Lecture Summary: Independence of Continuous Random Variables

Source: Lecture 5.5.docx

Key Points

- **Definition of Independence:**
  - Two continuous random variables X and Y are independent if their joint density function can be expressed as the product of their marginal densities:

    $$f_{X,Y}(x, y) = f_X(x)f_Y(y).$$

  - Independence implies that knowing the value of one variable does not provide information about the other.
- **Verification of Independence:**
  - To verify independence:
    1. Start with the joint density $f_{X,Y}(x, y)$.
    2. Compute the marginal densities:

       $$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx.$$

    3. Check if $f_{X,Y}(x, y) = f_X(x)f_Y(y)$.
- **Key Insights:**
  - Independence simplifies calculations since the joint density is the product of the marginals.
  - Assuming independence when variables are not independent can lead to incorrect conclusions.
- **Examples:**
  - **Uniform Distribution on a Unit Square:**
    * Joint density:

      $$f_{X,Y}(x, y) = \begin{cases} 1, & 0 \leq x, y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

    * Marginals:

      $$f_X(x) = \int_0^1 1\, dy = 1, \quad f_Y(y) = \int_0^1 1\, dx = 1.$$

    * Since $f_{X,Y}(x, y) = f_X(x)f_Y(y)$, X and Y are independent.
  - **Non-Uniform Example:**
    * Joint density:

      $$f_{X,Y}(x, y) = \begin{cases} 2, & 0 \leq x \leq y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

    * Marginals:
$$f_X(x) = \int_x^1 2\,dy = 2(1-x), \quad f_Y(y) = \int_0^y 2\,dx = 2y.$$
    * Here, $f_{X,Y}(x,y) \neq f_X(x)f_Y(y)$, so X and Y are not independent.

- **Applications:**
  - Independence is often assumed in practical models to simplify calculations.
  - When independence is valid, it facilitates easier computation of joint probabilities and densities.

- **Worked Example: Independent Exponentials:**
  - $X, Y \sim \text{Exponential}(\lambda)$ are independent:
$$f_X(x) = \lambda e^{-\lambda x}, \quad f_Y(y) = \lambda e^{-\lambda y}.$$
  - Joint density:
$$f_{X,Y}(x,y) = f_X(x)f_Y(y) = \lambda^2 e^{-\lambda(x+y)}, \quad x, y > 0.$$
  - Probability that $X + Y > c$:
$$P(X+Y>c) = \int_0^\infty \int_{c-y}^\infty \lambda^2 e^{-\lambda(x+y)}\,dx\,dy.$$
  - Solve step by step to obtain:
$$P(X+Y>c) = e^{-\lambda c}.$$

Simplified Explanation

Independence: Two variables are independent if their joint density equals the product of their marginals.

Key Idea: - Independence simplifies computations. - Assuming independence without validation can lead to errors.

Example: - Independent exponential variables yield:
$$f_{X,Y}(x,y) = \lambda^2 e^{-\lambda(x+y)}.$$

Conclusion

In this lecture, we:

- Defined independence for continuous random variables.
- Demonstrated verification and implications of independence.
- Worked through examples, highlighting practical use cases.

Independence is a critical concept that simplifies analysis and appears frequently in real-world probabilistic models.