Lecture Summary: Orthonormal Basis

Source: What is an orthonormal basis (1).pdf

Key Points

- **Orthonormal Sets:**
  - A set of vectors $\{v_1, v_2, \ldots, v_k\}$ in an inner product space is orthonormal if:
    1. The vectors are mutually orthogonal:
    $$\langle v_i, v_j \rangle = 0 \quad \text{for } i \neq j.$$
    2. Each vector has norm 1:
    $$\|v_i\| = 1 \implies \langle v_i, v_i \rangle = 1.$$
  - Example in $\mathbb{R}^4$ with the standard dot product:
  $$\left\{ \frac{1}{\sqrt{3}}(1,1,1,0), \frac{1}{\sqrt{42}}(2,1,1,6), \frac{1}{3}(2,0,2,-1) \right\}$$
  is an orthonormal set.
- **Orthonormal Basis:**
  - An orthonormal basis is an orthonormal set that is also a basis for the vector space.
  - Equivalently, it is an orthogonal basis where each vector has norm 1.
  - The standard basis in $\mathbb{R}^n$ with the dot product is an example of an orthonormal basis.
  - Example in $\mathbb{R}^3$:
  $$\left\{ \frac{1}{3}(1,2,2), \frac{1}{3}(-2,-1,2), \frac{1}{3}(2,-2,1) \right\}.$$
- **Constructing an Orthonormal Basis:**
  - From an orthogonal set $\{v_1, v_2, \ldots, v_k\}$, create an orthonormal set $\{u_1, u_2, \ldots, u_k\}$ by dividing each vector by its norm:
  $$u_i = \frac{v_i}{\|v_i\|}.$$
  - This method preserves orthogonality while ensuring each vector has norm 1.
  - Example in $\mathbb{R}^2$:
  $$\gamma = \{(1,3), (-3,1)\} \quad \text{becomes} \quad \beta = \left\{ \frac{1}{\sqrt{10}}(1,3), \frac{1}{\sqrt{10}}(-3,1) \right\}.$$
- **Importance of Orthonormal Bases:**
  - Given an orthonormal basis $\{v_1, v_2, \ldots, v_n\}$ for V, any vector $v \in V$ can be uniquely expressed as:
  $$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n,$$
  where:
  $$c_i = \langle v, v_i \rangle.$$
  - This simplifies the computation of coefficients in a linear combination, avoiding the need to solve systems of equations.

Simplified Explanation

Orthonormal Sets: A set of vectors is orthonormal if they are perpendicular (orthogonal) and have a length (norm) of 1.

Orthonormal Basis: An orthonormal basis is a maximal orthonormal set that spans the entire vector space. For example, the standard basis in $\mathbb{R}^n$ is orthonormal.

Constructing an Orthonormal Basis: Divide each vector in an orthogonal set by its norm to make it orthonormal.

Applications: With an orthonormal basis, coefficients in a linear combination can be computed as inner products.

Conclusion

In this lecture, we:

- Defined orthonormal sets and orthonormal bases.
- Showed how to construct an orthonormal basis from an orthogonal set.
- Highlighted the computational advantages of using orthonormal bases.

Orthonormal bases streamline vector space computations, making them crucial in linear algebra and applications like data science.