Vector Space

Defⁿ: Let V be a non-empty set and F be a field. Let, the binary operation '+' be defined on V, and, let scalar multiplication $(\cdot)$ be defined such that for every $\alpha \in F$, if the following axioms hold:

A_1 : $u + v \in V$ for all, $u, v \in V$.

A_2 : $(u+v)+w = u+(v+w)$ for all $u, v, w \in V$.

A_3 : $u+v = v+u$. for all, $u, v \in V$.

A_4 : There is a zero vector 0_V in V such that $u + 0_V = u$. $\forall u \in V$.

A_5 : For each $u \in V$, there exists $v \in V$ such that $u+v = 0_V$.

$\boxed{1 - 1 = 0}$

$v \rightarrow$ additive inverse

M_1 : $\alpha u \in V$, for all $\alpha \in F$, $u \in V$

M_2 : $\alpha(u+v) = \alpha u + \alpha v$, for all $\alpha \in F$, $u, v, \in V$

$$x(y+z) = xy + xz$$

M_3 : $(\alpha+\beta)\mu = \alpha\mu + \beta\mu$, $\quad \alpha, \beta \in F$, $\mu \in V$.

M_4 : $(\alpha\beta)(\mu) = \alpha(\beta\mu)$.

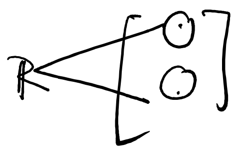
M_5 : $\mathbb{1}.\mu = \mu$, $\quad \mu \in V$.

M_6 : $0.v = 0$, $\quad v \in V$, $0 \in F$.

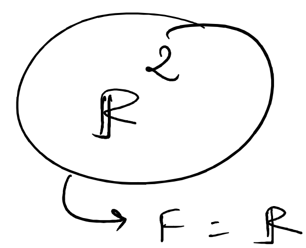
\# Verify that $\mathbb{R}^2$ is a vector space w.r.t operation component wise addition and scalar multiplication.

__Proof:__ Let $v_1, v_2 \in \mathbb{R}^2$

$$v_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}.$$



$$v_1 + v_2 = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix} \in \mathbb{R}^2.$$



Let, $v \in \mathbb{R}^2$, $\quad v = \begin{bmatrix} x \\ y \end{bmatrix}$.

$$cv = \begin{bmatrix} cx \\ cy \end{bmatrix} \in \mathbb{R}^2$$

$\mu, v \in V$.

$\mu + v \in V$, $\quad \mu . v \in V$ $\quad$ __multiplication__

They are closed under __addition__

So, $\mathbb{R}^2$ is closed w.r.t operation of vector addition

So, $\mathbb{R}^2$ is closed w.r.t operation of vector addition and scalar multiplication.

1. $u, v, w \in \mathbb{R}^2$.

$$u = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \quad v = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \quad w = \begin{bmatrix} x_3 \\ y_3 \end{bmatrix}$$

$$(u+v)+w = u+(v+w)$$

2. $e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} x \\ y \end{bmatrix} \in V$.

$$e+v = v = v+e$$

3. For, every $v = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$, $\exists \; (-v) = \begin{bmatrix} -x \\ -y \end{bmatrix}$.

$$v + (-v) = 0_v.$$

4. $u, v \in \mathbb{R}^2$

$$u+v = v+u.$$

6. $\alpha(u+v) = \alpha u + \alpha v$.

$$\alpha\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = \alpha\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \alpha\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

6. $\forall a, b \in \mathbb{R}$, $v = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$.

. . . .

$$(a+b)v = av + bv$$

7. $\forall a, b \in \mathbb{R}$, $v = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$.

$$(ab)v = a(bv) \longrightarrow a\begin{bmatrix} bx \\ by \end{bmatrix}$$

$$(ab)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} abx \\ aby \end{bmatrix}$$

$$= \begin{bmatrix} abx \\ aby \end{bmatrix}$$

8. $u = \begin{bmatrix} x \\ y \end{bmatrix} \in V$.

$$\mathbb{1} \cdot u = u.$$

$\mathbb{R}^2$ is a vector space over $\underline{\underline{\mathbb{R}}}$

<u>Note</u>: $\mathbb{R}^3, \mathbb{R}^4, \ldots \mathbb{R}^n$ all those are vector spaces over $\mathbb{R}$.

\# Consider $V = \mathbb{R}^3$ define:

$$(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$$

$$\alpha(x_1, x_2, x_3) = (x_1, \alpha x_2, x_3)$$

$$\forall \alpha \in \mathbb{R}$$

Is V a vector space?

$$\boxed{0 . v = 0}$$

$\boxed{0.v = 0}$

Soln: Let $(1, 1, 1) \in \mathbb{R}^3$.

$0.(1,1,1) = (1, 0, 1)$

$\underline{0+v}$

$\boxed{0.v = 0}$

$\boxed{1.u = u}$

$\mathbb{R}^3$.

\# Let, $V = \mathbb{R}^2$. Define addition and Scalar multiplication in V as follows:

For, $x, y \in \mathbb{R}^2$, $c \in \mathbb{R}$, $x = (x_1, x_2)$

$y = (y_1, y_2)$

$$x + y = (0, x_2 + y_2)$$

$$cx = (cx_1, cx_2).$$

Is V a vector space over $\mathbb{R}$?

Soln: Let $e = (e_1, e_2)$ be the zero vector of V.

Then for any $(x_1, x_2) \in V$

$$(x_1, x_2) + (e_1, e_2) = \underline{(x_1, x_2)}.$$

$$(0, x_2 + e_2) = (x_1, x_2).$$

which is not possible $x_1 \neq 0$?

So, V does not contain $\underline{\text{zero element}}$.

Hence V is not a vector space over $\mathbb{R}$.

Hence V is not a vector space over $\mathbb{R}$.
under given operation.

SUBSPACE

Let V be a vector space over a field F and W, a subset of V. Then W is said to be a subspace of V if and only if :

i) $0 \in W$, ii) $w_1 + w_2 \in W$, $\forall w_1, w_2 \in W$.

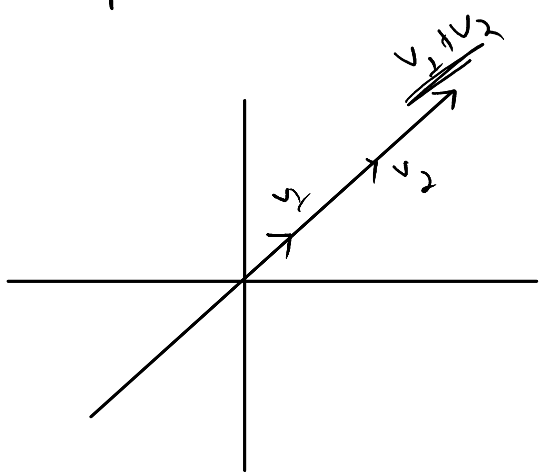
iii) $\alpha w \in W$ $\forall \alpha \in F$, $w \in W$.

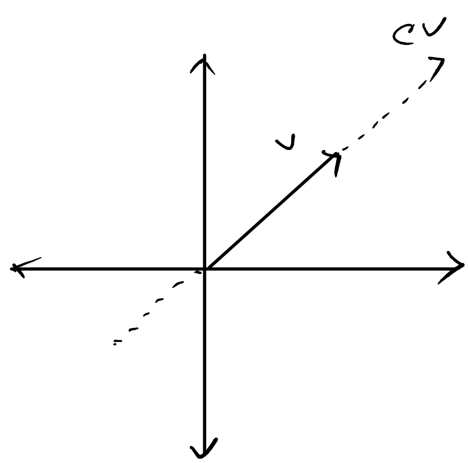
Eg: The subspaces in $\mathbb{R}$ is $\{0\}$, $\mathbb{R}$

Subspaces in $\mathbb{R}^2$.

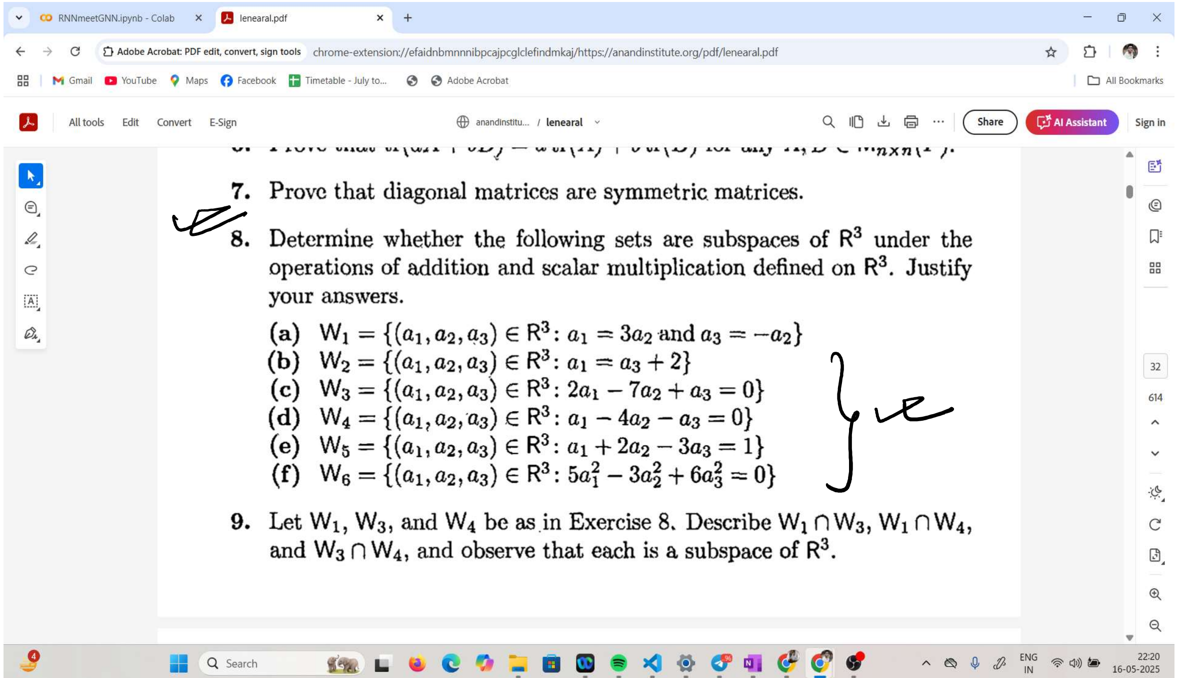
$w : \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$, $\mathbb{R}^2$.

* Any line passing through an origin is also a subspace of $\mathbb{R}^2$





Solve the question 8 :



7. Prove that diagonal matrices are symmetric matrices.

8. Determine whether the following sets are subspaces of R^3 under the operations of addition and scalar multiplication defined on R^3. Justify your answers.

(a) $\mathrm{W}_1 = \{(a_1, a_2, a_3) \in \mathrm{R}^3 : a_1 = 3a_2 \text{ and } a_3 = -a_2\}$
(b) $\mathrm{W}_2 = \{(a_1, a_2, a_3) \in \mathrm{R}^3 : a_1 = a_3 + 2\}$
(c) $\mathrm{W}_3 = \{(a_1, a_2, a_3) \in \mathrm{R}^3 : 2a_1 - 7a_2 + a_3 = 0\}$
(d) $\mathrm{W}_4 = \{(a_1, a_2, a_3) \in \mathrm{R}^3 : a_1 - 4a_2 - a_3 = 0\}$
(e) $\mathrm{W}_5 = \{(a_1, a_2, a_3) \in \mathrm{R}^3 : a_1 + 2a_2 - 3a_3 = 1\}$
(f) $\mathrm{W}_6 = \{(a_1, a_2, a_3) \in \mathrm{R}^3 : 5a_1^2 - 3a_2^2 + 6a_3^2 = 0\}$

9. Let W_1, W_3, and W_4 be as in Exercise 8. Describe $\mathrm{W}_1 \cap \mathrm{W}_3$, $\mathrm{W}_1 \cap \mathrm{W}_4$, and $\mathrm{W}_3 \cap \mathrm{W}_4$, and observe that each is a subspace of R^3.

8a) $W_1 = \{(a_1, a_2, a_3) \in R^3,\ a_1 = 3a_2 \ \& \ a_3 = -a_2\}$.

W_1 is a subspace of R^3?

$(0, 0, 0)$ $0 = 3.0$ ✓
$0 = -0$ ✓ $= 0$

$0 \in W_1$. ✓ — (i).

$w_1 = (a_1, a_2, a_3)$
$w_2 = (a_1', a_2', a_3')$
$> w_1 + w_2 = (a_1 + a_1', a_2 + a_2', a_3 + a_3')$

$a_1 = 3a_2$ ✓ | $a_1' = 3a_2'$ ✓
$a_3 = -a_2$ ✓ | $a_3' = -a_2'$.

$w_1 + w_2 \in W$. — (ii).

$a_1 + a_1' = 3(a_2 + a_2')$. | $a_3 + a_3' = -(a_2 + a_2')$.

$$a_1 + a_1' = 3(a_2 + a_2')$$
$$= 3a_2 + 3a_2'$$
$$= a_1 + a_1'$$

$$a_3 + a_3' = -(a_2 + a_2')$$
$$= -a_2 - a_2'$$
$$= a_3 + a_3'$$

$\alpha w \in W$. — (iii).

$w = (a_1, a_2, a_3)$

$$\alpha(a_1, a_2, a_3) = (\alpha a_1, \alpha a_2, \alpha a_3).$$

$$\alpha a_1 = 3(\alpha a_2).$$

$$\alpha a_3 = -\alpha a_2.$$

$a_1 = 3a_2$

$\Rightarrow \alpha a_1 = 3\alpha a_2$

$a_3 = -a_2$

$\Rightarrow \alpha a_3 = -\alpha a_2$

From -(i), (ii) and (iii), W_1 is a subspace of $\mathbb{R}^3$

[PROVED].