Lecture Summary: Determinants (Part 3)

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Key Points

- **Review of Determinants (Parts 1 and 2):**
  - Determinants were defined inductively for 1×1, 2×2, and 3×3 matrices.
  - Expansion along the first row was used to compute determinants, with minors and cofactors as key components:
$$\text{Cofactor: } C_{ij} = (-1)^{i+j} M_{ij}.$$
- **Expansion Along Any Row or Column:**
  - Determinants can be computed by expanding along any row or column.
  - Formula for expansion along the ith row:
$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}.$$
  - Formula for expansion along the jth column:
$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij}.$$
- **Properties of Determinants:**
  - **Multiplication:**
$$\det(AB) = \det(A) \cdot \det(B).$$
  - **Powers and Inverses:**
$$\det(A^n) = (\det(A))^n, \quad \det(A^{-1}) = \frac{1}{\det(A)}.$$
  - **Similarity Transformation:**
$$\det(P^{-1}AP) = \det(A).$$
  - **Transpose:**
$$\det(A^T) = \det(A), \quad \det(A^T A) = (\det(A))^2.$$
  - **Row/Column Operations:**
    - ∗ Swapping two rows or columns changes the determinant's sign.
    - ∗ Adding a multiple of one row (or column) to another does not change the determinant.

    * Multiplying a row or column by t scales the determinant by t.
    * Multiplying the entire matrix by t scales the determinant by t^n, where n is the matrix size.

- **Computational Tips:**
  - Determinants of matrices with a zero row or column are 0.
  - Determinants of matrices with a row (or column) that is a linear combination of others are 0.
  - Expand along a row or column with the most zeros for easier computation.

Simplified Explanation

Example: Determinant by Expansion Along the Second Row For $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$:

$$\det(A) = -a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}.$$

Example: Multiplying a Matrix by t If A is an $n \times n$ matrix and every element is multiplied by t:

$$\det(tA) = t^n \det(A).$$

Conclusion

In this lecture, we:

- Expanded determinants along any row or column, showing that the method generalizes beyond the first row.
- Explored additional determinant properties, including behavior under row/column operations, matrix transposes, and scalar multiplication.
- Reviewed computational tips to simplify determinant calculations.

In the next lecture, we will use determinants to solve systems of linear equations, showcasing their practical applications.