Lecture Summary: Linear Independence – Part 2

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Key Points

- **Review of Linear Independence:**
  - A set of vectors $v_1, v_2, \ldots, v_n$ is **linearly independent** if the only solution to:

    $$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$$

    is $a_1 = a_2 = \cdots = a_n = 0$.
  - A set is **linearly dependent** if there exist coefficients, not all zero, that satisfy the above equation.
- **Homogeneous System Approach:**
  - Linear independence can be tested by checking if the homogeneous system $Vx = 0$ has only the trivial solution.
  - Here, V is the matrix formed by using the vectors $v_1, v_2, \ldots, v_n$ as columns.
- **Key Results:**
  - If the number of vectors exceeds the dimension of the space ($n > m$), the vectors are always linearly dependent.
  - If the number of vectors equals the dimension of the space ($n = m$), linear independence depends on whether $\det(V) \neq 0$.
- **Relation to Determinants:**
  - If V is an $n \times n$ matrix:
    1. If $\det(V) = 0$, the vectors are linearly dependent.
    2. If $\det(V) \neq 0$, the vectors are linearly independent.

Simplified Explanation

Example 1: Checking Independence in $\mathbb{R}^2$ Given vectors $(5, 2)$ and $(1, 3)$:

$$x_1(5, 2) + x_2(1, 3) = (0, 0).$$

This results in the equations:

$$5x_1 + x_2 = 0, \quad 2x_1 + 3x_2 = 0.$$

The determinant of the corresponding matrix is:

$$\det \begin{bmatrix} 5 & 1 \\ 2 & 3 \end{bmatrix} = 13 \neq 0.$$

Hence, $(5, 2)$ and $(1, 3)$ are linearly independent.

Example 2: Dependency in $\mathbb{R}^2$ with Three Vectors Given $(1,2)$, $(1,3)$, and $(3,4)$:

$$x_1(1,2) + x_2(1,3) + x_3(3,4) = (0,0).$$

The augmented matrix is:

$$\begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 4 \end{bmatrix}.$$

Gaussian elimination reveals infinitely many solutions, indicating the vectors are linearly dependent.

Example 3: Independence in $\mathbb{R}^3$ Given $(1,2,0)$, $(0,2,4)$, and $(3,0,0)$:

$$x_1(1,2,0) + x_2(0,2,4) + x_3(3,0,0) = (0,0,0).$$

The determinant of the matrix:

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 2 & 0 \\ 0 & 4 & 0 \end{bmatrix} = 24 \neq 0.$$

Thus, the vectors are linearly independent.

Example 4: Dependency with Four Vectors in $\mathbb{R}^3$ Given $(1,2,0)$, $(0,2,4)$, $(3,0,0)$, and $(1,2,3)$:

$$x_1(1,2,0) + x_2(0,2,4) + x_3(3,0,0) + x_4(1,2,3) = (0,0,0).$$

Gaussian elimination yields a free variable, confirming the vectors are linearly dependent.

Conclusion

In this lecture, we:

- Discussed the homogeneous system approach to determining linear independence.
- Explored examples using both determinant checks and Gaussian elimination.
- Highlighted that more vectors than the space's dimension ($n > m$) always result in dependence.

This framework unifies linear independence testing, bridging algebraic and geometric interpretations.