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Linear Span of a Subset

* **Linear combination :-** Let $V(F)$ be a vector space and $S = \{v_1, \dots v_n\}$ be a subset of V. Then the vector $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$, α_i's $\in F$, $1 \le i \le n$ is called linear combination of the elements of S.

* **Linear Span :** Let $V(F)$ be a vector space and S be a non-empty subset of V. Then Linear span of S is the set of (all possible linear combination of finitely many elements of S.) $L(S)$

Note :- Let $V(F)$ be a vector space and S a finite subset of V. Then span(S) is the smallest subspace of V containing S.

$$\boxed{\text{Smallest subspace} = \text{Span}(S)}$$

Linear Independent Vectors :-

Let V is a vector space and $v_1, v_2, \dots v_n$ are the vectors of V are said to be linearly independent over the field F if there does not exists any linear combination

$$\boxed{c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0}$$

$$\Rightarrow c_i = 0$$

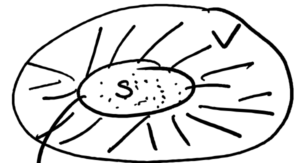
Linear dependent :- Atleast one c_i is non-zero.

Basis of a Vector space

Def^n: A subset S of a vector space $V(F)$ is said to be a basis of $V(F)$ if :

i) S spans V.

ii) S is linear independent.



$$V = \mathbb{R}^3 \qquad \boxed{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}}$$

$$V = \mathbb{R}^2 \qquad \left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right) \longrightarrow 2$$

$$V = \mathbb{R}^n$$

$$\text{Basis} : \left\{ \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}, \ldots, \begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix} \right\} \longrightarrow n$$

$\dim(V)$ = no. of elements in the Basis of V.

Note : Is zero vector linearly dependent or independent ?

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$\Rightarrow$ $0 \in V$, $\exists$ $\alpha \in F$ such that

$$\boxed{\alpha . 0 = 0 .}$$

$\alpha \neq 0$

$\alpha = 0$.

So, $\{0\}$ is linearly dependent set.

$\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ x_1 x_2 1.

$\boxed{x_1} . 0 + x_2 . 0 + 1 . 0 = 0$

$\hookrightarrow c_i \neq 0$.

Any non-zero vector is linearly dependent or independent?

$\Rightarrow$ $\alpha \in F$, $V \neq 0$. $V =$

$\alpha . V = 0$.

$\Rightarrow \alpha = 0$, $V \neq 0$

1 -1

$(-2, -1, 1, 2)$.

$( ) - 2 + ( )(-1) + ( ) 1$

$x\hat{i} + y\hat{j} + z\hat{k}$.

$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

$\alpha \begin{bmatrix} 1 \\ \vdots \end{bmatrix} = 0$

$\begin{bmatrix} \\ -1 \end{bmatrix}$

$$\alpha \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = 0$$

$$\alpha = 0$$

$R^n : \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ $\quad R^n = \begin{bmatrix} 1 \\ 2 \\ \vdots \\ n \end{bmatrix} \checkmark \rightarrow$ non-zero vector.

$R^2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. $\quad v = \begin{bmatrix} 1 \\ 0 \\ 3 \\ \checkmark \end{bmatrix} \rightarrow$ non zero vector.

$$\alpha V = 0$$

$\rightarrow \alpha = 0 \quad , V \neq 0$ $\quad \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$\{0\} \rightarrow$ dependent vector

$\{\text{non-zero}\} \rightarrow$ independent vector. $\quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

Subspaces Associated with Matrix :

Let $[A]_{m \times n}$ be a matrix. There are four subspaces associated with matrix.

a) Column Space $C(A)$.

b) Null Space $N(A)$

... Space $R(A)$

b) Null space

c) Row Space, $R(A)$

d) Null space of A^T, $N(A^T)$

Def^n:

$$A = \begin{array}{c} \begin{array}{cccc} c_1 & c_2 & c_3 & \dots c_n \end{array} \\ \begin{bmatrix} \vdots & \vdots & \vdots & \dots & \vdots \end{bmatrix} \end{array}$$

Set of all linear combination of those columns is called column space. $C(A)$.

i.e. every element of column space can be written as linear combination of columns of A.

Find the column space of the matrix:

$$A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ 1 & 3 & 4 & 1 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

$$\sim \begin{array}{c} \begin{array}{cccc} c_1 & & & c_4 \end{array} \\ \begin{bmatrix} \boxed{1} & 3 & 4 & 3 \\ 0 & 0 & 0 & \boxed{-2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \to \text{Row space}$$

$1(1\ 3\ 4\ 3) + 1(0\ 0\ 0\ {-2})$

$1 \quad 3 \quad 4 \quad 1$

$\Rightarrow 1$

$1 \times$

↳ where you get the pivot elements, those are linearly independent columns.

there are linearly independent columns.

$C(A)$ is spanned by column 1 and column 4 of original matrix.

System of Homogenous Equation: Null space of matrix $N(A)$.

$[A]_{m\times n}$ $\boxed{AX = 0.}$

$$N(A) = \{ X \in \mathbb{R}^n \mid AX = 0 \}.$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

$$\Rightarrow \left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{array}\right\}$$

Solve the system of equations:

$$x + 3y - 2z = 0$$

$$2x - y + 4z = 0$$

$$x - 11y + 14z = 0$$

$$x - 11y + 14z = 0$$

$$Ax = 0.$$

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix} \underset{R_3 \to R_3 - R_1}{\overset{R_2 \to R_2 - 2R_1}{\sim}} \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix}$$

$$\overset{R_3 \to R_3 - 2R_2}{\sim} \begin{bmatrix} \boxed{1} & 3 & -2 \\ 0 & \boxed{-7} & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \boxed{1} & 3 & -2 \\ 0 & \boxed{-7} & 8 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}$$

$z \to$ free-variable

$x, y \to$ Pivot variable.

$$x + 3y - 2z = 0$$
$$-7y + 8z = 0$$

$$x \cdot \text{col } 1 + y \cdot \text{col } 2 + z \cdot \text{col } 3 = 0.$$

$$z = 1, 2, \ldots k$$

$z = 1, \quad x + 3y - 2 = 0$

$-7y + 8 = 0.$

$\Rightarrow y = 8/7$

$x = -10/7$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -10/7 \\ 8/7 \\ 1 \end{bmatrix} \quad (*)$$

$z = k, \quad x = -\frac{10}{7}k, \quad y = \frac{8}{7}k.$

$$\left.\begin{bmatrix} x \\ y \\ z \end{bmatrix} = k \begin{bmatrix} -10/7 \\ 8/7 \\ 1 \end{bmatrix}\right\} \to \text{Total null space}$$

$\hookrightarrow$ General soln

Basis of Null space : $\left\{ \begin{bmatrix} -10/7 \\ 8/7 \\ 1 \end{bmatrix} \right\}$

\# Solve the following system of eqn :

$x + 2y + 3z = 0$

$3x + 4y + 4z = 0$

$7x + 10y + 12z = 0.$