Lecture Summary: Conditional Densities

Source: Lecture 5.6.docx

Key Points

- **Definition of Conditional Density:**
  - The conditional density of Y given $X = a$ is defined as:
  $$f_{Y|X}(y|a) = \frac{f_{X,Y}(a,y)}{f_X(a)}, \quad \text{for } f_X(a) > 0.$$
  - This density describes the distribution of Y for a fixed value of $X = a$.
  - Even though $P(X = a) = 0$ for continuous variables, the density is valid and derived as a limiting concept.
- **Properties of Conditional Density:**
  - $f_{Y|X}(y|a) \geq 0$ for all y.
  - The total integral over y equals 1:
  $$\int_{-\infty}^{\infty} f_{Y|X}(y|a)\, dy = 1.$$
  - The joint density can be expressed in terms of the conditional density:
  $$f_{X,Y}(x,y) = f_X(x) f_{Y|X}(y|x).$$
- **Examples:**
  - **Uniform Distribution on $[0,1] \times [0,1]$:**
    * Joint density: $f_{X,Y}(x,y) = 1$ for $0 \leq x, y \leq 1$.
    * Marginals: $f_X(x) = 1$ and $f_Y(y) = 1$.
    * Conditional densities:
    $$f_{Y|X}(y|x) = 1, \quad f_{X|Y}(x|y) = 1, \quad \text{for } 0 \leq x, y \leq 1.$$
  - **Non-Uniform Example:**
    * Joint density: $f_{X,Y}(x,y) = x + y$ for $0 \leq x, y \leq 1$.
    * Marginals:
    $$f_X(x) = \int_0^1 (x+y)\, dy = x + \frac{1}{2}, \quad f_Y(y) = \int_0^1 (x+y)\, dx = y + \frac{1}{2}.$$
    * Conditional density of $Y|X = a$:
    $$f_{Y|X}(y|a) = \frac{a+y}{a+\frac{1}{2}}, \quad 0 \leq y \leq 1.$$

- **Visualizing Conditional Densities:**
  - Conditional densities can be viewed as slices of the joint density at specific values of X or Y.
  - For example, fixing $X = a$ gives the distribution of Y conditional on $X = a$.
- **Applications:**
  - Conditional densities are essential for understanding relationships between variables.
  - Useful in regression, where Y is modeled given X.
  - Enables computation of probabilities in constrained scenarios.

Simplified Explanation

What is a Conditional Density? - Describes how one variable (e.g., Y) behaves given a specific value of another variable (e.g., $X = a$). - Formula:

$$f_{Y|X}(y|a) = \frac{f_{X,Y}(a,y)}{f_X(a)}.$$

Example: For a non-uniform joint density $f_{X,Y}(x,y) = x + y$:

$$f_{Y|X}(y|a) = \frac{a+y}{a+\frac{1}{2}}, \quad 0 \leq y \leq 1.$$

Key Idea: Conditional densities are derived by normalizing slices of the joint density.

Conclusion

In this lecture, we:

- Defined conditional densities for continuous random variables.
- Explored examples and derived properties.
- Highlighted practical applications in probability and statistics.

Understanding conditional densities is crucial for analyzing relationships and dependencies in continuous random variables.