Analytical treatement of free vibration, forced vibration and Resonance

* The phenomenon of setting a body into vibration with its own natural frequency is called 'free vibration'.

It has two types
1. Undamped free Vibration
2. Damped free Vibration

i) Undamped vibration –
When no resistances are present during the vibration of the body then, it is called undamped vibration.

ii) Damped free vibration –
When resistances are present during the vibration of the body then, it is called damped free vibration.

* Forced Vibration –
The phenomenon of setting a body into vibration with the application of a periodic force is called forced vibration.

* Resonance –
The phenomenon of setting a body into vibration with the application of a periodic force whose frequency is equal to natural frequency of the vibration of the body is called 'Resonance'.

At resonance the amplitude of vibration max^m. & a loud sound is heared.

⇒ Equation & Solution of undamped free vibration

Let y = displacement at any instant

$\frac{dy}{dt}$ = velocity at any instant

m = Mass of the particle

K.E of the vibration of the particle $= \frac{1}{2} m \left(\frac{dy}{dt}\right)^2$

Acceleration $= \omega^2 y$

$\therefore F = m\omega^2 y$

dy = Small displacement of the body

Work done $= m\omega^2 y \, dy$

$\therefore$ Total work done $= m\omega^2 \int_0^y y \, dy$

$= \frac{m\omega^2 y^2}{2}$

Let $m\omega^2 = K$

$\therefore$ Total work done $= \frac{1}{2} K y^2$

where K is a constant and is called restoring force per unit displacement.

This workdone is equivalent to potential energy of the vibration of particles of the body.

$\therefore$ Total energy $= \frac{1}{2} m \left(\frac{dy}{dt}\right)^2 + \frac{1}{2} K y^2$

In undamped free vibration total energy remains constant

$$\therefore \frac{1}{2} m \left(\frac{dy}{dt}\right)^2 + \frac{1}{2} k y^2 = C \quad \left(\begin{array}{l}\text{where } C \\ \text{is constant}\end{array}\right)$$

differentiate both side w.r to time

$$\frac{m}{\cancel{2}} \cancel{2} \frac{dy}{dt} \times \frac{d^2y}{dt^2} + \frac{k}{\cancel{2}} \cancel{2} y \cdot \frac{dy}{dt} = 0$$

$$\text{or, } \frac{dy}{dt}\left(m\frac{d^2y}{dt^2} + ky\right) = 0$$

$$\text{or, } \frac{d^2y}{dt^2} + \omega^2 y = 0 \quad \text{--- (1)} \quad \left(\begin{array}{l}\because \frac{k}{m} = \omega^2 \\ \Rightarrow k = m\omega^2\end{array}\right)$$

Eqn (1) is required eqn for undamped free vibration.

To solve this eqn multiple both side by $\frac{2dy}{dt}$ and integrate it w.r.t. time

$$\int 2\frac{d^2y}{dt^2} \cdot \frac{dy}{dt}\, dt + \omega^2 \int 2\frac{dy}{dt} \cdot y\, dt = C$$

$$\left(\frac{dy}{dt}\right)^2 + \omega^2 \cancel{2} \frac{y^2}{\cancel{2}} = C$$

where C is constant of integration

$$\left(\frac{dy}{dt}\right)^2 + \omega^2 y^2 = C \quad \text{--- (2)}$$

when $y = A$ (amplitude), $\frac{dy}{dt} = 0$

put this value is eqn (2)

or $\omega^2 A^2 = C$

$$0 + \omega^2 A^2 = C$$

$$C = \omega^2 A^2$$

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put the value of in eq^n (2)

$$\left(\frac{dy}{dt}\right)^2 = \omega^2 (A^2 - y^2)$$

or, $\frac{dy}{dt} = \omega \sqrt{A^2 - y^2}$

$$\frac{dy}{\sqrt{A^2-y^2}} = \omega dt$$

Integrate both side,

$$\int \frac{dy}{\sqrt{A^2-y^2}} = \int \omega dt$$

$$\sin^{-1}\frac{y}{A} = \omega t + \theta$$ [$\because$ where θ is constant of integration]

$$\Rightarrow \frac{y}{A} = \sin(\omega t + \theta)$$

$$\boxed{y = A \sin(\omega t + \theta)} \quad \text{--- (3)}$$

Eq^n (3) is the solution of undamped free vibration. This eq^n (3) represent eq^n of simple harmonic wave having amplitude 'A' & phase angle 'θ'.

This is an ideal case.

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Equation & Solution of damped free vibration

In case of damped free vibration a retarding force is directly proportional to velocity.

i.e, Retarding Force $\propto \frac{dy}{dt}$

Retarding Force $= \mu \frac{dy}{dt}$

where μ is constant of proportionality.

We know, that Equation for undamped free vibration is

$$m\frac{d^2y}{dt^2} + ky = 0 \quad \text{—— (1)}$$

∴ Retarding force $\mu \frac{dy}{dt}$ must be ~~inclined~~ included in eq^n (1) for damped free vibration.

∴ Eq^n for damped free vibration will be

$$m\frac{d^2y}{dt^2} + ky + \mu\frac{dy}{dt} = 0$$

or, $$\frac{d^2y}{dt^2} + \frac{k}{m}y + \frac{\mu}{m}\frac{dy}{dt} = 0$$

Let $\frac{\mu}{m} = 2b$, $\frac{k}{m} = \omega^2$

∴ Eq^n for damped free vibration becomes

$$\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + \omega^2 y = 0 \quad (11)$$

Eq^n (11) is required equation for damped free vibration.

Here, b is called damping coefficient & 2b is called retarding force per unit mass per unit velocity

Note – $$2b = \frac{\mu \frac{dy}{dt}}{m\frac{dy}{dt}} = \frac{\text{Retarding force}}{m \cdot \frac{dy}{dt}}$$

Let $y = e^{\lambda t}$ is the solution of eq^n (21) where λ is a constant.

$$\frac{dy}{dt} = \lambda e^{\lambda t}$$

$$\frac{d^2y}{dt^2} = \lambda^2 e^{\lambda t}$$

putting this values in eq^n (1)

$$\lambda^2 e^{\lambda t} + 2b\lambda e^{\lambda t} + \omega^2 e^{\lambda t} = 0$$

$$e^{\lambda t}(\lambda^2 + 2b\lambda + \omega^2) = 0$$

$$\lambda^2 + 2b\lambda + \omega^2 = 0$$

$$\lambda = \frac{-2b \pm \sqrt{4b^2 - 4\omega^2}}{2}$$

$$\lambda = -b \pm \sqrt{b^2 - \omega^2}$$

∴ Solution for damped free vibration becomes

$$y = A_1 e^{(-b+\sqrt{b^2-\omega^2})t} + A_2 e^{(-b-\sqrt{b^2-\omega^2})t} \quad \text{—— (III)}$$

Where A_1 And A_2 are two constant whose value is to be determine.

We have,

when $t=0$, $y = y_0$ (max^m displacement of vibration)

$\frac{dy}{dt}$ (velocity) $= 0$

put $t=0$ & $y = y_0$ in eq^n (III)

$y_0 = A_1 + A_2$ —— (IV)

from eq^n (III)

$$\frac{dy}{dt} = A_1(-b+\sqrt{b^2-\omega^2})e^{(-b+\sqrt{b^2-\omega^2})t} + A_2(-b-\sqrt{b^2-\omega^2}) e^{(-b-\sqrt{b^2-\omega^2})t} \quad \text{—— (V)}$$

putt $t=0$ & $\frac{dy}{dt} = 0$ in eq^n (V)

$0 = A_1(-b+\sqrt{b^2-\omega^2}) + A_2(-b-\sqrt{b^2-\omega^2})$

$0 = A_1(-b+\sqrt{b^2-\omega^2}) + A_2(-b-\sqrt{b^2-\omega^2})$

$A_1(-b+\sqrt{b^2-\omega^2}) = A_2(-b-\sqrt{b^2-\omega^2})$

$$A_1 - A_2 = \frac{by_0}{\sqrt{b^2-\omega^2}} \quad \text{—— (VI)}$$

(IV) + (VI) $= A_1$

(VI) − (IV) $= A_2$

we get, $A_1 = \frac{y_0}{2} + \frac{1}{2}\frac{by_0}{\sqrt{b^2-\omega^2}}$

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$$A_2 = \frac{y_0}{2} - \frac{1}{2}\frac{by_0}{\sqrt{b^2-\omega^2}}$$

putt the value of A_1 & A_2 in eqn (III)

$$y = \left(\frac{y_0}{2} + \frac{1}{2}\frac{by_0}{\sqrt{b^2-\omega^2}}\right)e^{(-b+\sqrt{b^2-\omega^2})t} + \left(\frac{y_0}{2} - \frac{1}{2}\frac{by_0}{\sqrt{b^2-\omega^2}}\right)e^{t(-b-\sqrt{b^2-\omega^2})}$$

After solving

$$y = y_0e^{-bt}\left\{\left(\frac{e^{(\sqrt{b^2-\omega^2})t} + e^{-\sqrt{b^2-\omega^2}\,t}}{2}\right) + \frac{b}{\sqrt{b^2-\omega^2}}\left(\frac{e^{\sqrt{b^2-\omega^2}\,t} - e^{\sqrt{b^2-\omega^2}\,t}}{2}\right)\right\} \quad \text{(VII)}$$

Eqn (VII) is the required solution for damped free vibration.

Musical sound & Noise

A sound wave which is pleasant to ear which is periodic is called musical sound & A sound wave ~~is~~ which is not pleasant to ear & which is not periodic ~~&~~ is called Noise.

There are 3 characterstisc of musical sound -

1) Intensity
ii) Pitch
iii) quality

i) Intensity -

Intensity is nothing but the loudness of sound.

The unit of Intensity in SI system is 'Bel' & unit of loudness is 'phon'.

→ Intensity depends upon following factors -

i) Intensity is directly proportional to square of the amplitude of sound wave i.e

$$I \propto A^2$$

ii) Intensity is directly proportional to square of the frequency of the sound wave i.e

$$I \propto n^2$$

iii) Intensity is inversely proportional to square of the distance b/w the source & the observer.

$$I \propto \frac{1}{d^2}$$

iv) Intensity ~~is pr~~ depends upon density of media

ii) Pitch :- It is the characteristics of musical sound by which a sound wave is heard sharp or dull. Pitch depends upon frequency of the sound wave.

iii) Quality - Quality is the characteristics of musical sound by which two or more sound wave having same amplitude same intensity & same pitch are differentiated from each other.

Quality depends upon wave shape of the sound wave.

Electrostatic

The carrier of electric current in case of metals are free electrons. The metals have large number of free electrons in them & when an electric field in applied to it then free electrons experience a force & get accelerated. Hence the free electrons flow through the metals when an external field is applied to each such type of metals are called Conductors. The materials which don't ~~oppoes~~ posses in them are called insulator,

Though insulator do not conduct electricity through it but when it is placed inside an electric field then induced changes are induced on its faces. such type of insulator are dielectric.

Non-Polar & Polar dielectric

In an atom the electrons in the form of electron cloud are distributed around the nucleus having equal positive charge. Therefore, centre of gravity of -ve & +ve charges in the atoms & molecules of a directory may or may not coincide. Therefore, the atoms & molecules of a dielectric are termed as non-polar of polar dielectric.

Non-Polar – when the centre of gravity of positive & negative charges of the atoms & molecules of a dielectric coincide then it is called Non-Polar dielectric.

In non-~~plo~~ polar dielectric the -ve & +ve charges are symmetrically distributed about their centeres. Since, -ve & +ve charge in Non-~~plo~~ polar ~~direclectr~~ dielectric concide Hence they posses zero electric dipole moment.

Ex – O_2, Benzene, N_2, H_2, Methene etc. are the example of Non-polar electric atom or molecule.

→ **Polar dielectric** –

When the centre of gravity of +ve & -ve charges of the atoms & molecules of a dielectric doesn't coincide, then it is called Polar dielectric.

Ex – NH_3, HCL

Numerical

Q. Write down the equation of wave travelling in the -ve x direction along x-axis & having an amplitude 0.01 m, a frequency ~~500~~ 550Hz & speed 330 m/s.

:– The equation of wave travelling in the -ve ~~direction~~ x-direction is given by

$$y = A \sin \frac{2\pi}{\lambda}(vt + x)$$

Now, $A = 0.01m$, $\eta = 550Hz$, $v = 330 m/s$

$$\lambda = \frac{v}{\eta} = \frac{33\cancel{0}}{55\cancel{0}} = \frac{3}{5} \text{ meter}$$

$$y = 0.01 \sin\left[\frac{2\pi 5}{3}(330t + x)\right]$$

$$\therefore y = 0.01 \sin 2\pi \left[550t + \frac{5}{3}x\right] \quad \text{Ans}$$

Q. Calculate the change in intensity level when the intensity of sound increases by 10^6 times its original intensity.

Sol — Let I_0 be the ~~intial~~ original intensity & I be the final intensity, then A/c to question

$$\frac{I}{I_0} = 10^6$$

Increase in intensity level $= 10 \log_{10}\left(\frac{I}{I_0}\right)$

$= 10 \log_{10} 10^6$

$= 10 \times 6 = 60$ decibel

⟹ **Note** — 1 decibel = 10 bels

Q. A hall of volume 5500 m^3 is found to have a reverberation time 2.3 sec. The sound absorbing surface of the hall has an area of 750 m^2. Calculate the Average absorbtion coefficient.

Sol — Let absorbtion Coefficient $= \alpha$

(Reverberation Time) $T = \frac{0.158\,V}{A\alpha}$

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$$\alpha = \frac{0.158\ V}{AT}$$

$$\alpha = \frac{0.158 \times 5500}{750 \times 2.3} = 0.504$$