Inner Approximations and a NISQ Algorithm for the Quantum Separability Problem

https://github.com/ankith-mohan/SEP

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Outline

Problem Statement

Given a density matrix $\rho \in Pos(\mathcal{A} \otimes \mathcal{B})$, we want to say whether or not ρ is entangled?

Formally, define

$$SEP(\mathcal{X} : \mathcal{Y}) = \left\{ \sum_{i=1}^{k} p_i \left|\psi_i\right\rangle \left\langle\psi_i\right| \otimes \left|\phi_i\right\rangle \left\langle\phi_i\right| : \sum_{i=1}^{k} p_i = 1, p_i \geq 0 \ \forall \ i \right\}$$

We want to compute

$$\alpha = \sup_{\rho_\alpha \in SEP} \langle \rho_\alpha, \mathbf{\Pi} \rangle$$

This is shown to be an NP-hard problem[1]

[1] [?]

Outline

Symmetric Extensions

Instead of asking whether or not ρ is entangled?

We ask whether ρ is symmetric extendible?

We compute

$$\beta_k = \sup_{\rho_{\beta_k} \in SymExt(k)} \langle \rho_{\beta_k}, \Pi \rangle$$

$\because SEP = \bigcap_{k=1}^{\infty} SymExt(k)$

$\implies \beta_1 \geq \beta_2 \geq \cdots \geq \beta_k \geq \alpha$

Summary

$$\alpha \leq \{\beta_r, \beta_1, \ldots, \beta_k\}$$

Outline

See-saw

Algorithm 1: see-saw

Input: $\sigma_A \in Pos(\mathcal{A})$, $\mathbf{\Pi}$

Output: γ

repeat

$$\sigma_B : \sup_{\sigma_B \in Pos(\mathcal{B})} \langle \sigma_A \otimes \sigma_B, \mathbf{\Pi} \rangle$$

$$\sigma_A : \sup_{\sigma_A \in Pos(\mathcal{A})} \langle \sigma_A \otimes \sigma_B, \mathbf{\Pi} \rangle$$

until *convergence or maximum number of iterations*;

$\gamma = \langle \sigma_A \otimes \sigma_B, \mathbf{\Pi} \rangle$

Maximally-mixed (MM) see-saw

$$\gamma_{MM} = \text{see-saw}\left(\frac{1}{\dim \mathcal{A}}\,\mathbb{I}, \mathbf{\Pi}\right)$$

Uniform superposition (US) see-saw

$$\gamma_{US} = \text{see-saw}\left(\frac{1}{\dim \mathcal{A}} \sum_i |i\rangle \langle i| , \mathbf{\Pi}\right)$$

Randomized see-saw

Algorithm 2: random see-saw

Input: $\mathbf{\Pi}$, N_{rand}

Output: γ_{rand}

$\gamma_{list} \rightarrow \{\emptyset\}$

for $i \rightarrow 1 : N_{rand}$ **do**

  Initialize σ_A to be a random matrix in $Pos(\mathcal{A})$

  $\gamma_{list}(i) = \text{see-saw}(\sigma_A, \mathbf{\Pi})$

$\gamma_{rand} = \max\{\gamma_{list}\}$

Summary

$$\{\gamma_{MM}, \gamma_{US}, \gamma_{rand}\} \leq \alpha$$

Outline

Idea

$$\sup_{\sigma_A \in Pos(\mathcal{A}), \sigma_B \in Pos(\mathcal{B})} \langle \sigma_A \otimes \sigma_B, \mathbf{\Pi} \rangle \leq \sup_{\sigma_A \in Pos(\mathcal{A}), \sigma_B \in Pos(\mathcal{B})} \langle \sqrt{\sigma_A} \otimes \sqrt{\sigma_B}, \mathbf{\Pi} \rangle$$

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[2] This inequality holds only when $\mathbf{\Pi}$ is a PSD matrix

Idea

$$\sup_{\sigma_A \in Pos(\mathcal{A}), \sigma_B \in Pos(\mathcal{B})} \langle \sigma_A \otimes \sigma_B, \mathbf{\Pi} \rangle \leq \sup_{\sigma_A \in Pos(\mathcal{A}), \sigma_B \in Pos(\mathcal{B})} \langle \sqrt{\sigma_A} \otimes \sqrt{\sigma_B}, \mathbf{\Pi} \rangle$$

This is not SDP :(
[3]

[3] $\mathbf{\Pi}$ is a PSD matrix

Idea

$$\sup_{\sigma_A \in Pos(\mathcal{A}), \sigma_B \in Pos(\mathcal{B})} \langle \sigma_A \otimes \sigma_B, \mathbf{\Pi} \rangle \leq \sup_{\sigma_A \in Pos(\mathcal{A}), \sigma_B \in Pos(\mathcal{B})} \langle \sqrt{\sigma_A} \otimes \sqrt{\sigma_B}, \mathbf{\Pi} \rangle$$

This is not SDP :([4]

This is SDP :)

[4] This inequality only holds when $\mathbf{\Pi}$ is a PSD matrix

How do we solve this?

$$\mu = \sup_{\mathbf{W} \in Herm(\mathcal{A} \otimes \mathcal{B})} \langle \mathbf{W}, \mathbf{\Pi} \rangle$$

$$\begin{aligned} &\text{subject to} \\ \begin{bmatrix} \sigma_{A_\delta} \otimes \mathbb{I}_{\dim \mathcal{A}} & \mathbf{W} \\ \mathbf{W} & \mathbb{I}_{\dim \mathcal{B}} \otimes \sigma_{B_\delta} \end{bmatrix} &\geq 0 \\ \sigma_{A_\delta} &\geq 0 \\ \mathrm{Tr}\{\sigma_{A_\delta}\} &= 1 \\ \sigma_{B_\delta} &\geq 0 \\ \mathrm{Tr}\{\sigma_{B_\delta}\} &= 1 \end{aligned}$$

σ_{A_δ} and σ_{B_δ} correspond to the optimal solution.

New bound

$$\langle \sigma_{A_\delta} \otimes \sigma_{B_\delta}, \mathbf{\Pi} \rangle \leq \alpha \leq \langle \sqrt{\sigma_{A_\delta}} \otimes \sqrt{\sigma_{B_\delta}}, \mathbf{\Pi} \rangle$$

where

$$\alpha = \sup_{\rho_\alpha \in SEP} \langle \rho_\alpha, \mathbf{\Pi} \rangle$$

New bound

$$\gamma_{sqrt} = \text{see-saw}(\sigma_{A_\delta}, \mathbf{\Pi}) \leq \alpha \leq \beta_{sqrt} = \langle \sqrt{\sigma_{A_\delta}} \otimes \sqrt{\sigma_{B_\delta}}, \mathbf{\Pi} \rangle$$

where

$$\alpha = \sup_{\rho_\alpha \in SEP} \langle \rho_\alpha, \mathbf{\Pi} \rangle$$

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[5] This inequality only holds when $\mathbf{\Pi}$ is a PSD matrix.

Summary

$$\{\gamma_{MM}, \gamma_{US}, \gamma_{rand}, \gamma_{sqrt}\} \leq \alpha \leq \{\beta_{sqrt}, \beta_r, \beta_1, \ldots, \beta_k\}$$

Outline

1. Problem
2. Upper Bounds
3. Lower Bounds
4. Square root
5. Quantification
6. NSS
7. Results

Quantification

How well are we really approximating α?

Define "Tightness of Bound" (TOB) as

$$ToB = \beta - \gamma$$

where

$$\beta = \min\{\beta_{sqrt}, \beta_r, \beta_1, \ldots, \beta_k\}$$

$$\gamma = \max\{\gamma_{MM}, \gamma_{US}, \gamma_{rand}, \gamma_{sqrt}\}$$

Outline

Hybrid density matrix ansatz

Alice: $\{|\psi_i\rangle\}_{i=1}^{N} \in \mathcal{A}$
Bob: $\{|\phi_j\rangle\}_{j=1}^{M} \in \mathcal{B}$

$$\mathbf{X} = \sum_{ijkl} \beta_{ijkl} \left|\psi_i\right\rangle \left\langle\psi_j\right| \otimes \left|\phi_k\right\rangle \left\langle\phi_l\right|$$

Original Problem

$$\sup \langle \mathbf{X}, \mathbf{\Pi} \rangle$$

subject to

$$Tr(\mathbf{X}) = 1$$

$$\mathbf{X} \in SEP(\mathcal{A} : \mathcal{B})$$

Lemma

$$\beta \in Herm(\mathcal{A}' \otimes \mathcal{B}')$$

where

$$dim(\mathcal{A}') << dim(\mathcal{A}),$$
$$dim(\mathcal{B}') << dim(\mathcal{B})$$

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$$dim(\mathcal{A}') << dim(\mathcal{A}),$$
$$dim(\mathcal{B}') << dim(\mathcal{B})$$

Lemma

$$\beta \in SEP(\mathcal{A}' \otimes \mathcal{B}') \implies \mathbf{X} \in SEP(\mathcal{A} \otimes \mathcal{B})$$

Problem (Update 1)

$\sup\langle \mathbf{X}, \mathbf{\Pi} \rangle$

subject to

$Tr(\mathbf{X}) = 1$

$\mathbf{X} \in SEP(\mathcal{A} : \mathcal{B}) \rightarrow \beta \in SEP(\mathcal{A}' : \mathbf{B}')$

Objective function

$$\mathbf{\Pi} = \sum_x \mathbf{C}_x \mathbf{U}_x$$

$$\begin{aligned}
\langle \mathbf{X}, \mathbf{\Pi} \rangle &= \langle \sum_{ijkl} \beta_{ijkl} \left|\psi_i\right\rangle \left\langle\psi_j\right| \otimes \left|\phi_k\right\rangle \left\langle\phi_l\right|, \sum_x \mathbf{C}_x \mathbf{U}_x \rangle \\
&= \sum_{ijklx} \beta_{ijkl} \mathbf{C}_x \langle \left\langle\psi_j\right| \left\langle\phi_l\right| \mathbf{U}_x \left|\psi_i\right\rangle \left|\phi_k\right\rangle \rangle \\
&= \sum_{ijkl} \langle \beta_{ijkl}, \sum_x \mathbf{C}_x d_{ijklx} \rangle \\
&= \langle \beta, \mathcal{D} \rangle
\end{aligned}$$

Problem (Update 2)

$$\sup\langle \mathbf{X}, \mathbf{\Pi} \rangle \to \sup\langle \beta, \mathcal{D} \rangle$$

subject to

$$Tr(\mathbf{X}) = 1$$

$$\mathbf{X} \in SEP(\mathcal{A} : \mathcal{B}) \to \beta \in SEP(\mathcal{A}' : \mathcal{B}')$$

Density matrix constraint

$$
\begin{aligned}
Tr(\mathbf{X}) &= Tr(\sum_{ijkl} \beta_{ijkl} |\psi_i\rangle \langle\psi_j| \otimes |\phi_k\rangle \langle\phi_l|) \\
&= \sum_{ijkl} \beta_{ijkl} \langle\psi_j|\psi_i\rangle \langle\phi_l|\phi_k\rangle \\
&= \sum_{ijkl} \langle\beta_{ijkl}, e_{ijkl}\rangle \\
&= \langle\beta, \mathcal{E}\rangle
\end{aligned}
$$

Modified Problem

$$\sup\langle \mathbf{X}, \mathbf{\Pi}\rangle \to \sup\langle \beta, \mathcal{D}\rangle$$

subject to

$$Tr(\mathbf{X}) = 1 \to \langle \beta, \mathcal{E}\rangle = 1$$

$$\mathbf{X} \in SEP(\mathcal{A} : \mathcal{B}) \to \beta \in SEP(\mathcal{A}' : \mathcal{B}')$$

Outline

Competing hypotheses

1. (Jamie): γ_{rand} should work better for smaller values of d. Intuitively, smaller values of d correspond to a smaller number of baskets. So starting from a set of random starting points, it is highly likely that we are going to reach the global optimum.
2. (Kishor): γ_{rand} might not work well for smaller values of d because here we will be dealing with a bad landscape. As the value of d increases, the landscape becomes more smooth and we would expect to reach the optimum more easily.