Lecture Summary: Independence of Multiple Random Variables and I.I.D. Case

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Key Points

- **Definition of Independence for Multiple Random Variables:**
  - n random variables $X_1, X_2, \ldots, X_n$ are independent if:

    $$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1) \cdot P(A_2) \cdot \cdots \cdot P(A_n),$$

    for any events $A_1, A_2, \ldots, A_n$ related to $X_1, X_2, \ldots, X_n$.
  - Alternatively, for discrete random variables, independence holds if the joint PMF factors as:

    $$f_{X_1,X_2,\ldots,X_n}(t_1, t_2, \ldots, t_n) = f_{X_1}(t_1) \cdot f_{X_2}(t_2) \cdot \cdots \cdot f_{X_n}(t_n).$$

- **Examples of Independence:**
  - **Example 1: Tossing a Fair Coin Thrice:**
    * Joint PMF: $f_{X_1,X_2,X_3}(t_1, t_2, t_3) = \frac{1}{8}$ for all $t_1, t_2, t_3 \in \{0, 1\}$.
    * Marginal PMFs: $f_{X_i}(t_i) = \frac{1}{2}$.
    * Check: The product of marginals equals the joint PMF, confirming independence.
  - **Example 2: Three-Digit Lottery Numbers:**
    * Random variables:
      · X: Hundreds place digit (uniform 0 to 9).
      · Y: Parity of the number (even or odd).
      · Z: Units place digit (uniform 0 to 9).
    * Observations:
      · X and Z are independent, as their joint PMF equals the product of their marginals.
      · Y and Z are dependent, as certain combinations (e.g., $Y = 1$ and $Z = 2$) are impossible.

- **Special Case: I.I.D. Random Variables:**
  - Independent and Identically Distributed (i.i.d.) random variables satisfy:
    * Independence: The joint PMF factors as a product of marginals.
    * Identical distribution: All marginals are the same, $f_{X_i}(t) = f_X(t)$.
  - Joint PMF simplifies to:

    $$f_{X_1,X_2,\ldots,X_n}(t_1, t_2, \ldots, t_n) = \prod_{i=1}^{n} f_X(t_i).$$

- **Applications of I.I.D. Random Variables:**
  - Simplifies probability calculations for complex events.

- **Example: Geometric Distribution:**
  * Let $X_1, X_2, \ldots, X_n \sim \text{Geometric}(p)$ be i.i.d.
  * Probability that all $X_i > j$:

$$P(X_1 > j, X_2 > j, \ldots, X_n > j) = (P(X > j))^n .$$

- **Example: Missing Value:**
  * Let X_i take values $0, 1, 2, 3, 4$ with known PMF. The probability that $X_i \neq 4$ for all i:

$$P(X_1 \neq 4, X_2 \neq 4, \ldots, X_n \neq 4) = (P(X \neq 4))^n .$$

- **Memoryless Property of Geometric Distributions:**
  - The Geometric distribution satisfies:

$$P(X > m + n \mid X > m) = P(X > n).$$

  - Interpretation: The future waiting time is independent of the past.

Simplified Explanation

Independence: Random variables are independent if the joint PMF equals the product of their marginals.

I.I.D. Random Variables: Independent and identically distributed variables share the same marginal PMF and simplify computations.

Example (Geometric): For $X \sim \text{Geometric}(p)$:

$$P(X_1 > j, \ldots, X_n > j) = (P(X > j))^n .$$

Conclusion

In this lecture, we:

- Defined independence for multiple random variables.
- Explored examples, including dependent and independent cases.
- Introduced the i.i.d. case and discussed its practical applications.
- Highlighted the memoryless property of the Geometric distribution.

Understanding independence and the i.i.d. case is essential for probabilistic modeling and statistical inference.