Lecture_14

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partition, partial order relation, poset, linear order relation.

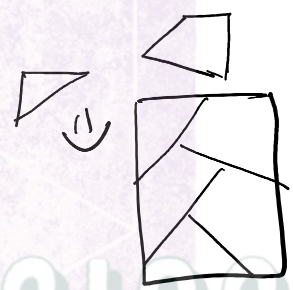
Partition, Partial Order Relation, Poset, Linear order relation.

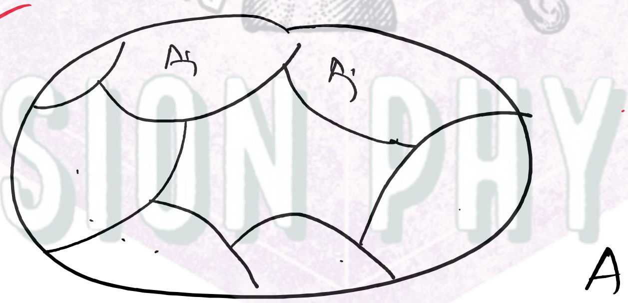
Partition of a Set:

Let, A be a non-empty set and $\mathcal{P}$ be a collection of non-empty subsets of A. The $\mathcal{P}$ is called partition of A, if the following properties hold:

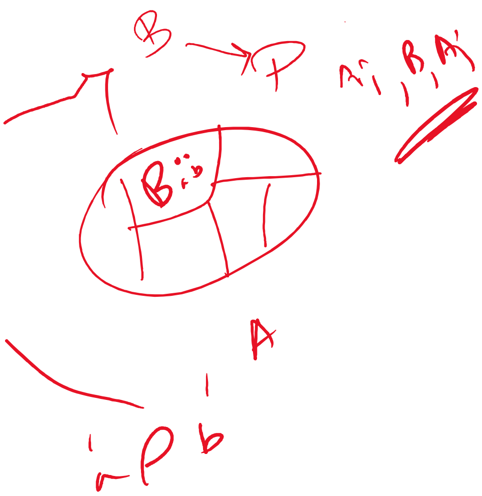
i) for all $A_i, A_j \in \mathcal{P}$, either $A_i = A_j$ or $A_i \cap A_j = \phi$.

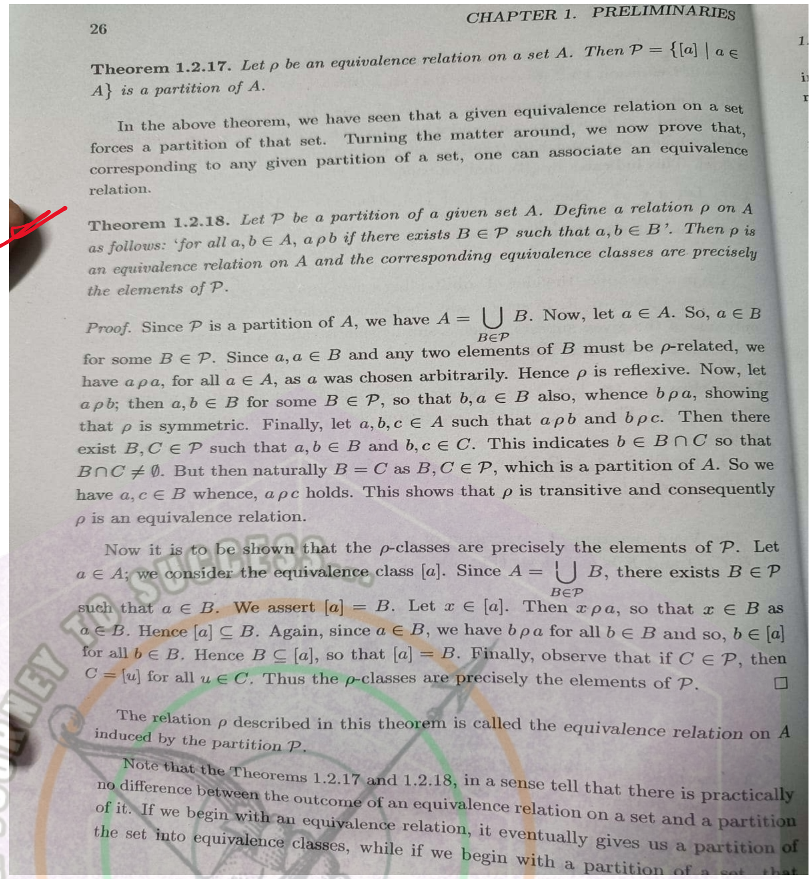
ii) $A = \bigcup_{A_i \in \mathcal{P}} A_i$





Thm:





Theorem 1.2.17. *Let ρ be an equivalence relation on a set A. Then $\mathcal{P} = \{[a] \mid a \in A\}$ is a partition of A.*

In the above theorem, we have seen that a given equivalence relation on a set forces a partition of that set. Turning the matter around, we now prove that, corresponding to any given partition of a set, one can associate an equivalence relation.

Theorem 1.2.18. *Let $\mathcal{P}$ be a partition of a given set A. Define a relation ρ on A as follows: 'for all $a, b \in A$, $a \,\rho\, b$ if there exists $B \in \mathcal{P}$ such that $a, b \in B$'. Then ρ is an equivalence relation on A and the corresponding equivalence classes are precisely the elements of $\mathcal{P}$.*

Proof. Since $\mathcal{P}$ is a partition of A, we have $A = \bigcup_{B \in \mathcal{P}} B$. Now, let $a \in A$. So, $a \in B$ for some $B \in \mathcal{P}$. Since $a, a \in B$ and any two elements of B must be ρ-related, we have $a \,\rho\, a$, for all $a \in A$, as a was chosen arbitrarily. Hence ρ is reflexive. Now, let $a \,\rho\, b$; then $a, b \in B$ for some $B \in \mathcal{P}$, so that $b, a \in B$ also, whence $b \,\rho\, a$, showing that ρ is symmetric. Finally, let $a, b, c \in A$ such that $a \,\rho\, b$ and $b \,\rho\, c$. Then there exist $B, C \in \mathcal{P}$ such that $a, b \in B$ and $b, c \in C$. This indicates $b \in B \cap C$ so that $B \cap C \neq \emptyset$. But then naturally $B = C$ as $B, C \in \mathcal{P}$, which is a partition of A. So we have $a, c \in B$ whence, $a \,\rho\, c$ holds. This shows that ρ is transitive and consequently ρ is an equivalence relation.

Now it is to be shown that the ρ-classes are precisely the elements of $\mathcal{P}$. Let $a \in A$; we consider the equivalence class $[a]$. Since $A = \bigcup_{B \in \mathcal{P}} B$, there exists $B \in \mathcal{P}$ such that $a \in B$. We assert $[a] = B$. Let $x \in [a]$. Then $x \,\rho\, a$, so that $x \in B$ as $a \in B$. Hence $[a] \subseteq B$. Again, since $a \in B$, we have $b \,\rho\, a$ for all $b \in B$ and so, $b \in [a]$ for all $b \in B$. Hence $B \subseteq [a]$, so that $[a] = B$. Finally, observe that if $C \in \mathcal{P}$, then $C = [u]$ for all $u \in C$. Thus the ρ-classes are precisely the elements of $\mathcal{P}$. □

The relation ρ described in this theorem is called the *equivalence relation on A induced by the partition $\mathcal{P}$*.

Note that the Theorems 1.2.17 and 1.2.18, in a sense tell that there is practically no difference between the outcome of an equivalence relation on a set and a partition of it. If we begin with an equivalence relation, it eventually gives us a partition of the set into equivalence classes, while if we begin with a partition of a set, that

Antisymmetric Relation.

A relation R on a set A is said to be antisymmetric if for all $a, b \in A$, whenever both $\boxed{a R b \text{ and } b R a}$, then $a = b$. (1, 2)

$1 R 1$ $\quad$ $1 R 1$

For example: $A = \{1, 2, 3, 4\}$.

Let R be a relation on A given by:

$R = \{(1, 1), (1, 2), (2, 3), (3, 4), (4, 4)\}$.

POSET (Partially Ordered Set):

A relation R on a set A is said to be a partial order on A if R is reflexive, antisymmetric and transitive. A partial order relation on a set A is usually denoted by $\leq$.

The set A along with the partial order defined on it is called a partially ordered set or poset.

Some examples on Poset :-

1) Let A be any non-empty set. $R : \subseteq$

$P(A) \rightarrow$ power set of A.

Reflexivity:

$X \subseteq X$ for all $X \in P(A)$.

'$\subseteq$' is reflexive.

Antisymmetric:

If, $X \subseteq Y$ and $Y \subseteq X$, $X, Y \in P(A)$.

$\Rightarrow X = Y$. (By def^n of equality of sets).

'$\subseteq$' is antisymmetric

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'$\subseteq$' is antisymmetric

<u>Transitivity</u>

$x \subseteq y$, $y \subseteq z$

$\Rightarrow x \subseteq z$.

'$\subseteq$' also transitive.

'$\subseteq$' is a partial order relation on set $P(A)$.

$(P(A), \subseteq)$ is a poset. $\boxed{\text{Proved.}}$

<u>H/W</u> 2. 'ρ' $\rightarrow$ relation defined $\mathbb{R}$ (Real).

$\rho = \{(a,b) \in \mathbb{R} \times \mathbb{R} \mid a-b \leq 0\}$. $\boxed{a-b \geq 0}$ $b \geq a$

Prove that : $(\mathbb{R}, \rho)$ is a poset.

3. 'R' $\rightarrow$ relation on set of integers. ($\mathbb{Z}$).

$R = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} \mid a \text{ divides } b \text{ in } \mathbb{Z}\}$.

Check whether. $(\mathbb{Z}, R)$ is a poset ?

$\Rightarrow (2,-2) \in R$

$(-2, 2) \in R$

$a|b$ $b|c$

$b = ak_1$ $c = bk_2$

$(-2, 2) \in R$

$b = ak_1 \quad , c = b k_2$

$= a k_1 k_2$

$-2 \neq 2$

$\hookrightarrow$ Then it is not antisymmetric

$c \mid a$

So, $(\mathbb{Z}, R)$ is not a poset.

Linearly Ordered Set

A poset (A, ρ) is called linearly ordered set or a chain if for all $a, b \in A$ either $a \rho b$ or $b \rho a$ must hold.

$a \leq b$ or $b \geq a$ or $a = b$

e.g.: $\{a\}, \{b\} \in P(A)$.

$\{a\} \subseteq \{b\}$ or $\{b\} \subseteq \{a\}$

$$\boxed{\{a, b\} \subseteq \{a, b, c\}}$$