Lecture Summary: Directional Derivatives

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Key Points

- **Definition of Directional Derivative:**
  - The directional derivative of a scalar function $f(x_1, x_2, \ldots, x_n)$ at a point $\vec{a}$ in the direction of a unit vector $\vec{u}$ is:
    $$D_{\vec{u}}f(\vec{a}) = \lim_{h \to 0} \frac{f(\vec{a} + h\vec{u}) - f(\vec{a})}{h}.$$
  - It measures the rate of change of f in the direction of $\vec{u}$.
- **Relation to Partial Derivatives:**
  - If $\vec{u} = \vec{e}_i$ (a standard unit vector), the directional derivative reduces to the partial derivative:
    $$D_{\vec{e}_i}f(\vec{a}) = \frac{\partial f}{\partial x_i}(\vec{a}).$$
  - Directional derivatives generalize partial derivatives to arbitrary directions.
- **Computing Directional Derivatives:**
  - If $\vec{u} = (u_1, u_2, \ldots, u_n)$ and $\vec{a} = (a_1, a_2, \ldots, a_n)$, then:
    $$D_{\vec{u}}f(\vec{a}) = \sum_{i=1}^{n} u_i \frac{\partial f}{\partial x_i}(\vec{a}).$$
  - This formula shows that directional derivatives are a weighted sum of partial derivatives.
- **Examples:**
  - Example 1: For $f(x, y) = x + y$ at $(0, 0)$ in the direction of $\vec{u} = \frac{1}{\sqrt{2}}(1, 1)$:
    $$D_{\vec{u}}f(0, 0) = \frac{1}{\sqrt{2}} \cdot 1 + \frac{1}{\sqrt{2}} \cdot 1 = \sqrt{2}.$$
  - Example 2: For $f(x, y, z) = xy + yz + zx$ at $(1, 2, 3)$ in the direction of $\vec{u} = \frac{1}{5}(4, 3, 0)$:
    $$D_{\vec{u}}f(1, 2, 3) = \frac{4}{5} \cdot 5 + \frac{3}{5} \cdot 4 = \frac{32}{5}.$$
- **Properties of Directional Derivatives:**
  - **Linearity:**
    $$D_{\vec{u}}(cf + g)(\vec{a}) = cD_{\vec{u}}f(\vec{a}) + D_{\vec{u}}g(\vec{a}).$$
  - **Product Rule:**
    $$D_{\vec{u}}(f \cdot g)(\vec{a}) = f(\vec{a})D_{\vec{u}}g(\vec{a}) + g(\vec{a})D_{\vec{u}}f(\vec{a}).$$

– **Quotient Rule:**

$$D_{\vec{u}} \left(\frac{f}{g} \right) (\vec{a}) = \frac{g(\vec{a}) D_{\vec{u}} f(\vec{a}) - f(\vec{a}) D_{\vec{u}} g(\vec{a})}{g(\vec{a})^2}.$$

- **Geometric Interpretation:**
  - The directional derivative is the slope of the graph of f along the line in the direction of $\vec{u}$.
  - It captures the instantaneous rate of change of f at a point in any direction.

Simplified Explanation

What is a Directional Derivative? It measures how a function changes as we move in a specific direction, generalizing the concept of partial derivatives.

How to Compute It? Use the formula:

$$D_{\vec{u}} f(\vec{a}) = \sum u_i \frac{\partial f}{\partial x_i}(\vec{a}),$$

where u_i are components of the unit vector $\vec{u}$.

Examples: For $f(x, y) = x + y$ at $(0, 0)$, moving diagonally at 45° (direction $\vec{u} = (1/\sqrt{2}, 1/\sqrt{2})$), the rate of change is $\sqrt{2}$.

Conclusion

In this lecture, we:

- Defined directional derivatives as a generalization of partial derivatives.
- Demonstrated computation techniques using both limits and formulas.
- Highlighted their geometric and practical significance.

Directional derivatives are fundamental tools in multivariable calculus, crucial for understanding gradients and optimization.