Set IV

CHANGE OF INTERVAL

In most engineering applications of Fourier series we require the representation of a given function over an interval different from $-\pi$ to π (or 0 to 2π). Suppose we wish to represent the function $f(x)$ in the interval $-c \leq x \leq c$ by Foourier series, or consider the periodic function defined in $(\alpha, \alpha+2c)$. To transform the problem to period of 2π, we use the substitution.

putting $\frac{z}{\pi} = \frac{x}{c}$ or $x = \frac{cz}{\pi}$

or $z = \frac{\pi x}{c}$

when $x = \alpha$, $z = \frac{\pi\alpha}{c} = \beta$ (say)

when $x = \alpha + 2c$, $z = \frac{\pi x}{c} = \frac{\pi}{c}(\alpha + 2c)$

$$z = \frac{\pi\alpha}{c} + 2\pi$$

$$z = \beta + 2\pi$$

Thus the function $f(x)$ of period $2c$ is transferred to the function $f\left(\frac{cz}{\pi}\right) = F(z)$ (say) of period 2π in $(\beta, \beta+2\pi)$.

Hence

$f\left(\frac{cz}{\pi}\right) = F(z)$ can be expanded in Fourier series

$$f\left(\frac{cz}{\pi}\right) = F(z) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nz + b_n \sin nz) \quad - (1)$$

where
$$\left.\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{\beta}^{\beta+2\pi} F(z)\,dz \\ a_n &= \frac{1}{\pi}\int_{\beta}^{\beta+2\pi} F(z)\cos nz\,dz \\ \text{and } b_n &= \frac{1}{\pi}\int_{\beta}^{\beta+2\pi} F(z)\sin nz\,dz \end{aligned}\right\} \quad (2)$$

If we make the inverse substitution $z = \frac{\pi x}{c}$ in (2), we set $dz = \frac{\pi}{c}dx$

②

$$a_0 = \frac{1}{2\pi}\int_{\alpha}^{\alpha+2c} f(x)\,\frac{\pi}{c}\,dx = \frac{1}{2c}\int_{\alpha}^{\alpha+2c} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{\beta}^{\beta+2\pi} f\left(\frac{ct}{\pi}\right)\cos nt\,dt = \frac{1}{\pi}\int_{\alpha}^{\alpha+2c} f(x)\cos\frac{n\pi x}{c}\times\frac{\pi}{c}\,dx$$

$$a_n = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\cos\left(\frac{n\pi x}{c}\right)dx$$

$$b_n = \frac{1}{\pi}\int_{\beta}^{\beta+2\pi} f\left(\frac{ct}{\pi}\right)\sin nt\,dt$$

$$= \frac{1}{\pi}\int_{\alpha}^{\alpha+2c} f(x)\sin\left(\frac{n\pi x}{c}\right)\frac{\pi}{c}\,dx$$

$$= \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\sin\left(\frac{n\pi x}{c}\right)dx$$

If we make inverse substitution in ① we get

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{c} + b_n\sin\frac{n\pi x}{c}\right)$$

where a_0, a_n and b_n are given above

This is the Fourier series for $f(x)$ in the interval $-c \le x \le c$.

Q. Find a Fourier series for $f(t) = 1 - t^2$ when $-1 \le t \le 1$.

Solution:- The required series is the form.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi t}{l} + b_n \sin\frac{n\pi t}{l} \right)$$

Here $l = 1$.

Then $a_0 = \int_{-1}^{1} (1-t^2)\,dt = \left[t - \frac{t^3}{3} \right]_{-1}^{1} = \left[\left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right) \right]$

$$= \left(1 - \frac{1}{3} + 1 - \frac{1}{3}\right) = \left(2 - \frac{2}{3}\right) = \frac{6-2}{3} = \frac{4}{3}$$

$$a_n = \int_{-1}^{1} \underset{I}{(1-t^2)}\, \underset{II}{\cos n\pi t}\, dt$$

$$= \left[(1-t^2) \frac{\sin n\pi t}{n\pi} - \int -2t \frac{\sin n\pi t}{n\pi}\, dt \right]_{-1}^{1}$$

$$= \left[(1-t^2) \frac{\sin n\pi t}{n\pi} \right]_{-1}^{1} + 2\int_{-1}^{1} t \frac{\sin n\pi t}{n\pi}\, dt$$

$$= 0 + \frac{2}{n\pi} \int_{-1}^{1} t \sin n\pi t\, dt$$

$$= -\left[\frac{2}{\pi n} \cdot t \frac{\cos n\pi t}{n\pi} \right]_{-1}^{1} + \frac{2}{n\pi} \int_{-1}^{1} \frac{\cos n\pi t}{n\pi}\, dt$$

$$= -\frac{2}{n^2\pi^2} \left[t \cos n\pi t \right]_{-1}^{1} + \frac{2}{n^2\pi^2} \left[\frac{\sin n\pi t}{n\pi} \right]_{-1}^{1}$$

$$= -\frac{2}{n^2\pi^2} \left[1 \cos n\pi - (-1) \cos n\pi(-1) \right] + \frac{2}{n^3\pi^3} \times 0$$

$$a_n = -\frac{2}{n^2\pi^2} \left[\cos n\pi + \cos n\pi \right] = -\frac{4}{n^2\pi^2} \cos n\pi$$

$$a_1 = \frac{4}{\pi^2}, \quad a_2 = \frac{-4}{2^2\pi^2}, \quad a_3 = \frac{4}{3^2\pi^2}$$

$$b_n = \int_{-1}^{1} (1-t^2) \sin n\pi t \, dt$$

$$\int_{-a}^{a} f(x)\,dx = 0 \quad \text{if } f(x) \text{ is odd.}$$

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \quad \text{if } f(x) \text{ is even}$$

$$f(t) = (1-t^2) \sin n\pi t$$

$$f(-t) = [1-(-t)^2] \sin n\pi(-t)$$

$$= -(1-t^2) \sin n\pi t = \cancel{\text{odd}} - f(t)$$

$$= \text{odd function.}$$

$$\therefore b_n = \int_{-1}^{1} (1-t^2) \sin n\pi t \, dt = 0$$

$$\therefore f(t) = (1-t^2) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi t + \sum_{n=1}^{\infty} b_n \sin n\pi t$$

$$= \frac{4}{3} \times \frac{1}{2} + \frac{4}{\pi^2}\left(\cos \pi t - \frac{\cos 2\pi t}{2^2} + \frac{\cos 3\pi t}{3^2} \cdots\right).$$

$$= \frac{4}{3} + \frac{4}{\pi^2}\left(\cos \pi t - \frac{\cos 2\pi t}{2^2} + \frac{\cos 3\pi t}{3^2} \cdots\right)$$

Q Develop $f(x)$ in Fourier series in the interval $(-2,2)$, if

$$f(x) = 0 \quad -2 < x < 0$$
$$= 1. \quad 0 < x < 2$$

Solution:- The required series of the form.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

Here $2l = 4 \quad l = 2$

Then $a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\left[\int_{-2}^{0} f(x)\,dx + \int_{0}^{2} f(x)\,dx\right]$

$$a_0 = \frac{1}{L}\left[\int_{-2}^{0} 0\, dx + \int_{0}^{2} 1 \cdot dx\right]$$

$$a_0 = \frac{1}{L}[x]_0^2 = \frac{1}{2}[2-0] = \frac{2}{2} = 1$$

$$a_n = \frac{1}{L}\int_{-2}^{2} f(x) \cos\frac{n\pi x}{L} = \frac{1}{L}\left[\int_{-L}^{0} 0\, dx + \int_{0}^{2} 1 \cdot (\cos\frac{n\pi x}{L})\, dx\right]$$

$$= \frac{1}{L}\left[\frac{\sin\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right]_0^2 = \frac{2n\pi}{2}\left(\sin n\pi - \sin 0\right]$$

$$a_n = 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) (\sin\frac{n\pi x}{L})\, dx$$

$$= \frac{1}{L}\int_{0}^{L} \sin\frac{n\pi x}{L}\, dx = -\frac{1}{L}\left(\frac{\cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right]_0^L$$

$$= -\frac{\cancel{2}}{\cancel{2}n\pi}\left[\cos\frac{n\pi \cancel{L}}{\cancel{L}} - \cos 0\right]$$

$$b_n = -\frac{1}{n\pi}\left[\cos n\pi - 1\right)$$

$$b_1 = -\frac{1}{\pi}\left[\cos\pi - 1\right] = -\frac{1}{\pi}\left[-1-1\right) = \frac{2}{\pi}$$

$$b_2 = -\frac{1}{2\pi}\left[\cos 2\pi - 1\right] = -\frac{1}{2\pi}\left(1-1\right) = 0$$

$$b_2 = 0$$

$$b_3 = -\frac{1}{3\pi}\left[\cos 3\pi - 1\right] = -\frac{1}{3\pi}\left[-1-1\right] = \frac{2}{3\pi}$$

$$\therefore f(x) = \frac{1}{2} + \frac{2}{\pi}\left[\frac{\sin\pi x}{2}\right) + \frac{\sin 3\pi x/2}{3} + \frac{\sin 5\pi x/2}{5} + \cdots)$$

Q Find the Fourier series for the function.

$f(x) = 2x - x^2, \quad 0 \le x \le 3$

and deduce that $\sum_{1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

Solution:- Here $2l = 3$ So $l = \frac{3}{2}$

Let $f(x) = 2x - x^2 = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$ — (1)

where $a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx = \frac{1}{\frac{3}{2}}\int_0^3 (2x - x^2)\,dx$

$= \frac{2}{3}\left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^3$

$= \frac{2}{3}\left[x^2 - \frac{x^3}{3}\right]_0^3 = \frac{2}{3}\left[9 - \frac{27}{3}\right] = \frac{3}{2}\left[\frac{27-27}{3}\right]$

$a_0 = 0$

and $a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$

$= \frac{1}{\frac{3}{2}}\int_0^3 (2x - x^2)\cos\frac{2n\pi x}{3}\,dx$

$= \frac{2}{3}\left[(2x - x^2)\frac{3}{2n\pi}\sin\left(\frac{2n\pi x}{3}\right)\right]_0^3 - \frac{2}{3}\int_0^3 (2-2x)\frac{3}{2n\pi}\sin\left(\frac{2n\pi x}{3}\right)dx$

$= 0 - \frac{2}{3}\times\frac{3}{2n\pi}\left[-(2-2x)\left(\frac{\cos\frac{2n\pi x}{3}}{\frac{2n\pi}{3}}\right)\right]_0^3 + \frac{1}{n\pi}\int_0^3 (-2)\left(-\cos\frac{2n\pi x}{3}\right)\frac{3}{2n\pi}\,dx$

$= -\frac{1}{n\pi}\left[-(2-2x)\cos\left(\frac{2n\pi x}{3}\right)\frac{3}{2n\pi}\right]_0^3 + \frac{1}{n\pi}\int_0^3 (-2)\left(-\cos\left(\frac{2n\pi x}{3}\right)\right)\times\frac{3}{2n\pi}\,dx$

$= -\frac{2\cdot 3}{n^2\pi^2}\cos 2n\pi - \frac{3}{n^2\pi^2}\cos 0 + \frac{3}{n^2\pi^2}\left[\sin\left(\frac{2n\pi x}{3}\right)\cdot\frac{3}{2n\pi}\right]_0^3$

$= -\frac{6}{n^2\pi^2} - \frac{3}{n^2\pi^2} = -\frac{9}{n^2\pi^2}$

and $b_n = \frac{1}{l}\int_0^{2l} f(x) \sin\frac{n\pi x}{l}\, dx$

$= \frac{1}{\frac{3}{2}}\int_0^{3} (2x-x^2) \sin\left(\frac{2n\pi x}{3}\right) dx$

$= \frac{2}{3}\left[-(2x-x^2)\cos\left(\frac{2n\pi x}{3}\right)\left(\frac{3}{2n\pi}\right)\right]_0^3 - \frac{2}{3}\int_0^3 -(2-2x)\cos\left(\frac{2n\pi x}{3}\right)\frac{3}{2n\pi}dx$

$= \frac{1}{n\pi}\left[[3\cos 2n\pi + 0] + \frac{2}{n\pi}\left[(1-x)\left(\sin\left(\frac{2n\pi x}{3}\right)\right)\frac{3}{2n\pi}\right]_0^3\right.$

$- \frac{2}{n\pi}\int_0^3 (-1)\left(-\cos\frac{2n\pi x}{3}\right)\left(\frac{3}{2n\pi}\right)dx$

$= \frac{3}{n\pi} + 0 + \frac{3}{n^2\pi^2}\,0 - \frac{3}{n^2\pi^2}\left[\sin\left(\frac{2n\pi x}{3}\right)\frac{3}{2n\pi}\right]_0^3$

$= \frac{3}{n\pi} + 0 = \frac{3}{n\pi}$

$b_n = \frac{3}{n\pi}$

putting the values of a_0, a_n and b_n in (1) we get

$2x-x^2 = 0 + \sum_{n=1}^{\infty}\left[-\frac{9}{n^2\pi^2}\cos\left(\frac{2n\pi x}{3}\right) + \frac{3}{n\pi}\sin\left(\frac{2n\pi x}{3}\right)\right]$ — (B)

At $x = \frac{3}{2}$ which lies in $(0,3)$ we get the above series as

$3 - \frac{9}{4} = \sum_{n=1}^{\infty}\left(-\frac{9}{n^2\pi^2}\cos n\pi + \frac{3}{n\pi}\sin n\pi\right)$

$\frac{3}{4} = -\frac{9}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$

$\frac{3}{4} = -\frac{9}{\pi^2}\left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} + \cdots\right]$

$\frac{3}{4} = \frac{9}{\pi^2}\left[\frac{1}{1^2} \cancel{+} \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \cdots\right]$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$

$$\frac{\pi^2}{12} = \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots\right) - 2\left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{8^2} + \cdots\right)$$

$$= \frac{1}{2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right)$$

$$\therefore \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

Q Obtain the Fourier series expansion of

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 2 & \text{if } 1 < x < 2 \end{cases}$$

Solution: Here $f(x)$ is defined in the interval $(0,2)$. Since the length $2l$ of the interval $(0,2)$ is 2, we have

$$2l = 2 \quad \therefore l = \frac{2}{2} = 1$$

The required series of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{--- (1)}$$

where $a_0 = \frac{1}{1}\int_0^2 f(x)\,dx = \int_0^1 1\cdot dx + \int_1^2 2\,dx$

$$= [x]_0^1 + 2[x]_1^2 = (1-0) + 2(2-1) = 1+2 = 3$$

$$a_n = \frac{1}{1}\int_0^2 f(x)\cos\frac{n\pi x}{l}\,dx = \left[\int_0^1 1\cdot\cos n\pi x\,dx + \int_1^2 2\cos n\pi x\,dx\right]$$

$$= \left[\frac{\sin n\pi x}{n\pi}\right]_0^1 + 2\left[\frac{\sin n\pi x}{n\pi}\right]_1^2$$

$$= \frac{1}{n\pi}[\sin n\pi - 0] + \frac{2}{n\pi}[\sin 2n\pi - \sin 0]$$

$$a_n = 0$$

Also $b_n = \frac{1}{1}\int_0^2 f(x) \sin\left(\frac{n\pi x}{1}\right) dx$

$= \int_0^1 1 \cdot \sin n\pi x \, dx + 2\int_1^2 \sin n\pi x \, dx$

$= -\left[\frac{\cos n\pi x}{n\pi}\right]_0^1 - 2\left[\frac{\cos n\pi x}{n\pi}\right]_1^2$

$= -\frac{1}{n\pi}\left[\cos n\pi - \cos 0\right] - \frac{2}{n\pi}\left[\cos 2n\pi - \cos n\pi\right]$

$= -\frac{1}{n\pi}\left[(-1)^n - 1\right] - \frac{2}{n\pi}\left[1 - (-1)^n\right]$

$b_n = -\frac{1}{n\pi}(-1)^n + \frac{1}{n\pi} - \frac{2}{n\pi} + \frac{2}{n\pi}(-1)^n$

$b_n = \frac{(-1)^n}{n\pi}\left[2-1\right] - \frac{1}{n\pi} = \frac{1}{n\pi}\left[(-1)^n - 1\right]$

$b_1 = \frac{1}{\pi}\left[-1-1\right] = -\frac{2}{\pi}$

$b_2 = \frac{1}{2\pi}(1-1) = 0$ $\quad b_2 = 0$

$b_3 = \frac{1}{3\pi}\left[-1-1\right] = -\frac{2}{3\pi}$

$b_4 = \frac{1}{4\pi}\left[1-1\right] = 0$

$b_5 = \frac{1}{5\pi}\left[-1-1\right] = -\frac{2}{5\pi}$

putting the values of a_0, a_n and b_n in (1) we get

$f(x) = \frac{3}{2} - \frac{2}{\pi}\left[\frac{\sin\pi x}{1} + \frac{\sin 3\pi x}{3} + \frac{\sin 5\pi x}{5} + \cdots\right]$

Q Find the Fourier series for the function

$f(x) = \begin{cases} x & \text{in } 0<x<1 \\ 1-x & \text{in } 1<x<2 \end{cases}$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

Solution:- Here $f(x)$ is defined in the interval $(0,2)$. ~~Hence~~
the length ~~is~~ $2l$ of the interval $(0,2)$ is 2. we have

$$2l = 2 \quad l = 1$$

The required series of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$a_0 = \frac{1}{1}\int_0^2 f(x)\,dx$$

$$= \int_0^1 x\,dx + \int_1^2 (1-x)\,dx$$

$$= \frac{1}{2}[x^2]_0^1 + \left[x - \frac{x^2}{2}\right]_1^2$$

$$= \frac{1}{2} + \left(2 - \frac{4}{2}\right) - \left(1 - \frac{1}{2}\right)$$

$$= \cancel{\frac{1}{2}} + 0 - \cancel{\frac{1}{2}} = 0$$

$$a_0 = 0$$

$$a_n = \frac{1}{1}\int_0^2 f(x)\left(\cos\frac{n\pi x}{1}\right)dx$$

$$= \int_0^1 x\cos n\pi x\,dx + \int_1^2 (1-x)\cos n\pi x\,dx$$

$$= \left[x\frac{\sin n\pi x}{n\pi}\right]_0^1 - \int_0^1 \frac{\sin n\pi x}{n\pi}\,dx + \int_1^2 \cos n\pi x\,dx$$
$$- \int_1^2 x\cos n\pi x\,dx$$

$$= \frac{1}{n\pi}[x\sin n\pi x]_0^1 + \frac{1}{n^2\pi^2}[\cos n\pi x]_0^1 + \left[\frac{\sin n\pi x}{n\pi}\right]_1^2$$
$$- \left[\left[x\frac{\sin n\pi x}{n\pi}\right]_1^2 - \int_1^2 \frac{\sin n\pi x}{n\pi}\,dx\right]$$

$$= \frac{1}{n\pi}\times 0 + \frac{1}{n^2\pi^2}[\cos n\pi - \cos 0] + \frac{1}{n\pi}[\sin 2n\pi - \sin 0]$$
$$- 0 - \frac{1}{n^2\pi^2}[\cos n\pi x]_1^2$$

$$\cancel{a_n = -\frac{2}{n^2\pi^2} + 0 - \frac{1}{n^2\pi^2}\left[\cos 2n\pi - \cos n\pi\right)}$$

$$a_n = \frac{1}{n^2\pi^2}\left[\cos n\pi - 1\right] - \frac{1}{n^2\pi^2}\left[\cos 2n\pi - \cos n\pi\right)$$

$$= \frac{1}{n^2\pi^2}\cos n\pi - \frac{1}{n^2\pi^2} - \frac{1}{n^2\pi^2}\cos 2n\pi + \frac{1}{n^2\pi^2}\cos n\pi$$

$$= \frac{2}{n^2\pi^2}\cos n\pi - \frac{1}{n^2\pi^2} - \frac{1}{n^2\pi^2}$$

$$a_n = \frac{2}{n^2\pi^2}\cos n\pi - \frac{2}{n^2\pi^2} = \frac{2}{n^2\pi^2}\left((-1)^n - 1\right)$$

$$a_1 = \frac{2}{\pi^2}(-1-1) = -\frac{4}{\pi^2} \qquad a_2 = \frac{2}{2^2\pi^2}(1-1)$$

$$a_1 = -\frac{4}{\pi^2}, \quad a_2 = 0 \qquad a_3 = \frac{2}{3^2\pi^2}(-1-1) = -\frac{4}{3^2\pi^2}$$

$$a_4 = \frac{2}{4^2\pi^2}(1-1) = 0 \qquad a_5 = \frac{2}{5^2\pi^2}(-1-1) = -\frac{4}{5^2\pi^2}$$

$$b_n = \int_0^1 x\sin n\pi x\,dx + \int_1^2 (1-x)\sin n\pi x\,dx$$

$$= \int_0^1 x\sin n\pi x\,dx + \int_1^2 \sin n\pi x\,dx - \int_1^2 x\sin n\pi x\,dx$$

$$= -\left[x\frac{\cos n\pi x}{n\pi}\right]_0^1 + \int_0^1 \frac{\cos n\pi x}{n\pi}\,dx - \left[\frac{\cos n\pi x}{n\pi}\right]_1^2$$

$$- \left\{-\left[x\frac{\cos n\pi x}{n\pi}\right]_1^2 + \int_1^2 \frac{\cos n\pi x}{n\pi}\,dx\right\}$$

$$= -\left[\frac{x\cos n\pi x}{n\pi}\right]_0^1 + \frac{1}{n^2\pi^2}\left[\sin n\pi x\right]_0^1 - \frac{1}{n\pi}\left[\cos 2n\pi - \cos n\pi\right]$$

$$- \left\{-\left[\frac{x\cos n\pi x}{n\pi}\right]_1^2 + \frac{1}{n^2\pi^2}\left[\sin n\pi x\right]_1^2\right\}$$

$$b_n = -\left[\frac{x\cos n\pi x}{n\pi}\right]_0^1 + 0 + \frac{1}{n\pi}\left[\cos 2n\pi - \cos n\pi\right]$$

$$+ \left[\frac{x\cos n\pi x}{n\pi}\right]_1^2 - 0$$

$$= -\cancel{\frac{1}{n\pi}\cos n\pi} + \frac{1}{n\pi}\cos 2n\pi + \cancel{\frac{1}{n\pi}\cos n\pi}$$

$$+ \frac{2\cos 2n\pi}{n\pi} - \frac{\cos n\pi}{n\pi}$$

$$b_n = \frac{\cos 2n\pi}{n\pi} - \frac{\cos n\pi}{n\pi} = \frac{1}{n\pi}\left[1-(-1)^n\right]$$

$$b_1 = \frac{1}{\pi}[1+1] = \frac{2}{\pi}, \quad b_2 = \frac{1}{2\pi}[1-1] = 0$$

$$b_3 = \frac{1}{3\pi}(1+1) = \frac{2}{3\pi} \quad b_4 = \frac{1}{4\pi}(1-1) = 0$$

$$b_5 = \frac{1}{5\pi}(1+1) = \frac{2}{5\pi}$$

$$\therefore f(x) = 0 + \frac{4}{\pi^2}\left(\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \cdots\right)$$
$$+ \frac{2}{\pi}\left(\frac{\sin \pi x}{1} + \frac{\sin 3\pi x}{3} + \frac{\sin 5\pi x}{5} + \cdots\right)$$

we now put $x=1$ in the given result. Since 1 is the point of discontinuity for $f(x)$

$$L.H.L\ f(1-0) = 1, \quad R.H.L = f(1+0) = 0$$

By ~~Dri~~ Dirichlet's theorem the sum of the series at $x=1$ is $\frac{f(1-0)+f(1+0)}{2} = \frac{1}{2}$

thus $$\frac{1}{2} = -\frac{4}{\pi^2}\left(-\frac{1}{1^2} - \frac{1}{3^2} - \frac{1}{5^2} \cdots\right) + \frac{2}{\pi}(0+0)$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$ proved