Lecture Summary: Finding Bases for Vector Spaces

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Key Points

- **Definition of Basis:**
  - A basis for a vector space V is a set of vectors that is:
    1. **Linearly Independent:** The only linear combination of the vectors that results in the zero vector has all coefficients equal to zero.
    2. **Spanning:** Any vector in V can be expressed as a linear combination of the basis vectors.
- **Equivalent Conditions for a Basis:**
  - A set B is a basis if:
    1. B is linearly independent and spans V.
    2. B is a **maximal linearly independent set**, meaning adding any vector to B makes it linearly dependent.
    3. B is a **minimal spanning set**, meaning removing any vector from B causes it to no longer span V.
  - Proof involves demonstrating how these conditions imply each other using properties of linear independence and span.
- **Methods to Find a Basis:**
  - **Appending Method:**
    1. Start with an empty set.
    2. Iteratively add vectors that are not in the span of the current set until the set spans V.
    3. Ensure the intermediate sets remain linearly independent.
  - **Deleting Method:**
    1. Start with a spanning set (e.g., a set of many vectors).
    2. Iteratively remove vectors that are linear combinations of others until no vector in the set can be expressed as a linear combination of the remaining vectors.
- **Examples:**
  - **Example 1 (Appending in $\mathbb{R}^2$):**
    * Start with $(1, 2)$. This spans a line in $\mathbb{R}^2$.
    * Add $(2, 3)$, which is not on the line. The set $\{(1, 2), (2, 3)\}$ spans $\mathbb{R}^2$ and is linearly independent, forming a basis.
  - **Example 2 (Deleting in $\mathbb{R}^3$):**
    * Start with a spanning set $S = \{(1, 0, 0), (1, 2, 0), (1, 0, 3), (0, 4, 2)\}$.
    * Observe that $(0, 4, 2)$ is a linear combination of the others. Remove it.
    * Next, remove $(0, 2, 3)$, which is also a linear combination of the remaining vectors.

    * The resulting set $\{(1,0,0),(1,2,0),(1,0,3)\}$ is a basis for $\mathbb{R}^3$.

- **Key Observations:**
  - The size of a basis for V (the number of vectors in the basis) is constant, irrespective of the method used to find it.
  - For $\mathbb{R}^n$, the standard basis $\{e_1, e_2, \ldots, e_n\}$ has size n.

Simplified Explanation

Example 1: Appending Method in $\mathbb{R}^2$ Start with the empty set:

- Add $(1,2)$, which spans a line.
- Add $(2,3)$, which is not on the line spanned by $(1,2)$. The set $\{(1,2),(2,3)\}$ spans $\mathbb{R}^2$ and is linearly independent.

Example 2: Deleting Method in $\mathbb{R}^3$ Start with $S = \{(1,0,0),(1,2,0),(1,0,3),(0,4,2)\}$:

- Remove $(0,4,2)$ since it is a linear combination of $(1,0,0)$, $(1,2,0)$, and $(1,0,3)$.
- Remove $(0,2,3)$ since it is also a linear combination of the remaining vectors.
- Resulting basis: $\{(1,0,0),(1,2,0),(1,0,3)\}$.

Conclusion

In this lecture, we:

- Defined equivalent conditions for a set to be a basis.
- Explained the appending and deleting methods for finding a basis.
- Provided examples in $\mathbb{R}^2$ and $\mathbb{R}^3$ to illustrate the process.

The basis of a vector space is a fundamental concept in linear algebra, providing an optimal representation for spanning the space.