Lecture Summary: Cumulative Distribution Function (CDF)

Source: Lecture 4.2.docx

Key Points

- **Definition of Cumulative Distribution Function (CDF):**
  - A CDF, denoted as $F_X(x)$, maps any real number x to the interval $[0, 1]$.
  - Formula:
  $$F_X(x) = P(X \leq x).$$
  - The CDF captures the probability that the random variable X takes a value less than or equal to x.
- **Properties of CDF:**
  - $F_X(x)$ is non-decreasing:
  $$\text{If } x_1 \leq x_2, \quad F_X(x_1) \leq F_X(x_2).$$
  - $F_X(x)$ starts at 0 as $x \to -\infty$:
  $$\lim_{x \to -\infty} F_X(x) = 0.$$
  - $F_X(x)$ approaches 1 as $x \to \infty$:
  $$\lim_{x \to \infty} F_X(x) = 1.$$
  - For $a < b$:
  $$P(a < X \leq b) = F_X(b) - F_X(a).$$
  - CDFs for discrete random variables have step-like behavior, while continuous random variables have smooth, continuous CDFs.
- **Example: Bernoulli Random Variable ($X \sim$ Bernoulli(p)):**
  - X takes values 0 with probability $1 - p$ and 1 with probability p.
  - CDF:
  $$F_X(x) = \begin{cases} 0, & x < 0, \\ 1 - p, & 0 \leq x < 1, \\ 1, & x \geq 1. \end{cases}$$
  - Graph: Stepwise increase at $x = 0$ and $x = 1$, reflecting the probabilities.
- **Example: Uniform Discrete Random Variable ($X \sim$ Uniform$\{1, 2, \ldots, 6\}$):**
  - PMF: $P(X = k) = \frac{1}{6}$ for $k = 1, 2, \ldots, 6$.
  - CDF:
  $$F_X(x) = \begin{cases} 0, & x < 1, \\ \frac{k}{6}, & k \leq x < k + 1 \ (k = 1, \ldots, 5), \\ 1, & x \geq 6. \end{cases}$$

- Graph: Stepwise increases with each step size corresponding to $\frac{1}{6}$.

- **Applications of CDF:**
  - Calculating probabilities for intervals:
  $$P(a < X \leq b) = F_X(b) - F_X(a).$$
  - Example:
    * $X \sim \text{Uniform}\{1, \ldots, 100\}$:
    $$P(3 \leq X \leq 10) = F_X(10) - F_X(3) = \frac{10}{100} - \frac{3}{100} = \frac{7}{100}.$$
    * For non-integer values like 3.2 or 10.6, the CDF output corresponds to the nearest integer boundary.
  - Probability for tail events:
  $$P(X > c) = 1 - F_X(c).$$

- **Connecting CDF to PMF:**
  - For discrete random variables:
  $$P(X = x) = F_X(x) - F_X(x^-),$$
  where $F_X(x^-)$ is the left-hand limit of F_X at x.
  - For continuous random variables, the derivative of $F_X(x)$ yields the probability density function (PDF):
  $$f_X(x) = \frac{d}{dx} F_X(x).$$

Simplified Explanation

What is the CDF? The cumulative distribution function describes the probability that a random variable is less than or equal to a given value:
$$F_X(x) = P(X \leq x).$$

Key Features: - Non-decreasing. - Ranges from 0 to 1. - Stepwise for discrete variables, smooth for continuous ones.

Example: For $X \sim \text{Bernoulli}(p)$: - $F_X(0) = 1 - p$. - $F_X(1) = 1$.

Conclusion

In this lecture, we:

- Defined the CDF and its key properties.
- Demonstrated examples with discrete random variables.
- Highlighted the use of CDFs in probability calculations and their relationship to PMFs and PDFs.

CDFs are crucial for understanding and calculating probabilities in both discrete and continuous frameworks.