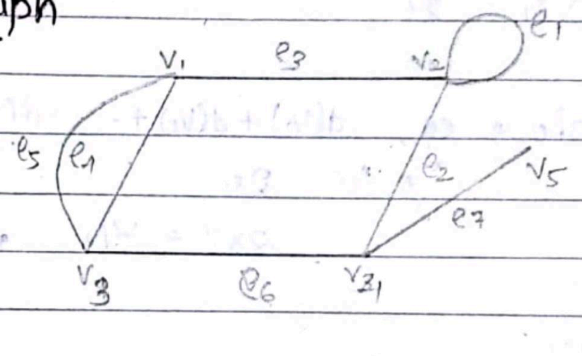
GRAPH THEORY

■ Simple Graph



* **Incidence and Degree**

When a vertex v_i is an end vertex of some edge e_j, v_i and e_j are said to be incident on each other.

Eg→ In this fig. edges e_2, e_6 and e_7 are incident on vertex v_4. Two non-parallel edges are said to be adjacent if they are incident on a common vertex. In this fig. e_2 and e_7 are adjacent. Similarly, two vertices are said to be adjacent if they are the end vertices of the same edge. In this fig. v_4 and v_5 are adjacent.

The number of edges incident on a vertex v_i with self loops counted twice is called degree - $d(v_i)$. $d(v_i)$ is the degree of v_i. from the fig,

$d(v_1) = 3$
$d(v_2) = 4$
$d(v_3) = 3$
$d(v_4) = 3$
$d(v_5) = 1$

The degree of vertex is sometimes also referred ~~as~~ to as its valency. Let us know concider a graph G. ~~where~~ with e edges and n vertices i.e. $v_1, v_2, v_3, \ldots, v_n$.

Since, each edge contributes to 2 degrees. The sum of the degrees of all vertices in graph G i.e.

$$\sum_{i=1}^{n} d(v_i) = 2e$$

In the above eg, $d(v_1) + d(v_2) + \dots + d(v_5) = 14$

$\sum v_i = 2e$

$= 2 \times 7 = 14.$ where $e = 7$.

* Theorem 1.1

The number of vertices of odd degree in a given graph is always even.

Proof → If we consider the vertices with odd and even degrees seperately then

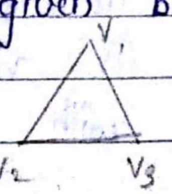
$$\sum_{i=1}^{n} d(v_i) = \sum_{even} d(v_j) + \sum_{odd} d(v_k)$$

Since, the LHS is even and the first expression on the RHS is even, the second expression must also be even. Therefore, $\sum_{odd} d(v_k)$ = an even numbers. Each individual $d(v_k)$ is odd, but the total number of terms in the sum must be even to make the sum an even number. Hence, the theorem is proved.

- Regular Graph

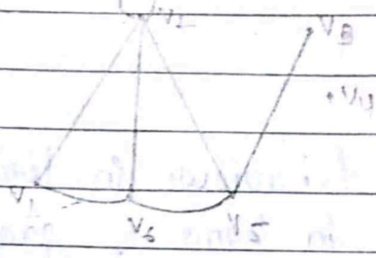
A graph in which all vertices are of equal degree, called a regular graph. In the fig. given below

$d(v_1) = d(v_2) = d(v_3)$



* Isolated Vertex

A vertex having no incident edge is an isolated vertex. In other words, isolated vertex are vertex with degree zero. In the fig. given, below, V_4 is isolated vertex.

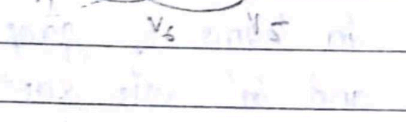


* Pendent Vertex

A vertex with degree one is called pendent vertex.
In the fig. given above, V_3 is the pendent ~~g~~ vertex.

■ Null Graph

In the defignition of a graph g, it is possible for the edge said e to be empty. Such a graph without any edges is called a null graph.



• Every vertex in a null graph is an isolated vertex and it is of degree zero.

• ~~All the~~ Although the edge said e may be empty but vertex said v must not be empty otherwise there is no graph.
In other words, by definition a graph must have alleast one vertex.

* Isomorphism

Two figures are considered to be equivalent if they

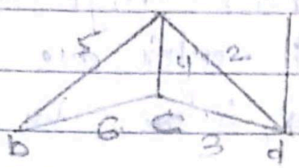


fig a G1

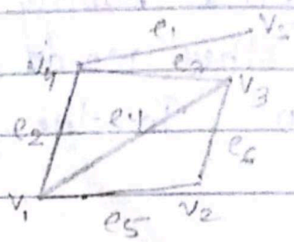


fig b G1'

have identical behaviour in terms of geometrical properties ie. in terms of graph theoretic properties. Two graphs G1 and G1' are said to be isomorphic if there edge is 1:1 correspondance ~~it means~~ b/w there vertices and b/w there edges such that the incidence relationship is preserved.

It is imidiately apparent by definition of isomorphism that two isomorphic must have the follow

1. Same of no. of vertices
2. The same no. of edges.
3. An equal no. of vertices with the given degree.
4. Same no. of connected components.
5. Same no. of cycles of the same length

In the given graphs above, there are three vertices of degree 3 in each graph ie.

G1'	G1
$d(v_4) = 3$	$d(a) = 3$
$d(v_1) = 3$	$d(c) = 3$
$d(v_3) = 3$	$d(d) = 3$

and they have one vertex of degree 2 in each graph

ie,
G' $\quad$ G

$d(v_2) = 2$ $\quad$ $d(b) = 2$

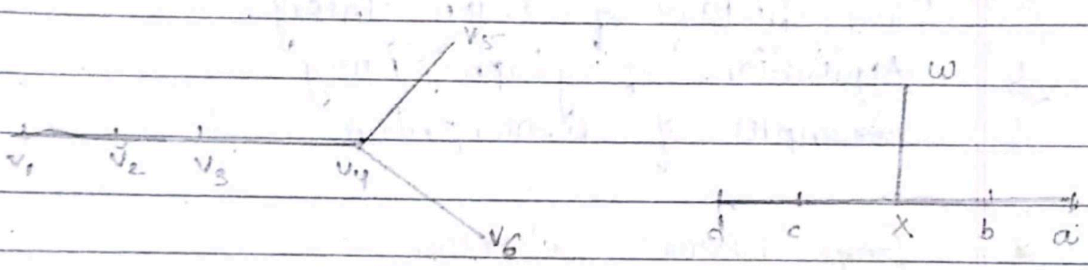
and have one vertex with degree 1 in each ie.

G' $\quad$ G

$d(v_5) = 1$ $\quad$ $d(e) = 1$

The graph G and G' have equal number of edges ie. 6 and equal number of vertices ie. 5. And hence, in the graph G and G' ~~are~~, the incidence relationship is preserved. Hence, the above graph are isomorphic in nature.

Q



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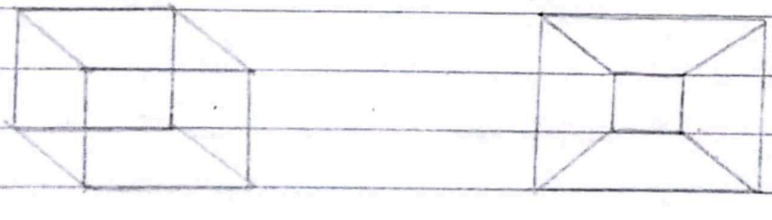
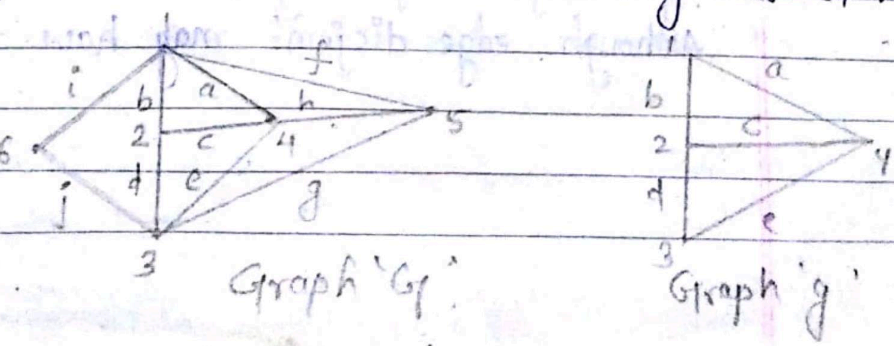


Sub-graph

A graph 'g' is said to be sub graph of 'G' if all the vertices and all the edge of 'g' are in 'G'. and each edge of 'g' has the same end vertices in 'g' as 'G'.

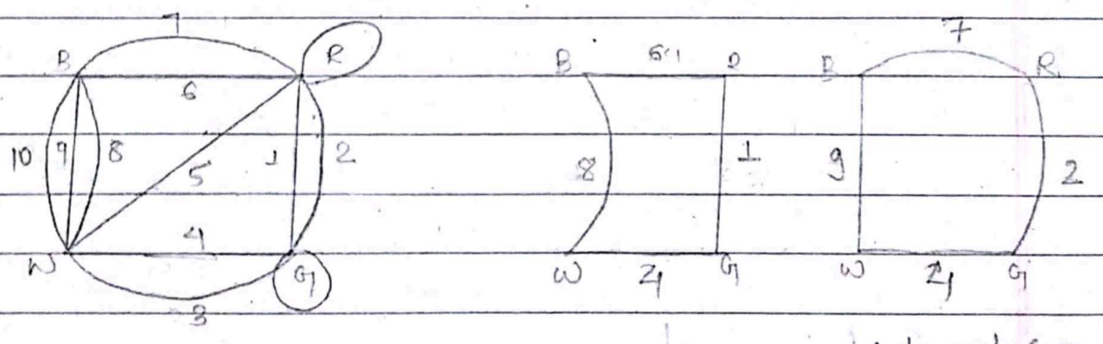


Graph 'G' $\qquad$ Graph 'g'

The following observation we can make immidiately -
1. Every graph is its own subgraph.
2. A subgraph of 'g' is also a subgraph 'G'.
3. A single vertex in a graph 'G' is a subgraph of 'G'.
4. A single edge in 'G' together with its end vertices is also a subgraph of 'G'.
The concept of subgraph in graph theory is similar to the concept of subset in set theory.

Q Brief history of graph theory.
Q. Applications of graph theory
Q Examples of isomorphism

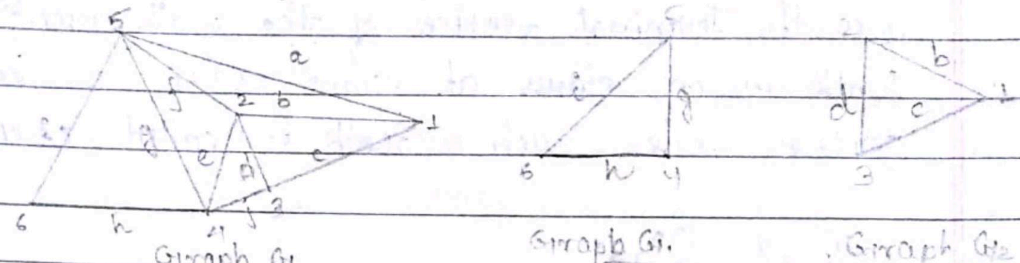
▪ Edge Disjoint Subgraph



graph G   Subgraph G2   Subgraph G1

Two subgraphs G_1 and G_2 of a graph G are said to be edge disjoint if they donot have any edges in common. Although edge disjoint may have vertices in common.

Vertex Disjoint Subgraph

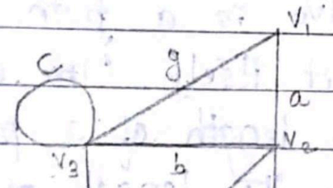


Graph G   Graph G1   Graph G2

Subgraphs that donot have vertices or vertex in common are said to be vertex disjoint subgraphs.

- Obviously graphs that do not have vertices in common cannot have edges in common as well.

WALK, PATHS And CIRCUITS



A walk is defined as a finite alternating sequences of vertices and edges ~~b~~ beginning and ending with vertices such that each edge is incident with the vertices preceed-ing and following it. No edge appears more than once in a walk but a vertex may appear more than once. for instance, $v_1 a v_2 b v_3 c v_3 d v_4 e v_2 f v_5$. A vertex however appeared more than once. A walk is also referred to a a edgetrain or a chain. The set of vertices and edges constituting a given walk ~~to~~ in a graph G is clearly a subgraph of G. vertices with which a walk begins and ends are called its terminal vertices. v_1 and v_5 are the terminal vertices of the walk mentioned above. Walk which starts at some vertex and ends at some another vertex such a walk is called open walk.

Closed Walk

It is possible for a walk to begin and end at the same vertex such a walk is called a closed walk. $v_1 a v_2 b v_3 g v_1$ is example of closed walk in the given above fig.

Path

An open walk in which no vertex appears more than once is called a path. $v_1 a v_2 b v_3 d v_4$ is a path. In other words, path doesnot intersect itself. The number of edges in a path is called the length of a path.
For eg. - $v_1 a v_2 b v_3 d v_4$ has 3 as its length of the path.
A self loop can be a part of a walk but not of a path. The terminal vertices of a path are of degree 1 and the rest of the vertices of degree 2. This degree is ofcourse counted only with respect to the edges included in the path and not the entire graph in the path may be contained.

A closed walk in which no vertex (except the initial and the final vertex) appears more than once is called a circuit. for eg. - $v_1 a v_2 b v_3 g v_1$ is a circuit. Every vertex in a circuit is of degree 2. A circuit is also called a cycle, elementary cycle, circle of path, and polygon. In Electrical engineering a circuit is sometimes referred to as a loop not to be confused as a self-loop.

Subgraph of G — Any connection of edges in G

↓

Walk in G — A non-edge retracing sequence of edges of G.

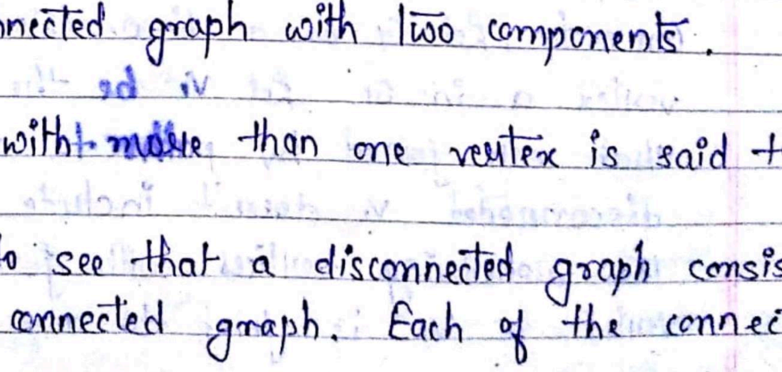
↓

Path of G — A non-intersecting open walk in G

Circuit in G — A non-intersecting closed-walk in G

Connected graphs, Disconnected graphs and Components

A graph G is said to be connected if there is atleast one path beth every pair of vertices in G. Otherwise, G is disconnected.



a disconnected graph with two components.

→ 1. A null graph with more than one vertex is said to be disconnected.

2. It is easy to see that a disconnected graph consists of two or more connected graph. Each of the connected

subgraph is called the component. The graph given above consists of two components which are individually connected. Another way of looking at the component →
(a) consider a vertex v_i in a disconnected graph G.
By definition not all vertices of G are joined by paths to ~~be~~ v_i. Vertex v_i and all the vertices of G that have paths to v_i together with all the edges incident on them ~~from~~ form a component.

Theorem 2.1

A graph G is disconnected if and only if its vertex ~~said~~ set V can be partitioned into two non-empty disjoint subsets V_1 & V_2 such that there exists no edge in G whose one end vertex is in subset V_1 and the other ~~ed~~ is in subset V_2.

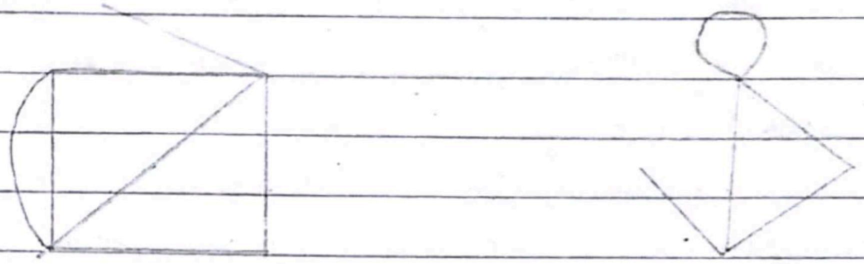
Proof →
Suppose that such ~~that~~ a partition exist consist two arbitrary vertices a and b such that $a \in V_1$ and $b \in V_2$.
No path can exist b/w vertices a and b otherwise, there would be atleast one edge whose one end vertex would be in V_1 and the other in V_2. Hence, if a partition exists, G is not connected.

Conversly, let G be a disconnected graph. Consider a vertex a in G. Let V_1 be the set of all vertices that are joined by paths to a. Since, G is disconnected V_1 doesn't include all vertices of G. The remaining vertices will form a set V_2. No vertex in V_1 is joint to any in V_2 by an edge.

hence partitioned.

Theorem 2.2

If a graph (connected or disconnected) has exactly two vertices of odd degree there must be path joining these two vertices.

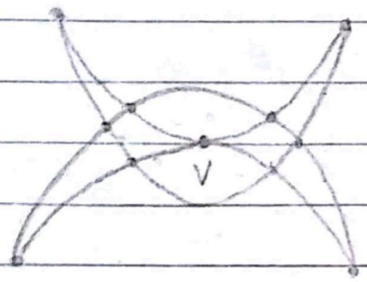


Proof → Let G be a graph with all even vertices except v_1 and v_2 which are odd. From theorem 1.1. which holds for every graph and therefore, every component of disconnected graph, no graph can have odd no. of vertices with odd degrees. Therefore, in graph G v_1 and v_2 must belong to the same component and hence there must be a path between them.

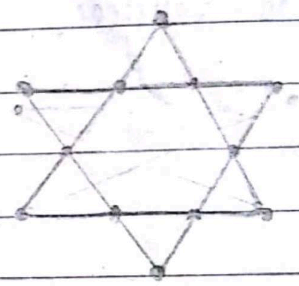
Euler Graph

~~A~~ If some closed walk in a graph contains all the edges of the graph then the walk is called an euler line and the graph is called the Euler graph

By definition a walk is always connected since the Euler line which is a walk contains all the edges of the graph. An Euler graph is always connected except for any isolated vertex, the graph may have since an isolated vertex donot contributed to an understanding to Euler graph. It is here after assume that an Euler graph do not have any isolated vertex and are therefore connected.



Mohd. Scimitars



star of David

- A given graph G is an Euler graph iff all the vertices of G are in even degree.

Theorem 2.4

A given connected graph G is an Eular graph iff all the vertices in G are of even degree.

Proof → Suppose that G is an Eular graph, it therefore contains an eular line. In tracing this walk we observe that everytime the walk ~~need~~ meets the vertex v it goes through two new edges incident on v with one ~~is~~ we entered v and the other be exited.

This is true not only ~~for~~ of all the intermediate vertices of the walk but also ~~for~~ the terminal vertices. Because the exited and entered the same vertex at the beginning and the end of the walk respectively. Thus if G is an Euler graph, the degree of every vertex is even.

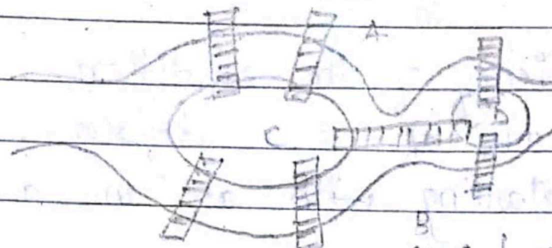
To prove the sufficiency of the condition, assume that all vertices of G are of even degree. Now we construct a walk starting ~~with~~ at an arbitrary constant ~~G~~.v (vertex).

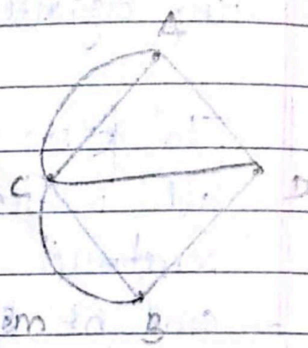
If a vertex v and traced going through the edge of G such that no edges is touched more than once. Since, every vertex is of even degree. We can exit from every vertex we entered tracing cannot stop at any vertex but vertex v. and hence v is also of even degree. In this closed walk, edges we traced includes all edges of G, G is an Euler graph.

Konigsberg Bridge Problem.

The konigsberg Bridge Problem is perhaps the best known eg. in graph theory. It was a long-standing problem until solved by Leonhard Euler (1707-1783) in 1736, by means of a graph. Euler wrote the first paper ever in graph theory and thus became the originator of the theory of graphs as well as of the rest of topology.

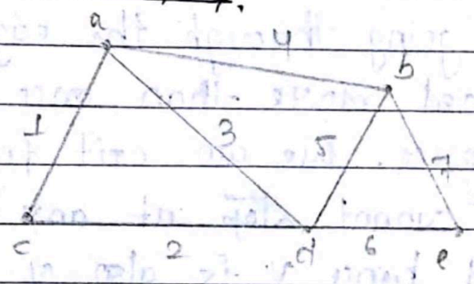
Two islands, C & D, formed by the Pregel River in konigsberg were connected to each other and to the banks A & B with seven bridges. The problem was to start at any of the four land areas of the city, A, B, C or D, ~~with~~ walk over each of the seven bridges exactly once, & return to the starting point. Euler represented this situation by means of a graph. The vertices represent the land areas & the edges represent the bridges.





Konigsberg Bridge Problem

- <u>Unicursal Graph</u>



The walk a1c2d3a4b5d6e7b which includes all the edges of the graph and ~~all~~ doesnot retrace any edge is not the closed walk. The initial vertex is a and the final is b we call such an open walk that includes all the edges of graph without retracing any edge a unicursal line or an open Euler line. A graph that has unicursal line is called a Unicursal graph. It is clear that by adding an edge b/w the initial & the final vertices of an unicursal line we get an Euler line. Thus, a connected graph is Unicursal iff it has exactly two vertices of odd degree.

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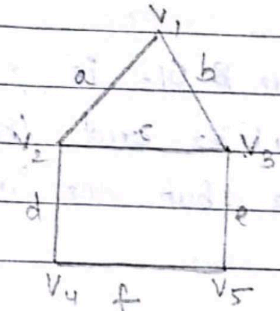
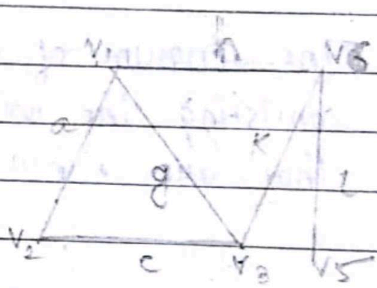
* Operations On Graph

Since, graphs are defined in terms of sets of edges and vertices. it is natural to apply set theoretical term and to define operations b/w graphs.

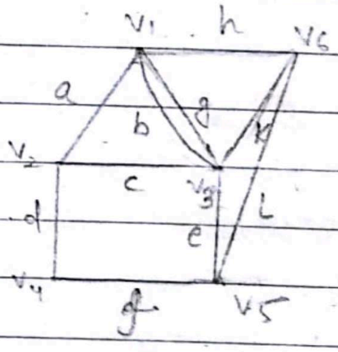
1. Union of two graphs.



Graph G_1.



Graph G_2



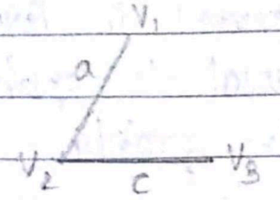
Graph G_3

The union of 2 graphs G_1 & G_2 is another graph G_3 whose vertex said V_3 is equal to $V_1 \cup V_2$ i.e. $V_3 = V_1 \cup V_2$ and the edge said e_3 is equal to $e_1 \cup e_2$ i.e. $e_3 = e_1 \cup e_2$.

2. Intersection of two graphs.

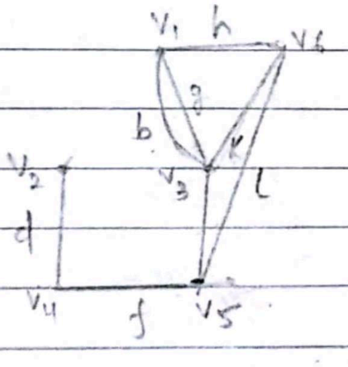
Intersection of two graphs G_1 and G_2 is written as G_4

G_4 is consist of vertices & edges that are in both G_1 and G_2.



3. Ringsum ($\oplus$) of two graphs.

The ringsum of two graphs G_1 & G_2 is a graph consisting the vertex said $V_1 \cup V_2$ and the edges that are either in G_1 or G_2. but not in both.



Graph $G_1 \oplus G_2$

Properties Of Operations On Graph

1) The three operations are commutated ie.
$G_1 \cup G_2 = G_2 \cup G_1$, $G_1 \cap G_2 = G_2 \cap G_1$ & $G_1 \oplus G_2 = G_2 \oplus G_1$

2) If G_1 and G_2 are edge disjoint then $G_1 \cap G_2$ is a null graph and $G_1 \oplus G_2 = G_1 \cup G_2$.

3) If G_1 and G_2 are vertex disjoint then $G_1 \cap G_2$ is empty

4) For any graph G, $G \cup G = G \cap G = G$ and $G \oplus G$ = null graph.

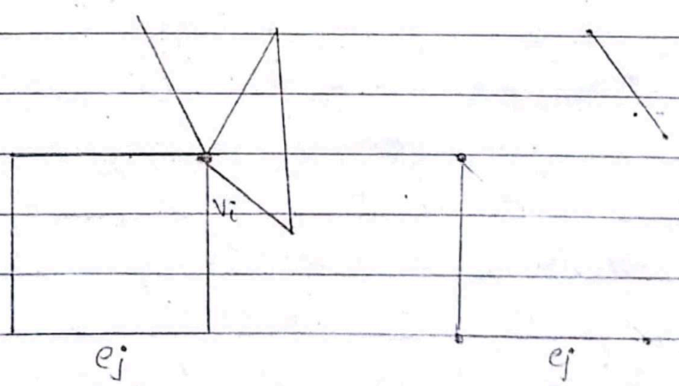
5) If g is a subgraph of G then $G \oplus g$ is by definition subgraph of g. which remains after all the edges in g have been removed from G. Therefore, $G \oplus g$ can be written as $G - g$. Whenever, $g \subseteq G$ because of this complementary nature $G \oplus g = G - g$ is often called the complement of g in G.

4. Decomposition.

→ The graph G is said to have decomposed ~~into~~ if two subgraphs g_1 and g_2 if $g_1 \cup g_2 = G$ and $g_1 \cap g_2$ = null graph. In other words, every edge of G occurs either in g_1 or in g_2 but not in both. But some of the vertices may occurs in g_1 and g_2. In decomposition isolated vertices are disregarded.

Deletion

→ ~~The word~~ Vertex Deletion

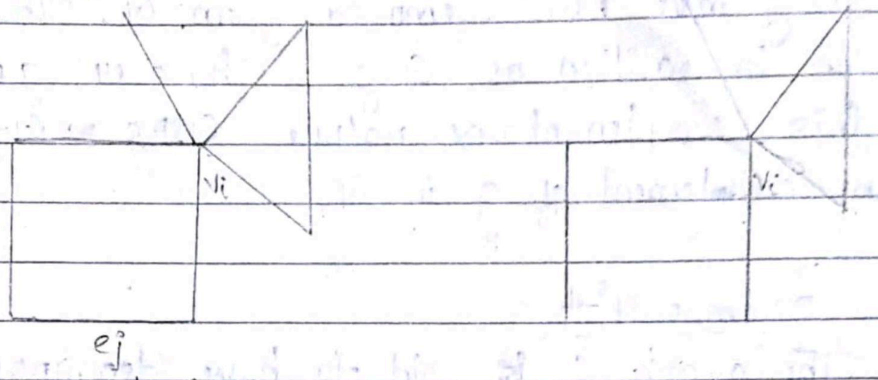


Graph G.

vertex deletion

If v_i is a vertex in graph G then $G - v_i$ denotes a subgraph of G obtain by deleting v_i from G. Deletion of a vertex always implies the deletion of all edges incident on that vertex.

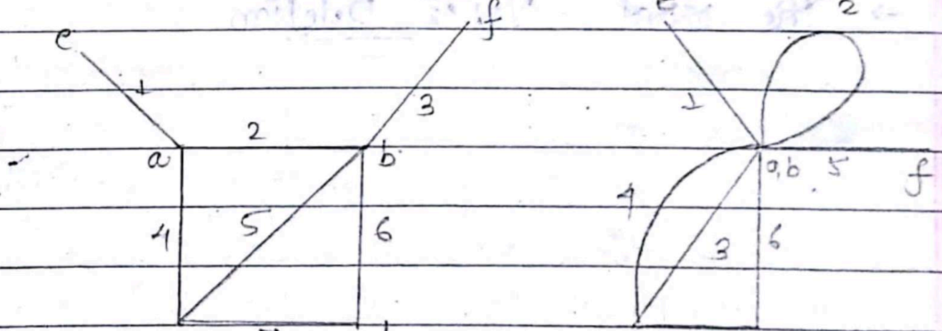
Edge Deletion



Graph G.

If e_j is an edge in graph G then $G-e_j$ is a subgraph of G obtained by deleting e_j from G. Deletion of an edge does not implies deletion of its end vertices. Therefore, $G-e_j = G \oplus e_j$

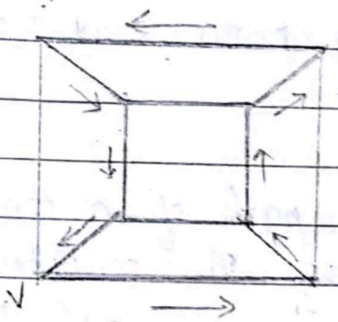
Fusion



Graph G

A pair of vertices a and b in graph G are said to be fused (merged or identified) if the two vertices are replaced by a single new vertex such that every edge that was incident on either a or b or on both is incident on the new vertex. The fusion of two vertices doesnot alter the no. of edges but it reduces the no. of vertices by one.

Hamiltonian paths and Circuits



A Euler line of a connected graph was characterised by the property being a closed walk that traverses every edge of the graph exactly once. A Hamiltonian circuit to a connected graph is defined as a closed walk that traverses every vertex of G exactly once except ofcourse the starting vertex at which the walk also terminates. for eg → when the graph given above, starting at vertex v, if one traverses along the edges as shown & heavy line passing through each vertex exactly once, one gets the Hamiltonian circuit.

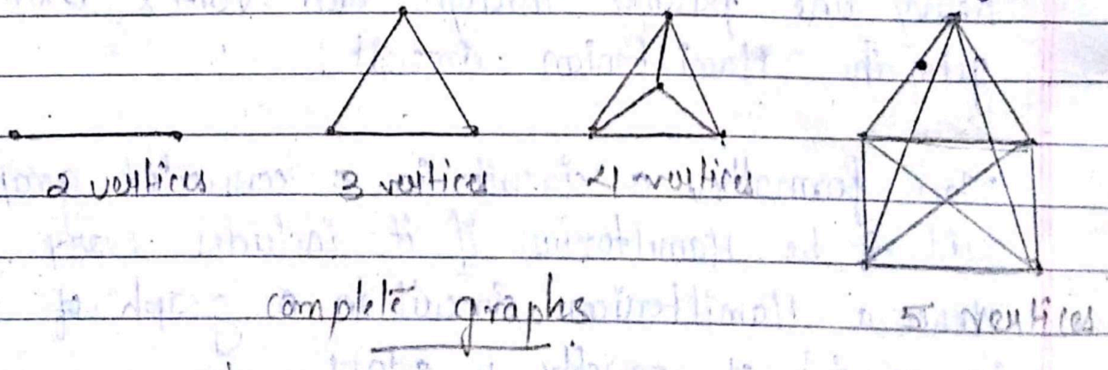
More formally, a circuit in a connected graph G is said to be Hamiltonian if it includes every vertex of G. Hence a Hamiltonian circuit in a graph of n vertices is consist of exactly n edges.

If we remove any one edge from a Hamiltonian circuit, we will get a path which is called Hamiltonian path. which clearly traverses every vertex of G. Since Hamiltonian path is a subgraph of Hamiltonian circuit (which in turn is a subgraph of another graph). Every graph that has a Hamiltonian circuit will have a Hamiltonian path. However, many graph with Hamiltonian paths may not have Hamiltonian circuit.

The length of a Hamiltonian path of a connected graph of n vertices are n-1 edges. In considering the existance of Hamiltonian circuit, we need to consider only simple graphs this is because a Hamiltonian circuit or Hamiltonian path traverses every vertices exactly once. Hence, it cannot include set of self loop or set of parallel edges.

Complete Graph

A simple graph in which there exists an edge b/w every pair of vertices is called a complete graph.



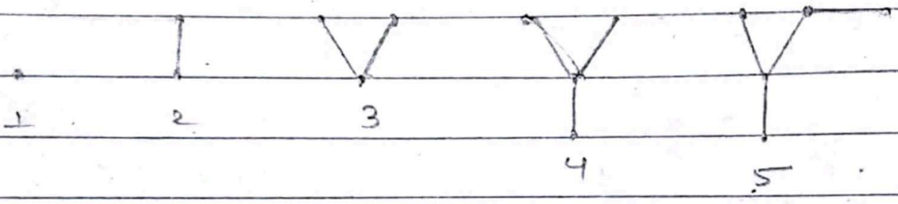
complete graphs

A complete graph is also referred to as a universal graph or a clique. Every vertex is referred as universal

Every vertex is joined to every other vertex through one edge. The degree of every vertex is $n-1$ in a complete graph of n vertices also the total no. of edges in G is $\frac{n(n-1)}{2}$.

TREES

A tree is a connected graph without any circuits. It follows from the definition that tree has to be a simple graph having neither a self loop nor parallel edges (because they both form circuits)



Trees with vertices

Properties of Trees

1. There is one and only one edge path b/w every pair of vertices in tree T.
2. If in a graph G there is one and only one path between every pair of vertices then G is a tree.
3. A tree with ~~and~~ n vertices has $n-1$ edges.
4. Any connected graph with n vertices and $n-1$ edges is a tree.
5. A graph is a tree if and only if it is minimally connected.
6. A graph with n vertices, $n-1$ edges and no circuits is connected.

ie. A graph G with n vertices is called a tree if

(i) G is connected and circuitless.

(ii) G is connected and has n-1 edges.
(iii) G is circuitless and has n-1 edges.
(iv) There is exactly one path b/w every pair of vertices in G.
(v) G is minimally connected.

Distance and Centre in a Tree

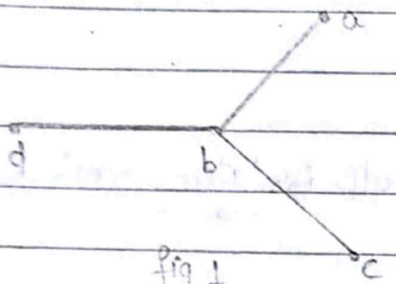


fig 1

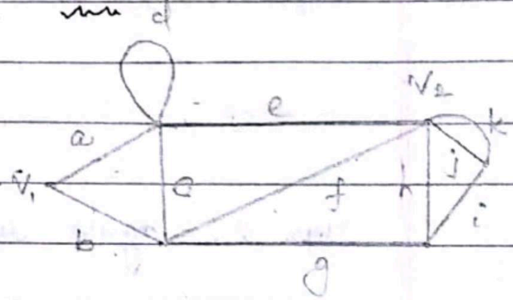
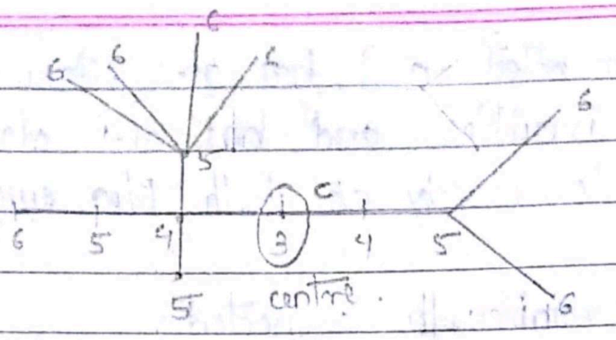


fig 2

In a connected graph G, the distance $d(v_i, v_j)$ b/w two vertices v_i and v_j is the length of the shortest path (i.e. the no. of edges in the shortest path) b/w them. In fig. 1 intutively it seems that vertex b is located more centrally than any of the three vertices. So, b can be said as the centre of the tree.

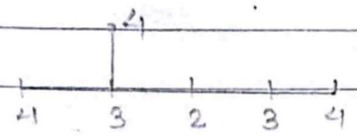
The definition of distance b/w any two vertices ~~an~~ is valid for any connected graph not necessarily for a tree. In a graph i.e. not a tree there are generally several paths b/w a pair of vertices. We have to enumerate all those paths and find the length of the shortest one.

In fig. 2 the distance b/w v_1 and v_2 is two.
In fig 1 $d(b,a)=1$, $d(b,c)=1$, $d(d,b)=1$, $d(d,a)=2$ & $d(d,c)=2$
Hence, it is clear that b is more centrally located.



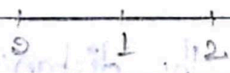
Graph T

Step 1, Removal of all ~~isolated~~ pendant vertices.



Graph T'

step 2. again removal of all ~~isolated~~ pendant vertices



graph T''

step 3, again removal of all ~~isolated~~ pendant vertices.

Graph T'''

A vertex with minimum eccentricity in a graph G is called a centre of G. The eccentricity E(v) of a vertex v in a given graph G is the distance from v to the vertex farthest from v in G. ie.

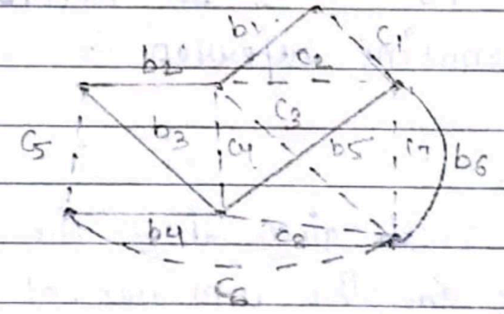
$$E(v) = \max_{v_i \in G} d(v, v_i)$$

The tree that has two vertices with having the same min.m eccentricity has two centre hence, these centres are called bicentres.

The distance b/w two spanning tree T_i and T_j of a graph G is defined as the no. of edges of G present in one tree but not in the other.

SPANNING TREE

A tree T is said to be spanning tree of a connected graph G if T is a subgraph of G and T contains all the vertices of G. Since, the vertices of G are barely hanging together in a spanning tree as a skeleton of the original graph G that's why a spanning tree is referred to as skeleton or a scaffolding. Since, spanning trees are the largest (with the max. no. of edges) trees among all trees in G it is also quiet appropriate to call a spanning tree. A maximal tree subgraph or maximal tree of G.



It is noted that a spanning tree is defined only for a connected graph of n-vertices, we cannot find a connected subgraph with n-vertices.

Each ~~compliment~~ component (by definition is connected) of a disconnected graph however that have a spanning tree. Thus, a disconnected graph with K spanning trees has a spanning forest consisting of K spanning trees. A collection of trees is called forest. finding a spanning tree of a connected graph G is simple. G has no circuit it is its own spanning tree. If G have a circuit delete an edge from the circuit this will still leave the graph connected. If there are more circuits, repeat the operation till an edge from the last circuit is deleted giving a connected circuit free graph that contains all the vertices of G.

Theorem

Every connected graph has atleast one spanning tree

Proof

An edge in a spanning tree T is called a branch of T. An edge of G that is not in a given spanning tree T is called a chord. In electroical engineering a chord is sometime referred to as a tie or a link.

for the eg., the ~~fig~~ graph given above the edges b_1, b_2, b_3, b_4, b_5 and b_6 are the branches of the ~~graph~~ spanning tree while the edges $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ and c_7 are the chords. It must be kept in mind that branches and chords are defined w.r.t the given particular spanning tree. An edge ie. a branch of a spanning tree T_1 in a graph G may be a chord w.r.to another spanning tree T_2. It is sometimes convenient to consider a connected graph G as a union of two subgraph T and T' ie $T \cup T' = G$. The T is the spanning tree and T' is the complement of T in G. Since the subgraph T' is the collection of chord it is quite appropriate to referred ~~it~~ to as chord set of T.

Such a circuit, formed by adding a chord to a spanning tree, is called a fundamental circuit

Theorem
With respect to any spanning tree, a connected graph of n vertices and e edges has $(n-1)$ tree branches and $(e-n+1)$ chords.

Proof. In the fig. given above, the no. of vertices n is equal to 7 and no. of edges is equal to 14. With respect to a particular spanning ~~gra~~ tree given in above graph, it has six branches ($b_1, b_2, b_3, b_4, b_5, b_6$) and 8 chords ($c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8$). No. of branches in the above spanning above is equal to $n-1$ ie. $7-1=6$.
No. of chords $= e-n+1$ ie. $14-7+1 = 8$. Hence proved.

Rank and nullity
→ When someone specifies a graph G, the first thing is most likely to be mentioned is n (no. of vertices in G). Immediately following thing comes e (no. of edges in G), then comes k (no. of components in G).

If $k=1$, G is connected. Since, every component of the graph must have atleast one vertex ie. $n \geq k$. Moreover, the no. of edges in component can be no less than $e \geq n-k$. Apart from the constraints $n-k \geq 0$ and $e-n+k \geq 0$. These three no.s e, n and k are independent and they are fundamental no.s in graph. From These 3 no.s mentioned above, we derived two more important no.s ie rank (r) and nullity (μ). ie $r = n-k$ and $\mu = e-n+k$. The rank of a connected graph is $n-1$ and the nullity is $e-n+1$.
cyclomatic no. and first Betti no.

The generation of one spanning tree from another, through addition of a chord and deletion of an appropriate branch is called a cyclic interchange or elementary tree transformation.

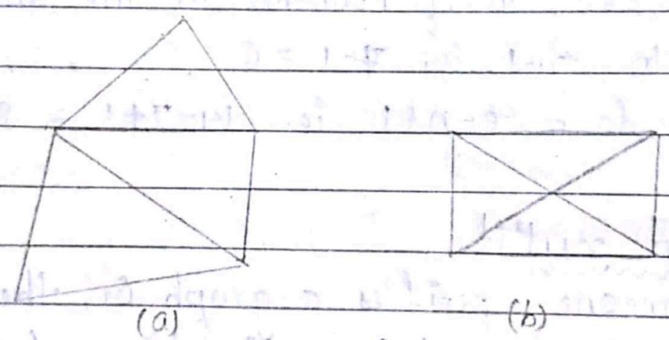
Date
Page

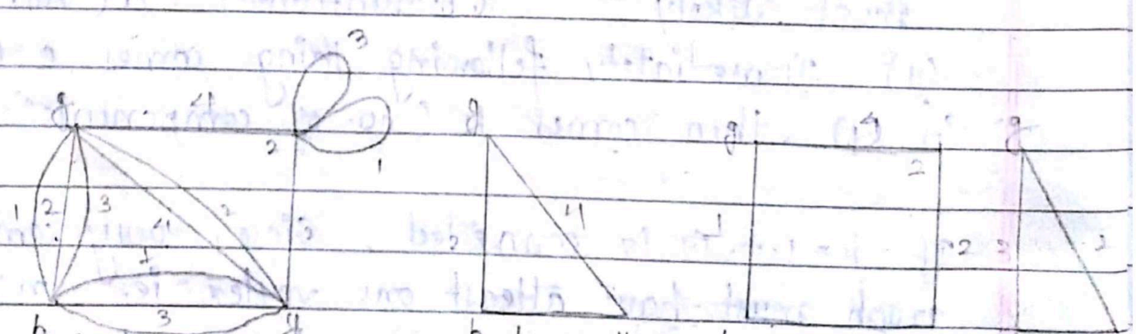
Rank of G is equal to no. of branches in any spanning tree of G. Nullity of G is equal to no. of chords in G.

Rank + Nullity = no. of edges in G.

Planar Graphs

A graph is said to be planar if it can be drawn in a plane in such a way that no edges cross one another except ofcourse at common vertices.



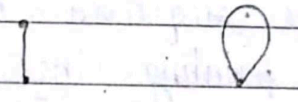


Suppose we draw the planar graph on the plane and take a sharper knife to cut, then the plane will be divided into pieces and that are called the region of the graph. To be more formal, a region of a graph is defined to be an area of a plane which is bounded by edges and it cannot be further divided into subareas

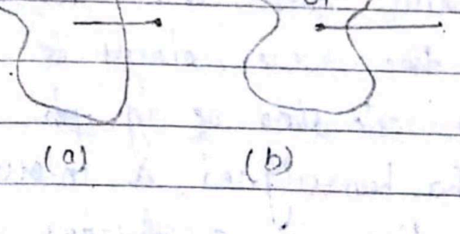
Theorem

for any connected planar graph $(v-e+r)=2$. where v, e and r are the no. of vertices, edges and regions of the planar graph respectively.

Proof → The proof proceed by induction on the no. of edges as the basis of induction we observe that for the two graphs with the single edge shown in the fig. below. ~~As~~ the induction says we assume that the equation ① $(v-e+r)=2$, satisfy in all



graphs with $(n-1)$ edges. Let G be a connected graph of n edges. If G has a vertex of degree 1, the removal of the vertex together with edges ~~connected~~ ~~with~~ incident on it will yield a connected graph G as illustrated in the fig. given below.



(a) (b)

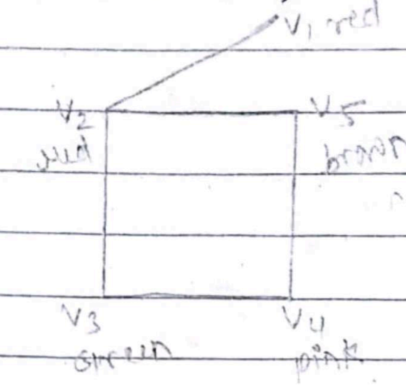
Since eqn ① is satisfied in G', it is also satisfied in G because putting the removed edge & vertex back into G will increase the count ~~the~~ of vertices & edges by 1 but will not change the count of region. If G has no vertex of degree 1, the removal of any edge of the boundary of a finite region will yield a connected graph G' as illustrated in fig. Since eq^n (1) is satisfied in G', is also satisfied in G because eq^n (1) is also known as Euler's formula for a planar graph.

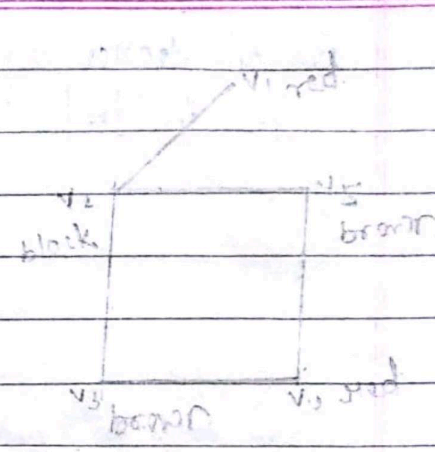
Graph Colouring

Suppose a given graph G with n nodes is given. It is required to paint its nodes such that no two adjacent nodes will be of same colour. What is the min^m no. of colours required. This constitute the colouring problem. After painting the nodes one can group them into different sets such that one set consisting of red nodes, another of green, another of yellow and so on. In the case of planar graph, This is the partitioning problem.

edges of graph

The colouring and partitioning can also be perform on the . In case of planar graph, the colouring can be done for the regions of graphs by Assigning all the nodes of a graph with the colours such that no two adjacent nodes are assigned the same colour is called proper colouring or simply colouring of graph. A graph in which every vertex has been assigned a colour according to a proper colouring is called a properly coloured graph. A graph G that requires min^m k diff. colours for its proper colouring is known as k-chromatic or k-colourable graph and the no. k is called the chromatic no. of G. Symbolically we denote chromatic no. of graph G is $K(G)$.

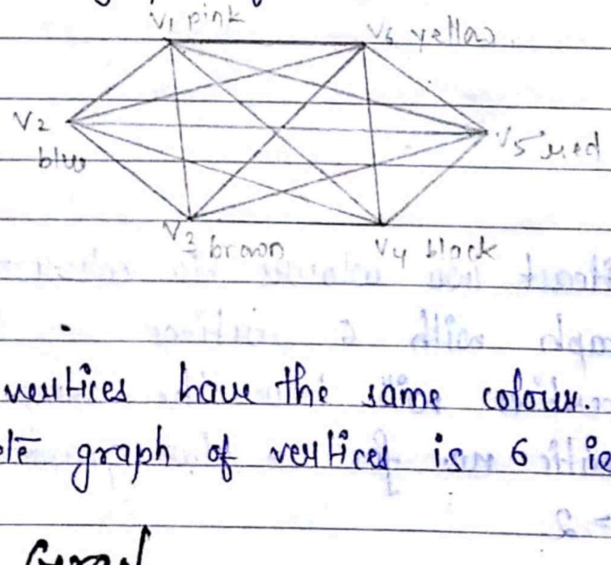




It may be noted that

Complete Graph (C_n)

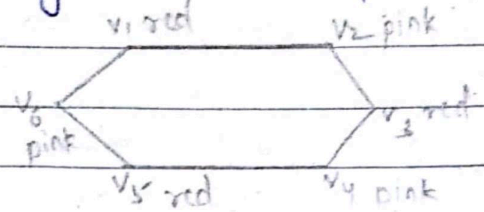
A complete graph is a graph where each vertex is connected to ~~anoth~~ every other in a graph by an edge. We require atleast 6 colours to colour the vertices of a complete graph of 6 vertices. so that no two



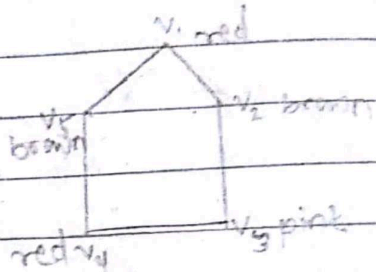
adjacent vertices have the same colour. So, the chromatic no. of complete graph of vertices is 6 ie. $k = 6$

Cycle Graph

It is a graph that consist of single cycle or in other words the no. of vertices connected in a closed chain. The cycle graph with n - vertices is denoted as C_n. The no. of vertices in C_n equals the no. of edges and every vertex is of degree 2 i.e. every vertex has exactly two edges incident upon it.



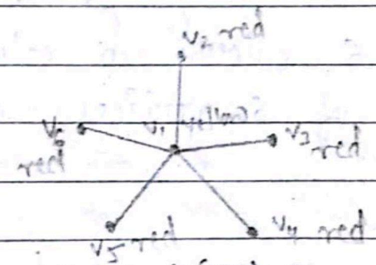
$K(C_6) = 2$.



$K(C_5) = 3$

The n-star Graph (S_n)

It is a graph consisting of n nodes with one node having degree $(n-1)$ and the other nodes having degree 1.

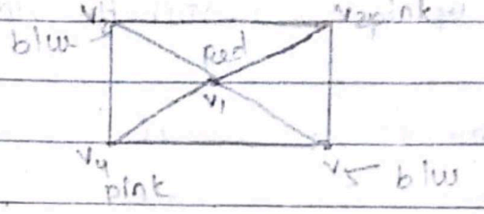


$K(S_6) = 2$

We need atleast two colours to colour the vertices of n-star graph with 6 vertices so that no two adjacent vertices will have the same colour. So the chromatic no. for n-star graph of 6 vertices is $K(S_6) = 2$.

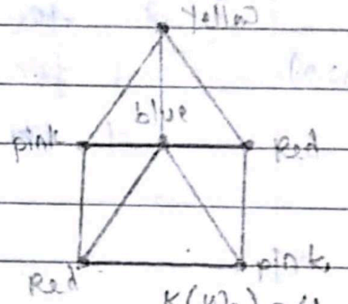
Wheel Graph (W_n)

A wheel graph (W_n) is a graph with n-vertices formed by connecting a single vertex to all vertices of a cycle having n-1 vertices. Let us consider a graph with 5 vertices i.e. $n=5$.



$K(W_5) = 3$

Lets draw a wheel graph with vertices 6.



$K(W_6) = 4$.

It may be observed that min. 3 colours are required to colour so that no two adjacent vertices have the same colour. So, the chromatic no. of a graph with 5 vertices is 3 i.e. $K(W_5) = 3$. Now, consider a wheel graph with 6 vertices, we need 4 colours to colour the vertices so that no two adjacent vertices will have the same colour. So, the chromatic no. of wheel with 6 vertices is 4 i.e. $K(W_6) = 4$.

In general, the chromatic no. of wheel graph with odd no. of vertices (for $n>1$) is 3 and for even no. of vertices it is 4.

Chromatic Number Of Some Commonly Used Graphs

	Type of graph G	chromatic no. K(G).
1.	Complete graph K_n	n
2.	Star graph S_n $(n>1)$	~~$n>1$~~ 2
3.	Cycle graph C_n $(n>1)$	3 ~~and~~ for n-odd and 2 for n-even
4.	Wheel graph W_n $(n>2)$	3 for n-odd 4 for n-even.

Note

~~Sub obs~~

Some observation which follow ~~from~~ ~~some~~ above dissuscions are given below :-

1. A graph which consist of only isolated vertices has chromatic no. 1.

2. A graph with one or more edges (without a self loop) has a min^m chromatic no. 2.

3. A graph consisting of simply n-circle for $n>3$ vertices has chromatic no. 2 if n-is even and 3 if n is odd.

4. ~~A complete~~ ~~graph~~
Every Tree with 2 or more vertices have chromatic no. 2

5. A graph has chromatic no. 2 iff it has no circuits of odd length.

6. If d is max^m no. of degree of vertices in a graph G then the chromatic no. is (d+1)

Propositional Calculus / propositional logic

(PL-1)

Proposition

→ A proposition is a declarative sentence ie. either true or false. for egs but not both.

→ A proposition that is true under all circumstance is referred as tautology.
A proposition that is false under all circumstances is referred to as contradiction.

→ We shall frequently refer proposition by symbolic names.
for eg → Let P denote the proposition.
P: Every student in sem I has passed the final exam.

→ The two possibilities of a proposition being true or false are also referred to as two possible values of a proposition might assume.

→ It is customary to use 'F' for false and 'T' for true. If the above statement is true, we simply write "P is T'. The area of logic which deals with proposition is called the propositional calculus.

Logical Connectives

There are two or more existing proposition can be combined to yield new proposition by using logical operators. These logical operator are also known as logical connectives.

$P \to q \Rightarrow$ p is called premise (Hypothesis)
q is called consequence

There are five logical connectives/operators.

$\wedge$ conjunction (and)
$\vee$ disjunction (or)
$\to$ Implication
$\leftrightarrow$ Bydirection
$\sim / \neg$ Negation (not)

Eg→ p: He is happy.
q: He is hard working
r: He is successfull.

$q \wedge r \to p$

p	q	$p \vee q$	$p \wedge q$	$p \to q$	$p \leftrightarrow q$	$\neg p$
F	F	F	F	T	T	T
F	T	T	F	T	F	T
T	F	T	F	F	F	T
T	T	T	T	T	T	F

Well formed Formula (WFF)

A statement formula is an expression which consist of variables, parenthesis and connective symbols. all such expressions are not statement formula. Well formed formula is produced by using follows -

1. A statement variable is itself a WFF.
2. If P is a well formed formula then $\neg P$ is also a WFF.
3. If P and q are two WFF then $P \vee q$, $P \wedge q$, $P \to q$, $P \leftrightarrow q$ are alse WFF.

And → when both p & q true then true otherwise false
or → when both p & q false then false otherwise true
exclusive or $\oplus$ → is true only when one of p or q is true.
implication → is false when p is true and q is false otherwise true
biconditional → is true when both have same values otherwise false.

4. A string of symbols consisting of the statement variables, connectives, paranthesis is said to WFF iff it can be produced by applying rules 1, 2, 3. finite no. of times.

Tautology

A truth table which is constructed for a given formula represents a truth value which results when the variables are replaced by statement formula. It is the final column of the truth table which decides the truth for the whole statement formula. The truth values in the last column of the Truth table generally depends upon the truth value of statement variable rather than the ~~variable~~ formula themselves.

Such a statement formula whose truth value is always true regardless of the truth values of the variable concerned is called 'c' (constants) i.e. Universally valid formula or a tautology.

The statement formula whose all truth values are false is called contradiction. It may be ~~asst~~ observed the consumption of two tautology is also a tautology.

The negation of a tautology is also a tautology.

A formula P is called a substitution instance of another formula q if P can be generated from q by substituting formula for some variables of q. Here.

the condition required is that the same formula is substi-
-tuted for some variables each time it occurs.
for eg → $P \wedge Q \rightarrow P$

The substitutions are performed only for the atomic formula but not for a compound formula. It may be noted that substitution instance of tautology is also a tautology.

Logical equivalence

Let 'p' and 'q' be two statement formulas and $a_1, a_2, \ldots, a_n$ represent all the variables contained in both p and q, if there exists an assignment of truth values, to the statement variables $a_1, a_2, \ldots, a_n$ and their resulting truth values are 'p' and 'q' such that the truth value of 'p' is equal to the truth value of 'q'. For each of the second possible set of truth values then 'p' and 'q' are said to be logically equivalent. In other words we can say that compound proposition which have the same values in all possible cases is called logical equivalence.

Let us assume that the variables and the assignment of the truth values to the variables appear in the same order in both the truth tables of 'p' and 'q'. It may be observed that if 'p' and 'q' are equivalent then the final columns in both the truth tables of 'p' and 'q' are identical.

for eg → (i) $P \wedge Q \equiv P$

(ii) $P \vee \bar{Q} \equiv \bar{Q} \vee P$

(iii) $\overline{Q \vee Q} = P \vee \bar{P}$

$p \rightarrow$ antecedent
$q \rightarrow$ consequent

from the definition of biconditional it can be observed that $p \leftrightarrow q$ is true whenever p and q have the same truth values. So we can say that p and q are logically equivalent if $p \leftrightarrow q$ is a tautology. Conversely, if $p \leftrightarrow q$ is a tautology then p and q are equivalent.

1. It can be proved that equivalence of two statement formula is symmetric in relation ie. $p \equiv q$ and $q \equiv p$

2. Also this relationship is transitive.
   ie. $p \equiv q$ and $q \equiv r$ then $p \equiv r$.

Rules of ~~eqv~~ inference

→ 1. One of the important function of logic is ~~to~~ provide principle of reasoning ~~and~~ or rules of inference. The theory associated with these rules is known as inference theory. since it deals with inferencing of a conclusion from subpremises of facts or assumes. The process of deriving a conclusion from a set of premises by using standard rule of inference is called as formal rule or deduction.

The rules of inference means the criteria for finding the validity of an arguement. In any arguement a conclusion is said to be true if the premises are considered to be true and the reasoning used in derivation of the conclusion follow some standard rule of conclusion. This type of arguement is called sound. In logic we give importance to the rules of inference because they help us to derive conclusions.

Any conclusion which is derived from these rules is called valid conclusion and the corresponding arguement is called the valid arguement.

Validity using the rules of inference

Let p and q be two statement formulas. We defined q is a valid conclusion of premise p. or q logically follows from p iff $p \rightarrow q$ is a tautology. We can extend this definition to include a set of formulas. We say that 'C' is a valid conclusion of the set of premises $A_1, A_2, \ldots, A_n$ iff $A_1 \wedge A_2, \ldots, \wedge A_n \rightarrow C$ is a tautology.

Rules of Inference

We present the derivation process by which one can justify that a particular formula is valid conclusion of a given set of premises. We describe three rules of inference which we call Rule P, Rule T and Rule CP.

Rule P → A premise can be inserted at any point in the derivation.

Rule T → If the formula q is tautoligicall implie by one and more of the previous formulas in a derivation then q can be inserted in the derivation.

Rule CP → It stands for 'conditional proof' which is also known as 'reduction thm'. It states that we can derive a formula q from p and set of premises then we can derive $p \rightarrow q$ from the set of premises alone. Rule CP is used in the cases where the conclusion is of the form p→q. In these cases, p is taken as an extra premise and q is derived from the given premises and from p.

The tautology $(p \wedge (p \to q)) \to q$ is the basis of rule of inference is called Modus Ponens or law of detachment

Ex. (i) Let the implication be -

If it rains today then I should carry an umbrell

and its hypothesis are -

P: It is rainy tody.

Q: I shall carry an umbrella

- Modus Ponen → The arguments are -

$S_1 = P$

$S_2 = P \to q$

$C = q$ (c is conclusion).

So, $S_1 \wedge S_2 \to q$

$(p \wedge (p \to q)) \to q$. $((P \to q) \wedge P) \to q$

also called forward chaining.

Modus Tollen

$S_1 = P \to q$

$S_2 = \neg q$

$C = \neg P$

$p \to q = (\sim p \vee q)$

Chain Rule / Transitivity

$(p \to q) \wedge (q \to r) = (p \to r)$

Q P: It is a ~~title~~ titan watch.
Q: It is a good watch.
R: It is a good watch therefore it is a titan watch.
→ Invalid using modus ponen.

Q P: All men are mortal
Q: Socrates is a man
R: Socrates is mortal.
→ Valid using chain rule.

— ∝ —

Rule Number	Tautology
IM_1	$p \wedge q \to p$
IM_2	$p \wedge q \to q$
IM_3	$p \to p \vee q$
IM_4	$q \to p \vee q$
IM_5	$\overline{p} \to (p \to q)$
IM_6	$q \to (p \to q)$
IM_7	$(\overline{p \to q}) \to p$
IM_8	$(\overline{p \to q}) \to \overline{q}$

Rule Number	Tautology
IM_9	$P, q \rightarrow P \wedge q$ (conjunctio
IM_{10}	$\overline{P}, (P \vee q) \rightarrow q$ (Disjunction, Syllogism)
IM_{11}	$P, (P \rightarrow q) \rightarrow q$ (Modus Ponen)
IM_{12}	$\overline{q}, (P \rightarrow Q) \rightarrow \overline{P}$ (Modus Ponen)
IM_{13}	$(P \rightarrow Q), (Q \rightarrow R) \rightarrow P \rightarrow R$
IM_{14}	$(P \vee q), P \rightarrow R, q \rightarrow R) \rightarrow R$ (Dislemma)
IM_{15}	$(P \vee q, \overline{P} \vee R) \rightarrow (q \vee R)$ Resolution
EQ_1	$\overline{\overline{P}} \equiv P$ (Double negation law)
EQ_2	$P \vee q \equiv q \vee P$ (commutative law)
EQ_3	$P \wedge q \equiv q \wedge P$ (commutative law)
EQ_4	$(P \vee q) \vee R \equiv P \vee (q \vee R)$ (Associative law)
EQ_5	$(P \wedge q) \wedge R \equiv P \wedge (q \wedge R)$ (Associative law)
EQ_6	$P \vee (q \wedge R) \equiv (P \vee q) \wedge (P \wedge R)$
EQ_7	$P \wedge (q \wedge R) \equiv (P \wedge q) \cancel{\times} (P \wedge R)$ (Distributive law)
EQ_8	$\overline{P \vee q} = \overline{P} \wedge \overline{q}$ (De Morgan's law)
EQ_9	$\overline{P \wedge q} = \overline{P} \vee \overline{q}$
EQ_{10}	$P \vee P = P$ (Idempotent law)
EQ_{11}	$P \wedge P = P$ (Idempotent law)
EQ_{12}	$R \vee (P \wedge \overline{P}) \equiv R$
EQ_{13}	$R \wedge (P \vee \overline{P}) \equiv R$
EQ_{14}	$R \vee (P \vee \overline{P}) \equiv T$
EQ_{15}	$P \rightarrow q \equiv \overline{P} \vee q$

EQ_{16}	$p \to q \equiv \overline{p} \vee q$
EQ_{17}	$\overline{p \to q} \equiv p \wedge \overline{q}$
EQ_{18}	$p \to q \equiv \overline{q} \to \overline{p}$.
EQ_{19}	$p \to (q \to R) \equiv (p \wedge q) \to R$.
EQ_{20}	$(\overline{p \leftrightarrow q}) \equiv p \leftrightarrow \overline{q}$
EQ_{21}	$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
EQ_{22}	$p \leftrightarrow q \equiv (p \wedge q) \vee (\overline{p} \wedge \overline{q})$
EQ_{23}	$p \vee (p \wedge q) \equiv p$ [absorption law]
EQ_{24}	$p \wedge (p \vee q) \equiv p$ [Absorption law]
EQ_{25}	$p \vee \overline{p} \equiv T$ [Negation law]
EQ_{26}	$p \wedge \overline{p} \equiv F$ [Negation law].

Q Show that $\bar{p}$ is tautologically implied by $\overline{p \wedge \bar{q}}$, $\bar{q} \vee r$, $\bar{r}$

Sol^n (i) $\overline{p \wedge \bar{q}}$ (Rule P)

(ii) Applying De morgan's law

$\overline{p \wedge \bar{q}} = \bar{p} \vee q$

(iii) $\bar{p} \vee q = p \to q$ (Rule T)

(iv) $\bar{q} \vee r = q \to r$ { Rule P, Rule T,

(v) $p \to r$ (using rule T, syllogicaly)

(vi) $\bar{r}$ (Rule P)

(vii) $\bar{p}$ (using rule T. modus Tollen)

Q ($\bar{p} \vee q$, $\bar{q} \vee r$, $r \to s$) $\to$ ($p \to s$)

Sol^n

(i) $\bar{p} \vee q$ (Rule P)

(ii) $\bar{p} \vee q = p \to q$ (Rule T)(EQ16., $p \to q = \neg p \vee q$)

(iii) ~~q~~ p (Rule T) (ii)(iii) Modus Ponen)

(iv) q (Rule T)

(v) $\bar{q} \vee r = q \to r$. (Rule P , EQ_{16})

(vi) r (Rule T (iv)(v) disjunctive syllogism)

(vii) $r \to s$ (Rule P)

(viii) s Rule (vii) (viii) Modus ponen.

(ix) $p \to s$ Rule chain rule.

Q Show that $\overline{\bar{P} \wedge Q}$ follow from $\bar{P} \wedge \bar{Q}$

Sol^n Assume $\overline{\bar{P} \wedge Q}$ as an additional premise. then

(1) $\overline{\bar{P} \wedge Q}$ Rule P.

(2) $P \wedge Q$ Rule T

(3) P Rule T

(4) $\overline{P \wedge Q}$ Rule P

(5) $\overline{P}$ Rule T

(6) $P \wedge \overline{P}$ Rule T.

(6) is a contradiction. Hence, by the indirect method of proof $\sim\overline{P \wedge Q}$ follows from $\overline{P} \wedge \overline{Q}$

Q Show that conclusion $c: \sim P$ follows the premises $H_1: \sim P \vee Q$; $H_2: \sim(Q \wedge \sim R)$ and $H_3: \sim R$.

Sol→ We get:

	1.	$\sim R$	Rule P (assumed premise)
	2.	$\sim(Q \wedge \sim R)$	Rule P
{2}	3.	$\sim Q \vee R$	Rule T
{3}	4.	$R \vee \sim Q$	Rule T
{4}	5.	$\sim R \rightarrow \sim Q$	Rule T
{1,5}	6.	$\sim Q$	Rule T
	7.	$\sim P \vee Q$	Rule P
{7}	8.	$\sim Q \rightarrow \sim P$	Rule T
{6, 8}	9.	$\sim P$	Rule T.

Here, c logically follows from H_1, H_2 and H_3.

Q Show that $S \vee R$ is tautologically implied by $(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow S)$

Sol→ We have,

	1.	$P \vee Q$	Rule P
{1}	2.	$\sim P \rightarrow Q$	Rule T
	3.	$Q \rightarrow S$	Rule P
{2,3}	4.	$\sim P \rightarrow S$	Rule T
	5.	$\sim S \rightarrow P$	Rule T (as $P \rightarrow Q \Leftrightarrow \sim Q \rightarrow \sim P$)

	6.	$P \to R$	Rule P
{5,6}	7.	$\sim S \to R$	Rule T
{7}	8.	$S \vee R$	Rule T.

Q Show that $R \to S$ can be derived from the premices $P \to (Q \to S)$, $\sim R \vee P$ and Q

Sol→ We get

	1.	R	Rule P (assumed premice)
	2.	$\sim R \vee P$	Rule P
{2}	3.	$R \to P$	Rule T
{1, 3}	4.	P	Rule T
{	5.	$P \to (Q \to S)$	Rule P
{4,5}	6.	$Q \to S$	Rule T
	7.	Q	Rule P
{7, 6}	8.	S	Rule T
	9.	$R \to S$	Rule CP.

Q Show that $\sim(P \wedge Q)$ follows from $\sim P \wedge \sim Q$.

Sol→ Assume $\sim(\sim P \wedge Q)$ as an additional premice then:

	1.	$\sim(\sim P(P \wedge Q))$	Rule P.
{1}	2.	$P \wedge Q$	Rule T
	3.	P	Rule T
	4.	$\sim P \wedge \sim Q$	Rule P
{4}	5.	$\sim P$	Rule T
{3,5}	6.	$P \wedge \sim P$	Rule T

$P \wedge \sim P$ is a contradiction. Hence, by the indirect method of proof $\sim(P \wedge Q)$ follows from $\sim P \wedge \sim Q$.

Q "If there was a meeting then catching the bus was difficult. If they arrived on time then catching the bus was not difficult. They arrived on time. Therefore, there was no meeting." Show that the statement constitute a valid arguement.

Sol→ Let:

P: There was a meeting

Q: Catching the bus was difficult

R: They arrived on time.

We have to prove $\sim P$ follows the premices $P \to Q$, $R \to \sim Q$ and R.

	1.	R	Rule P
	2.	$R \to \sim Q$	Rule P
{1, 2}	3.	$\sim Q$	Rule T
	4.	$P \to Q$	Rule P
{4}	5.	$\sim Q \to \sim P$	Rule T
{3, 5}	6.	$\sim P$	Rule T

Q Show that $\sim(P \vee (\sim P \wedge Q))$ and $\sim P \wedge \sim Q$ are logically equivalent

Sol:

$\sim(P \vee (\sim P \wedge Q)) \Leftrightarrow \sim P \wedge \sim(\sim P \wedge Q)$ by De morgan's law. I

$\Leftrightarrow \sim P \wedge (\sim(\sim P) \vee \sim Q)$ by De Morgan's law 2.

$\Leftrightarrow \sim P \wedge (P \vee \sim Q)$ by involution law

$\Leftrightarrow (\sim P \wedge P) \vee (\sim P \wedge \sim Q)$ by distribution law of '$\wedge$ over $\vee$'.

$\Leftrightarrow 0 \vee (\sim P \wedge \sim Q)$ by negation law 2.

$\Leftrightarrow (\sim P \wedge \sim Q) \vee 0$ by commutative law of $\vee$

$\Leftrightarrow \sim P \wedge \sim Q$ by identity law 2.

Hence, $\sim(P \vee (\sim P \wedge Q))$ and $\sim P \wedge \sim Q$ are logically equivalent.

Q Show that $(P\wedge Q) \rightarrow (P\vee Q)$ is a tautology.

Sol^n $(P\wedge Q) \rightarrow (P\vee Q) \Leftrightarrow \sim(P\wedge Q) \vee (P\vee Q)$ by implication law

$\Leftrightarrow (\sim P\vee \sim Q) \vee (P\vee Q)$ by De morgan's law. 2

$\Leftrightarrow \sim P\vee(\sim Q\vee P)\vee Q$ by associative law of $\vee$.

$\Leftrightarrow \sim P\vee(P\vee \sim Q)\vee Q$ by commutative law of $\vee$.

$\Leftrightarrow (\sim P\vee P)\vee(\sim Q\vee Q)$ by associative law of $\vee$

$\Leftrightarrow I \vee I$ by negation law 1.

$\Leftrightarrow I$ by domination law I

Hence, the result.

Q. Show that $\sim(P\wedge(\sim Q\wedge R)) \vee (Q\wedge R) \vee (P\wedge R) \Leftrightarrow R$

Sol^n $(\sim P\wedge(\sim Q\wedge R)) \vee (Q\wedge R) \vee (P\wedge R)$

$\Leftrightarrow (\sim P\wedge(\sim Q\wedge R)) \vee ((Q\vee P)\wedge R)$ by disstributive law of $\wedge$ over $\vee$.

$\Leftrightarrow ((\sim P\wedge \sim Q)\wedge R) \vee ((Q\vee P)\wedge R)$ by associative law of $\wedge$.

$\Leftrightarrow ((\sim P\wedge \sim Q)) \vee (Q\vee P)\wedge R$ by distributive law of $\wedge$ over $\vee$.

$\Leftrightarrow (\sim(P\vee Q)) \vee (P\vee Q)\wedge R$ by De Morgan's law I and commutative law of $\vee$

$\Leftrightarrow I\wedge R$ by negation law of I

$\Leftrightarrow R$ by idintity law I

Hence, the result.

Q Show that $((P\vee Q)\wedge\sim(\sim P\wedge(\sim Q\vee\sim R))) \vee (P\wedge \sim Q) \vee (\sim P\wedge \sim R)$ is a tautology.

Sol^n consider:

$(\sim P\wedge \sim Q) \vee (\sim P\wedge \sim R)$

$\Leftrightarrow \sim(P\vee Q) \vee \sim(P\vee R)$ by Demorgan's law I

$\Leftrightarrow \sim((P\vee Q)\wedge(P\vee R))$ by Demorgan's law 2.

$\Leftrightarrow \sim(\sim P\wedge(\sim Q\vee \sim R))$

$\Leftrightarrow$

$\Leftrightarrow \sim(\sim P \wedge \sim(Q \wedge R))$ by Demorgan's law 2.
$\Leftrightarrow \sim(\sim P) \vee \sim(\sim(Q \wedge R))$ by De morgan's law 2.
$\Leftrightarrow P \vee (Q \wedge R)$ by involution law.
$\Leftrightarrow (P \vee Q) \wedge (P \vee R)$ by distributive law of $\vee$ over $\wedge$.
Hence, the given formula is equivalent to
$((P \vee Q) \wedge (P \vee R)) \vee \sim((P \vee Q) \wedge (P \vee R))$

If we substitute $(P \vee Q) \wedge (P \vee R)$ for p in $p \vee \sim P$, we get the above formula. But $P \vee \sim P \Leftrightarrow 1$, the given formula is a tautology.

Q. Show the equivalence :-
(a) $P \to (Q \to P) \Leftrightarrow \sim P \to (P \to Q)$
Sol:> Consider :
$P \to (Q \to P) \Leftrightarrow \sim P \vee (\sim Q \vee P)$ by implication law.
$\Leftrightarrow (\sim P \vee P) \sim Q$ by associative and commutative laws of $\vee$
$\Leftrightarrow 1 \vee \sim Q$ by negation law 1
$\Leftrightarrow 1$ by null law 1.

Again : $\sim P \to (P \to Q) \Leftrightarrow \sim(\sim P) \vee (\sim P \vee Q)$ by implication law.
$\Leftrightarrow P \vee (\sim P \vee Q)$ by involution law.
$\Leftrightarrow P \vee (\sim P) \vee Q$ by associative law of $\vee$
$\Leftrightarrow 1 \vee Q$ by negation law 1
$\Leftrightarrow 1$ by null law 1

Hence, $P \to (Q \to P) \Leftrightarrow \sim P \to (P \to Q)$.

(b) $P \to (Q \vee R) \Leftrightarrow (P \to Q) \vee (P \to R)$
Sol:> $P \to (Q \vee R) \Leftrightarrow \sim P \vee (Q \vee R)$ by implication law
$\Leftrightarrow \sim P \vee \sim P \vee (Q \vee R)$ since $\sim P \Leftrightarrow \sim P \vee \sim P$.

$\Leftrightarrow (\sim P \vee Q) \vee (\sim P \vee R)$ by associative and commutative laws of $\vee$

$\Leftrightarrow (P \to Q) \vee (P \to R)$ by implication law.

(c) $(P \to Q) \wedge (R \to Q) \Leftrightarrow \cancel{(\sim P \wedge Q)} \Leftrightarrow (P \vee Q) \to Q$

Sol $\Rightarrow$ $(P \to Q) \wedge (R \to Q) \Leftrightarrow (\sim P \wedge Q) \wedge (\sim R \vee Q)$ by implication law.

$\Leftrightarrow (\sim P \wedge \sim R) \vee Q$ by distributive of $\vee$ over $\wedge$.

$\Leftrightarrow \sim (P \vee R) \vee Q$ by De Morgan's law 1.

$\Leftrightarrow (P \vee Q) \to Q$ by implication law.

(d) $\sim (P \leftrightarrow Q) \Leftrightarrow (P \vee Q) \wedge \sim (P \vee Q) \wedge (P \wedge Q)$

Sol $\Rightarrow$ $\sim (P \leftrightarrow Q) \Leftrightarrow \sim ((P \wedge Q) \vee \sim (P \vee Q))$ by bidirectional and De morgan's law 1.

$\Leftrightarrow \sim (P \wedge Q) \wedge \sim (\sim P \vee Q)$ by De morgan's law 1

$\Leftrightarrow \sim (P \wedge Q) \wedge (P \vee Q)$ by involution law.

$\Leftrightarrow (P \wedge Q) \wedge \sim (P \wedge Q)$ by commutative law of $\wedge$.

Q Show that $P \leftrightarrow Q$ logically implies $P \to Q$

Sol $\Rightarrow$ Consider the truth table of $P \leftrightarrow Q$ and $P \to Q$. Now, $P \to Q$ is true in line 1 and 4 and in these cases $P \leftrightarrow Q$ is also true. Hence, $P \leftrightarrow Q$ implies $P \to Q$

P	Q	$P \leftrightarrow Q$	$P \to Q$
0	0	1	1
0	1	0	1
1	0	0	0
1	1	1	1

Now two methods will be introduced to show $P \Rightarrow Q$

Method 1 → To show the implication $P \Rightarrow Q$, we assume that P has the truth value 1 and then show that the assumption leads to Q having the value 1. Then $P \rightarrow Q$ must have the value 1.

Method 2 → To show $P \Rightarrow Q$, we assume that Q has the truth ~~tab~~ value 0 and show that this assumption leads P having the value 0. Then $P \Rightarrow Q$ must have the truth value 1.

Q. Consider the implication $(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow R) \Leftrightarrow R$.

Soln → Now by method 1, we assume that $(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow R)$ is true. This assumption means that $P \vee Q$, $P \rightarrow R$ and $Q \rightarrow R$ are true. Since $P \vee Q$ is true, atleast one P or Q is true. If P is true then R must be true since $P \rightarrow R$ is true. If Q is true then R must be true since $Q \rightarrow R$ is true. So, R is true. Hence, the result.

Q Consider the implication $\sim Q \wedge (P \sim Q) \sim P$

Soln → Now by method 2, assume that $\sim P$ is false so P is true. If Q is true then $\sim Q$ is false. If Q is false then $P \rightarrow Q$ is then false. So in both cases, $\sim P \wedge (P \rightarrow Q)$ is false.

Q. Show the implications :-

(a) $(P \wedge Q) \Rightarrow P \rightarrow Q$

Soln → Assume the $P \wedge Q$ is true. This means that P and Q are true. So, $P \rightarrow Q$ must also be true. By method 1, the result is proved.

(b) $P \Rightarrow Q \rightarrow P$.

Solⁿ Assume that P is true. Then for all possible truth values of Q, $Q \rightarrow P$ is true. Hence, by method 1, the result follows.

(c) $P \rightarrow (Q \rightarrow R) \Rightarrow (P \rightarrow Q) \rightarrow (P \rightarrow R)$

Solⁿ Assume that $(P \rightarrow Q) \rightarrow (P \rightarrow R)$ is false. Then $P \rightarrow R$ is false, and $P \rightarrow Q$ is true. This means that P and Q are true. Since, R is false. So, $Q \rightarrow R$ is false and hence, by method 2 the implication follows.

Q Show the following implications without constructing the truth tables.

(a) $P \rightarrow Q \Rightarrow P \rightarrow (P \land Q)$

Solⁿ Assume that $P \rightarrow (P \land Q)$ is false. Then, P is true and $P \land Q$ is false. So, Q is false. Hence $P \rightarrow Q$ is false. So, by the method 2, the implication follows.

(b) $(P \rightarrow Q) \rightarrow Q \Rightarrow P \land Q$.

Solⁿ Assume that $P \lor Q$ is false. This means that both P and Q are false. So, $P \rightarrow Q$ is true and hence $(P \rightarrow Q) \rightarrow Q$ is false. Again by method 2, the result follows.

(c) $((P \lor \sim P) \rightarrow Q) \rightarrow ((P \lor \sim P) \rightarrow R) = Q \rightarrow R$

Solⁿ Suppose that $Q \rightarrow R$ is false. Then Q is true and R is false. This implies that $(P \lor \sim P) \rightarrow Q$ is true. and $P \lor \sim P \rightarrow R$ is false. Hence, $((P \lor \sim P) \rightarrow Q) \rightarrow ((P \lor \sim P) \rightarrow R)$ is false. By method 2, the result follows.

(d) $(Q \to (P \vee \sim P)) \to (R \to (P \vee \sim P)) \Rightarrow R \to Q.$

Sol^n Assume that $R \to Q$ is false. Then R is true and Q is false. So, $Q \to (P \wedge \sim P)$ is true and $R \to (P \wedge \sim P)$ is false. Hence $(Q \to (P \wedge \sim P)) \to (R \to (P \wedge \sim P))$ is false and the result follows.

Note → that $P \vee \sim P$ is always true and $P \wedge \sim P$ is always false.

Q If $P_1, P_2, \dots, P_n$ and P implies Q then $P_1, P_2, \dots, P_n$ implies $P \to Q$

Sol^n Let us assume that $P_1 \wedge P_2 \wedge \dots \wedge P_n \wedge P \Rightarrow Q$
This means $(P_1 \wedge P_2 \wedge \dots \wedge P_n \wedge P) \to Q$ is a tautology.
But we know that $(P \wedge Q) \to R \Leftrightarrow P \to (Q \to R)$
using this we get $(P_1 \wedge P_2 \wedge \dots \wedge P_n) \Rightarrow (P \to Q)$ is a tautology.
Hence, $(P_1 \wedge P_2 \wedge \dots \wedge P_n) \Rightarrow (P \to Q)$

Q Show that $R \to S$ can be derived from the premises $P \to (Q \to S)$, $\neg R \vee P$ and Q

Sol^n Instead of deriving $R \to S$, we shall include R as an additional premise and show S first.

$\{1\}$	1.	$\neg R \vee P$	Rule P
$\{2\}$	2.	R	Rule P (additional Premise)
$\{1,2\}$	3.	P	Rule T, (1), (2), and I_{10}
$\{4\}$	4.	$P \to (Q \to S)$	Rule P
$\{1,2,4\}$	5.	$Q \to S$	Rule T, (3), (4), and I_{11}
$\{6\}$	6.	Q	Rule P
$\{1,2,4,6\}$	7.	S	Rule T, (5), (6) & I_{11}
$\{1,4,8\}$	8.	$R \to S$	Rule CP.

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Q Show that $P \to S$ can be derived from the premises that $\neg P \vee Q$, $\neg Q \vee R$, $R \to S$.

Sol: We include P as an additional premise and derives S.

{1}	1. $\neg P \vee Q$	Rule P.
{2}	2. P	Rule P (additional premise)
{1,2}	3. Q	Rule T, (1), (2) and I_{10}
{4}	4. $\neg Q \vee R$	Rule P
{1, 2, 4}	5. R	Rule T, (3), (4) & I_{10}
{6}	6. $R \to S$	Rule P
{1, 2, 4, 6}	7. S	Rule T, (5), (6) and I_{11}
{1, 4, 6}	8. $P \to S$	Rule CP.

Q Show that $M \vee N$ is a valid arguement from the premises $\neg J \to (M \vee N)$, $(N \vee G) \to \neg J$, $N \vee G$.

Sol: {1}	1. $(N \vee G) \to \neg J$	Rule P
{2}	2. $N \vee G$	Rule P
{1, 2}	3. $\neg J$	Rule T, (1), (2), I_{11}
{4}	4. $\neg J \to (M \vee N)$	Rule P
{1, 2, 4}	5. $M \vee N$	Rule T, (3), (4), I_{11}

Q Show that $\neg S$ is a valid arguement from the premises $P \to Q$, $(\neg Q \vee R) \wedge (\neg R)$, $\neg(\neg P \wedge S)$

Sol: {1}	1. $P \to Q$	Rule P
{2}	2. $(\neg Q \vee R) \wedge (\neg R)$	Rule P
{2}	3. $\neg Q \vee R$	simplification, (2)
{2}	4. $Q \to R$	Rule T, (3), E_1, E_{16}
{1, 2}	5. $P \to R$	Rule T, (1), (4), I_{13}
{6}	6. $\neg(\neg P \wedge S)$	Rule P
{6}	7. $\neg(\neg P \wedge \neg\neg S)$	Rule T, E_1

{6}	8. $\neg(\neg(\neg P \to \neg S))$	Rule T, (7), E_{17}
{6}	9. $\neg(\neg(\neg S))$	Rule T, (8), I_8
{6}	10. $\neg S$	Rule T, E_1, (8)

Q Show that $R \wedge (P \vee Q)$ is a valid conclusion from the premises $P \vee Q$, $Q \to R$, $P \to$ and $\neg M$.

Sol^n

{1}	1. $P \to M$	Rule P.
{2}	2. $\neg M$	Rule P
{1, 2}	3. $\neg P$	Rule T, (1), (2) and I_{12}
{4}	4. $P \vee Q$	Rule P
{1, 2, 4}	5. Q	Rule T, (3), (4) and I_{10}
{6}	6. $Q \to R$	Rule P
{1, 2, 4, 6}	7. R	Rule T, (5), (6), I_{11}
{1, 2, 4, 6}	8. $R \wedge (P \vee Q)$	Rule T, (4), (7) and I_9.

Q Demonstrate that R is a valid inference from the premises $P \to Q$, $Q \to R$ and P.

Sol^n

{1}	1 $P \to Q$	Rule P
{2}	2. P	Rule P
{1, 2}	3. Q	Rule T, (1), (2) and I_{11}
{4}	4. $Q \to R$	Rule P
{1, 2, 4}	5. R	Rule T, (3), (4) and I_{11}

Therefore, R is a valid inference.

Q Show that $\neg P$ follows logicall from the premises $\neg(P \wedge \neg Q)$, $\neg Q \vee R$, $\neg R$

Sol^n

{1}	1. $\neg(P \wedge \neg Q)$	Rule P
{1}	2. $\neg P \vee Q$	Rule T, (1), E_{11}, E_1
{1, 2}	3. $P \to Q$	Rule T, (2), E_1, E_{16}

{4}	4.	$\neg Q \vee R$	Rule P.
{4}	5.	$Q \rightarrow R$	Rule T, (4), E_{16}
{1, 2, 4}	6.	$P \rightarrow R$	Rule T, (3), (5), I_{13}
{1, 2, 4}	7	$\neg R \rightarrow \neg P$	Rule T, (6), E_{18}
{8}	8.	$\neg R$	Rule P
{1, 2, 4, 8}	9.	$\neg P$	Rule T, (7), (8), I_{11}

Q Show that $R \vee S$ follows logically from the premises $C \vee D$, $(C \vee D) \rightarrow \neg H$, $\neg H \rightarrow (A \wedge \neg B)$ and $(A \wedge \neg B) \rightarrow (R \vee S)$

Solⁿ

{1}	1.	$(C \vee D) \rightarrow \neg H$	Rule P
{2}	2.	$\neg H \rightarrow (A \wedge \neg B)$	Rule P.
{1, 2}	3.	$(C \vee D) \rightarrow (A \wedge \neg B)$	Rule T, (1), (2), and I_{13}
{4}	4.	$(A \wedge \neg B) \rightarrow R \vee S$	Rule P
{1, 2, 4}	5	$(C \vee D) \rightarrow (R \vee S)$	Rule T, (3), (4) and I_{13}
{6}	6.	$C \vee D$	Rule P
{1, 2, 4, 6}	7.	$R \vee S$	Rule T, (5), (6) and I_{11}

The two tautologies frequently used in the above derivations are I_{13}, known as hypothetical syllogism and I_{11}, known as modus ponens.

Set Theory

Set

A set is a well defined collection of object known as member or elements of the set. We generally use capital letters to denote set and lower letters as the elements of the set. If 'x' is a element of 'A' or belongs to $x \notin A$, it is written as $x \in A$. If statement x is not an element of A then it is written as $x \notin A$.

Representation of a set

There are mainly two types of representation of sets :-

1. Roaster or Tabular form

→ All the no.s of the set are listed and are seperated by commas and enclosed within braces. for eg → set of vowels, $V = \{a, e, i, o, u\}$

2. Set builder or rule Method

→ In this method, we define certain properties which recognises elements in the set.

for eg → Set of real no.s i.e. b/w 1 and 8.

$x = \{x, x \text{ is a real no. } 1 < x < 8\}$

Set of Special Status

Z - set of all integer

Z^+ - set of all +ve integers.

N - set of all natural no.s

R - set of all real no.s

Q - set of rational no.s.
C - set of all complex no.s.

Universal Set
It is a fix set which contain each of the given set and is denoted by 'U'.
for eg→ $A = \{1,2,3,4\}$, $B = \{2,3,4,6\}$, $C = \{5,7,8,9\}$
then $U = \{1,2,3,4,5,6,7,8,9\}$.

Empty Set
A set containing no element is known as null or empty set. It is denoted by ϕ or $\{\ \}$.

A set which possesive atleast one element is called a 'non-empty set'. A set that contain a single element is called 'singleton set.'

Subset
If A and B are two sets such that every element of set A is also an element of set B then A is called a subset of B ie. $A \subseteq B$.
If $A \subseteq B$ then it is possible that $A = B$. But when $A \subseteq B$ and $A \neq B$ then we say that A is a proper subset of B ie. $A \subset B$

Important properties of Set

1. Every set is a subset of itself.
2. The empty set is a subset of every set.

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8. The total no. of all possible subset of a given set containing n elements is 2^n.

Eg→ $A = \{1, 2, 3\}$

⇒ $\{\phi\}, \{1,2,3\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}$

Power Set

Consider the set A, the subsets of A are $\phi, \{a\}, \{b\}, \{a,b\}$. then where $A = \{a, b\}$. Then the family of all subset of A is called the powerset of A which is denoted as $P(A)$ and is equal to

$P(A) = \{\phi, \{a\}, \{b\}, \{a,b\}\}$

Symbolically, $P(A) = \{x : x \text{ is a subset of } A\}$.

It may be noted that empty set and the set itself are the members of the powerset.

For eg→ $x = \{1, 2, 3\}$

$P(x) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3$

If a set is finite and contain n elements then the powerset of ~~A~~ will have 2^n elements.

Cardinality Of a Set

Number of distinct elements is called the cardinality or the cardinal no. of a set. It is denoted as $n(A)$.

For eg→ $A = \{1, 2, 3, 4, 5, 6\}$

then $n(A) = 6$.

Ordered Pair

An ordered pair of objects is a pair of object arranged in some order. Thus, in the sets ~~A~~ and ~~B~~ $\{a, b\}$

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a is the 1^{st} object and b is the 2^{nd} object of the fair. Essentially, $\{a,b\} \neq \{b,a\}$. Moreover, the two objects of the ordered fair need not be distinct. So, $\{b,b\}$ and $\{a,a\}$ is a well defined ordered fair.

An ordered triple is an order triple of objects.
for eg→ $\{a,b,c\}$ can also be written as $\{\{a,b\},c\}$.

Cartesian Product of Sets

Cartesian product is a mode by which two or more sets can be combined to obtain another one. If A and B are two non-empty sets, cartesian products of A and B is the set $A \times B$.

for eg→ $A = \{a\}$ and $B = \{b\}$
then $A \times B = \{(a,b) ; a \in A$ and $b \in B\}$

If A is ϕ and B is ϕ then $A \times B = \phi$.
If A and B are finite sets then $A \times B$ is also finite.
If A and B is infinite then $A \times B$ is also infinite.

Q $A = \{1, 2, 3\}$, $B = \{3, 4\}$, $C = \{4, 5, 6\}$
Find ① $A \times (B \cup C)$ ② $A \times (B \cap C)$ ③ $(A \times B) \cap (B \times C)$

Sol→ $A = \{1, 2, 3\}$
$B = \{3, 4\}$
$C = \{4, 5, 6\}$
then $A \times B = \{(1,3), (1,4), (2,3), (2,4), (3,3), (3,4)\}$
$B \times C = \{(3,4), (3,5), (3,6), (4,4), (4,5), (4,6)\}$
$B \cup C = \{3, 4, 5, 6\}$
$B \cap C = \{4\}$

(i) $A \times (B \cup C)$

$\rightarrow \{1,2,3\} \times \{3,4,5,6\}$

$\Rightarrow \{(1,3),(1,4),(1,5),(1,6),(2,3),(2,4),(2,5),(2,6),$
$(3,3),(3,4),(3,5),(3,6)\}$

(ii) $A \times (B \cap C)$

$\rightarrow \{1,2,3\} \times \{4\}$

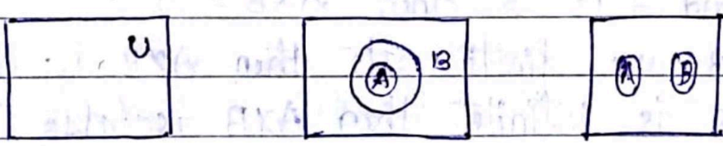
$\Rightarrow \{(1,4),(2,4),(3,4)\}$

(iii) $(A \times B) \cap (B \times C)$

$\rightarrow \{(3,4)\}$

Venn Diagram

To express the relationship among sets in a prospective way we will present them pictorically by means of diagrams. These diagrams are known as Venn diagrams

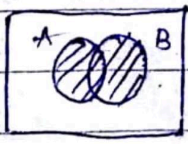


Operations On Sets

1. Union Of Sets

→ Union of two sets denoted by $A \cup B$ is a set of all those elements each one of which belongs to either A or B or both A and B.

i.e $A \cup B = \{x \in A \text{ or } x \in B\}$

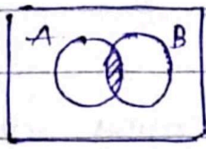


$A \cup B$

2. Intersection of Two sets

→ It is denoted as $A \cap B$ is the set of all those elements common to both A and B.

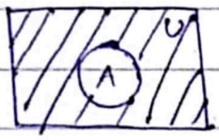
ie $A \cap B = \{x \in A \text{ and } x \in B\}$



$A \cap B$

3. Complement

→ Let U be universal set and let $A \subset U$, then complement of A is denoted as A' is the set of all those elements which belong to U but not A.

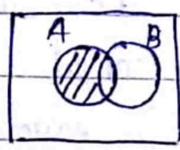


A'

4. Relative Complement

→ The relative complement of set A w.r.t a set A is simply the difference of A and B which is denoted as $A \sim B$. is the set of all elements which belong to A but does not belong to B.

ie $A \sim B = \{x : x \in A, x \notin B\}$



$A \sim B$

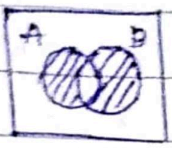
5. Symmetric Difference

→ The symmetric difference b/w A and B is denoted as $A \ominus B$ or $A \Delta B$. It is the set of all those element which belongs to A or belongs to B but not both.

$\boxed{A \ominus B = (A \cup B) \sim (A \cap B)}$.

successor - A^+ is a set that consists of all the elements $\exists$ - there exists of A together with an additional element which is set A.

eg:- $A = \{a, b\}$ so $A^+ = \{a, b, \{a, b\}\}$.

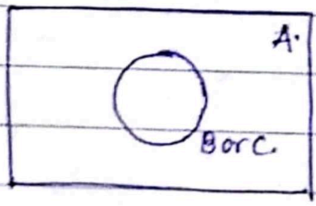


$A \Delta B$ or $A \ominus B$.

Q Express the relation which must be valid b/w the sets and also draw the Venn diagrams.

1. $A' \cap U = \phi$
2. $(A \cap B)' = B'$
3. $A \cap B = A \cap C$ and $A' \cap B = A' \cap C$.

Sol→ 1. $A = U$.
2. $B \subset A$
3. $B = C$.



Countable Set and Uncountable Set

If the rule is such that it associate with each element $a \in A$ and only one element $b \in B$, this rule is called one-to-one and also for each element $b \in B$, $\exists$ exactly only one element $a \in A$. then this rule is called one-to-one corresponding.

A set which is not finite is called an infinite set. A set is called countably infinite if $\exists$ one-to-one correspondance b/w the elements in the set and the element in N i.e. if the set is countabally

A set is said to be finite if there is one-to-one correspondance b/w the elements in the set and the element in some set n, where $n \in N$, n is said to be cardinality of the set.

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infinite we can make a list of no.s in such a way that each one correspond uniquely to a natural no.. A countabally infinite set is also termed as Denumer-able set.

The set of non-negative even integers is countabally infinite. A set which is either finite or Denumerable is called countable.

A set which is not countable is called uncountable. A set which is not countabally infinite is called uncoun-tabally infinite set or non-denumerable set or simply uncountable set.

Algebra of Sets

1. Identity laws

→ $A \cup \phi = A$

$A \cap U = A$

2. Domination Laws

$A \cup U = U$

$A \cap \phi = \phi$

3. Idempotency Laws

$A \cup A = A$

$A \cap A = A$.

4. Complement Laws

$A \cup A' = U$ and $A \cap A' = \phi$

A set is said to be uncountably infinite set if there exist an object which does not belong to set and disagree with each element of the set in ~~one or~~ atleast one way.

5. Involution or complementation Laws

→ $(A')' = A$.

6. Commutative Laws

→ $A \cup B = B \cup A$

$A \cap B = B \cap A$

7. Associative Laws

→ $(A \cup B) \cup C = A \cup (B \cup C)$

$(A \cap B) \cap C = A \cap (B \cap C)$

8. Distributive Laws

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

9. De-Morgan's Law

$(A \cup B)' = A' \cap B'$

$(A \cap B)' = A' \cup B'$.

10. Absorption Laws

$A \cup (A \cap B) = A$

$A \cap (A \cup B) = A$.

Multisets

→ A set is inherently an unordered collection of distinct elements. This unordered collection matters in case of repetative occurance of elements for instance, repeatation occurce when we count each mark obtained by the no. of students in an examination i.e. in case of counting frequency of numbers. This typical situation can be tackled by the concept of multiset.

Multisets are the unordered collection of elements in which an elements can occur more than once. Hence, multiset is a set in which elements are not necessarily distinct.

for eg → $A = \{1, 1, 2, 2, 3, 3, 4, 5, 6, 7\}$.

The notation used to represent a multiset is given by $S = \{m_1 a_1, m_2 a_2, m_3 a_3, \dots\dots, m_n a_n\}$
where the element a_1 appears m_1 times and so on.

The no. m_i is called the multiciplity of element a_i where $i = 1, 2, 3, \dots$

Actually, the multiplicity of an element in a multiset is defined to the no. of items, elements appear in multiset. ~~A~~ set is the special case of multiset in which the multiplicity of each element is 0 or 1.

Operations On Multiset

1. Union of multiset

→ Let A and B be two multisets ~~that's~~ thus the union of multiset A and B is the multiset. where the multiplicity of the elements is maximum of its multic-plities of A and B.

for eg → $A = \{a, a, b, c, d, d\}$
$B = \{a, b, b, c, c, d\}$
⇒ $A \cup B = \{a, a, b, b, c, c, d, d\}$

2. Intersection of Multisets

→ Intersection of multiset A and B is the multiset where multiplicity of element is minium of multiplicity of A and B.

for eg → A = {a, a, b, c, d, d}

B = {a, b, b, c, c, d}

A ∩ B = {a, b, c, d}

3. Sum of Multisets

→ Sum of A and B is denoted by A+B is the multiset in which the multiciplity of elements is the sum of multiciplities in set A and set B.

for eg → A = {a, a, b, b, c, c}

B = {a, b, b, d}

A+B = {a, a, a, b, b, b, c, c, d}

4. Difference of Multisets

→ ~~A and B~~ Difference of multisets A and B is denoted by A-B in which multiciplity of each element is equal to the multiciplities of elements in A minus multiciplity of elements in B. If difference is +ve, it is okay but if it is -ve then it is zero.

for eg → A = {a, a, b, b, c, c}

B = {a, b, b, d}

A-B = {a, c, c}

Q Let A and B be two multiset where

A = {2.a, 3.b, 1.c}

B = {1.a, 4.b, 1.d}

Find ① $A \cup B$ ③ $A + B$
② $A \cap B$ ④ $A - B$.

Sol→ $A = \{a, a, b, b, b, c\}$
$B = \{a, b, b, b, b, d\}$

① $A \cup B = \{a, a, b, b, b, b, c, d\} = \{2 \cdot a, 4 \cdot b, 1 \cdot c, 1 \cdot d\}$
② $A \cap B =$
③ $A + B = \{3 \cdot a, 7 \cdot b, 1 \cdot c, 1 \cdot d\}$
④ $A - B = \{1 \cdot a, 1 \cdot c\}$

Relations

→ A may be thought of a family type b/w, such that is the son of, is the mother of, etc. It may involve equality or inequality. The mathematical concept of relations deals with the way the variables are related or paired. In methametical expression such as <, >, is =, are relations. formally, relations in mathematics, describe connection b/w different elements of the same set whereas fn describe connection b/w two different sets.

for eg→ Set A is the no. of students.
Set B is actual age.

ie $A = \{$Kavita, Sonu, Monu, Rinky, Pinki$\}$
$B = \{14, 15, 13, 10, 12\}$.

The phrase "is younger than" defines a relation b/w any two relation b/w A. It is seem that Kavita is younger that Sonu But Monu is elder to Rinky. On the other hand if we actually want to know how old students are, we need to make connection b/w A & B.

So, age is a f^n which operate b/w A and B.

Binary Relations

Let A and B be two non-empty sets. Then any subset of R of $A \times B$ is called a Binary Relation 'R' from A to B. If $\{a, b\}$ belongs to R, one can say that A is related to B and can be written as the set $\{a \in A, (a,b) \in R$ for some $b \in B\}$ is called the domain of R. and it is denoted by $D_R(R)$. The set $\{b \in B, (a,b) \in R$ for some $a \in A\}$ is called the range of R and it's denoted by $R_R(R)$.

for eg → $A = \{3, 6, 9\}$
$B = \{4, 8, 12\}$

$A \times B = \{(3,4), (3,8), (3,12), (6,4), (6,8), (6,12), (9,4), (9,8), (9,12)\}$.

$R = \{(3,4), (3,8), (6,12)\}$

- If a relation R from $A \to B$ is defined by $(a,b) \in R$ such that 'a' divides 'b' by 0 remainder.

$R = \{(2,4), (2,6), (3,3), (3,6), (4,4)\}$

So, $D_R = \{2, 3, 4\}$
$R_R = \{3, 4, 6\}$.

If R_1 and R_2 are two relation with same domain D and range R, then one define relation as $R_1 \cup R_2$ and $R_1 \cap R_2$.

Q. $A = \{1, 2, 3, 4\}$
$B = \{3, 4, 5, 6\}$

1. $R = \{ a \in A \ R \ b \in B$ i.e. aRb iff $a < b \}$

→ $R = \{ (1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6), (4,5), (4,6) \}$

$D_R = \{1, 2, 3, 4\}$, $R_R = \{3, 4, 5, 6\}$

2. $R = \{ a \in A \ R \ b \in B$ i.e. aRb iff a and b both are odd no. $\}$

→ $R = \{ (1,3), (1,5), (3,3), (3,5) \}$

$D_R = \{1, 3\}$ and $R_R = \{3, 5\}$.

CLASSIFICATION OF RELATIONS

1. REFLEXIVE RELATION

Let R be a relation defined on a set A then R is reflexive if {aRa, holds for all $a \in A$} i.e. if {$(a,a) \in R$ for all $a \in A$}.

Eg → $A = \{1, 2, 3, 4\}$

$R = \{ (1,1), (2,2), (3,3), (4,4) \}$.

2. SYMMETRIC RELATION

A realation R defined on a set A is said to be symmetric if bRa holds whenever aRb for $a, b \in A$ i.e. R is symmetric on A if $(a,b) \in R \Rightarrow (b,a) \in R$.

Eg → $A = \{1, 2, 3\}$

$R = \{ (1,2), (2,1), (2,3), (3,2) \}$

3. ANTI-SYMMETRIC RELATION

A relation R on a set A is called anti-symmetric if for all $(a,b) \in A$, if $(a,b) \in R$ and $a \neq b$. ie $(a,b) \in A$ and $(b,a) \notin R$.

Eg → Let R be a relation on A defined by $(a,b) \in R$ is $a \leq b$, $(a,b) \in A$.

ie $A = \{1, 2, 3\}$

$R = \{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$

Here, $(1,2) \in R$ but $(2,1) \notin R$, so, the relation R is antisymmetric

4. TRANSITIVE RELATION

A relation R on a set A is said to be Transitive if $(a,b) \in R$, $(b,c) \in R \Rightarrow (a,c) \in R$. ie if aRb and bRc ⇒ aRc which means $(a,b,c) \in A$

Eg → Let $A = \{1, 2, 3\}$

$R = \{(1,2), (2,3), (1,3)\}$

Q. $A = \{a, b, c, d\}$

Sol^n $R_1 = \{(a,a), (b,c), (c,b), (d,b)\}$

→ Non-Transitive.

$R_2 = \{(b,b), (b,c), (c,b), (d,d)\}$

→ Transitive

$R_3 = \{(c,c), (c,d), (d,c), (d,d)\}$

→ Transitive.

- For eg → 1. Let A denote the set of straight line on a plane and R be a relation on A defined as 'is parallel to' then R is a Transitive relation on A.

Q. Set $A = \{1, 2, 3\}$

1. $R = \{(1,1), (2,2), (3,3), (1,3), (1,2)\}$

→ Reflexive and Transitive.

2. $R = \{(1,1), (2,2), (1,3), (3,1)\}$

→ Symmetric and Transitive.

3. $R = \{(1,1), (2,2), (3,3), (1,2), (2,1), (2,3), (3,2)\}$

→ Reflexive, symmetric and Transitive.

Equivalence Relation

A relation R on a set A is said to be an equivalence relation if R is reflexive, Transitive and symmetric.

for eg → $A = \{a, b, c\}$

$R = \{(a,a), (a,b), (b,a), (b,b), (b,c), (a,c), (c,a), (c,c)\}$

Thus, R is an equivalence relation on A.

Closure Of Relations

The closure of a relation is the smallest extension of a relation that has specific certain properties that has reflexivity, Transitivity and symmetry.

Let R be a relation on a set A, R may or may not have certain property P such as reflexivity, symmetry or Transitivity. ∃ a relation T with property P containing R such that T is a subset of every relation with property P containing R, then T is known as the

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closure of R w.r.t ~~R~~.P.

Reflexive Closure

A relation R-prime is the reflexive closure of a relati R iff

(i) R-prime is reflexive

(ii) $R \subseteq$ R-prime.

~~R~~ (iii) For any relation R'' if $R \subseteq R''$ is reflexive then R' is a subset R''. ie R' is the smallest relation that satisfies (i) and (ii). The reflexive of a relation R is denoted as $r(R)$.

Eg → $R = \{(a,b), (b,a), (b,b), (c,b)\}$

$A = \{a, b, c\}$

Now, let us consider R' which contains R as well as (a,a) and (c,c) ie $R' = \{(a,b), (b,a), (b,b), (c,b), (a,a), (c,c)\}$.

Furthermore any of the reflexive relation say R'' containing R must also contain {a,a} and {c,c} (otherwise it will not be reflexive). So, $R' \subseteq R''$.

As R' contains R is reflexive and is contained in every reflexive relation that contains R so, R' is the reflexive closure of R.

Symmetric Closure

~~Sy~~ A relation R-prime is a symmetric closure of a relation R iff

(i) R' is symmetric

(ii) $R \subseteq R'$

(iii) for any relation R'' if $R \subseteq R''$ and R'' is symmetric then $R' \subseteq R''$ i.e. R' is the smallest relation that satisfies both (i) and (ii). The symmetric closure of a relatic R is denoted as $s(R)$.

Eg → Let us consider a relation R.

$R = \{(a,a), (a,b), (c,c), (b,c), (b,a), (a,c)\}$ on the set $A = \{a,b,c\}$. Clearly R is not symmetric. To be symmetric R needs $\{c,a\}, \{c,b\}$ Let the Relation R' contain R as well as $\{(c,a), (c,b)\}$ i.e.

$R' = \{(a,a), (a,b), (c,c), (b,c), (b,a), (a,c), (c,a), (c,b)\}$.

Clearly R' is symmetric and $R \subseteq R'$. furthermore, any other symmetric relation ~~that contains~~ say R'' containing R must also contain (c,a) and (c,b). Now, R' contains R is symmetric as is contained within every symmetric relation that contains R i.e. $R' \subseteq R''$. So, R' is a symmetric closure of R.

Transitive Closure

A relation R' is a transitive closure of a relation R iff

(i) R' is transitive.

(ii) $R \subseteq R'$

(iii) For any relation R'', if $R \subseteq R''$ ~~for~~ and R'' is transitive then $R' \subseteq R''$ i.e. R' is the smallest relation that satisfies (i) and (ii). the transitive of a relation R is denoted as $t(R)$.

Eg → Let us consider a relation R.

$R = \{(a,a), (b,b), (a,c) (c,b), ~~(c,b)~~, (b,c)\}$ on the set $A = \{a, b, c\}$. Clearly R is not transitive. To be transitive R needs $\{a,b\}, \{c,c\}$ i.e.

$R' = \{(a,a), (b,b), (c,c), (a,c), (c,b), (a,b), (b,c)\}$

Clearly R' is Transitive and $R \subseteq R'$. Furthermore, any other transitive relation say R'' containing R must also contain (c,c) and (a,b). Now, R' contains R is transitive as is contained within every transitive relation that contains R ie $R' \subseteq R''$. So, R' is transitive closure of R.

Pigeon-hole Principale

A well-known proof technique in mathematics is the so called pigeon-hole principale also known as shoe-box arguement. The pigeon-hole principle says that if there are many pigeons and a few pigeon hole then there must be some pigeon hole occupied by two or more pigeon. Formally, let D and R be finite sets if $|D| > |R|$ then for any f^n f from $D \to R$ there exists $(d_1, d_2) \in D$ such that $f(d_1) = f(d_2)$.

Some applications of pigeon-hole principle.

1. Among 30 people there are atleast 2 of them who are born in the ~~same~~ same month. Here, 30 people are pigeon and 12 months are pigeon holes. Similarly if there are 8 people randomly selected from hole there are atleast two of them who are born on the same day. The pigeon hole principle can be stated in a slightly more general form.

" for any function f $D \to R$ $\exists$ i elements $d_1, d_2, \ldots, d_i$

$\lfloor \rfloor$ - largest integer

in D, $i = \lceil |D|/|R| \rceil$ such that $f(d_1) = f(d_2) = f(i)$

Extended pigeon hole principle

If the no. of pigeons is much larger then the no. of pigeon holes such as more than double, the no. of pigeon holes then extended. pigeon hole principle is applied. Say if they are x pigeons and y pigeon holes then 1 pigeon hole must be occupied by atleast $\lfloor (x-1)/y \rfloor + 1$

Q. If there are 30 magazines containing the no. of pages 61324 then one of the magazines must have atleast 2045 pages.

Sol^n → Let us consider the pages as pigeon and the magazines as pigeon holes. Assign each page to the magazine in which it appears then by extended pigeon hole principle, 1 magazine must contain atleast

$$\left\lfloor \frac{61324-1}{30} \right\rfloor + 1 = \left\lfloor \frac{61323}{30} \right\rfloor + 1$$

$$= \lfloor 2044.1 \rfloor + 1$$

$$= 2044 + 1$$

$$= 2045.$$

Q. Show that if 7 nos are taken from 1 - 12. And the choosing 2 no.s will add upto 30.

Q. 10 people came forward to volunteer for a 3 person commitee. Every possible commitee of 3 that can be formed from these 10 names is written on a slip of paper. One slip for each possible commitee and the slip had put in 10 hats. Show that atleast one hat contains 12 or more slips of paper.

Sol → Let slips be pigeons and hats be pigeon holes.
(Applying extended pigeon hole principle because no. of slips are 120.

$$^{12}C_3 = \frac{12!}{3! \times 9!} = 120.$$

$$\left\lfloor \frac{120-1}{10} \right\rfloor + 1$$

$$= \left\lfloor \frac{119}{10} \right\rfloor + 1$$

$$= \lfloor 11.9 \rfloor + 1$$

$$= 11 + 1$$

$$= 12.$$

Irreflexive Relation

→ A relation R defined on a set A is irreflexive if $a \not R a$ for every $a \in A$.

for eg. Let $A = \{1, 2, 3\}$
Let $R = \{(1,2), (2,3), (3,1), (2,1)\}$
then the relation R is irreflexive on A.

Assymmetric Relation

→ A relation R defined on a set A is assymmetric if whenever aRb then bRa is not possible i.e. $b \not R a$.

for eg → Set $A = \{a, b, c\}$
and $R = \{(a,b), (b,c)\}$
be a relation on A clearly R is assymmetric.

Mathematical Induction

Mathematical Induction

Suppose the statement to be proved can be proved in the form for all $n > n_0$. $P(n)$, where n_0 is a fixed integer ie suppose we wish to show that $P(n)$ is true for all integers $n \geq n_0$. The following result show how this can be done suppose that

(a) $P(n_0)$ is true.

(b) If $P(k)$ is true for some $k \geq n_0$ then $P(k+1)$ must also be true.

Then $P(n)$ is true $\forall\ n \geq n_0$. This principal is called Principle of mathematical Induction. Thus, to prove the truth of a statement $\forall\ n \geq n_0$, we must begin by/with proving directly that the first proposition $P(n_0)$ is true. This is called the basis step of the induction. Then we must prove that $P(k) \Rightarrow P(k+1)$ is tautology for any choice of $k \geq n_0$. Since, the only case where implication is false is if the ~~in~~ antecident is true and the consequence is false. This step is usually done by showing that if $P(k)$ is true then $P(k+1)$ would also have been true. This step is called the induction step. And the same work will usually be required to show that the implication will also be always true.

Q. Show by mathematical induction $\forall\ n \geq 1$.

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

Sol→ Let $p(n)$ be the predicate

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

Basis step

We must show that $p(1)$ is true. So,

$$p(1) = \frac{1(1+1)}{2} = 1.$$

which is clearly true.

Induction step

We must now show that for $K \geq 1$ if $p(K)$ is true then $p(K+1)$ must also be true. We assume that for some fixed $K \geq 1$

$$1+2+3+\cdots+K = \frac{K(K+1)}{2}$$

We now wish to show the truth of $p(K+1)$

$$1+2+3+\cdots+K+K+1 = \frac{K(K+1)}{2}+(K+1)$$

$$= (K+1)\left(\frac{K}{2}+1\right)$$

$$= \frac{(K+1)(K+2)}{2}$$

$$= \frac{(K+1)(K+1+1)}{2} \quad \left(\text{R.H.S of } p(K+1)\right)$$

~~R.H.S of p(K+1)~~

Thus, we have to show that L.H.S of $p(K+1)$ = R.H.S of $p(K+1)$. Hence, it follows $p(n)$ is also true for all n, $n \geq 1$.

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Q1. Show that $1^2+2^2+3^2+\ldots\ldots n^2 = \frac{n(n+1)(2n+1)}{6}$ $\forall$ $n \geq 1$.

Q2. Let $A_1+A_2+\ldots+A_n$. Show by mathematical induction that $\overline{\bigcup_{i=1}^{n} A_i} = \left(\bigcap_{i=1}^{n} \overline{A_i}\right)$

Q3. $2+4+6+\ldots+2n = n(n+1)$. Prove by mathematical induction.

Q4. Prove by mathematical induction that
$1^2+3^2+5^2+\ldots\ldots+(2n-1)^2 = \frac{n(2n+1)(2n-1)}{3}$.

~~Sol. Let P(n) be~~

Q5. $1+2+2^2+2^3+\ldots\ldots+2^n = 2^{n+1}-1$

Q6. $5+10+15+\ldots\ldots+5n = 5n(n+1)/2$

Q7. $1+a+a^2+\ldots+a^{n+1} = \frac{a^{n-1}}{a-1}$

Q8. ~~$1^3+2^3+3^3+\ldots+\alpha$~~

Sol 1. Let $P(n)$ be the predicate
$1^2+2^2+3^2+\ldots\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$

Basis step
We must show that $P(1)$ is True. So.
$P(1) = \frac{1(1+1)(1+2)}{6} = 1$.

which is clearly True.

Induction step

We must now show that for $k \geq 1$, if $p(k)$ is true then $p(k+1)$ must also be true. We assume that for some fixed $k \geq 1$

$$1^2+2^2+3^2+\cdots+k^2 = \frac{k(k+1)(2k+1)}{6} \text{ is true.}$$

We now wish to show the truth of $p(k+1)$

$$1^2+2^2+3^2+\cdots+k^2+(k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

$$= (k+1)\left(\frac{k\cdot(2k+1) + 6(k+1)}{6}\right)$$

$$= (k+1)\frac{(2k^2+k+6k+6)}{6}$$

$$= (k+1)\frac{(2k^2+7k+6)}{6}$$

$$= (k+1)\frac{(2k^2+4k+3k+6)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$= \frac{(k+1)(k+2)(2(k+1)+1)}{6}$$

Q. $2+4+6+\cdots+2n = n(n+1)$.

Sol. Let $p(n)$ be the predicate

$$2+4+6+\cdots+2n = n(n+1)$$

Basis step

We must show that $p(1)$ is true. So,

$P(1) = 1(1+1) = 2$
which is clearly true.

Induction step
We must now show that for $k \geq 1$, if $P(k)$ is true then $P(k+1)$ must also be true. We assume that for some fixed $k \geq 1$

$$2+4+6+\cdots+2k = k(k+1) \quad \text{is true.}$$

We now wish to show the truth of $P(k+1)$

$$\begin{aligned} 2+4+6+\cdots+2k+2(k+1) &= k(k+1)+2(k+1) \\ &= k^2+k+2k+2 \\ &= k^2+3k+2 = (k+1)(k+2) \\ &= (k+1)(k+1+1) \quad \text{proved.} \end{aligned}$$

Q $1^2+3^2+5^2+\cdots+(2n-1)^2 = \frac{n(2n+1)(2n-1)}{3}$

Sol: Let $P(n)$ be the predicate

$$1^2+3^2+5^2+\cdots+(2n-1)^2 = \frac{n(2n+1)(2n-1)}{3}$$

Basis step
We must show that $P(1)$ is true. So,

$$P(1) = \frac{1(2+1)(2-1)}{3} = \frac{3}{3} = 1$$

which is clearly true.

Induction step
We must now show that for $k \geq 1$, if $P(k)$ is true then $P(k+1)$ must also be true. We assume that for some fixed $k \geq 1$

$1^2+3^2+5^2+\cdots+(2k-1)^2 = \frac{k(2k+1)(2k-1)}{3}$ is true.

We now wish to show the truth of $P(k+1)$.

$1^2+3^2+5^2+\cdots+(2k-1)^2+(2k+1)^2 = \frac{k(2k+1)(2k-1)}{3}+(2k+1)^2$

$= \frac{k(2k+1)(2k-1)+3(2k+1)^2}{3}$

$= (2k+1)\left(\frac{k(2k-1)+3(2k+1)}{3}\right)$

$= \frac{(2k+1)(2k^2-k+6k+3)}{3}$

$= \frac{(2k+1)(2k^2+5k+3)}{3}$

$= \frac{(2k+1)(2k^2+2k+3k+3)}{3}$

$= \frac{(2k+1)(k+1)(2k+3)}{3}$

$= \frac{(k+1)(2(k+1)+1)(2(k+1)-1)}{3}$ proved.

Q $1+2+2^2+\cdots+2^n = 2^{n+1}-1$

Sol→ Let $P(n)$ be the predicate

$1+2+2^2+\cdots+2^n = 2^{n+1}-1$

Basis step

We must show that $P(1)$ is true. So,

$P(1) = 2^{1+1}-1 = 3$,

which is clearly true.

Induction step

We must now show that for $k \geq 1$, if $P(k)$ is true then $P(k+1)$ must also be true. We assume that for some fixed $k \geq 1$

$1+2+2^2+ \dots + 2^k = 2^{k+1}-1$ is true.

We now wish to show the truth of $P(k+1)$.

$$\begin{aligned} 1+2+2^2+ \dots + 2^k + 2^{k+1} &= 2^{k+1}-1 + 2^{k+1} \\ &= 2 \cdot 2^{k+1} - 1 \\ &= 2^{k+1+1} - 1 \end{aligned}$$ proved.

Q $5+10+15+ \dots + 5n = \frac{5n(n+1)}{2}$

Soln Let $P(n)$ be the predicate.

$5+10+15+ \dots + 5n = \frac{5n(n+1)}{2}$

Basis step

We must show that $P(1)$ is true. So,

$P(1) = \frac{5(1+1)}{2} = 5$.

which is clearly true.

Induction step

We must now show that for $k \geq 1$, if $P(k)$ is true then $P(k+1)$ must also be true. We assume that for some fixed $k \geq 1$

$5+10+15+ \dots + 5k = \frac{5k(k+1)}{2}$ is true.

We now wish to show the truth of $P(k+1)$.

$$5+10+15+\cdots+5k+5(k+1) = \frac{5k(k+1)}{2}+5(k+1)$$

$$= \frac{5k(k+1)+2.5(k+1)}{2}$$

$$= \frac{5(k+1)(k+2)}{2}$$

$$= \frac{5(k+1)(k+1+1)}{2}$$ proved

Q $1+a+a^2+\cdots+a^{n-1} = \frac{a^{n-1}}{a-1}$

Sol^n → Let $p(n)$ be the predicate.

$1+a+a^2+\cdots+a^{n-1} = \frac{a^{n-1}}{a-1}$

Basis Step

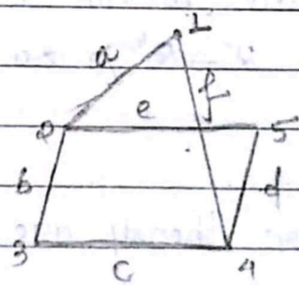
We must show that $P(1)$ is true. so,

$P(1) = \frac{a^{1-1}}{a-1}$

Test

Q1 What is incidence and degree. Explain with an example
Q2. Explain transitive closure with an example
Q3. Write applications of graph theory.
Q4. Explain the inference rule modus Ponen, ~~and~~ modus tollen and syllogism.

Ans1 When a vertex v_i is an end vertex of some edge e_j, v_i and e_j are said to be incident on each other.
The number of edges incident on a vertex v_i with self loops counted twice is called degree denoted by $d(v_i)$.
for eg →



Here, vertex 1 and edge a are incident on each other and vertex 1 ~~d~~ have degree 2 i.e $d(1) = 2$.

Q5 ~~Ans2~~ Explain multigraph and weighted graph.
Q6 Explain directed graph, undirected graph, symmetric digraph and asymmetric digraph.

Ans2 A relation R' is a transitive closure of a relation R iff
(i) R' is transitive
(ii) $R \subseteq R'$
(iii) for any relation R'' is the ~~smallest relation~~ that satisfies (i) and (ii) transitive then $R' \subseteq R''$ i.e. R' is

the smallest relation that satisfies (i) and (ii). The transitive of a relation R is denoted as $t(R)$.

For eg→ let us consider a relation R
$R = \{(a,a), (b,b), (a,c), (c,b), (b,c)\}$ on the set
$A = \{a,b,c\}$.
clearly R is not transitive. To be transitive R needs $\{a,b\}, \{c,c\}$ ie.
$R' = \{(a,a), (b,b), (c,c), (a,c), (c,b), (b,c), (a,b)\}$
Clearly R' is transitive and $R \subseteq R'$. furthermore any other relation say R'' containing R must also contain (c,c) and (a,b). Now, R' contains R is transitive as is contained within every transitive relation that contains R ie $R' \subseteq R''$. So, R' is transitive closure of R.

Ans 3 Applications of graph theory are :-

1. Konigsberg Bridge Problem →
→ Two islands C and D formed by the Pregel River in konigsberg were connected to each other and to the banks A and B with seven bridges. The problem was to start at any of the four land areas of the city A, B, C and D walk over each of the seven bridges exactly ones and return to the starting point (without swimming across the river).

2. Seating problem
→ 9 members of a new club meet each day for lunch at round table. They decide to sit such that

every member has different neighbours at each lunch.

3. Utilities Problem

→ There are 3 houses H_1, H_2, H_3 each to be connected to each of the utilities - water (W), gas (G) and electricity (E) by means of conduits.

4. Electrical Network Problem

→ Properties of an electrical network are f's of only 2 factors :-

(i) The nature and value of the elements forming the network such as resistors, inductors, transistors and so forth.

(ii) The way these elements are connected together, i.e., topology of the network.

Ans 4 The tautology $(p \land (p \to q)) \to q$ is the basis of rule of inference is called Modus Ponen

The arguements are -

$S_1 = p$, $S_2 = p \to q$, $c = q$ (c is conclusions

so, $S_1 \land S_2 \to q$

$(p \land (p \to q)) \to q$

The tautology $(\bar{q} \land (p \to q)) \to q$ is the basis of rule of inference is called Modus Tollen.

The tautology

Q7. Show that $\bar{P}$ is tautologically implied by $(\overline{P \wedge \bar{Q}})$, $\bar{Q} \vee r$, r

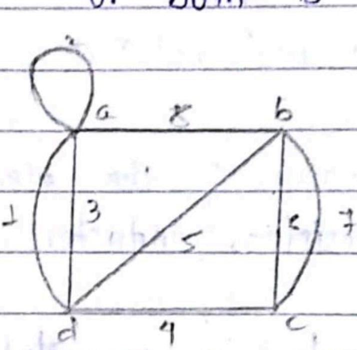
Q8. Show it by PMI.,

$$1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

Q9. Explain Pigeon hole principle with an example

Ans 5 <u>Multigraph</u> → Graph having parallel edges or self loop or both is called multigraph.

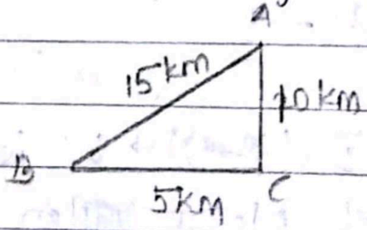
for eg.



i.e multigraph is a graph with parallel edges and several loops.

<u>Weighted graph</u> → A graph in which each edge or each vertex is associated with some value which may be cost, distance, etc is called weighted graph. When we add two information then graph is weighted.

Eg →



In above fig; A, B and C are cities and the edges represent distance between them.

Ans 7.

1.	$\overline{p \wedge \bar{q}}$	(Rule P)
2.	$\overline{p \wedge \bar{q}} = \bar{p} \vee q$	(De Morgan's law)
3.	$\bar{p} \vee q = p \to q$	(Rule T)
4.	$\bar{q} \vee r = q \to r$	{Rule P, Rule T}
5.	$p \to r$	{Rule T, syllogism}
6.	$\bar{r}$	(Rule P)
7.	$\bar{p}$	(using rule T, modus Tollen).

Soln 8. Let $P(n)$ be the predicate.

$$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

Basis rule

We must show that $P(1)$ is true, so,

$$P(1) = \frac{1(2-1)(2+1)}{3} = 1$$

which is clearly true.

Inductive step

We must now show for $k \geq 1$, if $P(k)$ is true then $P(k+1)$ must also be true. We assume that for some fixed $k \geq 1$

$$1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2 = \frac{k(2k-1)(2k+1)}{3}$$

is true.

We now wish to show the truth of $P(k+1)$.

$$1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2 + (2k+1)^2 = \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2$$

$$= \frac{k(2k-1)(2k+1) + 3(2k+1)^2}{3}$$

$$= (2k+1)\left(\frac{k(2k-1) + 3(2k+1)}{3}\right)$$

$$= (2k+1)\frac{(2k^2 - k + 6k + 3)}{3}$$

$$= (2k+1)\frac{(2k^2 + 5k + 3)}{3}$$

$$= (2k+1)\frac{(2k^2 + 2k + 3k + 3)}{3}$$

$$= (2k+1)\frac{(k+1)(2k+3)}{3}$$

$$= (k+1)\frac{(2(k+1)-1)(2(k+1)+1)}{3}$$

Ans 9 A well known proof technique in mathematics is the so ~~well~~ called pigeon hole principal. also known as shoe-box arguement. The pigeon hole principal says that if there are many pigeons and a few pigeon hole then there must be some pigeon hole occupied by two or more pigeon. formally, Let D and R be a finite set if $|D| > |R|$ then for any function f from $D \to R$ there exists $(d_1, d_2) \in D$ such that $f(d_1) = f(d_2)$

Eg→ Among 30 people there are atleast 2 of them who are born in the same month. Here, 30 people are pigeon and 12 monthe are pigeon hole.

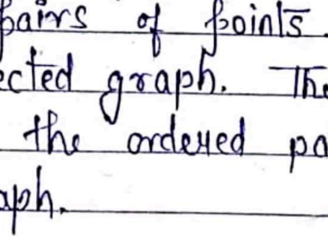
Q10 A store has an introductory sale on 12 types of chocolate. A customer may choose 1 chocolate from any of 5 choices. and He will be charged no more than Rs. 1.75.
Show that although diff choices may cause diff amount there must be atleast 2 diff ways so that cost will be same for both choices.

Q11 ~~Exp~~ Write short ^not^ on tautology and logical equivalence.
Q12 What is multiset. Explain with an example.

Ans12 Multisets are the unordered collection of elements in which an element can occur more than once. Hence, multset is a set in which elements are not necessarily distinct.
for eg → $A = \{1, 1, 2, 2, 3, 3, 4, 5, 6, 7\}$.

Ans6 Directed Graph

→ A directed graph or digraph G is defined abstractly as an ordered pair (V, E), where V is a set and E is a binary relation on V. A directed graph can be represented geometrically as a set of marked points V with a set of arrows E between pairs of points. for eg → Figure given below shows a directed graph. The elements in V are called vertices, and the ordered pairs in E are called edges of the directed graph.



fig

Undirected graph

→ A undirected graph G is defined abstractly as an ordered pair (V, E), where V is a set and E is a set of multisets of two elements from V.
for eg → $G = (\{a, b, c, d\}, \{\{a, b\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, c\}\})$ is an

undirected graph. As undirected graph can be represented geometrically as a set of marked points V with a set of line E between the points.

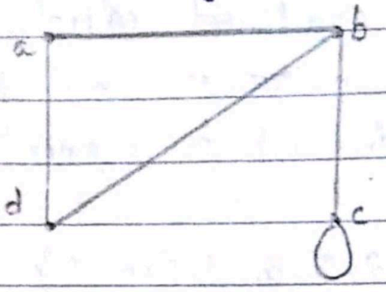
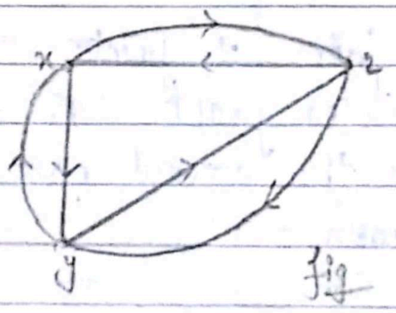


Fig:

Symmetric digraph

→ Digraphs in which for each edge (x,y), there is also an edge (y,x), are known as symmetric digraph.



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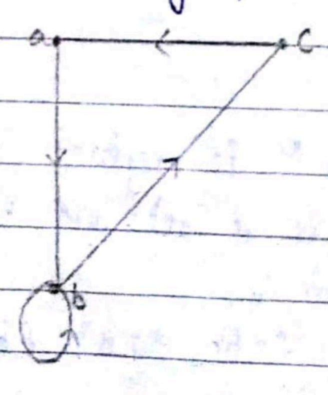
Asymmetric digraph

→ Digraphs having only one directed edge between ~~edo~~ each pair of vertices but can have self-loops are called asymmetric digraphs.



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Sol^n 10 Now the ~~con~~ customer can choose the chocolates (pigeons) in $^{12}C_5 = 792$ ways. The charge is Rs. 1.75 or 175 paisa.
Now, by extended pigeon hole principle, we have

$$\left\lfloor \frac{792-1}{175} \right\rfloor + 1$$

$= 5$ choices having the same cost.
So, there must be atleast two diff. ways to choose, so that the cost will be the same for both choices.

Ans 11 A statement formula that is true for all possible values of its propositional variables is called tautology.
for eg → $(p \vee q) \leftrightarrow (q \vee p)$ is tautology.

~~Two statement formula~~
Two propositions P and Q are said to be logically equivalent or simply equivalent if $P \rightarrow Q$ is a tautology.
Eg → $\sim(p \wedge q)$ and $\sim p \vee \sim q$ are logically equivalent

PERMUTATION AND COMBINATION

Rule of Product

If one experiment have 'm' possible outcomes and other experiment has 'n' possible outcome then there are $m \times n$ possible outcomes when both the experiment take place.

Rule of Sum

If one experiment has 'm' possible outcomes and another has 'n' possible outcome then there are $m+n$ possible outcomes when exactly one of the experiment takes place.

Q There are 52 ways to select a class representative of B.Sc IT and there are 49 ways to select a class representative of B.Sc. CA then by using rule of product How many ways to select CR.

→ There are 52×49 ways to select CR of both CA and IT.
On the other hand by using rule of sum there are $52 + 49$ ways to select a CR from CA or IT.

Permutation

Suppose we are to place 'r' distinct coloured balls in 'n' distinctly numbered boxes with the condition that a box can hold only one ball. Since the 1st balls can be placed in any of the n boxes then 2nd ball can be placed in any of the $n-1$ boxes and the r^{th} ball will be placed in $n-r+1$ boxes. Therefore,

total no. of distinct ways to place balls in boxes is
$n(n-1)(n-2) \cdots (n-r+1)$

ie $\frac{n!}{(n-r)!}$

Q1. In how many ways can 3 examination be scheduled within a 5 day, period so that no 2 exam will be scheduled on the same day.

Sol → $5 \times 4 \times 3 = 60$ ways Ans.

Q2. Suppose that you have 7 rooms and you want to assign 4 rooms to the programmers and remaining 3 rooms as office.

Sol →

Q3. Let us determine the no. of 4 digit decimal no. that contains no repeated digits.

Sol →

COMBINATION

Permutation with Like Elements

$$\frac{n!}{q_1!\, q_2!\, q_3! \ldots q_n!}$$

where

Circular Permutation

A circular Permutation of n objects is an arrangement of objects around a circle. In a circular arrangement we have to consider the relative position of the diff. things. The circular permutation is diff. only when the relative order of the objects are changed otherwise they will be same.

$(n-1)!$

Number of different circular permutation

We consider a order clockwise or anti-clockwise of objects around a circle as a same circular permutation Every arrangement of n object round the circle is counted twice in $(n-1)!$ circular permutation. Therefore, the total no. of diff. permutation of n distinct object when we consider the order is

$$\frac{(n-1)!}{2}$$

Sampling with Replacement

If the ball drawn will be replaced in the box before the next ball is drawn. There are n diff. ways each time to choose a ball. There are n^r diff. ordered sample with replacement of size r.

Sampling without Replacement

$$\frac{n!}{(n-r)!}$$

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Combination

Consider now a problem of placing three balls all of them coloured in red in 10 boxes numbered 1, 2, ⋯ 10. We want to know the no. of balls that can be placed when each box can hold only one ball.

→ $^{10}C_3 = \frac{10!}{7! \times 3!} = \frac{10 \times \cancel{9}^{3} \times \cancel{8}^{4}}{\cancel{3} \times \cancel{2}} = 120$.

Q. We want to determine the no. of ways to see 5 boys in a row of 12 chairs.

→ $^{12}C_5 = \frac{12!}{5! \times 7!} = \frac{\cancel{12} \times 11 \times \cancel{10} \times 9 \times 8}{\cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2}} = 792$

Q Suppose a housekeeper wants to schedule thrice a week as spegity dinner.

→ $^{7}C_3 = \frac{7!}{3! \times 4!} = \frac{7 \times \cancel{6} \times 5}{\cancel{6}} = 35$

Q 11 students are there to form 5 student committee.

→ $^{11}C_5 = \frac{11!}{5! \times 6!} = \frac{11 \times \cancel{10} \times \cancel{9}^{3} \times \cancel{8}^{2} \times 7}{\cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2}} = 462$

Q 11 students are there to form 5 student committee and student A must be there everytime.

→ $^{10}C_4 = \frac{10!}{4! \times 6!} = \frac{\cancel{1} \times 10 \times \cancel{9}^{3} \times \cancel{8} \times 7}{\cancel{4} \times \cancel{3} \times \cancel{2}} = \cancel{330}\ 210$

Q 11 students are there to form 5 student commitee and student A should not be there in committee.

→ $^{10}C_5 = \frac{10!}{5! \times 5!} = \frac{\cancel{10} \times 9 \times \cancel{8}^{2} \times 7 \times \cancel{6}^{2}}{\cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2}} = 252$

Q1 There are 100 students 40 boys and 60 girls. We want to form a committee of 10 students.
(i) It should have equal no. of boys and girls.
(ii) It should have 6 boys and 4 girls.

Q2. A student has to answer 8 questions out of 10.
(i) How many choices does the student have.
(ii) How many choices does the student have if she answers 1st 3 question.
(iii) How many choices does the student have if she has to answer 4 questions out of first 5 questions.

Solⁿ 1 There are 100 students with 40 boys and 60 girls.
(i) If committee should have equal no. of boys and girls then no. of boys and girls should be selected is 5.
So, No. of ways to select 5 boys out of 40 is $^{40}C_5$
and No. of ways to select 5 girls out of 60 is $^{60}C_5$
$\therefore$ Total no. of ways $= {}^{40}C_5 + {}^{60}C_5$

$$= \frac{40!}{5! \times 35!} + \frac{60!}{5! \times 55!}$$

$$= \frac{\overset{8}{\cancel{40}} \times \overset{13}{\cancel{39}} \times 38 \times 37 \times 36}{\cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2}} + \frac{\overset{12}{\cancel{60}} \times 59 \times \overset{29}{\cancel{58}} \times 57 \times 56}{\cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2}}$$

$$= 658008 + 5461512$$

$$= 6119520$$

(ii) No. of ways $= {}^{40}C_6 + {}^{60}C_4$

$$= \frac{40!}{6! \times 34!} + \frac{60!}{4! \times 56!}$$

$$= 3838380 + 487635$$

$$= 4326015$$

Sol^n (i) ${}^{10}C_8 = 45$

(ii) ${}^{7}C_5 = 21$

(iii) ${}^{5}C_4 = 5$

FUNCTION

Binary relation R from $A \rightarrow B$ is said to be a function if for every element a in A there is a unique element b in B such that (a,b) is in R, for a function R from $A \rightarrow B$ we use the notation $R(a) = b$, where b is called the image of a. The set a is called the domain of function R and set b is called range of f^n R.

for eg → Let A be the set of houses and B be the set of colours then a function from $A \rightarrow B$ is an assignment of colours for painting the house.

	α	β	γ	δ
a	✓			
b			✓	
c			✓	
d		✓		
e		✓		

fig. 1(b)

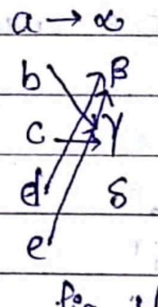

fig 1(a)

	Range
a	α
b	γ
c	γ
d	β
e	β

fig 1(c)

A f^n can be represented in a graphical form. fig. 1(a) shows a f^n R from $A \to B$ where $A = \{a, b, c, d, e\}$ and $B = \{\alpha, \beta, \gamma, \delta\}$ ① A function from $A \to B$ is said to be an <u>onto function</u> if every element of B is the image of one or more elements of A.

② <u>Into function</u> → A function from $A \to B$ is said to be into if there exist even in a single element in B having no preimage in A.

③ A function is said to be <u>one-to-one onto</u> function if its is both an onto and one-to-one function.

A function from $A \to B$ is said to be <u>one-to-one</u> if no two elements of A have the same image.

- An onto function is called a <u>surjection</u>.
- A one-to-one function is called an <u>injection</u>.
- One-to-one onto function is called a <u>bijection</u>.

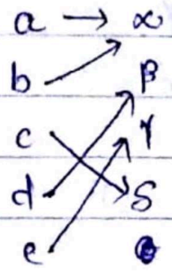

fig. 2(a)

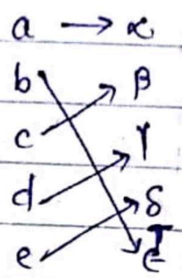

fig. 2(b)

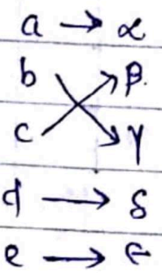

fig. 2(c)

fig 2(a) shows an onto function. fig. 2(b) is an example of one-to-one function. fig 2(c) is an example of one-to-one onto function.

Composition Of Function

A composite function, formed by the composition of one f^n to another, refers to the combining of functions in a manner where the output from one function becomes input for the next function. Let f be a function from $A \to B$ is g be a function from $B \to C$. The function $f: A \to B$ and $g: B \to C$ can be composed by first applying f to an arguement $a \in A$ and the applying g to the result. We can define a f^n h from $A \to C$ such that for every a in A $h(a) = g(f(a))$. The f^n h is called the composition of functions or relative product of f and g and it is denoted as gof. Mathematically, if $f: A \to B$ and $g: B \to C$ be two functions then $h = gof =$ $\{(a, c) : (a \in A) \wedge (c \in C) \wedge (\exists b)(b \in B \wedge b) = f(a) \wedge c = g(b)\}$

More precisely gof is called the left composition of g with f.

Q Let f and g be the f^n's from the set of integers defined by $f(x) = 2x+3$, $g(x) = 3x+2$. Determine fog and gof.

Sol → $f(x) = 2x+3$ and $g(x) = 3x+2$

$$fog = f\{3x+2\}$$
$$= 2(3x+2)+3$$
$$= 6x+7.$$

$$gof = g\{2x+3\} = 3(2x+3)+2 = 6x+11.$$

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Q. $f(x) = x+1$ and $g(x) = x^2+3$. find fog & gof

Sol^n→ $f(x) = x+1$

$g(x) = x^2+3$

So, $fog = f(x^2+3) = (x^2+3)+1 = x^2+4$

and

$gof = g\{x+1\} = (x+1)^2+3 = x^2+1+2x+3$

$= x^2+2x+4$.

~~Invertible~~ ~~function~~.

Q1. ~~Let~~ Suppose an urn contain 5 balls. find the no. of ordered sample of size 2

(i) with replacement.

(ii) without replacement.

Sol→ (i) There are 5 balls, and each ball in the ordered sample can be chosen in 5 ways

Hence, there are $5 \times 5 = 25$ samples with replacement.

(ii) The first ball in the ordered sample can be chosen in 5 ways and the next ball in the ordered sample can be chosen in 4 ways (when the first drawn ball is not replaced).

There are. $5 \times 4 = 20$, samples without replacement.

Q2 In how many ways can one chose two cards in succession from a deck of 52 cards, such that the first chosen card is not replaced.

Sol^n→ There are 52 cards in the deck of cards since the chosen card is not replaced the first can be chosen in 52 diff. ways and the second can be in 51 different ways.

∴ The number of ways in which the 2 cards are chosen
$= 52 \times 51 = 2652$.

Q3 A box contains 10 ~~digit~~ light bulbs. Find the no. n of ordered samples of :
(a) size 3 with replacement, and
(b) size 3 without replacement.

Sol→ (a) $n = 10$
$= 10^3 = 10 \times 10 \times 10 = 1000$

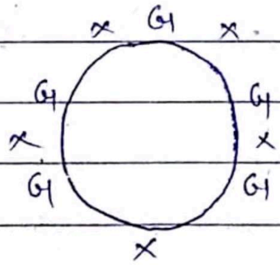
(b) ${}^{10}P_3 = 10 \times 9 \times 8 = 720$.

Q4 In how many ways can a party of 9 persons arrange themselves around a circular table.

Sol→ One person can sit at any place in the circular table. The other 8 persons can arrange themselves in 8! ways i.e. the 9 persons can be arranged among themselves round the table in $(9-1)! = 8!$ ways.

Q5 In how many ways 5 gents and 4 ladies dine at a round table, if no two ladies are to sit together?

Sol→



Since no two ladies are to sit together, they should seat themselves in betn gents i.e. lady is to be seated in betn two gents. The 5 gents can sit round the circular table in 5 positions (marked G in fig). They can be arranged in $(5-1)! = 4!$ ways. The ladies can sit in the 4 out of 5 seats (marked x in fig.). This can be done in $^5P_4$ ways.

The required no. of ways in which 5 gents and 4 ladies can sit round a table =

$= 4! \times {}^5P_4$

$= 4\times3\times2 \times 5\times4\times3\times2$

$= 2880$.

Q6. Twelve persons are made to sit around a round table. Find the no. of ways they can sit that two specified are not together.

Sol→ 12 persons can sit around a round table in $(12-1)! = 11!$ ways.

The total no. of ways in which 2 specified persons are together is $2!\,10!$.

The required no. of seating arrangements in which 2 specified persons are not together.

$= 11! - 2!\,10!$

$= 11\cdot10! - 2!\,10!$

$= 10!\,(11-2)$

$= 9\cdot10!$ Ans.

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Invertible Functions

Let $f: A \to B$ if $\exists$ a function $g: B \to A$ such that $gof = I_A$
$fog = I_B$
Then f is called an invertible function g and g is called inverse of f. We write $f^{-1} = g$.
$f^{-1}of = I_A$
$fof^{-1} = I_B$

Inverse function

Let f be a one-to-one corresponding from the set A to the set B. the inverse function of x is defined as the function that assigns to an element $b \in B$ the unique element $a \in A$ such that $f(a) = b$ then inverse of f is denoted as $f^{-1}(b) = a$. If a function is not one to one correspondance one cannot define the inverse of the function.

Partial Order Relation

A relation R on a set S is called an partial ordering or partial order of R is -

1. Reflexive
2. Antisymmetric
3. Transitive

A set S with partial orders R on A is called partially ordered set or order set as a poset. We write (S,R) to specify the partial order relation R usually, we denote a partial order relation by simple '$\preccurlyeq$'.

for eg→ Let S be a non-empty set and P(S) denote the powerset of S then the relation set inclusion denoted by '$\preccurlyeq$' in P(S) is a partial ordering

The set of natural number N forms a ~~poset~~ poset w.r.t a relation '$\preccurlyeq$' specifying $a \preccurlyeq a$ if $a \preccurlyeq b$ and $b \preccurlyeq a$ then $\underline{a=b}$ and finally if $a \preccurlyeq b$ and $b \preccurlyeq c$ then $a \preccurlyeq c$ for all $a, b, c \in N$

Asymptotic Notation

Complexity

Algorithm and complexity of algorithm comprise the study of computation. Complexity is the measure of resources required by the algorithm. The most commonly measured resource are time and space. The resource considered for space is memory of the computer

Asymptotic Notation

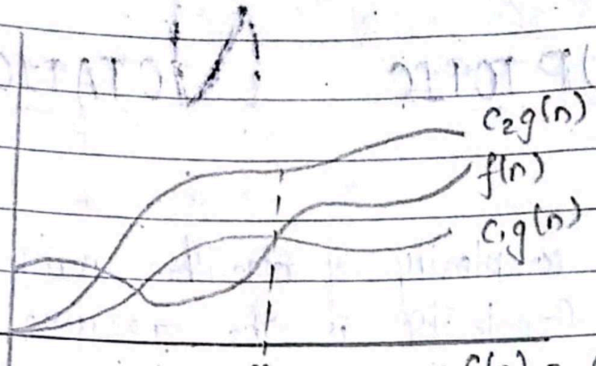
The notation used to describe the asymptotic running time of algorithm are defined in terms of function whose domains are set of natural nos, $N = \{0, 1, 2, \ldots\}$. Such notations are convenient describing the worst case running time farther function.

We use asymptotic notation primarily to describe the running time of algorithm. Asymptotic notation apply the f^n that characterise some other aspect of algorithm or even to f^n that are nothing what so seen to do with algorithm.

Types Of Notation

1. Θ Notation

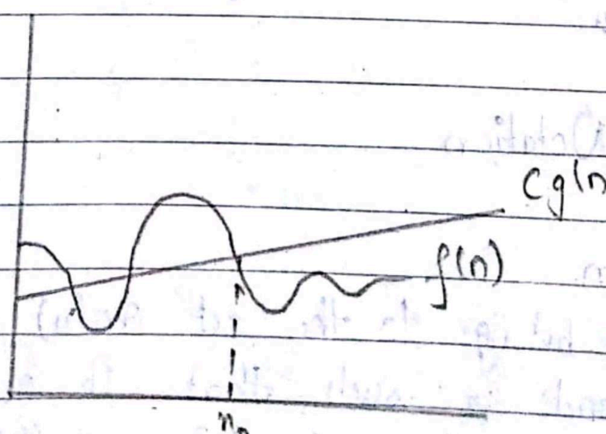
→ A f^n $f(n)$ belongs to the set $\Theta g(n)$ if $\exists$ positive constant c_1 and c_2 such that it can be sandwitched between $c_1 g(n)$ and $c_2 g(n)$ for sufficiently large ends because $\Theta(g(n))$ is a set we could write as $f(n) \in \Theta(g(n))$



$f(n) = \Theta(g(n))$

to indicate that $f(n)$ is a member of $\Theta(g(n))$. Fig. given above shows that $\forall$ values of n and to the right of n_0 value of $f(n)$ lies at or above $c_1(g(n))$ and at or below $c_2(g(n))$. In other words for all $n \geq n_0$ the f^n $f(n) = g(n)$. to within a constant factor we say that asympototic tight bound for $f(n)$. The definition of $\Theta(g(n))$ requires that every member of $f(n)$ belongs to $\Theta(g(n))$ be asymptotically non-negative i.e. $f(n)$ non-negative whereas n is sufficiently large. Consiquently the $f(n)$ f^n in $g(n)$ must be asymptotically non-negative or else $\Theta(g(n))$ is negative.

2. O Notation



$f(n) = O(g(n))$

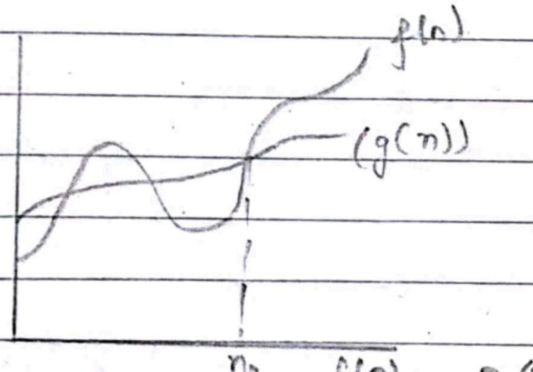
When we have an asymptotically upper bound we use (bigo) O-notation for a given f^n $g(n)$ we denote $O(g(n))$.

$O(g(n)) = \{ f(n)$: there exists (+)ve constant c such that $c \leq f(n) \leq c(g(n))\ \forall\ n \geq n_0 \}$

We use bigo notation to give upper bound on a f^n to within a constant f^n. figure given above shows that $\forall$ values of n at end & to the right of n_0 the value of the f^n $f(n)$ is on or below $c(g(n))$. We write $f(n) = O(g(n))$ to indicate that the f^n $f(n)$ is the member of the set $O(g(n))$.

3. Ω Notation

(Big omega or just omega also)



$f(n) = \Omega(g(n))$

for a given function $g(n)$ we denote $\Omega(g(n))$. The set of functions

$\Omega(g(n)) = \{ f(n) : \exists$ positive constant c and n such that $0 \leq (g(n)) \leq f(n)\ \forall\ n \geq n_0 \}$

Floor function or Ceiling function

Floor function

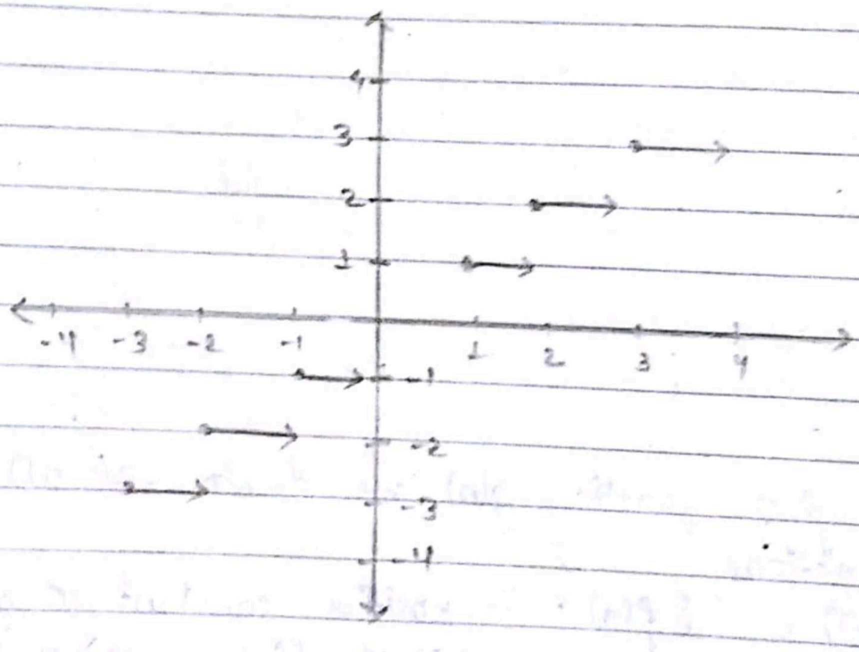
The floor function or floor of x is denoted by $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

for eg → $\left\lfloor \frac{1}{2} \right\rfloor = 0$.

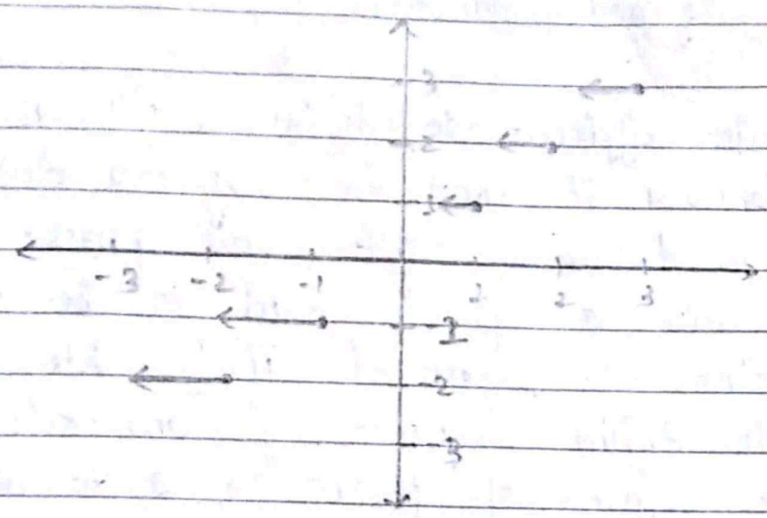
Ceiling function

The ceiling function or ceiling of x is denoted by $\lceil x \rceil$ is the least integer or greater than or equal to x.

for eg → $\left\lceil \frac{1}{2} \right\rceil = 1$



floor function $\lfloor x \rfloor$



Ceiling function $\lceil x \rceil$.

Mathematical Representation of floor and ceiling function.

1. $\lfloor x \rfloor = n$ iff $n < x < n+1$ (x is an integer)
2. $\lfloor x \rfloor = n$ iff $x-1 < n < x$
3. $\lceil x \rceil = n$ iff $n-1 < n \leq n$
4. $\lceil x \rceil = n$ iff $x \leq n < x+1$

5.a $\lfloor -x \rfloor = - \lceil x \rceil$

.b $\lceil -x \rceil = - \lfloor x \rfloor$

The above properties has been framed using the definition of floor and ceiling function. Another approach for defining the floor f^n is that to assume $n = n + \varepsilon$ where $n = \lfloor x \rfloor$ is an integer and ε is a fractional part of x. Certifies an inequality $0 \leq \varepsilon \leq 1$.

Similarly, to define ceiling f^n suppose $x = n - \varepsilon$ where $n = \lceil x \rceil$ is an integer and $0 \leq \varepsilon \leq 1$.

Reccursively defined Function

Sometimes it is difficult to define a f^n or a set explicitly. However, it may be easy to define this f^n or set in terms of itself. This process is called recursion. Thus a f^n is said to be recursively defined if the f^n refers to itself. We can use recursion to define sequences, f^ns and sets.

for eg→ The sequence of powers of 2 is given by

$$a_n = 2^n \quad for \quad n = 0, 1, 2, \ldots$$

Technique for recursively defined f^n when the argue-ment domain is inductively defined.

① Specify a value $f(x)$ for each basis element x of S.

② Specify rules that for each inductively defined element x in S, define the value $f(x)$ in terms of previously defined value of f.

Recurrence

When an algorithm contains recursive call to itself we can often describe by a reccurence eq^n or recurrence which describe the overall running time on a problem of size n. In terms of running time on n inputs. Reccurence goes hand in hand to divide an conquer paradime because they give a natural way to characterise the running time of divide and conquer algorithms.

A reccurence is an eqⁿ or an equality that describe the fⁿ in terms of its value of smaller inputs. Reccurences can take many forms. for eg→ A recursive algorithm by dividing some problem in unequal size such as 2/3 or 1/3 split. If we divide and combine steps it would take linear time, such an algorithm would give rise to the reccurence

eg→ $T(n) = \left[T\left(\frac{2n}{3}\right) + T\left(\frac{n}{3}\right) + \theta(n) \right]$

Subproblems are not necessarily constraints to being a contraint fraction of the original problem size. for eg→ A recursive version of linear search would create just one subproblem containing only one element fewer than original problem.

If each recursive call would take constant time plus the time for recursive calls.

$T(n) = T(n-1) + \theta(1)$

A reccurence for the running time of divide and conquer element falls out of 3 steps of the basic paradigme then $T(n)$ be the running time on a problem of size n. If the problem size is small enough say $n \geq c$ for some constant c. the straight forward soln take constant time $\theta(1)$.

Suppose, our division of problem is a subproblems each of which is $\frac{1}{b}$ the size of the original problem. It takes time $T(n/b)$ to solve one subproblem therefore to solve a subproblems we will require $a * T(n/b)$. If we take $D(n)$ time to divide the problem into subproblems and then $C(n)$ time to combine the subproblems into the sol^n to the original problem ~~to~~ we get the reccurence as

$$T(n) = \Theta(1) \quad \text{if } n \leq c$$
$$= a * T(n/b) + D(n) + C(n)$$

Merge Sort

Merge sort on just one element take some constant time i.e. $\Theta(1)$. When we have $n > 1$ we breakdown running time as follows -

① Divide

→ The divide step just compute the middle of the subarray which take some constant time. then $D(n) = \Theta(1)$

② Conquer

→ We reccursively solve two subproblems each of size $n/2$. which contributes $2 * T(n/2)$. this is the running time.

③ Combine

→ Merged procedure on an n element subarray takes $\Theta(n)$ therefore, combine time, $C(n) = \Theta(n)$. when we add the f^ns $T(n)$, $C(n)$ for the merge sort analysis. We are adding a f^n i.e $\Theta(n)$ and $\Theta(1)$. This is the linear f^n of n i.e. $\Theta(n)$. Adding it to the f^n $2 * T(n/2)$ term from the conquer step gives the reccurence for the worst ~~phase~~ case running time of the merge sort.

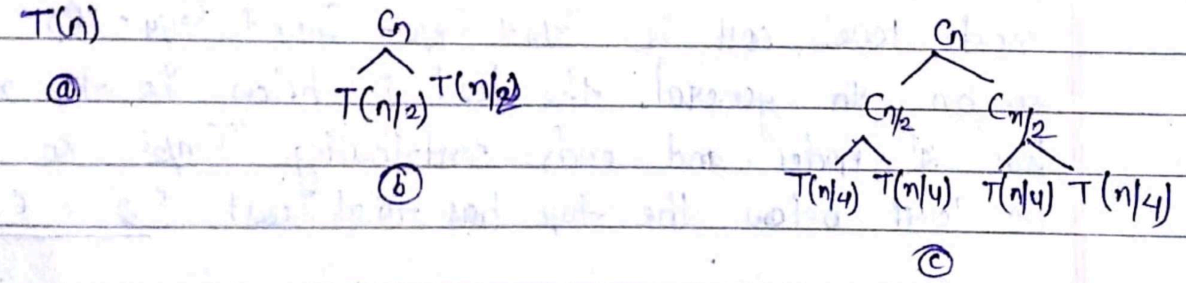
$T(n) = O(1)$
$\quad = 2 * T(n/2) + \underbrace{O(n)}_{\text{linear } f^n \text{ of } n}$ if $n > 1$

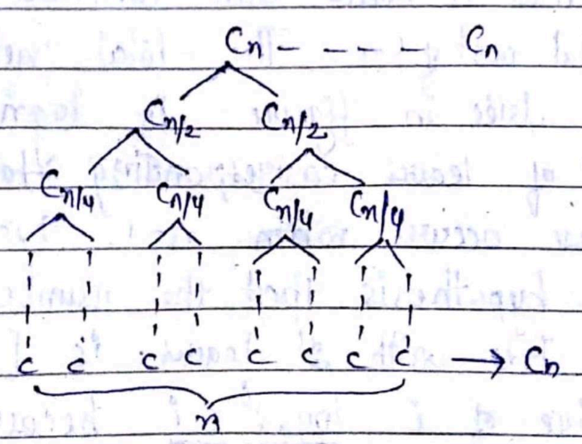
$T(n) = c$ if $n = 1$
$\quad = 2T(n/2) + O(n)$ if $n \geq 1$.

where constant c represents the time required to solve the problem of size 1 as well as the time for array element of the divide and combine steps.

Reccurence Tree

How to construct a reccursive tree for a reccurence $T(n) = 2T(n/2) + n$.





Total = Cnlogn + n

(d)

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Exp for convenience we assume that n is an exact power of 2. Part (a) of the diagram given above shows $T(n)$ which we will expand further. Part (b) is an equivalent tree representing the reccurence. The Cn term is the root and the two subtree of the root are 2 smaller reccurence that is $T(n/2)$. Part (c) of the diagram shows this process carried one step further by expanding $T(n/2)$ line. The cost encoured at each of the two subnode at the second level of reccursion is $Cn/2$. We continue expanding each node in the Tree by breaking it into constituent parts as determined by the reccurence until the problem size is get down to one, each be the cost of c. Part (d) shows the resulting reccursion tree. Next we add the cost across each level of the tree. The top level has cost Cn and the next level down has cost $Cn/2 + Cn/2 = Cn$ and again the next level cost is $Cn/4 + Cn/4 + Cn/4 + Cn/4 = Cn$ and so on. In general, the level i below the top node has 2^i nodes and each contributing $Cn/2^i$ so that the ith level below the top has total cost $[2^i \times C(n/2^i)]$

The bottom level has n nodes each contributing a cost of c for a total cost of Cn. The total number of levels of the reccursive tree in figure is $\log n + 1$ where n is the number of leaves corresponding to the input size. The base case occurs when $n=1$. Now assume as an inductive hypothesis that the number of levels of a reccursive Tree with 2^i leaves is $[\log 2^i + 1]$. Since for any value of i $\log_2 2^i = i$ because we are asuming the input size is the power of 2. Next the input size to consider is $[2^i+1]$. This is a tree where $n = 2^i+1$ leaves and has one more level $[\log_2 2^{i+1} + 1 = (i+1)+1]$. To compute the total cost represented by reccurence we simply add the total cost of all the levels. The reccursive tree has $(\log n + 1)$ leaves each costing Cn for a total cost of $Cn \times \log(n+1) = Cn\log n + Cn$. Ignoring the lower order terms and constant give us the desired result $\boxed{\Theta(n\log n)}$

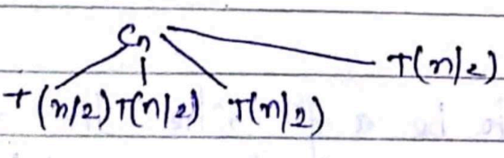
— x —

Assignment

Q1 Explain the principle of inclusion and exclusion (20-24)

Q2. Reccurence tree of $T(n) = 4T(\lfloor n/2 \rfloor) + \underset{\text{constant}}{Cn}$

→ $T(n) = 4T(\lfloor n/2 \rfloor) + Cn$



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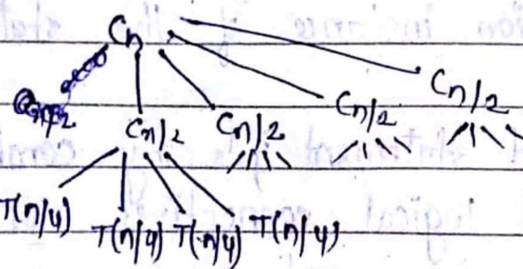

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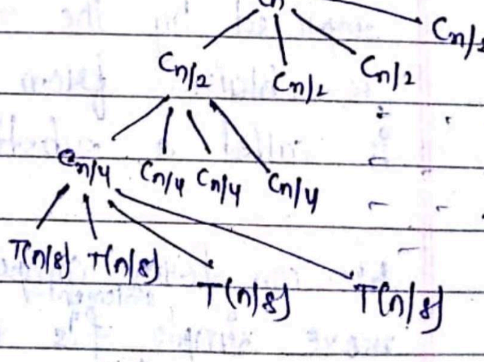

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PREDICATE CALCULUS

The logic associated with the predicate in any statement is called predicate logic.
for eg → ① Dog is an animal. ② Cat is an animal.
The part "is an animal" is called predicate.

A predicate may require more than one name to express a statement. A predicate that requires n $(n>0)$ names is known as an n place predicate.

The statement function, Variables and Quantifiers

~~The~~ We define a simple statement function of one variable as an expression which consist of a predicate symbol and an individual variable.
for eg → M(y) is a simple statement function of one variable.

The statement f^n turns to be a f^n when the variable is replaced by the name of an object. The statement which is obtained from the statement f^n by a replacement is called a substitution instance of the statement function.

We can form compound statement f's by combining one or more simple statement f's and logical connectives. for eg. → $M(x)$ represents 'x' is a man, and $V(x)$ represents 'x' is a vegetarian. then we can compose many compound statements such as $M(x) \wedge V(x)$, $\sim M(x) \vee V(x)$.

(2) It is also possible to form compound statement fⁿ by using ~~two~~ ~~variables~~ statement fⁿs of one variable.
for eg→ $M(x)$ represents x is a man
$V(y)$ represents y is a vegetarian.
we can compose compound statement fⁿ such as $M(x) \land V(y)$.

→ It is an extension of propositional logic in which every expression is a sentence that represents a fact.

→ Propositional logic deals with WFF that donot involve variables

→ In predicate calculus, variables are used which is similar to pronoun.
for eg→ Shweta is a girl.
girl(Shweta). → predicate calculus

→ Apart from sentences or statements, predicate calculus also has terms which represents objects.
for eg→ Left leg of Ram.
Left leg(Ram). → predicate calculus

→ Constant symbols, variables and functional fⁿ symbols are used to build terms.

→ Predicate symbols are used to build sentences.

Syntex of Predicate Calculus

- It consist of symbols and rules which are as follows-

1. Constant Symbol
→ It refers to a particular object, each constant symbol name exactly one object but not all objects require to have names and some can have several names.
for eg→ King of Ayodhya is Ram
King of Ayodhya (Ram) → PL2

2. Truth Symbol
→ T, F are used to denoted true or false and they are reserved symbols

3. Variables Symbol
It is also used such as p, q, r, s, t, ...

4. functional Symbol
→ It refers to a relation ie. a given object can be mapped exactly to another object defined by the relation
for eg→ $\cos 45° = 1/2$.

Connectives

1. Conjunction ($\wedge$)
2. disjunction ($\vee$)
3. bidirectional ($\leftrightarrow$)
4. Implication ($\rightarrow$)
5. Negation ($\sim, \neg$)

There are two quantifiers

① Universal quantifier ($\forall$)
② Extential quantifier ($\exists$)

Rules of predicate calculus tells us how to identify and recognise the legal or well formed stream out of all the finite stream.

Terms

It is either a constant or variable or functional expression. If $t_1, t_2, \ldots, t_n$ are terms and f is an n-placed f^n then it can be written as $f(t_1, t_2, \ldots, t_n)$.

~~Atomic~~ Atomic

Atomic Sentences

- ~~An atomic~~ These sentences can state facts and can be constructed using terms and predicate symbols.

- An atomic sentences are formed using predicate symbols followed by list of terms.

  for eg → Ram is brother of Laxman.

  Brother (Ram, Laxman) → PL2

- Truth values true ~~and~~ or false are atomic sentences

- Atomic sentences are also called atomic expression or proposition.

*

- We combine atomic sentences using logical operators to form sentences in predicate calculus.

- Atomic sentences can have arguements that have complex terms.

Eg → Laxman's father is married to Ram's mother.
Married (Laxman's father, Ram's mother) → PL2

- An atomic sentence is true if the relation is referred by the predicate symbol that hold b/w the object referred by the arguements.

Complex Sentences

We use logical connectives to construct complex sentence.
for eg → ① Ram is the brother of Laxman, Laxman is the brother of Ram.
→ Brother (Ram, Laxman) ∧ brother (Laxman , Ram)

② If Ram is older than 30 then he is not younger than 30.
→ older (Ram , 30) → ~younger (Ram, 30) → PL2

③ All Romans were either loyal to Caesar or hated him.
→ ∀(x): Roman (x) → loyal to (x, Caesar) ∨ hated (x, Caesar).

④ Everyone is loyal to someone.
→ ∀(x): ∃(y): Loyal (x, y)

Cartesian properties of Connectives and Quantifier

1. Properties of PL-1 and PL-2.

① $p \to q = \neg p \vee q$
② $p \leftrightarrow q = (\neg p \vee q) \vee (\neg q \vee p)$

2. For statements of PL-1 is valid in PL-2 but vice-versa is not true.

ⓐ $\neg(\forall(x) p(x)) = \exists(x)\, p(x)$
ⓑ $\neg(\exists(x).p(x)) = \forall(x)\, p(x)$.

3. ⓐ $\forall(x)[p(x) \vee Q(x)] = \forall(x)\, p(x) \vee \forall(x)\, Q(x)$
ⓑ $\exists(x)\,[P(x) \vee Q(x)] = \exists(x)\, P(x) \vee \exists(x)\, Q(x)$

4. ⓐ $\forall(x)\, p(x) = \forall(y) \cdot p(y)$
ⓑ $\exists(x)\, p(x) = \exists(y)\, P(y)$.

Free and Bound Variable

If any formula contains a part ~~like~~ like $(\exists x) M(x)$ or $(\forall x) M(x)$ then that part is called x-bound part of the formula. Any occurence of x in an x-bound part is known as a bound occurence of x. Any occeurence of x which is not a bound occurence is known as free occurence. The formula the immedietly follows the quantifier is known as the scope of the quantifiers. In other words, the scope of ~~the~~ a quantifier is a part of a logical expression to which a quantifier is applied. So, a variable is free if it lies outside the scope of all quantifiers in the formula which specifies the variable. for eg → The formula

$M(x)$ in $(\exists x)\, M(x)$ or $(\forall x)\, M(x)$ is the scope of the quantifier.

Eg → (i) $x\, M(x)$ here $M(x)$ is the scope of quantifiers

(ii) $\exists(x)(M(x) \wedge N(x))$

here $M(x) \wedge N(x)$ is the scope of the quantifier and all occurrences of x are bound.

(iii) $M(x) \wedge \exists(N(x))$

here the scope of $\exists(x)$ is $N(x)$ and it contains bound occurrence of x while occurrence of x in $M(x)$ is free.

→ It should be kept in mind the bound variable can be replaced by any other variable but not by a constant. Hence, the formulas

$x(M(y, x))$ and $z(M(y, z))$ are the same.

It may be observed that if there is a free variable in the formula then we have a statement f^n and there in the case when every occurrence of the variable is bound and no variable is free then we get a statement.

for eg → Someone in your school has visited Agra.

$S(x)$: x is in your school.

$A(x)$: x has visited Agra.

i.e. $\exists(x)\{S(x) \wedge A(x)\}$

x is bound with S but is free with A.

Methods of Proof

Direct Proof

Suppose the hypothesis P is true. then the implication $P \to Q$ can be proved if we can prove that Q is true by using the rules of inference and some other theorems. This represents that the combination P(true) and Q(false) never occurs.

Indirect Proof

The method is also known as direct proof of contrapositive. We know that the implication $p \to q$ is equivalent to its contrapositive ~~$\sim p \to \sim q$~~ $\sim q \to \sim p$ so we can prove the implication by showing that its contrapositive $\sim q \to \sim p$ is true. So, an indirect proof of $p \to q$ proceeds as follows -

1. First assume q is false.
2. Then prove on the basis of this assumption and other available information from the frame of reference p is false.

Conditional Proof

This method of proof is actually another form of elimination of cases.

We know that the two propositions $p \to (q \to r)$ and

$(p \wedge q) \rightarrow r$ are equivalent. So, a proof of the condition $p \rightarrow (q \rightarrow r)$ can proceed as follows -

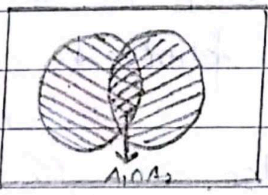
1. First join the two antecedents p and q.
2. Then prove are on the basis of this assumption and other available information.

Q Explain the principal of inclusion and exclusion.

→ Let A_1 and A_2 be two sets. We want to show that

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

ie. the number of elements in the union of two sets (say A_1 and A_2) is the sum of the numbers of elements in the two sets A_1 and A_2 minus the number of elements in the intersection of A_1 and A_2. This is known as the Principale of Inclusion and Exclusion. This equation can be represented by a Venn Diagram given below:-



Sets A_1 and A_2 might have some common elements. To be specific, the number of common elements A_1 and A_2 is $|A_1 \cap A_2|$. Each of these elements between is counted twice in $|A_1| + |A_2|$ (once in $|A_1|$ and once in $|A_2|$) although it should be counted as one element in $|A_1 \cup A_2|$. Therefore, the double count of these elements in $|A_1| + |A_2|$ should be adjusted by the subtraction of the Term $|A_1 \cap A_2|$ in the right hand side of eqn. As an example, suppose that the among a set of 12 books, 6 are novels, 7 were published in the year 1984 and 3 are novels published in 1984. Let A_1 denote the set of books that are novels, and A_2 denote the set of books published in 1984, we have

$|A_1| = 6$, $|A_2| = 7$, $|A_1 \cap A_2| = 3$

Consequently according to eq^n

$$|A_1 \cup A_2| = 6 + 7 - 3 = 10.$$

That is, there are 10 books which are either novels or 1984 publications, or both. Consequently, among the 12 books there are 2 non-novels that were not published in 1984.

Extending the result in eq^n, for three sets A_1, A_2 and A_3

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

Q Suppose we have 6 computers with the following specifications:

Computer	Floating point arithmetic unit	Magnetic disk Memory	Graphic display terminal
I	Yes	Yes	No
II	Yes	Yes	Yes
III	No	No	No
IV	No	Yes	Yes
V	No	Yes	No
VI	No	Yes	Yes

→ Let A_1, A_2 and A_3 be the sets of computers with a floating-point arithmetic unit, magnetic-disk storage, and graphic display terminal, respectively. We have

$|A_1| = 2$ $|A_2| = 5$ $|A_3| = 3$

$|A_1 \cap A_2| = 2$ $\quad |A_1 \cap A_3| = 1$ $\quad |A_2 \cap A_3| = 3$

$|A_1 \cap A_2 \cap A_3| = 1$

Consequently

$|A_1 \cup A_2 \cup A_3| = 2 + 5 + 3 - 2 - 1 - 3 + 1 = 5$

i.e. 5 of the six computers have one or more of the three kinds of hardware considered.

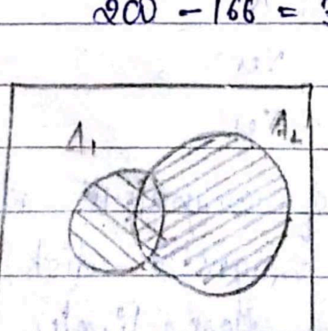
Q. Out of 200 students, 50 of them take the course Discrete Mathematics, 140 of them take the course Economics, and 24 of them take both the courses. Since both the course Economics and ~~24 of them take both the~~ or have scheduled examinations for the following day, only students who are not in either one of these courses will be able to go to the party, the night before. We want to know how many students will be able ~~to go to~~ at the party.

→ In the fig. drawn below, where A_1 is the set of students in the course Discrete Mathematics and A_2 is the set of students in the course Economics, we not that the number of students who take either one or both courses is equal to

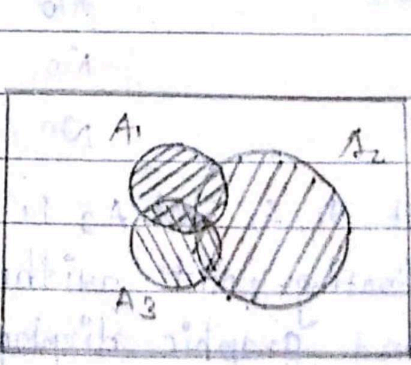
$50 + 140 - 24 = 166$

Consequently, the no. of students who will be at the party is

$200 - 166 = 34.$



(a)



(b)

Suppose that 60 of the 200 are underclass students. Among the underclass students, 20 of them take Discrete Mathematics, 45 of them take Economics and 16 of them take both. We want to know how many upperclass students will be at the party. According to the venn diagram in fig. 1.8b, where A_3 is the set of underclass students, we have

$|A_1 \cup A_2 \cup A_3| = 50 + 140 + 60 - 24 - 20 - 45 + 16 = 177$

Thus, the no. of upperclass students who will go to the party is $200 - 177 = 23$.

Q Thirty cars were assembled in a factory. The options available were a radio an air conditioner and white-wall tires. It is known that 15 of the cars have radios, 8 of them have air conditioners and 6 of them have white-wall tires. Moreover, 3 of them have all three options. We want to know at least how many cars do not have any options at all.

→ Let A_1, A_2 and A_3 be the sets of cars with a radio, an air conditioner and white-wall tires, respectively.

Since, $|A_1| = 15$ $|A_2| = 8$ $|A_3| = 6$

and $|A_1 \cap A_2 \cap A_3| = 3$

According to eq^n

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 - A_3| + |A_1 \cap A_2 \cap A_3| \longrightarrow (1)$$

So, $|A_1 \cup A_2 \cup A_3| = 15 + 8 + 6 - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + 3$

$= 32 - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3|$

Since, $|A_1 \cap A_2| \geqslant |A_1 \cap A_2 \cap A_3|$

$|A_1 \cap A_3| \geqslant |A_1 \cap A_2 \cap A_3|$

$|A_2 \cap A_3| \geqslant |A_1 \cap A_2 \cap A_3|$

We have,

$|A_1 \cup A_2 \cup A_3| \leq 32 - 3 - 3 - 3 = 23.$

That is, there are at most 23 cars that have one or more options. Consequently, there are atleast 27 cars that donot have any options.

* In the general case, for the sets $A_1, A_2, \dots, A_r$, we have

$$|A_1 \cup A_2 \cup \dots \cup A_r| = \sum_i |A_i| - \sum_{1 \leq i < j \leq r} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq r} |A_i \cap A_j \cap A_k| + \dots + (-1)^{r-1} |A_1 \cap A_2 \cap \dots \cap A_r| \quad \text{--- (2)}$$

Although the result in eq^n (1) is not difficult to visualize, the result in eq^n (2) is not as obvious. We now prove eq^n (1) by induction on the no. of sets r. Clearly, eq^n

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| \quad \text{--- (3)}$$

can serve as the basis of induction. As the induction step, we assume that eq^n (2) is valid for any $r-1$ sets. We note first that, viewing $(A_1 \cup A_2 \cup \dots \cup A_{r-1})$ and A_r are two sets, according to eq^n (3) we have

$$|A_1 \cup A_2 \cup \dots \cup A_r| = |A_1 \cup A_2 \cup \dots \cup A_{r-1}| + |A_r| - |A_r \cap (A_1 \cup A_2 \cup \dots \cup A_{r-1})| \quad \text{--- (4)}$$

Now,

$$|A_r \cap (A_1 \cup A_2 \cup \dots \cup A_{r-1})| = |(A_r \cap A_1) \cup (A_r \cap A_2) \cup \dots \cup (A_r \cap A_{r-1})|$$

According to the induction hypothesis, for the $r-1$ sets $A_r \cap A_1, A_r \cap A_2, \dots, A_r \cap A_{r-1}$, we have

$$|(A_r \cap A_1) \cup (A_r \cap A_2) \cup \dots \cup (A_r \cap A_{r-1})|$$

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$= |A_r \cap A_1| + |A_r \cap A_2| + \dots + |A_r \cap A_{r-1}| - |(A_r \cap A_1) \cap (A_r \cap A_2)| - |(A_r \cap A_1) \cap (A_r \cap A_3)| - \dots - |(A_r \cap A_1) \cap (A_r \cap A_2) \cap (A_r \cap A_3)|$
$+ \dots - \dots + (-1)^{r-2} |(A_r \cap A_1) \cap (A_r \cap A_2) \cap \dots \cap (A_r \cap A_{r-1})|$

$= |A_r \cap A_1| + |A_r \cap A_2| + \dots + |A_r \cap A_{r-1}| - |A_r \cap A_1 \cap A_2| - |A_r \cap A_1 \cap A_3|$
$- \dots + |A_r \cap A_1 \cap A_2 \cap A_3| + \dots - \dots$
$+ (-1)^{r-2} |A_r \cap A_1 \cap A_2 \cap \dots \cap A_{r-1}|$ —— (5)

Also, according to the induction hypothesis, for the $r-1$ sets $A_1, A_2, \dots, A_{r-1}$, we have,

$|A_1 \cup A_2 \cup A_3 \cup \dots \cup A_{r-1}| = |A_1| + |A_2| + \dots - |A_1 \cap A_2| - |A_1 \cap A_3| - \dots + \dots + (-1)^{r-2} |A_1 \cap A_2 \cap \dots \cap A_{r-1}|$ —— (6)

Substituting (5) and (6) into (4), we obtain (3)

Q Let us determine the no. of integers between 1 and 250 that are divisible by any of the integers 2, 3, 5 and 7.

→ Let A_1 denote the set of integers between 1 and 250 that are divisible by 2, A_2 denote the set of integers that are divisible by 3. A_3 denote the set of integers that are divisible by 5, and A_4 denote the set of integers that are divisible by 7. Since,

$|A_1| = \left\lfloor \frac{250}{2} \right\rfloor = 125$ $\quad$ $|A_2| = \left\lfloor \frac{250}{3} \right\rfloor = 83$

$|A_3| = \left\lfloor \frac{250}{5} \right\rfloor = 50$ $\quad$ $|A_4| = \left\lfloor \frac{250}{7} \right\rfloor = 35$

$|A_1 \cap A_2| = \left\lfloor \frac{250}{2 \times 3} \right\rfloor = 41$ $\quad$ $|A_1 \cap A_3| = \left\lfloor \frac{250}{2 \times 5} \right\rfloor = 25$

$|A_1 \cap A_4| = \left\lfloor \frac{250}{2\times7} \right\rfloor = 17$ $\quad$ $|A_2 \cap A_3| = \left\lfloor \frac{250}{3\times5} \right\rfloor = 16$

$|A_2 \cap A_4| = \left\lfloor \frac{250}{3\times7} \right\rfloor = 11$ $\quad$ $|A_3 \cap A_4| = \left\lfloor \frac{250}{5\times7} \right\rfloor = 7$

$|A_1 \cap A_2 \cap A_3| = \left\lfloor \frac{250}{2\times3\times5} \right\rfloor = 8$ $\quad$ $|A_1 \cap A_2 \cap A_4| = \left\lfloor \frac{250}{2\times3\times7} \right\rfloor = 5$

$|A_1 \cap A_3 \cap A_4| = \left\lfloor \frac{250}{2\times5\times7} \right\rfloor = 3$ $\quad$ $|A_1 \cap A_3 \cap A_4| = \left\lfloor \frac{250}{3\times5\times7} \right\rfloor = 2$

$|A_1 \cap A_2 \cap A_3 \cap A_4| = \left\lfloor \frac{250}{2\times3\times5\times7} \right\rfloor = 1$

we have,

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = 125 + 83 + 50 + 35 - 41 - 25 - 17 - 16 - 11 - 7 + 8 + 5 + 3 + 2 - 1$$

$$= 193$$

Master's Theorem Method for Solving Recurrences

Master book provides a book of methods for solving recurrences of the form $T(n) = aT(n/b) + f(n)$ where $a \geq 1$ and $b > 1$ are constants and $f(n)$ is an asymptotically (+) ve functions. The recurrence equation given above describe the running time of an algorithm that divides a problems. Each size 'n' into 'a' sub problems. Each of size $(\frac{n}{b})$ where a and b are (+) constants.

The 'a' subproblems are solved recursively each of time $T(n/b)$. The function $f(n)$ compasses the cost of dividing the problem and combining the results of the subproblems

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Master Theorems →

The master method depends on the following theorem.

Theorem 1 → Master theorem.

→ Let $a \geq 1$ and $b \geq 1$ constants. Let $f(n)$ be a function and $T(n)$ be defined on the non(-)ve integers by the recurrence

$$T(n) = aT(n/b) + f(n)$$

where we enterperate (n/b) either as $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$ then $T(n)$ has the following asymptotic bounds :-

(i) If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$ then $T(n) = \theta(n^{\log_b a})$

(ii) If $f(n) = \theta(n^{\log_b a})$ then $T(n) = \theta(n^{\log_b a} \log x)$

(iii) If $f(n) = \omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$ and $af(n/b) \leq c f(n)$ for some constant $c < 1$ and all sufficiently large 'n' then $T(n) = \theta(f(n))$

→ In each of the three cases, we compare the function $f(n)$ with the function $n^{\log_b a}$, intutively the larger of the two functions determine the solution to the recurrence.

In case 1, the function $n^{\log_b a}$ is larger then solution is $T(n) = \theta(n^{\log_b a})$ and $(n^{\log_b a} = n^{\log_2 2} = n)$

We might mistakenly think that the case (iii) is apply $\because f(n) = n\log n$ is asymptotically larger than

$n^{\log_b a} = n$. The problem is that, it is not polynomially larger. The ratio $\frac{f(n)}{\log_b a} = \frac{\cancel{n}\log n}{\cancel{n}} = \log n$.

* $\log n$, is asymptotically less than n^{ϵ} for any +ve constant 'ϵ'. Consequently the recurrence falls into the gap between case II and case III

Ex V $\Rightarrow$ $T(n) = 2T(\frac{n}{2}) + \theta(n)$

In the recurrence, $a=2$ $\quad f(n) = \theta(n)$

$b=2$

$$n^{\log_b a} = n^{\log_2 2} = n$$

<u>Case 2:</u> aplies $\because f(n) = \theta(n)$ and so we have the sol^n

$$T(n) = \theta(n\log n)$$

Ex VI :- $T(n) = 8T(\frac{n}{2}) + \theta(n^2)$

$\Rightarrow$ $a=8$ $\quad f(n) = \theta(n^2)$

$b=2$

$n^{\log_2 8} = n^3$, which is polynomially larger than $f(n)$.

$\because n^3$ is polynomially larger than $f(n)$ i.e.

$f(n) = O(n^{\log_2 8 - \epsilon})$

$= O(n^{3-\epsilon})$

for $\epsilon = 1$, case I applies and $T(n) = \theta(n^3)$

Ex VII : $T(n) = 7T(\frac{n}{2}) + \theta(n^2)$

$a=7$

$b=2$, $f(n) = \theta(n^2)$

$n^{\log_b a} = n^{\log_2 7}$

we writing it as log 7 and recalling it $2.80 < \log 7 < 2.81$ we see that $f(n) = O(n^{\log 7 - \epsilon})$, $\epsilon = 0.8$ again case 1 aplies and the sol^n is $T(n) = \Theta(n^{\log 7})$

In case 3, the f^n $f(n)$ is larger then $T(n) = \Theta(f(n))$
In case 2, the two f^n's are of same size, then we multiply by an logarithmic factor and the sol^n is $T(n) = \Theta(n^{\log_b a} \log n)$ which is equal ~~to~~ $\Theta(f(n) \cdot \log n)$.

- Using the Master Method :-

To use the method we simply determine which case of to the master theorem applies and write down the answer

i) $T(n) = 9T(\frac{n}{3} + n)$
- For this recurrence, $a = 9$
$b = 3$, $f(n) = n$

Now $n^{\log_b a} = n^{\log_3 9}$
$= n^2$
$= \Theta(n^2)$

$\because f(n) = O(n^{\log_3 9 - \epsilon})$, where $\epsilon = 1$
we can apply case 1 of master theorem and conclude that the sol^n is $T(n) = \Theta(n^2)$

ii) $T(n) = T(\frac{2n}{3}) + 1$
- For this recurrence, $a = 1$
$b = \frac{3}{2}$, $f(n) = 1$

$n^{\log_b a} = n^{\log_{\frac{3}{2}} 1}$
case 2 applies, $\because f(n) = \Theta(n^{\log_b a}) = \Theta(1)$
and the sol^n to the recurrence is

$T(n) = \Theta(1 \times \log n)$
$= \Theta(\log n)$

iii) $T(n) = 3T(\frac{n}{4} + n\log n)$

Here, $a = 3$, $b = 4$, $f(n) = n\log n$

$n^{\log_b a} = n^{\log_4 3}$

$\because f(n) = \omega(n^{\log_4 3 + \epsilon})$, where ϵ is $\epsilon \approx 0.2$

case 3 applies, if we can show that the regularity condition holds for $f(n)$. for sufficiently large 'n', we have that $af(\frac{n}{b}) = 3(\frac{n}{4})\log\frac{n}{4} \leq (\frac{3}{4})n\log n = cf(n)$

for $c = \frac{3}{4}$

Consequently by case 3, the solⁿ to the recurrence is
$T(n) = \Theta(n\log n)$

iv) $T(n) = 2T(\frac{n}{2}) + n\log n$

The master theorem does not apply to the recurrence even though it apply to have a proper form

$a = 2$
$b = 2$, $f(n) = n\log n$
$n^{\log_b a} = n^{\log_2 2} = n$

$\because f(n) = (n^{1+\epsilon})$

case 3 applies, if we can show that the regularity condition holds for $f(n)$. for sufficiently ~~large~~ 'n', we have that $af(\frac{n}{2}) = 2\frac{n}{2}\log\frac{n}{2} \leq n\log n = cf(n)$

$c = 1$

consequently by case 3, the solⁿ to the recurrence is
$T(n) = \Theta(n\log n)$.

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$T(n) = aT(n/b) + \theta(n^k \log^p n)$
where $a \geq 1$, $b > 1$, $k \geq 0$ and p is a real no.
a, b and k are constants

case ①: If $a > b^k$ then $T(n) = \theta(n^{\log_b a})$

case ②: If $a = b^k$
(2a) If $P > -1$ then $T(n) = \theta(n^{\log_b a} \cdot \log^{p+1} n)$

(2b) If $P = -1$ then $T(n) = \theta(n^{\log_b a} \cdot \log \log n)$

(2c) If $P < -1$ then $T(n) = \theta(n^{\log_b a})$.

Case ③: If $a < b^k$
(3a) If $P \geq 0$ then $T(n) = \theta(n^k \log^p n)$

(3b) If $P < 0$ then $T(n) = \theta(n^k)$

Ques 1. $T(n) = 3T(n/2) + n^2$
→ Here, $a = 3$, $b = 2$, $k = 2$, $P = 0$.
⇒ $a = 3$ and $b^2 = 4$
So, $a < b^k$.
According to case (3a), $T(n) = \theta(n^k \log^p n)$
$= \theta(n^2 \log^0 2)$
$= \theta(n^2)$

2. $T(n) = 4T(n/2) + n^2$
→ $a = 4$, $b = 2$, $k = 2$ and $p = 0$.
⇒ $a = 4$, $b^k = 4$
⇒ $a = b^k$.

$\log^2 n = \log . \log n$
$(\log n)^2 = \log n . \log n .$

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According to (2a), $T(n) = \theta(n^{\log_b a} . \log n)$
$= \theta(n^{\log_2 4} . \log n)$
$= \theta(n^2 . \log n)$.

3. $T(n) = T(n/2) + n^2$
→ $a = 1$, $b = 2$, $K = 2$, $p = 0$.
→ $a = 1$, $b^K = 4$.
→ $a < b^K$
According to case (3a), $T(n) = \theta(n^K \log^p n)$
$= \theta(n^2 \log^0 {}_2)$
$= \theta(n^2)$.

4 $T(n) = 2^n T(n/2) + n^n$
→ $a = 2^n$
Since, a is not constant, This problem cannot be solved by Master's theorem.

5. $T(n) = 16T(n/4) + n$
→ $a = 16$, $b = 4$, $K = 1$, $p = 0$.
→ $a = 16$ and $b^K = 4$.
→ $a > b^K$
According to case 1, $T(n) = \theta(n^{\log_b a})$
$= \theta(n^{\log_4 16})$
$= \theta(n^2)$.

6 $T(n) = 2T(n/2) + n \log n$
→ $a = 2$, $b = 2$, $K = 1$, $p = 1$
→ $a = 2$, $b^K = 2$
→ $a = b^K$

According to case (2a), $T(n) = \theta(n^{\log_b a} \cdot \log n)$
$= \theta(n^{\log_2 2} \cdot \log n)$
$= \theta(n \cdot \log^2 n)$.

7. $T(n) = 2T(n/2) + \frac{n}{\log n}$.

→ $a = 2$, $b = 2$, $k = 1$, $p = -1$.
→ $a = 2$, $b^k = 2$
→ $a = b^k$
→ According to case 2(b), $T(n) = \theta(n^{\log_b a} \cdot \log \log n)$
$= \theta(n^{\log_2 2} \cdot \log \log n)$
$= \theta(n \log \log n)$

8. $T(n) = 2T(n/4) + n^{0.51}$
→ $a = 2$, $b = 4$, $k = 0.51$, $p = 0$.
→ $a = 2$, $b^k = 4^{0.51}$
→ $a < b^k$
According to case 3a, $T(n) = \theta(n^k \log^p n)$
$= \theta(n^{0.51})$.

9. $T(n) = 0.5T(n/2) + \frac{1}{n}$

→ $a = 0.5$
Since, $a \not< 1$, so, it is not valid.

10. $T(n) = 6T(n/3) + n^2 \log n$
→ $a = 6$, $b = 3$, $k = 2$, $P = 1$
→ $a = 6$, $b^k = 9$
→ $a < b^k$

According to case 3a, $T(n) = \theta(n^k \log^p n)$

$= \theta(n^2 \log^0 n)$

$= \theta(n^2 \log n)$.