Lecture Summary: Marginal Densities and Independence

Source: Lecture 5.4.docx

Key Points

- **Marginal Densities:**
  - For two jointly continuous random variables X and Y with joint density $f_{X,Y}(x,y)$:
  $$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx.$$
  - Marginal densities describe individual distributions of X and Y, integrating out the other variable.
  - Visualization: Imagine a 3D density function where slicing along one axis yields a 2D marginal density function.
- **Independence:**
  - X and Y are independent if and only if:
  $$f_{X,Y}(x,y) = f_X(x)f_Y(y).$$
  - Independence implies that the joint density is the product of the marginals, with no dependency between variables.
- **Uniform Example:**
  - Joint density:
  $$f_{X,Y}(x,y) = \begin{cases} 1, & 0 \leq x, y \leq 1, \\ 0, & \text{otherwise}. \end{cases}$$
  - Marginal densities:
  $$f_X(x) = \int_0^1 1\,dy = 1, \quad f_Y(y) = \int_0^1 1\,dx = 1.$$
  - Both $f_X(x)$ and $f_Y(y)$ are uniform on $[0,1]$.
  - Marginal densities alone do not uniquely determine the joint density.
- **Non-Uniform Example:**
  - Joint density:
  $$f_{X,Y}(x,y) = \begin{cases} 2, & 0 \leq x \leq y \leq 1, \\ 0, & \text{otherwise}. \end{cases}$$
  - Marginal densities:
  $$f_X(x) = \int_x^1 2\,dy = 2(1-x), \quad f_Y(y) = \int_0^y 2\,dx = 2y.$$

  - This example illustrates non-uniform marginals and non-rectangular support regions.

- **Triangular Support Example:**
  - Joint density:
$$f_{X,Y}(x,y) = \begin{cases} 1, & 0 \le x \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$
  - Marginal densities:
$$f_X(x) = \int_x^1 1\,dy = 1 - x, \quad f_Y(y) = \int_0^y 1\,dx = y.$$
  - Triangular regions illustrate how support impacts marginal densities.

- **Key Insights:**
  - Marginal densities can be calculated from joint densities, but the reverse is not unique.
  - Joint densities must be carefully modeled as different joint densities can yield the same marginals.

Simplified Explanation

Marginal Densities: Describe the individual distributions of X and Y by integrating the joint density.

Independence: X and Y are independent if their joint density equals the product of their marginals.

Example: 1. Uniform on $[0,1]^2$: Marginals are uniform, but marginals alone don't define the joint density. 2. Non-uniform regions: Triangular support changes marginals.

Key Idea: Marginals describe individual variables, but the joint density shows how they are connected.

Conclusion

In this lecture, we:

- Defined marginal densities and their computation.
- Discussed independence for continuous random variables.
- Highlighted examples showing how joint densities impact marginals.

Understanding marginal densities and independence is essential for analyzing relationships between continuous random variables.