Lecture Summary: Properties of Vector Spaces

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Key Points

- **Cancellation Law:**
  - If $v_1, v_2, v_3 \in V$ and $v_1 + v_3 = v_2 + v_3$, then $v_1 = v_2$.
  - This follows directly from the vector space axioms using the existence of additive inverses and associativity of addition.
- **Uniqueness of the Zero Vector:**
  - The zero vector $0 \in V$, as defined in the axioms, is unique.
  - Proof: Suppose w also satisfies $v + w = v$ for all $v \in V$. Then $w = 0$ by the cancellation law.
- **Uniqueness of the Additive Inverse:**
  - For each $v \in V$, there exists a unique $-v$ such that $v + (-v) = 0$.
  - Proof: If another vector w satisfies $v + w = 0$, then $w = -v$ by the cancellation law.
- **Additional Properties:**
  - $0 \cdot v = 0$ for all $v \in V$.
  - $-c \cdot v = -(c \cdot v) = c \cdot (-v)$ for all $v \in V$ and scalars c.
  - $c \cdot 0 = 0$ for all scalars c.
- **Real-Life Application - Grocery Shop Example:**
  - Quantities of items (e.g., rice, dal, oil, biscuits, soap) are treated as vectors.
  - Addition corresponds to summing stocks or demands, and scalar multiplication adjusts quantities by a factor.
  - Negative quantities represent demand, while positive quantities represent supply.
  - This vector space behaves like $\mathbb{R}^5$.
- **Affine Flat Example:**
  - Consider a plane V parallel to the XY plane.
  - Scalar multiplication and addition involve projections onto the XY plane, performing operations in $\mathbb{R}^2$, and projecting back to V.
  - Visualization: Use arrows from a fixed point (e.g., the intersection of the Z-axis with V) to perform operations geometrically.

Simplified Explanation

Cancellation Law Example: For $v_1 + v_3 = v_2 + v_3$, subtract v_3 from both sides to get $v_1 = v_2$. This demonstrates the use of the axioms without additional structure.

Grocery Shop Vector Space: Items like rice (kg), dal (kg), oil (liters), biscuits (packets), and soap bars form vectors:

$$\begin{bmatrix} q_{\text{rice}} \\ q_{\text{dal}} \\ q_{\text{oil}} \\ q_{\text{biscuits}} \\ q_{\text{soap}} \end{bmatrix}.$$

Addition sums stocks or demands across items, while scalar multiplication adjusts quantities. Negative quantities represent demand.

Affine Flat: Operations on a plane parallel to XY involve projecting points to XY, performing $\mathbb{R}^2$ operations, and projecting back. For example:

- Scalar multiplication stretches or shrinks vectors.
- Addition uses the parallelogram law in $\mathbb{R}^2$ before projecting back to V.

Conclusion

In this lecture, we:

- Proved key properties of vector spaces, including the uniqueness of the zero vector and additive inverses.
- Demonstrated practical examples, such as the grocery shop vector space and affine flats, to illustrate abstract concepts.
- Highlighted the geometric intuition behind vector space operations.

These properties deepen our understanding of the structure and utility of vector spaces, preparing us for further exploration of linear transformations and geometry.