Lecture Summary: Determinants (Part 2)

Source: Lec 20.pdf

Key Points

- **Review of Determinants (Part 1):**
  - Determinants for 1×1, 2×2, and 3×3 matrices were defined.
  - For a 3×3 matrix, the determinant is calculated using expansion along the first row:

$$\det(A) = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

- **Special Matrices and Properties:**
  - **Upper Triangular and Lower Triangular Matrices:** - Determinant is the product of diagonal elements. - Example:

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 0 & 8 & 7 \\ 0 & 0 & 9 \end{bmatrix}, \quad \det(A) = 2 \cdot 8 \cdot 9 = 144.$$

  - **Transpose of a Matrix:** - Transpose reflects the matrix about its diagonal. - Determinant remains unchanged:

$$\det(A^T) = \det(A).$$

- **Minors and Cofactors:**
  - **Minor:** Determinant of the submatrix obtained by deleting the i-th row and j-th column. Denoted by M_{ij}.
  - **Cofactor:** Minor adjusted by a sign factor:

$$C_{ij} = (-1)^{i+j} M_{ij}.$$

  - Example: For $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$:

$$C_{11} = M_{11}, \quad C_{12} = -M_{12}.$$

- **Inductive Definition of Determinants:**
  - Determinants for $n \times n$ matrices are defined using determinants of $(n-1) \times (n-1)$ matrices.
  - Formula for expansion along the first row:

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} M_{1j} = \sum_{j=1}^{n} a_{1j} C_{1j}.$$

  - This process generalizes to any matrix size.

Simplified Explanation

Example: Determinant of a 4×4 Matrix For $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$:

$$\det(A) = a_{11} \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} + \cdots .$$

Identity Matrix Determinant: For the identity matrix I_n (with diagonal entries 1 and off-diagonal entries 0):

$$\det(I_n) = 1.$$

Conclusion

In this lecture, we:

- Reviewed determinants for small matrices and explored properties of special matrices.
- Defined minors and cofactors, essential for expanding determinants to larger matrices.
- Generalized determinants using an inductive approach, allowing computation for any $n \times n$ matrix.

Determinants play a crucial role in solving systems of linear equations, finding matrix inverses, and applications in higher mathematics like calculus.