Lecture Summary: Descriptive Statistics of Normal Samples

Source: Lecture 7.8.pdf

Key Points

- **Introduction to Normal Samples:**
  - Consider $X_1, X_2, \ldots, X_n$ as iid normal random variables with mean μ and variance σ^2.
  - Sample statistics such as mean $\bar{X}$ and variance S^2 are random variables derived from the data.
- **Sample Mean:**
  - Definition:
  $$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$
  - Properties:
    * $\bar{X}$ is normally distributed.
    * Mean of $\bar{X}$:
    $$\mathbb{E}[\bar{X}] = \mu.$$
    * Variance of $\bar{X}$:
    $$\mathrm{Var}(\bar{X}) = \frac{\sigma^2}{n}.$$
- **Sum of Squares:**
  - Definition:
  $$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$
  - For normal samples, S^2 is scaled chi-squared distributed:
  $$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}.$$
  - Intuition:
    * Only $n-1$ terms are independent due to the dependency introduced by $\bar{X}$.
    * The chi-squared distribution arises from the sum of squared normal variables.
- **Joint Distribution of $\bar{X}$ and S^2:**
  - Remarkable result: $\bar{X}$ and S^2 are independent.
  - Joint distribution is the product of their marginal distributions.
- **Special Cases:**

  - For standard normal samples ($\sigma^2 = 1$), the sum of squares is:

$$\sum_{i=1}^{n} X_i^2 \sim \chi_n^2.$$

  - This property is foundational in deriving confidence intervals and hypothesis tests for normal data.

- **Applications:**
  - Statistical inference: Building models assuming normality.
  - Hypothesis testing: Independence of $\bar{X}$ and S^2 simplifies many calculations.

Simplified Explanation

Key Results: 1. Sample mean $\bar{X}$ is normally distributed, with:

$$\mathbb{E}[\bar{X}] = \mu, \quad \mathrm{Var}(\bar{X}) = \frac{\sigma^2}{n}.$$

2. Sample variance S^2 follows a scaled chi-squared distribution:

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2.$$

3. $\bar{X}$ and S^2 are independent.

Why It Matters: - Normal distributions simplify statistical procedures. - Independence of $\bar{X}$ and S^2 aids in creating efficient models.

Conclusion

In this lecture, we:

- Explored properties of sample mean and variance for normal samples.
- Discussed the chi-squared distribution and its role in statistics.
- Highlighted the independence of $\bar{X}$ and S^2, a critical result for inference.

These concepts are foundational for advanced statistical methods, enabling accurate and efficient data analysis.