Problem 1

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Perspective projection

a) $(x, y, z) = (3, 4, 5)$
b) $(x, y, z) = (6, 8, 10)$
c) $(x, y, z) = (-1, -2, 6)$
d) $(x, y, z) = (0, 0, 6)$

$$x' = \frac{fx}{z} \qquad f = 0.028 \text{ m}$$

$$y' = \frac{fy}{z}$$

a) $x' = (0.028)\frac{3}{5} = \underline{0.017 \text{ m}}$ (rounded to nearest mm)

$y' = (0.028)\frac{4}{5} = \underline{0.022 \text{ m}}$

b) $x' = (0.028)\frac{6}{10} = \underline{0.017 \text{ m}}$

$y' = (0.028)\frac{8}{10} = \underline{0.022 \text{ m}}$

c) $x' = (0.028)\left(\frac{-1}{6}\right) = -\cancel{0.007}\text{ m}$ -0.0047 m

$y' = (0.028)\left(-\frac{2}{6}\right) = \underline{-0.009 \text{ m}}$

d) $x' = (0.028)\frac{0}{6} = \underline{0 \text{ m}}$

$y' = (0.028)\frac{0}{6} = \underline{0 \text{ m}}$

e) Referring to Figure 2.10, we have

$$\tan\frac{\theta}{2} = \frac{\left(\frac{W}{2}\right)}{f}$$

$$\theta = 2\tan^{-1}\frac{W}{2f} = 2\tan^{-1}\left(\frac{24 \text{ mm}}{(2)(28 \text{ mm})}\right)$$

$$\approx \underline{46.4^\circ}$$

Problem 2

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Given: $\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$

Then $(A^T)A = \begin{bmatrix} 29 & 36 & 43 \\ 36 & 45 & 54 \\ 43 & 54 & 65 \end{bmatrix}$

Using on-line matrix tools, we obtain the solution:

The eigenvalues are $\underline{\lambda \approx 0,\ 0.3896,\ 138.6}$

The unit eigenvector associated with $\lambda = 0$ is

$$\underline{x \approx \begin{bmatrix} 0.408 \\ -0.816 \\ 0.408 \end{bmatrix}}$$

Note: It will be good practice if you work out this solution by hand, and not rely entirely on the machine's answer.

Also note that the matrix (A^T)(A) is real and symmetric, and linear matrix theory tells us that such a matrix has many nice properties. In particular, the eigenvalues are all real, and the eigenvectors corresponding to different eigenvalues are orthogonal. If all elements of the matrix are also positive, as is the case here, then the inverse of the matrix must exist, and all eigenvalues will be distinct.

(Correction: the eigenvalues will be distinct in almost all cases; an exception is the case that all matrix elements are identical.)

Problem 3

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We want the intersection of these 2 lines in 2D:

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

First, some background. Chapter 2 of the textbook gives a terse (but fairly elegant) introduction to the concept of homogeneous coordinates.

Within this framework, a 2D point $\begin{bmatrix} x \\ y \end{bmatrix}$ is represented as $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ or (more generally) as $\begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{bmatrix}$.

Part of the idea is to add one more dimension to a point's representation. Whenever we want to convert back to normal 2D representation, we should divide by the scalar value that is stored as the component in that extra dimension:

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{bmatrix} \longrightarrow \begin{bmatrix} \tilde{x}/\tilde{w} \\ \tilde{y}/\tilde{w} \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

This representation allows us to represent a line in 2D using a dot-product notation, as in equation (2.3),

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \cdot \tilde{l} = 0$$

where a, b, and c are the coefficients that specify a line in 2D, similar to the equations at the top of this sheet.

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Problem 3 (cont.)

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a) The intersection of 2 lines is described in equation (2.4) using the cross product:

$$\tilde{x} = \tilde{l}_1 \times \tilde{l}_2$$

$$= \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} \times \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$$

$$= \begin{bmatrix} b_1 c_2 - b_2 c_1 \\ a_2 c_1 - a_1 c_2 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

To convert this vector from homogeneous coordinates to 2D representation, we divide by the 3rd component:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \dfrac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1} \\ \dfrac{a_2 c_1 - a_1 c_2}{a_1 b_2 - a_2 b_1} \end{bmatrix}, \text{ if } a_1 b_2 - a_2 b_1 \neq 0$$

solution

b) $a_1 = 3, b_1 = 4, c_1 = 5, a_2 = 6, b_2 = 7, c_2 = 8$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \dfrac{(4)(8)-(7)(5)}{(3)(7)-(6)(4)} \\ \dfrac{(6)(5)-(3)(8)}{(3)(7)-(6)(4)} \end{bmatrix} = \begin{bmatrix} \dfrac{32-35}{21-24} \\ \dfrac{30-24}{21-24} \end{bmatrix} = \begin{bmatrix} \dfrac{-3}{-3} \\ \dfrac{6}{-3} \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

c) In part (a) above, parallel lines correspond to a value of 0 in the denominator:

$$a_1 b_2 - a_2 b_1 = 0$$

Problem 4

If line AB is described by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} t + \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix},$$

then an arbitrary point on the line projects onto the image at $[x', y']^T$ according the the following:

$$x' = \frac{x}{z} f = \frac{at + x_0}{ct + z_0} f$$

$$y' = \frac{y}{z} f = \frac{bt + y_0}{ct + z_0} f$$

The vanishing point is the location in the image that corresponds to $t \to \infty$.

(l'Hopital's rule)

$$\lim_{t\to\infty} x' = \lim_{t\to\infty} \frac{at + x_0}{ct + z_0} f = \lim_{t\to\infty} \frac{\frac{d}{dt}(at + x_0)}{\frac{d}{dt}(ct + z_0)} f$$

$$= \frac{a}{c} f$$

similarly,

$$\lim_{t\to\infty} y' = \lim_{t\to\infty} \frac{bt + y_0}{ct + z_0} = \frac{b}{c} f$$

So, the vanishing point must lie at image location

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \frac{f}{c} \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{for } c \neq 0.$$

(For the degenerate case that line AB is parallel to the image plane, then $c = 0$ and no vanishing point exists.)