Lecture Summary: Probability Density Function (PDF)

Source: Lecture 4.5.docx

Key Points

- **Motivation for the PDF:**
  - In continuous random variables, the probability mass function (PMF) is replaced by the probability density function (PDF).
  - A PDF describes the "density" of a random variable over an interval rather than specific probabilities at individual points.
- **Definition of PDF:**
  - If X is a continuous random variable with CDF $F_X(x)$, then its PDF $f_X(x)$ is defined as:

    $$f_X(x) = \frac{d}{dx} F_X(x).$$

  - The CDF is obtained from the PDF by integration:

    $$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt.$$

- **Key Properties of PDF:**
  - $f_X(x) \geq 0$ for all x.
  - The total integral of $f_X(x)$ over its support equals 1:

    $$\int_{-\infty}^{\infty} f_X(x)\, dx = 1.$$

  - The PDF may exceed 1, but it represents probability density per unit length, not the probability itself.
- **Using the PDF for Probabilities:**
  - The probability of X falling within an interval $[a, b]$ is given by:

    $$P(a \leq X \leq b) = \int_{a}^{b} f_X(x)\, dx.$$

  - For any specific value c, $P(X = c) = 0$ for continuous random variables.
- **Examples:**
  - **Uniform Distribution:**
    * $f_X(x) = \frac{1}{5}$ for $x \in [0, 5]$, 0 otherwise.

    * Probability $P(1 \leq X \leq 3)$:

$$P(1 \leq X \leq 3) = \int_1^3 \frac{1}{5} \, dx = \frac{2}{5}.$$

- **Triangular Distribution:**
    * $f_X(x) = 2x$ for $x \in [0, 1]$, 0 otherwise.
    * Verify PDF: Check that $\int_0^1 2x \, dx = 1$.
    * Probability $P(0.1 \leq X \leq 0.3)$:

$$P(0.1 \leq X \leq 0.3) = \int_{0.1}^{0.3} 2x \, dx = \left[x^2\right]_{0.1}^{0.3} = 0.09 - 0.01 = 0.08.$$

- **Importance of PDF Over CDF:**
    - While the CDF describes cumulative probabilities, the PDF provides a direct view of the distribution's shape and density.
    - Example: For a bell curve (normal distribution), the PDF highlights the peak around the mean and symmetry, which is harder to deduce from the CDF.

Simplified Explanation

What is a PDF? The PDF describes the "density" of a random variable across its range. It replaces the PMF for continuous variables, focusing on intervals instead of specific points.

Key Formulae: - Probability over an interval:

$$P(a \leq X \leq b) = \int_a^b f_X(x) \, dx.$$

- CDF to PDF:

$$f_X(x) = \frac{d}{dx} F_X(x).$$

- PDF to CDF:

$$F_X(x) = \int_{-\infty}^{x} f_X(t) \, dt.$$

Examples: 1. Uniform: $f_X(x) = \frac{1}{5}$ for $x \in [0, 5]$. 2. Triangular: $f_X(x) = 2x$ for $x \in [0, 1]$.

Why Use PDFs? - PDFs offer a clear picture of distribution and density, enabling straightforward calculation of probabilities over intervals.

Conclusion

In this lecture, we:

- Defined the PDF and its relation to the CDF.
- Explored properties and practical use of PDFs for probability calculations.
- Illustrated examples of common distributions.

The PDF is an essential tool in continuous probability, offering an intuitive and powerful way to analyze distributions.