Lecture Summary: Equivalence and Similarity of Matrices

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Key Points

- **Equivalence of Matrices:**
  - Two matrices A and B of size $m \times n$ are **equivalent** if there exist invertible matrices Q (of size $m \times m$) and P (of size $n \times n$) such that:
  $$B = Q \cdot A \cdot P.$$
  - Equivalence can also be characterized through row and column operations:
    * Row operations correspond to multiplication by Q.
    * Column operations correspond to multiplication by P.
  - Equivalent matrices have the same rank.
- **Properties of Equivalence:**
  - **Reflexive:** A is equivalent to itself since Q and P can be chosen as identity matrices.
  - **Symmetric:** If A is equivalent to B, then B is equivalent to A:
  $$B = Q \cdot A \cdot P \implies A = Q^{-1} \cdot B \cdot P^{-1}.$$
  - **Transitive:** If A is equivalent to B and B is equivalent to C, then A is equivalent to C:
  $$B = Q \cdot A \cdot P,\, C = Q' \cdot B \cdot P' \implies C = (Q' \cdot Q) \cdot A \cdot (P \cdot P').$$
- **Similarity of Matrices:**
  - Two square matrices A and B of size $n \times n$ are **similar** if there exists an invertible matrix P of size $n \times n$ such that:
  $$B = P^{-1} \cdot A \cdot P.$$
  - Similar matrices represent the same linear transformation under different bases.
- **Properties of Similarity:**
  - **Reflexive:** A is similar to itself since P can be chosen as the identity matrix.
  - **Symmetric:** If A is similar to B, then B is similar to A:
  $$B = P^{-1} \cdot A \cdot P \implies A = (P^{-1})^{-1} \cdot B \cdot P^{-1}.$$
  - **Transitive:** If A is similar to B and B is similar to C, then A is similar to C:
  $$B = P^{-1} \cdot A \cdot P,\, C = Q^{-1} \cdot B \cdot Q \implies C = (Q^{-1} \cdot P^{-1}) \cdot A \cdot (P \cdot Q).$$
- **Why Similarity Matters:**
  - Similar matrices have the same determinant, rank, characteristic polynomial, minimal polynomial, and eigenvalues.
  - Some linear transformations can be diagonalized when expressed under an appropriate basis, simplifying their representation.

Simplified Explanation

Example 1: Equivalence of Rectangular Matrices Let:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

- Compute $B = Q \cdot A \cdot P$:

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

- The resulting B is equivalent to A.

Example 2: Similarity of Square Matrices Let:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

- Compute $B = P^{-1} \cdot A \cdot P$:

$$P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

- A and B are similar.

Conclusion

In this lecture, we:

- Defined equivalence and similarity of matrices.
- Highlighted properties and practical significance of these relations.
- Discussed examples demonstrating their application to linear transformations and basis changes.

Understanding equivalence and similarity helps analyze matrix representations of linear transformations under different bases, facilitating insights into their properties.