Lecture Summary: Marginals, Conditionals, and Joint PMFs

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Key Points

- **Overview of Marginals, Conditionals, and Joint PMFs:**
  - Previously, we derived joint PMFs from marginals and conditionals.
  - In practice, marginals and conditionals are often provided, and the joint PMF is derived from them using the factorization rule:

$$f_{XY}(t_1, t_2) = f_X(t_1) \cdot f_{Y|X}(t_2 \mid t_1).$$

- **Example 1: Dice Roll and Coin Toss:**
  - Experiment: Roll a die (X) and toss a coin X times. Define Y as the number of heads obtained.
  - **Marginal PMF:** $f_X(x) = \frac{1}{6},\ x \in \{1, 2, 3, 4, 5, 6\}$.
  - **Conditional PMF:** Given $X = x$, $Y \sim \text{Binomial}(x, \frac{1}{2})$:

$$f_{Y|X}(y \mid x) = \binom{x}{y} \left(\frac{1}{2}\right)^y \left(\frac{1}{2}\right)^{x-y},\ y \in \{0, 1, \dots, x\}.$$

  - **Joint PMF:**

$$f_{XY}(x, y) = f_X(x) \cdot f_{Y|X}(y \mid x) = \frac{1}{6} \cdot \binom{x}{y} \left(\frac{1}{2}\right)^x,\ y \in \{0, 1, \dots, x\}.$$

- **Example 2: Poisson Process and Coin Toss:**
  - Experiment: Generate a Poisson random variable $X \sim \text{Poisson}(\lambda)$. Toss a coin X times, and define Y as the number of heads.
  - **Marginal PMF:** $f_X(x) = \frac{\lambda^x e^{-\lambda}}{x!},\ x \in \{0, 1, 2, \dots\}$.
  - **Conditional PMF:** Given $X = x$, $Y \sim \text{Binomial}(x, \frac{1}{2})$:

$$f_{Y|X}(y \mid x) = \binom{x}{y} \left(\frac{1}{2}\right)^y \left(\frac{1}{2}\right)^{x-y}.$$

  - **Joint PMF:**

$$f_{XY}(x, y) = f_X(x) \cdot f_{Y|X}(y \mid x) = \frac{\lambda^x e^{-\lambda}}{x!} \cdot \binom{x}{y} \left(\frac{1}{2}\right)^x.$$

  - **Marginalization:**

$$f_Y(y) = \sum_{x=y}^{\infty} f_{XY}(x, y).$$

    This simplifies to:

$$f_Y(y) = \frac{(\lambda/2)^y e^{-\lambda}}{y!},\ y \in \{0, 1, 2, \dots\}.$$

    Hence, $Y \sim \text{Poisson}(\lambda/2)$.

- **Example 3: IPL Powerplay Runs and Wickets:**
  - Define X as the number of wickets in an over and Y as the runs scored.
  - **Marginal PMF of X:**
$$f_X(x) = \begin{cases} 0.6 & x = 0, \\ 0.3 & x = 1, \\ 0.1 & x = 2. \end{cases}$$
  - **Conditional PMF of $Y \mid X$:**
    * For $x = 0$: $Y \sim \text{Uniform}(0, 12)$.
    * For $x = 1$: Y follows a different distribution with lower means.
    * For $x = 2$: Y is further skewed toward smaller values.
  - **Joint PMF:**
$$f_{XY}(x, y) = f_X(x) \cdot f_{Y|X}(y \mid x).$$
  - **Marginalization:**
$$f_Y(y) = \sum_x f_{XY}(x, y).$$

    A graphical representation (e.g., a stem plot) helps visualize $f_Y(y)$.

Simplified Explanation

Key Idea: Start with marginals and conditionals, then derive joint PMFs. Marginalize joint PMFs to find distributions of individual variables.

Example: Poisson and Coin Toss - $X \sim \text{Poisson}(\lambda)$, Y is number of heads. - Result: $Y \sim \text{Poisson}(\lambda/2)$.

Conclusion

In this lecture, we:

- Explored deriving joint PMFs from marginal and conditional PMFs.
- Analyzed real-world examples, including IPL data and Poisson processes.
- Highlighted the practical and theoretical applications of these distributions.

This process is foundational for probabilistic modeling and inference.