Lecture Summary: Dimension and Rank of a Vector Space

Source: Lec34.pdf

Key Points

- **Definition of Basis:**
  - A basis for a vector space V is a set of vectors that is:
    1. **Linearly Independent:** The only solution to a linear combination equaling the zero vector is for all coefficients to be zero.
    2. **Spanning:** Every vector in V can be expressed as a linear combination of the basis vectors.
- **Dimension of a Vector Space:**
  - The **dimension** (or **rank**) of a vector space V is the number of vectors in any basis of V.
  - The dimension is denoted by $\dim(V)$ or $\text{rank}(V)$.
  - Every vector space has a basis, and all bases of a vector space have the same size.
- **Examples:**
  - The dimension of $\mathbb{R}^n$ is n, with the standard basis $\{e_1, e_2, \ldots, e_n\}$.
  - For subspaces of $\mathbb{R}^3$, the dimension depends on the basis of the subspace. For example:
    * The xy-plane in $\mathbb{R}^3$ has dimension 2, with basis $\{(1,0,0),(0,1,0)\}$.
    * A line through the origin in $\mathbb{R}^3$ has dimension 1, with a single basis vector.
- **Finding the Dimension:**
  - Two methods to compute the dimension:
    1. **Finding a Basis:** Use appending or deleting methods to construct a basis and count the number of vectors.
    2. **Using Matrices:** Write vectors that span the subspace as rows of a matrix and reduce it to row echelon form (or reduced row echelon form). The number of non-zero rows is the dimension.
- **Rank of a Matrix:**
  - The **rank** of a matrix A is defined as:
    * The dimension of the column space of A (span of the column vectors).
    * The dimension of the row space of A (span of the row vectors).
  - A fundamental result states that the column rank equals the row rank.
- **Example: Computing Rank and Dimension**
  - Consider $W \subseteq \mathbb{R}^3$ spanned by $\{(1,0,0),(0,1,0),(3,5,0)\}$:
    1. The vector $(3,5,0)$ is a linear combination of $(1,0,0)$ and $(0,1,0)$.
    2. Removing $(3,5,0)$ leaves a basis $\{(1,0,0),(0,1,0)\}$ for W.

3. Dimension of $W = 2$ (basis has 2 vectors).

– Using matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 5 & 0 \end{bmatrix}.$$

Row reduce:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The rank is 2 (2 non-zero rows), confirming $\dim(W) = 2$.

Simplified Explanation

Example 1: Subspace of $\mathbb{R}^3$ Given vectors $(1,0,0)$, $(0,1,0)$, and $(3,5,0)$:

- $(3,5,0) = 3(1,0,0) + 5(0,1,0)$, so it is dependent on the others.
- Removing $(3,5,0)$ leaves a basis $\{(1,0,0),(0,1,0)\}$.
- Dimension of the subspace is 2.

Example 2: Rank of a Matrix Matrix:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 5 & 0 \end{bmatrix}.$$

Row reduce:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Rank $= 2$ (2 non-zero rows).

Conclusion

In this lecture, we:

- Defined the dimension (rank) of a vector space as the number of elements in a basis.
- Showed methods for computing dimension via basis construction or matrix row reduction.
- Introduced the concept of matrix rank as the dimension of its row or column space.

Understanding dimension and rank is fundamental for analyzing vector spaces and matrices, bridging algebraic and geometric perspectives.