Lecture Summary: Linear Independence – Part 1

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Key Points

- **Definition of Linear Independence:**
  - A set of vectors $v_1, v_2, \ldots, v_n$ from a vector space V is **linearly independent** if:

    $$a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0 \implies a_1 = a_2 = \cdots = a_n = 0.$$

  - Linear independence means the only way to express the zero vector as a linear combination of $v_1, v_2, \ldots, v_n$ is with all coefficients $a_i = 0$.
- **Relation to Linear Dependence:**
  - A set of vectors is linearly dependent if there exist scalars $a_1, a_2, \ldots, a_n$, not all zero, such that:

    $$a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0.$$

  - Linear independence is the negation of linear dependence.
- **Key Properties:**
  - Any set of vectors containing the zero vector is linearly dependent.
  - Two non-zero vectors are linearly independent if and only if they are not scalar multiples of each other.
  - Three vectors in $\mathbb{R}^3$ are linearly independent if no vector is a linear combination of the other two.
- **Examples:**
  - In $\mathbb{R}^2$, vectors $(-1, 3)$ and $(2, 0)$ are linearly independent:

    $$a(-1, 3) + b(2, 0) = (0, 0) \implies a = 0, b = 0.$$

  - A set containing the zero vector, such as $\{(0, 0), (1, 2)\}$, is linearly dependent.
  - In $\mathbb{R}^3$, vectors $(1, 1, 2)$, $(1, 2, 0)$, and $(0, 2, 1)$ are linearly independent:

    $$a(1, 1, 2) + b(1, 2, 0) + c(0, 2, 1) = (0, 0, 0) \implies a = 0, b = 0, c = 0.$$

- **Geometric Interpretation:**
  - Two vectors are linearly independent if they point in different directions (not collinear).
  - Three vectors in $\mathbb{R}^3$ are linearly independent if they do not lie on the same plane.

Simplified Explanation

Example 1: Linear Independence in $\mathbb{R}^2$ Vectors $(-1, 3)$ and $(2, 0)$ satisfy:

$$a(-1, 3) + b(2, 0) = (0, 0) \implies -a + 2b = 0, \quad 3a = 0.$$

Solving gives $a = 0$, $b = 0$, so the vectors are linearly independent.

Example 2: Dependency with the Zero Vector If $v_1 = (0, 0)$ and $v_2 = (1, 2)$:

$$a(0, 0) + b(1, 2) = (0, 0) \implies b = 0, \quad a \text{ can be non-zero.}$$

The set is linearly dependent since the zero vector is included.

Example 3: Linear Independence in $\mathbb{R}^3$ Vectors $(1, 1, 2)$, $(1, 2, 0)$, and $(0, 2, 1)$:

$$a(1, 1, 2) + b(1, 2, 0) + c(0, 2, 1) = (0, 0, 0).$$

Solving gives $a = 0$, $b = 0$, $c = 0$, so the vectors are linearly independent.

Conclusion

In this lecture, we:

- Defined linear independence and its relationship with linear dependence.
- Presented key examples in $\mathbb{R}^2$ and $\mathbb{R}^3$ to illustrate the concept.
- Highlighted geometric interpretations and practical scenarios.

Understanding linear independence is essential for analyzing vector spaces, solving linear systems, and exploring higher-dimensional geometry.