Lecture Summary: Examples of Finding Bases for Kernel and Image of Linear Transformations

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Key Points

- **Review of Kernel and Image:**
  - The kernel $\ker(f)$ of a linear transformation $f : V \to W$ is:
  $$\ker(f) = \{v \in V \mid f(v) = 0\}.$$
  - The image $\mathrm{Im}(f)$ is:
  $$\mathrm{Im}(f) = \{w \in W \mid \exists v \in V,\, w = f(v)\}.$$
  - The kernel corresponds to the null space of the associated matrix, and the image corresponds to the column space.
- **Finding Bases Using Row Reduction:**
  - Row reduce the matrix representation of f to obtain:
    1. Basis for the null space (kernel) using the non-pivot columns.
    2. Basis for the column space (image) using the pivot columns.
- **Example 1: Transformation from $\mathbb{R}^4$ to $\mathbb{R}^3$:**
  - Transformation defined by:
  $$T(x_1, x_2, x_3, x_4) = \begin{bmatrix} 2x_1 + 4x_2 + 6x_3 + 8x_4 \\ x_1 + 3x_2 + 5x_4 \\ x_1 + x_2 + 6x_3 + 3x_4 \end{bmatrix}.$$
  - Matrix representation:
  $$A = \begin{bmatrix} 2 & 4 & 6 & 8 \\ 1 & 3 & 0 & 5 \\ 1 & 1 & 6 & 3 \end{bmatrix}.$$
  - Row reduce A:
  $$\begin{bmatrix} 1 & 0 & 9 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$
  - Kernel basis:
    * Non-pivot columns: x_3, x_4 are independent variables.
    * Solve for x_1, x_2:
    $$x_1 = -9x_3 - 2x_4, \quad x_2 = 3x_3 - x_4.$$

    * Basis vectors:

$$(-9, 3, 1, 0), \quad (-2, -1, 0, 1).$$

  - Image basis:
    * Pivot columns: 1 and 2.
    * Basis vectors from original matrix:

$$(2, 1, 1), \quad (4, 3, 1).$$

- **Example 2: Transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$:**
  - Transformation defined by:

$$T(x, y) = \begin{bmatrix} 0 \\ x + 2y \\ -x - 2y \end{bmatrix}.$$

  - Basis $\beta = \{(1, 1), (1, -1)\}$ for $\mathbb{R}^2$, $\gamma = \{(-1, 1, 0), (-1, 0, 1)\}$ for $\mathbb{R}^3$.
  - Compute $T(1, 1)$:

$$T(1, 1) = \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix} = 3(-1, 1, 0) + (-3)(-1, 0, 1).$$

  - Compute $T(1, -1)$:

$$T(1, -1) = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = -1(-1, 1, 0) + 1(-1, 0, 1).$$

  - Matrix representation:

$$A = \begin{bmatrix} 3 & -1 \\ -3 & 1 \end{bmatrix}.$$

  - Row reduce A:

$$\begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{bmatrix}.$$

  - Kernel basis:

$$\left(\frac{1}{3}, 1\right).$$

  - Image basis:

$$\{(-1, 1, 0)\}.$$

- **Rank-Nullity Theorem for Linear Transformations:**
  - Rank of T = dimension of image.
  - Nullity of T = dimension of kernel.
  - Rank-nullity theorem:

$$\text{rank}(T) + \text{nullity}(T) = \dim(V).$$

Simplified Explanation

Example 1: Basis for Kernel and Image Matrix:

$$A = \begin{bmatrix} 2 & 4 & 6 & 8 \\ 1 & 3 & 0 & 5 \\ 1 & 1 & 6 & 3 \end{bmatrix}.$$

- Row reduce to:

$$\begin{bmatrix} 1 & 0 & 9 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

- Kernel basis:

$$(-9, 3, 1, 0), \quad (-2, -1, 0, 1).$$

- Image basis:

$$(2, 1, 1), \quad (4, 3, 1).$$

Example 2: Custom Basis Matrix:

$$A = \begin{bmatrix} 3 & -1 \\ -3 & 1 \end{bmatrix}.$$

- Row reduce to:

$$\begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{bmatrix}.$$

- Kernel basis:

$$\left(\frac{1}{3}, 1\right).$$

- Image basis:

$$\{(-1, 1, 0)\}.$$

Conclusion

In this lecture, we:

- Demonstrated the computation of kernel and image bases for linear transformations.
- Illustrated the use of row reduction to find these bases.
- Revisited the rank-nullity theorem in the context of linear transformations.

These examples reinforce the practical application of kernel and image concepts in linear algebra.