Lecture Summary: Central Limit Theorem (CLT)

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Key Points

- **Definition of CLT:**
  - The Central Limit Theorem (CLT) states that for iid random variables $X_1, X_2, \ldots, X_n$ with mean μ and variance σ^2:
  $$Y = \frac{\sum_{i=1}^{n}(X_i - \mu)}{\sqrt{n}}$$
  converges in distribution to the standard normal distribution $N(0,1)$ as $n \to \infty$.
  - This result is remarkable because it holds regardless of the original distribution of X_i, provided $\mathbb{E}[X_i] = \mu$ and $\mathrm{Var}(X_i) = \sigma^2$.
- **Moment Generating Function (MGF):**
  - The MGF for a random variable X is defined as:
  $$M_X(\lambda) = \mathbb{E}[e^{\lambda X}].$$
  - Properties:
    * $M_X(\lambda)$ determines the distribution of X.
    * If $M_X(\lambda)$ exists for λ in an interval around 0, it can be expanded as:
    $$M_X(\lambda) = 1 + \lambda\mathbb{E}[X] + \frac{\lambda^2}{2}\mathbb{E}[X^2] + \cdots,$$
    where coefficients correspond to moments of X.
- **CLT and Scaling:**
  - When the sum $S = \sum_{i=1}^{n} X_i$ is scaled by $\sqrt{n}$:
  $$Y = \frac{S - n\mu}{\sqrt{n\sigma^2}},$$
  the resulting distribution approximates $N(0,1)$.
  - Scaling by $\sqrt{n}$ rather than n is crucial for obtaining a meaningful distribution in the limit.
- **Applications of CLT:**
  - **Probability Approximations:**
    * Example: Binomial(n, p)
    $$P(Y - n\mu > \delta n\mu) \approx 1 - F\left(\frac{\delta n\mu}{\sqrt{n\sigma^2}}\right),$$
    where $F(z)$ is the CDF of $N(0,1)$.

- **Continuous Distributions:**
  * Uniform$[-1,1]$ with $\mu = 0$ and $\sigma^2 = \frac{1}{3}$:

$$P\left(\frac{Y}{\sqrt{\frac{n}{3}}} > 0.1\sqrt{n}\right) \approx 1 - F(0.1\sqrt{3n}).$$

- **Comparison with Weak Law of Large Numbers (WLLN):**
  - WLLN:
$$\frac{S}{n} \to \mu \quad \text{in probability as } n \to \infty.$$
  - CLT:
$$\frac{S - n\mu}{\sqrt{n\sigma^2}} \to N(0,1) \quad \text{in distribution as } n \to \infty.$$
  - CLT provides a distributional approximation, while WLLN focuses on convergence to a constant.

Simplified Explanation

Central Limit Theorem: - As sample size increases, the sum (appropriately scaled) of iid random variables approaches a normal distribution, regardless of their original distribution.

Why It Matters: - Simplifies complex probability calculations. - Explains why normal distributions appear frequently in real-world data.

Applications: 1. Approximating probabilities for binomial or uniform distributions. 2. Estimating probabilities in high-dimensional or aggregated data.

Conclusion

In this lecture, we:

- Defined the Central Limit Theorem and its assumptions.
- Discussed the importance of scaling by $\sqrt{n}$ for distributional convergence.
- Applied CLT to practical examples, demonstrating its utility in probability approximations.

The CLT is a cornerstone of probability and statistics, explaining the ubiquity of the normal distribution in diverse contexts.