# Pascal's Triangle: Cellular Automata and Attractors

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*Mathematics is the art of giving the same name to different things*

— Henri Ponicaré

# Pascal's Triangle

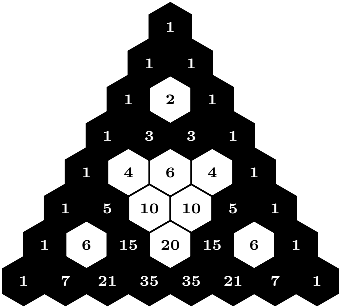
$$
\begin{array}{ccccccccccccccc}
&&&&&&& 1 &&&&&&& \\
&&&&&& 1 && 1 &&&&&& \\
&&&&& 1 && 2 && 1 &&&&& \\
&&&& 1 && 3 && 3 && 1 &&&& \\
&&& 1 && 4 && 6 && 4 && 1 &&& \\
&& 1 && 5 && 10 && 10 && 5 && 1 && \\
& 1 && 6 && 15 && 20 && 15 && 6 && 1 & \\
1 && 7 && 21 && 35 && 35 && 21 && 7 && 1
\end{array}
$$

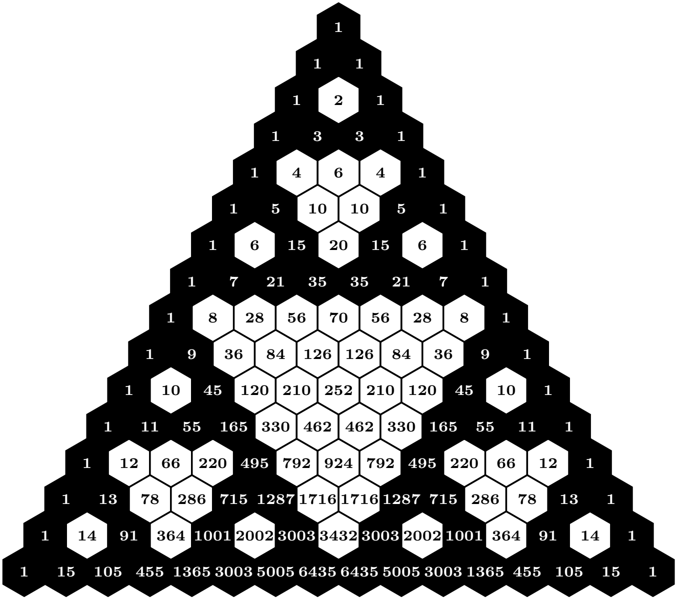
- One of the earliest mentions was in a Chinese document at around 1303 AD
- Looks pretty innocent right?

# Pascal's Triangle

Let's look at it in a few different ways
It is observed by coloring

- all odd numbers black
- all even numbers white




1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
1 11 55 165 330 462 462 330 165 55 11 1
1 12 66 220 495 792 924 792 495 220 66 12 1
1 13 78 286 715 1287 1716 1716 1287 715 286 78 13 1
1 14 91 364 1001 2002 3003 3432 3003 2002 1001 364 91 14 1
1 15 105 455 1365 3003 5005 6435 6435 5005 3003 1365 455 105 15 1

# Pascal's Triangle

- Can also be formulated through binomial coefficients

$$
\begin{aligned}
(1+x)^0 &= 1 \\
(1+x)^1 &= 1 + 1x \\
(1+x)^2 &= 1 + 2x + 1x^2 \\
&\vdots \\
(1+x)^n &= a_0 + a_1 x + \cdots + a_n x^n
\end{aligned}
$$

where coefficients are given by

$$a_k = \binom{n}{k} = \frac{n!}{(n-k)!k!}, \quad 0 \le k \le n$$

# Pascal's Triangle

- Understanding the divisibility of binomial coefficients by computing them is a bad idea

$$50! = 30414093201713378043612608166064768844377641568960512000000000000$$

Even if we use the recursive formula

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

$$\binom{40}{20} = 137846528820 > 2^{32}$$

which is a big number

- Fortunately, we don't need to compute these large numbers

# Pascal's Triangle

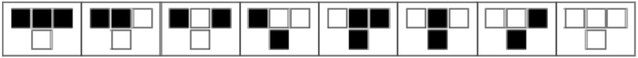
- Ex: Divisibility by 2 can be followed from addition rule

| $\binom{n+1}{k} =$ | $\binom{n}{k-1}+$ | $\binom{n}{k}$ |
|---|---|---|
| even | even | even |
| odd | odd | even |
| odd | even | odd |
| odd | odd | odd |

- Can we extend this idea of divisibility to other numbers ?
- What patterns do we get ? Global pattern? Why? ...IFS

# Cellular Automata

- Prefect feedback machines. They are mathematically finite state machines
- Each cell has one out of $p$ states. *p-state* automata
- Can be 1D, 2D . . .
- To run cellular automata, we need 2 pieces of information
  1. Initial state of cells
  2. Rules to describe new cell state from the states of a group of cells from the previous layer
- The rules should not depend on the position of the group of cells within the layer.



0 0 0 1 1 1 1 0

# Cellular Automata

- More kinds of rules

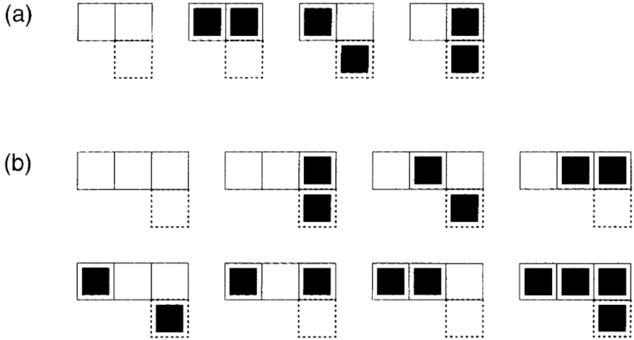



Figure: (a) is a four rule configuration and (b) is a 8 rule configuration

# Cellular Automata

So lets run some

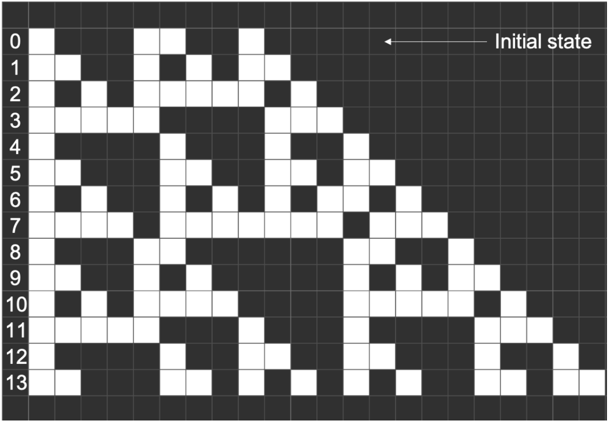



Figure: 13 Generations of Rule 60 Elementary CA

## Elementary Cellular Automata

- Simplest form of 1D Cellular Automata (CA)
- 2 State CA
- Next generation depends on itself,cell to the left and, cell to the right.
- Total number of rules are $2^{2^3} = 256$
- All the rules can be numbered in a nice way using binary

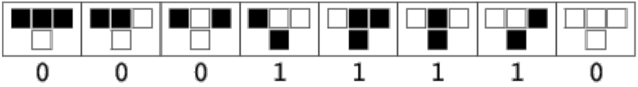



Figure: Elementary rule $30 = (00011110)_2$

- Mathematica can do this for you very easily using CellularAutomaton[ ]

# Elementary Cellular Automata

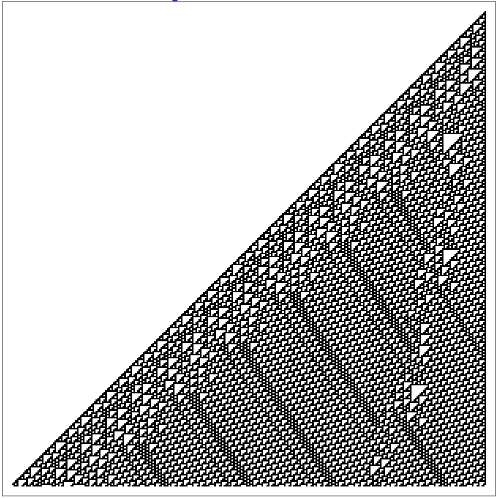


Figure: Rule 110, this might be special

# Elementary Cellular Automata

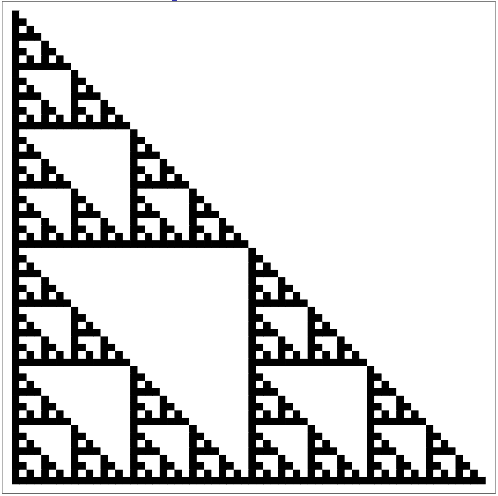


Figure: Rule 60. Sierpinski Triangle*

# Elementary Cellular Automata

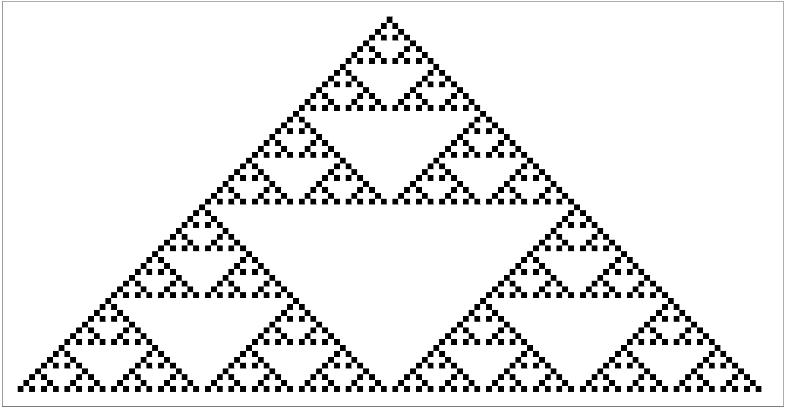


Figure: Rule 90.

# Elementary Cellular Automata

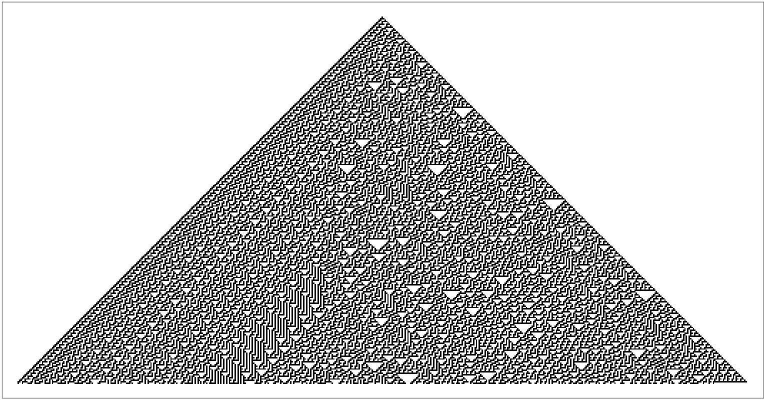


Figure: Rule 30

Kinda chaotic ?

# 2D CA: Game of life...

- Was made popular through the work of John Horton Conway
- 2-state CA
- Rules:
    1. Cell survives when exactly 2 or 3 of it's 8 neighbors are alive.
    2. If more than 3 neighbors are alive, the cell dies from over-crowdedness.
    3. If fewer than 2 neighbors are alive, the cell dies from loneliness.
    4. Dead cell comes back to life when surrounded by exactly 3 live neighbors.
- Makes some interesting patters and life like behavior
- **Note:** The position of the alive and dead cells w.r.t the center does not matter for this rule. Also known as Totalistic CA

Let's play the Game of Life......

# Cellular Automata

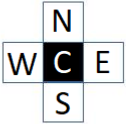
- Another variant of Game of Life: One out of eight rule
  1. Cell becomes alive if exactly one neighbor is alive.
  2. Otherwise unchanged.
- Has some nice self-similarity

# Cellular Automata

- Game of life is just one out of the imaginable set of rules
- For 2-state 2D automata and a neighborhood of eight cells, there are $2^{2^9} \approx 10^{154}$ different sets of rules!!
- Let's look at another rule: Majority Rule
  1. If 5 or more of the neighborhood of 9 cells (including itself) are alive, this cell lives or stays alive
  2. Otherwise dies or remains dead
- Here center cell adjusts to the majority
- This kind of rule is called Outer Totalistic
- Has some resemblance with percolation

## Cellular Automata with Different neighbors

- Let's consider only 4 neighbors.




- C = Center, N = North, E = East, S = South, W = West
- Can represent every neighbor in Binary $(CSWNE)_2$. Ex:$(01100)_2$ means S, W cells are alive, rest are dead
- Total number of sets of rules are $2^{2^5} \approx 4 \cdot 10^9$
- Two interesting examples are shown

# Cellular Automata

| CSWNE | C | CSWNE | C | CSWNE | C | CSWNE | C |
|---|---|---|---|---|---|---|---|
| 00000 | 0 | 01000 | 1 | 10000 | 1 | 11000 | 1 |
| 00001 | 0 | 01001 | 1 | 10001 | 1 | 11001 | 1 |
| 00010 | 0 | 01010 | 1 | 10010 | 1 | 11010 | 1 |
| 00011 | 0 | 01011 | 1 | 10011 | 1 | 11011 | 1 |
| 00100 | 1 | 01100 | 0 | 10100 | 1 | 11100 | 1 |
| 00101 | 1 | 01101 | 0 | 10101 | 1 | 11101 | 1 |
| 00110 | 1 | 01110 | 0 | 10110 | 1 | 11110 | 1 |
| 00111 | 1 | 01111 | 0 | 10111 | 1 | 11111 | 1 |

- This behaves like a 1D CA

# Cellular Automata

- Many interesting rules can be given by a simple formula
- Parity Rule:

$$C_{new} = C_{old} + N_{old} + E_{old} + S_{old} + W_{old} \mod 2$$

## CA, Pascal's Triangle and Polynomial

- Let's look at the powers of $r(x) = 1 + x$:

$$
\begin{aligned}
(r(x))^0 &= 1 \\
(r(x))^1 &= 1 + x \\
(r(x))^2 &= 1 + 2x + x^2 \\
(r(x))^3 &= 1 + 3x + 3x^2 + x^3 \\
&\vdots \\
(r(x))^n &= a_0(n) + a_1(n)x + a_2(n)x^2 + \cdots + a_n(n)x^n
\end{aligned}
$$

- By the addition rule of binomial coefficients

$$a_k(n) = a_{k-1}(n-1) + a_k(n-1)$$

- $a_k(n)$ gives the state of $\text{n}^{\text{th}}$ layer CA*

# CA, Pascal's Triangle and Polynomial

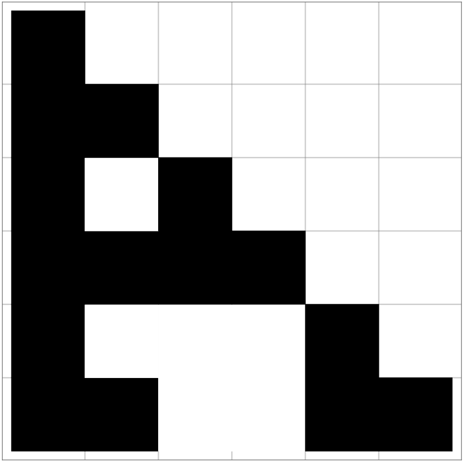
- Looking at the divisibility properties of $a_k(n)$ with 2

$$a_k(n) \equiv 0 \pmod 2 \qquad or \qquad a_k(n) \equiv 1 \pmod 2$$

- With this, the addition rules in mod 2 arithmetic simplifies to

| $\binom{n+1}{k} =$ | $\binom{n}{k-1}$ | $+ \binom{n}{k}$ |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 1 | 0 |
| 1 | 0 | 1 |
| 0 | 1 | 1 |

- Exactly same rule set used to make Pascal Triangle



- Thus, this figure shows the coefficients of the powers of $r(x) = 1 + x \mod 2$

## Generalizations

- There are 2 ways to generalize,
  1. A different coefficient modulo integer
  2. A different polynomial
- Ex: Let $r(x) = 1 + 2x$

$$\begin{aligned}(r(x))^0 &= 1\\ (r(x))^1 &= 1 + 2x\\ (r(x))^2 &= 1 + 4x + 4x^2\\ &\vdots\\ (r(x))^n &= a_0(n) + a_1(n)x + \cdots + a_n(n)x^n\end{aligned}$$

- By pattern,

$$a_k(n) = a_k(n-1) + 2a_{k-1}(n-1)$$

- By looking at the divisibility property with p = 3, we get

$$(r(x))^0 = 1$$
$$(r(x))^1 = 1\ 2$$
$$(r(x))^2 = 1\ 1\ 1$$
$$(r(x))^3 = 1\ 0\ 0\ 2$$

- The cellular automaton would be $a_{n,k} = a_{n-1,k} + 2a_{n-1,k-1} \mod 3$

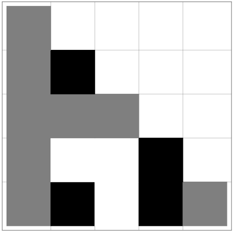


Figure: CA for $r(x) = 1 + 2x \mod 2$

# Linear Cellular Automata

- We can start with any polynomial $r(x) = a_0 + a_1 x + \cdots + a_d x^d$
- The coefficients of $(r(x))^n$ modulo some positive integer $p$ is obtained by an addition formula involving $d$ coefficients from $(r(x))^{n-1}$
- So, there is an associated Cellular Automata which generates coefficients modulo p of the powers of $(r(x))^n$
- A look-up table can be generated by an addition formula
- These are called *Linear Cellular Automata*
- Again, the choice of p determines the number of states. This opens up a lot of interesting problems

- Pattern Formation: Given a polynomial, what is the global pattern which evolves when the automaton has run for a long time?
- Colors: What is the relation between the global patterns which are obtained for different choices of p?
- Fractal Dimension: Fractal dimension of the global pattern ?
- Higher Dimension: What if we used polynomial of m variables ? (m-dimensional ?)
- Factorization: If a polynomial $r(x)$ can be factorized to two polynomials $s(x)$ and $t(x)$, how are the patters related ? Is that factorization unique ?
  In general, no, it depends on $p$
  Ex: $1 + x$ is irreducible with respect to integers. But in arithmetic modulo $p$, with $p$ not prime, there are non trivial factorization

$$1 + x \equiv (1 + 3x)(1 + 4x) \pmod 6$$

# Binomial Coefficients and Divisibility

- Discuss the question of whether a binomial coefficient is divisible by $p$ or not.
- Black and white coloring of the Pascal triangle depending on divisibility with p
- We will also see that in order to understand the patterns formed by mod $p$, we should look at the patters formed by the prime factors of $p$
- We will look at a direct, non-recursive computation of divisibility by p

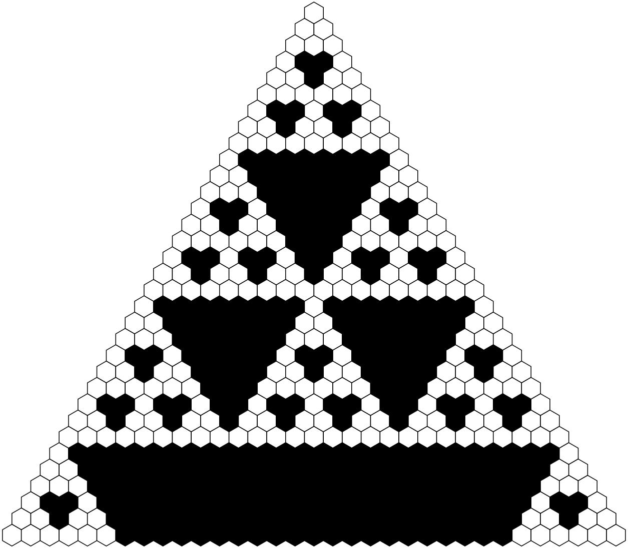


Figure: Pascal triangle Mod 3

- We define a new coordinate system such that, at position $(n, k)$ the binomial coefficient is

$$\binom{n+k}{k} = \frac{(n+k)!}{n!k!}$$

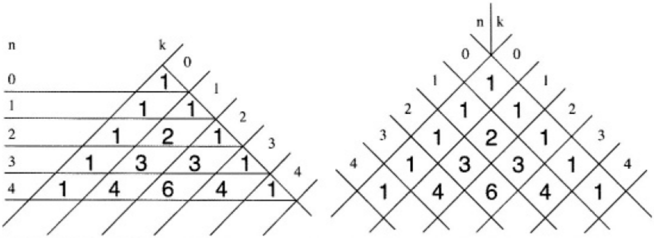



Figure: A new coordinate system

# Divisibility Sets

- We formally define our problem

$$P(r) = \left\{ (n,k) \,\middle|\, \binom{n+k}{n} \text{ is not divisible by } r \right\}$$

- Observe that if $p$ and $q$ are two different prime numbers and a given integer $r$ is **not** divisible by $p \cdot q$, then it is **not** divisible by either $p$ or $q$

$$P(pq) = P(p) \cup P(q), \text{ if } p \neq q,\ p, q \text{ prime}$$

- It is the negation of the statement, if $p$ and $q$ divides $r$, then $r$ is divisible by both $p$ and $q$.
- Ex: $P(6) = P(3) \cup P(2)$. This generalization can be extended to any integer

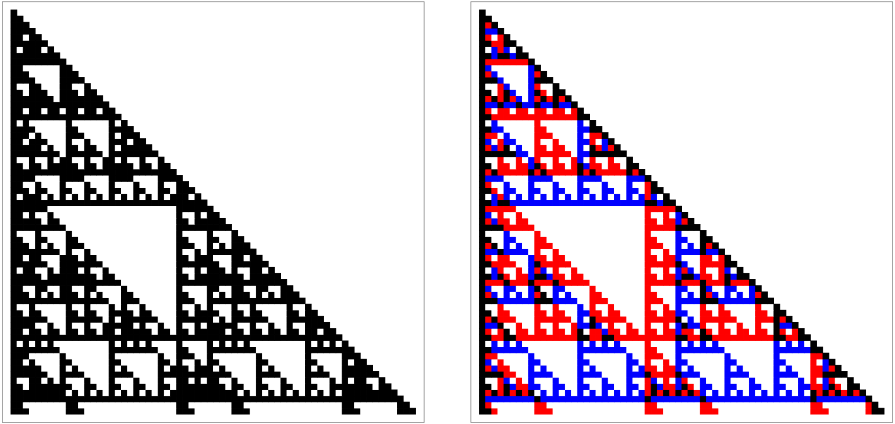


Figure: Right shows mod 6 pattern and left shows the union of mod 2 (Red) and mod 3 (Blue)

- For an integer $r = p_1^{n_1} \dots p_s^{n_s}$ where $p_1, \dots, p_s$ are primes

$$P(r) = P(p_1^{n_1}) \cup ... \cup P(p_s^{n_s})$$

- So to understand the pattern formed by $P(r)$, we just need to understand the pattern of $P(p^n)$

## Kummer's Result and p-adic numbers

- To understand Kummer's result, we need to consider numbers with base $p$, where $p$ is prime
- Like the decimal system, p-adic expansion of an integer is given by

$$n = a_0 + a_1 p + \cdots + a_m p^m$$

where $a_i \in \{0, 1, \ldots, p-1\}$

- The p-adic representation would be

$$n = (a_m a_{m-1} \ldots a_0)_p$$

- Ex: $15_{10} = (1111)_2 = (120)_3 = (30)_5 = (21)_7 = (14)_{11}$

- We define carry function $c_p$

$$c_p(n,k) = \text{number of carries in the p-adic addition of n and k}$$

- Ex: For $n = 15$ and $k = 8$

$$k = (08)_{10} = (1000)_2 = (022)_3 = (13)_5 = (11)_7 = (08)_{11}$$

- If we take the binary addition $\begin{array}{cccccc} & 0 & 1 & 1 & 1 & 1 \\ + & 0_1 & 1 & 0 & 0 & 0 \\ \hline & 1 & 0 & 1 & 1 & 1 \end{array}$

$$c_2(15,8) = 1$$

- Similarly, we can do the same for $p = 3, 5, 7 \ldots$

# Kummer's result

- Let $\tau = c_p(n,k)$, then $\binom{n+k}{k}$ is divisible by $p^\tau$ but not $p^{\tau+1}$
- So, prime factorization of $\binom{n+k}{k}$ contains exactly $c_p(n,k)$ factors of $p$
- This result is pretty amazing, since it gives a direct method to check if a binomial coefficient is divisible by $p$ or not.
- For $n = 15$ and $k = 8$

$$\binom{15+8}{8} = \binom{23}{8} = 2 \cdot 3 \cdot 11 \cdot 17 \cdot 19 \cdot 23$$

- So the Kummer's result implies

$$c_p(17,8) = \begin{cases} 1, & \text{for } p = 2, 3, 11, 17, 19 \\ 0, & \text{otherwise} \end{cases}$$

- Applying Kummer's result to Divisibility Set gives

$$P(p) = \{(n,k)|\ c_p(n,k) = 0\}$$

- That is, the number of carries $c_p(n,k)$ is only 0 if and only if

$$a_i + b_i < p, \quad i = 0, \dots, m$$

where $a_i$ and $b_i$ are the p-adic digits of n and k.

- This is called the *mod-p* condition.
- For prime powers,

$$P(p^\tau) = \{(n,k)|\ c_p(n,k) < \tau\}$$

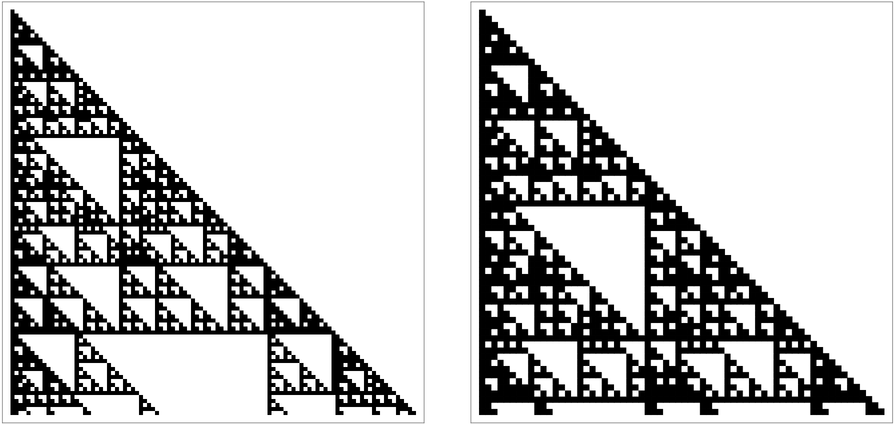


Figure: $P(6)$ generated using carry function (left) vs binomial divisibility (right)

# Iterated Function System

- It is a method of constructing fractals using a set of contraction mappings.
- A contraction mapping is an affine linear transformation

$$f(x,y) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$$

- Union of these contraction mapping gives the Hutchinson equation of the fractal
- If you put a probability factor to these contraction mappings, you get some pretty images.

- The Hutchinson operator for Sierpinski Triangle is:

$$S = w_{00}(S) \cup w_{01}(S) \cup w_{10}(S)$$

where $w_{ij}$ are contraction mappings of a unit square

$$\begin{aligned}
w_{00} &= (x/2, y/2) \\
w_{01} &= (x/2, y/2 + 1/2) \\
w_{10} &= (x/2 + 1/2, y/2) \\
w_{11} &= (x/2 + 1/2 + y/2 + 1/2)
\end{aligned}$$

- So how do we know for sure that our Pascal Sierpinski triangle is the same as the Hutchinson operator Sierpinski Triangle?
- Let's construct Mod 2 Pascal triangle in a unit square

- Let $Q$ be the unit square

$$Q = \{(x,y)|\ (x,y) \in [0,1] \times [0,1]\}$$

- Expand $x$ and $y$ in base 2

$$x = \sum_{i=1}^{\infty} a_i 2^{-i},\ \ a_i \in \{0,1\}$$
$$y = \sum_{i=1}^{\infty} b_i 2^{-i},\ \ b_i \in \{0,1\}$$

- From Kummer's result, we know that the coordinates of the points divisible by 2 are those which have no carries in the binary addition of the coordinates

$$S = \{(x,y) \in Q|\ a_i + b_i \leq 1\ \forall i\}$$

# Proof

- To show both are the same, we need to show

$$w_{00}(S) \cup w_{01}(S) \cup w_{10}(S) \subset S$$

and,

$$w_{00}(S) \cup w_{01}(S) \cup w_{10}(S) \supset S$$

- Take any point $(x, y) \in S$

$$(x, y) = (0, a_1 a_2 a_3 \ldots, 0.b_1 b_2 \ldots)$$

where $a_i + b_i \leq 1$

- Applying the 3 transformations gives

$$w_{00}(0, a_1 a_2 \ldots, 0.b_1 b_2 \ldots) = (0.0a_1 a_2 \ldots, 0.0b_1 b_2 \ldots)$$
$$w_{01}(0, a_1 a_2 \ldots, 0.b_1 b_2 \ldots) = (0.0a_1 a_2 \ldots, 0.1b_1 b_2 \ldots)$$
$$w_{10}(0, a_1 a_2 \ldots, 0.b_1 b_2 \ldots) = (0.1a_1 a_2 \ldots, 0.0b_1 b_2 \ldots)$$

- Clearly, all these points are also in $S$.
- To show the second relation, we take any point $(x, y) \in S$, and we have to provide another point $(x', y') \in S$ such that $(x, y)$ is one of the images of $w_{00}(x', y'), w_{10}(x', y')$, or $w_{01}(x', y')$. We choose

$$(x', y') = (0.a_2 \ldots, b_2 \ldots)$$

- We obtain

$$(x, y) = \begin{cases} w_{00}\left(x', y'\right) \text{ if } a_1 = 0 \text{ and } b_1 = 0 \text{ or} \\ w_{01}\left(x', y'\right) \text{ if } a_1 = 0 \text{ and } b_1 = 1 \text{ or} \\ w_{10}\left(x', y'\right) \text{ if } a_1 = 1 \text{ and } b_1 = 0 \end{cases}$$

- This completes our proof.

## Concequences

- The binary representation allows us to see Hutchinson operator in action applied to a point inside square
- If $(x, y)$ is an arbitrary point in $Q$, the applying the map $w_{00}, w_{01}, w_{10}$ again and again yields points with leading binary decimal points satisfying $a_i + b_i \leq 1$
- In symbols,

$$A_0 = Q$$

  Then running the IFS gives

$$A_n = w_{00}(A_{n-1}) \cup w_{01}(A_{n-1}) \cup w_{10}(A_{n-1})$$

  where the leading $n$ binary digits satisfy $a_i + b_i \leq 1$
- Finally, the sequence will lead to Sierpinski triangle

$$A_\infty = S$$

# IFS for other primes

- Now, we can finally look into the global pattern formation of divisibility of binomial coefficients with primes, i.e the global patterns formed by

$$P(r) = \left\{ (n,k) \,\middle|\, \binom{n+k}{n} \text{is not divisible by } r \right\}$$

- We construct an IFS: Divide the unit square $Q$ in $p^2$ congruent square $Q_{a,b}$ with $a, b \in \{0, \ldots, p-1\}$. We introduce the contraction mappings

$$w_{a,b}(x,y) = \left( \frac{x+a}{p}, \frac{y+b}{p} \right)$$

where

$$w_{a,b}(Q) = Q_{a,b}$$

- Then we set the restriction to define the set of transformation

$$a + b \leq p - 1$$

- This restriction follows from Kummer's result,i.e $\binom{n+k}{n}$ is indivisible by $p$ when there is no carries in the p-adic addition of $n, k$
- The Hutchinson operator for these contractions,

$$W_p(A) = \bigcup_{a+b<p} w_{a,b}(A)$$

need to improve this section, very less detail about the figures

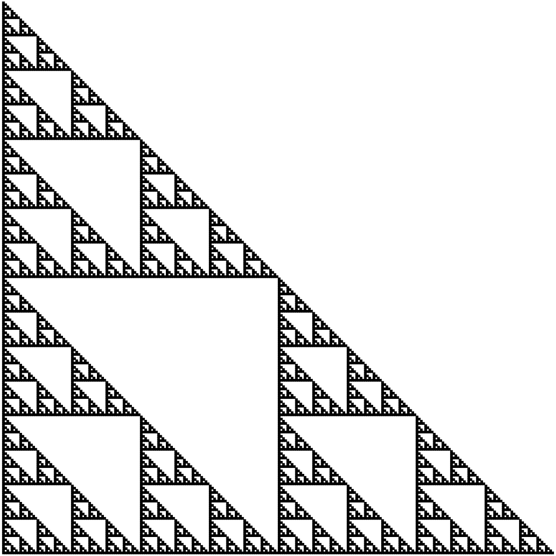


Figure: Limit of Mod 2 IFS

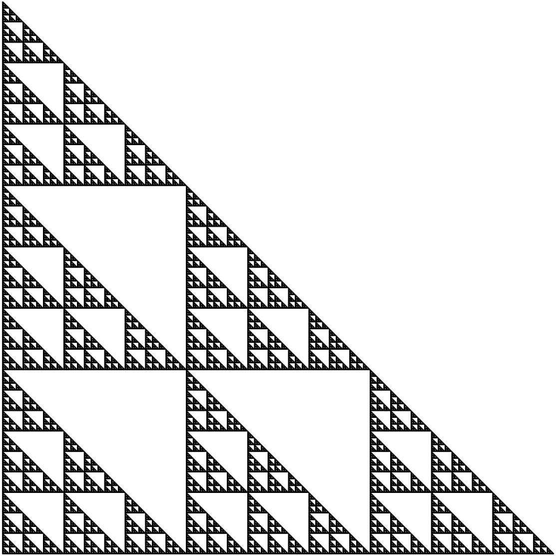


Figure: Limit of Mod 3 IFS.

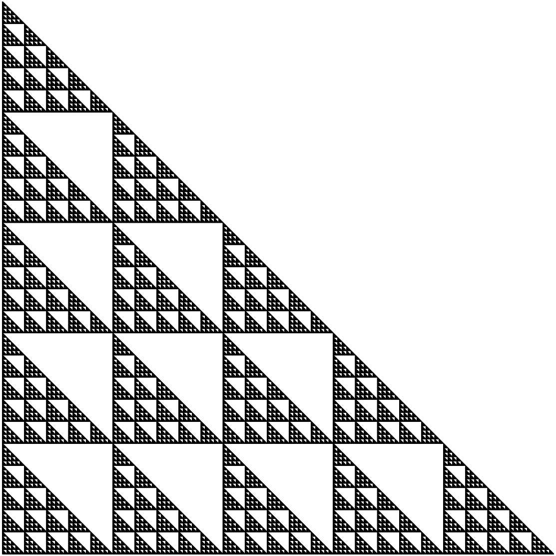


Figure: Limit of Mod 5 IFS.

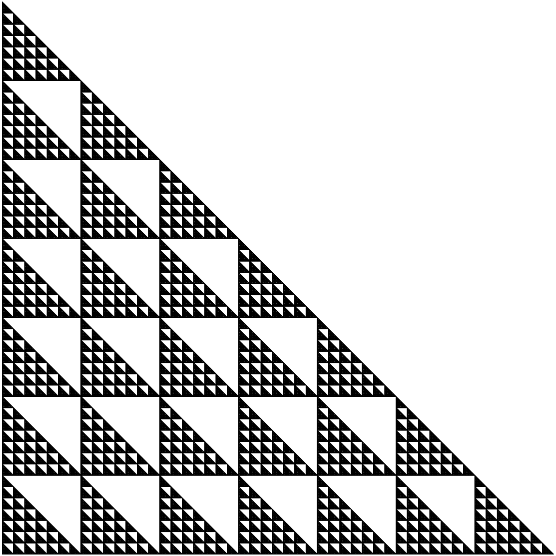


Figure: Limit of Mod 7 IFS.

# Conlcusion

- We approached the problem of coloring (or divisibility) of pascal's triangle in 3 different ways

  1. First, we looked at the macroscopic nature of divisibility, i.e how the neighbors affected the divisibility of a cell. We then explored the realm of cellular automata and how it extends this idea to general polynomials. We also peaked into some open ended problems
  2. Second, we looked at the microscopic nature of divisibility,i.e how the divisibility of a cell depends on its coordinate (Kummer's Result). We looked how p-adic numbers gave insight about binomial divisibility with primes. We also extended this idea of divisibility to all integers.
  3. Finally, we looked at the global pattern using Iterated function system. We showed that the limit of Pascal Mod 2 pattern was indeed the Sierpinski triangle. We also explored the global patterns of other primes and their IFS.

- Each method gave a different perspective of the same problem.
- This entire chapter is trying to solve the jig-saw puzzle that relates fractals, Pascal's triangle, and Cellular Automata.
- Understanding that these 3 very different things are all tied up by one common thread is, honestly, amazing.

# Thank You!