Lecture Summary: Bounds on Probabilities Using Mean and Variance

Source: Lecture 3.7.pdf

Key Points

- **Overview:**
  - The lecture explores the relationship between a random variable's mean (μ) and variance (σ^2) and the probabilities associated with its values.
  - Mean represents the "center" of a distribution, while variance measures the "spread."
  - Using μ and σ^2, we can establish bounds on probabilities without knowing the exact distribution.
- **Standard Units:**
  - A random variable X can be expressed in terms of standard deviations from the mean:

$$Z = \frac{X - \mu}{\sigma}.$$

  - Standard units help assess how extreme a value of X is relative to its distribution.
  - Example: $Z = 10$ indicates X is 10 standard deviations away from μ, typically an outlier.
- **Markov's Inequality:**
  - For a non-negative random variable X with finite mean μ:

$$P(X \geq c) \leq \frac{\mu}{c}.$$

  - Interpretation: The probability that X exceeds c is bounded by μ/c.
  - Example: If $\mu = 100$, then $P(X \geq 200) \leq 0.5$.
  - Limitation: Requires $X \geq 0$.
- **Chebyshev's Inequality:**
  - Applies to any random variable with finite mean μ and variance σ^2:

$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}.$$

  - Interpretation: The probability that X deviates from μ by $k\sigma$ decreases as k increases.
  - Complementary form:

$$P(|X - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}.$$

  - Example: For $k = 2$, $P(|X - \mu| \geq 2\sigma) \leq 0.25$.
- **Examples of Markov and Chebyshev Inequalities:**

- **Sum of Two Dice Rolls:**
  - $\mu = 7$, $\sigma = 2.42$.
  - $P(|X - \mu| \geq 2\sigma) \leq 0.25$, but actual calculation yields $P(|X - \mu| \geq 2\sigma) = 0.056$.
- **Uniform Distribution on $[1, 100]$:**
  - $\mu = 50.5$, $\sigma \approx 28.9$.
  - $P(|X - \mu| \geq 2\sigma) = 0$, satisfying Chebyshev's bound.

- **Applications:**
  - Assessing significance: Evaluate whether changes (e.g., reduction in accidents) are meaningful based on standard deviation.
  - Bounding probabilities in practical scenarios where distributions are unknown.

Simplified Explanation

Key Inequalities: - **Markov's Inequality:** Provides a bound for non-negative variables:

$$P(X \geq c) \leq \frac{\mu}{c}.$$

- **Chebyshev's Inequality:** Applies to all variables with finite variance:

$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}.$$

Why It Matters: - These bounds help estimate probabilities without requiring the exact distribution. **Example:** For dice rolls ($\mu = 7$, $\sigma = 2.42$):

$$P(|X - 7| \geq 2 \cdot 2.42) \leq 0.25.$$

Conclusion

In this lecture, we:

- Discussed bounds on probabilities using mean and variance.
- Introduced Markov's and Chebyshev's inequalities.
- Applied these concepts to practical examples.

These inequalities are fundamental tools in probability, providing insights into random variable behavior with minimal assumptions.