Lecture Summary: Basis for a Vector Space

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Key Points

- **Recap: Linear Dependence and Independence:**
  - A set of vectors $v_1, v_2, \ldots, v_n$ is **linearly dependent** if there exist scalars $a_1, a_2, \ldots, a_n$, not all zero, such that:
  $$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0.$$
  - A set is **linearly independent** if the only solution to the above equation is $a_1 = a_2 = \cdots = a_n = 0$.
- **Span of a Set of Vectors:**
  - The span of a set S is the set of all finite linear combinations of vectors in S:
  $$\text{span}(S) = \left\{ \sum_{i=1}^{n} a_i v_i \mid a_i \in \mathbb{R}, v_i \in S \right\}.$$
  - The span of a set is a subspace of the vector space.
  - Example: For $S = \{(1, 0)\}$ in $\mathbb{R}^2$, the span is the x-axis.
- **Spanning Set:**
  - A set S is a **spanning set** for a vector space V if $\text{span}(S) = V$.
  - Example: $\{(1, 0), (0, 1)\}$ is a spanning set for $\mathbb{R}^2$.
- **Basis:**
  - A **basis** for a vector space V is a set of vectors that is:
    1. **Spanning:** $\text{span}(B) = V$.
    2. **Linearly Independent:** No vector in B can be expressed as a linear combination of the others.
  - A basis is the optimal middle ground between having enough vectors to span V and avoiding redundancy by ensuring linear independence.
- **Examples of a Basis:**
  - The **standard basis** for $\mathbb{R}^n$ is:
  $$\epsilon = \{e_1, e_2, \ldots, e_n\}, \quad e_i = (0, \ldots, 0, 1, 0, \ldots, 0),$$
  where 1 is in the i-th position.
  - Any vector $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ can be written as:
  $$x_1 e_1 + x_2 e_2 + \cdots + x_n e_n.$$

  - Hence, ϵ is both spanning and linearly independent, making it a basis for $\mathbb{R}^n$.

- **Constructing a Basis:**
  - Start with an empty set and iteratively add vectors not in the span of the current set.
  - Example in $\mathbb{R}^3$:
    1. Start with $S_0 = \emptyset$ (span is $\{(0,0,0)\}$).
    2. Append $(0,2,1)$ to form S_1 (span is a line).
    3. Append $(2,2,0)$ to form S_2 (span is a plane).
    4. Append $(0,0,5)$ to form S_3 (span is all of $\mathbb{R}^3$).

Simplified Explanation

Example 1: Basis for $\mathbb{R}^2$

- $\{(1,0),(0,1)\}$ is a basis since:
  1. Any vector $(x,y) \in \mathbb{R}^2$ can be written as $x(1,0) + y(0,1)$.
  2. The set is linearly independent since the only solution to:

$$a(1,0) + b(0,1) = (0,0)$$

  is $a = b = 0$.

Example 2: Basis for $\mathbb{R}^3$

- $\{(1,0,0),(0,1,0),(0,0,1)\}$ is the standard basis.
- Any vector $(x,y,z) \in \mathbb{R}^3$ can be written as:

$$x(1,0,0) + y(0,1,0) + z(0,0,1).$$

Example 3: Constructing a Basis in $\mathbb{R}^3$ Start with an empty set:

- Add $(3,0,0)$ (span is x-axis).
- Add $(2,2,1)$ (span is a plane).
- Add $(1,3,3)$ (span is all of $\mathbb{R}^3$).

This results in a basis for $\mathbb{R}^3$.

Conclusion

In this lecture, we:

- Defined a basis as a linearly independent set that spans a vector space.
- Explored examples and constructed bases for $\mathbb{R}^2$ and $\mathbb{R}^3$.
- Highlighted the importance of the basis in understanding vector spaces and performing computations.

A basis provides a minimal and complete representation of a vector space, making it an essential concept in linear algebra.