Lecture Summary: Rank and Dimension Using Gaussian Elimination

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Key Points

- **Overview:**
  - Gaussian elimination can be used to compute the rank and dimension of a vector space or subspace and to find a basis.
  - This process is systematic and avoids ad-hoc methods, providing a clear algorithmic approach.
- **Row Method:**
  - **Steps:**
    1. Arrange the spanning set vectors as rows of a matrix.
    2. Perform row reduction to bring the matrix to row echelon form.
    3. Count the number of non-zero rows:
       * This is the dimension of the subspace.
       * The non-zero rows form a basis for the subspace.
  - **Example: Subspace W in $\mathbb{R}^3$ spanned by $\{(1,0,1),(-2,-3,1),(3,3,0)\}$:**
    1. Matrix:
    $$\begin{bmatrix} 1 & 0 & 1 \\ -2 & -3 & 1 \\ 3 & 3 & 0 \end{bmatrix}.$$
    2. Row reduction yields:
    $$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$
    3. Dimension $= 2$ (two non-zero rows).
    4. Basis: $\{(1,0,1),(0,1,-1)\}$.
- **Column Method:**
  - **Steps:**
    1. Arrange the spanning set vectors as columns of a matrix.
    2. Perform row reduction to row echelon form.
    3. Identify pivot columns (columns with leading ones).
    4. The original vectors corresponding to pivot columns form a basis.
  - **Example: Subspace W in $\mathbb{R}^3$ spanned by $\{(1,0,1),(-2,-3,1),(3,3,0)\}$:**
    1. Matrix:
    $$\begin{bmatrix} 1 & -2 & 3 \\ 0 & -3 & 3 \\ 1 & 1 & 0 \end{bmatrix}.$$

2. Row reduction yields:

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

3. Pivot columns: 1 and 2.
4. Basis: $\{(1,0,1), (-2,-3,1)\}$.

- **Comparison of Methods:**
  - **Row Method:**
    * Provides a basis directly from the rows of the row-reduced matrix.
    * The basis vectors may not belong to the original spanning set.
  - **Column Method:**
    * Provides a basis directly from the original spanning set.
    * Useful when you need the basis to be composed of specific original vectors.

Simplified Explanation

Example 1: Row Method in $\mathbb{R}^4$ Given vectors $\{(1,-2,0,4), (3,1,1,0), (-1,-5,-1,8), (3,8,2,-12)\}$:

- Row reduction:

$$\begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

- Dimension $= 2$.
- Basis: $\{(1,-2,0,4), (0,1,-1,2)\}$.

Example 2: Column Method in $\mathbb{R}^4$ Given the same vectors:

- Column matrix:

$$\begin{bmatrix} 1 & 3 & -1 & 3 \\ -2 & 1 & -5 & 8 \\ 0 & 1 & 2 & -12 \\ 4 & 0 & 8 & -12 \end{bmatrix}.$$

- Pivot columns: 1 and 2.
- Basis: $\{(1,-2,0,4), (3,1,1,0)\}$.

Conclusion

In this lecture, we:

- Explored two algorithmic methods for computing rank and dimension using Gaussian elimination.
- Compared the row method (basis from row-reduced matrix rows) and the column method (basis from original spanning set vectors).
- Highlighted examples in $\mathbb{R}^3$ and $\mathbb{R}^4$ to illustrate both methods.

Gaussian elimination provides a versatile approach for basis and dimension computation, adaptable to various applications.