Lecture Summary: Projections Using Inner Products

Source: Projections using inner products.pdf

Key Points

- **Definition of Projection:**
  - Let V be an inner product space, $v \in V$, and W a subspace of V.
  - The projection of v onto W is the vector in W that is closest to v, denoted $\text{proj}_W(v)$.
  - This is determined using the inner product:

    $$\text{proj}_W(v) = \sum_{i=1}^{n} \langle v, v_i \rangle v_i,$$

    where $\{v_1, v_2, \ldots, v_n\}$ is an orthonormal basis for W.
- **Properties of Projection:**
  - Projections minimize the distance:

    $$\|v - \text{proj}_W(v)\| \leq \|v - w\| \quad \text{for all } w \in W.$$

  - The formula for $\text{proj}_W(v)$ is independent of the choice of orthonormal basis.
  - If $v \in W$, then $\text{proj}_W(v) = v$.
  - If $v \perp W$, then $\text{proj}_W(v) = 0$.
- **Projection onto a Single Vector:**
  - For a subspace spanned by a single vector w, the projection is:

    $$\text{proj}_w(v) = \frac{\langle v, w \rangle}{\|w\|^2} w.$$

  - This is equivalent to scaling w to minimize the distance between v and the line through w.
- **Projection as a Linear Transformation:**
  - The projection operator $P_W : V \to V$ is defined as $P_W(v) = \text{proj}_W(v)$.
  - P_W is a linear transformation with the following properties:
    1. $P_W^2 = P_W$ (idempotence).
    2. $\text{Im}(P_W) = W$ (image of P_W is W).
    3. $\ker(P_W) = W^\perp$ (null space of P_W is the orthogonal complement of W).
- **Examples of Projections:**
  - **Example 1: Projection in $\mathbb{R}^2$:**
    * Subspace W spanned by $(3, 1)$.

    * Vector $v = (1, 3)$.
    * Orthonormal basis for W: $\frac{1}{\sqrt{10}}(3, 1)$.
    * Compute projection:

$$\text{proj}_W(v) = \frac{\langle v, (3,1) \rangle}{\|(3,1)\|^2}(3, 1).$$

      Result:

$$\text{proj}_W(v) = (1.8, 0.6).$$

  - **Example 2: Projection in $\mathbb{R}^3$:**
    * Subspace W spanned by $(1, 0, 0)$ and $(0, 1, 0)$ (the xy-plane).
    * Vector $v = (2, 3, 5)$.
    * Projection:

$$\text{proj}_W(v) = (2, 3, 0).$$

- **Projection Using Orthogonal Bases:**
  - If $\{w_1, w_2, \ldots, w_k\}$ is an orthogonal basis for W, normalize each vector:

$$u_i = \frac{w_i}{\|w_i\|}.$$

  - Compute projection as:

$$\text{proj}_W(v) = \sum_{i=1}^{k} \frac{\langle v, w_i \rangle}{\|w_i\|^2} w_i.$$

Simplified Explanation

Projection in Geometry: Projections minimize the distance between a vector v and a subspace W, like finding the "shadow" of v on W.

Key Formula: The projection of v onto W is:

$$\text{proj}_W(v) = \sum_{i=1}^{n} \langle v, v_i \rangle v_i,$$

where $\{v_1, v_2, \ldots, v_n\}$ is an orthonormal basis for W.

Applications: Projections simplify computations in vector geometry, linear transformations, and orthogonal decompositions.

Conclusion

In this lecture, we:

- Defined projections in inner product spaces.
- Demonstrated how projections work geometrically and algebraically.
- Showed how projections relate to linear transformations and their properties.

Projections are fundamental in linear algebra, with applications in optimization, computer graphics, and signal processing.