Lecture Summary: The Gram-Schmidt Process

Source: The Gram-Schmidt process.pdf

Key Points

- **Definition of Gram-Schmidt Process:**
  - The Gram-Schmidt process converts any basis $\{x_1, x_2, \ldots, x_n\}$ of an inner product space into an orthonormal basis $\{w_1, w_2, \ldots, w_n\}$.
  - An orthonormal basis is a set of mutually orthogonal vectors, each with norm 1:

$$\langle w_i, w_j \rangle = 0 \quad \text{for } i \neq j, \quad \|w_i\| = 1.$$

- **Procedure for Gram-Schmidt Process:**
  1. Start with a basis $\{x_1, x_2, \ldots, x_n\}$.
  2. Define $v_1 = x_1$ and $w_1 = \frac{v_1}{\|v_1\|}$.
  3. For $i \geq 2$, define:

$$v_i = x_i - \sum_{j=1}^{i-1} \langle x_i, w_j \rangle w_j, \quad w_i = \frac{v_i}{\|v_i\|}.$$

  4. The result is an orthonormal basis $\{w_1, w_2, \ldots, w_n\}$.

- **Key Concepts:**
  - The Gram-Schmidt process relies on projections to iteratively remove components of x_i in the direction of the previous orthonormal vectors.
  - Each v_i is orthogonal to all v_j for $j < i$, ensuring orthogonality.

- **Examples:**
  - Example in $\mathbb{R}^3$: Starting basis $\{(1, 2, 2), (-1, 0, 2), (0, 0, 1)\}$:
    * Step 1: $v_1 = (1, 2, 2)$, $w_1 = \frac{1}{3}(1, 2, 2)$.
    * Step 2: $v_2 = (-1, 0, 2) - \langle (-1, 0, 2), w_1 \rangle w_1$.

$$\langle (-1, 0, 2), w_1 \rangle = \frac{1}{3}(-1 \cdot 1 + 0 \cdot 2 + 2 \cdot 2) = \frac{3}{9}.$$

$$v_2 = (-1, 0, 2) - \frac{1}{3}(1, 2, 2) = (-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}), \quad w_2 = \frac{1}{3}(-4, -2, 4).$$

    * Step 3: $v_3 = (0, 0, 1) - \langle (0, 0, 1), w_1 \rangle w_1 - \langle (0, 0, 1), w_2 \rangle w_2$.

$$w_3 = \frac{1}{3}(2, -2, 1).$$

    * Orthonormal basis:

$$\left\{ \frac{1}{3}(1, 2, 2), \frac{1}{3}(-4, -2, 4), \frac{1}{3}(2, -2, 1) \right\}.$$

- **Applications of Gram-Schmidt Process:**
  - Converting any basis into an orthonormal basis for simplified computations in linear algebra.
  - Facilitating projections and decompositions, such as in QR decomposition of matrices.
  - Useful in functional analysis, signal processing, and data science.

Simplified Explanation

Gram-Schmidt Process: A step-by-step method to convert any basis into an orthonormal basis by iteratively removing components along previous directions.

Example in $\mathbb{R}^3$: Start with $(1, 2, 2)$, $(-1, 0, 2)$, $(0, 0, 1)$:

- Normalize $(1, 2, 2)$ to get the first orthonormal vector.
- Subtract projections to make $(-1, 0, 2)$ orthogonal to $(1, 2, 2)$, then normalize.
- Repeat for $(0, 0, 1)$ to make it orthogonal to both previous vectors, then normalize.

Applications: The Gram-Schmidt process is critical in transforming vector spaces, enabling simplified calculations, particularly with projections and decompositions.

Conclusion

In this lecture, we:

- Defined the Gram-Schmidt process.
- Demonstrated its use in generating orthonormal bases from arbitrary bases.
- Highlighted applications in linear algebra and computational mathematics.

The Gram-Schmidt process is a foundational tool in linear algebra, widely used in both theoretical and applied contexts.