Lecture Summary: Functions of a Continuous Random Variable

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Key Points

- **Introduction:**
  - A function of a continuous random variable can generate new random variables by applying transformations.
  - Common examples include scaling, translation, and nonlinear transformations.
- **Methodology:**
  - Start by finding the cumulative distribution function (CDF) of the transformed variable.
  - If differentiable, derive the probability density function (PDF) from the CDF:

$$f_Y(y) = \frac{d}{dy} F_Y(y).$$

- **Scaling Example:**
  - If $X \sim \text{Uniform}[0, 1]$ and $Y = 2X$, then:
    * $Y \sim \text{Uniform}[0, 2]$ with:

$$f_Y(y) = \begin{cases} \frac{1}{2}, & 0 \leq y \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

- **General Formula for Monotonic Functions:**
  - For monotonic functions $g(X)$ with inverse $g^{-1}(y)$, the PDF of $Y = g(X)$ is:

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{|g'(g^{-1}(y))|}.$$

  - Example:
    * If $X \sim \text{Exponential}(\lambda)$ and $Y = X^2$, then:

$$f_Y(y) = \frac{\lambda e^{-\lambda\sqrt{y}}}{2\sqrt{y}}, \quad y > 0.$$

- **Non-Monotonic Functions:**
  - When $g(X)$ is not monotonic, divide the domain into regions where $g(X)$ is monotonic.
  - Example:
    * If $X \sim \text{Uniform}[-3, 1]$ and $Y = X^2$, the PDF of Y must account for the non-monotonicity:

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}}, & 0 \leq y \leq 1, \\ \frac{1}{4\sqrt{y}}, & 1 < y \leq 9. \end{cases}$$

- **Special Case: Affine Transformations:**
  - If $Y = aX + b$:

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right).$$

  - Example:
    - $X \sim \text{Normal}(0, 1)$ and $Y = 2X + 3$, then $Y \sim \text{Normal}(3, 4)$.
- **Mixtures and Discontinuities:**
  - Functions like $g(X) = \max(X, 0)$ can produce distributions that are mixtures of discrete and continuous parts.
  - Example:
    - If $X \sim \text{Uniform}[-3, 1]$ and $Y = \max(X, 0)$, the resulting distribution is:

$$f_Y(y) = \begin{cases} \frac{3}{4}, & y = 0, \\ \frac{1}{4}, & 0 < y \leq 1. \end{cases}$$

- **General Approach:**
  - For any $g(X)$:
    1. Identify the regions of monotonicity.
    2. Compute the CDF by integrating the original PDF over appropriate regions.
    3. Derive the PDF if needed.

Simplified Explanation

Transforming Random Variables: Functions of a random variable, such as scaling or squaring, create new distributions.

Key Tools: - Monotonic functions: Use the formula for PDF transformation. - Non-monotonic functions: Divide into regions and compute probabilities separately.

Examples: 1. Scaling: $Y = 2X$ scales the range and flattens the PDF. 2. Squaring: $Y = X^2$ creates asymmetry in the PDF.

Applications: - Transformations allow modeling of derived quantities (e.g., area, volume, or thresholds).

Conclusion

In this lecture, we:

- Discussed transformations of continuous random variables.
- Introduced methods for deriving PDFs for transformed variables.
- Highlighted cases with monotonic, non-monotonic, and affine transformations.

Understanding transformations is crucial for modeling real-world scenarios involving derived quantities.