Lecture Summary: Limits for Scalar-Valued Multivariable Functions

Source: Limits for Scalar-Valued Multivariable Functions.pdf

Key Points

- **Limit of a Sequence in $\mathbb{R}^n$:**
  - A sequence $\{\vec{a}_n\}$ in $\mathbb{R}^n$ converges to $\vec{a} = (a_1, a_2, \ldots, a_n)$ if and only if each coordinate sequence converges:
  $$\lim_{n\to\infty} a_{n,i} = a_i, \quad \text{for all } i = 1, 2, \ldots, n.$$
  - Example:
  $$\left\{ \left(\frac{1}{n}, n \sin\left(\frac{1}{n}\right) \right) \right\}$$
  converges to $(0, 1)$ because $\frac{1}{n} \to 0$ and $n \sin\left(\frac{1}{n}\right) \to 1$.
- **Definition of Limit for a Multivariable Function:**
  - Let $f : D \subset \mathbb{R}^n \to \mathbb{R}$ be a scalar-valued multivariable function, and $\vec{a} \in \mathbb{R}^n$. Then:
  $$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L$$
  if for every sequence $\{\vec{a}_n\} \subset D$ with $\vec{a}_n \to \vec{a}$, we have $f(\vec{a}_n) \to L$.
  - This definition generalizes the notion of a limit from single-variable calculus.
- **Examples of Limits:**
  - Example 1: For $f(x, y) = x^2 + y^2$, as $(x, y) \to (0, 0)$:
  $$\lim_{(x,y)\to(0,0)} f(x, y) = 0.$$
  - Example 2: For $f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$, as $(x, y) \to (0, 0)$:
  $$\lim_{(x,y)\to(0,0)} f(x, y) \text{ does not exist,}$$
  because the limit depends on the path taken (e.g., along $x = y$ versus $x = -y$).
- **Properties of Limits:**
  - **Linearity:** For $c \in \mathbb{R}$:
  $$\lim_{\vec{x}\to\vec{a}} [c f(\vec{x}) + g(\vec{x})] = c \lim_{\vec{x}\to\vec{a}} f(\vec{x}) + \lim_{\vec{x}\to\vec{a}} g(\vec{x}).$$
  - **Product Rule:**
  $$\lim_{\vec{x}\to\vec{a}} [f(\vec{x}) \cdot g(\vec{x})] = \lim_{\vec{x}\to\vec{a}} f(\vec{x}) \cdot \lim_{\vec{x}\to\vec{a}} g(\vec{x}).$$

- **Quotient Rule:** If $\lim_{\vec{x}\to\vec{a}} g(\vec{x}) \neq 0$:

$$\lim_{\vec{x}\to\vec{a}} \frac{f(\vec{x})}{g(\vec{x})} = \frac{\lim_{\vec{x}\to\vec{a}} f(\vec{x})}{\lim_{\vec{x}\to\vec{a}} g(\vec{x})}.$$

- **Special Techniques:**
  - **Substitution:** For well-behaved functions, limits can often be computed by direct substitution.
  - **Sandwich (Squeeze) Theorem:** If $f(\vec{x}) \leq h(\vec{x}) \leq g(\vec{x})$ for all $\vec{x}$ near $\vec{a}$, and $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = \lim_{\vec{x}\to\vec{a}} g(\vec{x}) = L$, then $\lim_{\vec{x}\to\vec{a}} h(\vec{x}) = L$.
  - **Path Dependence:** To check if a limit exists, evaluate along different paths. If the results differ, the limit does not exist.

Simplified Explanation

What Are Limits for Multivariable Functions? Limits describe the behavior of a function $f(\vec{x})$ as $\vec{x}$ approaches a point $\vec{a}$ in $\mathbb{R}^n$.

How to Compute Limits? Check if the function approaches the same value along all paths leading to $\vec{a}$. If not, the limit does not exist.

Example: For $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$, paths $x = y$ and $x = -y$ yield different limits at $(0,0)$, so the limit does not exist.

Conclusion

In this lecture, we:

- Defined limits for scalar-valued multivariable functions.
- Explored techniques for computing and analyzing limits.
- Highlighted examples and the importance of path dependence.

Understanding limits in multivariable calculus is crucial for studying continuity, derivatives, and integrals in higher dimensions.