All of the names, notations and lemmas used here are taken from “Proofs and Types” book and lemmas and theorems will be used here without their proofs.

First we prove a result that the computational interpretation of types in L fragment as defined in Casinghino et al's paper ”Step-Indexed Normalization for a Language with General Recursion” are actually reducibility candidates of types they are defined for.

Lemma 1. $\mathscr{C}[\![A]\!]_k^L$ is a reducibility candidate for type A.
Proof. By definition of computational interpretation, $\mathscr{C}[\![A]\!]_k^L$ should satisfy conditions CR 1-3.

$$\mathscr{C}[\![A]\!]_k^L = \{a|.\vdash^L a : A \wedge a \rightsquigarrow {}^*v \in \mathscr{V}[\![A]\!]_k^L\}$$

- CR-1: by definition, terms in $\mathscr{C}[\![A]\!]_k^L$ reduce to a value
- CR-2: If $a \rightsquigarrow a'$ then $a' \in \mathscr{C}[\![A]\!]_k^L$ as $a' \rightsquigarrow {}^*v \in \mathscr{V}[\![A]\!]_k^L$
- CR-3: Similarly, if a is neutral, and whenever we convert a redex of a, we get a term $a' \in \mathscr{C}[\![A]\!]_k^L$, then it means, $\forall a',\ a \rightsquigarrow a' \rightsquigarrow {}^*v \in \mathscr{V}[\![A]\!]_k^L$, hence $a \in \mathscr{C}[\![A]\!]_k^L$.
  As we don't count computation steps in Logical fragment, k is constant in all cases above.

Parametric Reducibility and its lemmas are defined as in book “Proofs and Types”, section 14.

We define ρ, the mapping of variables to terms :
Well formed value substitutions :

$$\mathcal{G}[\![.]\!] = \{\phi\}$$

$$\mathcal{G}[\![\Gamma, x : \tau]\!]_k = \{\rho[x \to v] \mid \rho \in \mathcal{G}[\![\Gamma]\!]_k \wedge v \in \mathscr{V}[\![\tau]\!]_k\}$$

Theorem 1. Soundness : if $\Gamma \vdash^\theta a : A$ and $\Gamma \vDash_k \rho$ then

- If a has no type variables or in P fragment, $\rho a \in \mathscr{C}[\![A]\!]_k^\theta$
- If a has type variables and is in L fragment then,
  Suppose all the free variables of a are among $x_1, ..., x_n$ of types $U_1, ..., U_n$, and all the free type variables of $A, U_1, ..., U_n$ are among $X_1, ..., X_m$. If $\mathscr{C}[\![V_1]\!]_k^L, ..., \mathscr{C}[\![V_m]\!]_k^L$ are the reducibility candidates $R_1, ..., R_m$ of types $V_1, ..., V_m$ and $u_1, ..., u_n$ are terms of types $U_1[\underline{V}/\underline{X}], ..., Un[\underline{V}/\underline{X}]$ which are in $RED_{U_1}[\underline{R}/\underline{X}], ..., RED_{U_n}[\underline{R}/\underline{X}]$ then $a[\underline{V}/\underline{X}][\underline{u}/\underline{x}] \in RED_T[\underline{R}/\underline{X}]$.

Proof: By induction on typing derivations :

T-UNIV-LAM-L $\dfrac{\Delta, \alpha, \Gamma \vdash^L e : B}{\Delta\Gamma \vdash^L \Lambda\alpha.e : \forall\alpha.B}$

By I.H., $\forall\nu : Type,\ e[\underline{V}/\underline{X}][\nu/\alpha][\underline{u}/\underline{x}] \in RED_B[\underline{R}/\underline{X}][\mathscr{C}[\![\nu]\!]^L_k/\alpha]$ From Lemma 14.2.2, $\rho(\wedge\alpha.e[\underline{V}/\underline{X}]) \in RED_{\forall\alpha.B}[\underline{R}/\underline{X}]$

T-UNIV-APP-L $\dfrac{\Delta\Gamma \vdash^L e : \forall\alpha.A \quad \Delta \vdash B}{\Delta\Gamma \vdash^L e[B] : [B/\alpha]A}$

By I.H., $e[\underline{V}/\underline{X}][\underline{u}/\underline{x}] \in RED_{\forall\alpha.A}[\underline{R}/\underline{X}]$.
From Lemma 14.2.3, $\forall B : type, (e[B][\underline{V}/\underline{X}][\underline{u}/\underline{x}]) \in RED_{[B/\alpha]A}[\underline{R}/\underline{X}]$

In the P fragment, the value interpretation for universal type is defined as :

$$\mathscr{V}[\![\forall\alpha.A]\!]^P_k = \{\wedge\alpha.e|. \vdash^P \wedge\alpha.e : \forall\alpha.A \ \wedge \forall T : type, \forall j < k \ s.t. \Delta \vdash T, [T/\alpha]e \in \mathscr{C}[\![[T/\alpha]A]\!]^P_j\}$$

T-UNIV-LAM-P :
Proof: From the definition of $\mathscr{C}[\![\forall\alpha.B]\!]^P_k$, we need to prove

- $. \vdash^P \rho(\wedge\alpha.e) : \forall\alpha.B$ - by appeal to the substitution lemma

- $\rho(\wedge\alpha.e) = \wedge\alpha.(\rho(e))$.
  From definition of $\mathscr{C}[\![\forall\alpha.A]\!]^P_k$, if $\forall T : type,\ \rho([T/\alpha]e) \rightsquigarrow^j v$ for all $j < k$, then it suffices to show that
  $v \in \mathscr{V}[\![[T/\alpha]B]\!]^P_{k-j}$

  By Induction Hypothesis,

  $$\Delta, \alpha, \Gamma \vdash^P e : B$$

  $$\forall T : type, \rho([T/\alpha]e) \in \mathscr{C}[\![[T/\alpha]B]\!]^P_k$$

  Now, we know that $(\rho([T/\alpha]e)) \rightsquigarrow^j v$ and from definition of $\mathscr{C}[\![[T/\alpha]B]\!]^P_k$, we have $v \in \mathscr{V}[\![([T/\alpha]B]\!]^P_{k-j}$.
  Hence we get that $v \in \mathscr{V}[\![[T/\alpha]B]\!]^P_{k-j}$.