CS 754 Project Report

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Contents

1 Paper

https://arxiv.org/pdf/1006.3056.pdf

The given paper deals with solving inverse problems where the inverse is not well defined by using Gaussian Mixture Models (GMMs) which are estimated by MAP-EM (maximum a priori expectation-maximization) algorithm.

Image restoration amounts to estimating an image $\mathbf{f}$ subject to :

$$\mathbf{y} = \mathbf{U}\mathbf{f} + \mathbf{w}$$

where $\mathbf{f}$ is the image to be estimated, $\mathbf{U}$ is some operator which acts over the image, $\mathbf{w}$ is an additive noise, $\mathbf{y}$ is the measurement made.

2 Piecewise Linear Estimation using MAP-EM

An image is decomposed into $\sqrt{N} \times \sqrt{N}$ local patches, $\mathbf{y_i} = \mathbf{U_i}\mathbf{f_i} + \mathbf{w_i}$, where $\mathbf{U_i}$ is the degradation operator, $\mathbf{y_i}$ is the degraded image patch, $\mathbf{f_i}$ is the original image patch, $\mathbf{w_i}$ is noise (on patch i).

Consider k Gaussian distributions, $\{\mathcal{N}(\mu_k, \Sigma_k)\}_{1\leq k\leq K}$. The GMM models each image patch as being drawn independently from one of the Gaussians in this collection:

$$p(\mathbf{f_i}) = \frac{1}{(2\pi)^{N/2}|\Sigma_{k_i}|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{f_i} - \mu_k)^T \Sigma_{k_i}^{-1}(\mathbf{f_i} - \mu_k)\right)$$

The recovery of $\{\mathbf{f_i}\}_{1\leq i\leq I}$ from $\{\mathbf{y_i}\}_{1\leq i\leq I}$ consists of first estimating the parameters for the collection of k Gaussians, then identifying the distribution corresponding to each patch, and finally estimating $\mathbf{f_i}$ for each patch and reconstructing the original image.

This non-convex problem is solved using a 'maximum a posteriori expectation-maximization' (MAP-EM) algorithm. This alternates between two types of steps:

- E Step: The parameters for each distribution in the GMM are assumed to be known, and the MAP estimates for which patch is drawn from which Gaussian are determined, which also yields estimates for $\mathbf{f_i}$.
- M Step: The values of $\mathbf{f_i}$ are assumed to be known, along with the Gaussian model selection for each patch, and the Gaussian distributions' parameters are updated accordingly.

This process eventually converges on the original image, which is reconstructed by averaging over the patches obtained.

The paper further proposes an initialization strategy, involving a directional PCA basis, to improve this algorithm, which has also been implemented in our code.

3 Generalizing E step for $\mu_k \neq 0$

Since the paper left out (for simplicity) the mathematical expressions used in the E step of MAP-EM for the case of $\mu_k \neq 0$, we have derived them below:

$$\begin{aligned}
\tilde{f}_k &= \arg\min_{f_i}(||U_i f_i - y_i||_2^2 + \sigma^2 (f_i - \mu_k)^T \Sigma_k^{-1}(f_i - \mu_k)) \\
J_{i,k} &= ||U_i f_i - y_i||_2^2 + \sigma^2 (f_i - \mu_k)^T \Sigma_k^{-1}(f_i - \mu_k) \\
\Sigma_k^T &= \Sigma_k \\
\frac{\partial J_{i,k}}{\partial f_i} &= 0 \\
2(U_i f_i - y_i)^T (U) + 2\sigma^2 (f_i - \mu_k)^T \Sigma_k^{-1} &= 0 \\
(f_i^T U_i^T - y_i^T)(U) + \sigma^2 (f_i^T - \mu_k^T)\Sigma_k^{-1} &= 0 \\
f_i^T (U_i^T U_i + \sigma^2 \Sigma_k^{-1}) &= y_i^T U_i + \sigma^2 \mu_k^T \Sigma_k^{-1} \\
(f_i^T (U_i^T U_i + \sigma^2 \Sigma_k^{-1}))^T &= (y_i^T U_i + \sigma^2 \mu_k^T \Sigma_k^{-1})^T \\
(U_i^T U_i + \sigma^2 \Sigma_k^{-1})^T f_i &= (y_i^T U_i)^T + (\sigma^2 \mu_k^T \Sigma_k^{-1})^T \\
(U_i^T U_i + \sigma^2 \Sigma_k^{-1}) f_i &= U_i^T y_i + \sigma^2 \Sigma_k^{-1} \mu_k \\
f_i &= (U_i^T U_i + \sigma^2 \Sigma_k^{-1})^{-1}(U_i^T y_i + \sigma^2 \Sigma_k^{-1} \mu_k)
\end{aligned}$$

$$f_i = (U_i^T U_i + \sigma^2 \Sigma_k^{-1})^{-1}(U_i^T y_i + \sigma^2 \Sigma_k^{-1} \mu_k) \tag{1}$$

$$L_{i,k} = ||U_i f_i - y_i||_2^2 + \sigma^2 (f_i - \mu_k)^T \Sigma_k^{-1}(f_i - \mu_k) + \sigma^2 \det(\Sigma_k)$$

$$L_{i,k} = ||U_i f_i - y_i||_2^2 + \sigma^2 (f_i - \mu_k)^T \Sigma_k^{-1}(f_i - \mu_k) + \sigma^2 \det(\Sigma_k) \tag{2}$$

4 Inpainting Results

4.1 Image = Cameraman, Known pixels = 80%

RMSE = 0.0604
PSNR = 69.6311

Figure 1: Original Image

Figure 2: Corrupted / Masked Image

Figure 3: Reconstructed Image

4.2 Image = Lena, Known pixels = 50%

RMSE = 0.1278
PSNR = 55.9255

Figure 4: Original Image

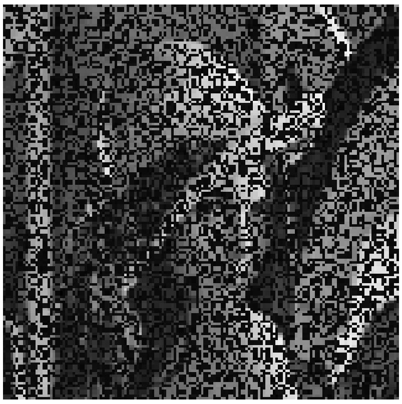

Figure 5: Corrupted / Masked Image

Figure 6: Reconstructed Image

4.3 Image = Peppers, Known pixels = 20%

RMSE = 0.2567 PSNR = 38.2660

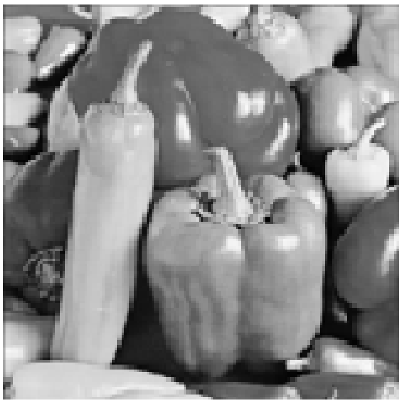

Figure 7: Original Image

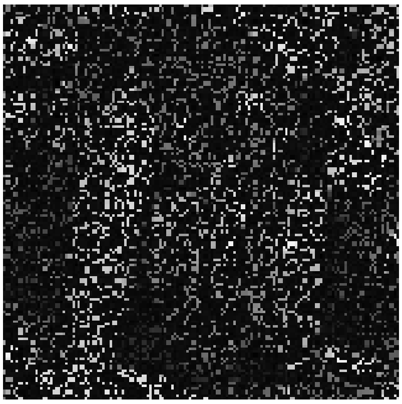

Figure 8: Corrupted / Masked Image

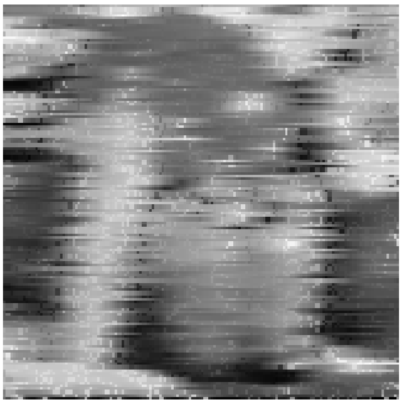

Figure 9: Reconstructed Image

5 Interpolation Zooming Results

5.1 Image = Lena

RMSE = 0.139175
PSNR = 59.433268

Figure 10: Original Image

Figure 11: Downsampled Image

Figure 12: Enlarged Uninterpolated Image (Modelled as Inpainting Problem)

Figure 13: Reconstructed Zoomed Image with Interpolation

5.2 Image = Cameraman

RMSE = 0.072461
PSNR = 61.394134

Figure 14: Original Image

Figure 15: Downsampled Image

Figure 16: Enlarged Uninterpolated Image (Modelled as Inpainting Problem)

Figure 17: Reconstructed Zoomed Image with Interpolation

5.3 Image = Peppers

RMSE = 0.081263
PSNR = 60.998690

Figure 18: Original Image

Figure 19: Downsampled Image

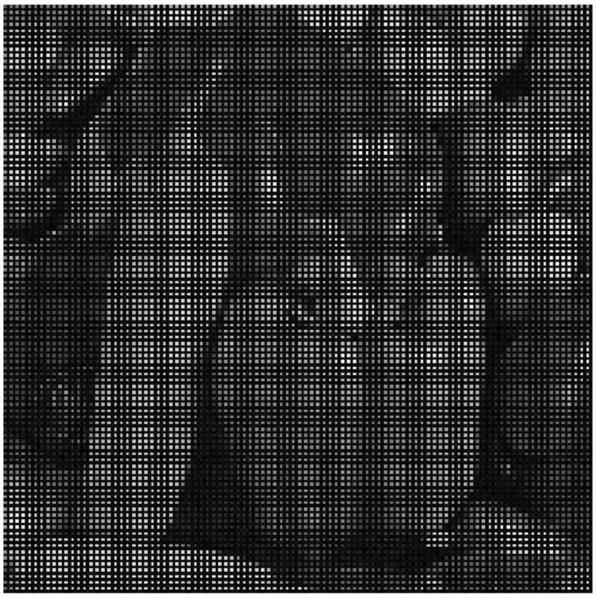

Figure 20: Enlarged Uninterpolated Image (Modelled as Inpainting Problem)

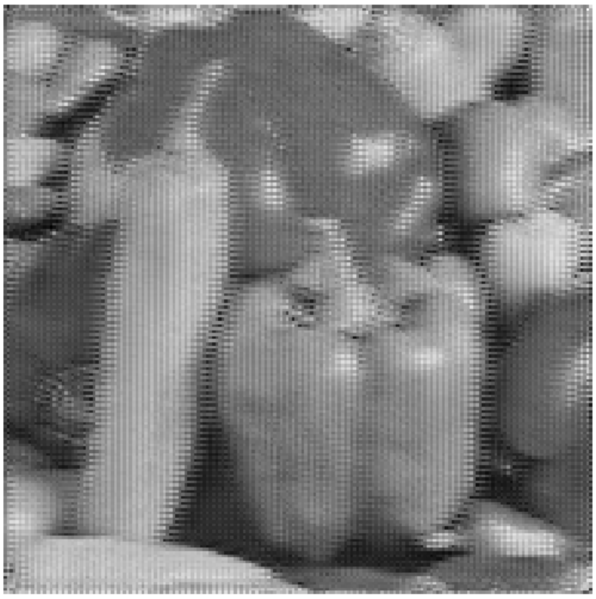

Figure 21: Reconstructed Zoomed Image with Interpolation