Lecture Summary: Tangent Hyperplanes and Linear Approximations

Source: Week 11 - Lec 03.pdf

Key Points

- **Tangent Hyperplanes:**
  - A tangent hyperplane generalizes the concept of a tangent plane (in $\mathbb{R}^3$) to higher dimensions.
  - For a scalar-valued function $f : D \subset \mathbb{R}^n \to \mathbb{R}$, the tangent hyperplane at $\vec{a} \in \mathbb{R}^n$ is the affine flat that contains all tangent lines at $\vec{a}$.
  - **Equation of the Tangent Hyperplane:**

    $$z = f(\vec{a}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a})(x_i - a_i),$$

    or equivalently,

    $$z = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a}),$$

    where ∇f is the gradient of f.

- **Interpretation in $\mathbb{R}^3$:**
  - For $f(x, y)$, the tangent plane at (a, b) is given by:

    $$z = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b).$$

  - Geometrically, all tangent lines at (a, b) lie on this plane.

- **Linear Approximation:**
  - The tangent hyperplane provides the best linear approximation to f near $\vec{a}$.
  - Define $L_f(\vec{x})$ as:

    $$L_f(\vec{x}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a}).$$

  - This is the linear approximation of f around $\vec{a}$.

- **Examples:**
  - **Example 1:** $f(x, y) = x + y$ at $(1, 1)$:

    $$\nabla f = (1, 1), \quad z = 2 + 1(x - 1) + 1(y - 1).$$

    Simplify to:

    $$z = x + y.$$

- **Example 2:** $f(x,y) = xy$ at $(1,1)$:

$$\nabla f = (y,x), \quad z = 1 + 1(x-1) + 1(y-1).$$

Simplify to:

$$z = x + y - 1.$$

- **Example 3:** $f(x,y,z) = x^2 + y^2 + z^2$ at $(2,3,-1)$:

$$\nabla f = (2x, 2y, 2z), \quad \nabla f(2,3,-1) = (4,6,-2).$$

Tangent hyperplane:

$$z = 14 + 4(x-2) + 6(y-3) - 2(z+1).$$

Simplify to:

$$z = 14 + 4x + 6y - 2z - 32 \implies z = 4x + 6y - 2z - 18.$$

- **Conditions for Existence:**
  - The gradient ∇f must exist and be continuous in a neighborhood of $\vec{a}$.
  - If ∇f is discontinuous or undefined, the tangent hyperplane may not exist.

Simplified Explanation

Tangent Hyperplanes: The tangent hyperplane is a generalization of the tangent plane for higher dimensions. It contains all tangent lines at a point and approximates the function locally.

Equation: For $f(x,y)$ at (a,b):

$$z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b).$$

Example: For $f(x,y) = x + y$ at $(1,1)$:

$$z = x + y.$$

Conclusion

In this lecture, we:

- Defined tangent hyperplanes and their role in multivariable calculus.
- Derived their equations and discussed their geometric significance.
- Demonstrated their use in linear approximations.

Tangent hyperplanes are essential for approximating and understanding the behavior of multivariable functions near a point.