Lecture Summary: Multivariable Functions and Visualization

Source: Multivariable Functions - Visualization.pdf

Key Points

- **Single-Variable Functions Recap:**
  - Functions of the form $f(x)$ map a domain $D \subset \mathbb{R}$ to $\mathbb{R}$.
  - Examples include:
    * Linear: $f(x) = ax + b$.
    * Polynomial: $f(x) = x^2 + x + 1$.
    * Rational: $f(x) = \frac{x}{x^2+1}$.
    * Trigonometric: $\sin(x), \cos(x)$, etc.
    * Exponential: e^x.
    * Logarithmic: $\log(x)$ (domain restricted to $x > 0$).
    * Compositions: $\log(x^2 + 1)$, $e^{\sin(x)}$.
  - Key operations include addition, multiplication, and composition of functions.
- **Scalar-Valued Multivariable Functions:**
  - A scalar-valued multivariable function maps $f : D \subset \mathbb{R}^n \to \mathbb{R}$.
  - Examples:
    * Linear: $f(x_1, x_2, \ldots, x_n) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$.
    * Polynomial: $f(x_1, x_2, x_3) = x_1^2 + x_2^3x_3 - x_3^6$.
    * Rational: $f(x, y) = \frac{x}{x^2+y^2}$ (domain excludes $x = y = 0$).
    * Combinations: $f(x, y) = \sin(x^2 + y^2)$.
- **Vector-Valued Multivariable Functions:**
  - A vector-valued multivariable function maps $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$.
  - Examples:
    * $f(x, y, z) = (x^2 + y^2, y^2 + z^2, z^2 + x^2)$.
    * $f(x, y, z) = (\sin(x), \cos(y), e^z)$.
  - Such functions represent vectors whose components are scalar-valued multivariable functions.
- **Operations on Multivariable Functions:**
  - Arithmetic: Addition, subtraction, and scalar multiplication extend naturally.
  - Products: $f \cdot g$ is defined for scalar-valued functions.
  - Division: f/g is defined where $g \neq 0$.
  - Composition: If $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$ and $g : E \subset \mathbb{R}^m \to \mathbb{R}^p$ such that $\text{range}(f) \subseteq E$, then $g \circ f$ is well-defined.

- **Visualization of Multivariable Functions:**
  - Graphs of $f : \mathbb{R}^2 \to \mathbb{R}$ are surfaces in $\mathbb{R}^3$.
  - Examples:
    * Linear: $f(x, y) = ax + by$ forms a plane.
    * Rational: $f(x, y) = \frac{xy}{x^2+y^2}$ forms a surface with undefined behavior at $(0, 0)$.
    * Trigonometric: $f(x, y) = \sin(x^2 + y^2)$ oscillates in concentric circles.
- **Curves in Multivariable Functions:**
  - Curves are special cases of multivariable functions: $f : \mathbb{R} \to \mathbb{R}^m$.
  - Examples:
    * Helix: $\gamma(t) = (\cos(t), \sin(t), t)$ in $\mathbb{R}^3$.
    * Circle: $\gamma(t) = (a \cos(t), b \sin(t))$ in $\mathbb{R}^2$.
  - Curves can be described parametrically or as sets of equations.

Simplified Explanation

Scalar-Valued Functions: Map from $\mathbb{R}^n$ to $\mathbb{R}$, such as $f(x, y) = x^2 + y^2$.

Vector-Valued Functions: Map from $\mathbb{R}^n$ to $\mathbb{R}^m$, such as $f(x, y, z) = (\sin(x), \cos(y), z^2)$.

Visualization: Graphs of scalar-valued functions in $\mathbb{R}^2 \to \mathbb{R}$ form surfaces, while higher dimensions are harder to visualize.

Curves: Special functions where $n = 1$, such as the helix $\gamma(t) = (\cos(t), \sin(t), t)$.

Conclusion

In this lecture, we:

- Introduced scalar- and vector-valued multivariable functions.
- Explored operations, including addition, multiplication, and composition.
- Visualized multivariable functions and curves with examples.

Multivariable functions generalize single-variable functions, forming the basis for multivariable calculus and its applications in geometry and data science.