Lecture Summary: Bias, Variance, and Risk of Estimators

Lecture: 9.4 - Bias, Variance, and Risk of an Estimator

Source: Lecture 8.4.pdf

Key Points

- **Point Estimation Problem:**
  - Given iid samples from a distribution described by a parameter θ, the goal is to estimate θ.
  - Estimators are functions of the samples that provide an approximation of θ.
- **Bias of an Estimator:**
  - Definition:
    $$\text{Bias}(\hat{\theta}, \theta) = \mathbb{E}[\hat{\theta}] - \theta.$$
  - Intuition:
    * Measures how far the expected value of the estimator is from the true parameter.
    * An unbiased estimator has Bias $= 0$.
- **Variance of an Estimator:**
  - Definition:
    $$\text{Var}(\hat{\theta}) = \mathbb{E}[(\hat{\theta} - \mathbb{E}[\hat{\theta}])^2].$$
  - Intuition:
    * Describes the spread of the estimator's distribution.
    * A low variance ensures consistency in estimates across different samples.
- **Risk of an Estimator:**
  - Squared Error Risk:
    $$R(\hat{\theta}, \theta) = \mathbb{E}[(\hat{\theta} - \theta)^2].$$
  - Alternate Terminology:
    * Also known as Mean Squared Error (MSE).
  - Components:
    $$R(\hat{\theta}, \theta) = \text{Bias}^2 + \text{Var}(\hat{\theta}).$$
- **Bias-Variance Tradeoff:**
  - To minimize risk:
    * Reduce bias (ensure $\mathbb{E}[\hat{\theta}] \approx \theta$).
    * Control variance (ensure estimates are consistent).
  - Tradeoff arises because reducing bias may increase variance and vice versa.

- **Examples:**
  1. **Estimator 1: $\hat{\theta}_1 = \frac{1}{2}$ (Constant)**
     - Bias: $\frac{1}{2} - \theta$.
     - Variance: 0 (constant value).
     - Risk: $(\frac{1}{2} - \theta)^2$.
  2. **Estimator 2: $\hat{\theta}_2 = \frac{X_1+X_2}{2}$ (Two Samples)**
     - Bias: 0 (unbiased).
     - Variance: $\frac{p(1-p)}{2}$.
     - Risk: $\frac{p(1-p)}{2}$.
  3. **Estimator 3: $\hat{\theta}_3 = \frac{\sum_{i=1}^{n} X_i}{n}$ (Sample Mean)**
     - Bias: 0 (unbiased).
     - Variance: $\frac{p(1-p)}{n}$.
     - Risk: $\frac{p(1-p)}{n}$.
- **Key Insights:**
  - Estimator $\hat{\theta}_3$ outperforms others because its risk decreases with n, ensuring better accuracy with larger sample sizes.
  - Adjustments to estimators (e.g., scaling terms) may alter bias, variance, and risk significantly.

Simplified Explanation

Key Idea: Estimators should have low bias, variance, and risk to reliably approximate the parameter θ.

Why It Matters: Bias and variance directly affect an estimator's accuracy, and their relationship highlights tradeoffs in estimation design.

Conclusion

In this lecture, we:

- Defined and analyzed bias, variance, and risk for parameter estimators.
- Explored their relationships through the Bias-Variance decomposition.
- Demonstrated calculations for common estimators in Bernoulli trials.

Understanding these concepts is essential for evaluating and designing effective statistical estimators.