Indirect Proofs

Outline for Today

- **Preliminaries**
  - Disproving statements
  - Mathematical implications
- **Proof by Contrapositive**
  - The basic method.
  - An interesting application.
- **Proof by Contradiction**
  - The basic method.
  - Contradictions and implication.
  - Contradictions and quantifiers.

Disproving Statements

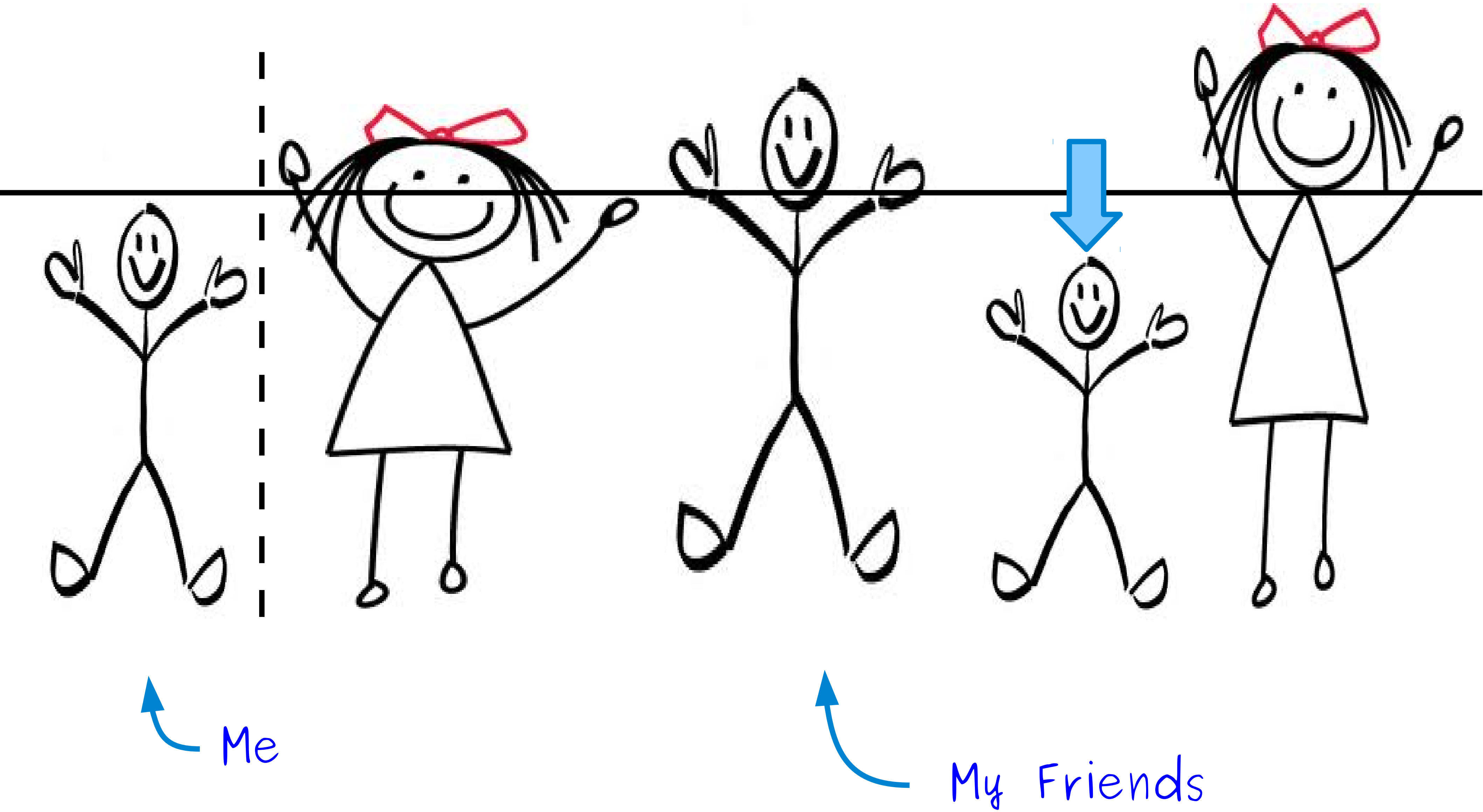
Proofs and Disproofs

- A proof is an argument establishing why a statement is true.
- A ***disproof*** is an argument establishing why a statement is *false*.
- Although proofs generally are more famous than disproofs, mathematics heavily relies on disproofs of conjectures that have turned out to be false.

Writing a Disproof

- The easiest way to disprove a statement is to write a proof of the opposite of that statement.
  - The opposite of a statement X is called the ***negation*** of statement X.
- A typical disproof is structured as follows:
  - Start by stating that you're going to disprove some statement X.
  - Write out the negation of statement X.
  - Write a normal proof that statement X is false.

"All My Friends Are Taller Than Me"



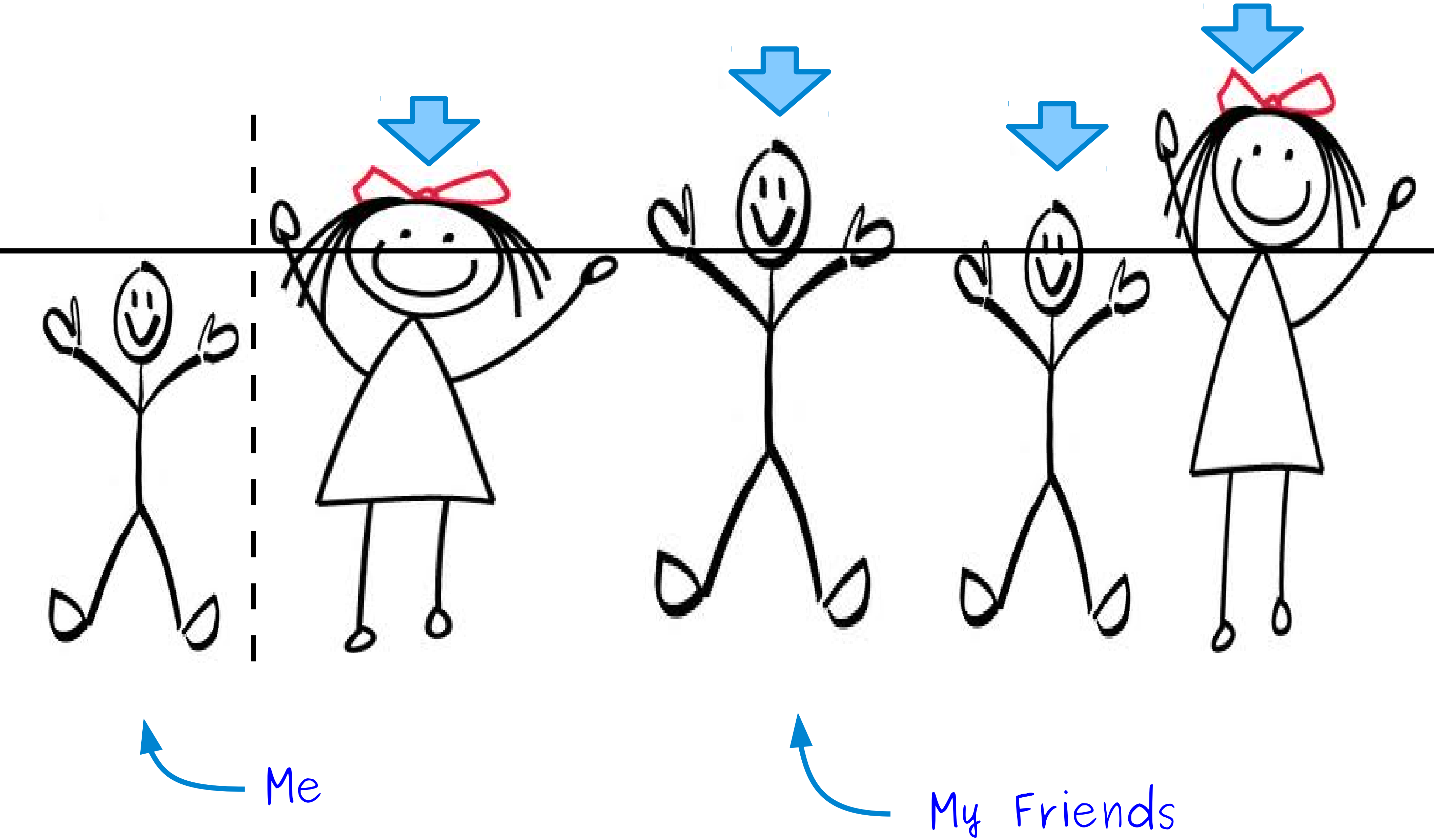
The negation of the *universal* statement

For all x, $P(x)$ is true.

is the *existential* statement

There exists an x where $P(x)$ is false.

“Some Friend Is Shorter Than Me”



The negation of the *existential* statement

There exists an x where $P(x)$ is true.

is the *universal* statement

For all x, $P(x)$ is false.

What would we have to show to disprove the following statement?

"Some set is the same size as its power set."

This is what we need to prove
to disprove the original statement.

"For any set S, the set S is ***not*** the same size as its power set."

Logical Implication

Implications

- An **implication** is a statement of the form

  If P is true, then Q is true.

- Some examples:
  - If n is an even integer, then n^2 is an even integer.
  - If $A \subseteq B$ and $B \subseteq A$, then $A = B$.
  - If you liked it, then you should've put a ring on it.

Implications

- An **implication** is a statement of the form

  If *P* is true, then *Q* is true.

- In the above statement, the term "*P* is true" is called the ***antecedent*** and the term "*Q* is true" is called the ***consequent***.

What Implications Mean

- Consider the simple statement

If I put fire near cotton, it will burn.

- Some questions to consider:
  - Does this apply to all fire and all cotton, or just some types of fire and some types of cotton? *(Scope)*
  - Does the fire cause the cotton to burn, or does the cotton burn for another reason? *(Causality)*
- These are deeper questions than they might seem.
- To mathematically study implications, we need to formalize what implications really mean.

What Implications Mean

- In mathematics, the statement

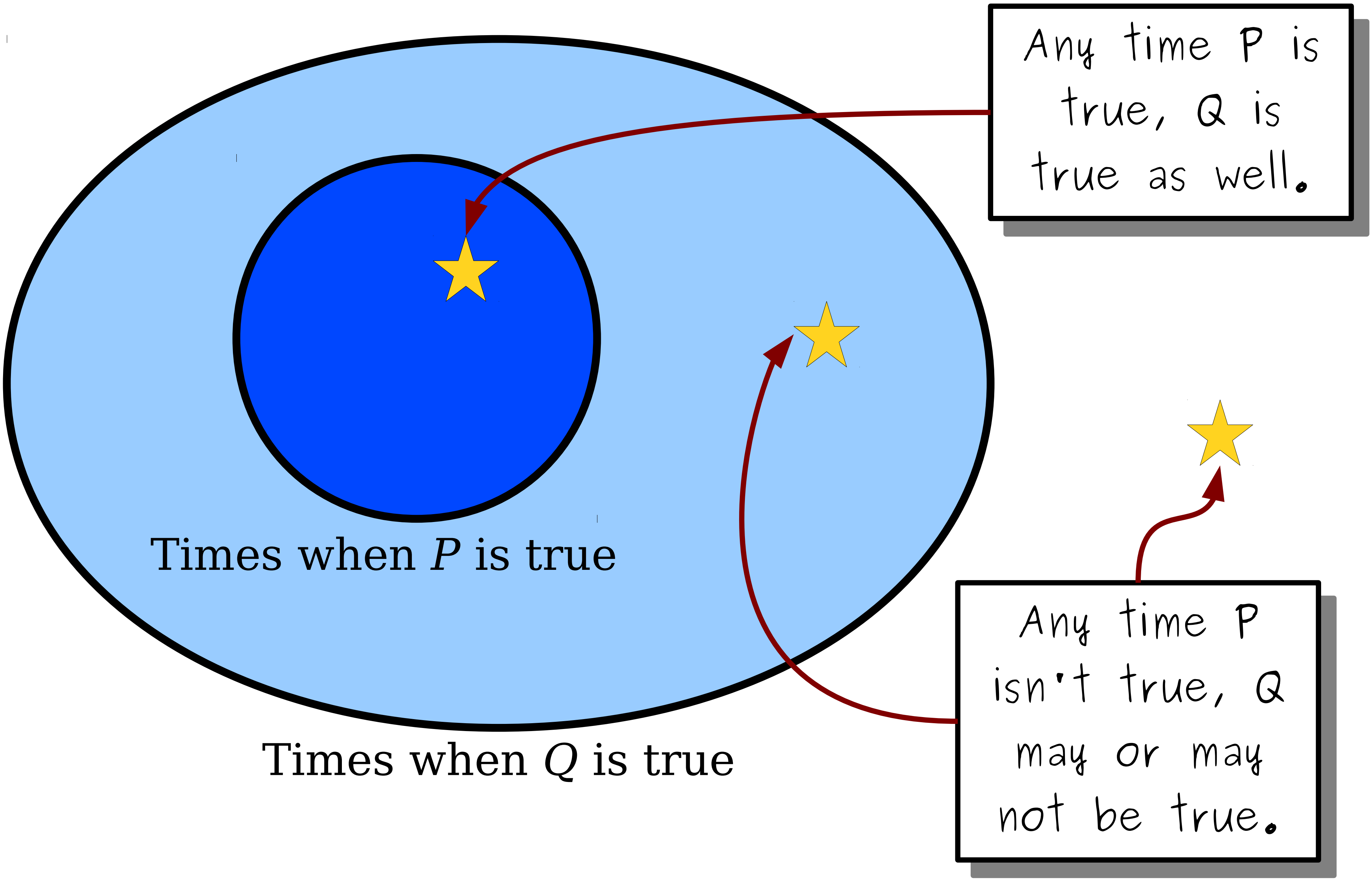
  If *P* is true, then *Q* is true.

  means *exactly* the following:

  Any time *P* is true, we are guaranteed that *Q* must also be true.

- There is no discussion of correlation or causation here. It simply means that if you find that *P* is true, you'll find that *Q* is true.

Implication, Diagrammatically



What Implication Doesn't Mean

- Implication is directional.
  - “If you die in Canada, you die in real life” doesn't mean that if you die in real life, you die in Canada.
- Implication only cares about cases where the antecedent is true.
  - “If an animal is a puppy, you should hug it” doesn't mean that if the animal is *not* a puppy, you *shouldn't* hug it.
- Implication says nothing about causality.
  - “If I like math, then 2 + 2 = 4” is true because any time I like math, we'll find that 2 + 2 = 4.
  - “If I hate math, then 2 + 2 = 4” is also true because any time I hate math, we'll find that 2 + 2 = 4.

Puppies Are Adorable

- Consider the statement

If *x* is a puppy, then I love *x*.

- Can you explain why the following statement is *not* the negation of the original statement?

If *x* is a puppy, then I don't love *x*.

- This second statement is too strong.
- Here's the correct negation:

There is some puppy *x* that I don't love.

The negation of the statement

“If *P* is true, then *Q* is true”

is the statement

“There are times when *P* is true and *Q* is false.”

Proof by Contrapositive

Honk **If** You Love Formal Logic

Suppose that you're driving this car and you *don't* get honked at.

What can you say about the people driving behind you?

The Contrapositive

- The ***contrapositive*** of the implication "If P, then Q" is the implication "If ***not*** Q, then ***not*** P."
- For example:
  - "If I store the cat food inside, then the raccoons will not steal my cat food."
  - Contrapositive: "If the raccoons stole my cat food, then I didn't store it inside."
- Another example:
  - "If Harry had opened the right book, then Harry would have learned about Gillyweed."
  - Contrapositive: "If Harry didn't learn about Gillyweed, then Harry didn't open the right book."

To prove the statement

If *P* is true, then *Q* is true

You may instead prove the statement

If *Q* is false, then *P* is false.

This is called a ***proof by contrapositive***.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: By contrapositive; we prove that if n is odd, then n^2 is odd.

Since n is odd, there is some integer k such that $n = 2k + 1$. Squaring both sides of this equality and simplifying gives the following:

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1. \end{aligned}$$

From this, we see that there is an integer m (namely, $2k^2 + 2k$) such that $n^2 = 2m + 1$. Therefore, n^2 is odd. ■

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: By contrapositive; we prove that if n is odd, then n^2 is odd.

Since n is odd, there is some integer k such that $n = 2$

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The general pattern here is the following:

1. Start by announcing that we're going to use a proof by contrapositive so that the reader knows what to expect.

2. Explicitly state the contrapositive of what we want to prove.

3. Go prove the contrapositive.

Biconditionals

- Combined with what we saw on Wednesday, we have proven that, if n is an integer:

If n is even, then n^2 is even.

If n^2 is even, then n is even.

- Therefore, if n is an integer:

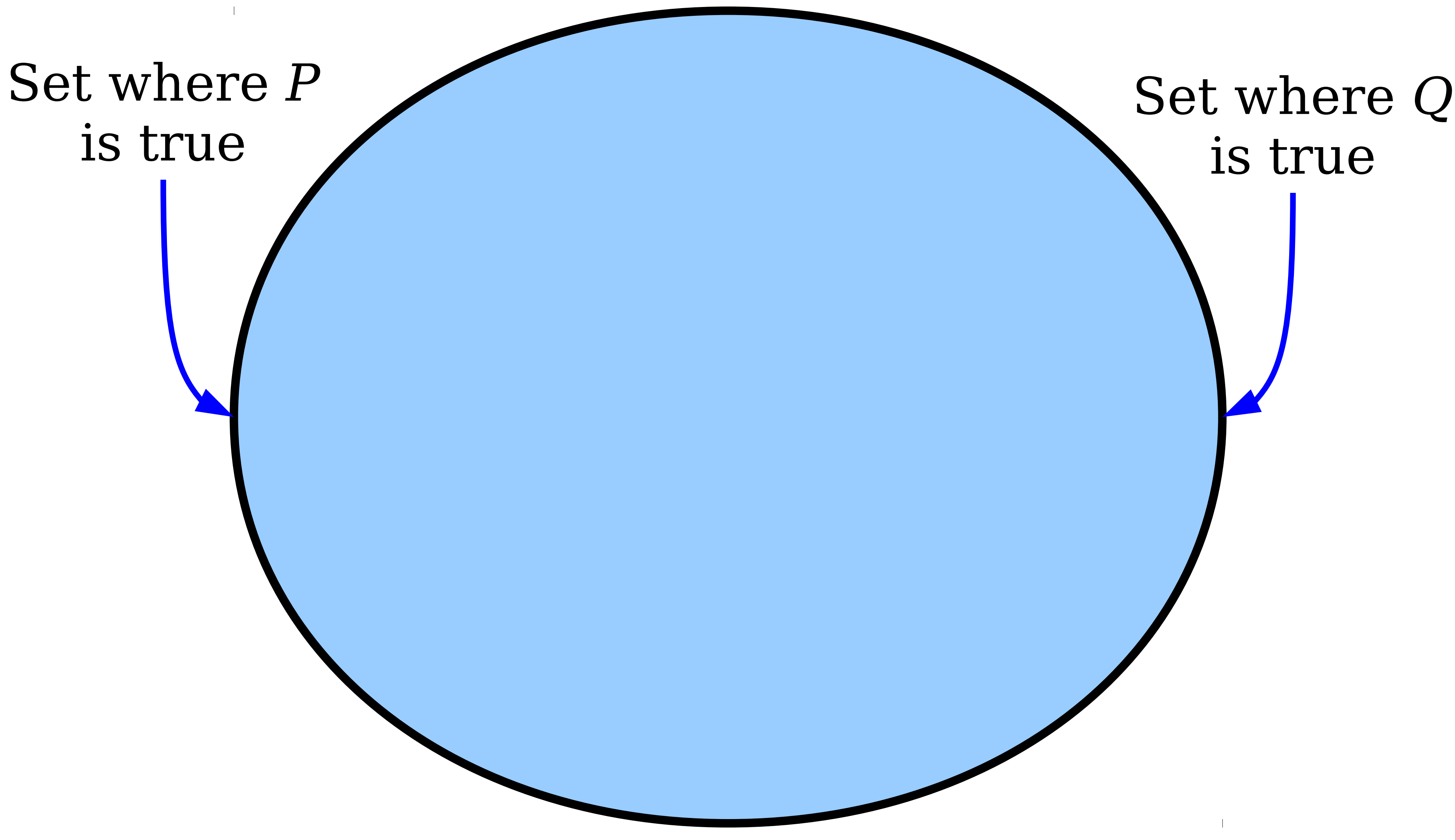
n is even if and only if n^2 is even.

- "If and only if" is often abbreviated **iff**:

n is even iff n^2 is even.

- This is called a **biconditional**.

P iff Q



Proving Biconditionals

- To prove ***P* iff *Q***, you need to prove that P implies Q and that Q implies P.
- You can any proof techniques you'd like to show each of these statements.
  - In our case, we used a direct proof and a proof by contrapositive.

Time-Out for Announcements!

Announcements

- Problem Set 1 out.
- **Checkpoint** due Monday, September 29.
  - Grade determined by attempt rather than accuracy. It's okay to make mistakes – we want you to give it your best effort, even if you're not completely sure what you have is correct.
  - We will get feedback back to you with comments on your proof technique and style.
  - The more an effort you put in, the more you'll get out.
- **Remaining problems** due Friday, October 3.
  - Feel free to email us with questions!

Submitting Assignments

- As a pilot for this quarter, we'll be using ***Scoryst*** for assignment submissions.
- All submissions should be electronic. If you're having trouble submitting, please contact the course staff.
- Signup link available at the course website.
- Late policy:
  - One “late period” that extends due date by one class period.
  - Work submitted late beyond a late period will have its score multiplied by 0.75.
  - No work accepted more than one class period after the due date.

Working in Groups

- You can work on the problem sets individually, in a pair, or in a group of three.
- Each group should only submit a single problem set, and should submit it only once.
- Full details about the problem sets, collaboration policy, and Honor Code can be found in Handouts 02 and 03.

Looking for a diverse community within engineering and science at Stanford? Join one of Stanford Engineering's diversity societies!

American Indian Science and Engineering Society

Meetings
Every Monday,
12:00 Noon,
at NACC

Society of Black Scientists and Engineers

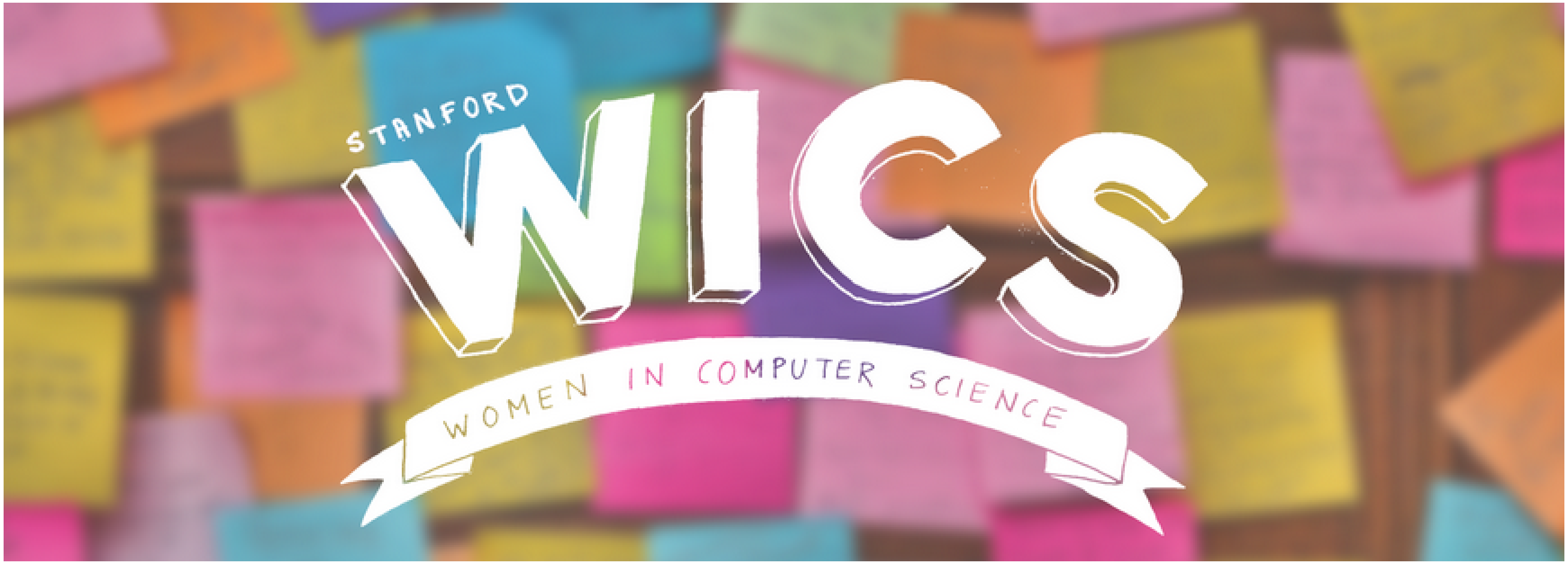
Meetings
Every Tuesday,
12:00 Noon,
at BCSC

Society of Women Engineers

Meetings
Every Wednesday,
12:00 Noon,
at MERL
(Building 660,
Second Floor)

Society of Latino Engineers

Meetings
Every Friday,
12:00 Noon,
at BCSC



Want a hand in **shaping** Stanford's own
Women in CS **community**?

Apply to be an **intern** for WICS by **Oct 4**!
Freshmen and grad students encouraged to apply!

Applications at http://bit.ly/1xjDMLB.
Questions? Email theodora@stanford.edu

Office hours start Monday.

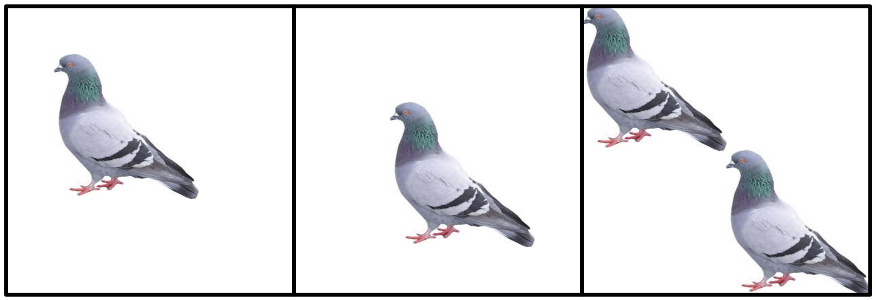
Schedule will be available on the course website.

Back to CS103!

The Pigeonhole Principle

The Pigeonhole Principle

- Suppose that you have n pigeonholes.
- Suppose that you have $m > n$ pigeons.
- If you put the pigeons into the pigeonholes, some pigeonhole will have more than one pigeon in it.



The Pigeonhole Principle

- Suppose that m objects are distributed into n bins.
- We want to prove the statement

  If $m > n$, then some bin contains at least two objects.

- What is the contrapositive of this statement?

  If every bin contains at most one object, then $m \leq n$.

- Look at the definitions of m and n. Does this make sense?

Theorem: Let m objects be distributed into n bins. If $m > n$, then some bin contains at least two objects.

Proof: By contrapositive; we prove that if every bin contains at most one object, then $m \leq n$.

Let x_i denote the number of objects in bin i. Since m is the number of total objects, we see that

$$m = x_1 + x_2 + \ldots + x_n.$$

We're assuming every bin has at most one object. In our notation, this means that $x_i \leq 1$ for all i. Using this inequalities, we get the following:

$$\begin{aligned} m &= x_1 + x_2 + \ldots + x_n \\ &\leq 1 + 1 + \ldots + 1 \qquad (n \textit{ times}) \\ &= n. \end{aligned}$$

So $m \leq n$, as required. ■

Some Simple Applications

- Any group of 367 people must have a pair of people that share a birthday.
  - 366 possible birthdays (pigeonholes)
  - 367 people (pigeons)
- Two people in San Francisco have the exact same number of hairs on their head.
  - Maximum number of hairs ever found on a human head is no greater than 500,000.
  - There are over 800,000 people in San Francisco.
- Each day, two people in New York City drink the same amount of water, to the thousandth of a fluid ounce.
  - No one can drink more than 50 gallons of water each day.
  - That's 6,400 fluid ounces. This gives 6,400,000 possible numbers of thousands of fluid ounces.
  - There are about 8,000,000 people in New York City proper.

Proof by Contradiction

“When you have eliminated all which is impossible, then whatever remains, however improbable, must be the truth.”

- Sir Arthur Conan Doyle*, The Adventure of the Blanched Soldier*

Proof by Contradiction

- A **proof by contradiction** is a proof that works as follows:
  - To prove that P is true, assume that P is *not* true.
  - Based on the assumption that P is not true, conclude something impossible.
  - Assuming the logic is sound, the only valid explanation is that the original assumption must have been wrong.
  - Therefore, P can't be false, so it must be true.

Theorem: No integer is both even and odd.

Proof: By contradiction; suppose some integer is both even and odd. Let that integer be k.

Since k is even, there is some $r \in \mathbb{Z}$ such that $k = 2r$. The integer k is also odd, so there is some $s \in \mathbb{Z}$ where $k = 2s+1$. Combining these equalities together tells us that

$$2r = 2s + 1.$$

Rearranging and simplifying this equation tells us the following:

$$\begin{aligned} 2r - 2s &= 1 \\ r - s &= \tfrac{1}{2} \end{aligned}$$

Because r and s are integers, their difference must also be an integer. However, we've just seen that $r - s = \tfrac{1}{2}$, which is not an integer.

We have reached a contradiction, so our assumption must have been wrong. Thus there is no integer that is both even and odd. ■

Theorem: No integer is both even and odd.

Proof: By contradiction; suppose some integer is both even and odd. Let that integer be k.

Since k is even, there is some $r \in \mathbb{Z}$ such that $k = 2r$. The integer k is also odd, so there is some $s \in \mathbb{Z}$ where $k = 2s+1$. Combining these equalities together tells us that

The three key pieces:

1. State that the proof is by contradiction.
2. State what you are assuming is the negation of the statement to prove.
3. State you have reached a contradiction and what the contradiction entails.

In CS103, please include all these steps in your proofs!

an integer.

We have reached a contradiction, so our assumption must have been wrong. Thus there is no integer that is both even and odd. ■

Rational and Irrational Numbers

Rational and Irrational Numbers

- A number r is called a ***rational number*** if it can be written as

$$r = \frac{p}{q}$$

  where p and q are integers and $q \neq 0$.

- A number that is not rational is called ***irrational***.
- Useful theorem: If r is rational, r can be written as p / q where $q \neq 0$ and where p and q have no common factors other than ± 1.

Theorem: $\sqrt{2}$ is irrational.

Proof: Assume for the sake of contradiction that $\sqrt{2}$ is rational. This means that there must be integers p and q where $q \neq 0$, where p and q have no common divisors other than 1 and -1, and where

$$p / q = \sqrt{2}. \qquad (1)$$

Multiplying both sides of equation (1) by q and squaring both sides shows us that

$$p^2 = 2q^2. \qquad (2)$$

From equation (2), we see that p^2 is even. Earlier, we proved that if p^2 is even, then p must also be even. Therefore, we know that there is some integer k such that $p = 2k$. Substituting this into equation (2) and simplifying gives us the following:

$$\begin{aligned} p^2 &= 2q^2 \\ (2k)^2 &= 2q^2 \\ 4k^2 &= 2q^2 \\ 2k^2 &= q^2 \qquad (3) \end{aligned}$$

Equation (3) shows that q^2 is even. Our earlier theorem tells us that, because q^2 is even, q must also be even. But this is not possible – we know that p and q have no common factors other than 1 and -1, but we've shown that p and q must have two as a common factor.

We have reached a contradiction, so our original assumption must have been wrong. Therefore, $\sqrt{2}$ is irrational. ■

Theorem: $\sqrt{2}$ is irrational.

Proof: Assume for the sake of contradiction that $\sqrt{2}$ is rational. This means that there must be integers p and q where $q \neq 0$, where p and q have no common divisors other than 1 and -1, and where

$$p / q = \sqrt{2}. \qquad (1)$$

Multiplying both sides of equation (1) by q and squaring both sides

The three key pieces:

1. State that the proof is by contradiction.
2. State what you are assuming is the negation of the statement to prove.
3. State you have reached a contradiction and what the contradiction entails.

In CS103, please include all these steps in your proofs!

Equation (3) shows that q^2 is even. Our earlier theorem tells us that, because q^2 is even, q must also be even. But this is not possible – we know that p and q have no common factors other than 1 and -1, but we've shown that p and q must have two as a common factor.

We have reached a contradiction, so our original assumption must have been wrong. Therefore, $\sqrt{2}$ is irrational. ■

Vi Hart on Pythagoras and the Square Root of Two:

http://www.youtube.com/watch?v=X1E7I7_r3Cw

Proving Implications

- To prove the implication

  "If *P* is true, then *Q* is true."

- you can use these three techniques:
  - **Direct Proof.**
    - Assume P and prove Q.
  - **Proof by Contrapositive**
    - Assume not Q and prove not P.
  - **Proof by Contradiction**
    - … what does this look like?

Negating Implications

- To prove the statement

  “If *P* is true, then *Q* is true”

  by contradiction, we do the following:
  - Assume this statement is false.
  - Derive a contradiction.
  - Conclude that the statement is true.
- What is the negation of this statement?

  “*P* is true and *Q* is false”

Contradictions and Implications

- To prove the statement

  “If *P* is true, then *Q* is true”

  using a proof by contradiction, do the following:

  - Assume that P is true and that Q is false.
  - Derive a contradiction.
  - Conclude that if P is true, Q must be as well.

Theorem: If n is an integer and n^2 is even, then n is even.

Proof: Assume for the sake of contradiction that n is an integer and that n^2 is even, but that n is odd.

Since n is odd we know that there is an integer k such that

$$n = 2k + 1 \tag{1}$$

Squaring both sides of equation (1) and simplifying gives the following:

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \end{aligned} \tag{2}$$

Equation (2) tells us that n^2 is odd, which is impossible; by assumption, n^2 is even.

We have reached a contradiction, so our assumption must have been incorrect. Thus if n is an integer and n^2 is even, n is even as well. ■

Theorem: If n is an integer and n^2 is even, then n is even.

Proof: Assume for the sake of contradiction that n is an integer and that n^2 is even, but that n is odd.

The three key pieces:

1. State that the proof is by contradiction.
2. State what the negation of the original statement is.
3. State you have reached a contradiction and what the contradiction entails.

In CS103, please include all these steps in your proofs!

Equation (2) tells us that n^2 is odd, which is impossible; by assumption, n^2 is even.

We have reached a contradiction, so our assumption must have been incorrect. Thus if n is an integer and n^2 is even, n is even as well. ■

Skills from Today

- Disproving statements
- Negating universal and existential statements.
- Negating implications.
- Determining the contrapositive of a statement.
- Writing a proof by contrapositive.
- Writing a proof by contradiction.

Next Time

- **Proof by Induction**
  - Proofs on sums, programs, algorithms, etc.

Appendices: Helpful References

Negating Implications

"If *P*, then *Q*"

becomes

"*P* but not *Q*"

Negating Universal Statements

"For all *x*, P(*x*) is true"

becomes

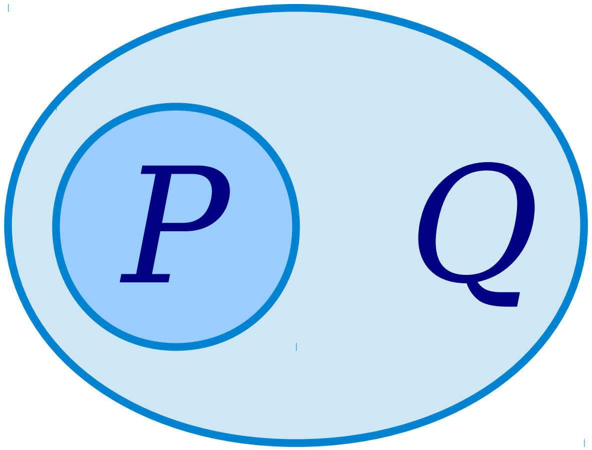
"There is an *x* where P(*x*) is false."

Negating Existential Statements

"There exists an *x* where P(*x*) is true"

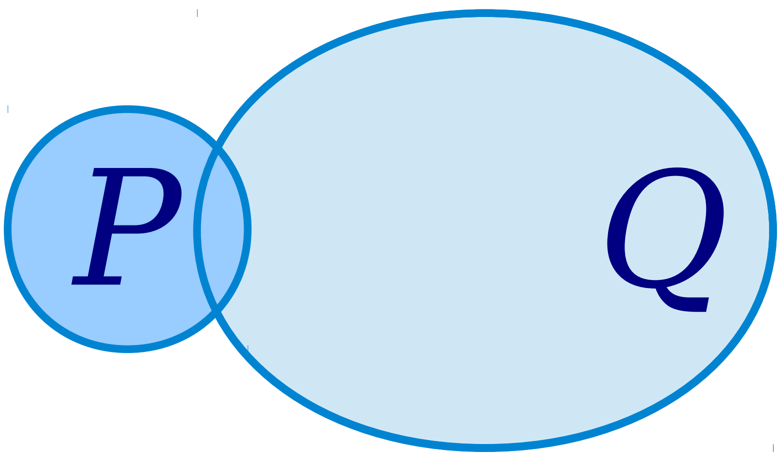
becomes

"For all *x*, P(*x*) is false."



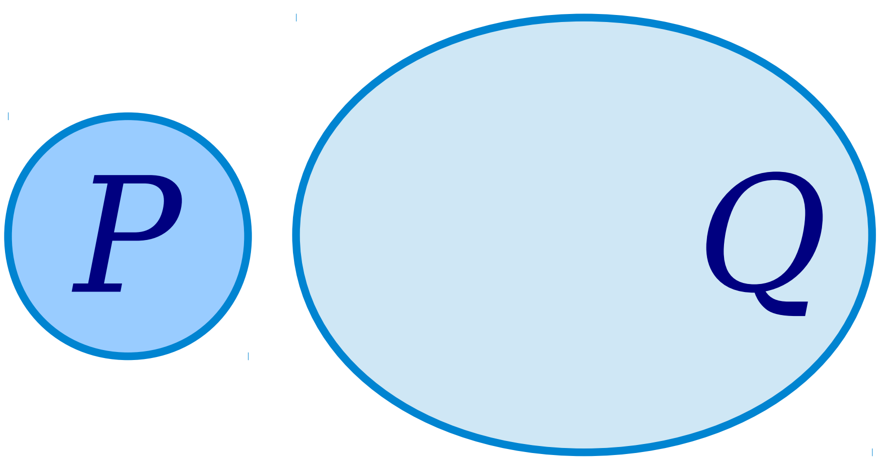
P* implies *Q

"If P is true, then Q is true."



P* does not imply *Q
-and-
P* does not imply not *Q

"Sometimes P is true and Q is true,
-and-
sometimes P is true and Q is false."



P* implies not *Q

If P is true, then Q is false