Lecture Summary: Estimator Design Approach - Method of Moments

Lecture: 9.5 - Estimator Design Approach - Method of Moments

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Key Points

- **Introduction to Method of Moments:**
  - A simple and widely used approach for designing estimators.
  - Relies on equating sample moments to distribution moments to solve for parameters.
- **Moments and Parameters:**
  - Moments are expectations of powers of the random variable, e.g., $\mathbb{E}[X]$, $\mathbb{E}[X^2]$.
  - Distribution moments are expressed as functions of parameters of the distribution.
  - Example distributions:
    * Bernoulli(p): $\mathbb{E}[X] = p$.
    * Poisson(λ): $\mathbb{E}[X] = \lambda$.
    * Exponential(λ): $\mathbb{E}[X] = \frac{1}{\lambda}$.
    * Normal(μ, σ^2): $\mathbb{E}[X] = \mu$, $\mathbb{E}[X^2] = \sigma^2 + \mu^2$.
    * Gamma(α, β): $\mathbb{E}[X] = \frac{\alpha}{\beta}$, $\mathbb{E}[X^2] = \frac{\alpha+\beta}{\beta^2}$.
- **Sample Moments:**
  - For n iid samples, the kth sample moment is:

$$M_k = \frac{1}{n} \sum_{i=1}^{n} X_i^k.$$

  - Sample moments are random variables and converge to their corresponding distribution moments as $n \to \infty$ (by the Weak Law of Large Numbers).
- **Method of Moments Procedure:**
  - Replace distribution moments with sample moments.
  - Solve the resulting equations for the parameters.
  - Example with one parameter:

$$\mathbb{E}[X] = g(\theta) \quad \to \quad M_1 = g(\hat{\theta}),$$

    where $\hat{\theta}$ is the method of moments estimator.
- **Examples:**

1. **Bernoulli(p):**
   - $\mathbb{E}[X] = p$.
   - Method of Moments Estimator:
   $$\hat{p} = M_1 = \frac{1}{n}\sum_{i=1}^{n} X_i.$$
2. **Poisson(λ):**
   - $\mathbb{E}[X] = \lambda$.
   - Method of Moments Estimator:
   $$\hat{\lambda} = M_1 = \frac{1}{n}\sum_{i=1}^{n} X_i.$$
3. **Exponential(λ):**
   - $\mathbb{E}[X] = \frac{1}{\lambda}$.
   - Method of Moments Estimator:
   $$\hat{\lambda} = \frac{1}{M_1} = \frac{n}{\sum_{i=1}^{n} X_i}.$$
4. **Normal(μ, σ^2):**
   - $\mathbb{E}[X] = \mu$, $\mathbb{E}[X^2] = \sigma^2 + \mu^2$.
   - Method of Moments Estimators:
   $$\hat{\mu} = M_1, \quad \hat{\sigma}^2 = M_2 - M_1^2.$$
5. **Gamma(α, β):**
   - $\mathbb{E}[X] = \frac{\alpha}{\beta}$, $\mathbb{E}[X^2] = \frac{\alpha+\beta}{\beta^2}$.
   - Solve for α and β using:
   $$M_1 = \frac{\alpha}{\beta}, \quad M_2 = \frac{\alpha+1}{\beta^2}.$$

- **Practical Applications:**
  - Bernoulli(p): Compute success probabilities using observed outcomes.
  - Poisson(λ): Estimate the average rate of events (e.g., radioactive decay).
  - Normal(μ, σ^2): Analyze measurement errors or natural variability.
  - Gamma(α, β): Model waiting times or continuous quantities.

Simplified Explanation

Key Idea: The Method of Moments estimates parameters by equating sample moments to their theoretical counterparts.

Steps: 1. Compute sample moments from data. 2. Solve equations relating moments to parameters. 3. Replace theoretical moments with sample moments to estimate parameters.

Examples: - For Bernoulli(p), $\hat{p} =$ sample mean. - For Exponential(λ), $\hat{\lambda} = \frac{1}{\text{sample mean}}$.

Conclusion

In this lecture, we:

- Introduced the Method of Moments for parameter estimation.
- Demonstrated examples across various distributions.
- Highlighted the method's simplicity and wide applicability.

The Method of Moments provides an intuitive starting point for parameter estimation, bridging sample data with theoretical models.