5554 - Fall2022 - HW2

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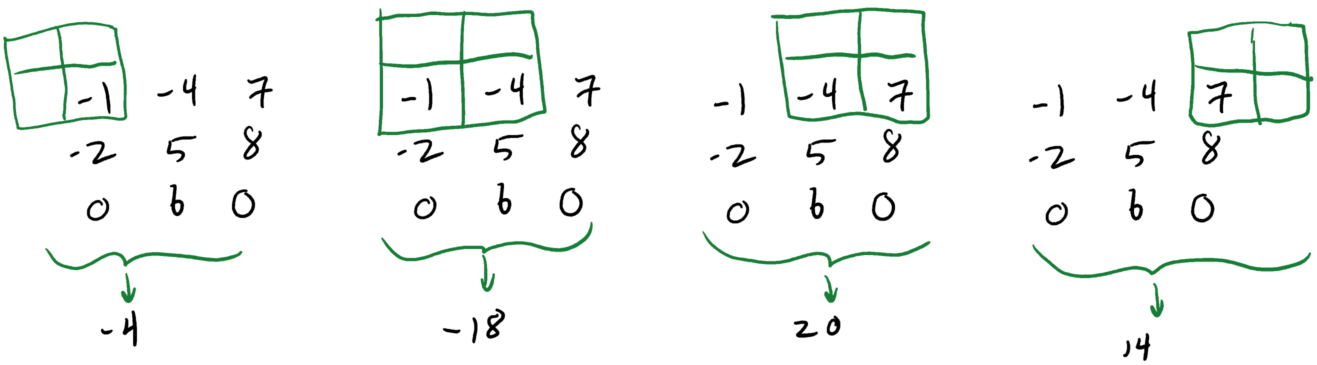
$$g = \begin{bmatrix} -1 & -4 & 7 \\ -2 & 5 & 8 \\ 0 & 6 & 0 \end{bmatrix} \qquad h = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

We consider g and h to be surrounded by zeros in all directions.
We can choose any convenient coordinate reference frames, as long as we are consistent.

a) $(g \otimes h)(i,j) = \sum_u \sum_v g(u,v)\, h(i+u, j+v)$

We could get the answer directly from the equation,
but it can also be visualized by "sliding" h across g and computing sums of products.
Nonzero values can occur only when both kernels overlap.



Continuing in this way, the answer is

$$g \otimes h = \begin{bmatrix} -4 & -18 & 20 & 14 \\ -11 & 3 & 59 & 23 \\ -6 & 37 & 41 & 8 \\ 0 & 18 & 6 & 0 \end{bmatrix}$$

b) Similarly, we can "slide" g across h and compute sums of products.

$$h \otimes g = \begin{bmatrix} 0 & 6 & 18 & 0 \\ 8 & 41 & 37 & -6 \\ 23 & 59 & 3 & -11 \\ 14 & 20 & -18 & -4 \end{bmatrix}$$

(Notice that $g \otimes h \neq h \otimes g$)

Problem 1 (continued)

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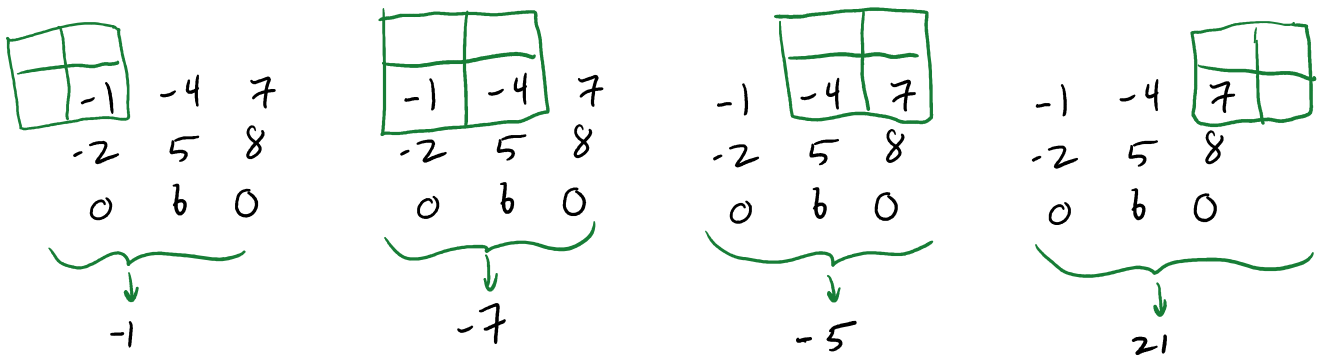
c) $$(g * h)(i,j) = \sum_u \sum_v g(u,v)\, h(i-u, j-v)$$

Once again, we could get the answer directly from the equation,
but it can also be visualized by "sliding" a reversed version of kernel h across g and computing sums of products. Nonzero values can occur only when both kernels overlap.

$$h(-u,-v) = \begin{array}{|c|c|} \hline 4 & 2 \\ \hline 3 & 1 \\ \hline \end{array}$$



Continuing in this way, the answer is

$$\underline{g * h = \begin{bmatrix} -1 & -7 & -5 & 21 \\ -4 & -13 & 21 & 52 \\ -4 & 8 & 54 & 32 \\ 0 & 12 & 24 & 0 \end{bmatrix}}$$

(Not required for this problem, but you could also use this same approach to demonstrate that g * h = h * g.)

Problem 2

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a) Let $h_1(y) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $h_2(x) = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}$.

Then $h(x,y) = h_1(y) * h_2(x) = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}$,

which is one of the Sobel filters, approximating $\frac{\partial}{\partial x}$.

b) We can choose $g_1(x) = \frac{1}{14}\begin{bmatrix} 1 & 3 & 6 & 3 & 1 \end{bmatrix}$

and $g_2(y) = \frac{1}{14}\begin{bmatrix} 1 \\ 3 \\ 6 \\ 3 \\ 1 \end{bmatrix}$. Then a discrete approximation

to a 2D Gaussian filter is given by

$$g(x,y) = g_1(x) * g_2(y) = \underbrace{\frac{1}{196} \cdot \begin{bmatrix} 1 & 3 & 6 & 3 & 1 \\ 3 & 9 & 18 & 9 & 3 \\ 6 & 18 & 36 & 18 & 6 \\ 3 & 9 & 18 & 9 & 3 \\ 1 & 3 & 6 & 3 & 1 \end{bmatrix}}$$

Problem 3

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Given: $G(x,y) = \frac{1}{2\pi\sigma^2} e^{\frac{-(x^2+y^2)}{2\sigma^2}}$

a) Suppose we convolve $f * G$, for some arbitrary 2D function f.

$$f(x,y) * G(x,y)$$

$$= \iint f(x-u, y-v)\, G(u,v)\, du\, dv$$

$$= \iint f(x-u, y-v) \frac{1}{2\pi\sigma^2} e^{-\frac{(u^2+v^2)}{2\sigma^2}}\, du\, dv$$

$$= \iint f(x-u, y-v) \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{u^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{v^2}{2\sigma^2}}\, du\, dv$$

$$= \int \left[\int f(x-u, y-v) \underbrace{\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{u^2}{2\sigma^2}}}_{G_1(u)} du \right] \underbrace{\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{v^2}{2\sigma^2}}}_{G_2(v)} dv$$

$$= \left[f(x,y) * G_1(x) \right] * G_2(y)$$

$$= f(x,y) * \left[G_1(x) * G_2(y) \right]$$

where $G_1(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}}$ and $G_2(y) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{y^2}{2\sigma^2}}$

Therefore $G(x,y) = G_1(x) * G_2(y)$,

which means that $G(x,y)$ must be separable.

Problem 3 (continued)

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b) Given:

$$G(x,y) \triangleq \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

$$\nabla^2 G \triangleq \frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2}$$

$$\frac{\partial G}{\partial x} = \frac{-2x}{2\sigma^2} \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}} = \frac{-x}{2\pi\sigma^4} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

$$\frac{\partial^2 G}{\partial x^2} = \left(\frac{-1}{2\pi\sigma^4} e^{-\frac{x^2+y^2}{2\sigma^2}}\right) + \left(\left[\frac{-x}{2\pi\sigma^4}\right]\left[\frac{-2x}{2\sigma^2} \cdot e^{-\frac{x^2+y^2}{2\sigma^2}}\right]\right)$$

$$= \left(\frac{x^2}{\sigma^4} - \frac{1}{\sigma^2}\right) \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

Similarly,

$$\frac{\partial^2 G}{\partial y^2} = \left(\frac{y^2}{\sigma^4} - \frac{1}{\sigma^2}\right) \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

Combining the two parts:

$$\nabla^2 G = \frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2}$$

$$= \boxed{\left(\frac{x^2+y^2}{\sigma^4} - \frac{2}{\sigma^2}\right) \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}}$$

$$= \left(\frac{x^2+y^2}{\sigma^4} - \frac{2}{\sigma^2}\right) G(x,y)$$

(As discussed in Section 7.2, this function can be split into two separable parts.)