Lecture Summary: Continuous Random Variables - Expected Value

Source: Lecture 4.8.docx

Key Points

- **Expected Value:**
  - Expected value provides the mean or central tendency of a distribution.
  - For continuous random variables, the expected value of a function $g(X)$ is:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx,$$

    where $f_X(x)$ is the PDF of X.
  - Integration is limited to the support of X where $f_X(x) > 0$.

- **Connection to Discrete Case:**
  - In discrete random variables:

$$E[g(X)] = \sum_x g(x) P(X = x).$$

  - In continuous variables, summation is replaced by integration, and the PDF replaces the PMF.

- **Special Cases:**
  - **Mean:** $g(X) = X$.

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx.$$

  - **Variance:** $g(X) = (X - \mu)^2$, where $\mu = E[X]$.

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2.$$

- **Examples:**
  - **Uniform Distribution:**

$$f_X(x) = \frac{1}{b-a}, \quad a \le x \le b.$$

    - Mean: $\mu = \frac{a+b}{2}$. - Variance: $\sigma^2 = \frac{(b-a)^2}{12}$.
  - **Exponential Distribution:**

$$f_X(x) = \lambda e^{-\lambda x}, \quad x \ge 0.$$

    - Mean: $\mu = \frac{1}{\lambda}$. - Variance: $\sigma^2 = \frac{1}{\lambda^2}$.
  - **Normal Distribution:**

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

    - Mean: μ. - Variance: σ^2.

- **Significance of Mean and Variance:**
  - Mean provides the central value.
  - Variance measures the spread around the mean.
  - Useful for comparing distributions and assessing data variability.
- **Practical Insights:**
  - Mean and variance are often easier to estimate than the entire distribution.
  - Understanding these properties helps in identifying the nature of the random variable (e.g., exponential distributions have variance equal to the square of the mean).

Simplified Explanation

Expected Value for Continuous Variables: - Formula:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx.$$

- Examples: - Uniform: $\mu = \frac{a+b}{2}$, $\sigma^2 = \frac{(b-a)^2}{12}$. - Exponential: $\mu = \frac{1}{\lambda}$, $\sigma^2 = \frac{1}{\lambda^2}$.

Why It Matters: - Mean: Central tendency of the distribution. - Variance: Spread or variability around the mean.

Conclusion

In this lecture, we:

- Extended the concept of expected value to continuous random variables.
- Derived formulas for mean and variance.
- Discussed common distributions and their properties.

Expected value and variance are foundational concepts for analyzing and understanding distributions, providing insights into their behavior and characteristics.