Lecture Summary: Linear Transformations and Ordered Bases

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Key Points

- **Linear Transformations: Recap and Basis Dependency:**
  - A linear transformation $f : V \to W$ satisfies:
  $$f(u+v) = f(u) + f(v), \quad f(c \cdot u) = c \cdot f(u),$$
  for $u, v \in V$ and scalar $c \in \mathbb{R}$.
  - The action of f is fully determined by its values on the basis vectors of V.
- **Isomorphism Using Basis:**
  - If V is an n-dimensional vector space with basis $\{v_1, v_2, \ldots, v_n\}$:
    1. Define $f(v_i) = e_i$ (the standard basis for $\mathbb{R}^n$).
    2. Extend this mapping linearly to V by:
    $$f\left(\sum_{i=1}^{n} c_i v_i\right) = \sum_{i=1}^{n} c_i e_i.$$
  - f is an isomorphism, meaning it is both one-to-one and onto.
- **Matrix Representation of Linear Transformations:**
  - For $f : V \to W$, let $\beta = \{v_1, v_2, \ldots, v_n\}$ be an ordered basis for V and $\gamma = \{w_1, w_2, \ldots, w_m\}$ for W.
  - Represent $f(v_j)$ as a linear combination of $\{w_1, w_2, \ldots, w_m\}$:
  $$f(v_j) = \sum_{i=1}^{m} a_{ij} w_i.$$
  - The coefficients a_{ij} form the ith row and jth column of the matrix representation of f.
- **Example: Linear Transformation on $\mathbb{R}^2$:**
  - Define $f(x, y) = (2x, y)$.
  - With standard basis $\{(1,0), (0,1)\}$, compute:
  $$f(1,0) = (2,0), \quad f(0,1) = (0,1).$$
  - Matrix representation:
  $$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$
- **Changing Ordered Basis:**

- Changing the basis changes the matrix representation.
- Example: If $\beta = \{(1,0),(1,1)\}$, then:

$$f(1,0) = (2,0), \quad f(1,1) = (2,1).$$

- Matrix representation becomes:

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}.$$

- The same linear transformation yields different matrices for different bases.

- **Bijection Between Linear Transformations and Matrices:**
  - For fixed ordered bases β and γ, there is a bijection between linear transformations $f : V \to W$ and $m \times n$ matrices.
  - The matrix A encodes the coefficients of $f(v_j)$ expressed in terms of γ.

Simplified Explanation

Example 1: Linear Transformation on $\mathbb{R}^3$ Let $W = \{(x,y,z) \in \mathbb{R}^3 \mid x + y + z = 0\}$ and $V = \mathbb{R}^2$:

- Basis for W: $\{(-1,1,0),(-1,0,1)\}$.
- Define $f(-1,1,0) = (1,0)$, $f(-1,0,1) = (0,1)$.
- Matrix representation (standard basis for V):

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Conclusion

In this lecture, we:

- Explored linear transformations and their dependency on ordered bases.
- Developed matrix representations and analyzed their changes with basis choice.
- Highlighted the bijection between linear transformations and matrices for fixed bases.

This framework unifies the concepts of linear algebra, emphasizing the interplay between transformations, bases, and matrix representations.