Lecture Summary: Cramer's Rule

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Key Points

- **Cramer's Rule:**
  - A method for solving systems of linear equations using determinants.
  - Applicable when the coefficient matrix A is invertible (i.e., $\det(A) \neq 0$).
- **Procedure for a 2×2 System:**
  - Consider the system:
  $$a_{11}x_1 + a_{12}x_2 = b_1, \quad a_{21}x_1 + a_{22}x_2 = b_2.$$
  - Matrix representation: $Ax = b$, where
  $$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$
  - Define A_{x_1} by replacing the first column of A with b:
  $$A_{x_1} = \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}.$$
  - Define A_{x_2} by replacing the second column of A with b:
  $$A_{x_2} = \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}.$$
  - Solutions are:
  $$x_1 = \frac{\det(A_{x_1})}{\det(A)}, \quad x_2 = \frac{\det(A_{x_2})}{\det(A)}.$$
- **Extension to 3×3 Systems:**
  - Consider the system $Ax = b$, where A is a 3×3 matrix, and b is a column vector.
  - Define matrices:
  $$A_{x_1} = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}, \quad A_{x_2} = \begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix}, \quad A_{x_3} = \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix}.$$
  - Solutions are:
  $$x_1 = \frac{\det(A_{x_1})}{\det(A)}, \quad x_2 = \frac{\det(A_{x_2})}{\det(A)}, \quad x_3 = \frac{\det(A_{x_3})}{\det(A)}.$$
- **Generalization to $n \times n$ Systems:**
  - For an $n \times n$ system $Ax = b$, A_{x_i} is defined by replacing the ith column of A with b.
  - The solution is:
  $$x_i = \frac{\det(A_{x_i})}{\det(A)}.$$

Simplified Explanation

Example: 2×2 **System** Given the system:

$$4x_1 - 3x_2 = 11, \quad 6x_1 + 5x_2 = 7,$$

matrix representation:

$$A = \begin{bmatrix} 4 & -3 \\ 6 & 5 \end{bmatrix}, \quad b = \begin{bmatrix} 11 \\ 7 \end{bmatrix}.$$

Compute:

$$\det(A) = (4)(5) - (-3)(6) = 38,$$

$$A_{x_1} = \begin{bmatrix} 11 & -3 \\ 7 & 5 \end{bmatrix}, \quad \det(A_{x_1}) = (11)(5) - (-3)(7) = 76,$$

$$A_{x_2} = \begin{bmatrix} 4 & 11 \\ 6 & 7 \end{bmatrix}, \quad \det(A_{x_2}) = (4)(7) - (11)(6) = -38.$$

Solutions:

$$x_1 = \frac{\det(A_{x_1})}{\det(A)} = \frac{76}{38} = 2, \quad x_2 = \frac{\det(A_{x_2})}{\det(A)} = \frac{-38}{38} = -1.$$

Conclusion

Cramer's Rule provides a straightforward algorithm for solving square systems of linear equations by leveraging determinants. It requires the coefficient matrix to be invertible ($\det(A) \neq 0$) and applies to $n \times n$ systems. This method is particularly useful for small systems but may be computationally intensive for large matrices.