Lecture Summary: Evaluation of Maximum Likelihood Estimators (MLE)

Lecture: 9.7 - Evaluation of Maximum Likelihood Estimators (MLE)

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Key Points

- **MLE for Poisson Distribution:**
  - PMF of Poisson:
  $$f_X(x;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x \in \{0, 1, 2, \dots\}.$$
  - Likelihood for n samples:
  $$L(\lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}.$$
  - Simplification (ignoring constants not dependent on λ):
  $$L(\lambda) \propto e^{-n\lambda}\lambda^{\sum_{i=1}^{n} x_i}.$$
  - Log-likelihood:
  $$\log L(\lambda) = -n\lambda + \left(\sum_{i=1}^{n} x_i\right)\log\lambda.$$
  - MLE for λ:
  $$\hat{\lambda}_{\text{MLE}} = \frac{\sum_{i=1}^{n} x_i}{n},$$
  which matches the sample mean and also agrees with the Method of Moments Estimator (MME).
- **MLE for Normal Distribution:**
  - PDF of Normal:
  $$f_X(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$
  - Likelihood for n samples:
  $$L(\mu,\sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}.$$
  - Log-likelihood (ignoring constants):
  $$\log L(\mu,\sigma^2) = -\frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2.$$

– MLE for μ:

$$\hat{\mu}_{\text{MLE}} = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}.$$

– MLE for σ^2:

$$\hat{\sigma}^2_{\text{MLE}} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2,$$

which differs slightly from the sample variance formula (division by $n-1$).

- **General Observations on MLE:**
  - **Advantages:**
    * Provides estimators that maximize the likelihood of observed data.
    * Often agrees with MME for common distributions (e.g., Poisson, Normal).
    * Consistent and asymptotically normal.
  - **Challenges:**
    * Requires knowledge of the underlying distribution.
    * Involves calculus (differentiation and solving equations) which can be non-trivial for complex models.

Simplified Explanation

Key Idea: MLE finds parameter values that make the observed data most likely, using the likelihood function.

Examples: - For Poisson, $\hat{\lambda}$ = sample mean. - For Normal, $\hat{\mu}$ = sample mean, $\hat{\sigma}^2 = \frac{\text{sum of squared deviations}}{n}$.

Insights: - MLE often aligns with MME but provides a formal likelihood-based foundation. - As sample size grows, MLE becomes more accurate and reliable.

Conclusion

In this lecture, we:

- Applied MLE to Poisson and Normal distributions.
- Discussed the recipe for deriving MLE, including simplifications and maximization steps.
- Highlighted the similarities and differences between MLE and MME.

MLE is a powerful and widely-used method for parameter estimation, balancing theoretical rigor with practical applicability.