# Lecture Summary: Joint Distributions of Discrete and Continuous Random Variables

## Source: Lecture 5.2.docx

## Key Points

- **Introduction to Joint Distributions:**
  - Describes relationships between two random variables where one is discrete, and the other is continuous.
  - Joint distributions model dependencies in real-world datasets, such as the Iris dataset (class as discrete, sepal length as continuous).
- **Notation and Conditional Densities:**
  - For discrete random variable $X$ with PMF $p_X(x)$, the conditional density of $Y$ given $X = x$ is denoted:
    $$f_{Y|X}(y|x).$$
  - Marginal density of $Y$ is obtained using the total probability law:
    $$f_Y(y) = \sum_x p_X(x) f_{Y|X}(y|x).$$
- **Gaussian Mixtures:**
  - Conditional densities can form a mixture distribution.
  - Example:
    * $X$ is uniform over $\{0, 1, 2\}$.
    * $Y|X = 0 \sim N(5, 0.4)$, $Y|X = 1 \sim N(6, 0.5)$, $Y|X = 2 \sim N(7, 0.6)$.
    * Marginal $f_Y(y)$ is a Gaussian mixture:
    $$f_Y(y) = \frac{1}{3}\left(f_{N(5,0.4)}(y) + f_{N(6,0.5)}(y) + f_{N(7,0.6)}(y)\right).$$
- **Reverse Problems and Bayes-like Rule:**
  - Conditional probability of the discrete variable given the continuous variable:
    $$P(X = x|Y = y) = \frac{p_X(x) f_{Y|X}(y|x)}{f_Y(y)}.$$
  - Analogous to Bayes' theorem, replacing probabilities with densities for continuous variables.
  - Example:
    * $X \in \{-1, 1\}$, $P(X = -1) = P(X = 1) = 0.5$.
    * $Y|X = -1 \sim \text{Uniform}(-2, 2)$, $Y|X = 1 \sim \text{Exp}(5)$.

    - Marginal $f_Y(y)$ combines these densities, and $P(X = x|Y = y)$ is computed using the formula.

- **Applications and Practical Examples:**
  - Classification:
    - Observing a continuous value $Y$ to predict the discrete class $X$.
    - Example: Height and gender prediction using height distribution for males and females.
  - Communication systems: Modeling signal distributions with discrete states and continuous noise.

## Simplified Explanation

**Joint Distributions:** Combines discrete and continuous random variables: - $X$: Discrete, with PMF $p_X(x)$. - $Y$: Continuous, with conditional density $f_{Y|X}(y|x)$.

**Key Formula:** Marginal density of $Y$:

$$f_Y(y) = \sum_x p_X(x) f_{Y|X}(y|x).$$

**Reverse Problem:** Find $P(X = x|Y = y)$:

$$P(X = x|Y = y) = \frac{p_X(x) f_{Y|X}(y|x)}{f_Y(y)}.$$

**Example:** Height of individuals in a population: - Male heights: $N(160, 10^2)$. - Female heights: $N(150, 5^2)$. - Observing a height of 155 cm, predict the gender using the conditional probabilities.

## Conclusion

In this lecture, we:

- Introduced joint distributions of discrete and continuous random variables.
- Discussed Gaussian mixtures and Bayes-like rules for reverse problems.
- Highlighted applications in classification and modeling.

These concepts are foundational in probability and statistics, offering tools for analyzing mixed-variable datasets.