Lecture Summary: Variance and Its Properties

Source: Lecture 3.4.pdf

Key Points

- **Motivation for Variance:**
  - Expected value provides a central measure but doesn't capture the spread of a random variable.
  - Example:
    * $X = 10$ (constant), $Y \in \{9, 11\}$ (equal probabilities), $Z \in \{0, 20\}$ (equal probabilities).
    * $E[X] = E[Y] = E[Z] = 10$, but their spreads differ.
- **Definition of Variance:**
  - Variance measures the spread of a random variable around its mean:
  $$\text{Var}(X) = E[(X - E[X])^2].$$
  - Standard deviation:
  $$\sigma(X) = \sqrt{\text{Var}(X)}.$$
- **Properties of Variance:**
  1. **Non-Negativity:**
  $$\text{Var}(X) \geq 0.$$
  2. **Scaling:**
  $$\text{Var}(aX) = a^2\text{Var}(X).$$
  3. **Translation:**
  $$\text{Var}(X + c) = \text{Var}(X).$$
  4. **Variance of the Sum (Independence):**
  $$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y), \quad \text{if } X \text{ and } Y \text{ are independent.}$$
- **Alternative Formula for Variance:**
  - Variance can also be computed as:
  $$\text{Var}(X) = E[X^2] - (E[X])^2.$$
- **Examples:**
  - **Die Roll:**
    * $X \sim \text{Uniform}\{1, 2, 3, 4, 5, 6\}$.
    * $E[X] = 3.5$, $\text{Var}(X) = \frac{35}{12}$, $\sigma(X) \approx 1.7078$.
  - **Bernoulli Random Variable:**

    - $X \sim \text{Bernoulli}(p)$.
    - $E[X] = p$, $\text{Var}(X) = p(1-p)$.
  - **Binomial Random Variable:**
    - $X \sim \text{Binomial}(n, p)$.
    - $E[X] = np$, $\text{Var}(X) = np(1-p)$.
  - **Poisson Random Variable:**
    - $X \sim \text{Poisson}(\lambda)$.
    - $E[X] = \lambda$, $\text{Var}(X) = \lambda$.

- **Standardization of Random Variables:**
  - A random variable X can be standardized as:

$$Y = \frac{X - E[X]}{\sigma(X)},$$

    making $E[Y] = 0$ and $\text{Var}(Y) = 1$.

- **Existence of Variance:**
  - Variance and expected value may not always exist for certain random variables.
  - Example: $X = 2^n$ with $P(X = 2^n) = 2^{-n-1}$ leads to divergence of $E[X]$.
  - Practical cases typically involve well-behaved random variables with finite mean and variance.

Simplified Explanation

Variance: Variance measures how spread out a random variable's values are around the mean. Formula:

$$\text{Var}(X) = E[(X - E[X])^2].$$

Key Properties: - Scaling: $\text{Var}(aX) = a^2\text{Var}(X)$. - Translation doesn't change variance: $\text{Var}(X + c) = \text{Var}(X)$. - Variance of independent sums: $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$.

Examples: - Die roll: $\text{Var}(X) = \frac{35}{12} \approx 2.916$. - Poisson random variable: $\text{Var}(X) = \lambda$.

Conclusion

In this lecture, we:

- Defined variance and its significance in measuring spread.
- Derived key properties and an alternative formula for variance.
- Standardized random variables for easier comparison.
- Highlighted rare cases where variance may not exist.

Variance complements expected value by providing a deeper understanding of a random variable's behavior.