Lecture Summary: Linear Transformations and Basis Dependency

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Key Points

- **What is a Linear Transformation?**
  - A **linear transformation** is a generalization of a linear mapping for arbitrary vector spaces V and W.
  - A function $f : V \to W$ is a linear transformation if:
  $$f(u + v) = f(u) + f(v), \quad f(c \cdot u) = c \cdot f(u),$$
  for all $u, v \in V$ and scalars $c \in \mathbb{R}$.
  - This definition is equivalent to the property of linearity.
- **Examples of Linear Transformations:**
  - Linear mappings such as cost functions and matrix-vector multiplications are examples of linear transformations.
  - Projections, rotations, and scaling in vector spaces are also linear transformations.
- **One-to-One (Injective) and Onto (Surjective) Transformations:**
  - A transformation $f : V \to W$ is **injective** if:
  $$f(v_1) = f(v_2) \implies v_1 = v_2.$$
  - A transformation $f : V \to W$ is **surjective** if:
  $$\forall w \in W, \exists v \in V \text{ such that } f(v) = w.$$
  - Injectivity ensures no two distinct vectors in V map to the same vector in W, while surjectivity ensures the entire range of W is covered.
- **Isomorphisms:**
  - A linear transformation is an **isomorphism** if it is both injective and surjective.
  - Example: The identity mapping $f(x) = x$ for $x \in \mathbb{R}^n$ is an isomorphism.
  - Example of non-isomorphism:
  $$f(x, y) = (2x, 0) \quad \text{is injective but not surjective.}$$
- **Role of Basis in Linear Transformations:**
  - A basis for V uniquely determines the action of a linear transformation $f : V \to W$.
  - If $\{v_1, v_2, \ldots, v_n\}$ is a basis for V, then f is fully determined by the values $f(v_1), f(v_2), \ldots, f(v_n)$.

- Changing the basis of V results in a different linear transformation.

- **Matrix Representation of Linear Transformations:**
  - A linear transformation can be represented as a matrix A:

    $$f(x) = A \cdot x,$$

    where x is the input vector and A encodes the transformation.
  - Different bases lead to different matrix representations.

Simplified Explanation

Example 1: Linear Transformation Defined by Basis Values Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by:

$$f(1,0) = (2,0), \quad f(0,1) = (0,1).$$

- For any $(x, y) \in \mathbb{R}^2$, write it as:
  $$(x, y) = x(1,0) + y(0,1).$$

- The transformation is:
  $$f(x, y) = x \cdot f(1,0) + y \cdot f(0,1) = (2x, y).$$

Example 2: Changing the Basis Using basis $\{(1,0), (1,1)\}$:

- Represent (x, y) as:
  $$(x, y) = a(1,0) + b(1,1),$$
  where $a = x - y, b = y$.

- Transformation with new basis:

  $$f(x, y) = (2a, b) = (2(x - y), y).$$

Conclusion

In this lecture, we:

- Defined linear transformations and explained their relationship with linear mappings.
- Discussed injectivity, surjectivity, and isomorphisms.
- Highlighted the role of basis in determining linear transformations and how changes in basis alter transformations.
- Illustrated matrix representations of linear transformations and their basis dependency.

Linear transformations provide a framework for understanding linear relationships in any vector space, and basis choice is crucial for their representation and computation.