Lecture Summary: Common Continuous Distributions

Source: Lecture 4.6.docx

Key Points

- **Introduction:**
  - This lecture introduces three common continuous distributions: Uniform, Exponential, and Normal.
  - These distributions are widely used in modeling practical scenarios and are essential to understand in probability theory.
- **Uniform Distribution:**
  - A random variable X is uniformly distributed on $[a, b]$ if its PDF is:

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b, \\ 0, & \text{otherwise.} \end{cases}$$

  - Properties:
    * The PDF is flat, indicating equal probability density over $[a, b]$.
    * CDF:

$$F_X(x) = \begin{cases} 0, & x < a, \\ \frac{x-a}{b-a}, & a \leq x \leq b, \\ 1, & x > b. \end{cases}$$

    * Probabilities are computed as:

$$P(a_1 \leq X \leq b_1) = \frac{b_1 - a_1}{b - a}.$$

  - Example:
    * $X \sim \text{Uniform}[-10, 10]$, PDF is $\frac{1}{20}$ for $-10 \leq X \leq 10$.
    * Probability of $-3 \leq X \leq 2$:

$$P(-3 \leq X \leq 2) = \frac{5}{20} = \frac{1}{4}.$$

- **Exponential Distribution:**
  - A random variable X follows an exponential distribution with parameter $\lambda > 0$ if its PDF is:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

  - Properties:
    * The PDF decays exponentially, with most probability density near $x = 0$.

* CDF:

$$F_X(x) = \begin{cases} 0, & x < 0, \\ 1 - e^{-\lambda x}, & x \geq 0. \end{cases}$$

- Example:
  * $X \sim \text{Exponential}(\lambda = 2)$.
  * Probability of $5 \leq X \leq 7$:

$$P(5 \leq X \leq 7) = \int_5^7 2e^{-2x}\, dx = e^{-10} - e^{-14}.$$

- Memoryless Property:

$$P(X > s + t \mid X > t) = P(X > s).$$

- **Normal (Gaussian) Distribution:**
  - A random variable X follows a normal distribution with mean μ and variance σ^2 if its PDF is:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

  - Properties:
    * Symmetric bell-shaped curve centered at μ.
    * Support: $(-\infty, \infty)$.
    * CDF does not have a closed form, requiring numerical methods or tables.
  - Standard Normal Distribution:

$$Z = \frac{X - \mu}{\sigma} \sim \text{Normal}(0, 1).$$

  - Example:
    * $X \sim \text{Normal}(\mu = 2, \sigma^2 = 5)$.
    * Probability of $X < 10$:

$$Z = \frac{10 - 2}{\sqrt{5}} \implies P(X < 10) = \Phi(3.58),$$

    where Φ is the standard normal CDF.

Simplified Explanation

Three Key Distributions: 1. Uniform: Equal density over an interval. 2. Exponential: Models waiting times; decays exponentially. 3. Normal: Bell-shaped curve; central to many natural phenomena.

Applications: - Uniform: Modeling equally likely outcomes (e.g., random sampling). - Exponential: Time until an event (e.g., bus arrival). - Normal: Modeling data with natural variability (e.g., heights).

Conclusion

In this lecture, we:

- Introduced the Uniform, Exponential, and Normal distributions.
- Defined their PDFs, CDFs, and key properties.
- Highlighted practical examples and computation techniques.

These distributions are fundamental in probability, appearing frequently in both theory and applications.