Coded Computation: Straggler Mitigation in Distributed Matrix Multiplication[1]

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[1] Yu et al., IEEE Transactions on Information Theory, 2020

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- This problem is solved by deploying the multiplication task over a large-scale distributed system, having several nodes.

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- Commonly tackled by adding redundant computations.
- Naturally, error correcting codes can be applied to introduce 'efficient redundancy' in computation.

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- Recovery threshold $K(\mathbf{f}, \mathbf{g}, \mathbf{d})$ is smallest k s.t. k-recoverable.

Redundant Code

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- Thus, the redundancy is $\frac{N}{r_1 r_2}$

Linear Codes

$$A = \begin{bmatrix} A_{0,0} & \cdots & A_{0,m-1} \\ \vdots & \ddots & \vdots \\ A_{p-1,0} & \cdots & A_{p-1,m-1} \end{bmatrix}, \quad B = \begin{bmatrix} B_{0,0} & \cdots & B_{0,n-1} \\ \vdots & \ddots & \vdots \\ B_{p-1,0} & \cdots & B_{p-1,n-1} \end{bmatrix}$$

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- Worker i returns

$$\begin{aligned}\tilde{C}_i &= \tilde{A}_i^T \tilde{B}_i \\ &= \sum_{j=0}^{p-1} \sum_{k=0}^{m-1} \sum_{j'=0}^{p-1} \sum_{k='0}^{n-1} A_{j,k}^T B_{j',k'} x_i^{(p-1+j-j')+kp+k'pm}\end{aligned}$$

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Theorem

$K_{entangled\text{-}poly} = pmn + p - 1.$

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If $\mathbb{F}$ *is a finite field, then* $\frac{1}{2}K_{entangled\text{-}poly} < K^* \leq K_{entangled\text{-}poly}$.

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- A fraction $f < 1$ of workers were set to be slow.
- We observed computation times and errors in the final result, for both types of codes, by varying matrix sizes, number of partitions, number of workers, etc.

Practical Issues with Entangled Polynomial Code - Numerical Instability

- **Interpolation Error** - Lagrange Interpolation is numerically very unstable. As soon as interpolation degree crosses 20, error starts rising very quickly (Runge's phenomenon[2]).

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- **Solution** - Use Chebyshev Nodes[3] for evaluation points.

$$x_k = \cos\left(\frac{2k-1}{2n}\pi\right)$$

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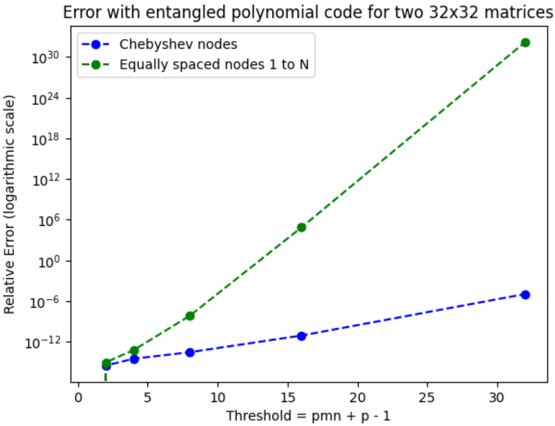

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- Empirical analysis validates this result.

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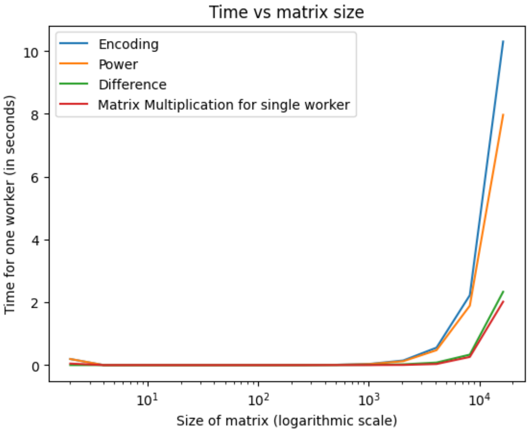

Execution times: Polynomial code vs. Redundant code

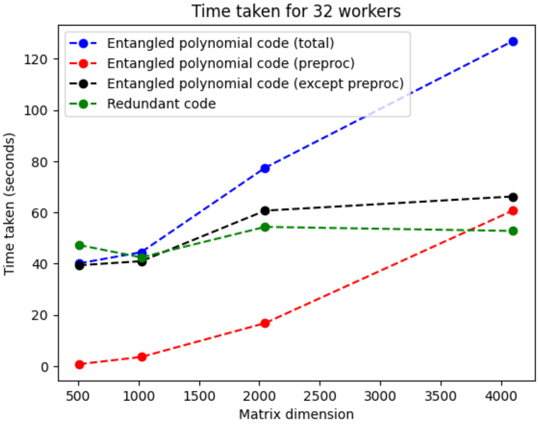

Figure: Caption

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  - Cost of preprocessing increases rapidly with matrix size, due to a large number of exponentiations, and causes a sharp increase in total computation time using entangled polynomial codes.
- This renders entangled polynomial codes unfit for practical use, unless these issues are fixed.

Thanks!

Any questions?