Pascal's Triangle: Cellular Automata and Attractors

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Mathematics is the art of giving the same name to different things

— Henri Ponicaré

Pascal's Triangle

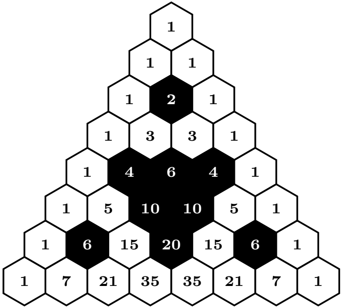
$$\begin{array}{ccccccccccccccc}
&&&&&&&1&&&&&&& \\
&&&&&&1&&1&&&&&& \\
&&&&&1&&2&&1&&&&& \\
&&&&1&&3&&3&&1&&&& \\
&&&1&&4&&6&&4&&1&&& \\
&&1&&5&&10&&10&&5&&1&& \\
&1&&6&&15&&20&&15&&6&&1& \\
1&&7&&21&&35&&35&&21&&7&&1
\end{array}$$

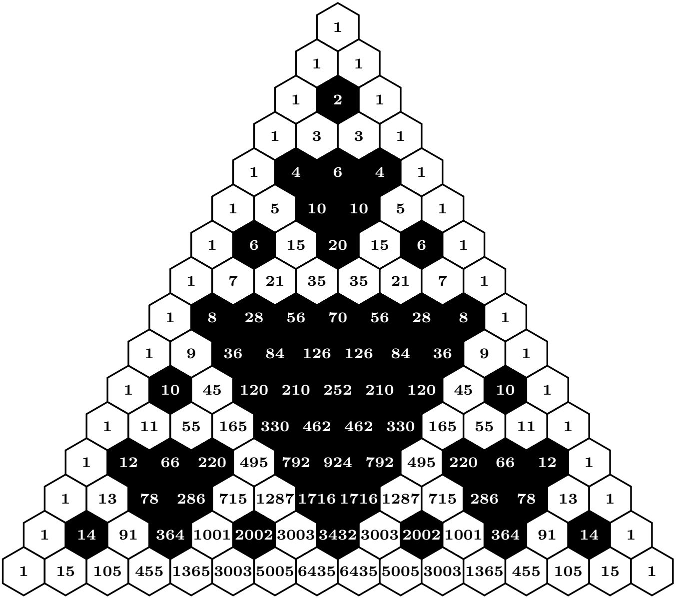
- One of the earliest mentions was in a Chinese document at around 1303 AD
- Looks pretty innocent right?

Pascal's Triangle

Let's look at it in a few different ways
It is observed by coloring

- all odd numbers white
- all even numbers black




1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
1 11 55 165 330 462 462 330 165 55 11 1
1 12 66 220 495 792 924 792 495 220 66 12 1
1 13 78 286 715 1287 1716 1716 1287 715 286 78 13 1
1 14 91 364 1001 2002 3003 3432 3003 2002 1001 364 91 14 1
1 15 105 455 1365 3003 5005 6435 6435 5005 3003 1365 455 105 15 1

Pascal's Triangle

- Can also be formulated through binomial coefficients

$$\begin{aligned}
(1+x)^0 &= \quad\quad 1 \\
(1+x)^1 &= \quad 1 + 1x \\
(1+x)^2 &= 1 + 2x + 1x^2 \\
&\vdots \\
(1+x)^n &= a_0 + a_1 x + \cdots + a_n x^n
\end{aligned}$$

where coefficients are given by

$$a_k = \binom{n}{k} = \frac{n!}{(n-k)!k!}, \quad 0 \leq k \leq n$$

Pascal's Triangle

- Understanding the divisibility of binomial coefficients by computing them is a bad idea

$$\begin{aligned}50! =&30414093201713378043612608166064768844377\\&641568960512000000000000\end{aligned}$$

Even if we use the recursive formula

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

$$\binom{40}{20} = 137846528820 > 2^{32}$$

which is a big number
Fortunately, we don't need to compute these large numbers

Pascal's Triangle

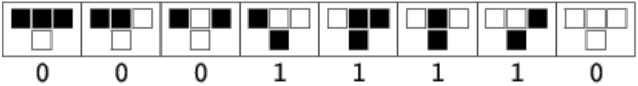
- Ex: Divisibility by 2 can be followed from addition rule

$\binom{n}{k-1}$	$\binom{n}{k}$	$\binom{n+1}{k}$
even	even	even
odd	even	odd
even	odd	odd
odd	odd	odd

- Can we extend this idea of divisibility to other numbers ?
- What patterns do we get ? Global pattern? Why? ...IFS

Cellular Automata

- Prefect feedback machines. They are mathematically finite state machines
- Each cell has one out of p states. *p-state* automata
- Can be 1D, 2D ...
- To run cellular automata, we need 2 pieces of information
  1. Initial state of cells
  2. Rules to describe new cell state from the states of a group of cells from the previous layer
- The rules should not depend on the position of the group of cells within the layer.



Cellular Automata

- More kinds of rules

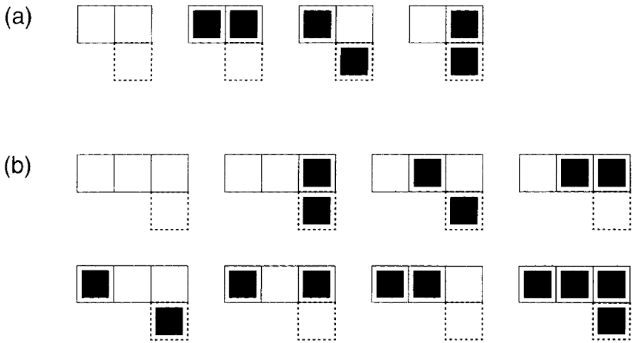


Figure: (a) is a four rule configuration and (b) is a 8 rule configuration

Cellular Automata

So lets run some

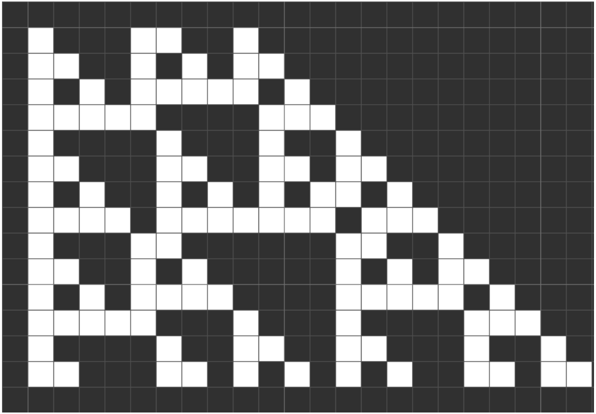


Figure: 13 Generations of Rule 60 Elementary CA

Elementary Cellular Automata

- Simplest form of 1D Cellular Automata (CA)
- 2 State CA
- Next generation depends on itself,cell to the left and, cell to the right.
- Total number of rules are $2^{2^3} = 256$
- All the rules can be numbered in a nice way using binary

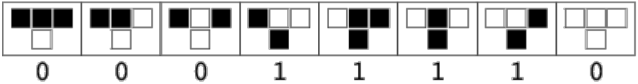


Figure: Elementary rule $30 = (00011110)_2$

- Mathematica can do this for you very easily using CellularAutomaton[ ]

Elementary Cellular Automata

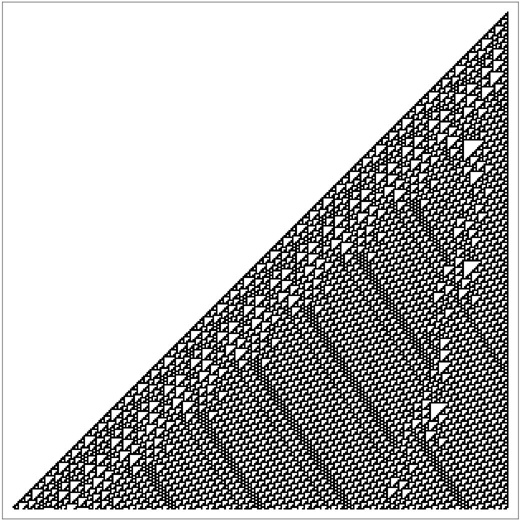


Figure: Rule 110, this might be special

Elementary Cellular Automata

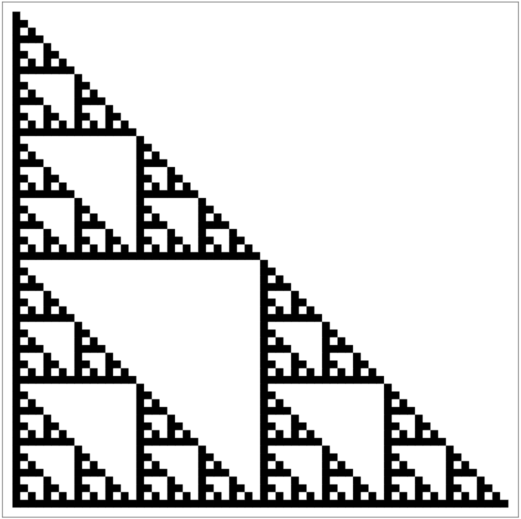


Figure: Rule 60. Sierpinski Triangle*

Elementary Cellular Automata

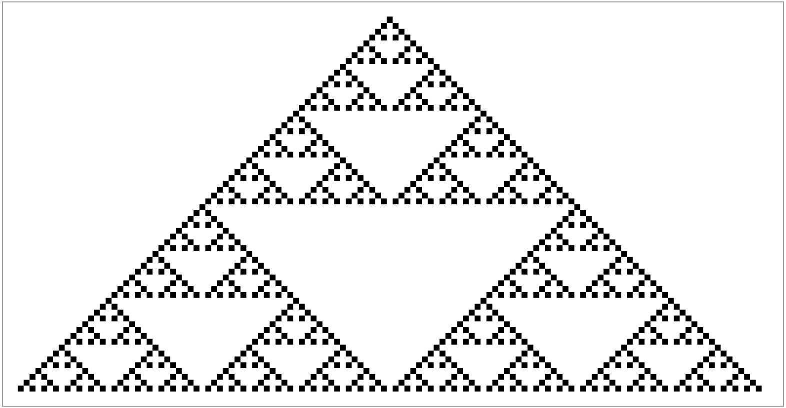


Figure: Rule 90.

Elementary Cellular Automata

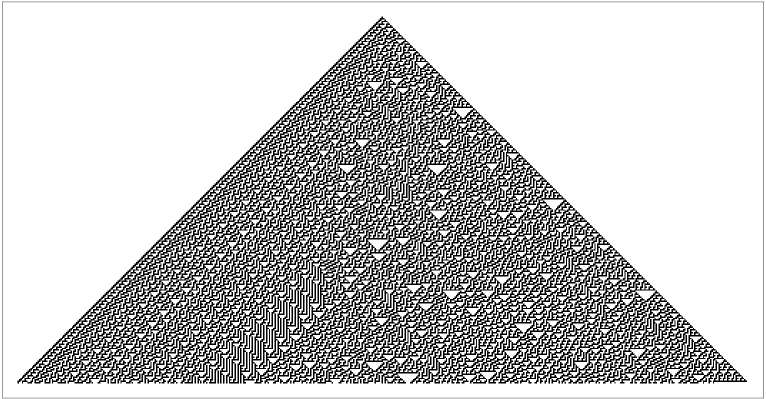


Figure: Rule 30

Kinda chaotic ?

2D CA: Game of life...

- Was very popular through the work of John Horton Conway in 1970's
- 2-state CA
- Rules:
  1. Cell survives when exactly 2 or 3 of it's 8 neighbors are alive.
  2. If more than 3 neighbors are alive, the cell dies from overcrowdedness.
  3. If fewer than 2 neighbors are alive, the cell dies from oneliness.
  4. Dead cell comes back to life when surrounded by exactly 3 live neighbors.
- Makes some interesting patters and life like behavior
- Note:The position of the alive and dead cells w.r.t the center does not matter for this rule. Also known as Totalistic CA

Let's play the Game of Life......

Cellular Automata

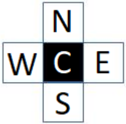
- Another variant of Game of Life: One out of eight rule
  1. Cell becomes alive if exactly one neighbor is alive.
  2. Otherwise unchanged.
- Has some nice self-similarity

Cellular Automata

- Game of life is just one out of the imaginable set of rules
- For 2-state 2D automata and a neighborhood of eight cells, there are $2^{2^9} \approx 10^{154}$ different sets of rules!!
- Let's look at another rule: Majority Rule
  1. If 5 or more of the neighborhood of 9 cells (including itself) are alive, this cell lives or stays alive
  2. Otherwise dies or remains dead
- Here center cell adjusts to the majority
- This kind of rule is called Outer Totalistic
- Has some resemblance with percolation

Cellular Automata with Different neighbors

- Let's consider only 4 neighbors.



- C = Center, N = North, E = East, S = South, W = West
- Can represent every neighbor in Binary $(CSWNE)_2$. Ex:$(01100)_2$ means S, W cells are alive, rest are dead
- Total number of sets of rules are $2^{2^5} \approx 4 \cdot 10^9$
- Two interesting examples are shown

Cellular Automata

CSWNE	C	CSWNE	C	CSWNE	C	CSWNE	C
00000	0	01000	1	10000	1	11000	1
00001	0	01001	1	10001	1	11001	1
00010	0	01010	1	10010	1	11010	1
00011	0	01011	1	10011	1	11011	1
00100	1	01100	0	10100	1	11100	1
00101	1	01101	0	10101	1	11101	1
00110	1	01110	0	10110	1	11110	1
00111	1	01111	0	10111	1	11111	1

- This behaves like a 1D CA

Cellular Automata

- Many interesting rules can be given by a simple formula
- Parity Rule:

$$C_{new} = C_{old} + N_{old} + E_{old} + S_{old} + W_{old} \mod 2$$

- Again, this is a 2-state CA

CA, Pascal's Triangle and Polynomial

- Let's look at the powers of $r(x) = 1 + x$:

$$
\begin{aligned}
(r(x))^0 &= 1 \\
(r(x))^1 &= 1 + x \\
(r(x))^2 &= 1 + 2x + x^2 \\
(r(x))^3 &= 1 + 3x + 3x^2 + x^3 \\
&\vdots \\
(r(x))^n &= a_0(n) + a_1(n)x + a_2(n)x^2 + \cdots + a_n(n)x^n
\end{aligned}
$$

- By the addition rule of binomial coefficients

$$a_k(n) = a_{k-1}(n-1) + a_k(n-1)$$

- $a_k(n)$ gives the state of $\mathrm{n^{th}}$ layer CA*

CA, Pascal's Triangle and Polynomial

- Looking at the divisibility properties of $a_k(n)$ with 2

$$a_k(n) \equiv 0 \pmod 2 \qquad \textit{or} \qquad a_k(n) \equiv 1 \pmod 2$$

- With this, the addition rules in mod 2 arithmetic simplifies to

$\binom{n}{k-1}$	$\binom{n}{k}$	$\binom{n+1}{k}$
0	0	0
1	0	1
0	1	1
1	1	0

- Exactly same rule set used to make Pascal Triangle
- Thus, that figure shows the coefficients of the power of $r(x) = 1 + x \mod 2$

Generalizations

- There are 2 ways to generalize,
  1. A different coefficient modulo integer
  2. A different polynomial
- Ex: Let $r(x) = 1 + 2x$

$$
\begin{aligned}
(r(x))^0 &= 1 \\
(r(x))^1 &= 1 + 2x \\
(r(x))^2 &= 1 + 4x + 4x^2 \\
&\vdots \\
(r(x))^n &= a_0(n) + a_1(n)x + \cdots + a_n(n)x^n
\end{aligned}
$$

- By pattern

$$a_k(n) = a_k(n-1) + 2a_{k-1}(n-1)$$

- By looking at the divisibility property with p = 3, we get

$$
\begin{aligned}
(r(x))^0 &= 1 \\
(r(x))^1 &= 1\ 2 \\
(r(x))^2 &= 1\ 1\ 1 \\
(r(x))^3 &= 1\ 0\ 0\ 2
\end{aligned}
$$

- The cellular automaton would be $a_{n,k} = a_{n-1,k} + 2a_{n-1,k-1} \mod 3$

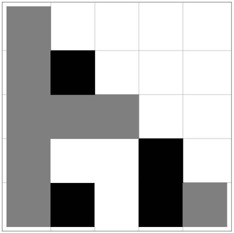


Figure: CA for $r(x) = 1 + 2x \mod 2$

Linear Cellular Automata

- We can start with any polynomial $r(x) = a_0 + a_1 x + \cdots + a_d x^d$
- The coefficients of $(r(x))^n$ modulo some positive integer p is obtained by an addition formula involving d coefficients from $(r(x))^{n-1}$
- So, there is an associated Cellular Automata which generates coefficients modulo p of the powers of $(r(x))^n$
- A look-up table can be generated by an addition formula
- These are called *Linear Cellular Automata*
- Again, the choice of p determines the number of states. This opens up a lot of interesting problems

- Pattern Formation: Given a polynomial, what is the global pattern which evolves when the automaton has run for a long time?
- Colors: What is the relation between the global patterns which are obtained for different choices of p?
- Fractal Dimension: Fractal dimension of the global pattern ?
- Higher Dimension: What if we used polynomial of m variables ? (m-dimensional ?)
- Factorization: If a polynomial $r(x)$ can be factorized to two polynomials $s(x)$ and $t(x)$, how are the patters related ? Is that factorization unique ?
  In general, no, it depends on p
  Ex: $1 + x$ is irreducible with respect to integers. But in arithmetic modulo p, with p not prime, there are non trivial factorization

$$1 + x \equiv (1 + 3x)(1 + 4x) \pmod 6$$

Binomial Coefficients and Divisibility

- Discuss the question of whether a binomial coefficient is divisible by p or not.
- Black and white coloring of the Pascal triangle depending on divisibility with p
- We will also see that in order to understand the patterns formed by mod p, we should look at the patters formed by the prime factors of p
- We will look at a direct, non-recursive computation of divisibility by p
- This was solved in an elegant manner by Ernest Eduard Kummer.

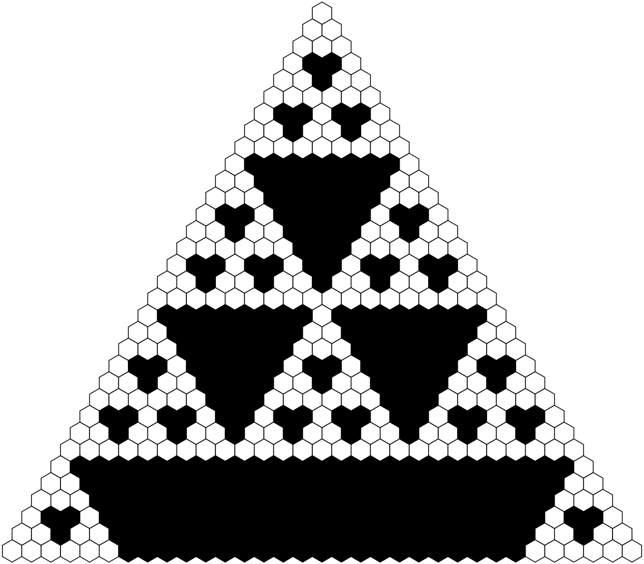


Figure: Pascal triangle Mod 3

- We define a new coordinate system such that, at position (n, k) the binomial coefficient is

$$\binom{n+k}{k} = \frac{(n+k)!}{n!k!}$$

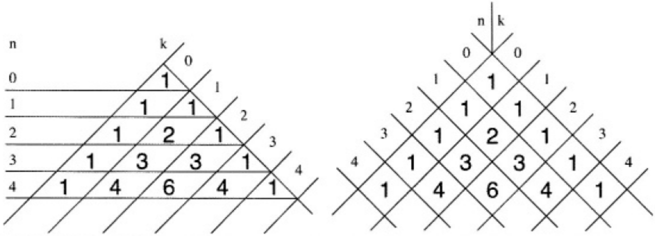


Figure: A new coordinate system

Divisibility Sets

- We formally define our problem

$$P(r) = \left\{ (n,k) \middle| \binom{n+k}{n} \text{ is not divisible by } r \right\}$$

- Observe that if p and q are two different prime numbers and a given integer r is **not** divisible by $p \cdot q$, then it is **not** divisible by either p or q

$$P(pq) = P(p) \cup P(q), \text{ if } p \neq q,\ p, q \text{ prime}$$

- It is the negation of the statement, if p and q divides r, then r is divisible by both p and q.
- Ex: $P(6) = P(3) \cup P(2)$. This generalization can be extended to any integer

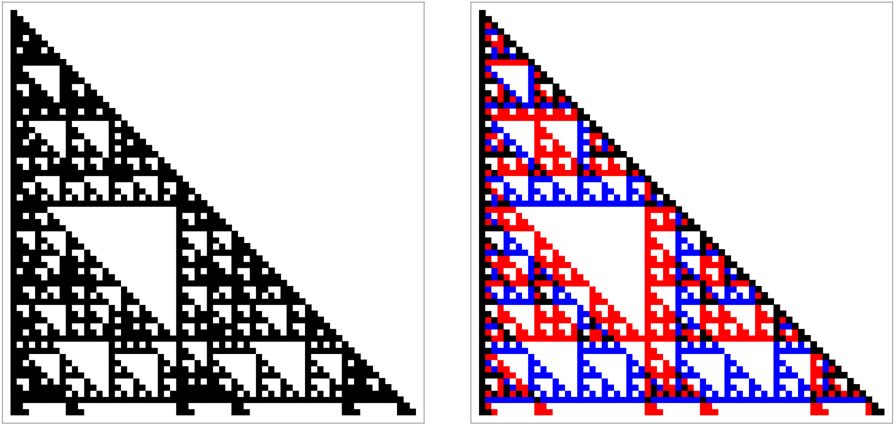


Figure: Right shows mod 6 pattern and left shows the union of mod 2 (Red) and mod 3 (Blue)

- For an integer $r = p_1^{n_1} \dots p_s^{n_s}$ where $p_1, \dots, p_s$ are primes

$$P(r) = P(p_1^{n_1}) \cup ... \cup P(p_s^{n_s})$$

- So to understand the pattern formed by $P(r)$, we just need to understand the pattern of $P(p^n)$

Kummer's Result and p-adic numbers

- To understand Kummer's result, we need to consider numbers with base p, where p is prime
- Like the decimal system, p-adic expansion of an integer is given by

$$n = a_0 + a_1 p + \cdots + a_m p^m$$

  where $a_i \in \{0, 1, \ldots, p-1\}$

- The p-adic representation would be

$$n = (a_m a_{m-1} \ldots a_0)_p$$

- Ex: $15_{10} = (1111)_2 = (120)_3 = (30)_5 = (21)_7 = (14)_{11}$

- We define carry function c_p

$$c_p(n,k) = \text{number of carries in the p-adic addition of n and k}$$

- Ex: For $n = 15$ and $k = 8$

$$k = (08)_{10} = (1000)_2 = (022)_3 = (13)_5 = (11)_7 = (08)_{11}$$

- If we take the binary addition $\begin{array}{cccccc} & 0 & 1 & 1 & 1 & 1 \\ + & 0_1 & 1 & 0 & 0 & 0 \\ \hline & 1 & 0 & 1 & 1 & 1 \end{array}$

$$c_2(15,8) = 1$$

- Similarly, we can do the same for $p = 3, 5, 7 \dots$

Kummer's result

- Let $\tau = c_p(n,k)$, then $\binom{n+k}{k}$ is divisible by p^τ but not $p^{\tau+1}$
- So, prime factorization of $\binom{n+k}{k}$ contains exactly $c_p(n,k)$ factors of p
- This result is pretty amazing, since it gives a direct method to check if a binomial coefficient is divisible by p or not.
- For $n = 15$ and $k = 8$

$$\binom{15+8}{8} = \binom{23}{8} = 2 \cdot 3 \cdot 11 \cdot 17 \cdot 19 \cdot 23$$

- So the Kummer's result implies

$$c_p(17,8) = \begin{cases} 1, & \text{for } p = 2,3,11,17,19 \\ 0, & \text{otherwise} \end{cases}$$

- Applying Kummer's result to Divisibility Set gives

$$P(p) = \{(n,k)|\ c_p(n,k) = 0\}$$

- That is, the number of carries $c_p(n,k)$ is only 0 if and only if

$$a_i + b_i < p, \quad i = 0, \ldots, m$$

  where a_i and b_i are the p-adic digits of n and k.
- This is called the *mod-p* condition.
- For prime powers,

$$P(p^\tau) = \{(n,k)|\ c_p(n,k) < \tau\}$$

Iterated Function System

- It is a method of constructing fractals using a set of contraction mappings.
- A contraction mapping is an affine linear transformation

$$f(x, y) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$$

- Union of these contraction mapping gives the Hutchinson equation of the fractal
- If you put a probability factor to these contraction mappings, you get some pretty images.

- The Hutchinson operator for Sierpinski Triangle is:

$$S = w_{00}(S) \cup w_{01}(S) \cup w_{10}(S)$$

  where w_{ij} are contraction mappings of a unit square

$$\begin{aligned} w_{00} &= (x/2, y/2) \\ w_{01} &= (x/2, y/2 + 1/2) \\ w_{10} &= (x/2 + 1/2, y/2) \\ w_{11} &= (x/2 + 1/2 + y/2 + 1/2) \end{aligned}$$

- So how do we know for sure that our Pascal Sierpinski triangle is the same as the Hutchinson operator Sierpinski Triangle?
- Let's construct Mod 2 Pascal triangle in a unit square

- Let Q be the unit square

$$Q = \{(x, y) | \ (x, y) \in [0, 1] \times [0, 1]\}$$

- Expand x and y in base 2

$$x = \sum_{i=1}^{\infty} a_i 2^{-i}, \ \ a_i \in \{0, 1\}$$
$$y = \sum_{i=1}^{\infty} b_i 2^{-i}, \ \ b_i \in \{0, 1\}$$

- From Kummer's result, we know that the coordinates of the points divisible by 2 are those which have no carries in the binary addition of the coordinates

$$S = \{(x, y) \in Q | \ a_i + b_i \leq 1 \ \forall i\}$$

Proof by picture

Proof by symbols

- To show both are the same, we need to show

  $$w_{00}(S) \cup w_{01}(S) \cup w_{10}(S) \subset S$$

  and,

  $$w_{00}(S) \cup w_{01}(S) \cup w_{10}(S) \supset S$$

- Take any point $(x, y) \in S$

  $$(x, y) = (0, a_1 a_2 a_3 \dots, 0.b_1 b_2 \dots)$$

  where $a_i + b_i \leq 1$

- Applying the 3 transformations gives

  $$\begin{aligned} w_{00}(0, a_1 a_2 \dots, 0.b_1 b_2 \dots) &= (0.0a_1 a_2 \dots, 0.0b_1 b_2 \dots) \\ w_{01}(0, a_1 a_2 \dots, 0.b_1 b_2 \dots) &= (0.0a_1 a_2 \dots, 0.1b_1 b_2 \dots) \\ w_{10}(0, a_1 a_2 \dots, 0.b_1 b_2 \dots) &= (0.1a_1 a_2 \dots, 0.0b_1 b_2 \dots) \end{aligned}$$

- Clearly, all these points are also in S.
- To show the second relation, we take any point $(x, y) \in S$, and we have to provide another point $(x', y') \in S$ such that (x, y) is one of the images of $w_{00}(x', y'), w_{10}(x', y')$, or $w_{01}(x', y')$. We choose

$$(x', y') = (0.a_2 \ldots, b_2 \ldots)$$

- We obtain

$$(x, y) = \begin{cases} w_{00}\left(x', y'\right) & \text{if } a_1 = 0 \text{ and } b_1 = 0 \text{ or} \\ w_{01}\left(x', y'\right) & \text{if } a_1 = 0 \text{ and } b_1 = 1 \text{ or} \\ w_{10}\left(x', y'\right) & \text{if } a_1 = 1 \text{ and } b_1 = 0 \end{cases}$$

- This completes our proof.

Concequences

- The binary representation allows us to see Hutchinson operator in action applied to a point inside square
- If (x, y) is an arbitrary point in Q, the applying the map w_{00}, w_{01}, w_{10} again and again yields points with leading binary decimal points satisfying $a_i + b_i \leq 1$
- In symbols,

  $$A_0 = Q$$

  Then running the IFS gives

  $$A_n = w_{00}(A_{n-1}) \cup w_{01}(A_{n-1}) \cup w_{00}(A_{n-1})$$

  where the leading n binary digits satisfy $a_i + b_i \leq 1$
- Finally, the sequence will lead to Sierpinski triangle

  $$A_\infty = S$$

IFS for other primes

- Now, we can finally look into the global pattern formation of divisibility of binomial coefficients with primes, i.e the global patterns formed by

$$P(r) = \left\{ (n,k) \,\middle|\, \binom{n+k}{n} \text{is not divisible by } r \right\}$$

- We construct an IFS: Divide the unit square Q in p^2 congruent square $Q_{a,b}$ with $a, b \in \{0, \dots, p-1\}$. We introduce the contraction mappings

$$w_{a,b}(x,y) = \left(\frac{x+a}{p}, \frac{y+b}{p} \right)$$

where

$$w_{a,b}(Q) = Q_{a,b}$$

- Then we set the restriction to define the set of transformation

$$a + b \leq p - 1$$

- This restriction follows from Kummer's result,i.e $\binom{n+k}{n}$ is indivisible by p when there is no carries in the p-adic addition of n, k
- The Hutchinson operator for these contractions,

$$W_p(A) = \bigcup_{a+b<p} w_{a,b}(A)$$

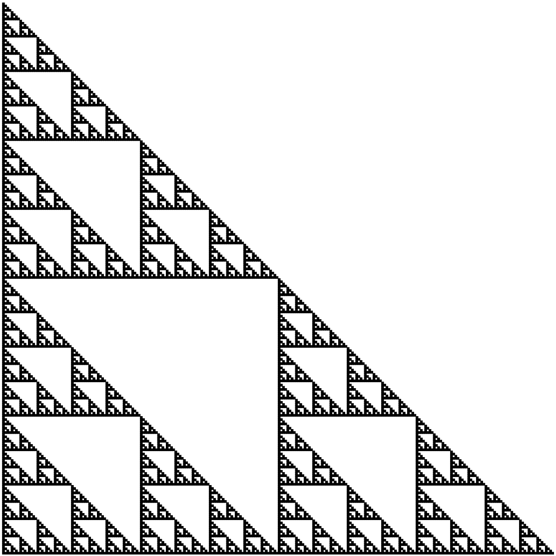


Figure: Limit of Mod 2 IFS

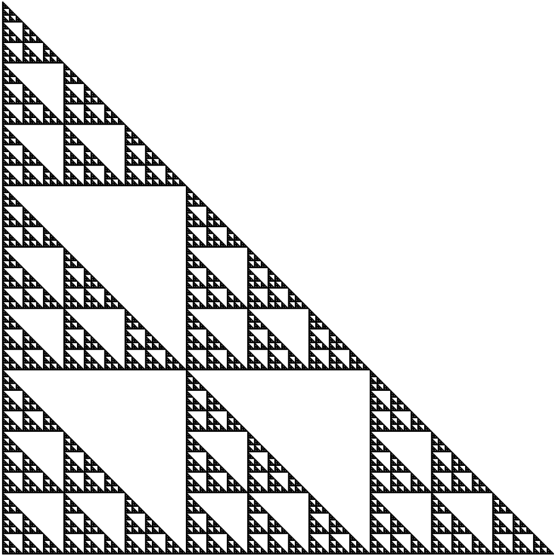


Figure: Limit of Mod 3 IFS

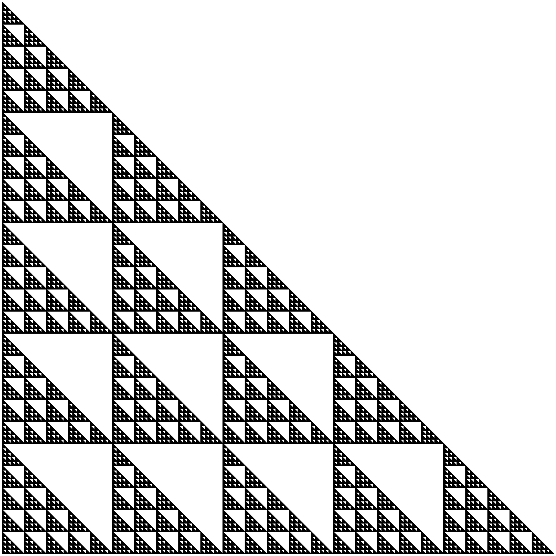


Figure: Limit of Mod 5 IFS

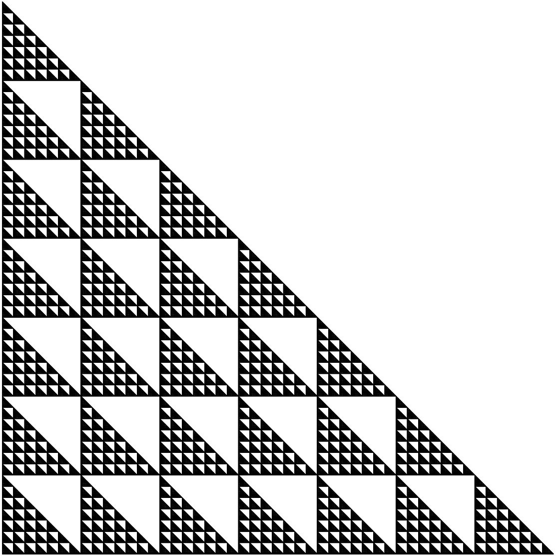


Figure: Limit of Mod 7 IFS

Conlcusion