Step-indexing in value interpretations can't accomodate T-Univ-App rule, hence we need to follow a different approach.
So, for the L-fragment we will use Tait-Gerard's Reducibility Theorem (Section 14.3 of "Proofs and Types" book) to prove strong normalization of L fragment in LTheta language.

All of the names, notations and lemmas used here are taken from "Proofs and Types" book and lemmas and theorems will be used here without their proofs.

We restate the **Reducibility Theorem** (Section 14.3):
Let t be a term of type T. Suppose all the free variables of t are among $x_1, ..., x_n$ of types $U_1, ..., U_n$, and all the free type variables of $T, U_1, ..., U_n$ are among $X_1, ..., X_m$. If $R_1, ..., R_m$ are reducibility candidates of types $V_1, ..., V_m$ and $u_1, ..., u_n$ are terms of types $U_1[\underline{V}/\underline{X}], ..., Un[\underline{V}/\underline{X}]$ which are in $RED_{U_1}[\underline{R}/\underline{X}], ..., RED_{U_n}[\underline{R}/\underline{X}]$ then $t[\underline{V}/\underline{X}][\underline{u}/\underline{x}] \in RED_T[\underline{R}/\underline{X}]$.

Proof: By induction on typing derivations in L fragment.

TLAM $\dfrac{\Delta\Gamma, y :^{\theta} A \vdash^L b : B}{\Delta\Gamma \vdash^L \lambda y.b : A^{\theta} \rightarrow B}$

By I.H., $b[\underline{V}/\underline{X}][\underline{u}/\underline{x}][v/y]$ is reducible forall v reducible of type A^{θ}
From lemma 6.3.2, $\lambda y.b[\underline{V}/\underline{X}][\underline{u}/\underline{x}]$ is also reducible.

TAPP $\dfrac{\Delta\Gamma \vdash^L a : A^{\theta'} \rightarrow B \quad \Delta\Gamma \vdash^L box\ b : A@\theta'}{\Delta\Gamma \vdash^L ab : B}$

By I.H., $a[\underline{V}/\underline{X}][\underline{u}/\underline{x}]$ is reducible of type $A^{\theta'} \rightarrow B$.
box $b[\underline{V}/\underline{X}][\underline{u}/\underline{x}]$ is reducible of type $A@\theta'$. Hence $b[\underline{V}/\underline{X}][\underline{u}/\underline{x}]$ is reducible of type $A^{\theta'}$ (inversion on TBOXL).
On application, $a[\underline{V}/\underline{X}][\underline{u}/\underline{x}]\ (b[\underline{V}/\underline{X}][\underline{u}/\underline{x}]) = ab[\underline{V}/\underline{X}][\underline{u}/\underline{x}]$ which is reducible.

T-UNIV-LAM $\dfrac{\Delta, \alpha, \Gamma \vdash^{\theta} e : B}{\Delta\Gamma \vdash^{\theta} \Lambda\alpha.e : \forall\alpha.B}$

By I.H., $e[\underline{V}/\underline{X}][\underline{u}/\underline{x}][\nu/\alpha] \in RED_B[\underline{R}/\underline{X}][S/\alpha]$ where S is the reducibility candidate for type ν.
From Lemma 14.2.2, $\wedge\alpha.e[\underline{V}/\underline{X}][\underline{u}/\underline{x}] \in RED_B[\underline{R}/\underline{X}]$

T-UNIV-LAM $\dfrac{\Delta\Gamma \vdash^L e : \forall\alpha.A \quad \Delta \vdash B}{\Delta\Gamma \vdash^L e[B] : [B/\alpha]A}$

By I.H., $e[\underline{V}/\underline{X}][\underline{u}/\underline{x}] \in RED_{\forall\alpha.A}[\underline{R}/\underline{X}]$.
From Lemma 14.2.3, $e[B][\underline{V}/\underline{X}][\underline{u}/\underline{x}] \in RED_{[B/\alpha]A}[\underline{R}/\underline{X}]\ \forall B : type$

TBOXL $\dfrac{\Delta\Gamma \vdash^L a : A}{\Delta\Gamma \vdash^L box\ a : A@\theta}$

By I.H., if a is reducible then so is box a.

TBOXLV $\dfrac{\Delta\Gamma \vdash^P v : A}{\Delta\Gamma \vdash^L box\ v : A@P}$

By I.H. $v \in \mathscr{V}[\![A]\!]_k^P$. Thus, $box\ v \in \mathscr{V}[\![A@P]\!]_k^P$.

As all values in $\mathscr{V}[\![A@P]\!]_k^P$ are values, it is a subset of $RED_{A@P}[\underline{R}/\underline{X}]$.
Hence $box\ v \in RED_{A@P}[\underline{R}/\underline{X}]$.

TFOVAL $\dfrac{\Delta\Gamma \vdash^P v : A \quad FO(A)}{\Delta\Gamma \vdash^L v : A}$

By I.H., $v \in \mathscr{V}[\![A]\!]^P$. Thus, $v \in RED_A[\underline{R}/\underline{X}]$.
$\Gamma \vDash_k \gamma$ when $x :^\theta A \in \Gamma$ implies $\rho x \in \mathscr{V}[\![A]\!]_k^\theta$.
Thus all substitutions on terms coming from P fragment, if well founded, will take them to reducible sets of their corresponding types in L fragment.

TINL $\dfrac{\Delta\Gamma \vdash^L a : A}{\Delta\Gamma \vdash^L inl\ a : A + B}$

By I.H., $a[\underline{V}/\underline{X}][\underline{u}/\underline{x}] \in RED_A[\underline{R}/\underline{X}]$.
So, $inl\ a[\underline{V}/\underline{X}][\underline{u}/\underline{x}] \in RED_A[\underline{R}/\underline{X}] \bigcup RED_B[\underline{R}/\underline{X}] = RED_{A+B}[\underline{R}/\underline{X}]$

Hence we can prove Strong Normalization in L fragment.