Lecture Summary: Systems of Linear Equations

Source: Lec 18.pdf

Key Points

- **Definition:**
  - A system of linear equations consists of multiple linear equations with the same set of unknowns.
  - A linear equation is of the form:

    $$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

    where $x_1, x_2, \ldots, x_n$ are unknowns, $a_1, a_2, \ldots, a_n$ are coefficients, and b is a constant.
- **Matrix Representation:**
  - Any system of linear equations can be represented in matrix form as:

    $$A \cdot x = b,$$

    where:

    $$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$
- **Solutions to Systems:**
  - A system of linear equations can have:
    1. No solution (inconsistent system),
    2. A unique solution,
    3. Infinitely many solutions.

Simplified Explanation

Example 1: Unique Solution Buyer A buys 2 kg of rice and 1 kg of dal, while buyer B buys 3 kg of rice and 1 kg of dal. They pay Rs. 215 and Rs. 260, respectively. The system of linear equations is:

$$2x + y = 215, \quad 3x + y = 260,$$

where x is the price of rice per kg and y is the price of dal per kg. Solving the equations:

$$x = 45, \quad y = 125.$$

Example 2: No Solution If buyer A pays Rs. 215 and buyer B pays Rs. 400 for the same quantities, the equations become:

$$2x + y = 215, \quad 4x + 2y = 400.$$

Doubling the first equation gives:

$$4x + 2y = 430,$$

which contradicts the second equation ($430 \neq 400$). Hence, no solution exists.

Example 3: Infinitely Many Solutions If buyer A pays Rs. 215 and buyer B pays Rs. 430, the equations are:

$$2x + y = 215, \quad 4x + 2y = 430.$$

The second equation is a multiple of the first, representing the same line. Thus, infinitely many solutions exist.

Connection with Geometry

- Each linear equation represents a line in 2D space.
- The number of solutions corresponds to the intersection of lines:
  - Unique solution: Lines intersect at a single point.
  - No solution: Lines are parallel and do not intersect.
  - Infinitely many solutions: Lines overlap completely.

Matrix Representation Examples

Example 1: The system:

$$8x + 8y + 4z = 1960, \quad 12x + 5y + 7z = 2215, \quad 3x + 2y + 5z = 1135$$

is represented as:

$$A = \begin{bmatrix} 8 & 8 & 4 \\ 12 & 5 & 7 \\ 3 & 2 & 5 \end{bmatrix}, \quad x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad b = \begin{bmatrix} 1960 \\ 2215 \\ 1135 \end{bmatrix}.$$

Example 2: The system:

$$3x + 2y + z = 6, \quad x - \frac{1}{2}y + \frac{2}{3}z = \frac{7}{6}, \quad 4x + 6y - 10z = 0$$

is represented as:

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & -\frac{1}{2} & \frac{2}{3} \\ 4 & 6 & -10 \end{bmatrix}, \quad x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad b = \begin{bmatrix} 6 \\ \frac{7}{6} \\ 0 \end{bmatrix}.$$

Conclusion

We studied systems of linear equations, their representation using matrices, and the types of solutions they can have. A system can have no solution, one unique solution, or infinitely many solutions. Matrix notation simplifies working with these systems, particularly for solving or analyzing them.