Lecture Summary: Minimum and Maximum of Random Variables

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Key Points

- **Introduction to Min and Max Functions:**
  - The minimum and maximum functions, $Z = \min(X, Y)$ and $Z = \max(X, Y)$, are common operations in probability and statistics.
  - These functions arise frequently in practical problems, such as tracking extremes in datasets.
- **PMF of the Minimum:**
  - For two random variables X and Y, the PMF of $Z = \min(X, Y)$ is derived by considering the following cases:
    1. Both X and Y are equal to z.
    2. $X = z$ and $Y > z$.
    3. $Y = z$ and $X > z$.
  - The PMF is given by:

$$P(Z = z) = P(X = z, Y = z) + P(X = z, Y > z) + P(Y = z, X > z).$$

- **PMF of the Maximum:**
  - Similarly, the PMF of $Z = \max(X, Y)$ considers the cases:
    1. Both X and Y are equal to z.
    2. $X = z$ and $Y < z$.
    3. $Y = z$ and $X < z$.
  - The PMF is derived using analogous logic to the minimum case.
- **Example: Minimum of Two Dice Rolls:**
  - Experiment: Roll two dice and let $Z = \min(X, Y)$.
  - For $Z = 1$, all outcomes where either die shows 1 contribute:

$$P(Z = 1) = \frac{11}{36}.$$

  - For $Z = 2$, outcomes where the smaller value is 2:

$$P(Z = 2) = \frac{9}{36}.$$

  - Similar computations for $Z = 3, 4, 5, 6$ yield decreasing probabilities.
- **CDF of Maximum for Independent Random Variables:**

- The cumulative distribution function (CDF) of $Z = \max(X, Y)$ for independent random variables simplifies:
$$F_Z(z) = F_X(z) \cdot F_Y(z).$$
- This arises from independence, as $P(\max(X, Y) \leq z) = P(X \leq z, Y \leq z)$.

- **CDF of Minimum:**
  - For $Z = \min(X, Y)$, the complementary CDF is useful:
  $$P(Z > z) = P(X > z) \cdot P(Y > z),$$
  leading to:
  $$F_Z(z) = 1 - (1 - F_X(z))(1 - F_Y(z)).$$

- **Special Case: Minimum of Two Geometric Distributions:**
  - For $X, Y \sim \text{Geometric}(p)$, independent:
  $$Z = \min(X, Y) \sim \text{Geometric}(2p - p^2).$$
  - The minimum retains a geometric distribution, whereas the maximum does not.

Simplified Explanation

Key Formulas: - PMF of $\min(X, Y)$ involves cases where one variable is smaller than or equal to the other. - For independent variables, $F_{\max}(z) = F_X(z) \cdot F_Y(z)$ and $P(\min(X, Y) > z) = P(X > z) \cdot P(Y > z)$.

Example: For two dice, $P(\min(X, Y) = 1) = \frac{11}{36}$ and $P(\min(X, Y) = 2) = \frac{9}{36}$.

Conclusion

In this lecture, we:

- Derived PMFs for the minimum and maximum of two random variables.
- Explored the role of independence in simplifying CDF calculations.
- Highlighted special cases like geometric distributions.

Min and max functions are essential in practical probability applications and statistical analysis.