Lecture Summary: Affine Subspaces and Affine Mappings

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Key Points

- **Definition of Affine Subspace:**
  - An affine subspace L of a vector space V is a subset of V defined as:
  $$L = v + U,$$
  where:
    * $v \in V$ is a fixed vector.
    * U is a vector subspace of V.
  - L is formed by translating the subspace U by the vector v.
- **Dimension of Affine Subspace:**
  - If U is n-dimensional, the corresponding affine subspace L is also considered n-dimensional.
  - The subspace U associated with L is unique.
- **Affine Subspaces in $\mathbb{R}^n$:**
  - In $\mathbb{R}^2$, affine subspaces include:
    * Points (shifting the origin by a vector).
    * Lines (translating a line passing through the origin).
    * The entire plane $\mathbb{R}^2$ (a subspace translated by 0).
  - In $\mathbb{R}^3$, affine subspaces include:
    * Points.
    * Lines.
    * Planes.
    * The entire space $\mathbb{R}^3$.
- **Affine Subspace as Solution Sets:**
  - For a linear system $Ax = b$, the solution set L is:
  $$L = v + n(A),$$
  where:
    * $n(A)$ is the null space of A.
    * v is any particular solution to $Ax = b$.
  - If $b = 0$, the system is homogeneous, and $L = n(A)$ is a subspace.
  - If $b \neq 0$ and lies in the column space of A, the solution set L is an affine subspace.

- **Affine Mappings:**
  - Let L and L' be affine subspaces of V and W, respectively.
  - A function $f : L \to L'$ is an **affine mapping** if:

    $$f(v+u) = f(v) + T(u),$$

    where:
    * $T : U \to U'$ is a linear transformation.
    * $v \in L$, $u \in U$, and $f(v) \in L'$.
  - Every affine mapping can be decomposed as a translation followed by a linear transformation.
- **Example: Affine Mapping in $\mathbb{R}^3$:**
  - Define $f(x, y, z) = (2x + 3y + 2, 4x - 5y + 3)$.
  - This is not a linear transformation since $f(0, 0, 0) \neq (0, 0)$.
  - However, f can be written as:

    $$f(x, y, z) = (2, 3) + (2x + 3y, 4x - 5y),$$

    where the second term is a linear transformation.
  - Therefore, f is an affine mapping.

Simplified Explanation

Affine Subspace Affine subspaces generalize vector subspaces by allowing translation:

- Example in $\mathbb{R}^2$: A line not passing through the origin is an affine subspace obtained by translating a line passing through the origin.
- Solution sets of non-homogeneous systems $Ax = b$ are affine subspaces when b lies in the column space of A.

Affine Mapping Affine mappings allow linear transformations with a translation:

- Example: $f(x, y) = (2x + 1, y - 1)$ translates by $(1, -1)$ and scales (x, y) by linear rules.

Conclusion

In this lecture, we:

- Introduced affine subspaces as translations of vector subspaces.
- Showed how solution sets of linear equations form affine subspaces.
- Defined affine mappings and demonstrated their decomposition into translations and linear transformations.

Affine subspaces and mappings extend linear algebra concepts, particularly in solving non-homogeneous systems and transformations in affine geometry.