First Order higher degree diff eqn

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \cdots + P_{n-1} p + P_n = 0$$

$$p = \frac{dy}{dx}, \quad P_i \to f(x, y)$$

Solvable for p, y, x.

Eqn solvable for p:

$p_1, p_2, \ldots p_n$

$$(p - p_1)(p - p_2)\ldots(p - p_n) = 0$$

Examples.

Solve the equation $p^2 + px + py + xy = 0$

Soln:

$$\left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx} + y \cdot \frac{dy}{dx} + xy = 0$$

$$p^2 + px + py + xy = 0$$

$$\Rightarrow p(p + x) + y(p + x) = 0$$

$$\Rightarrow (p + x)(p + y) = 0$$

$$\Rightarrow (p+x)(p+y)=0.$$

$$p+x=0. \Rightarrow \frac{dy}{dx}=-x \quad \text{---} (i)$$

$$p+y=0. \Rightarrow \frac{dy}{dx}=-y. \quad \text{---} (ii).$$

$$\left.\begin{array}{l} 2y=-x^2+c_1 \\ x=-\log y+c_2. \end{array}\right\} = \underline{\phi_1(x,y,c)}$$

$$\phi_2(\text{- - -})=0.$$

The complete primitive can be written in the form.

$$\underline{(2y+x-c)}(x+\log y-c)=0.$$

#2. Solve the eq^n $x^2p^2-2xyp+y^2=x^2y^2+x^4$.

$\underline{\underline{Sol^n}}$:
$$x^2p^2-2xyp+y^2-x^2y^2-x^4=0.$$

$$\Rightarrow (xp-y)^2=x^2y^2+x^4$$

$$\Rightarrow (xp-y)^2=x^2(y^2+x^2).$$

$$\Rightarrow xp-y=\pm\sqrt{x^2(y^2+x^2)}.$$

$$\Rightarrow \quad xp - y = \pm\sqrt{x^2(y^2+x^2)}.$$

$$\Rightarrow \quad xp - y = \pm x\sqrt{(y^2+x^2)}.$$

$$\Rightarrow \quad p = \frac{y \pm x\sqrt{y^2+x^2}}{x} \quad \|.$$

Put $y = vx$.

$$p = \frac{vx \pm x\sqrt{v^2x^2+x^2}}{x}.$$

$$\Rightarrow p = \frac{vx \pm x^2\sqrt{v^2+1}}{x}$$

$y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}.$

$\frac{dy}{dx} = \cancel{x}\frac{dv}{dx}$

$$\Rightarrow p = v \pm x\sqrt{v^2+1}.$$

$$\Rightarrow \frac{dy}{dx} = v \pm x\sqrt{v^2+1}.$$

$$\Rightarrow v + x\frac{dv}{dx} = v \pm x\sqrt{v^2+1}.$$

$$\Rightarrow \frac{dv}{dx} = \pm\sqrt{v^2+1}$$

$$\Rightarrow \frac{dv}{\sqrt{v^2+1}} = \pm\, dx .$$

$y = vx$

$$\left.\begin{aligned} v &= \sinh(x+c_1) \\ v &= \sinh(c_2 - x) \end{aligned}\right\} .$$

hen sol^n:

$$\left(\frac{y}{x} - \sinh(x+c)\right)\left(y/x - \sinh(c-x)\right) = 0$$

$$\Rightarrow \left\{y - x\sinh(x+c)\right\}\left\{y - x\sinh(c-x)\right\} = 0$$

Equations Solvable for y.

$$y = F(x, p) .$$

$$\Rightarrow y = F\left(x, \frac{dy}{dx}\right) .$$

Diff w.r.t x we get:

$$\frac{dy}{dx} = \frac{dF}{dx} + \frac{dF}{dp} \cdot \frac{dp}{dx} .$$

$$= \phi\left(x, p, \frac{dp}{dx}\right) .$$

$$p = \phi\left(x, p, dp/dx\right).$$

Example

#3. $y = px + p^2 x$.

Diff both sides w.r.t x, we get:

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + p^2 + 2xp\frac{dp}{dx}.$$

$$\Rightarrow -p^2 = x(1+2p)\frac{dp}{dx}.$$

$$\Rightarrow \frac{dx}{x} + \frac{(1+2p)}{p^2}dp = 0.$$

Integrating both sides —.

$$\log x + \log c = \frac{1}{p} - 2\log p.$$

$$\Rightarrow \log x + \log c + \log p^2 = 1/p$$

$$\Rightarrow \log(xcp^2) = 1/p.$$

$$\Rightarrow xcp^2 = e^{1/p}.$$

$1/p$

$$\Rightarrow x = \frac{e^{1/p}}{cp^2}$$

$$y = px + p^2 x$$
$$= x(p + p^2) = \frac{e^{1/p}}{cp^2} p(1+p)$$
$$= \frac{e^{1/p}}{cp}(1+p).$$

$$x = \frac{e^{1/p}}{cp^2}$$

$$y = \frac{e^{1/p}}{cp}(1+p)$$

Equations solvable for x.

$$x = F(y, p).$$

$$\hookrightarrow \frac{1}{p} = \phi\left(y, p, \frac{dp}{dy}\right).$$

#4 Solve $p^2 y + 2px = y$.

$$2px = y(1-p^2)$$

$$\Rightarrow x = \frac{y}{2p}\left[(1-p^2)\right]$$

Diff w.r.t to y,

$$\frac{dx}{dy} = \frac{(1-p^2)}{2p} + y\cdot\left\{-\frac{1}{2}\frac{(1+p^2)}{p^2}\cdot\frac{dp}{dy}\right\}.$$

$$\Rightarrow \frac{1}{p} = \frac{(1-p^2)}{2p} + y\cdot\left\{-\frac{1}{2}\frac{(1+p^2)}{p^2}\cdot\frac{dp}{dy}\right\}.$$

$$\Rightarrow \frac{1+p^2}{2p} = -\frac{1}{2}\left(\frac{1+p^2}{p^2}\right)y\cdot\frac{dp}{dy}.$$

$$\Rightarrow 1+p^2 = -\frac{(1+p^2)}{p}\cdot y\cdot\frac{dp}{dy}.$$

$$\Rightarrow 1+p^2 = -(1+p^2)\cdot\frac{y}{p}\left(\frac{dp}{dy}\right).$$

$$\Rightarrow \frac{(1+p^2)}{(1+p^2)} = -\frac{y}{p}\frac{dp}{dy}.$$

$$\Rightarrow \quad 1 = -\,y/p \,\frac{dp}{dy}.$$

$$\Rightarrow \quad \frac{dy}{y} + \frac{dp}{p} = 0.$$

$$\Rightarrow \quad \log y + \log p = \log c.$$

$$\Rightarrow \quad py = c.$$

$$\Rightarrow \quad p = c/y.$$

$$x = y(1-p^2)/2p.$$

$$= y\left(1-(c/y)^2\right)/2.c/y.$$

$$\Rightarrow y^2 = c^2 + 2cx \longrightarrow \text{Soln.}$$