Lecture Summary: Introduction to Vector Spaces

Source: Lec27.pdf

Key Points

- **Definition of Vector Spaces:**
  - A vector space is a set V with two operations:
    1. Vector addition: $+ : V \times V \to V$.
    2. Scalar multiplication: $\cdot : \mathbb{R} \times V \to V$.
  - These operations must satisfy 8 axioms, abstracted from $\mathbb{R}^n$.
- **Axioms of Vector Spaces:** For all $v, w, u \in V$ and scalars $a, b \in \mathbb{R}$:
  1. $v + w = w + v$ (commutativity).
  2. $(v + w) + u = v + (w + u)$ (associativity of addition).
  3. There exists a zero vector $0 \in V$ such that $v + 0 = v$.
  4. For every $v \in V$, there exists $-v \in V$ such that $v + (-v) = 0$.
  5. $1 \cdot v = v$, where 1 is the multiplicative identity in $\mathbb{R}$.
  6. $a \cdot (b \cdot v) = (a \cdot b) \cdot v$ (associativity of scalar multiplication).
  7. $a \cdot (v + w) = a \cdot v + a \cdot w$ (distributivity of scalar multiplication over vector addition).
  8. $(a + b) \cdot v = a \cdot v + b \cdot v$ (distributivity of scalar addition over scalar multiplication).
- **Examples of Vector Spaces:**
  - $\mathbb{R}^n$ with standard addition and scalar multiplication.
  - The set of $m \times n$ matrices with real entries, where:
    1. Matrix addition is element-wise.
    2. Scalar multiplication is element-wise scaling.
  - The solution set of a homogeneous system $Ax = 0$ forms a vector space.
- **Non-Examples of Vector Spaces:**
  - Redefining addition or scalar multiplication in $\mathbb{R}^2$ can lead to violations of the axioms. For example:
    1. Redefining addition such that the second component subtracts instead of adding can break commutativity.
    2. Redefining scalar multiplication to ignore one component can break distributivity.
- **Subspaces of a Vector Space:**
  - A subset $W \subseteq V$ is a subspace if:
    1. $0 \in W$.
    2. $v + w \in W$ for all $v, w \in W$.
    3. $a \cdot v \in W$ for all $v \in W$ and $a \in \mathbb{R}$.
  - Example: The solution set of a homogeneous system $Ax = 0$ is a subspace of $\mathbb{R}^n$.

Simplified Explanation

Example 1: $\mathbb{R}^2$ as a Vector Space For $v = (v_1, v_2)$ and $w = (w_1, w_2)$:

$$v + w = (v_1 + w_1, v_2 + w_2), \quad a \cdot v = (a \cdot v_1, a \cdot v_2).$$

These operations satisfy all 8 axioms.

Example 2: Matrices as a Vector Space For $A, B \in \mathbb{R}^{m \times n}$:

$$A + B = (A_{ij} + B_{ij}), \quad c \cdot A = (c \cdot A_{ij}),$$

where A_{ij} represents the (i, j) entry of A. These operations satisfy the vector space axioms.

Example 3: Homogeneous System The solution set of $Ax = 0$:

- If $v, w \in V$, then $v + w \in V$ (closure under addition).
- If $v \in V$, then $a \cdot v \in V$ for $a \in \mathbb{R}$ (closure under scalar multiplication).

Conclusion

In this lecture, we:

- Defined vector spaces as a generalization of $\mathbb{R}^n$.
- Presented examples (e.g., $\mathbb{R}^n$, matrices, and homogeneous systems) and non-examples.
- Introduced subspaces and their properties.

Vector spaces provide a unified framework for understanding and manipulating linear algebra concepts. This abstraction is crucial for advanced topics such as vector calculus and linear transformations.