Cubic Equation

$$a_0 x^3 + 3a_1 x^2 + 3a_2 x + a_3 = 0 \qquad x = y + h$$

$$G = (a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3), \quad H = (a_0 a_2 - a_1^2).$$

$$\left[y^3 + \frac{3H}{a_0} y + \frac{G}{a_0^3} = 0 \right]$$ Transformed Equation.

G, H

Nature of Roots in Cubic Equation :

Case - I, $G^2 + 4H^3 > 0$.

→ The cubic equation has two imaginary roots.

Case - II, $G^2 + 4H^3 < 0$ and $H < 0$.

→ It has no negative root, All its roots are real.

Case - III, $G^2 + 4H^3 = 0$.

α, β, γ, $(\alpha - \beta), (\beta - \gamma), (\gamma - \alpha)$ is zero.

$\alpha = \beta$; $\beta = \gamma$; $\gamma = \alpha$

Roots are repeating!

Case - IV, $G^2 + 4H^3 = 0$ and $H = 0$.

$\boxed{x^3 = 0}$ reduced, it has 3 equal roots.

Cardan's Method

1) $x^3 - 18x - 35 = 0$. — (i)

Step 1: $x = u + v$

or, $x^3 = (u+v)^3$ [cubing both sides]

or, $x^3 = u^3 + v^3 + 3u.v.x$. $3uv(u+v)$

or, $x^3 - 3uvx - (u^3+v^3) = 0$. — (ii)

Step - 2: Comparing the coefficients.

$-3uv = -18$. | $u^3 + v^3 = 35$.

$\Rightarrow uv = 6$ | $\Rightarrow u^3 + v^3 = 27 + 8$.

$\Rightarrow$

$$\left[u^3 = \frac{1}{2}\left(-G + \sqrt{G^2 + 4H^3}\right), \quad v^3 = \frac{1}{2}\left(-G - \sqrt{G^2 + 4H^3}\right) \right]$$

$$\boxed{G = a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3}$$

$H = a_0 a_2 - a_1^2$.

$x^3 - 18x - 35 = 0$ $\qquad a_0 x^3 + 3a_1 x^2 + 3a_2 x + a_3 = 0$

$a_0 = 1$, $\quad 3a_2 = -18 \Rightarrow a_2 = -6 \quad a_3 = -35$.

$a_1 = 0$.

By Putting $G = 35$, $H = \ldots$

$$u^3 = \frac{1}{2}\left(35 + \sqrt{35^2 - 864}\right) = 27$$

$$v^3 = \frac{1}{2}\left(35 - \sqrt{35^2 - 864}\right) = 8$$

$u^3 = 27$, $u = 3, 3\omega, 3\omega^2$.

$v^3 = 8$, $v = 2, 2\omega, 2\omega^2$.

$x = u + v$. $\qquad x = 5$ (an - I)

$u \cdot v = 6$. $\qquad x = \cancel{3\omega} \; 5\omega$

$\qquad x = 5\omega^2$. $\Big\}$ X

$3\omega \times 2\omega = 6\omega^2$ $\qquad x = 3\omega + 2\omega^2$

$3\omega \times 2\omega^3 = 6\omega^3 = 6$ $\qquad x = 3\omega^2 + 2\omega$

Transformation before Applying Cardan's Method :

$$x^3 - 15x^2 - 33x + 847 = 0.$$

$$x = y + h$$

$$(y+h)^3 - 15(y+h)^2 - 33(y+h) + 847 = 0$$

$$\Rightarrow y^3 + (3h-15)y^2 + (\ \cdots\ ) \cdots\ ' = 0$$

$\hookrightarrow$ need to make zero.

$$3h = 15$$
$$\Rightarrow h = 5$$

$$y^3 - 108y + 432 = 0 \quad \cdots \text{(i)}.$$

Apply cardans method.

$\rightarrow$ value of y

$x = y + h$, $h \rightarrow 5$ (Solve it!).

Bi-Quadratic Equation

$$a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0. \quad \text{(General form)}$$

Ferrari's sol^n of a biquadratic Equation,

It reduces the problem of solving a biquadratic equation to solve two quadratic equations. This is done by expressing the biquadratic as the difference of two perfect square.

$$a(ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0)$$

$$= \boxed{(ax^2 + 2bx + \lambda)^2 - (mx+n)^2}$$

$4b = 10$
$\Rightarrow b = \frac{5}{2}$
$4abx^3$

<u>Ex:</u> $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$.

$\hookrightarrow (x^2 - 5x + \lambda)^2 - (mx+n)^2$

λ, m, n are constants.

$35 = 25 + 2\lambda - m^2$
or, $m^2 = 2\lambda - 10$

$-50 = -10\lambda - 2mn$
$\Rightarrow mn = -5\lambda + 25$.

$24 = \lambda^2 - n^2 \Rightarrow n^2 = \lambda^2 - 24$.

Eliminate m, n from above equation

Construct equation in λ.

$$f(\lambda) = 0$$

$$\lambda = \dots$$

Find m, and n

$\left(x^2 - 5x + 5\right)^2 - 1 = 0$

$$\Rightarrow (x-2)(x-3)(x-1)(x-4) = 0$$