Algebraic Equations

$\boxed{f(x) = 0}$ $\rightarrow$ Algebraic equation / Polynomial eq^n.

$a_0 x + a_1 = 0 \qquad a_0 \neq 0$

$a_0 x^2 + a_1 x + a_2 = 0, \qquad a_0 \neq 0.$

Genaral form :-

$a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n = 0$

Fundamental theorem of Classical Algebra :

Every algebraic equation has a root, real or complex.

$f(x) = a_0 (x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_n)$

$ax^2 + bx + c = 0, \qquad x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$

$\hookrightarrow \left[x - \left(\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\right]\left[x - \left(\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}\right)\right]$

Note : $f(x) = 0$

$\hookrightarrow$ $\alpha + i\beta$ is a root

$=$

$\hookrightarrow \alpha + i\beta$ is a root
$\rightarrow \alpha - i\beta$ is also a root $\}$.

$f(x) = 0$

$\hookrightarrow \alpha + \sqrt{\beta}$

$\hookrightarrow \alpha - \sqrt{\beta} \longrightarrow$ conjugate surd.

Relation between roots & coefficients!

$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n, \quad a_0 \neq 0.$

Let, $\alpha_1, \ldots, \alpha_n$ be the roots of eq^n. $f(x) = 0$.

$$a_0 x^n + a_1 x^{n-1} + \cdots + a_n = a_0 (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)$$

$$= a_0 \left[x^n - \sum \alpha_1 x^{n-1} + \sum \alpha_1 \alpha_2 x^{n-2} - \cdots + (-1)^n (\alpha_1 \cdots \alpha_n) \right]$$

$a_0 (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)$ $\qquad \alpha_1 x$

$= a_0 \left[x^n \right.$

$$(x - \alpha_1)(x - \alpha_2)(x - \alpha_3).$$

$$= (x^2 - \alpha_2 x - \alpha_1 x + \alpha_1 \alpha_2)(x - \alpha_3)$$

$$= x^3 - \alpha_2 x^2 - \alpha_1 x^2 + \alpha_1 \alpha_2 x - \alpha_3 x^2 + \alpha_2 \alpha_3 x + \alpha_1 \alpha_3 x - \alpha_1 \alpha_2 \alpha_3$$

$$+\alpha_2\alpha_3 x + \alpha_1\alpha_3 x - \alpha_1\alpha_2\alpha_3$$

$$= x^3 - x^2\underbrace{(\alpha_2+\alpha_1+\alpha_3)}_{\sum \alpha_i} + x(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_1\alpha_3) - \alpha_1\alpha_2\alpha_3.$$

$\hookrightarrow n=3$ $\qquad \sum \alpha_i \alpha_j$

$\sum \alpha_1$

$$a_0\left[x^n - \sum \alpha_1 x^{n-1} + \sum \alpha_1\alpha_2 x^{n-2} - \dots + (-1)^n(\alpha_1 \dots \alpha_n)\right]$$

Sum of root $(\sum \alpha_i) = -\dfrac{a_1}{a_0}$

$$\sum\sum \alpha_i \alpha_j = \frac{a_2}{a_0}$$

$$\sum\sum\sum \alpha_i \alpha_j \alpha_k = -a_3/a_0$$

$\vdots$

(Product of roots) $\alpha_1\alpha_2 \dots \alpha_n = (-1)^n \dfrac{a_n}{a_0}$

$$f(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_n$$

Particular Cases :-

$$a_0 x^3 + a_1 x^2 + a_2 x + a_3 = 0$$

$\longrightarrow \alpha, \beta, \gamma$ roots.

$$\alpha+\beta+\gamma \text{ (Sum of roots)} = -\frac{a_1}{a_0}$$

$$\alpha\beta+\beta\gamma+\alpha\gamma = a_2/a_0.$$

$$\alpha\beta\gamma = -a_3/a_0$$

3 realations.

2) $a_0x^4+a_1x^3+a_2x^2+a_3x+a_4=0$

$\hookrightarrow \alpha, \beta, \gamma, \delta$

${}^4C_2$ ${}^4C_3 = \frac{4!}{3!}$

$$\alpha+\beta+\gamma+\delta = -\frac{a_1}{a_0}$$

$$\alpha\beta+\beta\gamma+\gamma\delta+\alpha\delta+\beta\delta+\alpha\gamma = \frac{a_2}{a_0}.$$

$$\alpha\beta\gamma+\alpha\beta\delta+\beta\gamma\delta+\alpha\gamma\delta = -\frac{a_3}{a_0}.$$

$$\alpha\beta\gamma\delta = \frac{a_4}{a_0}.$$

#1. $2x^3-x^2-18x+9=0.$

$\hookrightarrow$ Two roots are equal in magnitude but opposite in sign!

$\Rightarrow$ α, β, γ (Let), $\alpha=-\beta.$

$$\alpha+\beta+\gamma = -\frac{a_1}{a_0} = -\frac{(-1)}{2} = \frac{1}{2} \quad \text{—(i)}.$$

$$\alpha\beta + \beta\gamma + \alpha\gamma = -9 \quad \text{— (ii)}$$

$$\alpha\beta\gamma = -9/2 \quad \text{— (iii)}$$

$\alpha = -\beta$ $\qquad$ $\gamma = 1/2$

$\Rightarrow \alpha + \beta = 0$

$$\alpha\beta\gamma = -9/2$$

$$\Rightarrow \alpha(-\alpha) \cdot \frac{1}{2} = -9/2$$

$$\Rightarrow \alpha^2 = 9$$

$$\Rightarrow \alpha = \pm 3$$

$\{3, -3, 1/2\}$ (Ans)

2) $16x^4 - 64x^3 + 56x^2 + 16x - 15 = 0$

Their roots are in A.P.

Then determine the roots. $\qquad a-3d, a-d, a+d, a+3d$

$\alpha - 3\delta, \alpha - \delta, \alpha + \delta, \alpha + 3\delta$

roots of $f(x)$.

$$(\alpha - 3\delta) + (\alpha - \delta) + (\alpha + \delta) + (\alpha + 3\delta) = 4$$

$$\Rightarrow 4\alpha = 4$$

$\Rightarrow 4\alpha = 4$

$\Rightarrow \alpha = \underline{1}$

$(\alpha - 3\delta)(\alpha - \delta)(\alpha + \delta)(\alpha + 3\delta) = \frac{-15}{16}$

$\Rightarrow (\alpha^2 - 9\delta^2)(\alpha^2 - \delta^2) = -\frac{15}{16}$.

$\Rightarrow (1 - 9\delta^2)(1 - \delta^2) = \frac{-15}{16}$.

$\Rightarrow -1/2,\ 1/2,\ 3/2,\ 5/2$

Transformation of Equations

$\alpha_1, \ldots \alpha_n \qquad x^n + p_1 x^{n-1} + \ldots + p_n = 0$ ie

$m\alpha_1, \ldots m\alpha_n, \quad y = m\alpha_1 \qquad \alpha_1 = y/m$.

$\Rightarrow \left(\frac{y}{m}\right)^n + p_1\left(\frac{y}{m}\right)^{n-1} + \ldots + p_n = 0$

$$\Rightarrow \boxed{y^n + p_1 m y^{n-1} + p_2 m^2 y^{n-2} + \ldots + p_n m^n = 0.}$$

#3 Find the equation whose roots are the roots of the equations $x^4 - 8x^2 + 8x + 6 = 0$, ie — (1) each diminished by 2

$\Rightarrow \alpha, \beta, \gamma, \delta$

Transformed root $= \alpha - 2, \beta - 2, \gamma - 2, \delta - 2$

$$y = \alpha - 2 \Rightarrow \alpha = y + 2$$

α is a root of eq (i).

$$\alpha^4 - 8\alpha^2 + 8\alpha + 6 = 0$$

$$\Rightarrow (y+2)^4 - 8(y+2)^2 + 8(y+2) + 6 = 0$$

#4 α, β, γ are roots of the eq^n

$$x^3 + px^2 + qx + r = 0.$$

Find the eq^n such that $(\alpha\beta + \beta\gamma)$, $(\beta\gamma + \gamma\alpha)$, $(\gamma\alpha + \alpha\beta)$ are the roots.

$$\alpha + \beta + \gamma = -p.$$

$$\alpha\beta + \beta\gamma + \alpha\gamma = q.$$

$$\alpha\beta\gamma = -r.$$

$$\Rightarrow \alpha\gamma = -r/\beta.$$

$$\Rightarrow y = \alpha\beta + \beta\gamma$$

$$\Rightarrow y = q - \alpha\gamma.$$

$$\Rightarrow \boxed{y = q + \frac{r}{\beta}}$$

$$\Rightarrow \beta = \frac{r}{y-q}$$

$$x^3 + px^2 + qx + r = 0$$

$y = f(\alpha) / f(\beta) / f(\gamma)$

α, β

$$\Rightarrow \beta^3 + p\cdot\beta^2 + q\beta + r = 0$$

$$\Rightarrow \left(\frac{r}{y-q}\right)^3 + p\cdot\left(\frac{r}{y-q}\right)^2 + q\cdot\left(\frac{r}{y-q}\right) + r = 0.$$

$$\ldots^3 + q(y-q)^2 + pr(y-q) + r^2 = 0$$

$$\Rightarrow (y-q)^3 + q(y-q)^2 + pr(y-q) + r^2 = 0$$

This is the required equation!