Lecture Summary: Critical Points for Multivariable Functions

Source: Week 11 Lec 04.pdf

Key Points

- **Critical Points for Single-Variable Functions (Recap):**
  - A point a is a critical point of $f(x)$ if either:
  $$f'(a) = 0 \quad \text{or} \quad f'(a) \text{ does not exist.}$$
  - Critical points include:
    * Local maxima: $\frac{d^2 f}{dx^2}(a) < 0$.
    * Local minima: $\frac{d^2 f}{dx^2}(a) > 0$.
    * Saddle points: Neither maxima nor minima.
- **Extension to Multivariable Functions:**
  - For $f : \mathbb{R}^n \to \mathbb{R}$, a point $\vec{a}$ is a critical point if:
  $$\nabla f(\vec{a}) = 0 \quad \text{or} \quad \nabla f(\vec{a}) \text{ does not exist.}$$
  - Critical points can be:
    * Local maxima: $f(\vec{x}) \leq f(\vec{a})$ in a neighborhood of $\vec{a}$.
    * Local minima: $f(\vec{x}) \geq f(\vec{a})$ in a neighborhood of $\vec{a}$.
    * Saddle points: Not local extrema.
- **Examples:**
  - **Example 1:** $f(x, y) = \sin(xy)$:
    * Gradient $\nabla f = (y \cos(xy), x \cos(xy))$.
    * Setting $\nabla f = 0$ gives critical points:
    $$x = 0 \quad \text{or} \quad y = 0 \quad \text{or} \quad \cos(xy) = 0.$$
    * $\cos(xy) = 0$ yields infinitely many critical points.
  - **Example 2:** $f(x, y) = x^2 + 6xy + 4y^2 - 2x - 4y$:
    * Gradient $\nabla f = (2x + 6y + 2, 6x + 8y - 4)$.
    * Solving $\nabla f = 0$ gives $x = 2$ and $y = -1$ as the critical point.
- **Saddle Points:**
  - A critical point $\vec{a}$ is a saddle point if:
  $$\nabla f(\vec{a}) = 0 \quad \text{but} \quad \vec{a} \text{ is not a local extremum.}$$

- Example: $f(x, y) = x^2 - y^2$:
    * At $(0, 0)$, $\nabla f = 0$, but f is a maximum in some directions and a minimum in others.

- **Global Extrema:**
  - A point $\vec{a}$ is a global maximum if:

$$f(\vec{a}) \geq f(\vec{x}) \quad \text{for all } \vec{x} \in D.$$

  - A point $\vec{a}$ is a global minimum if:

$$f(\vec{a}) \leq f(\vec{x}) \quad \text{for all } \vec{x} \in D.$$

  - For continuous functions on closed and bounded domains, global extrema always exist.

- **Finding Global Extrema:**
  - Check critical points within the domain.
  - Evaluate f on the boundary and reduce dimensions iteratively.
  - Compare f values at all critical points and boundaries to determine global extrema.

- **Example of Global Extrema:**
  - Function: $f(x, y) = x^3 + y^3 - 3x - 3y^2 + 1$ on a square domain.
  - Steps:
    1. Find critical points inside the domain: $(1, 0)$ and $(1, 2)$.
    2. Evaluate f on the edges and corners of the square.
    3. Compare all values to determine:

$$\text{Absolute Maximum: } f(2, 0) = 3, \quad \text{Absolute Minimum: } f(1, 2) = -5.$$

Simplified Explanation

Critical Points: Points where the gradient is zero or undefined. These include potential maxima, minima, or saddle points.

Example: For $f(x, y) = x^2 + 6xy + 4y^2 - 2x - 4y$, solving $\nabla f = 0$ yields $(2, -1)$ as a critical point.

Global Extrema: To find the largest or smallest value of f over a domain, check critical points and boundary values.

Conclusion

In this lecture, we:

- Defined critical points and explored their significance in multivariable functions.
- Distinguished between local extrema, saddle points, and global extrema.
- Demonstrated techniques to find global extrema on closed domains.

Critical points and extrema are fundamental concepts in optimization and analysis of multivariable functions.