Relation and Functions.

Cartesian Product: X, Y non-empty set.

$$X \times Y = \{(x, y) \mid x \in X,\ y \in Y\}.$$

$$|X| = m, \quad |Y| = n$$

$$|X \times Y| = mn.$$

Relation

Defn:- A relation R from a set A into a set B is a subset of $A \times B$.

If $(a, b) \in R$, we say that a is R-related to b and denoted by aRb. or, $R(a) = b$.

e.g: $A = \{1, 2, 3\}$

$B = \{p, q, r\}$.

$R = \{(1, q), (2, r), (3, q), (1, p)\}$.

$A \times B = \{(1, p), (1, q), (1, r), (2, p), (2, q), (2, r), (3, p), (3, q), (3, r)\}$

$R \subset A \times B$

hence it is a relation from A to B.

$2 R \gamma$, $3 \not R \gamma$.

Note: '$A \times A$' and null set ϕ is always a relation on A.

Domain and Range of R (relation)

$\operatorname{dom}(R) = \{ a \mid a \in A , \exists\, b \in B ,$ such that $(a, b) \in R \}$

$\operatorname{range}(R) = \{ b \mid b \in B , \exists\, a \in A ,$ such that $(a, b) \in R \}$.

Example:

$A = \{ 4, 5, 6, 9 \}$.

$B = \{ 20, 22, 24, 28, 30 \}$.

Let us define a relation R from A to B by 'aRb if and only if a divides b', when $a \in A$, $b \in B$

$R = \{ (4, 20), (4, 24), (4, 28), (5, 20), (5, 30), (6, 24), (6, 30) \}$.

$D(R) = \{4, 5, 6\}$

$I(R) = \{20, 24, 28, 30\}$.

Inverse of a Relation:

R, $R^{-1} = \{(b, a) \mid (a, b) \in R\}$

example: Find R^{-1}.

e.g.: $A = \{1, 2, 3\}$

$B = \{p, q, r\}$.

$R = \{(1, q), (2, r), (3, q), (1, p)\}$.

$R^{-1} = \{(q, 1), (r, 2), (q, 3), (p, 1)\}$.

\# $(R^{-1})^{-1} = R$.

Equivalence relation

Let A be a set and R be a relation on A. The R is called:

i) reflexive, if for all $a \in A$, aRa

$a, b \in A$ whenever aRb

i) reflexive , if ,

ii) symmetric, if for all $a, b \in A$, whenever aRb implies bRa must hold.

$$(a,b) \in R \Rightarrow (b,a) \in R.$$

iii) transitive; if $a, b, c \in A$, $(a,b) \in R$, $(b,c) \in R$, $(a,c) \in R$ must hold

If all three happened, then R is equivalence relation.

Eg 1: Let R_1 be a relation on $\mathbb{Z}$ defined as:
'for all $a, b \in \mathbb{Z}$, $a R_1 b$ if and only if $ab \geq 0$'.

Solⁿ: Reflexivity:-

$a \in \mathbb{Z}$, $a \times a = a^2 \geq 0$.

$(a,a) \in R_1$ $\quad a R_1 a$.

Symmetric:-

$a, b \in \mathbb{Z}$, $ab \geq 0$.

$ba \geq 0$.

$$(a,b) \in R_1 \Rightarrow (b,a) \in R_1.$$

Transitivity:

$-5, 0$, $-5 \times 0 = 0 \geq 0 \rightarrow -5 R_1 0$
(a b)

$(0, 7)$, $0 \times 7 = 0 \geq 0 \rightarrow 0 R_1 7$.

$(-5, 7)$, $-5 \times 7 = -35 \ngeq 0$.

$\rightarrow -5 \not{R}_1 7$.

So, it is not transitive.

Hence, it is not an equivalence relation.

\# Eg 2: The following relations are defined on the set $\mathbb{R}$ of real numbers, Find whether these relations are reflexive, symmetric or Transitive.

i) $a R b$ iff $|a-b| > 0$

ii) $a R b$ iff $1 + ab > 0$.

iii) $a R b$ iff $|a| \leq b$

Soln:

$$\left.\begin{array}{l}|a-b| > 0 \\ |b-c| > 0\end{array}\right\} \rightarrow |a-c| > 0.$$

$$|a-b| + |b-c| \geq |a-b+b-c| > 0$$
$$= |a-c| > 0.$$

So, it is not reflexive, but symmetric and transitive.

ii) aRb, iff $ab+1>0$
$$\Rightarrow b > -1/a.$$

Reflexivity,
$$a^2+1>0$$

Symmetric,
$$ba+1>0$$

$$1+ab>0 \quad -(i)$$
$$1+bc>0 \quad -(ii).$$

To Prove: $(1+ac)>0$?

$a=-1$, $b=0$, $c=3$.

$$1-3=-2 \not> 0.$$

So, it is not transitive.

iii) aRb iff $|a| \leq b$

$|-3| \leq 4$

$a=-3$

$b=4$

$|b| \leq a$

$a = -3$

$b = 4$

$|b| \le a$

$|4| \le -3$

$|p| \le q$

$|q| \le r$

$|p| \le r$

$\longrightarrow$ So, it is transitive relation.

Equivalence Class

Let, ρ be an equivalence relation on set X. For all $x \in X$, Let $[x]$ denote the set,

$$[x] = \{ y \in X \mid y \rho x \}$$

The set $[x]$ is the equivalence class determined by x w.r.t ρ.

e.g: $S = \{1, 2, 3, 4, 5\}$.

$$\rho = \{(1,1), (2,2), (1,3), (1,2), (2,1)\}.$$

$[1] = \{1, 3, 2\}$

$[2] = \{1, 2\}$

$[3] = \phi$ $\quad$ $4 = \phi$.

$[3) = \bar{\phi} \quad , \quad 4 = \phi .$