LECTURE NOTES

ON

MECHANICAL VIBRATIONS
(A70346)

IV B.Tech I Semester (JNTUH-R15)

Prof.VVSH. PRASAD

DEPARTMENT OF MECHANICALENGINEERING

INSTITUTE OF AERONAUTICAL ENGINEERING
(Autonomous)

DUNDIGAL, HYDERABAD-500043

MECHANICAL VIBRATIONS

Introduction: When an elastic body such as, a spring, a beam and a shaft are displaced from the equilibrium position by the application of external forces, and then released, they execute a vibratory motion, due to the elastic or strain energy present in the body. When the body reaches the equilibrium position, the whole of the elastic or stain energy is converted into kinetic energy due to which the body continues to move in the opposite direction. The entire KE is again converted into strain energy due to which the body again returns to the equilibrium position. Hence the vibratory motion is repeated indefinitely.

Oscillatory motion is any pattern of motion where the system under observation moves back and forth across some equilibrium position, but does not necessarily have any particular repeating pattern.

Periodic motion is a specific form of oscillatory motion where the motion pattern repeats itself with a uniform time interval. This uniform time interval is referred to as the *period* and has units of seconds per cycle. The reciprocal of the period is referred to as the *frequency* and has units of cycles per second. This unit of combination has been given a special unit symbol and is referred to as *Hertz* (Hz)

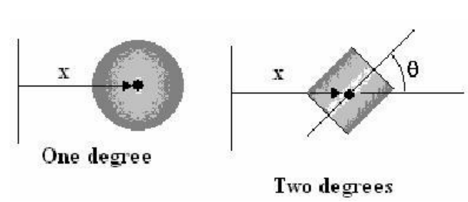
Harmonic motion is a specific form of periodic motion where the motion pattern can be describe by either a sine or cosine. This motion is also sometimes referred to as simple harmonic motion. Because the sine or cosine technically used angles in *radians*, the frequency term expressed in the units radians per seconds (*rad/sec*). This is sometimes referred to as the *circular frequency.* The relationship between the frequency in Hz (cps) and the frequency in *rad/sec* is simply the relationship. *2π rad/sec*.

Natural frequency is the frequency at which an undamped system will tend to oscillate due to initial conditions in the absence of any external excitation. Because there is no damping, the system will oscillate indefinitely.

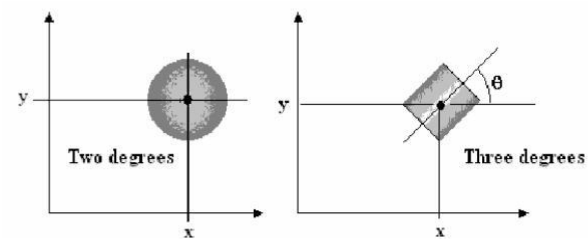
Damped natural frequency is frequency that a damped system will tend to oscillate due to initial conditions in the absence of any external excitation. Because there is damping in the system, the system response will eventually decay to rest.

Resonance is the condition of having an external excitation at the natural frequency of the system. In general, this is undesirable, potentially producing extremely large system response.

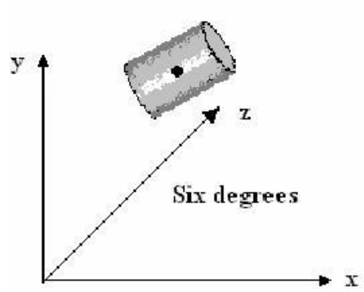
Degrees of freedom: The numbers of degrees of freedom that a body possesses are those necessary to completely define its position and orientation in space. This is useful in several fields of study such as robotics and vibrations. Consider a spherical object that can only be positioned somewhere on the *x* axis.



This needs only one dimension, 'x' to define the position to the centre of gravity so it has one degree of freedom. If the object was a cylinder, we also need an angle 'θ' to define the orientation so it has two degrees of freedom.



Now consider a sphere that can be positioned in Cartesian coordinates anywhere on the z plane. This needs two coordinates 'x' and 'y' to define the position of the centre of gravity so it has two degrees of freedom. A cylinder, however, needs the angle 'θ' also to define its orientation in that plane so it has three degrees of freedom.



In order to completely specify the position and orientation of a cylinder in Cartesian space, we would need three coordinates x, y and z and three angles relative to each angle. This makes six degrees of freedom. A rigid body in space has ($x,y,z,\theta_x\ \theta_y\ \theta_z$).
In the study of free vibrations, we will be constrained to one degree of freedom.

Types of Vibrations:
Free or natural vibrations: A free vibration is one that occurs naturally with no energy being added to the vibrating system. The vibration is started by some input of energy but the vibrations die away with time as the energy is dissipated. In each case, when the body is moved away from the rest position, there is a natural force that tries to return it to its rest position. Free or natural vibrations occur in an elastic system when only the internal restoring forces of the system act upon a body. Since these forces are proportional to the displacement of the body from the equilibrium position, the acceleration of the body is also proportional to the displacement and is always directed towards the equilibrium position, so that the body moves with SHM.

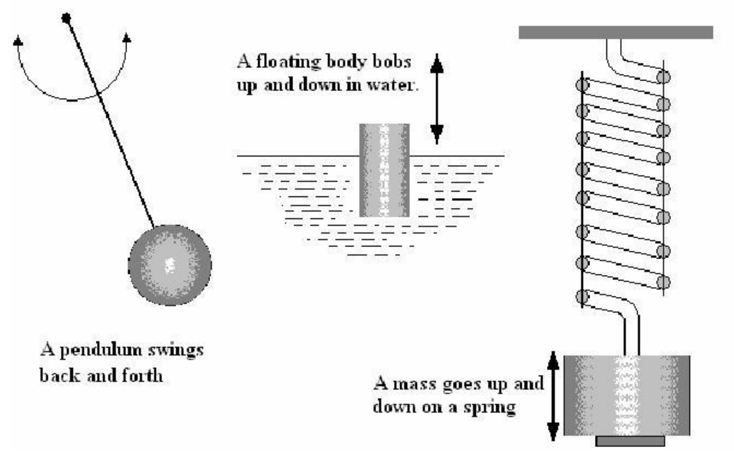


Figure 1. Examples of vibrations with single degree of freedom.

Note that the mass on the spring could be made to swing like a pendulum as well as bouncing up and down and this would be a vibration with two degrees of freedom. The number of degrees of freedom of the system is the number of different modes of vibration which the system may posses.

The motion that all these examples perform is called SIMPLE HARMONIC MOTION (S.H.M.). This motion is characterized by the fact that when the displacement is plotted against time, the resulting graph is basically sinusoidal. Displacement can be linear (e.g. the distance moved by the mass on the spring) or angular (e.g. the angle moved by the simple pendulum). Although we are studying natural vibrations, it will help us understand S.H.M. if we study a forced vibration produced by a mechanism such as the Scotch Yoke.

Simple Harmonic Motion

The wheel revolves at ω radians/sec and the pin forces the yoke to move up and down. The pin slides in the slot and Point *P* on the yoke oscillates up and down as it is constrained to move only in the vertical direction by the hole through which it slides. The motion of point *P* is simple harmonic motion. Point *P* moves up and down so at any moment it has a displacement *x*, velocity *v* and an acceleration *a*.

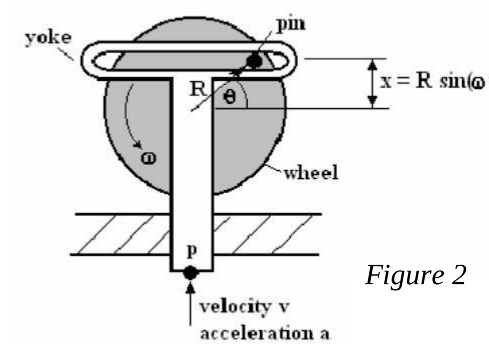
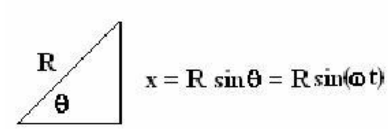


Figure 2

The pin is located at radius *R* from the centre of the wheel. The vertical displacement of the pin from the horizontal centre line at any time is *x*. This is also the displacement of point *P*. The yoke reaches a maximum displacement equal to *R* when the pin is at the top and –*R* when the pin is at the bottom.



This is the amplitude of the oscillation. If the wheel rotates at ω radian/sec then after time *t* seconds the angle rotated is $\theta = \omega t$ radians. From the right angle triangle we find $x = R\,Sin(\omega t)$ and the graph of $x - \theta$ is shown on *figure 3a*.

Velocity is the rate of change of distance with time. The plot is also shown on figure 3*a*. $v = dx/dt = \omega R\,Cos(\omega t)$.

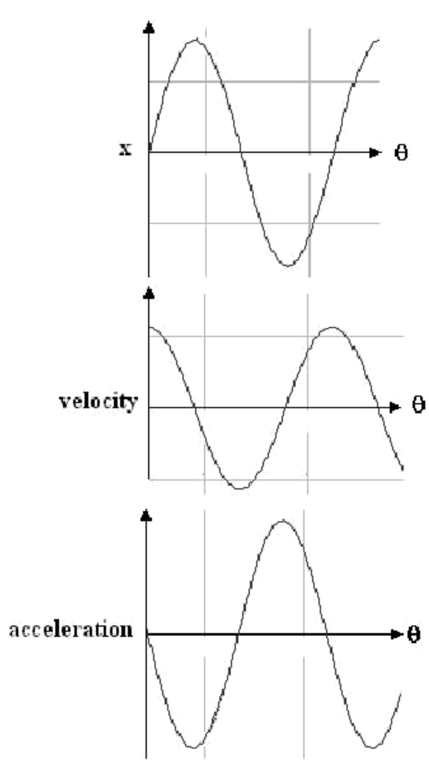
The maximum velocity or amplitude is ωR and this occurs as the pin passes through the horizontal position and is plus on the way up and minus on the way down. This makes sense since the tangential velocity of a point moving in a circle is $v = \omega R$ and at the horizontal point they are the same thing.

Acceleration is the rate of change of velocity with time. The plot is also shown on *figure 3a*. $a = dv/dt = -\omega^2 R\,Sin(-\omega^2 R)$

The amplitude is $\omega^2 R$ and this is positive at the bottom and minus at the top (when the yoke is about to change direction)

Since $R\,Sin(\omega R) = x$; then substituting *x* we find $a = -\omega^2 x$

This is the usual definition of S.H.M. The equation tells us that any body that performs sinusoidal motion must have an acceleration that is directly proportional to the displacement and is always directed to the point of zero displacement. The constant of proportionality is ω^2. Any vibrating body that has a motion that can be described in this way must vibrate with S.H.M. and have the same equations for displacement, velocity and acceleration.



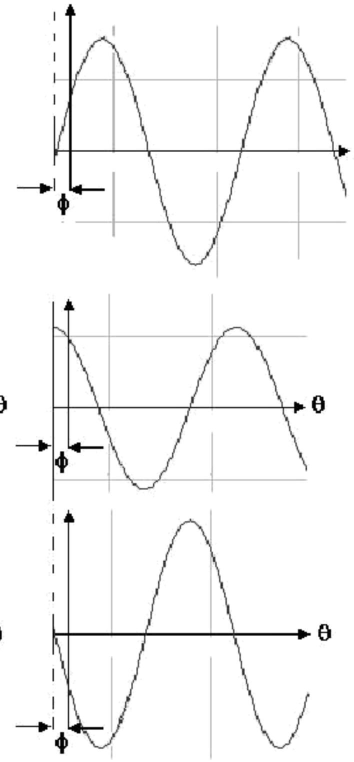


FIGURE 3a *FIGURE 3b*

Angular Frequency, Frequency and Periodic time

ω is the angular velocity of the wheel but in any vibration such as the mass on the spring, it is called the angular frequency as no physical wheel exists.

The frequency of the wheel in revolutions/second is equivalent to the frequency of the vibration. If the wheel rotates at 2 rev/s the time of one revolution is ½ seconds. If the wheel rotates at 5 rev/s the time of one revolution is $^1/_5$ second. If it rotates at f rev/s the time of one revolution is $^1/_f$. This formula is important and gives the periodic time.

Periodic Time T = time needed to perform one cycle.
f is the frequency or number of cycles per second.

It follows that: $T = {^1/_f}$ and $f = {^1/_T}$

Each cycle of an oscillation is equivalent to one rotation of the wheel and 1 revolution is an angle of 2π radians.

When $\theta = 2\pi$ and $t = T$.

It follows that since $\theta = \omega t$; then $2\pi = \omega T$

Rearrange and $\theta = {^{2\pi}/_T}$. Substituting $T = {^1/_f}$, then $\omega = 2\pi f$

Equations of S.H.M.

Consider the three equations derived earlier.

Displacement $x = R\ Sin(\omega t)$.

Velocity $v = dx/dt = \omega R\ Cos(\omega t)$ and Acceleration $a = dv/dt = -\omega^2 R\ Sin(\omega t)$

The plots of *x, v* and *a* against angle θ are shown on figure 3*a*. In the analysis so far made, we measured angle θ from the horizontal position and arbitrarily decided that the time was zero at this point.

Suppose we start the timing after the angle has reached a value of φ from this point. In these cases, φ is called the phase angle. The resulting equations for displacement, velocity and acceleration are then as follows.

$x = R$ D

$v = dx/dt = \omega R\ Cos(\omega t$ i

$+ \varphi)$. $a = dv/dt = -\omega^2 R$ s

placement

Velocity

Acceleration

The plots of *x, v* and *a* are the same but the vertical axis is displaced by φ as shown on figure 3*b*. A point to note on figure 3*a and* 3*b* is that the velocity graph is shifted ¼ cycle (90^o) to the left and the acceleration graph is shifted a further ¼ cycle making it ½ cycle out of phase with *x*.

Forced vibrations: When the body vibrates under the influence of external force, then the body is said to be under forced vibrations. The external force, applied to the body is a periodic disturbing force created by unbalance. The vibrations have the same frequency as the applied force.
(Note: When the frequency of external force is same as that of the natural vibrations, resonance takes place)

Damped vibrations: When there is a reduction in amplitude over every cycle of vibration, the motion is said to be damped vibration. This is due to the fact that a certain amount of energy possessed by the vibrating system is always dissipated in overcoming frictional resistance to the motion.

Types of free vibrations:

Linear / Longitudinal vibrations: When the disc is displaced vertically downwards by an external force and released as shown in the *figure 4*, all the particles of the rod and disc move parallel to the axis of shaft. The rod is elongated and shortened alternately and thus the tensile and compressive stresses are induced alternately in the rod. The vibration occurs is know as *Linear/Longitudinal vibrations*.

Transverse vibrations: When the rod is displaced in the transverse direction by an external force and released as shown in the *figure 5*, all the particles of rod and disc move approximately perpendicular to the axis of the rod. The shaft is straight and bends alternately inducing bending stresses in the rod. The vibration occurs is know as *transverse vibrations*.

Torsional vibrations: When the rod is twisted about its axis by an external force and released as shown in the *figure 6*, all the particles of the rod and disc move in a circle about the axis of the rod. The rod is subjected to twist and torsional shear stress is induced. The vibration occurs is known as *torsional vibrations.*

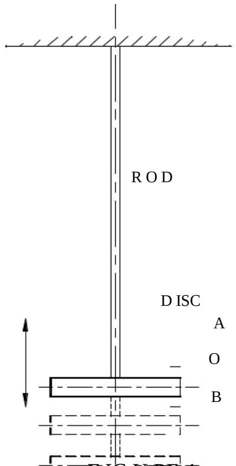


FIGURE 4

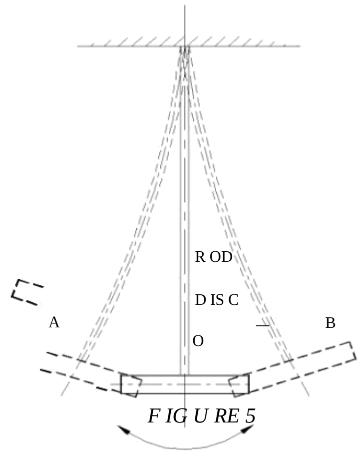


FIGURE 5

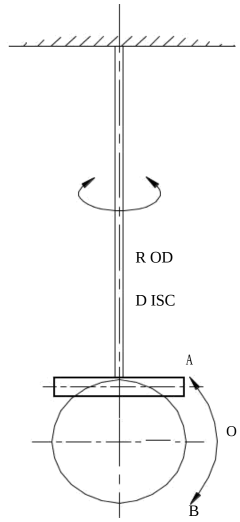
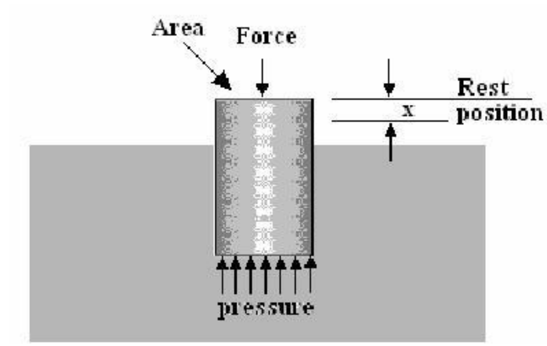


FIGURE 6

***Oscillation of a floating body*:**



You may have observed that some bodies floating in water bob up and down. This is another example of simple harmonic motion and the restoring force in this case is **buoyancy**.

Consider a floating body of mass M kg. Initially it is at rest and all the forces acting on it add up to zero. Suppose a force F is applied to the top to push it down a distance x. The applied force F must overcome this buoyancy force and also overcome the inertia of the body.

Buoyancy force:

The pressure on the bottom increases by $\Delta p = \rho g x$.

The buoyancy force pushing it up on the bottom is F_b and this increases by $\Delta p A$.

Substitute for Δp and $F_b = \rho g x A$

Inertia force:

The inertia force acting on the body is $F_i = M a$

Balance of forces:

The applied force must be $F = F_i + F_b$ -this must be zero if the vibration is free. $0 = Ma + \rho g x A$

$$a = -\frac{\rho A g}{M} x$$

This shows that the acceleration is directly proportional to displacement and is always directed towards the rest position so the motion must be simple harmonic and the constant of proportionality must be the angular frequency squared.

$$\omega^2 = \frac{\rho A g}{M}$$

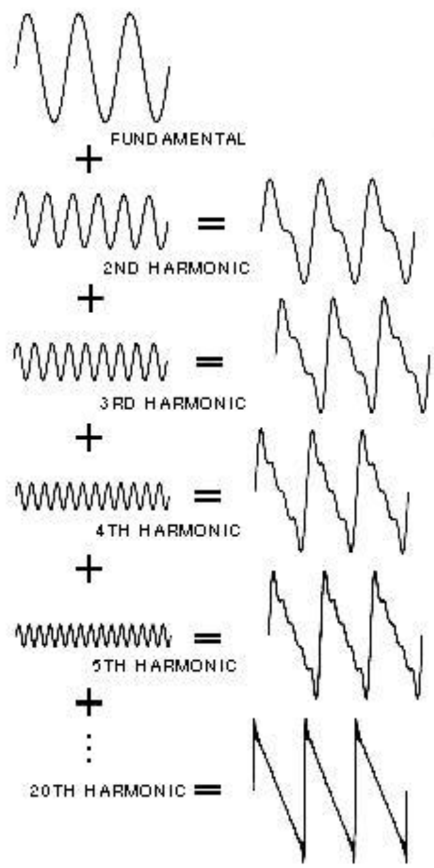
$$\omega = \sqrt{\frac{\rho A g}{M}}$$

$$f_n = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{\rho A g}{M}}$$

Example: A cylindrical rod is 80 mm diameter and has a mass of 5 kg. It floats vertically in water of density 1036 kg/m3. Calculate the frequency at which it bobs up and down. (Ans. 0.508 Hz)

Principal of super position:

The principal of super position means that, when TWO or more waves meet, the wave can be added or subtracted.



Two waveforms combine in a manner, which simply adds their respective Amplitudes linearly at every point in time. Thus, a complex SPECTRUM can be built by mixing together different Waves of various amplitudes.

The principle of superposition may be applied to waves whenever two (or more) waves traveling through the same medium at the same time. The waves pass through each other without being disturbed. The net displacement of the medium at any point in space or time, is simply the sum of the individual wave dispacements.

General equation of physical systems is:

$m\ddot{x} + c\dot{x} + kx = F(t)$ - This equation is for a linear system, the inertia, damping and spring force are linear function $\ddot{x}$, $\dot{x}$ *and* x respectively. This is not true case of non-linear systems.

$m\ddot{x} + \varphi(\dot{x}) + f(x) = F(t)$ - Damping and spring force are not linear functions of $\dot{x}$ *and* x

Mathematically for linear systems, if x_1 is a solution of;

$$m\ddot{x} + c\dot{x} + kx = F_1(t)$$

and x_2 is a solution of;

$$m\ddot{x} + c\dot{x} + kx = F_2(t)$$

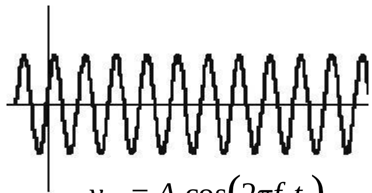
then $(x_1 + x_2)$ is a solution of;

$$m\ddot{x} + c\dot{x} + kx = F_1(t) + F_2(t)$$

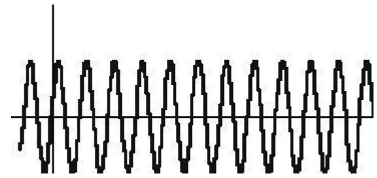
Law of superposition does not hold good for non-linear systems.

If more than one wave is traveling through the medium: The resulting net wave is given by the *Superposition Principle given by the sum of the individual waveforms"*

Beats: When two harmonic motions occur with the same amplitude '*A*' at different frequency is added together a phenomenon called "beating" occurs.



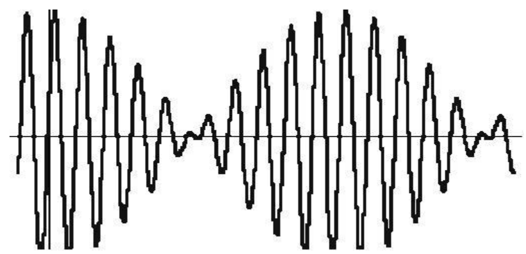
$y_1 = A\cos(2\pi f_1 t)$



$y_2 = A\cos(2\pi f_2 t)$

The resulting motion is:

$$y = (y_1 + y_2) = A[\cos(2\pi f_1 t) + \cos(2\pi f_2 t)]$$



with trigonometric manipulation, the above equation can be written as:

$$y = 2A\cos 2\pi\frac{f_1 - f_2}{2}t \times \cos 2\pi\frac{f_1 + f_2}{2}t$$

The resultant waveform can be thought of as a wave with frequency $f_{ave} = (f_1 + f_2)/2$ which is constrained by an envelope with a frequency of $f_b = |f_1 - f_2|$. The envelope frequency is called the beat frequency. The reason for the name is apparent if you listen to the phenomenon using sound waves.

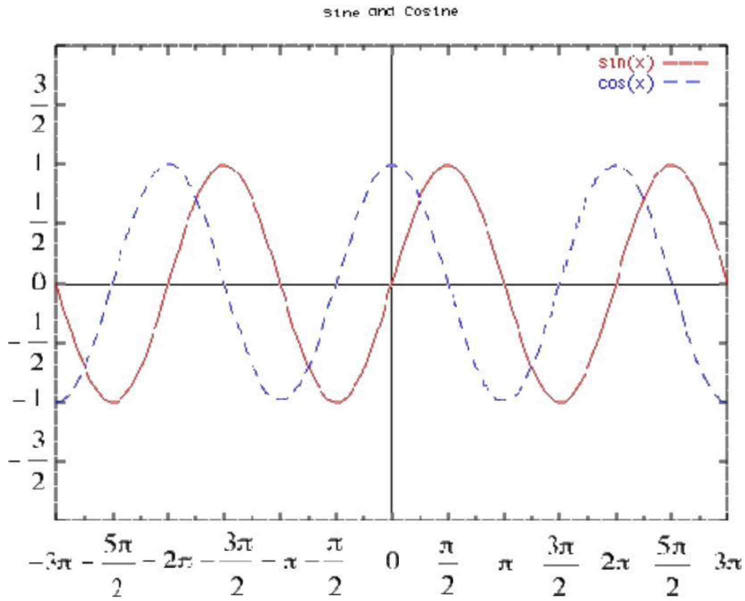
(Beats are often used to tune instruments. The desired frequency is compared to the frequency of the instrument. If a beat frequency is heard the instrument is "out of tune". The higher the beat frequency the more "out of tune" the instrument is.)

Fourier series: decomposes any periodic function or periodic signal into the sum of a (possibly infinite) set of simple oscillating functions, namely sines and cosines (or complex exponentials).

Fourier series were introduced by Joseph Fourier (1768–1830) for the purpose of solving the heat equation in a metal plate.

The Fourier series has many such applications in electrical engineering, vibration analysis, acoustics, optics, signal processing, image processing, quantum mechanics, thin-walled shell theory,etc.

Sin & Cos functions



J. Fourier, developed a periodic function in terms of series of Sines and Cosines.
The vibration results obtained experimentally can be analysed analytically.
If ***x(t)*** is a periodic function with period *T*, the Fourier Series can be written as:

$x(t) = \frac{a_o}{2}$

$+ + a_1 \cos \omega t + a_2 \cos 2\omega t + a_3 \cos 3\omega t +$ $b_1 \sin \omega t + b_2 \sin 2\omega t + b_3 \sin 3\omega t$

+ ....

$$x(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t)$$

Where $\omega = \frac{2\pi}{T}$ is the fundamental frequency and

$a_0, a_1, a_2, a_3 ....b_1, b_2, b_3.........$ are constant coefficients.

The term $(a_1 \cos \omega t + b_1 \sin \omega t)$ is called the Fundamental or First Harmonic.

The term $(a_2 \cos \omega t + b_2 \sin \omega t)$ is called the second Harmonic and so on.

Undamped Free Vibrations:

NATURAL FREQUENCY OF FREE LONGITUDINAL VIBRATION

Equilibrium Method: Consider a body of mass '*m*' suspended from a spring of negligible mass as shown in the *figure 4*.

Let m = Mass of the body

W = Weight of the body = mg

K = Stiffness of the spring

δ = Static deflection of the spring due to 'W'

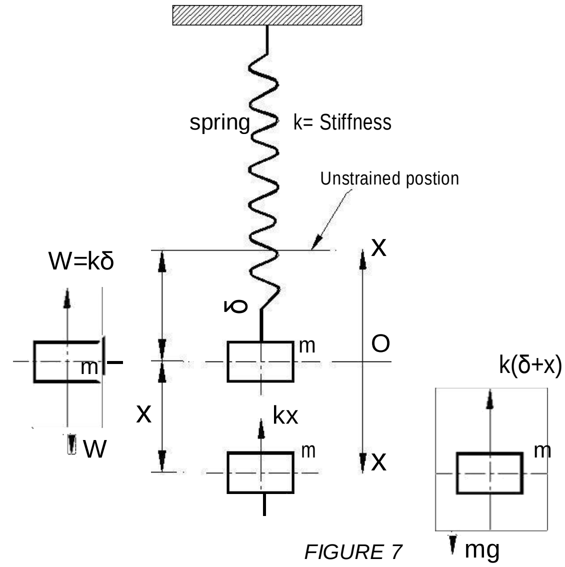


FIGURE 7

By applying an external force, assume the body is displaced vertically by a distance '*x*', from the equilibrium position. On the release of external force, the unbalanced forces and acceleration imparted to the body are related by Newton Second Law of motion.

$\therefore$ The restoring force = $F = -k \cdot x$

(-ve sign indicates, the restoring force 'k.x' is opposite to the direction of the displacement 'x')

By Newton's Law; $F = m \cdot a$

$$\therefore F = -k\,x = m\frac{d^2x}{dt^2}$$

$\therefore$ The differential equation of motion, if a body of mass '*m*' is acted upon by a restoring force '*k*' per unit displacement from the equilibrium position is;

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \qquad \text{– This equation represents SHM}$$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \qquad \omega^2 = \frac{k}{m} \quad \text{– for SHM}$$

The natural period of vibration is $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$ *Sec*

The natural frequency of vibration is $f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ *cycles* / sec

From the *figure 7*; when the spring is strained by an amount of 'δ' due to the weight $W = mg$

$$\delta k = mg$$

Hence $$\frac{k}{m} = \frac{g}{\delta}$$

$$\therefore f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}} \quad Hz \ \ or\ cps$$

Energy method: The equation of motion of a conservative system may be established from energy considerations. If a conservative system set in motion, the mechanical energy in the system is partially kinetic and partially potential. The *KE* is due to the velocity of mass and the *PE* is due to the stain energy of the spring by virtue of its deformation.

Since the system is conservative; and no energy is transmitted to the system and from the system in the free vibrations, the sum of *PE* and *KE* is constant. Both velocity of the mass and deformation of spring are cyclic. Thus, therefore be constant interchange of energy between the mass and the spring.

(KE is maximum, when PE is minimum and PE is maximum, when KE is minimum - so system goes through cyclic motion)

$$KE + PE = Constant$$

$$\frac{d}{dt}\left[KE + PE\right] = 0 \qquad -(1)$$

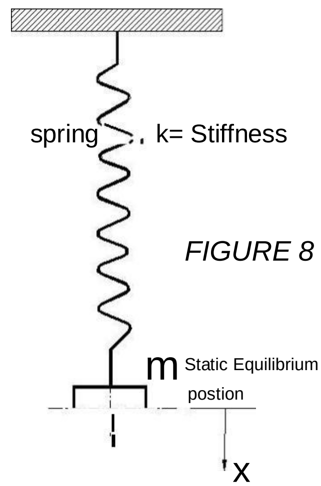


FIGURE 8

We have $KE = \frac{1}{2} m v^2 = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 \qquad -(2)$

Potential energy due to the displacement is equal to the strain energy in the spring, minus the *PE* change in the elevation of the mass.

$$\therefore PE = \int_0^x (Total\ spring\ forece)dx - mg\ dx$$

$$= \int_0^x (mg + kx - mg)dx = \frac{1}{2}kx^2 \qquad -(3)$$

$$PE = \frac{0 + kx}{2}x = \frac{1}{2}kx^2$$

Equation (1) becomes

$$\frac{d}{dt}\left[\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}kx^2\right] = 0$$

$$m\frac{d^2x}{dt^2}\frac{dx}{dt} + kx\frac{dx}{dt} = 0$$

$$\frac{d^2x}{dt^2} + kx = 0 \qquad OR \qquad \frac{dx}{dt} = 0$$

But velocity $\frac{dx}{dt}$ *can be zero for all values of time*.

$$\therefore m\frac{d^2x}{dt^2} + kx = 0 \qquad \left[m\ddot{x} + kx = 0\right]$$

$$\Rightarrow \frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \qquad -\ Equation\ represents\ SHM$$

$\therefore$ *Time period* $= T = 2\pi\sqrt{\frac{m}{k}}$ sec *and*

Natural frequency of vibration $= f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ *cycles / sec*

(The natural frequency is inherent in the system. It is the function of the system parameters 'k' and 'm' and it is independent of the amplitude of oscillation or the manner in which the system is set into motion.)

Rayleigh's Method: The concept is an extension of energy method. We know, there is a constant interchange of energy between the *PE* of the spring and *KE* of the mass, when the system executes cyclic motion. At the static equilibrium position, the *KE* is maximum and *PE* is zero; similarly when the mass reached maximum displacement (amplitude of oscillation), the *PE* is maximum and *KE* is zero (velocity is zero). But due to conservation of energy total energy remains constant.

Assuming the motion executed by the vibration to be simple harmonic, then;

$$x = A\ Sin\omega t$$

x = displacement of the body from the mean position after time 't' sec and A = Maximum displacement from the mean position

$$\dot{x} = A\ Sin\omega t$$

At mean position, t = 0; Velocity is maximum

$$\therefore v_{max} = \left(\frac{dx}{dt}\right)_{max} = \dot{x}_{max} = \omega A$$

$$\therefore \text{Maximum K. E} = \frac{1}{2} m\,\omega^2 A^2$$

$$\text{Maximum P.E} = \frac{1}{2} k\,x_{max}^2 \qquad x_{max} = A$$

$$\therefore \text{Maximum P.E} = \frac{1}{2} kA^2$$

$$\text{We know} \quad (KE)_{max} = (PE)_{max}$$

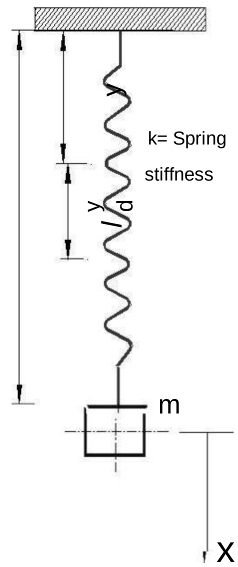
$$m\,\omega^2 = k$$

$$= k^{1/2}$$

$$\because \delta k = m g$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad \Rightarrow \frac{k}{m} = \frac{g}{\delta}$$

1. Determine the natural frequency of the spring-mass system, taking mass of the spring into account.



Let l = Length of the spring
under equilibrium condition

ρ = Mass/unit length of the spring

= Mass of the spring = $\rho \cdot l$

m_s

Consider an elemental length of 'dy' of the

spring at a distance 'y' from support.

∴ Mass of the element = ρ dy

At any instant, the mass 'm' is displaced by a

$$\therefore PE = \frac{1}{2} k x^2$$

K E of the system at this instant,

is the sum of (KE) mass and (KE) spring

$$KE = \frac{1}{2} m \dot{x}^2 + \int_0^l \frac{1}{2} (\rho\, dy) \left(\frac{y}{l}\dot{x}\right)^2$$

$$= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} \frac{\rho \dot{x}^2}{l^2} \int_0^l y^2\, dy$$

$$= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} \frac{\rho \dot{x}^2}{l^2} \frac{l^3}{3}$$

$$= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} \dot{x}^2 \frac{m_s}{3} = \frac{1}{2}\left(m + \frac{m_s}{3}\right)\dot{x}^2$$

$$\frac{1}{2} k x^2 + \frac{1}{2}\left(m + \frac{m_s}{3}\right)\dot{x}^2 = 0$$

Differentiating with respect to 't' ; $\frac{d}{dt}(PE + KE) = 0$

$$k x \dot{x} + \left(m + \frac{m_s}{3}\right)\dot{x}\ddot{x} = 0 \quad \text{- Differential equation}$$

$$\ddot{x} + \frac{k}{m + \frac{m_s}{3}} x = 0$$

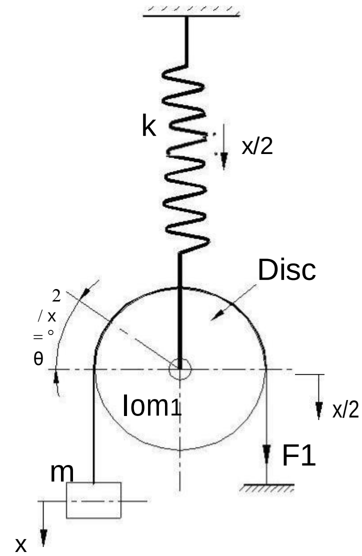
$$\therefore f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m + \frac{m_s}{3}}} \text{ cps} \qquad [\rho l = m_s]$$

OR

$$\therefore f_n = \sqrt{\frac{k}{m + \frac{m_s}{3}}} \text{ radians / sec}$$

We know that PE + KE = Constant

2. Determine the natural frequency of the system shown in figure by Energy and Newton's method.



When mass 'm' moves down a distance 'x' from its equilibrium position, the center of the disc if mass m_1 moves down by x and rotates though and angle θ.

$$\therefore \frac{x}{2} = r\theta$$

$$\Rightarrow \dot{\theta} = \frac{\dot{x}}{2r}$$

$$KE = (KE)_{Tr} + (KE)_{rot}$$

$$= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m_1 \frac{\dot{x}^2}{4} + \frac{1}{2} I_o \dot{\theta}^2$$

$$= \frac{1}{2} m \dot{x}^2 + \frac{1}{8} m_1 \dot{x}^2 + \frac{1}{4} m_1 r^2 \frac{\dot{x}^2}{4r^2}$$

$$= \frac{1}{2} m \dot{x}^2 + \frac{3}{16} m_1 \dot{x}^2$$

$$PE = \frac{1}{2} k \left(\frac{x}{2}\right)^2$$

$$\frac{d}{dt}(KE + PE) = 0$$

$$\Rightarrow m \dot{x}\ddot{x} + \frac{3}{8} m_1 \dot{x}\ddot{x} + \frac{1}{4} k \dot{x} x = 0$$

$$\frac{8m + 3m_1}{8} \ddot{x} + \frac{1}{4} kx = 0$$

$$\therefore f_n = \frac{1}{2\pi}\sqrt{\frac{2k}{8m + 3m_1}} \text{ cps}$$

$$\text{or } f_n = \sqrt{\frac{2k}{8m + 3m_1}} \text{ rad/ sec}$$

Newton's Method: Use x, as co-ordinate

$$\text{or mass } m; \quad m\ddot{x} = -F \qquad -1$$

$$\text{for disc } m_1; \quad m_1 \frac{\ddot{x}}{2} = F + F_1 - \frac{kx}{2} \qquad -2$$

$$I_O \ddot{\theta} = Fr - F_1 r \qquad -3$$

$$\text{Substituting (1) in (2) and (3) and replace } \ddot{\theta} \text{ by } \frac{\ddot{x}}{2r}$$

$$m_1 \frac{\ddot{x}}{2} = -m\ddot{x} + F_1 - \frac{kx}{2} \qquad -4$$

$$I_O \frac{\ddot{x}}{2r} = -m\ddot{x} r - F_1 r \qquad -5$$

$$I_O \frac{\ddot{x}}{2r^2} = -m\ddot{x} - F_1 \qquad -6$$

Adding equations (4) *and* (6)

$$\frac{1}{2} m_1 \ddot{x} + I_O \frac{\ddot{x}}{2r^2} = -2m\ddot{x} - \frac{kx}{2} \qquad \left\{ I_O = \frac{1}{2} m_1 r^2 \right.$$

$$\ddot{x}\left(m_1 + \frac{m_1}{2} + 4m\right) + kx = 0$$

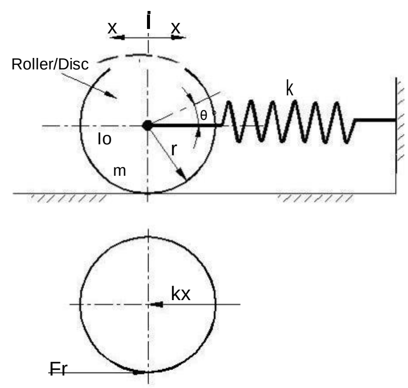
$$f_n = \sqrt{\frac{2k}{3m_1 + 8m}} \ \ rad/\sec$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{2k}{3m_1 + 8m}} \ \ cps \ or \ Hz$$

3. Determine the natural frequency of the system shown in figure by Energy and Newton's method. Assume the cylinder rolls on the surface without slipping.

a) Energy method:

When mass 'm' rotates through an angle θ, the center of the roller move distance 'x'



$$KE = (KE)_{Tr} + (KE)_{rot}$$

$$= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} I_O \dot{\theta}^2$$

$$= \frac{1}{2} m \dot{x}^2 + \frac{1}{2}\left(\frac{1}{2} m r^2\right)\frac{\dot{x}^2}{r^2}$$

$$\therefore KE = \frac{3}{4} m \dot{x}^2$$

$$PE = \frac{1}{2} k x^2$$

$$\frac{d}{dt}(KE + PE) = 0$$

$$\Rightarrow \frac{3}{2} m \ddot{x} + k x = 0$$

$$Natural \ frequency = f_n = \frac{1}{2\pi}\sqrt{\frac{2k}{3m}}$$

Newton's method:

$$m a = \sum F$$

$$m\ddot{x} = -kx + F_r$$

using torque equation $\quad I_O \ddot{\theta} = -F_r \cdot r$

$$F_r = -\frac{m\ddot{x}}{2}$$

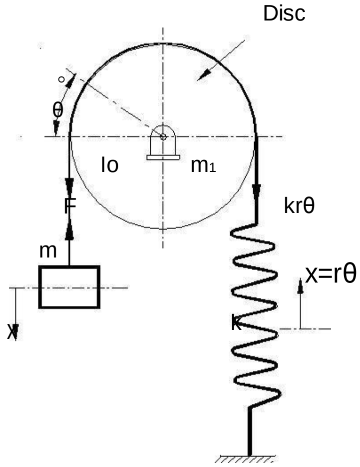
$$m\ddot{x} = -kx - \frac{1}{2} m \ddot{x}$$

$$\frac{3}{2} m\ddot{x} + kx = 0$$

$$\therefore f_n = \frac{1}{2\pi}\sqrt{\frac{2k}{3m}} \ \ cps \quad or \quad f_n = \sqrt{\frac{2k}{3m}} \ \ rad/\sec$$

4. Determine the natural frequency of the system shown in figure by Energy and Newton's method.

Energy Method: Use θ or x as coordinate



$$KE = (KE)_{Tr} + (KE)_{rot}$$

$$= \frac{1}{2} m\dot{x}^2 + \frac{1}{2} I_o \dot{\theta}^2$$

$$= \frac{1}{2} m\dot{x}^2 + \frac{1}{2}\cdot\frac{1}{2} m_1 r^2 \left(\frac{\dot{x}}{r}\right)^2$$

$$\therefore KE = \frac{1}{2} m\dot{x}^2 + \frac{1}{4} m_1 \dot{x}^2$$

$$PE = \frac{1}{2} k x^2$$

$$\frac{d}{dt}(KE + PE) = 0$$

$$\Rightarrow \ddot{x}\left(\frac{1}{2} m + \frac{1}{4} m_1\right) \dot{x} + kx\dot{x} = 0$$

$$\ddot{x} + \frac{k}{\frac{1}{2} m + \frac{1}{4} m_1} x = 0$$

$$\text{Natural frequency} = f_n = \sqrt{\frac{2k}{2m + m_1}} \text{ rad / sec} = \frac{1}{2\pi}\sqrt{\frac{2k}{2m + m_1}} \text{ cps OR } \frac{1}{2\pi}\sqrt{\frac{kr^2}{I_o + mr^2}} \text{ cps or Hz}$$

Newton's Method: Consider motion of the disc with 'θ' as coordinate.

For the mass 'm' : $m\ddot{x} = -F r \quad -(1)$

For the disc : $I_o\ddot{\theta} = -F r - k r^2 \theta \quad -(2)$

Substitute (1) in (2):

$$I_o\ddot{\theta} = -m r \ddot{x} - k r^2 \theta$$

$$I_o\ddot{\theta} = -m r^2 \ddot{\theta} - k r^2 \theta$$

$$(I_o + m r^2)\ddot{\theta} + k r^2 \theta = 0$$

$$\therefore f_n = \sqrt{\frac{k r^2}{I_o + m r^2}} \text{ rad / sec}$$

$$= \frac{1}{2\pi}\sqrt{\frac{k r^2}{I_o + m r^2}} \text{ cps or Hz}$$

$$= \frac{1}{2\pi}\sqrt{\frac{2k}{m_1 + 2m}} \text{ cps} \qquad I_o = \frac{1}{2} m_1 r^2$$

5. Determine the natural frequency of the system shown in figure 5 by Energy and Newton's method. Assume the cylinder rolls on the surface without slipping.

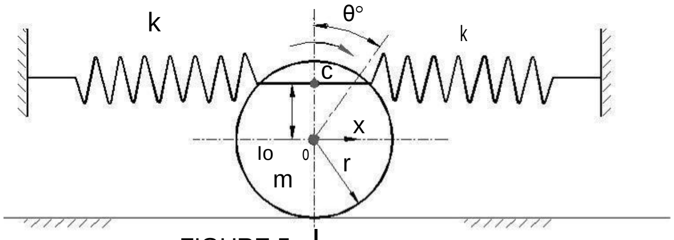


FIGURE 5

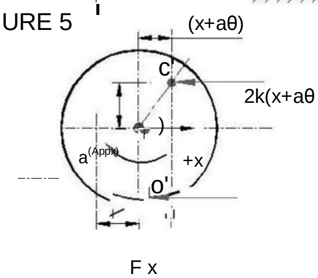


Figure 'a'

Energy Method:

$$KE = (KE)_{Tr} + (KE)_{rot}$$

$$= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} I_o \dot{\theta}^2$$

$$= \frac{1}{2} m r^2 \dot{\theta}^2 + \frac{1}{2}\cdot\frac{1}{2} m r^2 \dot{\theta}^2$$

$$\therefore KE = \frac{3}{4} m r^2 \dot{\theta}^2$$

$$PE = 2\cdot\frac{1}{2} k x^2 = k(r+a)^2 \theta^2$$

$$\frac{d}{dt}(KE + PE) = 0$$

$$\Rightarrow \frac{3}{2} m r^2 \dot{\theta}\ddot{\theta} + 2k(r+a)^2 \theta\dot{\theta} = 0$$

$$3 m r^2 \ddot{\theta} + 4k(r+a)^2 \theta = 0$$

$$\text{Natural frequency} = f_n = \sqrt{\frac{4k(r+a)^2}{3mr^2}}\ \text{rad/sec} = \frac{1}{2\pi}\sqrt{\frac{4k(r+a)^2}{3mr^2}}\ \text{cps or Hz}$$

Newton's Method: Considering combined translation and rotational motion as shown in Figure 'a'.

Hence it must satisfy:

$$m\ddot{x} = \sum \text{Force in x direction}$$

$$m\ddot{x} = -F - 2k(x + a\theta)$$

$$\text{and} \quad I_o\ddot{\theta} = \sum \text{Torque about } '\theta'$$

$$= Fr - 2k(x + a\theta)a$$

$$m r \ddot{\theta} = -F - 2k(r+a)\theta \qquad -(1)$$

$$\text{and} \quad \frac{1}{2} m r^2 \ddot{\theta} = Fr - 2k(r+a)a\theta \qquad -(2)$$

Multiply equation (1) by 2*r* and (2) by 2. Then add

$$3mr^2\ddot{\theta} = -4kr(r+a)\theta - 4k(r+a)a\theta$$

$$3mr^2\ddot{\theta} + 4k[r(r+a) + (r+a)a]\theta = 0$$

$$3mr^2\ddot{\theta} + 4k(r+a)^2\theta = 0$$

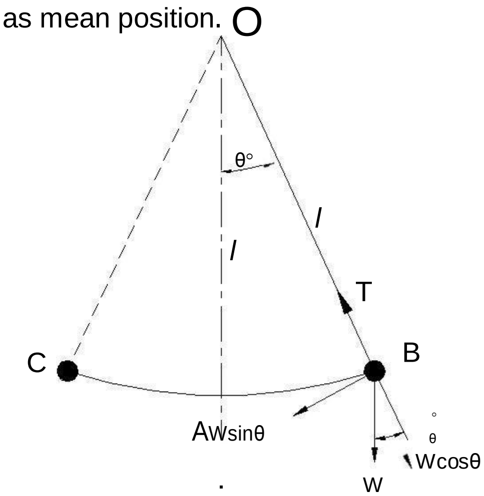
$$\text{Natural frequency} = f_n = \sqrt{\frac{4k(r+a)^2}{3mr^2}}\ \text{rad/sec} = \frac{1}{2\pi}\sqrt{\frac{4k(r+a)^2}{3mr^2}}\ \text{cps or Hz}$$

Refer PPT – For more problems

Oscillation of a simple pendulum:

Figure shows the arrangement of simple pendulum, which consists of a light, inelastic (inextensible), flexible string of length '*l*' with heavy bob of weight *W* (*m*·*g*) suspended at the lower end and the upper end is fixed at 'O'. The bob oscillates freely in a vertical plane.

The pendulum is in equilibrium, when the bob is at 'A'. If the bob is brought at B or C and released, it will start oscillating between B and C with 'A' as mean position.



Oscillation of a simple pendulum

Let θ be a very small angle ($< 4^o$), the bob will have SHM.

Consider the bob at 'B', the forces acting on the bob are:

i) weight of the bob = $W = mg$ ⇒ acting downwards vertically.

ii) tension 'T' in the string

The two components of the weight '*W*'

i) along the string = $W\cos\theta$

ii) normal to the string = $W\sin\theta$

The component W sinθ acting towards 'A' will be unbalanced and will give rise to an acceleration '*a*' in the direction of 'A'.

$$\therefore \text{Acceleration} = a = \frac{\text{Force}}{\text{Mass}} = \frac{W\sin\theta}{m}$$

$$= \frac{mg\sin\theta}{m} = g\sin\theta$$

Since θ is very small; $\sin\theta = \theta$

$$\therefore a = g\,\theta$$

$$= g\,\frac{Arc\ length\ AB}{Radius}$$

$$= g\,\frac{Arc\ AB}{L} \qquad -(1)$$

Acceleration of the body with SHM is given by; [$a = -\omega^2 \cdot$ Distance form centre]

Numerically, $a = \omega^2 \cdot \text{Arc AB}$ -(2)

From (1) and (2)

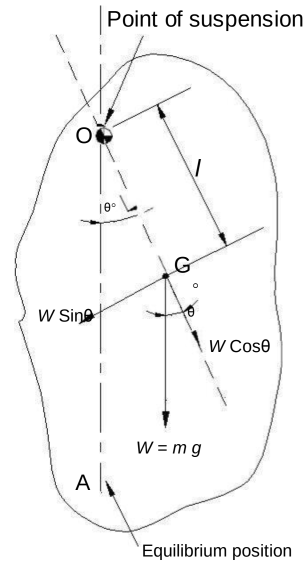
$$\omega = \sqrt{\frac{g}{L}}$$

$$\textit{Time period of oscillation} = T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$$

$$\textit{Beat} = \textit{Beat is half of the oscillation} = \frac{T}{2}$$

Second's pendulum: is defined as that pendulum which has one beat per second. Thus the time period for second's pendulum will be equal to two seconds.

Compound pendulum:



When a rigid body is suspended vertically, and it oscillates with a small amplitude under the effect of force of gravity, the body is known as compound pendulum.

Let W = weight of the pendulum

$= \text{mass} \cdot g = m \cdot g$ N

k_G = radius of gyration

l = distance from point of suspension to 'G' (CG) of the body.

The components of the force W, when the pendulum is given a small angular displacement 'θ' are:

1. $W \cos\theta$ - along the axis of the body.
2. $W \sin\theta$ - along normal to the axis of the body.

The component $W \sin\theta$ acting towards equilibrium position (couple tending to restore) of the pendulum:

$$C = W \sin\theta \cdot l = m\, g\, l \sin\theta$$

Since θ is very small $\sin\theta = \theta$

$$\therefore C = m\, g\, l\, \theta$$

Mass moment of inertia about the axis of suspension 'O':

$$I = I_G + m\, l^2 \quad \text{(parallel axis theorem)}$$

$$= m\,(k_G^2 + l^2)$$

$\therefore$ Angular acceleration of the pendulum $= \alpha = \dfrac{C}{I}$

$$\alpha = \frac{mgl\,\theta}{m\left(k_G^2 + l^2\right)}$$

$$\frac{\theta}{\alpha} = \frac{k_G^2 + l^2}{g\, l}$$

$$\frac{\theta}{\alpha} = \frac{\text{Displacement}}{\text{Acceleration}}$$

We know that time period $= T = 2\pi \sqrt{\dfrac{\text{Displacement}}{\text{Acceleration}}} = 2\pi \sqrt{\dfrac{\theta}{\alpha}} = 2\pi \sqrt{\dfrac{k_G^2 + l^2}{g\, l}}$ sec

Frequency of oscillation $= \dfrac{1}{T} = n = \dfrac{1}{2\pi}\sqrt{\dfrac{g\, l}{k_G^2 + l^2}}$

Comparing the above equation with simple pendulum, the equivalent length of simple pendulum

$$L = \frac{k_G^2 + l^2}{l} = \frac{k_G^2}{l} + l$$

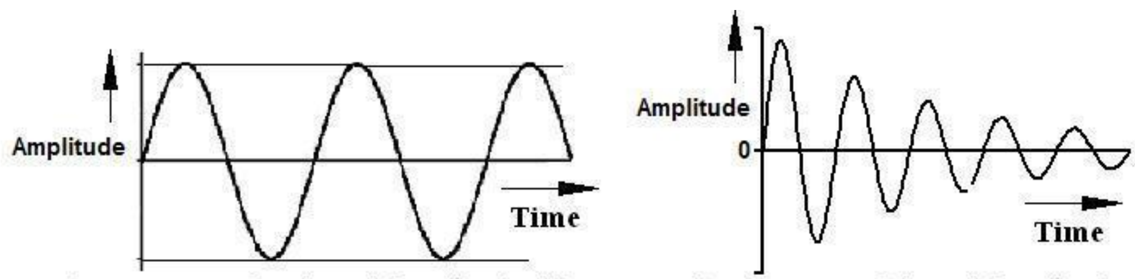
Damped Free Vibrations
Single Degree of Freedom Systems

Introduction:

Damping – dissipation of energy.

For a system to vibrate, it requires energy. During vibration of the system, there will be continuous transformation of energy. Energy will be transformed from potential/strain to kinetic and vice versa.

In case of undamped vibrations, there will not be any dissipation of energy and the system vibrates at constant amplitude. Ie, once excited, the system vibrates at constant amplitude for infinite period of time. But this is a purely hypothetical case. But in an actual vibrating system, energy gets dissipated from the system in different forms and hence the amplitude of vibration gradually dies down. Fig.1 shows typical response curves of undamped and damped free vibrations.



Response curve of undamped free vibration Time   Response curve of damped free vibration

Fig 1

Types o damping:
(i) Viscous damping

In this type of damping, the damping resistance is proportional to the relative velocity between the vibrating system and the surroundings. For this type of damping, the differential equation of the system becomes linear and hence the analysis becomes easier. A schematic representation of viscous damper is shown in Fig.2.

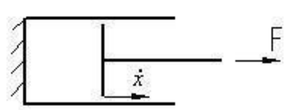


Fig 2

Here, $F \propto \dot{x}$ or $F = c\dot{x}$, where, F is damping resistance, $\dot{x}$ is relative velocity and c is the damping coefficient.

(ii) Dry friction or Coulomb damping

In this type of damping, the damping resistance is independent of rubbing velocity and is practically constant.

(iii) Structural damping

This type of damping is due to the internal friction within the structure of the material, when it is deformed.

Spring-mass-damper system:

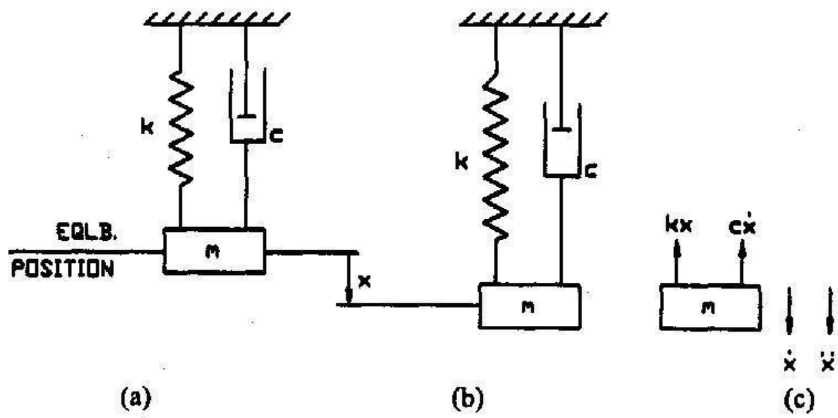


Fig 3

Fig.3 shows the schematic of a simple spring -mass -damper system, where, m is the mass of the system, k is the stiffness of the system and c is the damping coefficient. If x is the displacement of the system, from Newton's second law of motion, it can be written

$$m\ddot{x} = -c\dot{x} - kx$$

Ie $m\ddot{x} + c\dot{x} + kx = 0$ (1)

This is a linear differential equation of the second order and its solution can be written as

$$x = e^{st} \qquad (2)$$

Differentiating (2), $\frac{dx}{dt} = \dot{x} = se^{st}$

$$\frac{d^2x}{dt^2} = \ddot{x} = s^2e^{st}$$

Substituting in (1), $ms^2e^{st} + cse^{st} + ke^{st} = 0$

$$\left(ms^2 + cs + k\right)e^{st} = 0$$

Or $ms^2 + cs + k = 0$ (3)

Equation (3) is called the characteristic equation of the system, which is quadratic in s. The two values of s are given by

$$s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \qquad (4)$$

The general solution for (1) may be written as

$$x = C_1 e^{s_1 t} + C_2 e^{s_2 t} \qquad (5)$$

Where, C1 and C2 are arbitrary constants, which can be determined from the initial conditions.

In equation (4), the values of $s_1 = s_2$, when $\left(\frac{c}{2m}\right)^2 = \frac{k}{m}$

Or, $$\frac{c}{2m} = \sqrt{\frac{k}{m}} = \omega_n \qquad (6)$$

Or $c = 2m\omega_n$, which is the property of the system and is called critical damping coefficient and is represented by c_c.

Ie, critical damping coefficient = $c_c = 2m\omega_n$

The ratio of actual damping coefficient c and critical damping coefficient c_c is called damping factor or damping ratio and is represented by ζ.

$$\text{Ie, } \zeta = \frac{c}{c_c} \qquad (7)$$

In equation (4), $\frac{c}{2m}$ can be written as $\frac{c}{2m} = \frac{c}{c_c} \cdot \frac{c_c}{2m} = \zeta . \omega_n$

Therefore, $$s_{1,2} = -\zeta . \omega_n \pm \sqrt{(\zeta . \omega_n)^2 - \omega_n^2} = \left[-\zeta \pm \sqrt{\zeta^2 - 1}\right]\omega_n \qquad (8)$$

The system can be analyzed for three conditions.

- ii $\zeta > 1$, ie, $c > c_c$, which is called over damped system.
- iii $\zeta = 1$. ie, $c = c_c$, which is called critically damped system.
- iv $\zeta < 1$, ie, $c < c_c$, which is called under damped system.

Depending upon the value of ζ, value of s in equation (8), will be real and unequal, real and equal and complex conjugate respectively.

(i) Analysis of over-damped system ($\zeta > 1$).

In this case, values of s are real and are given by

$$s_1 = \left[-\zeta + \sqrt{\zeta^2 - 1}\right]\omega_n \text{ and } s_2 = \left[-\zeta - \sqrt{\zeta^2 - 1}\right]\omega_n$$

Then, the solution of the differential equation becomes

$$x = C_1 e^{\left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t} + C_2 e^{\left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t} \qquad (9)$$

This is the final solution for an over damped system and the constants C_1 and C_2 are obtained by applying initial conditions. Typical response curve of an over damped system is shown in fig.4. The amplitude decreases exponentially with time and becomes zero at $t = \infty$.

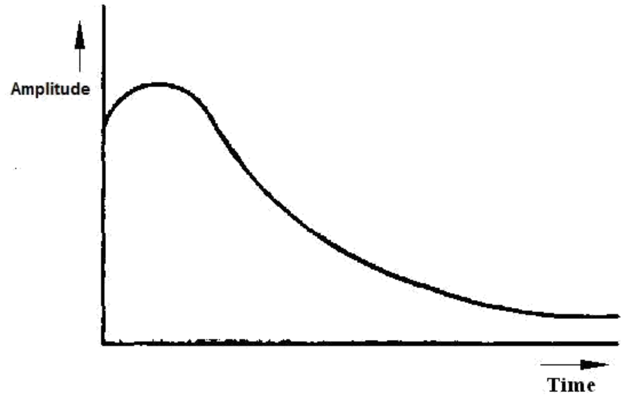


Fig 4 Typical response curve of an over damped system

(ii) Analysis of critically damped system ($\zeta = 1$).

In this case, based on equation (8), $s_1 = s_2 = -\omega_n$

The solution of the differential equation becomes

$$x = C_1 e^{s_1 t} + C_2 t e^{s_2 t}$$

Ie, $x = C_1 e^{-\omega_n t} + C_2 t e^{-\omega_n t}$

Or, $x = (C_1 + C_2 t) e^{-\omega_n t}$ (10)

This is the final solution for the critically damped system and the constants C_1 and C_2 are obtained by applying initial conditions. Typical response curve of the critically damped system is shown in fig.5. In this case, the amplitude decreases at much faster rate compared to over damped system.

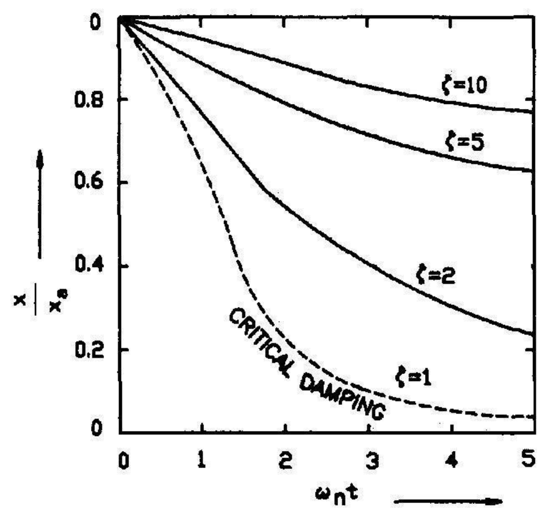


Fig 5 **Displacement-time plots of over-damped and critically damped systems with zero starting velocity**

(iii) Analysis of under damped system ($\zeta < 1$).

In this case, the roots are complex conjugates and are given by $s_1 = \left[-\zeta + j\sqrt{1-\zeta^2}\right]\omega_n$

$$s_2 = \left[-\zeta - j\sqrt{1-\zeta^2}\right]\omega_n$$

The solution of the differential equation becomes

$$x = C_1 e^{\left(-\zeta + j\sqrt{1-\zeta^2}\right)\omega_n t} + C_2 e^{\left(-\zeta - j\sqrt{1-\zeta^2}\right)\omega_n t}$$

This equation can be rewritten as

$$x = e^{-\zeta\omega_n t}\left[C_1 e^{j\sqrt{1-\zeta^2}\,\omega_n t} + C_2 e^{-j\sqrt{1-\zeta^2}\,\omega_n t}\right] \quad (11)$$

Using the relationships

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

Equation (11) can be written as

$$x = e^{-\zeta\omega_n t}\left[C_1\left\{\cos\sqrt{1-\zeta^2}\,\omega_n t + j\sin\sqrt{1-\zeta^2}\,\omega_n t\right\} + C_2\left\{\cos\sqrt{1-\zeta^2}\,\omega_n t - j\sin\sqrt{1-\zeta^2}\,\omega_n t\right\}\right]$$

Or $x = e^{-\zeta\omega_n t}\left[(C_1 + C_2)\left\{\cos\sqrt{1-\zeta^2}\,\omega_n t\right\} + j(C_1 - C_2)\left\{\sin\sqrt{1-\zeta^2}\,\omega_n t\right\}\right]$ (12)

In equation (12), constants (C_1+C_2) and j(C_1-C_2) are real quantities and hence, the equation can also be written as

$$x = e^{-\zeta\omega_n t}\left[A\left\{\cos\sqrt{1-\zeta^2}\,\omega_n t\right\} + B\left\{\sin\sqrt{1-\zeta^2}\,\omega_n t\right\}\right]$$

Or, $x = A_1 e^{-\zeta\omega_n t}\left[\left\{\sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \varphi_1\right)\right\}\right]$ (13)

The above equations represent oscillatory motion and the frequency of this motion is represented by $\omega_d = \sqrt{1-\zeta^2}\,\omega_n$ (14)

Where, ω_d is the damped natural frequency of the system. Constants A_1 and Φ_1 are determined by applying initial conditions. The typical response curve of an under damped system is shown in Fig.6.

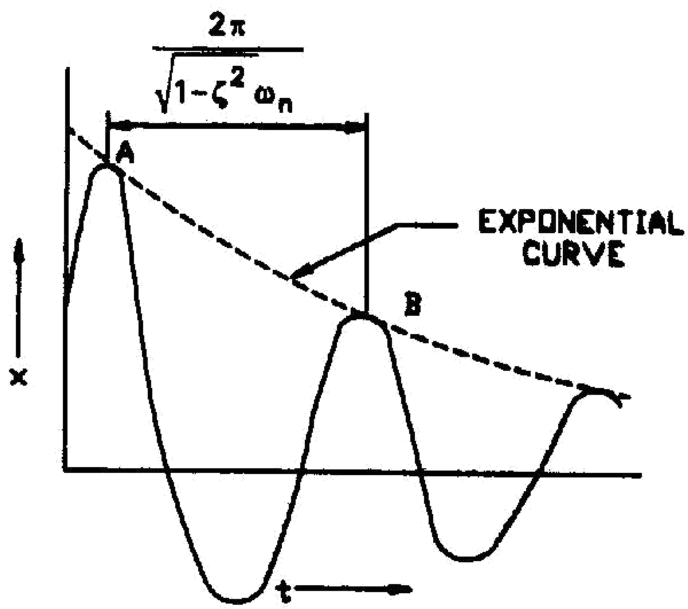


Fig 6 Typical response curve of an under damped system

Applying initial conditions,

$x = X_o$ at $t = 0$; and $\dot{x} = 0$ at $t = 0$; and finding constants A_1 and Φ_1, equation (13) becomes

$$x = \frac{X_o}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \quad (15)$$

The term $\frac{X_0}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}$ represents the amplitude of vibration, which is observed to decay exponentially with time.

LOGARITHMIC DECREMENT

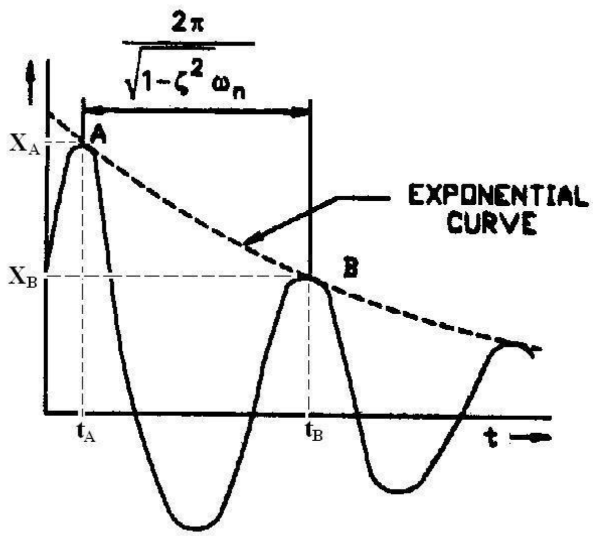


Fig.7 Logarithmic decrement

Referring to Fig.7, points A & B represent two successive peak points on the response curve of an under damped system. X_A and X_B represent the amplitude corresponding to points A & B and t_A & t_B represents the corresponding time.

We know that the natural frequency of damped vibration = $\omega_d = \sqrt{1-\zeta^2}\,\omega_n$ rad/sec.

$$\text{Therefore, } f_d = \frac{\omega_d}{2\pi} \text{ cycles/sec}$$

Hence, time period of oscillation = $t_B - t_A = \frac{1}{f_d} = \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{1-\zeta^2}\,\omega_n}$ sec (16)

From equation (15), amplitude of vibration

$$X_A = \frac{X_o}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t_A}$$

$$X_B = \frac{X_o}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t_B}$$

Or, $$\frac{X_A}{X_B} = e^{-\zeta\omega_n(t_A - t_B)} = e^{\zeta\omega_n(t_B - t_A)}$$

Using eqn. (16), $$\frac{X_A}{X_B} = e^{\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}}$$

Or, $$\log_e \frac{X_A}{X_B} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

This is called logarithmic decrement. It is defined as the logarithmic value of the ratio of two successive amplitudes of an under damped oscillation. It is normally denoted by δ.

$$\text{Therefore, } \delta = \log_e \frac{X_A}{X_B} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \qquad (17)$$

This indicates that the ratio of any two successive amplitudes of an under damped system is constant and is a function of damping ratio of the system.

For small values of ζ, $\delta \approx 2\pi\zeta$

If X_0 represents the amplitude at a particular peak and X_n represents the amplitude after 'n' cycles, then, logarithmic decrement = $\delta = \log_e \frac{X_0}{X_1} = \log_e \frac{X_1}{X_2} = \ldots\ldots = \log_e \frac{X_{n-1}}{X_n}$

Adding all the terms, $n\delta = \log_e \frac{X_0}{X_1} \cdot \frac{X_1}{X_2} \ldots\ldots \frac{X_{n-1}}{X_n}$

$$\text{Or, } \delta = \frac{1}{n} \log_e \frac{X_0}{X_n} \qquad (18)$$

Solved problems

1) The mass of a spring-mass-dashpot system is given an initial velocity 5ω_n, where ω_n is the undamped natural frequency of the system. Find the equation of motion for the system, when (i) ζ = 2.0, (ii) ζ = 1.0, (i) ζ = 0.2.

Solution:

Case (i) For ζ = 2.0 – Over damped system

For over damped system, the response equation is given by

$$x = C_1 e^{\left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t} + C_2 e^{\left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t}$$

Substituting ζ = 2.0, $x = C_1 e^{[-0.27]\omega_n t} + C_2 e^{[-3.73]\omega_n t}$ (a)

Differentiating, $\dot{x} = -0.27\omega_n C_1 e^{-0.27\omega_n t} - 3.73\omega_n C_2 e^{-3.73\omega_n t}$ (b)

Substituting the initial conditions

x = 0 at t = 0; and $\dot{x} = 5\omega_n$ at t = 0 in (a) & (b),

$0 = C_1 + C_2$ (c)

$5\omega_n = -0.27\,\omega_n C_1 - 3.73\,\omega_n C_2$ (d)

Solving (c) & (d), $C_1 = 1.44$ and $C_2 = -1.44$.

Therefore, the response equation becomes

$$x = 1.44\left(e^{[-0.27]\omega_n t} - e^{[-3.73]\omega_n t}\right) \qquad (e)$$

Case (ii) For ζ = 1.0 – Critically damped system

For critically damped system, the response equation is given by

$$x = (C_1 + C_2 t)e^{-\omega_n t} \quad \text{(f)}$$

Differentiating, $\dot{x} = -(C_1 + C_2 t)\omega_n e^{-\omega_n t} + C_2 e^{-\omega_n t}$ (g)

Substituting the initial conditions

x = 0 at t = 0; and $\dot{x} = 5\omega_n$ at t = 0 in (f) & (g),

$C_1 = 0$ and $C_2 = 5\omega_n$

Substituting in (f), the response equation becomes

$$x = (5\omega_n t)e^{-\omega_n t} \quad \text{(h)}$$

Case (iii) For $\zeta = 0.2$ – under damped system

For under damped system, the response equation is given by

$$x = A_1 e^{-\zeta\omega_n t}\left[\left\{\sin\sqrt{1-\zeta^2}\,\omega_n t + \varphi_1\right\}\right]$$

Substituting $\zeta = 0.2$, $x = A_1 e^{-0.2\omega_n t}\left[\left\{\sin(0.98\omega_n t + \varphi_1)\right\}\right]$ (p)

Differentiating,

$$\dot{x} = -0.2\omega_n A_1 e^{-0.2\omega_n t}\left[\left\{\sin(0.98\omega_n t + \varphi_1)\right\}\right] + 0.98\omega_n A_1 e^{-0.2\omega_n t}\cos(0.98\omega_n t + \varphi_1) \quad \text{(q)}$$

Substituting the initial conditions

x = 0 at t = 0; and $\dot{x} = 5\omega_n$ at t = 0 in (p) & (q),

$A_1 \sin\Phi_1 = 0$ and $A_1 \cos\Phi_1 = 5.1$

Solving, $A_1 = 5.1$ and $\Phi_1 = 0$

Substituting in (p), the response equation becomes

$$x = 5.1e^{-0.2\omega_n t}\left[\left\{\sin(0.98\omega_n t)\right\}\right] \quad \text{(r)}$$

2) A mass of 20kg is supported on two isolators as shown in fig.Q.2. Determine the undamped and damped natural frequencies of the system, neglecting the mass of the isolators.

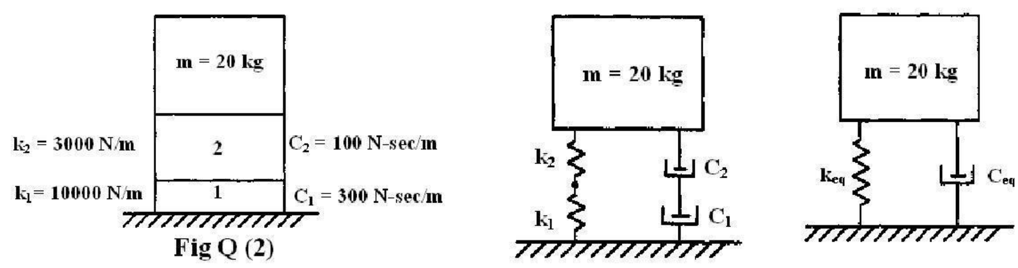


Fig Q (2)

Solution:

Equivalent stiffness and equivalent damping coefficient are calculated as

$$1 = 1 + 1 = 1 + 1 = 13$$

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} = \frac{1}{10000} + \frac{1}{3000} = \frac{13}{30000}$$

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{300} + \frac{1}{100} = \frac{4}{300}$$

Undamped natural frequency = $\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{30000/13}{20}} = 10.74\, rad/\sec$

$$f_n = \frac{10.74}{2\pi} = 1.71 cps$$

Damped natural frequency = $\omega_d = \sqrt{1-\zeta^2}\;\omega_n$

$$\zeta = \frac{C_{eq}}{2\sqrt{k_{eq}m}} = \frac{300/4}{2\cdot\sqrt{\frac{30000}{13}\cdot 20}} = 0.1745$$

$$\therefore \omega_d = \sqrt{1-0.1745^2}\cdot 10.74 = 10.57\, rad/\sec$$

Or, $f_d = \frac{10.57}{2\cdot\pi} = 1.68 cps$

3) A gun barrel of mass 500kg has a recoil spring of stiffness 3,00,000 N/m. If the barrel recoils 1.2 meters on firing, determine,

(a) initial velocity of the barrel

(b) critical damping coefficient of the dashpot which is engaged at the end of the recoil stroke

(c) time required for the barrel to return to a position 50mm from the initial position.

Solution:

(a) Strain energy stored in the spring at the end of recoil:

$$P = \frac{1}{2}kx^2 = \frac{1}{2}\cdot 300000\cdot 1.2^2 = 216000 N-m$$

Kinetic energy lost by the gun barrel:

$T = \frac{1}{2}mv^2 = \frac{1}{2}\cdot 500\cdot v^2 = 250v^2$, where v = initial velocity of the barrel

Equating kinetic energy lost to strain energy gained, ie

T = P, $250v^2 = 216000$

v = 29.39m/s

(b) Critical damping coefficient = $C_c = 2\sqrt{km} = 2\sqrt{300000\cdot 500} = 24495 N-\sec/m$

(c) Time for recoiling of the gun (undamped motion):

Undamped natural frequency = $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{300000}{500}} = 24.49 r/s$

Time period = $\tau = \frac{2\pi}{\omega_n} = \frac{2\pi}{24.49} = 0.259$ sec

ω_n 24.29

Time of recoil = $\frac{\tau}{4} = \frac{0.259}{4} = 0.065$ sec

Time taken during return stroke:

Response equation for critically damped system = $x = (C_1 + C_2 t)e^{-\omega_n t}$

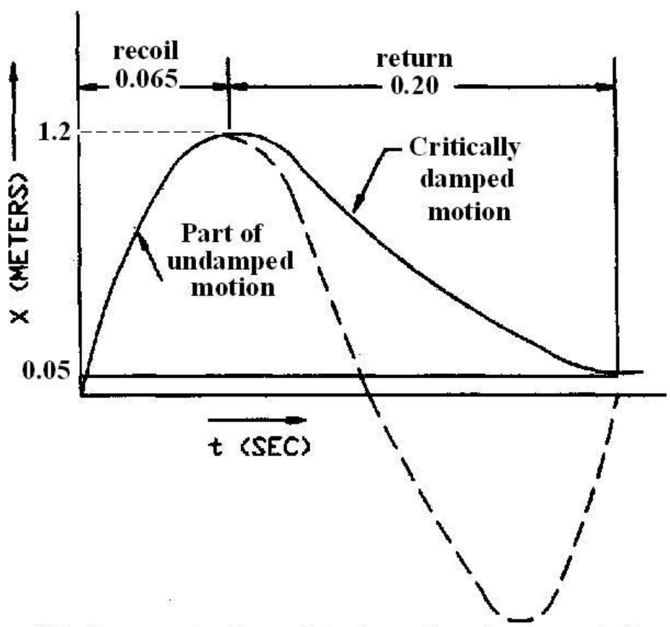
Differentiating, $\dot{x} = C_2 e^{-\omega_n t} - (C_1 + C_2 t)\omega_n e^{-\omega_n t}$

Applying initial conditions, x = 1.2, at t = 0 and $\dot{x} = 0$ at t = 0, C_1 = 1.2, & C_2 = 29.39

Therefore, the response equation = $x = (1.2 + 29.39t)e^{-24.49t}$

When x = 0.05m, by trial and error, t = 0.20 sec

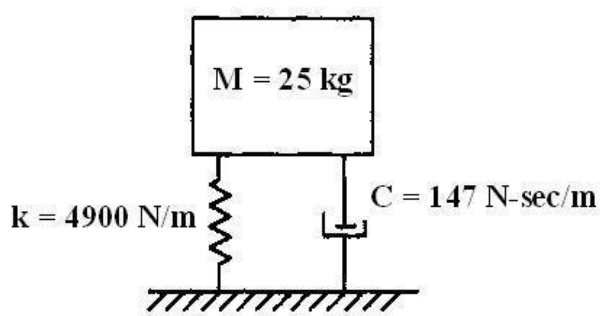
Therefore, total time taken = time for recoil + time for return = 0.065 + 0.20 = 0.265 sec The displacement – time plot is shown in the following figure.



Displacement – time plot of recoil and return stroke

4) A 25 kg mass is resting on a spring of 4900 N/m and dashpot of 147 N-se/m in parallel. If a velocity of 0.10 m/sec is applied to the mass at the rest position, what will be its displacement from the equilibrium position at the end of first second?

Solution:



The above figure shows the arrangement of the system.

Critical damping coefficient = $c_c = 2m\omega_n$

Where $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4900}{25}} = 14 r/s$

Therefore, $c_c = 2 \cdot 25 \cdot 14 = 700\ N-\sec/m$

Since C< C_c, the system is under damped and $\zeta = \frac{c}{c_c} = \frac{147}{700} = 0.21$

Hence, the response equation is $x = A_1 e^{-\zeta\omega_n t}\left[\left\{\sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \varphi_1\right)\right\}\right]$

Substituting ζ and ω_n, $x = A_1 e^{-0.21 \cdot 14 t}\left[\left\{\sin\left(\sqrt{1-0.21^2}\,14t + \varphi_1\right)\right\}\right]$ $x = A_1 e^{-2.94t}\left[\left\{\sin(13.7t + \varphi_1)\right\}\right]$

Differentiating, $\dot{x} = -2.94 A_1 e^{-2.94t}\left[\left\{\sin(13.7t + \varphi_1)\right\}\right] + 13.7 A_1 e^{-2.94t}\cos(13.7t + \varphi_1)$

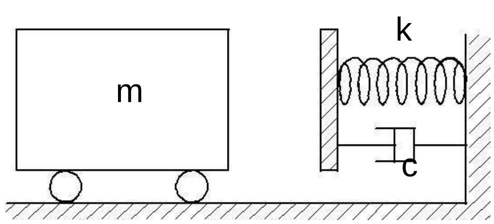
Applying the initial conditions, x = 0, at t = 0 and $\dot{x} = 0.10 m/s$ at t = 0

$\Phi_1 = 0$

$0.10 = -2.94 A_1\left[\left\{\sin(\varphi_1)\right\}\right] + 13.7 A_1 \cos(\varphi_1)$ Since, $\Phi_1 = 0$, 0.10 = 13.7 A_1; A_1 = 0.0073

Displacement at the end of 1 second = $x = 0.0073 e^{-2.94}\left[\left\{\sin(13.7)\right\}\right] = 3.5 \cdot 10^{-4}\ m$

5) A rail road bumper is designed as a spring in parallel with a viscous damper. What is the bumper's damping coefficient such that the system has a damping ratio of 1.25, when the bumper is engaged by a rail car of 20000 kg mass. The stiffness of the spring is 2E5 N/m. If the rail car engages the bumper, while traveling at a speed of 20m/s, what is the maximum deflection of the bumper?



Solution: Data = m = 20000 kg; k = 200000 N/m; $\zeta = 1.25$

Critical damping coefficient =

$c_c = 2 \cdot \sqrt{m \cdot k} = 2 \cdot \sqrt{20000 \cdot 200000} = 1.24 \cdot 10^5\ N-\sec/m$

Damping coefficient $C = \zeta \cdot C_C = 1.25 \cdot 1.24 \cdot 10^5 = 1.58 \cdot 10^5\ N-\sec/m$

Undamped natural frequency = $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{200000}{20000}} = 3.16 r/s$

Since $\zeta = 1.25$, the system is over damped.

For over damped system, the response equation is given by

$$x = C_1 e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} + C_2 e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}$$

Substituting $\zeta = 1.25$, $\quad x = C_1 e^{[-0.5]\omega_n t} + C_2 e^{[-2.0]\omega_n t}$ (a)

Differentiating, $\quad \dot{x} = -0.5\omega_n C_1 e^{-0.5\omega_n t} - 2.0\omega_n C_2 e^{-2.0\omega_n t}$ (b)

Substituting the initial conditions

x = 0 at t = 0; and $\dot{x} = 20 m/s$ at t = 0 in (a) & (b),

$0 = C_1 + C_2$ (c)

$20 = -0.5\,\omega_n C_1 - 2.0\,\omega_n C_2$ (d)

Solving (c) & (d), $\quad C_1 = 4.21$ and $C_2 = -4.21$

Therefore, the response equation becomes

$$x = 4.21\left(e^{[-1.58 \cdot t]} - e^{[-6.32 \cdot t]}\right) m \qquad \text{(e)}$$

The time at which, maximum deflection occurs is obtained by equating velocity equation to zero.

Ie, $\dot{x} = -0.5\omega_n C_1 e^{-0.5\omega_n t} - 2.0\omega_n C_2 e^{-2.0\omega_n t} = 0$

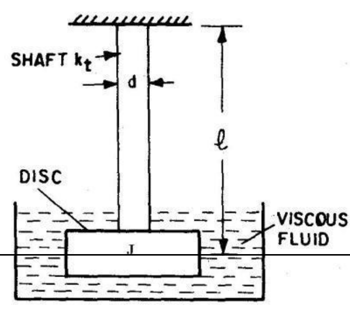
Ie, $-6.65 e^{-1.58t} + 26.61 e^{-6.32t} = 0$

Solving the above equation, t = 0.292 secs.

Therefore, maximum deflection at t = 0.292secs,

Substituting in (e), $x = 4.21\left(e^{-[1.58 \cdot 0.292]} - e^{-[6.32 \cdot 0.292]}\right) m$, = 1.99m.

6) A disc of a torsional pendulum has a moment of inertia of 6E-2 kg-m^2 and is immersed in a viscous fluid. The shaft attached to it is 0.4m long and 0.1m in diameter. When the pendulum is oscillating, the observed amplitudes on the same side of the mean position for successive cycles are 9^0, 6^0 and 4^0. Determine (i) logarithmic decrement (ii) damping torque per unit velocity and (iii) the periodic time of vibration. Assume G = 4.4E10 N/m^2, for the shaft material.



Shaft dia. = d = 0.1m

Shaft length = l = 0.4m

Moment of inertia of disc = J = 0.06 kg-m^2.

Modulus of rigidity = G = 4.4E10 N/m^2

Solution:
The above figure shows the arrangement of the system.

$$\delta = \log_e \frac{9}{6} = \log_e \frac{6}{4} = 0.405$$

We know that logarithmic decrement = $\delta = \frac{\pi\zeta}{\sqrt{1-\zeta^2}}$, rearranging which, we get

Damping factor $\zeta = \frac{\delta}{\sqrt{4\pi^2+\delta^2}} = \frac{0.405}{\sqrt{4\pi^2+0.405^2}} = 0.0645$

Also, $\zeta = \frac{C}{C_C}$, where, critical damping coefficient = $C_C = 2\sqrt{k_t J}$

Torsional stiffness = $k_t = \frac{GI_p}{l} = \frac{G}{l}\cdot\frac{\pi d^4}{32} = \frac{4.4\cdot 10^{10}}{0.4}\cdot\frac{\pi\cdot 0.1^4}{32} = 1.08\cdot 10^6\ N-m/rad$

Critical damping coefficient = $C_C = 2\sqrt{k_t J} = 2\sqrt{1.08\cdot 10^6\cdot 0.06} = 509N-m/rad$

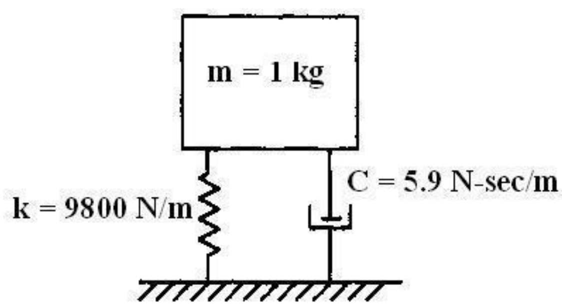
Damping coefficient of the system = $C = C_C\cdot\zeta = 509\cdot 0.0645 = 32.8N-m/rad$

(iii) Periodic time of vibration = $\tau = \frac{1}{f_d} = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$

Where, undamped natural frequency = $\omega_n = \sqrt{\frac{k}{J}} = \sqrt{\frac{1.08\cdot 10^6}{0.06}} = 4242.6rad/\sec$

Therefore, $\tau = \frac{2\pi}{4242.6\cdot\sqrt{1-0.0645^2}} = 0.00148\sec$

7) A mass of 1 kg is to be supported on a spring having a stiffness of 9800 N/m. The damping coefficient is 5.9 N-sec/m. Determine the natural frequency of the system. Find also the logarithmic decrement and the amplitude after three cycles if the initial displacement is 0.003m.



Solution:

Undamped natural frequency = $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{9800}{1}} = 99 r/s$

Damped natural frequency = $\omega_d = \sqrt{1-\zeta^2}\,\omega_n$

Critical damping coefficient = $c_c = 2 \cdot m \cdot \omega_n = 2 \cdot 1 \cdot 99 = 198 N-\sec/m$

Damping factor = $\zeta = \frac{c}{c_c} = \frac{5.9}{198} = 0.03$

Hence damped natural frequency = $\omega_d = \sqrt{1-\zeta^2}\,\omega_n = \sqrt{1-0.013^2} \cdot 99 = 98.99 rad/\sec$

Logarithmic decrement = $\delta = \dfrac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \dfrac{2\cdot\pi\cdot 0.03}{\sqrt{1-0.03^2}} = 0.188$

Also, $\delta = \dfrac{1}{n}\log_e \dfrac{x_0}{x_n}$; if $x_0 = 0.003$,

then, after 3 cycles, $\delta = \dfrac{1}{n}\log_e \dfrac{x_0}{x_n}$ ie, $0.188 = \dfrac{1}{3}\cdot\log_e \dfrac{0.003}{x_3}$

ie, $x_3 = \dfrac{0.003}{e^{3\cdot 0.188}} = 1.71\cdot 10^{-3}\,m$

8) The damped vibration record of a spring-mass-dashpot system shows the following data.
Amplitude on second cycle = 0.012m; Amplitude on third cycle = 0.0105m; Spring constant k = 7840 N/m; Mass m = 2kg. Determine the damping constant, assuming it to be viscous.

Solution:

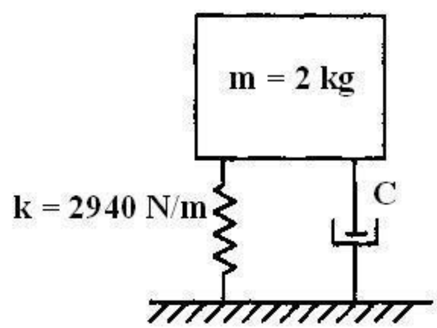
Here, $\delta = \log_e \dfrac{x_2}{x_3} = \log_e \dfrac{0.012}{0.0105} = 0.133$

Also, $\delta = \dfrac{2\pi\zeta}{\sqrt{1-\zeta^2}}$, rearranging, $\zeta = \dfrac{\delta}{\sqrt{4\pi^2+\delta^2}} = \dfrac{0.133}{\sqrt{4\pi^2+0.133^2}} = 0.021$

Critical damping coefficient = $c_c = 2\cdot\sqrt{m\cdot k} = 2\cdot\sqrt{2\cdot 7840} = 250.4 N-\sec/m$

Damping coefficient $C = \zeta\cdot C_C = 0.021\cdot 250.4 = 5.26 N-\sec/m$

9) A mass of 2kg is supported on an isolator having a spring scale of 2940 N/m and viscous damping. If the amplitude of free vibration of the mass falls to one half its original value in 1.5 seconds, determine the damping coefficient of the isolator.



Solution:

Undamped natural frequency = $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{2940}{2}} = 38.34 r/s$

Critical damping coefficient = $c_C = 2 \cdot m \cdot \omega_n = 2 \cdot 2 \cdot 38.34 = 153.4 N - \sec/m$

Response equation of under damped system = $x = A_1 e^{-\zeta \omega_n t} \left[\left\{ \sin\left(\sqrt{1-\zeta^2}\, \omega_n t + \varphi_1 \right) \right\} \right]$

Here, amplitude of vibration = $A_1 e^{-\zeta \omega_n t}$

If amplitude = X0 at t = 0, then, at t = 1.5 sec, amplitude = $\frac{X_0}{2}$

Ie, $A_1 e^{-\zeta \omega_n \cdot 0} = X_0$ or $A_1 = X_0$

Also, $A_1 e^{-\zeta \omega_n \cdot 1.5} = \frac{X_0}{2}$ or $X_0 \cdot e^{-\zeta \cdot 38.34 \cdot 1.5} = \frac{X_0}{2}$ or $e^{-\zeta \cdot 38.34 \cdot 1.5} = \frac{1}{2}$

Ie, $e^{\zeta \cdot 38.34 \cdot 1.5} = 2$, taking log, $\zeta \cdot 38.34 \cdot 1.5 = 0.69 \therefore \zeta = 0.012$

Damping coefficient $C = \zeta \cdot C_C = 0.012 \cdot 153.4 = 1.84 N - \sec/m$

Forced Vibrations

Introduction:

In free un-damped vibrations a system once disturbed from its initial position executes vibrations because of its elastic properties. There is no damping in these systems and hence no dissipation of energy and hence it executes vibrations which do not die down. These systems give natural frequency of the system.

In free damped vibrations a system once disturbed from its position will execute vibrations which will ultimately die down due to presence of damping. That is there is dissipation of energy through damping. Here one can find the damped natural frequency of the system.

In forced vibration there is an external force acts on the system. This external force which acts on the system executes the vibration of the system. The external force may be harmonic and periodic, non-harmonic and periodic or non periodic. In this chapter only external harmonic forces acting on the system are considered. Analysis of non harmonic forcing functions is just an extension of harmonic forcing functions.

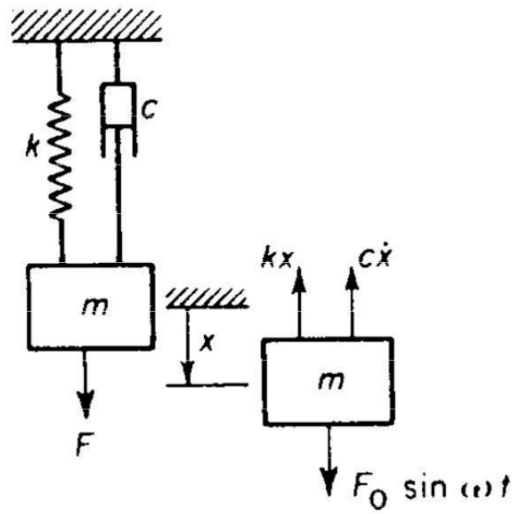
Examples of forced vibrations are air compressors, I.C. engines, turbines, machine tools etc,.

Analysis of forced vibrations can be divided into following categories as per the syllabus.

1. Forced vibration with constant harmonic excitation
2. Forced vibration with rotating and reciprocating unbalance
3. Forced vibration due to excitation of the support A: Absolute amplitude B: Relative amplitude
4. Force and motion transmissibility

For the above first a differential equation of motion is written. Assume a suitable solution to the differential equation. On obtaining the suitable response to the differential equation the next step is to non-dimensional the response. Then the frequency response and phase angle plots are drawn.

1. **Forced vibration with constant harmonic excitation**



From the figure it is evident that spring force and damping force oppose the motion of the mass. An external excitation force of constant magnitude acts on the mass with a frequency ω. Using Newton's second law of motion an equation can be written in the following manner.

$$m\ddot{x} + c\dot{x} + kx = F_o Sin\omega t ------1$$

Equation 1 is a linear non homogeneous second order differential equation. The solution to eq. 1 consists of complimentary function part and particular integral. The complimentary function part of eq, 1 is obtained by setting the equation to zero. This derivation for complementary function part was done in damped free vibration chapter.

$$x = x_c + x_p -----2$$

The complementary function solution is given by the following equation.

$$x_c = A_2 e^{-\zeta\omega_n t} Sin[\sqrt{1-\xi^2}\,\omega_n t + \varphi_2] --- 3$$

Equation 3 has two constants which will have to be determined from the initial conditions. But initial conditions cannot be applied to part of the solution of eq. 1 as given by eq. 3. The complete response must be determined before applying the initial conditions. For complete response the particular integral of eq. 1 must be determined. This particular solution will be determined by vector method as this will give more insight into the analysis.

Assume the particular solution to be

$$x_p = XSin(\omega t - \varphi) ----4$$

Differentiating the above assumed solution and substituting it in eq. 1

π

&

$$x_p = \omega X Sin\ \omega t - \varphi +$$

$$\ddot{x}_p = \omega^2 XSin(\omega t - \varphi + \pi)$$

$$F_o Sin\omega t - kXSin(\omega t - \varphi) - c\omega x\ \omega t - \varphi + \frac{\pi}{2}$$

$$- m\omega^2 XSin(\omega t - \varphi + \pi) = 0 - - - - - 5$$

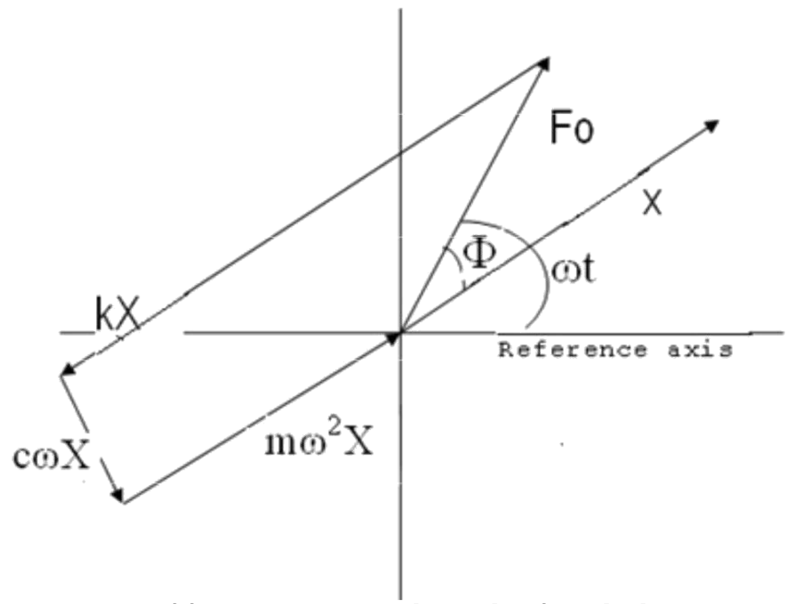


Fig. Vector representation of forces on system subjected to forced vibration

Following points are observed from the vector diagram

1. The displacement lags behind the impressed force by an angle Φ.
2. Spring force is always opposite in direction to displacement.
3. The damping force always lags the displacement by 90°. Damping force is always opposite in direction to velocity.
4. Inertia force is in phase with the displacement.

The relative positions of vectors and heir magnitudes do not change with time.

From the vector diagram one can obtain the steady state amplitude and phase angle as follows

$$X = F_0 \Big/ \sqrt{[(k - m\omega^2)^2 + (c\omega)^2]} \quad ---6$$

$$\varphi = \tan^{-1}\left[c\omega \Big/ (k - m\omega^2)\right] \quad ---7$$

The above equations are made non-dimensional by dividing the numerator and denominator by ***K.***

$$X = \frac{X_{st}}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [2\xi(\omega/\omega_n)]^2}} \quad ---8$$

$$\varphi = \tan^{-1}\frac{2\xi(\omega/\omega_n)}{1 - (\omega/\omega_n)^2} \quad ---9$$

where, $X_{st} = F_o/k$ is zero frequency deflection

Therefore the complete solution is given by

$$x = x_c + x_p$$

$$x = A_2 e^{-\zeta\omega_n t} Sin[\sqrt{1 - \xi^2}\, \omega_n t + \varphi_2] + \frac{X_{st} Sin(\omega t - \varphi)}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [2\xi(\omega/\omega_n)]^2}} \quad ---10$$

The two constants $A2$ and φ2 have to be determined from the initial conditions.

The first part of the complete solution that is the complementary function decays with time and vanishes completely. This part is called transient vibrations. The second part of the complete solution that is the particular integral is seen to be sinusoidal vibration with constant amplitude and is called as steady state vibrations. Transient vibrations take place at damped natural frequency of the system, where as the steady state vibrations take place at frequency of excitation. After transients die out the complete solution consists of only steady state vibrations.

In case of forced vibrations without damping equation 10 changes to

$$x = A_2 Sin[\omega_n t + \varphi_2] + \frac{X_{st} Sin(\omega t)}{\left|1 - (\omega / \omega_n)^2\right|} \quad ---11$$

$\Phi 2$ is either 0° or 180° depending on whether $\omega<\omega_n$ or $\omega>\omega_n$

Steady state Vibrations: The transients die out within a short period of time leaving only the steady state vibrations. Thus it is important to know the steady state behavior of the system, Thus Magnification Factor (M.F.) is defined as the ratio of steady state amplitude to the zero frequency deflection.

$$M.F. = \frac{X}{X_{st}} = \frac{1}{\sqrt{[1 - (\omega / \omega_n)^2]^2 + [2\xi(\omega / \omega_n)]^2}} \quad ---12$$

$$\varphi = tan^{-1} \frac{2\xi(\omega / \omega_n)}{1 - (\omega / \omega_n)^2} \quad ---13$$

Equations 12 and 13 give the magnification factor and phase angle. The steady state amplitude always lags behind the impressed force by an angle Φ. The above equations are used to draw frequency response and phase angle plots.

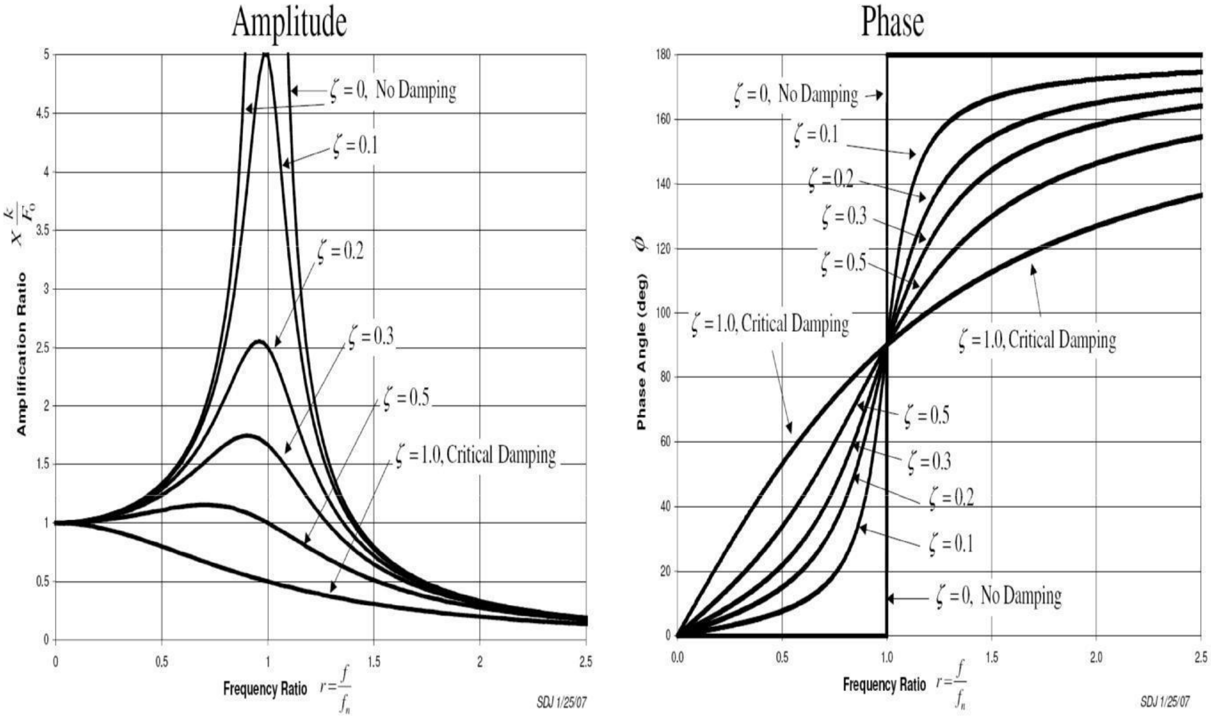


Fig. Frequency response and phase angle plots for system subjected forced vibrations.

Frequency response plot: The curves start from unity at frequency ratio of zero and tend to zero as frequency ratio tends to infinity. The magnification factor increases with the increase in frequency ratio up to 1 and then decreases as frequency ratio is further increased. Near resonance the amplitudes are very high and decrease with the increase in the damping ratio. The peak of magnification factor lies slightly to the left of the resonance line. This tilt to the left increases with the increase in the damping ratio. Also the sharpness of the peak of the curve decreases with the increase in the damping.

Phase angle plot: At very low frequency the phase angle is zero. At resonance the phase angle is 90°. At very high frequencies the phase angle tends to 180°. For low values of damping there is a steep change in the phase angle near resonance. This decreases with the increase in the damping. The sharper the change in the phase angle the sharper is the peak in the frequency response plot.

The amplitude at resonance is given by equation 14

$$c\omega X_r = F_o$$

$$X_r = X_{st} \big/ 2\xi \; --- \; 14$$

The frequency at which maximum amplitude occurs is obtained by differentiating the magnification factor equation with respect to frequency ratio and equating it to zero.

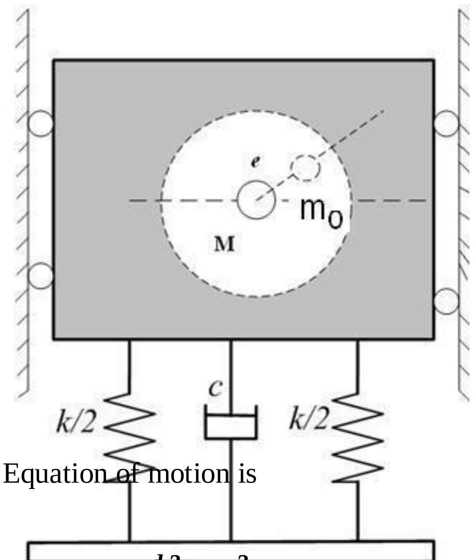
$$\frac{\omega_p}{\omega_n} = \sqrt{1 - 2\xi^2} \; --- \; 15$$

Also no maxima will occur for $\xi > \frac{1}{\sqrt{2}}$ or $\xi > 0.707$

2. Rotating and Reciprocating Unbalance

Machines like electric motors, pumps, fans, clothes dryers, compressors have rotating elements with unbalanced mass. This generates centrifugal type harmonic excitation on the machine.

The final unbalance is measured in terms of an equivalent mass m_o rotating with its c.g. at a distance e from the axis of rotation. The centrifugal force is proportional to the square of frequency of rotation. It varies with the speed of rotation and is different from the harmonic excitation in which the maximum force is independent of the frequency.



Let m_o = Unbalanced mass
e = eccentricity of the unbalanced mass M= Total mass of machine including unbalanced mo
m_o makes an angle ωt with ref. axis. $M_o e\omega^2$ is the centrifugal force that acts radially outwards.

Equation of motion is

$$(M - m_o)\frac{d^2x}{dt^2} + m_o\frac{d^2}{dt^2}(x + e\sin\omega t) = -kx - c\frac{dx}{dt}$$

$$M\ddot{x} + c\dot{x} + kx = m_o e\omega^2 \sin\omega t \quad ---1$$

The solution of following equation 2 is given by

$$m\ddot{x} + c\dot{x} + kx = F_o \sin\omega t \quad ---2$$

$$x = A_2 e^{-\zeta\omega_n t}\, Sin[\sqrt{1-\xi^2}\,\omega_n t + \varphi_2] + \frac{X_{st}\, Sin(\omega t - \varphi)}{\sqrt{[1-(\omega/\omega_n)^2]^2 + [2\xi(\omega/\omega_n)]^2}}$$

Compare eq. 1 with eq.2 the only change is *Fo* is replaced by $m_o e\omega^2$

The transient part of the solution remains the same. The only change is in the steady state part of the solution.

Therefore the steady state solution of eq.1 can be written as

$$x = X\sin(\omega t - \varphi)$$

$$where...X = \frac{\dfrac{m_o e\omega^2}{k}}{\sqrt{\left(1 - \dfrac{M\omega^2}{k}\right)^2 + \left(\dfrac{c\omega}{k}\right)^2}}$$

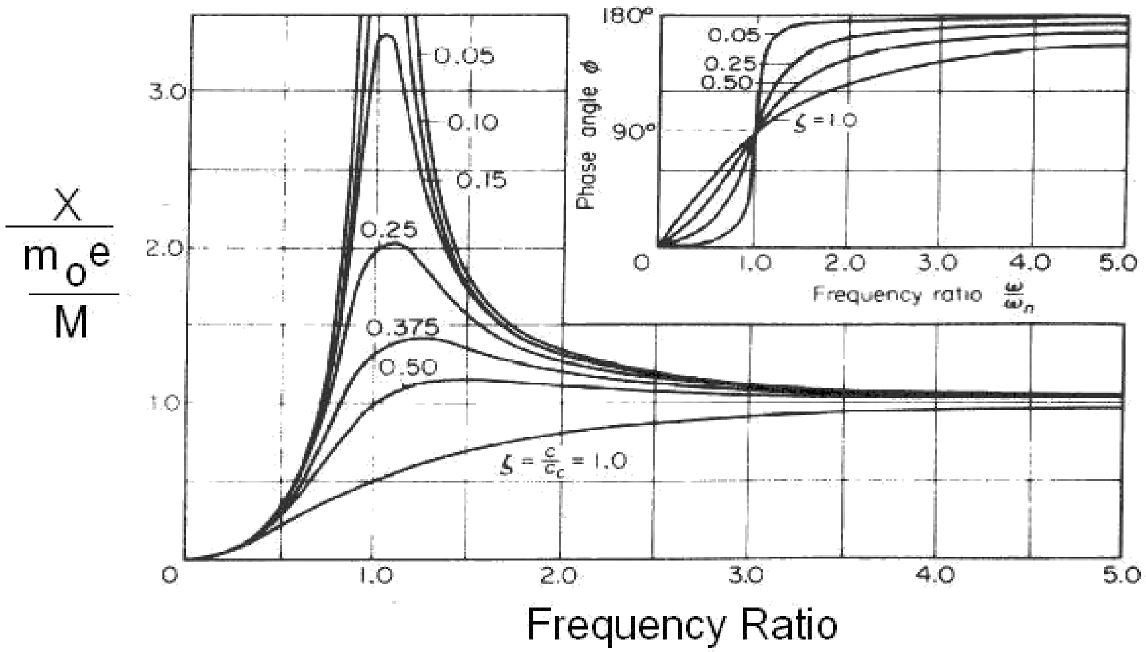
The above equation reduces to dimensionless form as

$$\frac{X}{\dfrac{m_o e}{M}} = \frac{\left(\dfrac{\omega}{\omega_n}\right)^2}{\sqrt{\left(1 - \left(\dfrac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\xi\dfrac{\omega}{\omega_n}\right)^2}}$$

The phase angle equation and its plot remains the same as shown below

$$\varphi = \tan^{-1}\frac{2\xi\dfrac{\omega}{\omega_n}}{1 - \left(\dfrac{\omega}{\omega_n}\right)^2}$$

$^{\omega}\boldsymbol{n}$



Frequency response and phase angle plots (Unbalance)

At low speeds $m_o e\omega^2$ is small, hence all response curves start from zero.

At resonance $\omega / \omega_n = 1$, therefore

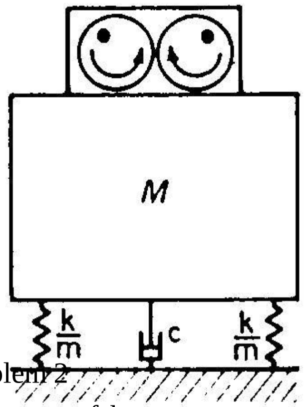
$$\frac{X}{\frac{m_o e}{M}} = \frac{1}{2\xi}$$

And amplitude is limited due to damping present in the system. Under these conditions the motion of main mass (M-m_o) lags that of the mass m_o by 90°. When ω / ω_n is very large the ratio $X/(m_o e/M)$ tends to unity and the main mass (M-m_o) has an amplitude of $X = m_o e/M$. This motion is 180° out of phase with the exciting force. That is when unbalanced mass moves up, the main mass moves down and vice versa.

Problem 1

A counter rotating eccentric weight exciter is used to produce the forced oscillation of a spring-supported mass as shown in Fig. By varying the speed of rotation, a resonant amplitude of 0.60 cm was recorded. When the speed of rotation was increase considerably beyond the resonant frequency, the amplitude appeared to approach a fixed value of 0.08 cm. Determine the

damping factor of the system.



$$X = \frac{\frac{m\,e}{M}}{2\xi} = \mathbf{0.6cm}$$

$$X = \frac{m\,e}{M} = \mathbf{0.08cm}$$

Dividing one by the other

$\xi = \mathbf{0.0667}$

(Answer)

Problem 2

A system of beam supports a mass of 1200 kg. The motor has an unbalanced mass of 1 kg located at 6 cm radius. It is known that the resonance occurs at 2210 rpm. What amplitude of vibration can be expected at the motors operating speed of 1440 rpm if the damping factor is assumed to be less than 0.1

Solution:

Given: M = 1200 kg, m_O= 1 kg, eccentricity = e = 0.06m, Resonance at 2210 rpm, Operating speed = 1440 rpm, ξ = 0.1, X = ?.

$$\omega = \frac{2\pi N}{60} = \mathbf{231.43\ rad / s} \qquad \omega = \frac{2\pi N_{op}}{60} = \mathbf{\mathit{150.79 rad/s}}$$

$$\frac{\omega}{\omega_n} = r = 0.652$$

$$\frac{X}{\frac{m_o e}{M}} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}}$$

Substituting the appropriate values in the above eq.

$X = 0.036$ ***mm*** (Answer)

However, if ξ is made zero, the amplitude $X = 0.037$ ***mm*** (Answer)

This means if the damping is less than 0.1, the amplitude of vibration will be between 0.036 mm and 0.037 mm. (Answer)

Problem 3

An eccentric mass exciter is used to determine the vibratory characteristics of a structure of mass 200 kg. At a speed of 1000 rpm a stroboscope showed the eccentric mass to be at the bottom position at the instant the structure was moving downward through its static equilibrium position and the corresponding amplitude was 20 mm. If the unbalance of the eccentric is 0.05 kg-m, determine, (a) un damped natural frequency of the system (b) the damping factor of the structure (c) the angular position of the eccentric at 1300 rpm at the instant when the structure is moving downward through its equilibrium position.

Solution:

Given: M = 200 kg, Amplitude at 1000 rpm = 20 mm, *moe* = 0.05 *kg-m* At 1000 rpm the eccentric mass is at the bottom when the structure was moving downward – This means a there is phase lag of 90° (i.e., at resonance). At resonance $\omega = \omega_n$.

$$\omega = \omega_n = \frac{2\pi N}{60} = 104.72 \, rad/s$$

$$\frac{X}{\frac{m_o e}{M}} = \frac{1}{2\xi}$$

$\xi = 0.00625$ (Answer)

$$\varphi = \tan \frac{2\xi \frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

$\varphi = 176.189^{o}$ (Answer)

Problem 4

A 40 kg machine is supported by four springs each of stiffness 250 N/m. The rotor is unbalanced such that the unbalance effect is equivalent to a mass of 5 kg located at 50mm from the axis of rotation. Find the amplitude of vibration when the rotor rotates at 1000 rpm and 60 rpm. Assume damping coefficient to be 0.15

Solution:

Given: M = 40 kg, m_o= 5 kg, e = .05 m, ξ = 0.15, N = 1000 rpm and 60 rpm. When N = 1000 rpm

$$\omega = \frac{2\pi N}{60} = 104.67 \text{rad/s}$$

$$\omega_n = \sqrt{\frac{k}{M}} = 5 \text{rad/s} \qquad \frac{\omega}{\omega_n} = 20.934$$

$$\frac{X}{\frac{m_o e}{M}} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\frac{\omega}{\omega_n}\right]^2}} = 1.256$$

X at 1000 rpm = 6.26 mm (Solution)

When N = 60 rpm

$$\omega = \frac{2\pi N}{60} = 6.28 \text{rad/s}$$

$$\frac{\omega}{\omega_n}$$

Using the same eq. X at 60 rpm = 14.29 mm (Solution)

Problem 5

A vertical single stage air compressor having a mass of 500 kg is mounted on springs having a stiffness of 1.96×10^5 N/m and a damping coefficient of 0.2. The rotating parts are completely balanced and the equivalent reciprocating parts have a mass of 20 kg. The stroke is 0.2 m. Determine the dynamic amplitude of vertical motion and the phase difference between the motion and excitation force if the compressor is operated at 200 rpm. Solution

Given: M = 500 kg, $k = 1.96\times10^5$ N/m, ξ = 0.2, m_o = 20 kg, stroke = 0.2 m, N = 200 rpm, X = ?.

Stroke = 0.2 m, i.e. eccentricity e = stroke/2 = 0.1 m

Using the equations $\boldsymbol{X}$ = **10.2** mm and φ **=** ***105.9°*** (Solution)

(a)

$$c = 500/4 = 125N - s/m$$

$$c_c = 2\sqrt{kM} = 2\sqrt{6400 \times 20} = 715.54N - s/m$$

$$\xi = \frac{c}{c_c} = \frac{125}{715.5} = 0.175 \text{ (Solution)}$$

(b)

$$\frac{X}{\frac{m_o e\omega}{M}} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\frac{\omega}{\omega_n}\right]^2}} \qquad \varphi = \tan^{-1}\frac{2\xi\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}$$

$\boldsymbol{X}$ = 0.15 cm and φ = 169° (Solution)

(c)

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{6400}{20}} = 17.88rad/s$$

$$N_r = \frac{60 \times \omega_n}{2 \times \pi} = 170.74rpm$$

$$X_r = \frac{m_o e}{2\xi M} = 0.357cm$$

(d)

$$c\omega X = 125 \times \frac{2 \times \pi \times 400}{60} \times 0.15 \times 10^{-2} = 7.85N$$

$$kX = 6400 \times 0.15 \times 10^{-2} = 9.6N$$

$$F = \sqrt{(c\omega X)^2 + (kX)^2} = 12.4N$$

Conclusions on rotating and reciprocating unbalance

- Unbalance in machines cannot be made zero. Even small unbalanced mass can produce high centrifugal force. This depends on the speed of operation.
- Steady state amplitude is determined for a machine subjected unbalanced force excitation.
- For reciprocating machines, the eccentricity can be taken as half the crank radius.
- Frequency response plot starts from zero at frequency ratio zero and tends to end at unity at very high frequency ratios.

2. Response of a damped system under the harmonic motion of the base

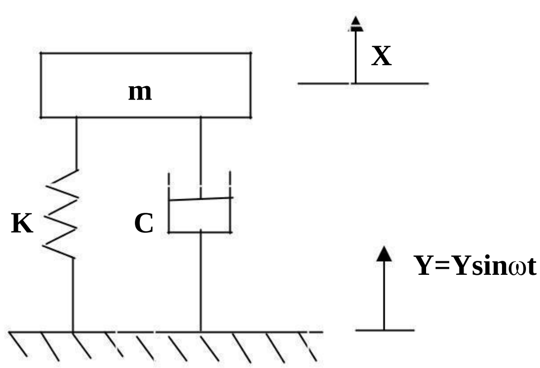
In many cases the excitation of the system is through the support or the base instead of being applied through the mass. In such cases the support will be considered to be excited by a regular sinusoidal motion.

Example of such base excitation is an automobile suspension system excited by a road surface, the suspension system can be modeled by a linear spring in parallel with a viscous damper. such model is depicted in Figure 1.

There are two cases: (a) Absolute Amplitude of mass m

(b) Relative amplitude of mass m

(a) Absolute Amplitude of mass m



$$m\ddot{x} = -k(x-y) - c(\dot{x}-\dot{y})$$
$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky$$

It is assumed that the base moves harmonically, that is

$$y(t) = Y\sin\omega t$$

where *Y* denotes the amplitude of the base motion and ω represents the frequency of the base excitation

Substituting Eq.2 in Eq. 1

$$m\ddot{x} + c\dot{x} + kx = c\omega Y\cos\omega t + kY\sin\omega t$$

The above equation can be expressed as

$$m\ddot{x} + c\dot{x} + kx = Y\sqrt{k^2 + (c\omega)^2}\,\sin(\omega t + \alpha)$$
$$where$$
$$\alpha = \tan^{-1}\left(\frac{c\omega}{k}\right)$$

where $Y[k^2 + (c\,\omega)^2]^{1/2}$ is the amplitude of excitation force. Examination of equation 3 reveals that it is identical to an Equation developed during derivation for M.F. The solution is:

$mx\&\&+ cx\& + kx = F_o \sin\omega t$ **x = Xsin (ωt − φ)**

$$X = \frac{F/k}{\sqrt{[1-(\omega/\omega_n)^2]^2 + [2\xi(\omega/\omega_n)]^2}}$$

$$m\ddot{x}+c\dot{x}+kx = Y\sqrt{k^2+(c\omega)^2}\,\sin(\omega t+\alpha)$$

where

$$\alpha = \tan^{-1}\left(\frac{c\omega}{k}\right)$$

$$x = X\sin(\omega t + \alpha - \varphi)$$

Therefore the steady state amplitude and phase angle to eq. 3 is

$$X = \frac{Y\sqrt{k^2+(c\omega)^2}}{\sqrt{(k-m\omega^2)^2+(c\omega)^2}} \qquad \tan\phi = \frac{c\omega}{k-m\omega^2}$$

The above equations can be written in dimensionless form as follows

$$\frac{X}{Y} = \frac{\sqrt{1+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\xi\frac{\omega}{\omega_n}\right]^2}}$$

$$\varphi = \tan^{-1}\frac{2\xi\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}$$

n n n

ω

2ξ

ω

ω

φ − α = tan − − tan − 2ξ

1 n1 ω

ω

1 −

ω

The motion of the**n** mass**2** *m* lags that**n** of the support by an angle (φ − α*)* as shown by equation 6.

Equation 5 which gives the ratio of (***X/Y***) is also known as motion transmissibility or displacement transmissibility.

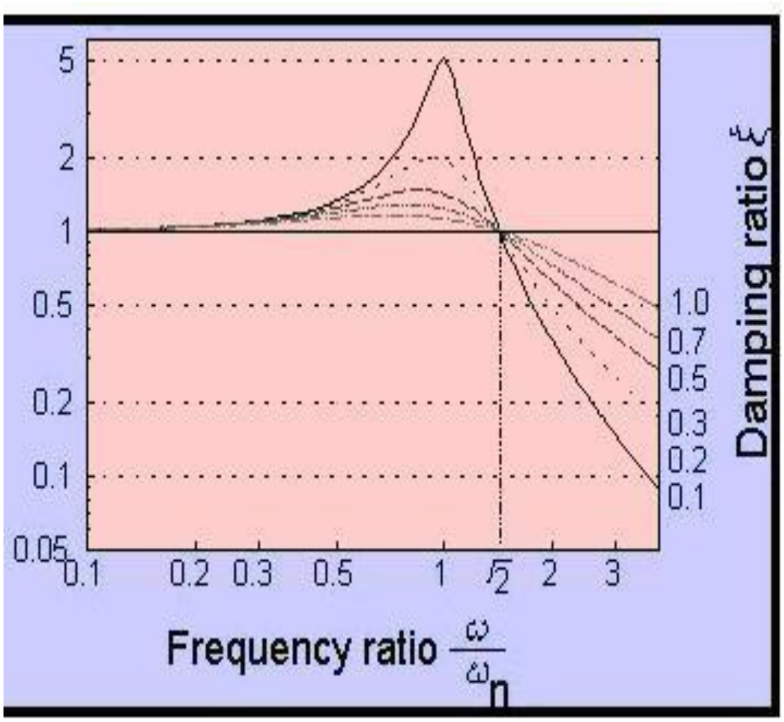


Fig. gives the frequency response curve for motion transmissibility.

1. For low frequency ratios the system moves as a rigid body and ***X/Y*** 1.
2. At resonance the amplitudes are large
3. For very high frequency ratios the body is almost stationary (***X/Y*** 0) It will seen later that the same response curve is also used for Force Transmissibility.

(b) Relative Amplitude of mass m

Here amplitude of mass m relative to the base motion is considered. The equations are basically made use in the
Seismic instruments. If z represents the relative motion of the mass
w.r.t. support,

$$z = x - y$$

$$x = y + z$$

Substituting the value of x in eq.

$$m\ddot{x} + c\dot{x} + kx = c\omega Y \cos \omega t + kY \sin \omega t$$

$$m\ddot{z} + c\dot{z} + kz = m\omega^2 Y \sin \omega t$$

The above equation is similar to the equation developed for rotating and reciprocating unbalances. Thus the relative steady state amplitude can written as

$$\frac{Z}{Y} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi \frac{\omega}{\omega_n}\right]^2}} \qquad \varphi = \tan^{-1} \frac{2\xi (\omega/\omega_n)}{1 - (\omega/\omega_n)^2}$$

Thus eq. 7 and eq.8 are similar to the one developed during the study of rotating an reciprocating unbalances. Frequency and phase response plots will also remain same.

Problem 1

The support of a spring mass system is vibrating with an amplitude of 5 mm and a frequency of 1150 cpm. If the mass is 0.9 kg and the stiffness of springs is 1960 N/m, Determine the amplitude of vibration of mass. What amplitude will result if a damping factor of 0.2 is included in the system. Solution:

Given: Y = 5 mm, f = 1150cpm, m =0.9 kg, k = 1960 N/m, X = ? ξ = 0.2, then X = ?

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1960}{0.9}} = 46.67 rad/s$$

$$\omega = 2 \times \pi \times f = 2 \times \pi \times 1150/60 = 120.43 rad/s$$

$$\frac{\omega}{\omega_n} = r = 2.58$$

$$\frac{X}{Y} = \frac{\sqrt{1 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}}$$

When ξ = 0, X = 0.886 mm (Solution)

When ξ = 0.2, X = 1.25 mm (Solution)

Observe even when damping has increased the amplitude has not decreased but it has increased.

Problem 2

The springs of an automobile trailer are compressed 0.1 m under its own weight. Find the critical speed when the trailer is travelling over a road with a profile approximated by a sine wave of amplitude 0.08 m and a wavelength of 14 m. What will be the amplitude of vibration at 60 km/hr. Solution:

Given: Static deflection = d_{st} = 0.1 m, Y = 0.08 m, γ = 14 m, Critical Speed = ?, X_{60} = ?.
Critical speed can be found by finding natural frequency.

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{d_{st}}} = \sqrt{\frac{9.81}{.1}} = 9.9 rad/s$$

$$f_n = \frac{\omega_n}{2\pi} = 1.576 cps$$

$$V = wavelength \times f_n = 14 \times 1.576 = 22.06 m/s$$

Corresponding V = 22.06 m/s = 79.4 km/hr Amplitude
X at 60 km/hr
V_{60} = 16.67 m/s

$$f = velocity / wavelength = \frac{16.67}{14} = 1.19 cps$$

$$\frac{\omega}{\omega_n} = r = .756$$

$$\omega = 2\pi f = 7.48 rad/s$$

When damping is

$$\frac{X}{Y} = \sqrt{\frac{1 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}}$$

Problem 3

A heavy machine of 3000 N, is supported on a resilient foundation. The static deflection of the foundation due to the weight of the machine is found to be 7.5 cm. It is observed that the machine vibrates with an amplitude of 1 cm when the base of the machine is subjected to harmonic oscillations at the undamped natural frequency of the system with an amplitude of 0.25 cm. Find (a) the damping constant of the foundation (b) the dynamic force amplitude on the base (c) the amplitude of the displacement of the machine relative to the base. Solution

Given: mg = 3000 N, Static deflection = d_{st} = 7.5 cm, X = 1 cm, Y = 0.25 cm, $\omega = \omega_n$, ξ = ?, F_{base} = ?, Z = ?

(a)

$$\omega = \omega_n$$

$$\frac{X}{Y} = \frac{0.010}{0.0025} = 4 = \sqrt{\frac{1 + (2\xi)^2}{(2\xi)^2}}$$

Solving for $\xi = 0.1291$

$$c = \xi c_c = \xi \times 2\sqrt{km} = 903.05 N - s/m$$

(c)

$$\frac{Z}{Y} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\frac{\omega}{\omega_n}\right]^2}}$$

Note: Z = 0.00968 m, X = 0.001m, Y = 0.0025 m, Z is not equal to X-Y due phase difference between x, y, z.

Using the above eq. when $\omega = \omega_n$, the relative amplitude is Z = 0.00968 m (Solution)

(b)

$$F_d = \sqrt{(c\omega Z)^2 + (kZ)^2} \quad \omega = \omega_n = 3.65 rad/s$$

$$F_d = 388.5N$$ (Solution)

Problem 4

The time of free vibration of a mass hung from the end of a helical spring is 0.8 s. When the mass is stationary, the upper end is made to move upwards with displacement y mm given by y = 18 sin 2πt, where t is time in seconds measured from the beginning of the motion. Neglecting the mass of spring and damping effect, determine the vertical distance through which the mass is moved in the first 0.3 seconds.

Solution:

Given: Time period of free vibration = 0.8 s., y = 18 sin 2πt, ξ = 0, x at the end of first 0.3 s. = ?

$$m\ddot{x} + kx = ky$$

$$m\ddot{x} + kx = kY\sin\omega t$$ where, Y = 18 mm, and ω = 2π rad/s.

The complete solution consists of Complementary function and Particular integral part.

$$x = x_c + x_p$$

$$x_c = A\cos\omega_n t + B\sin\omega_n t$$

$$x_p = X\sin(\omega t + \alpha - \varphi)$$

where, $X = \dfrac{Y}{\left|1 - \left(\dfrac{\omega}{\omega_n}\right)^2\right|}$

$$\omega_n = \frac{2\pi}{\tau} = \frac{2\pi}{0.8}$$

$$\omega = 2\pi$$

$$\frac{\omega}{\omega_n} = 0.8$$

and $\varphi - \alpha = 0$

$$(\varphi - \alpha) = 0, \text{ if } \frac{\omega}{\omega_n} < 1$$

$$(\varphi - \alpha) = 180^o, \text{ if } \frac{\omega}{\omega_n} > 1$$

$$\omega_n = \frac{2\pi}{\tau} = \frac{2\pi}{0.8}$$

$$\omega = 2\pi$$

$$\frac{\omega}{\omega_n} = 0.8$$

and $\varphi - \alpha = 0$

$$\text{Hence, } x_p = \frac{Y}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \sin\omega t$$

The complete solution is given by

$$x = A\cos\omega_n t + B\sin\omega_n t + \frac{Y}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \sin\omega t$$

Substituting the initial conditions in the above eq. constants A and B can be obtained

$x = 0;$ *at* $t = 0$ *gives* $A = 0$

&

$x = 0,$ *at* $t = 0$ *and* $B = -\dfrac{Y\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$

Thus the complete solution after substituting the values of A and B

$$x = \frac{Y}{1 - \frac{\omega}{\omega_n}} \qquad \frac{\omega}{\omega_n}$$

when t = 0.3 s, the value of x from the above eq. is x = 19.2 mm (Solution)

Conclusions on Response of a damped system under the harmonic motion of the base

- Review of forced vibration (constant excitation force and rotating and reciprocating unbalance).
- Steady state amplitude and phase angle is determined when the base is excited sinusoidally. Derivations were made for both absolute and relative amplitudes of the mass.

4. Vibration Isolation and Force Transmissibility

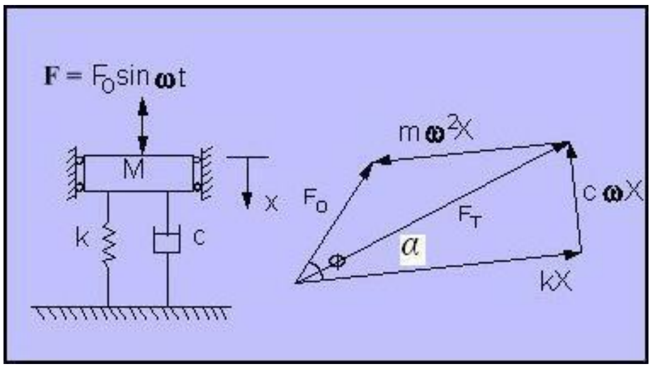
- Vibrations developed in machines should be isolated from the foundation so that adjoining structure is not set into vibrations. (Force isolation)
- Delicate instruments must be isolated from their supports which may be subjected to vibrations. (Motion Isolation)
- To achieve the above objectives it is necessary to choose proper isolation materials which may be cork, rubber, metallic springs or other suitable materials.

- Thus in this study, derivations are made for force isolation and motion isolation which give insight into response of the system and help in choosing proper isolation materials.

Transmissibility is defined as the ratio of the force transmitted to the foundation to that impressed upon the system.

$$m\ddot{x} + c\dot{x} + kx = F_o \sin \omega t \quad ---1$$

$$x_p = XSin(\omega t - \varphi)$$



The force transmitted to the base is the sum of the spring force and damper force. Hence, the amplitude of the transmitted force is:

$$F_T = \sqrt{(kX)^2 + (c\omega X)^2}$$

$$= kX\sqrt{1 + \left(\frac{c\omega}{k}\right)^2}$$

$$= kX\sqrt{1 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2}$$

$$\frac{Xk}{F_o} = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2}}$$

Substituting the value of *X* from Eq. 2 in Eq. 3 yields

$$F_T = \frac{F_o\sqrt{1 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2}}$$

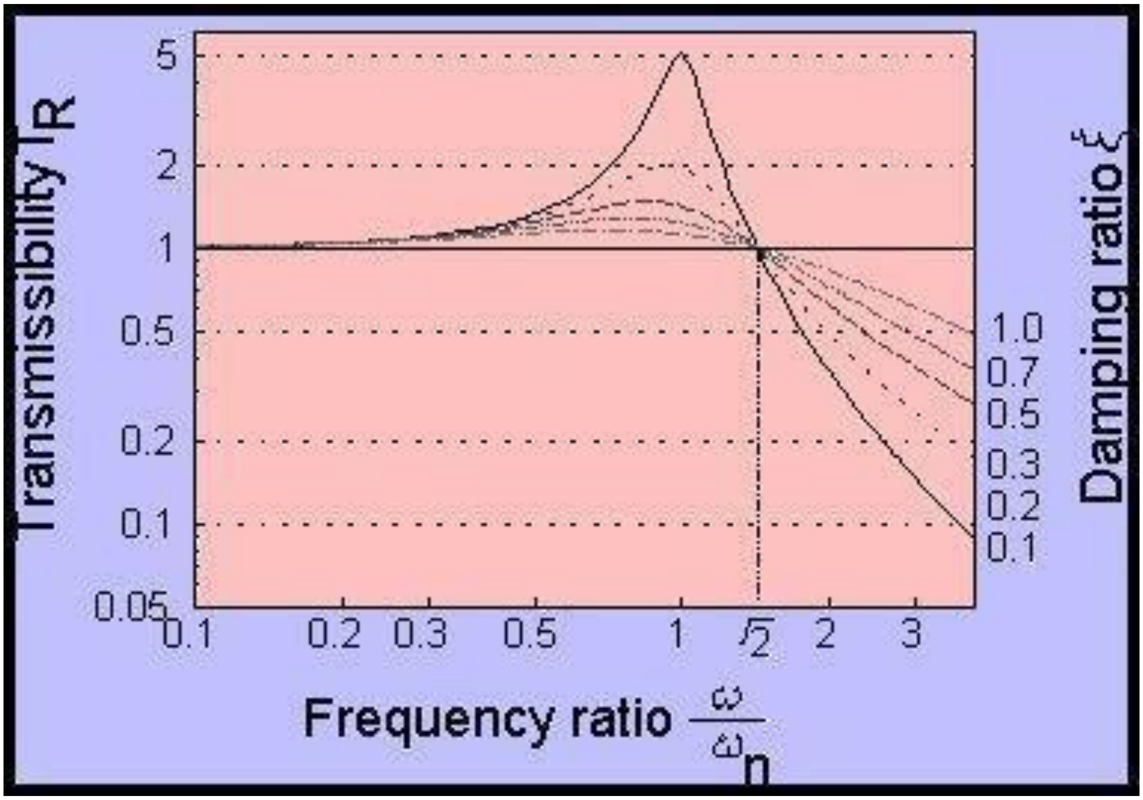
Hence, the force transmission ratio or transmissibility, TR is given by

$$TR = \frac{F_T}{F_0} = \frac{\sqrt{1 + \left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right]^2}}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right]^2}}$$

Eq. 6 gives phase angle relationship between Impressed force and transmitted force.

$$\varphi - \alpha = \tan^{-1}\left[\frac{2\xi\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right] - \tan^{-1}\left[2\xi\frac{\omega}{\omega_n}\right]$$

n^2 $_n$



$$TR = \frac{F_T}{F_o} = \frac{\sqrt{1 + \left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right]^2}}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right]^2}}$$

• Curves start from unity value of transmissibility pass through unit value of transmissibility at (ω / ωn) = and after that they Tend to zero as (ω / ωn) $\rightarrow$.

- Response plot can be divided into three regions depending on the control of spring, damper and mass.
- When (ω / ω_n) is large, it is mass controlled region. Damping in this region deteriorates the performance of machine.
- When (ω / ω_n) is very small, it is spring controlled region.
- When (ω / ω_n) ranges from 0.6 to , it is damping controlled region.
- For effective isolation (ω / ω_n) should be large. It means it will have spring with low stiffness (hence large static deflections).

Motion Transmissibility

Motion transmissibility is the ratio of steady state amplitude of mass m (X) to the steady amplitude (Y) of the supporting base.

$$\frac{X}{Y} = \frac{\sqrt{1 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}}$$

$$\varphi - \alpha = \tan^{-1}\left(\frac{2\xi \frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right) - \tan^{-1}\left(2\xi \frac{\omega}{\omega_n}\right)$$

The equations are same as that of force transmissibility. Thus the frequency response and phase angle plots are also the same.

Problem 1

A 75 kg machine is mounted on springs of stiffness $k=11.76\times10^6$ N/m with a damping factor of 0.2. A 2 kg piston within the machine has a reciprocating motion with a stroke of 0.08 m and a speed of 3000 cpm. Assuming the motion of the piston to be harmonic, determine the amplitude of vibration of machine and the vibratory force transmitted to the foundation.

Solution:

Given: M = 75 kg, ξ = 0.2, m_o = 2 kg, stroke = 0.08 m, N = 3000 cpm, X = ?, F_T = ?.

ω $= k =$ $\dfrac{11.76 \times 10^{5}}{75}$ $=125$ rad/s

m

$\omega = \dfrac{2\pi \times 3000}{60} = 314.16$ rad/s

ω_n $\sqrt{\ }$ $\sqrt{\ }$ $\underline{0.08}$

$\omega = 2.51$ $e = _2$ $= 0.04$ m

ω

$$\frac{X}{M_0} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}}$$

Using the above eq. X = 1.25 mm (Solution)

$F = m\,e\,\omega^2 = 7900$ N

$$TR = \frac{F_T}{F_o} = \frac{\sqrt{1 + \left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right]^2}}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right]^2}}$$

o o 2

Using the above eq. F_T = 2078 N (Solution)

Solution by Complex Algebra

Here steady state solution is obtained for a system subjected to constant excitation force using complex algebra

$$m\ddot{x} + c\dot{x} + kx = F_o \sin \omega t \; - - - 1$$

Let the harmonic forcing function be represented by in complex form as

$$F(t) = F e^{\omega}$$

Now the equation of motion becomes

$$m\ddot{x} + c\dot{x} + kx = F e^{i\omega t} \; - - - 2$$

Actual excitation is given by real part of *F(t)*, he response will also be given by real part of *x(t)*, where *x(t)* is a complex quantity

satisfying the differential equation**o**1. **i t**

Assuming a particular solution as

$$x(t) = X e^{i\omega t} \; - - - 3 \; o$$

Substituting eq. 3 in eq. 2

$$X = \frac{F}{(k - m\omega \qquad) + ic\omega} \; - - - 4$$

o

2

Multiplying the numerator and denominator of eq.4 by

And separating real and imaginary parts

$$X = F_o\left[\frac{k - m\omega^2}{(k - m\omega^2)^2 + c^2\omega^2} - i\frac{c\omega}{(k - m\omega^2)^2 + c^2\omega^2}\right] \quad ---5$$

Multiplying and dividing the above expression by $\sqrt{(k - m\omega^2)^2 + c^2\omega^2}$

$$X = \frac{F_o}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}}\left[\frac{k - m\omega^2}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} - i\frac{c\omega}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}}\right]$$

$$X = \frac{F_o}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}}\, e^{-i\varphi} \quad ---6$$

where, $$\tan\varphi = \frac{c\omega}{k - m\omega^2} \quad ---7$$

Thus the steady state solution becomes

$$x = \frac{F_o}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}}\, e^{i(\omega t - \varphi)} \quad ---8$$

The real part of the above equation 8 gives the steady state response

$$x_p = \frac{F_o \cos(\omega t - \varphi)}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} \quad ---9$$

Dividing the numerator and denominator of this eq. by k

X = $\sqrt{[- (/)2]F2ko+ [(/)]2}$

Problem 1

A torsional system**1** consists**ωωn** of a disc**2ξ** of**ω** mass**ωn** moment of inertia J = 10 kg- m^2, a torsional damper of damping coefficient c = 300 N-m-s/rad, and steel shaft of diameter 4 cm a d length 1 m (fixed at one end and attached to the disc at the other end). A steady angular oscillation of amplitude 2^o is observed when a harmonic torque of mag itude 1000 N-m is applied to the disc. (a) Find the frequency of the applied torque, (b) find the maximum torque transmitted to the support. Assume modulus of rigidity for the steel rod to be 0.83×1011 N/m^2.

Solution:

Given: J = 10 kg-m^2, c = 300 N-m-s/rad, l = 1 m, d = 4 cm,

Θ = 2^o, To = 1000 N-m, G = 0.83×1011 N/m^2, ω = ?, $Tmax$=? Stiffness of the shaft is given by

= 20860 N − m rad

T GI 0.83 ×10 ^{11} **×π ×0.04** **4**

k = θ = L = 1 × 32

k 20860

ω = $\sqrt{J}$ = $\sqrt{10}$ = 45.67 rad/s

t p ×π

t = $\sqrt{[- (/)2]2Fko+ [(/)]2}$ **θ = $\sqrt{[- (/)2]Tk2ot+ [(/)]2}$**

1 ω ωn 2ξ ω ωn

θ inn radians = 180 = 0.035rad

T 1000

~~k θ~~ = ~~20860~~ ×~~0.035~~ =1.369

1 ω ωn 2ξ ω ωn

o
t

ω_2

$-\ r\ )2 + 4\xi$

$(1$ $\quad r = 1.876$ **where,** $r =$

2 $\quad\omega$

$\omega \quad c \quad 300 \quad n\ 2$

$\xi = 2J\omega \ = \ \cancel{2 \times 10 \times 45.67} = 0.328$

$r \quad -1.569r - 0.876 = 0$

$\omega \ = 1.416 \qquad \omega = 64.68$ **rad/s**

ω

$1 + \quad 2\xi$

$T \quad \omega$

$T\ = \qquad T \quad 1000N \qquad m$

$tr = \qquad -$

$\omega \qquad \omega$

$tr \quad 1 - \qquad + \ 2\xi$

$o \qquad 2 \quad 2 \quad n\ 2 \qquad 2$

$\omega \qquad \omega$

$n \qquad n$

Vibration Measuring Instruments

Need for vibration measuring instruments

1.To detect shifts in the natural frequencies – could indicate a possible failure or need for maintenance

2.To select operational speeds to avoid resonance 3.Theoretically estimated values may be different from the actual values due to assumptions.

4.To design active vibration isolation systems 5.To validate the approximate model

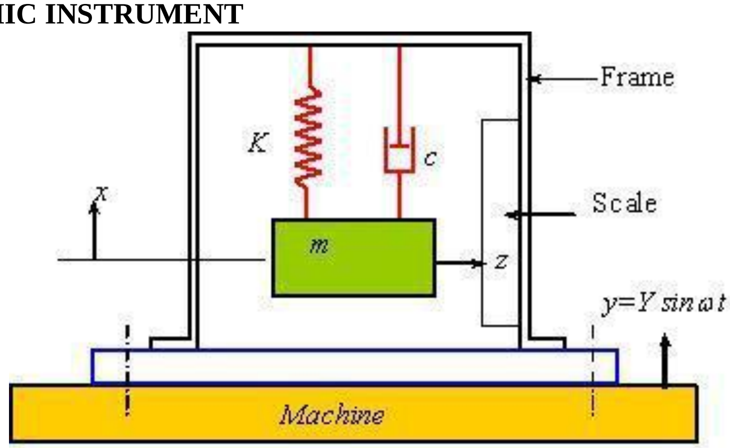
6.To identify the system in terms of mass, stiffness and damper. When a transducer is used in conjunction with another device to measure vibrations it is called vibration pickup. Commonly used vibration pickups are seismic instruments.

If the seismic instrument gives displacement of the vibrating body – It is known as VIBROMETER.

If the seismic instrument gives velocity of the vibrating body – It is known as VELOMETER.

If the seismic instrument gives acceleration of the vibrating body – It is known as ACCELEROMETER.

SEISMIC INSTRUMENT



Vibrating body is assumed to have a Harmonic Motion given by

$\mathbf{y = Y \sin \omega t} \quad --- 1$

Eq. of motion for the mass m can be written as

$\mathbf{m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0} \quad --- 2$

Defining relative displacement as

$\mathbf{z = x - y} \quad --3$

or, $\mathbf{x = y + z}$

Substituting this value of *x* in equation 2

$\mathbf{m\ddot{z} + c\dot{z} + kz = -m\ddot{y}}$

$\mathbf{m\ddot{z} + c\dot{z} + kz = m\omega^2 Y \sin \omega t} \quad - - - 4$

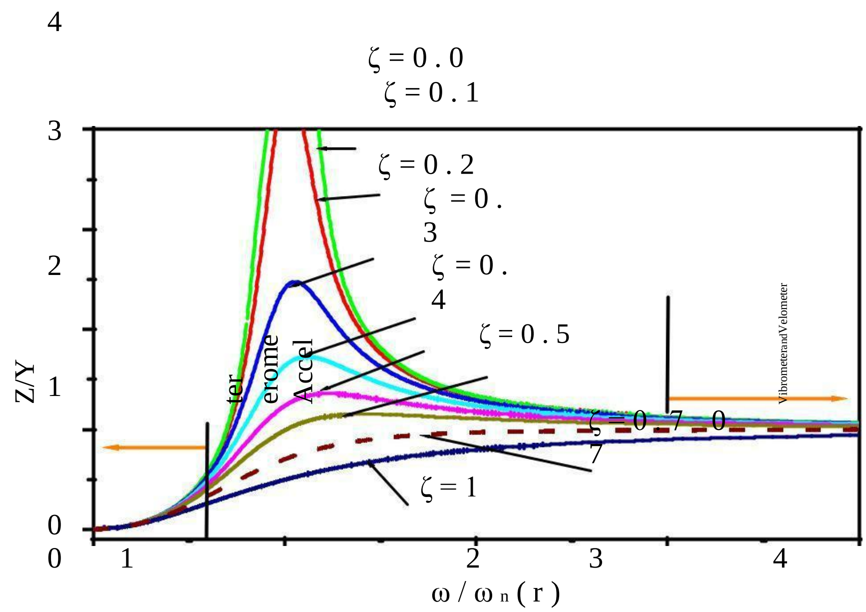
Steady state solution of eq. 4 is given by

$\mathbf{z = Z \sin(\omega t - \varphi)}$

$$\frac{Z}{Y} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi \frac{\omega}{\omega_n}\right]^2}} = \frac{r^2}{\sqrt{[1 - r^2]^2 + (2\xi r)^2}} \quad \textit{where } r = \frac{\omega}{\omega_n}$$

$$\varphi = \tan^{-1} \frac{2\xi \frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2} = \tan^{-1} \frac{2\xi r}{1 - r^2}$$

The frequency response plot is shown in the fig. The type of instrument is determined by the useful range of frequency



The phase angle plot shown below indicates the phase lag of the seismic mass with respect to vibrating base of machine

1 8 0

1 6 0

1 4 0

φ

ξ = 1 . 0

1 2 0

ξ = 0 . 7

ξ = 0 .0 7 5

angle,

1 0 0

Phase

8 0

6 0

4 0

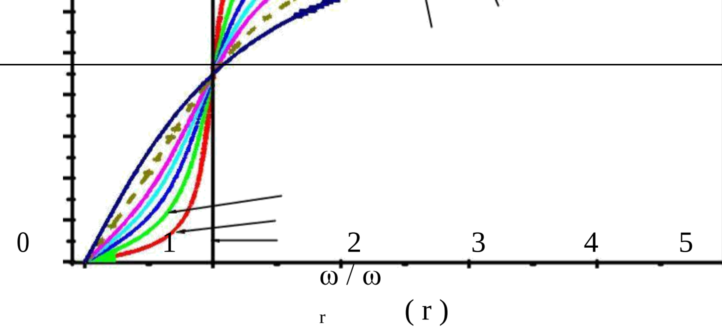
ξ = 0 . 2

2 0

ξ = 0 . 1

0

ξ = 0


0
1
2
3
4
5
ω / ω
r
(r)

Vibrometers

Used for measurement of displacement of vibrating body.

It means when $Z / Y \sim 1$, the observed reading on the scale directly gives the displacement of the vibrating body. For this to happen $r = \omega / \omega n \geq 3$.

$$\mathbf{z = Z\sin(\omega t - \varphi)} \qquad \frac{r^2}{\sqrt{[1 - r^2]^2 + (2\xi r)^2}} = 1 = \frac{Z}{Y}$$

$$\mathbf{y = Y\sin\omega t}$$

Thus when $r = \omega / \omega n \geq 3$, $Z / Y \sim 1$, but there is a phase lag. Z lags behind Y by an angle φ or by time lag of $t = \varphi / \omega$. This time lag is not important if the input consists of single harmonic component.

Thus for vibrometers the *range of frequency lies on the right hand side of frequency response plot*. It can also be seen from the plot a better approximation can be obtained if ξ is less than 0.707.

Also for fixed value of ω, and for $r = \omega / \omega n \geq 3$, the value of ωn must be small. It means mass must be very large and stiffness must be small. This results in bulky instrument which may not be desirable in many applications.

Accelerometer

Accelerometer measures the acceleration of a vibrating body.

They are widely used for measuring acceleration of vibrating bodies and earthquakes. Integration of acceleration record provides displacement and velocity.

$$\frac{Z}{Y} = \frac{r}{\sqrt{1[-r] + (2\xi r)}}$$

$$\frac{1}{\sqrt{1[-r]^2 + 2(2\xi^2 2 r)^2}} = 1_2$$

then $Z = r^2 Y$, $Z_2 = \dfrac{\omega^2 Y}{\omega n^2}$

The expression **ω Y** is equal to the acceleration amplitude of the

Vibrating body. The amplitude recorded by the instrument is proportional to the acceleration of vibrating body since ω_n is constant for the instrument.

For an accelerometer the ratio ω / ω_n << 1 and the range of operation lies on the extreme left hand side of the frequency response curve.

Since ω / ω_n << 1, the natural frequency must be high. That is mass must be as low as possible. Thus the instrument is small in size and is preferred in vibration measurement.

Frequency ratio, *r* should be between 0 and 0.6 Damping ratio, ξ , should be less than 0.707

Problem 1 The static deflection of the vibrometer mass is 20 mm. The instrument when attached to a machine vibrating with a frequency of 125 cpm records a relative amplitude of 0.03 cm. Determine (a) the amplitude of vibration (b) the maximum velocity of vibration (c) the maximum acceleration.

Solution:

Given: *dst* = 20 mm, *f* = 125 cpm, *Z* = 0.03 cm, *Y* = ?, *Vmax* = ?, *a* = ?, ξ = 0.0

$$\omega_n = \sqrt{\frac{g}{d_{st}}} = \sqrt{\frac{9.81}{0.02}} = 22.15 \text{ rad/s} \qquad \omega = \frac{2\pi N}{60} = 13.09 \text{ rad/s}$$

$$r = \frac{\omega}{\omega_n} = 0.59$$

$$\frac{Z}{Y} = \frac{r}{\sqrt{[1 - r]^2 + (2\xi r)^2}} = \frac{r}{|1 - r|} \quad \text{as } \xi = 0$$

$$\frac{Z}{Y} = \frac{0.59}{|1 - 0.59|} = 0.534$$

$$Y = \frac{0.03}{0.534} = 0.56\text{cm}$$

Max Velocity =Yω = 0.733 cm/s

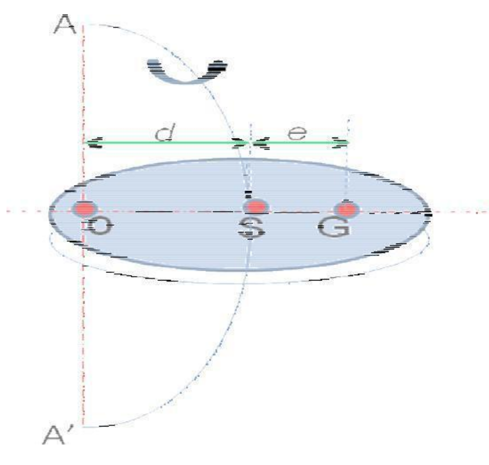
Max Acceleration =Yω = 95.95 cm/s

Critical speed of shaft

- There are many engineering applications in which shafts carry disks (turbines, compressors, electric motors, pumps etc.,)
- These shafts vibrate violently in transverse directions at certain speed of operation known as critical speed of shaft.
- Among the various causes that create critical speeds, the mass unbalance is the most important.
- The unbalance cannot be made zero. There is always some unbalance left in rotors or disks.

- Whirling is defined as the rotation of the plane containing the bent shaft about the bearing axis.
- The whirling of the shaft can take place in the same or opposite direction as that of the rotation of the shaft.
- The whirling speed may or may not be equal to the rotation speed.

Critical speed of a light shaft having a single disc – without damping



Consider a light shaft carrying a single disc at the centre in deflected position. ‘S’ is the geometric centre through which centre line of shaft passes. ‘G’ is centre of gravity of disc. ‘O’ intersection of bearing centre line with the disc ‘*e’* is distance between c.g. ‘G’ and the geometric centre ‘S’. ‘*d*’ is displacement of the geometric centre ‘S’ from the un deflected position ‘O’. ‘*k*’ is the stiffness of the shaft in the lateral direction.

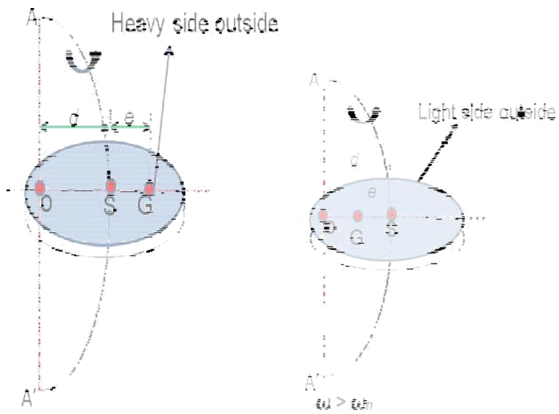
The forces acting on the disc are

- Centrifugal force ‘$m\omega^2(d+e)$’ acts radially outwards at ‘G’
- Restoring force *‘kd’* acts radially inwards at ‘S’
- For equilibrium the two forces must be equal and act along the same line

$$m\omega^2(d+e) = kr \; --- 1$$

$$d = \frac{m\omega^2 e}{k - m\omega^2} = \frac{\left(\frac{\omega}{\omega_n}\right)^2 e}{1 - \left(\frac{\omega}{\omega_n}\right)^2} = \frac{r^2 e}{(1 - r^2)} \; --- 2$$

- Deflection *‘d’* tends to infinity when $\omega = \omega_n$
- *‘d’* is positive below the critical speed, the disc rotates with heavy side outside when $\omega < \omega_n$
- *‘d’* is negative above the critical speed, the disc rotates with light side outside when $\omega > \omega_n$
- When $\omega >> \omega_n$, ‘$d \to -e$’, which means point ‘G’ approaches point ‘O’ and the disc rotates about its centre of gravity.



Problem 1

A rotor having a mass of 5 kg is mounted midway on a 1 cm shaft supported at the ends by two bearings. The bearing span is 40 cm. Because of certain manufacturing inaccuracies, the centre of gravity of the disc is 0.02 mm away from the geometric centre of the rotor. If the system rotates at 3000 rpm find the amplitude of steady state vibrations and the dynamic force transmitted to the bearings. Assume the rotor to be simply supported. Take E = 1.96×10^{11} N/m^2. Solution:

Given: m = 5 kg, d = 1 cm, l = 40 cm, e = 0.02 mm,

N = 3000 rpm, $E = 1.96\times10^{11}$ N/m^2 , d = ?, Simply supported

$$\frac{d}{e} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{1-\left(\frac{\omega}{\omega_n}\right)^2}$$

In the above eq. e and ω are known, ω_n has to be found out in order to find d. To find ω_n, stiffness has to be determined.

For a simply supported shaft the deflection at the mid point is given by the following equation.

$$\delta = \frac{mgl^3}{48EI}, \quad k = \frac{mg}{\delta} = \frac{48EI}{l^3} = \frac{48\times1.96\times10^{11}\times\pi\times0.01^4}{64\times0.4^3} = 72000 \text{ N/m}$$

$$\omega = \frac{2 \times \pi \times N}{60} = 314.16 \text{ rad/s}, \quad \omega_n = \sqrt{\frac{k}{m}} = 120 \text{ rad/s}, \quad \frac{\omega}{\omega_n} = 2.168$$

$$\frac{d}{e} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{1 - \left(\frac{\omega}{\omega_n}\right)^2} = \frac{2.168^2}{(1 - 2.168^2)} = -1.1558,$$

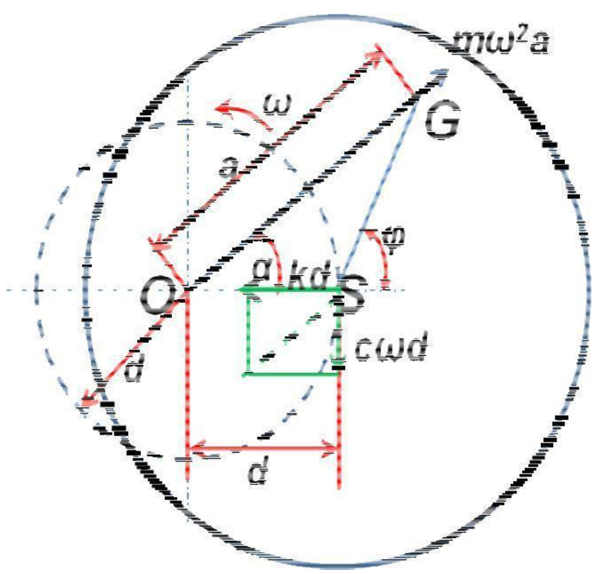
$$d = -1.1558 \times 0.02 = -0.023\text{mm}$$

- Sign implies displacement is out of phase with centrifugal force

Dynamic on bearing = kd = 1.656 N load

on each bearing = 0.828 N

Critical speed of light shaft having a single disc – with damping



O' is intersection of bearing centre line with the disc 'S' is geometric centre of the disk

‘G’ is centre of gravity of disc

The forces acting on the disc are

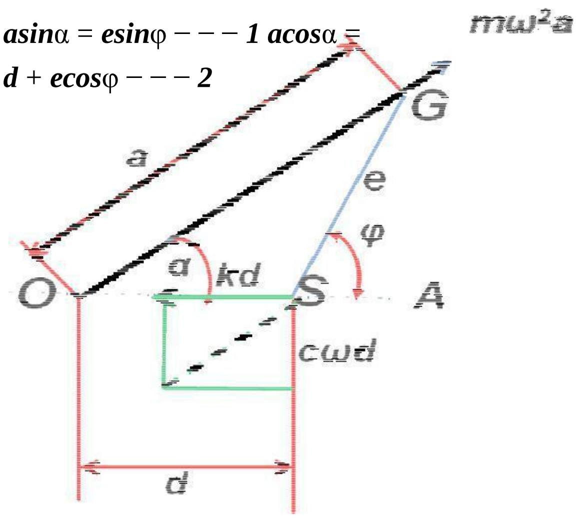
1. Centrifugal force $m\omega^2 a$ at G along OG produced
2. Restoring force ***kd*** at S along SO
3. Damping force ***cωd*** at S in a direction opposite to velocity at S
4. The points O,S and G no longer lie on straight line

Let,

OG = ***a***,SG = ***e***,OS = ***d***, ∟GOS = α ∟*GSA* = φ

From the geometry

$$a\sin\alpha = e\sin\varphi - - - 1 \quad a\cos\alpha = d + e\cos\varphi - - - 2$$



$$\sum X = 0,$$

$$-kd + m\omega^2 a = 0 - - - 3$$

$$\sum Y = 0,$$

$$-c\omega d + m\omega^2 a\sin\alpha = 0 - - - 4$$

Eliminating *a* and α from equations 3 and 4 with the help of equations 1 and 2

$$-kd + m\omega^2 (d + e\cos\varphi) = 0 - - - 5$$

$$-c\omega d + m\omega^2 (e\sin\varphi) = 0 - - - 6$$

$$\text{From eq.5 } \cos\varphi = \frac{kd - m\omega^2 d}{m\omega^2 e} \quad ---7$$

$$\text{From eq.6 } \sin\varphi = \frac{c\omega d}{m\omega^2 e} \quad ---8$$

Squaring and adding the equations 7 and 8

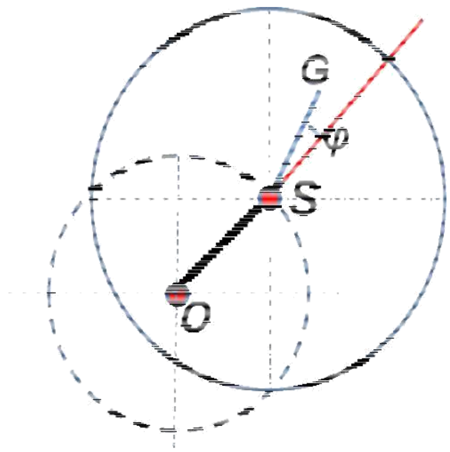
$$\frac{d}{e} = \frac{m\omega^2}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$

$$\frac{d}{e} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\frac{\omega}{\omega_n}\right]^2}} = \frac{r^2}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}} \quad ---9$$

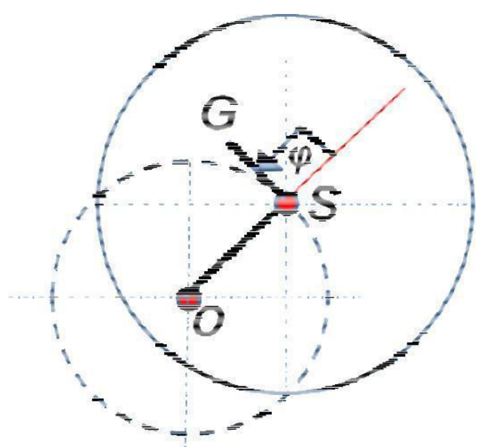
$$\varphi = \tan^{-1}\frac{2\xi\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2} = \tan^{-1}\frac{2\xi r}{1 - r^2} \quad ---10$$

From the curves or the equation it can be seen that (Discussions of speeds above and below critical speed)

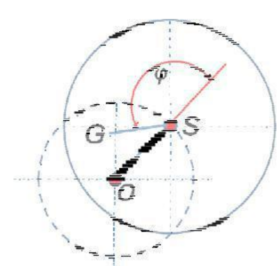
(a) $\Phi \sim 0$ when $\omega \ll \omega_n$ **(Heavy side out)**
(b) $0 < \varphi < 90°$ when $\omega < \omega_n$ **(Heavy side out)**



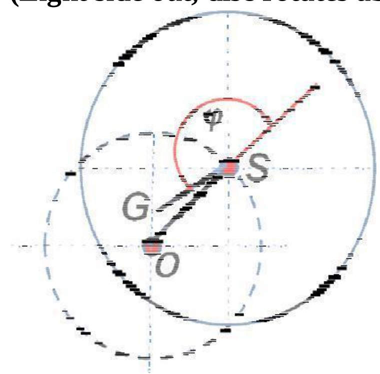
(c) Φ *= 90°* **when** $\omega = \omega_n$



(d) 90° **<** Φ **<** *180°* when $\omega > \omega_n$ **(Light side out)**



***(e)* Φ ~ *180°* when ω >> ω*n*, (Light side out, disc rotates about its centre of gravity)**



Critical speed of shaft may be placed above or below the operating speed. 1. If the unit is to operate at high speeds, that do not vary widely, the critical speed may be below the operating speed, and the shaft is then said to be flexible.

2 In bringing the shaft up to the operating speed, the critical speed must be passed through. If this is done rapidly, resonance conditions do not have chance to build up.

3. If the operating speed is low or if speeds must vary through wide ranges, the critical speed is placed over the operating speed and the shaft is said to be rigid or stiff.

4. Generally the running speed must at least 20% away from the critical speed.

Problem 1

A disk of mass 4 kg is mounted midway between the bearings

Which may be assumed to be simply supported. The bearing span is 48 cm. The steel shaft is 9 mm in diameter. The c.g. of the disc is displaced 3 mm from the geometric centre. The equivalent viscous damping at the centre of the disc- shaft may be taken as 49 N-s/m. If the shaft rotates at 760 rpm, find the maximum stress in the shaft and compare it with the dead load stress in the shaft when the shaft is horizontal. Also find the power required to drive the shaft at this speed. Take E = 1.96×10^{11} N/m^2.

Solution:

Given: m = 4kg, l = 48 cm, e = 3 mm, c = 49 N-s/m, N = 760rpm

$E = 1.96 \times 10^{11}$ N/m^2, s_{max} = ?. Dia = 9 mm, $E = 1.96 \times 10^{11}$ N/m^2.

$$k = \frac{48EI}{l^3} = 27400 N/m$$

$$\omega_n = \sqrt{\frac{k}{m}} = 82.8 rad/s \qquad \frac{d}{e} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\frac{\omega}{\omega_n}\right]^2}}$$

$$\omega = \frac{2\pi N}{60} = 79.5 rad/s$$

$$\xi = \frac{c}{2\sqrt{km}} = 0.074$$

d = 0.017 m (solution)

The dynamic load on the bearings is equal to centrifugal force of the disc which is equal to the vector sum of spring force and damping force.

$$F_{dy} = \sqrt{(kd)^2 + (c\omega d)^2} = d\sqrt{k^2 + (c\omega)^2} = 470N$$

The total maximum load on the shaft under dynamic conditions is the sum of above load and the dead load.

$$F_{max} = 470 + (4 \times 9.81) = 509.2N$$

The load under static conditions is

$$F_s = 4 \times 9.81 = 39.2N$$

The maximum stress, due to load acting at the centre of a simply supported shaft is

$$s = \frac{M \times dia}{I \times 2} = \frac{Fl \times dia \times 64}{4 \times 2 \times \pi \times dia^4} = 168 \times 10^4 F$$

The total maximum stress under dynamic conditions

$$s_{max} = 168 \times 10^4 \times F = 8.55 \times 10^8 \ N/m^2$$

Maximum stress under dead load

$$s_{max} = 168 \times 10^4 \times 39.2 = 6.59 \times 10^7 \ N/m^2$$

Damping torque $= T = (c\omega d)d = 1.125N - m$

Power $= \frac{2\pi NT}{60} = 90W$

MULTI DEGREE OF FREEDOM SYSTEMS

Approximate methods

4. Dunkerley's method
5. Rayleigh's method

Influence co-efficients

Numerical methods

5. Matrix iteration method
6. Stodola's method
7. Holzar's method

1. Influence co-efficents

It is the influence of unit displacement at one point on the forces at various points of a multi-DOF system.

OR

It is the influence of unit Force at one point on the displacements at various points of a multi-DOF system.

The equations of motion of a multi-degree freedom system can be written in terms of influence co-efficients. A set of influence co-efficents can be associated with each of matrices involved in the equations of motion.

$$[M]\{\ddot{x}\} + [K]\{x\} = [0]$$

For a simple linear spring the force necessary to cause unit elongation is referred as stiffness of spring. For a multi-DOF system one can express the relationship between displacement at a point and forces acting at various other points of the system by using influence co-efficents referred as stiffness influence coefficents

The equations of motion of a multi-degree freedom system can be written in terms of inverse of stiffness matrix referred as flexibility influence co-efficients.

Matrix of flexibility influence co-efficients = $[K]^{-1}$

The elements corresponds to inverse mass matrix are referred as flexibility mass/inertia co-efficients.

Matrix of flexibility mass/inertia co-efficients = $[M]^{-1}$

The flexibility influence co-efficients are popular as these coefficents give elements of inverse of stiffness matrix. The flexibility mass/inertia co-efficients give elements of inverse of mass matrix

Stiffness influence co-efficents.

For a multi-DOF system one can express the relationship between displacement at a point and forces acting at various other points of the system by using influence co-efficents referred as stiffness influence coefficents.

$$\{F\} = [K]\{x\}$$

$$[K] = \begin{bmatrix} k_{11} & {}_{12} & {}_{13} \\ k_{21} & {}_{22} & {}_{32} \\ {}_{31} & k_{32} & {}_{33} \end{bmatrix}$$

wher, k_{11}, ……..k_{33} are referred as stiffness influence coefficients k_{11}-stiffness influence coefficient at point 1 due to a unit deflection at point 1 k_{21}-stiffness influence coefficient at point 2 due to a unit deflection at point 1 stiffness influence coefficient at point 3 due to a k_{31}-unit deflection at point 1

Example-1.

Obtain the stiffness coefficients of the system shown in Fig.1.

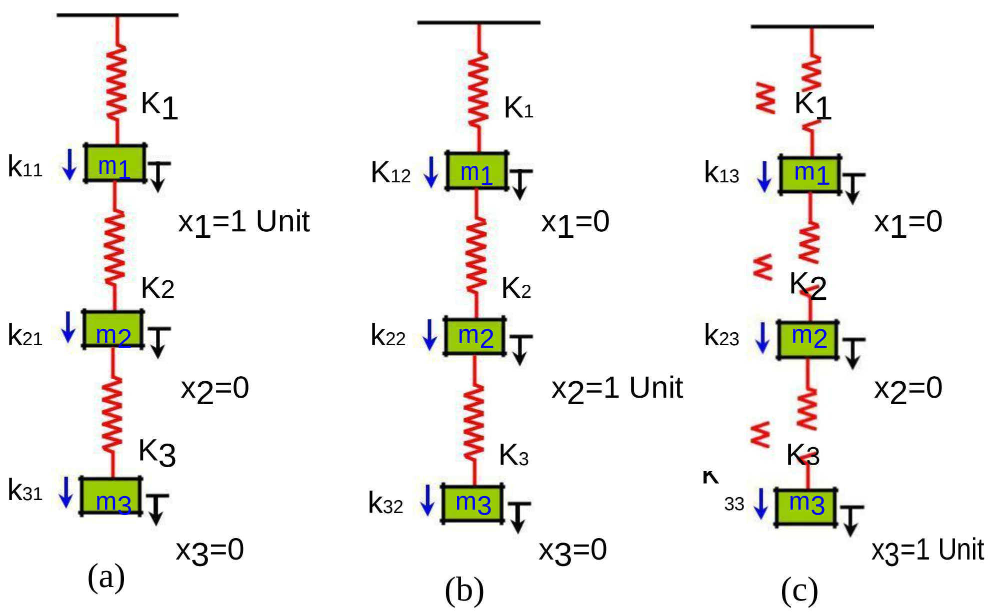


Fig.1 Stiffness influence coefficients of the system

I-step:

Apply 1 unit deflection at point 1 as shown in Fig.1(a) and write the force equilibrium equations. We get,

$k_{11} = K_1 + K_2$

5. $_{21} = -K_2$

6. $_{31} = 0$

II-step:

Apply 1 unit deflection at point 2 as shown in Fig.1(b) and write the force equilibrium equations. We get,

$$k_{12} = -K_2$$

$$\text{ii}\ {}_{22} = K_2 + K_3$$

$$\text{iii}\ {}_{31} = -K_3$$

III-step:

Apply 1 unit deflection at point 3 as shown in Fig.1(c) and write the force equilibrium equations. We get,

$$k_{13} = 0$$

$$\bullet\ {}_{23} = -K_3$$

$$\bullet\ {}_{33} = K_3$$

$$[K] = \begin{bmatrix} k_{11} & {}_{12} & {}_{13} \\ k_{21} & {}_{22} & {}_{32} \\ {}_{31} & k_{32} & {}_{33} \end{bmatrix}$$

$$[K] = \begin{bmatrix} (K_1 + K_2) & -K_2 & 0 \\ -K_2 & (K_2 + K_3) & {}_3 \\ 0 & -K_3 & {}_3 \end{bmatrix}$$

From stiffness coefficients K matrix can be obtained without writing Eqns. of motion.

Flexibility influence co-efficents.

$$\{F\} = [K]\{x\}$$

$$\{x\} = [K]^{-1}\{F\}$$

$$\{x\} = [\alpha]\{F\}$$

where, $[\alpha]$ - Matrix of Flexibility influence co-efficents given by

$$[\alpha] = \begin{bmatrix} \alpha_{11} & {}_{12} & {}_{13} \\ \alpha_{21} & \alpha_{22} & {}_{32} \\ {}_{31} & \alpha_{32} & {}_{33} \end{bmatrix}$$

wher, α_{11}, ……..α_{33} are referred as stiffness influence coefficients

α_{11}-flexibility influence coefficient at point 1 due to a unit force at point 1

α_{21}- flexibility influence coefficient at point 2 due to a unit force at point 1

α_{31}- flexibility influence coefficient at point 3 due to a unit force at point 1

Example-2.

Obtain the flexibility coefficients of the system shown in Fig.2.

I-step:

Apply 1 unit Force at point 1 as shown in Fig.2(a) and write the force equilibrium equations. We get,

$$\alpha_{11} = \alpha_{21} = \alpha_{31} = \frac{1}{K_1}$$

II-step:

Apply 1 unit Force at point 2 as shown in Fig.2(b) and write the force equilibrium equations. We get,

$$\alpha_{22} = \alpha_{32} = \frac{1}{K_1} + \frac{1}{K_2}$$

III-step:

Apply 1 unit Force at point 3 as shown in Fig.2(c) and write the force equilibrium equations. We get,

$$\alpha_{23} = \frac{1}{K_1} + \frac{1}{K_2}$$

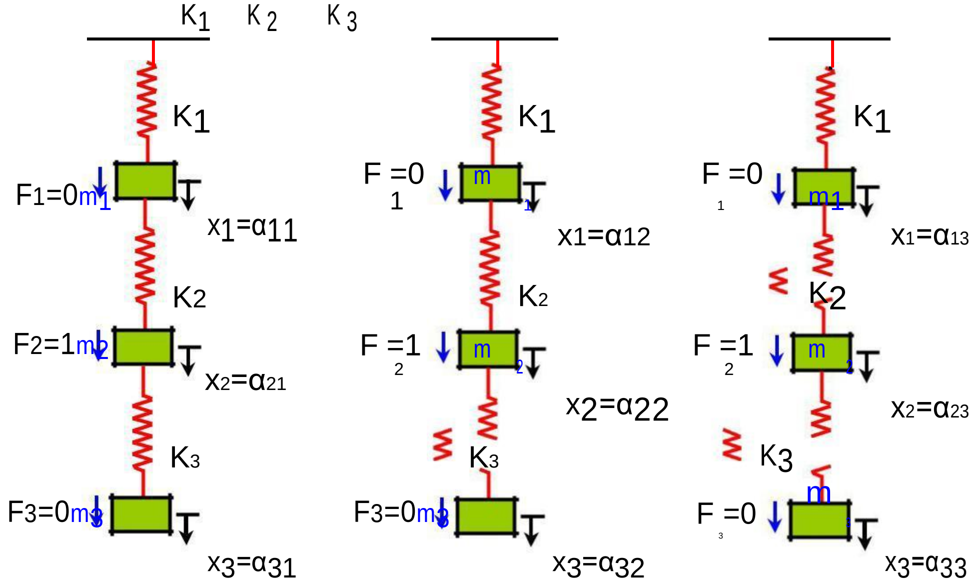
$$\alpha_{33} = \frac{1}{K_1} + \frac{1}{K_2} + \frac{1}{K_3}$$

Therefore,

$$\alpha_{11} = \alpha_{21} = \alpha_{12} = \alpha_{31} = \alpha_{13} = \frac{1}{K_1}$$

$$\alpha_{22} = \alpha_{32} = \alpha_{23} = \frac{1}{K_1} + \frac{1}{K_2}$$

$$\alpha_{33} = \frac{1}{K_1} + \frac{1}{K_2} + \frac{1}{K_3}$$



(a) (b) (c)

Fig.2 Flexibility influence coefficients of the system

For simplification, let us consider : $K_1 = K_2 = K_3 = K$

$$\alpha_{11} = \alpha_{21} = \alpha_{12} = \alpha_{31} = \alpha_{13} = \frac{1}{K_1} = \frac{1}{K}$$

$$\alpha_{22} = \alpha_{32} = \alpha_{23} = \frac{1}{K} + \frac{1}{K} = \frac{2}{K}$$

$$\alpha_{33} = \frac{1}{K} + \frac{1}{K} + \frac{1}{K} = \frac{3}{K}$$

$$[\alpha] = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix}$$

$$[\alpha] = \frac{1}{K}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$[\alpha] = [K]^{-1}$$

In Vibration analysis if there is need of $[K]^{-1}$ one can use flexibility co-efficent matrix.

Example-3

Obtain of the Flexibility influence co-efficents of the pendulum system shown in the Fig.3.

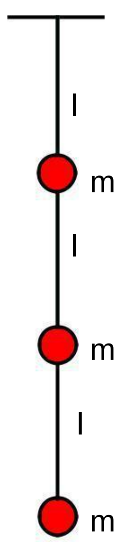


Fig.3 Pendulum system

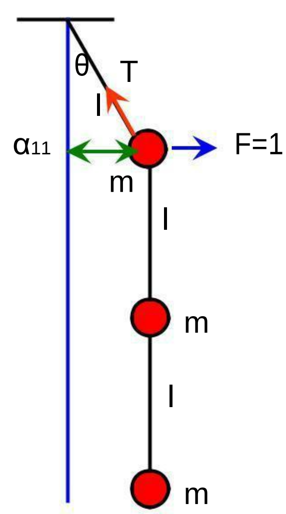


Fig.4 Flexibility influence co-efficents

I-step:

Apply 1 unit Force at point 1 as shown in Fig.4 and write the force equilibrium equations. We get,

$T \sin\theta = l$

$T \cos\theta = g(m + m + m) = 3mg$

$\tan\theta = \dfrac{1}{3mg}$

θ is small, $\tan\theta = \sin\theta$

$\sin\theta = \dfrac{\alpha_{11}}{l}$

$\alpha_{11} = l \sin\theta$

$\alpha_{11} = \dfrac{l}{3mg}$

Similarly apply 1 unit force at point 2 and next at point 3 to obtain,

$\alpha_{22} = \dfrac{l}{5mg}$

the influence coefficients are:

$\alpha_{11} = \alpha_{21} = \alpha_{12} = \alpha_{31} = \alpha_{13} = \dfrac{l}{5mg}$

$\alpha_{22} = \alpha_{32} = \alpha_{23} = \dfrac{11l}{6mg}$

$\alpha_{33} = \dfrac{11l}{6mg}$

Approximate methods

In many engineering problems it is required to quickly estimate the first (fundamental) natural frequency. Approximate methods like Dunkerley's method, Rayleigh's method are used in such cases.

(i) Dunkerley's method

Dunkerley's formula can be determined by frequency equation,

(c) $\omega^2 [M] + [K] = [0]$

(d) $[K] + \omega^2 [M] = [0]$ (e)

$\dfrac{1}{\omega_1^2}[I] + [K]^{-1}[M] = [0]$

(f) $\dfrac{1}{\omega_1^2}[I] + [\alpha][M] = [0]$

For n DOF systems,

$$-\frac{1}{\omega^2}\begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix} + \begin{vmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & & & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{vmatrix} \begin{vmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & m_n \end{vmatrix} = [0]$$

$$\begin{vmatrix} -\frac{1}{\omega^2} + \alpha_{11}m_1 & \alpha_{12}m_2 & \cdots & \alpha_{1n}m_n \\ \alpha_{21}m_1 & -\frac{1}{\omega^2} + \alpha_{22}m_2 & \cdots & \alpha_{2n}m_n \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{n1}m_1 & \alpha_{n2}m_2 & \cdots & -\frac{1}{\omega^2} + \alpha_{nn}m_n \end{vmatrix} = [0]$$

Solve the determinant

$$\left(\frac{1}{\omega^2}\right)^n - (\alpha_{11}m_1 + \alpha_{22}m_2 + \dots + \alpha_{nn}m_n)\left(\frac{1}{\omega^2}\right)^{n-1} + (\alpha_{11}\alpha_{22}m_1m_2 + \alpha_{11}\alpha_{33}m_1m_3 + \dots + \alpha_{nn}m_n + \dots) = [0] \qquad (1)$$

It is the polynomial equation of nth degree in $(1/\omega^2)$. Let the roots of above Eqn. are:

$$\frac{1}{\omega_1^2}, \frac{1}{\omega_2^2}, \dots\dots \frac{1}{\omega_n^2}$$

$$\left(\frac{1}{\omega^2} - \frac{1}{\omega_1^2}\right)\left(\frac{1}{\omega^2} - \frac{1}{\omega_2^2}\right)\dots\left(\frac{1}{\omega^2} - \frac{1}{\omega_n^2}\right)$$

$$= \left(\frac{1}{\omega^2}\right)^n - \left(\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \dots\dots + \frac{1}{\omega_n^2}\right)\left(\frac{1}{\omega^2}\right)^{n-1} - \dots = 0 \qquad (2)$$

Comparing Eqn.(1) and Eqn. (2), we get,

$$\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \dots\dots + \frac{1}{\omega_n^2} = (\alpha_{11}m_1 + \alpha_{22}m_2 + \dots + \alpha_{nn}m_n)$$

In mechanical systems higher natural frequencies are much larger than the fundamental (first) natural frequencies. Approximately, the first natural frequency is:

$$\frac{1}{\omega_1^2} \cong (\alpha_{11}m_1 + \alpha_{22}m_2 + \dots + \alpha_{nn}m_n)$$

The above formula is referred as Dunkerley's formula, which can be used to estimate first natural frequency of a system approximately.

The natural frequency of the system considering only mass m_1 is:

$$\omega_{1n} = \sqrt{\frac{1}{\alpha_{11}m_1}} = \sqrt{\frac{K_1}{m_1}}$$

The Dunkerley's formula can be written as:

$$\frac{1}{\omega_1^2} \cong \frac{1}{\omega_{1n}^2} + \frac{1}{\omega_{2n}^2} + \dots\dots + \frac{1}{\omega_{nn}^2} \qquad (3)$$

where, ω_{1n} , ω_{2n} , ... are natural frequency of single degree of freedom system considering each mass separately.

The above formula given by Eqn. (3) can be used for any mechanical/structural system to obtain first natural frequency

Examples: 1

Obtain the approximate fundamental natural frequency of the system shown in Fig.5 using Dunkerley's method.

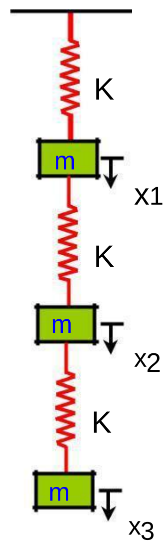


Fig.5 Linear vibratory system

Dunkerley's formula is:

$$\frac{1}{\omega_1^2} \cong \left(\alpha_{11}m_1 + \alpha_{22}m_2 + \ldots + \alpha_{nn}m_n\right) \text{ OR}$$

$$\frac{1}{\omega_1} \cong \frac{1}{\omega_{1n}} + \frac{1}{\omega_{2n}} + \ldots\ldots + \frac{1}{\omega_{nn}}$$

Any one of the above formula can be used to find fundamental natural frequency approximately.

Find influence flexibility coefficients.

$$\alpha_{11} = \alpha_{21} = \alpha_{12} = \alpha_{31} = \alpha_{13} = \frac{1}{K}$$

$$\alpha_{22} = \alpha_{32} = \alpha_{23} = \frac{2}{K}$$

$$\alpha_{33} = \frac{3}{K}$$

Substitute all influence coefficients in the Dunkerley's formula.

$$\frac{1}{\omega_1^2} \cong \left(\alpha_{11}m_1 + \alpha_{22}m_2 + \ldots + \alpha_{nn}m_n\right)$$

$$\frac{1}{\;} \cong \frac{m}{\;} + \frac{2m}{\;} + \frac{3m}{\;} = \frac{6m}{\;}$$

ω_1^2

K K K K

$\omega_1 = 0.40\sqrt{K/m}$ rad/s

Examples: 2

Find the lowest natural frequency of the system shown in Figure by Dunkerley's method. Take m_1=100 kg, m_2=50 kg

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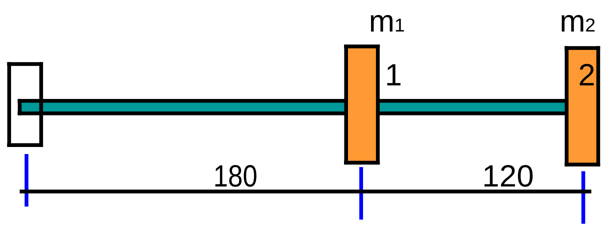

Fig.6 A cantilever rotor system.

Obtain the influence co-efficents:

$$\alpha_{11} = \frac{1.944\times10^{-3}}{EI}$$

$$\alpha_{22} = \frac{9\times10^{-3}}{EI}$$

$$\frac{1}{\omega_n^2} \cong (\alpha_{11}m_1 + \alpha_{22}m_2)$$

$$\omega_n = 1.245 \text{ rad/s}$$

(ii) Rayleigh's method

It is an approximate method of finding fundamental natural frequency of a system using energy principle. This principle is largely used for structural applications.

Principle of Rayleigh's method

Consider a rotor system as shown in Fig.7. Let, m_1, m_2 and m_3 are masses of rotors on shaft supported by two bearings at A and B and y_1, y_2 and y_3 are static deflection of shaft at points 1, 2 and 3.

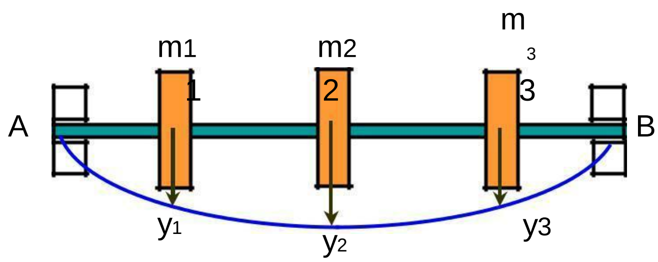

Fig.7 A rotor system.

For the given system maximum potential energy and kinetic energies are:

$$V_{max} = \frac{1}{2}\sum_{i=1}^{n} m_i g y_i \tag{4}$$

$$T_{max} = \frac{1}{2}\sum_{i=1}^{n} m_i \dot{y}_i^2$$

where, m_i- masses of the system, y_i –displacements at mass points. Considering the system vibrates with SHM,

$$\dot{y}_i = \omega^2 y_i$$

From above equations

$$T_{max} = \frac{\omega^2}{2}\sum_{i=1}^{n} m_i y_i^2 \tag{5}$$

According to Rayleigh's method,

$$V_{max} = T_{max} \tag{6}$$

substitute Eqn. (4) and (5) in (6)

$$\omega^2 = \frac{\sum_{i=1}^{n} m_i g y_i}{\sum_{i=1}^{n} m_i y_i^2} \tag{7}$$

The deflections at point 1, 2 and 3 can be found by.

$$y_1 = \alpha_{11}m_1g + \alpha_{12}m_2g + \alpha_{13}m_3g$$

$$\omega_2 = \alpha_{21}m_1g + \alpha_{22}m_2g + \alpha_{23}m_3g$$

$$\omega_3 = \alpha_{31}m_1g + \alpha_{32}m_2g + \alpha_{33}m_3g$$

Eqn.(7) is the Rayleigh's formula, which is used to estimate frequency of transverse vibrations of a vibratory systems.

Examples: 1

Estimate the approximate fundamental natural frequency of the system shown in Fig.8 using Rayleigh's method. Take: m=1kg and K=1000 N/m.

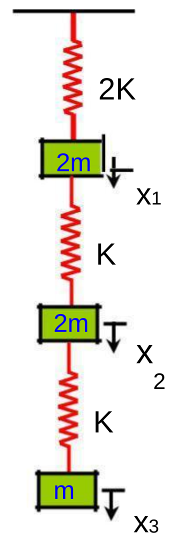

Fig.8 Linear vibratory system

Obtain influence coefficients,

$$\alpha_{11} = \alpha_{21} = \alpha_{12} = \alpha_{31} = \alpha_{13} = \frac{1}{2K}$$

$$\alpha_{22} = \alpha_{32} = \alpha_{23} = \frac{3}{2K}$$

$$\alpha_{33} = \frac{5}{2K}$$

Deflection at point 1 is:

$$y_1 = \alpha_{11}m_1g + \alpha_{12}m_2g + \alpha_{13}m_3g$$

$$y_1 = \frac{mg}{2K}(2+2+1) = \frac{5mg}{2K} = \frac{5g}{2000}$$

Deflection at point 2 is:

$$y_2 = \alpha_{21}m_1g + \alpha_{22}m_2g + \alpha_{23}m_3g$$

$$y_2 = \frac{mg}{2K}(2+6+3) = \frac{11mg}{2K} = \frac{11g}{2000}$$

Deflection at point 3 is:

$$y_3 = \alpha_{31}m_1g + \alpha_{32}m_2g + \alpha_{33}m_3g$$

$$y_3 = \frac{mg}{2K}(2+6+5) = \frac{13mg}{2K} = \frac{13g}{2000}$$

Rayleigh's formula is:

$$\omega^2 = \frac{\sum_{i=1}^{n} {}_{ii}}{\sum_{i=1}^{n} m_i \, y_i{}^2}$$

$$\omega^2 = \frac{2\times\frac{5}{2000} + 2\times\frac{11}{2000} + 2\times\frac{13}{2000}\, g}{2\left(\frac{5}{2000}\right)^2 + 2\left(\frac{11}{2000}\right)^2 + 2\left(\frac{13}{2000}\right)^2 g}$$

$$\omega = 12.41 \text{ rad/s}$$

Examples: 2

Find the lowest natural frequency of transverse vibrations of the system shown in Fig.9 by Rayleigh's method.

E=196 GPa, I=10-6 m4, m1=40 kg, m2=20 kg

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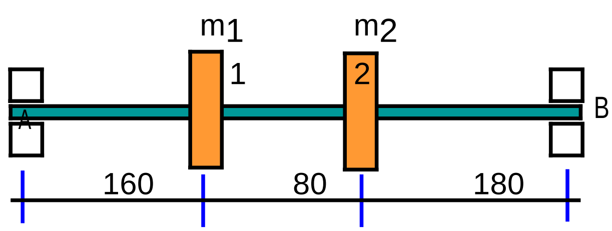

Fig.9 A rotor system.

Step-1:

Find deflections at point of loading from strength of materials principle.

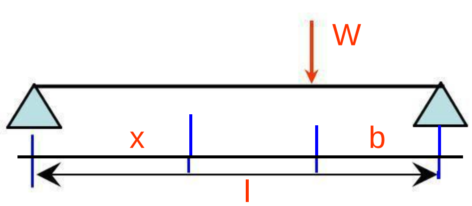

Fig.10 A simply supported beam

For a simply supported beam shown in Fig.10, the deflection of beam at distance x from left is given by:

$$y = \frac{Wbx}{6EIl}\left(l^2 - x^2 - b^2\right) \text{ for } x \le (l - b)$$

For the given problem deflection at loads can be obtained by superposition of deflections due to each load acting separately.

Deflections due to 20 kg mass

$$y_1' = \frac{(9.81x20)x0.18x0.16}{6EI0.42}\left(0.42^2 - 0.16^2 - 0.18^2\right) = \frac{0.265}{EI}$$

$$y_2' = \frac{(9.81x20)x0.18x0.24}{6EIx0.42}\left(0.42^2 - 0.24^2 - 0.18^2\right) = \frac{0.29}{EI}$$

Deflections due to 40 kg mass

$$y_1'' = \frac{(9.81x40)x0.16x0.26}{6EIx0.42}\left(0.42^2 - 0.26^2 - 0.16^2\right) = \frac{0.538}{EI}$$

$$y_2'' = \frac{(9.81x40)x0.16x0.18}{6EIx0.42}\left(0.42^2 - 0.18^2 - 0.16^2\right) = \frac{0.53}{EI}$$

The deflection at point 1 is:

$$y_1 = y_1' + y_1'' = \frac{0.803}{EI}$$

The deflection at point 2 is:

$$y_2 = y_2' + y_2'' = \frac{0.82}{EI}$$

$$\omega^2 = \frac{\sum_{i=1}^{n} m_i g y_i}{\sum_{i=1}^{n} m_i y_i^2}$$

$$X^2 = \frac{9.81(40x0.803 + 20x0.82)}{(40x0.803^2) + (20x0.82^2)}$$

$$\omega_n = 1541.9 \text{ rad/s}$$

Numerical methods

(i) Matrix iteration method

Using this method one can obtain natural frequencies and modal vectors of a vibratory system having multi-degree freedom.

It is required to have $\omega_1 < \omega_2 < \ldots\ldots\ldots < \omega_n$

Eqns. of motion of a vibratory system (having n DOF) in matrix form can be written as:

$$[M]\{\ddot{x}\} + [K]\{x\} = [0]$$

where,

$$\{x\} = \{A\}\sin(\omega t + \varphi) \tag{8}$$

substitute Eqn.(8) in (9)

$$-\omega^2 [M]\{A\} + [K]\{A\} = [0] \tag{9}$$

For principal modes of oscillations, for r^{th} mode,

$$-\omega_r^2 [M]\{A\}_r + [K]\{A\}_r = [0]$$

$$[K]^{-1}[M]\{A\}_r = \frac{1}{\omega_r^2}\{A\}_r$$

$$[D]\{A\}_r = \frac{1}{\omega_r^2}\{A\}_r \tag{10}$$

where, $[D]$ is referred as Dynamic matrix.

Eqn.(10) converges to first natural frequency and first modal vector.
The Equation,

$$[M]^{-1}[K]\{A\}_r = \omega_r^2\{A\}_r$$

$$[D_1]\{A\}_r = \omega_r^2\{A\}_r \tag{11}$$

where, $[D_1]$ is referred as inverse dynamic matrix.

Eqn.(11) converges to last natural frequency and last modal vector.

In above Eqns (10) and (11) by assuming trial modal vector and iterating till the Eqn is satisfied, one can estimate natural frequency of a system.

Examples: 1

Find first natural frequency and modal vector of the system shown in the Fig.10 using matrix iteration method. Use flexibility influence co-efficients.

Find influence coefficients.

$$\alpha_{11} = \alpha_{21} = \alpha_{12} = \alpha_{31} = \alpha_{13} = \frac{1}{2K}$$

$$\alpha_{22} = \alpha_{32} = \alpha_{23} = \frac{3}{2K}$$

$$\alpha_{33} = \frac{5}{2K}$$

$$[\alpha] = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{32} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix}$$

$$[\alpha] = [K]^{-1} = \frac{1}{2K}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 5 \end{bmatrix}$$

First natural frequency and modal vector

$$[K]^{-1}[M]\{A\}_r = \frac{1}{\omega_r}\{A\}_r$$

$$[D]\{A\}_r = \frac{1}{\omega_r}\{A\}_r$$

Obtain Dynamic matrix $[D] = [K]^{-1}[M]$

$$[D] = \frac{m}{2K}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 5 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{m}{2K}\begin{bmatrix} 2 & 2 & 1 \\ 2 & 6 & 3 \\ 2 & 6 & 5 \end{bmatrix}$$

Use basic Eqn to obtain first frequency

$$[D]\{A\}_1 = \frac{1}{\omega_r}\{A\}_1$$

Assume trial vector and substitute in the above Eqn.

Assumed vector is: $\{u\}_1 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$

First Iteration

$$[D]\{u\}_1 = \frac{m}{2K}\begin{bmatrix} 2 & 2 & 1 \\ 2 & 6 & 3 \\ 2 & 6 & 5 \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \frac{5m}{2K}\begin{Bmatrix} 1 \\ 2.2 \\ 2.6 \end{Bmatrix}$$

As the new vector is not matching with the assumed one, iterate again using the new vector as assumed vector in next iteration.

Second Iteration

$$[D]\{u\}_2 = \frac{m}{2K}\begin{bmatrix}2 & 2 & 1\\ 2 & 6 & 3\\ & 6 & 5\end{bmatrix}\begin{Bmatrix}1\\ 2.2\\ 2.6\end{Bmatrix} = \frac{4.5m}{K}\begin{Bmatrix}1\\ 2.55\\ 3.13\end{Bmatrix}$$

Third Iteration

$$[D]\{u\}_3 = \frac{m}{2K}\begin{bmatrix}2 & 2 & 1\\ 2 & 6 & 3\\ & 6 & 5\end{bmatrix}\begin{Bmatrix}1\\ 2.555\\ 3.133\end{Bmatrix} = \frac{5.12m}{K}\begin{Bmatrix}1\\ 2.61\\ 3.22\end{Bmatrix}$$

Fourth Iteration

$$[D]\{u\}_4 = \frac{m}{2K}\begin{bmatrix}2 & 2 & 1\\ 2 & 6 & 3\\ & 6 & 5\end{bmatrix}\begin{Bmatrix}1\\ 2.61\\ 3.22\end{Bmatrix} = \frac{5.22m}{K}\begin{Bmatrix}1\\ 2.61\\ 3.23\end{Bmatrix}$$

As the vectors are matching stop iterating. The new vector is the modal vector. To obtain the natural frequency,

$$[D]\begin{Bmatrix}1\\ 2.61\\ 3.22\end{Bmatrix} = \frac{5.22m}{K}\begin{Bmatrix}1\\ 2.61\\ 3.23\end{Bmatrix}$$

Compare above Eqn with with basic Eqn.

$$[D]\{A\}_1 = \frac{1}{\omega_1^2}\{A\}_1$$

$$\frac{1}{\omega_1^2} = \frac{5.22m}{K}$$

$$\omega_1^2 = \frac{1}{5.22}\frac{K}{m}$$

$$\omega_1 = 0.437\sqrt{\frac{K}{m}} \text{ Rad/s}$$

Modal vector is:

$$\{A\}_1 = \begin{Bmatrix}1\\ 2.61\\ 3.23\end{Bmatrix}$$

Method of obtaining natural frequencies in between first and last one (Sweeping Technique)

For understanding it is required to to clearly understand Orthogonality principle of modal vectors.

Orthogonality principle of modal vectors

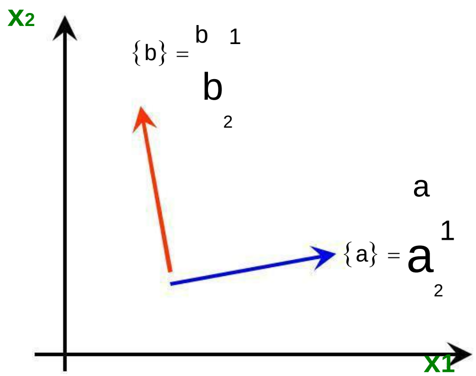

Fig.11 Vector representation graphically

Consider two vectors shown in Fig.11. Vectors $\{a\}$ and $\{b\}$ are orthogonal to each other if and only if

$$\{a\}^T \{b\} = 0$$

$$\{a_1 \;\; a_2\} \begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix} = 0$$

$$\{a_1 \;\; a_2\} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix} = 0$$

$$\{a\}^T [I]\{b\} = 0 \tag{12}$$

where, $[I]$ is Identity matrix.

From Eqn.(12), Vectors $\{a\}$ and $\{b\}$ are orthogonal to each other with respect to identity matrix.

Application of orthogonality principle in vibration analysis

Eqns. of motion of a vibratory system (having n DOF) in matrix form can be written as:

$$[M]\{\ddot{x}\} + [K]\{x\} = [0]$$

$$\{x\} = \{A\}\sin(\omega t + \varphi)$$

$$-\omega^2 [M]\{A\}_1 + [K]\{A\}_1 = [0]$$

$$\omega^2 [M]\{A\}_1 = [K]\{A\}_1$$

If system has two frequencies ω_1 and ω_2

$$\omega_1^2 [M]\{A\}_1 = [K]\{A\}_1 \qquad (13)$$

$$\omega_2^2 [M]\{A\}_2 = [K]\{A\}_2 \qquad (14)$$

Multiply Eqn.(13) by $\{A\}_2^T$ and Eqn.(14) by $\{A\}_1^T$

$$\omega_1^2 \{A\}_2^T [M]\{A\}_1 = \{A\}_2^T [K]\{A\}_1 \qquad (15)$$

$$\omega_2^2 \{A\}_1^T [M]\{A\}_2 = \{A\}_1^T [K]\{A\}_2 \qquad (16)$$

Eqn.(15)-(16)

$$\{A\}_1^T [M]\{A\}_2 = 0$$

Above equation is a condition for mass orthogonality.

$$\{A\}_1^T [K]\{A\}_2 = 0$$

Above equation is a condition for stiffness orthogonality.

By knowing the first modal vector one can easily obtain the second modal vector based on either mass or stiffness orthogonality. This principle is used in the matrix iteration method to obtain the second modal vector and second natural frequency. This technique is referred as **Sweeping technique**

Sweeping technique

After obtaining $\{A\}_1$ and ω_1 to obtain $\{A\}_2$ and ω_2 choose a trial vector $\{V\}_1$ orthogonal to $\{A\}_1$,which gives constraint Eqn.:

$$\{V\}_1^T [M]\{A\}_1 = 0$$

$$\{V_1 \quad V_2 \quad V_3\}_1 \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix}_1 = 0$$

$$\{(V_1 m_1 A_1) + (V_2 m_2 A_2) + (V_3 m_3 A_3)\} = 0$$

$$\{(m_1 A_1)V_1 + (m_2 A_2)V_2 + (m_3 A_3)V_3\} = 0$$

$$V_1 = \alpha V_2 + \beta V_3$$

where α and β are constants

$$\alpha = -\frac{m_2 A_2}{m_1 A_1}$$

$$\beta = -\frac{m_3 A_3}{m_1 A_1}$$

Therefore the trial vector is:

$$\begin{Bmatrix} V_1 \\ V_2 \\ V_3 \end{Bmatrix} = \begin{Bmatrix} (\alpha V_2 + \beta V_3) \\ V_2 \\ V_3 \end{Bmatrix}$$

$$= \begin{bmatrix} 0 & \alpha & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} V_1 \\ V_2 \\ V_3 \end{Bmatrix}$$

$$= [S]\{V\}_1$$

where = $[S]$ is referred as Sweeping matrix and $\{V\}_1$ is the trial vector. New dynamics matrix is:

$$[D_s] = [D][S]$$

$$[D_s]\{V\}_1 = \frac{1}{\omega_2^2}\{A\}_2$$

The above Eqn. Converges to second natural frequency and second modal vector. This method of obtaining frequency and modal vectors between first and the last one is referred as sweeping technique.

Examples: 2

For the Example problem 1, Find second natural frequency and modal vector of the system shown in the Fig.10 using matrix iteration method and Sweeping technique. Use flexibility influence co-efficients.

For this example already the first frequency and modal vectors are obtained by matrix iteration method in Example 1. In this stage only how to obtain second frequency is demonstrated.

First Modal vector obtained in Example 1 is:

$$\{A\}_1 = \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2.61 \\ 3.23 \end{Bmatrix}$$

$$[M] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ is the mass matrix

Find sweeping matrix

$$[S] = \begin{bmatrix} 0 & \alpha & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\alpha = -\frac{m_2 A_2}{m_1 A_1} = -\frac{2(2.61)}{2(1)} = -2.61$$

$$\beta = -\frac{m_3 A_3}{m_1 A_1} = -\frac{1(3.23)}{2(1)} = -1.615$$

Sweeping matrix is:

$$[S] = \begin{bmatrix} 0 & -2.61 & -1.615 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

New Dynamics matrix is:

$$[D_S] + [D][S]$$

$$[D_S] = \frac{m}{2K}\begin{bmatrix} 2 & 2 & 1 \\ 2 & 6 & 3 \\ 1 & 3 & 5 \end{bmatrix}\begin{bmatrix} 0 & -2.61 & -1.615 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{m}{K}\begin{bmatrix} 0 & -1.61 & -1.11 \\ 0 & 0.39 & -0.11 \\ 0 & 0.39 & 1.89 \end{bmatrix}$$

First Iteration

$$[D_s]\{V_1\} = \frac{1}{\omega^2}\{A_2\}$$

$$\frac{m}{K}\begin{bmatrix} 0 & -1.61 & -1.11 \\ 0 & 0.39 & -0.11 \\ 0 & 0.39 & 1.89 \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \frac{m}{K}\begin{Bmatrix} -2.27 \\ 0.28 \\ 2.28 \end{Bmatrix} = \frac{0.28m}{K}\begin{Bmatrix} -9.71 \\ 1 \\ 8.14 \end{Bmatrix}$$

Second Iteration

$$\frac{m}{K}\begin{bmatrix} 0 & -1.61 & -1.11 \\ 0 & 0.39 & -0.11 \\ 0 & 0.39 & 1.89 \end{bmatrix}\begin{Bmatrix} -9.71 \\ 1 \\ 8.14 \end{Bmatrix} = \frac{m}{K}\begin{Bmatrix} -10.64 \\ -0.50 \\ 15.77 \end{Bmatrix} = \frac{0.5m}{K}\begin{Bmatrix} -21.28 \\ -1 \\ 31.54 \end{Bmatrix}$$

Third Iteration

$$\frac{m}{K}\begin{bmatrix} 0 & -1.61 & -1.11 \\ 0 & 0.39 & -0.11 \\ 0 & 0.39 & 1.89 \end{bmatrix}\begin{Bmatrix} -21.28 \\ -1 \\ 31.54 \end{Bmatrix} = \frac{m}{K}\begin{Bmatrix} -33.39 \\ -3.85 \\ 59.52 \end{Bmatrix} = \frac{3.85m}{K}\begin{Bmatrix} -8.67 \\ -1 \\ 15.38 \end{Bmatrix}$$

Fourth Iteration

$$\frac{m}{K}\begin{bmatrix} 0 & -1.61 & -1.11 \\ 0 & 0.39 & -0.11 \\ 0 & 0.39 & 1.89 \end{bmatrix}\begin{Bmatrix} -8.67 \\ -1 \\ 15.38 \end{Bmatrix} = \frac{m}{K}\begin{Bmatrix} -18.68 \\ -2.08 \\ 28.67 \end{Bmatrix} = \frac{2.08m}{K}\begin{Bmatrix} -8.98 \\ -1 \\ 13.78 \end{Bmatrix}$$

Fifth Iteration

$$\frac{m}{K}\begin{bmatrix} 0 & -1.61 & -1.11 \\ 0 & 0.39 & -0.11 \\ 0 & 0.39 & 1.89 \end{bmatrix}\begin{Bmatrix} -8.98 \\ 1 \\ 13.78 \end{Bmatrix} = \frac{m}{K}\begin{Bmatrix} -13.68 \\ -1.90 \\ 25.65 \end{Bmatrix} = \frac{1.90m}{K}\begin{Bmatrix} -7.2 \\ 1 \\ 13.5 \end{Bmatrix}$$

Sixth Iteration

$$\frac{m}{K}\begin{bmatrix} 0 & -1.61 & -1.11 \\ 0 & 0.39 & -0.11 \\ 0 & 0.39 & 1.89 \end{bmatrix}\begin{Bmatrix} -7.2 \\ -1 \\ 13.5 \end{Bmatrix} = \frac{m}{K}\begin{Bmatrix} -13.24 \\ -1.87 \\ 25.12 \end{Bmatrix} = \frac{1.87m}{K}\begin{Bmatrix} -7.08 \\ -1 \\ 13.43 \end{Bmatrix}$$

$$\frac{1}{\omega^2} = \frac{1087m}{K}$$

$$\omega_1^2 = \frac{1}{1.87}\frac{K}{m}$$

$$\omega_1 = 0.73\sqrt{\frac{K}{m}}$$

Modal vector

$$\{A\}_2 = \begin{Bmatrix} -1 \\ -0.14 \\ 1.89 \end{Bmatrix}$$

Similar manner the next frequency and modal vectors can be obtained.

(ii) Stodola's method

It is a numerical method, which is used to find the fundamental natural frequency and modal vector of a vibratory system having multi-degree freedom. The method is based on finding inertia forces and deflections at various points of interest using flexibility influence coefficents.

Principle / steps

1. Assume a modal vector of system. For example for 3 dof systems:

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

- Find out inertia forces of system at each mass point, $F_1 = m_1 \omega^2 x_1$ for Mass 1

  $F_2 = m_2 \omega^2 x_2$ for Mass 2

  $F_3 = m_3 \omega^2 x_3$ for Mass 3

- Find new deflection vector using flexibility influence coefficients, using the formula,

$$\begin{Bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{Bmatrix} = \begin{Bmatrix} F_1\alpha_{11} + F_2\alpha_{12} + F_3\alpha_{13} \\ F_1\alpha_{21} + F_2\alpha_{22} + F_3\alpha_{23} \\ F_1\alpha_{31} + F_2\alpha_{32} + F_3\alpha_{33} \end{Bmatrix}$$

4. If assumed modal vector is equal to modal vector obtained in step 3, then solution is converged. Natural frequency can be obtained from above equation, i.e

If $\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \cong \begin{Bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{Bmatrix}$ Stop iterating.

Find natural frequency by first equation,

$$x'_1 = 1 = F_1\alpha_{11} + F_2\alpha_{12} + F_3\alpha_{13}$$

5. If assumed modal vector is not equal to modal vector obtained in step 3, then consider obtained deflection vector as new vector and iterate till convergence.

Example-1

Find the fundamental natural frequency and modal vector of a vibratory system shown in Fig.10 using Stodola's method.

First iteration

1. Assume a modal vector of system $\{u\}_1 = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$

- Find out inertia forces of system at each mass point

$$F_1 = m_1\omega^2 x_1 = 2m\omega^2$$

$$F_2 = m_2\omega^2 x_2 = 2m\omega^2$$

$$F_3 = m_3\omega^2 x_3 = m\omega^2$$

• Find new deflection vector using flexibility influence coefficients Obtain flexibility influence coefficients of the system:

$$\alpha_{11} = \alpha_{21} = \alpha_{12} = \alpha_{31} = \alpha_{13} = \frac{1}{2K}$$

$$\alpha_{22} = \alpha_{32} = \alpha_{23} = \frac{3}{2K}$$

$$\alpha_{33} = \frac{5}{2K}$$

$$x'_1 = F_1\alpha_{11} + F_2\alpha_{12} + F_3\alpha_{13}$$

Substitute for F’s and α,s

$$x'_1 = \frac{m\omega^2}{K} + \frac{m\omega^2}{K} + \frac{m\omega^2}{2K} = \frac{5m\omega^2}{2K}$$

$$x'_2 = F_1\alpha_{21} + F_2\alpha_{22} + F_3\alpha_{23}$$

Substitute for F’s and α,s

$$x'_2 = \frac{m\omega^2}{K} + \frac{6m\omega^2}{2K} + \frac{3m\omega^2}{2K} = \frac{11m\omega^2}{2K}$$

$$x'_3 = F_1\alpha_{31} + F_2\alpha_{32} + F_3\alpha_{33}$$

Substitute for F’s and α,s

$$x'_3 = \frac{m\omega^2}{K} + \frac{6m\omega^2}{2K} + \frac{5m\omega^2}{2K} = \frac{13m\omega^2}{2K}$$

4. New deflection vector is:

$$\begin{Bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{Bmatrix} = \frac{m\omega^2}{2K}\begin{Bmatrix} 5 \\ 11 \\ 13 \end{Bmatrix}$$

$$\begin{Bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{Bmatrix} = \frac{5m\omega^2}{2K}\begin{Bmatrix} 1 \\ 2.2 \\ 2.6 \end{Bmatrix} = \{u\}_2$$

The new deflection vector $\{u\}_2 \neq \{u\}_1$. Iterate again using new deflection vector $\{u\}_2$

Second iteration

1. Initial vector of system $\{u\}_2 = \begin{Bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2.2 \\ 2.6 \end{Bmatrix}$

2. Find out inertia forces of system at each mass point

$$F'_1 = m_1 \omega^2 x'_1 = 2m\omega^2$$

$$F'_2 = m_2 \omega^2 x'_2 = 4.4m\omega^2$$

$$F'_3 = m_3 \omega^2 x'_3 = 2.6m\omega^2$$

3. New deflection vector,

$$x''_1 = F'_1\alpha_{11} + F'_2\alpha_{12} + F'_3\alpha_{13}$$

Substitute for F’s and α,s

$$x''_1 = \frac{m\omega^2}{K} + \frac{4.4m\omega^2}{2K} + \frac{2.6m\omega^2}{2K} = \frac{9m\omega^2}{2K}$$

$$x''_2 = F'_1\alpha_{21} + F'_2\alpha_{22} + F'_3\alpha_{23}$$

Substitute for F’s and α,s

$$x''_2 = \frac{m\omega^2}{K} + \frac{13.2m\omega^2}{2K} + \frac{7.8m\omega^2}{2K} = \frac{23m\omega^2}{2K}$$

$$x''_3 = F'_1\alpha_{31} + F'_2\alpha_{32} + F'_3\alpha_{33}$$

Substitute for F’s and α,s

$$x''_3 = \frac{m\omega^2}{K} + \frac{13.2m\omega^2}{2K} + \frac{13m\omega^2}{2K} = \frac{28.2m\omega^2}{2K}$$

4. New deflection vector is:

$$\begin{Bmatrix} x''_1 \\ x''_2 \\ x''_3 \end{Bmatrix} = \frac{m\omega^2}{2K} \begin{Bmatrix} 9 \\ 23 \\ 28.2 \end{Bmatrix}$$

$$\begin{Bmatrix} x''_1 \\ x''_2 \\ x''_3 \end{Bmatrix} = \frac{9m\omega^2}{2K} \begin{Bmatrix} 1 \\ 2.55 \\ 3.13 \end{Bmatrix} = \{u\}_3$$

The new deflection vector $\{u\}_3 \neq \{u\}_2$. Iterate again using new deflection vector $\{u\}_3$

Third iteration

1. Initial vector of system $\{u\}_3 = \begin{Bmatrix} x''_1 \\ x''_2 \\ x''_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2.55 \\ 3.13 \end{Bmatrix}$

2. Find out inertia forces of system at each mass point

$$F''_1 = m_1 \omega^2 x''_1 = 2m\omega^2$$

$$F''_2 = m_2 \omega^2 x''_2 = 5.1m\omega^2$$

2 2 2

$F''_3 = m\,\omega^2 x''_3 = 3.13m\omega^2$

3. new deflection vector,

$x'''_1 = F''_1\alpha_{11} + F''_2\alpha_{12} + F''_3\alpha_{13}$

Substitute for F's and α,s

$$x'''_1 = \frac{m\omega^2}{K} + \frac{5.1m\omega^2}{2K} + \frac{3.13m\omega^2}{2K} = \frac{10.23m\,\omega^2}{2K}$$

$x'''_2 = F''_1\alpha_{21} + F''_2\alpha_{22} + F''_3\alpha_{23}$

Substitute for F's and α,s

$$x'''_2 = \frac{m\omega^2}{K} + \frac{15.3m\,\omega^2}{2K} + \frac{9.39m\,\omega^2}{2K} = \frac{26.69m\,\omega^2}{2K}$$

$x'''_3 = F''_1\alpha_{31} + F''_2\alpha_{32} + F''_3\alpha_{33}$

Substitute for F's and α,s

$$x'''_3 = \frac{m\omega^2}{K} + \frac{15.3m\omega^2}{2K} + \frac{16.5m\omega^2}{2K} = \frac{28.2m\omega^2}{2K}$$

stop

4. New deflection vector is:

$$\begin{Bmatrix} x'''_1 \\ x'''_2 \\ x'''_3 \end{Bmatrix} = \frac{m\omega}{2K}\begin{Bmatrix} 10.23 \\ 26.69 \\ 33.8 \end{Bmatrix}$$

$$\begin{Bmatrix} x'''_1 \\ x'''_2 \\ x'''_3 \end{Bmatrix} = \frac{10.23m\omega^2}{2K}\begin{Bmatrix} 1 \\ 2.60 \\ 3.30 \end{Bmatrix} = \{u\}_4$$

The new deflection vector $\{u\}_4 \cong \{u\}_3$

Fundamental natural frequency can be obtained by.

$$\frac{10.23m\,\omega^2}{2K} = 1$$

$$\omega = 0.44\sqrt{\frac{K}{m}}\ \text{rad/s}$$

Modal vector is:

$$A_1 = \begin{Bmatrix} 1 \\ 2.60 \\ 3.30 \end{Bmatrix}$$

Example-2

For the system shown in Fig.12 find the lowest natural frequency by Stodola's method (carryout two iterations)

July/Aug 2005 VTU for 10 marks

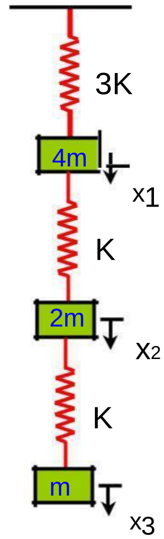

Fig.12 Linear vibratory system

Obtain flexibility influence coefficients,

$$\alpha_{11} = \alpha_{21} = \alpha_{12} = \alpha_{31} = \alpha_{13} = \frac{1}{3K}$$

$$\alpha_{22} = \alpha_{32} = \alpha_{23} = \frac{4}{3K}$$

$$\alpha_{33} = \frac{7}{3K}$$

First iteration

1. Assume a modal vector of system $\{u\}_1 = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$

- Find out inertia forces of system at each mass point

$$F_1 = m_1\omega^2 x_1 = 4m\omega^2$$

$$F_2 = m_2\omega^2 x_2 = 2m\omega^2$$

$$F_3 = m_3\omega^2 x_3 = m\omega^2$$

- New deflection vector using flexibility influence coefficients, $x'_1 = F_1\alpha_{11} + F_2\alpha_{12} + F_3\alpha_{13}$

$$x'_1 = \frac{4m\omega^2}{3K} + \frac{2m\omega^2}{3K} + \frac{m\omega^2}{3K} = \frac{7m\omega^2}{3K}$$

$$x'_2 = F_1\alpha_{21} + F_2\alpha_{22} + F_3\alpha_{23}$$

$$x'_2 = \frac{4m\omega^2}{3K} + \frac{8m\omega^2}{3K} + \frac{4m\omega^2}{3K} = \frac{16m\omega^2}{3K}$$

$$x'_3 = F_1\alpha_{31} + F_2\alpha_{32} + F_3\alpha_{33}$$

$$x'_3 = \frac{4m\omega^2}{3K} + \frac{8m\omega^2}{3K} + \frac{7m\omega^2}{3K} = \frac{19m\omega^2}{3K}$$

4. New deflection vector is:

$$\begin{Bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{Bmatrix} = \frac{m\omega^2}{3K}\begin{Bmatrix} 7 \\ \\ 19 \end{Bmatrix}$$

$$\begin{Bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{Bmatrix} = \frac{7m\omega^2}{3K}\begin{Bmatrix} 1 \\ 2.28 \\ 2.71 \end{Bmatrix} = \{u\}_2$$

The new deflection vector $\{u\}_2 \neq \{u\}_1$. Iterate again using new deflection vector $\{u\}_2$

Second iteration

1. Initial vector of system $\{u\}_2 = \begin{Bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2.28 \\ \end{Bmatrix}$

$_3$ 2.71

2. Find out inertia forces of system at each mass point

$$F'_1 = m_1 \omega^2 x'_1 = 4m\omega^2$$

$$F'_2 = m_2 \omega^2 x'_2 = 4.56m\omega^2$$

$$F'_3 = m_3 \omega^2 x'_3 = 2.71m\,\omega^2$$

3. New deflection vector

$$x''_1 = F'_1\alpha_{11} + F'_2\alpha_{12} + F'_3\alpha_{13}$$

$$x''_1 = \frac{4m\omega^2}{3K} + \frac{4.56m\omega^2}{3K} + \frac{2.71m\omega^2}{3K} = \frac{11.27m\,\omega^2}{3K}$$

$$x''_2 = F'_1\alpha_{21} + F'_2\alpha_{22} + F'_3\alpha_{23}$$

$$x''_2 = \frac{4m\omega^2}{3K} + \frac{18.24m\,\omega^2}{3K} + \frac{10.84m\,\omega^2}{3K} = \frac{33.08m\,\omega^2}{3K}$$

$$x''_3 = F'_1\alpha_{31} + F'_2\alpha_{32} + F'_3\alpha_{33}$$

$$x''_3 = \frac{4m\omega^2}{3K} + \frac{18.24m\,\omega^2}{3K} + \frac{18.97m\,\omega^2}{3K} = \frac{41.21m\,\omega^2}{3K}$$

4. New deflection vector is:

$$\begin{Bmatrix} x''_1 \\ x''_2 \\ x''_3 \end{Bmatrix} = \frac{m\omega}{3K} \begin{Bmatrix} 11.27 \\ 33.08 \\ 41.21 \end{Bmatrix}$$

$$\begin{Bmatrix} x''_1 \\ x''_2 \\ x''_3 \end{Bmatrix} = \frac{3.75m\omega^2}{K} \begin{Bmatrix} 1 \\ 2.93 \\ 3.65 \end{Bmatrix} = \{u\}_3$$

Stop Iterating as it is asked to carry only two iterations. The Fundamental natural frequency can be calculated by,

$$\frac{3.75m\omega^2}{2K} = 1$$

$$\omega = 0.52\sqrt{\frac{K}{m}}$$

Modal vector,

$$\{A\}_1 = \begin{Bmatrix} 1 \\ 2.93 \\ 3.65 \end{Bmatrix}$$

Disadvantage of Stodola’s method

Main drawback of Stodola’s method is that the method can be used to find only fundamental natural frequency and modal vector of vibratory systems. This method is not popular because of this reason.

(iii) Holzar's method

It is an iterative method, used to find the natural frequencies and modal vector of a vibratory system having multi-degree freedom.

Principle

Consider a multi dof semi-definite torsional semi-definite system as shown in Fig.13.

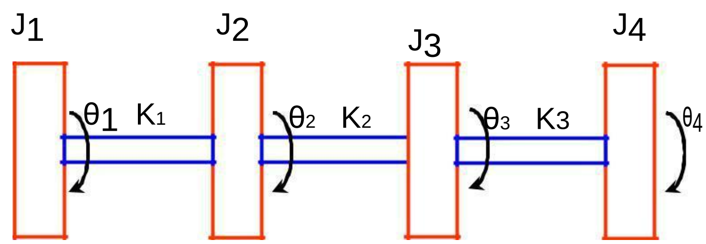

Fig.13 A torsional semi-definite system

The Eqns. of motions of the system are:

$$J_1\ddot{\theta}_1 + K_1(\theta_1 - \theta_2) = 0$$

$$J_2\ddot{\theta}_2 + K_1(\theta_2 - \theta_1) + K_2(\theta_2 - \theta_3) = 0$$

$$J_3\ddot{\theta}_3 + K_2(\theta_3 - \theta_2) + K_3(\theta_3 - \theta_4) = 0$$

$$J_4\ddot{\theta}_4 + K_3(\theta_4 - \theta_3) = 0$$

The Motion is harmonic,

$$\theta_i = \varphi_i \sin(\omega t) \tag{17}$$

where i=1,2,3,4

Substitute above Eqn.(17) in Eqns. of motion, we get,

$$\omega^2 J_1\varphi_1 = K_1(\varphi_1 - \varphi_2) \tag{18}$$

$$\omega^2 J_2\varphi_2 = K_1(\varphi_2 - \varphi_1) + K_2(\varphi_2 - \varphi_3)$$

$$\omega^2 J_3\varphi_3 = K_2(\varphi_3 - \varphi_2) + K_3(\varphi_3 - \varphi_4)$$

$$\omega^2 J_4\varphi_4 + K_3(\varphi_4 - \varphi_3) \tag{19}$$

Add above Eqns. (18) to (19), we get

$$\sum_{i=1}^{4}\omega^2 J_i\varphi_i = 0$$

For n dof system the above Eqn changes to,

$$\sum_{i=1}^{n}\omega^2 J_i\varphi_i = 0 \tag{20}$$

The above equation indicates that sum of inertia torques (torsional systems) or inertia forces (linear systems) is equal to zero for semi-definite systems.

In Eqn. (20) ω and φi both are unknowns. Using this Eqn. one can obtain natural frequencies and modal vectors by assuming a trial frequency ω and amplitude φ_1 so that the above Eqn is satisfied.

Steps involved

- Assume magnitude of a trial frequency ω
- Assume amplitude of first disc/mass (for simplicity assume $\varphi_1=1$
- Calculate the amplitude of second disc/mass φ_2 from first Eqn. of motion

$$\omega^2 J_1\varphi_1 = K_1(\varphi_1 - \varphi_2) = 0$$

$$\varphi_2 = \varphi_1 - \frac{\omega^2 J_1 \varphi_1}{K_1}$$

4. Similarly calculate the amplitude of third disc/mass φ_3 from second Eqn. of motion.

$$\omega^2 J_2\varphi_2 = K_1(\varphi_2 - \varphi_1) + K_2(\varphi_2 - \varphi_3) = 0$$

$$\omega^2 J_2 \varphi_2 = K_1\left(\varphi_1 - \frac{\omega^2 J_1 \varphi_1}{K_1} - \varphi_1\right) + K_2(\varphi_2 - \varphi_3) = 0$$

$$\omega^2 J_2 \varphi_2 = -\omega^2 J_1 \varphi_1 + K_2(\varphi_2 - \varphi_3) = 0$$

$$K_2(\varphi_2 - \varphi_3) = \omega^2 J_1 \varphi_1 + \omega^2 J_2 \varphi_2$$

$$\varphi_3 = \varphi_2 - \frac{\omega^2 J_1 \varphi_1 + \omega^2 J_2 \varphi_2}{K_2} \qquad (21)$$

The Eqn (21) can be written as:

$$\varphi_3 = \varphi_2 - \frac{\sum_{i=1}^{2} J_i \varphi_i \omega^2}{K_2}$$

5. Similarly calculate the amplitude of nth disc/mass φ_n from (n-1)th Eqn. of motion is:

$$\varphi_n = \varphi_{n-1} - \frac{\sum_{i=1}^{n} J_i \varphi_i \omega^2}{K_n}$$

6. Substitute all computed φi values in basic constraint Eqn.

$$\sum_{i=1}^{n} \omega^2 J_i \varphi_i = 0$$

- If the above Eqn. is satisfied, then assumed ω is the natural frequency, if the Eqn is not satisfied, then assume another magnitude of ω and follow the same steps.

For ease of computations, Prepare the following table, this facilitates the calculations.

Table-1. Holzar’s Table

1	2	3	4	5	6	7	8
ω	S No	J	φ	$J\omega^2\varphi$	$\sum J\omega^2\varphi$	K	$\frac{1}{K}\sum J\omega^2\varphi$

Example-1

For the system shown in the Fig.16, obtain natural frequencies using Holzar’s method**.**

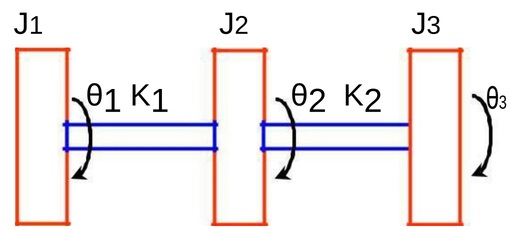

Fig.14 A torsional semi-definite system

Make a table as given by Table-1, for iterations, follow the steps discussed earlier. Assume ω from lower value to a higher value in proper steps.

Table-2. Holzar’s Table for Example-1

1	2	3	4	5	6	7	8
ω	S No	J	φ	$J\omega^2\varphi$	$\sum J\omega^2\varphi$	K	$\frac{1}{K}\sum J\omega^2\varphi$
I-iteration							
0.25	1	1	1	0.0625	0.0625	1	0.0625
	2	1	0.9375	0.0585	0.121	1	0.121
	3	1	0.816	0.051	**0.172**		
II-iteration							
0.50	1	1	1	0.25	0.25	1	0.25
	2	1	0.75	0.19	0.44	1	0.44
	3	1	0.31	0.07	**0.51**		
III-iteration							
0.75	1	1	1	0.56	0.56	1	0.56
	2	1	0.44	0.24	0.80	1	0.80
	3	1	-0.36	-0.20	**0.60**		
IV-iteration							
1.00	1	1	1	1	1	1	1
	2	1	0	0	1	1	1

	3	1	-1	-1	**0**		
V-iteration							
1.25	1	1	1	1.56	1.56	1	1.56
	2	1	-0.56	-0.87	0.69	1	0.69
	3	1	-1.25	-1.95	**-1.26**		
VI-iteration							
1.50	1	1	1	2.25	2.25	1	2.25
	2	1	-1.25	-2.82	-0.57	1	-0.57
	3	1	-0.68	-1.53	**-2.10**		
VII-iteration							
1.75	1	1	1	3.06	3.06	1	3.06
	2	1	-2.06	-6.30	-3.24	1	-3.24
	3	1	1.18	3.60	**0.36**		

Table.3 Iteration summary table

ω	$\sum J\omega^2\varphi$
0	0
0.25	0.17
0.5	0.51
0.75	0.6
1	0
1.25	-1.26
1.5	-2.1
1.75	0.36

.

The values in above table are plotted in Fig.15.

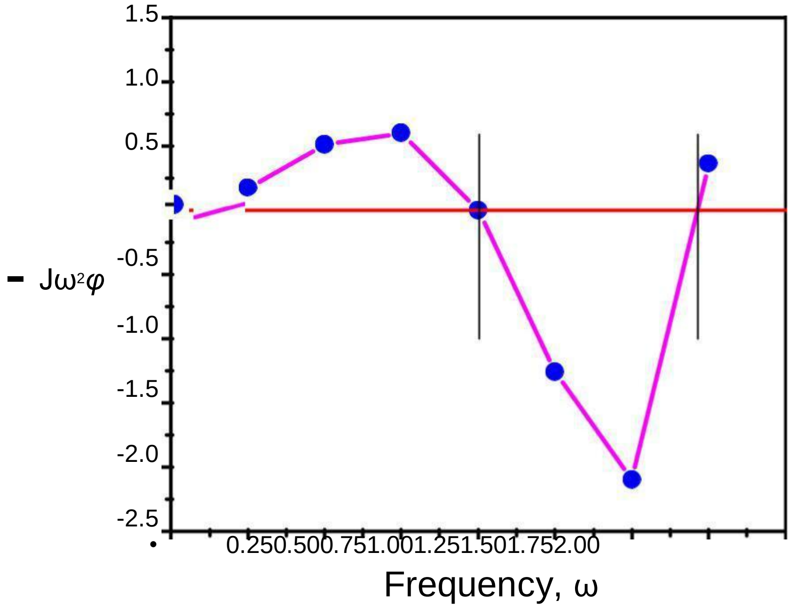

Fig.15. Holzar's plot of Table-3

From the above Graph, the values of natural frequencies are:

ω_1 = 0 rad/s ω_2

= 1 rad/s ω_3 =

1.71 rad/s

Definite systems

The procedure discussed earlier is valid for semi-definite systems. If a system is definite the basic equation Eqn. (20) is not valid. It is well-known that for definite systems, deflection at fixed point is always ZERO. This principle is used to obtain the natural frequencies of the system by iterative process. The Example-2 demonstrates the method.

Example-2

For the system shown in the figure estimate natural frequencies using Holzar's method**.**

July/Aug 2005 VTU for 20 marks

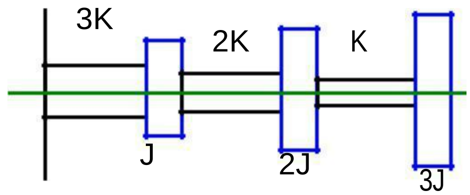

Fig.16 A torsional system

Make a table as given by Table-1, for iterations, follow the steps discussed earlier. Assume ω from lower value to a higher value in proper steps.

Table-4. Holzar's Table for Example-2

1	2	3	4	5	6	7	8
ω	S No	J	φ	$J\omega^2\varphi$	$\sum J\omega^2\varphi$	K	$\frac{1}{K}\sum J\omega^2\varphi$
I-iteration							
0.25	1	3	1	0.1875	0.1875	1	0.1875
	2	2	0.8125	0.1015	0.289	2	0.1445
	3	1	0.6679	0.0417	0.330	3	0.110
	4		**0. 557**				
II-iteration							
0.50	1	3	1	0.75	0.75	1	0.75
	2	2	0.25	0.125	0.875	2	0.437
	3	1	-0.187	-0.046	0.828	3	0.27
	4		**-0.463**				
III-iteration							
0.75	1	3	1	1.687	1.687	1	1.687
	2	2	-0.687	-0.772	0.914	2	0.457
	3	1	-1.144	-0.643	0.270	3	0.090
	4		**-1.234**				
IV- iteration							
1.00	1	3	1	3	3	1	3
	2	2	-2	-4	-1	2	-0.5

	3	1	-1.5	-1.5	-2.5	3	-0.833
	4		**-0.667**				
V-iteration							
1.25	1	3	1	4.687	4.687	1	4.687
	2	2	-3.687	-11.521	-6.825	2	-3.412
	3	1	-0.274	-0.154	-6.979	3	-2.326
	4		**2.172**				
VI-iteration							
1.50	1	3	1	6.75	6.75	1	6.75
	2	2	-5.75	-25.875	-19.125	2	-9.562
	3	1	3.31	8.572	-10.552	3	-3.517
	4		**7.327**				
1	2	3	4	5	6	7	8
ω	S No	J	φ	$J\omega^2\varphi$	$\sum J\omega^2\varphi$	K	$\frac{1}{K}\sum J\omega^2\varphi$
VII-iteration							
1.75	1	3	1	9.18	9.18	1	9.18
	2	2	-8.18	-50.06	-40.88	2	-20.44
	3	1	12.260	37.515	-3.364	3	-1.121
	4		**13.38**				
VIII-iteration							
2.0	1	3	1	12	12	1	12
	2	2	-11	-88	-76	2	-38
	3	1	-27	108	32	3	10.66
	4		**16.33**				
IX-iteration							
2.5	1	3	1	18.75	18.75	1	18.75
	2	2	-17.75	-221.87	-203.12	2	-101.56
	3	1	83.81	523.82	320.70	3	106.90
	4		**-23.09**				

Table.5 Iteration summary table

ω	φ4
0	0
0.25	0.557
0.5	-0.463
0.75	-1.234
1	-0.667
1.25	2.172
1.5	7.372
1.75	13.38
2	16.33
2.5	-23.09

The values in above table are plotted in Fig.17.

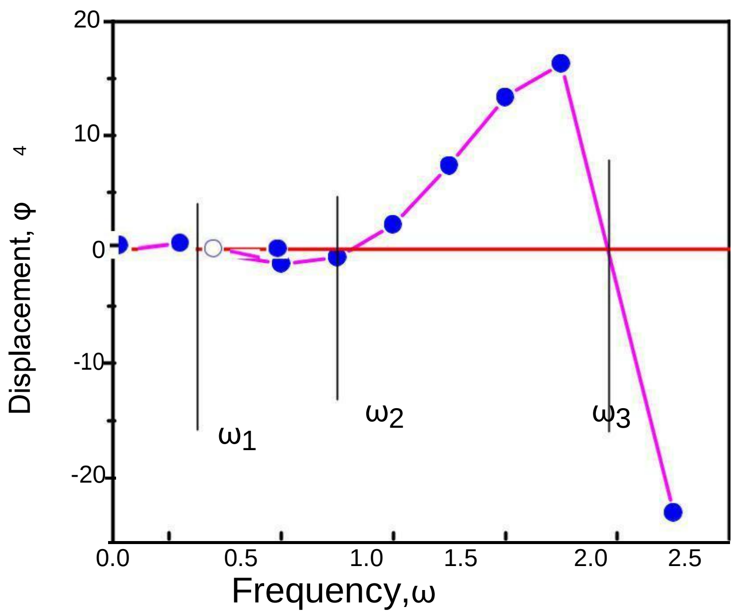

Fig.17. Holzar's plot of Table-5

From the above Graph, the values of natural frequencies are:

ω_1 = 0.35 rad/s

ω_2 = 1.15 rad/s

ω_3 = 2.30 rad/s