Lecture Summary: Determinants (Part 1)

Source: Lec 19.pdf

Key Points

- **Definition of Determinants:**
  - The determinant is a scalar value associated with a square matrix.
  - Notation: $\det(A)$ or $|A|$.
  - Applications:
    * Solving systems of linear equations.
    * Finding the inverse of a matrix.
    * Applications in calculus and other advanced topics.
- **Determinants of Small Matrices:**
  - **1x1 Matrix:** For $A = [a]$, $\det(A) = a$.
  - **2x2 Matrix:**
  $$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \det(A) = ad - bc.$$
  - **3x3 Matrix:**
  $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$
  $$\det(A) = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$
- **Properties of Determinants:**
  - **Multiplication:** $\det(AB) = \det(A) \cdot \det(B)$.
  - **Inverse:** $\det(A^{-1}) = \frac{1}{\det(A)}$.
  - **Row/Column Operations:**
    * Swapping two rows or columns changes the determinant's sign.
    * Adding a multiple of one row (or column) to another does not change the determinant.
    * Multiplying all entries of a row or column by a constant t scales the determinant by t.

Simplified Explanation

Example: Determinant of a 2x2 Matrix For $A = \begin{bmatrix} 2 & 3 \\ 6 & 10 \end{bmatrix}$,

$$\det(A) = (2 \cdot 10) - (3 \cdot 6) = 20 - 18 = 2.$$

Example: Determinant of a 3x3 Matrix For $A = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 8 & 7 \\ 5 & 6 & 9 \end{bmatrix}$:

$$\det(A) = 2 \cdot \begin{vmatrix} 8 & 7 \\ 6 & 9 \end{vmatrix} - 4 \cdot \begin{vmatrix} 3 & 7 \\ 5 & 9 \end{vmatrix} + 1 \cdot \begin{vmatrix} 3 & 8 \\ 5 & 6 \end{vmatrix}.$$

$$\det(A) = 2 \cdot (72 - 42) - 4 \cdot (27 - 35) + 1 \cdot (18 - 40).$$

$$\det(A) = 2 \cdot 30 - 4 \cdot (-8) + 1 \cdot (-22) = 70.$$

Special Matrices and Their Determinants

- **Identity Matrix:** For $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\det(I_2) = 1$. Similarly, for $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\det(I_3) = 1$.

Conclusion

In this lecture, we explored:

- The concept of determinants for 1x1, 2x2, and 3x3 matrices.
- Properties of determinants, including their behavior under multiplication, inversion, and row/column operations.
- Determinants are computed inductively, with larger matrices reduced to smaller ones.

These foundational properties of determinants will be critical in solving linear equations, understanding matrix behavior, and various applications in higher mathematics.