Lecture Summary: Simulations and Applications of Expected Value

Source: Lecture 3.3.pdf

Key Points

- **Expected Value in Practice:**
  - The expected value represents the average value of a random variable over a large number of trials.
  - Simulations are a powerful tool to observe how the expected value aligns with average outcomes in experiments.
- **Casino Dice Game Simulation:**
  - Game: A player bets on the outcome of the sum of two dice rolls being under 7, over 7, or equal to 7.
  - Simulated returns match closely with theoretical expected values:

$$\text{Expected Gain} \approx \text{Simulated Average Gain}.$$

  - Example: For specific betting strategies, simulations consistently show a negative expected gain, reflecting the casino's advantage.
- **Simulating Common Distributions:**
  - Python's `scipy.stats` library generates samples for distributions like Binomial, Geometric, and Poisson.
  - Simulated averages align closely with theoretical expectations:
    * Binomial$(20, 0.3)$: $E[X] = 6$.
    * Geometric(0.3): $E[X] = \frac{1}{0.3} \approx 3.33$.
    * Poisson(λ): $E[X] = \lambda$.
  - Highlights the utility of simulations in verifying theoretical results.
- **Balls and Bins Problem:**
  - Problem: m balls are thrown into n bins uniformly at random. Compute the expected number of empty bins.
  - Theoretical result:

$$E[\text{Empty Bins}] = n\left(1 - \frac{1}{n}\right)^m \approx ne^{-\frac{m}{n}}.$$

  - Simulation steps:
    1. Repeat the experiment 1000 times (Monte Carlo method).
    2. Assign each ball to a random bin and track the count.
    3. Calculate the average number of empty bins.
  - Simulated results closely match theoretical values.

- **Importance of Simulations:**
  - Provides a practical way to verify theoretical concepts like expected value.
  - Demonstrates the law of large numbers: Simulated averages converge to expected values as sample size increases.

Simplified Explanation

Key Idea: Simulations show that the average outcomes in repeated trials align with the expected value.

Examples: - Casino game: Expected gains and simulated gains match closely, highlighting the casino's long-term profit. - Balls and bins: Simulated and theoretical counts of empty bins are nearly identical.

Why It Matters: Simulations provide a tangible way to observe and validate probabilistic concepts.

Conclusion

In this lecture, we:

- Used Python simulations to observe the behavior of expected value.
- Verified theoretical results for distributions like Binomial, Geometric, and Poisson.
- Demonstrated the practical utility of expected value in games and probabilistic problems.

Simulations bridge theory and practice, making abstract concepts like expected value accessible and applicable.