Lecture Summary: Independence of Two Random Variables

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Key Points

- **Definition of Independence:**
  - Two random variables X and Y are independent if for all events A related to X and B related to Y:
  $$P(A \cap B) = P(A) \cdot P(B).$$
  - This definition extends the concept of independence from events to random variables by using their respective probability measures.
- **Joint PMF and Independence:**
  - Two random variables X and Y are independent if and only if their joint PMF satisfies:
  $$f_{XY}(t_1, t_2) = f_X(t_1) \cdot f_Y(t_2), \quad \forall t_1, t_2.$$
  - When X and Y are independent, conditioning does not affect their distributions:
  $$f_{X|Y}(t_1 \mid t_2) = f_X(t_1), \quad f_{Y|X}(t_2 \mid t_1) = f_Y(t_2).$$
- **Examples of Independence:**
  - **Example 1: Uniform Joint PMF:**
    * Joint PMF: $f_{XY}(t_1, t_2) = \frac{1}{4}$ for all $t_1, t_2 \in \{0, 1\}$.
    * Marginal PMFs: $f_X(t_1) = f_Y(t_2) = \frac{1}{2}$.
    * Check: $f_{XY}(t_1, t_2) = f_X(t_1) \cdot f_Y(t_2)$.
    * Conclusion: X and Y are independent.
  - **Example 2: Dependent Case:**
    * Joint PMF: Values deviate from the product of marginals (e.g., a zero in the joint PMF but non-zero marginals).
    * Example: $f_{XY}(1, 1) = 0$ but $f_X(1) > 0$ and $f_Y(1) > 0$.
    * Conclusion: X and Y are dependent.
- **Detecting Independence:**
  - To verify independence, compute the joint PMF and compare it to the product of marginals for all possible values of t_1 and t_2.
  - Independence holds if and only if this equality is true for all combinations.
  - Dependency can be proven by finding a single violation of this condition.
- **Practical Example: IPL Powerplay Overs:**
  - Random variables:

    * X: Total runs scored in an over.
    * Y: Number of wickets lost.
  - Independence is unlikely because runs and wickets are typically correlated (e.g., higher wickets often result in lower runs).
  - This needs empirical validation using data.

Simplified Explanation

Independence of Random Variables: X and Y are independent if knowing Y gives no information about X, and vice versa.

Key Formula: Independence means:

$$f_{XY}(t_1, t_2) = f_X(t_1) \cdot f_Y(t_2), \quad \forall t_1, t_2.$$

Example: For $f_{XY}(0,0) = \frac{1}{4}$ and $f_X(0) = \frac{1}{2}$, $f_Y(0) = \frac{1}{2}$:

$$f_{XY}(0,0) = f_X(0) \cdot f_Y(0) = \frac{1}{4}.$$

Thus, X and Y are independent.

Conclusion

In this lecture, we:

- Defined independence for random variables.
- Derived the joint PMF conditions for independence.
- Explored examples to distinguish independent and dependent cases.

Independence is a cornerstone of probabilistic analysis and is critical for simplifying computations and modeling real-world scenarios.