Lecture Summary: Image and Kernel of Linear Transformations

Source: Lec42.pdf

Key Points

- **Definition of Kernel:**
  - For a linear transformation $f : V \to W$, the **kernel** of f is defined as:
  $$\ker(f) = \{v \in V \mid f(v) = 0\}.$$
  - The kernel is a subspace of V:
    * Closure under addition: If $v_1, v_2 \in \ker(f)$, then $v_1 + v_2 \in \ker(f)$.
    * Closure under scalar multiplication: If $v \in \ker(f)$ and $\alpha \in \mathbb{R}$, then $\alpha v \in \ker(f)$.
- **Definition of Image:**
  - For $f : V \to W$, the **image** of f (also called the range) is:
  $$\mathrm{Im}(f) = \{w \in W \mid \exists v \in V,\, w = f(v)\}.$$
  - The image is a subspace of W:
    * Closure under addition: If $w_1, w_2 \in \mathrm{Im}(f)$, then $w_1 + w_2 \in \mathrm{Im}(f)$.
    * Closure under scalar multiplication: If $w \in \mathrm{Im}(f)$ and $\alpha \in \mathbb{R}$, then $\alpha w \in \mathrm{Im}(f)$.
- **Kernel and Injectivity:**
  - A linear transformation f is injective if and only if $\ker(f) = \{0\}$ (the zero subspace).
- **Image and Surjectivity:**
  - A linear transformation f is surjective if and only if $\mathrm{Im}(f) = W$.
- **Relation to Matrices:**
  - For $f : V \to W$, let $\beta = \{v_1, \ldots, v_n\}$ and $\gamma = \{w_1, \ldots, w_m\}$ be ordered bases for V and W, respectively.
  - The corresponding matrix A is defined by expressing $f(v_j)$ as a linear combination of $\{w_1, \ldots, w_m\}$:
  $$f(v_j) = \sum_{i=1}^{m} a_{ij} w_i.$$
  - The jth column of A consists of the coefficients a_{ij}.
- **Kernel and Null Space:**
  - $\ker(f)$ corresponds to the null space of A, which is:
  $$\mathrm{Null}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}.$$

  - The kernel is isomorphic to the null space of the associated matrix A.

- **Image and Column Space:**
  - $\mathrm{Im}(f)$ corresponds to the column space of A, which is the span of the columns of A.
  - Finding a basis for $\mathrm{Im}(f)$ involves identifying pivot columns of A in row-reduced form.

- **Summary of Isomorphisms:**
  - $\ker(f)$ is isomorphic to the null space of A.
  - $\mathrm{Im}(f)$ is isomorphic to the column space of A.

Simplified Explanation

Example 1: Linear Transformation on $\mathbb{R}^2$ Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as:

$$f(x, y) = (2x, y).$$

- Kernel: Solve $f(x, y) = (0, 0)$:

$$(2x, y) = (0, 0) \implies x = 0, y = 0.$$

- $\ker(f) = \{(0, 0)\}$ (the zero subspace).
- Image: Check if every $(u, v) \in \mathbb{R}^2$ can be written as:

$$(2x, y) = (u, v).$$

- Solve:

$$x = \frac{u}{2},\ y = v.$$

- $\mathrm{Im}(f) = \mathbb{R}^2$ (entire space).

Example 2: Linear Transformation on $\mathbb{R}^2$ Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as:

$$f(x, y) = (2x, 0).$$

- Kernel: Solve $f(x, y) = (0, 0)$:

$$(2x, 0) = (0, 0) \implies x = 0.$$

- $\ker(f) = \{(0, y) \mid y \in \mathbb{R}\}$ (the y-axis).
- Image: Check if every $(u, v) \in \mathbb{R}^2$ can be written as:

$$(2x, 0) = (u, v).$$

- $v = 0$, $u = 2x \implies x = \frac{u}{2}$.
- $\mathrm{Im}(f) = \{(u, 0) \mid u \in \mathbb{R}\}$ (the x-axis).

Conclusion

In this lecture, we:

- Defined the kernel and image of linear transformations.
- Connected kernels to null spaces and images to column spaces.
- Established isomorphisms between these subspaces and their matrix representations.
- Demonstrated examples to compute kernel and image using matrices.

This lays the foundation for understanding the structure of linear transformations and their interplay with matrices.