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# Lecture - 20

Let's start with some Problems!

\# If $2^n - 1$ be a prime. P.T 'n' is a prime

$\Rightarrow$ Let $n$ be a composite number.

$n = p \cdot q$, when $p, q > 1$. $p, q \in \mathbb{Z}$

$$2^n - 1 = 2^{pq} - 1$$

$$= (2^p)^q - (1)^q$$

$$= (2^p - 1)\left(2^{p(q-1)} + 2^{p(q-2)} + \dots + 2^p + 1\right)$$

$$a^n - b^n$$

$$= (a-b)(\dots)$$

$$= (a-b)\left(a^{n-1} + a^{n-2}b + \dots b^{n-1}\right)$$

So, $2^n - 1$ is composite.

P. 1

Hence contradiction!

'n' is prime.

PROVED

\# Prove that $n^4 + 4^n$ is a composite number for all $n > 1$.

for all $n > 1$.

$n^4 + 4^n = (\quad)(\quad)$ $\quad > 1$

$(n^2)^2 + (2^n)^2$ $\quad (2^2)^n = 2^{2n}$

Case - I,

$n \to$ even.

$n^4 + 4^n \longrightarrow$ divisible by 4

$n^4 + 4^n = 4 \times (\; - - \cdots)$

Hence it is composite.

Case - II,

$n \to$ odd, $\quad n = 2k+1$. , $k \in \mathbb{N}$

$$n^4 + 4^n = (2k+1)^4 + 4^{2k+1}.$$

$$= (2k+1)^4 + 4^{2k} \cdot 4.$$

$$= (2k+1)^4 + (2^2)^{2k+1}$$

$$= \left[2\left(k + \tfrac{1}{2}\right)\right]^4 + 2^{4k+2}$$

$4k+2$ ]

$$= 2^4 \left[ \left(k + \frac{1}{2}\right)^4 + \frac{2^{4k+2}}{2^4} \right]$$

$$= (16) \underbrace{\left[ \left(k + \frac{1}{2}\right)^4 + 2^{4k-2} \right]}$$

So, it is obviously composite!

So, $n^4 + 4^n$ is composite for all $n > 1$.

$\boxed{\text{PROVED.}}$

## Fundamental Theorem of Arithmetic.

Any positive integer is either 1 or a prime, or it can be expressed as product of prime, and the representation must be unique.

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$

$p_i$'s $\longrightarrow$ prime $\qquad \alpha_i$'s $\in \mathbb{N}$

$n = 2052.$

$$\begin{array}{r|l} 2 & 2052 \\ \hline 2 & 1026 \\ \hline 3 & 513 \end{array}$$

$2052 = 2 \times 2 \times 3 \times 3 \times 19 \times 3$

$$2052 = 2 \times 2 \times 3 \times 3 \times 19 \times 3$$

$$= \underbrace{2^2 \times 3^3 \times 19^1}$$

$$\begin{array}{r|l} 3 & 513 \\ \hline 3 & 171 \\ \hline 19 & 57 \\ \hline 3 & 3 \\ \hline & 1 \end{array}$$

$n = 1065$

$$\begin{array}{r|l} 5 & 1065 \\ \hline 3 & 213 \\ \hline & 71 \end{array}$$

$$1065 = \underbrace{5^1 \times 3^1 \times 71}$$

$\hookrightarrow$ square free.

# If $2^n + 1$ is an odd prime for some integers $n$. Prove that '$n$' is a power of 2.

$\Rightarrow$ $n = 1$, $2^1 + 1 = 3 \longrightarrow$ odd prime

$6 = 2^1 \cdot 3$ $\quad n = 2$, $2^2 + 1 = 5 \longrightarrow$ odd prime

$n = 3$, $8 + 1 = 9 \not\rightarrow$ prime.

$n = 2^K$, $\underline{2^n + 1} \longrightarrow \underline{\underline{\text{prime}}}$.

Let, $n = \text{odd} \longrightarrow 2^n + 1 \not\rightarrow$ prime

$n = 2K + 1$ $\qquad \underbrace{(\quad)}\ \underbrace{(\quad)}$.

$n = 2k+1$

$$2^{2k+1} + 1 = 2^{2k} \cdot 2 + 1$$

$$= 2^{2k} \cdot 2 + 1 + 3 - 3$$

$$= 2^{2k} \cdot 2 + 4 - 3$$

$$= 2^{2k} \cdot 2 + (2)^2 - 3$$

$$=$$

$$\underline{\underline{a^n + b^n}} = (a-b)(a^{n-1} + a^{n-2}b + \cdots +$$

$$= (a+b)\left( \underline{- - - -} \right)$$

$$a^n + b^n = (a+b)\left(a^{n-1} - a^{n-2}b + a^{n-3}b^2 + \cdots b^{n-1}\right).$$

$$2^{2k+1} + 1 = 2^{2k} \cdot 2 + 1$$

$$= 2\left(2^{2k} + \tfrac{1}{2}\right)$$

$$= 2\left(\left(2^k\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2\right)$$

$$= 2\left[\left(2^k + \frac{1}{\sqrt{2}}\right)^2 - 2 \cdot 2^k \cdot \frac{1}{\sqrt{2}}\right]$$

$$= 2\left[\left(2^k + \frac{1}{\sqrt{2}}\right)^2 - 2^{1/2} \cdot 2^k\right]$$

$$= 2\underbrace{\left[\left(2^k + \frac{1}{\sqrt{2}}\right)^2 - 2^{k+1/2}\right]}_{> 1}.$$

$$6 = 2^k \times 3$$

$n \to$ odd

$$a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b - \dots + b^{n-1})$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$a^4 + b^4 = (\quad)(\quad) \not\to \text{Real field}.$$

When, $n = 2k + 1$.

$n = 2k$

$$2^n + 1 = (2)^n + (1)^n$$

$2^n + 1 \to$ prime

$$= (2+1)(2^{n-1} + 2^{n-2} + \dots 1^{n-1})$$

$n \to$ can be even

$= (2+1)(2 \quad + 2 \quad + \cdots + \quad )$

$n \to$ can be even

$= (3)( - - - )$

So, $3 \mid 2^n + 1$.

$2^n + 1$ is not prime when $n$ is odd.

$n$ must be even.

So, when $n$ is even,

$$\boxed{n = 2^k \cdot \underbrace{\underbrace{p_1^{\alpha_1} \cdots p_k^{\alpha_k}}}}$$

$p_i$'s are prime.

$n = 2^k \, z \to$ odd int.

If $z > 1$, $2^n + 1 = 2^{2^k \cdot z} + 1$

$= \left(2^{2^k}\right)^z + 1.$

$= \underline{\underline{\left(2^{2^k} + 1\right)}} ( - - - - )$

So, $2^{2^k} + 1 \mid 2^n + 1$

So, it is not prime then.

$n = 2^k \cdot \underbrace{p_1^{\alpha_1} \cdots p_k^{\alpha_k}}$ $\to$ It must be even or 1.

$\hookrightarrow$ It must be even or $\underline{1}$.

So, $Z = \underline{1}$ or $2^{k_1}$.

$n = 2^k \cdot 2^{k_1} = 2^{k+k_1} \rightarrow$ even power.

<u>Euclid's Theorem</u>: The no of primes are <u>infinite</u>.

<u>Proof</u>: <u>Contradiction</u>.

Let the no. of primes be finite. Let $p$ be the greatest prime.

$$2, 3, 5, 7, \ldots p.$$

$$Z = 2 \cdot 3 \cdot 5 \cdot 7 \cdots p$$

$$2 \mid Z, \quad 3 \mid Z, \quad 5 \mid Z, \quad \ldots \quad p \mid Z.$$

$$K = (2 \cdot 3 \cdot 5 \cdot 7 \cdots p + 1).$$

$K$ is not divisible by any of the primes

$$2, 3, \ldots p$$

So, $K$ is a prime.

In all cases '$p$' fails to be a greatest prime.

Thm: There are infinitely many primes of $4n-1$.

How to find number of positive divisors of a positive integer?

$\rightarrow$ Let, '$n$' $\in \mathbb{Z}^+ > 1$.

$$n = p_1^{\alpha_1} \cdots p_k^{\alpha_k} \qquad p_1 < p_2 < \cdots p_k$$

$$\alpha_i\text{'s} \in \mathbb{N}$$

$$\tau(n) = \underbrace{(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_k+1)}$$

$\hookrightarrow$ 1 and $n$

$$n = 36$$
$$= 12 \times 3$$
$$= 4 \times 3 = 2^2 \times 3^1 \times 3$$

$$\alpha_1 = 2, \quad \alpha_2 = 1.$$

$$\tau(36) = (2+1).(2+1)$$
$$= 3 \times 3 = 9$$

$$\underbrace{1, 2, 3, 4, 6, 9, 12, 18, 36}. \rightarrow 9.$$

$\tau(48) = ?\ 10.$

**Th$^m$:** The total number of positive divisors of a positive integer $n$ is odd if and only if '$n$' is a perfect square.

$n = k^2$, $\tau(n) \to$ odd

$\alpha_i \to$ even

$$\tau(n) = \underbrace{(\alpha_1+1)}_{\text{odd}} \cdots\cdots \underbrace{(\alpha_k+1)}_{\text{odd}}.$$

$\tau(n) \to$ odd.

$3 \times 3 = 9$

$3 \times 5 = 15$

Converse, $(\alpha_1+1) \cdots (\alpha_k+1) \to$ odd.

$(\alpha_1+1) \to$ odd

$\alpha_1 \to$ even $\cdots$ $\alpha_k \to$ even

$(p_1^{\alpha_1} \cdots p_k^{\alpha_k}) \to$ Perfect sq.

H/W

\# Find $\tau(360)$ and $\tau(900)$

\# Find the number of odd positive divisors of 2700.