Lecture Summary: Properties of Expected Value

Source: Lecture 3.2.pdf

Key Points

- **Definition Recap:**
  - The expected value, $E[X]$, represents the weighted average of a random variable X.
  - Formula:
    $$E[X] = \sum_{t \in T_X} t \cdot P(X = t).$$
- **Properties of Expected Value:**
  1. **Expected Value of a Constant:**
     $$E[c] = c.$$
  2. **Non-Negativity for Non-Negative Random Variables:**
     $$E[X] \geq 0 \quad \text{if } X \geq 0.$$
  3. **Linearity of Expectation:**
     - For any random variables X and Y, and constants a, b:
       $$E[aX + bY] = aE[X] + bE[Y].$$
     - This property holds regardless of whether X and Y are independent.
  4. **Expectation of a Function:**
     - For a function $g(X)$:
       $$E[g(X)] = \sum_{t \in T_X} g(t) \cdot P(X = t).$$
- **Applications and Examples:**
  - **Casino Math:**
    * A betting strategy on outcomes "under 7," "over 7," and "equal to 7" yields:
      $$E[\text{Gain}] = \frac{-2 + 3p_1}{6}.$$
    * Result: The expected gain is negative regardless of p_1, illustrating how casinos structure bets to ensure long-term losses for players.
  - **Linearity of Expectation Example:**
    * Sum of two dice rolls:
      $$E[X + Y] = E[X] + E[Y] = 3.5 + 3.5 = 7.$$
    * No need to compute the joint distribution of X and Y.

- **Binomial Distribution:**
  - For $X \sim \text{Binomial}(n, p)$:
  $$E[X] = np,$$
  derived easily using linearity by summing n independent Bernoulli trials.
- **Centering a Random Variable:**
  - To create a random variable with mean 0:
  $$Y = X - E[X].$$
  - Applications include machine learning and data normalization.
- **Balls and Bins Problem:**
  - Throw 10 balls into 3 bins. Let $X_i = 1$ if bin i is empty, 0 otherwise.
  - Expected number of empty bins:
  $$E[X_1 + X_2 + X_3] = E[X_1] + E[X_2] + E[X_3] = 3 \cdot \left(\frac{2}{3}\right)^{10}.$$
  - Highlights the simplicity of expectation calculations using linearity.

Simplified Explanation

Key Properties: - $E[c] = c$ for a constant c. - $E[X] \geq 0$ if $X \geq 0$. - $E[aX + bY] = aE[X] + bE[Y]$ simplifies calculations, even for dependent variables.

Applications: - Casino games ensure negative expected gains for players. - Simplified expectation computations for the sum of random variables, like dice rolls or binomial distributions. - Data normalization by centering a random variable: $Y = X - E[X]$.

Conclusion

In this lecture, we:

- Examined key properties of expected value, including linearity.
- Applied these properties to real-world problems such as games, distributions, and data normalization.
- Demonstrated the power of expected value in simplifying calculations and gaining insights into probabilistic phenomena.

Expected value is an essential tool in probability, offering a balance between theoretical analysis and practical utility.