Lecture Summary: Linear Dependence

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Key Points

- **Definition of Linear Dependence:**
  - A set of vectors $v_1, v_2, \ldots, v_n$ in a vector space V is **linearly dependent** if there exist scalars $a_1, a_2, \ldots, a_n$, not all zero, such that:
  $$a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0.$$
  - Equivalently, the zero vector is a linear combination of $v_1, v_2, \ldots, v_n$ with at least one non-zero coefficient.
- **Key Observations:**
  - If two vectors are linearly dependent, one is a scalar multiple of the other, and they lie on the same line.
  - If three vectors in $\mathbb{R}^3$ are linearly dependent, they lie on the same plane.
  - If a set of vectors is linearly dependent, any superset of that set is also linearly dependent.
- **Linear Combinations:**
  - A vector v is a linear combination of $v_1, v_2, \ldots, v_n$ if:
  $$v = a_1v_1 + a_2v_2 + \cdots + a_nv_n,$$
  where $a_1, a_2, \ldots, a_n$ are scalars.
  - Example:
  $$2(1,2) + (2,1) = (4,5).$$
- **Testing for Linear Dependence:**
  - Form the equation:
  $$a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0.$$
  - Solve for $a_1, a_2, \ldots, a_n$. If a non-trivial solution exists (i.e., not all a_i are zero), the vectors are linearly dependent.

Simplified Explanation

Example 1: Linear Dependence in $\mathbb{R}^2$ Given vectors $(1,2)$, $(2,1)$, and $(4,5)$:

- $(4,5)$ is a linear combination of $(1,2)$ and $(2,1)$:
$$2(1,2) + (2,1) = (4,5).$$

- Rearranging:
$$2(1,2) + (2,1) - (4,5) = (0,0).$$
- The zero vector is a linear combination of these vectors with non-zero coefficients, so they are linearly dependent.

Example 2: Linear Dependence in $\mathbb{R}^3$ Given vectors $(2,1,2)$, $(3,0,1)$, and $(10,-4,-2)$:

- Equation:
$$2(2,1,2) - 3(3,0,1) + \frac{1}{2}(10,-4,-2) = (0,0,0).$$
- Since non-zero coefficients exist, these vectors are linearly dependent.

Example 3: Linear Independence Given $(0,2,1)$, $(2,2,0)$, and $(1,2,0)$:

- Suppose:
$$a(0,2,1) + b(2,2,0) + c(1,2,0) = (0,0,0).$$
- Solving gives $a = 0, b = 0, c = 0$. Hence, these vectors are linearly independent.

Conclusion

In this lecture, we:

- Defined linear dependence and provided the geometric intuition for $\mathbb{R}^2$ and $\mathbb{R}^3$.
- Demonstrated how to check for linear dependence using linear combinations.
- Highlighted that if a set of vectors is linearly dependent, any superset is also linearly dependent.

Linear dependence is a foundational concept for understanding the structure of vector spaces and solving systems of equations.