Let $T[\underline{X}]$ be a type where $\underline{X}$ contains all the free variables of T. Let $\underline{U}$ be a sequence of types of the same length; then we can define a type $T[\underline{U}/\underline{X}]$ by simultaneous substitution of types. Let $PAR\mathscr{C}[\![U_i]\!]_k^\theta$ and $PAR\mathscr{V}[\![U_i]\!]_k^\theta$ be the parametrised value and parametrised computational interpretations of the type U. Let $\underline{R}$ stand for the parametrised interpretations (P.I.) for the sequence of types. Then we define the set $PAR\mathscr{V}[\![T]\!]_k^\theta[\underline{R}/\underline{X}]$ and $PAR\mathscr{C}[\![T]\!]_k^\theta[\underline{R}/\underline{X}]$ for terms of type $T[\underline{U}/\underline{X}]$ as follows :

- $T = X_i$, then $PAR\mathscr{V}[\![T]\!]_k^\theta[\underline{R}/\underline{X}]= \mathscr{V}[\![U_i]\!]_k^\theta$
  and $PAR\mathscr{C}[\![T]\!]_k^\theta[\underline{R}/\underline{X}] = \mathscr{C}[\![U_i]\!]_k^\theta$

- $T = Unit$
  $PAR\mathscr{V}_k^\theta[\underline{R}/\underline{X}] = \{()\}$

- $T = A + B$
  $PAR\mathscr{V}[\![T]\!]_k^\theta[\underline{R}/\underline{X}] = \{inl\ v|v \in PAR\mathscr{V}[\![A]\!]_k^\theta[\underline{R}/\underline{X}]\} \bigcup \{inr\ v|v \in PAR\mathscr{V}[\![B]\!]_k^\theta[\underline{R}/\underline{X}]\}$

- $T = A@\theta'$
  $PAR\mathscr{V}[\![T]\!]_k^L[\underline{R}/\underline{X}] = \{box\ v|v \in PAR\mathscr{V}[\![A]\!]_k^{\theta'}[\underline{R}/\underline{X}]\}$

- $T = A^{\theta'} \rightarrow B$ then,
  $PAR\mathscr{V}[\![T]\!]_k^L[\underline{R}/\underline{X}] = \{rec\ f\ x.a|.\vdash^L rec\ f\ x.a : T[\underline{U}/\underline{X}]\ and\ \forall j \leq k,\ if\ v \in PAR\mathscr{V}[\![A]\!]_j^{\theta'}[\underline{R}/\underline{X}]\ then\ [v/x]a \in PAR\mathscr{C}[\![B]\!]_j^L[\underline{R}/\underline{X}]\}$
  $PAR\mathscr{V}_k^P[\![T]\!][\underline{R}/\underline{X}] = \{rec\ f\ x.a|.\vdash^P rec\ f\ x.a : T[\underline{U}/\underline{X}]\ and\ \forall j < k,\ if\ v \in PAR\mathscr{V}[\![A]\!]_j^{\theta'}[\underline{R}/\underline{X}]\ then\ [v/x][rec\ f\ x.a/f]a \in PAR\mathscr{C}[\![B]\!]_j^P[\underline{R}/\underline{X}]\}$

- $T = \mu\alpha.a$
  $PAR\mathscr{V}[\![T]\!]_k^P[\underline{R}/\underline{X}] = \{roll\ v|.\vdash^P roll\ v : \mu\alpha.a[\underline{U}/\underline{X}]\ and\ \forall j < k,\ v \in PAR\mathscr{V}[\![[\mu\alpha.a/\alpha]A]\!]_j^P[\underline{R}/\underline{X}]\}$
  $PAR\mathscr{V}[\![T]\!]_k^L[\underline{R}/\underline{X}] = \phi$

- $T = \forall\alpha.A$
  $PAR\mathscr{V}[\![\forall\alpha.a]\!]_k^\theta[\underline{R}/\underline{X}] = \{t|\forall V : type\ s.t.\ S\ is\ P.I.\ of\ V,\ tV \in PAR\mathscr{C}[\![A]\!]_k^\theta[\underline{R}/\underline{X}, S/Y]\}$

And parametrised computational interpretations are defined as :

$$PAR\mathscr{C}[\![T]\!]_k^P[\underline{R}/\underline{X}] = \{a|.\vdash^P a : T[\underline{U}/\underline{X}]\ and\ \forall j \leq k,\ if\ a \rightsquigarrow{}^j v\ then\ v \in PAR\mathscr{V}[\![T]\!]_{k-j}^P[\underline{R}/\underline{X}]\}$$

$$PAR\mathscr{C}[\![T]\!]_k^L[\underline{R}/\underline{X}] = \{a|.\vdash^P a : T[\underline{U}/\underline{X}]\ and\ a \rightsquigarrow{}^* v \in PAR\mathscr{V}[\![T]\!]_k^L[\underline{R}/\underline{X}]\}$$

Observe that Computational Interpretations in the L fragment satisfy all the CR conditions as given in proofs and types book :

**Lemma 1.** $\mathscr{C}[\![A]\!]_k^L$ is a reducibility candidate for a type A. for type A.
**Proof.** By definition of computational interpretation, $\mathscr{C}[\![A]\!]_k^L$ should satisfy conditions CR 1-3.

$$\mathscr{C}[\![A]\!]_k^L = \{a|.\vdash^L a : A \wedge a \rightsquigarrow{}^* v \in \mathscr{V}[\![A]\!]_k^L\}$$

- CR-1: by definition, terms in $\mathscr{C}[\![A]\!]_k^L$ reduce to a value

- CR-2: If $a \rightsquigarrow a'$ then $a' \in \mathscr{C}[\![A]\!]_k^L$ as $a' \rightsquigarrow {}^*v \in \mathscr{V}[\![A]\!]_k^L$
- CR-3: Similarly, if a is neutral, and whenever we convert a redex of $a$, we get a term $a' \in \mathscr{C}[\![A]\!]_k^L$, then it means, $\forall a',\ a \rightsquigarrow a' \rightsquigarrow {}^*v \in \mathscr{V}[\![A]\!]_k^L$, hence $a \in \mathscr{C}[\![A]\!]_k^L$.
  As we don't count computation steps in Logical fragment, k is constant in all cases above.
  Hence Parametric Interpretations of types in L fragment are actually reducibility candidates.
  We can use Lemmas for Substitution, universal abstraction and universal application as given in proofs and types book, section 14.2.

We now define well formed type and value substitutions :

**Well formed type substitutions :**

$$\mathcal{D}[\![.]\!] = \{\phi\}$$
$$\mathcal{D}[\![\Delta, \alpha]\!] = \{\rho[\alpha \to X] \mid \rho \in \mathcal{D}[\![\Delta]\!] \wedge FTV(X) \in dom(\rho)\}$$

**Well formed value substitutions :**

$$\mathcal{G}[\![.]\!] = \{\phi\}$$
$$\mathcal{G}[\![\Gamma, x : \tau]\!]_k^\rho = \{\gamma[x \to v] \mid \gamma \in \mathcal{G}[\![\Gamma]\!]_k^\rho \wedge v \in \mathscr{V}[\![\rho(\tau)]\!]_k\}$$

**New Soundness Theorem :** If $\Delta, \Gamma \vdash^\theta t : T$ and $\Gamma \vDash_k \rho$, $\Delta \vDash \gamma$ then, $\rho(\gamma(t)) \in PAR\mathscr{C}[\![T]\!]_k^\theta[\underline{R}/\underline{X}]$

**Proof :** By induction on typing derivation.

- T-UNIV-APP : $\dfrac{\Delta\Gamma \vdash^L e : \forall\alpha.A \quad \Delta \vdash B}{\Delta\Gamma \vdash^L e[B] : [B/\alpha]A}$

  By def.

$$PAR\mathscr{C}[\![T]\!]_k^L[\underline{R}/\underline{X}] = \{a |. \vdash a : T[\underline{V}/\underline{X}] \ and \ a \rightsquigarrow {}^*v \in PAR\mathscr{V}[\![T]\!]_k^L[\underline{R}/\underline{X}]\}$$

**(i) :** $\rho(\gamma(t)) : T[\underline{V}/\underline{X}]$ by appeal to the substitution lemma.
**(ii) :** for $T = \forall Y.A$ we need to prove that $\rho(\gamma(t)) \rightsquigarrow {}^*v \in PAR\mathscr{V}[\![T]\!]_k^L[\underline{R}/\underline{X}]$
Or, from the definition of $PAR\mathscr{V}[\![T]\!]_k^L[\underline{R}/\underline{X}]$, $\forall V : type\ s.t.\ S\ is\ C.R.\ of\ V, vV \in PAR\mathscr{V}[\![T]\!]_k^L[\underline{R}/\underline{X}, S/Y]$

From Induction Hypothesis, on T-Univ-App :
$\Delta\Gamma \vdash^L t : \forall Y.A$
So, $\rho(\gamma(t)) \in PAR\mathscr{C}[\![\forall Y.A]\!]_k^L[\underline{R}/\underline{X}]$
$\Rightarrow \wedge Y.\rho(\gamma(e)) \in PAR\mathscr{V}[\![\forall Y.A]\!]_k^L[\underline{R}/\underline{X}]$
then from definition, $\forall V : type$ and its C.R. $S$,
$\rho(\gamma[Y \to V](e)) \in PAR\mathscr{V}[\![\forall Y.A]\!]_k^L[\underline{R}/\underline{X}, S/Y]$
proved.