Lecture Summary: Solution to a System of Linear Equations with an Invertible Coefficient Matrix

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Key Points

- **Invertibility of Square Matrices:**
  - A square matrix A is invertible if there exists a matrix A^{-1} such that:
    $$A \cdot A^{-1} = A^{-1} \cdot A = I,$$
    where I is the identity matrix.
  - Invertibility is equivalent to $\det(A) \neq 0$. If $\det(A) = 0$, A is not invertible.
- **Inverse of a 2×2 Matrix:**
  - For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:
    $$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \quad \text{where } \det(A) = ad - bc.$$
- **Adjugate Matrix:**
  - The adjugate (or adjoint) matrix, $\operatorname{adj}(A)$, is the transpose of the cofactor matrix of A.
  - If A is invertible:
    $$A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A).$$
- **Solution of a Linear System ($Ax = b$):**
  - If A is invertible, the solution is:
    $$x = A^{-1}b.$$
- **Homogeneous Systems:**
  - A system is homogeneous if $Ax = 0$.
  - For a square matrix A:
    1. If $\det(A) \neq 0$, the only solution is $x = 0$.
    2. If $\det(A) = 0$, the system has infinitely many solutions.

Simplified Explanation

Example 1: Solving a System Using Inverses Given the system:

$$8x_1 + 8x_2 + 4x_3 = 1960, \quad 12x_1 + 5x_2 + 7x_3 = 2215, \quad 3x_1 + 2x_2 + 5x_3 = 1135,$$

matrix representation:

$$A = \begin{bmatrix} 8 & 8 & 4 \\ 12 & 5 & 7 \\ 3 & 2 & 5 \end{bmatrix}, \quad b = \begin{bmatrix} 1960 \\ 2215 \\ 1135 \end{bmatrix}.$$

Compute $\det(A)$:

$$\det(A) = -188 \neq 0 \quad \Rightarrow A \text{ is invertible.}$$

Using:

$$A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A),$$

and multiplying $A^{-1}b$, the solution is:

$$x_1 = 45, \quad x_2 = 125, \quad x_3 = 150.$$

Example 2: Homogeneous System If $Ax = 0$:

- If $\det(A) \neq 0$, $x = 0$ is the only solution.
- If $\det(A) = 0$, there are infinitely many solutions.

Conclusion

In this lecture, we:

- Explored the relationship between the determinant and invertibility of square matrices.
- Derived the formula for A^{-1} using the adjugate matrix and applied it to solve linear systems.
- Introduced homogeneous systems and their solution behavior based on $\det(A)$.

These concepts are foundational for solving linear equations and understanding matrix properties in linear algebra.