# Lecture Summary: Null Space, Nullity, and Basis

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## Key Points

- **Definition of Null Space:**
  - Let $A$ be an $m \times n$ matrix. The **null space** of $A$ is defined as:
$$\text{Null}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}.$$
  - It represents the solution space of the homogeneous system $Ax = 0$.
  - Null space is a subspace of $\mathbb{R}^n$.
- **Nullity:**
  - The **nullity** of $A$ is the dimension of the null space of $A$.
  - Denoted as $\text{nullity}(A)$, it measures the number of independent solutions to $Ax = 0$.
- **Subspace Verification:**
  - To confirm that the null space is a subspace:
    1. If $x, y \in \text{Null}(A)$, then $x + y \in \text{Null}(A)$ (closure under addition).
    2. If $x \in \text{Null}(A)$ and $\lambda \in \mathbb{R}$, then $\lambda x \in \text{Null}(A)$ (closure under scalar multiplication).
- **Finding the Basis for the Null Space:**
  - Use Gaussian elimination to row reduce $A$.
  - Identify independent and dependent variables:
    1. Columns with leading 1s correspond to dependent variables.
    2. Other columns correspond to independent variables.
  - Assign parameters (e.g., $t_1, t_2$) to independent variables.
  - Solve for dependent variables in terms of the independent variables.
  - Substitute values for one parameter at a time (e.g., $t_i = 1$, all others $= 0$) to construct basis vectors.
- **Rank-Nullity Theorem:**
  - The rank-nullity theorem states:
$$\text{rank}(A) + \text{nullity}(A) = n,$$
    where $n$ is the number of columns of $A$.

## Simplified Explanation

**Example: Null Space and Basis in** $\mathbb{R}^3$ Given:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}.$$

Steps:

1. Augmented matrix:
$$[A|0] = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 0 \\ 3 & 3 & 3 & 0 \end{bmatrix}.$$

2. Row reduce:
$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

3. Identify variables:
   - $x_1$ is dependent.
   - $x_2, x_3$ are independent.

4. Solve: $x_1 = -x_2 - x_3$.

5. Basis for null space:
$$\{(-1, 1, 0), (-1, 0, 1)\}.$$

**Example 2: Applying the Rank-Nullity Theorem**

- $A$ has 3 columns.
- Row-reduced matrix has 1 non-zero row $\Rightarrow \text{rank}(A) = 1$.
- Nullity:
$$\text{nullity}(A) = 3 - 1 = 2.$$

## Conclusion

In this lecture, we:

- Defined the null space and nullity of a matrix.
- Demonstrated how to find the basis and nullity using Gaussian elimination.
- Applied the rank-nullity theorem to relate rank and nullity.

Null space analysis is critical in solving systems of equations and understanding the structure of matrices in linear algebra.