Lecture Summary: Echelon Form

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Key Points

- **Definition of Echelon Form:**
  - A matrix is in **Row Echelon Form (REF)** if:
    1. The first non-zero element in each row (called the *leading entry*) is 1.
    2. Each leading entry is to the right of the leading entry in the row above.
    3. Rows with all zero elements are at the bottom.
  - A matrix is in **Reduced Row Echelon Form (RREF)** if, in addition to being in REF:
    1. Each column containing a leading 1 has all other entries in that column as 0.
- **Properties of Echelon Forms:**
  - Solutions to a system of linear equations can be easily obtained when the coefficient matrix is in echelon form.
  - Independent variables are those corresponding to columns without leading 1s.
  - Dependent variables correspond to columns containing leading 1s.
- **Procedure to Solve $Ax = b$:**
  - Convert the coefficient matrix A to RREF.
  - Identify independent and dependent variables.
  - Assign arbitrary values to independent variables and compute dependent variables using the equations.
- **Homogeneous Systems:**
  - For $Ax = 0$, if A is in RREF:
    1. If there are no zero rows, the system has a unique solution ($x = 0$).
    2. If there are zero rows, the system has infinitely many solutions, parameterized by the independent variables.
- **Inconsistent Systems:**
  - If a row in A is all zeros, but the corresponding entry in b is non-zero, the system has no solution.

Simplified Explanation

Example: REF and RREF For $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$:

- REF:

1. The first non-zero element in each row is 1.
2. The leading 1 in the second row is to the right of the leading 1 in the first row.
3. There are no zero rows.

- RREF:

1. Each column containing a leading 1 has all other entries as 0.

Example: Solving $Ax = b$ Given $Ax = b$ with A in RREF:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

Equations:

$$x_1 + 2x_3 = b_1, \quad x_2 + 3x_3 = b_2.$$

- Assign $x_3 = c$ (independent variable).
- Solve for dependent variables:
$$x_1 = b_1 - 2c, \quad x_2 = b_2 - 3c.$$
- General solution:
$$x = \begin{bmatrix} b_1 - 2c \\ b_2 - 3c \\ c \end{bmatrix}.$$

Inconsistent Case: If a row in A is all zeros but the corresponding $b_i \neq 0$, then the system is inconsistent (no solution).

Conclusion

In this lecture, we:

- Defined Row Echelon Form (REF) and Reduced Row Echelon Form (RREF).
- Showed how to use RREF to solve systems of linear equations efficiently.
- Analyzed solutions for consistent, inconsistent, and homogeneous systems.

This approach provides an algorithmic method to find all solutions to $Ax = b$, leveraging the structure of echelon forms.