Lecture Summary: Lengths and Angles in Vector Spaces

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Key Points

- **Dot Product in $\mathbb{R}^n$:**
  - The dot product of two vectors in $\mathbb{R}^n$, $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$, is defined as:
  $$u \cdot v = \sum_{i=1}^{n} u_i v_i.$$
  - Example in $\mathbb{R}^2$: For $u = (3, 4)$ and $v = (2, 7)$:
  $$u \cdot v = 3 \cdot 2 + 4 \cdot 7 = 6 + 28 = 34.$$
- **Length (Norm) of a Vector:**
  - The length (or norm) of a vector $v = (v_1, v_2, \ldots, v_n)$ is:
  $$\|v\| = \sqrt{v \cdot v} = \sqrt{\sum_{i=1}^{n} v_i^2}.$$
  - Example in $\mathbb{R}^2$: For $v = (3, 4)$:
  $$\|v\| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = 5.$$
  - Example in $\mathbb{R}^3$: For $v = (4, 3, 3)$:
  $$\|v\| = \sqrt{4^2 + 3^2 + 3^2} = \sqrt{16 + 9 + 9} = \sqrt{34}.$$
- **Angle Between Two Vectors:**
  - The angle θ between two vectors u and v in $\mathbb{R}^n$ is given by:
  $$\cos(\theta) = \frac{u \cdot v}{\|u\|\|v\|}.$$
  - θ can be calculated as:
  $$\theta = \cos^{-1}\left(\frac{u \cdot v}{\|u\|\|v\|}\right).$$
  - Example in $\mathbb{R}^3$: For $u = (1, 0, 0)$ and $v = (1, 0, 1)$:
  $$u \cdot v = 1, \quad \|u\| = 1, \quad \|v\| = \sqrt{2}.$$
  $$\cos(\theta) = \frac{1}{1 \cdot \sqrt{2}} = \frac{1}{\sqrt{2}} \implies \theta = \frac{\pi}{4}.$$

- **Key Observations:**
  - The length of a vector v is the square root of the dot product of v with itself:
  $$\|v\| = \sqrt{v \cdot v}.$$
  - The dot product and angle are related through cosine, providing a geometric interpretation of their relationship.

Simplified Explanation

Dot Product The dot product of two vectors is a scalar obtained by multiplying their corresponding components and summing them:

$$u = (3, 4),\ v = (2, 7) \implies u \cdot v = 3 \cdot 2 + 4 \cdot 7 = 34.$$

Length of a Vector The length (norm) of a vector is computed as the square root of the sum of the squares of its components:

$$v = (3, 4) \implies \|v\| = \sqrt{3^2 + 4^2} = 5.$$

Angle Between Vectors The angle between two vectors is computed using the cosine formula:

$$\cos(\theta) = \frac{u \cdot v}{\|u\|\|v\|}.$$

For $u = (1, 0, 0)$ and $v = (1, 0, 1)$:

$$\cos(\theta) = \frac{1}{\sqrt{2}} \implies \theta = \frac{\pi}{4}.$$

Conclusion

In this lecture, we:

- Defined the dot product, length, and angle between vectors in $\mathbb{R}^n$.
- Demonstrated the geometric significance of these operations.
- Provided examples to compute lengths and angles in various dimensions.

These concepts are foundational in linear algebra, bridging algebraic and geometric interpretations of vector spaces.