Value interpretation for universal type is defined as :

$$\mathscr{V}[\![\forall\alpha.A]\!]_k^\theta = \{\wedge\alpha.e|.\vdash^\theta \wedge\alpha.e : \forall\alpha.A \;\wedge \forall T : type, \forall j \leq k \; s.t.\Delta \vdash T, [T/\alpha]e \in \mathscr{C}[\![[T/\alpha]A]\!]_j^\theta\}$$

This definition can be used in proof of the Downward Closure Lemma as shown below :

To Prove : if $v \in \mathscr{V}[\![\tau]\!]_k^\theta$ and $j \leq k$ then $v \in \mathscr{V}[\![\tau]\!]_j^\theta$
Proof : By induction on types :

- $\tau =$ Unit
  Trivial, will satisfy all j as indices are not even part of value interpretation.
- $\tau = A^{\theta'} \to B$ and $\theta =$ L
  Suppose $i \leq j$ and $v' \in \mathscr{V}[\![A^{\theta'}]\!]_i^L$.
  Then $i \leq k$ also. Hence, By definition of $\mathscr{V}[\![A^{\theta'} \to B]\!]_k^L$,
  $[v'/x]a \in \mathscr{C}[\![B]\!]_i^L$. This is true by definition of $\mathscr{V}[\![A^{\theta'} \to B]\!]_j^L$ also. Hence proved.
- $\tau = \forall\alpha.A$
  let $i \leq j$ then $i \leq k$ too. if $\wedge\alpha.e \in \mathscr{V}[\![\forall\alpha.A]\!]_k^\theta$ then by definition, $\forall T : type \; s.t.\Delta \vdash T, [T/\alpha]e \in \mathscr{C}[\![[T/\alpha]A]\!]_i^\theta$
  Again, this is the same by definition for $\mathscr{V}[\![\forall\alpha.A]\!]_j^\theta$ also, hence proved.

Well-formed substitutions are defined as follows :
Well formed type substitutions :

$$\mathcal{D}[\![.]\!] = \{\phi\}$$
$$\mathcal{D}[\![\Delta, \alpha]\!] = \{\rho[\alpha \to X] \mid \rho \in \mathcal{D}[\![\Delta]\!] \wedge FTV(X) \in dom(\rho)\}$$

Well formed value substitutions :

$$\mathcal{G}[\![.]\!] = \{\phi\}$$
$$\mathcal{G}[\![\Gamma, x : \tau]\!]_k^\rho = \{\gamma[x \to v] \mid \gamma \in \mathcal{G}[\![\Gamma]\!]_k^\rho \wedge v \in \mathscr{V}[\![\rho(\tau)]\!]_k\}$$

T-Univ-Lam rule:

$$\frac{\Delta, \alpha, \Gamma \vdash^\theta e : B}{\Delta\Gamma \vdash^\theta \Lambda\alpha.e : \forall\alpha.B}$$

To prove the soundness theorem using T-Univ-Lam case in P fragment:
Given: $\Gamma \vdash^P \wedge\alpha.e : \forall\alpha.B,\; \Gamma \vDash_k \gamma,\; \Delta \vDash \rho$
To Prove: $\gamma(\rho(\wedge\alpha.e)) \in \mathscr{C}[\![\forall\alpha.B]\!]_k^P$
Proof: From the definition of $\mathscr{C}[\![\forall\alpha.B]\!]_k^P$, we need to prove

- $. \vdash^P \gamma(\rho(\wedge\alpha.e)) : \forall\alpha.B$ - by appeal to the substitution lemma

- $\gamma(\rho(\wedge\alpha.e)) = \wedge\alpha.\gamma(\rho(e))$ if α doesn't occur in $dom(\rho)$.
  From definition of $\mathscr{C}[\![\forall\alpha.A]\!]^P_k$, if $\forall T : type,\ \gamma(\rho([T/\alpha]e)) \leadsto^j v$ for all $j < k$, $\Delta \vdash \rho(\alpha), \Delta \vdash \Gamma$, then it suffices to show that
  $v \in \mathscr{V}[\![\rho([T/\alpha]B)]\!]^P_{k-j}$

  By Induction Hypothesis,

  $$\Delta, \alpha, \Gamma \vdash^P e : B$$

  $$\forall T : type, \gamma(\rho([T/\alpha]e)) \in \mathscr{C}[\![\rho([T/\alpha]B)]\!]^P_k$$

  Now, we know that $\gamma(\rho([T/\alpha]e)) \leadsto^j v$ and from definition of $\mathscr{C}[\![\rho([T/\alpha]B)]\!]^P_k$, we have $v \in \mathscr{V}[\![\rho([T/\alpha]B)]\!]^P_{k-j}$.
  Hence we get that $v \in \mathscr{V}[\![[\rho([T/\alpha]B)]\!]^P_{k-j}$.

T-Univ-App rule

$$\frac{\Delta\Gamma \vdash^\theta e : \forall\alpha.A \quad \Delta \vdash B}{\Delta\Gamma \vdash^\theta e[B] : [B/\alpha]A}$$

To prove the soundness theorem using T-Univ-App case in L fragment:
Given: $\Gamma \vdash^L e[B] : [B/\alpha]A,\ \Gamma \vDash_k \gamma,\ \Delta \vDash \rho$

To Prove: $\gamma(\rho(e[B])) \in \mathscr{C}[\![[\rho(B)/\alpha]A]\!]^L_k$

Proof: From the definition of $\mathscr{C}[\![[\rho(B)/\alpha]A]\!]^L_k$, we need to prove

- $. \vdash^L \gamma(\rho(e[B])) : [\rho(B)/\alpha]A$ - By appeal to the substitution lemma

- if $\gamma(\rho([B/\alpha]e)) \leadsto^* v$, $\Delta \vdash B, \Delta \vdash \Gamma$, then it suffices to show that
  $v \in \mathscr{V}[\![[\rho(B)/\alpha]A]\!]^L_k$

  $$\gamma(\rho(e[B])) = \gamma(\rho(e))[\rho(B)] \tag{1}$$

  if α doesn't occur in $dom(\rho)$.

  By Induction Hypothesis,

  $$\Delta\Gamma \vdash^L e : \forall\alpha.A$$

$$\gamma(\rho(e)) \in \mathscr{C}[\![\forall\alpha.A]\!]_k^L$$

By the definition of $\mathscr{C}[\![\forall\alpha.A]\!]_k^L$, $\gamma(\rho(e)) \leadsto^* v_1 \in \mathscr{V}[\![\forall\alpha.A]\!]_k^L$.
Hence $\gamma(\rho(e))[\rho(B)] \leadsto^* v_1[\rho(B)]$ from eq.1.
Now as $v_1 \in \mathscr{V}[\![\forall\alpha.A]\!]_k^L$,it will look like $\wedge\alpha.v'$ for some v' s.t. $\forall T : Type, v'[T/\alpha] \in \mathscr{C}[\![[T/\alpha]A]\!]_k^L$.
Initialising $\rho(B)$ in T above, we get $v'[\rho(B)/\alpha] \in \mathscr{C}[\![[\rho(B)/\alpha]A]\!]_k^L$.
By definition of $\mathscr{C}[\![[\rho(B)/\alpha]A]\!]_k^L$, $v'[\rho(B)/\alpha] \leadsto^* v'' \in \mathscr{V}[\![[\rho(B)/\alpha]A]\!]_k^L$.
But v'' is actually v, hence proved.