Inequalities

Syllabus : AM - GM - HM, Cauchy - Schwarz

Introduction : $a > b$, $a < b$, $a \geq b$, $a \leq b$

Properties:

$a, b, c \in \mathbb{R}$

i) $a > b$, $b > c \Rightarrow a > c$

ii) $a > b \Rightarrow a + c > b + c$

iii) $a > b$ & $c > 0 \Rightarrow ac > bc$

iv) $a > b$, $c < 0 \Rightarrow ac < bc$

v) $a > b$, $c = 0 \Rightarrow ac = bc$

Thm: $\{a_1, a_2, \ldots a_n\}$; $\{b_1, b_2, \ldots b_n\} \in \mathbb{R}$.

$a_i > b_i \quad \forall i \quad i \in \mathbb{Z}$

$$\sum_{i=1}^{n} a_i > \sum_{i=1}^{n} b_i$$

$$\Rightarrow a_1 + a_2 + \ldots + a_n > b_1 + b_2 + \ldots + b_n$$

Thm: $a_i > b_i$ $\quad \prod^{n} a_i > \prod^{n} b_i$

Th^m: $a_i > b_i$ $\quad \prod_{i=1}^{n} a_i > \prod_{i=1}^{n} b_i$

Th^m: $a, b \in \mathbb{R}^+$, $a > b$, $n \in \mathbb{Z}^+$.

$$a^n > b^n.$$

Arithmetic, Geometric & Harmonic Mean:

$$AM = \frac{a_1 + a_2 + \dots + a_n}{n} \qquad a_i \in \mathbb{R}^+$$

$$GM = \sqrt[n]{a_1 a_2 \dots a_n}, \qquad a_i \in \mathbb{R}^+$$

$$HM = \frac{n}{1/a_1 + 1/a_2 + \dots + 1/a_n}, \qquad a_i \in \mathbb{R}^+$$

Weighted Arithmetic Mean:-

Let, $p_1, p_2, \dots p_n$ be n positive reational number.

$$W.A.M = \frac{p_1 a_1 + p_2 a_2 + \dots + p_n a_n}{p_1 + p_2 + \dots + p_n}.$$

Weighted Geometric Mean:

$$W.G.M = \left(a_1^{p_1} a_2^{p_2} \dots a_n^{p_n} \right)^{1/(p_1 + p_2 + \dots + p_n)}.$$

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Weighted Harmonic Mean:

$$W.H.M = \frac{p_1+p_2+\cdots+p_n}{p_1/a_1 + p_2/a_2 + \cdots + p_n/a_n}.$$

AM - GM Inequality:

If $a_1, a_2, \ldots a_n$ be n positive real numbers then

$$\frac{a_1+a_2+\cdots+a_n}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_n}.$$

Equality condition: $a_1 = a_2 = \cdots = a_n$

$$\frac{n a_1}{n} \geq \sqrt[n]{a_1^n}$$

$$a_1 \geq a_1 \quad =$$

AM - GM - HM.

$$\boxed{AM \geq GM \geq HM}, \quad G \geq H$$

H/W Proof: $GM \geq HM$.

Weighted case:

Weighted case :

$$\frac{p_1 a_1 + p_2 a_2 + \dots + p_n a_n}{p_1 + p_2 + \dots + p_n} \geq \left(a_1^{p_1} \dots a_n^{p_n}\right)^{1/(p_1 + \dots + p_n)}$$

W.A.M $\geq$ W.G.M .

Problems :-

$a, b, c, d \in \mathbb{R}^+$, $a \neq b \neq c \neq d$.

P.T : $a^5 + b^5 + c^5 + d^5 > abcd(a+b+c+d)$.

Sol^n : a^5, b^5, c^5, d^5 $\quad \frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$.

$\Rightarrow (a+b+c+d)^4 \geq 16\,abcd$.

$\Rightarrow (a+b+c+d)^5 \geq 16\,abcd\,(a+b+c+d)$

$a, b, c, d \in \mathbb{R}^+$

By A.M . G.M Inequality,

$$\frac{a+b+c+d}{4} > \sqrt[4]{abcd}\ .$$

$$\Rightarrow \left(\frac{a+b+c+d}{4}\right)^4 > abcd\ .$$

$\frac{64}{4}$

$\Rightarrow \left(\frac{a+b+c+d}{4}\right) \geq$

$64 \times 4 = 256$

$\Rightarrow (a+b+c+d)^4 > 256abcd$

$\Rightarrow (a+b+c+d)^4 \cdot (a+b+c+d) > 256abcd(a+b+c+d)$.

$\Rightarrow (a+b+c+d)^5 > 256abcd(a+b+c+d)$.

$(a+b+c+d)^5 \geq a^5 + b^5 + c^5 + d^5 >$

$(a+b)^2 \geq a^2 + b^2$ $\quad (a+b+c+d)^5$

$(a+b)^3 \geq a^3 + b^3$ $\quad \geq abcd$.

$(a+b)^n \geq a^n + b^n$ $\quad b > c$, $b > 2$ $\Rightarrow c > 2$

$a^5 + b^5 + c^5 + d^5 > 256\, abcd(a+b+c+d)$.

$$\boxed{257 > 256 \times 1 > 1 \times 1}$$

$\rightarrow a^5 + b^5 + c^5 + d^5 > abcd(a+b+c+d)$.

d 4 $\quad a \neq b \neq c \neq d$ $\quad b > 2$, $b > c$.

$5 > 2$ $\quad 5 > 3$

$5 > 3$ $\quad 5 > 2$

$5 > 3 \qquad\qquad 5 > 2$

Result:

$$\frac{a^{p+q} + b^{p+q} + c^{p+q} + d^{p+q}}{4} > \left(\frac{a^p + b^p + c^p + d^p}{4}\right) \cdot \left(\frac{a^q + b^q + c^q + d^q}{4}\right)$$

$\{a^p, b^p, c^p, d^p\} \rightarrow$ no's

$\{a^q, b^q, c^q, d^q\} \longrightarrow$ weight's

$$\frac{a^p \cdot a^q + b^p \cdot b^q + c^p \cdot c^q + d^p \cdot d^q}{a^q + b^q + c^q + d^q}$$

$$\geq \left[(a^p)^{a^q} \cdot (b^p)^{b^q} \cdot (c^p)^{c^q} \cdot (d^p)^{d^q}\right]^{\frac{1}{a^q + b^q + c^q + d^q}}$$

$$\Rightarrow \frac{a^{p+q} + b^{p+q} + c^{p+q} + d^{p+q}}{a^q + b^q + c^q + d^q} \geq a^{pa^q} \cdot b^{pb^q} \cdot c^{pc^q} \cdot d^{pd^q}$$

\# $n \in \mathbb{Z}^+$, > 1, Prove that $n(n+1)^2 > 4(n!)^{3/n}$.

$\approx O(n^3)$.

$\Rightarrow 1^3, 2^3, \ldots n^3$.

AM > GM

$$\Rightarrow \frac{1^3+2^3+\ldots n^3}{n} \geq (1^3 \cdot 2^3 \ldots n^3)^{1/n}.$$

$$\Rightarrow \frac{1^3+2^3+\ldots+n^3}{n} \geq (1 \cdot 2 \ldots n)^{3/n}$$

$$\Rightarrow \frac{n^2(n+1)^2}{4n} \geq (n!)^{3/n}$$

$$\Rightarrow n(n+1)^2 \geq 4(n!)^{3/n}$$

$\boxed{\text{PROVED.}}$

\# $a_1, \ldots a_5 \in \mathbb{R}^+$ P.T

$$\left(\frac{a_1+a_2+\ldots+a_5}{5}\right)^5 \geq \left(\frac{a_1+a_2}{2}\right)^2 \left(\frac{a_3+a_4+a_5}{3}\right)^3.$$

Sol^n: $\left(\frac{a_1+a_2}{2}\right)$, $\left(\frac{a_3+a_4+a_5}{3}\right)$; $(2, 3)$

$$\frac{2 \cdot \left(\frac{a_1+a_2}{2}\right) + 3 \cdot \left(\frac{a_3+a_4+a_5}{3}\right)}{} \quad . \qquad 1/_{2+3}$$

3.

$$\left(\frac{\quad}{2}\right) \quad \left(\quad\right) \quad /2+3$$

$$\frac{\quad}{2+3} \geq \left[\left(\frac{a_1+a_2}{2}\right)^2 \cdot \left(\frac{a_3+a_4+a_5}{3}\right)^3\right]$$

Solve it and check it!

Cauchy – Schwarz Inequality :-

a_i's $\in \mathbb{R}$, b_i's $\in \mathbb{R}$.

$$\left(a_1^2+a_2^2+\dots+a_n^2\right)\left(b_1^2+b_2^2+\dots+b_n^2\right) \geq \left(a_1b_1+a_2b_2+\dots+a_nb_n\right)^2$$

Equality : # $a_i = 0$ or $b_i = 0$

$a_i = b_i k$ $k \to$ non-zero real.

$x, y \in \mathbb{R}$.

P.T $-\frac{1}{2} \leq \boxed{\frac{(x+y)(1-xy)}{(1+x^2)(1+y^2)}} \leq \frac{1}{2}$.

$a_i = \{\underset{a_1}{2x}, \underset{a_2}{(1-x^2)}\}$ $b_i = \{\underset{b_1}{1-y^2}, \underset{b_2}{2y}\}$.

$$\left[(2x)^2+(1-x^2)^2\right]\left[(1-y^2)^2+(2y)^2\right]$$

$$\left[(2x)^2+(1-x^2)^2\right]\left[(1-y^2)^2+(2y)^2\right] \geq \left[2x(1-y^2)+(1-x^2)2y\right]^2$$

$$\Rightarrow \left[4x^2+1-2x^2+x^4\right]\left[1-2y^2+y^4+4y^2\right] \geq \left[2x-2xy^2+2y-2x^2y\right]^2$$

$$\Rightarrow \left[1+2x^2+x^4\right]\left[1+2y^2+y^4\right] \geq \left[2(x+y)-2xy(x+y)\right]^2$$

$$\Rightarrow (1+x^2)^2(1+y^2)^2 \geq \left[2(1-xy)(x+y)\right]^2$$

$$\Rightarrow (1+x^2)^2(1+y^2)^2 \geq 4\left[(1-xy)(x+y)\right]^2$$

$$\Rightarrow \left[\frac{(x+y)(1-xy)}{(1+x^2)(1+y^2)}\right]^2 \leq \frac{1}{4}$$

$x^2 \leq 4$

$-2 \leq x \leq 2$

$$\Rightarrow -\frac{1}{2} \leq \quad '' \quad \leq \frac{1}{2}$$