COSETS

Left coset: Let G be a group and H be a subgroup of G. Let 'a' be an element of G. For all h in H, $ah \in G$.

The subset $\{ah : h \in H\}$ is called a left coset of H in G and is denoted by aH.

$b, c, \ldots \in G$, $bh, ch \ldots \in bH, cH, \ldots$

Note: In an additive group G, a left coset of H is denoted by $a+H = \{a+h : h \in H\}$

$H+a$

Example:

① $G = (\mathbb{Z}, +)$ and $H = (3\mathbb{Z}, +)$. $3n$

The left coset $0+H = \{3n : n \in \mathbb{Z}\} = H$.

The left coset $1+H = \{3n+1 : n \in \mathbb{Z}\}$.

② $G = S_3$ and $H = \{\rho_0, \rho_3\}$.

$\rho_0 H = \{\rho_0, \rho_3\}$ $\quad\quad \{\rho_0, \ldots \rho_5\}$

$P_0 H = \{P_0, P_3\}$ $\{P_0, \ldots, P_5\}$

$P_1 H = \{P_1, P_5\}$ ...

$P_5 H = \{ \quad \}$.

$P_0 = e$

$P_1 = (12)$

$P_2 = (13)$

$P_3 \ (23)$

$P_4 = (123)$

$P_5 \ (132)$

$H = \{e, (23)\}$.

$(132)e = (132)$.

$(132)(23) = (13)$.

P_5 (132).

$\left.\begin{matrix} 1 \to 1 \\ 1 \to 3 \end{matrix}\right\}$. $\left.\begin{matrix} 2 \to 3 \\ 3 \to 2 \end{matrix}\right\}$ $\left.\begin{matrix} 3 \to 2 \\ 2 \to 1 \end{matrix}\right\}$

$1 \to 3$ $\qquad 2 \to 2$ $\qquad 3 \to 1$

(13)

$\{P_5, P_2\}$. $||$.

S_3

[5 , 2] .

S_3

1 2 3 .

Re y.

ρ_1 (1 2) $\rho_4 = (1\,2\,3)$

ρ_2 (1 3) $\rho_5 = (1\,3\,2)$.

ρ_3 (2 3)

<u>Thm.</u> Let G be a group and H be a subgroup of G. Let $h \in H$. Then $hH = H$.

<u>Thm:</u> Let G be a group and H be a subgroup of G. Any two left cosets of H in G are either identical or they have no common element.

[If $a, b \in G$, then either $aH = bH$ or $aH \cap bH = \phi$].

<u>Thm:</u> Let G be a group and H be a subgroup of G. Let $a, b \in G$. Then $aH = bH$ iff $a^{-1}b \in H$.

$\Rightarrow$ $aH = bH$, Then $ah_1 = bh_2$ $\forall\, h_1, h_2 \in H$.

$\therefore$ $a^{-1}b = h_1 h_2^{-1} \in H$, since H is a subgroup.

Conversely let $a^{-1}b \in H$. Then $a^{-1}b = h_3$ for

Conversely, Let $a^{-1}b \in H$, Then $a^{-1}b = h_3$ for some $h_3 \in H$.

$\therefore b = a h_3$ $\quad b \in aH$. But $b \in bH$.

So, Left coset of aH and bH have a common element b

$aH = bH$. (by above thm).

Th^m: Any two left cosets of H in a group G have the same cardinality.

LAGRANGE'S Th^m:

The order of every subgroup of a finite group G is a divisor of the order of G.

$$o(H) \mid o(G).$$

FERMAT TH^M:

If p be a prime and a be an integer. such that p is not a divisor of a. then

$$a^{p-1} \equiv 1 \pmod{p}.$$

Th^m: Every group of prime order is cyclic.

Th^m: Every group of prime order is cyclic.

Th^m: The order of each element in a finite group G is a divisor of $o(G)$.

Right coset

$$\{ha : h \in H\}$$

Additive: $H + a$.

Example $G = (\mathbb{Z}, +)$ $H = (3\mathbb{Z}, +)$.

$H + 0 = \{3n$

$H + 1 = \{3n + 1$

$H + 2 = \{3n + 2$

Th^m: Let H be a subgroup of a group G. Then the set of all distinct left cosets of H in G and the set of all distinct right cosets in G have same cardinality.

$$|L| = |R|.$$

Index: $[G : H]$ (Index of H in G)

$$= \frac{o(G)}{o(H)}.$$

$$= \frac{\quad}{o(H)}$$

$G = S_3 \quad H = A_3$.

$$[G:H] = \frac{o(G)}{o(H)} = \frac{6}{3} = 2.$$

$G = (\mathbb{Z}, +)$, $H = (2\mathbb{Z}, +)$.

$$[G:H] =$$

(iv) $aH = bH$ if and only if $b \in aH$,

(v) $a \in bH \Leftrightarrow b \in aH$.

✓ **2.** Describe the left cosets and the right cosets of H in G and find $[G:H]$.

(i) $G =$ Klein's 4-group with elements e, a, b, c and $H = \langle a \rangle$.

(ii) $G = S_3$ with elements $\rho_0, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5$ and $H = \langle \rho_1 \rangle$.

Normalizer: For a subgroup H in G the normalizer of H in G is defined by:

$$N_G(H) = \{ g \in G : gHg^{-1} \in H \}$$