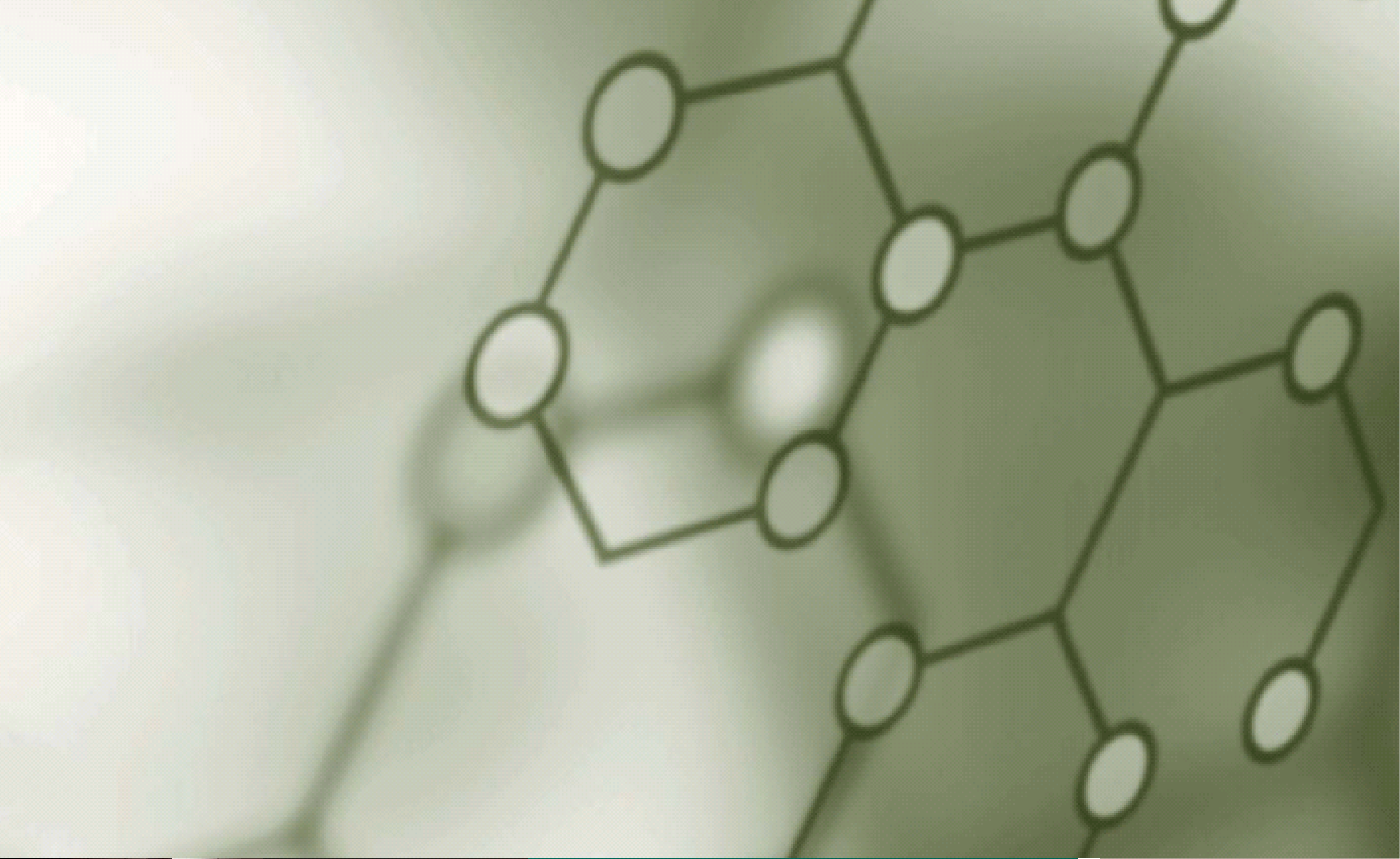

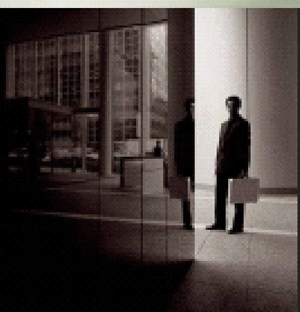

SUNGARD ADAPTIV

Risk Management and Operations Solutions

Adaptiv

Finance Guide

Last updated: November 24, 2014

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Preface

This document is a commentary on the financial principles and mathematical approaches used in Adaptiv. Its purpose is to explain any modelling assumptions thoroughly so that readers can appreciate and feel comfortable with all of the financial methods and be able to use them to the full. It also gives the user the ability to test all functions against their stated specifications.

The aim of this guide is not necessarily to cover all forms of analysis that are present in Adaptiv; many of these are covered by the Adaptiv User's Guide. This guide focuses on those forms of analysis that require financial modelling.

Some of the formulae are illustrated with spreadsheets that generate the same values as Adaptiv. This enables readers to explore the behaviour of the functions independently of the product and, perhaps, to extend the functionality into proprietary activity. The spreadsheets can be found on the Adaptiv CD.

The chapters of this document are summarized below; and the dependencies between chapters are illustrated in Figure 0.1.

- Chapter 1 shows how input market rates are transformed into the market data structures that are used for pricing deals and supplying market data.
- Chapter 2 details the pricing models used for each type of instrument.
- Chapter 4 describes the market rate sensitivities returned by the pricing models and how the portfolio reports use this information to calculate sensitivity to interest rate and FX risk factors.
- Chapter 6 describes the calculation of market value-at-risk via the variance-covariance, historical-simulation, and Monte Carlo-simulation methods. This chapter also covers: multi-step evolution of market rates, instrument ageing, and backtesting of value-at-risk against actual P&L.
- Chapter 7 covers the following topics: traditional calculation of credit exposure using add-on factors; calculation of credit exposure by multi-step Monte Carlo simulation; models of credit events and the calculation of credit value-at-risk.

Chapter 3 is an independent account of Adaptiv's fixed-income calculations.

Chapter 5 describes two hedging methods available in Adaptiv: hedging by matching sensitivities, and hedging by minimizing the variance of the portfolio's value.

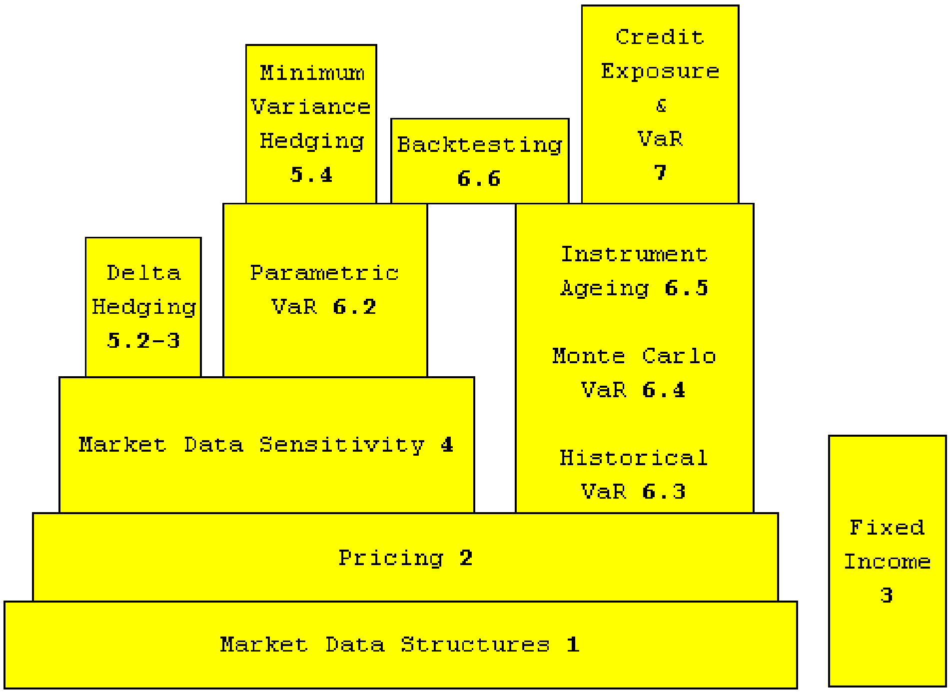


FIGURE 0.1. chapter dependencies.

CHAPTER 1

Market Data Structures

1.1. The Role of Market Data Structures

Pricing is a much bigger issue for derivatives than it is for, say, equities, commodities or even securities because there is not a consensus regarding the price of most deals. The range of traded items is much greater because most details specifying a trade are negotiable between the counterparties rather than accepted from standard contract terms. This in turn means that the process of comparing the price of a particular deal with benchmark deals, for which there is an approximate consensus on price, has a greater role.

In Adaptiv, market data structures are those objects that relate the price of illiquid deals to standard, liquid ones. Examples of market data structures are given in Table 1.1.

Many aspects of instrument definition that are needed for pricing are defined in its terms and conditions without reference to any market data. Thus, a swap can generate the coupon amounts on the fixed side without reference to the yield curve. However, to calculate the present value of these amounts, the swap must obtain discount factors from the yield curve. Similarly, the swap knows the index and reset date of each floating rate but to estimate a floating rate in advance of the reset date, it must consult the yield curve.

A market data structure provides services to instruments that need non-contractual data for pricing and analysis. These services take the form of provision of various kinds of derived market data which we refer to as *pricing factors*. Each type of market data structure has a natural, preferred way of storing its market data as pricing factors. For yield curves, for example, this is as discount factors (which are basis-independent).

The most elementary kind of market data structure is one in which the user supplies the structure with predigested pricing factors. For example, spot FX rates on the FX grid. More commonly, the market data structure comprises a set of reference

	Yield Curve	**FX Grid**	**Cap Volatility Surface**
Liquid Instrument	Cash Future or FRA Swap or Bond	Spot FX	Future Option Cap
Pricing Factors	Discount Factor	Spot FX Rate	Caplet Volatility
Risk Factors	Zero-coupon Rate	Spot FX Rate	ATM Caplet Volatility

TABLE 1.1. examples of market data structures.

instruments, together with information on their current market price, from which it derives the pricing factors.

The pricing model of an instrument requests the pricing factors it requires from the available set of market data structures. The price of some instruments, such as a vanilla swap, only depends on discount factors and spot FX rates. For more complicated instruments, the required pricing factors depend on the chosen pricing model. For example, a Bermuda swaption may request the volatility and mean-reversion parameters of the Hull-White model from a suitable market data structure. In this case, the market data structure would have derived these parameters from the prices of liquid swaptions (calibration of the model).

The process of deriving pricing factors from quoted market rates involves inverting the pricing process by solving for a set of pricing factors that give the observed market prices. In the case of yield curves, solving for the pricing factors (discount factors) is usually done by a simple iterative calculation called 'bootstrapping' or 'stripping the curve'. The pricing process for instruments on the yield curve is explained in Chapter 2.

It is not only instrument pricing models that can subscribe to market data structure services. Other possible requesters of market data include:

- Other market data structures. For example, the cap volatility surface needs discount factors and forward rates from the yield curve in order to generate forward rate volatilities.
- External applications can make use of market data structures using Script+ and other COM-based technology.
- Solvers in trading screens, for example, a forward rate calculator or an implied volatility calculator.
- Graphs request large data blocks from the market data structure in order to present the data graphically.

The requester of services may specify *pricing rules* for the market data structure. Pricing rules specify how to generate the pricing factors. For example, the yield curve bootstrapping method is set in the pricing rules.

Another important role played by a market data structure is to provide risk factors to the market data archive. The term *risk factor* is used here to distinguish a third type of market data. Risk factors are archived on a daily basis and used for value-at-risk (VaR) analysis (see Chapter 6). For example, the user may choose zero-coupon rates as the risk factors to be returned by a yield curve. Zero-coupon rates have the advantage of being independent of the instruments on the yield curves (which may change from one day to the next) and therefore may be compared over time. As another example, the cap volatility surface calculates forward-rate volatilities on a strike axis, and these are the pricing factors. Adaptiv uses a subset of these pricing factors as risk factors, namely the at-the-money (ATM) forward rate volatilities. During simulation VaR analysis, the market data structure is responsible for applying perturbations to the risk factors. For example, the cap volatility surface shifts itself in parallel along the strike axis in response to a perturbation in the ATM forward rate volatilities.

These various roles of a market data structure are illustrated in Figure 1.1.

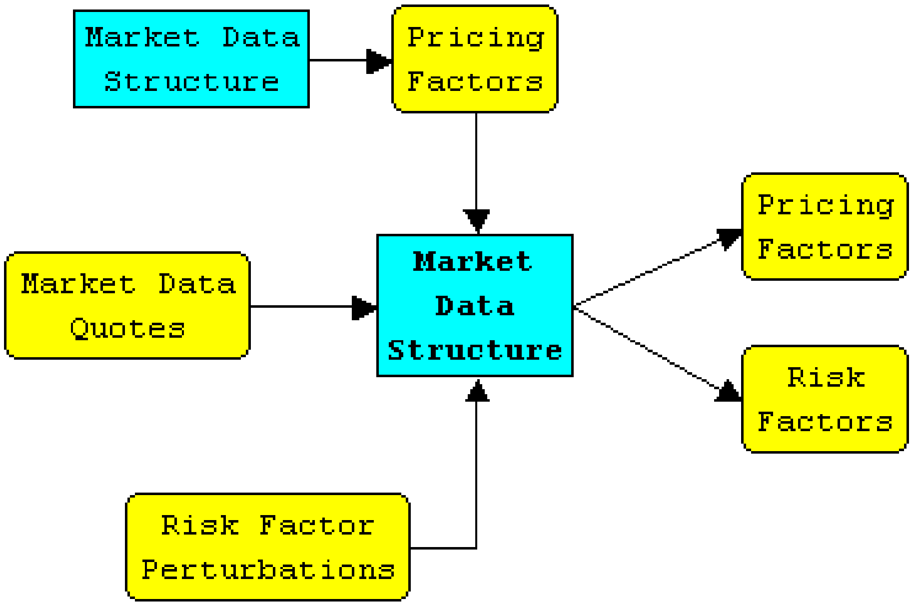


FIGURE 1.1. market data structures.

1.2. Yield Curves

1.2.1. Endogenous and Exogenous Discounting. Yield curves provide two main services.

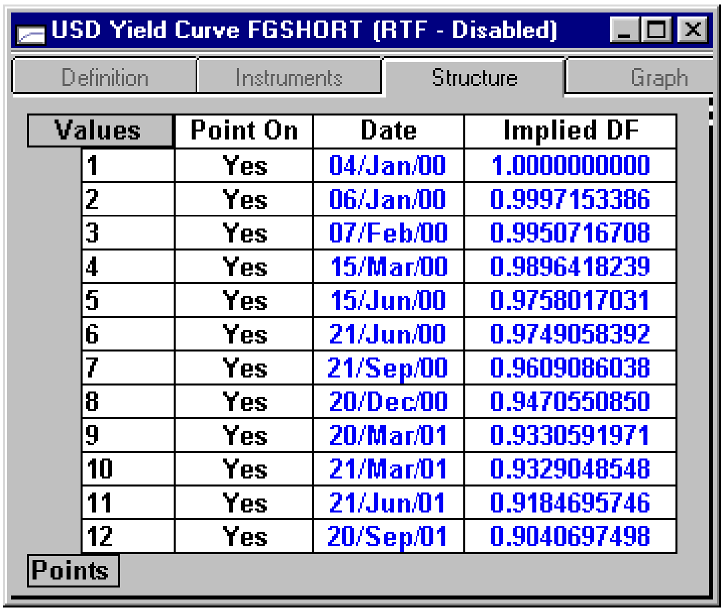
Discount: The provision of discount factors for discounting payments.
Reference: The estimation of forward deposit rates.

In Adaptiv, this is recognized in the pricing rules where separate 'Discount' and 'Reference' yield curves are specified. Interest rate sensitivities are calculated separately for the two curves. Two examples of the use of different discount and reference curves are:

- US Commercial Paper (CP), in which the forward rates are estimated from a special CP curve, usually defined as a spread over LIBOR, while discounting is done at LIBOR rates.
- Situations in which a bank specifies its own borrowing and lending curve for discounting payments in all markets.

Analogous to this, there are two ways that a yield curve can be constructed: by endogenous or exogenous discounting.

Endogenous Discounting: This is the normal case in which the yield curve produces discount factors that account for both the discounting and rate estimation components of the prices of the yield curve instruments.
Exogenous Discounting: Exogenous discounting would be done in a situation where the discount curve was already known and the yield curve rates were taken to explain only rate estimation information. For example, it could be argued that a CP curve should be calculated in this way using



Values	Point On	Date	Implied DF
1	Yes	04/Jan/00	1.0000000000
2	Yes	06/Jan/00	0.9997153386
3	Yes	07/Feb/00	0.9950716708
4	Yes	15/Mar/00	0.9896418239
5	Yes	15/Jun/00	0.9758017031
6	Yes	21/Jun/00	0.9749058392
7	Yes	21/Sep/00	0.9609086038
8	Yes	20/Dec/00	0.9470550850
9	Yes	20/Mar/01	0.9330591971
10	Yes	21/Mar/01	0.9329048548
11	Yes	21/Jun/01	0.9184695746
12	Yes	20/Sep/01	0.9040697498

FIGURE 1.2. table of discount factors generated by bootstrapping.

a previously calculated LIBOR yield curve to provide discount factors for discounting.

1.2.2. Yield Curve Calculation. Each instrument entered on the curve contributes one market price. The goal of the yield curve calculation is to derive from these prices a complete discount factor curve, that is, a discount factor for each future date. Since zero-coupon rates and discount factors are equivalent ways of expressing the time-value-of-money at a future date (see Section 1.2.3), this is equivalent to deriving a curve of zero-coupon rates. The calculation of the yield curve takes place in two stages:

Bootstrapping: Constructing a table of discount factors at the cashflow dates of the instruments, and usually at other intermediate dates between the cashflow dates. This table of discount factors is shown on the Structure Page of the yield curve — see Figure 1.2. When the yield curve instruments are priced from the table, their prices are required to match[1] the entered market prices from which the discount factors were derived. In some cases, for example if a swap rate is provided at each fixed coupon date, the market prices determine a unique set of discount factors at the cashflow dates. More typically, a methodology is required to derive the discount factors at the intermediate cashflow dates (for example the 18m point where no market price is available). In Adaptiv, this methodology is referred to as the bootstrapping method.

Interpolation: In the Adaptiv context, yield curve interpolation means filling the curve between the dates of the table of discount factors generated by the bootstrapping calculation.

[1] If too many instruments are entered on the curve then an exact match may not be possible. In such cases, a best fitting set of discount factors can be obtained by minimizing the differences between the market and calculated prices.

1.2.3. Interpolation.

1.2.3.1. *Standard Methods.* In this section, it is assumed the bootstrapping method has already calculated a table of discount factors. Adaptiv provides three standard methods for interpolating the table of discount factors: constant daily forward rate, linear on zero-coupon rates, and cubic spline. Other methods can be added using Adaptiv Deal+ tools (VB and C++).

1.2.3.2. *Constant Daily Forward Rate.* This method fills in the forward curve in a stepwise fashion.

Given two known discount factors Z_s and Z_e at dates D_s and D_e (where D_s is before D_e), the constant one-day forward rate (daily forward rate) r between these dates is given by

$$Z_e = \frac{Z_s}{(1 + r/B)^{D_e - D_s}} \tag{1.1}$$

where $B = 365.25$. Thus, the discount factor at the end date is equal to the discount factor at the start date discounted again at rate r with daily compounding. Once r has been calculated, the interpolated discount factor Z at some intermediate date D, is given by

$$Z = \frac{Z_s}{(1 + r/B)^{D - D_s}} \tag{1.2}$$

Exactly the same formula can be used to extrapolate a discount factor, either before D_s or after D_e. Note that the interpolated discount factors are independent of the choice of basis B. This is because Equations (1.2) and (1.1) combine to give

$$Z = Z_s \left(\frac{Z_e}{Z_s}\right)^{(D-D_s)/(D_e-D_s)} = Z_s^{(D_e-D)/(D_e-D_s)} Z_e^{(D-D_s)/(D_e-D_s)} \tag{1.3}$$

This is also equivalent to

$$\log Z = \frac{(D_e - D)\log Z_s + (D - D_s)\log Z_e}{D_e - D_s}$$

and hence constant daily forward rate interpolation is linear interpolation on the log of the discount factor.

1.2.3.3. *Linear on Zero-Coupon Rates.* This method fills between the discount factors by linear interpolation of the equivalent continuously compounded zero-coupon rates.

Given two known discount factors Z_s and Z_e at dates D_s and D_e (where D_s is before D_e), the corresponding zero-coupon rates z_s and z_e are given by

$$z_s = B(-\log Z_s)/(D_s - D_0)$$
$$z_e = B(-\log Z_e)/(D_e - D_0)$$

where D_0 is the reference date (i.e., today) and $B = 365.25$. The interpolated zero-coupon rate z at an intermediate date D is

$$z = \frac{(D_e - D)z_s + (D - D_s)z_e}{D_e - D_s}$$

and the interpolated discount factor Z at date D is given by

$$Z = \exp(-z(D - D_0)/B) \tag{1.4}$$

Since the zero-coupon rate at the reference date is zero, linear interpolation may give rates that are too small near the reference date. To avoid this problem, Adaptiv uses constant daily forward rate interpolation between the reference date and the first date on the table of discount factors. An alternative method is to extrapolate the first zero-coupon rate back to the reference date.

As with the constant daily forward rate method, the interpolated discount factor is independent of the choice of basis B because it can be expressed more compactly as

$$Z = Z_s^{(D-D_0)(D_e-D)/(D_s-D_0)(D_e-D_s)} Z_e^{(D-D_0)(D-D_s)/(D_e-D_0)(D_e-D_s)}$$

for $D_0 < D_s$. In the case that $D_s = D_0$, the constant daily forward rate method is used, so that

$$Z = Z_e^{(D-D_0)/(D_e-D_0)}.$$

1.2.3.4. *Extrapolation.* Yield curves are extrapolated using the constant daily forward rate method described here, except when cubic spline interpolation is selected, in which case the slope of the log discount factor curve is extrapolated.

Equation (1.1) is used to obtain the constant daily forward rate r between the last two dates, D_s and D_e, from the table of discount factors. For any date D beyond the final date D_e, the extrapolated discount factor is given by

$$Z = \frac{Z_e}{(1 + r/B)^{D-D_e}}. \tag{1.5}$$

The spreadsheet `FG_YCExtrap.xls` contains an example of this calculation.

1.2.3.5. *Cubic Spline.* Cubic spline interpolation is applied to the log of the discount-factor curve in order to give a quadratic curve for the instantaneous forward rates.

Let $Z(t)$ denote the discount factor for maturity time t. The instantaneous forward rate $f(t)$ is defined to be minus the derivative of $\log Z(t)$ with respect to t, so that $f(t) = -(\log Z)'(t) = -Z'(t)/Z(t)$, or equivalently

$$Z(t) = \exp\left(-\int_0^t f(s)ds\right).$$

Hence $f(t)$ is approximately equal to the one-day forward rate $(Z(t)/Z(t+\delta) - 1)/\delta$, where $\delta = 1/365$.

Suppose the outcome of the bootstrapping calculation is a table of discount factors at times $t_0, t_1, \ldots, t_n$, where t_0 is the current time and $Z(t_0) = 1$. Cubic spline interpolation gives a curve joining these points that is cubic on each subinterval $[t_i, t_{i+1}]$ and has a continuous derivative on the whole interval $[t_0, t_n]$, and hence the derivative is continuous and quadratic on each interval $[t_i, t_{i+1}]$. Therefore, cubic spline interpolation between the points $(t_i, \log Z(t_i))$ constructs a piecewise quadratic instantaneous forward-rate curve $f(t)$ and piecewise cubic log discount-factor curve $\log Z(t)$.

Cubic spline interpolation is described in Appendix A. The second-order derivative of $\log Z$ is set to zero at the end points t_0 and t_n. For dates beyond t_n, the curve is extrapolated using the slope of the curve at t_n.

The spreadsheet `FG_YCCubicSpline.xls` calculates interpolated discount factors from a given table of discount factors and shows the resulting forward-rate and zero-rate curves.

1.2.4. Cash Bootstrapping. Each cash deposit rate entered on the yield curve is related to two discount factors: the discount factor at the start of the deposit period, and the discount factor at the end of the deposit period. Medium-term deposit rates are usually quoted for a period starting at the *spot date*. In most markets, the spot date is two business days from today; in some markets (such as Sterling LIBOR), the spot date is today.

First consider the case when the start of the deposit period is today; this is the case for the overnight deposit rate, and in markets for which the spot date is today. The discount factor at the end of the deposit period is given by

$$Z_e = \frac{1}{1+\alpha r}, \tag{1.6}$$

where r is the deposit rate and α is the accrual period (for example, 91/360). Hence the discount factor at the end date is derived directly from the quoted deposit rate.

In the general case, the discount factor at the end of the deposit period is given by

$$Z_e = \frac{Z_s}{1+\alpha r}, \tag{1.7}$$

where Z_s is the discount factor at the start of the period. This formula, of which Equation (1.6) is a special case, is the well-known no-arbitrage relationship between forward rates and zero-coupon bond prices (see Section 2.5.2). It shows that the end-date discount factor for each cash point can be calculated from the quoted deposit rate once the discount factor at spot is known.

For the discount factor at the spot date, there are two cases to consider.

(1) An overnight deposit rate r_1 is given. The tomorrow discount factor is $Z_1 = 1/(1+\alpha_1 r_1)$ and is extrapolated using the constant daily forward rate method to give the discount factor at spot:

$$Z_{\text{spot}} = Z_1^{(D_{\text{spot}}-D_0)/(D_1-D_0)}, \tag{1.8}$$

where D_0 is today and D_1 is the next business day. (This is an application of Equation (1.3) with D_s equal to today and $D = D_{\text{spot}}$).

(2) There is no overnight deposit rate. In this case the discount factor at spot is derived by extrapolating back from the first deposit rate. Again, this is done using the constant daily forward rate method. Let r denote the first cash rate with spot date D_{spot} and end date D_e. Applying the interpolation formula (1.3) with $D_s = D_{\text{spot}}$, $D = D_0$ and Z_s/Z_e replaced by $(1+\alpha r)$ and D equal to today gives

$$1 = Z_{\text{spot}}(1+\alpha r)^{(D_{\text{spot}}-D_0)/(D_e-D_{\text{spot}})}, \tag{1.9}$$

from which Z_{spot} can be calculated.

1.2.5. Discounting to Spot versus Today. Adaptiv calculates discount factors to today but some systems calculate discount factors to the spot date. Discounting to spot simplifies the calculations of Section 1.2.4 because the discount factor at the spot date is 1. However, there are at least two drawbacks of this approach.

(1) It is impossible to recognize the value of putting money on overnight deposit, or to integrate overnight rates into the curve. Most people would accept that it is more valuable to put money on deposit than not to, and that the extent of this value is dependent on the available overnight rate.

Like any other deposit rate this information must be included in the yield curve to give an accurate picture of portfolio value.

(2) Most operations have to deal from time to time in currencies that have a spot offset of zero, as well as those with spot offsets of one or two days. Given this fact, it is difficult to see how valuation of cashflows (as opposed to straight FX deals) arising from these different currencies could be treated consistently if discounting is always done to a spot offset that is two days out.

1.2.6. Futures and Forward Rates. Before proceeding to the bootstrapping calculations, we must first consider what information about forward rates or discount factors can be derived from a single futures price. A futures contract differs from a forward contract because the profit and loss from daily price changes is credited to the margin account. Thus, the holder of the futures contract can at any time close-out their position and settle the difference between the current price and contract price. For this reason, it follows (see Cox, Ingersoll and Ross [**10**]) that, under real-world (stochastic interest rate) conditions and in the absence of arbitrage, the forward and futures price of the same underlying will differ. For example, if the futures price is 95% then it is not correct to conclude that the corresponding forward rate is 5%. Typically, the forward rate (or equivalently, FRA price) would trade a number of basis points lower than the futures rate (100% minus the futures price). This difference is know as the *futures/forward convexity correction.* The size of the convexity correction increases with the maturity date of contract.

On first reading, the reader may wish to assume that the forward rate *is* equal to 100% minus the futures price and skip to Section 1.2.9.

To see why the convexity correction exists, consider the following strategy. Suppose that trading takes place at times $t_0, t_1, \ldots, t_n$, where t_0 is today and t_n is the futures maturity date, and suppose the size of the futures contract is one million. Let R_i, r_i, Z_i respectively denote the value at time t_i of the futures rate, forward rate and zero-coupon bond price maturing at the end the deposit period. At time t_0: trade a single period swap covering the deposit period, receiving the fixed rate on a principal of N millions; and take a short position in NZ_0 futures contracts. The swap is traded at the market rate r_0 and therefore costs nothing. At time t_1, closeout the futures position and execute the opposite swap at the new market rate r_1. Invest the profit from the futures closeout in a zero-coupon bond maturing at the end of the deposit period. Then the net cashflow received at the end of the deposit period is

$$\frac{NZ_0(R_1 - R_0)}{Z_1} - N(r_1 - r_0)$$

and this can be rewritten

$$\frac{N(R_1 - R_0)(Z_0 - Z_1)}{Z_1} + N(b_1 - b_0)$$

where b_i denotes $R_i - r_i$. Repeating this at each time t_{i-1} gives a total payoff of

$$-Nb_0 + \sum\nolimits_{i=1}^{n} \frac{N(R_i - R_{i-1})(Z_{i-1} - Z_i)}{Z_i}. \tag{1.10}$$

Under the assumption that movements in r and Z are negatively correlated, each term $(R_i - R_{i-1})(Z_{i-1} - Z_i)$ is positive. If the initial convexity correction b_0 were negative or zero then this strategy would create an arbitrage. Therefore, the futures rate is strictly greater than the forward rate. Equation (1.10) also suggests that the size of convexity correction is an increasing function of the futures maturity date.

Days to Start	Covariance Factor
22	1.00E-06
34	2.00E-06
45	3.00E-06
56	4.00E-06
79	5.00E-06
...	...

TABLE 1.2. the standard table of futures/forward covariance factors.

1.2.7. Futures/Forward Convexity Correction. Adaptiv supports two methods of calculating the futures/forward convexity correction: a general method and a specific method.

The general method allows the user to specify the value of the covariance in Equation (2.18). Adaptiv's interpretation of this equation is

$$0.25(R-r) = (1-0.25R)c \tag{1.11}$$

where r is the forward rate, R is the futures rate (100% minus the futures price) and c is a *covariance factor*. Rearranging Equation (1.11) gives the forward rate in terms of the futures rate:

$$r = R(1+c) - 4c$$

Table 1.2 shows the first few entries in the standard table of covariance factors shipped with Adaptiv. The table is interpolated by reading the covariance factor from the last row in the table with Days to Start less than or equal to the number of days to the start of the futures contract.

The specific method is to use a formula for convexity correction that can be obtained under the Hull-White model. In the one-factor Hull-White model with constant short-rate volatility σ and constant mean reversion-rate a, the convexity correction is given by

$$1+\alpha R = (1+\alpha r)\exp\left(\zeta^2 + C\right), \tag{1.12}$$

where

$$\zeta^2 = \sigma^2\left(\frac{1-\mathrm{e}^{-2aT_s}}{2a}\right)\left(\frac{1-\mathrm{e}^{-a(T_e-T_s)}}{a}\right)^2 \tag{1.13}$$

$$C = \zeta^2\frac{\left(1-\mathrm{e}^{-aT_s}\right)^2}{\left(1-\mathrm{e}^{-2aT_s}\right)\left(1-\mathrm{e}^{-a(T_e-T_s)}\right)} \tag{1.14}$$

and T_s and T_e are the times to the start and the end of the deposit period. This formula is given in Kirikos and Novak [**22**].

The value of ζ can be derived from cap prices because under the Hull-White model, $\log(1+\alpha r(T_s))$ is normal with variance ζ^2 and the price of a caplet on the deposit rate r is

$$\left[(1+\alpha r)\mathcal{N}(d_2+\zeta) - (1+\alpha K)\mathcal{N}(d_2)\right] Z_e,$$

where K is the caplet strike, Z_e is the discount factor at the end of the deposit period and

$$d_2 = \frac{1}{\zeta}\log\left(\frac{1+\alpha r}{1+\alpha K}\right) - \frac{\zeta}{2}.$$

The market value of the caplet is obtained by interpolating a Black caplet volatility σ_B, and Section 1.4.4 showed how caplet volatilities can be derived from quoted cap volatilities. Equating the Hull-White value of the at-the-money ($K = r$) caplet with its Black-formula value gives

$$(2\mathcal{N}(\zeta/2) - 1)(1 + \alpha r) = V_{\text{ATM}} = (2\mathcal{N}(\sigma_B\sqrt{T_s}/2) - 1)\alpha r$$

and hence $\zeta = 2\mathcal{N}^{-1}((V_{\text{ATM}}/(1 + \alpha r) + 1)/2)$.

If the mean-reversion rate a is given and ζ is calculated from cap prices then Equation (1.13) gives the Hull-White volatility σ, and then C can be calculated. Alternatively, using the approximation $1 - \mathrm{e}^{-x} \simeq x$ in Equation (1.14) gives

$$C \simeq \zeta^2 \frac{T_s}{2(T_e - T_s)}. \tag{1.15}$$

This provides a good approximation of the Hull-White convexity correction that *depends only* on ζ, and ζ is obtained directly from cap prices.

However, when calculating ζ, the futures rate R and the Black volatility σ_B are known, but the forward r is not. This difficultly is easily overcome by an iterative calculation. On the first iteration, set $r = R$ to calculate ζ, then calculate C using either a known mean-reversion rate or the approximation (1.15), and then calculate r from Equation (1.12). On the next iteration, use the value of r from the previous iteration to calculate ζ, and then calculate C and r again. And so on. Three iterations are usually sufficient.

1.2.8. Bootstrapping the Futures Strip. Futures contracts do not directly provide any discount factors but they do, of course, give forward rates. In this section, it is assumed the futures/forward convexity correction has already been applied so that the forward rate is known for each contract period. If we know the discount factor at the start of the contract period then Equation (1.7) gives the discount factor at the end of the contract period. This means that we know the discount factor at the start of the next contract and hence the discount factor at the end of the next contract by applying Equation (1.7) again. In this way, the discount factor at each date in the futures strip can be calculated from the forward rates and an initial discount factor from the start of the first contract.

In fact, the above description oversimplifies the calculation because a futures strip is not a contiguous strip of start and end dates; there are gaps and overlaps in the strip. As an example, consider three-month US LIBOR futures traded on CME. The start of the contract period is determined by the exchange calendar—in this case, the third Wednesday of the contract month. The end of the contract period is the start date plus three calendar months (corresponding to a three-month deposit period). Therefore the end of the contract period will typically be a few days before or a few days after the start of the next contract. Hence there are gaps and overlaps in the strip.

So once again, given the discount factor at the start of the current contract, the discount factor at the end of the contract is calculated from the forward rate using Equation (1.7). To calculate the discount at the start of the next futures contract, Adaptiv takes the view that the current contract has greater weight, owing to its greater liquidity, than the next contract and therefore interpolates or extrapolates from the current end date.

In the case of an overlap (next start date before current end date):

- the next contract does not affect the curve until after the current end date
- the interpolation method set in the pricing rules is used

- only the current end date is added to the table of discount factors (see Section 1.2.2).

In the case of a gap (current end date before next start):

- the influence of the current contract continues until the next start date
- the constant daily forward rate method is used to extrapolate to the next start date
- both the current end date and next start date are added to the table of discount factors.

1.2.9. Cash/Futures Crossover. If both cash and futures points are on the curve (and the pricing rules specify that both are to be used) then there will typically be conflicts over which points to use and how to use them. Adaptiv takes the view that futures contracts have greater liquidity than deposits and so in case of conflicts, the futures override the cash. This has three consequences:

(1) All of the futures on the futures strip will be used.
(2) The first cash point inside the futures strip may be used; all other cash points inside the strip are overridden.
(3) The first futures price may affect discount factors before the start of the strip.

The futures strip needs to be anchored with an initial discount factor at the start date of the first contract. Adaptiv provides two methods for calculating this discount factor. The choice of method is determined by the choice of interpolation method.

Constant Daily Forward Rate.: Let D_{cash} denote the last cash point which is on or before the start of the first futures contract. Extrapolate from D_{cash} to the start of the first futures contract D_s using the constant daily forward rate derived from the futures price. Thus, the discount factor at the start of the futures strip is given by

$$Z_s = Z_{\text{cash}}(1+\alpha r)^{-(D_s - D_{\text{cash}})/(D_e - D_s)}, \tag{1.16}$$

where Z_{cash} is the discount factor at D_{cash} and r is the forward rate for the first contract period, from D_s to D_e. In the case that there are no cash points on or before D_s, extrapolate from the reference date D_0 by setting $Z_{\text{cash}} = 1$ and $D_{\text{cash}} = D_0$.

Linear on Zero-Coupon Rates.: The discount factor at the start of the futures strip is obtained by interpolating between its nearest cash points using linear interpolation on zero-coupon rates, as described in Section 1.2.3. If there are no cash points before D_s, or no cash points after D_s, then the Constant Daily Forward Rate method is used.

1.2.10. Bootstrapping the FRA Strip. FRA strips are treated in the same way as futures strips. In the FRA case, the calculations are more straightforward because:

(1) the forward rates are the quoted FRA prices (hence no convexity correction is required)
(2) the start and end dates are contiguous.

1.2.11. Short End Example. The spreadsheet `FG_YCShort.xls` replicates the cash and futures bootstrapping calculations.

1.2.12. Bootstrapping on Yield versus Forwards. When bootstrapping over sparse regions of the yield curve (for example, between 7 and 10 years), it has traditionally been common to interpolate using 'filler' swaps or bonds whose yield lies linearly between rates of quoted points. This is referred to as bootstrapping on yields. However, the effect of interpolating in this way means that the work that the individual forward rates (at say 8 and 9 years) have to do to generate a linear growth in the yield of the entire instrument is huge. This leads to a see-saw pattern on the forward curve that is usually very marked. Since the forward rates are the main inputs (together with their volatilities) into the valuation of most interest rate products this leads to distortion and instability in pricing.

A preferable approach is to interpolate so that the forward rates over the sparse periods are in a linear relationship. This leads to smoother forward curves. Note that this is very different from 'bendy ruler' style curve fitting algorithms (for example, cubic spline). Using linear interpolation of forwards generally removes the interpolation spikes of yields-based bootstrapping without recourse to methods that can cause strange slope changes to show up in unpredictable ways.

The main reason that yield based methods are still in use is that they are easier to implement. Forwards-based methods give rise to high-order polynomial equations which require numerical methods for their solution.

1.2.13. Swap Bootstrapping Equation. Each new swap point on the curve contributes a swap rate s. The bootstrapping process must generate discount factors such that the swap traded at rate s has zero net present value. Thus, the bootstrapping process must solve

$$\sum_i r_i \alpha_i^{\text{flt}} Z_i^{\text{flt}} = \sum_j s \alpha_j^{\text{fxd}} Z_j^{\text{fxd}} \tag{1.17}$$

where r_i is the forward rate for the i^{th} floating coupon, the α_i are the accrual periods and the Z_i are the discount factors at the coupon payment dates. In this equation, some of the forward rates and discount factors are already known; they are interpolated from the discount factors derived from previous points on the curve. The unknown payment date discount factors Z_i are first expressed in terms of the unknown forward rates. This is done by extracting the end date discount factors using Equation (1.7). Often the payment date will coincide with the end of a rate period. If this is not the case then the payment date discount factor is obtained by interpolation or extrapolation. Thus, Equation (1.17) becomes a (polynomial) equation in a certain number of unknown forward rates. The number of unknowns equals the number of coupon periods between the current swap point and the last known forward rate.

1.2.14. Solving the Equation—On Yields. In this approach, extra swap points are added so that there is a swap point at every coupon date. In this case, Equation (1.17) has one unknown forward rate and is usually linear (interpolation of the payment date discount factor may make it slightly non-linear). A swap rate must be assigned to each extra swap point and this is done by linear interpolation between the known swap rates. To be precise, the interpolated swap s rate is given by

$$s = \frac{s_{\text{lower}}(D_{\text{upper}} - D) + s_{\text{upper}}(D - D_{\text{lower}})}{D_{\text{upper}} - D_{\text{lower}}}$$

where s_{lower} and s_{upper} are the known swap rates on either side and D_{lower}, D and D_{upper} are the end dates of the last forward period in the swap. For example, if

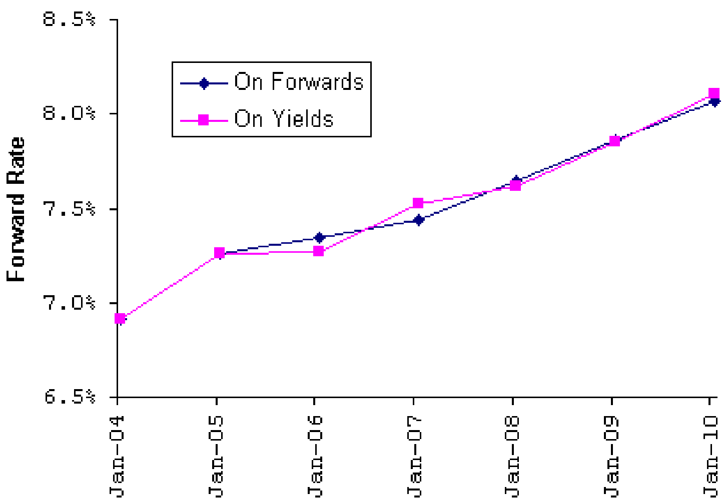


FIGURE 1.3. 1y forward rates from two bootstrapping methods.

the 7 and 10 year swap rates entered on the curve are 5.7% and 6% then On Yields bootstrapping will internally construct 8 and 9 year swap points with rates 5.8% and 5.9% respectively.

In the case that there is no lower swap rate r_{lower} (for example, the 18 month point on a semi-annual curve) then r_{lower} is set to the last known cash or futures rate.

1.2.15. Solving the Equation—On Forwards. In this approach, a linear relationship is imposed on the unknown forward rates. The unknown forward rates are joined in a straight line by extrapolating the slope between the last known and the first unknown forward rate. Thus, the second and subsequent unknown forward rates are given by

$$r = r_0 + \left(\frac{D - D_0}{D_1 - D_0}\right)(r_1 - r_0)$$

where r_0 is the last known forward rate with start date D_0, r_1 is the first unknown forward rate with start date D_1 and D is the start date for r. In this way, solving Equation (1.17) reduces to solving a polynomial equation in one unknown and this is done using the Newton-Raphson method. The result is a piecewise linear forward curve.

A significant difference between this forwards-based approach and traditional yield-based methodologies is that it is not generally possible to write down an algebraic solution to Equation (1.17). This is because of the presence of the unknown forward rate in the denominator of the expression for the discount factors Z_i. This raises the degree of the equation to be solved by one degree for each coupon period so that on a semi-annual yield curve with no points between 7 and 10 years we have an equation of degree 6. There is no general solution for polynomials of degree 5 or higher.

1.2.16. Swap Bootstrapping Example. For both methods, the bootstrapping calculations can easily be replicated in Excel using the Solver. This is done in the spreadsheet `FG_YCLong.xls`.

For this example, Figure 1.3 shows a graph of the one-year forward rates.

1.2.17. Swap versus Bond Methodologies. Bond points on yield curves can be viewed in two different ways:

(1) Swap traders often use generic-dated bonds without regard to exact dates of the actively traded issues. This makes comparison of swap and bond prices easier.
(2) Dealers trading bonds require that bonds price back to par from the yield curve. This in turn requires that the bootstrapping method uses the actual cashflow dates of the bonds.

Adaptiv supports either approach and terms them 'swap methodology' and 'bond methodology' respectively. If swap methodology is on then the bond instrument gets the same cashflow structure as a generic swap; the rate for this swap is set to the bond yield plus the swap spread. If bond methodology is on then the bond instrument gets the cashflow structure specified in the bond issues table.

1.3. FX Rates

1.3.1. Spot FX Rates. Spot FX rates are entered in the FX Grid against any specified base currency, typically against US dollar. Given the spot rate for each non-base currency against the base (for example, CHF:USD, USD:EUR or JPY:USD), the cross spot FX rates (for example, JPY:EUR) are implied using the no-arbitrage (or parity) relationship

$$S_{\mathrm{X:Y}} = S_{\mathrm{X:B}} S_{\mathrm{B:Y}}$$

where $S_{\mathrm{X:Y}}$ denotes the price of one unit of currency Y denominated in units of currency X and B denotes the base (or cross-fill) currency.

If Y is a Euro legacy currency and X is any other currency (including the cross-fill currency) then

$$S_{\mathrm{X:Y}} = S_{\mathrm{X:EUR}} / S_{\mathrm{Y:EUR}}$$

where $S_{\mathrm{Y:EUR}}$ is the Euro locking rate, for example $S_{\mathrm{DEM:EUR}} = 1.95583$.

1.3.2. Forward FX Rates and Yield Curves. The short end of the yield curve may be calculated from a known yield curve in another currency (typically the dollar curve) and the market-quoted spot and forward FX rates between the two currencies. Forward FX rates are entered in the FX Grid as spreads above or below the spot rate.

The no-arbitrage (or parity) relationship between FX rates and yield curves is

$$F = S_0 \frac{\widetilde{Z}}{Z} \tag{1.18}$$

where F is the forward FX rate, S_0 is the today FX rate, Z is the discount factor at the forward settlement date for the quoting currency (the currency in which the rate is quoted, for example, JPY is the quoting currency of the JPY:USD rate) and $\widetilde{Z}$ is the discount factor for the other currency. Note that we make a distinction between the FX rate for settlement at the spot date (today + two business days) and FX rate for settlement today. The relationship between these two rates is a special case of Equation (1.18):

$$S = S_0 \frac{\widetilde{Z}_{\mathrm{spot}}}{Z_{\mathrm{spot}}} \tag{1.19}$$

where S denotes the spot rate.

The calculation of the yield curve proceeds as follows. One of the yield curves is known and the other is to be calculated—either the Z's are known and the $\widetilde{Z}$'s are to calculated or vice versa. The today FX rate is calculated by linear extrapolation from the spot rate and the first forward rate. Thus, the today FX rate is given by

$$S_0 = S + (F_1 - S)\left(\frac{D_0 - D_{\text{spot}}}{D_1 - D_{\text{spot}}}\right)$$

where F_1 is the first market-quoted forward FX rate. Once the today rate S_0 is known, Equation (1.18) is used to calculate the unknown discount factor from the known discount factor and the forward FX rate. This is done for the spot date and each of the forward dates.

1.4. Volatility Structures

1.4.1. Black Formula and Implied Volatility. It is widely accepted that the lognormal distribution of prices assumed by the Black model does not adequately cover options that are deeply in- or out-of-the-money. This is because it is more likely that a large jump will bring an out-of-the-money option into the money (or vice versa) than the distribution suggests. For example, a dramatic political or economic event can shock the markets into an extraordinary rise in interest rates that could heavily penalize sellers of caps. Using volatility figures that are perfectly suited to pricing options close to the money, the probability of this happening under a lognormal distribution is extremely low. This leads to the conclusion that the distribution has kurtosis, or fat tails, with large movements having a higher probability than predicted by the lognormal distribution.

This section uses option pricing theory to clarify the relationship between distribution of the underlying price and the implied Black volatility as a function of the strike.

Consider a European option on a forward rate. The payoff of the option is $\max(F(T) - K, 0)$ at time T_1, where K is the strike price, T is the exercise date and $F(t)$ denotes the forward rate at time t. It is well-known (see, for example, Jamshidian [**18**]) that the no-arbitrage price of the option is

$$\mathbb{E}[\max(F(T) - K, 0)]Z(0, T_1)$$

where $\mathbb{E}$ is the expectation in the forward measure and $Z(t, T)$ denotes the price at time t of a zero-coupon bond maturing at time T. Let $V(K)$ denote the non-discounted option price $\mathbb{E}[\max(F(T) - K, 0)]$. Differentiating under the expectation gives

$$-V'(K) = \mathbb{P}(F(T) > K)$$

This shows that the distribution of the terminal forward rate $F(T)$ is directly related to the variation of the option price with strike (compare with the fourth equation of Dupire [**5**]). We now show that the distribution of $F(T)$ is lognormal if and only if the option is given by Black formula.

First suppose that $F(T)$ is lognormal with volatility σ, that is, $\log F(T)$ is normal with variance $\sigma^2 T$. Since $F(t)$ is a martingale in the forward measure, we have $\mathbb{E}[F(T)] = F(0)$. It follows (see Appendix B) that the expectation $\mathbb{E}[\max(F(T) - K, 0)]$ can be calculated explicitly and is given by the Black formula

$$V_B(\sigma, K) = F(0)\mathcal{N}\left(d_2(\sigma, K) + \sigma\sqrt{T}\right) - K\mathcal{N}\left(d_2(\sigma, K)\right)$$

where $\mathcal{N}$ denotes the normal distribution function and

$$d_2(\sigma, K) = \frac{1}{\sigma\sqrt{T}}\log\left(\frac{F(0)}{K}\right) - \frac{\sigma\sqrt{T}}{2}$$

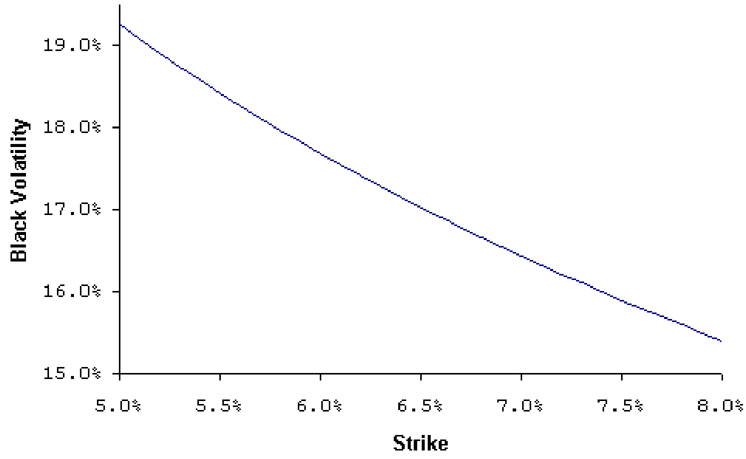


FIGURE 1.4. volatility skew for a caplet priced under the Hull-White model.

Conversely, if $V(K)$ is given by the Black formula $V_B(K, \sigma)$ then

$$\mathbb{P}(F(T) > K) = -\frac{\partial V_B(K, \sigma)}{\partial K} = \mathcal{N}\left(d_2(\sigma, K)\right)$$

and it follows that $F(T)$ is lognormal with volatility σ.

Returning to the general setting, the implied Black volatility $\sigma_B(K)$ is defined to be the value of σ in the Black formula that attains the option price. Thus, $\sigma_B(K)$ is the solution of

$$V(K) = V_B(\sigma_B(K), K)$$

and is constant as a function of K if and only if $F(T)$ is lognormal. The variation of σ_B with strike K is called the *volatility skew*.

Figure 1.4 shows the implied volatility skew of a caplet priced under the Hull-White model (or any Gaussian HJM model). In this example: the rate period runs from 1y to 2y, the discount factor at the start is 0.95, the forward rate is 6% and standard deviation of $\log(Z_e/Z_s)$ is 1%.

1.4.2. Adaptiv Volatility Surfaces. Adaptiv provides separate market data structures for pricing caps, swaptions, FX options and bond options. These market data structures will be referred to as *volatility surfaces*. Features common to all the volatility surfaces are described below and in Section 1.4.3. Sections 1.4.4–1.4.7 cover each of the volatility surfaces in turn.

Prices of liquid options are entered on the volatility surface. The prices are entered as either premiums or Black volatilities. The Black volatilities are stored by exercise date, strike and, in the case of the swaption and bond volatility surfaces, by tenor. The resulting surface of Black volatilities constitutes the set of pricing factors returned by the market data structure (see the discussion of Section 1.1).

The Adaptiv volatility surfaces allow the volatility skew to be *specified* by entering Black volatilities against various strikes. This approach, where Black volatility is entered and interpolated, should be contrasted with model-based methods where prices of liquid instruments are used to calibrate the model parameters that allow other instruments to be priced within the model. A model-based approach can be implemented in Adaptiv by creating a market data structure which stores the liquid option prices and performs the model calibration. Adaptiv's COM-based development tools allow new market data structures to be added to the system.

1.4.3. Interpolation. The pricing model may request a volatility from the surface for a given exercise date and strike. The surface is interpolated and the volatility returned. Interpolation along the exercise date axis is linear. Interpolation along the strike is either linear or cubic spline (see Appendix A). Because the at-the-money (ATM) strike varies along the exercise date axis, interpolating on the raw strike would lead to distortions. For this reason, the interpolation is done on K/F, the ratio of the strike to the forward price (ATM strike). The ratio K/F is a simple measure of in- or out-the-moneyness that is independent of the current level of the forward price.

The following example illustrates the method. Suppose the pricing model requests a volatility at exercise 4m and strike K, and the volatility surface has grid points at 3m and 6m. Let $F_{4\text{m}}$ denote the forward price out of 4m. The first step is to calculate the volatilities at 3m and 6m by either linear or cubic spline interpolation on the strike/forward ratios.

Let $K_1, \ldots, K_n$ denote the range of strikes on the surface at time point 3m, and let $\sigma_1, \ldots, \sigma_n$ denote the corresponding range of volatilities. The moneyness level for the required volatility is $\gamma = K/F_{4\text{m}}$. The moneyness levels at the 3m time point are $\gamma_1, \ldots, \gamma_n$, where $\gamma_i = K_i/F_{3\text{m}}$. If $\gamma < \gamma_1$ then the 3m volatility is given by $\sigma_{3\text{m}} = \sigma_1$. If $\gamma > \gamma_n$ then $\sigma_{3\text{m}} = \sigma_n$. Otherwise, $\gamma_i \le \gamma \le \gamma_{i+1}$ for some $i < n$. In this case, and if interpolation is linear, the 3m volatility is given by

$$\sigma_{3\text{m}} = \frac{(\gamma_{i+1} - \gamma)\sigma_i + (\gamma - \gamma_i)\sigma_{i+1}}{\gamma_{i+1} - \gamma_i}.$$

The volatility $\sigma_{6\text{m}}$ is calculated in the same way. The final interpolated volatility σ is obtained by linear interpolation on the exercise date between $\sigma_{3\text{m}}$ and $\sigma_{6\text{m}}$:

$$\sigma = \frac{(D_{6\text{m}} - D_{4\text{m}})\sigma_{3\text{m}} + (D_{4\text{m}} - D_{3\text{m}})\sigma_{6\text{m}}}{D_{6\text{m}} - D_{3\text{m}}}.$$

1.4.4. Cap Volatility Surface.

1.4.4.1. *Par Volatility.* The cap volatility surface is responsible for deriving the volatilities of forward deposit rates. A cap volatility surface is defined with respect to a tenor (1m, 3m, 6m or 12m) which is the term of the underlying deposit rates. The forward rate volatilities are derived from cap prices entered on the surface. If the tenor of the surface is 3m then both cap and futures option prices may be entered. Caps of different maturities (1y, 2y, 3y etc.) and varying strikes are entered on the surface. Typically, the price of a cap is entered as a *par volatility.* The par volatility of a cap is the equivalent flat volatility that gives the market value of the cap. Thus, the market value of the cap V and the par volatility $\tilde{\sigma}$ are related by

$$V = \sum_i V_B(r_i, K, \tilde{\sigma}, T_i) Z_i \tag{1.20}$$

where V_B denotes the Black formula (defined in Section 1.4.1) and for the i^{th} caplet: r_i is the forward deposit rate; Z_i the discount factor at the payment date; and T_i the time to the exercise date.

Par volatilities are calculated from cap prices by solving Equation (1.20) using the Newton-Raphson method.

1.4.4.2. *Deriving the Forward Rate Volatilities.* The forward rate volatilities are derived from the cap prices via a bootstrapping calculation. The calculation is illustrated with the following example. To simplify the discussion, we first consider the case of one par volatility (and strike) for each cap maturity.

Suppose 1y, 2y and 3y par volatilities, $\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3$, are entered on a semi-annual surface at strikes K_1, K_2, K_3. The 1y cap has only one caplet running from 6m to 1y. The forward rate volatility out of 6m is given by

$$V_B(r_2, K_1, \sigma_2, T_1)Z_2 = V_B(r_2, K_1, \tilde{\sigma}_1, T_1)Z_2$$

where: r_i and σ_i denote the i^{th} semi-annual forward rate and its volatility; Z_i and T_i denote the payment date discount factor and time to exercise date of the i^{th} caplet. Clearly, the unique solution to this first bootstrapping equation is that: $\sigma_2 = \tilde{\sigma}_1$, the 6m forward rate volatility equals the 1y par volatility.

The forward rate volatilities out of 1y and 1y6m are derived from the 2y cap price by equating the sum of three caplet prices:

$$\sum_{i=2}^{4} V_B(r_i, K_2, \sigma_i, T_{i-1})Z_i = \sum_{i=2}^{4} V_B(r_i, K_2, \tilde{\sigma}_2, T_{i-1})Z_i$$

The first forward rate volatility σ_2 is known from the 1y cap price. There are two unknown volatilities σ_3 and σ_4. In general, there is not a unique choice of forward rate volatilities to satisfy the bootstrapping equation. Adaptiv offers two bootstrapping methods whereby the number of unknowns in the bootstrapping equation is reduced to one so that it can be solved using the Newton-Raphson method.

Step Method: The unknown volatilities are all equated. Thus, we set $\sigma_4 = \sigma_3$ and solve for σ_3. This method gives a step pattern in the volatility curve.

Slope Method: The second and subsequent unknown volatilities are extrapolated in a straight line from the last known and first unknown volatility. Thus, we set

$$\sigma_4 = \sigma_2 + \left(\frac{T_3 - T_1}{T_2 - T_1}\right)(\sigma_3 - \sigma_2)$$

and solve for σ_3. This method gives a piecewise linear volatility curve.

In general, the slope method improves on the step method. However, the slope method converges under certain conditions (steeply sloping volatility curves) where the step method does not converge.

The last step in the calculation is to derive σ_5 and σ_6, the volatilities out of 2y and 2y6m, from the 3y par volatility. This is identical to the previous step.

The general bootstrapping equation for the n-year cap is

$$\sum_{i=2}^{pn} V_B(r_i, K_n, \sigma_i, T_{i-1})Z_i = \sum_{i=2}^{pn} V_B(r_i, K_n, \tilde{\sigma}_n, T_{i-1})Z_i \tag{1.21}$$

where p is the number of forward-rate periods per year (for example, $p = 2$ for semi-annual). The last known volatility in Equation (1.21) is $\sigma_{p(n-1)}$ and the remaining volatilities are solved for using either the step or the slope method.

1.4.4.3. *Volatility Skew.* In the above example, one cap was added to the surface at each of the maturity points. The result of bootstrapping calculation was a volatility *curve*. In the general case, caps with a range of strikes and the *same* maturity may be added to the surface. The bootstrapping calculation described above is easily adapted to incorporate the additional points on the surface. The result is a surface of forward rates volatilities (by maturity and strike). The 1y points are processed first, followed by the 2y points and so on; the order in which the different strikes are processed is not important. Equation (1.21) is applied to each instrument on the surface as follows. The volatilities $\sigma_2, \ldots, \sigma_{p(n-1)}$ at strike K_n are interpolated from the known volatilities derived from caps with earlier maturity dates. This is done using the interpolation method described in Section 1.4.3. Then the bootstrapping equation is solved for the remaining volatilities $\sigma_{p(n-1)+1}, \ldots, \sigma_{pn}$ which are added to the surface at strike K_n. Each n-year cap on the surface contributes volatilities between $(n-1)$ and n years at its own strike level. In this way, the surface is extended to n years and is then available to be interpolated for the $(n+1)$-year caps.

1.4.4.4. *Future Options.* On quarterly surfaces, a strip of futures options can be added. The futures options override any caps with maturity date before the end of the strip. Each futures option contributes a single point to the forward rate volatility surface. The volatility is either entered directly or implied from the entered premium by inverting the pricing formula of Section 2.12.

1.4.4.5. *Different Tenors.* Strictly, it is necessary to set up different volatility surfaces for each tenor (1m, 3m, 6m and 12m). However, in most markets, prices of liquid caps will only be available for one tenor, either 3m or 6m. It is possible to assign (in the pricing rules) the same surface to all four tenors but it is preferable to construct separate surfaces for each tenor, even if the same par volatilities are used.

1.4.4.6. *Example.* The forward rate volatility bootstrapping calculations are replicated in the spreadsheet `FG_CapVolSurface.xls`. The spreadsheet uses the Excel Solver in conjunction with some VB functions that perform the caplet pricing and the cubic spline interpolation. For this example, Figure 1.5 shows that the slope method produces a smoother forward rate volatility surface.

1.4.5. Swaption Volatility Surface.

1.4.5.1. *Black Volatility.* Swaption prices are entered on the swaption volatility surface as either premiums or Black volatilities. The premium V of a payer's swaption and the Black volatility σ are related by the Black formula

$$V = V_{\text{flt}}\mathcal{N}\left(d_2 + \sigma\sqrt{T}\right) - V_{\text{fxd}}\mathcal{N}(d_2)$$

where V_{flt} and V_{fxd} denote the present value of the floating and fixed side of the underlying swap respectively,

$$d_2 = \frac{1}{\sigma\sqrt{T}}\log\left(\frac{V_{\text{flt}}}{V_{\text{fxd}}}\right) - \frac{\sigma\sqrt{T}}{2}$$

and T is the time to the exercise date.

1.4.5.2. *Volatility Interpolation.* The swaption volatility surface has three axes: exercise date, swap tenor and strike. Hence it requires 3-dimensional interpolation. This is done by a generalization of the method described in Section 1.4.3.

The following example illustrates the method. The swaption pricing model requests a volatility at exercise 4m, tenor 4y and strike K. The volatility surface has exercise

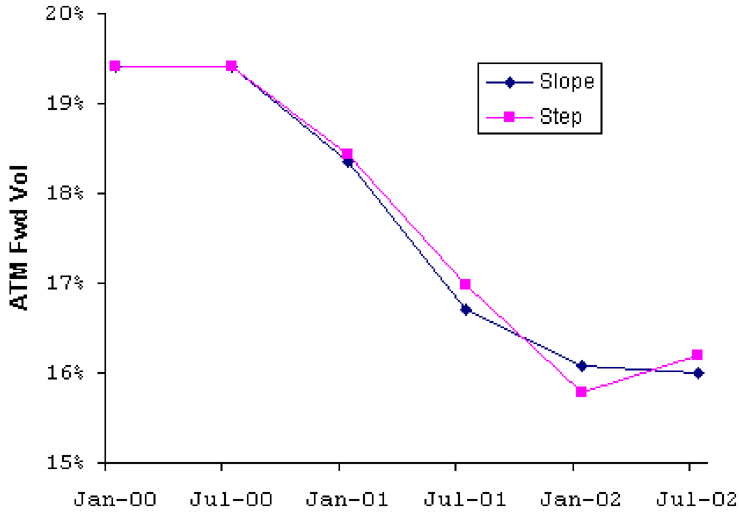


FIGURE 1.5. implied at-the-money forward rate volatility under the two bootstrapping methods.

grid points at 3m and 6m and tenor grid points at 3y and 5y. Hence there are four adjacent (exercise, tenor) points on the surface: (3m,3y), (3m,5y), (6m,3y) and (6m,5y). The volatility at each of these points is calculated by interpolation of strike/forward ratio using either linear or cubic spline interpolation. The final interpolated volatility σ is obtained by linear interpolation between the four points. Thus

$$\sigma = \frac{(D_{5\mathrm{y}} - D_{4\mathrm{y}})\sigma_{3\mathrm{y}} + (D_{4\mathrm{y}} - D_{3\mathrm{y}})\sigma_{5\mathrm{y}}}{D_{5\mathrm{y}} - D_{3\mathrm{y}}}$$

where $\sigma_{3\mathrm{y}}$ is the linear interpolation of $\sigma_{3\mathrm{m},3\mathrm{y}}$ and $\sigma_{6\mathrm{m},3\mathrm{y}}$:

$$\sigma_{3\mathrm{y}} = \frac{(D_{6\mathrm{m}} - D_{4\mathrm{m}})\sigma_{3\mathrm{m},3\mathrm{y}} + (D_{4\mathrm{m}} - D_{3\mathrm{m}})\sigma_{6\mathrm{m},3\mathrm{y}}}{D_{6\mathrm{m}} - D_{3\mathrm{m}}}$$

and $\sigma_{5\mathrm{y}}$ is the linear interpolation of $\sigma_{3\mathrm{m},5\mathrm{y}}$ and $\sigma_{6\mathrm{m},5\mathrm{y}}$.

The spreadsheet `FG_SwapVolSurface.xls` contains a worked example of this calculation.

1.4.6. FX Volatility Surface.

1.4.6.1. *Black Formula.* The price V of a European FX option and the Black volatility σ are related by the Black formula

$$V = \pm\left(F\mathcal{N}(\pm d_1) - K\mathcal{N}(\pm d_2)\right)Z$$

where

- F is the forward FX rate to the option settlement date and K is the strike rate
- Z is the discount factor at the option settlement date
- $\pm$ is takes the value $+1$ for a call and -1 for a put
- $d_1 = d_2 + \sigma\sqrt{T}$ and

$$d_2 = \frac{1}{\sigma\sqrt{T}}\log\left(\frac{F}{K}\right) - \frac{\sigma\sqrt{T}}{2}$$

- T is the time to the exercise date.

In this context, a call option is the right to pay the currency in which the rate is quoted and receive the other currency. The discount factor Z is for the currency in which the FX rate is quoted.

See Section 2.14.2 for further discussion of the pricing of European FX options.

1.4.6.2. *Entry of Prices via Strike or Delta.* FX option prices are entered as either premiums or Black volatilities. To define a volatility skew, option prices may be entered against either strike or delta. In this context, delta is the hedge ratio of the option, $\mathcal{N}(\pm d_1)$. The surface has two price entry modes: *by delta* and *by strike*.

If prices are entered by strike then the order of calculation is as follows.

(1) If volatility is entered then premium and delta are recalculated with strike held constant.
(2) If premium is entered then volatility and delta are recalculated with strike held constant. This is the standard implied volatility calculation — solving the Black formula for σ using the Newton-Raphson method.

Alternatively, if prices are entered by delta then the order of calculation is as follows.

(1) If volatility is entered then premium and strike are recalculated with delta held constant. The strike is calculated by inverting $\Delta = \mathcal{N}(\pm d_1)$, using a polynomial approximation of the inverse normal distribution function.
(2) If premium is entered then volatility and strike are recalculated with delta held constant. The first step in this calculation is to invert $\Delta = \mathcal{N}(\pm d_1)$ to obtain d_1. The volatility is obtained by substituting

$$K = F \exp\left[\sigma\sqrt{T}\left(\sigma\sqrt{T}/2 - d_1\right)\right]$$

into

$$V = \pm\left(F\Delta - K\mathcal{N}(\pm(d_1 + \sigma\sqrt{T})\right) Z$$

and then solving for σ using the Newton-Raphson method.

1.4.6.3. *Puts and Calls.* The FX volatility surface allows the different Black volatilities to be entered for put and call options. Of course, this gives implied premiums that violate the no-arbitrage put-call parity relationship

$$V_{\text{call}} - V_{\text{put}} = (F - K)Z$$

but allows the trader to capture the actual volatility spreads traded in the market.

1.4.6.4. *Volatility Interpolation.* The is done by the generic method described in Section 1.4.3.

1.4.7. Bond Option Volatility Surface.

1.4.7.1. *Price Volatility and the Black Formula.* The price of a bond option can be entered on the surface expressed in one of three ways: a premium, a price volatility, or a yield volatility. The option price V and price volatility σ are related by Black formula

$$V = \pm\left(F\mathcal{N}\left(\pm d_1\right) - K\mathcal{N}(\pm d_2)\right) Z$$

where

- F is the (dirty) forward bond price to the option settlement date and K is the (dirty) strike price

- Z is the discount factor at the option settlement date
- $\pm$ is takes the value $+1$ for a call and -1 for a put
- $d_1 = d_2 + \sigma\sqrt{T}$ and

$$d_2 = \frac{1}{\sigma\sqrt{T}} \log\left(\frac{F}{K}\right) - \frac{\sigma\sqrt{T}}{2}$$

- T is the time to the exercise date.

The forward bond price F is calculated from the spot price and repo rate using the no-arbitrage formula of Section 2.15.3.

1.4.7.2. *Yield Volatility.* Given a bond price F, the yield-to-maturity y is given by $F = P(y)$, where the $P(y)$ is the value of bond cashflows discounted at rate y (the exact details of the price formula $P(y)$ do not concern us here). Differentiating the price formula gives a relationship between the return on the price and the return on the yield:

$$\frac{\Delta F}{F} \simeq \left(\frac{P'(y)y}{P(y)}\right)\frac{\Delta y}{y}$$

Hence the price and yield volatilities are related by

$$\sigma_{\text{price}} \simeq My\sigma_{\text{yield}}$$

where $M = |P'(y)/P(y)|$ is the modified duration of the bond.

1.4.7.3. *Volatility Interpolation.* The bond volatility surface has three axes: exercise date, bond maturity date and strike. The interpolation method is essentially the same as for the swaption volatility surface (see Section 1.4.5.2) with bond maturity date playing the role of swap tenor.

The surface can be interpolated on either price or yield volatility. In the case of price volatility interpolation, interpolation of the volatility skew is done on the ratio of price strike to forward bond price (both excluding the accrual). For yield volatility interpolation, the price strikes and forward bond prices are converted to yields and interpolation of the volatility skew is done on the ratio of yield strike to yield.

One reason for preferring yield volatility interpolation is that bond prices are coupon-dependent. The difference between the price volatility of two issues with similar maturity comes from the difference in maturity date *and* the difference in coupon. Therefore, interpolating between two such price volatilities could give a misleading result.

1.4.7.4. *Bond Futures Options.* A bond futures option volatility entered on the surface treated as the volatility of the cheapest-to-deliver bond.

CHAPTER 2

Pricing

2.1. Fair Price

This chapter gives an account of Adaptiv's core pricing models and pricing formulae for derivative instruments. The key ideas in the theory of pricing such instruments are: *relative pricing* and *absence of arbitrage*. To obtain the fair price of a derivative instrument we seek to relate its price to the price of simpler instruments. This relationship is derived on the assumption that the market does not allow arbitrage to exist. Typically, we can argue that: if the price of a derivative instrument is *not* a certain function of the prices of simpler instruments then an arbitrage opportunity would exist. Hence the fair (or no-arbitrage) price is determined by such an argument.

As an example, Section 2.3 shows how the fair price of a fixed-coupon bond is related to the prices of zero-coupon bonds maturing at the coupon payment dates. In this case, the fair price is *model-independent* in the sense that it depends only on the current zero-coupon bonds prices (that is, the current yield curve). The same is true of a FRA, vanilla swap and FX transaction. It will be shown that the fair prices of these instruments depend only on the current yield curves and current spot FX rates.

More generally, the fair price is derived under certain assumptions about the future evolution of prices. These assumptions constitute the *pricing model*. For example, the Garman-Kohlhagen model for FX option pricing assumes that interest rates are constant and that the volatility of the spot rate is constant. In such cases, the derivative instrument does not have a unique fair price because the fair price depends on the chosen pricing model. This multiplicity of fair prices should not concern us too much; it reflects the fact that there is not enough information in the market-quoted prices to uniquely determine the price of every derivative instrument.

2.2. PV, Price and Discount Factor

The *present value* (PV) of an instrument is defined to be the fair price of its future cashflows, where the cashflows may be either fixed or contingent on future rates. Adaptiv adopts the convention that cashflows settled today are excluded from the present value. The *price* of an instrument is its present value expressed in a more natural, convenient or market-standard fashion. For example, the (clean) price of a bond is its present value with the accrued interest subtracted, expressed as a percentage of the principal amount.

The price of a zero-coupon bond paying one unit at date D will be referred to as the *discount factor* at D.

2.3. Fixed Payments

This section covers the pricing of instruments paying fixed cashflows in a single currency. Examples of these are fixed-interest loans and deposits, fixed-coupon bonds and floating rate spreads.

Suppose the cashflows of the instrument are $c_1, \ldots, c_n$ at dates $D_1, \ldots, D_n$, where c_i may be positive or negative. For each cashflow, sell an amount c_i of zero-coupon bond maturing at D_i. This offsets all the future cashflow and therefore, to avoid arbitrage, the fair price of the instrument equals the total proceeds from selling the zero-coupon bonds. Thus the present value is

$$c_1 Z_1 + \cdots + c_n Z_n \tag{2.1}$$

where Z_i denotes the discount factor at D_i.

2.4. Foreign Currency

Let $\widetilde{V}$ denote the present value of an instrument, where this value is expressed in a foreign currency. This means that selling the instrument yields a foreign currency amount $\widetilde{V}$. Selling the instrument and executing a FX transaction gives an amount $S_0\widetilde{V}$ in the base currency, where S_0 denotes the foreign exchange rate for settlement today. Therefore, the fair price of the instrument in base currency is

$$S_0\widetilde{V} \tag{2.2}$$

Consider a forward FX transaction to purchase one unit of foreign currency at price F. Then the foreign currency cashflow is 1 and the base currency cashflow is $-F$. The present value of these cashflows is $S_0\widetilde{Z} - FZ$, where $\widetilde{Z}$ and Z are respectively the foreign and base currency discount factors at the settlement date. The forward FX rate is value of F for which the PV of this transaction is zero. Hence the forward FX rate is

$$F = S_0 \frac{\widetilde{Z}}{Z} \tag{2.3}$$

As a special case, the spot FX rate and today FX rate are related by

$$S = S_0 \frac{\widetilde{Z}_{\text{spot}}}{Z_{\text{spot}}} \tag{2.4}$$

This equation allows the today FX rate to be implied from the market-quoted spot FX rate.

2.5. Floating Cashflows

2.5.1. Forward Deposit Rate. Consider a fixed-interest loan. The principal amount P is received at the start date D_s in exchange for the principal and interest amounts at the end date D_e. The present value of the transaction is

$$PZ_s - P(1 + \alpha r)Z_e$$

where r is the fixed interest rate, Z_s and Z_e are respectively the discount factors at D_s and D_e, and α is the year fraction from D_e to D_s (calculated according to market convention, for example Act/360). The forward deposit rate is the value of r for which the PV of this transaction is zero. Hence the forward deposit rate is

$$r = \frac{1}{\alpha}\left(\frac{Z_s}{Z_e} - 1\right) \tag{2.5}$$

2.5.2. Standard Floating Cashflow. A standard floating cashflow is fixed upfront and paid in arrears. Consider a deposit period starting at date D_s and ending at date D_e. The forward deposit rate at time t is given by

$$r(t) = \frac{1}{\alpha}\left(\frac{Z(t, D_s)}{Z(t, D_e)} - 1\right) \tag{2.6}$$

where $Z(t, D)$ denotes the price at time t of a zero-coupon bond with maturity date D. The floating cashflow pays an amount $P\alpha_{\text{int}} r(T)$ at the end date D_e, where P is the principal amount, α_{int} is the year fraction[1]and T is the fixing date. Therefore, the floating cashflow has present value $P\alpha_{\text{int}} r(T)Z(T, D_e)$ at time T; and this equal to

$$\frac{P\alpha_{\text{int}}}{\alpha}\left(Z(T, D_s) - Z(T, D_e)\right)$$

which is the PV at time T of two principal cashflows. Thus, the floating cashflow and the principal cashflows have the same PV at the fixing date. To avoid arbitrage, it follows that they have the same PV at any time before the fixing date. Therefore, the PV of the floating cashflow at time $t \le T$ is

$$\frac{P\alpha_{\text{int}}}{\alpha}\left(Z(t, D_s) - Z(t, D_e)\right) = P\alpha_{\text{int}} r(t)Z(t, D_e)$$

Dropping the time dependency in our notation, we can now state that the present value of standard floating cashflow is $P\alpha_{\text{int}} rZ_e$, where the forward deposit rate is given by Equation (2.5).

2.5.3. Discounted Floating Cashflows. Forward rate agreements and Australian swap cashflows are discounted and paid upfront. A discounted and upfront cashflow pays an amount of the form

$$\frac{F(r(T))}{1 + \alpha r(T)}$$

at the start date D_s, where $F(r)$ is a function of the rate. The present value of such a cashflow at time T is

$$\frac{F(r(T))}{1 + \alpha r(T)} Z(T, D_s) = F(r(T))Z(T, D_e)$$

This shows that the PV of the discounted and upfront cashflow is the same as the PV of the standard floating cashflow that pays $F(r(T))$ at the end date D_e.

A standard FRA pays

$$\frac{P\alpha_{\text{int}}(r(T) - r_0)}{1 + \alpha r(T)}$$

at the start date, where r_0 is the fixed rate. Hence its PV is $P\alpha_{\text{int}}(r - r_0)Z_e$.

An Australian-style floating cashflow pays

$$\frac{P\alpha_{\text{int}} r(T)}{1 + \alpha r(T)}$$

at the start and hence its PV is $P\alpha_{\text{int}} rZ_e$. The same is true of the floating side cashflow in an Australian or South African FRA.

2.5.4. Compounding Swaps.

[1]There is a distinction between the year fraction for the rate, α, and the year fraction for calculating the interest cashflow, α_{int}. For example, LIBOR rates are quoted either Act/360 or Act/365 but interest may be calculated on a 30/360 basis.

2.5.4.1. *Introduction.* Many swap structures have floating interest that is fixed more frequently than the payment frequency of the fixed side, for example quarterly LIBOR against annual fixed rate. In a compounding swap, floating interest amounts that are fixed within the same fixed-side period are combined into a single cashflow paid at the end of the fixed-side period. One way to combine the interest amounts is to simply add them up. More typically, the interest is compounded at the prevailing deposit rate.

2.5.4.2. *Without Spread.* Suppose there are n amounts of floating interest to be compounded. Then the k^{th} interest cashflow pays

$$P(1+\alpha_n r_n(T_n))\cdots(1+\alpha_{k+1}r_{k+1}(T_{k+1}))\alpha_k r_k(T_k)$$

at the *end* of the n^{th} interest period, where P is the principal amount and, for the i^{th} interest period, α_i is the year fraction, r_i is the deposit rate and T_i is the fixing date. Section 2.5.3 shows that a cashflow of the form $F(r(T))$ paid at the start of the deposit period has the same PV the cashflow $F(r(T))(1+\alpha r(T))$ paid at the end of the deposit period. Thus, the PV of the k^{th} interest cashflow is the same as the PV of a cashflow that pays

$$P(1+\alpha_{n-1}r_{n-1}(T_{n-1}))\cdots(1+\alpha_{k+1}r_{k+1}(T_{k+1}))\alpha_k r_k(T_k)$$

at the *start* of the n^{th} interest period. Assuming the interest periods are contiguous, we can repeat this argument and conclude that the PV of the k^{th} interest cashflow is equal to the PV of an amount $P\alpha_k r_k(T_k)$ paid at the end of the k^{th} period. But this is the PV of a standard swap cashflow, equal to

$$P\alpha_k r_k Z_k$$

where Z_k is the discount factor at the end of the k^{th} period.

If the deposit rates $r_k, r_{k+1}, \ldots, r_l$ have already been fixed then the same argument shows that the PV of the k^{th} interest cashflow is equal to

$$P(1+\alpha_l r_l)\cdots(1+\alpha_{k+1}r_{k+1})\alpha_k r_k Z_l$$

2.5.4.3. *With Spread.* There are several accepted methods of including spread rates in the calculation of the compounded interest cashflow. Adaptiv supports the following four methods.

No Compounding.: The interest amounts are added without compounding. The k^{th} interest cashflow is

$$P\alpha_k(r_k+s_k)$$

where s_k denotes the spread rate.

Compound and Add Spread: The spread rate is not compounded. The k^{th} interest cashflow is

$$P(1+\alpha_n r_n)\cdots(1+\alpha_{k+1}r_{k+1})\alpha_k r_k + P\alpha_k s_k$$

Flat Compounding: The interest amount including the spread is compounded at the rates *excluding* the spread. The k^{th} interest cashflow is

$$P(1+\alpha_n r_n)\cdots(1+\alpha_{k+1}r_{k+1})\alpha_k(r_k+s_k)$$

Add Spread and Compound: The interest amount including the spread is compounded at the rates *including* the spread. The k^{th} interest cashflow is

$$P[1+\alpha_n(r_n+s_n)]\cdots[1+\alpha_{k+1}(r_{k+1}+s_{k+1})]\alpha_k(r_k+s_k)$$

In each case, the k^{th} interest cashflow is of the form $F(r_k(T_k), \ldots, r_n(T_n))$ and Adaptiv assigns it the PV

$$F(r_k, \ldots, r_n)Z_n$$

where Z_n is the discount factor at the end of the n^{th} period. This PV formula is naive in the sense that it is plausible but has not been derived from no-arbitrage principles. See Section 2.9 for further discussion of this point.

2.6. Discount and Reference Curves

So far, the market's preference for credit and liquidity has been neglected when applying no-arbitrage principles to derive the fair price of an instrument. Adaptiv allows credit and liquidity effects to be represented in the pricing formulae by means of the discount and reference yield curves. Forward deposit rates are obtained from the reference curve via Equation (2.5). The discount curve is used to discount future cashflows. Thus, the PV of a standard swap cashflow becomes $P\alpha_{\text{int}} r Z_e^{\text{disc}}$, where

$$r = \frac{1}{\alpha}\left(\frac{Z_s^{\text{ref}}}{Z_e^{\text{ref}}} - 1\right)$$

and Z^{disc} and Z^{ref} denote discount factors taken from the discount and reference curves respectively.

2.7. Caps

2.7.1. Caps, Floors and Collars.

A cap is an instrument consisting of *caplets*. A caplet is a European call option on a deposit rate. Typically, the deposit periods of the caplets are contiguous so that the cap is said to consist of a strip of caplets. A floor consists of floorlets and a floorlet is a European put option on a deposit rate. A collar is a long position in a cap combined with a short position in a floor.

2.7.2. Black formula for Standard and Digital Caplets.

A standard caplet pays an amount

$$P\alpha_{\text{int}} \max(r(T) - K, 0)$$

at the end of the deposit period, where K is the strike rate and T is the time to fixing date for the deposit rate (two days before the start of the deposit period in most markets). A standard floorlet pays an amount $P\alpha_{\text{int}} \max(K - r(T), 0)$.

A digital caplet [floorlet] pays a fixed amount at the end of the deposit period if $r(T)$ is greater [less] than K and otherwise pays nothing. Typically, the payoff amount is of the form $P\alpha_{\text{int}} r_0$ for some fixed rate r_0.

Adaptiv prices standard caplets and floorlets using the Black formula

$$\pm P\alpha_{\text{int}} \left(r\mathcal{N}(\pm d_1) - K\mathcal{N}(\pm d_2)\right) Z_e$$

for the standard payoff and

$$\mathcal{N}(\pm d_2)Z_e$$

for a unit digital payoff, where

- $\pm$ takes the value $+1$ for a caplet and -1 for a floorlet
- r is the forward deposit rate
- $\sigma^2 T$ is the variance of $\log r(T)$
- Z_e is the discount factor at the end of the deposit period

- $d_1 = d_2 + \sigma\sqrt{T}$ and

$$d_2 = \frac{1}{\sigma\sqrt{T}} \log\left(\frac{r}{K}\right) - \frac{\sigma\sqrt{T}}{2}.$$

The value of σ is read from the Cap Volatility Surface; see Section 1.4.4.

2.7.3. Models Consistent with Black Formula. No-arbitrage pricing theory shows that the present value of the caplet is given by

$$\mathbb{E}[\max(r(T) - K, 0)]P\alpha_{\text{int}} Z_e$$

where $\mathbb{E}$ is the expectation in the equivalent martingale measure induced by the numeraire $Z_e(t)$ (the forward measure). It was shown in Section 1.4.1 that the caplet price is given by the Black formula if and only if $r(T)$ has a lognormal distribution under the forward measure.

There are classes of no-arbitrage models (known as *market models*) in which the deposit rates are lognormal martingales under their respective forward measures. With such a model, all the caplets in the portfolio can be priced consistently using the Black formula. Brace, Gatarek and Musiela (BGM) [**6**] have defined a model where, for a fixed tenor δ, all the δ-period rates are simultaneously lognormal. Jamshidian [**19**] has defined a discrete date version of the BGM model using a discrete cash account (risk-neutral) numeraire.

Another class of no-arbitrage models (known as *Markov-functional models*) consistent with Black formula have been defined by Hunt, Kennedy and Pelsser [**14**]. In these models, the 'terminal' deposit rates $r(T)$ are lognormal but the process $r(t)$ is not necessarily lognormal. Markov-functional models are computationally more tractable than market models because the rates are functions of a low-dimensional Markov process.

2.7.4. Discounted (Australian) Caplets. An Australian caplet pays upfront, the difference between the discounted rate and discounted strike rate. Thus, it pays an amount

$$P\alpha_{\text{int}} \max\left(\frac{r(T)}{1+\alpha r(T)} - \frac{K}{1+\alpha K}, 0\right)$$

at the start of the deposit period. This is equal to

$$\frac{P\alpha_{\text{int}}}{(1+\alpha K)} \frac{\max(r(T) - K, 0)}{(1+\alpha r(T))}$$

and the argument of Section 2.5.3 shows that the Australian caplet has the same PV as a cashflow that pays

$$\frac{P\alpha_{\text{int}}}{(1+\alpha K)} \max(r(T) - K, 0)$$

at the *end* of the deposit period. Hence the PV of the Australian caplet is equal to the PV of the corresponding standard caplet discounted at the strike rate.

2.7.5. Average-Rate Caps. An average-rate caplet is a call option on the average of a fixed-tenor deposit rate observed at regular intervals throughout the caplet period, for example 6 month LIBOR observed weekly over the 6 month interest period of the caplet. Adaptiv prices average-rate caplets using an adjusted volatility in the Black formula.

Suppose the caplet fixes to the average of the deposit rates $r_1, \ldots, r_n$ with fixing dates $T_1 < \cdots < T_n$, and the average rate is given by $\bar{r} = \sum_{i=1}^{n} \omega_i r_i$, where $\omega_1, \ldots, \omega_n$ are positive weighting factors satisfying $\sum_{i=1}^{n} \omega_i = 1$. The caplet is priced using a modified form of the Black formula of Section 2.7.2:

$$P\alpha_{\text{int}} \left[\bar{r}\mathcal{N}\left(d_2 + \bar{\sigma}\sqrt{T_n}\right) - K\mathcal{N}(d_2) \right] Z, \tag{2.7}$$

where Z is the discount factor at the payment date and

$$d_2 = \frac{1}{\bar{\sigma}\sqrt{T_n}} \log\left(\frac{\bar{r}}{K}\right) - \frac{\bar{\sigma}\sqrt{T_n}}{2}.$$

The effective volatility of the average rate is given by

$$\bar{\sigma}^2 T_n = \sum\nolimits_{i,j>k}^{n} \left(\omega_i \sigma_i \sqrt{T_i}\right) \left(\omega_j \sigma_j \sqrt{T_j}\right) \rho_{ij}, \tag{2.8}$$

where $r_1, \ldots, r_k$ are the known rates, σ_i is the implied Black volatility of r_i, and ρ_{ij} is a correlation coefficient. The correlation coefficients can either be obtained from historical data; or set to

$$\rho_{ij} = \frac{\sigma_i \sqrt{T_i}}{\sigma_j \sqrt{T_j}}$$

for $i < j$, in which case

$$\bar{\sigma}^2 T_n = \sum\nolimits_{i>k}^{n} \omega_i \left(\omega_i + 2\sum\nolimits_{j>i}^{n} \omega_j\right) \sigma_i^2 T_i. \tag{2.9}$$

The origin of these formulae is explained in Appendix D.

A special case of this is for a caplet starting today, with equally spaced observations $T_i = i\Delta t$, constant volatility $\sigma_i = \sigma$ and equal weights $\omega_i = 1/n$. Under these conditions, Equation (2.9) becomes[2]

$$\bar{\sigma}^2 = \frac{\sigma^2}{n^3} \sum\nolimits_{i=1}^{n} \left(1 + 2(n-i)\right) i = \sigma^2 \frac{(2n+1)(n+1)}{6n^2},$$

and this is approximately $\sigma^2/3$ for large n. Hence the average-rate volatility is damped by a factor of $1/\sqrt{3}$ — the same factor as for the continuous geometric average of a lognormal rate with constant volatility (see for example Wilmott [39]).

2.8. European Swaptions

2.8.1. Swap Rate and PVBP. Consider a swap structure with n interest cashflows on the fixed side. Let $\Phi(t)$ denote the PVBP (present value of a basis point) of the fixed side. Then

$$\Phi(t) = \alpha_1 Z(t, D_1) + \cdots + \alpha_n Z(t, D_n) \tag{2.10}$$

where α_i is the year fraction and D_i the payment date of the i^{th} cashflow. The fixed side PV is given by $V_{\text{fxd}}(t) = r_{\text{fxd}}\Phi(t)$ where r_{fxd} is the fixed rate. The breakeven swap rate $r(t)$ is the value of r_{fxd} which makes $V_{\text{fxd}}(t)$ equal to the PV of the floating side, $V_{\text{flt}}(t)$. Thus, the swap rate is given by $V_{\text{flt}}(t) = r(t)\Phi(t)$.

[2] using the identities $\sum_{i=1}^{n} i = n(n+1)/2$ and $\sum_{i=1}^{n} i(i+1) = n(n+1)(n+2)/3$

2.8.2. Black Formula. The value of a swaption on its exercise date is

$$\max\left[\pm\left(V_{\text{flt}}(T) - V_{\text{fxd}}(T)\right), 0\right] = \max\left[\pm\left(r(T) - r_{\text{fxd}}\right)\Phi(T), 0\right]$$

where T is the time to the exercise date and $\pm$ takes the value $+1$ if the option is to pay the fixed rate (payer's swaption) and -1 if the option is receive fixed (receiver's swaption). Adaptiv prices European swaptions using the Black formula

$$\pm\left[V_{\text{flt}}\mathcal{N}(\pm d_1) - V_{\text{fxd}}\mathcal{N}(\pm d_2)\right]$$

where

$$d_2 = \frac{1}{\sigma\sqrt{T}}\log\left(\frac{V_{\text{flt}}}{V_{\text{fxd}}}\right) - \frac{\sigma\sqrt{T}}{2}$$

$d_1 = d_2 + \sigma\sqrt{T}$ and $\sigma^2 T$ is the variance of the $\log r(T)$.

The value of σ is read from the Swaption Volatility Surface; see Section 1.4.5.

2.8.3. Pricing Theory. No-arbitrage pricing theory shows that the present value of the swaption is given by

$$\mathbb{E}[\max(r(T) - r_{\text{fxd}}, 0)]\Phi$$

where $\mathbb{E}$ is the expectation in the equivalent martingale measure induced by the numeraire $\Phi(t)$ (the forward swap measure). If $r(T)$ is lognormal under the forward swap measure then the present value of the swaption evaluates to the Black formula (see Appendix B):

$$\pm\left[r\mathcal{N}(\pm d_1) - r_{\text{fxd}}\mathcal{N}(\pm d_2)\right]\Phi \tag{2.11}$$

where

$$d_2 = \frac{1}{\sigma\sqrt{T}}\log\left(\frac{r}{r_{\text{fxd}}}\right) - \frac{\sigma\sqrt{T}}{2},$$

$d_1 = d_2 + \sigma\sqrt{T}$ and $\sigma^2 T$ is the variance of $\log r(T)$. This is equivalent to the formula stated in Section 2.8.2.

Models with lognormal swap rates are described in Jamshidian [**19**] and Hunt and Kennedy [**14**].

2.9. Convexity and Quanto Corrections

2.9.1. A Generic LIBOR/CMS Cashflow. This section covers the pricing of cashflows that depend on the future value of a single deposit rate or swap rate. We consider a cashflow that pays an amount

$$\Pi(r(T))$$

where $r(T)$ is the value at time T of the rate (either deposit or swap) and Π is a known function of the rate. Examples of instruments containing such cashflows are:

- Arrears-fixing LIBOR swaps. These have cashflows with $\Pi(r) = r$ and a payment date at the start of the rate period.
- Average LIBOR rate swaps. These decompose into cashflows with $\Pi(r) = r$ but with a payment date between the start and end of the rate period.
- Constant maturity swaps (CMS). These have cashflows with $\Pi(r) = r$, where r is a swap rate.
- LIBOR and constant maturity caps. These have cashflows with $\Pi(r) = \max(r - K, 0)$ where K is the strike rate.

- LIBOR and constant maturity digital caps. These have cashflows with

$$\Pi(r) = \begin{cases} 1 & \text{if } r \geq K \\ 0 & \text{if } r < K \end{cases}$$

where K is the strike rate.

2.9.2. Naive Pricing and Convexity Correction. A pay-upfront (equivalently, fix-in-arrears) LIBOR cashflow pays an amount $r(T)$ at the start of the deposit period, where T is the time to the fixing date. Suppose the price of such a cashflow is rZ_s, where $r = r(0)$ is the forward deposit rate and Z_s is the discount factor at the start of the deposit period. This price will be referred to as *naive* because the formula is plausible but has not been derived using no-arbitrage principles.

The naive price is not correct because it leads to an arbitrage opportunity. To see this, consider the following strategy.

- Buy one unit of the pay-upfront forward transaction at the naive price r. This transaction has a net cashflow of $(r(T) - r)$ at the start of the deposit period, which is then invested at the prevailing rate $r(T)$ to obtain $(1 + \alpha r(T))(r(T) - r)$ at the end of the period.
- Sell $(1 + \alpha r)$ units of the standard forward transaction at the no-arbitrage price r. This transaction has a net cashflow of $(1 + \alpha r)(r - r(T))$ at the end of the period.

The net cashflow at the end of the period is

$$(1 + \alpha r(T))(r(T) - r) + (1 + \alpha r)(r - r(T)) = \alpha(r(T) - r)^2 \geq 0$$

Since this cashflow is never negative, paying the naive price leads to an arbitrage opportunity. This argument shows that the naive price is too low. Under a lognormal model, the no-arbitrage price of the pay-upfront cashflow is

$$\left(\frac{1 + \exp(\sigma^2 T)\alpha r}{1 + \alpha r}\right) r,$$

where σ is the implied Black volatility of the rate. This is a special case of the formula stated in Section 2.9.3.

Paying $r(T)$ at the start of the period is equivalent to paying $(1 + \alpha r(T))r(T)$ at the end of period. The equivalent pay-in-arrears amount is a convex function of the rate (see Figure 2.1). For this reason, the difference between the no-arbitrage price and the naive price is referred to as the *convexity correction*.

2.9.3. No-Arbitrage Pricing. Consider again the cashflow $\Pi(r(T))$ where $r(t)$ is a swap rate with PVBP given by Equation (2.10). LIBOR-based cashflows are a special case because a deposit rate is the same as a one-period swap rate. To price such a cashflow, Hunt and Kennedy [**14**] use their *linear swap rate model* (LSRM) under which $r(T)$ is lognormal in the forward swap measure and the zero-coupon bond prices are of the form

$$Z(t, S) = \Phi(t)(A + B(S)r(t))$$

for $t \leq T \leq S$, where A is constant and $B(S)$ is a deterministic function and $\Phi(t)$ is the PVBP defined in Section 2.8.1. Let π denote the expected payoff function defined by

$$\pi(r) = \mathbb{E}(\Pi(r \exp(\sigma\sqrt{T}W - \sigma^2 T/2))),$$

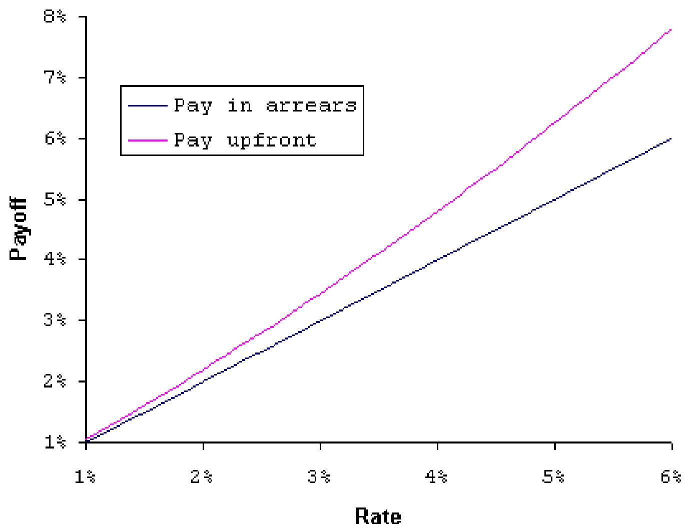


FIGURE 2.1. convexity of a pay-upfront (fix-in-arrears) cashflow.

where W is a standard normal variable and $\sigma^2 T$ is the variance of $\log r(T)$. Under the LSRM, the PV of cashflow is given by the formula

$$\left[\omega\pi(r) + (1-\omega)\pi\left(r\exp(\sigma^2 T)\right)\right] Z \tag{2.12}$$

where Z is the discount factor at the payment date and

$$\omega = \frac{\alpha_1 Z_1 + \cdots + \alpha_n Z_n}{\alpha_1 Z + \cdots + \alpha_n Z} \tag{2.13}$$

is the ratio of the PVBP of the swap rate to the PVBP with all the discount factors replaced by the payment-date discount factor. The price formula can be rewritten as

$$\pi(r)Z + (1-\omega)\left[\pi\left(r\exp(\sigma^2 T)\right) - \pi(r)\right] Z.$$

The first term $\pi(r)Z$ is the naive price and the second term is the convexity correction.

In the LIBOR case ($n = 1$), $\omega = Z_e/Z$ where Z_e is the discount factor at the end of the rate period.

2.9.4. Application of the General Formula.

2.9.4.1. *LIBOR with Non-Standard Payment Date.* Arrears-fixing and average-rate swaps decompose into LIBOR cashflows with a payment date between the start and end of the rate period. For these cashflows, $\Pi(r) = r$ and hence $\pi(r) = r$. Therefore, the PV of the cashflow is

$$rZ + \left(\exp(\sigma^2 T) - 1)(Z - Z_e\right) r. \tag{2.14}$$

For a pay-upfront cashflow, $Z = Z_s$ and the PV becomes

$$r\left(\frac{1 + \exp(\sigma^2 T)\alpha r}{1 + \alpha r}\right) Z.$$

2.9.4.2. *LIBOR and CMS Caplets.* For these cashflows, $\Pi(r) = \max(r - K, 0)$ and it follows from the results of Appendix B that $\pi(r)$ is given by the Black formula

$$\pi(r) = r\mathcal{N}(d_2 + \sigma\sqrt{T}) - K\mathcal{N}(d_2),$$

where

$$d_2 = \frac{1}{\sigma\sqrt{T}} \log\left(\frac{r}{K}\right) - \frac{\sigma\sqrt{T}}{2}.$$

2.9.5. Quanto Cashflows. Consider a general quanto cashflow that pays an amount $\Pi(\tilde{r}(T))$ in domestic currency, where $\tilde{r}(T)$ is the value at time T of a foreign-currency deposit or swap rate, and Π is a known function of the rate. Hunt and Kennedy [**14**] define a multi-currency linear swap rate model which can be applied to quanto cashflows, and leads to simple pricing formulae for quanto swaps and caps on LIBOR and constant-maturity rates.

Before stating the general formula, the following argument, similar to that given by Baxter and Rennie [**9**], shows that the value of a quanto swap cashflow is not given by the naive formula $\tilde{r}Z$. Suppose that foreign-currency interest rates and the foreign-to-domestic FX rate are known to be negatively correlated. Then consider the strategy:

- Buy F units of a quanto forward LIBOR transaction at the naive price $\tilde{r}$, where F is the current forward FX rate for settlement at the end of the deposit period. This transaction pays an amount $(\tilde{r}(T) - \tilde{r})F$ of domestic currency, where T is the fixing date.
- Sell one unit of the corresponding (standard) foreign-currency forward transaction at the market rate $\tilde{r}$. This transaction pays an amount $\tilde{r} - \tilde{r}(T)$ of foreign currency.
- Execute a forward FX transaction at time T to convert the foreign-currency payoff into domestic currency.

The cost of each standard forward transaction is zero, and the cost of the quanto transaction is zero if the counterparty agrees to trade at the naive price. The net payoff in domestic currency is

$$(\tilde{r}(T) - \tilde{r})F + (\tilde{r} - \tilde{r}(T))F(T) = (F - F(T))(\tilde{r}(T) - \tilde{r}).$$

Thus, under the assumption that $\tilde{r}$ and F are negatively correlated, this payoff is positive; and hence trading at the naive price leads to an arbitrage opportunity.

Under the multi-currency LSRM, the quanto forward rate is given by

$$\tilde{r} \exp\left(-\hat{\sigma}\tilde{\sigma}\rho T\right),$$

where $\tilde{\sigma}$ is the volatility of the LIBOR rate, $\hat{\sigma}$ is the volatility of the FX rate, and ρ is the correlation coefficient of these two rates. This formula shows explicitly what can be deduced from the above strategy: that the quanto forward rate is greater than the standard forward rate when the FX and LIBOR rates are negatively correlated.

2.9.6. No-Arbitrage Pricing of Quanto Cashflows. The generalization of the pricing formula of Section 2.9.3 is that the PV of the quanto cashflow is given by

$$[\omega\pi\left(\tilde{r}\exp(-\hat{\sigma}\tilde{\sigma}\rho T)\right) + (1-\omega)\pi\left(\tilde{r}\exp(\sigma\tilde{\sigma}\theta T - \hat{\sigma}\tilde{\sigma}\rho T)\right)] Z, \tag{2.15}$$

where: $\tilde{r}$ is the foreign-currency rate; $\tilde{\sigma}$ is the volatility of the foreign-currency rate, σ is the volatility of the corresponding domestic-currency rate, θ is the correlation of these two rates; $\hat{\sigma}$ is the volatility of the foreign-to-domestic FX rate, and ρ is the correlation of the foreign-to-domestic FX rate and the foreign-currency rate. The weighting factor ω is a function of domestic-currency discount factors and is given by Equation (2.13).

As in Section 2.9.4, swap cashflows have $\Pi(r) = r$ and $\pi(r) = r$; and caplets have $\Pi(r) = \max(r - K, 0)$ and $\pi(r)$ given by the Black formula.

In the LIBOR case, $\omega = Z_e/Z$ where Z_e is the discount factor at the end of the rate period; and typically, the payment date coincides with the end of the rate period so that $\omega = 1$ and the PV of the cashflow is $\pi\left(\tilde{r}\exp(-\hat{\sigma}\tilde{\sigma}\rho T)\right)Z$.

2.10. Interest-Rate Spread Options

2.10.1. Introduction. A spread option is an option on the difference (spread) between two asset prices. This section covers the pricing options on spreads of the form $ar_1 - r_2$, where r_1 and r_2 are deposit or swap rates and a is a constant. For example, an option on the difference between 0.75 times the 5y GBP swap rate and the 7y USD swap rate with strike 50 basis points. Adaptiv prices European options with either a standard or digital payoff. The standard payoff is

$$\max(ar_1 - r_2 - K, 0),$$

where K is the strike rate, and the digital payoff is equal to 1 if $ar_1 - r_2 > K$ and otherwise 0. The settlement date of the option can be any date after the exercise date, and the payoff can be in a third currency — not necessarily the currency of rate r_1 or the currency of rate r_2.

2.10.2. Lognormal Model. This type of spread option can be priced using the multi-currency linear swap rate model of Hunt and Kennedy [**14**], under which the rates at the exercise date have a lognormal distribution. The model leads to a closed-form formula for the spread options comprising a weighted sum of two expectations of the form $\mathbb{E}(\max(X - Y - K, 0)$ where X and Y are lognormal variables and K is a constant (the strike). Evaluating this expectation requires a one-dimensional numerical integration, except in the special case that the strike is zero when it is given by the Margrabe formula.

Consider a payoff of the form $\Pi(r_1(T), r_2(T))$ where Π is a known function and $r_1(T)$ and $r_2(T)$ are lognormal variables with $\mathrm{var}(\log r_i(T)) = \sigma_i^2 T$ and $\mathrm{cov}(\log r_1(T), \log r_2(T)) = \sigma_1\sigma_2\rho T$. Let π denote the expected payoff function defined by $\pi(r_1, r_2) = \mathbb{E}(\Pi(X_1, X_2))$, where $X_i = r_i \exp(\sigma_i\sqrt{T}W_i - \sigma_i^2 T/2)$ and W_1 and W_2 are standard normal variables with correlation ρ. The value of such a payoff under the multi-currency LSRM is given by

$$[\omega\pi(\tilde{r}_1, \tilde{r}_2) + (1 - \omega)\pi(\hat{r}_1, \hat{r}_2)]\, Z, \tag{2.16}$$

where $\tilde{r}_i = r_i \exp(-\tilde{\rho}_i\tilde{\sigma}_i\sigma_i T)$ and $\hat{r}_i = \tilde{r}_i \exp(\rho_i\sigma_i\sigma T)$, and the various volatilities and correlations are defined as follows.

- σ_i is the volatility of rate r_i (as defined above).
- σ is the volatility of the *settlement*-currency rate r corresponding to r_1.
- $\tilde{\sigma}_i$ is the volatility of the *FX rate* between the currency of r_i and the settlement currency.
- ρ_i is the correlation between r_i and the settlement-currency rate r.
- $\tilde{\rho}_i$ is the correlation between r_i and the FX rate *from* the currency of r_i *to* the settlement currency.

The weighting factor ω is the function of settlement-currency discount factors given by Equation (2.13), and Z is the discount factor at the settlement date.

It remains to evaluate the expectation of the payoff function. For a spread option with constant rate multiplier a, the payoff is $\Pi(ar_1, r_2)$ where $\Pi(r_1, r_2) = \max(r_1 - r_2 - K, 0)$. For the special case that the strike K is zero, the expectation of the payoff function is given by Margrabe's formula

$$\pi(r_1, r_2) = r_1\mathcal{N}(d + v/2) - r_2\mathcal{N}(d - v/2),$$

where $d = \log(r_1/r_2)/v$ and $v^2 = (\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2)T$. This is proved in Appendix B. For the case that the strike is not zero, consider the general linear payoff $\Pi(r_1, r_2) = (A_1r_1 + A_2r_2 + A)1_{\{r_1-r_2>K\}}$, where A_1, A_2 and A are constants. The required expectation is $\mathbb{E}(\Pi(X_1, X_2))$ where $X_i = r_i \exp(\sigma_i\sqrt{T}W_i - \sigma_i^2T/2)$ and W_1 and W_2 are standard normal variables with correlation ρ. Define $\kappa = \sqrt{1-\rho^2}$ and $W = (W_1 - \rho W_2)/\kappa$ so that W and W_2 are independent standard normal variables and $W_1 = \kappa W + \rho W_2$. Write $u = \rho\sigma_1\sqrt{T}$ and $v = \kappa\sigma_1\sqrt{T}$. Then $X_1 = R_1\exp(vW - v^2/2)$ where $R_1 = r_1\exp(uW_2 - u^2/2)$ is a function of W_2 only. Then

$$\begin{aligned}\mathbb{E}\left(\Pi(X_1, X_2)\right) &= \mathbb{E}\left(\mathbb{E}\left([A_1X_1 + A_2X_2 + A]\,1_{\{X_1>X_2+K\}}|W_2\right)\right)\\ &= \mathbb{E}\left(A_1R_1\mathcal{N}(d + v/2) + [A_2X_2 + A]\mathcal{N}(d - v/2)\right),\end{aligned}$$

where $d = \log(R_1/(X_2 + K))/v$. This shows that $\mathbb{E}(\Pi(X_1, X_2))$ is equal to the expectation of a known function of W_2 and hence can be evaluated by a one-dimensional numerical integration, for example by Simpson's Rule as described in Press et al. [**29**].

2.10.3. Normal Model. The normal model is an adaptation of the Hunt-Kennedy formula (2.16) where the expected payoff $\pi(r_1, r_2)$ is calculated under the assumption that the rates $r_1(T)$ and $r_2(T)$ are normally distributed. In this context, the expected payoff function is defined by $\pi(r_1, r_2) = \mathbb{E}(\Pi(X_1, X_2))$, where $X_i = r_i + v_i\sqrt{T}W_i$, v_i^2T is the variance of $r_i(T)$, and W_1 and W_2 are standard normal variables with correlation ρ.

For the standard payoff function $\Pi(r_1, r_2) = \max(r_1 - r_2 - K, 0)$, the expected payoff is given by

$$\pi(r_1, r_2) = (r_1 - r_2 - K)\mathcal{N}(d) + v\sqrt{T}\mathcal{N}'(d), \tag{2.17}$$

where $\mathcal{N}'(d) = \exp(-d^2/2)/\sqrt{2\pi}$, $d = (r_1 - r_2 - K)/v\sqrt{T}$ and $v\sqrt{T}$ is the standard deviation of $r_1 - r_2$ given by $v^2 = v_1^2 - 2\rho v_1v_2 + v_2^2$. This is proved in Appendix B.

The source for the normal volatility v_i is the corresponding lognormal (Black) volatility σ_i from the volatility surface. For spread options the lognormal volatility is converted to a normal volatility using the formula

$$v_i = r_i\sigma_i.$$

For single-rate options the conversion formula is strike-dependent and is given in Section 2.10.4. Sensitivity to volatility is expressed with respect to the lognormal volatilities.

The model described here is not theoretically consistent because the quanto and convexity-correction rates $\tilde{r}_i$ and $\hat{r}_i$ that appear in Formula (2.16) are features of the lognormal model. It is possible to suppress the quanto and convexity corrections by setting to zero the reference/discount correlations ρ_i and the reference/FX correlations $\tilde{\rho}_i$. In this case the PV of the cashflow becomes simply $\pi(r_1, r_2)Z$, where Z is the discount factor at the settlement date.

The normal model is also available for single-rate payoffs, such as standard caplets and swaptions. The single-rate case is obtained by setting $\sigma_2 = 0$ and $r_2 = 0$ in Equation (2.17).

2.10.4. Normal Volatility and Skew. For single-rate options the normal model is applied using a normal volatility converted from a strike-dependent lognormal volatility taken from the cap or swaption volatility surface. The conversion formula is the lognormal case of Equation (A24b) from Hagan and Woodward [**13**]. The formula is

$$v = \left[\frac{(r+K)}{2} - \frac{(r-K)^2}{12K} - \frac{K\sigma^2 T}{24} + \dots \right] \sigma,$$

where v is the normal volatility, σ is the lognormal volatility, K is the strike of the option and T is the time to expiry. The ellipsis $(\dots)$ denotes higher-order terms in $(r-K)$ and $\sigma\sqrt{T}$.

2.10.5. Normal Model and Negatives Rates. The normal model is applied automatically when the rate r or the strike K are negative. In this case the absolute value of the rate is used in the conversion formula, so that the normal volatility is given by

$$v = |r|\sigma,$$

where σ is the lognormal volatility. If the rate is zero then the formula $v = |K|\sigma$ is used.

2.11. Futures

2.11.1. Present Value of a Futures Contract. The holder of a futures contract can choose to close-out their position and realize an amount of cash proportional to the difference between the current price and the contract price. Hence the present value of long position in a futures contract is

$$n(V/\varepsilon)(F-K)$$

where F is the current market price, K is the contract price, n is the number of contracts, V is the value of a one tick movement in the price and ε is one price tick.

2.11.2. Futures versus Forward Price. A futures contract differs from a forward contract because the profit and loss from daily price changes is credited to the margin account. Thus, the holder of the futures contract can, at any time, close-out their position and settle the difference between the current price and contract price. For this reason, the forward and futures price of the same underlying will differ.

Consider a futures contract with maturity date T. The price of the contract at maturity is fixed at the forward price $f(T)$ of some underlying asset. For example, for Eurodollar futures, T is the Monday before the third Wednesday and $f(T)$ is the three-month forward LIBOR starting on the third Wednesday. No-arbitrage pricing theory (see Cox, Ingersoll and Ross [**10**] and Shreeve [**36**]) shows that the futures price F is given by

$$F = \widetilde{\mathbb{E}}(f(T))$$

where $\widetilde{\mathbb{E}}$ is the expectation taken in the *risk-neutral measure*. The forward price f is given by

$$f = \mathbb{E}(f(T))$$

where $\mathbb{E}$ is taken in the *forward measure*. If interest rates are deterministic then the risk-neutral and forward measures coincide and hence the futures and forward prices are the same. More generally, the difference between the futures and forward price is given by

$$(f - F)Z = \widetilde{\text{cov}}\left(f(T), 1/\beta(T)\right) \tag{2.18}$$

where β is the value of the cash account (the value of one currency unit invested at the short-term rate) and Z is the discount factor at T.

2.11.3. Implied Futures Price. The implied futures price is the theoretical futures price derived from the corresponding forward price. Implied futures prices are used to calculate the sensitivity of the PV to underlying market risk factors; see Chapter 4. For short-dated contracts, the difference between the futures and forward price is small. For futures on bonds and FX, Adaptiv takes the implied futures price to be the same as the forward price. Deposit futures can have much longer maturity dates. For these contracts, Adaptiv uses the methods described in Section 1.2.7 to calculate the futures price from the forward rate.

2.12. Options on Futures

2.12.1. European Option with Premium Paid Upfront. This section covers European futures options where the exercise date coincides with the futures maturity date. For such options, the futures price and forward price are equal on the exercise date: $F(T) = f(T)$. Hence the option payoff is

$$\max\left(\pm(f(T) - K), 0\right)$$

where K is the strike price. If $f(T)$ has a lognormal distribution under the forward measure then the option price is given by the Black formula:

$$\pm\left[f\mathcal{N}(\pm d_1) - K\mathcal{N}(\pm d_2)\right]Z$$

(see Section 1.4.1) where Z is the discount factor at the forward settlement date,

$$d_2 = \frac{1}{\sigma\sqrt{T}}\log\left(\frac{f}{K}\right) - \frac{\sigma\sqrt{T}}{2}$$

$d_1 = d_2 + \sigma\sqrt{T}$ and $\sigma^2 T$ is the variance of $\log f(T)$.

2.12.2. American Option with Premium Paid Upfront. American options on futures are priced using the general binomial tree method described in Section 2.13. Early exercise can be advantageous for both put and call options.

2.12.3. Option with Premium in Margin. Futures options traded on the LIFFE have their value credited to the margin account. The premium is not settled until the option is exercised or the position is closed-out.

These options are nominally American but early exercise is never advantageous to holder of option because the payoff from early exercise is less than or equal to the payoff from closing-out the option position. The payoff from early exercise at time t is $U(t) - P$, where $U(t) = \pm(F(t) - K)$ is the value of the underlying futures contract and P is the contract price. The payoff from closing-out the option contract is $P(t) - P$, where $P(t)$ is the value of the option; and $P(t)$ is greater than or equal to $U(t)$.

Therefore, margin-style futures options are essentially European.

The option price behaves in the same way as a futures price; it is a martingale under the risk-neutral measure. At the exercise date T, the option price is $\max(\pm[F(T) - K], 0)$. If $F(T)$ has a lognormal distribution under the risk-neutral measure then it follows that the option price is given by the Black formula without discounting:

$$\pm[F\mathcal{N}(\pm d_1) - K\mathcal{N}(\pm d_2)]$$

where

$$d_2 = \frac{1}{\sigma\sqrt{T}} \log\left(\frac{F}{K}\right) - \frac{\sigma\sqrt{T}}{2}$$

$d_1 = d_2 + \sigma\sqrt{T}$ and $\sigma^2 T$ is the variance of $\log F(T)$.

2.12.4. Volatility. The volatility σ used to price a futures option is the caplet volatility obtained from the Cap Volatility Surface (see Section 1.4.4) by interpolation at the last trading date.

A mid-curve option is a futures option with exercise date in the short term on a futures contract with underlying deposit period starting in the medium term. In this case, the exercise date used to interpolate the Cap Volatility Surface is the start of the deposit period.

2.13. Generic American Options

Adaptiv prices American options using binomial tree methods. This section describes the generic method that is applied to American options on futures, FX and bonds. More details on pricing with binomial trees can be found in Hull [**15**].

The tree is used to evolve a forward price f under its forward measure. Let Π denote the payoff function. Standard options have a payoff function $\Pi(x) = \max(\pm(x - K), 0)$ where K is the strike and $\pm$ takes the value $+1$ for a call option and -1 for a put option. Digital options have $\Pi(x) = 1$ if $\pm(x - K) > 0$ and $\Pi(x) = 0$ otherwise. There are two types of option to consider:

- Option on futures. f is the forward price to the futures maturity date. At each time step, the futures price F is recovered from the forward price. If exercised, the option pays an amount $\Pi(F)$ on the exercise date.
- Options on spot transactions. f is the forward price to the spot settlement date of the last exercise date (the last spot date). At each time step, the spot price S is recovered from the forward price. If exercised, the option pays an amount $\Pi(S)$ on the spot settlement date of the exercise date.

The time from the first exercise date to the last exercise date is divided into n equal time steps of length Δt. At each node there are two possible price changes: up by a factor u with probability p or down by a factor $d = 1/u$ with probability $q = (1 - p)$. Hence the forward price at time $i\Delta t$ and price level j, where $0 \le j \le i \le n$, is given by $f(i, j) = fu^j d^{i-j}$. The up movement is

$$u = \exp\left(\sigma\sqrt{\Delta t}\right).$$

The probability of an up movement is given by $pu + qd = 1$, or equivalently

$$p = \frac{1 - d}{u - d}.$$

These values are chosen so that f has the martingale property:

$$pf(i + 1, j + 1) + qf(i + 1, j) = f(i, j)$$

and has constant volatility σ:

$$p(\log u)^2 + q(\log d)^2 = \sigma^2 \Delta t$$

The numeraire for the forward measure is the price of the zero-coupon bond maturing on the futures maturity date or last spot date. This is assumed to be a deterministic function of time i and is denoted $Z(i)$. The price of the zero-coupon bond maturing on the next spot settlement date is assumed to be constant and is denoted Z_{spot}.

Let $V(i,j)$ denote the value of the option at node (i,j). First, consider the value of the option at nodes on the last exercise date $(i = n)$. For options on futures, the forward price coincides with the futures price and the value of the option is given by

$$V(n,j) = \Pi(F(n,j)).$$

For options on spot, the forward price coincides with the spot price and the value of the option is given by

$$V(n,j) = \Pi(S(n,j))Z_{\text{spot}}.$$

The value of the option at any of the previous nodes $(i < n)$ is the maximum of the value of payoff at the node and the discounted expected values at the successor nodes. Hence

$$V(i,j) = \max\left(\Pi\left(F(i,j)\right), \left[pV(i+1,j+1) + qV(i+1,j)\right] Z(i)/Z(i+1)\right)$$

for options on futures and

$$V(i,j) = \max\left(\Pi(S(i,j))Z_{\text{spot}}, \left[pV(i+1,j+1) + qV(i+1,j)\right] Z(i)/Z(i+1)\right)$$

for options on spot. The current option value $V(0,0)$ is calculated by applying this recursive formula.

The owner of the option would choose to exercise early when the value of the option no longer exceeds the *intrinsic value*. The intrinsic value is the value of the payoff obtained by exercising the option today. On the binomial tree, this happens when $V(i,j) = \Pi(F(i,j))$ for options on futures, or $V(i,j) = \Pi(S(i,j))Z_{\text{spot}}$ for options on spot.

The corresponding European option is used as a *control variate* in order to reduce discretisation error. The tree is used to calculate the European option price given by $V_{\text{Euro}}(n,j) = V(n,j)$ and

$$V_{\text{Euro}}(i,j) = \left[pV_{\text{Euro}}(i+1,j+1) + qV_{\text{Euro}}(i+1,j)\right] Z(i)/Z(i+1)$$

The price of the American option is taken to be

$$V_{\text{Euro}} + \max\left(V(0,0) - V_{\text{Euro}}(0,0), 0\right)$$

where V_{Euro} is the analytic European option price calculated from the Black formula. Using the difference $V(0,0) - V_{\text{Euro}}(0,0)$ reduces the discretisation error. Discretisation error may cause $V(0,0)$ to be less than $V_{\text{Euro}}(0,0)$ in some cases; hence the positive part is taken.

2.14. FX Options

2.14.1. Quote and Other Currency. For a given pair of currencies, the currency in which the FX rate is quoted will be referred to as the *quoting currency* and the other member of the currency pair will be referred to as the *other currency*. Adaptiv's convention is to express the PV of over-the-counter FX options in the currency in which the rate is quoted. For example, the PV of EUR:USD option is expressed in USD and the PV of JPY:GBP option is expressed in JPY.

2.14.2. European Option Payoff. In this section, it will be convenient to consider a European option on a forward FX rate with a generalized payoff function. On its settlement date, the option pays an amount A of other currency and an amount B of quoting currency; these cashflows are contingent on the forward rate at the exercise date being above the strike rate (call option) or below the strike rate (put option). Thus, the condition for a non-zero payoff is

$$\pm(F(T) - K) \geq 0$$

where

- F is the forward rate to the option settlement date
- K is the strike rate
- T is the time to the exercise date
- $\pm$ takes the value $+1$ for a call option and -1 for a put option.

The standard and digital payoff options are special cases. For a standard payoff, $B = -AK$. For a digital payoff in the other currency, $B = 0$; and for a digital payoff in the quoting currency, $A = 0$.

2.14.3. Black Formula. If $F(T)$ is lognormal under the quoting currency forward measure then the price of the option, expressed in quoting currency, is given by the Black formula

$$\pm\left(AF\mathcal{N}(\pm d_1) + B\mathcal{N}(\pm d_2)\right)Z$$

where Z is the quoting-currency discount factor at the settlement date,

$$d_2 = \frac{1}{\sigma\sqrt{T}} \log\left(\frac{F}{K}\right) - \frac{\sigma\sqrt{T}}{2}$$

$d_1 = d_2 + \sigma\sqrt{T}$ and $\sigma^2 T$ is variance of $\log F(T)$.

Under the assumption that interest rates are deterministic, the forward rate $F(t)$ is equal to the today rate $S(t)$ multiplied by a deterministic factor and hence the variance of $\log F(T)$ is the same as the variance of $\log S(T)$. From Equations (2.3) and (2.4), the forward rate and spot rate are related by

$$F = S\frac{\widetilde{Z}/\widetilde{Z}_{\text{spot}}}{Z/Z_{\text{spot}}} \tag{2.19}$$

where Z and $\widetilde{Z}$ denote, respectively, the quoting-currency and other-currency discount factors at the settlement date, and Z_{spot} and $\widetilde{Z}_{\text{spot}}$ denote the discount factors at the spot date. The Garman-Kohlhagen formula is obtained by substituting this expression into the Black formula.

2.14.4. Single Barrier.

2.14.4.1. *Payoff.* A single barrier FX option pays an amount A of other currency and an amount B of quoting currency; these cashflows are contingent on the forward rate being above/below the strike rate at the expiry date and contingent on the spot rate attaining/not attaining the barrier rate before the expiry date. The precise conditions for payoff are specified in Table 2.1 for each type of barrier option. The notation used in the table is as follows: T is the time to exercise date, S is the spot rate, F is the forward rate to the settlement date, K is the strike rate, H is the barrier rate and $\pm$ takes the $+1$ for a call option and -1 for a put option.

Type	**Conditions for Payoff:** $\pm(F(T) - K) \geq 0$ **and**
Down-and-in	$\min\{S(t) : t < T\} \leq H$
Up-and-in	$\max\{(t) : t < T\} \geq H$
Down-and-out	$\min\{S(t) : t < T\} > H$
Up-and-out	$\max\{S(t) : t < T\} < H$

TABLE 2.1. single barrier option payoff conditions.

Type	**Price**
Down-in call $H \leq K$ Up-in put $K \leq H$	$(H/S)^{2a} E\left(H^2/S, K\right)$
Down-in call $K \leq H$ Up-in put $H \leq K$	$E(S,K) - E(S,H) + (H/S)^{2a} E\left(H^2/S, H\right)$
Up-in call $H \leq K$ Down-in put $K \leq H$	$E(S,K)$
Up-in call $K \leq H$ Down-in put $H \leq K$	$E(S,H) + (H/S)^{2a}\left[E\left(H^2/S, K\right) - E\left(H^2/S, H\right)\right]$

TABLE 2.2. single barrier knock-in prices.

2.14.4.2. *Price.* Closed-form formulae for the price of single barrier options can be obtained under the assumptions that the continuously compounded interest rates are constant and the volatility of the spot rate process is constant.

Let $E(S, K)$ denote the price of the European option as a function of the spot rate and strike rate. This is obtained by substituting Equation (2.19) into the Black formula of Section 2.14.3. The price of each type of knock-in option is given in Table 2.2, where a denotes

$$a = -\frac{1}{2} + \frac{\log \widetilde{Z} - \log Z}{\sigma^2 T_1}$$

and T_1 is the time to the settlement date. These results are derived in Appendix E, or in Rubinstein and Reiner [**34**].

A portfolio comprising one knock-in option and one knock-out option, both with strike K and barrier H, has the same payoff as a European with strike K. Hence the knock-in price $V_{\text{in}}(S, K, H)$ and the knock-out price $V_{\text{out}}(S, K, H)$ are related by

$$V_{\text{in}}(S, K, H) + V_{\text{out}}(S, K, H) = E(S, K) \tag{2.20}$$

2.14.4.3. *Knock-in Rebate.* A single barrier knock-in rebate pays an amount A of other currency and an amount B of quoting currency contingent on the barrier *not* being broken before the expiry date. Let H denote the barrier rate and T denote the time to the expiry date. The payoff condition for a down-and-in rebate is $\min\{S(t) : t < T\} > H$ and the payoff condition for up-and-in rebate is $\max\{S(t) : t < T\} < H$. A down-and-in rebate is the same as a down-and-out call with $K = H$; an up-and-in rebate is the same as an up-and-out put with $K = H$. It follows from Table 2.2 and Equation (2.20) that the price of a knock-in rebate is

$$E(S, H) - (H/S)^{2a} E\left(H^2/S, H\right)$$

where the sign in the Black formula takes value $+1$ for down and -1 for up.

Type	Conditions for Payoff: $\pm(F(T) - K) \geq 0$ and
Knock-out	$H_1 < \min\{S(t) : t < T\}$ and $\max\{S(t) : t < T\} < H_2$
Knock-in	$H_1 \geq \min\{S(t) : t < T\}$ or $\max\{S(t) : t < T\} \geq H_2$

Table 2.3. double barrier option payoff conditions.

2.14.4.4. *Knock-out Rebate.* A single barrier knock-out rebate pays an amount A of other currency and an amount B of quoting currency contingent on the barrier being broken before the expiry date. Payment occurs on the date that the barrier is broken. Let H denote the barrier rate and T denote the time to the expiry date. The payoff condition for a down-and-out rebate is $\min\{S(t) : t < T\} \leq H$ and the payoff condition for up-and-out rebate is $\max\{S(t) : t < T\} \geq H$.

The price of a knock-out rebate is

$$(AH + B)\left[\left(\frac{H}{S}\right)^{a+b}\mathcal{N}(\pm d_1) + \left(\frac{H}{S}\right)^{a-b}\mathcal{N}(\pm d_2)\right] \tag{2.21}$$

where $\pm$ takes the value $+1$ for down and -1 for up,

$$a = -\frac{1}{2} + \frac{\log\widetilde{Z} - \log Z}{\sigma^2 T_1}$$
$$b = \left(a^2 - \frac{2\log Z}{\sigma^2 T_1}\right)^{1/2}$$
$$d_2 = \frac{1}{\sigma\sqrt{T}}\log\left(\frac{H}{S}\right) - b\sigma\sqrt{T}$$

and $d_1 = d_2 + 2b\sigma\sqrt{T}$. This result is derived in Appendix E.

2.14.5. Double Barriers.

2.14.5.1. *Payoff.* A double barrier option pays an amount A of other currency and an amount B of quoting currency; these cashflows are contingent on the forward rate being above/below the strike rate at the expiry date and contingent on the spot rate attaining one of the barriers or not attaining either barrier before the expiry date. The conditions for payoff are specified in Table 2.3, where H_1 denotes the lower barrier rate and H_2 denotes the upper barrier rate.

2.14.5.2. *Price.* There is no closed-form formula for the price of the double barrier option but a Fourier series expansion can be obtained. This Fourier series expansion is stated here and derived in Appendix E. It is usually sufficient to sum the first few terms of the expansion. If the valuation date is close to the exercise date or if the volatility is very low, the price is close to the payoff function, which is discontinuous, and therefore more terms from the Fourier series are required to approximate it.

The knock-in and knock-out price are related by

$$V_{\text{in}}(S, K, H_1, H_2) + V_{\text{out}}(S, K, H_1, H_2) = E(S, K)$$

and the Fourier series expansion of the knock-out price is given by

$$V_{\text{out}}(S, K, H_1, H_2) = Z\sum_{n=1}^{\infty} b_n \exp\left(-ax - \frac{a\sigma^4 T^2}{2} - \frac{n^2\sigma^2 T}{2c^2}\right)\sin\left(\frac{nx}{c}\right)$$

where $x = \log S - \log H_1$,

$$
\begin{aligned}
a &= -\frac{1}{2} + \frac{\log \widetilde{Z} - \log Z}{\sigma^2 T_1} \\
c &= \frac{\log H_2 - \log H_1}{\pi} \\
b_n &= \frac{2}{\pi}\left[A\rho H_1 \int_L^U dy \exp\left((a+1)cy\right)\sin(ny) + B\int_L^U dy \exp(ay)\sin(ny)\right]
\end{aligned}
$$

and $\rho = (\tilde{Z}/Z)^{(T_1-T)/T_1}$. For a call option, the bounds on the integral are $L = 0$ and $U = \min(\kappa/c, \pi)$, where $\kappa = \log K - \log(\rho H_1)$. For a put option, the bounds are $L = \max(0, \kappa/c)$ and $U = \pi$. The integrals are evaluated using the identities:

$$
\begin{aligned}
(p^2+q^2)\int \mathrm{e}^{py}\cos(qy)dy &= \mathrm{e}^{py}\left[p\cos(qy) + q\sin(qy)\right] \\
(p^2+q^2)\int \mathrm{e}^{py}\sin(qy)dy &= \mathrm{e}^{py}\left[p\sin(qy) - q\cos(qy)\right]
\end{aligned}
$$

2.14.5.3. *Knock-in Rebate.* A double barrier knock-in rebate pays an amount A of other currency and an amount B of quoting currency contingent on neither barrier being attained before the expiry date. This is the same payoff as a knock-out call with $K = 0$.

2.14.6. American. American options on spot FX transactions are priced by applying the general method described in Section 2.13 with a payoff function $\Pi(S)$ expressed in quoting currency. The zero-coupon bond prices for maturity at the last spot date are modelled as

$$
\begin{aligned}
Z(i) &= Z_{\text{spot}}\left(\frac{Z}{Z_{\text{spot}}}\right)^{(n-i)/n} \\
\widetilde{Z}(i) &= \widetilde{Z}_{\text{spot}}\left(\frac{\widetilde{Z}}{\widetilde{Z}_{\text{spot}}}\right)^{(n-i)/n}
\end{aligned}
$$

where Z_{spot} and $\widetilde{Z}_{\text{spot}}$ are the zero-coupon bond prices for maturity at spot, assumed to be constant. From Equations (2.3) and (2.4), the spot rate at node (i, j) is given by

$$
S(i,j) = F(i,j)\frac{Z(i)/Z_{\text{spot}}}{\widetilde{Z}(i)/\widetilde{Z}_{\text{spot}}}
$$

2.15. Bond Forward Price

2.15.1. Clean versus Dirty Price. The *dirty* price of a bond is the price of its future cashflows. On a coupon date, the dirty price decreases by an amount equal to the coupon rate. To avoid such jumps in the price, bond prices are usually quoted exclusive of the interest accrued in the current coupon period; this is referred to as the *clean* price.

The dirty price is most relevant to pricing and therefore all bond prices referred to in this chapter are dirty.

2.15.2. Discounted Cashflow Formula. The holder of a forward contract on a bond receives fixed cashflows $c_{m+1}, \ldots, c_n$ on dates $D_{m+1}, \ldots, D_n$ in exchange for the contract price F paid on the forward settlement date D, where $D < D_{m+1} < \cdots < D_n$ (c_n includes the redemption cashflow). The PV of the contract is

$$-FZ + \sum_{i=m+1}^{n} c_i Z_i$$

where Z_i denotes the discount factor at D_i and Z denotes the discount factor at D. The forward price is the value of F which makes the PV of the forward contract equal to zero. Hence the forward price is

$$F = \frac{1}{Z} \sum_{i=m+1}^{n} c_i Z_i \tag{2.22}$$

2.15.3. Spot Price Formula. In Section 2.15.2, the absence of arbitrage is used to relate the forward price of a bond to zero-coupon bond prices. Another no-arbitrage argument relates the forward price to the spot price and repo rate. This relationship derives from the following strategy.

> Sell the forward contract at price F. Buy the bond at spot price S and finance the purchase by borrowing at the repo rate r from the spot date D_{spot} to the forward settlement date D. Sell zero-coupon bonds to offset any coupon cashflows received from the bond during repo period; invest the proceeds from these zero-coupon bonds in another zero-coupon bond maturing at D.

Let $c_1, \ldots, c_m$ denote the coupons received at dates $D_1, \ldots, D_m$, where $D_{\text{spot}} < D_1 < \cdots < D_m \le D$. To avoid arbitrage, the net cashflow generated by this strategy should be zero. Hence

$$F - S(1 + \alpha r) + \frac{1}{Z}(c_1 Z_1 + \cdots + c_m Z_m) = 0 \tag{2.23}$$

where α is the year fraction for the repo period, Z_i is the discount factor at D_i and Z is the discount factor at D. In the case that there are no coupons in the repo period, the forward price is simply given by

$$F = S(1 + \alpha r)$$

The advantage of Equation (2.23) is that it relates the forward price directly to the market-quoted spot price rather than deriving the forward price entirely from implied zero-coupon bond prices. The spot price reflects the market's view of the credit and liquidity of an individual bond issue; this information cannot easily be captured in the yield curve from which the zero-coupon bond prices are derived.

2.16. Bond Options

2.16.1. European. A European bond option pays an amount

$$P \max(\pm (F(T) - K), 0)$$

on the option settlement date, where

- P is the principal amount
- F is the forward bond price to the option settlement date
- K is the strike price
- T is the time to the exercise date

- $\pm$ takes the value $+1$ for a call option and -1 for a put option[3].

Adaptiv prices European bond options using the Black formula

$$\pm P\left(F\mathcal{N}(\pm d_1) - K\mathcal{N}(\pm d_2)\right) Z$$

where Z is the discount factor at the settlement date,

$$d_2 = \frac{1}{\sigma\sqrt{T}}\log\left(\frac{F}{K}\right) - \frac{\sigma\sqrt{T}}{2}$$

$d_1 = d_2 + \sigma\sqrt{T}$ and $\sigma^2 T$ is the variance of $\log F(T)$.

The value of σ is the price volatility read from the Bond Volatility Surface; see Section 1.4.7.

2.16.2. Pricing Theory. No-arbitrage pricing theory shows that the present value of the a European bond option is given by

$$P\mathbb{E}[\max(F(T) - K, 0)]Z$$

where $\mathbb{E}$ is the expectation in the equivalent martingale measure induced by the numeraire $Z(t)$ (the forward measure). It was shown in Section 1.4.1 that the option price is given by the Black formula if and only if $F(T)$ has a lognormal distribution under the forward measure.

2.16.3. Relation to Swaptions. Consider a European call option on a bond. The value of the option on its exercise date T is given by

$$Z(T)\max\left(F(T) - K, 0\right) = \max\left(c_1 Z_1(T) + \cdots + c_n Z_n(T) - KZ(T), 0\right)$$

where $c_1, \ldots, c_n$ are the bond cashflows at dates $D_1 < \cdots < D_n$ and Z_i denotes the discount factor at date D_i. However, this is equal to K multiplied by the value of an option on a swap with fixed side cashflows $c_1/K, \ldots, c_{n-1}/K$ and $c_n/K - 1$. Therefore, the price of a European bond option is the same as the price of a European swaption with the same term and time to exercise.

In practice, the market does not price bond options as equivalent swaptions, or vice versa, for two reasons:

(1) the credit spread between the swap and bond yield curves
(2) for swaptions, the Black formula is obtained under the assumption that the swap rate is a lognormal variable, whereas for bond options, the bond price is assumed to be lognormal.

Nevertheless, Adaptiv allows European bond options to be priced as their equivalent European swaptions, using an interpolated swap-rate volatility. The advantage of this method is that the Swaption Volatility Surface provides a single source of volatility risk for both types of option.

[3] The payoff function $\max(F - K, 0)$ is independent of whether the forward and strike prices are expressed inclusive or exclusive of accrued interest (dirty or clean). However, it is the dirty prices rather than the clean prices that appear in the Black formula.

2.16.4. American. American bond options are priced by applying the general method described in Section 2.13.

The zero-coupon bond price for maturity at the last spot date is modelled as the deterministic function $Z(i) = Z^{(n-i)/n}$. Let D_i denote the spot settlement date corresponding to the time step $i\Delta t$. The spot price at node (i, j) is given by

$$Z(i)F(i,j) = Z_{\text{spot}}S(i,j) - Z(i)\sum\nolimits_{k=i+1}^{n} \frac{c_k}{Z(k)}$$

where c_k is the sum of any coupon cashflows with cashflow date D satisfying $D_{k-1} < D \leq D_k$ (compare with Section 2.15).

2.17. Exotic Interest-Rate Derivatives

2.17.1. Term-Structure Models. This section outlines the general mathematical framework of no-arbitrage pricing and term-structure models. For further explanation, the reader should consult texts such as Baxter and Rennie [**9**], Hunt and Kennedy [**14**], and Rebonato [**30**] which also covers tree methods.

A term-structure model is a model of the future evolution of the yield curve. The state of the yield curve at time t is given by the zero-coupon bond prices $Z(t, T)$ for each maturity date $T \geq t$, where $Z(t, T)$ is defined to be the price at time t of a unit cashflow paid at time T. The zero-coupon bond price can be expressed as

$$Z(t,T) = \exp\left(-\int_t^T f(t,u)du\right)$$

where $f(t, u)$ is the value at time t of instantaneous forward rate for the infinitesimal period u to $u + du$.

No-arbitrage pricing theory expresses the relationship of the price of one contingent claim relative to the price of another; it shows that if $N(t)$ is the price of a positive contingent claim N, then there exists a measure $\tilde{\mathbb{P}}$ such that the price of any contingent claim X is given by

$$X(t) = N(t)\tilde{\mathbb{E}}_t\left(\frac{X}{N}\right),$$

where $\tilde{\mathbb{E}}_t$ denotes the conditional expectation at time t under $\tilde{\mathbb{P}}$; and hence $X(t)/N(t)$ is a martingale under $\tilde{\mathbb{P}}$. In this context, N is called a numeraire and $\tilde{\mathbb{P}}$ an equivalent martingale measure.

The numeraire $N(t) = Z(t, T)$, for a fixed maturity date T, was used in Section 2.7.3 to obtain the Black formula for caplets. The equivalent martingale measure corresponding to this choice of numeraire is known as the *forward measure*. Another important choice of numeraire is the *cash account*, which is defined by

$$N(t) = \exp\left(\int_0^t r(s)ds\right),$$

where $r(t) = f(t, t)$, and is equal to the value of one unit invested at instantaneous short-term rate $r(t)$. Under the corresponding equivalent martingale measure, the zero-coupon bond prices are given by

$$Z(t,T) = \tilde{\mathbb{E}}_t\left[\exp\left(-\int_t^T r(s)ds\right)\right],$$

and the price of a contingent claim $X(t)$ has drift equal to the risk-free rate $r(t)$. For this reason, this equivalent martingale measure is referred to as the *risk-neutral measure*.

The most commonly used term-structure models are defined by specifying the evolution of the short rate as a Markovian process

$$dr(t) = a(r(t), t)dt + \sigma(r(t), t) \cdot dW(t),$$

where $W(t)$ is a Brownian motion under the risk-neutral measure. If n is the dimension of $W(t)$ then the model is said to be a *n-factor* model.

2.17.2. A Class of One-Factor Models. Adaptiv's tree construction methodology is applicable to a general class of one-factor short-rate models described as follows. The short rate is given by

$$f(r(t)) = \alpha(t) + x(t),$$

where $f(r)$ is a deterministic and invertible function, $\alpha(t)$ is a deterministic function and $x(t)$ is a process satisfying the Ornstein-Ulhenbeck stochastic differential equation:

$$dx(t) = -a(t)x(t)dt + \sigma(t)dW(t) \tag{2.24}$$

for time-dependent but deterministic mean reversion $a(t)$ and volatility $\sigma(t)$. The function $\alpha(t)$ is not a parameter of the model; it is shown in Section 2.17.4 that $\alpha(t)$ is determined by the initial yield curve.

For such a model, it follows that the short rate process satisfies the equation

$$df(r(t)) = [\theta(t) - a(t)f(r(t))]\, dt + \sigma(t)dW(t),$$

where $\theta(t) = \alpha'(t) + a(t)\alpha(t)$. The two most commonly used short-rate models are:

- the Hull-White model (see [**17**]), which has $f(r) = r$
- the Black-Karasinski model (see [**7**]), which has $f(r) = \log r$.

The solution of Equation (2.24) is

$$x(t) = \mathrm{e}^{-A(t)}\, x(0) + \int_0^t \mathrm{e}^{-[A(t)-A(s)]}\, \sigma(s)dW(s).$$

where $A(t)$ denotes $\int_0^t a(s)ds$. This follows by considering the process $y(t) = \mathrm{e}^{A(t)}\, x(t)$, which satisfies the equation $dy(t) = \mathrm{e}^{A(t)}\, \sigma(t)dW(t)$. It also follows that the mean and variance of $x(T)$ conditional on $x(t)$ are given by

$$\mathbb{E}_t(x(T)) = \mathrm{e}^{-[A(T)-A(t)]}\, x(t) \tag{2.25}$$

$$\mathrm{var}_t(x(T)) = \int_t^T \mathrm{e}^{-2[A(T)-A(s)]}\, \sigma(s)^2 ds. \tag{2.26}$$

2.17.3. Tree Construction.

2.17.3.1. *Introduction.* Adaptiv provides an advanced alternative to standard trinomial-tree implementations of the Hull-White and Black-Karasinski models. The standard method, as described in Hull and White [**16**], uses evenly spaced points on the time axis and this has two main disadvantages. Firstly, the need for interpolation when dates in the instrument structure fall between time steps. Secondly, many derivative instruments have both short-term events, such as rate fixings and exercise of options, which require a small time step for sufficient accuracy, and long-term events, such as cashflows. As a consequence, the total number of time steps is very large and the computation is very lengthy. In the standard tree, the position of nodes on the rate axis is strictly related to the size of the time step, and the width of the tree spreads out in proportion to time. This is faster than the variance of the rate, which grows approximately as the square root of time (depending on the mean-reversion rate). As a result, many very low probability events are included in the tree, and this significantly reduces its efficiency.

Adaptiv uses a methodology which eliminates these inefficiencies through the following features.

- Variable time steps and full control over the placement of time nodes. In particular, nodes can be placed at all significant dates of the instrument structure: rate fixing dates, exercise dates, rate-period start and end dates, and cashflow dates.
- At each time step, the nodes lie within a specified number of standard deviations of the distribution of the rate. Thus, low-probability states are avoided leading to a significant gain in efficiency.
- A rate transition is allowed from the root node to any of the nodes at the first time step. This initial multinomial transition accurately reproduces the probability distribution of the underlying rate, and therefore gives accurate pricing of short-term events but without affecting the number of time steps in the remainder of the tree.

2.17.3.2. *Tree Structure.* A separate tree is constructed for each instrument to be priced. The nodes are placed at dates $t_0, t_1, \ldots, t_n$ which include the valuation date t_0 and all the significant dates in the instrument's structure. A parameter of the tree construction called the *period splitter*, denoted by S, allows the number of time steps to be increased by subdividing each of the original time steps into S periods of each length.

The Ornstein-Ulhenbeck process $x(t)$ described in Section 2.17.2 has standard deviation $\mathrm{e}^{-A(t)}\, B(t)^{1/2}$, where $B(t)$ denotes $\int_0^t \mathrm{e}^{2A(s)}\, \sigma(s)^2 ds$. At each time step, the nodes are evenly spaced and placed within C standard deviations of the underlying process, where C is called the *tail cutoff*. This is achieved by creating $2N_i + 1$ nodes at t_i, and by setting the value of the process to

$$x(i,j) = C\,\mathrm{e}^{-A(t_i)}\, B(t_i)^{1/2} \frac{j}{N_i} \tag{2.27}$$

for $-N_i \le j \le N_i$. The probability that $x(t_i)$ lies below $x(i, -N_i)$ or above $x(i, N_i)$ is equal to $2\mathcal{N}(-C)$, where $\mathcal{N}$ is the standard normal distribution function. For example, setting $C = 6$ gives the probability $2\mathcal{N}(-6) \simeq 2 \times 10^{-9}$.

The positioning of the nodes and effect of the tail cutoff is illustrated in Figure 2.2.

2.17.3.3. *Transition out of the Root Node.* A transition is allowed from the root node at t_0 to any of the $2N_1 + 1$ nodes at t_1. Suppose all transitions from the root node to non-central nodes are equally probable. Let p denote the probability of a transition from $(0, 0)$ to $(1, j)$, for $j \neq 0$. Then the probability of a transition from $(0, 0)$ to $(1, 0)$ is $1 - 2N_1p$, and hence p must satisfy $2N_1p < 1$. Equating the variance of the discrete and continuous process gives

$$2p \sum_{j=1}^{N_1} x(1,j)^2 = \mathrm{e}^{-2A(t_1)}\, B(t_1),$$

which is equivalent to

$$2N_1 p = \frac{N_1^3}{C^2 \sum_{j=1}^{N_1} j^2} = \frac{6N_1^2}{C^2(2N_1+1)(N_1+1)}. \tag{2.28}$$

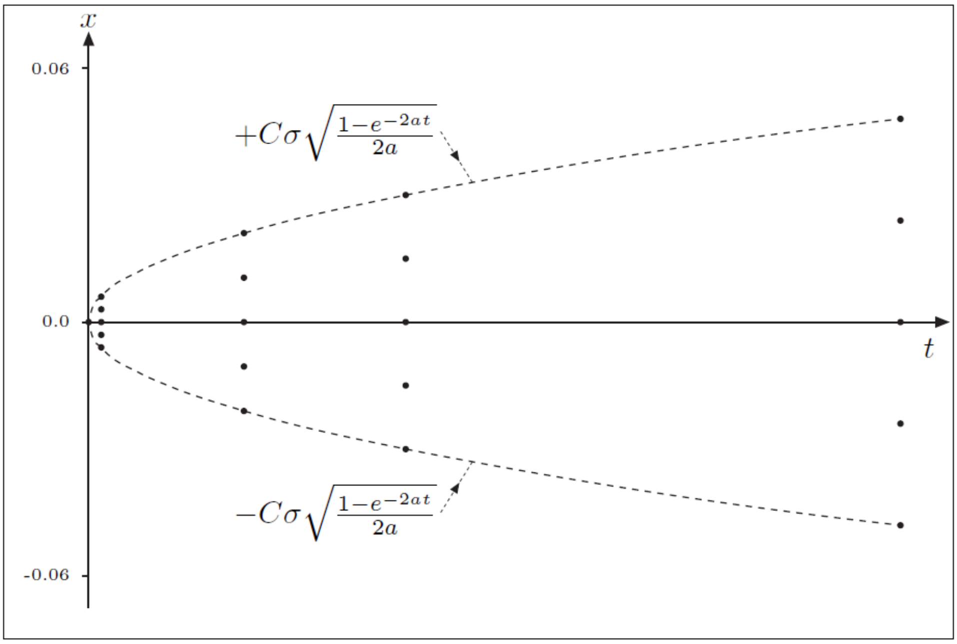


FIGURE 2.2. example node structure for the process $dx(t) = -ax(t)dt + \sigma dW(t)$ with number of nodes $2N + 1 = 5$, tail cutoff $C = 6$, standard deviation $\sigma = 0.01$ and mean reversion $a = 0.01$

The right-hand side of Equation (2.28) is less than $3/C^2$ and hence $2N_1 p < 1$ is satisfied for all $C \geq 2$.

2.17.3.4. *Trinomial Transitions.* The transitions out of all other nodes are trinomial. The transition from node (i, j) is to one of three nodes: $(i + 1, j_1)$, $(i + 1, j_2)$ or $(i + 1, j_3)$. Let p_k denote the probability of a transition to j_k. Then $p_1 + p_2 + p_3 = 1$. Equating the conditional mean of the discrete process with the conditional mean of the continuous process, given by Equation (2.25), gives

$$\sum_{k=1}^{3} p_k x(i+1, j_k) = \bar{x} = \mathrm{e}^{-[A(t_{i+1})-A(t_i)]}\, x(i, j).$$

Similarly, equating the conditional variance of the discrete and continuous process, given by Equation (2.26), gives

$$\sum_{k=1}^{3} p_k \left[x(i+1, j_k) - \bar{x}\right]^2 = \mathrm{e}^{-2A(t_{i+1})} \left[B(t_{i+1}) - B(t_i)\right].$$

Substituting in the values of $x(i, j)$ given by Equation (2.27) leads to the following system of linear equations in p_1, p_2 and p_3:

$$\begin{aligned} p_1 + p_2 + p_3 &= 1 \\ \frac{p_1 j_1 + p_2 j_2 + p_3 j_3}{N_{i+1}} &= \mu \frac{j}{N_i} \\ \frac{p_1 j_1^2 + p_2 j_2^2 + p_3 j_3^2}{N_{i+1}^2} &= \left(\mu \frac{j}{N_i}\right)^2 + \frac{1 - \mu^2}{C^2}, \end{aligned}$$

where μ denotes $\sqrt{B(t_i)/B(t_{i+1})}$.

Now suppose that the number of nodes at each time step is constant: $N_i = N$ for all i. Then the theorem of Appendix F shows that this system of equations has a solution, subject to the constraint that $p_k \geq 0$, provided C and N are chosen such that $C \geq 2$ and $N \geq C^2$; for example by choosing $C = 6$ and $N = 36$. Moreover, j_2 can be chosen to be the closest integer to $j\mu$; and this is equivalent to choosing $x(i+1, j_2)$ as close as possible to the conditional mean $\mathrm{e}^{-[A(t_{i+1})-A(t_i)]}\, x(i,j)$.

This theorem on tree construction holds in the general case of time-dependent volatility σ and mean-reversion rate a. Adaptiv implements the case of constant σ and a, under which $A(t) = at$ and

$$B(t) = \left(\frac{\mathrm{e}^{2at} - 1}{2a}\right)\sigma^2.$$

2.17.3.5. *Short Rate.* Once the tree for the Ornstein-Ulhenbeck process $x(t)$ has been constructed, the corresponding tree for short-rate process $r(t)$ is obtained from

$$r(i,j) = g(\alpha(i) + x(i,j)), \tag{2.29}$$

where g is the inverse of the function f which characterises the model. In the case of the Hull-White model, $f(r) = r$ and hence $g(x) = x$; and for the Black-Karasinski model, $f(r) = \log r$ and $g(x) = \mathrm{e}^x$.

2.17.4. Matching the Initial Yield Curve. The initial curve of zero-coupon bond prices $Z(t_0, t)$ is derived from market-quoted cash, futures, swap and bond prices using a combination of no-arbitrage arguments, interpolation and/or best-fit techniques. With the exception of futures[4], the price of each market-quoted instrument depends only on the current zero-coupon bond prices, and is therefore model-independent and independent of volatility.

The values of $\alpha(i)$ are chosen so that the model predicts initial zero-coupon bond prices that agree with the initial curve. Let $Z(i)$ denote the price at time t_0 of the zero-coupon bond paying a unit cashflow at time t_i. Let $Q(i,j)$ denote the price of the tree-specific security paying a unit cashflow at node (i,j), known as an Arrow-Debreu price. Then the zero-coupon bond prices are given by

$$Z(i) = \sum\nolimits_j Q(i,j); \tag{2.30}$$

and the price of a unit cashflow at a given node (i,j) is related to the prices of unit cashflows at its predecessor nodes by

$$Q(i,j) = \sum\nolimits_k p_{(i-1,k)\to(i,j)}\, \mathrm{e}^{-r(i-1,k)\Delta(i-1)}\, Q(i-1,k), \tag{2.31}$$

where $p_{(i-1,k)\to(i,j)}$ denotes the transition probability from node $(i-1,k)$ to node (i,j) and $\Delta(i)$ denotes the time interval $t_{i+1} - t_i$. Substituting Equations (2.31) and (2.29) into Equation (2.30), and summing the transition probabilities to 1, gives

$$Z(i) = \sum\nolimits_k \mathrm{e}^{-g(\alpha(i-1)+x(i-1,k))\Delta(i-1)}\, Q(i-1,k). \tag{2.32}$$

The values of $\alpha(i)$ and the Arrow-Debreu prices are derived by a forward inductive procedure. At each step, $\alpha(i-1)$ is calculated by solving Equation (2.32) from known values of $Q(i-1,k)$. The values of $Q(i,j)$ are then obtained from Equation (2.31) for the next step.

Since $Z(0) = Q(0,0) = 1$ and $x(0,0) = 0$, the value of $\alpha(0)$ is given by $Z(1) = \mathrm{e}^{-g(\alpha(0))\Delta(0)}$, which is equivalent to $\alpha(0) = f(-\log Z(1)/\Delta(0))$. In the case of the

[4]The price of a deposit future includes a convexity-correction term that depends on the volatility of the rate.

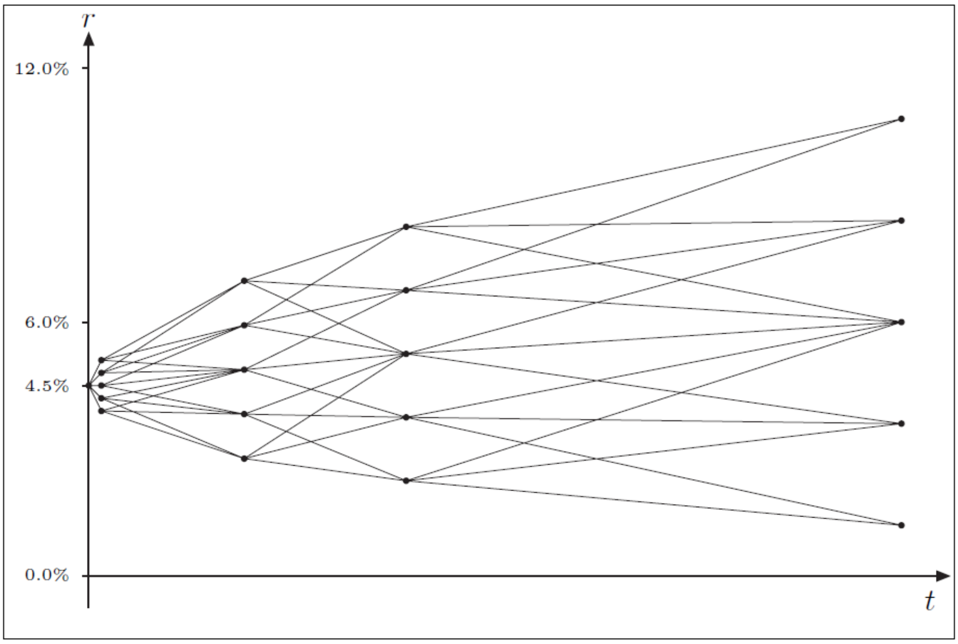


FIGURE 2.3. example short-rate tree adjusted to match the initial yield curve for the Hull-White model $dr(t) = (\theta(t) - ar(t))dt + \sigma dW(t)$.

Hull-White Model, $g(x) = x$ and hence rearranging Equation (2.32) gives

$$\alpha(i-1) = \frac{1}{\Delta(i-1)} \log\left(\frac{\sum_k e^{-x(i-1,k)\Delta(i-1)} Q(i-1,k)}{Z(i)} \right).$$

For the Black-Karasinski model, $g(x) = e^x$ and Equation (2.32) is solved for $\alpha(i-1)$ using the Newton-Raphson method.

Figure 2.3 shows an example of the final tree structure for the short rate.

2.17.5. Tree Pricing.

2.17.5.1. *Pricing Fixed Cashflows.* The Arrow-Debreu prices calculated in Section 2.17.4 give the current zero-coupon bond prices and hence the current value of any fixed cashflow. Alternatively, the price of a zero-coupon bond maturing at time step k can be calculated by backward induction via the equation

$$Z_k(i,j) = e^{-r(i,j)\Delta(i)} \sum_\ell p_{(i,j)\to(i+1,\ell)} Z_k(i+1,\ell), \tag{2.33}$$

where $Z_k(i,j)$ denotes the value at node (i,j) of a unit cashflow paid at time step k; so that, in particular, $Z_k(k,j) = 1$ for all j.

2.17.5.2. *Rate Fixings.* Consider a rate fixing at time step i. The value of a deposit rate, or constant maturity rate, is a function of zero-coupon bond prices. Hence, the value of the rate at node (i,j) is a function of certain $Z_k(i,j)$, each of which can be calculated from the backward induction Equation (2.33).

2.17.5.3. *Options.* The general method is illustrated by two examples. The first example is a callable bond. Let $V(i,j)$ denote the value of the bond on reaching node (i,j), including any cashflows realised at time i. The value of the bond is equal to 1 at maturity, so $V(n,j) = 1$ for all rate nodes j. At time i, the issuer has the right to call back the bond in exchange for an amount K_i. Set $K_i = \infty$ if the bond cannot be called at time i. Then $V(i,j)$ is the minimum of the call price K_i and sum of the coupon cashflow and the discounted value of the callable bond at the successor nodes of (i,j):

$$V(i,j) = \min\left(K_i, C_i + \mathrm{e}^{-r(i,j)\Delta(i)} \sum\nolimits_k p_{(i,j)\to(i+1,k)} V(i+1,k)\right),$$

where C_i is the amount of any coupon cashflow received at time i. Starting at $i = n$, $V(i,j)$ is calculated by backward induction using this equation.

The second example is a Bermuda option on an underlying instrument whose value is denoted $U(i,j)$. If the option is exercised at time i then the holder of the option receives the underlying and pays an amount K_i. The strike amount K_i is set to ∞ if no exercise is allowed at time i. Then the value of the option $V(i,j)$ is given by

$$V(i,j) = \max\left(U(i,j) - K_i, \mathrm{e}^{-r(i,j)\Delta(i)} \sum\nolimits_k p_{(i,j)\to(i+1,k)} V(i+1,k)\right).$$

In the case of a Bermuda swaption, $U(i,j)$ is a simple function of zero-coupon bond prices, each of which is calculated using Equation (2.33). For a payer's swaption,

$$U(i,j) = Z_{i_0}(i,j) - Z_{i_m}(i,j) - r\alpha_1 Z_{i_1}(i,j) - r\alpha_m Z_{i_m}(i,j)$$

where: i_0 is the start date of the underlying swap and $i_1, \ldots, i_m$ are the fixed-side cashflow dates (so that $i \le i_0 < i_1 < \cdots < i_m$), r is the fixed rate, and $\alpha_1, \ldots, \alpha_m$ are the fixed-side interest periods.

2.17.6. Model Calibration. Calibration is the process of deriving the parameters of an interest-rate model from market prices of liquid options, usually European caps or swaptions. The calibration calculation works by pricing a given set of liquid options under the model, and then adjusting the model parameters until they fit, or best-fit, their quoted market prices. The best fit is achieved by minimizing the sum of the squares of the pricing errors, where the pricing errors are the differences between the quoted market prices and the corresponding prices calculated under the model. For the minimization, Adaptiv uses a modified and enhanced Levenberg-Marquardt method (see Press et al. [**29**]).

After calibration, the model gives prices of exotic or illiquid instruments, such as Bermuda swaptions, that are consistent with the subset of quoted market prices to which the model was calibrated.

Market prices of European swaptions and caps are quoted in terms of implied Black volatility. In the US market, brokers usually quote volatilities for around 50 to 80 at-the-money swaptions with varying time-to-expiration and time-to-maturity, so that the volatilities can be arranged in a matrix with time-to-expiration ranging from 1 day to 10 years and time-to-maturity ranging from 1 year to 30 years (see Section 1.4.5.2).

The successful completion of the calibration calculation is dependent on a careful choice of liquid options. It not usually possible to calibrate one-factor interest-rate models *globally*, so that they fit the full set of quoted cap and swaption prices. This is because such models can produce only certain variance-covariance patterns of interest-rate curves; and hence cannot fully explain the difference in volatility between caps and swaptions, between swaptions with different periods, and between swaptions with the same time-to-expiration but different time-to-maturity. Moreover, in the case of the constant-parameter Hull-White and Black-Karasinski models, there are only two model parameters, the volatility σ and mean-reversion rate a, and these cannot be chosen to fit all the quoted option prices. The calibration is typically *local* to a set of exotic instruments having similar underlying instruments (for example, swaps of similar maturity), with the calibrating instruments being European options on those underlying instruments. An example of a good set of calibrating swaptions is 1y $\times$ 4y, 2y $\times$ 3y, 3y $\times$ 2y, and 4y $\times$ 1y, which are suitable for calibrating the model for pricing a Bermuda cancellable swap with 5 years to maturity.

CHAPTER 3

Fixed Income

3.1. Introduction

This chapter covers the following elementary fixed income calculations:

- yield-to-maturity
- convexity, duration and modified duration
- price factor, implied repo rate and cheapest-to-deliver

Many of the calculations described here are implemented in the spreadsheet `FG_Bond.xls`.

3.2. Yield

3.2.1. Yield-to-Maturity. Let F denote the dirty price of a bond; this is the price inclusive of the interest accrued up to the value date. Given an interest rate y, let $P(y)$ denote the value obtained by discounting the bond cashflows back to the value date at the rate y. The *yield-to-maturity* (or *internal rate of return*) is the value of y for which the discounted value of the cashflows equals the price of the bond:

$$P(y) = F$$

The precise method used to discount the cashflows depends on the conventions of the local bond market. Adaptiv supports many of these methods and they are described in Sections 3.2.3–3.2.5.

Yield-to-maturity allows the market price of one bond to be compared with another bond of similar maturity but different coupon. The bond with the greater yield-to-maturity is cheaper relative to the other bond because the market is discounting its cashflows at higher rate.

3.2.2. Modified Duration and Convexity. Modified duration and convexity provide an estimate of the sensitivity of the bond price to small parallel shifts in the yield curve. Let $P'(y)$ and $P''(y)$ respectively denote the first and second order derivatives of the price formula $P(y)$ with respect to the yield y. Suppose F is the current market price and y is the yield-to-maturity given by $P(y) = F$. If the market price shifts by a small amount ΔF then the yield-to-maturity shifts by Δy, where $\Delta F = P(y + \Delta y) - P(y)$. The proportional price change $\Delta F/F$ is approximately equal to

$$\left(\frac{P'(y)}{P(y)}\right)\Delta y + \frac{1}{2}\left(\frac{P''(y)}{P(y)}\right)\Delta y^2$$

(the linear and quadratic terms in the Taylor expansion of $P(y+\Delta y)$). The derivative $P'(y)$ is negative because price decreases as yield increases (see Figure 3.1). The

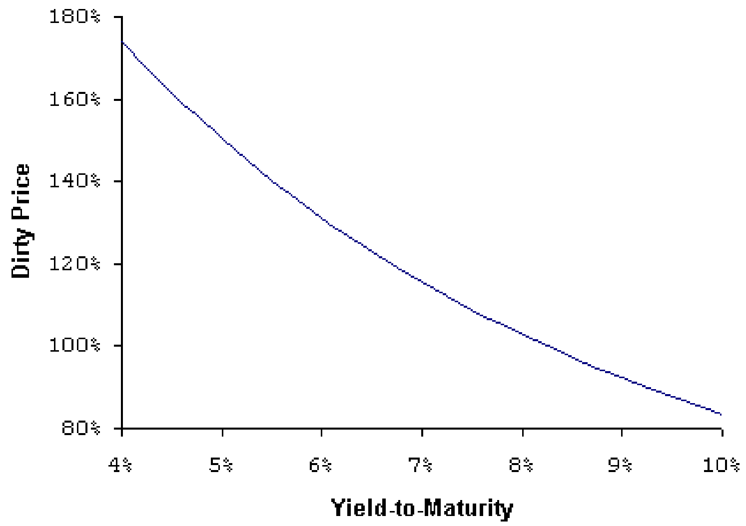


FIGURE 3.1. price as a function of yield-to-maturity for a 30y bond with 8.25% coupon.

quantity

$$\frac{-P'(y)}{P(y)}$$

is the *modified duration*. The second order derivative $P''(y)$ is positive showing that $P(y)$ is a convex function. For this reason,

$$\frac{P''(y)}{P(y)}$$

is referred to as the *convexity*.

3.2.3. ISMA Method. For a given value date D, let $c_1, \ldots, c_n$ denote the coupon cashflows on the dates $D_1, \ldots, D_n$, where $D < D_1 < \cdots < D_n$. Let C denote the redemption cashflow on date D_{n+1}. The ISMA yield-to-maturity y is defined by

$$P = \sum_{i=1}^{n} c_i \nu^{t_i} + C\nu^T$$

where

- P is the dirty price
- ν is the annual discount factor $\nu = 1/\left(1 + y/h\right)^h$
- h is the number of coupons per year
- $t_i = \alpha + (i-1)/h$
- $T = \alpha + (n-1)/h + \beta$
- α is the year fraction from the value date D to first coupon date D_1, calculated according to the yield basis of the bond issue. If the yield basis is Act/Act then $\alpha = (D_1 - D)/(D_1 - D_0)$ where D_0 is previous coupon date.
- β is the year fraction from the last coupon date D_n to the redemption cashflow D_{n+1} (this is usually zero). If the yield basis is Act/Act then $\beta = (D_{n+1} - D_n)(D_n - D_{n-1})$.

In the formula, the yield is compounded h times per year and each cashflow is discounted by $(1+y/h)^{-ht}$, where t is the time to the cashflow date.

The *duration* of a bond is defined to be the modified duration multiplied by the per coupon yield factor $(1+y/h)$. Under the ISMA method, duration is given by

$$\frac{1}{P}\left(\sum_{i=1}^{n} t_i c_i \nu^{t_i} + TC\nu^T\right)$$

The term in parenthesis is the sum of time periods $t_1, \ldots, t_n, T$ weighted by the discounted cashflows $c_1\nu^{t_1}, \ldots, c_n\nu^{t_n}, C\nu^T$ (hence 'duration').

3.2.4. Other Fixed-coupon Methods.

3.2.4.1. *Moosmuller.* This method is used in Germany for bonds with annual coupon frequency. The yield-to-maturity y is defined by

$$P = \frac{1}{1+\alpha y}\left(\sum_{i=1}^{n} c_i \nu^{i-1} + C\nu^{n-1}\right)$$

where P is the dirty price, $\nu = 1/(1+y)$ and α is the year fraction from the value date to the first coupon date.

3.2.4.2. *Euro Commercial Paper (ECP).* The yield-to-maturity y is defined by

$$P = \frac{c+C}{1+\alpha y}$$

where

- P is the dirty price
- c is the coupon cashflow
- C is the redemption cashflow
- α is the year fraction from the value date to the maturity date of the issue, calculated according to the yield basis of the issue.

3.2.4.3. *Japanese Government Bond (JGB).* The yield-to-maturity y is given by the price formula

$$P_{\text{clean}} = \frac{\alpha r + 1}{1+\alpha y}$$

where

- P_{clean} is the clean price
- r is the annual coupon rate
- α is the year fraction from the value date to the maturity date of the issue, calculated using the 365/365 method.

3.2.5. Discount Bills. Yield methods for discount bills are specified in Table 3.1 where y denotes the yield and α denotes the year fraction from the value date to the maturity date, calculated according to the yield basis of the issue.

3.2.6. FRN Discount Margin. The discount margin of a floating rate note (FRN) is analogous to the yield-to-maturity of a fixed interest bond.

Given the deposit rate for the period to the next coupon date and an assumed forward deposit rate for the remaining coupon periods, the known and estimated

Method	**Price Formula**	**Adaptiv ID**
US Treasury Bill	$1 - \alpha y$	UsDti
Money Market	$1/(1 + \alpha y)$	CanDti, AusDti, OTC
Zero-coupon	$1/(1 + \alpha y)$ maturity $\leq$ 1y $1/(1 + y)^\alpha$ maturity $>$ 1y	EspDti, ItlCtz

TABLE 3.1. yield methods for discount bills.

cashflows are discounted at these rates plus the discount margin in order to obtain the dirty price. Adaptiv defines the discount margin ε by the following equation:

$$P = \frac{1}{1 + \alpha(r + \varepsilon)} \left(c_0 + \left(\frac{R + s}{h} \right) (\nu + \nu^2 + \cdots + \nu^n) + C\nu^n \right)$$

where

- P is the dirty price
- c_0 is the known coupon cashflow
- R is the assumed forward deposit rate for the unknown cashflows
- s is the margin over the floating index
- h is the number of coupons per year.
- ν is the discount factor
$$\nu = \frac{1}{1 + (R + \varepsilon)/h}$$
- n is the number of unknown coupon cashflows
- C is the redemption cashflow
- r is the deposit rate from the value date to next coupon date
- α is the year fraction from the value date to next coupon date calculated according to the basis of the floating index

3.3. Bond Futures

3.3.1. Price Factors. A bond futures contract specifies the coupon and number of years to maturity of the underlying bond to be delivered by the holder of the short position. In practice there does not usually exist a security with exactly the required coupon and maturity date. Therefore, the exchange permits the holder of the short position to deliver a security chosen from a basket of acceptable bond issues. Each bond in the basket has approximately the required number of years to maturity. On the delivery date, the holder of the short position receives an amount equal to the futures price[1] multiplied by a *price factor*. The price factor is the price of the bond obtained by discounting its cashflows using the notional coupon rate as the yield. The purpose of the price factor is approximately to convert the price of the notional underlying security to the price of the actual security that is delivered.

3.3.2. Price Factor Calculation. Price factors are specified by the exchange via a standardized calculation of the price of the deliverable security. A general formula for the price factor is

$$c_1 \left(-a + \nu^\alpha \right) + c\nu^\alpha \left(\nu + \nu^2 + \cdots + v^{n-1} \right) + v^{\alpha+n-1}$$

where

- c_1 is the coupon per unit of principal for the next coupon

[1] The exchange delivery settlement price (EDSP)

- c is the coupon per unit of principal for the remaining coupons
- ν is the discount factor $\nu = 1/(1 + C/h)$
- C is the notional annual coupon rate
- h is the number of coupons per year
- α is the remaining fraction of the current coupon period
- a is the proportion of the next coupon accrued up to the delivery date.

This formula is applicable to all price factor calculations; the exact method used to calculate α and a is specified by the exchange. See the spreadsheet `FG_Bond.xls`.

Note that $\nu + \nu^2 + \cdots + v^{n-1}$ is equal to

$$\frac{v - \nu^n}{1 - \nu}$$

3.3.3. Cheapest-to-deliver. Among the basket of deliverable bond issues, the cheapest-to-deliver is the one that yields greatest profit to a trader who adopts the following strategy:

> Sell one bond futures contract at the market price F. Buy one contract unit of the deliverable bond, funding the purchase at the market repo rate r. Invest any coupons received until the delivery date at the prevailing money-market rate. At the delivery date: repay the borrowing, deliver the bond and receive the exchange delivery settlement price plus the variation margin.

The net cashflow at the delivery date is

$$(\phi F + A) + K - S(1 + \alpha r)$$

where ϕ is the price factor, A is the accrued interest to the delivery date, S is the dirty spot price of the bond, α is the year fraction for the repo period and K denotes the return on any invested coupons payments (compare with Section 2.15.3).

The *implied repo rate* for the deliverable bond is defined to be the repo rate implied by taking ϕF to be the forward bond price. Hence the implied repo rate $\tilde{r}$ is given by

$$S(1 + \alpha\tilde{r}) = (\phi F + A) + K$$

and the profit generated by the strategy can be rewritten

$$S\alpha(\tilde{r} - r)$$

Thus, the cheapest-to-deliver is the deliverable bond with the maximum implied repo rate.

CHAPTER 4

Market Data Sensitivity

4.1. Interest Rate

4.1.1. Delta-equivalent Cashflows. The PV of a transaction may depend on discount factors from one or more yield curves. For each yield curve, its present value V is a function of discount factors $Z_1, \ldots, Z_N$ at future dates $D_1, \ldots, D_N$. Typically, each D_i is either the payment date of a cashflow, or the start or end date of a floating rate period.

Let $\Delta Z_1, \ldots, \Delta Z_N$ denote the changes in the discount factors for a small perturbation of yield curve. The corresponding change in present value is approximately

$$\frac{\partial V}{\partial Z_1}\Delta Z_1 + \cdots + \frac{\partial V}{\partial Z_N}\Delta Z_N \tag{4.1}$$

The first-order derivatives

$$\frac{\partial V}{\partial Z_i}$$

are referred to as *delta-equivalent cashflows* because the formula (4.1) shows that V has approximately the same sensitivity to small shifts in the yield curve as a portfolio of fixed cashflow amounts $\partial V/\partial Z_i$ with payment dates D_i.

Thus, the effect on the portfolio's PV of a small perturbation in the yield curve is approximately equal to the change in PV of the portfolio of delta-equivalent cashflows. Delta-equivalent cashflows are returned by the transaction's pricing model. Once calculated, they can be used to obtain an accurate estimate of the PV changes from any number of small shifts in the yield curve. Adaptiv uses this method to calculate first-order interest rate sensitivity (see Section 4.1.4)

Delta-equivalent cashflows are also known as *zero positions*. The delta-equivalent cashflows with respect to discount and reference yield curves are displayed on the Flow Profiles report under the headings Discount Zero and Reference Zero.

4.1.2. Example Delta-equivalent Cashflows Calculations.

4.1.2.1. *Fixed Cashflows.* For a portfolio comprising only fixed cashflows, the delta-equivalent cashflows are identical to the actual cashflows. This intuitively obvious result follows the mathematical definition in Section 4.1.1. The PV of a fixed cashflow is of the form CZ, where C is the cashflow amount and Z is the discount factor at the payment date. Hence the delta-equivalent cashflow is given by $\partial(CZ)/\partial Z = C$.

4.1.2.2. *Floating Cashflows.* In Section 2.5.2 it was shown that the PV[1] of a floating cashflow is $P\alpha_{\text{int}}rZ$, where Z is the discount factor at the payment date

[1]If the payment date does not coincide with the end of the deposit period then Equation 2.14 should be used. For this example, the convexity correction term has been neglected.

and r is the forward deposit rate, given by

$$r = \frac{1}{\alpha}\left(\frac{Z_s}{Z_e} - 1\right)$$

Thus, the present value V has a dependence on three discount factors: the payment date discount factor Z from the discount curve; and the start and end date discount factors Z_s and Z_e from the reference curve. The discount-curve delta-equivalent cashflow is

$$\frac{\partial V}{\partial Z} = \frac{V}{Z} \tag{4.2}$$

and the reference-curve delta-equivalent cashflows are

$$\frac{\partial V}{\partial Z_s} = \frac{1}{\alpha Z_e}\frac{V}{r} \tag{4.3}$$

$$\frac{\partial V}{\partial Z_e} = \frac{-Z_s}{\alpha Z_e^2}\frac{V}{r} \tag{4.4}$$

In the typical case: the discount and reference curves are the same; the payment date coincides with the end of the deposit-rate period, so that $Z = Z_e$; and the interest basis is the same as the basis for the deposit rate, so that $\alpha_{\text{int}} = \alpha$. In this case, the value of the floating cashflow is given by

$$PZ_s - PZ_e.$$

The delta-equivalent cashflow at the start date is the principal amount P, and the total delta-equivalent cashflow at the end date is $-P$. Thus, the floating cashflow is equivalent in value, and hence in risk, to a positive principal cashflow at the start date and a negative principal cashflow at the end date.

4.1.2.3. *Caplet.* The PV of a standard caplet is given by the Black formula

$$P\alpha_{\text{int}}\left[r\mathcal{N}(d_1) - K\mathcal{N}(d_2)\right]Z$$

(see Section 2.7.2). The discount-curve delta-equivalent cashflow is V/Z and the reference-curve delta-equivalent cashflows are given by Equations (4.3) and (4.4) with V/r replaced by

$$\frac{\partial V}{\partial r} = P\alpha_{\text{int}}\mathcal{N}(d_1)Z$$

In the typical case, as defined in Section 4.1.2.2, the delta-equivalent cashflow at the start date is given by

$$\frac{\partial V}{\partial Z_s} = P\mathcal{N}(d_1),$$

and the total delta-equivalent cashflow at the end date is given by

$$\frac{\partial V}{\partial Z_e} + \frac{\partial V}{\partial Z} = -P\left[\mathcal{N}(d_1) + K\alpha\mathcal{N}(d_2)\right].$$

Thus, neglecting the small term $K\alpha\mathcal{N}(d_2)$, the delta-equivalent cashflows of the caplet are the delta-equivalent cashflows of the underlying floating cashflow, P at the start date and $-P$ at the end date, weighted by the hedge ratio $\mathcal{N}(d_1)$ (see Section C.3).

4.1.2.4. *European FX Option.* The PV, expressed in the quoting currency, of a call option on one unit of the other currency is given by the Black formula

$$[F\mathcal{N}(d_1) - K\mathcal{N}(d_2)]\, Z$$

where the forward rate F is equal to

$$S\frac{\widetilde{Z}/\widetilde{Z}_{\text{spot}}}{Z/Z_{\text{spot}}}$$

(see Section 2.14.3). Hence the PV is a function of: the spot rate S; the settlement and spot date discount factors, Z and Z_{spot}, from the quoting-currency curve; and the settlement and spot date discount factors $\widetilde{Z}$ and $\widetilde{Z}_{\text{spot}}$ from the other-currency curve. The delta-equivalent cashflows with respect to the quoting-currency curve are

$$\frac{\partial V}{\partial Z} = -K\mathcal{N}(d_2)$$
$$\frac{\partial V}{\partial Z_{\text{spot}}} = \mathcal{N}(d_1)S\frac{\widetilde{Z}}{\widetilde{Z}_{\text{spot}}}$$

The delta-equivalent cashflows with respect to the other-currency curve are

$$\frac{\partial V}{\partial \widetilde{Z}} = \mathcal{N}(d_1)S\frac{Z_{\text{spot}}}{\widetilde{Z}_{\text{spot}}}$$
$$\frac{\partial V}{\partial \widetilde{Z}_{\text{spot}}} = -\mathcal{N}(d_1)S\frac{\widetilde{Z}Z_{\text{spot}}}{\left(\widetilde{Z}_{\text{spot}}\right)^2}$$

4.1.3. Gamma-equivalent Cashflows. In Section 4.1.1, the delta-equivalent cashflows were defined as the first-order derivatives $\partial V/\partial Z_i$. By analogy, the second-order derivatives

$$\frac{\partial^2 V}{\partial Z_i \partial Z_j}$$

will be referred to as *gamma-equivalent cashflows* (although they are *not* cashflows). Gamma-equivalent cashflows are returned by the transaction's pricing model and used by the IR Sensitivity report to calculate second-order sensitivity to interest rates. The PV of a transaction leg may be depend on both the discount and reference yield curve. In this case, the pricing model returns cross-derivatives $\partial^2 V/\partial Z_i^{\text{disc}}\partial Z_j^{\text{ref}}$.

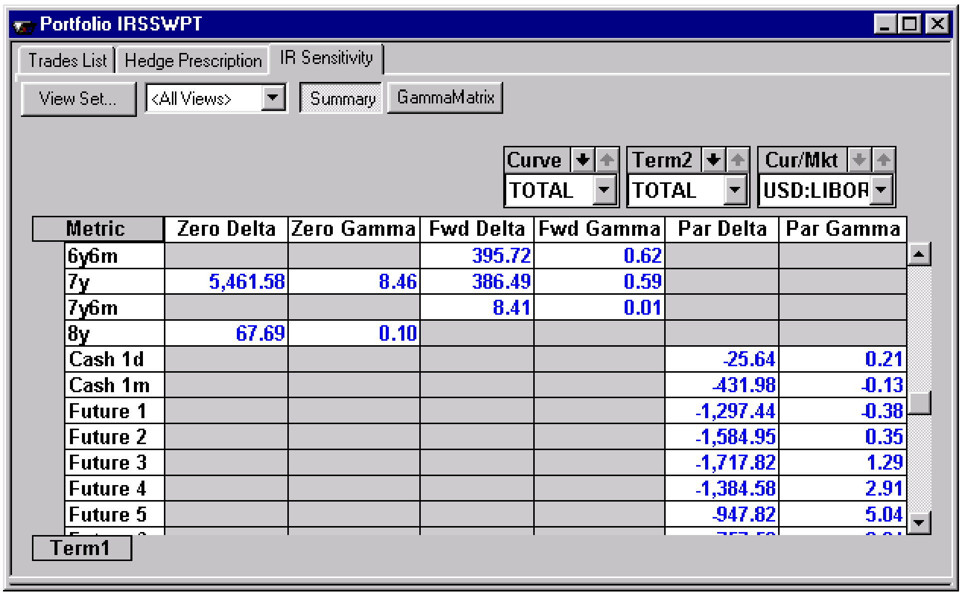
For many portfolios, the number of gamma-equivalent cashflows is very large and it is necessary to bucket them on to grid before calculating interest-rate gamma: see Appendix G.

4.1.4. Par Rate Sensitivity.

4.1.4.1. *Par Rates.* The market-quoted rates entered on the yield curve are referred to as the *par rates*. These are either: deposit rates, futures rates (100% minus futures price), swap rates or bond yields.

Consider the portfolio's present value V as a function of discount factors $Z_1, \ldots, Z_N$ from one particular yield curve. Then each discount factor Z_i is a function of the yield curve's par rates $r_1, \ldots, r_n$.

The first-order derivatives $\partial V/\partial r_k$ multiplied by 0.01% (one basis point) are displayed on the IR Sensitivity report under the heading Par Delta and on the Hedge



Metric	Zero Delta	Zero Gamma	Fwd Delta	Fwd Gamma	Par Delta	Par Gamma
6y6m			395.72	0.62		
7y	5,461.58	8.46	386.49	0.59		
7y6m			8.41	0.01		
8y	67.69	0.10				
Cash 1d					-25.64	0.21
Cash 1m					-431.98	-0.13
Future 1					-1,297.44	-0.38
Future 2					-1,584.95	0.35
Future 3					-1,717.82	1.29
Future 4					-1,384.58	2.91
Future 5					-947.82	5.04

FIGURE 4.1. the IR Sensitivity report.

Prescription report under the heading Perfect Sens. The second-order derivatives

$$\frac{\partial^2 V}{\partial r_k \partial r_\ell}$$

multiplied by $(0.01\%)^2$ are displayed on the IR Sensitivity report under the heading Par Gamma.

4.1.4.2. *Par Delta.* The calculation of $\partial V/\partial r_k$ is broken into two stages: the derivatives of V with respect to the Z_i; and the derivatives of each Z_i with respect to the r_k. Differentiating the composite function $V(Z(r))$ gives

$$\frac{\partial V}{\partial r_k} = \sum\nolimits_{i=1}^{N} \frac{\partial V}{\partial Z_i} \frac{\partial Z_i}{\partial r_k} \tag{4.5}$$

The derivatives $\partial V/\partial Z_i$ are the portfolio's delta-equivalent cashflows, returned by the transaction pricing models. The dependence of the discount factors on the par rates is usually complicated and depends on the details of the bootstrapping method (see Section 1.2.2). Therefore, the derivatives $\partial Z_i/\partial r_k$ are calculated by discrete rather than analytical methods.

Given a perturbation ε of the current par rate r, the shifted discount factor is given by the Taylor's series

$$Z(r+\varepsilon) = Z(r) + \varepsilon \frac{\partial Z}{\partial r}(r) + \frac{1}{2}\varepsilon^2 \frac{\partial^2 Z}{\partial r^2}(r) + \cdots \tag{4.6}$$

Subtracting the expansions of $Z(r+\varepsilon)$ and $Z(r-\varepsilon)$ gives

$$\frac{\partial Z}{\partial r}(r) = \frac{Z(r+\varepsilon) - Z(r-\varepsilon)}{2\varepsilon} + O(\varepsilon^3)$$

Hence, the derivative $\partial Z_i/\partial r_k$ is calculated as follows: shift r_k up by one basis point, recalculate the yield curve and hence Z_i; shift r_k down by one basis point (from its original value) and recalculate Z_i; then take the difference between the up-shifted Z_i and the down-shifted Z_i, and divide by two basis points.

4.1.4.3. *Par Gamma.* Equation (4.5) was obtained applying the chain rule to the composite function $V(Z(r))$. Another application of the chain rules gives

$$\frac{\partial^2 V}{\partial r_k \partial r_\ell} = \sum\nolimits_{i,j=1}^{N} \frac{\partial^2 V}{\partial Z_i \partial Z_j} \frac{\partial Z_i}{\partial r_k} \frac{\partial Z_j}{\partial r_\ell} + \sum\nolimits_{i=1}^{N} \frac{\partial V}{\partial Z_i} \frac{\partial^2 Z_i}{\partial r_k \partial r_\ell} \tag{4.7}$$

The delta-equivalent cashflows $\partial V/\partial Z_i$ and the gamma-equivalent cashflows $\partial^2 V/\partial Z_i \partial Z_j$ are returned by the transaction pricing models. The derivatives $\partial^2 Z_i/\partial r_k \partial r_\ell$ are calculated by an extension of the method used for $\partial Z_i/\partial r_k$.

For the calculation of $\partial^2 Z_i/\partial r_k^2$, Equation (4.6) gives

$$\varepsilon^2 \frac{\partial^2 Z}{\partial r^2} = Z(r+\varepsilon) + Z(r-\varepsilon) - 2Z(r)$$

For the calculation of $\partial^2 Z_i/\partial r_k \partial r_\ell$, where $k \neq \ell$, consider the discount factor Z as a function of two par rates s and r. The perturbed discount factor $Z(s+\delta, r+\varepsilon)$ is given by Taylor's series

$$Z + \delta \frac{\partial Z}{\partial s} + \varepsilon \frac{\partial Z}{\partial r} + \frac{1}{2}\left(\delta^2 \frac{\partial^2 Z}{\partial s^2} + 2\delta\varepsilon \frac{\partial^2 Z}{\partial s \partial r} + \varepsilon^2 \frac{\partial^2 Z}{\partial r^2}\right) + \cdots$$

It follows that the mixed second-order derivatives are given by

$$4\varepsilon^2 \frac{\partial^2 Z}{\partial s \partial r} = Z(s+\varepsilon, r+\varepsilon) - Z(s+\varepsilon, r-\varepsilon) - Z(s-\varepsilon, r+\varepsilon) + Z(s-\varepsilon, r-\varepsilon)$$

Hence: the diagonal terms $\partial^2 Z_i/\partial r_k^2$ can be calculated by making an upward and a downward shift of the rate r_k; the off-diagonal terms $\partial^2 Z_i/\partial r_k \partial r_\ell$ can be calculated by making all four combinations of upward and downward shifts of the two rates r_k and r_ℓ.

4.1.5. Zero-coupon Rate Sensitivity.

4.1.5.1. *Zero-coupon Rates.* Adaptiv calculates sensitivity to the zero-coupon rates at specified grid dates $D_1, \ldots, D_n$. Let P_k denote the discount factor at D_k. The corresponding zero-coupon rate z_k is defined by

$$P_k = \left(1 + \frac{z_k}{h}\right)^{-ht_k}$$

for discrete compounding at frequency h per year, and

$$P_k = \exp(-t_k z_k)$$

for continuous compounding; where t_k is the year fraction from the valuation date D_0 to grid date D_k, calculated according to the basis specified for the zero-coupon rates.

4.1.5.2. *Measuring Sensitivity to Zero-coupon Rates.* For a given yield curve, the portfolio's present value V is a function of certain discount factors $Z_1, \ldots, Z_N$. However, Z_i is not necessarily a function of the discount factors at the grid dates[2], and hence not necessarily a function of the corresponding zero-coupon rates $z_1, \ldots, z_n$. Adaptiv calculates sensitivity to zero-coupon rates by replacing each Z_i with the corresponding discount factor obtained by interpolating the discount factors at the grid dates.

For each discount factor Z_i, let $\widetilde{Z}_i$ denote the discount factor obtained by interpolating the grid-date discount factors. Then $\widetilde{Z}_i$ is a function of the zero-coupon rates $z_1, \ldots, z_n$ and its derivatives $\partial \widetilde{Z}_i/\partial z_k$ and $\partial^2 \widetilde{Z}_i/\partial z_k \partial z_\ell$ are calculated by

[2] In the special case that the grid is *finer* than the grid of dates generated by the bootstrapping process, any discount factor on the curve can be obtained by interpolating between the grid dates.

making upward and downward shifts of the zero-coupon rates and recalculating the interpolated discount factor (see Section 4.1.4). The pseudo-derivative $\partial V/\partial z_k$ is defined as

$$\frac{\partial V}{\partial z_k} = \sum\nolimits_{i=1}^{N} \frac{\partial V}{\partial Z_i} \frac{\partial \widetilde{Z}_i}{\partial z_k} \tag{4.8}$$

by analogy with Equation (4.5). In this definition, the delta-equivalent cashflow $\partial V/\partial Z_i$ is calculated with respect to the *original* discount factors but $\partial \widetilde{Z}/\partial z_k$ is the derivative of the *interpolated* discount factor. Similarly, the pseudo-derivative $\partial^2 V/\partial z_k \partial z_\ell$ is defined as

$$\frac{\partial^2 V}{\partial z_k \partial z_\ell} = \sum\nolimits_{i,j=1}^{N} \frac{\partial^2 V}{\partial Z_i \partial Z_j} \frac{\partial \widetilde{Z}_i}{\partial z_k} \frac{\partial \widetilde{Z}_j}{\partial z_\ell} + \sum\nolimits_{i=1}^{N} \frac{\partial V}{\partial Z_i} \frac{\partial^2 \widetilde{Z}_i}{\partial z_k \partial z_\ell} \tag{4.9}$$

by analogy with Equation (4.7).

The first-order derivatives multiplied by 0.01% are displayed on the Hedge Prescription report under the heading Zero Sensitivity and on the IR Sensitivity report under the heading Zero Delta. The second-order derivatives multiplied by $(0.01\%)^2$ are displayed on the IR Sensitivity report under the heading Zero Gamma.

4.1.6. Forward Rate Sensitivity (Portfolio-Level). Adaptiv calculates sensitivity to the forward deposit rates between specified grid dates $D_1, \ldots, D_n$. Let f_k denote the forward rate from D_{k-1} to D_k and let P_k denote the discount factor at D_k. The forward rates and discount factors are related by $D_k = D_{k-1}/(1+\alpha_k f_k)$, where α_k is the year fraction from D_{k-1} to D_k (see Section 2.5.1). It follows that the grid-date discount factors can be expressed as functions of the forward rates:

$$P_k = \frac{1}{(1+\alpha_1 f_1)\cdots(1+\alpha_k f_k)}$$

Sensitivity to the forward rates is calculated in the same as Section 4.1.5, with f_k replacing z_k in Equations (4.8) and (4.9).

The IR Sensitivity report displays the first-order derivatives multiplied by 0.01% under the heading Fwd Delta, and the second-order derivatives multiplied by $(0.01\%)^2$ under the heading Fwd Gamma.

4.1.7. Forward Rate Sensitivity (Trade-Level). The pricing models for interest-rate products make a direct calculation of forward-rate sensitivity, and the results are displayed on the Trades List and Trade Inspector. Sensitivities to both reference and discount yield curves are calculated.

Suppose the trade value V depends on certain forward deposit rates $r_1, \ldots, r_n$ from the reference yield curve. The *reference delta* is defined to be $\sum_i \partial V/\partial r_i$, which is the sensitivity of the trade value to a parallel shift in the forward rates from the reference curve. Similarly, *reference gamma* is defined to be the total second-order sensitivity $\sum_{i,j} \partial^2 V/\partial r_i \partial r_j$, which is the sensitivity of the reference delta to a parallel shift in the reference-curve forward rates. For example, the value of the floating side of a vanilla interest-rate swap is given by $V = \sum_i P_i \alpha_i r_i Z_i$, where P_i are the principal amounts, α_i are year fractions, and Z_i are cashflow-date discount factors from the discount curve. In this case the reference delta is $\sum_i P_i \alpha_i Z_i$, which is the PVBP of the floating side, and the reference gamma is zero.

Sensitivities to the forward rates from the discount curve are calculated by expressing the cashflow-date discount factors as functions of the forward rates between the cashflow dates. Suppose trade value V depends on the discount factors from the discount curve at dates $D_1, \ldots, D_m$; typically, these are cashflow dates. Let Z_i denote the discount factor at D_i. The pricing model calculates the delta-equivalent cashflows $\partial V/\partial Z_i$. For vanilla interest-rate products, such as swaps and caps, $\partial V/\partial Z_i$ is equal to the value of the cashflow at D_i divided by Z_i. The discount factors can be written

$$Z_i = \frac{1}{(1+\beta_1 s_1)\cdots(1+\beta_i s_i)},$$

where β_i is the year fraction for the period from D_{i-1} to D_i and s_i is the forward rate for that period. The initial date D_0 is the valuation date. The year fractions β_i are calculated using the interest basis of the floating cashflows, if any, and otherwise using the basis of any fixed-interest cashflows. The *discount delta* can now be defined as $\sum_i \partial V/\partial s_i$, which is the sensitivity of the trade value to a parallel shift in the forward rates from the discount curve. Similarly, *discount gamma* is defined to be the total second-order sensitivity $\sum_{i,j} \partial^2 V/\partial s_i \partial s_j$, which is the sensitivity of the discount delta to a parallel shift in the discount-curve forward rates. The first-order derivatives of Z_i with respect to the forward rates are given by $\partial Z_i/\partial s_j = -\beta_j Z_i/(1+\beta_j s_j) = -\beta_j Z_i (Z_j/Z_{j-1})$, where $j \le i$.

For display on the Trades List and Trade Inspector, reference and discount delta are multiplied by 0.01% to give the sensitivity of the trade value to a basis point shift in rates. Reference gamma is multiplied by $(0.01\%)^2$ to give the sensitivity of the scaled reference delta to a basis point shift in rates; and similarly for discount gamma.

4.2. FX Rate

4.2.1. Portfolio FX Sensitivity. Let V denote the PV of the portfolio in base currency. Then V can be expressed as a function of $S_1, \ldots, S_n$; where each S_i is a spot exchange rate against a specified *reference* currency. The non-reference currencies will be referred to as *foreign* currencies. The rate S_i is quoted in either the reference currency or the foreign currency, depending on the quoting convention defined in the FX Grid. The exchange rate between two foreign currencies will be referred to as a *cross rate*. The change in PV corresponding to spot rate shifts $\Delta S_1, \ldots, \Delta S_n$ is given by Taylor's series

$$\Delta V = \sum\nolimits_{i=1}^{n} \Delta S_i \frac{\partial V}{\partial S_i} + \frac{1}{2} \sum\nolimits_{i,j=1}^{n} \Delta S_i \Delta S_j \frac{\partial^2 V}{\partial S_i \partial S_j} + \cdots$$

Cross gamma terms $\partial^2 V/\partial S_i \partial S_j$ (for $i \neq j$) usually arise from transactions with an intrinsic sensitivity to a cross rate; this is explained in Section 4.2.3.

4.2.2. FX Sensitivity Report. The FX Sensitivity report (see Figure 4.2) calculates the first-order and second-order derivatives of V with respect to $S_1, \ldots, S_n$ and displays them under the headings FXDelta and FXGamma. The FXDelta values are the first-order derivatives multiplied by a 0.01 shift in the spot rate ($\Delta S_i = 0.01$):

$$0.01 \frac{\partial V}{\partial S_i}$$

The FXGamma values are the second-order derivatives multiplied by 0.01 shifts in the spot rates ($\Delta S_i = 0.01$ and $\Delta S_j = 0.01$):

$$(0.01)^2 \frac{\partial^2 V}{\partial S_i \partial S_j}$$

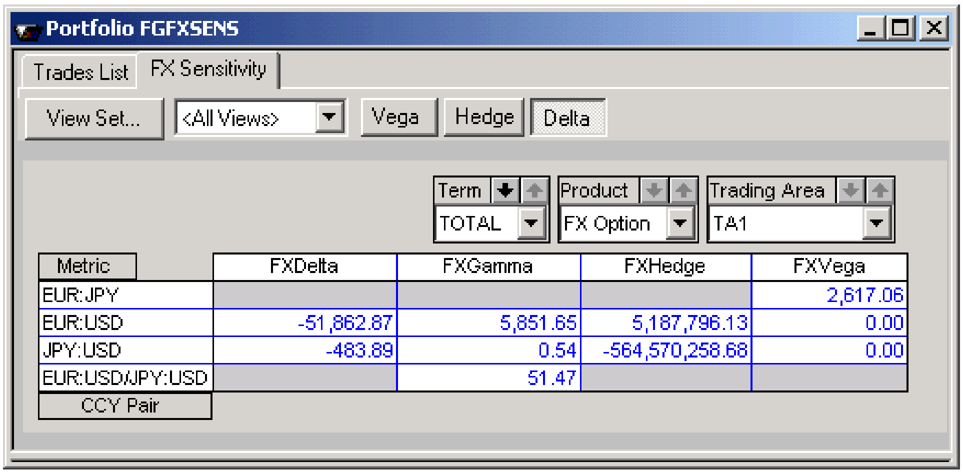


FIGURE 4.2. the FX Sensitivity report.

Values under the headings %FXDelta and %FXGamma are the derivatives of V with respect to the log of the spot rates. These derivatives are scaled by 0.01 corresponding to a 1% shift in the spot rate ($\Delta S_i = S_i/100$). The %FXDelta values are

$$0.01\frac{\partial V}{\partial \log S_i} = (S_i/100)\frac{\partial V}{\partial S_i}$$

and the %FXGamma values are

$$(0.01)^2\frac{\partial^2 V}{\partial \log S_i \partial \log S_j} = (S_i/100)(S_j/100)\frac{\partial^2 V}{\partial S_i \partial S_j} + (i=j)(S_i/100)\frac{\partial V}{\partial S_i}$$

4.2.3. Calculation of FX Sensitivity.

4.2.3.1. *Resolving Spot Rate Dependencies.* Let V denote the PV of a transaction leg[3] in local currency. If the transaction is an FX option, or other instrument with an intrinsic FX sensitivity, then V is a function of a spot exchange rate S. The transaction pricing model is responsible for returning the present value $V(S)$ and its derivatives $V'(S)$ and $V''(S)$ with respect to S.

If S is the exchange rate between two foreign currencies then the parity relationship of Section 1.3.1 gives either $S = S_iS_j$, $S = S_i/S_j$ or $S = 1/S_iS_j$.

Let $\tilde{S}$ denote the spot FX rate between the currency of the transaction leg and the portfolio's base currency. The PV of the transaction leg in base currency is $V(S)\tilde{T}$, where $\tilde{T}$ is the implied today FX rate from the transaction-leg currency to the base currency. From Section 2.4, $\tilde{T}$ is equal to either $\tilde{S}Z/\tilde{Z}$ or $(1/\tilde{S})Z/\tilde{Z}$, where Z and $\tilde{Z}$ denote the discount factors at the spot date for the base and transaction-leg currencies respectively.

Thus, the PV of the transaction leg in base currency is of the form

$$(S_1)^\alpha (S_k)^\beta \left(\frac{Z}{\tilde{Z}}\right) V\left((S_i)^\delta (S_j)^\varepsilon\right)$$

[3] Multi-currency transactions are split into single-currency transaction legs. Examples of multi-currency transactions are: spot and forward FX; cross-currency interest rate swaps; and FX options. In the case of an FX option, only one transaction leg returns a PV.

where: S_k is the spot rate between the transaction-leg currency and the reference currency; S_1 is the spot rate between the base currency and the reference currency; and α, β, δ and ε are constants taking the value 1, -1 or 0. Differentiating this gives the transaction leg's contributions to the FX Sensitivity report. These calculations are illustrated by the following example.

4.2.3.2. *Example.* Consider an option on the JPY:EUR rate in a portfolio with base currency USD, and suppose the reference currency is also USD. By convention, the FX option pricing model returns results in the currency in which the rate is quoted. Thus, the pricing model returns the local-currency present value V_{JPY} and its derivatives V'_{JPY} and V''_{JPY} with respect to $S_{\text{JPY:EUR}}$. The base-currency PV is

$$V = \left(\frac{1}{S_{\text{JPY:USD}}}\frac{Z_{\text{USD}}}{Z_{\text{JPY}}}\right)V_{\text{JPY}}.$$

V_{JPY} is a function of $S_{\text{JPY:EUR}}$ and $S_{\text{JPY:EUR}} = S_{\text{JPY:USD}}S_{\text{USD:EUR}}$. Hence, differentiating the composite function gives

$$\begin{aligned}\frac{\partial V}{\partial S_{\text{USD:EUR}}} &= V'_{\text{JPY}}\frac{Z_{\text{USD}}}{Z_{\text{JPY}}}\\ \frac{\partial V}{\partial S_{\text{JPY:USD}}} &= \left(\frac{V'_{\text{JPY}}S_{\text{USD:EUR}}}{S_{\text{JPY:USD}}} - \frac{V_{\text{JPY}}}{S^2_{\text{JPY:USD}}}\right)\frac{Z_{\text{USD}}}{Z_{\text{JPY}}}.\end{aligned}$$

These are the contributions to FXDelta. Differentiating again gives three contributions to FXGamma: $\partial^2 V/\partial S^2_{\text{USD:EUR}}$, $\partial^2 V/\partial S^2_{\text{JPY:USD}}$ and the cross gamma $\partial^2 V/\partial S_{\text{USD:EUR}}\partial S_{\text{JPY:USD}}$. A numerical example is given in the spreadsheet `FG_FXSens.xls`.

4.3. Vega

4.3.1. FX Rate Volatility. Let $\sigma_{\text{X:Y}}(D)$ denote the implied Black volatility of the exchange rate for currency pair X:Y and option exercise date D. If the present value V of a transaction depends on FX volatility then its pricing model returns one or more derivatives

$$\frac{\partial V}{\partial \sigma_{\text{X:Y}}(D)}$$

These are aggregated by the FX Sensitivity report and displayed under the heading FXVega. The report allocates the derivatives to a fixed grid of exercise dates $D_1, \ldots, D_n$. A derivative v returned with exercise date D is allocated to the adjacent grid dates in linear proportions: λv to the lower date D_{k-1} and $(1-\lambda)v$ to the upper date D_k, where

$$\lambda = \frac{D_k - D}{D_k - D_{k-1}}$$

By convention, the derivatives are multiplied by a 0.1% (ten basis point) shift in volatility.

4.3.2. Interest Rate Volatility. Let $\sigma_\delta(D)$ denote the implied Black volatility of the δ-period rate for option exercise date D. If the present value V of a transaction depends on deposit rates (caps and floors) or swap rates (swaptions and CMS products) then its pricing model returns one or more derivatives

$$\frac{\partial V}{\partial \sigma_\delta(D)}$$

These are aggregated by the IR Vega report. The derivatives can be allocated to a fixed grid of exercise dates. This is usually done in linear proportions, as in

Section 4.3.1. The derivatives are multiplied by a 0.1% (ten basis point) shift in volatility.

For the term-structure models of Section 2.17.2, vega is calculated by applying an additive shift of 0.1% to the volatility parameter σ and then recalculating the value of the transaction. For the Black-Karasinski model, this gives the sensitivity to the lognormal volatility of the short rate, and is approximately comparable to the cap and swaption lognormal volatilities on the IR Vega report. For the Hull-White model, the sensitivity to σ is multiplied by the short rate in order to convert it from a normal to a lognormal volatility. The short rate r is given by $Z_1 = \exp(-r/B)$, where Z_1 is the one-day discount factor and $B = 365.25$.

4.4. Theta

4.4.1. Analytic Theta.

4.4.1.1. *Introduction.* Theta is the sensitivity of present value to a small change in the valuation date. Increasing the valuation date has two main effects.

- Cashflows are closer to the valuation date and are therefore discounted by a smaller factor. The effect of this on the PV is referred to as *discount theta*.
- For standard options, the decrease in the time to the exercise date reduces the value of an option because the underlying price has less time over which to fluctuate. A similar effect applies to other volatility-dependent instruments, for example constant maturity swaps. For constant maturity swaps, the decrease in the time to the fixing date has the effect of reducing the convexity correction (see Section 2.9.4). More generally, for any claim whose value is contingent on market variables at some future date, the effect of moving closer to this date will be referred to as *option theta*.

These two theta effects are independent and can be added together.

4.4.1.2. *Option Theta.* Option theta is the derivative of the present value V with respect to the time to exercise T, multiplied by a negative shift of one day:

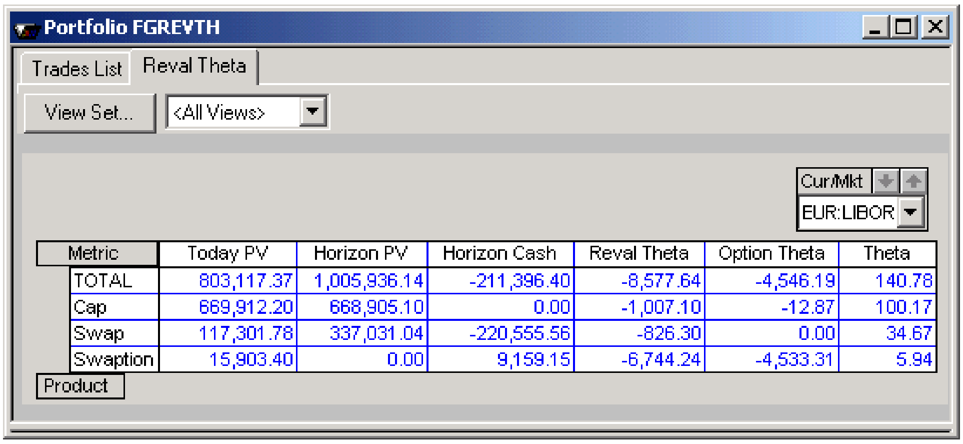
$$\frac{-1}{365.25}\frac{\partial V}{\partial T}$$

This gives an estimate of the change in V for a one-day decrease in T while other market variables (such as forward prices, discount factors and volatility) remain constant.

4.4.1.3. *Discount Theta.* Discount theta is usually calculated by approximating the decrease in each payment-date discount factor as the difference between the discount factor at the payment date and the discount factor at one day before the payment date.

For example, suppose the PV of a cashflow is CZ, where: Z is the discount factor at the payment date; and C is either a fixed cashflow, or estimated cashflow independent of the discount yield curve. Then the discount theta is equal to $C(Z - Z_{-1})$, where Z_{-1} denotes the discount factor at the day before the payment date.

4.4.2. Revaluation Theta. Revaluation theta is the difference between: the PV calculated from the market data structures calibrated for today; and the PV calculated from market data structures having today's market data *inputs* but recalibrated for tomorrow. In the case of a yield curve, this means that the discount



Metric	Today PV	Horizon PV	Horizon Cash	Reval Theta	Option Theta	Theta
TOTAL	803,117.37	1,005,936.14	-211,396.40	-8,577.64	-4,546.19	140.78
Cap	669,912.20	668,905.10	0.00	-1,007.10	-12.87	100.17
Swap	117,301.78	337,031.04	-220,555.56	-826.30	0.00	34.67
Swaption	15,903.40	0.00	9,159.15	-6,744.24	-4,533.31	5.94

FIGURE 4.3. the Reval Theta report.

factor table is recalculated for tomorrow using the same yield curve rates as inputs to the bootstrapping process.

Revaluation theta is calculated by the Reval Theta report. The valuation at the horizon date (tomorrow) is adjusted for realised cashflows, rate fixings, option exercise and other trade events via the process of *instrument ageing* described in Section 6.7. The report also displays discount theta (under the heading Theta) and option theta.

CHAPTER 5

Hedging

5.1. Introduction

Hedging can be defined as the process of minimizing the sensitivity of the portfolio value to changes in market rates. Adaptiv provides three hedging methods.

Exact Delta Matching: The first-order sensitivities of the portfolio are exactly neutralized by adding hedging transactions to the portfolio. The advantage of this method is that the delta of the hedged portfolio is zero. However, it usually requires many hedging transactions to completely neutralize the delta, and therefore: the cost of hedging is high; and there may be liquidity problems with some of the required hedging transactions.

Approximate Delta Matching: This is a variation of exact delta matching whereby the delta of the portfolio is approximated. This is done so that the approximate delta can be exactly hedged by a smaller set of hedging instruments.

Minimum Variance: The variance of the PV of the hedged portfolio is minimized by adjusting the amounts of each hedging instrument. This method can be applied to *any* set of hedging instruments and any type of market risk (but is usually applied to hedging interest-rate risk).

5.2. Yield Curve Hedging

5.2.1. Exact Delta Matching. For each instrument used to construct the yield curve (cash, futures, FRA, swap or bond), the portfolio has a sensitivity to the market rate of that instrument. In this section, it is shown that each first-order sensitivity can be hedged, independently of the others, with an amount of the corresponding instrument on the curve.

Let $\mathbf{r} = (r_1, \ldots, r_n)$ denote the vector of market rates, one for each instrument on the curve. Consider a hedging transaction derived from one of the instruments on the curve yield. Let $S_i(\mathbf{r}; K)$ denote the PV of the transaction as a function the market rates $\mathbf{r}$ and its traded rate K. The PV of the transaction is zero when traded at the market rate:

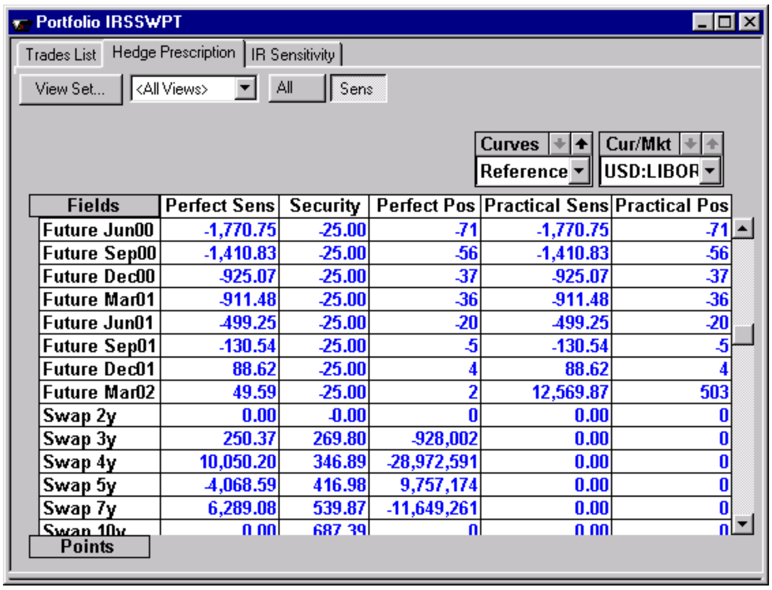
$$S_i(\mathbf{r}; r_i) = 0$$

and the PV remains at zero if any other market rate r_j is shifted[1]. Hence

$$\frac{\partial S_i}{\partial r_j}(\mathbf{r}; r_i) = 0 \tag{5.1}$$

for $i \neq j$. This property of the hedging transaction leads to a simple formula for the hedge prescription.

[1]It is assumed that all the instruments on the curve have PV = 0 when *priced back* from the curve. This is true for the conventional bootstrapping methods described in Section 1.2 but not, for example, if the curve is built by minimizing the pricing error.



Fields	Perfect Sens	Security	Perfect Pos	Practical Sens	Practical Pos
Future Jun00	-1,770.75	-25.00	-71	-1,770.75	-71
Future Sep00	-1,410.83	-25.00	-56	-1,410.83	-56
Future Dec00	-925.07	-25.00	-37	-925.07	-37
Future Mar01	-911.48	-25.00	-36	-911.48	-36
Future Jun01	-499.25	-25.00	-20	-499.25	-20
Future Sep01	-130.54	-25.00	-5	-130.54	-5
Future Dec01	88.62	-25.00	4	88.62	4
Future Mar02	49.59	-25.00	2	12,569.87	503
Swap 2y	0.00	-0.00	0	0.00	0
Swap 3y	250.37	269.80	-928,002	0.00	0
Swap 4y	10,050.20	346.89	-28,972,591	0.00	0
Swap 5y	-4,068.59	416.98	9,757,174	0.00	0
Swap 7y	6,289.08	539.87	-11,649,261	0.00	0
Swap 10y	0.00	687.39	0	0.00	0

FIGURE 5.1. the Hedge Prescription report.

Let $V(\mathbf{r})$ denote the PV of the portfolio. To hedge the portfolio, an amount λ_i of each hedging transaction is added so that the PV of the hedged portfolio is given by

$$H(\mathbf{r};\mathbf{K}) = V(\mathbf{r}) + \sum_{i=1}^{n} \lambda_i S_i(\mathbf{r}; K_i)$$

where $\mathbf{K} = (K_1, \ldots, K_n)$ is the vector of traded rates. The hedge is to be constructed from at-market transactions and therefore the hedged portfolio should satisfy

$$\frac{\partial H}{\partial r_i}(\mathbf{r},\mathbf{r}) = 0$$

It follows from Equation (5.1) that

$$\frac{\partial H}{\partial r_i}(\mathbf{r},\mathbf{r}) = \frac{\partial V}{\partial r_i}(\mathbf{r}) + \lambda_i \frac{\partial S_i}{\partial r_i}(\mathbf{r}, r_i)$$

and hence the hedge prescription λ_i is the ratio of the par delta of the portfolio to the par delta of the at-market hedging transaction:

$$\lambda_i = -\frac{\partial V/\partial r_i}{\partial S_i/\partial r_i}$$

The Hedge Prescription report (shown in Figure 5.1) displays the par delta of the portfolio under the heading Perfect Sens and the par delta of the hedging transactions under the heading Security. The hedge prescription amounts λ_i are shown under the heading Perfect Pos.

5.2.2. Approximate Delta Matching. Exact delta matching may require *all* the instruments on the curve to be included in the hedge. For a more practical hedge prescription, Adaptiv allows a subset of these instruments to be selected. For each instrument on the curve, the par delta is allocated (bucketed) to the instrument from the subset with the closest maturity. The calculation of the hedge prescription then proceeds as in Section 5.2.1 with the bucketed delta used in place of the actual delta. This gives a hedge prescription in terms of the chosen subset of instruments from the curve.

The Hedge Prescription report shows the bucketed par delta under the heading Practical Sens and the corresponding hedge prescription under the heading Practical Pos.

5.3. FX Hedging

Continuing in the notation of Section 4.2.1, the first-order derivative $\partial V/\partial S_i$ can be exactly hedged with a corresponding spot FX transaction: an amount A_i of foreign currency is traded in exchange for an amount B_i of reference currency. The transaction is executed at the market rate so that $B_i = A_i S_i$ or A_i/S_i. Let H_i denote the PV of the hedging transaction, given by

$$H_i = \left(A_i(S_i)^\delta - B_i\right)(S_1)^\alpha Z_{\text{spot}}$$

where: $\delta = 1$ if the rate S_i is quoted in reference currency, otherwise $\delta = -1$; S_1 is the spot rate between the reference currency and the base currency, $\alpha = 1$ if this rate is quoted in base currency, otherwise $\alpha = -1$; and Z_{spot} is the base-currency discount factor at the spot date. The first-order sensitivity to S_i is hedged when

$$\frac{\partial V}{\partial S_i} + \frac{\partial H_i}{\partial S_i} = 0,$$

and $\partial H_i/\partial S_i = A_i \delta (S_i)^{\delta-1}(S_1)^\alpha Z_{\text{spot}}$. It follows that the first-order sensitivity to S_i is hedged by trading an amount

$$A_i = -\frac{1}{\delta(S_i)^{\delta-1}(S_1)^\alpha Z_{\text{spot}}}\frac{\partial V}{\partial S_i}$$

of the foreign currency at spot.

In the special case of hedging the sensitivity to S_1, the PV of the hedging transaction is given by $H_1 = \left(A_1 - B_1(S_1)^{-\delta}\right) Z_{\text{spot}}$; and it follows that the required amount of base currency is

$$A_1 = -\frac{S_1}{\delta Z_{\text{spot}}}\frac{\partial V}{\partial S_1}.$$

These amounts are displayed on the FX Sensitivity report under the heading FXHedge (see Section 4.2.2).

5.4. Minimum Variance Hedging

5.4.1. Introduction. Section 5.2 describes two methods of hedging interest rate risk.

- Exact delta matching generates an accurate but inefficient hedge. The hedge is unnecessarily accurate because the portfolio becomes hedged against arbitrary movements in the curve, at the expense of employing all the instruments on the curve, whereas movements in yield-curve rates are highly correlated.
- Approximate delta matching gives more flexibility in the choice of hedging instruments, although they still have to be chosen from the yield curve. Bucketing delta from one yield curve point to another does not affect the accuracy of the hedge for a *parallel shift* in the curve. However, accuracy is diminished under non-parallel shifts.

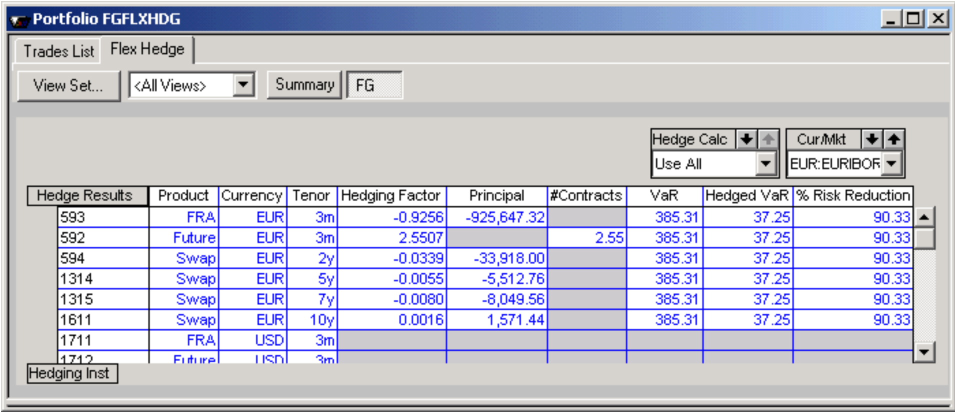


FIGURE 5.2. the Flex Hedge report.

Minimum variance hedging overcomes some of the disadvantages of delta matching because: any set of hedging instruments can be used; it takes account of the correlation between market rate movements; the reduction in risk is measured and therefore the accuracy of different combinations of hedging instruments can be judged.

Minimum variance hedging is implemented in the Flex Hedge report, from which some sample results are shown in Figure 5.2.

5.4.2. Variance of the Hedged Portfolio. Let V denote the value of the portfolio to be hedged, and let $V_1, \dots, V_m$ denote the values of the hedging instruments. The value of the hedged portfolio is

$$V_\lambda = V + \lambda_1 V_1 + \cdots + \lambda_m V_m,$$

where $\lambda_1, \dots, \lambda_m$ are the amounts of the hedging instruments. Then the objective is to minimize $\mathrm{var}(\Delta V_\lambda)$, the variance of the change in value of the hedged portfolio.

5.4.3. Minimizing the Variance. It follows from the identity $\mathrm{var}(X+Y) = \mathrm{var}(X) + 2\,\mathrm{cov}(X,Y) + \mathrm{var}(Y)$ that $\mathrm{var}(\Delta V_\lambda)$ is a quadratic function of λ and is equal to

$$\alpha + 2a \cdot \lambda + \lambda \cdot A\lambda,$$

where: $\alpha = \mathrm{var}(\Delta V)$, $a_k = \mathrm{cov}(\Delta V, \Delta V_k)$ and $A_{k\ell} = \mathrm{cov}(\Delta V_k, \Delta V_\ell)$. These coefficients may be calculated using the analytical methods of variance-covariance value-at-risk described in Section 6.2. The symmetric matrix A is positive because

$$\lambda \cdot A\lambda = \mathrm{var}\left(\Delta V_\lambda - \Delta V\right).$$

It can be shown[2] that there exists a vector b such that $a = Ab$. Then $\mathrm{var}(\Delta V_\lambda)$ can be written

$$[\alpha - b \cdot a] + (b + \lambda) \cdot A(b + \lambda).$$

It follows that the minimum value is $\alpha - b \cdot a$, and is attained at $\lambda = -b$.

[2] The proof is as follows. Let $Q(\lambda)$ denote $\mathrm{var}(\Delta V_\lambda)$. Then $Q(\lambda) \geq 0$ for all λ. The space $\mathbb{R}^m$ can be decomposed into the direct sum of the kernel of A and its orthogonal complement, and the orthogonal complement of the kernel of a symmetric matrix is equal to its range. Hence there exists $b, c \in \mathbb{R}^m$ such that $a = Ab + c$ and $Ac = 0$. It follows that

$$Q(\mu c) = \alpha + 2\mu \|c\|^2$$

for all $\mu \in \mathbb{R}$. This shows that Q cannot be bounded below if $c \neq 0$. Hence $c = 0$ and $a = Ab$.

Thus, the variance of the hedged portfolio is minimized by solving the linear equation $Ab = a$, a system of m linear equations in m variables. This is a straightforward computation because the dimension m is small (typically less than 10). A solution always exists and the matrix A is not required to be invertible. If it is singular then there is more than one solution for b, and hence more than one hedge that minimizes the variance.

In practice, Adaptiv solves the linear equation using singular value decomposition, as described in Press et al. [**29**].

CHAPTER 6

Value-at-Risk

6.1. Portfolios and Risk

Consider the value of the portfolio V as a function of the vector of market risk factors $x = (x_1, \ldots, x_n)$ and time t. Let Δx denote a stochastic shift in the risk-factor values over a given risk horizon Δt. Let ΔV denote the corresponding change in the portfolio value defined by

$$\Delta V = V(x + \Delta x, t + \Delta t) - V(x, t)$$

The value change ΔV will be referred to as the *profit-and-loss* (P&L).

In this mathematical setting, the risk on the portfolio value can be measured in two ways: the standard deviation of ΔV, and the *value-at-risk*. The value-at-risk (VaR) for a given confidence probability p $(0 < p < 1)$ is defined to be value of v for which

$$\mathbb{P}(\Delta V \leq v) = 1 - p \tag{6.1}$$

That is, the probability of a *loss worse than the value-at-risk* is one minus the confidence probability. Typical values for the confidence probability are 95% and 99%. Let F denote the distribution function of the P&L, defined by $F(x) = \mathbb{P}(\Delta V \leq x)$. Then the value-at-risk for the confidence probability p is equal to $F^{-1}(1-p)$. Thus, in order to calculate VaR, the inverse of the P&L distribution function must be determined.

The return on the portfolio value is defined by $R = \Delta V/V$. In applications to asset management, it is usual to express risk relative to a benchmark portfolio with return $\hat{R}$. The standard deviation of the relative return $R - \hat{R}$ is known as the *tracking error*. The relative return can be written

$$R - \hat{R} = \frac{\Delta V - \Delta \hat{V}}{V}$$

by scaling the benchmark portfolio to have the same value as the actual portfolio, so that $\hat{V} = V$. Adaptiv constructs a *netted portfolio* comprising the actual portfolio and a short position in the scaled benchmark portfolio; the tracking error is then calculated as the standard deviation of the netted portfolio value divided by the value of the actual portfolio.

6.2. Variance-Covariance VaR

6.2.1. Risk Factor and P&L Distribution. The *variance-covariance*, or *parametric*, method of calculating VaR is an analytical approximation of the statistics (mean, variance, skewness, kurtosis, etc.) of the P&L distribution based on a statistical model of the daily movements of the market risk factors.

Each risk factor x_i is *nominally* classified as either lognormal or normal:

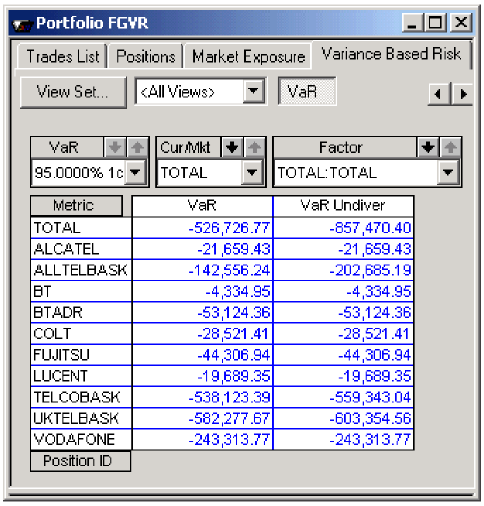


FIGURE 6.1. value-at-risk on the Variance Risk report.

- **Lognormal:** The factor takes only positive values and the increment in $\log x_i$ is normally distributed. The increment $\Delta(\log x_i)$ is approximately equal to the proportional return $\Delta x_i/x_i$ when Δx_i is small, and hence is referred to as the logarithmic return.
- **Normal:** The increment in x_i is normally distributed, and the factor may take positive or negative values.

Let y_i denote either $\log x_i$ for a lognormal factor, or x_i for a normal factor. Then the increment Δy_i is normally distributed. Section 6.2.3 describes a model under which the increments Δy_i are normally distributed and consistent with observed changes in market data. For this section, it suffices to assume that the distribution of the vector of increments Δy is normal with mean and variance determined from historical data.

If the portfolio value V is a linear function of y then the distribution of ΔV is also normal and hence determined by its mean and variance.

More generally, if V is a quadratic function of y then there are explicit formulae for the mean, variance, skewness, kurtosis and higher-order moments of the P&L distribution (see Section 6.2.4). VaR can be calculated by fitting the first four moments to a suitable distribution function.

Variance-covariance VaR is calculated by the Variance Risk report (see Figure 6.1). The spreadsheet `FG_VarRisk.xls` can be used to replicate the calculations on the Variance Risk report from a given set of risk-factor sensitivities and risk-factor statistics.

6.2.2. Linear VaR. Consider the portfolio value V as a function of the vector $y = (y_1, \ldots, y_n)$, where $y_i = \log x_i$ for lognormal factors or $y_i = x_i$ for normal factors. Let L denote the first-order approximation of ΔV:

$$L = \delta \cdot \Delta y + \theta \Delta t,$$

where δ is the *exposure* vector defined by $\delta_i = \partial V/\partial y_i$, and $\theta = \partial V/\partial t$ is the sensitivity of the portfolio value to elapse of time. Since Δy is normal, the distribution of L is normal with mean $\delta \cdot \mathbb{E}(\Delta y) + \theta \Delta t$ and variance $\delta \cdot C\delta$, where C is the covariance

Risk Factor Type	**Market Data Structure**	**Sensitivity Source**
Interest Rate	Yield Curve	IR Sensitivity (Par or Zero Delta)
Deposit Rate Volatility	Cap Volatility Surface	IR Vega (Caplet Vega)
Swap Rate Volatility	Swaption Volatility Surface	IR Vega (Swaption Vega)
FX Rate	FX Grid	FX Sensitivity (FXDelta)
FX Rate Volatility	FX Volatility Surface	FX Sensitivity (FXVega)
Equity Price & Volatility	Equity Market Parameters	Pricing Model

TABLE 6.1. risk factor types, origin and sensitivity.

matrix of Δy:

$$C_{ij} = \mathbb{E}(\Delta y_i \Delta y_j) - \mathbb{E}(\Delta y_i)\mathbb{E}(\Delta y_j).$$

It follows that the value-at-risk for confidence probability p is

$$[\delta \cdot \mathbb{E}(\Delta y) + \theta \Delta t] + \mathcal{N}^{-1}(1-p)\sqrt{\delta \cdot C\delta}, \tag{6.2}$$

where $\mathcal{N}^{-1}$ is the inverse of the standard normal distribution function.

The covariance matrix can be written

$$C_{ij} = \sigma_i \sigma_j \rho_{ij} \Delta t,$$

where σ_i is the volatility of the factor x_i defined by $C_{ii} = \sigma_i^2 \Delta t$, and $\rho_{ij} = C_{ij}/\sqrt{C_{ii}C_{jj}}$ is the correlation of the increments Δy_i and Δy_j.

For short-term horizons, the mean of the P&L distribution, $\delta \cdot \mathbb{E}(\Delta y) + \theta \Delta t$, is dominated by the standard deviation $\sqrt{\delta \cdot C\delta}$, and for this reason the mean is usually neglected. Adaptiv gives the user the option to include the term $\delta \cdot \mathbb{E}(\Delta y)$ in the value-at-risk.

For lognormal factors, $y_i = \log x_i$ and hence $\partial V/\partial y_i = x_i \partial V/\partial x_i$; for normal factors, $\partial V/\partial y_i = \partial V/\partial x_i$. The derivatives of V with respect to x and t are obtained either directly from the pricing model, or from the market data sensitivity reports described in Chapter 4. Table 6.1 indicates the origin of the sensitivity to interest-rate, FX and equity risk factors.

In the case that portfolio value depends on a single risk factor x, the standard deviation of the P&L distribution is equal to $|\delta|\,\sigma\sqrt{\Delta t}$ and hence the value-at-risk (neglecting the mean of the P&L distribution) is given by

$$\mathcal{N}^{-1}(1-p)\left|\frac{\partial V}{\partial y}\right|\sigma\sqrt{\Delta t} \tag{6.3}$$

The sum of the single-factor VaR amounts given by Equation (6.3) is referred to as the *undiversified* VaR. The undiversified VaR is an upper bound on the (diversified) VaR. It is equivalent to the VaR obtained by setting the correlation between each pair of risk factors x_i and x_j to: $+1$ if the exposures $\partial V/\partial x_i$ and $\partial V/\partial x_j$ have the same sign; or -1 if the exposures have the opposite sign. The difference between the undiversified and diversified VaR gives a measure of how much the correlation between risk factors reduces value-at-risk.

6.2.3. Covariance from Historical Data.

6.2.3.1. *SMA and EWMA Models.* Thus, variance-covariance VaR requires the covariance of the risk-factor returns over the risk horizon. Adaptiv supports two statistical models under which the covariance matrix can be derived from the historical risk-factor values. Following the terminology of the RiskMetrics technical document [32], the models will be referred to as: *simple moving average* (SMA) and *exponentially weighted moving average* (EWMA).

Let $x(t)$ denote the vector of risk-factor values $(x_1(t), \ldots, x_n(t))$ at time t, where time is discrete and scaled in business days, so that $t \in \{0, 1, 2, \ldots\}$. Let $r(t)$ denote the vector of returns defined by $r_i(t) = \log x_i(t) - \log x_i(t-1)$ for a lognormal factor or $r_i(t) = x_i(t) - x_i(t-1)$ for a normal factor.

The returns are modelled by the process

$$r(t) = \mu(t) + \Sigma(t)\varepsilon(t),$$

where the vector $\varepsilon(t)$ has the standard normal distribution, $\varepsilon(s)$ and $\varepsilon(t)$ are independent for $s < t$, and $\Sigma(t)$ is a positive matrix known at time $t-1$. Then the conditional covariance matrix $C(t) = \Sigma(t)^2$ satisfies

$$C_{ij}(t+1) = \mathbb{E}_t \left[(r-\mu)_i(t+1)(r-\mu)_j(t+1) \right],$$

where $\mathbb{E}_t$ denotes the expectation conditional on the information[1] available at time t. The drift vector $\mu(t+1)$ is defined to be the sample mean of the returns known at time t:

$$\mu(t+1) = \frac{1}{t} \sum\nolimits_{s=1}^{t} r(s). \tag{6.4}$$

The EWMA model defines the conditional covariance matrix by the recursive formula

$$C_{ij}(t+1) = (1-\lambda)(r-\mu)_i(t)(r-\mu)_j(t) + \lambda C_{ij}(t) \tag{6.5}$$

where λ is a constant parameter satisfying $0 < \lambda < 1$. It follows from this definition that $C(t)$ is a positive matrix and hence $\Sigma(t)$ is defined to be its unique positive square root. The parameter λ is referred to as the *decay factor* and is usually chosen close to 1. Thus, $C(t+1)$ is the sum of the most recent information about the covariance, weighted by $1-\lambda$, and the previous covariance matrix $C(t)$, weighted by λ. Since $C(1) = 0$, Equation (6.5) is equivalent to

$$C_{ij}(t+1) = (1-\lambda) \sum\nolimits_{s=1}^{t} \lambda^{t-s} (r-\mu)_i(s)(r-\mu)_j(s) \tag{6.6}$$

This shows the most recent data is weighted by 1, the previous day's data is weighted by λ, the data previous to that is weighted by λ^2, and so on.

For the choice of λ, consider the total of the weighting factors for historical data before time $t+1-w$; this is equal to

$$(1-\lambda) \sum\nolimits_{s=1}^{t-w} \lambda^{t-s}$$

and is bounded by $(1-\lambda)(\lambda^w + \lambda^{w+1} + \cdots) = \lambda^w$. Thus, for a given historical window of w days and a given tolerance level τ, choosing $\lambda = \tau^{1/w}$ ensures that the total weight of historical data more than w days from the current date is less than the required tolerance τ. For example, for a window of 90 days and tolerance level 1%, set the decay factor to be $0.01^{1/90} \simeq 0.95$.

[1] More precisely, we assume that $\varepsilon(t)$ is adapted to a filtration $\mathcal{F}(t)$ and $\varepsilon(t)$ is independent of $\mathcal{F}(t-1)$, and that $\Sigma(t)$ is $\mathcal{F}(t-1)$-measurable.

The SMA model defines the conditional covariance matrix to be the sample covariance:

$$C_{ij}(t+1) = \frac{1}{t-1}\sum\nolimits_{s=1}^{t}(r-\mu)_i(s)(r-\mu)_j(s) \tag{6.7}$$

where $\mu(t)$ is the sample mean given by Equation (6.4). Then $C(t)$ satisfies the following analogue of Equation (6.5):

$$C_{ij}(t+1) = \omega_t(r-\mu)_i(t)(r-\mu)_j(t) + (1-\omega_t)C_{ij}(t) \tag{6.8}$$

where ω_t denotes $1/(t-1)$.

6.2.3.2. *Multiple Day Horizon.* Value-at-risk is often calculated for a risk horizon of more than one day. A horizon of h days requires an estimate of the covariance of the h-day returns $\sum_{k=1}^{h} r(t+k)$. Let $C(t+h,t)$ denote the covariance matrix of h-day returns, defined by

$$C_{ij}(t+h,t) = \mathbb{E}_t\left[\left(\sum\nolimits_{k=1}^{h}(r-\mu)_i(t+k)\right)\left(\sum\nolimits_{k=1}^{h}(r-\mu)_j(t+k)\right)\right]$$

From the independence of $\varepsilon(s)$ and $\varepsilon(t)$ for $s<t$, it follows that

$$\mathbb{E}_t\left[(r-\mu)_i(t+k)(r-\mu)_j(t+\ell)\right] = 0$$

for $k \neq \ell$, and hence that

$$\begin{aligned} C_{ij}(t+h,t) &= \sum\nolimits_{k=1}^{h}\mathbb{E}_t\left[(r-\mu)_i(t+k)(r-\mu)_j(t+k)\right] \\ &= \sum\nolimits_{k=1}^{h}\mathbb{E}_t\left[\mathbb{E}_{t+k-1}\left[(r-\mu)_i(t+k)(r-\mu)_j(t+k)\right]\right] \\ &= \sum\nolimits_{k=1}^{h}\mathbb{E}_t\left[C_{ij}(t+k)\right] \end{aligned}$$

where $C(s+1) = C(s+1,s)$ is the covariance matrix of the one-day returns. But in the EWMA model:

$$\begin{aligned} \mathbb{E}_t\left[C_{ij}(s+1)\right] &= (1-\lambda)\mathbb{E}_t\left[\mathbb{E}_{s-1}\left[(r-\mu)_i(s)(r-\mu)_j(s)\right]\right] + \lambda\mathbb{E}_t\left[C_{ij}(s)\right] \\ &= (1-\lambda)\mathbb{E}_t\left[C_{ij}(s)\right] + \lambda\mathbb{E}_t\left[C_{ij}(s)\right] \\ &= \mathbb{E}_t\left[C_{ij}(s)\right] \end{aligned}$$

for $s>t$; and the same identity holds in the SMA model. Since the first conditional expectation $\mathbb{E}_t\left[C_{ij}(t+1)\right]$ is equal to $C_{ij}(t+1)$, it now follows that

$$C_{ij}(t+h,t) = hC_{ij}(t+1)$$

Thus, in the SMA and EWMA models, the h-day covariance is simply h multiplied by the one-day covariance. In particular, the h-day volatility is $\sqrt{h}$ multiplied by the one-day volatility. This fact is referred to as the *square root of time rule*.

Adaptiv provides two methods for calculating the multiple-day covariance matrix. The first method is to apply the square root of time rule.

The second method is to model the h-day returns $r(t) = \log x(t) - \log x(t-h)$ by the process $r(t) = \mu(t) + \Sigma(t)\varepsilon(t)$, where: $\varepsilon(t)$ has the standard normal distribution and $\Sigma(t)$ is the square root of the conditional covariance matrix $C(t)$, given by either Equation (6.6) or Equation (6.7); and $\mu(t)$ is given by Equation (6.4). The $\varepsilon(t)$ are not assumed to be independent because $r(s)$ and $r(t)$ cover *overlapping* periods when $|t-s|<h$ and are therefore correlated. The correlation between $r(s)$ and $r(t)$ is known as *autocorrelation*. Under these assumptions, the covariance matrix of $r(t+1)$ is known at time t. Adaptiv then substitutes covariance of $r(t+1)$ for the covariance of $r(t+h)$ in the calculation of variance-covariance VaR.

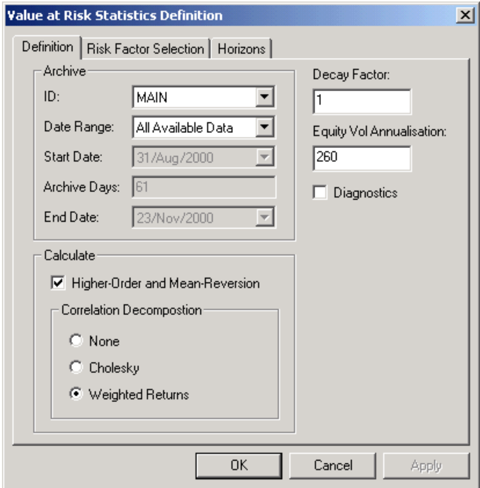


FIGURE 6.2. the Statistics Set.

6.2.3.3. *Missing Data.* The covariance matrix is calculated from data taken from a specified historical period. A lognormal factor has a valid value on a given date if a value exists (is not missing) and is positive; for a normal factor, any non-missing value is valid. For each risk factor and each date in the historical period, the h-day return can be calculated if:

(1) there is a valid risk-factor value on this date, and
(2) there are at least h valid risk-factor values after this date.

If the return cannot be calculated then it will be described as *missing*.

The covariance of two risk factors is calculated as the dot product of two vectors of weighted returns. The entries in these vectors are of the form $\omega_k(r_k - \mu)$ where r_k is the historical return on day k in the historical period, μ is the average return, and ω_k is a weighting factor. The return r_k is set to zero when it is missing. In the case of the simple moving average model, $\omega_k = 1/N$ where N is the number of non-missing returns; and for the exponentially weighted moving average model with decay factor λ, $\omega_k = 1$ for the first non-missing return, $\omega_k = \lambda$ for the second non-missing return, $\omega_k = \lambda^2$ for the third non-missing return, and so on. Hence missing returns do not contribute to the weighting factors. Similarly, missing returns do not contribute to the average return μ. This method ensures that the covariance matrix has the decomposition $C = RR^*$, where R is the matrix whose rows are the vectors of weighted returns — see Section 6.4.2.

In the special case of the variance of a single risk factor (and hence its volatility), the calculation described above is equivalent to discarding any missing or non-positive risk-factor values before proceeding to calculate the returns. Hence, days on which there is not a valid risk-factor value are treated as non-business days.

The risk-factor statistics are calculated by a component of Adaptiv called the Statistics Set. Figure 6.2 shows the Definition page of the Statistics Set. See Appendix H and the spreadsheet `FG_VaRStats.xls` for more details of the calculations.

6.2.4. Quadratic VaR. The variance-covariance VaR formula (6.2) is based on two main assumptions: the distribution of the risk-factor returns is normal and the P&L is a linear function of the returns. In this section, the second assumption is relaxed. If the P&L is assumed to be a quadratic function of the returns then the moments of the P&L distribution can be calculated and used to estimate the value-at-risk.

Let Q denote the second-order approximation of the P&L:

$$Q = \theta\Delta t + \delta \cdot \Delta y + \tfrac{1}{2}\Delta y \cdot \Gamma\Delta y$$

where θ and δ were defined in Section 6.2.2 and Γ is the exposure matrix defined by

$$\Gamma_{ij} = \frac{\partial^2 V}{\partial y_i \partial y_j}.$$

It is shown in Appendix I that the mean, variance, skewness and kurtosis of Q are given by

$$\begin{aligned}
\mathbb{E}(Q) &= \theta\Delta t + \tfrac{1}{2}\operatorname{tr}(\Gamma C) \\
\mathbb{E}\left[(Q-\mathbb{E}(Q))^2\right] &= \delta \cdot C\delta + \tfrac{1}{2}\operatorname{tr}\left((\Gamma C)^2\right) \\
\mathbb{E}\left[(Q-\mathbb{E}(Q))^3\right] &= 3\delta \cdot C\Gamma C\delta + \operatorname{tr}\left((\Gamma C)^3\right) \\
\mathbb{E}\left[(Q-\mathbb{E}(Q))^4\right] - 3\mathbb{E}\left[(Q-\mathbb{E}(Q))^2\right] &= 12\delta \cdot C(\Gamma C)^2\delta + 3\operatorname{tr}\left((\Gamma C)^4\right)
\end{aligned}$$

respectively, where tr denotes the trace of a matrix (sum of the diagonal entries). These formulae are stated for the case $\mathbb{E}(y) = 0$; the general case is included in Proposition I.3.

It is possible to fit the first four moments to a distribution, for example to the RTDM distribution of Ramberg et al. [**31**]. In the case of the RTDM distribution, the value-at-risk for confidence probability p is

$$\lambda_1 + \frac{(1-p)^{\lambda_3} - p^{\lambda_4}}{\lambda_2}$$

where the parameters λ_1, λ_2, λ_3 and λ_4 are determined from mean, variance, skewness and kurtosis (see Appendix J).

6.3. Historical VaR

6.3.1. Introduction. Historical VaR is the method of estimating the P&L distribution by valuing the portfolio under a number of simulated movements in market risk factors, where the simulated risk factors are obtained by applying historical changes in risk-factor values to the current risk-factor values.

The simulated risk-factor values are calculated by a component of Adaptiv called the Historical Scenario Generator, shown in Figure 6.3. The calculations are replicated in the spreadsheet `FG_HSG.xls`.

The simulated valuations are shown on the Market Simulation report.

6.3.2. Historical Sampling. Let $\hat{x}(1), \ldots, \hat{x}(N)$ denote the vectors of simulated risk-factor values generated from historical data. Each $\hat{x}(t)$ is referred to as a *historical scenario*. The methods by which the scenarios are generated are described in Section 6.3.3.

Let x denote the vector of risk-factor values at the risk horizon. It is assumed that the historical scenarios are independent samples of the distribution of x. If g is a real-valued function of x then

$$\frac{1}{N}\sum_{t=1}^{N} g(\hat{x}(t))$$

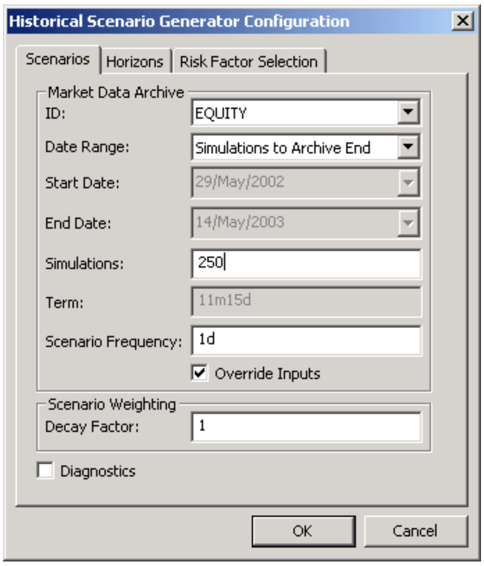


FIGURE 6.3. the Historical Scenario Generator.

is an unbiased estimator of the expectation $\mathbb{E}(g(x))$ and has variance

$$\frac{1}{N}\,\mathrm{var}(g(x))$$

In particular, taking $g(x)$ to be the indicator function of the set $\{V(x) \le v\}$, where V is the value of the portfolio, it follows that the distribution function of $V(x)$ is estimated by

$$\mathbb{P}(V(x) \le v) = \mathbb{E}(1_{\{V(x)\le v\}}) \simeq \frac{1}{N}\,|\{t : V(\hat{x}(t)) \le v\}| \tag{6.9}$$

Let V_t denote $V(\hat{x}(t))$ and let τ be a permutation of $\{1, \ldots, N\}$ that orders the simulated portfolio values, so that $V_{\tau(1)} \le \cdots \le V_{\tau(N)}$. Equation (6.9) shows that $V_{\tau(k)}$ is the value-at-risk for confidence probability $1 - k/N$. For example, if there are 100 simulated risk-factor values then $V_{\tau(5)}$, the fifth smallest portfolio value, is the value-at-risk for confidence probability 95%.

The advantage of the historical method over variance-covariance VaR is that the P&L distribution is derived without having to make a linear or quadratic approximation of the portfolio PV, and without having to assume that the risk-factor returns are normally distributed. However, a large number of market-rate scenarios are required to estimate the P&L distribution and this has both a practical and a theoretical disadvantage: firstly, the time taken to value the portfolio N times; secondly, that all the historical scenarios are treated equally and hence the distant historical data has the same influence as recent data. The latter problem can be overcome by skewing the historical sampling so that recent data is more heavily weighted. Such a method is described in Section 6.3.5.

6.3.3. Simulated Risk Factors. Adaptiv provides three methods for generating simulated risk-factor values from historical data: *proportional*, *additive* and *historical value*. The proportional method is suitable for positive risk factors; the additive method can be applied to risk factors taking either positive or negative values. The proportional and additive methods are either *non-cumulative* (typical) or *cumulative*. Under the proportional methods, zero and negative values are treated as missing, as described in Section 6.3.4.

Let $x(t)$ denote the vector of risk-factor values at times $t = 1, \dots, T$ where time is scaled in business days (as in Section 6.2.3). Let h denote the number of days to the risk horizon. The scenarios are generated at a frequency of f days so that there are scenarios for each of the historical start dates $t_1, \dots, t_N$ where $t_N + h = T$ and $t_k = t_{k-1} + f$.

Under the non-cumulative proportional method, each simulated risk-factor value $\hat{x}_i(t)$ is the current value x_i multiplied by the proportional h-day change in the historical values:

$$\hat{x}_i(t) = x_i \frac{x_i(t+h)}{x_i(t)}.$$

Under the non-cumulative additive method, the simulated risk-factor value is the sum of the current value and the h-day historical change:

$$\hat{x}_i(t) = x_i + [x_i(t+h) - x_i(t)].$$

The historical-value method sets each simulated risk-factor value to its corresponding historical value at the risk horizon, so that $\hat{x}(t) = x(t+h)$.

Under the cumulative proportional method, the simulated value at time t is equal to the previous simulated value, at time $t - f$, multiplied by h-day proportional change at time t:

$$\hat{x}_i(t) = \hat{x}_i(t-f) \frac{x_i(t+h)}{x_i(t)}.$$

Similarly, the cumulative-additive simulated values are given by

$$\hat{x}_i(t) = \hat{x}_i(t-f) + [x_i(t+h) - x_i(t)].$$

6.3.4. Missing Historical Data. When calculating each historical return $R = x_i(t+h)/x_i(t)$, or difference $D = x_i(t+h) - x_i(t)$, the following rules are applied when missing historical values are encountered. The effect is to skip over missing data where possible, or substitute a null return or difference. Under the proportional methods, non-positive values of x_i are treated as missing.

If the start value $x_i(t)$ is missing then set $R = 1$ or $D = 0$. If the end value $x_i(t+h)$ is missing then set the end value to the first non-missing value after $t + h$ if one exists, or otherwise set $R = 1$ or $D = 0$.

6.3.5. Weighted Historical Sampling. The historical sampling described in Section 6.3.2 can be generalized by assigning a probability weighting $\mu(t)$ to each simulated risk-factor value $\hat{x}(t)$. If $0 \le \mu(t) \le 1$ for all t and $\sum_{t=1}^{N} \mu(t) = 1$ then

$$\sum_{t=1}^{N} \mu(t) g(\hat{x}(t))$$

is an unbiased estimator of $\mathbb{E}(g(x))$ and has variance

$$\mathrm{var}(g(x)) \sum_{t=1}^{N} \mu(t)^2.$$

The distribution of $V(x)$ is estimated by

$$\mathbb{P}(V(x) \le v) = \mathbb{E}(1_{\{V(x) \le v\}}) \simeq \sum_{\{t : V_t \le v\}} \mu(t).$$

Hence the value-at-risk for confidence probability p is equal to $V_{\tau(k)}$, where k is greatest integer for which

$$\sum\nolimits_{\ell=1}^{k} \mu(\tau(\ell)) < 1 - p.$$

The variation reduction factor $\sum_{t=1}^{N} \mu(t)^2$ is minimized by choosing unweighted sampling, that is $\mu(t) = 1/N$ for all t, but this has the disadvantage of giving equal weight to all scenarios. An alternative weighting is defined by

$$\mu(t) = \frac{\lambda^{N-t}}{\sum_{s=1}^{N} \lambda^{N-s}} = \lambda^{N-t} \left(\frac{1-\lambda}{1-\lambda^N} \right),$$

where $0 < \lambda < 1$. The factor λ reduces of the weighting of distant historical data relative to recent data in a similar way to the EWMA model of Section 6.2.3. The variance reduction factor converges downward to $(1-\lambda)/(1+\lambda)$ as $N \to \infty$, and hence the variance of the estimator cannot be made arbitrarily small by increasing the number of historical samples. For example, with $N = 250$ and $\lambda = 0.985$, the most recent data ($t = N$) has weight 1.535%, the most distant data ($t = 1$) has weight 0.036%, and the variance reduction factor is 0.79% (with lower bound 0.76% for this value of λ) compared with the unweighted variance reduction of $1/250 = 0.4\%$.

This method increases the overall noise in the VaR estimate but the noise is concentrated in the recent history. The effect of recent large market-rate movements is magnified, and the effect of distant historical events is greatly diminished. In particular, large market-rate movements do not cause the VaR to jump when they fall out of the 250-day sample.

6.4. Monte Carlo VaR

6.4.1. Introduction. Monte Carlo value-at-risk is the method of estimating the P&L distribution by valuing the portfolio under a number of simulated movements in market risk-factors, where the simulated risk-factor values are random samples of the historical risk-factor distribution. The calculation of the simulated risk-factor values proceeds in several stages; these are described in Sections 6.4.2–6.4.7, illustrated in Figure 6.4, and the calculations are replicated in the spreadsheet `FG_MCRFOneStep.xls`. Once the simulated risk-factor values have been obtained, the P&L distribution can be estimated. This is done either directly (the non-parametric method) or by fitting to a standard distribution.

6.4.2. Generating Correlated Uniform Variables.

6.4.2.1. *Independent Uniform Variables.* Adaptiv's random number generator (described in Appendix K) is used to generate a sequence of random n-dimensional vectors $U(1), \dots, U(N)$, where $U(j) = (U_1(j), \dots, U_n(j))$ and n is the number of risk factors. In the following sections, the $U_i(j)$ are assumed to be independent samples of the uniform distribution on the unit interval $[0, 1]$.

Figure 6.5 shows a random sequence of 1000 samples from the unit square $[0, 1] \times [0, 1]$.

6.4.2.2. *Linear Correlation.* The joint distribution of a vector of standard normal variables $Z = (Z_1, \dots, Z_n)$ is determined by its correlation matrix $\mathrm{cov}(Z_i, Z_j) = \mathbb{E}(Z_i Z_j)$. The standard method of generating correlated random variables from independent random variables is to map them to independent normal variables and

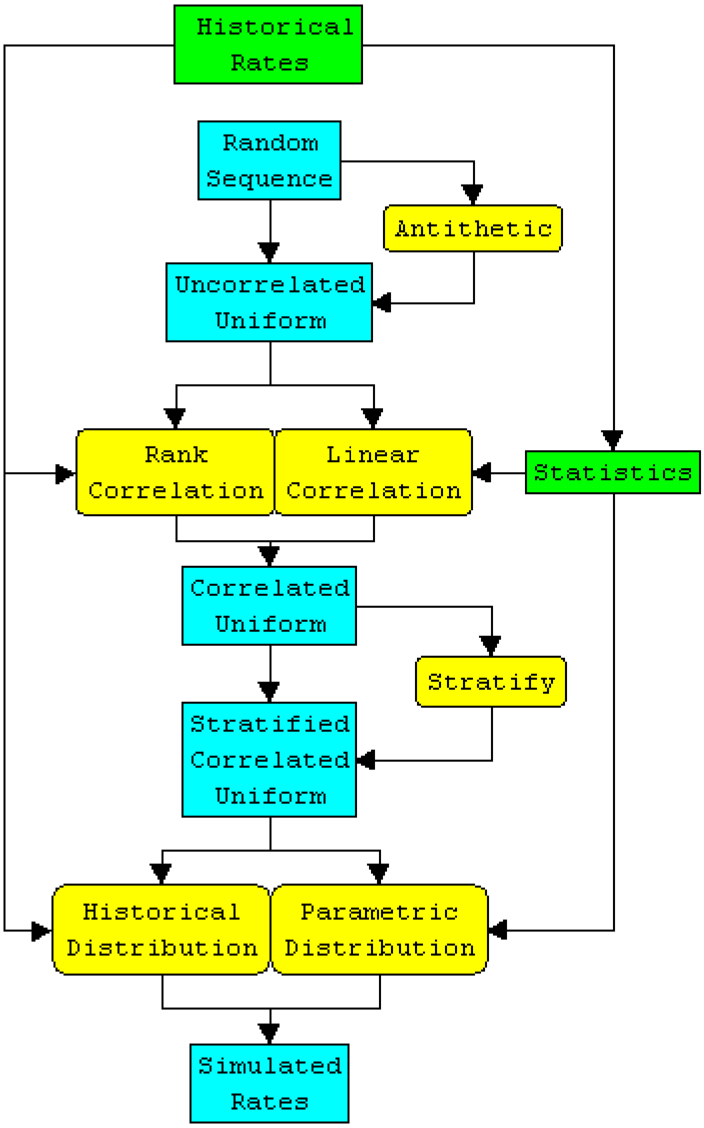


FIGURE 6.4. Monte Carlo simulation of risk-factor values for a single time step.

then take linear combinations of these to obtain normal variables with the required correlation matrix.

Let F be a continuous distribution function. If U is uniformly distributed on $[0, 1]$ then the random variable $F^{-1}(U)$ has the distribution F. Conversely, if X has the distribution function F then $F(X)$ is uniformly distributed on $[0, 1]$. In particular, the standard normal distribution function $\mathcal{N}$ maps standard normal variables to uniform variables, and its inverse $\mathcal{N}^{-1}$ maps uniform variables to standard normal variables.

Let C be a $n \times n$ correlation matrix. Then C is a positive matrix and has the decomposition $C = DD^*$, where D is an $n \times m$ matrix, m is the rank of C and D^* denotes the transpose of D. It follows that if Z is a vector of m independent

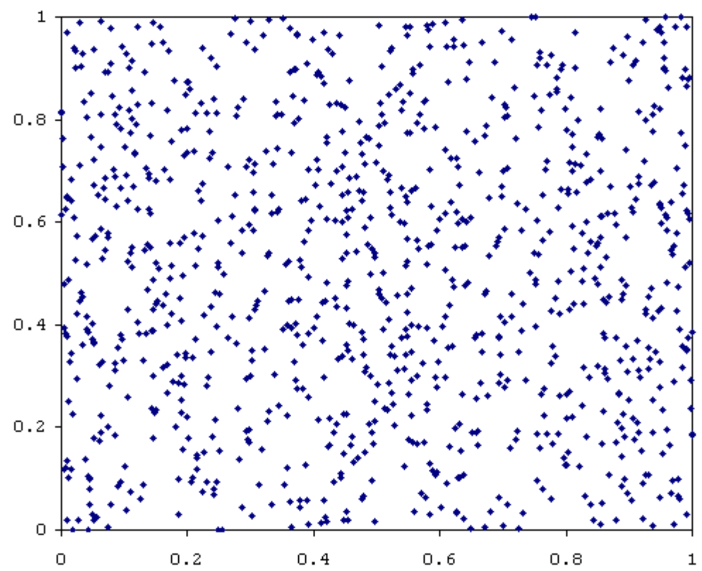


FIGURE 6.5. a two-dimensional random sequence.

standard normal variables then $\tilde{Z} = DZ$ is vector of n standard normal variables with $\operatorname{cov}(\tilde{Z}_i, \tilde{Z}_j) = C_{ij}$. The decomposition $C = DD^*$ is not unique but there are two unique decompositions of particular interest. Firstly, there is a unique *positive* D such that $C = D^2$. Secondly, if C is positive definite (positive with rank equal to n) then there is a unique lower-triangular matrix L such that $C = LL^*$. The lower-triangular decomposition is known as Cholesky decomposition and is much simpler to calculate than the positive decomposition.

Adaptiv provides two decomposition methods: Cholesky decomposition and weighted-return decomposition. If C is positive definite then Cholesky decomposition can be used. If C is the correlation matrix of n risk factors and is derived from a series of N historical returns then there exists a $n \times N$ matrix of weighted returns R such that $C = RR^*$. If $n > N$ then C is not positive definite and weighted-return decomposition should be used; it can also be applied when $n \leq N$ and C is not positive definite for other reasons.

Thus, suppose the correlation matrix C has the decomposition $C = DD^*$, where D is a $n \times m$ matrix. Let U be a vector of m independent uniform variables. Then $Z_i = \mathcal{N}^{-1}(U_i)$ defines a vector of independent standard normal variables; $\tilde{Z} = DZ$ is a vector of n standard normal variables with $\operatorname{cov}(\tilde{Z}_i, \tilde{Z}_j) = C_{ij}$; and $\tilde{U}_i = \mathcal{N}(\tilde{Z}_i)$ defines a vector of correlated uniform variables. This mapping, of independent to correlated uniform variables, is summarized as follows:

$$U \xrightarrow{\mathcal{N}^{-1}} Z \xrightarrow{D} \tilde{Z} \xrightarrow{\mathcal{N}} \tilde{U}.$$

The normal distribution function $\mathcal{N}$ and its inverse $\mathcal{N}^{-1}$ can be evaluated by polynomial approximations (see the Visual Basic functions embedded in the spreadsheet `FG_MCRFOneStep.xls`).

6.4.2.3. *Cholesky Decomposition.* If C is a positive definite matrix then its Cholesky decomposition is the unique lower-triangular matrix L satisfying $C = LL^*$.

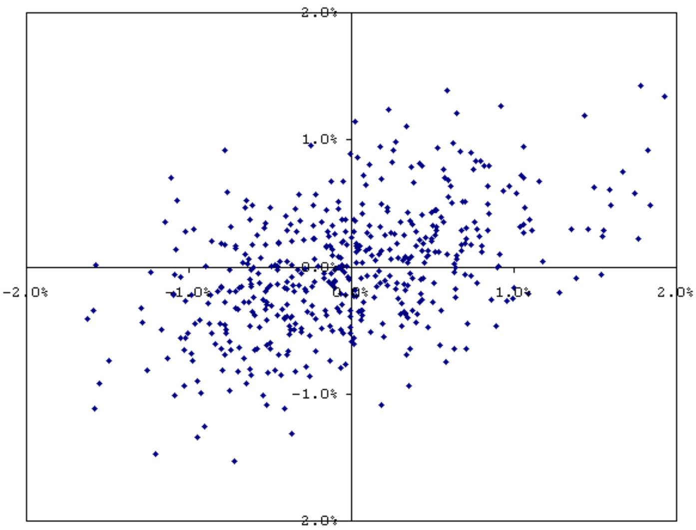


FIGURE 6.6. 500 simulated returns for EUR:USD and GBP:USD with linear correlation of 0.52.

The decomposition is given explicitly by

$$L_{ii} = \left(C_{ii} - \sum\nolimits_{k<i} L_{ik}^2\right)^{1/2}$$
$$L_{ji} = \frac{1}{L_{ii}} \left(C_{ij} - \sum\nolimits_{k<i} L_{ik} L_{jk}\right) \quad \text{for} \quad j > i.$$

Press et al. [29] remark that attempting to calculate the Cholesky decomposition provides an efficient test for positive definiteness. The calculation fails at the first i for which $C_{ii} \leq \sum_{k<i} L_{ik}^2$, in which case the matrix is not positive definite. Adaptiv tests for positive definiteness in this way: by reporting any failure in the calculation of Cholesky decomposition.

Cholesky decomposition is implemented in the spreadsheet `FG_Cholesky.xls`.

6.4.2.4. *Weighted-Return Decomposition.* Suppose the correlation matrix is given by

$$C_{ij} = \frac{1}{\sigma_i \sigma_j} \sum\nolimits_{k=1}^{N} \omega_k r_i(k) r_j(k),$$

where $r_i(k)$ are the historical returns centred about the average return, and

$$\sigma_i^2 = \sum\nolimits_{k=1}^{N} \omega_k r_i(k)^2.$$

In the simple moving average model, $\omega_k = 1/N$ so that the returns are equally weighted. In the exponentially weighted moving average model, the weighting factors are given by $\omega_k = \lambda^{N-k}$, where λ is the decay factor (see Section 6.2.3).

Then it follows that the correlation matrix has the decomposition $C = RR^*$, where R is the $n \times N$ matrix defined by

$$R_{ik} = \frac{\sqrt{\omega_k} r_i(k)}{\sigma_i}.$$

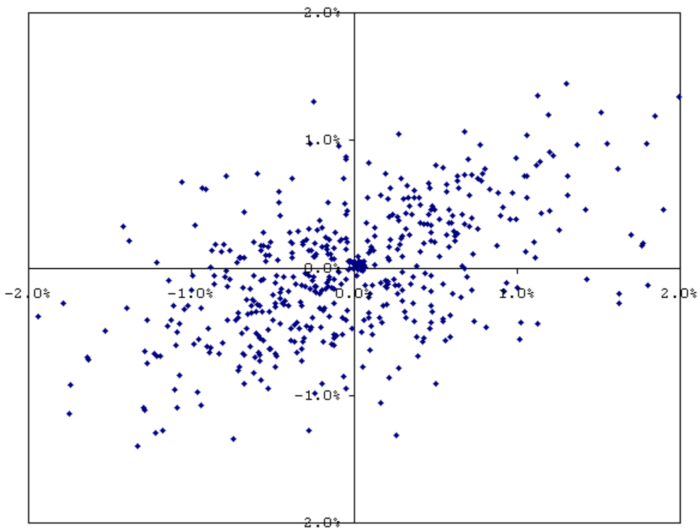


FIGURE 6.7. simulated returns for EUR:USD and GBP:USD with rank correlation based on 500 days of historical data.

The decomposition remains valid when there is missing data because the correlation matrix is originally calculated from vectors of weighted returns that are adjusted for missing data, as described in Section 6.2.3.

C is a positive matrix because $v \cdot Cv = R^*v \cdot R^*v \geq 0$ for all vectors v. If $N < n$ then the n rows of R are linearly dependent so that there exists a non-zero vector v such that $R^*v = 0$, and hence $Cv = R(R^*v) = 0$. Therefore C is not invertible, and hence not positive definite, when the number of returns is less than the number of factors. Conversely, it does *not* follow that C is automatically positive definite when $N \geq n$ because there may be linear dependence in the series of returns (and hence in the rows of R).

The weighted-return decomposition is implemented in the spreadsheet `FG_VaRStats.xls`.

6.4.2.5. *Rank Correlation.* Linear correlation is based on pairwise correlation coefficients and fails to capture two effects: higher-order correlation, where groups of risk factors tend to move together; and conditional correlation, where market rates become strongly correlated during large movements such as market crashes. In order to model these more complicated non-linear effects, Adaptiv offers the option of capturing *rank correlation*. In this approach, suggested by Shaw [**35**], the ordering of historical rate movements is replicated in the simulated rates. For example, consider two rates that usually behave independently but tend to crash together. This behaviour is captured by generating independent simulations of each rate and then ensuring that large downward movements are paired together in the same scenario. Figures 6.6 and 6.7 show the contrast between linear and rank correlation for two FX rates; rank correlation generates a greater concentration of simulated returns along the diagonal.

For a precise description of the method, consider a single risk factor and let $r(t)$ denote the historical h-day return $\log x(t) - \log x(t-h)$ defined for $t \geq 1$, where h is a specified horizon. Let ρ be a permutation of $\{1, \dots, N\}$ that orders the historical

		Scenario					
		1	**2**	**3**	**4**	**5**	$\cdots$
Risk Factor	**1**	15	21	35	23	18	$\cdots$
	2	40	48	12	24	2	$\cdots$
	3	7	23	27	13	15	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

FIGURE 6.8. ranking historical returns.

risk-factor returns, so that

$$r(\rho(1)) \leq \cdots \leq r(\rho(N)).$$

The *rank* of $r(t)$ is its sequence number in the ordered list of returns. The smallest value of $r(t)$ has rank 1 and the largest has rank N. The rank of $r(t)$ is equal to $\rho^{-1}(t)$, where ρ^{-1} denotes the inverse of the permutation ρ. Let υ be a permutation of $\{1, \ldots, N\}$ that orders the independent uniform variables:

$$U(\upsilon(1)) \leq \cdots \leq U(\upsilon(N)).$$

The correlated uniform variables are defined by

$$\tilde{U}(t) = U(\upsilon(\rho^{-1}(t))),$$

so that $\tilde{U}(t)$ is the value of $U(j)$ that has the same rank as the historical return $r(t)$.

The same mapping is applied to each risk factor in turn. In this way, the ranking of the historical returns is replicated in the ranking of the uniform variables. This is illustrated in Figure 6.8, where the ranks are: 15 for the first risk factor in the first scenario, 40 for the second risk factor in the first scenario, and so on.

There are two limitations to this method. Firstly, the number of scenarios that can be generated is bounded by the number of days of historical data. In practice, this is the number of days of historical data that the user judges it reasonable to use. Thus, rank correlation suffers from the same disadvantage as 'pure' historical value-at-risk. Secondly, unlike linear correlation, rank correlation disrupts the independence of the sample vectors; that is, $\tilde{U}(k)$ is not independent of $\tilde{U}(\ell)$.

6.4.3. Multi-Step Evolution of Risk-Factor Values.

6.4.3.1. *Introduction.* The risk-factor values or log risk-factor values are evolved in discrete steps according to one of the following methods: Normal, Mean Reverting, Non-Normal and Historical. The Normal and Mean Reverting methods are discrete approximations of continuous-time normal processes. Typically, equity and FX rates are evolved lognormally, and interest rates are evolved with mean-reverting lognormal processes. The calculation of the simulated risk-factor values is replicated in the spreadsheet `FG_VarRisk.xls`.

6.4.3.2. *Statistics.* Let $x = (x_1, \ldots, x_n)$ denote the vector of risk-factor values. Define $y_i = \log x_i$ for lognormal factors or $y_i = x_i$ for normal factors.

Let C and m denote the covariance matrix and mean vector of the historical h-day changes in y, where h belongs to the set of horizons for which statistics have been derived. For each time step in the evolution, Adaptiv uses the horizon h which is closest to the length of the time step. The drift vector μ, volatility vector σ and correlation matrix ρ are then defined by $m_i = \mu_i h$, $C_{ii} = \sigma_i^2 h$ and $C_{ij} = \sigma_i \sigma_j \rho_{ij} h$.

The correlation matrix can be written $\rho_{ij} = \lambda_i \cdot \lambda_j$, where λ_i is the i^{th} row of the $n \times m$ decomposition matrix. If Cholesky decomposition is used then $m = n$; otherwise, weighted-return decomposition is used and m is the number of historical returns.

6.4.3.3. *Normal.* For each risk factor, the value y_i is evolved according to the normal process

$$y_i(t) = y_i(0) + \mu_i t + \sigma_i \lambda_i \cdot W(t), \tag{6.10}$$

where $W(t)$ is a standard m-dimensional Brownian motion. Hence a discrete change in y_i is given by

$$\Delta y_i = \mu_i \Delta t + \sigma_i \sqrt{\Delta t} \tilde{Z}_i, \tag{6.11}$$

where Δt is the time step, $\tilde{Z}_i = \lambda_i \cdot Z$ and Z is a vector of m independent normal variables.

The discrete evolution of y is implemented by generating the vector $\tilde{Z}$ of correlated normal variables using either linear or rank correlation, as described in Section 6.4.2.

The user may choose to suppress the historical drift; in which case μ_i is replaced by $-\sigma_i^2/2$ so that the proportional change in x_i is unbiased, that is $\mathbb{E}(\exp(\Delta y_i)) = 1$.

6.4.3.4. *Non-Normal.* By analogy with Equation (6.11), the value y_i is evolved as

$$\Delta y_i = \mu_i \Delta t + \sigma_i \sqrt{\Delta t} F^{-1}(\tilde{U}_i),$$

where F^{-1} is the inverse distribution function of a RTDM distribution with mean equal to 0 and variance equal to 1 (see Appendix J), and $\tilde{U}$ is a vector of correlated uniform variables generated using either linear or rank correlation. The parameters of the RTDM distribution are derived from the historical skewness and kurtosis of h-day changes in y_i.

The user may choose to set the historical drift to zero. In this case, the evolution is defined by

$$\Delta y_i = X_i - \log \mathbb{E}(\exp(X_i)),$$

where X_i denotes $\sigma_i \sqrt{\Delta t} F^{-1}(\tilde{U}_i)$, so that the proportional change in x_i is unbiased. The value of $\mathbb{E}(\exp(X_i))$ is calculated using the approximation

$$\mathbb{E}(\exp(X_i)) \simeq 1 + \frac{1}{2!}\mathbb{E}(X_i^2) + \frac{1}{3!}\mathbb{E}(X_i^3) + \frac{1}{4!}\mathbb{E}(X_i^4).$$

6.4.3.5. *Mean Reverting.* The value y_i is evolved according to the Ornstein-Ulhenbeck process

$$dy_i(t) = \alpha_i(\beta_i - y_i)dt + \gamma_i \lambda_i \cdot dW(t), \tag{6.12}$$

where α, β and γ are constant vectors and $\alpha_i > 0$. This process is mean-reverting in the sense that the drift term tends to pull y_i back to the *long-run mean* β_i. Moreover, the stochastic differential equation has the explicit solution

$$y_i(t) = \mathrm{e}^{-\alpha_i t}\, y_i(0) + \left(1 - \mathrm{e}^{-\alpha_i t}\right)\beta_i + \gamma_i \int_0^t \mathrm{e}^{-\alpha_i (t-s)}\, \lambda_i \cdot dW(s)$$

(see, for example, Lamberton and Lapeyre [**23**]) and it follows that $y_i(t)$ is a normal variable with mean

$$\mathbb{E}(y_i(t)) = \beta_i + \mathrm{e}^{-\alpha_i t}(y_i(0) - \beta_i)$$

converging to the long-run mean β_i as $t \to \infty$, and variance

$$\mathrm{var}(y_i(t)) = \gamma_i^2 \left(\frac{1 - \mathrm{e}^{-2\alpha_i t}}{2\alpha_i}\right)$$

converging to $\gamma_i^2/2\alpha_i$ as $t \to \infty$. The discrete change in y_i is simulated as

$$y_i(t+\Delta t) = \mathrm{e}^{-\alpha_i \Delta t}\, y_i(t) + \left(1 - \mathrm{e}^{-\alpha_i \Delta t}\right) \beta_i + \gamma_i \sqrt{\frac{1-\mathrm{e}^{-2\alpha_i \Delta t}}{2\alpha_i}}\, \tilde{Z}_i, \tag{6.13}$$

where $\tilde{Z}_i = \lambda_i \cdot Z$ and Z is a vector of independent normal variables.

For each risk factor, the values of the mean-reversion rate α_i, long-run mean β_i, and volatility γ_i are derived from historical changes in y_i by calculating the linear regression of Δy_i with respect to y_i; this is because the discrete h-day approximation of Equation (6.12) is

$$\Delta y_i = \alpha_i(\beta_i - y_i)h + \gamma_i \sqrt{h} e,$$

where e is a variable with $\mathbb{E}(e) = 0$, $\mathrm{var}(e) = 1$ and $\mathrm{cov}(e, y_i) = 0$ (see Appendix H).

For lognormal factors, the mean of the factor value $x_i(t) = \exp(y_i(t))$ is given by

$$\mathbb{E}(x_i(t)) = \exp\left(\mathbb{E}(y_i(t)) + \tfrac{1}{2}\,\mathrm{var}(y_i(t))\right),$$

and hence the long-run mean of $x_i(t)$ is

$$\lim_{t\to\infty} \mathbb{E}(x_i(t)) = \exp\left(\beta_i + \gamma_i^2/4\alpha_i\right).$$

6.4.3.6. *Historical. Historical distribution* is the method of generating simulated risk-factor values by taking correlated random selections from the historical risk-factor returns.

The simulated values are based on the random samples $\tilde{U}(j)$ of correlated uniform variables and the historical h-day returns $r_i(t) = x_i(t)/x_i(t-h)$. A random index between 1 and N is generated by rounding up $\tilde{U}_i(j)N$ to the nearest integer. The simulated return on the i^{th} risk-factor value on the j^{th} scenario is defined to be $r_i(k+1)$, where k is the integer part of $\tilde{U}_i(j)N$.

6.4.4. Yield Curve Factors.

6.4.4.1. *Scenarios on Zero-Coupon Rates.* The risk factors that describe movements in the yield curve are zero-coupon rates at certain grid points on the curve. The grid points are specified as a set of terms, for example: the 3m zero rate, the 6m zero rate, etc. The yield curve itself is represented by the table of discount factors returned by the bootstrapping method (see Section 1.2.2).

The following method is used to apply a perturbation of the risk factors to the yield curve. Suppose the grid dates for the risk factors are $D_1, \ldots, D_n$, and the shifts in the zero-coupon rates at these dates are $\Delta z_1, \ldots, \Delta z_n$. The calculation of the shifted discount factor at date D proceeds as follows.

(1) Obtain the discount factor Z at date D using the interpolation method assigned to the yield curve (see Section 1.2.3).
(2) Calculate the corresponding zero-coupon rate z at date D. The zero-coupon rates are defined by either $Z = \exp(-zt)$ for continuously compounded rates, or
$$Z = \left(1 + \frac{z}{h}\right)^{-ht}$$
for rates discretely compounded at frequency h, where t is the time to date D.

(3) Calculate the shift in the zero-coupon rate at date D by linear interpolation between the shifts at adjacent grid dates:

$$\Delta z = \frac{(D_{i+1} - D)\Delta z_i + (D - D_i)\Delta z_{i+1}}{D_{i+1} - D_i},$$

where $D_i \leq D \leq D_{i+1}$.

(4) Convert the shifted zero-coupon rate $z + \Delta z$ back to a discount factor $Z + \Delta Z$ using the above formulae.

6.4.4.2. *Mean Reversion Parameters.* The mean-reverting lognormal process of Section 6.4.3 is suitable for yield-curve risk factors. The following method can be used to calibrate the long-run mean and mean-reversion rate of each zero-rate factor to the observed forward rates.

Let z_τ denote a risk factor, where τ is the term of the zero-coupon rate in years (for example, $\tau = 1/2$ for a 6m rate). Then $\log z_\tau(t)$ is assumed to follow an Ornstein-Ulhenbeck process with mean-reversion rate α_τ and long-run mean β_τ, and the mean of $\log z_\tau(t)$ is given by

$$\mathbb{E}(\log z_\tau(t)) = \beta_\tau + \mathrm{e}^{-\alpha_\tau t}(z_\tau(0) - \beta_\tau)$$

Let Z_t denote the discount factor from the current yield curve at time t. The forward starting zero-coupon rate $f_\tau(t)$ for the period from t to $t + \tau$ is given by

$$\frac{Z_{t+\tau}}{Z_t} = \begin{cases} \exp\left(-f_\tau(t)\tau\right) & \text{continuous compounding, or} \\ \left(1 + f_\tau(t)/h\right)^{-h\tau} & \text{discrete compounding at frequency } h. \end{cases}$$

The difference between the expected log rates $\mathbb{E}(\log z_\tau(t))$ and log forward rates $\log f_\tau(t)$ can be minimized for a discrete sample of time points $t_1, \ldots, t_n$. The error

$$\sum\nolimits_i \left(\mathbb{E}(\log z_\tau(t_i)) - \log f_\tau(t_i)\right)^2$$

is a known function of α_τ and β_τ, and can be minimized using a suitable numerical method, such as downhill simplex or Powell's method (see Press et al. [**29**]).

6.4.5. Estimating the P&L Distribution.

6.4.5.1. *Monte Carlo Estimates.* Let x denote the random vector of risk-factor values at the risk horizon. The simulated risk factors $\hat{x}(1), \ldots, \hat{x}(N)$ are generated as independent samples of the distribution of x.

Let g be a real-valued function of x with finite variance. The Monte Carlo (MC) estimate of $\mathbb{E}(g(x))$ is defined by

$$\langle g \rangle = \frac{1}{N} \sum\nolimits_{j=1}^{N} g(\hat{x}(j)) \tag{6.14}$$

and has the following two properties. Firstly, $\langle g \rangle$ is an unbiased estimator of $\mathbb{E}(g(x))$, that is $\mathbb{E}\langle g \rangle = \mathbb{E}(g(x))$, and secondly the variance of $\langle g \rangle$ is given by

$$\operatorname{var}\langle g \rangle = \frac{1}{N} \operatorname{var}(g(x))$$

Hence $\langle g \rangle$ converges $\mathbb{E}(g(x))$ in the following sense: the standard deviation of the error $\langle g \rangle - \mathbb{E}(g(x))$ converges to zero like $1/\sqrt{N}$.

6.4.5.2. *Non-Parametric Distribution.* The distribution function of the portfolio value $V(x)$ is given by

$$\mathbb{P}(V(x) \leq v) = \mathbb{E}(\chi(x))$$

where $\chi(x)$ is the indicator function of the set $\{V(x) \leq v\}$. Therefore $\langle\chi\rangle$ is the Monte Carlo estimate of $\mathbb{P}(V(x) \leq v)$, and $\langle\chi\rangle$ is equal to the fraction of simulated portfolio values less or equal to v:

$$\frac{1}{N}\left|\{j : V(\hat{x}(j)) \leq v\}\right| \tag{6.15}$$

Let V_j denote $V(\hat{x}(\pi(j)))$, where π is a permutation of $\{1, \ldots, N\}$ that orders the simulated portfolio values, so that $V_1 \leq \cdots \leq V_N$. Then V_k is the value-at-risk for confidence probability $1 - k/N$. For example, if there are 1000 simulated risk-factor values then V_{10}, the tenth smallest portfolio value, is the value-at-risk for confidence probability 99%.

6.4.5.3. *Normal Distribution.* The value-at-risk is calculated under the assumption that P&L distribution is normal with mean and variance obtained from Monte Carlo estimates. The MC estimate of $\mu = \mathbb{E}(V(x))$ is $\hat{\mu} = \langle V\rangle$, and the MC estimate of $\sigma^2 = \mathbb{E}\left((V(x) - \mu)^2\right)$ is

$$\hat{\sigma}^2 = \frac{N}{N-1}\left(\langle V^2\rangle - \langle V\rangle^2\right)$$

The factor $N/(N-1)$ makes $\hat{\sigma}^2$ an unbiased estimator of σ^2. Then, under a normal P&L distribution, the value-at-risk for confidence level p is given by

$$\hat{\mu} + \mathcal{N}^{-1}(1-p)\hat{\sigma}$$

where $\mathcal{N}$ is the standard normal distribution function.

6.4.5.4. *Non-Normal Distribution.* The value-at-risk is calculated under the assumption that P&L distribution has the RTDM distribution with mean, variance, skewness and kurtosis obtained from Monte Carlo estimates.

The MC estimate of the skewness $\mathbb{E}\left((V(x) - \mu)^3\right)$ is

$$\langle V^3\rangle - 3\langle V^2\rangle\langle V\rangle + 2\langle V\rangle^3$$

and the MC estimate of the kurtosis $\mathbb{E}\left((V(x) - \mu)^4\right) - 3\mathbb{E}\left((V(x) - \mu)^2\right)^2$ is

$$\langle V^4\rangle - 4\langle V^3\rangle\langle V\rangle - 3\langle V^2\rangle^2 + 12\langle V^2\rangle\langle V\rangle^2 - 6\langle V\rangle^4$$

The value-at-risk for confidence probability p is

$$\lambda_1 + \frac{(1-p)^{\lambda_3} - p^{\lambda_4}}{\lambda_2}$$

where λ_1, λ_2, λ_3 and λ_4 are the parameters of the RTDM distribution determined from MC estimates of the mean, variance, skewness and kurtosis (see Appendix J).

6.4.5.5. *Extreme Value Distribution.* Since value-at-risk is usually calculated for a 95% confidence or greater, only the tail of the P&L distribution is required for its calculation. Adaptiv provides a method of fitting the simulated portfolio values in the tail to a standard fat-tailed distribution.

Let $V_1 \leq \cdots \leq V_N$ denote the ordered simulated portfolio values. The extent of the tail is specified by a parameter called the threshold number, denoted by m. The lower tail of the P&L distribution is fitted to a Pareto distribution, given by

$$\mathbb{P}\left(V(x) \leq v\right) = \frac{m}{N}\left(\frac{V_{m+1} - \bar{V}}{v - \bar{V}}\right)^{\alpha} \tag{6.16}$$

for $v < V_{m+1}$, where $\bar{V} = V_{N/2}$ is the median of the simulated portfolio values. The total probability below the start of the tail V_{m+1} is equal to m/N. The parameter α is Hill's estimator and is defined by

$$\frac{1}{\alpha} = \frac{1}{m} \sum\nolimits_{i=1}^{m} \log\left(\frac{V_i - \bar{V}}{V_{m+1} - \bar{V}}\right)$$

(see, for example, Danielsson and de Vries [12]). Under this distribution, the VaR for confidence probability p is equal to

$$\bar{V} + (V_{m+1} - \bar{V})\left(\frac{m}{(1-p)N}\right)^{1/\alpha}$$

6.4.5.6. *Expected Shortfall.* The value-at-risk is a probabilistic upper bound on the potential loss in portfolio value. Shortfall analysis provides an estimate of the average of the losses that exceed this bound.

The *expected shortfall* for confidence probability p is defined (for example, in Jorion [21]) to be the expected P&L given that the P&L is less then the value-at-risk, that is

$$\mathbb{E}(V(x)|V(x) \leq v) = \frac{\mathbb{E}(V(x)1_{\{V(x)\leq v\}})}{1-p}$$

where $\mathbb{P}(V(x) \leq v) = 1 - p$. If the P&L distribution has a density function f then the expected shortfall is given by $(1-p)^{-1}\int_{-\infty}^{v} uf(u)du$.

For the non-parametric distribution, the expected shortfall is equal to

$$\frac{1}{(1-p)N} \sum\nolimits_{j=1}^{(1-p)N} V_j$$

which is the average of the simulated portfolio values less than or equal to the value-at-risk $V_{(1-p)N}$.

For the normal distribution, $\hat{\sigma} f(\hat{\sigma} z + \mu) = \exp(-z^2/2)/\sqrt{2\pi}$ and the expected shortfall is equal to

$$\hat{\mu} - \frac{\hat{\sigma}\exp(-(v-\hat{\mu})^2/2\hat{\sigma}^2)}{(1-p)\sqrt{2\pi}}$$

In the case of the RTDM distribution, the inverse distribution function F^{-1} is given by Equation (J.1) and it follows from the identity $\int_{-\infty}^{v} uf(u)du = \int_{0}^{1-p} F^{-1}(q)dq$ that the expected shortfall is equal to

$$\lambda_1 + \frac{1}{(1-p)\lambda_1}\left(\frac{(1-p)^{1+\lambda_3}}{1+\lambda_3} + \frac{p^{1+\lambda_4} - 1}{1+\lambda_4}\right)$$

The extreme value distribution has density function $f(u) = \alpha F(u)\left(\bar{V} - u\right)^{-1}$ for $u < V_{m+1}$, where $F(v)$ is the right-hand side of Equation (6.16). If $\alpha > 1$ then $\int_{-\infty}^{v} uf(u)du$ is finite and the expected shortfall is equal to

$$\bar{V} + \frac{\alpha}{\alpha - 1}(V_{m+1} - \bar{V})\left(\frac{m}{(1-p)N}\right)^{1/\alpha}$$

The spreadsheet `FG_MCPLDist.xls` replicates Adaptiv's calculation of P&L distribution and expected shortfall.

6.4.6. Theta and Instrument Ageing. The Market Simulation report calculates the simulated valuations at a future horizon date, and hence captures theta effects such as option time decay. The valuation at the horizon date is adjusted for realised cashflows, rate fixings, option exercise and other trade events via the process of *instrument ageing* described in Section 6.7.

6.4.7. Variance Reduction.

6.4.7.1. *Introduction.* Recall from Section 6.4.5 that the Monte Carlo estimate of $\mathbb{E}(g(x))$ with N simulations has variance $\text{var}(g(x))/N$. *Variance reduction* is the generic name given to a set of techniques that may, under certain conditions, give an estimate with lower variance than the standard Monte Carlo estimate for the same number of simulations, or improve on the convergence of the standard estimate by other means. Adaptiv offers two such methods: antithetic sampling, and Latin hypercube sampling.

6.4.7.2. *Antithetic Sampling.* Let x be a random vector whose distribution function F is symmetric about its mean μ, that is $F(\mu+\xi) = 1 - F(\mu-\xi)$ for all vectors ξ. Suppose $x(1), \ldots, x(N)$ are independent random samples of this distribution. The corresponding antithetic (mirror-image) samples are defined by $\bar{x}(j) = 2\mu - x(j)$. The antithetic MC estimate of $\mathbb{E}(g(x))$ is defined by

$$\frac{1}{2N} \sum\nolimits_{j=1}^{N} g(x(j)) + g(\bar{x}(j))$$

and has variance less than $\text{var}(g(x))/2N$, the variance of the standard MC estimate with $2N$ independent samples, if and only if $g(x)$ and $g(\bar{x})$ are negatively correlated. There are two extreme cases: if g is linear or antisymmetric about μ, that is $g(\mu+x) - g(\mu) = g(\mu) - g(\mu - x)$, then the variance of the antithetic estimate is zero and hence only one random sample is required to estimate $\mathbb{E}(g(x))$; if g is symmetric about μ, that is $g(\mu+x) = g(\mu-x)$, then the variance of the antithetic estimate is double that of the standard MC estimate with $2N$ samples.

The distribution of the risk-factor vector x is not necessarily symmetric about its mean but if linear correlation is used then each sample $x(j)$ is a deterministic function of the vector $U(j)$ of independent uniform variables. Hence antithetic samples are generated by setting $\bar{U}_i(j) = 1 - U_i(j)$.

Antithetic sampling must be used with care because there is no guarantee that the variance of the antithetic estimate is lower than that of the standard MC estimate.

6.4.7.3. *Latin Hypercube Sampling.* Latin hypercube sampling takes a sequence of N random samples of a n-dimensional distribution and *stratifies* the samples by dividing the n-dimensional hypercube into N^n cells and placing each sample in its own cell. This is done in such a way that variance reduction is achieved and the multivariate distribution is preserved (see Stein [**37**]).

Adaptiv applies Latin hypercube sampling to the random samples $\tilde{U}(j)$ of correlated uniform variables. Let π_i be a permutation of $\{1, \ldots, N\}$ that orders the i^{th} components of the sample, so that

$$\tilde{U}_i(\pi_i(1)) \leq \cdots \leq \tilde{U}_i(\pi_i(N))$$

Then $\pi_i^{-1}(j)$ is the rank of $\tilde{U}_i(j)$. The Latin hypercube samples $H(j)$ are defined by

$$H_i(j) = \frac{\pi_i^{-1}(j) - 1 + E_i(j)}{N}$$

where $E_i(j)$ are independent and uniformly distributed on the unit interval $[0, 1]$, and generated independently of $U_i(j)$. Hence $H_i(j)$ is a random selection from the sub-interval $((\rho - 1)/N, \rho/N)$, where ρ denotes the rank of $\tilde{U}_i(j)$. The risk-factor evolution then proceeds as in Section 6.4.3 using H in place of $\tilde{U}$.

6.5. Marginal, Component and Incremental Risk

6.5.1. Marginal Decomposition of Standard Deviation. Suppose a random variable X has the decomposition $X = \omega_1 X_1 + \cdots + \omega_m X_m$ where each X_k is a random variable and the ω_k are weighting factors. Two examples of such a decomposition are:

(1) The decomposition of the value change $\Delta V = \sum_k \Delta V_k$, where V is the value of the total portfolio and V_k is the value of a sub-portfolio, and in this case $\omega_k = 1$.
(2) The decomposition of the return on the total portfolio as $R = \sum_k \omega_k R_k$, where R_k is the return on a sub-portfolio and ω_k is the ratio of the value of the sub-portfolio to the value of the total portfolio.

Let σ denote the standard deviation of X. The *marginal standard deviations* with respect to this decomposition of X are the derivatives $\partial\sigma/\partial\omega_k$ of the standard deviation with respect to the weights. It follows from the identity

$$\sigma(X + \varepsilon X_k)^2 = \sigma(X)^2 + 2\varepsilon \operatorname{cov}(X, X_k) + \varepsilon^2 \sigma(X_k)^2 \tag{6.17}$$

that the marginal standard deviations are given by

$$\frac{\partial\sigma}{\partial\omega_k} = \frac{\operatorname{cov}(X, X_k)}{\sigma} = \beta_k \sigma,$$

where β_k is the linear regression coefficient of X_k with respect to X. For a small change in the weight ω_k,

$$\frac{\Delta\sigma}{\sigma} = \beta_k \Delta\omega_k,$$

and hence the proportional change in the total standard deviation is approximately equal to the beta multiplied by the change in weight.

The total standard deviation is the weighted sum of the marginal standard deviations:

$$\sigma = \omega_1 \frac{\partial\sigma}{\partial\omega_1} + \cdots + \omega_m \frac{\partial\sigma}{\partial\omega_m}.$$

Equivalently, the weighted sum of the linear regression coefficients is equal to 1:

$$1 = \omega_1 \beta_1 + \cdots + \omega_m \beta_m.$$

The *component standard deviation* is defined as $\omega_k \sigma_k$, so that the total standard deviation is the sum of its component standard deviations.

A marginal standard deviation may be negative, in which case the total standard deviation decreases as the corresponding weight increases; it may also be positive and greater than the total standard deviation, and such a large, positive marginal standard deviation is known as a *hot spot* (see Litterman [**24**]).

6.5.2. Simulation Marginal Standard Deviation. Consider a partition of the portfolio into sub-portfolios. The total value change has the decomposition

$$\Delta V = \Delta V_1 + \cdots + \Delta V_m,$$

where V is the value of the total portfolio and the V_k are the values of the sub-portfolios. This is a weighted decomposition of the random variable ΔV with weighting factors set to 1.

Monte Carlo or historical simulation provides an estimate of the covariance of ΔV and ΔV_k for each sub-portfolio; from which we calculate the marginal standard deviations

$$\frac{\operatorname{cov}(\Delta V, \Delta V_k)}{\sigma(\Delta V)},$$

where the total standard deviation $\sigma(\Delta V)$ is estimated in the same way.

The return on the portfolio $R = \Delta V/V$ has the decomposition

$$R = \omega_1 R_1 + \cdots + \omega_m R_m,$$

where $R_k = \Delta V_k/V_k$ is the return on the sub-portfolio and $\omega_k = V_k/V$ is the weight of the sub-portfolio by value. The marginal standard deviations of the return are

$$\frac{\operatorname{cov}(R, R_k)}{\sigma(R)}.$$

The marginal standard deviations of value changes and returns are closely related, and one can be derived from the other, because $\sigma(R) = \sigma(\Delta V)/V$ and

$$\operatorname{cov}(R, R_k) = \frac{\operatorname{cov}(\Delta V, \Delta V_k)}{V V_k}.$$

6.5.3. Relative Risk. To calculate risk relative to a benchmark portfolio, Adaptiv calculates a netted portfolio comprising the actual portfolio and a short position in the benchmark portfolio, where the benchmark portfolio is scaled to have the same value as the actual portfolio. Given a partitioning of the actual portfolio into sub-portfolios, Adaptiv generates a corresponding partition of the benchmark portfolio into sub-portfolios. The value change of the netted portfolio has the decomposition

$$\Delta V = \Delta V^a - \Delta V^b = \sum\nolimits_k \Delta V_k^a - \sum\nolimits_k \Delta V_k^b,$$

where V^a denotes the value of the actual portfolio and V^b denotes the value of the benchmark portfolio, and similarly for the sub-portfolios. The benchmark portfolio is scaled to have $V^b = V^a$, and hence the value of the netted portfolio is zero: $V = V^a - V^b = 0$.

Dividing all the value changes by $V^a = V^b$ gives the following decomposition of the relative return

$$R = R^a - R^b = \sum\nolimits_k \omega_k^a R_k^a - \sum\nolimits_k \omega_k^b R_k^b,$$

where $R_k^a = \Delta V_k^a / V_k^a$ is the return on the actual sub-portfolio and $\omega_k^a = V_k^a/V^a$ is its weight, and similarly for the benchmark portfolio.

The tracking error is the standard deviation of the relative return R. It has marginal standard deviations

$$\frac{\operatorname{cov}(R_k^a, R)}{\sigma(R)} \quad \text{and} \quad \frac{-\operatorname{cov}(R_k^b, R)}{\sigma(R)};$$

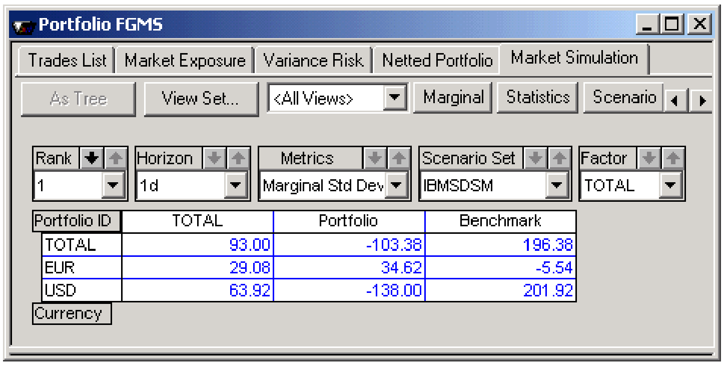


FIGURE 6.9. marginal standard deviation on the Market Simulation report.

and component standard deviations

$$\frac{\omega_k^a \operatorname{cov}(R_k^a, R)}{\sigma(R)} \quad \text{and} \quad \frac{-\omega_k^b \operatorname{cov}(R_k^b, R)}{\sigma(R)}.$$

Figure 6.9 shows a decomposition of tracking error into its component standard deviations. In this example, the returns are annualised and expressed in basis points.

6.5.4. Variance-Covariance Marginal Standard Deviation. Consider the linear approximation $\delta \cdot \Delta y$ of the total value change ΔV and suppose the exposure vector δ has the decomposition

$$\delta = \omega_1 \delta_1 + \cdots + \omega_m \delta_m,$$

where the δ_i are exposure vectors and the ω_i are weighting factors. Then the standard deviation of the linear approximation is $\sigma = \sqrt{\delta \cdot C\delta}$ and the marginal standard deviations with respect to the decomposition of δ are

$$\frac{\partial \sigma}{\partial \omega_k} = \frac{\delta_k \cdot C\delta}{\sigma}.$$

The Variance Risk report calculates marginal standard deviations with respect to two types of decomposition.

Portfolio Decomposition: The decomposition of the exposure vector is with respect to the partitioning of the portfolio into sub-portfolios, for example partitioning into currency and trader sub-portfolios.

Risk-Factor Decomposition: The exposure vector is decomposed into its risk-factor components:

$$\delta = (\delta_1, \ldots, 0) + \cdots + (0, \ldots, \delta_n).$$

In this case, the marginal standard deviations show the effect of increasing the exposure to individual risk factors. The marginal standard deviation for the i^{th} risk factor is equal to $\delta_i \sum_j C_{ij}\delta_j/\sigma$.

Figure 6.10 shows the risk-factor decomposition for a portfolio of equity positions.

The analysis of standard deviation into marginal contributions extends to the quadratic case. The quadratic approximation of ΔV is

$$\delta \cdot \Delta y + \tfrac{1}{2}\Delta y \cdot \Gamma \Delta y,$$

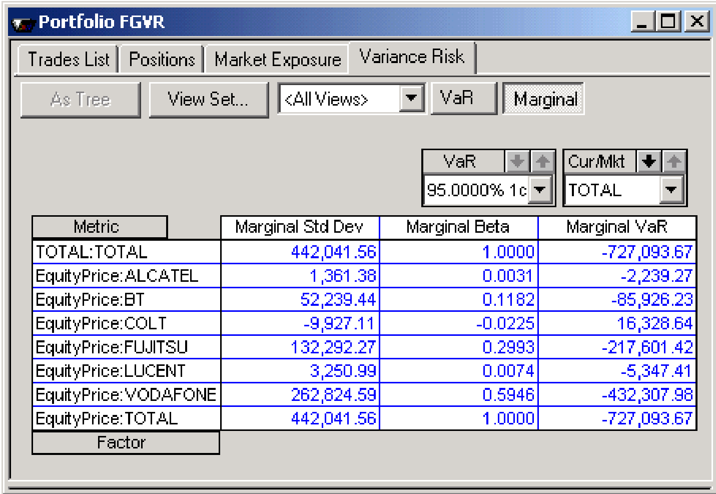


FIGURE 6.10. risk-factor decomposition on the Variance Risk report.

and its standard deviation is given by

$$\sigma^2 = \delta \cdot C\delta + \tfrac{1}{2}\operatorname{tr}\left((\Gamma C)^2\right)$$

(see Section 6.2.4). Suppose that the exposure vector has the decomposition $\delta = \omega_1\delta_1 + \cdots + \omega_m\delta_m$ and the exposure matrix has a corresponding decomposition

$$\Gamma = \omega_1\Gamma_1 + \cdots + \omega_m\Gamma_m.$$

Then the marginal standard deviations with respect to this decomposition are given by

$$\frac{\partial\sigma}{\partial\omega_k} = \frac{\delta_k \cdot C\delta + \frac{1}{2}\operatorname{tr}(\Gamma_k C\Gamma C)}{\sigma}.$$

The quadratic approximation of the return $R = \Delta V/V$ is

$$\hat{\delta} \cdot \Delta y + \tfrac{1}{2}\Delta y \cdot \hat{\Gamma}\Delta y,$$

where $\hat{\delta}_i = \delta_i/V$ and $\hat{\Gamma}_{ij} = \Gamma_{ij}/V$. Therefore the standard deviation and the marginal standard deviations can be calculated by replacing δ by $\hat{\delta}$ and Γ by $\hat{\Gamma}$.

For a portfolio of stocks, $V = \sum_i n_i S_i$, where n_i is the number of shares held in the stock with price S_i. In this case:

$$\delta_i = \frac{\partial V}{\partial(\log S_i)} = S_i\frac{\partial V}{\partial S_i} = n_i S_i$$

and hence $\hat{\delta}_i = n_i S_i/V$ is the fraction of the portfolio value held in the i^{th} stock (also known as the percentage *weight* of the stock).

6.5.5. Marginal and Component VaR. Let X denote either the value change ΔV or the return R, and suppose X has a weighted decomposition $X = \sum_k \omega_k X_k$.

Suppose X is normally distributed. Then the value-at-risk is given by

$$v(p) = \mu + \sigma\mathcal{N}^{-1}(1-p),$$

where μ and σ are the mean and standard deviation of X. The marginal standard deviations are given by $\partial\sigma/\partial\omega_k = \beta_k\sigma$, where $\beta_k = \operatorname{cov}(X, X_k)/\sigma^2$. The mean of

X has the decomposition $\mu = \sum_k \omega_k \mu_k$, where μ_k is the mean of X_k, and hence $\partial\mu/\partial\omega_k = \mu_k$. It follows that

$$\frac{\partial v(p)}{\partial \omega_k} = \mu_k + \beta_k(v(p) - \mu)$$

and

$$v(p) = \sum\nolimits_k \omega_k \frac{\partial v(p)}{\partial \omega_k}.$$

More generally, for non-normal P&L distributions, the marginal contributions to value-at-risk can be defined by

$$v_k(p) = \mu_k + \beta_k(v(p) - \mu),$$

which is an extension of the formula for $\partial v(p)/\partial\omega_k$ in the normal case. Then by definition, the value-at-risk has the decomposition $v(p) = \sum_k \omega_k v_k(p)$.

6.5.6. Incremental Risk. The *incremental* risk of a sub-portfolio with respect to the total portfolio is defined as the change in total risk that would occur if the sub-portfolio were removed from the total portfolio. Thus, incremental risk is the change in risk due to a discrete change in the portfolio content; whereas marginal risk is the sensitivity of the risk to small changes in the portfolio composition.

The incremental standard deviation of the total portfolio, with value change ΔV, with respect to the sub-portfolio, with value change ΔV_k, is given by

$$\sigma(\Delta V) - \sigma(\Delta V - \Delta V_k).$$

The standard deviation of $\Delta V - \Delta V_k$ can be expanded as

$$\begin{aligned}\sigma(\Delta V - \Delta V_k)^2 &= \sigma(\Delta V)^2 - 2\operatorname{cov}(\Delta V, \Delta V_k) + \sigma(\Delta V_k)^2 \\ &= (1 - 2\beta_k)\sigma(\Delta V)^2 + \sigma(\Delta V_k)^2.\end{aligned}$$

This shows that the discrete change in the standard deviation depends only on the marginal standard deviation (covariance) and the standard deviation of the sub-portfolio — a fact which is also evident from Equation (6.17).

If the sub-portfolio is removed from the total portfolio then the return $R = \Delta V/V$ becomes $(\Delta V - \Delta V_k)/(V - V_k) = (R - \omega_k R_k)/(1 - \omega_k)$, where $\omega_k = V_k/V$ is the weight of the sub-portfolio. Hence the incremental standard deviation of the return is

$$\sigma(R) - \frac{\sigma(R - \omega_k R_k)}{1 - \omega_k},$$

and the standard deviation of $R - \omega_k R_k$ is given by

$$\sigma(R - \omega_k R_k)^2 = (1 - 2\omega_k\beta_k)\sigma(R)^2 + \omega_k^2\sigma(R_k)^2.$$

The incremental value-at-risk at confidence probability p is

$$v(p) - \tilde{v}_k(p)$$

where $v(p)$ denotes the VaR of the total portfolio and $\tilde{v}_k(p)$ is the VaR of the portfolio with the sub-portfolio removed; these VaRs are defined by

$$\mathbb{P}(\Delta V \le v(p)) = 1 - p = \mathbb{P}(\Delta V - \Delta V_k \le \tilde{v}_k(p)).$$

If ΔV and ΔV_k are normally distributed then it follows that the incremental VaR is given by the following formula involving only the mean of ΔV_k and the incremental standard deviation:

$$\mu(\Delta V_k) + [\sigma(\Delta V) - \sigma(\Delta V - \Delta V_k)]\,\mathcal{N}^{-1}(1 - p).$$

6.6. Equity Risk

6.6.1. Total and General Market Risk. For portfolios containing equity products, Adaptiv supports two independent calculations of risk: *total* risk and *general market* risk.

The total risk includes both risk from movements in the market indices and specific risk due to idiosyncratic movements in stock prices. Total risk is calculated by treating each stock price as a risk factor; simulated returns on the stock prices are calculated in the same way as for other risk factors. In particular, the covariance matrix of the risk-factor returns includes entries for all the stock prices.

The general market risk (or β-risk) is the risk attributed to movements in market indices. Each stock is associated with a benchmark index, either a real market index (for example S&P500 and FTSE100), or a synthetic one. The index prices are treated as risk factors and the simulated return on a stock price is generated by multiplying the simulated return on its benchmark index by the linear regression coefficient (β) of the stock-price returns with respect to the index-price returns.

6.6.2. Linear Regression. Given a random variable r, and another random variable q having non-zero variance, there are constants α and β such that:

$$r = \alpha + \beta q + e$$

where e is a random variable that satisfies $\mathbb{E}(e) = 0$ and $\text{cov}(e, q) = 0$. The constants are unique and given by

$$\beta = \frac{\text{cov}(r, q)}{\text{var}(q)}$$

and $\alpha = \mathbb{E}(r) - \beta\mathbb{E}(q)$. This decomposition is called the *linear regression* of the *dependent* variable r with respect to the *independent* variable q, and e is called the error.

6.6.3. Equity Beta. In Adaptiv, each stock is associated with a benchmark index, for example IBM can be associated with the S&P500 index. Adaptiv calculates the linear regression of the stock-price return $r(t) = \log S(t) - \log S(t-1)$ with respect to its corresponding index-price return $q(t) = \log I(t) - \log I(t-1)$. The covariance $\text{cov}(r(t), q(t))$ and variance $\text{var}(q(t))$ are calculated within either the SMA model (Equation 6.7), or the EWMA model (Equation 6.6).

6.6.4. Calculation of General Market Risk. Let S denote an equity price with associated benchmark index price I. Then the logarithmic return on S has the decomposition:

$$\Delta \log S = \alpha + \beta(\Delta \log I) + e$$

where β is the linear regression coefficient of $\Delta \log S$ with respect to $\Delta \log I$. The general market risk is based on the value change ΔV which is obtained by replacing each stock-price return $\Delta \log S$ with its corresponding index-price return $\Delta \log I$ multiplied by β.

In simulation risk calculations, it is conventional to use proportional returns rather than log returns. Thus, the simulated changes in stock prices are given by

$$\frac{\Delta S}{S} = \beta \left(\frac{\Delta I}{I} \right).$$

In variance-covariance calculations, the general market risk is obtained by replacing each $\Delta(\log S)$ by its corresponding $\beta\Delta(\log I)$ in the linear or quadratic approximation of ΔV. This is equivalent to a standard variance-covariance risk calculation with the index prices as risk factors and equivalent sensitivities to changes in the log index prices. Each stock price S contributes

$$\beta\frac{\partial V}{\partial(\log S)}$$

to the equivalent first-order sensitivity to $\log I$. A pair of stock prices S and $\tilde{S}$ with corresponding benchmark indices I and $\tilde{I}$ contributes

$$\beta\tilde{\beta}\frac{\partial^2 V}{\partial(\log S)\partial(\log \tilde{S})}$$

to the equivalent second-order sensitivity to $\log I$ and $\log \tilde{I}$.

6.7. Instrument Ageing

6.7.1. Introduction. Instrument ageing is the process of evolving the state of an instrument through time so that its value can be estimated on one or more future dates. The ageing process estimates the outcome of trade events such as rate fixings and option exercise, and hence estimates contingent cashflows. Actual and estimated cashflows are added to the instrument's *cash balance*, and then accrue interest at the prevailing market rate. The total value of the instrument at a given date in the future is the sum of the cash balance and the present value of the aged instrument at the future date.

Instrument ageing is used whenever the value of an instrument is required at some future date. The ageing is performed with respect to a set of horizon dates at which the market data has been determined, usually by simulation. These dates define time steps, and the instrument is aged through each step in turn. The future state of the instrument is *path dependent* because it may depend on the complete history of the market rates up to the current time step.

A simple application of instrument ageing is within the Reval Theta report. Each instrument is aged in a single step from today to the horizon date, which is typically the next business day. The instrument's theta is defined to be

$$V_1 + C - V_0,$$

where V_0 is the value of the instrument today, V_1 is the value of the aged instrument at the horizon date, and C is the estimated cash balance.

For Monte Carlo value-at-risk, instrument ageing is performed for each scenario on the market rates. Typically, the horizon for market value-at-risk is short-term, for example ten days, and hence instrument ageing is performed in a single step.

Multi-step instrument ageing is important for Monte Carlo-based credit risk because of the need to calculate the future value, and hence potential future exposure, at long-term horizons, typically of one year or longer (see Section 7.3.2).

6.7.2. Time Steps and Market Data. Let $T_0, T_1, \ldots, T_m$ denote the horizon dates through which the instrument is aged, where T_0 is the reference date of the portfolio. The instrument is aged from T_0 to T_1, and then from T_1 to T_2, and so on.

The simulated market rates are evolved through the same time steps. The methods used to generate multi-step scenarios on the market risk factors are described in

Section 6.4.3. At each time step, the market data structures are recalibrated with the evolved risk factors. Calibration of a market data structure means calculating its pricing factors from a given set of risk factors (see Section 1.1). For example, a yield curve is calibrated by deriving its table of discount factors from a given set of cash, futures and swap rates.

6.7.3. Ageing Methodology. The methodology is best described by considering an instrument to consist of one or more cashflows, where the cashflow amount is either fixed or contingent on one or more market rates. Thus, each cashflow can be written

$$\Pi(r_1, \dots, r_k),$$

where Π denotes a payoff function and $r_1, \dots, r_k$ are the market rates on which the cashflow depends.

For example, a floating swap cashflow has $\Pi(r) = P\alpha_{\text{int}}r$, where r is the deposit rate, P is the principal amount and α_{int} is the interest period.

Options fit within this framework because it is assumed that they are always exercised into the cash value of the underlying instrument. Hence they become contingent cashflows. For example, a European FX option has

$$\Pi(S) = P \max(S - K, 0),$$

where P is the foreign-currency principal amount, S is the spot FX rate and K is the strike rate.

For each period T_i to T_{i+1}, the ageing calculation proceeds as follows.

(1) For each cashflow, estimate the value of the market rates whose observation dates lie in this period. These rate observations affect the value of the cashflow at any of the future horizon dates, and the final estimate of the cashflow amount. The methods used to estimate rates are described in Section 6.7.6.
(2) Calculate the value of any contingent cashflows that are realised in this period — by evaluating the payoff function Π from the known and estimated rates $r_1, \dots, r_k$.
(3) Accrue interest on the start-date cash balance from the start date T_i to the end date T_{i+1}, and add this to the end-date cash balance. For each cashflow realised in the period, calculate the interest accrued from its payment date to the end date T_{i+1}; and add the cashflow and interest to the end-date cash balance. More details of these calculations are given Section 6.7.7.

A rate observation is deemed to belong to the period T_i to T_{i+1} if the observation date t satisfies $T_i < t \le T_{i+1}$. Similarly, a cashflow falls within the period T_i to T_{i+1} if its payment date t satisfies $T_i < t \le T_{i+1}$. Thus, cashflows on the end date are excluded from the present value but included in the cash balance; and rate observations are estimated up to and including the end date.

6.7.4. Barrier Options. Continuous barrier options are approximated as discrete barrier options with observations at the horizon dates and at the expiry date. If a barrier is broken before the expiry date then the state of the instrument is updated for pricing at the next and following horizon dates.

Consider a knock-out option which is not knocked out at the start of the ageing period T_i. If the option expires after T_{i+1} and the rate at T_{i+1} breaks one of

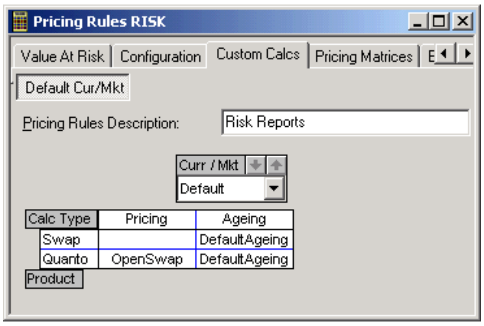


FIGURE 6.11. pricing models and instrument ageing calculators.

the barriers then: the rebate cashflow is added to the cash balance at T_{i+1}; the instrument state is updated so it returns a zero value when priced at T_{i+1} and subsequent horizon dates. If the option expires before T_{i+1} then an observation of the rate at the expiry date is obtained by interpolation between the rates at T_i and T_{i+1}. If the interpolated rate breaks one of the barriers then the cashflow at expiry is the rebate, otherwise the cashflow at expiry is the value of the underlying option at expiry. The cashflow at expiry plus accrued interest is added to the cash balance at T_{i+1}.

Similarly for a knock-in option which is not knocked in at the start of the ageing period. If the option expires after T_{i+1} and the rate at T_{i+1} breaks one of the barriers then the instrument state is updated so it prices as the underlying option at T_{i+1} and subsequent horizon dates. If the option expires before T_{i+1} and the interpolated rate at the expiry date does not break the barriers then the cashflow at expiry is the rebate, otherwise the cashflow at expiry is the value of the underlying option at expiry.

6.7.5. Alternative Methodologies. The methodology set out in Section 6.7.3 is mainly passive: the position in each security is constant; the value of derivative instruments is eventually realised as cash. Active methodologies incorporate trading strategies into the ageing process; for example, a stop-loss strategy, whereby the position in a security is sold if its price drops below a specified level.

This type of analysis can be developed as a customised *ageing calculator*. An ageing calculator is a component of the system that provides an implementation of Adaptiv's instrument ageing interface. In a similar way to pricing models, ageing calculators are assigned by instrument type and currency in the portfolio pricing rules (see Figure 6.11).

6.7.6. Estimating Rates. Consider an ageing step from time T_s to time T_e. Suppose the instrument has a cashflow that is contingent on the value of a market rate r at time t, where $T_s < t \leq T_e$. The required rate r is estimated by interpolating between the rate r_s derived from the market data with reference date T_s, and the rate r_e derived from the market data with reference date T_e:

$$r = \frac{r_s(T_e - t) + r_e(t - T_s)}{T_e - T_s}. \tag{6.18}$$

A simple example is the estimation of the spot FX rate on the exercise date of a FX option that lies between T_s and T_e. In this case, r_s is the spot FX rate from the market data at time T_s, and r_e is the spot FX rate from the market data at time T_e.

To estimate a deposit (LIBOR) rate, the rates r_s and r_e are defined to have the same term and the same number of days to their start date as the rate r. For example, if the required rate r has a period of 91 days and starts on $t+2$, then r_s is a 91-day rate starting on $T_s + 2$, and r_e is a 91-day rate starting on $T_e + 2$. For CMS rates, r_s and r_e have same nominal term (for example, 5y) and the same number of days to their start date as the rate r.

To estimate the d-day discount factor Z at time t, the linear interpolation formula (6.18) is applied to $r = \log Z$, $r_s = \log Z_s$ and $r_e = \log Z_e$, where Z_s is the d-day discount factor from the market data at time T_s, and Z_e is the d-day discount factor from the market data at time T_e. Thus, the interpolated discount factor is given by

$$Z = Z_s^{(T_e - t)/(T_e - T_s)} Z_e^{(t - T_s)/(T_e - T_s)},$$

which is superficially similar to the formula for constant daily forward interpolation in Section 1.2.3, and can be interpreted as follows. The daily forward rate derived from Z_s is compounded over λd days, where $\lambda = (T_e - t)/(T_e - T_s)$, and the daily forward rate derived from Z_e is compounded over $(1 - \lambda)d$ days.

6.7.7. Cash Balance. Consider an ageing period from time T_s to time T_e. Suppose the instrument has a cashflow at time t, where $T_s < t \leq T_e$. Let C denote the cashflow amount, which may be either known or estimated.

The cashflow will start to accrue interest after its settlement date; its contribution to the cash balance at the end date T_e is estimated as C/Z, where Z is an interpolated $(T_e - t)$-day discount factor; and Z is interpolated by the method described in Section 6.7.6. For this calculation, the discount factors are obtained from the discount curve assigned to the instrument leg to which the cashflow belongs.

The interest accrued on the initial cash balance is estimated in the same way as a cashflow paid at time T_s. The initial cash balance C contributes C/Z_s to the final cash balance, where Z_s is the $(T_e - T_s)$-day discount factor from the yield curve at time T_s.

6.8. Backtesting

6.8.1. Backtesting Analysis. Backtesting is the comparison of the value-at-risk predicted at the start of a risk horizon with the actual P&L over this period. The comparison is made over a specified *analysis* period, a typical period being 250 days. A day on which the P&L is less than the VaR is called an *exception*. Section 6.8.2 shows how the number of exceptions can be used to measure the accuracy of the VaR.

For the purpose of backtesting analysis, trading effects are excluded from the P&L, and the adjusted P&L is referred to as *no-action* P&L. Adaptiv calculates no-action P&L using one of the following two methods.

OTC: For OTC instruments, the effect of new, amended and voided trades is subtracted from the total P&L. The effect of an amended trade on the P&L is estimated as the difference between the PV of the new trade and the PV of the old trade, where both valuations are under the same set of market data: the market data at the end of the P&L period.

Securities and Exchange-traded: For these instruments, the no-action P&L is calculated directly as the principal amount, or number of contracts, multiplied by the change in market price.

Range of Cumulative Probability	Colour	Scaling Factor
$B_{N,p}(e) < B_{250,99\%}(5) \simeq 95.88\%$	Green	3
$B_{250,99\%}(5) \leq B_{N,p}(e) < B_{250,99\%}(6)$	Yellow	3.4
$B_{250,99\%}(6) \leq B_{N,p}(e) < B_{250,99\%}(7)$	Yellow	3.5
$B_{250,99\%}(7) \leq B_{N,p}(e) < B_{250,99\%}(8)$	Yellow	3.65
$B_{250,99\%}(8) \leq B_{N,p}(e) < B_{250,99\%}(9)$	Yellow	3.75
$B_{250,99\%}(9) \leq B_{N,p}(e) < B_{250,99\%}(10)$	Yellow	3.85
$B_{N,p}(e) \geq B_{250,99\%}(10) \simeq 99.99\%$	Red	4

TABLE 6.2. bands for the observed number of exceptions e.

6.8.2. Statistical Test of VaR Model. Suppose the VaR, for confidence level p and risk horizon h days, is compared with h-day P&L over a period of N business days. Under the model used to calculate the value-at-risk, each day in the analysis period is an independent trial with one of two possible outcomes: an exception with probability $1 - p$, or not an exception with probability p. The total number of exceptions in the analysis period has the binomial distribution: the probability of n or fewer exceptions is given by

$$B_{N,p}(n) = \sum\nolimits_{m=0}^{n} \frac{N!}{(N-m)!m!}(1-p)^m p^{N-m}$$

Let e denote the observed number of exceptions. If $B_{N,p}(e)$ is close to 1 then the model is likely to be *underestimating* the VaR because it predicts a low probability of obtaining more than the observed number of exceptions. Conversely, if $B_{N,p}(e)$ is close to 0 then the model is likely to be *overestimating* the VaR (too conservative) because it predicts a low probability of obtaining the observed number of exceptions or fewer.

The Basle committee report [**3**] specifies bands for the observed number of exceptions, with each band having a colour (green, yellow or red) and a scaling factor for the calculation of regulatory capital. Adaptiv assigns the colour and scaling factor according to the rules defined in Table 6.2; these are generalized from the rules in Table 2 of [**3**]. The model is deemed to be acceptable if the observed number of exceptions is in the green band. A result in the red band indicates that the model has underestimated the VaR.

The spreadsheet `FG_Backtest.xls` evaluates $B_{N,p}(e)$ and assigns the colour and scaling factor.

CHAPTER 7

Credit Risk

7.1. Introduction

Adaptiv provides two methods of reporting credit risk.

Traditional Credit Risk: Exposure-at-default is the sum of the positive market value and the principal amount weighted by an add-on factor. Expected loss is estimated as the exposure-at-default E multiplied by a probability of default p, multiplied by $(1 - R)$, where R is a recovery rate. The calculations are described in Section 7.2 and follow the methodology set out in the New Basel Capital Accord [4]. Exposure-at-default calculated under this methodology is shown on the Credit Risk report.

Monte Carlo Exposure Profiling and Credit Value-at-Risk: The distribution of the credit exposure is calculated by valuation under multi-step Monte Carlo simulation of the market rates. Monte Carlo simulations of credit events are combined with the simulated credit exposures to give the distribution of the credit loss. The distribution of the credit events is derived within one of the three credit VaR models supported by Adaptiv. The calculations are described in Section 7.3. Results are shown on the MC Credit Risk report.

7.2. Traditional Credit Risk

7.2.1. Obligors and Exposures. Let ABC denote an organization; for example, a bank, government or corporation. Suppose we have a position in a security issued by ABC, or have executed a derivative transaction with ABC as the counterparty. In either case, there is a *credit exposure* to ABC if the security or transaction has some positive market value, or has some potential positive market value in the future. If ABC were to become bankrupt then it would default on its obligation to make future contractual payments, and hence the value of the instrument would either be lost or only partially recoverable from ABC. In this context, the organization ABC is referred to as an *obligor*.

7.2.2. Exposure at Default. For a single instrument, the estimated exposure-at-default is the sum of the positive market value and a proportion of the principal amount to reflect the potential future exposure:

$$\max(V, 0) + \alpha P,$$

where V is the present value of the instrument and P is its actual or notional principal amount. The weighting factor α is referred to as an *add-on* factor. The Basle committee report [2] specifies rules such that each instrument is assigned an add-on factor in the range 0%–15% according to its type (interest rate, FX, equity, etc.) and its remaining time to maturity.

Now consider a set of instruments with exposure to the same obligor. In the absence of any *netting agreements*, the total exposure is

$$\sum_i \max(V_i, 0) + \alpha_i P_i,$$

where the sum is over all instruments i in the set. Suppose that there is a netting agreement in effect such that there is a set of instruments $\mathcal{A}$ for which it permissible to net credit exposures; and let $\mathcal{B}$ denote the remaining set of instruments whose exposures cannot be netted. Adaptiv calculates the total exposure using the formula described in the Basle committee report [2]. Before stating the formula, let R, $\tilde{R}$ and $\tilde{F}$ denote the net replacement cost, gross replacement cost, and gross potential future exposure respectively; these are defined by

$$R = \max\left(\sum_{i\in\mathcal{A}} V_i, 0\right) + \sum_{i\in\mathcal{B}} \max(V_i, 0) \tag{7.1}$$

$$\tilde{R} = \sum_i \max(V_i, 0) \tag{7.2}$$

$$\tilde{F} = \sum_i \alpha_i P_i. \tag{7.3}$$

The total exposure is $E = R + F$, where F is the net potential future exposure defined by

$$F = \omega \tilde{F} + (1-\omega)\left(\frac{R}{\tilde{R}}\right)\tilde{F} \tag{7.4}$$

and ω is a factor in the range 0% to 100%. The fraction $R/\tilde{R}$ is referred to as the *net-to-gross* ratio. In the Basle committee report [2], ω is set to 40%. If $\omega = 100\%$ then $E = R + \tilde{F}$, so that the total exposure is the sum of the net replacement cost and the gross potential future exposure. The gross replacement cost $\tilde{R}$ is zero if and only if all the instrument values V_i are zero or negative. In this case, the net-to-gross ratio is set to 1, and hence the net potential future exposure is equal to the gross potential future exposure.

So far, we have considered only one subset of nettable instruments. In general, a set of instruments with exposures to the same obligor can be partitioned into subsets $\mathcal{A}_1, \mathcal{A}_2, \ldots$ such that it is permissible to net exposures within each subset $\mathcal{A}_j$, but not permissible to net exposures belonging to different subsets. In this setting, the net replacement cost is given by

$$R = \sum_j \max\left(\sum_{i\in\mathcal{A}_j} V_i, 0\right). \tag{7.5}$$

This generalizes Equation (7.1) because each non-nettable instrument $i \in \mathcal{B}$ has a corresponding singleton subset $\mathcal{A}_j = \{i\}$.

Adaptiv provides an alternative method that allows negative present value to offset potential future exposure. Under this method, the exposure of a single instrument is

$$\max\left(V + \alpha P, 0\right);$$

and in the general case, the total exposure is

$$\sum_j \max\left(\sum_{i\in\mathcal{A}_j} (V_i + \alpha_i P_i), 0\right).$$

7.2.3. Expected Loss. Suppose p is the probability that an obligor will default. Let E denote the estimated exposure-at-default for this obligor. Let R denote the estimated recovery rate; the recovery rate is the proportion of the market value that would be recovered if the obligor defaulted. It is assumed that there are two possible events: a default and consequent loss equal to $E(1-R)$, or no default and loss equal to zero. The expected loss arising from the exposure is therefore

$$E(1-R)p.$$

The probability of default is assigned to the obligor according to its credit rating. A recovery rate is assigned to each instrument according to its seniority class (see Section 7.3.5).

7.3. Exposure Profiling and Credit VaR

7.3.1. Introduction. In Section 7.2.3, the *expected* credit loss due to default of an individual obligor was calculated from a given probability of default and recovery rate. Adaptiv's credit VaR models calculate the *distribution* of the credit loss based on a statistical model of events under which the credit worthiness of an obligor changes, which will be referred to as *credit events*. The models take into account the correlation between credit events of related obligors, which is known as *concentration risk*. Before proceeding to describe the models, we shall first distinguish three type of credit-related risk:

- **Default:** The risk of an obligor defaulting on its financial obligations because of bankruptcy. The consequence of such a state of default is the complete or partial loss of the market value of any financial instrument transacted with the obligor.
- **Rating Transition:** The risk of downgrade, or upgrade, of the credit rating of an obligor. After such a transition, a security issued by the obligor would be assigned a credit spread consistent with the obligor's new rating. Hence there is a consequent change in market value of any securities issued by the obligor.
- **Spread:** The risk arising from systematic changes in credit spreads. A credit spread reflects the market's view of the credit riskiness of an obligor, or class of obligors; however, a credit spread is also another type of variable that affects the market value of an instrument. In this context, Adaptiv treats credit spreads as market variables and hence changes in credit spreads as market risk.

Adaptiv provides three credit VaR models: the actuarial model, similar to the CreditRisk+ methodology of Credit Suisse First Boston [**11**]; the microeconomic model, similar to the methodology of J.P. Morgan's CreditMetrics [**33**]; and the macroeconomic model, which is similar to the CreditPortfolioView methodology of McKinsey & Company as described in Wilson [**40**]. The actuarial and macroeconomic models provide a distribution of credit loss due to default events; the microeconomic model includes both rating-transition and default events as causes of loss. All three models generate a Monte Carlo simulation of the distribution of credit loss, where the credit loss is the change in the portfolio value over some specified horizon, typically one year.

In the CreditRisk+ methodology, analytical methods are used to derive the distribution of the credit loss. However, there are two advantages of using Monte Carlo methods. Firstly, more complicated distributions can be used, for example, stochastic default rates can be combined with stochastic recovery rates. Secondly, credit and market risk can be integrated in the following sense: the market and credit variables are simulated simultaneously, with the credit variables assumed to

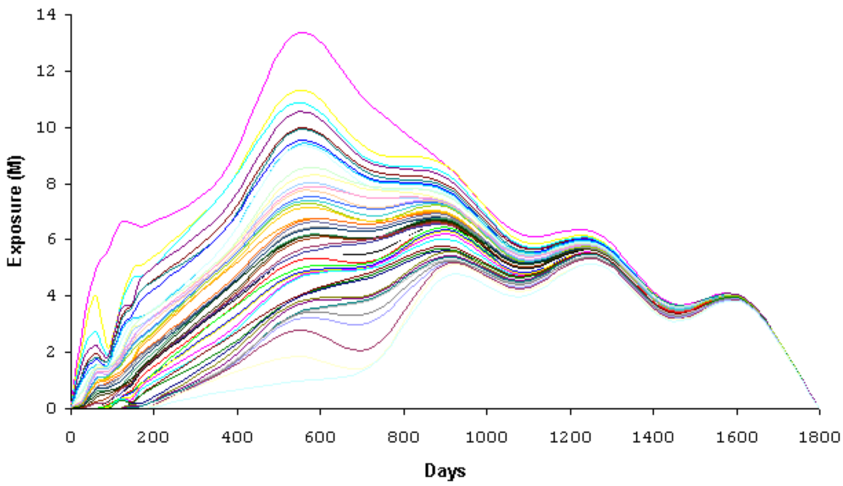


FIGURE 7.1. simulated exposure profiles of a 5y swap.

be independent of the market variables, so that the credit exposure varies according to the simulated market rates.

Thus, the first step in simulating the credit loss is to calculate the simulated credit exposure.

7.3.2. Credit Exposure Profile. A single profile of the credit exposure comprises a set of future horizon dates and, for each obligor, a credit exposure value at each date. Hence the profile constitutes a graph of credit exposure against time for each obligor. In general, the credit exposure at a future date is a function of the market rates at that date.

Adaptiv simulates the exposure profile using Monte Carlo methods. The calculation proceeds in three stages and is summarized as follows.

(1) The future values of the market rates are obtained by Monte Carlo simulation. The rates are evolved in multiple steps. The profile dates are a subset of the sequence of dates through which the rates are evolved.
(2) Each instrument contributing to the exposure is aged through the market-rate dates, and its market value is calculated at each profile date.
(3) For each obligor, the market values are netted according to any netting agreements in effect. The credit exposure is the sum of the positive netted market values.

Figure 7.1 shows some simulated exposure profiles for a 5y interest-rate swap; where the profiles were generated by the MC Credit Risk report.

The market rates are evolved through a sequence $T_1 < \cdots < T_m$ of horizon dates, and instrument ageing is performed with respect to the same sequence of dates. The methods used to generate multi-step scenarios on the market rates are described in Section 6.4.3; and the methodology of instrument ageing is explained in Section 6.7. The profiles dates are a subset of $\{T_1, \ldots, T_m\}$, and valuation is performed only at the profile dates.

The credit exposures are calculated from the market valuations as follows. Let $\mathcal{A}$ denote the set of instruments with an exposure to a given obligor. This set can

Rating	Mean	Standard Deviation
AAA	1.5%	0.75%
AA	2.0%	1.00%
A	3.0%	1.50%
BAA	5.0%	2.50%
BA	7.5%	3.75%
B	10.0%	5.00%

TABLE 7.1. example statistics of the default rate.

be partitioned into subsets $\mathcal{A}_1, \mathcal{A}_2, \dots$ such that it is permissible to net exposures within each subset $\mathcal{A}_j$, but not permissible to net exposures belonging to different subsets. Let V_i denote the market value of instrument $i \in \mathcal{A}$. Then the total credit exposure to the obligor is given by

$$E = \sum\nolimits_j \max\left(\sum\nolimits_{i \in \mathcal{A}_j} V_i, 0\right). \tag{7.6}$$

Monte Carlo simulation gives a non-parametric estimate of the distribution of credit exposure (see Section 6.4.5). Let $E_1, \dots, E_N$ denote the simulated credit exposures in descending order, so that $E_{i+1} \leq E_i$. Then k/N is the probability of an exposure greater than or equal to E_k. For example, for 1000 scenarios, there is a 1% probability of a credit exposure greater than or equal to E_{10}.

7.3.3. Simulating Default Risk. Let $0 \leq p \leq 1$ and U be a random variable uniformly distributed on the unit interval $[0, 1]$. Then the indicator function of the event $\{U \leq p\}$, which will be denoted $1_{\{U \leq p\}}$, is a Bernoulli variable with mean p; that is, it takes the value 1 with probability p, and the value 0 with probability $1 - p$. Now suppose D is a random variable taking values in $[0, 1]$ and U is a uniform variable on $[0, 1]$ independent of D. Let I denote $1_{\{U \leq D\}}$. Then $\mathbb{E}(I|D) = D$ and hence, conditional on D, the variable I has a Bernoulli distribution with mean D.

The credit loss due to default can now be defined by

$$L = E(1 - R)1_{\{U \leq D\}}, \tag{7.7}$$

where E is the credit exposure, R is the recovery rate, D is the default rate and U is a uniform variable independent of D. In all the models, it is assumed that E, R, D and U are independent random variables. It follows that the expected credit loss is given by

$$\mathbb{E}(L) = \mathbb{E}(E)[1 - \mathbb{E}(R)]\mathbb{E}(D).$$

The distribution of the credit loss is estimated by Monte Carlo simulation of the independent variables E, R, D and U.

The distribution of the recovery rates is common to all of Adaptiv's models and is described in Section 7.3.5; the distribution of the default rates is specific to the model.

7.3.4. Actuarial Model. In the actuarial model, the default rates are driven by a set of random variables $X_1, \dots, X_n$ that represent country and industry sectors, such as US Telecommunications, US Chemicals, UK Insurance etc. Each obligor has a participation rate in each sector, and the sum of its participation rates over all sectors is 100%. Let θ_{Ai} denote the participation rate of obligor A in sector i;

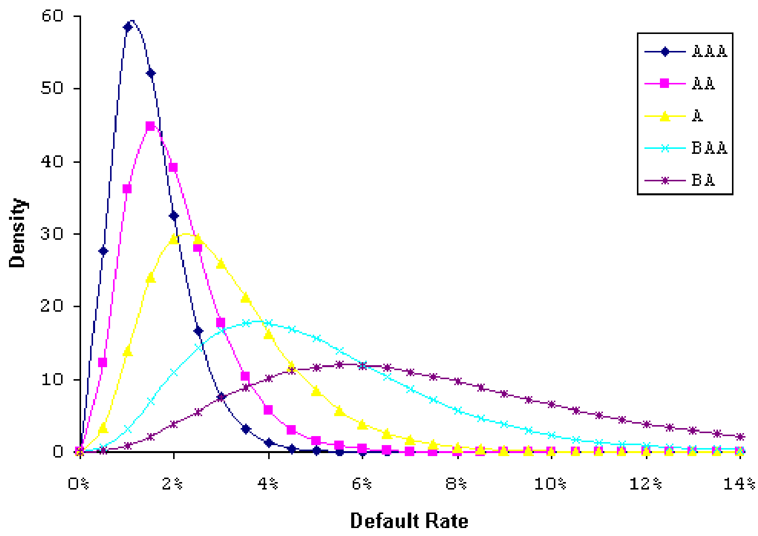


FIGURE 7.2. gamma distributions of the default rate.

then the θ_{Ai} are non-negative and

$$\sum\nolimits_i \theta_{Ai} = 1.$$

Each obligor A is assigned an expected default rate $\tilde{\mu}_A$ and a standard deviation of its default rate $\tilde{\sigma}_A$. These are assigned according to the credit rating of the obligor.

The sector variables are independent, and each variable X_i has the gamma distribution with mean

$$\mu_i = \frac{\sum_A \theta_{Ai}\tilde{\mu}_A}{\sum_A \theta_{Ai}} \tag{7.8}$$

and standard deviation

$$\sigma_i = \frac{\sum_A \theta_{Ai}\tilde{\sigma}_A}{\sum_A \theta_{Ai}}. \tag{7.9}$$

Thus, μ_i is a weighted average of the $\tilde{\mu}_A$ over all obligors participating in the sector; and similarly for σ_i. The sector variables are simulated from independent uniform variables $U_1, \ldots, U_n$ by setting $X_i = F_i^{-1}(U_i)$, where F_i is the gamma distribution function with mean μ_i and standard deviation σ_i.

The density function of the gamma distribution with parameters α and β is given by

$$f(x) = \frac{(x/\beta)^{\alpha-1}\exp(-x/\beta)}{\Gamma(\alpha)\beta},$$

where $x > 0$ and Γ is the standard gamma function. The distribution is determined by its mean $\mu = \alpha\beta$ and variance $\sigma^2 = \alpha\beta^2$. If X has the gamma distribution with parameters α and β, and c is a positive constant, then cX has the gamma distribution with parameters α and $c\beta$.

Table 7.1 shows some example statistics of the default rates for various ratings; Figure 7.2 shows the corresponding density functions of the gamma distributions with these statistics.

Seniority Class	**Mean**	**Standard Deviation**
Senior Secured	53.80	26.86
Senior Unsecured	51.13	25.45
Senior Subordinated	38.52	23.81
Subordinated	32.74	20.08
Junior Subordinated	17.09	10.90

TABLE 7.2. example statistics of the recovery rate.

For each obligor A, the default rate is defined as a weighted sum of the sector variables:

$$D_A = \tilde{\mu}_A \sum_i \theta_{Ai} \left(\frac{X_i}{\mu_i} \right).$$

The expectation of the obligor's default rate is given by $\mathbb{E}(D_A) = \tilde{\mu}_A$, and hence agrees with the given expected default rate. The normalized sector variable X_i/μ_i has the gamma distribution with mean 1 and standard deviation σ_i/μ_i. Note that D_A is independent of the denominator in Equations (7.8) and (7.9) because it cancels out in σ_i/μ_i.

The spreadsheet `FG_CVARAct.xls` contains an example simulation of default events and recovery rates under the actuarial model.

A special case of the actuarial model is obtained by partitioning the obligors into *disjoint* sectors. In this case, the participation rate θ_{Ai} is equal to 1 if the obligor A belongs to sector i, and otherwise $\theta_{Ai} = 0$. Hence the obligor default rates are given by $D_A = \tilde{\mu}_A X_i/\mu_i$, where i is the unique sector to which A belongs, and the normalized sector variable X_i/μ_i has standard deviation

$$\frac{\sigma_i}{\mu_i} = \frac{\sum_{A \in i} \tilde{\sigma}_A}{\sum_{A \in i} \tilde{\mu}_A}.$$

7.3.5. Recovery Rate. If an organization becomes bankrupt then its creditors may be able to recover a proportion of their exposure to it. Recovery rates are uncertain; and they depend on the seniority of each exposure relative to other claims on the organization's assets. For example, among loans, bonds and stock, the seniority of loans is greater than bonds, and that of bonds is greater than stock.

Within Adaptiv's credit VaR models, each security or transaction is assigned a *seniority class*, and each combination of country and seniority class is assigned a random recovery rate. The recovery rates are modelled as independent beta variables.

The beta distribution with parameters a and b has the density function

$$f(x) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)},$$

where $0 < x < 1$ and B is the standard beta function. The mean μ and standard deviation σ are related to the parameters by $\mu = a/(a+b)$ and $\sigma^2/\mu^2 + 1 = (a+1)/(a+\mu)$.

Hence the recovery-rate variables are specified by their mean and standard deviation.

Table 7.2 shows some example statistics of the recovery rates for various seniority classes; Figure 7.3 shows the corresponding density functions of the beta distributions with these statistics.

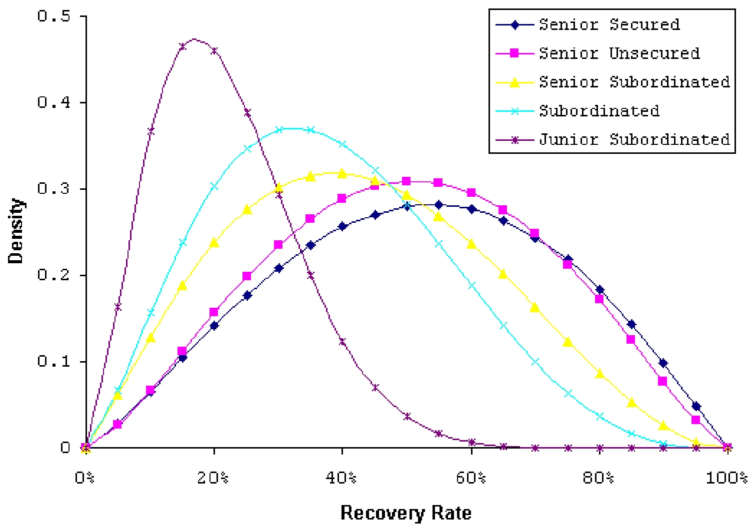


FIGURE 7.3. beta distributions of the recovery rate.

The total exposure to an obligor is given by Equation (7.6). Since two instruments within the same netting group $\mathcal{A}_j$ may have different recovery rates, Adaptiv calculates an exposure-weighted average recovery rate for each netting group; this is given by

$$\bar{R}_j = \frac{\sum_{i \in \mathcal{A}_j} \max(V_i, 0) R_i}{\sum_{i \in \mathcal{A}_j} \max(V_i, 0)},$$

where R_i is the recovery rate assigned to instrument i. The recovered amount for this netting group is $E_j \bar{R}_j$, where

$$E_j = \max\left(\sum_{i \in \mathcal{A}_j} V_i, 0\right).$$

7.3.6. Microeconomic Model. The microeconomic model provides a model of rating transitions and default events. The model will be described in two stages. Firstly, the return on the value of an obligor's assets over the horizon period determines the rating of the obligor at the horizon date. Secondly, the asset returns of all obligors in the model are driven by a common set of explanatory variables corresponding to microeconomic indices.

The model is defined with respect to a rating system comprising $m + 1$ possible rating states, numbered 0 to m. State m is the highest rating (for example, AAA), state 1 is lowest rating (for example CCC), and state 0 is the state of default. Consider an obligor A and let k denote its current rating state, where $1 \leq k \leq m$, so that A is not in a state of default. The return on the obligor's assets over the horizon period is modelled by a normal variable Z_A whose standard deviation will be denoted σ_A. The central assumption of the model is the existence of threshold returns $z_{k,0} < z_{k,1} < \ldots < z_{k,n} = \infty$ such that the obligor's final state is l if and only if $z_{k,l-1} < Z_A \leq z_{k,l}$, where $z_{k,-1}$ is set to $-\infty$. Thus, the final state is l if and only if the asset return Z_A ends up at or below the threshold $z_{k,l}$ but above the next lowest threshold $z_{k,l-1}$. The probability of a transition to rating l is given by

$$p_{k,l} = \mathbb{P}(z_{k,l-1} < Z_A \leq z_{k,l}) = \mathcal{N}(z_{k,l}/\sigma_A) - \mathcal{N}(z_{k,l-1}/\sigma_A),$$

Final Rating	Transition Probability	Threshold Return
AAA	0.03	
AA	0.14	3.43σ
A	0.67	2.93σ
BBB	7.73	2.39σ
BB	80.53	1.37σ
B	8.84	-1.23σ
CCC	1.00	-2.04σ
Default	1.06	-2.30σ

TABLE 7.3. transition probabilities from rating BBB and corresponding threshold returns, where σ is the standard deviation of the asset return.

and hence

$$z_{k,l} = \sigma_A \mathcal{N}^{-1}\left(p_{k,0} + \cdots + p_{k,l}\right). \tag{7.10}$$

A further assumption of the model is that the transition probabilities $p_{k,l}$ are independent of the obligor, or equivalently that the normalized threshold values $z_{k,l}/\sigma_A$ are independent of A.

Table 7.3 shows some example transition probabilities and the corresponding threshold returns given by Equation (7.10).

Let E_{Al} denote the exposure to obligor A in final state l. The total credit loss is given by

$$L_A = E_{Ak} - \sum_{l=0}^{m} E_{Al} 1_{\{z_{k,l-1} < Z_A \leq z_{k,l}\}}.$$

For $l > 0$, the exposure E_{Al} is calculated using simulated yield-curve rates that include the credit spreads assigned to rating l. The exposure in the default state is $E_{A0} = R_A E_{Ak}$, where R_A is the obligor's recovery rate. Recovery rates are modelled as independent beta variables, as described in Section 7.3.5. The asset returns, exposures and recovery rates are assumed to be independent; hence the expected credit loss is given by

$$\mathbb{E}(L_A) = \mathbb{E}(E_{Ak}) - \sum_{l=0}^{m} \mathbb{E}(E_{Al}) p_{k,l}.$$

The asset returns of the obligors in the model are constructed from a set of correlated normal variables $X_1, \ldots, X_n$ corresponding to microeconomic indices. Typically, the microeconomic indices are of country and industry sectors, such as US Telecommunications, US Chemicals, UK Insurance etc. Let σ_i denote the standard deviation of X_i, and let ρ_{ij} denote the correlation of X_i and X_j. Each obligor A has a participation rate θ_{Ai} in each index variable X_i, and a specific coefficient θ_A; these are non-negative and satisfy $\sum_i \theta_{Ai} + \theta_A = 1$. The asset return Z_A has an obligor-specific component driven by a standard normal variable Y_A that is independent of the index variables. The asset returns are defined by

$$Z_A = \sum_i \theta_{Ai} X_i + \alpha_A \sigma_A Y_A, \tag{7.11}$$

where $\alpha_A^2 = 1 - (1 - \theta_A)^2$ and

$$\sigma_A^2 = \frac{\operatorname{var}\left(\sum_i \theta_{Ai} X_i\right)}{(1 - \theta_A)^2} = \sum_{ij} \rho_{ij} (\hat{\theta}_{Ai} \sigma_i)(\hat{\theta}_{Aj} \sigma_j), \tag{7.12}$$

where $\hat{\theta}_{Ai} = \theta_{Ai}/(1 - \theta_A)$. Since Y_A is independent of X_i, it follows that σ_A is the standard deviation of Z_A.

The model construction is summarized as follows. The normalized index variables X_i/σ_i are constructed by taking linear combinations of independent normal variables with coefficients from the Cholesky decomposition of the correlation matrix ρ (see Section 6.4.2). Each obligor-specific variable Y_A is generated independently of the index variables and independently of the other obligor-specific variables in the model. Each obligor has a participation rate θ_{Ai} in each index, and a specific coefficient θ_A. The asset returns are given by Equation (7.11), and their standard deviations by Equation (7.12). Thus, the simulated asset returns are obtained by simulating $n + N$ independent normal variables, where n is the number indices and N is the number of obligors. Finally, the simulated asset returns are translated into credit events according to threshold values defined by Equation (7.10); in which $p_{k,l}$ is the probability of transition from rating state k to rating state l.

The spreadsheet `FG_CVARMicro.xls` contains an example simulation of default events, rating-transition events and recovery rates under the microeconomic model.

7.3.7. Macroeconomic Model. In the macroeconomic model, each country/industry sector is assigned a default rate, which has a systematic component and a sector-specific component. The systematic component is a function of a set of macroeconomic variables, such as GNP growth rate, unemployment rate and long-term interest rates. The value of each macroeconomic variable at the risk horizon has some dependency on its current and historical values. In this way, the model captures the effect of cycles in the economy, which are known to influence default rates. In particular, it predicts higher default rates in times of recession.

Let n denote the number of sectors in the model. The obligor default rates are defined by

$$D_A = \sum_{j=1}^{n} \theta_{Aj} D_j, \tag{7.13}$$

where θ_{Aj} is the participation rate of obligor A in sector j, and D_j is the default rate of sector j. The participation rates satisfy $\theta_{Aj} \geq 0$ and $\sum_j \theta_{Aj} = 1$.

The sector default rates are given

$$D_j = \frac{1}{1 + \exp(-Y_j)}, \tag{7.14}$$

where Y_j is a normal variable. The function $y \mapsto 1/(1 + \mathrm{e}^{-y})$ is known as the *logit* function; it maps $(-\infty, \infty)$ onto $(0, 1)$ and hence maps a normally distributed variable to a variable whose distribution is suitable for a default rate.

Each sector variable Y_j is a linear function of the model's macroeconomic variables $X_1, \ldots, X_m$ and a sector-specific variable Z_j:

$$Y_j = \gamma_j + \beta_{j,1} X_1 + \cdots + \beta_{j,m} X_m + \alpha_j Z_j, \tag{7.15}$$

where γ_j, $\beta_{j,i}$ and α_j are constant parameters. The sector-specific variables $Z_1, \ldots, Z_n$ are independent standard normal variables.

The macroeconomic variables satisfy an auto-regressive process, such that X_i, the value of the macroeconomic index at the horizon date, is given by

$$X_i = a_i + \sum_{\ell=0}^{k} b_{i,\ell} X_i(-\ell) + \sigma_i W_i \tag{7.16}$$

where: a_i, $b_{i,\ell}$ and σ_i are constant parameters; W_i is a standard normal variable; $X_i(0)$ is the current value of the macroeconomic index, and $X_i(-1), \ldots, X_i(-k)$ are its historical values up to k time intervals (years) since today. The variables

$W_1, \ldots, W_m$ generate a random shock in the macroeconomic variables and may be correlated with a specified correlation matrix ρ.

The model construction is summarized as follows. Macroeconomic shock variables W_i are constructed by taking linear combinations of independent normal variables with coefficients from the Cholesky decomposition of the correlation matrix ρ (see Section 6.4.2). The simulated macroeconomic variables X_i are then given by Equation (7.16). Independent sector variables Z_j are generated; then the sector variables Y_j are given by Equation (7.15), and the sector default rates D_j by the logit transformation of Equation (7.14). Finally, the participation rates θ_{Aj} give the obligor default rates D_A according to Equation (7.13). Recovery rates are modelled as independent beta variables, as described in Section 7.3.5.

The spreadsheet `FG_CVARMacro.xls` contains an example simulation of default events and recovery rates under the macroeconomic model.

7.3.8. Loss Statistics and Marginal Contributions. Let L denote the total credit loss over all obligors, and let $L_1, \ldots, L_N$ denote simulated values of L under the joint Monte Carlo simulation of the market movements and credit events. The Monte Carlo estimates of the mean and standard deviation of L are given by

$$\hat{\mu} = \frac{1}{N} \sum\nolimits_j L_j$$

and

$$\hat{\sigma}^2 = \frac{1}{N-1} \sum\nolimits_j (L_j - \hat{\mu})^2.$$

Now let L_A denote the credit loss due to obligor A, so that L has the decomposition $L = \sum_A L_A$. In was shown in Section 6.5 that the marginal contribution of L_A to the standard deviation of L is given by

$$\frac{\text{cov}(L_A, L)}{\sigma},$$

where σ is the standard deviation of L. The Monte Carlo estimate of the covariance of L and L_A is

$$\frac{1}{N} \sum\nolimits_j (L_j - \hat{\mu})(L_{A,j} - \hat{\mu}_A),$$

where $L_{A,j}$ are the simulated values of L_A and

$$\hat{\mu}_A = \frac{1}{N} \sum\nolimits_j L_{A,j}.$$

Hence the Monte Carlo simulation gives an estimate of the marginal contribution of each obligor to the standard deviation of the total loss.

APPENDIX A

Cubic Spline Interpolation

The implementation of cubic spline interpolation in Adaptiv follows Press et al. [29]. The method has been implemented in some VB code within the spreadsheets `FG_YCCubicSpline.xls` and `FG_CapVolSurface.xls`. The theory of cubic spline interpolation is summarized below.

Suppose we have a sequence of points $(x_1, y_1), \ldots, (x_n, y_n)$, where $x_i < x_{i+1}$. An interpolation function is any function, $y(x)$, satisfying $y(x_i) = y_i$ for $i = 1, \ldots n$. A cubic spline interpolation function is one for which the second derivative $y''(x)$ exists, is continuous on the interval $[x_1, x_n]$, and linear on the subintervals $[x_i, x_{i+1}]$. These conditions can be satisfied by setting

$$y(x) = A_i(x)y_i + B_i(x)y_{i+1} + C_i(x)\lambda_i + D_i(x)\lambda_{i+1}$$

for $x_i \leq x \leq x_{i+1}$, where the λ_i are constant and

$$\begin{aligned}
A_i(x) &= \frac{x_{i+1} - x}{x_{i+1} - x_i} \\
B_i(x) &= \frac{x - x_i}{x_{i+1} - x_i} = 1 - A_i(x) \\
C_i(x) &= \frac{1}{6}\left(A_i(x)^3 - A_i(x)\right)(x_{i+1} - x_i)^2 \\
D_i(x) &= \frac{1}{6}\left(B_i(x)^3 - B_i(x)\right)(x_{i+1} - x_i)^2.
\end{aligned}$$

From this definition it follows that $y(x)$ is a cubic polynomial on each subinterval $[x_i, x_{i+1}]$ with first-order derivative

$$y'(x) = \frac{y_{i+1} - y_i}{x_{i+1} - x_i} + \frac{x_{i+1} - x_i}{6}\left((3B_i(x)^2 - 1)\lambda_{i+1} - (3A_i(x)^2 - 1)\lambda_i\right),$$

and second-order derivative

$$y''(x) = A_i(x)\lambda_i + B_i(x)\lambda_{i+1}.$$

Hence $y''(x)$ is continuous and piecewise linear as required, and $y''(x_i) = \lambda_i$. Imposing continuity on $y'(x)$ leads to the linear equations

$$\frac{x_i - x_{i-1}}{6}\lambda_{i-1} + \frac{x_{i+1} - x_{i-1}}{3}\lambda_i + \frac{x_{i+1} - x_i}{6}\lambda_{i+1} = \frac{y_{i+1} - y_i}{x_{i+1} - x_i} - \frac{y_i - y_{i-1}}{x_i - x_{i-1}}$$

for $i = 2, \ldots, n-1$. If two of the λ_i are specified then this tridiagonal system of linear equations has a unique solution. In the Adaptiv implementations, the second-order derivative is set to zero at the boundary points x_1 and x_n.

Adaptiv uses either a constant or linear method to extrapolate the curve. The constant method sets $y(x) = y_1$ for $x < x_1$ and $y(x) = y_n$ for $x > x_n$. This method is used when interpolating the strike axis of a volatility surface. Yield curves are extrapolated using the linear method, which extends the gradient of the curve at the end points: $y(x) = y_1 + y'(x_1)(x - x_1)$ for $x < x_1$, and $y(x) = y_n + y'(x_n)(x - x_n)$

for $x > x_n$. Since $\lambda_1 = 0 = \lambda_n$ and $B_1(x_1) = 0 = A_{n-1}(x_n)$, the gradients at the end points are given by

$$y'(x_1) = \frac{y_2 - y_1}{x_2 - x_1} - \frac{(x_2 - x_1)\lambda_2}{6}$$

$$y'(x_n) = \frac{y_n - y_{n-1}}{x_n - x_{n-1}} + \frac{(x_n - x_{n-1})\lambda_{n-1}}{6}.$$

APPENDIX B

European Option Payoff

B.1. Lognormal Assets (Black and Margrabe)

The Black formula is obtained by evaluating the expectation of $\max(X-K,0)$ for a lognormal variable X and constant K. This appendix gives some general results on evaluating expectations of this type. Let $\mathcal{B}(F,K,v)$ denote the 'Black' function

$$\mathcal{B}(F,K,v) = F\mathcal{N}(d+v/2) - K\mathcal{N}(d-v/2),$$

where $d = \log(F/K)/v$ and $\mathcal{N}$ denotes the standard normal distribution function $\mathcal{N}(x) = \int_{-\infty}^{x} dz \exp(-z^2/2)/\sqrt{2\pi}$.

Lemma B.1. *Let Z_1 and Z_2 be standard normal variables with $\mathbb{E}(Z_1Z_2) = \rho$. Define $X_i = x_i \exp(-\sigma_i^2/2 + \sigma_i Z_i)$, where the x_i and σ_i are positive constants. Then*

$$\begin{aligned}
\mathbb{P}(X_1 > X_2) &= \mathcal{N}\left(d - (\sigma_1^2 - \sigma_2^2)/2v\right)\\
\mathbb{E}\left(X_2 1_{\{X_1>X_2\}}\right) &= x_2\mathcal{N}(d-v/2)\\
\mathbb{E}\left(X_1 1_{\{X_1>X_2\}}\right) &= x_1\mathcal{N}(d+v/2)
\end{aligned}$$

where $v^2 = \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2$ is the variance of $\log(X_1/X_2)$ and $d = \log(x_1/x_2)/v$.

A special case of this Lemma is $\mathbb{E}(\max(X_1 - X_2, 0)) = \mathcal{B}(x_1, x_2, v)$, which is the Margrabe formula for the value of an option on the spread $X_1 - X_2$ (see Margrabe [**25**]). Furthermore, when X_2 is constant, so that $X_2 = x_2$ and $\sigma_2 = 0$, then we have the Black formula for a call option with strike x_2. Similarly, making X_1 constant gives the Black formula for a put option with strike x_1.

Proof. For the probability of $\{X_1 > X_2\}$, observe that $X_1 > X_2$ if and only if

$$\sigma_1 Z_1 - \sigma_2 Z_2 > -\log(x_1/x_2) + (\sigma_1^2 - \sigma_2^2)/2$$

and $\sigma_1 Z_1 - \sigma_2 Z_2$ is normal with standard deviation v^2.

For the second result, define $W_1 = (Z_1 - \rho Z_2)/\kappa$, where $\kappa = \sqrt{1-\rho^2}$. Then W_1 and Z_2 are independent standard normal variables and $Z_1 = \kappa W_1 + \rho Z_2$. Some algebraic manipulation gives that $X_1 > X_2$ if and only if

$$\frac{\kappa\sigma_1 W_1 + vd - v^2/2}{\sigma_2 - \rho\sigma_1} > Z_2 - \sigma_2.$$

Hence the expectation of $X_2 1_{\{X_1>X_2\}}/x_2$ is given by

$$\mathbb{E}\left(\mathbb{E}\left(\exp(\sigma_2 Z_2 - \sigma_2^2/2)1_{\{X_1>X_2\}}|W_1\right)\right) = \mathbb{E}\left(\mathcal{N}\left(\frac{\kappa\sigma_1 W_1 + vd - v^2/2}{\sigma_2 - \rho\sigma_1}\right)\right)$$

That $\mathbb{E}\left(X_2 1_{\{X_1>X_2\}}\right) = x_2\mathcal{N}(d-v/2)$ now follows with an application of Lemma B.2.

The expectation of $X_1 1_{\{X_1>X_2\}}$ can be deduced by writing

$$X_1 1_{\{X_1>X_2\}} = X_1 - X_1 1_{\{X_2>X_1\}}$$

and then interchanging the roles of X_1 and X_2 in the second result to give

$$\mathbb{E}\left(X_1 1_{\{X_2 > X_1\}}\right) = x_1 \mathcal{N}(-d - v/2).$$

Lemma B.2. *If Z is a standard normal variable and a and b are constants then*

$$\mathbb{E}\left(\mathcal{N}(a + bZ)\right) = \mathcal{N}\left(a(1 + b^2)^{-1/2}\right).$$

Proof. Choose a standard normal variable W that is independent of Z. Then

$$\mathbb{E}\left(\mathcal{N}(a + bZ)\right) = \mathbb{E}\left(\mathbb{E}\left(1_{\{W \le a + bZ\}} | Z\right)\right) = \mathbb{E}\left(1_{\{W - bZ \le a\}}\right)$$

and the standard deviation of $W - bZ$ is $(1 + b^2)^{1/2}$.

B.2. Normal Assets

The following result is the analogue of Lemma B.1 for normally-distributed asset prices.

Let $\mathcal{N}'$ denote the normal density function $\mathcal{N}'(z) = \exp(-z^2/2)/\sqrt{2\pi}$.

Lemma B.3. *Let Z_1 and Z_2 be standard normal variables with $\mathbb{E}(Z_1 Z_2) = \rho$. Define $X_i = x_i + \sigma_i Z_i$, where the x_i are constant and σ_i are positive constants. Let K be another constant (the strike). Then $\mathbb{P}(X_1 - X_2 > K) = \mathcal{N}(d)$ and*

$$\mathbb{E}\left(X_i 1_{\{X_1 - X_2 > K\}}\right) = x_i \mathcal{N}(d) + \frac{\operatorname{cov}(X_i, X_1 - X_2)}{v} \mathcal{N}'(d),$$

where $v^2 = \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2$ is the variance of $X_1 - X_2$ and $d = (x_1 - x_2 - K)/v$.

The covariances are given by $\operatorname{cov}(X_1, X_1 - X_2) = \sigma_1^2 - \rho\sigma_1\sigma_2$, and similarly $\operatorname{cov}(X_2, X_1 - X_2) = \rho\sigma_1\sigma_2 - \sigma_2^2$. Combining the results for X_1, X_2 and K gives the following formula for the expectation of the standard spread-option payoff:

$$\mathbb{E}\left(\max(X_1 - X_2 - K, 0)\right) = (x_1 - x_2 - K)\mathcal{N}(d) + v\mathcal{N}'(d).$$

Proof. Let Z be a standard normal variable and let X be a normal variable. Then $\mathbb{P}(\mu + \sigma Z > 0) = \mathcal{N}(\mu/\sigma)$ and

$$\mathbb{E}\left(X 1_{\{\mu + \sigma Z > 0\}}\right) = \mathbb{E}(X)\mathcal{N}(\mu/\sigma) + \operatorname{cov}(X, Z)\mathcal{N}'(\mu/\sigma).$$

To prove the second result, first prove the case $X = Z$ by integrating the normal density function $z \exp(-z^2/2)$ from $-\mu/\sigma$ to ∞. Then the general case follows by writing $X = \mathbb{E}(X) + \operatorname{cov}(X, Z)Z + Y$ so that Y is normal, independent of Z and $\mathbb{E}(Y) = 0$.

APPENDIX C

European Option Sensitivity

C.1. Generic Black Formula

Adaptiv prices standard-payoff European options using the Black formula. This appendix provides a summary of the commonly used derivatives of the Black formula.

The generic Black formula is

$$V = \pm \left[F\mathcal{N}(\pm d_1) - K\mathcal{N}(\pm d_2) \right] Z$$

where V is the value of the option, F is the underlying forward price or rate, K is the strike, Z is a discount factor, σ is the volatility, T is the time to the exercise date, $d_1 = d_2 + \sigma\sqrt{T}$,

$$d_2 = \frac{1}{\sigma\sqrt{T}} \log\left(\frac{F}{K}\right) - \frac{\sigma\sqrt{T}}{2}$$

and $\pm$ takes the value $+1$ for a call option and -1 for a put.

C.2. Delta

The first-order derivative of V with respect to the price:

$$\frac{\partial V}{\partial F} = \pm\mathcal{N}(\pm d_1)Z$$

C.3. Hedge Ratio

The ratio of the delta of the option to the delta of the underlying forward transaction:

$$\frac{\partial V/\partial F}{\partial U/\partial F} = \pm\mathcal{N}(\pm d_1)$$

where $U = (F - K)Z$ is the value of the underlying forward transaction.

C.4. Gamma

The second-order derivative of V with respect to the price:

$$\frac{\partial^2 V}{\partial F^2} = \frac{\mathcal{N}'(d_1)}{F\sigma\sqrt{T}}Z$$

where the derivative of the normal distribution function is given by

$$\mathcal{N}'(x) = \frac{\exp(-x^2/2)}{\sqrt{2\pi}}$$

C.5. Vega

The first-order derivative of V with respect to the volatility:

$$\frac{\partial V}{\partial \sigma} = F\sqrt{T}\mathcal{N}'(d_1)Z$$

For interest-rate and FX options, the derivative is multiplied by 0.001, a ten basis-point upward shift in volatility; for equity options the derivative is multiplied by 0.01, a percentage point shift in volatility.

C.6. Option Theta

The first-order derivative of V with respect to the time-to-exercise:

$$\frac{\partial V}{\partial T} = \frac{F\sigma\mathcal{N}'(d_1)}{2\sqrt{T}}Z$$

The derivative is usually multiplied by $-1/365.25$, a downward shift of one day.

APPENDIX D

Average-Rate Volatility

The BGM model (see Section 2.7.3) provides a convenient theoretical framework in which to analyse the volatility of an average deposit rate and derive formula (2.8) of Section 2.7.5. However, the reader should understand that the pricing formula (2.7) is based on an analogy with the Black formula for standard caplets, and is *not* a closed-form solution derived within the BGM model.

For a fixed tenor δ, let $r_T(t)$ denote the value at time t of the deposit rate for the period T to $T+\delta$. Under the BGM model, these δ-period rates are simultaneously lognormal and given by

$$r_T(t) = r_T(0)\exp\left(-\frac{1}{2}\int_0^t \lambda_T(s)^2 ds + \int_0^t \lambda_T(s)\cdot[\Sigma_{T+\delta}(s)ds + dW(s)]\right),$$

where: $\lambda_T(t)$ is a deterministic volatility for $r_T(t)$, $\Sigma_{T+\delta}(t)$ is the (stochastic) volatility of the $(T+\delta)$-maturity zero-coupon bond price, and $W(t)$ is a Brownian motion under the risk-neutral measure. Moreover, $dW_{T+\delta}(t) = \Sigma_{T+\delta}(t)dt + dW(t)$ is a Brownian motion under the $(T+\delta)$-forward measure, and it follows that the price of any δ-period caplet is given by the Black formula of Section 2.7.2 with $\sigma^2 T = \int_0^T \lambda_T(s)^2 ds$.

Let $T_1 < \cdots < T_n$ denote the fixing dates of the constituent rates of the average. Define $r_i(t) = r_{T_i}(\min(t, T_i))$; let $\chi_i(t)$ denote the indicator function of the interval $(-\infty, T_i)$, which takes the value 1 if $t \le T_i$, and otherwise 0; and, for the sake of brevity, let $\lambda_i(t)$ denote $\lambda_{T_i}(t)$. Let $\omega_1, \ldots, \omega_n$ be positive weighting factors satisfying $\sum_i \omega_i = 1$. The arithmetic average $\bar{r}(t) = \sum_i \omega_i r_i(t)$ will be approximated by the geometric average $\tilde{r}(t) = \prod_i r_i(t)^{\omega_i}$, which is given by

$$\tilde{r}(t) = \tilde{r}(0)\exp\left(\int_0^t \mu(s)ds + \sum\nolimits_{i>k} \omega_i \int_0^t \chi_i(s)\lambda_i(s)\cdot dW(s)\right),$$

where $r_1(0), \ldots, r_k(0)$ are the known rates (having $T_i \le 0$ for $i \le k$) and $\mu(t)$ is a stochastic drift term. Hence the geometric average has a deterministic volatility vector $\sum_{i>k} \omega_i \chi_i(t)\lambda_i(t)$; and the variance of $\log \tilde{r}(T_n)$ is equal to

$$\sum\nolimits_{i,j>k} \omega_i\omega_j \int_0^{\min(T_i,T_j)} \lambda_i(s)\cdot\lambda_j(s)ds = \sum\nolimits_{i,j>k} \left(\omega_i\sigma_i\sqrt{T_i}\right)\left(\omega_j\sigma_j\sqrt{T_j}\right)\rho_{ij},$$

where $\sigma_i^2 T_i = \int_0^{T_i} \lambda_i(s)^2 ds$ is the variance of $\log r_i(T_i)$ and the correlation coefficients ρ_{ij} are defined by

$$\rho_{ij} = \frac{\int_0^{\min(T_i,T_j)} \lambda_i(s)\cdot\lambda_j(s)ds}{\left(\int_0^{T_i} \lambda_i(s)^2 ds\right)^{1/2}\left(\int_0^{T_j} \lambda_j(s)^2 ds\right)^{1/2}}.$$

In the case that the model's volatility parameter $\lambda_T(t)$ is independent of T, that is $\lambda_T(t) = \lambda(t)$, then $\rho_{ij} = (\sigma_i\sqrt{T_i})/(\sigma_j\sqrt{T_j})$ for $i < j$.

APPENDIX E

Barrier Options

E.1. Black-Scholes PDE

The simplest way to derive the price formula of Section 2.14 is to exhibit them as solutions of the Black-Scholes partial differential equation satisfying the appropriate terminal and boundary conditions.

It will be assumed that:

- the volatility $\sigma(t)$ of the spot FX rate is deterministic
- the zero-coupon bond prices are given by $Z(t,T_1) = \exp\left(-\int_t^{T_1} r(s)ds\right)$ and $\widetilde{Z}(t,T_1) = \exp\left(-\int_t^{T_1} \tilde{r}(s)ds\right)$, where the short-term rates r and $\tilde{r}$ are deterministic.

Under these conditions, the option price $V(S,t)$ satisfies the Black-Scholes PDE:

$$\frac{\partial V}{\partial t} + [r(t) - \tilde{r}(t)]S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma(t)^2 S^2 \frac{\partial^2 V}{\partial S^2} = r(t)V \tag{E.1}$$

which is equivalent to

$$\frac{\partial U}{\partial t} + \frac{1}{2}\frac{\partial^2 U}{\partial F^2} = 0 \tag{E.2}$$

where $U(F,t)Z(t,T_1) = V(S,t)$ and F is the forward rate

$$F = S\frac{\widetilde{Z}(t,T_1)}{Z(t,T_1)}$$

E.2. European Option

Consider the European option described in Section 2.14.2. Under the change of variables $F = K\,\mathrm{e}^x$ and $\tau = \int_t^T \sigma(s)^2 ds$, the Black-Scholes equation (E.2) becomes

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\right) \tag{E.3}$$

where $u(x,\tau) = U(F,t)$. The terminal condition for the European option is

$$U(F,T) = \begin{cases} AF + B & \text{if } \pm(F-K) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

which becomes

$$u(x,0) = \begin{cases} AK\,\mathrm{e}^x + B & \text{if } \pm x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{E.4}$$

It is easily verified that

$$u_2(x,\tau) = \mathcal{N}\left(\pm\left(\frac{x}{\sqrt{\tau}} - \frac{\sqrt{\tau}}{2}\right)\right)$$

Type	**Terminal Condition**	**Boundary Conditions**
Down	$V(S,T) = 0$ for $H < S$	$V(H,t) = E_K(H,t)$ $V(S,t) \to 0$ as $S \to \infty$
Up	$V(S,T) = 0$ for $S < H$	$V(H,t) = E_K(H,t)$ $V(S,t) \to 0$ as $S \to 0$
Down Rebate	$V(S,T) = \Phi(T)$ for $H < S$	$V(H,t) = 0$ $V(S,t) \to \Phi(t)$ as $S \to \infty$
Up Rebate	$V(S,T) = \Phi(T)$ for $S < H$	$V(H,t) = 0$ $V(S,t) \to \Phi(t)$ as $S \to 0$

TABLE E.1. terminal and boundary conditions for single barrier knock-in options.

is a solution to Equation (E.3). If $u(x,\tau)$ is a solution then $\mathrm{e}^x\, u(-x,\tau)$ is also a solution (this transformation corresponds to interchanging the order of the currencies). Define

$$u_1(x,\tau) = K\,\mathrm{e}^x\,[1 - u_2(-x,\tau)] = K\,\mathrm{e}^x\,\mathcal{N}\left(\pm\left(\frac{x}{\sqrt{\tau}} + \frac{\sqrt{\tau}}{2}\right)\right)$$

Then $Au_1(x,\tau) + Bu_2(x,\tau)$ is a solution to Equation (E.3) satisfying the initial condition (E.4).

E.3. Single Barrier

E.3.1. Black-Scholes PDE. The closed-form pricing formulae of Section 2.14.4 are derived under the assumption that $\sigma(t)$, $r(t)$ and $\tilde{r}(t)$ are constant. Under the change of variables $S = H\,\mathrm{e}^x$ and $\tau = \sigma^2(T-t)$, the Black-Scholes equation (E.1) is equivalent to

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + a\frac{\partial u}{\partial x} \tag{E.5}$$

or

$$\frac{\partial v}{\partial \tau} + \frac{r}{\sigma^2}v = \frac{1}{2}\frac{\partial^2 v}{\partial x^2} + a\frac{\partial v}{\partial x} \tag{E.6}$$

where $u(x,\tau)Z(t,T_1) = V(S,t) = v(x,\tau)$ and $a = -1/2 + (r-\tilde{r})/\sigma^2$.

E.3.2. Knock-in Option and Rebate. If $u(x,\tau)$ is a solution to Equation (E.5) then $\mathrm{e}^{-2ax}\,u(c-x,\tau)$ is also a solution for any constant c. It follows that

$$\left(\frac{H}{S}\right)^{2a} E_K(H^2/S,t)$$

is a solution of Equation (E.1), where $E_K(S,t)$ denotes the price of the European option described in Section 2.14.2. Let $\Phi(t)$ denote the price of the underlying cashflows:

$$\Phi(t) = A\widetilde{Z}(t,T_1)S + BZ(t,T_1)$$

Then the pricing formulae in Table 2.2 are solutions of the Black-Scholes equation (E.1) satisfying the terminal and boundary conditions given in Table E.1.

Type	Terminal Condition	Boundary Conditions
Down	$V(S,T)=0$ for $H<S$	$V(H,t)=AH+B$ $V(S,t)\to 0$ as $S\to\infty$
Up	$V(S,T)=0$ for $S<H$	$V(H,t)=AH+B$ $V(S,t)\to 0$ as $S\to 0$

TABLE E.2. terminal and boundary conditions for single barrier knock-out rebates.

E.3.3. Knock-out Rebate. The knock-out rebate formula of Section 2.14.4 satisfies the terminal and boundary conditions given in Table E.2. It is also easily verified that

$$\mathrm{e}^{-(a+k)x}\,\mathcal{N}\left(\pm\left(-\frac{x}{\sqrt{\tau}}+k\sqrt{\tau}\right)\right)$$

is a solution to Equation (E.6) if and only if $k^2=a^2+2r/\sigma^2$. This gives two solutions of Equation (E.6) corresponding to $k=b$ and $k=-b$. It follows that formula (2.21) of Section 2.14.4 satisfies the Black-Scholes equation (E.1).

E.4. Double Barrier

The pricing formula of Section 2.14.5 is derived under the assumption that $\sigma(t)$, $r(t)$ and $\tilde{r}(t)$ are constant. Under the change of variables $S=H_1\,\mathrm{e}^x$ and $\tau=\sigma^2(T-t)$, where H_1 denotes the lower barrier rate, the Black-Scholes equation (E.1) becomes

$$\frac{\partial u}{\partial \tau}=\frac{1}{2}\frac{\partial^2 u}{\partial x^2}+a\frac{\partial u}{\partial x}$$

where $u(x,\tau)Z(t,T_1)=V(S,t)$ and $a=-1/2+(r-\tilde{r})/\sigma^2$. Defining $w(x,\tau)=\exp(ax+a\tau^2/2)u(x,\tau)$, the PDE simplifies to

$$\frac{\partial w}{\partial \tau}=\frac{1}{2}\frac{\partial^2 w}{\partial x^2} \tag{E.7}$$

In the new variables, the barriers are located at $x=0$ and $x=c\pi$, hence the boundary conditions for the knock-out option are $w(0,\tau)=0=w(c\pi,\tau)$. A solution is required for the region $0\le x/c\le\pi$. The initial condition can be extended to make an odd function on the interval $-\pi\le x/c\le\pi$ in order to seek a solution of the form

$$w(x,\tau)=\sum_{n=1}^{\infty}b_n(\tau)\sin(nx/c)$$

Substituting this series into Equation (E.7) gives

$$b_n'(\tau)=-\frac{1}{2}\left(\frac{n}{c}\right)^2 b_n(\tau)$$

and hence $b_n(\tau)=b_n(0)\,\mathrm{e}^{-n^2\tau/c^2}$. It remains to calculate the Fourier coefficients of the initial function $w(x,0)$; these are given by

$$b_n(0)=\frac{2}{\pi}\int_0^{\pi}w(cy,0)\sin(ny)dy$$

The terminal condition for the knock-out option is

$$V(S,T)=\begin{cases}(AF+B)Z(T,T_1) & \text{if } \pm(F-K)\ge 0 \text{ and } H_1<S<H_2\\ 0 & \text{otherwise}\end{cases}$$

where $F=S\exp\left[(r-\tilde{r})(T_1-T)\right]$. This is equivalent to the initial condition

$$w(x,0)=\begin{cases}A\rho H_1\,\mathrm{e}^{(a+1)x}+B\,\mathrm{e}^{x} & \text{if } \pm(x/c-\kappa)\ge 0\\ 0 & \text{otherwise}\end{cases}$$

where $\rho = \exp\left[(r - \tilde{r})(T_1 - T)\right]$ and $c\kappa = \log K - \log(\rho H_1)$.

APPENDIX F

Tree Transitions

This appendix states and proves the theorem referred to in Section 2.17.3, which justifies the method used by Adaptiv to construct a tree for pricing interest-rate derivatives under the Hull-White or Black-Karasinski models.

Consider the following system of linear equations

$$p_1 + p_2 + p_3 = 1 \tag{F.1}$$

$$j_1 p_1 + j_2 p_2 + j_3 p_3 = j\mu \tag{F.2}$$

$$j_1^2 p_1 + j_2^2 p_2 + j_3^2 p_3 = (j\mu)^2 + (1-\mu^2)(N/C)^2 \tag{F.3}$$

in the variables p_1, p_2, p_2, where N and C are positive integers, j_1, j_2, j_3 and j are integers, and $0 < \mu < 1$.

Theorem F.1. *If $C \geq 2$, $N \geq C^2$ and $-N \leq j \leq N$ then there exists j_1, j_2, j_3 satisfying*

$$-N \leq j_1 < j_2 < j_3 \leq N$$

such that j_2 is the closest integer to $j\mu$ and the system of linear equations (F.1), (F.2) and (F.3) has a non-negative solution: $p_1 \geq 0$, $p_2 \geq 0$ and $p_3 \geq 0$.

Proof. The determinant and inverse of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ j_1 & j_2 & j_3 \\ j_1^2 & j_2^2 & j_3^2 \end{pmatrix}$$

are given by $\det A = (j_2 - j_1)(j_3 - j_2)(j_3 - j_1)$ and

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} j_2 j_3 \,(j_3 - j_2) & j_2^2 - j_3^2 & j_3 - j_2 \\ j_1 j_3 \,(j_1 - j_3) & j_3^2 - j_1^2 & j_1 - j_3 \\ j_1 j_2 \,(j_2 - j_1) & j_1^2 - j_2^2 & j_2 - j_1 \end{pmatrix}.$$

It follows that if j_1, j_2, j_3 are distinct then equations (F.1), (F.2) and (F.3) have the unique solution

$$p_1 = \frac{(f_3 - f\mu)\,(f_2 - f\mu) + \nu}{(f_2 - f_1)\,(f_3 - f_1)}$$

$$p_2 = \frac{-\,(f_3 - f\mu)\,(f_1 - f\mu) - v}{(f_3 - f_2)\,(f_2 - f_1)}$$

$$p_3 = \frac{(f_1 - f\mu)\,(f_2 - f\mu) + v}{(f_3 - f_1)\,(f_3 - f_2)},$$

where $\nu = (1-\mu^2)/C^2$, $f = j/N$ and $f_\ell = j_\ell/N$ for $\ell = 1, 2, 3$. This solution is non-negative if and only if

$$(-f_2 + f\mu)\,(f_3 - f\mu) \leq \nu \tag{F.4}$$

$$(-f_1 + f\mu)\,(f_3 - f\mu) \geq v \tag{F.5}$$

$$(-f_1 + f\mu)\,(f_2 - f\mu) \leq \nu. \tag{F.6}$$

Since equations (F.1), (F.2) and (F.3) are invariant under a change in sign of j_1, j_2, j_3 and j, it suffices to prove the theorem for $j \geq 0$.

Case 1. *$j = 0$ (transition from $x = 0$).*

Choose $j_2 = 0$ and set $j_1 = -j_3$. Then the transition probabilities are $p_2 = 1 - \nu/f_3^2$ and $p_1 = p_3 = \nu/2f_3^2$, and are non-negative if and only if

$$j_3^2 \geq \frac{N^2(1-\mu^2)}{C^2}.$$

Since $C \geq 1$ and $0 < \mu < 1$, this inequality is satisfied when $j_3 = N$. In practice, the value of j_3 is chosen so that p_2 is as close as possible to $2/3$.

Case 2. *$j = N$ and $\mu \geq 1 - 1/2N$ (mean close to the boundary).*

Choose $j_3 = N$ and $j_2 = N-1$. Then inequality (F.6) is satisfied for any $j_1 \leq N-2$.

Since $\nu = (1-\mu)(1+\mu)/C^2$ and $\mu < 1$, inequality (F.4) becomes

$$\frac{1}{N} \leq \left(1 + \frac{1}{C^2}\right) - \mu\left(1 - \frac{1}{C^2}\right). \tag{F.7}$$

The right-hand side of inequality (F.7) is greater than $2/C^2$ and therefore this inequality is satisfied when $N \geq C^2/2$.

Inequality (F.5) is equivalent to

$$\frac{j_1}{N} \leq \mu\left(1 - \frac{1}{C^2}\right) - \frac{1}{C^2}. \tag{F.8}$$

The right-hand side of inequality (F.8) is greater than $-1/C^2$ and therefore this inequality is satisfied when $j_1 = -N$. In practice, j_1 is chosen to be the largest integer satisfying (F.8) and $j_1 \leq N-2$.

Case 3. *$0 < j < N$ and $\mu \geq 1 - 1/2N$.*

Choose $j_2 = j$. Then inequality (F.4) is satisfied for any $j_3 > j_2$. Choose $j_3 = N$ in order to maximize the left-hand side of inequality (F.5). Then inequalities (F.5) and (F.6) become

$$A = f\mu - \frac{1+\mu}{fC^2} \leq f_1 \leq f\mu - \frac{\nu}{1-f\mu} = B$$

To prove the existence of a $f_1 = j_1/N$ satisfying these bounds, it suffices to show that $B \geq -1$ and $B - A \geq 1/N$. For the first estimate:

$$(1-f\mu)(B+1)C^2 = C^2\left(1-(f\mu)^2\right) - \left(1-\mu^2\right) \geq \left(C^2-1\right)\left(1-\mu^2\right)$$

and hence $B + 1 \geq 0$. The second estimate can be obtained as follows:

$$\begin{aligned}
N(B-A) &\geq C^2(B-A) \\
&= (1+\mu)\left(\frac{1}{f} - \frac{1-\mu}{1-f\mu}\right) \\
&\geq (2-1/2N)\left(\frac{1}{1-1/N} - \frac{1/2N}{1-(1-1/N)}\right) \\
&= \frac{(N-1/4)(N+1)}{(N-1)N} \\
&> 0.
\end{aligned}$$

Case 4. *$\mu < 1 - 1/2N$ and there exists a j_2 such that $0 \leq j_2 - j\mu \leq 1/2$ (closest integer to $j\mu$ is above).*

Then it follows that $j_2 < N$ and inequality (F.4) is satisfied for any $j_3 > j_2$. Choose $j_3 = N$ to maximize the left-hand side of inequality (F.5). Then inequalities (F.5) and (F.6) become

$$A = f\mu - \frac{\nu}{f_2 - f\mu} \le f_1 \le f\mu - \frac{\nu}{1 - f\mu} = B$$

To prove the existence of a $f_1 = j_1/N$ satisfying these bounds, it suffices to show that $B \ge -1$ and $B - A \ge 1/N$. The first estimate is identical to Case 3. The second estimate can be obtained as follows:

$$\begin{aligned} C^2(B-A) &= (1-\mu^2)\left(\frac{1}{f_2 - f\mu} - \frac{1}{1-f\mu}\right) \\ &= \frac{(1-\mu^2)(1-f_2)}{(f_2 - f\mu)(1-f\mu)} \\ &\ge \frac{(1-\mu^2)(1-f\mu-1/2N)}{(1/2N)(1-f\mu)} \\ &= (1-\mu^2)\left(2N - \frac{1}{1-f\mu}\right) \qquad \text{(F.9)} \\ &\ge \left(1-\left(1-\frac{1}{2N}\right)^2\right)\left(2N - \frac{1}{1-(1-1/N)(1-1/2N)}\right) \\ &= \left(1-\frac{1}{4N}\right)\left(2 - \frac{2}{3-1/N}\right) \\ &\ge \frac{21}{20} \end{aligned}$$

for $N \ge 2$. Hence $N(B-A) \ge C^2(B-A) > 1$ for $N \ge C^2$ and $C \ge 2$.

Case 5. *$\mu < 1 - 1/2N$ and there exists a j_2 such that $0 \le j\mu - j_2 \le 1/2$ (closest integer to $j\mu$ is below).*

Then $j_2 < N$ and inequality (F.6) is satisfied for any $j_1 < j_2$. Choose $j_1 = -N$ in order to maximize the left-hand side of inequality (F.5). Then inequalities (F.4) and (F.5) become

$$A = f\mu + \frac{\nu}{1 + f\mu} \le f_3 \le f\mu + \frac{\nu}{f\mu - f_2} = B$$

To prove the existence of a $f_3 = j_3/N$ satisfying these bounds, it suffices to show that $A \le 1$ and $B - A \ge 1/N$. The first estimate follows from

$$(1+f\mu)(1-A)C^2 = C^2\left(1-(f\mu)^2\right) - \left(1-\mu^2\right) \ge \left(C^2-1\right)\left(1-\mu^2\right).$$

The second estimate follows because

$$\begin{aligned} C^2(B-A) &= (1-\mu^2)\left(\frac{1}{f\mu - f_2} - \frac{1}{1+f\mu}\right) \\ &= \frac{(1-\mu^2)(1+f_2)}{(f\mu - f_2)(1+f\mu)} \\ &\ge \frac{(1-\mu^2)(1+f\mu-1/2N)}{(1/2N)(1+f\mu)} \\ &= (1-\mu^2)\left(2N - \frac{1}{1+f\mu}\right) \end{aligned}$$

and the last expression is greater than (F.9) of Case 4.

APPENDIX G

GECF Bucketing

G.1. Large GECF Matrices

The size of a portfolio's matrix of gamma-equivalent cashflows (GECF) is of the order of the number of cashflows squared. For example, a portfolio of 100 caps and swaptions will return at least $10,000$ gamma-equivalent cashflows. Therefore, the calculation of interest rate gamma by a direct implementation of Equation (4.7) would be too slow, even for a medium-sized portfolio of options.

To improve performance, the gamma-equivalent cashflows are first allocated to a two-dimensional grid of dates. This limits the size of the GECF matrix to the number of grid points. Moreover, with knowledge of discount factor interpolation formula, the bucketing can be done without significant loss of accuracy. The process of allocating results to a grid of dates is usually referred to as *bucketing*.

The IR Sensitivity report implements two bucketing methods corresponding to the two standard yield curve interpolation methods described in Section 1.2.3.

G.2. Bucketing Methods

G.2.1. Bucketing Grid. It is assumed that the grid to which the cashflows are allocated is at least as fine as the grid of dates generated by the bootstrapping process. Then any discount factor returned by the yield curve can be obtained by interpolating between discount factors at adjacent grid dates.

G.2.2. Constant Daily Forward Rate. In order to simplify the analysis, suppose the present value V is a function of a single discount factor Z at date D, where D lies between two adjacent grid dates D_k and D_{k+1}. Then Z is given by

$$\log Z = \lambda_k \log P_k + \lambda_{k+1} \log P_{k+1} \tag{G.1}$$

where P_k denotes the discount factor at D_k,

$$\lambda_k = \frac{D_{k+1} - D}{D_{k+1} - D_k}$$

and $\lambda_{k+1} = 1 - \lambda_k$. Thus, the delta-equivalent cashflow $V'(Z)$ can be replaced by two bucketed delta-equivalent cashflows: $\partial V/\partial P_k$ and $\partial V/\partial P_{k+1}$. These are given by

$$\frac{\partial V}{\partial P_\ell} = V'(Z)\frac{\partial Z}{\partial P_\ell} = V'(Z)\frac{\lambda_\ell Z}{P_\ell}$$

for $\ell = k$ and $\ell = k + 1$.

For the general case, suppose V is a function of discount factors $Z_1, \ldots, Z_n$. Each discount factor Z_i is a function of two discount factors $P_{k(i)}$ and $P_{k(i)+1}$ at adjacent grid dates. The bucketed delta-equivalent cashflows are given by

$$\frac{\partial V}{\partial P_k} = \sum\nolimits_{i=1}^{n} \frac{\partial V}{\partial Z_i} \frac{\partial Z_i}{\partial P_k}$$

Hence, each original delta-equivalent cashflow $\partial V/\partial Z_i$ contributes two bucketed delta-equivalent cashflows:

$$\frac{\partial V}{\partial Z_i} \frac{\lambda_k Z_i}{P_k} \tag{G.2}$$

for $k = k(i)$ and $k = k(i) + 1$.

The bucketed gamma-equivalent cashflows are given by

$$\frac{\partial^2 V}{\partial P_k \partial P_\ell} = \sum\nolimits_{i,j=1}^{n} \frac{\partial^2 V}{\partial Z_i \partial Z_j} \frac{\partial Z_i}{\partial P_k} \frac{\partial Z_j}{\partial P_\ell} + \sum\nolimits_{i=1}^{n} \frac{\partial V}{\partial Z_i} \frac{\partial^2 Z_i}{\partial P_k \partial P_\ell}$$

Hence, each original gamma-equivalent cashflow $\partial^2 V/\partial Z_i \partial Z_j$ contributes four bucketed gamma-equivalent cashflows:

$$\frac{\partial^2 V}{\partial Z_i \partial Z_j} \frac{\lambda_k Z_i}{P_k} \frac{\lambda_\ell Z_j}{P_\ell} \tag{G.3}$$

for: $k = k(i)$ and $\ell = k(j)$; $k = k(i) + 1$ and $\ell = k(j)$; $k = k(i)$ and $\ell = k(j) + 1$; $k = k(i) + 1$ and $\ell = k(j) + 1$. The contribution of the delta-equivalent cashflow $\partial V/\partial Z_i$ to the bucketed gamma-equivalent cashflows is small and is neglected by the IR Sensitivity report.

G.2.3. Linear on Zero-coupon Rates. Under this method, the interpolated discount factor is also given by Equation (G.1) with

$$\lambda_k = \frac{(D - D_0)(D_{k+1} - D)}{(D_k - D_0)(D_{k+1} - D_k)}$$
$$\lambda_{k+1} = \frac{(D - D_0)(D - D_k)}{(D_{k+1} - D_0)(D_{k+1} - D_k)}$$

where D_0 is the valuation date. Hence the equivalent cashflows are allocated to the grid using formulae (G.2) and (G.3) with different values for the λ_k.

APPENDIX H

Risk-Factor Statistics

H.1. Single Risk-Factor Statistics

H.1.1. Drift, Volatility, Skewness and Kurtosis. Let $r(t)$ denote the h-day risk-factor returns defined by $r(t) = y(t) - y(t-h)$, where x is the factor value and either $y = \log x$ for lognormal factors or $y = x$ for normal factors, and $t = 1, \ldots, T$.

The drift μ, volatility σ, skewness α_3 and kurtosis α_4 are given by

$$\begin{aligned}
\mu &= \frac{1}{T} \sum\nolimits_{t=1}^{T} r(t) \\
\sigma^2 &= \frac{1}{W(\lambda, T)} \sum\nolimits_{t=1}^{T} \lambda^{T-t} \left(r(t) - \mu\right)^2 \\
\sigma^3 \alpha_3 &= \frac{1}{\sum_{t=1}^{T} \lambda^{T-t}} \sum\nolimits_{t=1}^{T} \lambda^{T-t} \left(r(t) - \mu\right)^3 \\
\sigma^4 (\alpha_4 + 3) &= \frac{1}{\sum_{t=1}^{T} \lambda^{T-t}} \sum\nolimits_{t=1}^{T} \lambda^{T-t} \left(r(t) - \mu\right)^4
\end{aligned}$$

where λ is the decay factor and

$$W(\lambda, T) = \begin{cases} T - 1 & \text{if } \lambda = 1 \\ \sum_{t=1}^{T} \lambda^{T-t} = (1 - \lambda^T)/(1 - \lambda) & \text{for } 0 < \lambda < 1 \end{cases}$$

Note that Adaptiv uses the factor $\sum_{t=1}^{T} \lambda^{T-t}$ rather than $1/(1 - \lambda)$; this ensures that reasonable values are returned when T is small.

H.1.2. Mean Reversion. The mean-reversion statistics are obtained by calculating the linear regression of the returns $r(t)$ with respect to the (logarithmic) factor values $y(t)$ according to the equation $r = \alpha(\beta - y) + \gamma e$, where $\mathbb{E}(e) = 0$, $\mathrm{var}(e) = 1$ and $\mathrm{cov}(e, y) = 0$ (see Section 6.4.3). The mean-reversion rate α, long-run mean β and mean-reversion volatility γ are given by

$$\begin{aligned}
\alpha \sigma_y^2 &= -C_{ry} \\
\alpha \beta &= \mu_r + \alpha \mu_y \\
\gamma^2 &= \sigma_r^2 + \alpha^2 \sigma_y^2
\end{aligned}$$

where

$$\begin{aligned}
C_{ry} &= \frac{1}{\sum_{t=1}^{T} \lambda^{T-t}} \sum\nolimits_{t=1}^{T} \lambda^{T-t} \left(r(t) - \mu_r\right) \left(y(t) - \mu_y\right) \\
\mu_y &= \frac{1}{T} \sum\nolimits_{t=1}^{T} y(t) \\
\sigma_y^2 &= \frac{1}{\sum_{t=1}^{T} \lambda^{T-t}} \sum\nolimits_{t=1}^{T} \lambda^{T-t} \left(y(t) - \mu_y\right)^2
\end{aligned}$$

and similarly for μ_r and σ_r.

H.1.3. RTDM Parameters. The parameters λ_1, λ_2, λ_3 and λ_4 are derived from the drift, volatility, skewness and kurtosis (see Appendix J).

H.2. Correlation

The correlation of the risk factor x_i with the risk factor x_j is given by

$$\rho_{ij} = \frac{\sum_{t=1}^{T} \lambda^{T-t} \left(r_i(t) - \mu_i\right)\left(r_j(t) - \mu_j\right)}{\left[\sum_{t=1}^{T} \lambda^{T-t} \left(r_i(t) - \mu_i\right)^2\right]^{1/2} \left[\sum_{t=1}^{T} \lambda^{T-t} \left(r_j(t) - \mu_j\right)^2\right]^{1/2}}$$

H.3. Equity Beta

The linear regression coefficient β of the stock-price returns $r(t) = S(t)/S(t-1)$ with respect to the index-price returns $q(t) = I(t)/I(t-1)$ (see Section 6.6) is given by

$$\beta = \frac{\sum_{t=1}^{T} \lambda^{T-t} \left(r(t) - \mu_r\right)\left(q(t) - \mu_q\right)}{\sum_{t=1}^{T} \lambda^{T-t} \left(q(t) - \mu_q\right)^2}$$

where $\mu_r = (1/T)\sum_{t=1}^{T} r(t)$ and $\mu_q = (1/T)\sum_{t=1}^{T} q(t)$. Note that Adaptiv uses proportional returns for this calculation rather than logarithmic returns.

APPENDIX I

Quadratic Variance-Covariance VaR

I.1. Introduction

The main result of this appendix is Proposition I.3, which gives the formulae for the cumulants of a quadratic function of Gaussian variables. There are two preliminary results. Lemma I.1 is the one-variable case. In this case, the quadratic has a non-central χ^2 distribution and its cumulants are calculated by expanding the log of the characteristic function. Lemma I.2 is some linear algebra, which is used in Proposition I.3 simultaneously to diagonalize the gamma matrix and the covariance matrix.

I.2. Cumulants of Random Variables

This section establishes notation and reviews some facts about cumulants.

Let $\mathcal{A}$ denote the algebra of random variables X for which $\mathbb{E}\left(|X|^n\right) < \infty$ for all n. The cumulant functions are a family of multi-linear functions on $\mathcal{A}$. The N-fold cumulant function is defined by

$$\langle X_1, \ldots, X_N \rangle = \sum (-1)^{n-1}(n-1)! \mathbb{E}(\Delta_1) \cdots \mathbb{E}(\Delta_n)$$

where the sum is over all $\Delta_1, \ldots, \Delta_n$ such that $\{1, \ldots, N\}$ is a disjoint union of $\Delta_1, \ldots, \Delta_n$ and $\min \Delta_1 < \cdots < \min \Delta_n$; and the notation $\mathbb{E}(\{i_1, \ldots, i_k\})$ means $\mathbb{E}(X_{i_1} \cdots X_{i_k})$, where $i_1 < \cdots < i_k$. The sequence $\Delta_1, \ldots, \Delta_n$ will be referred to as an ordered partition of $\{1, \ldots, N\}$. The definition can be inverted to obtain the following decomposition

$$\mathbb{E}(X_1 \cdots X_N) = \sum \langle \Delta_1 \rangle \cdots \langle \Delta_n \rangle$$

where the sum is over all ordered partitions $\Delta_1, \ldots, \Delta_n$ of $\{1, \ldots, N\}$; and the notation $\langle \{i_1, \ldots, i_k\} \rangle$ means $\langle X_{i_1}, \ldots, X_{i_k} \rangle$, where $i_1 < \cdots < i_k$. The first three cumulant functions are: the mean $\langle X_1 \rangle = \mathbb{E}(X_1)$, the covariance

$$\langle X_1, X_2 \rangle = \mathbb{E}(X_1 X_2) - \mathbb{E}(X_1)\mathbb{E}(X_2)$$

and

$$\begin{aligned} \langle X_1, X_2, X_3 \rangle = \mathbb{E}(X_1 X_2 X_3) &- \mathbb{E}(X_1 X_3)\mathbb{E}(X_2) - \mathbb{E}(X_1)\mathbb{E}(X_2 X_3) \\ &- \mathbb{E}(X_1 X_3)\mathbb{E}(X_2) + 2\mathbb{E}(X_1)\mathbb{E}(X_2)\mathbb{E}(X_3) \end{aligned}$$

The N-order *cumulant* of a random variable X is defined to be $\langle X, \ldots, X \rangle$, the N-fold cumulant function evaluated at $X, \ldots, X$. The first four cumulants of X are its mean, variance, skewness and kurtosis.

Recall that a Gaussian variable has skewness and kurtosis equal to zero. More generally, a set of random variables has a joint Gaussian distribution if and only if all its higher-order cumulants are zero. Thus, $X_1, \ldots, X_N$ have a joint Gaussian distribution if and only $\langle \Delta \rangle = 0$ for all subsets Δ of $\{1, \ldots, N\}$ with more than two elements. Hence, in the case of Gaussian variables with zero mean, the expectation $\mathbb{E}(X_1 \cdots X_N)$ can be decomposed into sums of products of the covariance $\langle X_i, X_j \rangle$.

Another important property of the truncated covariance functions is that $\langle X_1, \dots, X_N \rangle = 0$ if the variables X_i are pairwise independent. This fact is used in Proposition I.3.

Finally, it can be shown that the characteristic function of $X_1, \dots, X_N$, defined by

$$\mathbb{E}\left[\exp\left(\mathrm{i}\lambda_1 X_1 + \cdots + \mathrm{i}\lambda_N X_N\right)\right]$$

is equal to

$$\exp\left(\sum_{n_1,\dots,n_N=0}^{\infty} \frac{(\mathrm{i}\lambda_1)^{n_1}\cdots(\mathrm{i}\lambda_N)^{n_N}}{n_1!\cdots n_N!}\langle\overbrace{X_1,\dots,X_1}^{n_1},\dots,\overbrace{X_N,\dots,X_N}^{n_N}\rangle\right)$$

Hence, the cumulants of a variable can be calculated by taking derivatives of the log of its characteristic function. This is used in Lemma I.1.

I.3. Cumulants of a Gaussian Quadratic

Lemma I.1. *Let Z be a Gaussian variable with $\mathbb{E}(Z) = 0$ and $\mathbb{E}(Z^2) = 1$, and let $\alpha, \beta, \gamma \in \mathbb{R}$. Then the n-order cumulant of $\alpha + \beta Z + \gamma Z^2/2$ is equal to $\alpha + \gamma/2$ for $n = 1$, and equal to*

$$\frac{1}{2}n!\beta^2\gamma^{n-2} + \frac{1}{2}(n-1)!\gamma^n$$

for $n > 1$.

Proof. If μ is a constant, then $(Z + \mu)^2$ has the non-central χ^2 distribution with 1 degree of freedom and non-central parameter μ^2 and its characteristic function is given by

$$\mathbb{E}\left[\exp\left(\mathrm{i}\lambda(Z+\mu)^2\right)\right] = \left(\frac{1}{1-2\mathrm{i}\lambda}\right)^{1/2} \exp\left(\frac{\mathrm{i}\lambda\mu^2}{1-2\mathrm{i}\lambda}\right)$$

Writing

$$\mathrm{i}\lambda\left(\beta Z + \frac{\gamma Z^2}{2}\right) = \frac{\mathrm{i}\lambda\gamma}{2}\left(Z + \frac{\beta}{\gamma}\right)^2 - \frac{\mathrm{i}\lambda\beta^2}{2\gamma}$$

it follows that

$$\mathbb{E}\left[\exp\left(\mathrm{i}\lambda\left[\beta Z + \frac{\gamma Z^2}{2}\right]\right)\right] = \left(\frac{1}{1-\mathrm{i}\lambda\gamma}\right)^{1/2} \exp\left(\frac{(i\lambda)^2\beta^2}{2(1-\mathrm{i}\lambda\gamma)}\right)$$

Therefore, the log of the characteristic function of $\alpha + \beta Z + \gamma Z^2/2$ is equal to

$$\begin{aligned}
&\mathrm{i}\lambda\alpha - \frac{1}{2}\log(1-\mathrm{i}\lambda\gamma) + \frac{(\mathrm{i}\lambda)^2\beta^2}{2(1-\mathrm{i}\lambda\gamma)} \\
&= \mathrm{i}\lambda\alpha + \frac{1}{2}\sum\nolimits_{n=1}^{\infty}\frac{(\mathrm{i}\lambda\gamma)^n}{n} + \frac{1}{2}(\mathrm{i}\lambda)^2\beta^2\sum_{n=0}^{\infty}(\mathrm{i}\lambda\gamma)^n \\
&= \mathrm{i}\lambda\left(\alpha + \frac{\gamma}{2}\right) + \sum\nolimits_{n=2}^{\infty}\left[\frac{1}{2}n!\beta^2\gamma^{n-2} + \frac{1}{2}(n-1)!\gamma^n\right]\frac{(\mathrm{i}\lambda)^n}{n!}
\end{aligned}$$

Lemma I.2. *Let $X_1, \dots, X_n$ be Gaussian variables with $\mathbb{E}(X_j) = 0$. Define $C_{ij} = \mathrm{cov}(X_i, X_j)$ and let m denote the rank of the matrix C. Then there exists a $m \times n$ matrix D such that $C = D^*D$; and there exist Gaussian variables $Z_1, \dots, Z_m$ such that $\mathbb{E}(Z_i) = 0$, $\mathrm{cov}(Z_i, Z_j) = 1_{ij}$ and D^*Z has the same distribution as X.*

Proof. Since $C_{ij} = C_{ji}$ and

$$\sum_{ij} C_{ij}\alpha_i\alpha_j = \mathbb{E}\left[\left(\sum_i \alpha_i X_i\right)^2\right] \geq 0$$

the matrix C is symmetric and non-negative and therefore can be diagonalized by an orthogonal transformation. Thus, there exists an $n \times n$ matrix V such that $V^*V = 1 = VV^*$ and $C = V^*(\Lambda^*\Lambda)V$, where Λ is a $m \times n$ matrix satisfying $\Lambda_{ij} = 0$ for $i \neq j$ and $\Lambda_{ii} > 0$. Define $D = \Lambda V$ and $Y = VX$. Then $C = D^*D$ and

$$\operatorname{cov}(Y_i, Y_j) = \sum_{k,\ell} V_{ik} \operatorname{cov}(X_k, X_\ell) V_{j\ell} = (VCV^*)_{ij} = (\Lambda^*\Lambda)_{ij}$$

Hence define $Z_i = \Lambda_{ii}^{-1} Y_i$ for $i \leq m$, and let $\tilde{X}$ denote D^*Z. Then $\tilde{X}_1, \ldots, \tilde{X}_n$ are Gaussian, $\mathbb{E}(\tilde{X}_j) = 0$ and

$$\operatorname{cov}(\tilde{X}_i, \tilde{X}_j) = \sum_{k,\ell} D_{ki} \operatorname{cov}(Z_k, Z_\ell) D_{\ell j} = (D^*D)_{ij} = C_{ij}$$

Therefore $\tilde{X}$ has the same distribution as X.

Proposition I.3. *Let $X_1, \ldots, X_n$ be Gaussian variables with $\mathbb{E}(X_i) = c_i$ and $\operatorname{cov}(X_i, X_j) = C_{ij}$. Let $\alpha \in \mathbb{R}$, $a \in \mathbb{R}^n$ and let A be an $n \times n$ symmetric matrix. Then the N-order cumulant of $\alpha + a \cdot X + (X \cdot AX)/2$ is equal to*

$$\alpha + a \cdot c + \frac{1}{2} c \cdot Ac + \frac{1}{2} \operatorname{tr}(AC)$$

for $N = 1$, and equal to

$$\frac{1}{2} N!(a + Ac) \cdot C(AC)^{N-2}(a + Ac) + \frac{1}{2}(N-1)! \operatorname{tr}\left((AC)^N\right)$$

for $N > 1$.

Proof. The general case can be deduced from the special case $c = 0$ by setting $X = Y + c$ and using the identity

$$a \cdot X + \frac{1}{2} X \cdot AX = a \cdot c + \frac{1}{2} c \cdot Ac + (a + Ac) \cdot Y + \frac{1}{2} Y \cdot AY$$

So, without loss of generality, assume $c = 0$. First apply Lemma I.2. There exists a $m \times n$ matrix D such that $C = D^*D$. Choose the $m \times m$ orthogonal matrix U to diagonalize DAD^*:

$$UDAD^*U^* = B$$

where $B_{ij} = 0$ for $i \neq j$. There exist Gaussian variables $Z_1, \ldots, Z_m$ such that $\mathbb{E}(Z_i) = 0$, $\operatorname{cov}(Z_i, Z_j) = 1_{ij}$ and $\tilde{X} = D^*U^*Z$ has the same distribution as X. Define $b = UDa$. Then it follows that

$$a \cdot \tilde{X} + (\tilde{X} \cdot A\tilde{X})/2 = b \cdot Z + (Z \cdot BZ)/2$$

Since the $Z_1, \ldots, Z_m$ are independent,

$$\langle \alpha + b \cdot Z + (Z \cdot BZ)/2 \rangle_N = \langle \alpha \rangle_N + \sum_{i=1}^m \langle b_i Z_i + B_{ii} Z_i^2/2 \rangle_N$$

Now apply Lemma I.1 and the required formulae follow from the identities

$$\sum_{i=1}^m B_{ii}^N = \operatorname{tr}(B^N) = \operatorname{tr}\left(UD(AC)^{N-1}AD^*U^*\right) = \operatorname{tr}\left((AC)^N\right)$$

$$\sum_{i=1}^m b_i^2 B_{ii}^N = b \cdot B^N b = UDa \cdot UD(AC)^{N-1}AD^*U^*UDa = a \cdot C(AC)^N a$$

APPENDIX J

The RTDM Distribution

This appendix presents a summary of the four-parameter distribution of Ramberg et al. [31]. Adaptiv uses the RTDM distribution as an approximate P&L distribution for value-at-risk calculations, and also as a risk-factor distribution in Monte Carlo simulation.

The inverse of the RTDM distribution is defined by

$$R(p) = \lambda_1 + \frac{p^{\lambda_3} - (1-p)^{\lambda_4}}{\lambda_2} \tag{J.1}$$

where λ_1, λ_2, λ_3 and λ_4 are the parameters of the distribution. The distribution function F satisfies $F(R(p)) = p$ and hence the density function $f(x) = F'(x)$ is given by

$$f(R(p)) = \frac{1}{R'(p)} = \frac{\lambda_2}{\lambda_3 p^{\lambda_3 - 1} + \lambda_4 (1-p)^{\lambda_4 - 1}}$$

If X has the RTDM distribution then its mean, variance, skewness and kurtosis are given by

$$\mu = \mathbb{E}(X) = \lambda_1 + \frac{A}{\lambda_2} \tag{J.2}$$

$$\sigma^2 = \mathbb{E}((X-\mu)^2) = \frac{(B - A^2)}{\lambda_2^2} \tag{J.3}$$

$$\alpha_3 = \frac{\mathbb{E}((X-\mu)^3)}{\sigma^3} = \frac{C - 3AB + 2A^3}{(B-A^2)^{3/2}} \tag{J.4}$$

$$\alpha_4 = \frac{\mathbb{E}((X-\mu)^4)}{\sigma^4} = \frac{D - 4AC + 6A^2B - 3A^4}{(B-A^2)^2} \tag{J.5}$$

where

$$\begin{aligned}
A &= \frac{1}{1+\lambda_3} - \frac{1}{1+\lambda_4} \\
B &= \frac{1}{1+2\lambda_3} - 2\beta(1+\lambda_3, 1+\lambda_4) + \frac{1}{1+2\lambda_4} \\
C &= \frac{1}{1+3\lambda_3} - 3\beta(1+2\lambda_3, 1+\lambda_4) + 3\beta(1+\lambda_3, 1+2\lambda_4) - \frac{1}{1+3\lambda_4} \\
D &= \frac{1}{1+4\lambda_3} - 4\beta(1+3\lambda_3, 1+\lambda_4) + 6\beta(1+2\lambda_3, 1+2\lambda_4) \\
&\quad - 4\beta(1+\lambda_3, 1+3\lambda_4) + \frac{1}{1+4\lambda_4}
\end{aligned}$$

and β denotes the beta function defined by $\beta(x,y) = \int_0^1 p^{x-1}(1-p)^{y-1}dp$.

The procedure for fitting the distribution to given values of μ, σ, μ_3 and μ_4 is as follows. The beta function is evaluated using the identity $\beta(x,y)\Gamma(x+y) = \Gamma(x)\Gamma(y)$, where $\Gamma(x) = \int_0^\infty t^{x-1}\,\mathrm{e}^{-t}\,dt$ is the gamma function, which can be evaluated using the Lanczos approximation given in Press et al. [29]. The values of λ_3 and λ_4 are obtained by numerical solution of Equations (J.4) and (J.5), which are independent of λ_1 and λ_2; for this, Adaptiv uses the downhill simplex method described in [**29**]. The values of λ_1 and λ_2 are then given by $\lambda_2 = \sqrt{(B - A^2)}/\sigma$ and $\lambda_1 = \mu - A/\lambda_2$.

APPENDIX K

Random Number Generation

K.1. Random Sequences

A uniform random sequence is a sequence of numbers that are, in some sense, indistinguishable from a sequence of random variables that are independent and uniformly distributed on an interval of the real line (usually chosen to be the unit interval $[0,1]$). In practice, such a sequence is generated by a deterministic algorithm and will exhibit some degree of non-uniformity and correlation; various statistical tests can be used to measure this (see, for example, Niederreiter [**27**]).

A simple way to generate a uniform random sequence is to use a linear congruence relation, such as

$$x_n = (a x_{n-1} + c) (\bmod N) \tag{K.1}$$

where a and c are constant integers and N is a large integer. Commonly used parameters are: $a = 7^5$, $c = 0$ and $N = 2^{31} - 1$. Starting with a seed value x_0, Equation (K.1) generates a sequence of numbers between 0 and N. Setting $u_n = x_n/N$ gives a uniform random sequence on the unit interval $[0,1]$.

For applications to value-at-risk, one can afford to use a higher quality random number generator (RNG) than the simple linear congruence method described above. This is because the computational cost of the RNG is dwarfed by the time taken to prepare the market-data scenarios and value the portfolio under each scenario.

Adaptiv uses the Data Encryption Standard (DES) adapted to random number generation; this algorithm is well-studied, fast enough for VaR and *scaleable* in the sense that it can easily be altered to produce random numbers of any desired level of statistical quality. Adaptiv's implementation of DES closely follows Press et al. [**29**].

K.2. The DES Algorithm

The input to DES is a 64-bit integer that is treated as a concatenation of two 32-bit integers: the left (or high) word L and the right (or low) word R.

The algorithm proceeds by iterating through a series of rounds. In each round R is passed into a highly non-linear function (known as the *cipher* function) and the output is XORed with L; this result is assigned to the right word, and the original right word is assigned to the left. Thus, the input to each round (L, R) is mapped to $(R, L \oplus g(R))$, where $\oplus$ denotes bitwise XOR and g denotes the cipher function. Starting from two seed words, an element of the random sequence is generated after *four* rounds by setting

$$u = \frac{L \oplus R}{2^{32}}$$

The new values of L and R are then used as the seed for another four rounds, and the next element of the random sequence is generated in the same way.

The cipher function is defined by the following procedure: XOR the input word R with a constant word a; split the result into two 16-bit half-words h (high) and l (low); evaluate the expression $\neg(h * h) + l * l$, where the arithmetical operations $+$ and $*$ are modulo 2^{32} and $\neg$ denotes bitwise NOT of 32-bit words; then reverse the half-words of this value; XOR this result with another constant b; and finally, add $h * l$. Thus, $g(R)$ is given by

$$g(R) = ((\lambda|\eta) \oplus b) + h * l$$

where $|$ denotes concatenation of 16-bit words and the half-words h, l and η, λ are defined by:

$$R \oplus a = h|l$$
$$\neg(h * h) + l * l = \eta|\lambda$$

In fact, the cipher function is parameterized by the round number because different constants a and b are used in each of the four rounds.

The spreadsheet `FG_RNG.xls` contains an implementation of this algorithm in VB.

Bibliography

[1] *International Convergence of Capital Measurement and Capital Standards.* Basle Committee on Banking Supervision. July 1988. Bank for International Settlements, www.bis.org/publ.

[2] *Treatment of Potential Exposure for Off-Balance-Sheet Items.* Basle Committee on Banking Supervision. April 1995. Bank for International Settlements, www.bis.org/publ. 117, 118

[3] *Supervisory framework for the use of "Backtesting" in conjunction with the internal models approach to market risk capital requirements.* Basle Committee on Banking Supervision. January 1996. Bank for International Settlements, www.bis.org/publ. 116

[4] *The New Basel Capital Accord. January 2001.* Bank for International Settlements, www.bis.org/publ. 117

[5] B. Dupire. *Pricing with a Smile.* Risk, January 1994. 25

[6] A. Brace, D. Gatarek, M. Musiela. 1997. *The Market Model of Interest Rate Dynamics.* Mathematical Finance, Vol. 7, No. 2, 127–155. 38

[7] F. Black, P. Karasinski. 1991. *Bond and Option Pricing when Short Rates are Lognormal.* Financial Analysts Journal, July-August 1991, 52–59. 57

[8] A. Brace, M. Musiela. *Swap Derivatives in a Gaussian HJM Framework*, Mathematics of Derivative Securities. Cambridge University Press, 1997.

[9] M. Baxter, A. Rennie. *Financial Calculus.* Cambridge University Press, 1996. 43, 56

[10] J. Cox, J. Ingersoll, S. Ross. 1981. *The Relation Between Forward Prices and Futures Prices.* Journal of Financial Economics, 9, 321–346. 18, 46

[11] CreditRisk+. A Credit Risk Management Framework. 1997. 119

[12] J. Danielsson, Caspar G. de Vries. 1997. *Tail index and quantile estimation with very high frequency data*, Journal of Empirical Finance, 4, 241–257. 104

[13] P. Hagan, D. Woodward. 1999. *Equivalent Black Volatilities.* Applied Mathematical Finance 6, 147–157. 46

[14] P. Hunt, J. Kennedy. *Financial Derivatives in Theory and Practice.* Wiley, 2000. 38, 40, 41, 43, 44, 56

[15] J. Hull. Option, Futures and Other Derivative Securities. Prentice Hall, 1997. 48

[16] J. Hull, A. White. *Branching Out.* Risk, July 1994. 57

[17] J. Hull, A. White. 1990. *Pricing Interest Rate Derivative Securities.* Review of Financial Studies, Vol 3, 573–592. 57

[18] F. Jamshidian. *Options and Futures Evaluation with Deterministic Volatilities.* Mathematical Finance, Vol 3, No. 2, 149–159, April 1993. 25

[19] F. Jamshidian. 1997. *Libor and Swap Market Models and Measures.* Finance Stochastics, Vol. 1, 293-330. 38, 40

[20] F. Jamshidian. *Sorting Out Swaptions.* Risk, March 1996.

[21] P. Jorion. *Value at Risk: the new benchmark for measuring financial risk.* McGraw-Hill, Second Edition, 2001. 104

[22] G. Kirikos, D. Novak. *Convexity Conundrums.* Risk, March 1997. 19

[23] D. Lamberton, B. Lapeyre. *Introduction to Stochastic Finance Applied to Finance.* Chapman & Hall, 1996. 100

[24] R. Litterman, *Hot Spots^{TM} and Hedges*, Goldman Sachs Risk Management Series, October 1996. 106

[25] W. Margrabe. 1978. *The value of an option to exchange one asset for another.* Journal of Finance, 33, 177–186. 130

[26] M. Musiela, M Rutkowski. *Martingale Methods in Financial Modelling.* Springer, 1998.

[27] H. Niederreiter. *Random Number Generation and Quasi-Monte Carlo Methods.* Society for Industrial and Applied Mathematics, 1992. 151

[28] A. Owen, D. Tavella, *Scrambled Nets for Value-at-Risk Calculations*; from *VAR: Understanding and Applying Value-at-Risk*, Risk Publications, 1997.

[29] W. Press, S. Teukolsky, W. Vetterling, B. Flannery. *Numerical Recipes in C.* Cambridge University Press, Second Edition, 1992. 45, 62, 84, 97, 102, 128, 149, 150, 151

[30] R. Rebonato. *Interest-Rate Option Models.* Wiley, Second Edition, 1998. 56

[31] J. Ramberg, E. Dudewicz, P. Tadikamalla, F. Mykytka. 1979. *A Probability Distribution and Its Uses in Fitting Data*. Technometrics, Vol. 21, No. 2, 201–214. 91, 149

[32] RiskMetrics Technical Document. Fourth Edition 1996. 88

[33] CreditMetrics Technical Document. April 1997. 119

[34] M. Rubinstein, E. Reiner. *Breaking Down the Barriers*. Risk, September 1991. 51

[35] J. Shaw. *Beyond VaR and Stress Testing*; from *VAR: Understanding and Applying Value-at-Risk*, Risk Publications, 1997. 98

[36] S. Shreeve. 2004. *Stochastic Calculus for Finance II, Continuous-Time Models*, Springer. 46

[37] M. Stein. 1987. *Large Sample Properties of Simulations Using Latin Hypercube Sampling*. Technometrics, Vol. 29, No. 2, 143–151. 105

[38] P. Wilmott, S. Howison, J. Dewynne. *The Mathematics of Financial Derivatives*. Cambridge University Press, 1995.

[39] P. Wilmott. *Derivatives*. Wiley. 1998. 39

[40] T. Wilson. *Portfolio Credit Risk*. Risk, September and October 1997. 119