Lecture Summary: Inner Products and Norms on a Vector Space

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Key Points

- **Definition of Inner Product:**
  - An inner product on a vector space V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ that satisfies:
    1. **Positivity:** $\langle v, v \rangle > 0$ for all $v \neq 0$, and $\langle v, v \rangle = 0$ if $v = 0$.
    2. **Linearity in the first argument:**
    $$\langle v_1 + v_2, v_3 \rangle = \langle v_1, v_3 \rangle + \langle v_2, v_3 \rangle, \quad \langle cv_1, v_2 \rangle = c\langle v_1, v_2 \rangle,$$
    where $c \in \mathbb{R}$.
    3. **Symmetry:**
    $$\langle v_1, v_2 \rangle = \langle v_2, v_1 \rangle.$$
  - A vector space equipped with an inner product is called an **inner product space**.
- **Examples of Inner Products:**
  - The **dot product** in $\mathbb{R}^n$:
  $$\langle u, v \rangle = u \cdot v = \sum_{i=1}^{n} u_i v_i.$$
  - Non-standard inner product in $\mathbb{R}^2$:
  $$\langle u, v \rangle = x_1 y_1 - (x_1 y_2 + x_2 y_1) + 2x_2 y_2,$$
  where $u = (x_1, x_2)$ and $v = (y_1, y_2)$. This inner product is verified using matrix representation:
  $$\langle u, v \rangle = u^T \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} v.$$
- **Definition of Norm:**
  - A norm on a vector space V is a function $\| \cdot \| : V \to \mathbb{R}$ satisfying:
    1. **Positivity:** $\|v\| \geq 0$ for all v, and $\|v\| = 0$ if and only if $v = 0$.
    2. **Homogeneity:** $\|cv\| = |c|\|v\|$ for all scalars c.
    3. **Triangle inequality:** $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$.
  - The **Euclidean norm** (length) in $\mathbb{R}^n$:
  $$\|v\| = \sqrt{\sum_{i=1}^{n} v_i^2}.$$

  - L^1 norm in $\mathbb{R}^n$:

$$\|v\|_1 = \sum_{i=1}^{n} |v_i|.$$

- **Relation Between Inner Product and Norm:**
  - If V is an inner product space, a norm can be defined as:

$$\|v\| = \sqrt{\langle v, v\rangle}.$$

  - This norm satisfies all norm axioms:
    * **Positivity:** $\|v\| = 0$ if and only if $v = 0$.
    * **Homogeneity:** $\|cv\| = |c|\|v\|$ due to properties of the inner product.
    * **Triangle inequality:** $\|v + w\|^2 \leq (\|v\| + \|w\|)^2$ follows from expanding and bounding $\langle v + w, v + w\rangle$.

Simplified Explanation

Inner Product: A generalization of the dot product:

$$\langle u, v\rangle = u \cdot v.$$

For $\mathbb{R}^2$, another example:

$$\langle u, v\rangle = x_1y_1 - (x_1y_2 + x_2y_1) + 2x_2y_2.$$

Norm: The length of a vector can be defined as:

$$\|v\| = \sqrt{\langle v, v\rangle}.$$

Other norms, like L^1, measure length differently:

$$\|v\|_1 = \sum |v_i|.$$

Connection: An inner product induces a norm that satisfies all the standard properties of length.

Conclusion

In this lecture, we:

- Defined inner products and norms on vector spaces.
- Demonstrated examples of inner products and norms in $\mathbb{R}^n$.
- Highlighted the relationship between inner products and norms, showing how an inner product induces a norm.

Inner products and norms extend the concepts of length and angles to general vector spaces, forming the foundation of functional analysis and advanced geometry.