# Lecture Summary: Tangents for Scalar-Valued Multivariable Functions

## Source: Week 11 - Lec 02.pdf

## Key Points

- **Tangents in One Variable:**
  - In single-variable calculus, the tangent line to a curve $C$ at a point $P$ represents the instantaneous direction of the curve at $P$.
  - The equation of the tangent line is given by:

    $$y - f(a) = f'(a)(x - a),$$

    where $f'(a)$ is the derivative of $f$ at $a$.
- **Extension to Multivariable Functions:**
  - For a scalar-valued multivariable function $f : D \subset \mathbb{R}^n \to \mathbb{R}$, the tangent line is generalized using directional derivatives.
  - Directional derivatives compute the rate of change of $f$ along a given direction, forming the slope of the tangent line in that direction.
- **Tangents in Two Dimensions:**
  - Consider $f(x, y)$ defined on $D \subset \mathbb{R}^2$. At $(a, b)$, take a line $L$ in the direction of a unit vector $\vec{u} = (u_1, u_2)$.
  - Restrict $f$ to $L$ and compute the directional derivative $f_u(a, b)$, which is the slope of the tangent line.
  - The tangent line passing through $(a, b, f(a, b))$ is given parametrically by:

    $$\begin{aligned} x(t) &= a + tu_1, \\ y(t) &= b + tu_2, \\ z(t) &= f(a, b) + tf_u(a, b). \end{aligned}$$

  - Alternatively, in vector form:

    $$(x(t), y(t), z(t)) = (a, b, f(a, b)) + t(u_1, u_2, f_u(a, b)).$$
- **Tangents in Higher Dimensions:**
  - For $f : \mathbb{R}^n \to \mathbb{R}$, consider a line through $\vec{a} \in \mathbb{R}^n$ in the direction of $\vec{u} = (u_1, u_2, \ldots, u_n)$.
  - Parametric equations of the tangent line:

    $$x_i(t) = a_i + tu_i, \quad z(t) = f(\vec{a}) + tf_u(\vec{a}),$$

    where $f_u(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{u}$.

- **Examples:**
  - **Example 1:** $f(x, y) = x + y$ at $(1, 1)$ in the direction of $(1, 0)$:

    $$\nabla f = (1, 1), \quad f_u(1, 1) = 1.$$

    Parametric form:

    $$x(t) = 1 + t, \quad y(t) = 1, \quad z(t) = 2 + t.$$
  - **Example 2:** $f(x, y) = xy$ at $(1, 1)$ in the direction of $(3, 4)$:

    $$\vec{u} = \left(\frac{3}{5}, \frac{4}{5}\right), \quad \nabla f = (y, x), \quad f_u(1, 1) = \frac{7}{5}.$$

    Parametric form:

    $$x(t) = 1 + \frac{3t}{5}, \quad y(t) = 1 + \frac{4t}{5}, \quad z(t) = 1 + \frac{7t}{5}.$$
- **When Tangents Fail to Exist:**
  - Tangents may not exist if the gradient is not continuous at the point of interest.
  - Examples include piecewise-defined functions with discontinuities or corners, such as $f(x, y) = |x| + |y|$.
- **General Conditions for Tangents:**
  - If $\nabla f$ exists and is continuous in an open neighborhood of $\vec{a}$, then:

    $$f_u(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{u},$$

    ensuring the existence of tangents in all directions at $\vec{a}$.

# Simplified Explanation

**Tangents in Multivariable Functions:** Generalize the concept of tangent lines from single-variable calculus using directional derivatives.

**Parametric Form:** The tangent line to $f(x, y)$ at $(a, b)$ in the direction of $\vec{u}$ is:

$$x(t) = a + tu_1, \quad y(t) = b + tu_2, \quad z(t) = f(a, b) + tf_u(a, b).$$

**Example:** For $f(x, y) = x + y$ at $(1, 1)$ in the direction of $(1, 0)$:

$$x(t) = 1 + t, \quad y(t) = 1, \quad z(t) = 2 + t.$$

# Conclusion

In this lecture, we:

- Extended the concept of tangents from single-variable to multivariable functions.
- Explored examples and derived parametric equations for tangent lines.
- Discussed conditions under which tangents exist.

Understanding tangents in multivariable calculus is essential for geometric and analytic interpretations of scalar fields.