Lecture Summary: Expected Value

Source: Lecture 3.1.pdf

Key Points

- **Definition of Expected Value:**
  - The expected value (denoted as $E[X]$) is a measure of the "average" value a random variable X takes, weighted by its probabilities.
  - Formula:
    $$E[X] = \sum_{t \in T_X} t \cdot f_X(t),$$
    where T_X is the range of X and $f_X(t) = P(X = t)$ is the probability mass function (PMF).
  - Interpretation: Over repeated trials, the arithmetic mean of X approaches $E[X]$.
- **Importance of Expected Value:**
  - Summarizes the central tendency of a random variable.
  - Provides insights into long-term outcomes (e.g., average gains in a casino game).
  - Useful in practical decision-making and theoretical analysis.
- **Examples:**
  - **Bernoulli Random Variable:**
    * PMF: $P(X = 1) = p$, $P(X = 0) = 1 - p$.
    * Expected value:
      $$E[X] = 0 \cdot (1 - p) + 1 \cdot p = p.$$
  - **Uniform Random Variable (1 to 6):**
    * PMF: $P(X = x) = \frac{1}{6}$ for $x \in \{1, 2, 3, 4, 5, 6\}$.
    * Expected value:
      $$E[X] = \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) = 3.5.$$
  - **Lottery Example:**
    * PMF: $P(X = 200) = \frac{1}{1000}$, $P(X = 20) = \frac{27}{1000}$, $P(X = 0) = \frac{972}{1000}$.
    * Expected value:
      $$E[X] = 200 \cdot \frac{1}{1000} + 20 \cdot \frac{27}{1000} + 0 \cdot \frac{972}{1000} = 0.56.$$
- **Properties of Expected Value:**
  - Units of $E[X]$ match the units of X.
  - $E[X]$ may not be a value in the range of X.
- **Advanced Examples:**

  - **Geometric Distribution:**
    * PMF: $P(X = k) = (1 - p)^{k-1}p$ for $k = 1, 2, \ldots$.
    * Expected value:
$$E[X] = \frac{1}{p}.$$
  - **Poisson Distribution:**
    * PMF: $P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$ for $k = 0, 1, \ldots$.
    * Expected value:
$$E[X] = \lambda.$$
  - **Binomial Distribution:**
    * PMF: $P(X = k) = \binom{n}{k}p^k(1 - p)^{n-k}$ for $k = 0, 1, \ldots, n$.
    * Expected value:
$$E[X] = np.$$

- **Key Techniques:**
  - Use summation formulas for uniform and geometric progressions to simplify calculations.
  - Understand difference equations for summation proofs.

Simplified Explanation

Expected Value: The "average" value of a random variable, found by summing each possible value weighted by its probability.

Examples: - Bernoulli(p): $E[X] = p$. - Uniform(1 to 6): $E[X] = 3.5$. - Geometric(p): $E[X] = \frac{1}{p}$.

Why It Matters: Expected value helps understand long-term outcomes and guides practical decisions (e.g., betting strategies).

Conclusion

In this lecture, we:

- Defined expected value and its practical significance.
- Demonstrated calculations for common distributions.
- Highlighted techniques for simplifying summations.

Expected value bridges probability and real-world applications, providing insights into the central tendency of random variables.