Lecture Summary: Gaussian Elimination Method

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Key Points

- **Purpose of Gaussian Elimination:**
  - Solve systems of linear equations $Ax = b$.
  - Determine if a solution exists.
  - Find the determinant of a square matrix.
  - Compute the inverse of an invertible square matrix.
- **Augmented Matrix:**
  - The system $Ax = b$ is written as an augmented matrix $[A|b]$.
  - A is an $m \times n$ matrix of coefficients, and b is an $m \times 1$ column vector of constants.
  - The augmented matrix $[A|b]$ is of size $m \times (n + 1)$.
- **Gaussian Elimination Steps:**
  1. Form the augmented matrix $[A|b]$.
  2. Use elementary row operations to bring the matrix to Row Echelon Form (REF):
     - Create leading 1s in the rows.
     - Eliminate entries below the leading 1s.
  3. Further reduce to Reduced Row Echelon Form (RREF) by:
     - Making all entries above and below each leading 1 equal to 0.
- **Interpreting Solutions:**
  - If the last row of $[A|b]$ is $[0 \cdots 0|c]$ where $c \neq 0$, the system is inconsistent (no solution).
  - If the system is consistent:
    1. Identify dependent variables corresponding to leading 1s.
    2. Assign arbitrary values to independent variables (free variables).
    3. Solve for dependent variables by back substitution.
- **Homogeneous Systems ($Ax = 0$):**
  - $x = 0$ (trivial solution) is always a solution.
  - If A has more variables than equations ($n > m$), there are infinitely many solutions, parameterized by free variables.
- **Advantages of Gaussian Elimination:**
  - Algorithmic and efficient for solving linear systems.
  - Avoids repeated determinant calculations (as in Cramer's Rule).
  - Simplifies finding matrix inverses and determinants.

Simplified Explanation

Example 1: Augmented Matrix and Solution System of equations:

$$3x_1 + 2x_2 + x_3 = 6, \quad x_1 + x_2 = 2, \quad 7x_2 + x_3 + x_4 = 8.$$

Augmented matrix:

$$[A|b] = \begin{bmatrix} 3 & 2 & 1 & 0 & 6 \\ 1 & 1 & 0 & 0 & 2 \\ 0 & 7 & 1 & 1 & 8 \end{bmatrix}.$$

Steps:

1. Divide R_1 by 3 to make the first pivot 1.
2. Eliminate entries below the pivot in the first column.
3. Repeat for subsequent columns to reach REF:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Interpretation:

$$x_1 = 1, \quad x_2 = 1, \quad x_3 + x_4 = 1.$$

General solution:

$$x_1 = 1, \quad x_2 = 1, \quad x_3 = 1 - c, \quad x_4 = c \quad (c \in \mathbb{R}).$$

Example 2: Homogeneous System For $Ax = 0$, with A having more variables than equations:

$$\text{Infinitely many solutions exist, parameterized by free variables.}$$

Conclusion

In this lecture, we:

- Introduced Gaussian elimination as a systematic method for solving $Ax = b$.
- Defined the augmented matrix and illustrated its use in solving linear systems.
- Demonstrated Gaussian elimination for both consistent and inconsistent systems.
- Explored the special case of homogeneous systems and their solution properties.

Gaussian elimination is a versatile and efficient tool for solving linear systems, computing determinants, and finding matrix inverses.