We have the T-Univ-App rule in L fragment:

$$\frac{\Delta\Gamma \vdash^L e : \forall\alpha.A \quad \Delta \vdash B}{\Delta\Gamma \vdash^L e[B] : [B/\alpha]A}$$

Where well-formed substitutions are defined as follows :

Well formed type substitutions :

$$\mathcal{D}[\![.]\!] = \{\phi\}$$
$$\mathcal{D}[\![\Delta, \alpha]\!] = \{\rho[\alpha \to X] \mid \rho \in \mathcal{D}[\![\Delta]\!] \land FTV(X) \in dom(\rho)\}$$

Well formed value substitutions :

$$\mathcal{G}[\![.]\!] = \{\phi\}$$
$$\mathcal{G}[\![\Gamma, x : \tau]\!]^\rho_k = \{\gamma[x \to v] \mid \gamma \in \mathcal{G}[\![\Gamma]\!]^\rho_k \land v \in \mathscr{V}[\![\rho(\tau)]\!]_k\}$$

To prove the soundness theorem using T-Univ-App case:
Given: $\Gamma \vdash^L e[B] : [B/\alpha]A,\ \Gamma \vDash_k \gamma,\ \Delta \vDash \rho$

To Prove: $\gamma(\rho(e[B])) \in \mathscr{C}[\![[\rho(B)/\alpha]A]\!]^L_k$

Proof: From the definition of $\mathscr{C}[\![[\rho(B)/\alpha]A]\!]^L_k$, we need to prove

- $. \vdash^L \gamma(\rho(e[B])) : [\rho(B)/\alpha]A$ - By appeal to the substitution lemma
- if $\gamma(\rho([B/\alpha]e)) \rightsquigarrow^* v$, $\Delta \vdash B, \Delta \vdash \Gamma$, then it suffices to show that $v \in \mathscr{V}[\![[\rho(B)/\alpha]A]\!]^L_k$

$$\gamma(\rho(e[B])) = \gamma(\rho(e))[\rho(B)] \tag{1}$$

if α doesn't occur in $dom(\rho)$.

By Induction Hypothesis,

$$\Delta\Gamma \vdash^L e : \forall\alpha.A$$

$$\gamma(\rho(e)) \in \mathscr{C}[\![\forall\alpha.A]\!]^L_k$$

By the definition of $\mathscr{C}[\![\forall\alpha.A]\!]^L_k$, $\gamma(\rho(e)) \rightsquigarrow^* v_1 \in \mathscr{V}[\![\forall\alpha.A]\!]^L_k$.
Hence $\gamma(\rho(e))[\rho(B)] \rightsquigarrow^* v_1[\rho(B)]$ from eq.1.

Now as $v_1 \in \mathscr{V}[\![\forall\alpha.A]\!]_k^L$, it will look like $\wedge\alpha.v'$ for some v' s.t. $\forall T : Type, v'[T/\alpha] \in \mathscr{C}[\![[T/\alpha]A]\!]_k^L$.
Initialising $\rho(B)$ in T above, we get $v'[\rho(B)/\alpha] \in \mathscr{C}[\![[\rho(B)/\alpha]A]\!]_k^L$.
By definition of $\mathscr{C}[\![[\rho(B)/\alpha]A]\!]_k^L$, $v'[\rho(B)/\alpha] \rightsquigarrow^* v'' \in \mathscr{V}[\![[\rho(B)/\alpha]A]\!]_k^L$.
But v'' is actually v, hence proved.