Properties on composition of Functions.

Thm 1: Let $f : A \to B$, $g : B \to C$, $h : C \to D$, be three mappings. Then $h \circ (g \circ f) = (h \circ g) \circ f$.

Proof: $h(g(f(x))) = hg(f(x))$

$h \circ g \circ f : A \to D$. | $(h \circ g) \circ f : A \to D$.

Let, $x \in A$ and let $f(x) = y$, $g(y) = z$, $h(z) = w$.

$$g \circ f(x) = g(y) = z$$

$$h \circ g(y) = h(z) = w.$$

$$h \circ (g \circ f)(x) = h(z) = w, \quad x \in A$$

$$(h \circ g) \circ f(x) = h \circ g(y) = w, \quad x \in A.$$

$$h \circ (g \circ f)(x) = (h \circ g) \circ f(x) \quad \forall x \in A.$$

[PROVED]

Thm 2: If $f : A \to B$ and $g : B \to C$ both injective mappings, then the composite mapping $g \circ f : A \to C$ is injective.

Thm 3: If $f : A \to B$ and $g : B \to C$ be both surjective, then $g \circ f : A \to C$ is surjective.

Thm 3: If $f : A \to B$ and $g : B \to C$ are surjective, then $g \circ f : A \to C$ is surjective.

Thm 4: If $f : A \to B$ and $g : B \to C$ be two mappings such that $g \circ f : A \to C$ is surjective, then g is surjective.

For example: $f : \mathbb{Z} \to \mathbb{Z}$

$f(x) = 2x$, $x \in \mathbb{Z}$ and

$g : \mathbb{Z} \to \mathbb{Z}$

$g(x) = \left[\frac{x}{2}\right]$, $x \in \mathbb{Z}$

$g \circ f : \mathbb{Z} \to \mathbb{Z}$. $f(x) = x$

$g \circ f(x) = x$. $\to$ identity f^n.

Inverse Mapping.

Def^n: Let $f : A \to B$ be a mapping. f is said to be invertible if there exist a mapping $g : B \to A$ such that $g \circ f = i_A$ and $f \circ g = i_B$. In this case g is said to be an inverse of f.

Thm: If $f : A \to B$ be an invertible mapping, then its inverse is unique.

inverse is unique.

$$f^{-1} \circ f = i_A \qquad f \circ f^{-1} = i_B.$$

Ex: $f : \mathbb{R} \to \mathbb{R}$

$f(x) = 3x \quad , \quad g(x) = x/3 \qquad x \in \mathbb{R}.$

S.T g is inverse of f.

$\Rightarrow \quad g \circ f : \mathbb{R} \to \mathbb{R} \quad , \quad g \circ f(x) = g(3x)$

$= 3x/3 = x.$

$f \circ g : \mathbb{R} \to \mathbb{R}, \quad , f(g(x)) = 3 \cdot \frac{x}{3} = x.$

So, $f \circ g = g \circ f = x = i_{\mathbb{R}}$

So, g is the inverse of f.

Thm: A mapping $f : A \to B$ is invertible if and only if f is a bijection.

$\Rightarrow \quad f : A \to B$ is invertible,

$g : B \to A, \quad g \circ f = i_A$ and $f \circ g = i_B.$

$i_A \to$ injective, So f is injective.

$i_B \to$ surjective, so f is also surjective.

$\therefore$ f is a bijection.

Conversely, Let $f: A \to B$ be a bijection.

Let, $y \in B$, since f is a bijection, y has a unique pre-image 'x' in A. $\to f(x) = y$.

Define a mapping $g: B \to A$ $\quad g(y) = x$, $y \in B$.

Then $g \circ f(x) = g(y) = x$, $x \in A$.

$f \circ g(y) = f(x) = y$, $y \in B$.

$\therefore$ $g \circ f = i_A$; $f \circ g = i_B$

So, f is invertible.