Proof. Let $f : A \to B$ be invertible. ... $g : B \to A$ such that $g\circ f = i_A$ and $f\circ g = i_B$.
Since i_A is injective and $g\circ f = i_A$, f is an injection.
Since i_B is surjective and $f\circ g = i_B$, f is a surjection.
Therefore f is a bijection.

Conversely, let $f : A \to B$ be a bijection.
Let $y \in B$. Since f is a bijection, y has a unique pre-image x in A.
Define a mapping $g : B \to A$ by $g(y) = x$ (the pre-image of y under f), $y \in B$. Then $g\circ f(x) = g(y) = x$, $x \in A$ and $f\circ g(y) = f(x) = y$, $y \in B$.
Here $g\circ f = i_A$, $f\circ g = i_B$ and therefore f is invertible.

Note. The theorem gives a clue how to determine f^{-1}. To each element y in B, f^{-1} assigns the pre-image of y under f.

Theorem 1.12.3. Let $f : A \to B$ be a bijective mapping. Then the mapping $f^{-1} : B \to A$ is also a bijection and $(f^{-1})^{-1} = f$.

Proof. Since $f : A \to B$ is a bijection, $f^{-1} : B \to A$ exists and $f^{-1}\circ f = i_A$ and $f\circ f^{-1} = i_B$. (By definition).
Let y_1, y_2 be two distinct elements in B and $f^{-1}(y_1) = x_1$, $f^{-1}(y_2) = x_2$.
Then $f(x_1) = f[f^{-1}(y_1)] = f\circ f^{-1}(y_1) = y_1$ and $f(x_2) = f[f^{-1}(y_2)] = f\circ f^{-1}(y_2) = y_2$, since $f\circ f^{-1} = i_B$.
Since f is a mapping, $y_1 \neq y_2 \Rightarrow x_1 \neq x_2$.
That is, $y_1 \neq y_2 \Rightarrow f^{-1}(y_1) \neq f^{-1}(y_2)$ and therefore f^{-1} is injective.
To establish that f^{-1} is surjective, let $x \in A$. Let $f(x) = y$.

SETS 43

Then $f^{-1}(y) = f^{-1}[f(x)] = f^{-1}\circ f(x) = x$, since $f^{-1}\circ f = i_A$.
This shows that y is a pre-image of x under the mapping f^{-1}. Therefore f^{-1} is surjective.
Since f^{-1} is injective as well as surjective, f^{-1} is a bijection.

Second part.
Since $f : A \to B$ is a bijection, $f^{-1} : B \to A$ exists and $f^{-1}\circ f = i_A$ and $f\circ f^{-1} = i_B$. Since $f^{-1} : B \to A$ is a bijection, $(f^{-1})^{-1} : A \to B$ exists and $f^{-1}\circ (f^{-1})^{-1} = i_A$ and $(f^{-1})^{-1}\circ f^{-1} = i_B$.
Let $x \in A$ and $f(x) = y$.
$f^{-1}(y) = f^{-1}[f(x)] = f^{-1}\circ f(x) = x$, since $f^{-1}\circ f = i_A$.
$(f^{-1})^{-1}(x) = (f^{-1})^{-1}[f^{-1}(y)] = (f^{-1})^{-1}\circ f^{-1}(y) = y$, since $(f^{-1})^{-1}\circ f^{-1} = i_B$.
Therefore $f(x) = (f^{-1})^{-1}(x)$ for all $x \in A$.
Consequently, $f = (f^{-1})^{-1}$. This completes the proof.

Theorem 1.12.4. Let $f : A \to B$ and $g : B \to C$ be both bijective mappings. Then the mapping $g\circ f : A \to C$ is invertible and $(g\circ f)^{-1} = f^{-1}\circ g^{-1}$.

Proof. Since $f : A \to B$ and $g : B \to C$ are both bijective, the composite mapping $g\circ f : A \to C$ is a bijection, by Theorem 1.11.6.
Hence $(g\circ f)^{-1} : C \to A$ exists, by Theorem 1.12.2.
Since $f : A \to B$ is a bijection, $f^{-1} : B \to A$ exists.
Since $g : B \to C$ is a bijection, $g^{-1} : C \to B$ exists.
Then the composite mapping $f^{-1}\circ g^{-1} : C \to A$ exists.
Let $z \in C$. Let y be the pre-image of z under the bijective mapping g and x be the pre-image of y under the bijective mapping f.
Then $g\circ f(x) = z$. Since $g\circ f$ is invertible, $(g\circ f)^{-1}(z) = x$. — (i)
Since f is invertible and $f(x) = y$, $f^{-1}(y) = x$.
Since g is invertible and $g(y) = z$, $g^{-1}(z) = y$.
Therefore $f^{-1}\circ g^{-1}(z) = f^{-1}[g^{-1}(z)] = f^{-1}(y) = x$. — (ii)
Thus $(g\circ f)^{-1}(z) = (f^{-1}\circ g^{-1})(z)$ for all $z \in C$.
Consequently, $(g\circ f)^{-1} = f^{-1}\circ g^{-1}$. This completes the proof.

Worked Examples.
1. Let A, B be both finite sets of n elements and a mapping $f : A \to B$ is injective. Prove that f is a bijection.

Let $A = \{a_1, a_2, \ldots, a_n\}$. Then $f(a_1), f(a_2), \ldots, f(a_n)$ all belong to B. Since f is injective, $f(a_1), f(a_2), \ldots, f(a_n)$ are all distinct elements of B. As they are n in number, they are all the elements of B.

$f \to$ Bijection
$\hookrightarrow f^{-1} \to$ Bijection
and $(f^{-1})^{-1} = f$

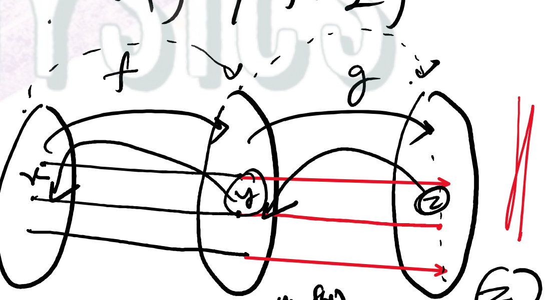
$f^m = f.$

$\boxed{f\circ f\circ \ldots f = f}$

$f^{-1}(y_1) = x_1$
$\Rightarrow f(x_1) = y_1$
$f(x_2) = y_2$

$f(x_1) = f(x_2)$
$\Rightarrow x_1 = x_2$

$x_1 \neq x_2$
$\Rightarrow f(x_1) \neq f(x_2)$



$g(f(x)) : A \to C$

P.T it is invertible $[g(f(x))]^{-1}$ exist.

P.T it is invertible — $[g(f(x))]^{-1}$ exist.

$$gof^{-1} = f^{-1} o\, g^{-1}$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

I, O, C, D — output O

$f(\ldots) = P$ ....

S → P? — 85%

W → P? 10%

G → P? 5%

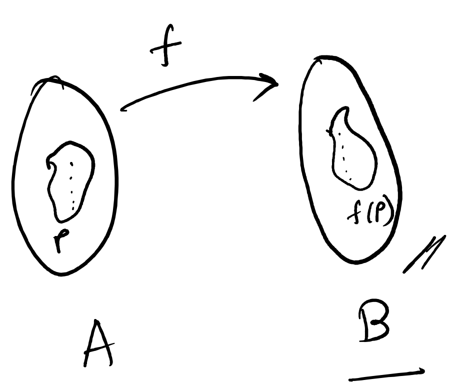
I will complete this chapter

$f(\ldots)$

$$f(x) = |x|$$

Direct and Inverse image :

$f : A \to B$ $\quad$ $P \to$ non-empty subset of A.



$$f(P) = \{f(x) : x \in P\}$$

↳ direct image of P under f.

A B

Suppose S be a non-empty subset of B. Then the inverse image of S under f is the subset $f^{-1}(S)$ of A given by $f^{-1}(S) = \{x : f(x) \in S\}$.

well as the inverse image of S under f.

Examples.

$\to 0$ $\to 4$

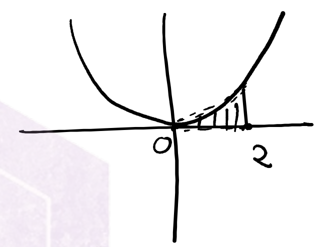
1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2, x \in \mathbb{R}$.

Let $P = \{x \in \mathbb{R} : 0 \le x \le 2\}$. Then the direct image $f(P)$ is given by $f(P) = \{y \in \mathbb{R} : 0 \le y \le 4\}$. The inverse image of the set $f(P)$, i.e., $f^{-1}(f(P))$ is given by $f^{-1}(f(P)) = \{x \in \mathbb{R} : -2 \le x \le 2\}$.

Here $f^{-1}(f(P)) \ne P$.

Let $S = \{x \in \mathbb{R} : -1 \le x \le 4\}$. Then $f^{-1}(S) = \{x \in \mathbb{R} : -2 \le x \le 2\}$ and $ff^{-1}(S) = \{x \in \mathbb{R} : 0 \le x \le 4\}$.

Here $ff^{-1}(S) \ne S$.



$P \subseteq \mathbb{R}$

$f^{-1}(f(x)) = x$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 3x, x \in \mathbb{R}$.

Let $P = \{x \in \mathbb{R} : 0 \le x \le 2\}$. Then the direct image $f(P)$ is given by $f(P) = \{y \in \mathbb{R} : 0 \le y \le 6\}$. The inverse image of the set $f(P)$, i.e., $f^{-1}(f(P))$ is given by $f^{-1}(f(P)) = \{x \in \mathbb{R} : 0 \le x \le 2\}$.

Therefore $f^{-1}(f(P)) = P$.

Here f is a bijection and therefore the inverse mapping f^{-1} exists.

The direct image of the set $f(P)$ under f^{-1} is $\{x \in \mathbb{R} : 0 \le x \le 2\}$.

Therefore the direct image of the set $S = \{y \in \mathbb{R} : 0 \le y \le 6\}$ under f^{-1} is same as the inverse image of the set S under f.

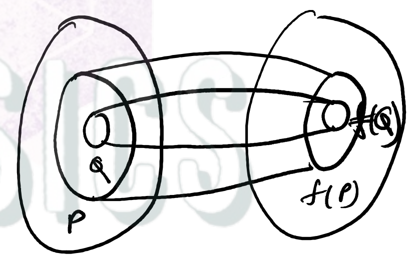
Theorem 1.13.1. Let $f : A \to B$ be a mapping and P, Q be non-empty subsets of A. Then

(a) $P \subset Q \Rightarrow f(P) \subset f(Q)$;

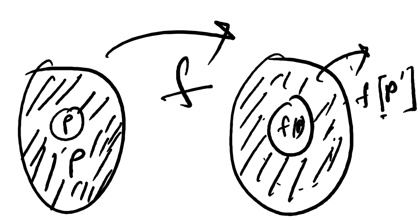
(b) $f(P \cup Q) = f(P) \cup f(Q)$;

(c) $f(P \cap Q) \subset f(P) \cap f(Q)$; $*$

(d) $f(P \cap Q) = f(P) \cap f(Q)$, if f is injective;



A B



(e) $[f(P)]' \subset f(P')$, if f is surjective;

(e) $[f(P)]' \subset f(P')$, if f is surjective;

(f) $[f(P)]' = f(P')$, if f is bijective.

Proof. **(a)** Let $y \in f(P)$. Then there is an element x in P such that $f(x) = y$.

$x \in P \Rightarrow x \in Q$. Therefore $f(x) \in f(Q)$, i.e., $y \in f(Q)$.

Consequently, $f(P) \subset f(Q)$.

(b) $P \subset P \cup Q$. $Q \subset P \cup Q$.

By **(a)**, $f(P) \subset f(P \cup Q)$ and $f(Q) \subset f(P \cup Q)$.

Consequently, $f(P) \cup f(Q) \subset f(P \cup Q)$ ... (i)

Let $y \in f(P \cup Q)$. Then there exists an element $x \in P \cup Q$ such that $y = f(x)$. $x \in P \cup Q \Rightarrow x \in P$ or $x \in Q$

$\Rightarrow f(x) \in f(P)$ or $f(x) \in f(Q)$

$\Rightarrow f(x) \in f(P) \cup f(Q)$.

Therefore $y \in f(P \cup Q) \Rightarrow y \in f(P) \cup f(Q)$.

Consequently, $f(P \cup Q) \subset f(P) \cup f(Q)$ ... (ii)

From (i) and (ii) it follows that $f(P \cup Q) = f(P) \cup f(Q)$.

(c) $P \cap Q \subset P$ and $P \cap Q \subset Q$.

By **(a)**, $f(P \cap Q) \subset f(P)$ and $f(P \cap Q) \subset f(Q)$.

Consequently, $f(P \cap Q) \subset f(P) \cap f(Q)$.

Note. $f(P \cap Q) \neq f(P) \cap f(Q)$, in general.

For example, let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$, $x \in \mathbb{R}$. Let $P = \{1, 2, 3, 4\}$, $Q = \{-1, -2, 3, 4\}$.

Then $P \cap Q = \{3, 4\}$. $f(P) \cap f(Q) = \{1, 4, 9, 16\}$.

Here $f(P \cap Q) = \{9, 16\} \neq f(P) \cap f(Q)$.

(d) By **(c)**, $f(P \cap Q) \subset f(P) \cap f(Q)$ ... (i)

Let $y \in f(P) \cap f(Q)$. Then $y \in f(P)$ and $y \in f(Q)$.

Since f is injective, y has a unique pre-image, say x, in P and Q both. So $x \in P \cap Q$.

This implies $f(x) \in f(P \cap Q)$, i.e., $y \in f(P \cap Q)$.

Consequently, $f(P) \cap f(Q) \subset f(P \cap Q)$ ... (ii)

From (i) and (ii) it follows that $f(P \cap Q) = f(P) \cap f(Q)$.

(e) Let $y \in [f(P)]'$. Then $y \notin f(P)$. Since f is onto, y has at least one pre-image. Since $y \notin f(P)$, y has no pre-image in P.

Let x be a pre-image of y. Then $x \in P'$ and therefore $f(x) \in f(P')$. That is, $y \in f(P')$.

Consequently, $[f(P)]' \subset f(P')$.

(f) Since f is surjective, $[f(P)]' \subset f(P')$.

Let $y \in f(P')$. Since f is injective, y has a unique pre-image, say x and since $y \in f(P'), x \in P'$ and therefore $x \notin P$.

This implies $f(x) \notin f(P)$, i.e., $y \in [f(P)]'$. Therefore, $f(P') \subset [f(P)]'$.

Consequently, $[f(P)]' = f(P')$.

Theorem 1.13.2. Let $f : A \to B$ be an *onto* mapping and S, T be subsets of B. Then

(a) $S \subset T \Rightarrow f^{-1}(S) \subset f^{-1}(T)$;

(b) $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$;

(c) $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$;

(d) $f^{-1}(S') = [f^{-1}(S)]'$.

Proof. **(a)** Let $x \in f^{-1}(S)$. Then $f(x) \in S$.

Since $S \subset T, f(x) \in S \Rightarrow f(x) \in T$. Therefore $x \in f^{-1}(T)$.

Thus $x \in f^{-1}(S) \Rightarrow x \in f^{-1}(T)$. Consequently, $f^{-1}(S) \subset f^{-1}(T)$.

(b) $S \subset S \cup T, T \subset S \cup T$.

By **(a)**, $f^{-1}(S) \subset f^{-1}(S \cup T)$ and $f^{-1}(T) \subset f^{-1}(S \cup T)$.

Consequently, $f^{-1}(S) \cup f^{-1}(T) \subset f^{-1}(S \cup T) \quad \ldots \quad$ (i)

Let $x \in f^{-1}(S \cup T)$. Then $f(x) \in S \cup T$ and this implies $f(x) \in S$ or $f(x) \in T$. That is, $x \in f^{-1}(S)$ or $x \in f^{-1}(T)$, i.e., $x \in f^{-1}(S) \cup f^{-1}(T)$.

Therefore $x \in f^{-1}(S \cup T) \Rightarrow x \in f^{-1}(S) \cup f^{-1}(T)$.

Consequently, $f^{-1}(S \cup T) \subset f^{-1}(S) \cup f^{-1}(T) \quad \ldots \quad$ (ii)

From (i) and (ii) it follows that $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$.

(c) $S \cap T \subset S$ and $S \cap T \subset T$.

By (**a**), $f^{-1}(S \cap T) \subset f^{-1}(S)$ and $f^{-1}(S \cap T) \subset f^{-1}(T)$.

Consequently, $f^{-1}(S \cap T) \subset f^{-1}(S) \cap f^{-1}(T) \quad \ldots \quad$ (i)

Let $x \in f^{-1}(S) \cap f^{-1}(T)$. Then $x \in f^{-1}(S)$ and $x \in f^{-1}(T)$.

$x \in f^{-1}(S) \Rightarrow f(x) \in S$; $x \in f^{-1}(T) \Rightarrow f(x) \in T$.

So $x \in f^{-1}(S) \cap f^{-1}(T) \Rightarrow f(x) \in S \cap T \Rightarrow x \in f^{-1}(S \cap T)$.

Consequently, $f^{-1}(S) \cap f^{-1}(T) \subset f^{-1}(S \cap T) \quad \ldots \quad$ (ii)

From (i) and (ii) it follows that $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$.

(d) Let $x \in f^{-1}(S')$. Then $f(x) \in S'$, i.e., $f(x) \notin S$. This implies $x \notin f^{-1}(S)$, i.e., $x \in [f^{-1}(S)]'$. Therefore $f^{-1}(S') \subset [f^{-1}(S)]' \quad \ldots$ (i)

Let $x \in [f^{-1}(S)]'$. Then $x \notin f^{-1}(S)$. So $f(x) \notin S$, i.e., $f(x) \in S'$. This implies $x \in f^{-1}(S')$. Therefore $[f^{-1}(S)]' \subset f^{-1}(S') \quad \ldots$ (ii)