Lecture Summary: Null Space and Basis – Part 2

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Key Points

- **Recap: Null Space and Nullity:**
  - The null space of an $m \times n$ matrix A is:
  $$\text{Null}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}.$$
  - Nullity is the dimension of the null space.
  - Rank-Nullity Theorem:
  $$\text{rank}(A) + \text{nullity}(A) = n.$$
- **Procedure to Find the Null Space and Basis:**
  1. Form the augmented matrix $[A \mid 0]$.
  2. Perform row reduction to bring A to reduced row echelon form (RREF).
  3. Identify dependent and independent variables:
     - Columns with leading 1s correspond to dependent variables.
     - Other columns correspond to independent variables.
  4. Assign parameters to independent variables and solve for dependent variables in terms of the parameters.
  5. Basis for the null space consists of vectors generated by setting one parameter to 1 and the others to 0.
- **Example:** 3×4 **Matrix**
  - Given:
  $$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 3 & 0 & 3 \\ 1 & 1 & 1 & 2 \end{bmatrix}.$$
  - Row reduction to RREF:
  $$\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$
  - Dependent variables: x_1, x_2, x_3.
  - Independent variable: x_4.
  - Solve for x_1, x_2, x_3 in terms of $x_4 = t$:
  $$x_1 = 3t, \quad x_2 = -3t, \quad x_3 = -2t.$$
  - General solution:
  $$\{(3t, -3t, -2t, t) \mid t \in \mathbb{R}\}.$$

  - Basis vector for the null space:
$$(3, -3, -2, 1).$$
  - Nullity = 1 (one independent variable).

- **Rank and Nullity Relationship:**
  - Rank is the number of non-zero rows in the row-reduced matrix.
  - Nullity is the number of free (independent) variables.
  - For A with 4 columns:
$$\text{rank}(A) = 3, \quad \text{nullity}(A) = 1, \quad \text{rank}(A) + \text{nullity}(A) = 4.$$

Simplified Explanation

Example 1: Null Space and Basis in $\mathbb{R}^3$ For A:
$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix},$$
RREF:
$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

- x_1 is dependent, x_2, x_3 are independent.
- Solve for x_1:
$$x_1 = -x_2 - x_3.$$
- Basis vectors:
$$(-1, 1, 0), \quad (-1, 0, 1).$$
- Nullity = 2.

Example 2: Basis Verification Using Determinants Given vectors $(1, 2, 3)$, $(0, 1, 2)$, and $(1, 3, 0)$ in $\mathbb{R}^3$:

- Form matrix A with these as columns:
$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 3 & 2 & 0 \end{bmatrix}.$$
- Compute determinant:
$$\det(A) = -5 \neq 0.$$
- Non-zero determinant implies linear independence, so these vectors form a basis.

Conclusion

In this lecture, we:

- Computed null space and nullity using Gaussian elimination.
- Verified rank and nullity using the rank-nullity theorem.
- Connected linear independence to determinant checks for basis validation.

Understanding null spaces and bases enhances our ability to analyze matrix transformations and solve systems of equations.