Lecture Summary: Orthogonality and Linear Independence

Source: Lec48.pdf

Key Points

- **Orthogonality in $\mathbb{R}^n$:**
  - Two vectors u and v in $\mathbb{R}^n$ are orthogonal if the angle θ between them is $90°$, which implies:
  $$\cos(\theta) = 0 \implies u \cdot v = 0.$$
  - Example: The vectors $(1, 2, 3)$ and $(2, 2, -2)$ are orthogonal because:
  $$1 \cdot 2 + 2 \cdot 2 + 3 \cdot (-2) = 2 + 4 - 6 = 0.$$
- **General Definition of Orthogonality:**
  - In an inner product space V, two vectors u and v are orthogonal if:
  $$\langle u, v \rangle = 0,$$
  where $\langle \cdot, \cdot \rangle$ is the inner product.
  - Example in $\mathbb{R}^2$ with a non-standard inner product:
  $$\langle u, v \rangle = x_1 y_1 - (x_1 y_2 + x_2 y_1) + 2x_2 y_2.$$
  Vectors $(1, 1)$ and $(1, 0)$ are orthogonal:
  $$\langle (1, 1), (1, 0) \rangle = 1 \cdot 1 - (1 \cdot 0 + 1 \cdot 1) + 2 \cdot 1 \cdot 0 = 0.$$
  - Orthogonality depends on the chosen inner product.
- **Orthogonal Sets:**
  - A set of vectors $\{v_1, v_2, \ldots, v_k\}$ in V is orthogonal if:
  $$\langle v_i, v_j \rangle = 0 \quad \text{for all } i \neq j.$$
  - Example: In $\mathbb{R}^3$ with the dot product, the set $\{(4, 3, -2), (-3, 2, -3), (-5, 18, 17)\}$ is orthogonal:
  $$\begin{aligned} (4, 3, -2) \cdot (-3, 2, -3) &= -12 + 6 + 6 = 0, \\ (4, 3, -2) \cdot (-5, 18, 17) &= -20 + 54 - 34 = 0, \\ (-3, 2, -3) \cdot (-5, 18, 17) &= 15 + 36 - 51 = 0. \end{aligned}$$
- **Orthogonality and Linear Independence:**
  - An orthogonal set of non-zero vectors is always linearly independent.
  - Proof sketch:

    * Assume $\sum_{i=1}^{k} c_i v_i = 0$.
    * Taking the inner product with v_1, only the term $c_1\langle v_1, v_1\rangle$ remains (since $\langle v_i, v_1\rangle = 0$ for $i \neq 1$).
    * $\langle v_1, v_1\rangle > 0 \implies c_1 = 0$.
    * Repeating for all v_i, we conclude $c_i = 0$ for all i.

- **Orthogonal Basis:**
  - A basis $\{v_1, v_2, \ldots, v_n\}$ of V is orthogonal if it is an orthogonal set.
  - A basis is orthogonal if and only if it is a maximal orthogonal set.
  - Example:
    * In $\mathbb{R}^3$, $\{(4, 3, -2), (-3, 2, -3), (-5, 18, 17)\}$ is an orthogonal basis since it is orthogonal and has size 3, the dimension of $\mathbb{R}^3$.
    * In $\mathbb{R}^2$ with the non-standard inner product:

      $$\langle u, v\rangle = x_1y_1 - (x_1y_2 + x_2y_1) + 2x_2y_2,$$

      the set $\{(1, 1), (1, 0)\}$ forms an orthogonal basis.

Simplified Explanation

Orthogonal Vectors: Vectors are orthogonal if their inner product is 0. In $\mathbb{R}^n$, this corresponds to being at right angles.

Orthogonal Sets: A set of vectors is orthogonal if every pair in the set is orthogonal. Orthogonal sets are always linearly independent.

Orthogonal Basis: An orthogonal basis is an orthogonal set that spans the entire vector space. For example, the standard basis in $\mathbb{R}^n$ is an orthogonal basis with the dot product.

Conclusion

In this lecture, we:

- Defined orthogonality using inner products.
- Explored the relationship between orthogonal sets and linear independence.
- Introduced orthogonal bases and provided examples in different inner product spaces.

Orthogonality simplifies the study of vector spaces, making linear independence checks and computations more efficient.