Lecture Summary: Directional Derivatives in Terms of the Gradient

Source: Directional derivatives in terms of the gradient.pdf

Key Points

- **Definition of Gradient:**
  - The gradient of a scalar function $f(x_1, x_2, \ldots, x_n)$ is:

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right).$$

  - It is a vector-valued function from $\mathbb{R}^n$ to $\mathbb{R}^n$.
  - At a point $\vec{a} \in \mathbb{R}^n$, the gradient vector is $\nabla f(\vec{a})$.
- **Directional Derivatives in Terms of the Gradient:**
  - The directional derivative of f at $\vec{a}$ in the direction of a unit vector $\vec{u}$ is:

$$D_{\vec{u}} f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{u}.$$

  - This simplifies the computation of directional derivatives compared to using limits directly.
- **Examples:**
  - Example 1: For $f(x, y) = x + y$, the gradient is $\nabla f = (1, 1)$.

$$D_{\vec{u}} f(x, y) = (1, 1) \cdot \vec{u} = u_1 + u_2.$$

  - Example 2: For $f(x, y, z) = xy + yz + zx$, the gradient is $\nabla f = (y + z, x + z, x + y)$.

$$D_{\vec{u}} f(x, y, z) = (y + z, x + z, x + y) \cdot \vec{u} = u_1(y + z) + u_2(x + z) + u_3(x + y).$$

  - Example 3: For $f(x, y) = \sin(xy)$, the gradient is $\nabla f = (y \cos(xy), x \cos(xy))$.

$$D_{\vec{u}} f(x, y) = (y \cos(xy), x \cos(xy)) \cdot \vec{u} = u_1 y \cos(xy) + u_2 x \cos(xy).$$

- **Properties of the Gradient:**
  - **Linearity:**

$$\nabla(cf + g) = c\nabla f + \nabla g.$$

  - **Product Rule:**

$$\nabla(fg) = g\nabla f + f\nabla g.$$

  - **Quotient Rule:**

$$\nabla \left(\frac{f}{g} \right) = \frac{g\nabla f - f\nabla g}{g^2}.$$

- **Importance of the Gradient:**

- The gradient vector ∇f at $\vec{a}$ indicates the direction of steepest ascent of f at that point.
- The magnitude $\|\nabla f(\vec{a})\|$ gives the rate of steepest ascent.

- **Continuity of the Gradient:**
  - If ∇f is continuous around $\vec{a}$, then $D_{\vec{u}} f(\vec{a})$ exists and equals $\nabla f(\vec{a}) \cdot \vec{u}$.
  - If ∇f is not continuous, the formula $D_{\vec{u}} f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{u}$ may fail.

Simplified Explanation

Gradient Vector: A vector of partial derivatives that points in the direction of steepest ascent.

Directional Derivative: Instead of using limits, compute the directional derivative using:

$$D_{\vec{u}} f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{u}.$$

Example: For $f(x, y) = x + y$ and $\vec{u} = (1/\sqrt{2}, 1/\sqrt{2})$:

$$D_{\vec{u}} f(0, 0) = (1, 1) \cdot (1/\sqrt{2}, 1/\sqrt{2}) = \sqrt{2}.$$

Conclusion

In this lecture, we:

- Defined the gradient and its role in computing directional derivatives.
- Highlighted the connection between gradients and rates of change.
- Demonstrated examples to illustrate the use of gradients in multivariable calculus.

The gradient simplifies directional derivative calculations and provides geometric insights into the behavior of multivariable functions.