# Exponential Function

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$z = x + iy$. $\exp(z)$

$$\Rightarrow \exp(z) = \exp(x+iy)$$
$$= e^{x+iy}$$
$$= e^x \cdot e^{iy}$$
$$= e^x(\cos y + i\sin y)$$
$$\| $$
$$z = re^{i\theta}$$



$\exp(z) \longrightarrow$ Modulus $= e^x$

Amplitude $= y$.

$u + iv \longrightarrow$ non-zero complex number

It's polar representation,

$$u + iv = r(\cos\theta + i\sin\theta)$$

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$$u + iv = e^{\log r}(\cos\theta + i\sin\theta)$$

$$= e^{\log r} \cdot e^{i\theta}$$

$$= e^{(\log r + i\theta)}$$

$$= \boxed{\exp(\log r + i\theta)}.$$

## Properties

1) $\exp z_1 \cdot \exp z_2 = \exp(z_1 + z_2)$

2) $\dfrac{\exp(z_1)}{\exp(z_2)} = \exp(z_1 - z_2)$.

3) $(\exp z)^n = \exp(nz)$.

4) $\exp(z + 2n\pi i) = \exp z$, $n$ is an Integer.

$$= \exp(z) \cdot \exp(2n\pi i)$$

$$= \exp(z) \; 1.$$

$$= \exp(z).$$

Ex: Solve : $\exp(z) = 1 + \sqrt{3}i$

$$z = \ln(2) + i\frac{\pi}{3}.$$

$$\hookrightarrow z = \ln(2) + i\left(2n\pi + \frac{\pi}{3}\right).$$

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**Ex** : Find all complex number $z$ such that $\exp(2z+1) = i$.

Sol$^n$ :- $z = x+iy$.

$$2z+1 = 2(x+iy) + 1 = (2x+1) + i2y.$$

$$\exp(2z+1) = i$$

$$\Rightarrow e^{2x+1}(\cos 2y + i\sin 2y) = i.$$

$$\Rightarrow \cos 2y \cdot e^{2x+1} + i e^{2x+1} \cdot \sin 2y = i.$$

$$e^{2x+1} \cdot \cos 2y = 0 \qquad e^{2x+1} \cdot \sin 2y = 1$$

$$x = -\tfrac{1}{2} \qquad \sin 2y = 1$$

or, $\sin 2y = \sin\frac{\pi}{2}$

or, $y = (4n+1)\pi/4$.

$$\boxed{z = -\frac{1}{2} + i(4n+1)\pi/4}$$

## Logarithmic function

Let, $z$ be a non-zero complex number, then

Let $z$ be a non-zero complex number, then $\exists$ a complex number $w$, $\exp(w) = z$.

$\hookrightarrow$ $\boxed{w \text{ is a logarithm of } z.}$

$$\exp(w) = \exp(\underbrace{w + 2n\pi i}_{})$$

$\hookrightarrow$ is a logarithm of $z$.

## Principal Logarithm of $z$.

$$z = r(\cos\theta + i\sin\theta), \quad -\pi < \theta \leq \pi$$

$w = u + iv$ is a logarithm of $z$.

$$\exp(w) = z.$$

$$\Rightarrow \exp(u + iv) = z.$$

$$\Rightarrow e^{u}(\cos v + i\sin v) = r(\cos\theta + i\sin\theta)$$

$\Rightarrow$ Equating real and imaginary parts,

$$e^{u}\cos v = r\cos\theta \quad \Big| \quad e^{u}\sin v = r\sin\theta.$$

$$\Rightarrow e^{u} = r \quad \Big| \quad \cos v = \cos\theta \quad \Big| \quad \sin v = \sin\theta$$

$$u = \log r, \quad v = \theta + 2n\pi.$$

'$z$' $\rightarrow r$, $\rightarrow \theta$

$$\boxed{w = \log r + i(\theta + 2n\pi)}$$

$w \rightarrow$ logarithm of $z$

$\hookrightarrow \mathrm{Log}(z)$

$$\hookrightarrow \boxed{\log(z) = \log|z| + i(\arg z + 2n\pi)}$$

$$\boxed{\text{Log}(z) = \log|z| + i\,(\arg z + 2n\pi)}.$$

$\hookrightarrow$ $\text{Log}(z)$ not $\log(z)$

'z'

$\hookrightarrow \log_2(z)$

$\log_e(z)$

Ex Find $\text{Log}\, z$ and $\log z$.

of i) $z = 1$, ii) $z = i$

$$\text{Log}(z) = \log|z| + i(\arg z + 2n\pi)$$

$$= \log|1| + i(\arg 1 + 2n\pi).$$

$$= 0 + i\,2n\pi \qquad \log(z) = 0.$$

$$= 2in\pi.$$

ii) $z = i$

$$i = 1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$$\text{Log}(i) = \log 1 + i\left(\frac{\pi}{2} + 2n\pi\right).$$

$$= \frac{i\pi}{2} + 2n\pi = (4n+1)\frac{\pi}{2}i.$$

$$\log(i) = \log_e\left(e^{i\pi/2}\right) = i\cdot\frac{\pi}{2}.$$

$$i = 1.e^{i\pi/2}$$

$$= 1.\left(\cos\pi/2 + i\sin\pi/2\right)$$

# verify that:

$$\boxed{\text{Log}(-i)^{1/2} = \frac{1}{2}\text{Log}(-i)}.$$

$$\boxed{Log(-i) = \frac{1}{2} \ldots}$$

$$\Rightarrow \quad -i = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\pi/2\right)$$

$$\frac{1}{2}Log(-i) = \frac{1}{2}\left[\left(2n\pi - \frac{\pi}{2}\right)i\right]$$

$$\boxed{\frac{-\pi + 4n\pi}{2} = \frac{\pi}{2}(4n-1)}$$

$$Log(-i) = \log(1) + i\left(-\frac{\pi}{2} + 2n\pi\right)$$

$$\Rightarrow Log(-i) = (4n-1)\frac{\pi}{2} i$$

## Trignometric Functions

$$\cos x = \frac{\exp(ix) + \exp(-ix)}{2}.$$

$$\sin x = \frac{\exp(ix) - \exp(-ix)}{2i}.$$

$$\cos n\theta = \cos^n\theta - {}^nC_2 \cos^{n-2}\theta \sin^2\theta + {}^nC_4 \cos^{n-4}\theta \sin^4\theta - \ldots$$

$$\sin n\theta = {}^nC_1 \cos^{n-1}\theta \sin\theta - {}^nC_3 \cos^{n-3}\theta \cdot \sin^3\theta + {}^nC_5 \cos^{n-5}\theta \sin^5\theta - \ldots$$

$$\tan n\theta = \frac{\sin n\theta}{\cos n\theta}$$

$n-3$ $\quad$ $3$

$$\tan n\theta = \frac{\ }{\cos n\theta}$$

$$= \frac{{}^nC_1 \cos^{n-1}\theta \cdot \sin\theta - {}^nC_3 \cos^{n-3}\theta \cdot \sin^3\theta + \ldots}{\cos^n\theta - {}^nC_2 \cos^{n-2}\theta \sin^2\theta + \ldots}$$

$$= \frac{{}^nC_1 \frac{\cos^{n-1}\theta}{\cos^n\theta} \cdot \sin\theta - {}^nC_3 \frac{\cos^{n-3}\theta}{\cos^n\theta} \cdot \sin^3\theta + \ldots}{1 - {}^nC_2 \frac{\cos^{n-2}\theta}{\cos^n\theta} \sin^2\theta - \ldots}$$

$$= \frac{{}^nC_1 \frac{1}{\cos\theta} \cdot \sin\theta - {}^nC_3 \frac{1}{\cos^3\theta} \cdot \sin^3\theta + \ldots}{1 - {}^nC_2 \frac{1}{\cos^2\theta} \cdot \sin^2\theta - \ldots}$$

$$= \frac{{}^nC_1 \tan\theta - {}^nC_3 \cdot \tan^3\theta + \ldots}{1 - {}^nC_2 \tan^2\theta + {}^nC_4 \tan^4\theta - \ldots}$$