Lecture Summary: Functions of Random Variables and Convolutions

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Key Points

- **Functions of Random Variables:**
  - A function of random variables creates a new random variable. For instance:
  $$Z = g(X, Y).$$
  - The probability mass function (PMF) of the new variable can be derived by summing the joint PMF of X and Y over all values that map to Z:
  $$P(Z = z) = \sum_{(x,y):g(x,y)=z} P(X = x, Y = y).$$
- **Example: Sum of Two Dice Rolls:**
  - Let X and Y be results of two dice rolls.
  - $Z = X + Y$, and the range of Z is $\{2, 3, \ldots, 12\}$.
  - PMF:
  $$P(Z = 2) = P(X = 1, Y = 1), \quad P(Z = 3) = P(X = 1, Y = 2) + P(X = 2, Y = 1), \text{ etc.}$$
  - Visualization involves diagonal contours where $x + y = z$.
- **Example: Area of a Rectangle:**
  - Let X and Y be the length and breadth of a rectangle.
  - $Z = XY$ gives the area.
  - Compute the PMF of Z using the joint PMF of X and Y, summing probabilities for all (x, y) pairs that satisfy $XY = z$.
- **General Formula for Functions:**
  - For n random variables $X_1, X_2, \ldots, X_n$ and a function $Z = g(X_1, X_2, \ldots, X_n)$:
  $$P(Z = z) = \sum_{(x_1,x_2,\ldots,x_n):g(x_1,x_2,\ldots,x_n)=z} P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n).$$
- **Convolution: Sums of Independent Random Variables:**
  - If X and Y are independent and integer-valued:
  $$P(Z = z) = \sum_{x} P(X = x) \cdot P(Y = z - x).$$

– This operation is called a convolution of $P(X)$ and $P(Y)$.

- **Example: Sum of Poisson Random Variables:**
  – If $X \sim \text{Poisson}(\lambda_1)$ and $Y \sim \text{Poisson}(\lambda_2)$ are independent:

$$Z = X + Y \sim \text{Poisson}(\lambda_1 + \lambda_2).$$

- **Conditional Distributions:**
  – For $Z = X + Y$, the conditional distribution of X given $Z = z$ is:

$$P(X = x \mid Z = z) = \frac{P(X = x) \cdot P(Y = z - x)}{P(Z = z)}.$$

  – Example: For independent Poisson variables:

$$X \mid Z = z \sim \text{Binomial}\left(z, \frac{\lambda_1}{\lambda_1 + \lambda_2}\right).$$

- **Functions and Independence:**
  – If X and Y are independent, then $f(X)$ and $g(Y)$ are also independent for any functions f and g.

Simplified Explanation

Key Ideas: - Functions of random variables transform distributions. - The PMF of a new variable is computed by summing over joint PMFs. - Convolution is used to compute sums of independent random variables.

Example: Convolution for Poisson Variables: If $X \sim \text{Poisson}(\lambda_1)$ and $Y \sim \text{Poisson}(\lambda_2)$:

$$Z = X + Y \sim \text{Poisson}(\lambda_1 + \lambda_2).$$

Conclusion

In this lecture, we:

- Derived PMFs for functions of random variables using summation.
- Introduced convolution for sums of independent variables.
- Highlighted applications such as Poisson sums and conditional distributions.

These concepts form the basis for analyzing complex relationships between random variables.