Updated Syntax :

Types $\quad A, B ::= \text{Unit} \mid A^{\theta} \to\ B \mid A + B \mid A@\theta \mid \alpha \mid \mu\alpha.A \mid \forall\alpha.A$

Terms $\quad a, b ::= x \mid \text{rec } f\ x.a \mid a\ b \mid \text{box } a \mid \text{unbox } x = a \text{ in } b \mid () \mid \text{inl } a \mid \text{inr } a \mid \text{case } a \text{ of}$ inl $x \Rightarrow a_1$;inr $x \Rightarrow a_2 \mid \text{roll } a \mid \text{unroll } a \mid \Lambda\alpha.a \mid a[\tau]$

Language Classifiers $\quad \theta ::= \text{L} \mid \text{P}$

Environments $\Delta\Gamma ::= .\ \mid \Delta\Gamma, x :^{\theta} A$

$$\Delta ::= . \mid \Delta, \alpha$$

Values v::= $x \mid () \mid \text{inl } v \mid \text{inr } v \mid \text{rec } f\ x.a \mid \text{box } v \mid \text{roll } v \mid \Lambda\alpha$.t

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TUnivApp

$$\frac{\Delta\Gamma \vdash^{L} e : \forall\alpha.A \quad \Delta \vdash B \quad neutral(B)}{\Delta\Gamma \vdash^{L} e[B] : [B/\alpha]A}$$

TUnivLam

$$\frac{\Delta, \alpha, \Gamma \vdash^{\theta} e : B}{\Delta\Gamma \vdash^{\theta} \Lambda\alpha.e : \forall\alpha.B}$$

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1 Good terms

We define a set ${}_kGOOD_T$ of terms by induction on type T :

- t of atomic type T , then t doesn't get stuck for k steps
- t of arrow type $V \to W$, $\forall uj, s.t.\ . \vdash u : V, u \rightsquigarrow^{j} v \wedge irred(v), tv \in\ {}_{k-j}GOOD_W$

Now we define Well Behaved candidates of type T, for k steps as set ${}_kW_T$ such that,
(CW 1) If $t \in\ {}_kW_T$ then t doesn't get stuck before k steps of computation
(CW 2) If $t \in\ {}_kW_T$, $t \rightsquigarrow^{i} t'$ then $t \in\ {}_{k-i}W_T$.
(CW 3) If t is neutral and whenever we convert a redex of t, we obtain some term $t' \in\ {}_aW_T$ where a is the lowest in all of the conversions, we have $t \in\ {}_{a+1}W_T$.

So, we have

$${}_kW_T = \{t \mid . \vdash t : T,\ t\ satisfies\ CW\ 1,\ CW\ 2\ and\ CW\ 3\}$$

$${}_kW_X \to\ {}_{k-j}W_Y = \{t \mid . \vdash t : X \to Y, \forall j, x, j \leq k, x \in X s.t. x \to^{j} v \wedge irred(v),\ tx \in\ {}_{k-j}W_Y\}$$

To Prove: Well behaved terms (which don' get stuck for k steps) satisfy the given 3 conditions.
Proof: By induction on types.
1. Atomic Types :

- CW 1 : tautology
- CW 2 : if a computation has run on a term for i steps, only k-i steps would remain till it is well. Hence, if $t \in {}_kW_T$, $t \rightsquigarrow^i t'$ then $t \in {}_{k-i}W_T$.
- CW 3 : If t is neutral and whenever we convert a redex of t, we obtain some term $t' \in {}_aW_T$, where a is lowest in all conversions, then it is guaranteed that after any conversion, the term will remain well behaved for atleast a steps. Hence, $t \in {}_{a+1}W_T$.

2. Arrow types :

- CW 1: Let u be a variable of type U. So, u is neutral and normal, hence well behaved for any number of steps. So, $tu \in_k GOOD_V$. As u doesn't consume any steps, hence t is well behaved for k steps.
- CW 2 : Let u be a variable of type U. So, u is neutral and normal, hence well behaved for any number of steps. Now, $tu \rightsquigarrow^i t'u$, so, by I.H. CW 2on V type, $t'u \in_{k-i} GOOD_V$. As u didn't consume any step, hence, $t \rightsquigarrow^i t'$.
- CW 3 : Let u be a variable of type U. So, u is neutral and normal, hence well behaved for any number of steps. So, by I.H. on the smaller type W, $tu \in {}_{a+1}W_T$ where a is lowest in all conversions. As no computation in u, hence, $t \in {}_{a+1}W_T$.

Now, we define Candidates of Goodness as

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\begin{aligned}
{}_kGOOD_{X_i}[\underline{\mathrm{G}}/\underline{\mathrm{X}}] &= {}_kW_i \\
{}_kGOOD_{Y \to Z}[\underline{\mathrm{G}}/\underline{\mathrm{X}}] &= {}_kGOOD_Y[\underline{\mathrm{G}}/\underline{\mathrm{X}}] \to {}_{k-j}GOOD_Z[\underline{\mathrm{G}}/\underline{\mathrm{X}}] \\
{}_kGOOD_{\forall Y.Z}[\underline{\mathrm{G}}/\underline{\mathrm{X}}] &= \{t | \forall V, tV \in {}_kGOOD_Z[\underline{\mathrm{G}}/\underline{\mathrm{X}}, {}_kW_V/Y]\}
\end{aligned}
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To Prove: ${}_kGOOD_{\forall Y.Z}[\underline{\mathrm{G}}/\underline{\mathrm{X}}]$ is a goodness candidate for type $T[\underline{\mathrm{U}}/\underline{\mathrm{X}}]$.
Proof : If ${}_kGOOD_{\forall Y.Z}[\underline{\mathrm{G}}/\underline{\mathrm{X}}]$ is a goodness candidate for type $T[\underline{\mathrm{U}}/\underline{\mathrm{X}}]$ then it should fulfill CW 1, CW 2 and CW 3 :
(CW 1) Let $t \in_k GOOD_{\forall Y.Z}[\underline{\mathrm{G}}/\underline{\mathrm{X}}]$. Then for any type V whose goodness candidate is ${}_kW_V$, $tV \in {}_kGOOD_Z[\underline{\mathrm{G}}/\underline{\mathrm{X}}, {}_kW_V/Y]$ as defined.
So, by Induction Hypothesis on smaller type, tV is good. As there is no computation within V, hence t is good for k steps.
(CW 2) Let $t \in_k GOOD_{\forall Y.Z}[\underline{\mathrm{G}}/\underline{\mathrm{X}}]$ and $t \rightsquigarrow^i t'$. Then for all types V and their goodness candidate ${}_kW_V$, $tV \in {}_kGOOD_Z[\underline{\mathrm{G}}/\underline{\mathrm{X}}, {}_kW_V/Y]$ and $tV \rightsquigarrow^i t'V$.
By Induction Hypothesis on smaller type, $t'V \in {}_{k-i}GOOD_Z[\underline{\mathrm{G}}/\underline{\mathrm{X}}, {}_kW_V/Y]$.
Hence, $t' \in {}_{k-i}GOOD_{\forall Y.Z}[\underline{\mathrm{G}}/\underline{\mathrm{X}}]$. As t' will be good only for $k-i$ more steps of computations, so, it satisfies CW 2.
(CW 3) Let $t \in_k GOOD_{\forall Y.Z}[\underline{\mathrm{G}}/\underline{\mathrm{X}}]$. Then for any type V whose goodness candidate is

${}_kW_V$, $tV \in \ {}_kGOOD_Z[\underline{\mathrm{G}}/\underline{\mathrm{X}},\ {}_kW_V/Y]$ as defined. Now by I.H. on CW 3, we know that if there are several redexes from tV, then $tV \in_{a+1} GOOD_Z[\underline{\mathrm{G}}/\underline{\mathrm{X}},\ {}_kW_V/Y]$ where a is the minimum of all the other redexes. As no computation occurs in V, we can argue that $t \in_{a+1} GOOD_{\forall Y.Z}[\underline{\mathrm{G}}/\underline{\mathrm{X}},\ {}_kW_V/Y]$.