

# Is Pascal's Triangle a Fractal ?

Ankith A Das

The University of Sydney

May 15, 2019

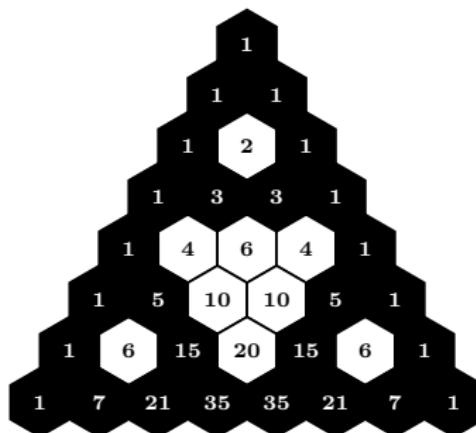
## Pascal's Triangle

$$\begin{array}{ccccccc} & & & & 1 & & \\ & & & & 1 & 1 & \\ & & & & 1 & 2 & 1 \\ & & & & 1 & 3 & 3 & 1 \\ & & & & 1 & 4 & 6 & 4 & 1 \\ & & & & 1 & 5 & 10 & 10 & 5 & 1 \\ & & & & 1 & 6 & 15 & 20 & 15 & 6 & 1 \\ & & & & 1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 \end{array}$$

- ▶ One of the earliest mentions was in a Chinese document at around 1303 AD
- ▶ Looks pretty innocent right?

# Pascal's Triangle

- It is observed by coloring
  - 1. all odd numbers black
  - 2. all even numbers white



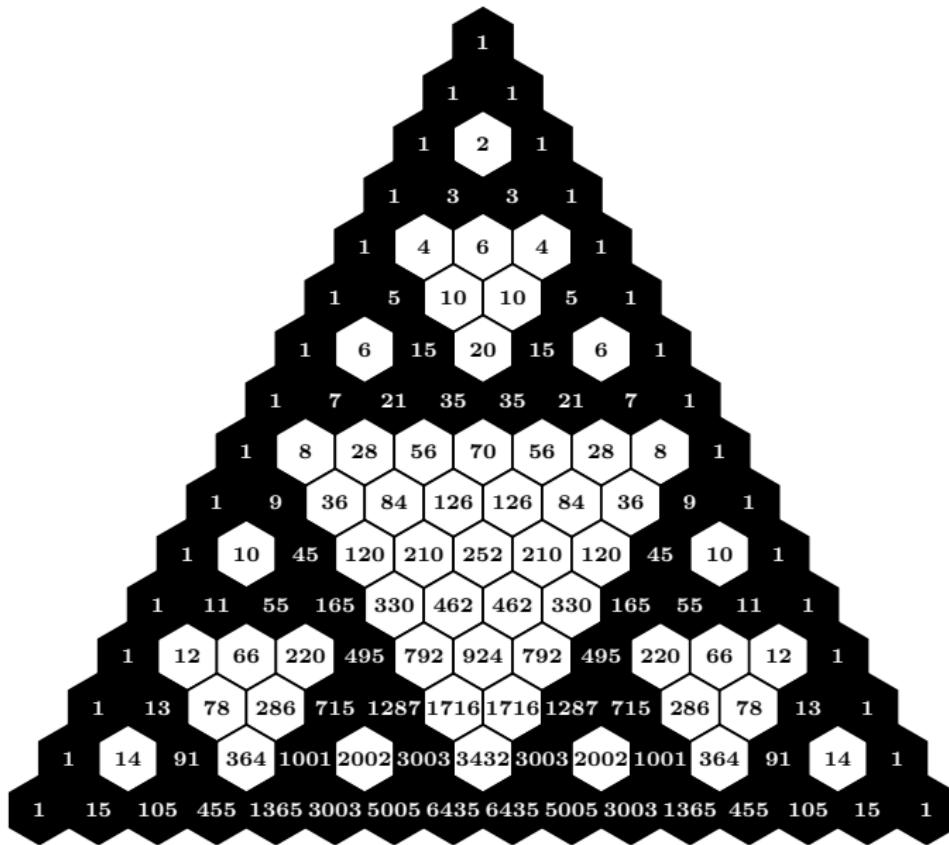


Figure: A bigger picture

## Pascal's Triangle

- So how can we formulate this pattern ? We can begin by looking at binomial coefficients

$$(1 + x)^0 = 1$$

$$(1 + x)^1 = 1 + 1x$$

$$(1 + x)^2 = 1 + 2x + 1x^2$$

⋮

$$(1 + x)^n = a_0 + a_1x + \cdots + a_nx^n$$

where coefficients are given by

$$a_k = \binom{n}{k} = \frac{n!}{(n - k)!k!}, \quad 0 \leq k \leq n$$

## Pascal's Triangle

- ▶ Checking if a binomial coefficient is odd or even by computing  $\frac{n!}{(n-k)!k!}$  is a bad idea

$$50! = 3041409320171337804361260816606476884437$$

$$7641568960512000000000000$$

- ▶ Even if we use the recursive formula

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

$$\binom{40}{20} = 137846528820 > 2^{32}$$

which is a big number (computer has fixed precision)

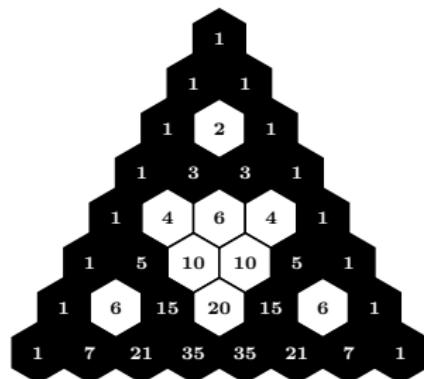
- ▶ Fortunately, we don't need to compute these large numbers

# Pascal's Triangle

- In order to color a row in the triangle, we only need to know the color of the previous row

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

even	even	even
odd	odd	even
odd	even	odd
even	odd	odd



- The idea of looking at neighbor cells to determine the state of a new cell is the core idea of cellular automata

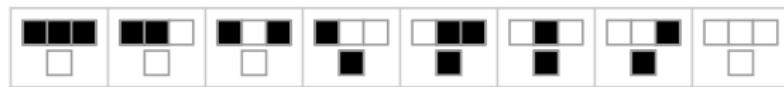
## The Macroscopic view

# Cellular Automata

- ▶ Consists of a grid of cells, each one with a finite number of states, like 1 or 0.



- ▶ A set of rules to describe new cell state from the states of a group of cells from the previous layer
- ▶ Next generation depends on itself, cell to the left and, cell to the right (Neighbors).



- ▶ Ex:



# Cellular Automata

So lets run some

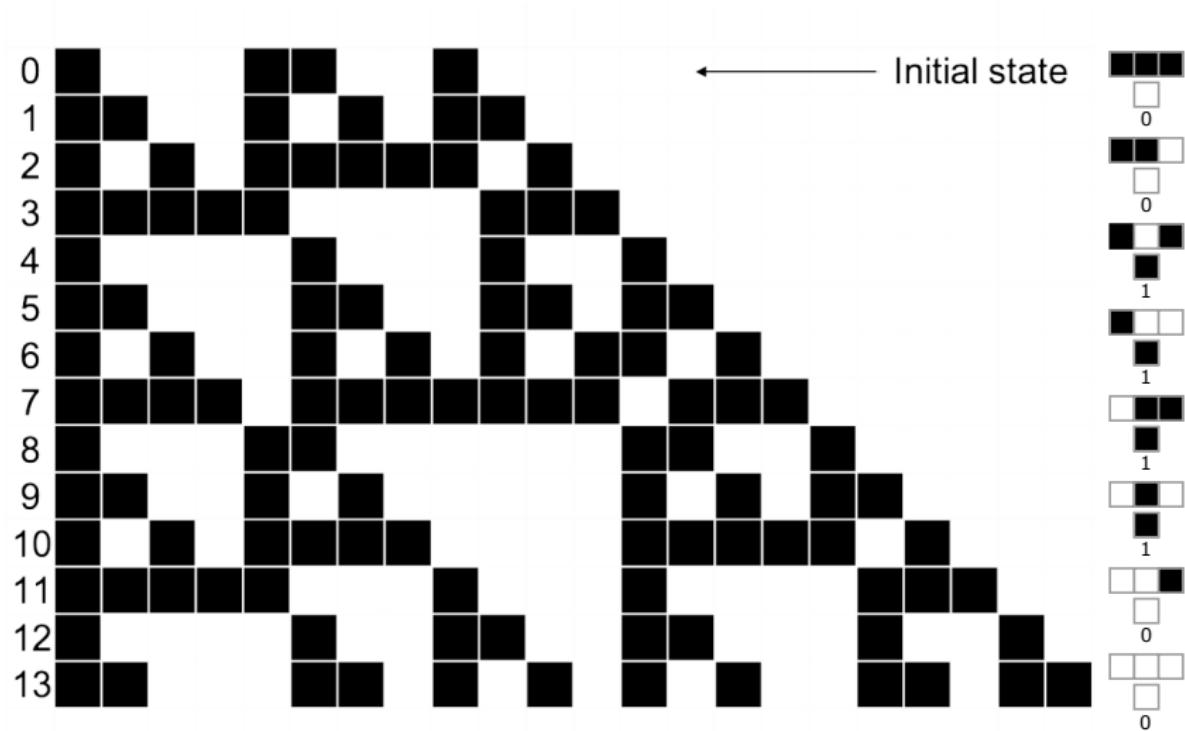


Figure: 13 Generations of a 1-D Cellular Automata

- ▶ There  $2 \times 2 \times 2 = 8$  binary states for each 3 cells.
- ▶ So, there are a total of  $2^8 = 256$  possible rule sets.
- ▶ Each rule can be index with 8-bit binary

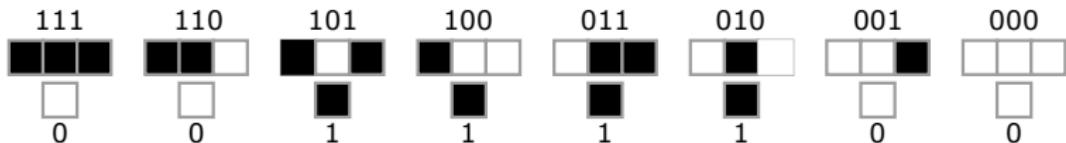


Figure: Rule  $(00111100)_2 = 60$

# Cellular Automata

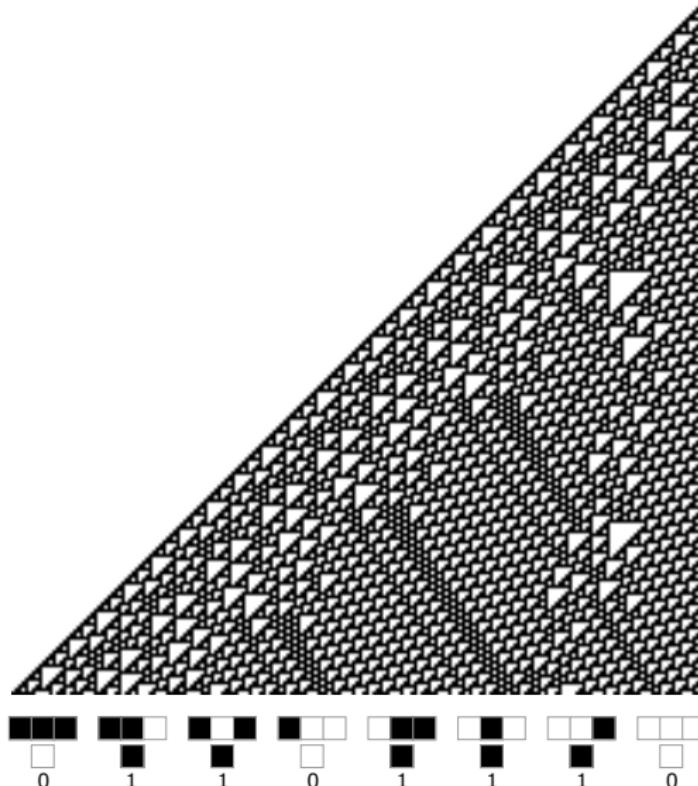


Figure: Rule 110, this might be special

# Cellular Automata

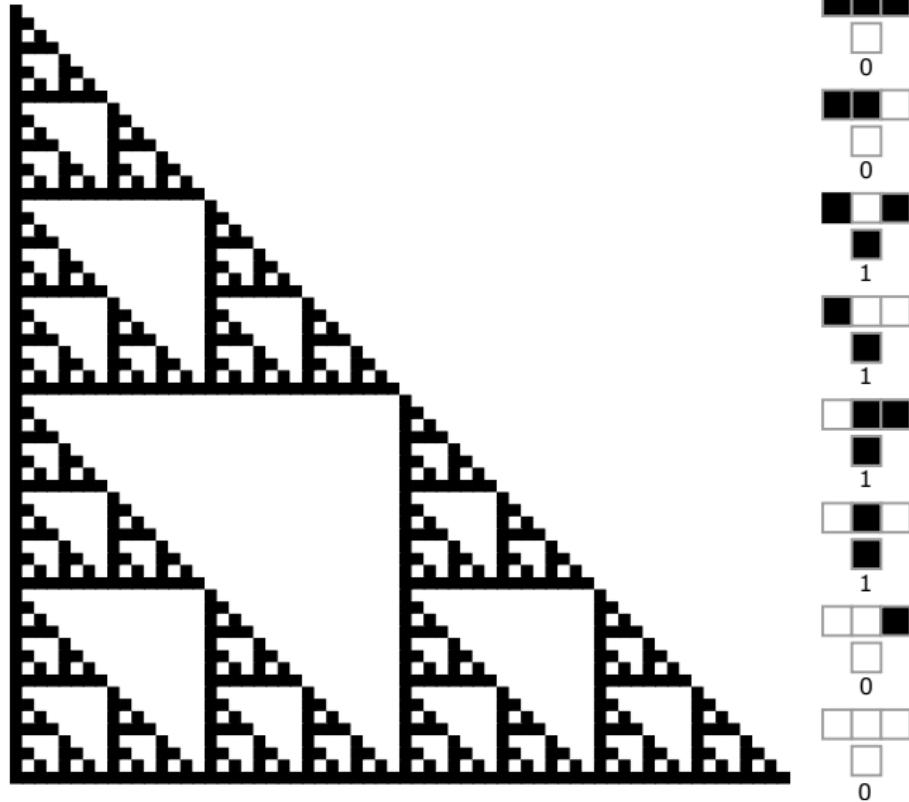


Figure: Rule 60. Sierpinski Triangle\*

# Cellular Automata

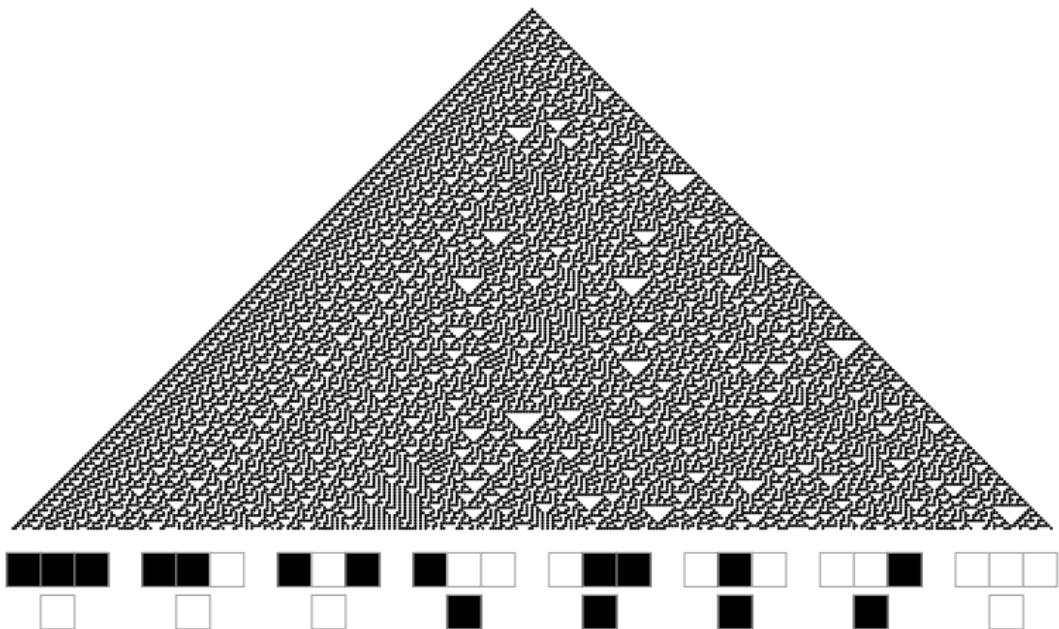


Figure: Rule 30

- ▶ Has been used as a random number generator in Mathematica
- ▶ Sensitive to initial conditions

## CA and Polynomials

- ▶ Let's look at the powers of  $r(x) = 1 + x$ :

$$(r(x))^0 = 1$$

$$(r(x))^1 = 1 + x$$

$$(r(x))^2 = 1 + 2x + x^2$$

⋮

$$(r(x))^n = a_0(n) + a_1(n)x + a_2(n)x^2 + \cdots + a_n(n)x^n$$

- ▶ By the addition rule of binomial coefficients

$$a_k(n) = a_{k-1}(n-1) + a_k(n-1)$$

- ▶ This is a Cellular Automata kind of behavior

## CA and polynomials

- ▶ To color the triangle, we could look at divisibility of  $a_k(n)$  with respect to 2
- ▶ Now, there are 2 possibilities

$$a_k(n) \equiv 0 \pmod{2} \quad \text{or} \quad a_k(n) \equiv 1 \pmod{2}$$

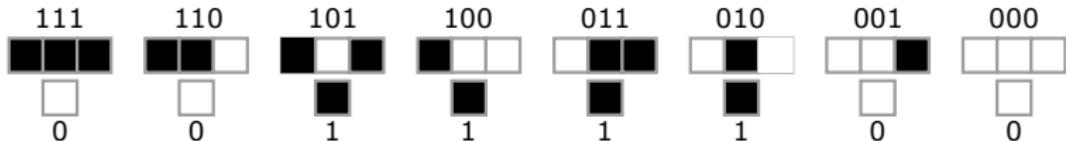
- ▶ The recursive addition rules in mod 2 arithmetic simplifies to

$\binom{n+1}{k}$	$\binom{n}{k-1}$	$\binom{n}{k}$
0	0	0
1	1	0
1	0	1
0	1	1

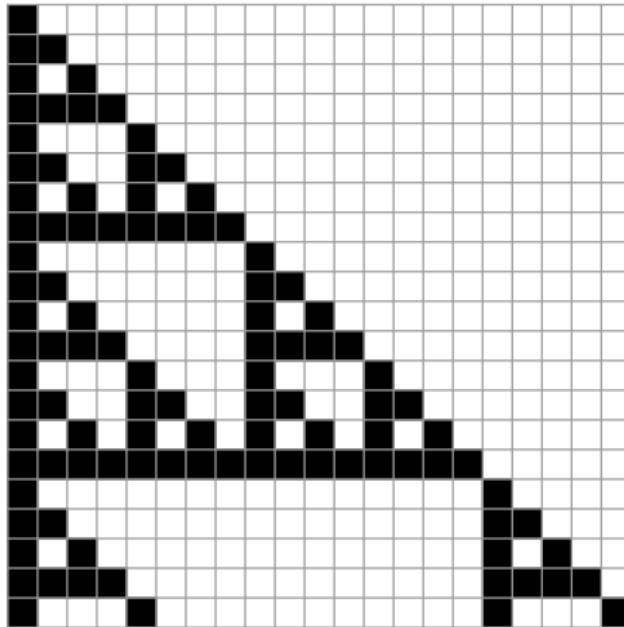
- If we set  $1 = \text{Black}$ ,  $0 = \text{White}$



- If we add a right neighbor to the above rule such that it doesn't affect the output



- This is rule  $00111100_2 = 60$  Cellular Automata



- ▶ Thus, this figure shows the coefficients of the powers of  $r(x) = 1 + x \bmod 2$

## Generalizations

- There are 2 ways to generalize,

1. A different integer modulus
2. A different polynomial

- Ex: Let  $r(x) = 1 + 2x$

$$(r(x))^0 = 1$$

$$(r(x))^1 = 1 + 2x$$

$$(r(x))^2 = 1 + 4x + 4x^2$$

$$(r(x))^3 = 1 + 6x + 12x^2 + 8x^3$$

⋮

$$(r(x))^n = a_0(n) + a_1(n)x + \cdots + a_n(n)x^n$$

- By observation,

$$a_k(n) = a_k(n-1) + 2a_{k-1}(n-1)$$

- ▶ By looking at the remainder with  $p = 3$ , i.e.  $a_k(n) \bmod 3$

$$(r(x))^0 = 1$$

$$(r(x))^1 = 1 \ 2$$

$$(r(x))^2 = 1 \ 1 \ 1$$

$$(r(x))^3 = 1 \ 0 \ 0 \ 2$$

- ▶ The cellular automaton would be  $a_{n,k} = a_{n-1,k} + 2a_{n-1,k-1} \bmod 3$

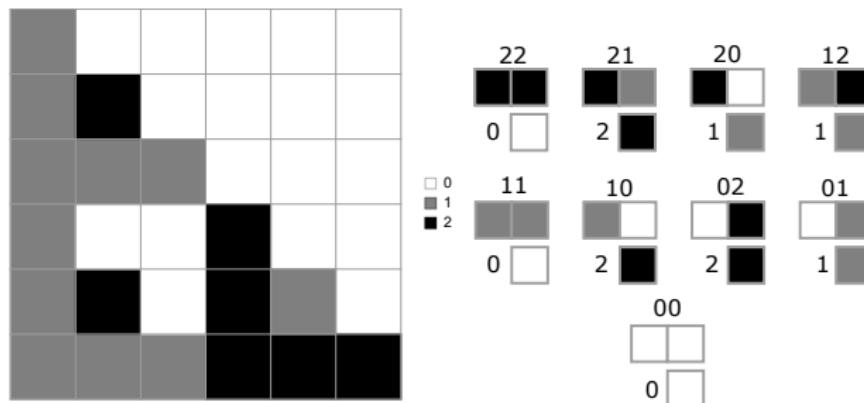
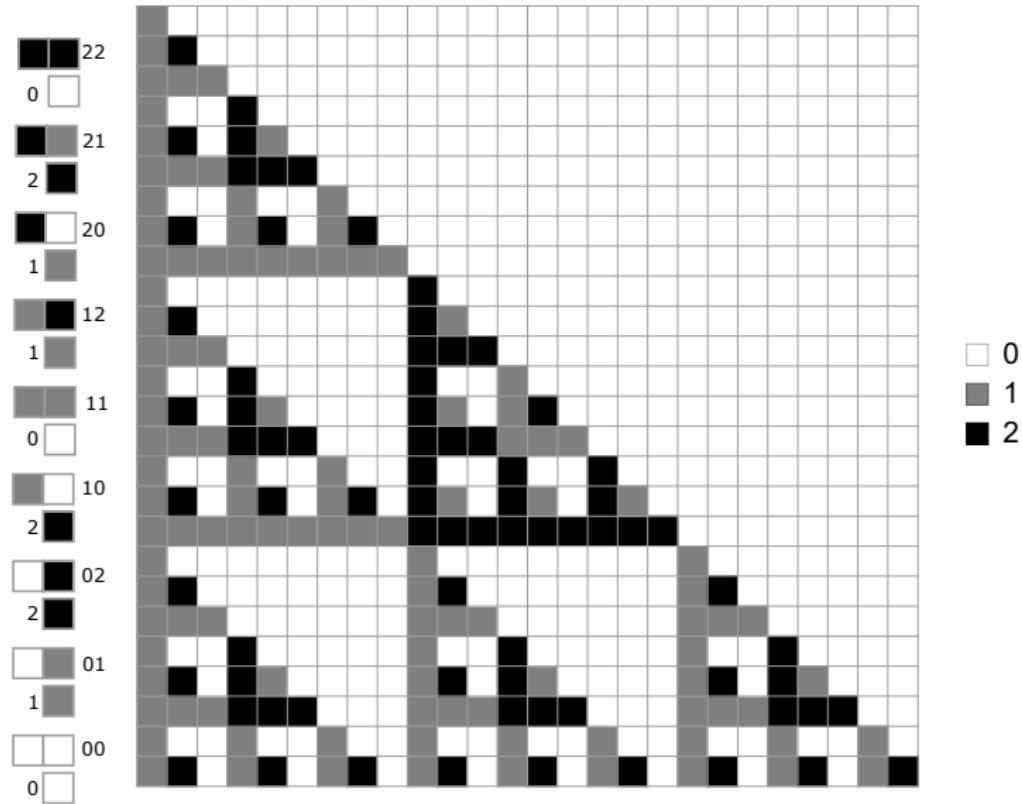


Figure: CA for  $r(x) = 1 + 2x \bmod 3$  and rule table



**Figure:** A bigger picture for rule  $(02110221)_3 = 5421$

## Generalizations

- ▶ Another quick example:  $r(x) = 1 + x \pmod{3}$



- ▶ The CA rule is  $a_{n,k} = a_{n-1,k} + a_{n-1,k-1} \pmod{3}$

## The Microscopic view

## Binomial Coefficients and Divisibility

- ▶ Black and white coloring of the Pascal triangle depending on divisibility with  $p \pmod{p}$
- ▶ We will also see that in order to understand the patterns formed by  $\pmod{p}$ , we should look at the patterns formed by the prime factors of  $p$
- ▶ We will look at a direct, non-recursive computation of binomial divisibility by  $p$

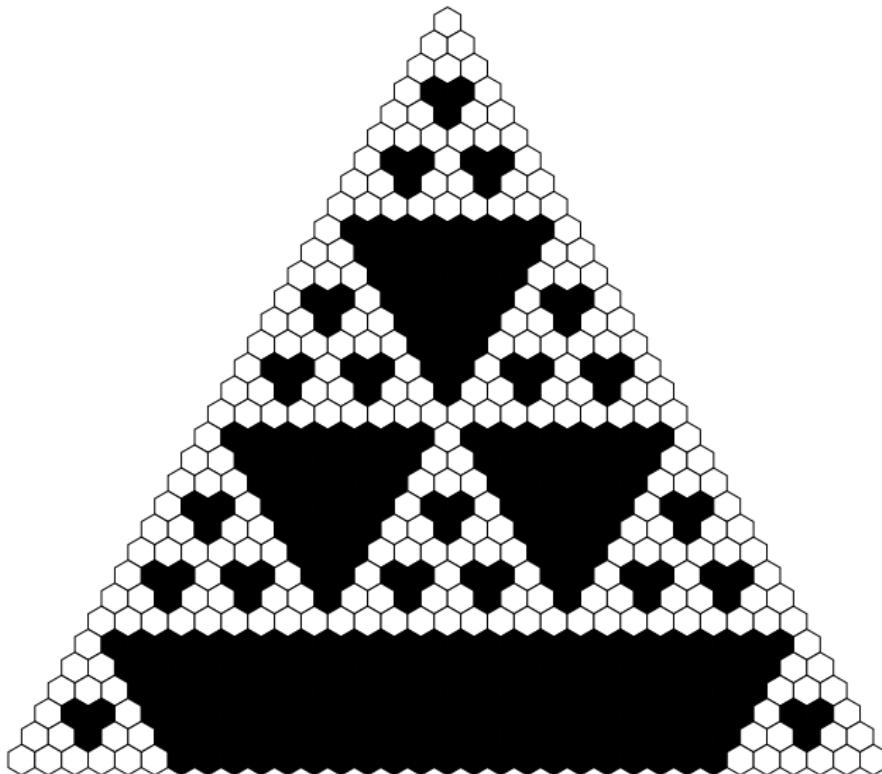


Figure: Pascal triangle Mod 3

- We define a new coordinate system such that, at position  $(n, k)$  the binomial coefficient is

$$\binom{n+k}{k} = \frac{(n+k)!}{n!k!}$$

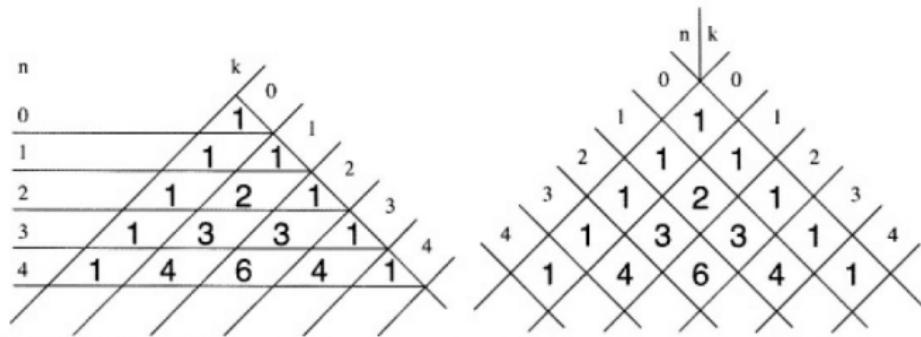


Figure: A new coordinate system

## Divisibility Sets

- We formally define our problem

$$P(r) = \left\{ (n, k) \mid \binom{n+k}{n} \text{ is not divisible by } r \right\}$$

- Plotting the points in this set gives the same Pascal Triangle pattern

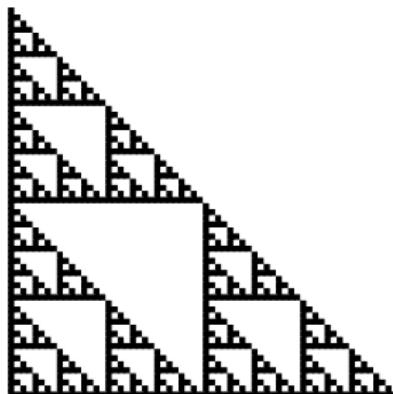


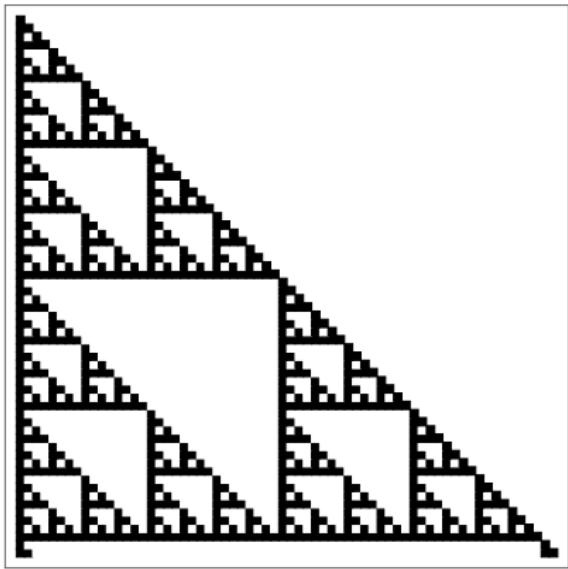
Figure:  $P(2)$

## Divisibility Sets

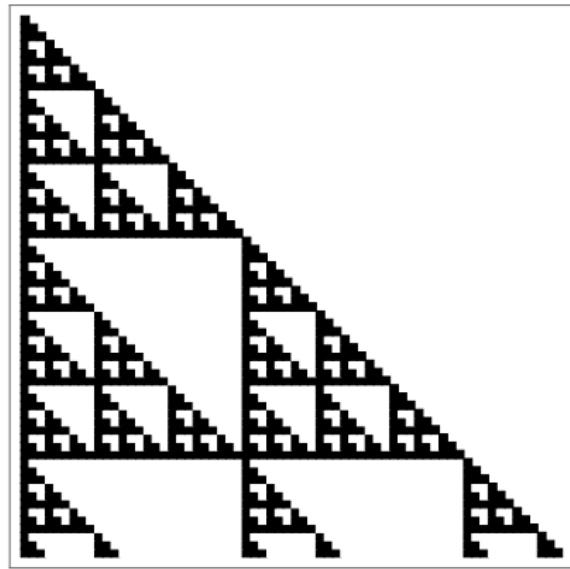
- ▶ Observe that if  $p$  and  $q$  are two different prime numbers and a given integer  $r$  is **not** divisible by  $p \cdot q$ , then it is **not** divisible by either  $p$  or  $q$

$$P(pq) = P(p) \cup P(q), \text{ if } p \neq q, p, q \text{ prime}$$

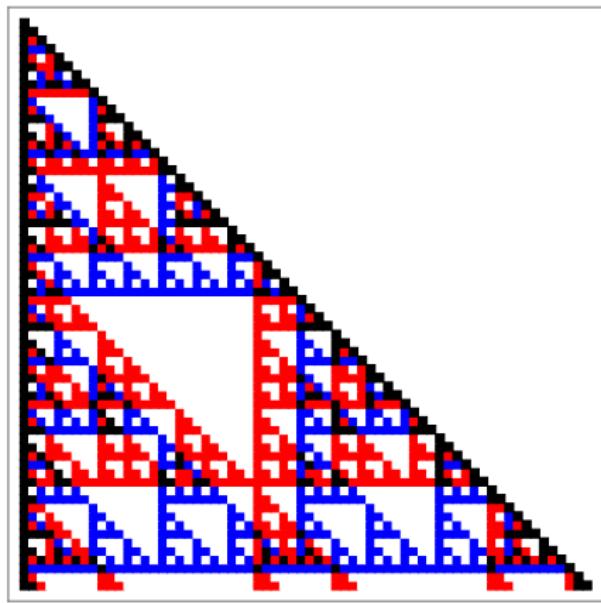
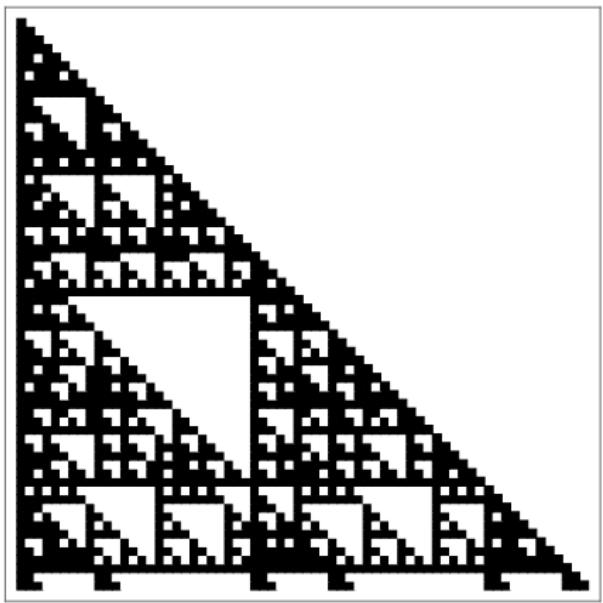
- ▶ It is the negation of the statement, if  $p$  and  $q$  divides  $r$ , then  $r$  is divisible by both  $p$  and  $q$ .
- ▶ Ex:  $P(6) = P(2) \cup P(3)$



Mod 2



Mod 3



**Figure:** Right shows mod 6 pattern and left shows the union of mod 2 (Red) and mod 3 (Blue)

- ▶ For an integer  $r = p_1^{n_1} \dots p_s^{n_s}$  where  $p_1, \dots, p_s$  are primes

$$P(r) = P(p_1^{n_1}) \cup \dots \cup P(p_s^{n_s})$$

- ▶ So to understand the pattern formed by  $P(r)$ , we just need to understand the pattern of  $P(p^n)$

## Kummer's Result and p-adic numbers

- ▶ To understand Kummer's result, we need to consider numbers with base  $p$ , where  $p$  is prime
- ▶ Like the decimal system, p-adic expansion of an integer is given by

$$n = a_0 + a_1p + \cdots + a_mp^m$$

where  $a_i \in \{0, 1, \dots, p - 1\}$

- ▶ The p-adic representation would be

$$n = (a_ma_{m-1} \dots a_0)_p$$

- ▶ Ex:  $15_{10} = (1111)_2 = (120)_3 = (30)_5 = (21)_7 = (14)_{11}$

- We define carry function  $c_p$

$c_p(n, k) = \text{number of carries in the } p\text{-adic addition of } n \text{ and } k$

- Ex: For  $n = 15$  and  $k = 8$

$$k = (08)_{10} = (1000)_2 = (022)_3 = (13)_5 = (11)_7 = (08)_{11}$$

- If we take the binary addition 
$$\begin{array}{r} 0 & 1 & 1 & 1 & 1 \\ + 0_1 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 1 & 1 & 1 \end{array}$$

$$c_2(15, 8) = 1$$

- Similarly, we can do the same for  $p = 3, 5, 7 \dots$

## Kummer's result

- ▶ Let  $\tau = c_p(n, k)$ , then  $\binom{n+k}{k}$  is divisible by  $p^\tau$  but not  $p^{\tau+1}$
- ▶ So, prime factorization of  $\binom{n+k}{k}$  contains exactly  $c_p(n, k)$  factors of  $p$
- ▶ This result is pretty amazing, since it gives a direct method to check if a binomial coefficient is divisible by  $p$  or not.
- ▶ For  $n = 5$  and  $k = 4$ , Kummer's result says

$$c_p(5, 4) = \begin{cases} 1, & \text{for } p = 2, 7 \\ 2, & \text{for } p = 3 \\ 0 & \text{otherwise} \end{cases}$$

- ▶ This implies

$$\binom{5+4}{4} = \binom{9}{4} = 2^1 \cdot 3^2 \cdot 7^1$$

- ▶ Applying Kummer's result to Divisibility Set gives

$$P(p) = \{(n, k) \mid c_p(n, k) = 0\}$$

- ▶ That is, the number of carries  $c_p(n, k)$  is only 0 if and only if

$$a_i + b_i < p, \quad i = 0, \dots, m$$

where  $a_i$  and  $b_i$  are the  $p$ -adic digits of  $n$  and  $k$ .

- ▶ This is called the *mod-p* condition.
- ▶ For prime powers,

$$P(p^\tau) = \{(n, k) \mid c_p(n, k) < \tau\}$$

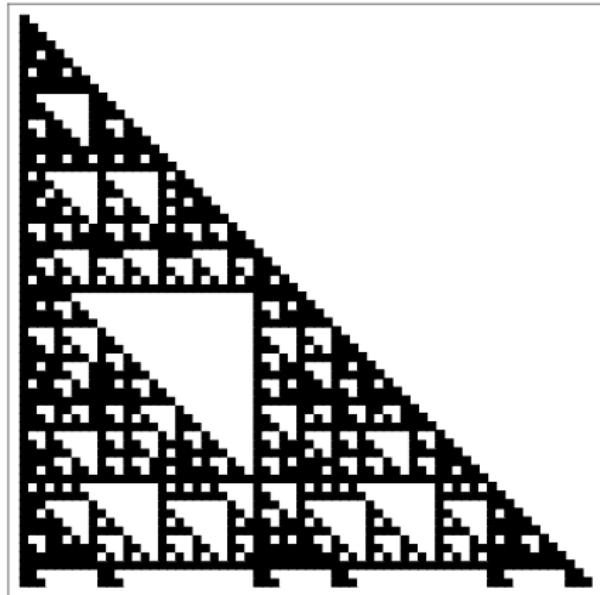
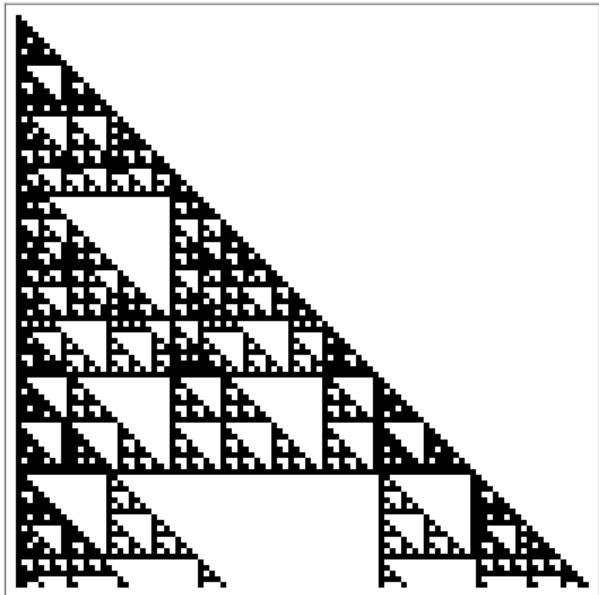


Figure:  $P(6)$  generated using carry function (left) vs binomial divisibility (right)

The Big Picture

## Iterated Function System

- ▶ It is a method of constructing fractals using a set of contraction mappings.
- ▶ A contraction mapping is an affine linear transformation

$$f(x, y) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$$

where

$$\det \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) < 1$$

- ▶ Union of these contraction mapping gives the Hutchinson equation of the fractal

- The Hutchinson operator for Sierpinski Triangle is:

$$S = w_{00}(S) \cup w_{01}(S) \cup w_{10}(S)$$

where  $w_{ij}$  are contraction mappings of a unit square

$$w_{00} = (x/2, y/2)$$

$$w_{01} = (x/2, y/2 + 1/2)$$

$$w_{10} = (x/2 + 1/2, y/2)$$

$$w_{11} = (x/2 + 1/2 + y/2 + 1/2)$$

- Let  $Q$  be a unit square



Figure: Initial region,  $Q = S^0$



Figure:  $S^1 = S(Q)$

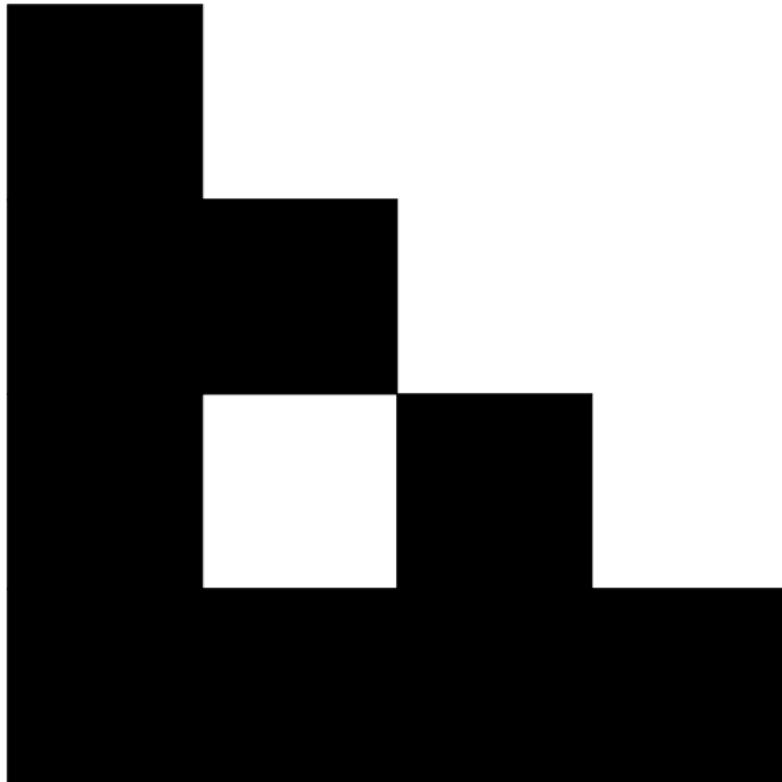


Figure:  $S^2 = S(S(Q))$

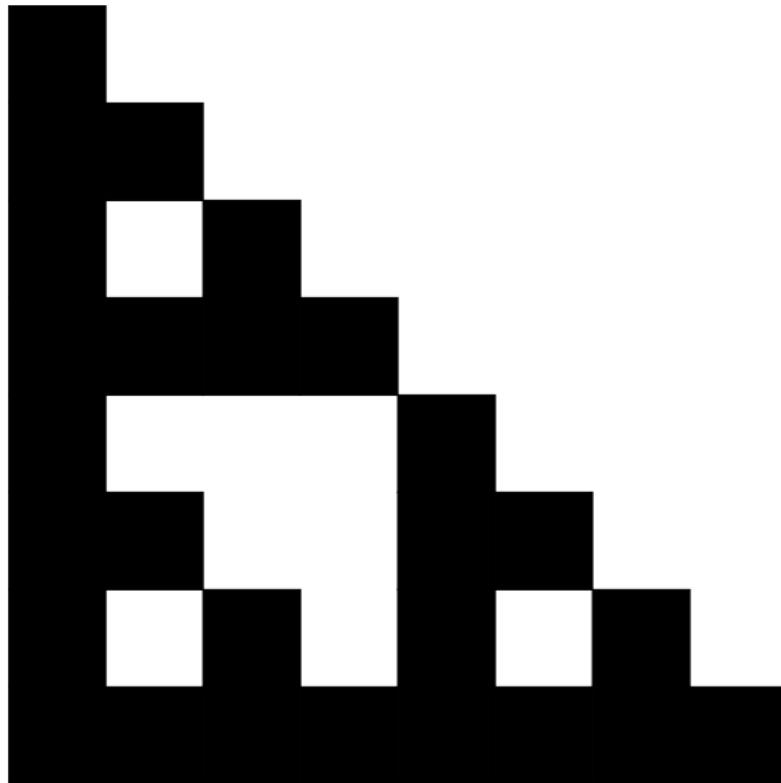


Figure:  $S^3 = S(S(S(Q)))$

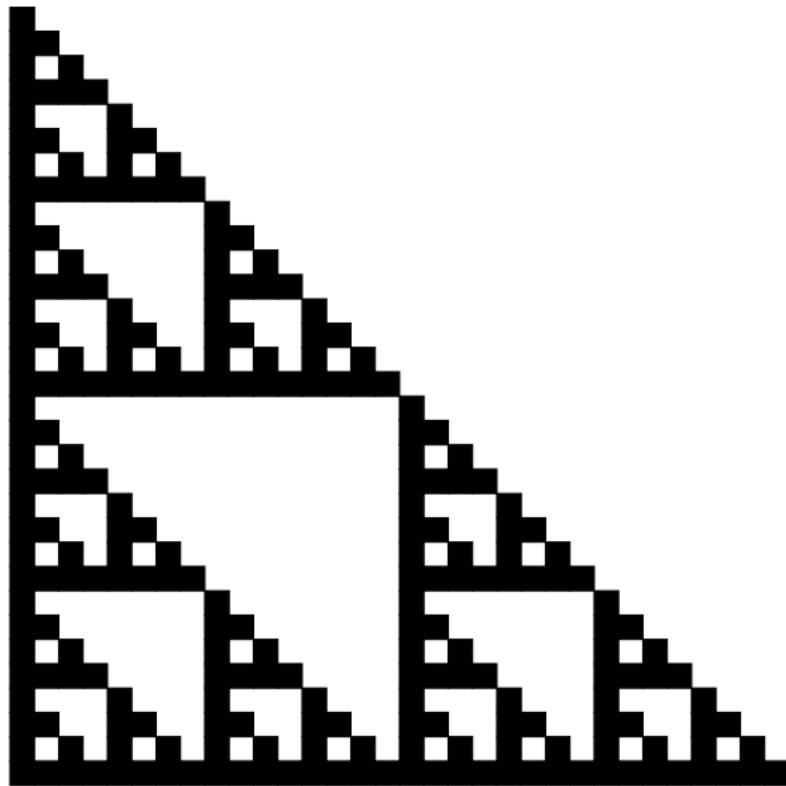


Figure:  $S^5$

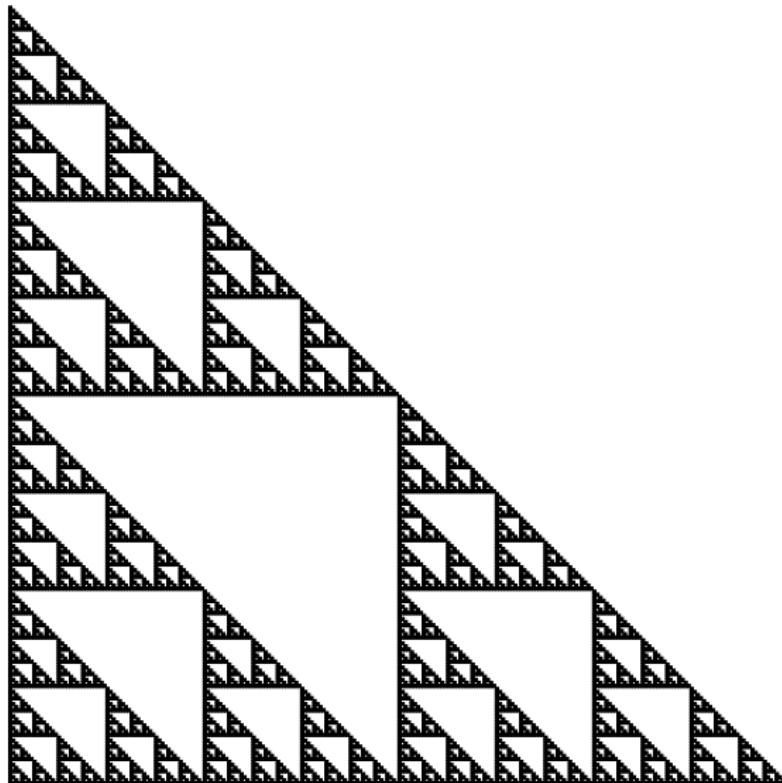


Figure:  $S^\infty$

So how do we know for sure that our Pascal Triangle pattern is the Sierpinski Triangle ?

- ▶ Let's construct Mod 2 Pascal triangle in a unit square
- ▶ Let  $Q$  be the unit square

$$Q = \{(x, y) \mid (x, y) \in [0, 1] \times [0, 1]\}$$

- ▶ Expand  $x$  and  $y$  in base 2

$$x = \sum_{i=1}^{\infty} a_i 2^{-i}, \quad a_i \in \{0, 1\}$$

$$y = \sum_{i=1}^{\infty} b_i 2^{-i}, \quad b_i \in \{0, 1\}$$

- ▶ From Kummer's result, we know that the coordinates of the points divisible by 2 are those which have no carries in the binary addition of the coordinates

$$S = \{(x, y) \in Q \mid a_i + b_i \leq 1 \ \forall i\}$$

## Proof

- To show both are the same, we need to show

$$w_{00}(S) \cup w_{01}(S) \cup w_{10}(S) \subset S$$

and,

$$w_{00}(S) \cup w_{01}(S) \cup w_{10}(S) \supset S$$

- Take any point  $(x, y) \in S$

$$(x, y) = (0, a_1 a_2 a_3 \dots, 0.b_1 b_2 \dots)$$

where  $a_i + b_i \leq 1$

- Applying the 3 transformations gives

$$w_{00}(0.a_1 a_2 \dots, 0.b_1 b_2 \dots) = (0.0a_1 a_2 \dots, 0.0b_1 b_2 \dots)$$

$$w_{01}(0.a_1 a_2 \dots, 0.b_1 b_2 \dots) = (0.0a_1 a_2 \dots, 0.1b_1 b_2 \dots)$$

$$w_{10}(0.a_1 a_2 \dots, 0.b_1 b_2 \dots) = (0.1a_1 a_2 \dots, 0.0b_1 b_2 \dots)$$

- ▶ Clearly, all these points are also in  $S$ .
- ▶ To show the second relation, we take any point  $(x, y) \in S$ , and we have to provide another point  $(x', y') \in S$  such that  $(x, y)$  is one of the images of  $w_{00}(x', y')$ ,  $w_{10}(x', y')$ , or  $w_{01}(x', y')$ . We choose

$$(x', y') = (0.a_2 \dots, b_2 \dots)$$

- ▶ We obtain

$$(x, y) = \begin{cases} w_{00}(x', y') & \text{if } a_1 = 0 \text{ and } b_1 = 0 \text{ or} \\ w_{01}(x', y') & \text{if } a_1 = 0 \text{ and } b_1 = 1 \text{ or} \\ w_{10}(x', y') & \text{if } a_1 = 1 \text{ and } b_1 = 0 \end{cases}$$

- ▶ This completes our proof.

## Consequences

- ▶ The binary representation allows us to see Hutchinson operator in action applied to a point inside square
- ▶ If  $(x, y)$  is an arbitrary point in  $Q$ , then applying the map  $w_{00}, w_{01}, w_{10}$  again and again yields points with leading binary decimal points satisfying  $a_i + b_i \leq 1$
- ▶ In symbols,

$$A_0 = Q$$

Then running the IFS gives

$$A_n = w_{00}(A_{n-1}) \cup w_{01}(A_{n-1}) \cup w_{10}(A_{n-1})$$

where the leading  $n$  binary digits satisfy  $a_i + b_i \leq 1$

- ▶ Finally, the sequence will lead to Sierpinski triangle

$$A_\infty = S$$

## IFS for other primes

- ▶ Now, we can finally look into the global pattern formation of divisibility of binomial coefficients with primes, i.e the global patterns formed by

$$P(r) = \left\{ (n, k) \mid \binom{n+k}{n} \text{ is not divisible by } r \right\}$$

- ▶ We construct an IFS: Divide the unit square  $Q$  in  $p^2$  congruent square  $Q_{a,b}$  with  $a, b \in \{0, \dots, p-1\}$ . We introduce the contraction mappings

$$w_{a,b}(x, y) = \left( \frac{x+a}{p}, \frac{y+b}{p} \right)$$

where

$$w_{a,b}(Q) = Q_{a,b}$$

- ▶ Then we set the restriction to define the set of transformation

$$a + b \leq p - 1$$

- ▶ This restriction follows from Kummer's result, i.e.  $\binom{n+k}{n}$  is indivisible by  $p$  when there are no carries in the  $p$ -adic addition of  $n, k$
- ▶ The Hutchinson operator for these contractions,

$$W_p(A) = \bigcup_{a+b < p} w_{a,b}(A)$$

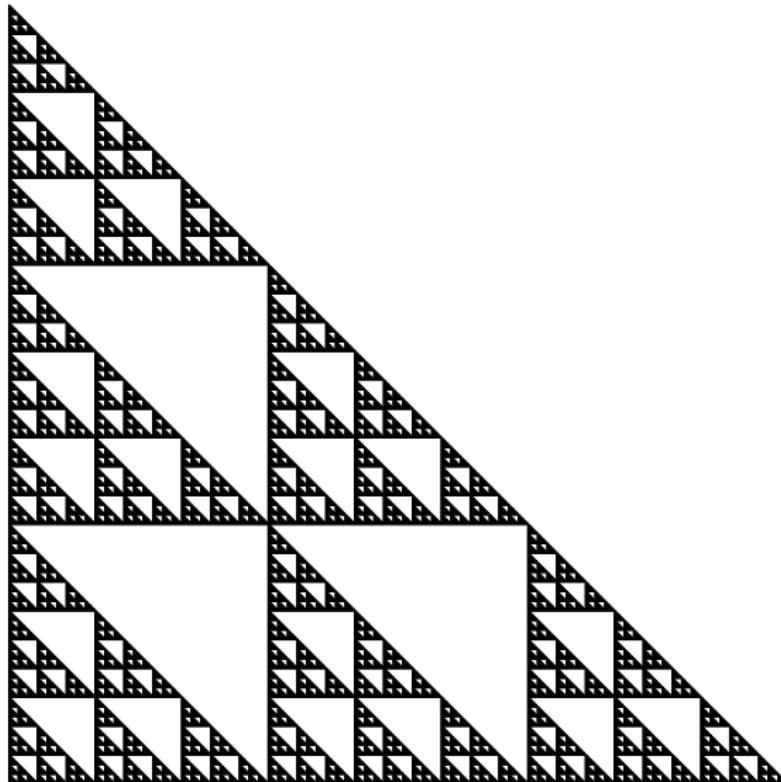


Figure: Limit of Mod 3 IFS.

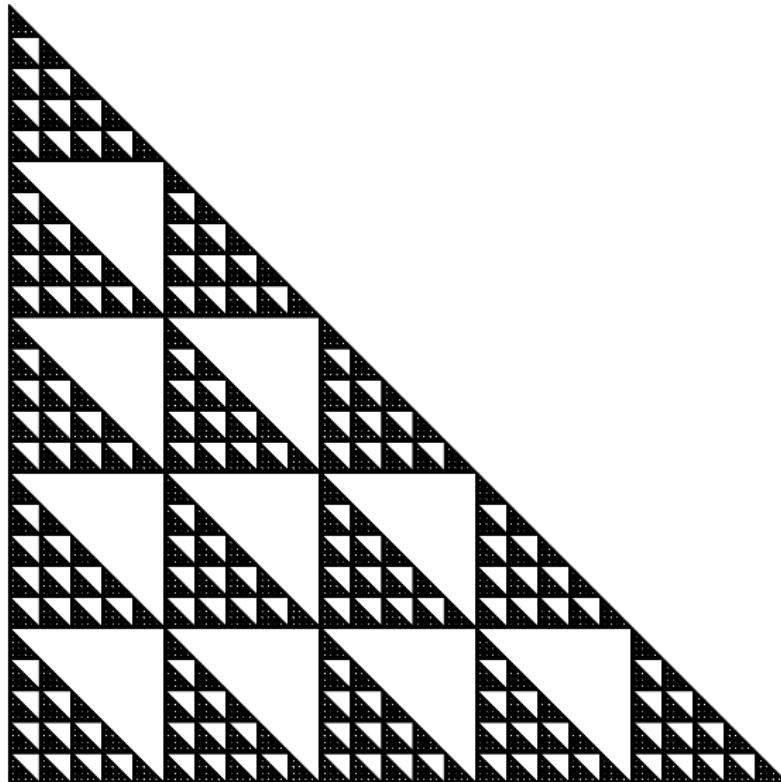


Figure: Limit of Mod 5 IFS.

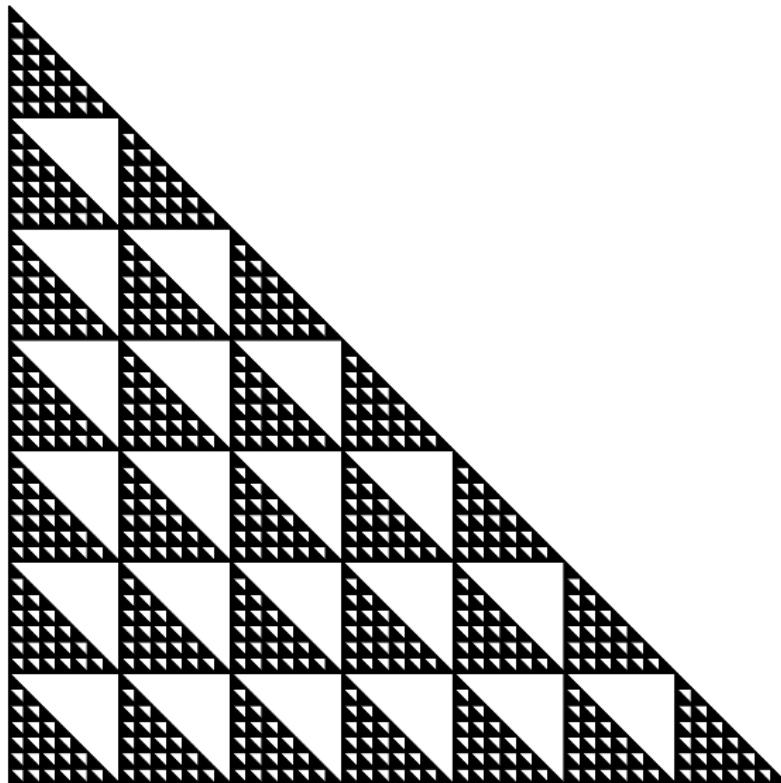


Figure: Limit of Mod 7 IFS.

## Summary

- ▶ We approached the problem of coloring (or divisibility) of pascal's triangle in 3 different ways
  1. First, we looked at the macroscopic nature of divisibility, i.e how the neighbors affected the divisibility of a cell. We then explored the realm of cellular automata and how it extends this idea to general polynomials.
  2. Second, we looked at the microscopic nature of divisibility,i.e how the divisibility of a cell depends on its coordinate (Kummer's Result). We looked how p-adic numbers gave insight about binomial divisibility with primes.
  3. Finally, we looked at the global pattern using Iterated function system. We showed that the limit of Pascal Mod 2 pattern was indeed the Sierpinski triangle. We also explored the global patterns of other primes and their IFS.

- ▶ Each method gave a different perspective of the same problem.
- ▶ This entire chapter is trying to solve the jig-saw puzzle that relates fractals, Pascal's triangle, and Cellular Automata.
- ▶ Understanding that these 3 very different things are all tied up by one common thread is, honestly, amazing.

Thank You!