12)  $5 + 6 = 6 + 5$       $5/w, 6/z$  1

     (13)  $8 + (5 - 2) = 6 + 5$     I  11, 12

     (14)  $8 + (5 - 2) = (2 \cdot 3) + 5$    I  13, 4

**6.4, Exercise 9, pages 237-239** (*continued*)

**A.**   7. Prove: $(1+0)+1=2$

    (1) $(\forall x)(x+0=x)$          P

    (2) $1+1=2$              P

    (3) $1+0=1$              $1/x$   1

    (4) $(1+0)+1=(1+0)+1$    L

    (5) $(1+0)+1=1+1$      I 4, 3

    (6) $(1+0)+1=2$        I 5, 2


    8. Prove: $2+(2+1)=5$

    (1) $(\forall x)(\forall y)(x+y=y+x)$    P

    (2) $3=2+1$              P

    (3) $3+2=5$              P

    (4) $2+(2+1)=2+(2+1)$    L

    (5) $2+(2+1)=2+3$      I 4, 2

    (6) $2+3=3+2$         $2/x,\ 3/y$   1

    (7) $2+(2+1)=3+2$      I 5, 6

    (8) $2+(2+1)=5$        I 7, 3


    9. Prove: $2\cdot(5\cdot7)=70$

    (1) $(\forall x)(\forall y)(x\cdot y=y\cdot x)$    P

    (2) $5\cdot7=35$             P

    (3) $35\cdot2=70$           P

    (4) $2\cdot(5\cdot7)=2\cdot(5\cdot7)$    L

    (5) $2\cdot(5\cdot7)=2\cdot35$     I 4, 2

    (6) $2\cdot35=35\cdot2$      $2/x,\ 35/y$   1

    (7) $2\cdot(5\cdot7)=35\cdot2$     I 5, 6

    (8) $2\cdot(5\cdot7)=70$      I 7, 3


   10. Prove: $13-(1+2)=2\cdot5\ \rightarrow\ 10=4+6$

    (1) $13-3=10$                 P

    (2) $2\cdot(2+3)=4+6$         P

    (3) $1+2=3$                 P

    (4) $2+3=5$                 P

    (5)         $13-(1+2)=2\cdot5$      P

    (6)         $13-3=2\cdot(2+3)$      I 5; 3, 4

    (7)             $10=4+6$         I 6; 1, 2

    (8) $13-(1+2)=2\cdot5\ \rightarrow\ 10=4+6$    CP 5, 7


**B.**   1. Prove: $\mathsf{P} \vee \mathsf{Q} \rightarrow (\mathsf{R}\ \&\ \mathsf{Q}) \vee (\mathsf{P} \vee \mathsf{Q})$

    (1)        $\mathsf{P} \vee \mathsf{Q}$                P

    (2)        $(\mathsf{R}\ \&\ \mathsf{Q}) \vee (\mathsf{P} \vee \mathsf{Q})$     LA 1

    (3) $\mathsf{P} \vee \mathsf{Q} \rightarrow (\mathsf{R}\ \&\ \mathsf{Q}) \vee (\mathsf{P} \vee \mathsf{Q})$    CP 1, 2

**B.**   2. Prove: P & (Q ∨ R) → (P & Q) ∨ (P & R)

| | | |
|---|---|---|
| (1) | P & (Q ∨ R) | P |
| (2) | ¬((P & Q) ∨ (P & R)) | P |
| (3) | ¬(P & Q) & ¬(P & R) | DL 2 |
| (4) | ¬(P & Q) | S 3 |
| (5) | ¬(P & R) | S 3 |
| (6) | ¬P ∨ ¬Q | DL 4 |
| (7) | ¬P ∨ ¬R | DL 5 |
| (8) | P | S 1 |
| (9) | ¬Q | TP 6, 8 |
| (10) | ¬R | TP 7, 8 |
| (11) | Q ∨ R | S 1 |
| (12) | R | TP 11, 9 |
| (13) | R & ¬R | A 12, 10 |
| (14) | (P & Q) ∨ (P & R) | RAA 2, 13 |
| (15) | P & (Q ∨ R) → (P & Q) ∨ (P & R) | CP 1, 14 |

3. Prove: [P → (Q → R)] → [P & Q → R]

| | | |
|---|---|---|
| (1) | P → (Q → R) | P |
| (2) | P & Q | P |
| (3) | P | S 2 |
| (4) | Q | S 2 |
| (5) | Q → R | PP 1, 3 |
| (6) | R | PP 5, 4 |
| (7) | P & Q → R | CP 2, 6 |
| (8) | [P → (Q → R)] → [P & Q → R] | CP 1, 7 |

4. Prove: P & Q → ¬P ∨ Q

| | | |
|---|---|---|
| (1) | P & Q | P |
| (2) | Q | S 1 |
| (3) | ¬P ∨ Q | LA 2 |
| (4) | P & Q → ¬P ∨ Q | CP 1, 3 |

5. Prove: ¬(¬P → ¬Q) → ¬P & Q

| | | |
|---|---|---|
| (1) | ¬(¬P → ¬Q) | P |
| (2) | ¬(¬P & Q) | P |
| (3) | P ∨ ¬Q | DL 2 |
| (4) | ¬P | P |
| (5) | ¬Q | TP 3, 4 |
| (6) | (¬P → ¬Q) | CP 4, 5 |
| (7) | (¬P → ¬Q) & ¬(¬P → ¬Q) | A 6, 1 |
| (8) | ¬P & Q | RAA 2, 7 |
| (9) | ¬(¬P → ¬Q) → ¬P & Q | CP 1, 8 |

**6.4, Exercise 9, pages 237–239** (*continued*)

**B.**　6. Prove: $(P \leftrightarrow Q) \leftrightarrow (\neg P \leftrightarrow \neg Q)$

| | | | |
|---|---|---|---|
| (1) | $P \leftrightarrow Q$ | | P |
| (2) | | $\neg P$ | P |
| (3) | | $Q \to P$ | LB 1 |
| (4) | | $\neg Q$ | TT 3, 2 |
| (5) | $\neg P \to \neg Q$ | | CP 2, 4 |
| (6) | | $\neg Q$ | P |
| (7) | | $P \to Q$ | LB 1 |
| (8) | | $\neg P$ | TT 7, 6 |
| (9) | $\neg Q \to \neg P$ | | CP 6, 8 |
| (10) | $\neg P \leftrightarrow \neg Q$ | | LB 5, 9 |
| (11) | $(P \leftrightarrow Q) \to (\neg P \leftrightarrow \neg Q)$ | | CP 1, 10 |
| (12) | $\neg P \leftrightarrow \neg Q$ | | P |
| (13) | $\neg P \to \neg Q$ | | LB 12 |
| (14) | $\neg Q \to \neg P$ | | LB 12 |
| (15) | | P | P |
| (16) | | Q | TT 14, 15 |
| (17) | $P \to Q$ | | CP 15, 16 |
| (18) | | Q | P |
| (19) | | P | TT 13, 18 |
| (20) | $Q \to P$ | | CP 18, 19 |
| (21) | $P \leftrightarrow Q$ | | LB 17, 20 |
| (22) | $(\neg P \leftrightarrow \neg Q) \to (P \leftrightarrow Q)$ | | CP 12, 21 |
| (23) | $(P \leftrightarrow Q) \leftrightarrow (\neg P \leftrightarrow \neg Q)$ | | LB 11, 22 |

## Chapter 6, Review Test, pages 239–240

**I.**　a.　$Rxy \leftrightarrow x$ rides with $y$
　　　$c = $ Charlotte
　　　$a = $ Ann
　　　$Rca$

　　c.　$b = $ Beethoven
　　　$c = $ the composer of 'Fidelio'
　　　$b = c$

　　b.　$Rx \leftrightarrow x$ can run twenty miles an hour
　　　Domain restricted to people
　　　$(\forall x)(\neg Rx)$

　　d.　$Dx \leftrightarrow x$ is a Democrat
　　　$Rx \leftrightarrow x$ is a Republican
　　　$xAy \leftrightarrow x$ agrees with $y$
　　　$(\forall x)(\forall y)(Dx \ \& \ Ry \ \to \ \neg(xAy))$

　　e.　$j = $ Thomas Jefferson
　　　$a = $ the author of the Declaration of Independence
　　　$j = a$

　　f.　$Rxyz \leftrightarrow x$ represents $y$ in cases before $z$
　　　$s = $ the Solicitor General
　　　$g = $ the government
　　　$c = $ the Supreme Court
　　　$Rsgc$

**I.** f. (alternative)

$Sx \leftrightarrow x$ is the Solicitor General

$Rx \leftrightarrow x$ represents the government before the Supreme Court

$(\forall x)(Sx \leftrightarrow Rx)$

f. (alternative)

$Rxy \leftrightarrow x$ represents the government before $y$

$s =$ the Solicitor General

$c =$ the Supreme Court

$Rsc$

g. $Tx \leftrightarrow x$ is a turtle

$Rx \leftrightarrow x$ is a reptile

$(\forall x)(Tx \leftrightarrow Rx)$

h. $Nx \leftrightarrow x$ was an American naturalist famous for his study of birds

$a =$ Audubon

$Na$

**II.** a. $(2 \cdot 5) - 3 = 7$          c. $4 + (5^2) = 29$          e. $(24 - 6) - 2 = 2$

b. $3 \cdot (4 + 5) = 27$          d. $(4 + 5)^2 = 81$          f. $24 - (6 - 2) = 6$

**III.** a. Prove: $Mp \rightarrow Rp \lor Sp$

| | | |
|---|---|---|
| (1) | $(\forall x)(Mx \rightarrow Px \lor Bx)$ | P |
| (2) | $(\forall x)(Px \rightarrow Rx)$ | P |
| (3) | $(\forall x)(Bx \rightarrow Sx)$ | P |
| (4) | $Mp \rightarrow Pp \lor Bp$ | $p/x$   1 |
| (5) | $Pp \rightarrow Rp$ | $p/x$   2 |
| (6) | $Bp \rightarrow Sp$ | $p/x$   3 |
| (7) | $\quad Mp$ | P |
| (8) | $\quad Pp \lor Bp$ | PP 4, 7 |
| (9) | $\quad Rp \lor Sp$ | DS 8, 5, 6 |
| (10) | $Mp \rightarrow Rp \lor Sp$ | CP 7, 9 |

b. Prove: $\neg Mfg$

| | | |
|---|---|---|
| (1) | $(\forall x)(\forall y)(Bx \ \& \ Fyx \rightarrow Yxy)$ | P |
| (2) | $Bc \ \& \ \neg Ycf$ | P |
| (3) | $(\forall x)(Mxg \rightarrow Fxc)$ | P |
| (4) | $Mfg \rightarrow Ffc$ | $f/x$   3 |
| (5) | $Bc \ \& \ Ffc \rightarrow Ycf$ | $c/x, f/y$   1 |
| (6) | $\neg Ycf$ | S 2 |
| (7) | $\neg(Bc \ \& \ Ffc)$ | TT 5, 6 |
| (8) | $\neg Bc \lor \neg Ffc$ | DL 7 |
| (9) | $Bc$ | S 2 |
| (10) | $\neg Ffc$ | TP 8, 9 |
| (11) | $\neg Mfg$ | TT 4, 10 |

**Chapter 6, Review Test, pages 239–240** (*continued*)

**III.** c. Prove: $e \neq r$

| | | |
|---|---|---|
| (1) | $(\forall x)(Gx \rightarrow Hx)$ | P |
| (2) | $Ge$ | P |
| (3) | $\neg Hr$ | P (See note.) |
| (4) | $Ge \rightarrow He$ | $e/x$   1 |
| (5) | $He$ | PP 4, 2 |
| (6) | $\quad e = r$ | P |
| (7) | $\quad Hr$ | I 5, 6 |
| (8) | $\quad Hr \,\&\, \neg Hr$ | A 7, 3 |
| (9) | $e \neq r$ | RAA 6, 8 |

*Note.* Some students may translate this as $(\forall x)(Mx \rightarrow \neg Bx)$. That would be 'No winner of the horse show has baked a cake'. This is not quite the same thing as '*The* winner of the horse show has not baked a cake'.

**IV.** a. Prove: $\neg Ra \lor Pb$

| | | |
|---|---|---|
| (1) | $(\forall x)(Rx \rightarrow Sx)$ | P |
| (2) | $\neg Sa$ | P |
| (3) | $Ra \rightarrow Sa$ | $a/x$   1 |
| (4) | $\neg Ra$ | TT 3, 2 |
| (5) | $\neg Ra \lor Pb$ | LA 4 |

b. Prove: $\sqrt{25} < 0 \lor \sqrt{25} > 0$

| | | |
|---|---|---|
| (1) | $(\forall x)(x < 0 \leftrightarrow Nx)$ | P |
| (2) | $(\forall x)(x > 0 \leftrightarrow Px)$ | P |
| (3) | $P\sqrt{25} \lor N\sqrt{25}$ | P |
| (4) | $\sqrt{25} < 0 \leftrightarrow N\sqrt{25}$ | $25/x$   1 |
| (5) | $\sqrt{25} > 0 \leftrightarrow P\sqrt{25}$ | $25/x$   2 |
| (6) | $P\sqrt{25} \rightarrow \sqrt{25} > 0$ | LB 5 |
| (7) | $N\sqrt{25} \rightarrow \sqrt{25} < 0$ | LB 4 |
| (8) | $\sqrt{25} < 0 \lor \sqrt{25} > 0$ | DS 3, 6, 7 |

c. Prove: $3 + 5 > 2 + 2$

| | | |
|---|---|---|
| (1) | $(\forall x)(x > 5 \lor x < 7)$ | P |
| (2) | $3 + 5 \not< 7$ | P |
| (3) | $(\forall y)(y > 5 \rightarrow y > 2 + 2)$ | P |
| (4) | $3 + 5 > 5 \lor 3 + 5 < 7$ | $3+5/x$   1 |
| (5) | $3 + 5 > 5$ | TP 4, 2 |
| (6) | $3 + 5 > 5 \rightarrow 3 + 5 > 2 + 2$ | $3+5/y$   3 |
| (7) | $3 + 5 > 2 + 2$ | PP 6, 5 |

**IV.** d. Prove: $Mca \rightarrow Pec$

    (1) $(\forall x)(\forall y)(\neg Mxy \lor Syx)$          P

    (2) $Ba$          P

    (3) $(\forall u)(\forall z)(Szc \rightarrow (Bz \rightarrow Puc))$          P

    (4) $\neg Mca \lor Sac$          $c/x,\ a/y$   1

    (5) $Sac \rightarrow (Ba \rightarrow Pec)$          $e/u,\ a/z$   3

    (6)        $Mca$          P

    (7)        $Sac$          TP 4, 6

    (8)        $Ba \rightarrow Pec$          PP 5, 7

    (9)        $Pec$          PP 8, 2

    (10) $Mca \rightarrow Pec$          CP 6, 9

e. Prove: $3+4>3$

    (1) $(\forall x)(\forall y)(x>y+3 \rightarrow x>y)$          P

    (2) $(\forall z)(\forall u)(u-3<z \rightarrow 3+z>u)$          P

    (3) $(3+3)-3<4$          P

    (4) $(3+3)-3<4 \rightarrow 3+4>3+3$          $4/z,\ 3+3/u$   2

    (5) $3+4 > 3+3$          PP 4, 3

    (6) $3+4>3+3 \rightarrow 3+4>3$          $3+4/x,\ 3/y$

    (7) $3+4>3$          PP 6, 5

f. Prove: $5+2=2+(2+3)$

    (1) $(\forall x)(\forall y)(x+y=y+x)$          P

    (2) $2+3=5$          P

    (3) $5+2=2+5$          $5/x,\ 2/y$   1

    (4) $5+2=2+(2+3)$          I 3, 2

g. Prove: $5-4=1 \rightarrow 6\cdot(5-4)=6$

    (1) $(\forall v)(v\cdot 1=v)$          P

    (2)        $5-4=1$          P

    (3)        $6\cdot(5-4)=6\cdot(5-4)$          L

    (4)        $6\cdot(5-4)=6\cdot 1$          I 3, 2

    (5)        $6\cdot 1=6$          $6/v$   1

    (6)        $6\cdot(5-4)=6$          I 4, 5

    (7) $5-4=1 \rightarrow 6\cdot(5-4)=6$          CP 2, 6

# A Simple Mathematical System:
# Axioms for Addition

▶ 7.1 *Commutative Axiom*

**7.1, Exercise 1, page 244**

**A.** Theorem 4: $3 = 1 + (1 + 1)$

    (1) $3 = 2 + 1$                         Def. of 3

    (2) $2 + 1 = 1 + 2$                 $2/x$, $1/y$ Comm. Axiom

    (3) $3 = 1 + 2$                      I 1, 2

    (4) $3 = 1 + (1 + 1)$             I 3, Def. of 2

An alternative, more elegant proof is the following single line:

    (1) $3 = 1 + (1 + 1)$             I Th. 1, Def. of 2

Theorem 5: $5 = 1 + (1 + (1 + 2))$

    (1) $5 = 4 + 1$                     Def. of 5

    (2) $4 + 1 = 1 + 4$                $4/x$, $1/y$ Comm. Axiom

    (3) $5 = 1 + 4$                      I 1, 2

    (4) $5 = 1 + (1 + (1 + 2))$     I 3, Th. 2

Theorem 6: $4 = (1 + 2) + 1$

    (1) $4 = 3 + 1$                     Def. of 4

    (2) $4 = (2 + 1) + 1$             I 1, Def. of 3

    (3) $2 + 1 = 1 + 2$               $2/x$, $1/y$ Comm. Axiom

    (4) $4 = (1 + 2) + 1$           I 2, 3

Alternative:

    (1) $1 + (1 + 2) + 1 = (1 + 2) + 1$     $1/x$, $1 + 2/y$ Comm. Axiom

    (2) $4 = (1 + 2) + 1$          I 1, Th. 2

Alternative:

    (1) $4 = (1 + 2) + 1$          I Def. of 4, Th. 1

Theorem 7: $5 = 1 + ((2 + 1) + 1)$

    (1) $5 = 1 + (1 + (1 + 2))$       Th. 5

    (2) $1 + (1 + 2) = (1 + 2) + 1$     $1/x$, $1 + 2/y$ Comm. Axiom

    (3) $5 = 1 + ((1 + 2) + 1)$       I 1, 2

    (4) $1 + 2 = 2 + 1$              $1/x$, $2/y$ Comm. Axiom

    (5) $5 = 1 + ((2 + 1) + 1)$       I 3, 4

**A.** Theorem 8: $2+(1+3)=4+2$

    (1) $2+(1+3)=2+(1+3)$

    (2) $1+3=3+1$

    (3) $1+3=4$

    (4) $2+(1+3)=2+4$

    (5) $2+4=4+2$

    (6) $2+(1+3)=4+2$

  L

  $1/x$, $3/y$ Comm. Axiom

  I 2, Def. of 4

  I 1, 3

  $2/x$, $4/y$ Comm. Axiom

  I 4, 5

Theorem 9: $((1+2)+1)+2=(1+1)+4$

    (1) $((1+2)+1)+2=((1+2)+1)+2$

    (2) $1+2=2+1$

    (3) $1+2=3$   (See note.)

    (4) $((1+2)+1)+2=(3+1)+2$

    (5) $((1+2)+1)+2=4+2$

    (6) $4+2=2+4$

    (7) $((1+2)+1)+2=2+4$

    (8) $((1+2)+1)+2=(1+1)+4$

  L

  $1/x$, $2/y$ Comm. Axiom

  I 2, Def. of 3

  I 1, 3

  I 4, Def. of 4

  $4/x$, $2/y$ Comm. Axiom

  I 5, 6

  I 7, Def. of 2

*Note.* There are almost always choices as to when to apply an identity. (3) could have been $((1+2)+1)+2=((2+1)+1)+2$ by identity of (1), (2).

**B.** The variables such as $u$, $v$, $w$, $x$, $y$, $z$ are used in two ways by mathematicians. In the strictly correct sense, a variable may take on any appropriate value in a discourse. For example,

(1) $$(x\geq0 \ \lor \ x<0)$$

is true for any number. This fact is symbolized by the universal quantifier

(2) $$(\forall x)(x\geq0 \ \lor \ x<0).$$

In Chapters 5 and 6, variables were used in this strict logical sense.

However, in mathematics the letters $u$, $v$, $w$, $x$, $y$, $z$ are also frequently used and called variables in cases where they can have only certain values. This is done when we do not yet know, or for convenience do not wish to write out those certain values. For example,

(3) $$3x+5=5x-3$$

is true only for $x=4$. The $x$ cannot take on just any value, although before this equation is solved we might not know just what its true value is. In many of the exercises in the sentential logic in Chapters 1-4 we used these letters in this way. And in the exercises that follow (note problems 2 and 3 on the next page) they will often be used in this way. Whenever they appear in premises without corresponding quantifiers, we must assume they are such disguised particular values. We cannot apply universal specification or (see Chapter 8) universal generalization.

Unfortunately, mathematicians often leave universal quantifiers understood where they are intended. It then becomes a subtle job to decide whether their $x$'s, $y$'s, and so forth, are true variables. One clue is any stated or implied claim that some statement is true *in general*, or for any number, as (1) above. Another method of distinguishing is used when a statement is in the form of an identity (as mathematical statements frequently are). Consider

(4) $$2x+(4-x)=3+(x+1).$$

Simplifying each side, this becomes

(5) $$x+4=x+4,$$

which is a law of logic true for all $x$. Such a statement is called an 'identity' as contrasted to (3), which is called an 'equation'. So calling such an expression an identity is another way of saying that universal quantifiers are understood and that the $x$'s, and so forth, are true variables.

**7.1, Exercise 1, page 244** (*continued*)

**B.**  1. Prove: 4 > 3
      (1) $(1 + 2) + 1 > (1 + 2)$        P
      (2) $1 + 2 = 2 + 1$        $1/x, 2/y$ Comm. Axiom
      (3) $1 + 2 = 3$        I 2, Def. of 3
      (4) $3 + 1 > 3$        I 1, 3
      (5) $4 > 3$        I 4, Def. of 4

  2. Prove: $x \not> 1 + 1$
      (1) $x > 2 \rightarrow x = (1 + 2) + 1$        P
      (2) $x \neq 4$        P
      (3) $1 + 2 = 2 + 1$        $1/x, 2/y$ Comm. Axiom
      (4) $1 + 2 = 3$        I 3, Def. of 3
      (5) $x > 2 \rightarrow x = 3 + 1$        I 1, 4
      (6) $x > 2 \rightarrow x = 4$        I 5, Def. of 4
      (7) $x \not> 2$        TT 6, 2
      (8) $x \not> 1 + 1$        I 7, Def. of 2

  3. Prove: $x > (1 + 2) + 1$
      (1) $x = 5$        P
      (2) $x > 4$ & $x < 6 \leftrightarrow x = 1 + (1 + (1 + 2))$        P
      (3) $x > 4$ & $x < 6 \leftrightarrow x = 5$        I 2, Th. 5
      (4) $x = 5 \rightarrow x > 4$ & $x < 6$        LB 3
      (5) $x > 4$ & $x < 6$        PP 4, 1
      (6) $x > 4$        S 5
      (7) $x > (1 + 2) + 1$        I 6, Th. 6

  4. Prove: 3 > 2
      (1) $(\forall x)(x + 1 > x)$        P
      (2) $2 + 1 > 2$        $2/x$ 1
      (3) $3 > 2$        I 2, Def. of 3

  5. Prove: 5 > 2
      (1) $(\forall x)(\forall y)(y > 0 \rightarrow x + y > x)$        P
      (2) $3 > 0$        P
      (3) $3 + 2 = 4 + 1$        P
      (4) $3 > 0 \rightarrow 2 + 3 > 2$        $2/x, 3/y$, 1
      (5) $2 + 3 > 2$        PP 4, 2
      (6) $2 + 3 = 3 + 2$        $2/x, 3/y$ Comm. Axiom
      (7) $3 + 2 > 2$        I 5, 6
      (8) $4 + 1 > 2$        I 7, 3
      (9) $5 > 2$        I 8, Def. of 5

**C.**  Subtraction is not commutative.

**D.**  Multiplication is also commutative.

**E.**  Division is not commutative.

▶ 7.2 *Associative Axiom*

**7.2, Exercise 2, pages 246–247**

**A.** Multiplication is associative.

**B.** Subtraction and division are not associative.

**C.** Theorem 11: $5 = 3 + 2$

Th. 7

  (1) $5 = 1 + ((2+1) + 1)$    I 1, Def. of 3

  (2) $5 = 1 + (3 + 1)$    $3/x$, $1/y$ Comm. Axiom

  (3) $3 + 1 = 1 + 3$    I 2, 3

  (4) $5 = 1 + (1 + 3)$    $1/x$, $1/y$, $3/z$ Assoc. Axiom

  (5) $(1 + 1) + 3 = 1 + (1 + 3)$    I 4, 5

  (6) $5 = (1 + 1) + 3)$    I 6, Def. of 2

  (7) $5 = 2 + 3$    $2/x$, $3/y$, Comm. Axiom

  (8) $2 + 3 = 3 + 2$    I 7, 8

  (9) $5 = 3 + 2$

There are many ways to prove any of these theorems. For example, note alternative that follows.

Alternative:

Def. of 5

  (1) $5 = 4 + 1$    Def. of 4

  (2) $4 = 3 + 1$    I 1, 2

  (3) $5 = (3 + 1) + 1$    $3/x$, $1/y$, $1/z$ Assoc. Axiom

  (4) $(3 + 1) + 1 = 3 + (1 + 1)$    I 3, 4

  (5) $5 = 3 + (1 + 1)$    I 5, Def. of 2

  (6) $5 = 3 + 2$

Theorem 12: $3 = (1 + 1) + 1$

I Def. of 3, Def. of 2

  (1) $3 = (1 + 1) + 1$

Theorem 13: $4 = ((1 + 1) + 1) + 1$

I Def. of 4, Def. of 3

  (1) $4 = (2 + 1) + 1$    I 1, Def. of 2

  (2) $4 = ((1 + 1) + 1) + 1$

Theorem 14: $4 = (1 + (1 + 1)) + 1$

  (1) $(1 + 1) + 1 = 1 + (1 + 1)$    $1 + 1/x$, $1/y$ Comm. Axiom
       or   $1/x$, $1/y$, $1/z$ Assoc. Axiom

  (2) $4 = (1 + (1 + 1)) + 1$    I 1, Th. 13

Theorem 15: $5 = 2 + (1 + 2)$

  (1) $3 + 2 = 2 + 3$    $3/x$, $2/y$ Comm. Axiom

  (2) $5 = 2 + 3$    I Th. 11, 1

  (3) $5 = 2 + (1 + 2)$    I 2, Th. 1

Theorem 16: $2 + 5 = 3 + 4$

L

  (1) $2 + 5 = 2 + 5$    I 1, Th. 11

  (2) $2 + 5 = 2 + (3 + 2)$    $2/x$, $3 + 2/y$ Comm. Axiom

  (3) $2 + (3 + 2) = (3 + 2) + 2$    I 2, 3

  (4) $2 + 5 = (3 + 2) + 2$    $3/x$, $2/y$, $2/z$ Assoc. Axiom

  (5) $(3 + 2) + 2 = 3 + (2 + 2)$    I 4, 5

  (6) $2 + 5 = 3 + (2 + 2)$    I 6, Th. 10

  (7) $2 + 5 = 3 + 4$

**7.2, Exercise 2, pages 246–247** (*continued*)

**C.**  Theorem 17:  $5+3=(4+2)+2$

| | |
|---|---|
| (1) $5+3=5+3$ | L |
| (2) $5+3=(4+1)+3$ | I 1, Def. of 5 |
| (3) $(4+1)+3=4+(1+3)$ | $4/x,\ 1/y,\ 3/z$ Assoc. Axiom |
| (4) $5+3=4+(1+3)$ | I 2, 3 |
| (5) $1+3=3+1$ | $1/x,\ 3/y$ Comm. Axiom |
| (6) $4=1+3$ | I 5, Def. of 4 |
| (7) $5+3=4+4$ | I 4, 6 |
| (8) $5+3=4+(2+2)$ | I 7, Th. 10 |
| (9) $(4+2)+2=4+(2+2)$ | $4/x,\ 2/y,\ 2/z$ Assoc. Axiom |
| (10) $5+3=(4+2)+2$ | I 8, 9 |

**7.2, Exercise 3, page 248**

On these long proofs it may be necessary to give several hints and considerable help. Remind the students to use theorems whenever possible. Actually, these are not particularly difficult if systematically handled; but some are long.

1.  Prove:  $(1+2)+3>5$

| | |
|---|---|
| (1) $(\forall x)(x+1>x)$ | P |
| (2) $5+1>5$ | $5/x,\ 1$ |
| (3) $1+5=5+1$ | $1/x,\ 5/y$ Comm. Axiom |
| (4) $1+5>5$ | I 2, 3 |
| (5) $3+2=2+3$ | $3/x,\ 2/y$ Comm. Axiom |
| (6) $5=2+3$ | I 5, Th. 11 |
| (7) $1+(2+3)>5$ | I 4, 6 |
| (8) $(1+2)+3=1+(2+3)$ | $1/x,\ 2/y,\ 3/z$ Assoc. Axiom |
| (9) $(1+2)+3>5$ | I 7, 8 |

2.  Prove:  $y=(3+z)+w$

| | |
|---|---|
| (1) $y=3+(z+w)\ \ \lor\ \ (z+w)+2\neq 2+(w+z)$ | P |
| (2) $(z+w)+2=2+(z+w)$ | $z+w/x,\ 2/y$ Comm. Axiom |
| (3) $z+w=w+z$ | $z/x,\ w/y$ Comm. Axiom |
| (4) $(z+w)+2=2+(w+z)$ | I 2, 3 |
| (5) $y=3+(z+w)$ | TP 1, 4 |
| (6) $(3+z)+w=3+(z+w)$ | $3/x,\ z/y,\ w/z$ Assoc. Axiom |
| (7) $y=(3+z)+w$ | I 5, 6 |

3.  Prove:  $(1+2)+(2+4)>(2+1)+2$

| | |
|---|---|
| (1) $4>0$ | P |
| (2) $(\forall x)(\forall y)(y>0\ \ \rightarrow\ \ x+y>x)$ | P |
| (3) $4>0\ \ \rightarrow\ \ ((2+1)+2)+4>(2+1)+2$ | $(2+1)+2/x,\ 4/y$ 2 |
| (4) $((2+1)+2)+4>(2+1)+2$ | PP 3, 1 |
| (5) $2+1=1+2$ | $2/x,\ 1/y$ Comm. Axiom |
| (6) $((1+2)+2)+4>(2+1)+2$ | I 4, 5 |
| (7) $((1+2)+2)+4=(1+2)+(2+4)$ | $1+2/x,\ 2/y,\ 4/z$ Assoc. Axiom |
| (8) $(1+2)+(2+4)>(2+1)+2$ | I 6, 7 |

4. Prove: $x \neq 2+2$ & $x=1+((2+1)+1$

| | | |
|---|---|---|
| (1) | $x=4 \rightarrow x+y=1+(1+(1+2))$ | P |
| (2) | $x=(2+y)+1 \rightarrow x=2+(1+2)$ | P |
| (3) | $\neg(x+y=2+3 \lor x\neq3+y)$ | P |
| (4) | $2+3=3+2$ | $2/x, 3/y$ Comm. Axiom |
| (5) | $2+3=5$ | I 4, Th. 11 |
| (6) | $\neg(x+y=5 \lor x\neq3+y)$ | I 3, 5 |
| (7) | $x+y\neq5$ & $x=3+y$ | DL 6 |
| (8) | $x+y\neq5$ | S 7 |
| (9) | $x=4 \rightarrow x+y=5$ | I 1, Th. 5 |
| (10) | $x\neq4$ | TT 9, 8 |
| (11) | $(2+y)+1=1+(2+y)$ | $2+y/x, 1/y$ Comm. Axiom |
| (12) | $(1+2)+y=1+(2+y)$ | $1/x, 2/y, 4/z$ Assoc. Axiom |
| (13) | $(2+y)+1=(1+2)+y$ | I 11, 12 |
| (14) | $(2+y)+1=3+y$ | I 13, Th. 1 |
| (15) | $x=3+y$ | S 7 |
| (16) | $x=(2+y)+1$ | I 14, 15 |
| (17) | $x=2+(1+2)$ | PP 2, 16 |
| (18) | $x=5$ | I 17, Th. 15 |
| (19) | $x=1+((2+1)+1)$ | I 18, Th. 7 |
| (20) | $x\neq2+2$ | I 10, Th. 10 |
| (21) | $x\neq2+2$ & $x=1+((2+1)+1$ | A 20, 19 |

5. Prove: $a>y+2 \lor a<y+4$

| | | |
|---|---|---|
| (1) | $(\forall x)(x\neq2+3 \leftrightarrow x>(1+2)+2 \lor x<5)$ | P |
| (2) | $a>3+2 \rightarrow a>(y+1)+1$ | P |
| (3) | $a<1+((2+1)+1) \rightarrow a<(y+3)+1$ | P |
| (4) | $a\neq5$ | P |
| (5) | $a\neq2+3 \leftrightarrow a>(1+2)+2 \lor a<5$ | $a/x$ 1 |
| (6) | $a\neq2+3 \rightarrow a>(1+2)+2 \lor a<5$ | LB 5 |
| (7) | $a\neq3+2$ | I 4, Th. 11 |
| (8) | $3+2=2+3$ | $3/x, 2/y$, Comm. Axiom |
| (9) | $a\neq2+3$ | I 7, 8 |
| (10) | $a>(1+2)+2 \lor a<5$ | PP 6, 9 |
| (11) | $((2+1)+1)+1=(2+1)+(1+1)$ | $2+1/x, 1/y, 1/z$ Assoc. Axiom |
| (12) | $2+1=1+2$ | $2/x, 1/y$ Comm. Axiom |
| (13) | $((2+1)+1)+1=(1+2)+2$ | I 11; 12, Def. of 2 |
| (14) | $a>((2+1)+1)+1 \lor a<3+2$ | I 10; 13, Th. 11 |
| (15) | $a<(y+3)+1 \lor a>(y+1)+1$ | DS 14, 3, 2 |
| (16) | $(y+3)+1=y+(3+1)$ | $y/x, 3/y, 1/z$ Assoc. Axiom |
| (17) | $(y+3)+1=y+4$ | I 16, Def. of 4 |
| (18) | $(y+1)+1=y+(1+1)$ | $y/x, 1/y, 1/z$ Assoc. Axiom |
| (19) | $(y+1)+1=y+2$ | I 18, Def. of 2 |
| (20) | $a<y+4 \lor a>y+2$ | I 15; 17, 19 |
| (21) | $a>y+2 \lor a<y+4$ | CL 20 |

**7.2, Exercise 3, page 248** (*continued*)

6. Prove: $x < 4$

| | |
|---|---|
| (1) $x = 1 + 2 \;\rightarrow\; x < (1 + (1 + 1)) + 1$ | P |
| (2) $x = 2 \;\rightarrow\; x < (2 + 1) + 1$ | P |
| (3) $(x^2 - 5x) + (2 + 4) = 0$ | P |
| (4) $(x^2 - 5x) + (3 + 5) = 0 \;\rightarrow\; x = 3 \;\lor\; x = 1 + 1$ | P |
| (5) $2 + 4 = 2 + 4$ | L |
| (6) $2 + 4 = 2 + (3 + 1)$ | I 5, Def. of 4 |
| (7) $2 + (3 + 1) = (3 + 1) + 2$ | $2/x$, $3 + 1/y$ Comm. Axiom |
| (8) $(3 + 1) + 2 = 3 + (1 + 2)$ | $3/x$, $1/y$, $2/z$ Assoc. Axiom |
| (9) $(3 + 1) + 2 = 3 + 3$ | I 8, Th. 1 |
| (10) $2 + (3 + 1) = 3 + 3$ | I 7, 9 |
| (11) $2 + 4 = 3 + 3$ | I 6, 10 |
| (12) $(x^2 - 5x) + (3 + 3) = 0$ | I 3, 11 |
| (13) $x = 3 \;\lor\; x = 1 + 1$ | PP 4, 12 |
| (14) $x = 1 + 2 \;\lor\; x = 2$ | I 13; Th. 1, Def. of 2 |
| (15) $x < (1 + (1 + 1)) + 1 \;\lor\; x < (2 + 1) + 1$ | DS 14, 1, 2 |
| (16) $x < (1 + 2) + 1 \;\lor\; x < 3 + 1$ | I 15, Def. of 2, Def. of 3 |
| (17) $x < 3 + 1 \;\lor\; x < 3 + 1$ | I 16, Th. 1 |
| (18) $x < 3 + 1$ | DP 17 |
| (19) $x < 4$ | I 18, Def. of 4 |

7. Prove: $x = 5 + 3 \;\rightarrow\; x > 4$

| | |
|---|---|
| (1) $x = (4 + 2) + 2 \;\rightarrow\; x > 4 + 2$ | P |
| (2) $x < 3 + 2 \;\rightarrow\; x \not> 1 + 5$ | P |
| (3) $x > 3 + 1 \;\lor\; x < 5$ | P |
| (4) $\quad x = 5 + 3$ | P |
| (5) $\quad x = (4 + 2) + 2$ | I 4, Th. 17 |
| (6) $\quad x > 4 + 2$ | PP 1, 5 |
| (7) $\quad 4 + 2 = 4 + 2$ | L |
| (8) $\quad 4 + 2 = 4 + (1 + 1)$ | I 7, Def. of 2 |
| (9) $\quad (4 + 1) + 1 = 4 + (1 + 1)$ | $4/x$, $1/y$, $1/z$ Assoc. Axiom |
| (10) $\quad 4 + 2 = (4 + 1) + 1$ | I 8, 9 |
| (11) $\quad 4 + 2 = 5 + 1$ | I 10, Def. of 5 |
| (12) $\quad 5 + 1 = 1 + 5$ | $5/x$, $1/y$ Comm. Axiom |
| (13) $\quad 4 + 2 = 1 + 5$ | I 11, 12 |
| (14) $\quad x > 1 + 5$ | I 6, 13 |
| (15) $\quad x \not< 3 + 2$ | TT 2, 14 |
| (16) $\quad x \not< 5$ | I 15, Th. 11 |
| (17) $\quad x > 3 + 1$ | TP 3, 16 |
| (18) $\quad x > 4$ | I 17, Def. of 4 |
| (19) $x = 5 + 3 \;\rightarrow\; x > 4$ | CP 4, 18 |

**7.2, Exercise 4, page 250**

Theorem 18: $5 + (6 + (4 + 3)) = (5 + 6) + (4 + 3)$

| | |
|---|---|
| (1) $(5 + 6) + (4 + 3) = 5 + (6 + (4 + 3))$ | $5/x$, $6/y$, $4 + 3/z$ Assoc. Axiom |
| (2) $(5 + 6) + (4 + 3) = (5 + 6) + (4 + 3)$ | L |
| (3) $5 + (6 + (4 + 3)) = (5 + 6) + (4 + 3)$ | I 2, 1 |

Theorem 19: $2+(3+1)=3+(2+1)$

| | | |
|---|---|---|
| (1) | $2+(3+1)=2+(3+1)$ | L |
| (2) | $(2+3)+1=2+(3+1)$ | $2/x,\ 3/y,\ 1/z$ Assoc. Axiom |
| (3) | $2+(3+1)=(2+3)+1$ | I 1, 2 |
| (4) | $2+3=3+2$ | $2/x,\ 3/y$ Comm. Axiom |
| (5) | $2+(3+1)=(3+2)+1$ | I 3, 4 |
| (6) | $(3+2)+1=3+(2+1)$ | $3/x,\ 2/y,\ 1/z$ Assoc. Axiom |
| (7) | $2+(3+1)=3+(2+1)$ | I 5, 6 |

Theorem 20: $((2+5)+1)+7=1+(2+(5+7))$

| | | |
|---|---|---|
| (1) | $((2+5)+1)+7=(2+5)+(1+7)$ | $2+5/x,\ 1/y,\ 7/z$ Assoc. Axiom |
| (2) | $1+7=7+1$ | $1/x,\ 7/y$ Comm. Axiom |
| (3) | $((2+5)+1)+7=(2+5)+(7+1)$ | I 1, 2 |
| (4) | $((2+5)+7)+1=(2+5)+(7+1)$ | $2+5/x,\ 7/y,\ 1/z$ Assoc. Axiom |
| (5) | $((2+5)+1)+7=((2+5)+7)+1$ | I 3, 4 |
| (6) | $((2+5)+7)+1=1+((2+5)+7)$ | $(2+5)+7/x,\ 1/y$ Comm. Axiom |
| (7) | $((2+5)+1)+7=1+((2+5)+7)$ | I 5, 6 |
| (8) | $(2+5)+7=2+(5+7)$ | $2/x,\ 5/y,\ 7/z$ Assoc. Axiom |
| (9) | $((2+5)+1)+7=1+(2+(5+7))$ | I 7, 8 |

## 7.2, Exercise 5, pages 250–251

**A.**   1. Theorem: $(\forall x)(Fx) \rightarrow Fa$

| | | |
|---|---|---|
| (1) | $(\forall x)(Fx)$ | P |
| (2) | $Fa$ | $a/x$  1 |
| (3) | $(\forall x)(Fx) \rightarrow Fa$ | CP 1, 2 |

2. Theorem: $P \rightarrow Q \lor P$

| | | |
|---|---|---|
| (1) | $P$ | P |
| (2) | $Q \lor P$ | LA 1 |
| (3) | $P \rightarrow Q \lor P$ | CP 1, 2 |

3. Theorem: $a=b$  &  $Ga \rightarrow Gb$

| | | |
|---|---|---|
| (1) | $a=b$  &  $Ga$ | P |
| (2) | $Ga$ | S 1 |
| (3) | $a=b$ | S 1 |
| (4) | $Gb$ | I 2, 3 |
| (5) | $a=b$  &  $Ga \rightarrow Gb$ | CP 1, 4 |

4. Theorem: $P \lor \neg P$

| | | |
|---|---|---|
| (1) | $\neg(P \lor \neg P)$ | P |
| (2) | $\neg P$ & $\neg\neg P$ | DL 1 |
| (3) | $P \lor \neg P$ | RAA 1, 2 |

**7.2, Exercise 5, pages 250–251** (*continued*)

**A.**  5. Theorem: $3 = 2+1$  &  $b = 3$  →  $2+1 = b$

| | | |
|---|---|---|
| (1) | $3 = 2+1$  &  $b = 3$ | P |
| (2) | $3 = 2+1$ | S 1 |
| (3) | $b = 3$ | S 1 |
| (4) | $b = 2+1$ | I 2, 3 |
| (5) | $2+1 = 2+1$ | L |
| (6) | $2+1 = b$ | I 5, 4 |
| (7) | $3 = 2+1$  &  $b = 3$  →  $2+1 = b$ | CP 1, 6 |

**B.**  Theorem 21: $(\forall x)(x+3 > x)$  →  $(1+(2+3))+4 > 2+5$

| | | |
|---|---|---|
| (1) | $(\forall x)(x+3 > x)$ | P |
| (2) | $(2+5)+3 > 2+5$ | $2+5/x$  1 |
| (3) | $(2+5)+3 = 2+(5+3)$ | $2/x$, $5/y$, $3/z$ Assoc. Axiom |
| (4) | $2+(5+3) > 2+5$ | I 2, 3 |
| (5) | $5+3 = 3+5$ | $5/x$, $3/y$ Comm. Axiom |
| (6) | $2+(3+5) > 2+5$ | I 4, 5 |
| (7) | $(2+3)+5 = 2+(3+5)$ | $2/x$, $3/y$, $5/z$ Assoc. Axiom |
| (8) | $(2+3)+5 > 2+5$ | I 6, 7 |
| (9) | $(2+3)+(4+1) > 2+5$ | I 8, Def. of 5 |
| (10) | $((2+3)+4)+1 = (2+3)+(4+1)$ | $2+3/x$, $4/y$, $1/z$ Assoc. Axiom |
| (11) | $((2+3)+4)+1 > 2+5$ | I 9, 10 |
| (12) | $((2+3)+4)+1 = 1+((2+3)+4)$ | $(2+3)+4/x$, $1/y$ Comm. Axiom |
| (13) | $1+((2+3)+4) > 2+5$ | I 11, 12 |
| (14) | $(1+(2+3))+4 = 1+((2+3)+4)$ | $1/x$, $2+3/y$, $4/z$ Assoc. Axiom |
| (15) | $(1+(2+3))+4 > 2+5$ | I 13, 14 |
| (16) | $(\forall x)(x+3 > x)$  →  $(1+(2+3))+4 > 2+5$ | CP 1, 15 |

Theorem 22: $(3+1)+(2+4) = 5+5$

| | | |
|---|---|---|
| (1) | $(3+1)+(2+4) = (3+1)+(2+4)$ | L (See note.) |
| (2) | $3+(1+(2+4)) = (3+1)+(2+4)$ | $3/x$, $1/y$, $2+4/z$ Assoc. Axiom |
| (3) | $(3+1)+(2+4) = 3+(1+(2+4))$ | I 1, 2 |
| (4) | $(1+2)+4 = 1+(2+4)$ | $1/x$, $2/y$, $4/z$ Assoc. Axiom |
| (5) | $(3+1)+(2+4) = 3+((1+2)+4)$ | I 3, 4 |
| (6) | $1+2 = 2+1$ | $1/x$, $2/y$ Comm. Axiom |
| (7) | $(3+1)+(2+4) = 3+((2+1)+4)$ | I 5, 6 |
| (8) | $(2+1)+4 = 2+(1+4)$ | $2/x$, $1/y$, $4/z$ Assoc. Axiom |
| (9) | $(3+1)+(2+4) = 3+(2+(1+4))$ | I 7, 8 |
| (10) | $(3+2)+(1+4) = 3+(2+(1+4))$ | $3/x$, $2/y$, $1+4/z$ Assoc. Axiom |
| (11) | $(3+1)+(2+4) = (3+2)+(1+4)$ | I 9, 10 |
| (12) | $1+4 = 4+1$ | $1/x$, $4/y$ Comm. Axiom |
| (13) | $(3+1)+(2+4) = (3+2)+(4+1)$ | I 11, 12 |
| (14) | $(3+1)+(2+4) = 5+(4+1)$ | I 13, Th. 11 |
| (15) | $(3+1)+(2+4) = 5+5$ | I 14, Def. of 5 |

*Note.* In this proof we recognize that interchanging the '1' and '2' will make it possible to use Theorem 11 and the definition of 5. The following alternative proof avoids so many applications of associativity and commutativity.

**B.**  Theorem 22:  $(3+1)+(2+4)=5+5$

   (1)  $(3+1)+(2+4)=(3+1)+(2+4)$                              L

   (2)  $(3+1)+(2+4)=4+(2+4)$                               I 1, Def. of 4

   (3)  $(3+1)+(2+4)=4+((1+1)+4)$                       I 2, Def. of 2

   (4)  $(1+1)+4=1+(1+4)$                            $1/x,\ 1/y,\ 4/z$ Assoc. Axiom

   (5)  $(3+1)+(2+4)=4+(1+(1+4))$                      I 3, 4

   (6)  $4+(1+(1+4))=(4+1)+(1+4)$                    $4/x,\ 1/y,\ 1+4/z$ Assoc. Axiom

   (7)  $(3+1)+(2+4)=(4+1)+(1+4)$                    I 5, 6

   (8)  $1+4=4+1$                                       $1/x,\ 4/y$ Comm. Axiom

   (9)  $(3+1)+(2+4)=(4+1)+(4+1)$                    I 7, 8

 (10)  $(3+1)+(2+4)=5+5$                              I 9, Def. of 5

Similarly we might have applied the definition of 2 to line (2) of the proof for Theorem 21.

## 7.2, Exercise 6, pages 254–255

  1.  Prove:  $x>4+4\ \rightarrow\ x>2+(1+2)$

    (1)  $x>2+(3+3)\ \rightarrow\ x>5$                     P

    (2)  $x>(4+2)+2\ \rightarrow\ x>5+3$              P

To see the sentential form of this argument, simplify the arithmetic terms.

    Prove:  $x>8\ \rightarrow\ x>5$

    (1)  $x>8\ \rightarrow\ x>5$

    (2)  $x>8\ \rightarrow\ x>8$

This shows us that premise (1) is sufficient to prove the conclusion if we first prove the theorems $2+(3+3)=4+4$ and $5=2+(1+2)$.  (However, it might be that the proofs of these could be long and premise (2) could shorten the total proof).  It is often more elegant to prove such needed theorems in separate proofs. Mathematicians and logicians avoid confusing such a special theorem with their set of numbered theorems by calling it a *lemma*.

  1.  Lemma 1:  $2+(3+3)=4+4$

    (1)  $2+(3+3)=2+(3+3)$                         L

    (2)  $(2+3)+3=2+(3+3)$                       $2/x,\ 3/y,\ 3/z$ Assoc. Axiom

    (3)  $2+(3+3)=(2+3)+3$                       I 1, 2

    (4)  $2+3=3+2$                               $2/x,\ 3/y$ Comm. Axiom

    (5)  $2+(3+3)=(3+2)+3$                       I 3, 4

    (6)  $2+(3+3)=(3+(1+1))+3$                 I 5, Def. of 2

    (7)  $(3+1)+1=3+(1+1)$                       $3/x,\ 1/y,\ 1/z$ Assoc. Axiom

    (8)  $2+(3+3)=((3+1)+1)+3$                I 6, 7

    (9)  $2+(3+3)=(4+1)+3$                       I 8, Def. of 4

  (10)  $(4+1)+3=4+(1+3)$                       $4/x,\ 1/y,\ 3/z$ Assoc. Axiom

  (11)  $2+(3+3)=4+(1+3)$                      I 9, 10

  (12)  $1+3=3+1$                              $1/x,\ 3/y$ Comm. Axiom

  (13)  $2+(3+3)=4+(3+1)$                      I 11, 12

  (14)  $2+(3+3)=4+4$                         I 13, Def. of 4

    Lemma 2:  $5=2+(1+2)$

    (1)  $3+2=2+3$                               $3/x,\ 2/y$ Comm. Axiom

    (2)  $5=2+3$                                 I 1, Th. 11

    (3)  $5=2+(1+2)$                           I 2, Th. 1

**7.2, Exercise 6, pages 254–255** (*continued*)

Prove: $x > 4+4 \;\rightarrow\; x > 2+(1+2)$

| | | |
|---|---|---|
| (1) $x > 2+(3+3) \;\rightarrow\; x > 5$ | | P |
| (2) $x > (4+2)+2 \;\rightarrow\; x > 5+3$ | | P |
| (3) $x > 4+4 \;\rightarrow\; x > 5$ | | I 1, lemma 1 |
| (4) $x > 4+4 \;\rightarrow\; x > 2+(1+2)$ | | I 3, lemma 2 |

An alternative approach (6 fewer lines using premise (2)):

Prove: $x > 4+4 \;\rightarrow\; x > 2+(1+2)$

| | | |
|---|---|---|
| (1) | $x > 2+(3+3) \;\rightarrow\; x > 5$ | P |
| (2) | $x > (4+2)+2 \;\rightarrow\; x > 5+3$ | P |
| (3) | $x > 4+4$ | P |
| (4) | $x > 4+(2+2)$ | I 3, Th. 10 |
| (5) | $(4+2)+2 = 4+(2+2)$ | 4/$x$, 2/$y$, 2/$z$ Assoc. Axiom |
| (6) | $x > (4+2)+2$ | I 4, 5 |
| (7) | $x > 5+3$ | PP 2, 6 |
| (8) | $x > (3+2)+3$ | I 7, Th. 11 |
| (9) | $3+2 = 2+3$ | 3/$x$, 2/$y$ Comm. Axiom |
| (10) | $x > (2+3)+3$ | I 8, 9 |
| (11) | $(2+3)+3 = 2+(3+3)$ | 2/$x$, 3/$y$, 3/$z$ Assoc. Axiom |
| (12) | $x > 2+(3+3)$ | I 10, 11 |
| (13) | $x > 5$ | PP 1, 12 |
| (14) | $x > 2+(1+2)$ | I 13, Th. 15 |
| (15) | $x > 4+4 \;\rightarrow\; x > 2+(1+2)$ | CP 3, 14 |

2. Let us first examine the sentential form of the argument.

Prove: $y = 4$

| | | |
|---|---|---|
| (1) $x = 10 \;\leftrightarrow\; y = 13$ | | P |
| (2) $x = 10 \;\lor\; y = 4$ | | P |
| (3) $\lnot(x = 10 \;\lor\; y = 13)$ | | P |
| (4) $x \neq 10 \;\;\&\;\; y \neq 13$ | | DL 3 |
| (5) $x \neq 10$ | | S 4 |
| (6) $y = 4$ | | TP 2, 5 |

To follow this form of sentential proof we need the following lemma:

Lemma 3: $(4+3)+3 = 5+(1+4)$

| | | |
|---|---|---|
| (1) $(4+3)+3 = (4+3)+3$ | | L |
| (2) $(4+3)+3 = (4+(1+2))+3$ | | I 1, Th. 1 |
| (3) $(4+1)+2 = 4+(1+2)$ | | 4/$x$, 1/$y$, 2/$z$ Assoc. Axiom |
| (4) $(4+3)+3 = ((4+1)+2)+3$ | | I 2, 3 |
| (5) $(4+1)+(2+3) = ((4+1)+2)+3$ | | 4+1/$x$, 2/$y$, 3/$z$ Assoc. Axiom |
| (6) $(4+3)+3 = (4+1)+(2+3)$ | | I 4, 5 |
| (7) $2+3 = 3+2$ | | 2/$x$, 3/$y$ Comm. Axiom |
| (8) $(4+3)+3 = (4+1)+(3+2)$ | | I 6, 7 |
| (9) $(4+3)+3 = (4+1)+5$ | | I 8, Th. 11 |
| (10) $(4+1)+5 = 5+(4+1)$ | | 4+1/$x$, 5/$y$ Comm. Axiom |
| (11) $(4+3)+3 = 5+(4+1)$ | | I 9, 10 |
| (12) $4+1 = 1+4$ | | 4/$x$, 1/$y$ Comm. Axiom |
| (13) $(4+3)+3 = 5+(1+4)$ | | I 11, 12 |

Prove: $y = 4$

| | |
|---|---|
| (1) $x = ((1+2)+3)+4 \leftrightarrow y = 5+(3+5)$ | P |
| (2) $x = 5+(1+4) \lor y = 2+2$ | P |
| (3) $\neg[x = (4+3)+3 \lor y = 2+((3+3)+5)]$ | P |
| (4) $x \neq (4+3)+3 \ \& \ y \neq 2+((3+3)+5)$ | DL 3 |
| (5) $x \neq (4+3)+3$ | S 1 |
| (6) $x \neq 5+(1+4)$ | I 5, Lemma 3 |
| (7) $y = 2+2$ | TP 2, 6 |
| (8) $y = 4$ | I 7, Th. 10 |

## ▶ 7.3 *Axiom for Zero*

**7.3, Exercise 7, pages 256–257**

**A.** Theorem 26: $(2+0)+0 = 2$

| | |
|---|---|
| (1) $(2+0)+0 = (2+0)+0$ | L |
| (2) $2+0 = 2$ | $2/x$ Zero Axiom |
| (3) $(2+0)+0 = 2+0$ | I 1, 2 |
| (4) $(2+0)+0 = 2$ | I 3, 2 |

Theorem 27: $3+(0+0) = 3$

| | |
|---|---|
| (1) $3+0 = 3$ | $3/x$ Zero Axiom |
| (2) $(3+0)+0 = (3+0)+0$ | L |
| (3) $(3+0)+0 = 3+0$ | I 2, 1 |
| (4) $(3+0)+0 = 3$ | I 3, 1 |
| (5) $(3+0)+0 = 3+(0+0)$ | $3/x, 0/y, 0/z$ Assoc. Axiom |
| (6) $3+(0+0) = 3$ | I 4, 5 |

Theorem 28: $4 = 2+(0+2)$

| | |
|---|---|
| (1) $2+0 = 2$ | $2/x$ Zero Axiom |
| (2) $2+0 = 0+2$ | $2/x, 0/y$ Comm. Axiom |
| (3) $0+2 = 2$ | I 1, 2 |
| (4) $4 = 2+(0+2)$ | I Th. 10, 3 |

Theorem 29: $5 = 2+(0+(3+0))$

| | |
|---|---|
| (1) $5 = 2+3$ | I Th. 15, Th. 1 |
| (2) $3+0 = 3$ | $3/x$ Zero Axiom |
| (3) $3+0 = 0+3$ | $3/x, 0/y$ Comm. Axiom |
| (4) $0+3 = 3$ | I 2, 3 |
| (5) $5 = 2+(0+3)$ | I 1, 4 |
| (6) $5 = 2+(0+(3+0))$ | I 5, 2 |

**7.3, Exercise 7, pages 256–257** (*continued*)

**A.** Theorem 30: $4 + (0 + 3) = ((0 + 5) + 0) + 2$

| | | |
|---|---|---|
| (1) | $4 + 3 = 3 + 4$ | $4/x$, $3/y$ Comm. Axiom |
| (2) | $4 + 3 = 2 + 5$ | I 1, Th. 16 |
| (3) | $3 + 0 = 3$ | $3/x$ Zero Axiom |
| (4) | $4 + (3 + 0) = 2 + 5$ | I 2, 3 |
| (5) | $3 + 0 = 0 + 3$ | $3/x$, $0/y$ Comm. Axiom |
| (6) | $4 + (0 + 3) = 2 + 5$ | I 4, 5 |
| (7) | $2 + 5 = 5 + 2$ | $2/x$, $5/y$ Comm. Axiom |
| (8) | $4 + (0 + 3) = 5 + 2$ | I 6, 7 |
| (9) | $5 + 0 = 5$ | $5/x$ Zero Axiom |
| (10) | $5 + 0 = 0 + 5$ | $5/x$, $0/y$ Comm. Axiom |
| (11) | $0 + 5 = 5$ | I 9, 10 |
| (12) | $4 + (0 + 3) = (0 + 5) + 2$ | I 8, 11 |
| (13) | $(0 + 5) + 0 = 0 + 5$ | $0 + 5/x$ Zero Axiom |
| (14) | $4 + (0 + 3) = ((0 + 5) + 0) + 2$ | I 12, 13 |

**B.**    1. Prove: $1 > 0$

| | | |
|---|---|---|
| (1) | $(\forall x)(x + 1 > x)$ | P |
| (2) | $0 + 1 > 0$ | $0/x$   1 |
| (3) | $1 + 0 = 0 + 1$ | $1/x$, $0/y$ Comm. Axiom |
| (4) | $1 + 0 = 1$ | $1/z$ Zero Axiom |
| (5) | $0 + 1 = 1$ | I 4, 3 |
| (6) | $1 < 0$ | I 2, 5 |

     2. Prove: $x = y + 4$

| | | |
|---|---|---|
| (1) | $(x + 3) + 0 = (y + 2) + (3 + 2) \;\; \leftrightarrow \;\; x = (y + 0) + 4$ | P |
| (2) | $y \neq 7$ | P |
| (3) | $1 + (2 + x) = 4 + (3 + y) \;\; \vee \;\; y = 7$ | P |
| (4) | $(x + 3) + 0 = x + 3$ | $x + 3/x$ Zero Axiom |
| (5) | $(x + 3) + 0 = x + (1 + 2)$ | I 1, Th. 1 |
| (6) | $x + (1 + 2) = (1 + 2) + x$ | $x/x$, $1 + 2/y$ Comm. Axiom |
| (7) | $(x + 3) + 0 = (1 + 2) + x$ | I 5, 6 |
| (8) | $(1 + 2) + x = 1 + (2 + x)$ | $1/x$, $2/y$, $x/z$ Assoc. Axiom |
| (9) | $(x + 3) + 0 = 1 + (2 + x)$    (See note.) | I 7, 8 |
| (10) | $(y + 2) + (3 + 2) = (y + 2) + (3 + 2)$ | L |
| (11) | $3 + 2 = 2 + 3$ | $3/x$, $2/y$ Comm. Axiom |
| (12) | $(y + 2) + (3 + 2) = (y + 2) + (2 + 3)$ | I 10, 11 |
| (13) | $(y + 2) + (2 + 3) = y + (2 + (2 + 3))$ | $y/x$, $2/y$, $2 + 3/z$ Assoc. Axiom |
| (14) | $(y + 2) + (3 + 2) = y + (2 + (2 + 3))$ | I 12, 13 |
| (15) | $(2 + 2) + 3 = 2 + (2 + 3)$ | $2/x$, $2/y$, $3/z$ Assoc. Axiom |
| (16) | $(y + 2) + (3 + 2) = y + ((2 + 2) + 3)$ | I 14, 15 |
| (17) | $(y + 2) + (3 + 2) = y + (4 + 3)$ | I 16, Th. 10 |
| (18) | $y + (4 + 3) = (4 + 3) + y$ | $y/x$, $4 + 3/y$ Comm. Axiom |
| (19) | $(y + 2) + (3 + 2) = (4 + 3) + y$ | I 17, 18 |
| (20) | $(4 + 3) + y = 4 + (3 + y)$ | $4/x$, $3/y$, $4/z$ Assoc. Axiom |
| (21) | $(y + 2) + (3 + 2) = 4 + (3 + y)$    (See note.) | I 19, 20 |
| (22) | $y + 0 = y$ | $y/x$ Zero Axiom |
| (23) | $y + 4 = y + 4$ | L |

(24) $(y+0)+4=y+4$   (See note.)  
(25) $1+(2+x)=4+(3+y)$   $\leftrightarrow$   $x=y+4$  
(26) $1+(2+x)=4+(3+y)$   $\rightarrow$   $x=y+4$  
(27) $1+(2+x)=4+(3+y)$  
(28) $x=y+4$

I 23, 22  
I 1; 9, 21, 24  
LB 25  
TP 3, 2  
PP 26, 27

*Note.* These are the required identities. They could have been proved as separate lemmas. In line (25) they are applied to line (1). The rest of the proof is then sentential.

**B.** 3. Prove: $x+2=y+(4+2)$   &   $y=3$  
(1) $x+2\neq y+(2+(1+3))$   $\rightarrow$   $(x+0)+1\neq(2+(2+y))+1$   P  
(2) $x+1=(3+0)+(y+2)$   &   $y+0=(1+0)+2$   P  
(3) $(x+0)+1=(x+0)+1$   L  
(4) $x+0=x$   $x/x$ Zero Axiom  
(5) $(x+0)+1=x+1$   I 3, 4   (See preceding note.)  
(6) $(2+(2+y))+1=1+(2+(2+y))$   $2+(2+y)/x$, $1/y$ Comm. Axiom  
(7) $(1+2)+(2+y)=1+(2+(2+y))$   $1/x$, $2/y$, $2+y/z$ Assoc. Axiom  
(8) $(2+(2+y))+1=(1+2)+(2+y)$   I 6, 7  
(9) $(2+(2+y))+1=3+(2+y)$   I 8, Th. 1  
(10) $3+0=3$   $3/x$ Zero Axiom  
(11) $(2+(2+y))+1=(3+0)+(2+y)$   I 9, 10  
(12) $2+y=y+2$   $2/x$, $y/y$ Comm. Axiom  
(13) $(2+(2+y))+1=(3+0)+(y+2)$   I 11, 12   (See preceding note.)  
(14) $y+0=y$   $y/x$ Zero Axiom   (See preceding note.)

(15) $1+0=1$   $1/x$ Zero Axiom  
(16) $3=(1+0)+2$   I Th. 1, 15  
(17) $x+2\neq y+(4+2)$   $\rightarrow$   $x+1\neq(3+0)+(y+2)$   I 1; Th. 8, 5, 13  
(18) $x+1=(3+0)+(y+2)$   S 2  
(19) $x+2=y+(4+2)$   TT 17, 18  
(20) $y+0=(1+0)+2$   S 2  
(21) $y=3$   I 20; 14, 16  
(22) $x+2=y+(4+2)$   &   $y=3$   A 19, 21

**C.**  1. False  
2. True  
3. False  

4. False  
5. False  
6. True  

7. False  
8. True  
9. False  

▶ 7.4 *Axiom for Negative Numbers*

**7.4, Exercise 8, pages 260–261**

**A.** Theorem 33: $2+(-1)=1$  
(1) $2+(-1)=2+(-1)$   L  
(2)         $=(1+1)+(-1)$   Def. of 2  
(3)         $=1+(1+(-1))$   $1/x$, $1/y$, $-1/z$ Assoc. Axiom   (See note.)  
(4)         $=1+0$   $1/x$ Neg. Axiom  
(5)         $=1$   $1/x$ Zero Axiom

*Note.* Here and in the next problem the strict form is followed of parentheses around negative terms that follow an addition sign: $(1+(-1))$. This is not necessary since no ambiguity results if they are omitted, thus $(1+-1)$. When many parentheses occur together they are difficult to read. Note that line (4) of the proof of Theorem 35 would be easier to read if written $=-5+(1+(4+-4))$.

**7.4, Exercise 8, pages 260–261** (*continued*)

**A.**    Theorem 34: $-1+3=2$

| | | |
|---|---|---|
| (1) | $-1+3=-1+3$ | L |
| (2) | $=3+(-1)$ | $-1/x,\ 3/y$ Comm. Axiom |
| (3) | $=(2+1)+(-1)$ | Def. of 3 |
| (4) | $=2+(1+(-1))$ | $2/x,\ 1/y,\ -1/z$ Assoc. Axiom |
| (5) | $=2+0$ | $1/x$ Neg. Axiom |
| (6) | $=2$ | $2/x$ Zero Axiom |

Theorem 35: $-5+1=-4$

| | | |
|---|---|---|
| (1) | $-5+1=-5+1$ | L |
| (2) | $=(-5+1)+0$ | $-5+1/x$ Zero Axiom |
| (3) | $=(-5+1)+(4+(-4))$ | $4/x$ Neg. Axiom |
| (4) | $=-5+(1+(4+(-4)))$ | $-5/x,\ 1/y,\ 4+(-4)/z$ Assoc. Axiom |
| (5) | $=-5+((1+4)+(-4))$ | $1/x,\ 4/y,\ -4/z$ Assoc. Axiom |
| (6) | $=-5+((4+1)+(-4))$ | $1/x,\ 4/y$ Comm. Axiom |
| (7) | $=-5+(5+(-4))$ | Def. of 5 |
| (8) | $=(-5+5)+(-4)$ | $-5/x,\ 5/y,\ -4/z$ Assoc. Axiom |
| (9) | $=(5+(-5))+(-4)$ | $-5/x,\ 5/y$ Comm. Axiom |
| (10) | $=0+(-4)$ | $5/x$ Neg. Axiom |
| (11) | $=-4+0$ | $0/x,\ -4/y$ Comm. Axiom |
| (12) | $=-4$ | $-4/x$ Zero Axiom |

Theorem 36: $2+(1+-2)=1$

| | | |
|---|---|---|
| (1) | $2+(1+-2)=2+(-2+1)$ | $1/x,\ -2/y$ Comm. Axiom |
| (2) | $=(2+-2)+1$ | $2/x,\ -2/y,\ 1/z$ Assoc. Axiom |
| (3) | $=0+1$ | $2/x$ Neg. Axiom |
| (4) | $=1+0$ | $0/x,\ 1/y$ Comm. Axiom |
| (5) | $=1$ | $1/x$ Zero Axiom |

Theorem 37: $-4+(3+4)=3$

| | | |
|---|---|---|
| (1) | $-4+(3+4)=-4+(4+3)$ | $3/x,\ 4/y$ Comm. Axiom |
| (2) | $=(-4+4)+3$ | $-4/x,\ 4/y,\ 3/z$ Assoc. Axiom |
| (3) | $=(4+-4)+3$ | $-4/x,\ 4/y$ Comm. Axiom |
| (4) | $=0+3$ | $4/x$ Neg. Axiom |
| (5) | $=3+0$ | $0/x,\ 3/y$ Comm. Axiom |
| (6) | $=3$ | $3/x$ Zero Axiom |

Theorem 38: $3+(-5+-3)=-5$

| | | |
|---|---|---|
| (1) | $3+(-5+-3)=3+(-3+-5)$ | $-5/x,\ -3/y$ Comm. Axiom |
| (2) | $=(3+-3)+(-5)$ | $3/x,\ -3/y,\ -5/z$ Assoc. Axiom |
| (3) | $=0+(-5)$ | $3/x$ Neg. Axiom |
| (4) | $=-5+0$ | $0/x,\ -5/y$ Comm. Axiom |
| (5) | $=-5$ | $-5/x$ Zero Axiom |

**A.** Theorem 39: $(2+(1+-1))+-2=0$

(1) $(2+(1+-1))+(-2)=(2+0)+(-2)$ — $1/x$ Neg. Axiom

(2) $\qquad\qquad\qquad =2+(-2)$ — $2/x$ Zero Axiom

(3) $\qquad\qquad\qquad =0$ — $2/x$ Neg. Axiom

Theorem 40: $1+-5=-((1+2)+1)$

(1) $1+-5=-5+1$ — $1/x,\ -5/y$ Comm. Axiom

(2) $\qquad =-4$ — Th. 35

(3) $\qquad =-((1+2)+1)$ — Th. 6

Theorem 41: $-2+5=1+2$

(1) $-2+5=5+-2$ — $-2/x,\ 5/y$ Comm. Axiom

(2) $\qquad =(3+2)+-2$ — Th. 11

(3) $\qquad =3+(2+-2)$ — $3/x,\ 2/y,\ -2/z$ Assoc. Axiom

(4) $\qquad =3+0$ — $2/x$ Neg. Axiom

(5) $\qquad =3$ — $3/x$ Zero Axiom

(6) $\qquad =1+2$ — Th. 1

Theorem 42: $-(2+(0+2))+(2+5)=(1+0)+(4+(2+-4))$

(1) $-(2+(0+2))+(2+5)=-(2+(0+2))+(2+5)$ — L

(2) $\qquad\qquad\qquad\qquad =-4+(2+5)$ — Th. 28

(3) $\qquad\qquad\qquad\qquad =-4+(3+4)$ — Th. 16

(4) $\qquad\qquad\qquad\qquad =3$ — Th. 37

(5) $\qquad\qquad\qquad\qquad =1+2$ — Th. 1

(6) $\qquad\qquad\qquad\qquad =(1+0)+2$ — $1/x$ Zero Axiom

(7) $\qquad\qquad\qquad\qquad =(1+0)+(2+0)$ — $2/x$ Zero Axiom

(8) $\qquad\qquad\qquad\qquad =(1+0)+(2+(4+-4))$ — $4/x$ Neg. Axiom

(9) $\qquad\qquad\qquad\qquad =(1+0)+(2+(-4+4))$ — $4/x,\ -4/y$ Comm. Axiom

(10) $\qquad\qquad\qquad\qquad =(1+0)+((2+-4)+4)$ — $2/x,\ -4/y,\ 4/z$ Assoc. Axiom

(11) $\qquad\qquad\qquad\qquad =(1+0)+(4+(2+-4))$ — $2+-4/x,\ 4/y$ Comm. Axiom

**B.** 1. Prove: $(7+0)+(x+-7)=(-4+x)+(2+(0+2))$

(1) $(7+0)+(x+-7)=(x+-7)+(7+0)$ — $7+0/x,\ x+-7/y$ Comm. Axiom

(2) $\qquad\qquad\qquad =(x+-7)+7$ — $7/x$ Zero Axiom

(3) $\qquad\qquad\qquad =x+(-7+7)$ — $x/x,\ -7/y,\ 7/z$ Assoc. Axiom

(4) $\qquad\qquad\qquad =x+(7+-7)$ — $-7/x,\ 7/y$ Comm. Axiom

(5) $\qquad\qquad\qquad =x+0$ — $7/x$ Neg. Axiom

(6) $\qquad\qquad\qquad =x+(4+-4)$ — $4/x$ Neg. Axiom

(7) $\qquad\qquad\qquad =(4+-4)+x$ — $x/x,\ 4+-4/y$ Comm. Axiom

(8) $\qquad\qquad\qquad =4+(-4+x)$ — $4/x,\ -4/y,\ x/z$ Assoc. Axiom

(9) $\qquad\qquad\qquad =(-4+x)+4$ — $4/x,\ -4+x/y$ Comm. Axiom

(10) $\qquad\qquad\qquad =(-4+x)+(2+(0+2))$ — Th. 28

**7.4, Exercise 8, pages 260–261** (*continued*)

**B.**    2. Prove: $(1+0)+(1+x)=(5-3)+(2+y)$

| | | |
|---|---|---|
| (1) | $x=y+2 \quad \leftrightarrow \quad -1+(3+x)=0+(y+4)$ | P |
| (2) | $x=(y+4)+-2$ | P |
| (3) | $-1+(3+x)=-1+(3+x)$ | L |
| (4) | $=(-1+3)+x$ | $-1/x,\ 3/y,\ x/z$ Assoc. Axiom |
| (5) | $=2+x$ | Th. 34 |
| (6) | $=(1+1)+x$ | Def. of 2 |
| (7) | $=1+(1+x)$ | $1/x,\ 1/y,\ x/z$ Assoc. Axiom |
| (8) | $=(1+0)+(1+x)$ | $1/x$ Zero Axiom    (See note.) |
| (9) | $0+(y+4)=0+(y+4)$ | L |
| (10) | $=(3+-3)+(y+4)$ | $3/x$ Neg. Axiom |
| (11) | $=(3+-3)+(4+y)$ | $y/x,\ 4/y$ Comm. Axiom |
| (12) | $=((3+-3)+4)+y$ | $3+-3/x,\ 4/y,\ y/z$ Assoc. Axiom |
| (13) | $=((3+-3)+(2+2))+y$ | Th. 10 |
| (14) | $=(((3+-3)+2)+2)+y$ | $3+-3/x,\ 2/y,\ 2/z$ Assoc. Axiom |
| (15) | $=((3+-3)+2)+(2+y)$ | $3+-3/x,\ 2/y,\ 2/z$ Assoc. Axiom |
| (16) | $=(3+(-3+2))+(2+y)$ | $3/x,\ -3/y,\ 2/z$ Assoc. Axiom |
| (17) | $=(3+(2+-3))+(2+y)$ | $-3/x,\ 2/y$ Comm. Axiom |
| (18) | $=((3+2)+-3)+(2+y)$ | $3/x,\ 2/y,\ -3/z$ Assoc. Axiom |
| (19) | $=(5-3)+(2+y)$ | Th. 11, Def. 1    (See note.) |
| (20) | $y+2=(y+2)+0$ | $y+2/x$ Zero Axiom |
| (21) | $=(y+2)+(2+-2)$ | $2/x$ Neg. Axiom |
| (22) | $=((y+2)+2)+-2$ | $y+2/x,\ 2/y,\ -2/z$ Assoc. Axiom |
| (23) | $=(y+(2+2))+-2$ | $y/x,\ 2/y,\ 2/z$ Assoc. Axiom |
| (24) | $=(y+4)+-2$ | Th. 10 |
| (25) | $x=y+2$ | I 2, 24 |
| (26) | $x=y+2 \quad \rightarrow \quad -1+(3+x)=0+(y+4)$ | LB 1 |
| (27) | $-1+(3+x)=0+(y+4)$ | PP 26, 25 |
| (28) | $(1+0)+(1+x)=(5-3)+(2+y)$ | I 27; 8, 19 |

3. Prove: $x^2 \neq 4+5 \quad \rightarrow \quad x=3+-1$

| | | |
|---|---|---|
| (1) | $(x^2+-x)+-6=-3+(-2+5)$ | P |
| (2) | $x=-(2+1) \quad \rightarrow \quad x^2=(5+1)+(4+-1)$ | P |
| (3) | $(x^2+-x)+-6=0 \quad \rightarrow \quad \neg(x\neq-3 \ \ \& \ \ x\neq1+1)$ | P |
| (4) | $0=3+3$ | $3/x$ Neg. Axiom |
| (5) | $=-3+3$ | $3/x,\ -3/y$ Comm. Axiom |
| (6) | $=-3+(1+2)$ | Th. 1 |
| (7) | $=-3+(-2+5)$ | Th. 41    (See note.) |
| (8) | $3+-1=-1+3$ | $3/x,\ -1/y$ Comm. Axiom |
| (9) | $=2$ | Th. 34 |
| (10) | $=1+1$ | Def. of 2    (See note.) |
| (11) | $4+5=5+4$ | $4/x,\ 5/y$ Comm. Axiom |
| (12) | $=(5+0)+4$ | $5/x$ Zero Axiom |
| (13) | $=(5+(1+-1))+4$ | $1/x$ Neg. Axiom |
| (14) | $=((5+1)+-1)+4$ | $5/x,\ 1/y,\ -1/z$ Assoc. Axiom |
| (15) | $=(5+1)+(-1+4)$ | $5+1/x,\ -1/y,\ 4/z$ Assoc. Axiom |
| (16) | $=(5+1)+(4+-1)$ | $-1/x,\ 4/y$ Comm. Axiom    (See note.) |

| | |
|---|---|
| (17) $x=-3 \rightarrow x^2=4+5$ | I 2; Def. of 3, 16 |
| (18) $(x^2+-x)+-6=0$ | I 1, 7 |
| (19) $\qquad x^2 \neq 4+5$ | P |
| (20) $\qquad x \neq -3$ | TT 17, 19 |
| (21) $\qquad \neg(x \neq -3 \ \& \ x \neq 1+1)$ | PP 3, 18 |
| (22) $\qquad x=-3 \ \lor \ x=1+1$ | DL 21 |
| (23) $\qquad x=1+1$ | TP 22, 20 |
| (24) $\qquad x=3+-1$ | I 23, 10 |
| (25) $x^2 \neq 4+5 \rightarrow x=3+-1$ | CP 19, 24 |

*Note.* These are the identities required for reducing the problem to a sentential argument.

**C.**  1. 1
2. $(\forall x)(x \cdot 1 = x)$

3. $1/x$
4. $(\forall x)(x \cdot 1/x = 1)$

**D.**  6, 4, 3, 2, 0, $-4$, $-5$, $-7$, $-9$, $-15$

**E.**  1. $-5$
2. $-2$
3. $-4$
4. 6

5. $-8$
6. 3
7. 9
8. 0

9. 7
10. $-1$

**F.**  1. $1/5$
2. $1/2$
3. $1/4$
4. $1/-6$

5. $1/8$
6. $1/-3$
7. $1/-9$
8. There is none.

9. $1/-7$
10. 1 or $1/1$

## Chapter 7, Review Test, page 262

Closed book test on Sections **I**, **II**, and **III**.

**I.**  a. $(\forall x)(\forall y)(x+y=y+x)$

b. $(\forall x)(\forall y)(\forall z)((x+y)+z=x+(y+z))$

c. $(\forall x)(x+0=x)$

d. $(\forall x)(x+(-x)=0)$

**II.**  a. T     c. F    e. T

b. F    d. F    f. T

**III.**  a. (1) $3=2+1$
$\qquad = 2+(1+0)$
$\qquad = 2+(0+1)$

b. (1) $(2+0)+0=2+0$
(2) $\qquad = 2$
(3) $\qquad = 1+1$

c. (1) $-2+(0+2)=-2+(2+0)$
(2) $\qquad = (-2+2)+0$
(3) $\qquad = (2+-2)+0$
(4) $\qquad = 0+0$
(5) $\qquad = 0$

Def. of 3
$1/x$ Zero Axiom
$1/x$, $0/y$ Comm. Axiom

$2+0/x$ Zero Axiom
$2/x$ Zero Axiom
Def. of 2

$0/x$, $2/y$ Comm. Axiom
$-2/x$, $2/y$, $0/z$ Assoc. Axiom
$-2/x$, $2/y$ Comm. Axiom
$2/x$ Zero Axiom
$0/x$ Zero Axiom

**Chapter 7, Review Test, page 262** *(continued)*

**III.** d. (1) $5 = 4+1$            Def. of 5
     (2)     $= (3+1)+1$       Def. of 4
     (3)     $= 3+(1+1)$       3/x, 1/y, 1/z Assoc. Axiom
     (4)     $= (2+1)+2$       Def. of 3, Def. of 2
     (5)     $= 2+(1+2)$       2/x, 1/y, 2/z Assoc. Axiom

*Note.* This can be proved in one step using Theorem 15.

e. (1) $2+(2-2)=2+0$      2/x Neg. Axiom
  (2)       $= 2$           2/x Zero Axiom

**IV.**     Open book, so theorems can be used.
       Prove: $\neg(y=-4 \;\rightarrow\; x=-2)$

(1) $\neg(y=(2+0)+1 \;\vee\; y+-x \neq -5)$      P
(2) $x=-(-1+3) \;\rightarrow\; y=-4+(3+4)$      P
(3) $y+-x=3+(-5+-3) \;\rightarrow\; y=-5+1$      P
(4) $(2+0)+1=(2+0)+1$      L
(5)       $=2+1$      2/x Zero Axiom
(6)       $=3$      Def. of 3
(7)       $-4+(3+4)=$      Th. 37
(8) $y \neq (2+0)+1 \;\&\; y+-x=-5$      DI 1
(9) $y \neq (2+0)+1$      S 8
(10) $y \neq -4+(3+4)$      I 9, 7
(11) $x \neq -(-1+3)$      TT 2, 10
(12) $x \neq -2$      Th. 34
(13) $y+-x=-5$      S 8
(14) $y+-x=3+(-5+-3)$      I 13, Th. 38
(15) $y=-5+1$      PP 3, 14
(16) $y=-4$      I 15, Th. 35
(17) $y=-4 \;\rightarrow\; x=-2$      P
(18) $x=-2$      PP 17, 16
(19) $x=-2 \;\&\; x \neq -2$      A 18, 12
(20) $\neg(y=-4 \;\rightarrow\; x=-2)$      RAA 17, 19

# CHAPTER EIGHT

# Universal Generalization

On page 264 of the text it is remarked that there are some conditions under which rule UG cannot be applied. A careful study of the example on page 265 can make it clear that UG is a valid rule but that there is indeed need for a restriction on its use. Recall that on page 171 it was shown that an argument corresponds to a conditional saying that if the premises are true, then so is the conclusion. It does not just say the conclusion is true. So the argument here amounts to saying the following:

Suppose no fish are mammals and all dogs are mammals. If these are true, then no fish are dogs.

Let us see how UG allows us to prove this. Recall that on page 192 we said that variables correspond to pronouns in English. On line (3), page 265, specifying $x$ for $x$ we use the statement 'No fish are mammals' to infer 'If $it$ is a fish, then $it$ is not a mammal.' Similarly (4) says 'If $it$ is a dog, then $it$ is a mammal' and (8) infers from these that 'If $it$ is a fish, then $it$ is not a dog.' First, if this is to be a proper inference, it is necessary that all the occurrences of the pronoun 'it' refer to the same thing so that the rules of sentential inference can apply. This is indicated by using the same variable, $x$, just as the same constant, say Cleo, could have been specified throughout.

Second, to use (8), 'If $it$ is a fish, then $it$ is not a dog' to conclude 'No fish are dogs' requires that the argument could be carried through for every possible $it$, that is, it must be general with respect to the variable chosen for the quantifier. The only chance that the variable might not be general occurs when that variable appears in a premise without the corresponding universal quantifier governing it. Without a quantifier it might be a disguised particular (see discussion page 145 of this manual), rather than a variable for which the formula is true in general. An example is the added premise of line (5) $Fx$. This says, 'Suppose $it$ is a fish.' On the next two lines we infer '$It$ is not a mammal' and '$It$ is not a dog.' But from these we cannot validly add universal quantifiers by UG and conclude 'Everything is a fish', 'Nothing is a mammal' or 'Nothing is a dog'. But line (8) does not depend on the premise added in the subordinate proof, so it remains general with respect to $x$. UG with respect to $x$ is valid. This gives us the restriction which prevents invalid application of universal generalization:

Universal Generalization with respect to a given variable cannot be applied to a formula if that formula depends on a premise in which that variable appeared without the corresponding universal quantifier. (See footnote, page 267.)

Fortunately, this restriction seldom applies. In arguments where we want to apply UG, premises containing unquantified variables do not normally occur. If they do, universal generalization is invalid and that is the reason for the restriction. Frequently such a premise is added in a subordinate proof to carry out a conditional proof or indirect proof. But here, there is normally no reason to apply UG in the subordinate proof, and the following lines in the main proof do not depend logically on the added premises.

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▶ *8.1 Theorems with Variables*

### 8.1, Exercise 1, pages 265–266

**A.** 1. Prove: $(\forall x)(Sx \rightarrow Vx)$

| | | |
|---|---|---|
| (1) | $(\forall x)(Sx \rightarrow Rx)$ | P |
| (2) | $(\forall y)(Ry \rightarrow Vy)$ | P |
| (3) | $Sx \rightarrow Rx$ | 1 $x/x$ |
| (4) | $Rx \rightarrow Vx$ | 2 $x/y$ |
| (5) | $Sx \rightarrow Vx$ | HS 3, 4 |
| (6) | $(\forall x)(Sx \rightarrow Vx)$ | UG 5 |

2. Prove: $(\forall x)(Vx \rightarrow \neg Ox)$

| | | |
|---|---|---|
| (1) | $(\forall x)(Vx \rightarrow \neg Wx)$ | P |
| (2) | $(\forall x)(Ox \rightarrow Wx)$ | P |
| (3) | $Vx \rightarrow \neg Wx$ | $x/x$ 1 (See note.) |
| (4) | $Ox \rightarrow Wx$ | $x/x$ 2 (See note.) |
| (5) | $Vx$ | P |
| (6) | $\neg Wx$ | PP 3, 5 |
| (7) | $\neg Ox$ | TT 4, 6 |
| (8) | $Vx \rightarrow \neg Ox$ | CP 5, 7 |
| (9) | $(\forall x)(Vx \rightarrow \neg Ox)$ | UG 8 |

*Note.* Usual strategy: Do universal specification before starting subordinate proof. This keeps subordinate proofs short and lines (3) and (4) could be used again if needed if the proof extended for several more lines.

3. Prove: $(\forall x)(Dx \rightarrow \neg Rx)$

| | | |
|---|---|---|
| (1) | $(\forall x)(Rx \rightarrow Mx)$ | P |
| (2) | $(\forall x)(Dx \rightarrow \neg Mx)$ | P |
| (3) | $Rx \rightarrow Mx$ | 1 $x/x$ |
| (4) | $Dx \rightarrow \neg Mx$ | 2 $x/x$ |
| (5) | $Dx$ | P |
| (6) | $\neg Mx$ | PP 4, 5 |
| (7) | $\neg Rx$ | TT 3, 6 |
| (8) | $Dx \rightarrow \neg Rx$ | CP 5, 7 |
| (9) | $(\forall x)(Dx \rightarrow \neg Rx)$ | UG 8 |

4. Prove: $(\forall x)(Lx \rightarrow \neg Fx)$

| | | |
|---|---|---|
| (1) | $(\forall x)(Lx \rightarrow Cx)$ | P |
| (2) | $(\forall x)(Cx \rightarrow \neg Fx)$ | P |
| (3) | $Lx \rightarrow Cx$ | 1 $x/x$ |
| (4) | $Cx \rightarrow \neg Fx$ | 2 $x/x$ |
| (5) | $Lx \rightarrow \neg Fx$ | HS 3, 4 |
| (6) | $(\forall x)(Lx \rightarrow \neg Fx)$ | UG 5 |

5. Prove: $(\forall x)(Sx \rightarrow Vx)$

| | | |
|---|---|---|
| (1) | $(\forall x)(Sx \rightarrow Ax)$ | P |
| (2) | $(\forall x)(Ax \rightarrow Vx)$ | P |
| (3) | $Sx \rightarrow Ax$ | 1 $x/x$ |
| (4) | $Ax \rightarrow Vx$ | 2 $x/x$ |
| (5) | $Sx \rightarrow Vx$ | HS 3, 4 |
| (6) | $(\forall x)(Sx \rightarrow Vx)$ | UG 5 |

6. Prove: $(\forall x)(Sx \rightarrow \neg Mx)$

| | | |
|---|---|---|
| (1) | $(\forall x)(Mx \rightarrow \neg Bx)$ | P |
| (2) | $(\forall x)(Sx \rightarrow Bx)$ | P |
| (3) | $Mx \rightarrow \neg Bx$ | 1 $x/x$ |
| (4) | $Sx \rightarrow Bx$ | 2 $x/x$ |
| (5) | $Sx$ | P |
| (6) | $Bx$ | PP 4, 5 |
| (7) | $\neg Mx$ | TT 3, 6 |
| (8) | $Sx \rightarrow \neg Mx$ | CP 5, 7 |
| (9) | $(\forall x)(Sx \rightarrow \neg Mx)$ | UG 8 |

**A.**  7. Prove: $(\forall x)(Kx \rightarrow \neg Dx)$

| | | |
|---|---|---|
| (1) | $(\forall x)(Kx \rightarrow \neg Cx)$ | P |
| (2) | $(\forall x)(Dx \rightarrow Cx)$ | P |
| (3) | $Kx \rightarrow \neg Cx$ | 1  x/x |
| (4) | $Dx \rightarrow Cx$ | 2  x/x |
| (5) | $Kx$ | P |
| (6) | $\neg Cx$ | PP 3, 5 |
| (7) | $\neg Dx$ | TT 4, 6 |
| (8) | $Kx \rightarrow \neg Dx$ | CP 5, 7 |
| (9) | $(\forall x)(Kx \rightarrow \neg Dx)$ | UG 8 |

8. Prove: $(\forall y)(Ry \rightarrow Ly)$

| | | |
|---|---|---|
| (1) | $(\forall x)(Rx \rightarrow Px)$ | P |
| (2) | $(\forall y)(Py \rightarrow Ly)$ | P |
| (3) | $Ry \rightarrow Py$ | 1  x/y |
| (4) | $Py \rightarrow Ly$ | 2  y/y |
| (5) | $Ry \rightarrow Ly$ | HS 3, 4 |
| (6) | $(\forall y)(Ry \rightarrow Ly)$ | UG 5 |

9. Prove: $(\forall x)(Tx \rightarrow Px)$

| | | |
|---|---|---|
| (1) | $(\forall z)(Dz \rightarrow Pz)$ | P |
| (2) | $(\forall u)(Tu \rightarrow Du)$ | P |
| (3) | $Tx \rightarrow Dx$ | 2  u/x |
| (4) | $Dx \rightarrow Px$ | 1  z/x |
| (5) | $Tx \rightarrow Px$ | HS 3, 4 |
| (6) | $(\forall x)(Tx \rightarrow Px)$ | UG 5 |

10. Prove: $(\forall x)(Lx \rightarrow \neg Sx)$

| | | |
|---|---|---|
| (1) | $(\forall x)(Sx \rightarrow Px)$ | P |
| (2) | $(\forall x)(Lx \rightarrow \neg Px)$ | P |
| (3) | $Sx \rightarrow Px$ | 1  x/x |
| (4) | $Lx \rightarrow \neg Px$ | 2  x/x |
| (5) | $Lx$ | P |
| (6) | $\neg Px$ | PP 4, 5 |
| (7) | $\neg Sx$ | TT 3, 6 |
| (8) | $Lx \rightarrow \neg Sx$ | CP 5, 7 |
| (9) | $(\forall x)(Lx \rightarrow \neg Sx)$ | UG 8 |

11. Prove: $(\forall x)(x + 3 > x)$

| | | |
|---|---|---|
| (1) | $(\forall x)(\forall y)(\forall z)(x > y \ \& \ y > z \rightarrow x > z)$ | P |
| (2) | $(\forall x)(x + 1 > x)$ | P |
| (3) | $x + 1 > x$ | 2  x/x |
| (4) | $(x + 1) + 1 > x + 1$ | 2  x+1/x |
| (5) | $(x + 2) + 1 > x + 2$ | 2  x+2/x |
| (6) | $(x+1)+1 > x+1 \ \& \ x+1 > x \rightarrow (x+1)+1 > x$ | 1  (x+1)+1/x, x+1/y, x/z |
| (7) | $(x+2)+1 > x+2 \ \& \ x+2 > x \rightarrow (x+2)+1 > x$ | 1  (x+2)+1/x, x+2/y, x/z |
| (8) | $(x + 1) + 1 > x + 1 \ \& \ x + 1 > x$ | A 4, 3 |
| (9) | $(x + 1) + 1 > x$ | PP 6, 8 |
| (10) | $x + (1 + 1) > x$ | x/x, 1/y, 1/z  Assoc. Axiom |
| (11) | $x + 2 > x$ | Def. of 2 |
| (12) | $(x + 2) + 1 > x + 2 \ \& \ x + 2 > x$ | A 5, 11 |
| (13) | $(x + 2) + 1 > x$ | PP 7, 12 |
| (14) | $x + (2 + 1) > x$ | x/x, 2/y, 1/z  Assoc. Axiom |
| (15) | $x + 3 > x$ | Def. of 3 |
| (16) | $(\forall x)(x + 3 > x)$ | UG 15 |

**B.**  Notice that in the first steps of these proofs we have combined two steps, first introducing an identity consisting usually of the left side of the desired conclusion set equal to itself. In Theorem 38 for example, this would be $x + (-y) = x + (-y + -x)$ by Rule L. Since this step is very obvious, we can combine it with another. This other step should be one that begins the process of getting the right side of the identity into the desired form. The steps can be combined because the page shows what has been done: the left side of the $=$ shows what both sides of the original identity consisted of, and the right side shows what was produced from this by the next step.

| | |
|---|---|
| Theorem 5: $(\forall x)(\forall y)(\forall z)(x = y + (y + (y + z)))$ | not true |
| Theorem 10: $(\forall y)(\forall x)(x = y + y)$ | not true |
| Theorem 15: $(\forall x)(\forall y)(\forall z)(x = y(z + y))$ | not true |
| Theorem 31: $(\forall x)(\forall y)(x + (y + (-x)) = y)$ | true, proof in text pages 263-264 |

**8.1, Exercise 1, pages 265–266** (*continued*)

**B.**   Theorem 32: $(\forall x)(\forall y)(-x+(y+x)=y)$       true

     (1)   $-x+(y+x)=-x+(x+y)$       $y/x,\ x/y,$ Comm. Axiom

     (2)           $=(-x+x)+y$       $-x/x,\ x/y,\ y/z$ Assoc. Axiom

     (3)           $=(x+-x)+y$       $-x/x,\ x/y$ Comm. Axiom

     (4)           $=0+y$       $x/x$ Neg. Axiom

     (5)           $=y+0$       $0/x,\ y/y$ Comm. Axiom

     (6)           $=y$       $y/x$ Zero Axiom

     (7)   $(\forall x)(\forall y)(-x+(y+x)=y)$       UG 6

Theorem 33: $(\forall x)(\forall y)(x+(-y)=y)$       not true

Theorem 34: $(\forall x)(\forall y)(\forall z)(-x+y=z)$       not true

Theorem 35: $(\forall x)(\forall y)(\forall z)(-x+y=-z)$       not true

Theorem 36: $(\forall x)(\forall y)(x+(y+-x)=y)$       true

     (1)   $x+(y+-x)=x+(-x+y)$       $y/x,\ -x/y$ Comm. Axiom

     (2)           $=(x+-x)+y$       $x/x,\ -x/y,\ y/z$ Assoc. Axiom

     (3)           $=0+y$       $x/x$ Neg. Axiom

     (4)           $=y+0$       $0/x,\ y/y$ Comm. Axiom

     (5)           $=y$       $y/x$ Zero Axiom

     (6)   $(\forall x)(\forall y)(x+(y+-x)=y)$       UG 5

Theorem 37: $(\forall x)(\forall y)(-x+(y+x)=y)$       true

     (1)   $-x+(y+x)=-x+(x+y)$       $y/x,\ x/y$ Comm. Axiom

     (2)           $=(-x+x)+y$       $-x/x,\ x/y,\ y/z$ Assoc. Axiom

     (3)           $=(x+-x)+y$       $-x/x,\ x/y$ Comm. Axiom

     (4)           $=0+y$       $x/x$ Neg. Axiom

     (5)           $=y+0$       $0/x,\ y/y$ Comm. Axiom

     (6)           $=y$       $y/x$ Zero Axiom

     (7)   $(\forall x)(\forall y)(-x+(y+x)=y)$       UG 6

Theorem 38: $(\forall x)(\forall y)(x+(-y+-x)=-y)$       true

     (1)   $x+(-y+-x)=x+(-x+-y)$       $-y/x,\ -x/y$ Comm. Axiom

     (2)           $=(x+-x)+-y$       $x/x,\ -x/y,\ -y/z$ Assoc. Axiom

     (3)           $=0+-y$       $x/x$ Neg. Axiom

     (4)           $=-y+0$       $0/x,\ -y/y$ Comm. Axiom

     (5)           $=-y$       $-y/x$ Zero Axiom

     (6)   $(\forall x)(\forall y)(x+(-y+-x)=-y)$       UG 5

Theorem 39: $(\forall x)(\forall y)(\forall z)((x+(y+-y))+-x=z)$       not true

Theorem 40: $(\forall x)(\forall y)(\forall z)(x+-y=-((x+z)+x))$       not true

Theorem 41: $(\forall x)(\forall y)(\forall z)(-x+y=z+x)$       not true

Theorem 42: $(\forall x)(\forall y)(\forall z)(\forall w)(\forall v)(-(x+(y+x))+(x+z)=(w+y)+(v+(x+-v)))$       not true

**C.** 1. Define: $Fx \leftrightarrow x$ is a frog

$Ax \leftrightarrow x$ is an amphibian

$Vx \leftrightarrow x$ is a vertebrate

Prove: $(\forall x)(Fx \rightarrow Vx)$

| | | |
|---|---|---|
| (1) | $(\forall x)(Fx \rightarrow Ax)$ | P |
| (2) | $(\forall y)(Ay \rightarrow Vy)$ | P |
| (3) | $Fx \rightarrow Ax$ | $x/x$  1 |
| (4) | $Ax \rightarrow Vx$ | $x/y$  2 |
| (5) | $Fx \rightarrow Vx$ | HS 3, 4 |
| (6) | $(\forall x)(Fx \rightarrow Vx)$ | UG 5 |

5. Define: $Rv \leftrightarrow v$ is a boy in Room 5

$Bx \leftrightarrow x$ is on the basketball team

$Tx \leftrightarrow x$ is tall

Prove: $(\forall z)(Rz \rightarrow Tz)$

| | | |
|---|---|---|
| (1) | $(\forall x)(Rx \rightarrow Bx)$ | P |
| (2) | $(\forall y)(By \rightarrow Ty)$ | P |
| (3) | $Rz \rightarrow Bz$ | $z/x$  1 |
| (4) | $Bz \rightarrow Tz$ | $z/y$  2 |
| (5) | $Rz \rightarrow Tz$ | HS 3, 4 |
| (6) | $(\forall z)(Rz \rightarrow Tz)$ | UG 5 |

7. Prove: $\neg Bp$

| | | |
|---|---|---|
| (1) | $(\forall x)(Bx \rightarrow Tx)$ | P |
| (2) | $\neg Tp$ | P |
| (3) | $Bp \rightarrow Tp$ | $p/x$  1 |
| (4) | $\neg Bp$ | TT 3, 2 |

9. Prove: $\neg Rj$

| | | |
|---|---|---|
| (1) | $(\forall x)(Rx \rightarrow \neg Fx)$ | P |
| (2) | $Fj$ | P |
| (3) | $Rj \rightarrow \neg Fj$ | $j/x$  1 |
| (4) | $\neg Rj$ | TT 3, 2 |

11. Prove: $\neg As$

| | | |
|---|---|---|
| (1) | $(\forall y)(Dy \rightarrow \neg Ay)$ | P |
| (2) | $Ds$ | P |
| (3) | $Ds \rightarrow \neg As$ | $s/y$  1 |
| (4) | $\neg As$ | PP 3, 2 |

13. Methods for proving the invalidity of fallacious arguments in predicate logic are given in a *Second Course in Mathematical Logic*. The different argument formed by replacing the second premise of problem 13 of Exercise 1, Chapter 5, page 182, with the premise 'All bills supported by the President are likely to pass' is valid. This new argument is more difficult to symbolize and prove than the others in this section and might be interesting to give to the abler students.

Define: $Tx \leftrightarrow x$ is today's (that is, $x$ is considered today)

$Bx \leftrightarrow x$ is a bill

$Lx \leftrightarrow x$ is likely to pass

$Sx \leftrightarrow x$ is supported by the President

Prove: $(\forall x)(Tx \ \& \ Bx \rightarrow \neg Sx)$

| | | |
|---|---|---|
| (1) | $(\forall x)(Tx \ \& \ Bx \rightarrow \neg Lx)$ | P |
| (2) | $(\forall x)(Bx \ \& \ Sx \rightarrow Lx)$ | P |
| (3) | $Tx \ \& \ Bx \rightarrow \neg Lx$ | $x/x$  1 |
| (4) | $Bx \ \& \ Sx \rightarrow Lx$ | $x/x$  2 |
| (5) | $Tx \ \& \ Bx$ | P |
| (6) | $\neg Lx$ | PP 3, 5 |
| (7) | $\neg(Bx \ \& \ Sx)$ | TT 4, 6 |
| (8) | $\neg Bx \lor \neg Sx$ | DL 7 |
| (9) | $Bx$ | S 5 |
| (10) | $\neg Sx$ | TP 8, 9 |
| (11) | $Tx \ \& \ Bx \rightarrow \neg Sx$ | CP 5, 10 |
| (12) | $(\forall x)(Tx \ \& \ Bx \rightarrow \neg Sx)$ | UG 11 |

**8.1, Exercise 1, pages 265-266** (*continued*)

**C.** 14. Define: $Tx \leftrightarrow x$ is a third grader

$Fx \leftrightarrow x$ is a fourth grader ($x$ is in fourth grade)

$Mx \leftrightarrow x$ is a member of the Girl Scout troop

$Gxy \leftrightarrow x$ is a girl friend of $y$ ⎫     could be simply:

$\phantom{Gxy \leftrightarrow} j = $ Jean             ⎭ $Gx \leftrightarrow x$ is a girl friend of Jean

Prove: $(\forall x)(Gxj \rightarrow \neg Fx)$

| | | |
|---|---|---|
| (1) | $(\forall x)(Tx \lor Fx \rightarrow \neg Mx)$ | P |
| (2) | $(\forall x)(Gxj \rightarrow Mx)$ | P |
| (3) | $Tx \lor Fx \rightarrow \neg Mx$ | $x/x$   1 |
| (4) | $Gxj \rightarrow Mx$ | $x/x$   2 |
| (5) | $\quad Gxj$ | P |
| (6) | $\quad Mx$ | PP 4, 5 |
| (7) | $\quad \neg(Tx \lor Fx)$ | TT 3, 6 |
| (8) | $\quad \neg Tx \ \& \ \neg Fx$ | DL 7 |
| (9) | $\quad \neg Fx$ | S 8 |
| (10) | $Gxj \rightarrow \neg Fx$ | CP 5, 9 |
| (11) | $(\forall x)(Gxj \rightarrow \neg Fx)$ | UG 10 |

15. Prove: $(\forall x)(Ax \rightarrow Cx)$

| | | |
|---|---|---|
| (1) | $(\forall x)(Ax \rightarrow Bx)$ | P |
| (2) | $(\forall x)(Bx \rightarrow Cx)$ | P |
| (3) | $Ax \rightarrow Bx$ | $x/x$   1 |
| (4) | $Bx \rightarrow Cx$ | $x/x$   2 |
| (5) | $Ax \rightarrow Cx$ | HS 3, 4 |
| (6) | $(\forall x)(Ax \rightarrow Cx)$ | UG 5 |

20. Prove: $(\forall x)(Tx \rightarrow \neg Wx)$

| | | |
|---|---|---|
| (1) | $(\forall x)(Tx \rightarrow \neg Sx)$ | P |
| (2) | $(\forall x)(Wx \rightarrow Sx)$ | P |
| (3) | $Tx \rightarrow \neg Sx$ | $x/x$   1 |
| (4) | $Wx \rightarrow Sx$ | $x/x$   2 |
| (5) | $\quad Tx$ | P |
| (6) | $\quad \neg Sx$ | PP 3, 5 |
| (7) | $\quad \neg Wx$ | TT 4, 6 |
| (8) | $Tx \rightarrow \neg Wx$ | CP 5, 7 |
| (9) | $(\forall x)(Tx \rightarrow \neg Wx)$ | UG 8 |

▶ 8.2   *Theorems with Universal Quantifiers*

**8.2, Exercise 2, pages 270-271**

At many points the proofs of the following theorems require inventive and clever steps. Without extensive experience it is difficult to visualize the specification possibilities of the theorems and axioms and the points where identity may be applied. It may be necessary to go slowly, giving hints one at a time when students are stuck.

**A.**   Theorem 6: $(\forall x)(x - 0 = x)$

| | | |
|---|---|---|
| (1) | $x - 0 = x + (-0)$ | $x/x, \ 0/y$   Def. 1 |
| (2) | $\phantom{x-0} = x + 0$ | Th. 5 |
| (3) | $\phantom{x-0} = x$ | $x/x$   Zero Axiom |
| (4) | $(\forall x)(x - 0 = x)$ | UG 3 |

**A.** Theorem 7: $(\forall x)(0-x=-x)$

 (1) $0-x=0+-x$       $0/x$, $x/y$ Def. 1

 (2) $\quad\ = -x+0$       $0/x$, $-x/y$ Comm. Axiom

 (3) $\quad\ = -x$        $-x/x$ Zero Axiom

 (4) $(\forall x)(0-x=-x)$      UG 3

Theorem 8: $(\forall x)(\forall y)(\forall z)((x-y)+(y-z)=x-z)$

 (1) $(x-y)+(y-z)=(x+-y)+(y+-z)$   $x/x$, $y/y$, Def. 1; $y/x$, $z/y$ Def. 1

 (2) $\qquad\qquad\ =x+(-y+(y+-z))$   $x/x$, $-y/y$, $y+-z/z$ Assoc. Axiom

 (3) $\qquad\qquad\ =x+((-y+y)+-z)$   $-y/x$, $y/y$, $-z/z$ Assoc. Axiom

 (4) $\qquad\qquad\ =x+((y+-y)+-z)$   $-y/x$, $y/y$ Comm. Axiom

 (5) $\qquad\qquad\ =x+(0+-z)$     $y/x$ Neg. Axiom

 (6) $\qquad\qquad\ =x+(-z+0)$     $0/x$, $-z/y$ Comm. Axiom

 (7) $\qquad\qquad\ =x+-z$      $-z/x$ Zero Axiom

 (8) $\qquad\qquad\ =x-z$      $x/x$, $-z/y$ Def. 1

 (9) $(\forall x)(\forall y)(\forall z)((x-y)+(y-z)=x-z)$   UG 8

Theorem 9: $(\forall x)(\forall y)(-x=y \ \leftrightarrow\ x=-y)$

 (1) $\qquad\qquad -x=y$      P

 (2) $\qquad\qquad x=-(-x)$     $x/x$ Th. 2  (See note.)

 (3) $\qquad\qquad x=-y$      I 2, 1

 (4) $-x=y\ \rightarrow\ x=-y$     CP 1, 3

 (5) $\qquad\qquad x=-y$      P

 (6) $\qquad\qquad -(-y)=y$     $y/x$ Th. 2

 (7) $\qquad\qquad -x=y$      I 6, 5

 (8) $x=-y\ \rightarrow\ -x=y$     CP 5, 7

 (9) $-x=y\ \leftrightarrow\ x=-y$     LB 4, 8

 (10) $(\forall x)(\forall y)(-x=y\ \leftrightarrow\ x=-y)$   UG 9

*Note.* Something is needed into which $-x$ can be substituted to use the premise.

Theorem 10: $(\forall x)(\forall y)(\forall z)(x+y=z+y\ \rightarrow\ x=z)$

 (1) $\qquad\quad x+y=z+y$     P

 (2) $\qquad\quad x=x+0$      $x/x$ Zero Axiom  (See note.)

 (3) $\qquad\quad\ =x+(y+-y)$    $y/x$ Neg. Axiom

 (4) $\qquad\quad\ =(x+y)+-y$    $x/x$, $y/y$, $-y/z$ Assoc. Axiom

 (5) $\qquad\quad\ =(z+y)+-y$    I 4, 1

 (6) $\qquad\quad\ =z+(y+-y)$    $z/x$, $y/y$, $-y/z$ Assoc. Axiom

 (7) $\qquad\quad\ =z+0$      $y/x$ Neg. Axiom

 (8) $\qquad\quad\ =z$       $z/x$ Zero Axiom

 (9) $x+y=z+y\ \rightarrow\ x=z$    CP 1, 8

 (10) $(\forall x)(\forall y)(\forall z)(x+y=z+y\ \rightarrow\ x=z)$   UG 9

*Note.* Since a conditional is to be proved, line (9), a conditional proof is used and the antecedent is added as a premise. Next, the consequent $x=z$ is to be proved. So $x=x$ is considered as a starter. The adding of zero by the zero axiom and then replacing the zero by some application of the negative axiom (or some theorem depending on it like Th. 11) is the usual strategem for increasing the number of terms. Similarly, reducing the number of terms is accomplished by obtaining something in the form $x+-x$, replacing it by zero (negative axiom) and dropping the zero by the zero axiom. Reduction in terms can also be accomplished by Theorem 1 which essentially depends on these two axioms.

**8.2, Exercise 2, pages 270-271** (*continued*)

**A.** Theorem 11: $(\forall x)(\forall y)((-y+-x)+(x+y)=0)$

| | | | |
|---|---|---|---|
| (1) | $(-y+-x)+(x+y)=-y+(-x+(x+y))$ | (See note.) | $-y/x,\ -x/y,\ x+y/z$ Assoc. Axiom |
| (2) | $=-y+((-x+x)+y)$ | | $-x/x,\ x/y,\ y/z$ Assoc. Axiom |
| (3) | $=-y+((x+-x)+y)$ | | $-x/x,\ x/y$ Comm. Axiom |
| (4) | $=-y+(0+y)$ | | $x/x$ Neg. Axiom |
| (5) | $=-y+(y+0)$ | | $0/x,\ y/y$ Comm. Axiom |
| (6) | $=-y+y$ | | $y/x$ Zero Axiom |
| (7) | $=y+-y$ | | $-y/x,\ y/y$ Comm. Axiom |
| (8) | $=0$ | | $y/x$ Neg. Axiom |
| (9) | $(\forall x)(\forall y)((-y+-x)+(x+y)=0)$ | | UG 8 |

Theorem 12: $(\forall x)(\forall y)(-(x+y)=-x+-y)$

| | | |
|---|---|---|
| (1) | $-(x+y)=0+-(x+y)$  (See note.) | $x+y/x$ Th. 7; $0/x,\ x+y/y$ Def. 1 |
| (2) | $=((-y+-x)+(x+y))+-(x+y)$ | $x/x,\ y/y$ Th. 11 |
| (3) | $=(-y+-x)+((x+y)+-(x+y))$ | $-y+-x/x,\ x+y/y,\ -(x+y)/z$ Assoc. Axiom |
| (4) | $=(-y+-x)+0$ | $x+y/x$ Neg. Axiom |
| (5) | $=-y+-x$ | $-y+-x/x$ Zero Axiom |
| (6) | $=-x+-y$ | $-y/x,\ -x/y$ Comm. Axiom |
| (7) | $(\forall x)(\forall y)(-(x+y)=-x+-y)$ | UG 6 |

*Note.* These specifications and application of identity require some inventiveness. It may be necessary to give the students hints. If the hints have to go as far as stating the specifications, the students should then write out the results and figure out how to apply identity for themselves.

Theorem 13: $(\forall x)(\forall y)(\forall z)(\forall w)((x-y)+(z-w)=(x+z)-(y+w))$

| | | |
|---|---|---|
| (1) | $(x-y)+(z-w)=(x+-y)+(z+-w)$ | $x/x,\ y/y$, Def. 1; $z/x,\ w/y$ Def. 1 |
| (2) | $=x+(-y+(z+-w))$ | $x/x,\ -y/y,\ z+-w/z$ Assoc. Axiom |
| (3) | $=x+((-y+z)+-w)$ | $-y/x,\ z/y,\ -w/z$ Assoc. Axiom |
| (4) | $=x+((z+-y)+-w)$ | $-y/x,\ z/y$ Comm. Axiom |
| (5) | $=x+(z+(-y+-w))$ | $z/x,\ -y/y,\ -w/z$ Assoc. Axiom |
| (6) | $=(x+z)+(-y+-w)$ | $x/x,\ z/y,\ -y+-w/z$ Assoc. Axiom |
| (7) | $=(x+z)+-(y+w)$ | $y/x,\ w/y$ Th. 12 |
| (8) | $=(x+z)-(y+w)$ | $x+z/x,\ y+w/y$ Def. 1 |
| (9) | $(\forall x)(\forall y)(\forall z)(\forall w)((x-y)+(z-w)=(x+z)-(y+w))$ | UG 8 |

Theorem 14: $(\forall x)(\forall y)(\forall z)(\forall w)((x-y)-(z-w)=(x+w)-(y+z))$   (See note.)

| | | |
|---|---|---|
| (1) | $(x-y)-(z-w)=(x+-y)+-(z+-w)$ | three applications of Def. 1 |
| (2) | $=(x+-y)+(-z+-(-w))$ | $z/x,\ -w/y$ Th. 12 |
| (3) | $=(x+-y)+(w+-z)$ | $-z/x,\ --w/y$ Comm. Axiom; $w/x$, Th. 2 |
| (4) | $=x+(-y+(w+-z))$ | $x/x,\ -y/y,\ w+-z/z$ Assoc. Axiom |
| (5) | $=x+((-y+w)+-z)$ | $-y/x,\ w/y,\ -z/z$ Assoc. Axiom |
| (6) | $=x+((w+-y)+-z)$ | $-y/x,\ w/y$ Comm. Axiom |
| (7) | $=x+(w+(-y+-z))$ | $w/x,\ -y/y,\ -z/z$ Assoc. Axiom |
| (8) | $=(x+w)+(-y+-z)$ | $x/x,\ w/y,\ -y+-z/z$ Assoc. Axiom |
| (9) | $=(x+w)+-(y+z)$ | $y/x,\ z/y$ Th. 12 |
| (10) | $=(x+w)-(y+z)$ | $x+w/x,\ y+z/y$ Def. 1 |
| (11) | $(\forall x)(\forall y)(\forall z)(\forall w)((x-y)-(z-w)=(x+w)-(y+z))$ | UG 10 |

*Note.* If the student had difficulty with Theorem 13, he should have less trouble with Theorem 14, that is, if he has a proof of Theorem 13 as a model.

**A.** Theorem 15: $(\forall x)(\forall y)(\forall z)(x-y=z \ \leftrightarrow \ x-z=y)$ — P

| | | |
|---|---|---|
| (1) | $x-y=z$ | $x/x,\ z/y$ Def. 1 (See note.) |
| (2) | $x-z=x+-z$ | (1); $x/x,\ y/y$ Def. 1 |
| (3) | $=x+-(x+-y)$ | $x/x,\ -y/y$ Th. 12 |
| (4) | $=x+(-x+-(-y))$ | $y/x$ Th. 2; $x/x,\ -x/y,\ y/z$ Assoc. Axiom |
| (5) | $=(x+-x)+y$ | $x/x$ Neg. Axiom |
| (6) | $=0+y$ | $0/x,\ y/y$ Comm. Axiom |
| (7) | $=y+0$ | $y/x$ Zero Axiom |
| (8) | $=y$ | CP 1, 8 |
| (9) | $x-y=z \ \rightarrow \ x-z=y$ | P |
| (10) | $x-z=y$ | $x/x,\ y/y$ Def. 1 |
| (11) | $x-y=x+-y$ | (1); $x/x,\ z/y$ Def. 1 |
| (12) | $=x+-(x+-z)$ | $x/x,\ -z/y$ Th. 12 |
| (13) | $=x+(-x+--z)$ | $z/x$ Th. 2; $x/x,\ -x/y,\ z/z$ Assoc. Axiom |
| (14) | $=(x+-x)+z$ | $x/x$ Neg. Axiom |
| (15) | $=0+z$ | $0/x,\ z/y$ Comm. Axiom |
| (16) | $=z+0$ | $z/x$ Zero Axiom |
| (17) | $=z$ | CP 10, 17 |
| (18) | $x-z=y \ \rightarrow \ x-y=z$ | LB 9, 18 |
| (19) | $x-y=z \ \leftrightarrow \ x-z=y$ | UG 19 |
| (20) | $(\forall x)(\forall y)(\forall z)(x-y=z \ \leftrightarrow \ x-z=y)$ | |

*Note.* Line (1) is the usual introduced premise for CP. Line (2) is equally atomatic since the consequent needed is an identity; $x-z=x-z$ is the automatic start for proving that identity. Line (3) simply substitutes $x-y$ from line (1) for $z$ of line (2) and changes form by Definition 1.

Theorem 16: $(\forall x)(\forall y)(\forall z)(x+y=z \ \leftrightarrow \ x-z=-y)$ — P

| | | |
|---|---|---|
| (1) | $x+y=z$ | $x/x,\ z/y$ Def. 1 |
| (2) | $x-z=x+-z$ | (1) |
| (3) | $=x+-(x+y)$ | $x/x,\ y/y$ Th. 12 |
| (4) | $=x+(-x+-y)$ | $x/x,\ -x/y,\ -y/z$ Assoc. Axiom |
| (5) | $=(x+-x)+-y$ | $x/x$ Neg. Axiom |
| (6) | $=0+-y$ | $0/y,\ -y/y$ Comm. Axiom |
| (7) | $=-y+0$ | $-y/x$ Zero Axiom |
| (8) | $=-y$ | CP 1, 8 |
| (9) | $x+y=z \ \rightarrow \ x-z=-y$ | P |
| (10) | $x-z=-y$ | $y/x$ Th. 2 |
| (11) | $-(-y)=y$ | I 11, 10 |
| (12) | $-(x-z)=y$ | L |
| (13) | $x+y=x+y$ | I 13, 12; $x/x,\ z/y$ Def. 1 |
| (14) | $x+y=x+-(x+-z)$ | $x/x,\ -z/y$ Th. 12 |
| (15) | $=x+(-x+--z)$ | $z/x$, Th. 2; $x/x,\ -x/y,\ z/z$ Assoc. Axiom |
| (16) | $=(x+-x)+z$ | $x/x$ Neg. Axiom |
| (17) | $=0+z$ | $0/x,\ z/y$ Comm. Axiom |
| (18) | $=z+0$ | $z/x$ Zero Axiom |
| (19) | $=z$ | CP 10, 19 |
| (20) | $x-z=y \ \leftrightarrow \ x+y=z$ | LB 9, 20 |
| (21) | $x+y=z \ \leftrightarrow \ x-z=-y$ | UG 21 |
| (22) | $(\forall x)(\forall y)(\forall z)(x+y=z \ \leftrightarrow \ x-z=-y)$ | |

**8.2, Exercise 2, pages 270–271** (*continued*)

**A.** Theorem 17: $(\forall x)(\forall y)(\forall z)(\forall w)(x-y=z-w \ \leftrightarrow \ x+w=y+z)$

| | | |
|---|---|---|
| (1) | $x-y=z-w$ | P |
| (2) | $x-y=z-w \ \leftrightarrow \ x-(z-w)=y$ | $x/x,\ y/y,\ z-w/z$ Th. 15    (See Note 1.) |
| (3) | $x-(z-w)=y$ | PP 1, 2   (See Note 2.) |
| (4) | $x+w=(x+w)+0$ | $x+w/x$   Zero Axiom |
| (5) | $\quad =(x+w)+(z+-z)$ | $z/x$   Neg. Axiom |
| (6) | $\quad =(x+w)+(-z+z)$ | $z/x,\ -z/y$   Comm. Axiom |
| (7) | $\quad =x+(w+(-z+z))$ | $x/x,\ w/y,\ -z+z/z$   Assoc. Axiom |
| (8) | $\quad =x+((w+-z)+z)$ | $w/x,\ -z/y,\ z/z$   Assoc. Axiom |
| (9) | $\quad =(x+(w+-z))+z$ | $x/x,\ w+-z/y,\ z/z$   Assoc. Axiom |
| (10) | $\quad =(x+(--w+-z))+z$ | $w/x,$   Th. 2 |
| (11) | $\quad =(x-(w+z))+z$ | $w/x,\ z/y$   Th. 12 |
| (12) | $\quad =y+z$ | (3) |
| (13) | $x-y=z-w \ \rightarrow \ x+w=y+z$ | CP 1, 12 |
| (14) | $x+w=y+z$ | P |
| (15) | $x+w=y+z \ \leftrightarrow \ x-(y+z)=-w$ | $x/x,\ w/y,\ y+z/z$ Th. 16 |
| (16) | $x+w=y+z \ \rightarrow \ x-(y+z)=-w$ | LB 15 |
| (17) | $x-(y+z)=-w$ | PP 14, 16 |
| (18) | $x-y=(x-y)+0$ | $x-y/x$ Zero Axiom |
| (19) | $\quad =(x-y)+(z+-z)$ | $z/x$ Neg. Axiom |
| (20) | $\quad =(z+-z)+(x-y)$ | $x-y/x,\ z+-z/y$ Comm. Axiom |
| (21) | $\quad =z+(-z+(x-y))$ | $z/x,\ -z/y,\ x-y/z$ Assoc. Axiom |
| (22) | $\quad =z+((x-y)+-z)$ | $-z/x,\ x-y/y$ Comm. Axiom |
| (23) | $\quad =z+((x+-y)+-z)$ | $x/x,\ y/y$ Def. 1 |
| (24) | $\quad =z+(x+(-y+-z))$ | $x/x,\ -y/y,\ -z/z$ Assoc. Axiom |
| (25) | $\quad =z+(x+-(y+z))$ | $y/x,\ z/y$ Th. 12 |
| (26) | $\quad =z+(x-(y+z))$ | $x/x,\ y+z/y$ Th. 12 |
| (27) | $\quad =z+-w$ | (17) |
| (28) | $\quad =z-w$ | $z/x,\ w/y$ Def. 1 |
| (29) | $x+w=y+z \ \rightarrow \ x-y=z-w$ | CP 14, 28 |
| (30) | $x-y=z-w \ \leftrightarrow \ x+w=y+z$ | LB 13, 29 |
| (31) | $(\forall x)(\forall y)(\forall z)(\forall w)(x-y=z-w \ \leftrightarrow \ x+w=y+z)$ | UG 30 |

*Note 1.* Since the premise added for CP could not combine with the obvious identity, $x+w=x+w$, to get the desired consequent, $x+w=y+z$, as in the previous two theorems, another approach is needed. By the specifications indicated, Theorem 15 exactly fits the added premise, thus giving something to work with.

*Note 2.* This gives an expression for $y$, but what is needed for $x+w=y+z$ is an expression for $y+z$. That would be $x-(z-w)+z$ by line (3). So $(x+w)+(z+-z)$, i.e. $(x+w)+0$. This indicates the next step.

**B.**    1. Prove: $-3<-2$

| | | |
|---|---|---|
| (1) | $(\forall x)(x<x+1)$ | P |
| (2) | $-3<-3+1$ | $-3/x$   1 |
| (3) | $-3+1=-3+1$ | L |
| (4) | $\quad =-(2+1)+1$ | Def. of 3 |
| (5) | $\quad =(-2+-1)+1$ | $2/x,\ 1/y$ Th. 12 |
| (6) | $\quad =-2+(-1+1)$ | $-2/x,\ -1/y,\ -1/z$ Assoc. Axiom |
| (7) | $\quad =-2+(1+-1)$ | $-1/x,\ 1/y$ Comm. Axiom |

|  |  |  |
|---|---|---|
| (8) | $= -2 + 0$ | $1/x$  Neg. Axiom |
|  |  | $-2/x$  Zero Axiom |
| (9) | $= -2$ | 2, 9 |
| (10) | $-3 < -2$ |  |

**B.**  2. Prove: $x + x = 0$

(1) $x = 0$                                     P

(2) $x + x = 0$                            I $x/x$ Zero Axiom, 1

3. Prove: $x = -4$

(1) $x + 5 = 1$                             P (See note.)

(2) $5 + x = 1$                             $x/x$, $5/y$ Comm. Axiom

(3) $1 = 1 + (4 + -4)$                $1/x$ Zero Axiom, $4/x$ Neg. Axiom

(4) $\quad = (1 + 4) + -4$          $1/x$, $4/y$, $-4/z$ Assoc. Axiom

(5) $\quad = (4 + 1) + -4$          $1/x$, $4/y$ Comm. Axiom

(6) $\quad = 5 + -4$                     Def. of 5

(7) $5 + x = 5 + -4$                    2, 6

(8) $3 + x = 5 + -4 \;\rightarrow\; x = -4$    $5/x$, $x/y$, $-4/z$ Th. 1

(9) $x = -4$                                 PP 8, 7

*Note.* Recognizing that with $5 + x = 5 + -4$, we could reach our conclusion by the cancellation theorem, Theorem 1, if we had $1 = 5 + -4$.

4. Prove: $(\forall x)(x < 0 \;\leftrightarrow\; 0 < -x)$

(1) $(\forall x)(\forall y)(\forall z)(y < z \;\rightarrow\; x + y < x + z)$      P

(2) $x < 0 \;\rightarrow\; -x + x < -x + 0$        $-x/x$, $x/y$, $0/z$  1  (See note.)

(3) $x < 0 \;\rightarrow\; x + -x < -x + 0$        $-x/x$, $x/y$ Comm. Axiom

(4) $x < 0 \;\rightarrow\; 0 < -x + 0$               $x/x$ Neg. Axiom

(5) $x < 0 \;\rightarrow\; 0 < -x$                      $-x/x$ Zero Axiom

(6) $x < -x \;\rightarrow\; x + 0 < x + -x$        $x/x$, $0/y$, $-x/z$  1

(7) $0 < -x \;\rightarrow\; x < 0$                      $x/x$ Zero Axiom; $x/x$ Neg. Axiom

(8) $x < 0 \;\leftrightarrow\; x < -x$                  LB 5, 7

(9) $(\forall x)(x < 0 \;\leftrightarrow\; 0 < -x)$      UG 8

*Note.* $x/y$, $0/z$ are easily seen to make the antecedent of the premise match the antecedent of the conclusion. The consequent is then $? + x < ? + 0$ where the question now is what one thing could be filled in for the '?' to make this match the desired consequent. Looked at this way, $-x/x$ is easily selected.

5. Prove: $(\forall x)\neg(x < x)$

(1) $(\forall x)(\forall y)(x < y \;\rightarrow\; \neg(y < x))$      P

(2) $x < x \;\rightarrow\; \neg(x < x)$              $x/x$, $x/y$  1

(3) $\quad\quad x < x$                                  P

(4) $\quad\quad \neg(x < x)$                           PP 2, 3

(5) $\quad\quad (x < x) \;\&\; \neg(x < x)$       A 3, 4

(6) $\neg(x < x)$                                          RAA 3, 5

(7) $(\forall x)\neg(x < x)$                            UG 6