Lecture Summary: Continuity for Multivariable Functions

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Key Points

- **Limits for Scalar-Valued Multivariable Functions (Recap):**
  - A scalar function $f : D \subset \mathbb{R}^n \to \mathbb{R}$ has a limit L at $\vec{a} \in \mathbb{R}^n$ if for all sequences $\{\vec{a}_n\} \subset D$ converging to $\vec{a}$, $f(\vec{a}_n) \to L$.
  - If the limit exists, it is independent of the path taken to approach $\vec{a}$.
- **Limits for Vector-Valued Multivariable Functions:**
  - A vector-valued function $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$ has a limit $\vec{L} = (L_1, L_2, \ldots, L_m)$ at $\vec{a} \in \mathbb{R}^n$ if:

    $$\lim_{\vec{x} \to \vec{a}} f_i(\vec{x}) = L_i \quad \text{for each component } f_i.$$

  - If any component's limit does not exist, the limit of f does not exist.
- **Limits Along a Curve:**
  - Consider a scalar function $f : D \subset \mathbb{R}^n \to \mathbb{R}$ and a curve C passing through $\vec{a}$.
  - The limit of f at $\vec{a}$ along C exists and equals L if for every sequence $\{\vec{a}_n\} \subset C$ converging to $\vec{a}$, $f(\vec{a}_n) \to L$.
  - If limits along different curves yield different values, the global limit at $\vec{a}$ does not exist.
- **Continuity of a Multivariable Function:**
  - A function $f : D \subset \mathbb{R}^n \to \mathbb{R}$ is continuous at $\vec{a} \in D$ if:

    $$\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = f(\vec{a}).$$

  - This is equivalent to saying that for every sequence $\{\vec{a}_n\} \subset D$ converging to $\vec{a}$, $f(\vec{a}_n) \to f(\vec{a})$.
  - A function is continuous on D if it is continuous at every point $\vec{a} \in D$.
- **Examples:**
  - **Example 1:** For $f(x, y) = x^2 + y^2$, the limit at $(0, 0)$ exists and equals 0. The function is continuous everywhere.
  - **Example 2:** For $f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$:
    - Along x-axis ($y = 0$): $\lim_{(x,y)\to(0,0)} f(x, y) = 1$.
    - Along y-axis ($x = 0$): $\lim_{(x,y)\to(0,0)} f(x, y) = -1$.

    The global limit does not exist due to path dependence.
  - **Example 3:** For $f(x, y) = \frac{xy}{x^2 + y^2}$:
    - Along x-axis ($y = 0$) and y-axis ($x = 0$): $\lim_{(x,y)\to(0,0)} f(x, y) = 0$.

        * Along $y = x$: $\lim_{(x,y)\to(0,0)} f(x,y) = \frac{1}{2}$.

    Since limits along different paths do not match, the global limit does not exist.

- **Connection Between Curve Limits and Global Limits:**
    - The global limit $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L$ exists if and only if:

$$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) \text{ along every curve } C \text{ passing through } \vec{a} \text{ equals } L.$$

    - If limits along any two curves differ, the global limit does not exist.

Simplified Explanation

What Are We Studying? How to determine if a multivariable function has a limit or is continuous at a point.

Key Points: - A limit exists if the function approaches the same value along all paths to a point. - Continuity requires the function's limit at a point to equal its value at that point.

Example: For $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$, limits along different axes differ, so the limit does not exist globally.

Conclusion

In this lecture, we:

- Defined limits for scalar- and vector-valued multivariable functions.
- Introduced continuity for multivariable functions.
- Explored examples to illustrate limits and continuity.

Understanding limits and continuity is foundational for analyzing multivariable functions in higher-dimensional calculus.