08 January 2026 20:57

Lecture - 17

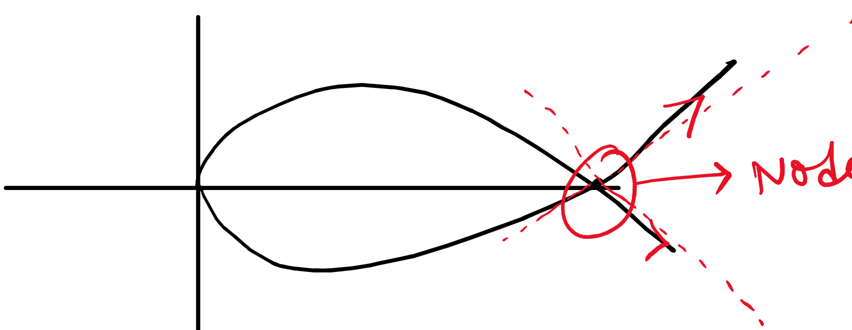
Extraneous Loci

* Tac - locus.
* Nodal Locus
* cuspidal Locus
* Envelope locus.

Double point

→ Node

→ cusp.

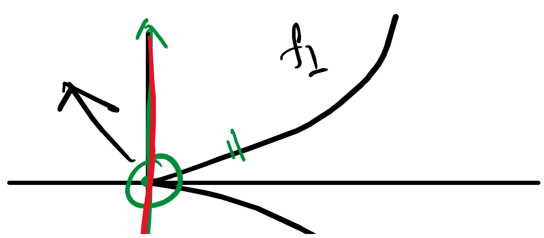


→ Node.

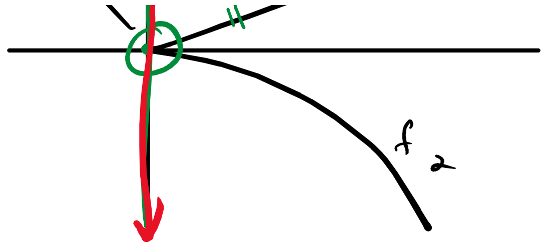
So, the tangent are distinct

cusp → Tangents are co-incidental.

~~cusp~~



So, the tangents are co-incidental.



P- discriminant

It is obtained by eliminating the p.

$$f(x, y, p) = 0$$

$$\& \quad \frac{\partial f}{\partial p} = 0.$$

C - discriminant

Eliminating 'c'.

$$\phi(x, y, c) = 0$$

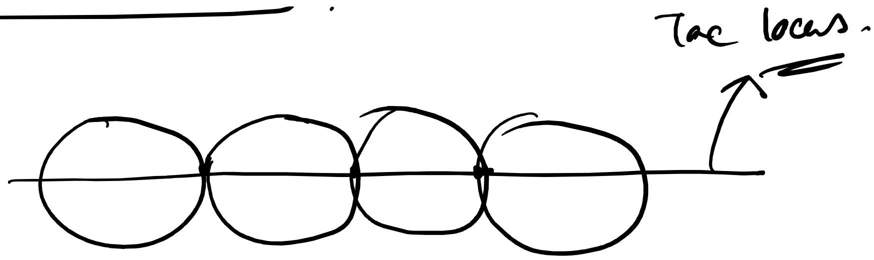
$$\& \quad \frac{\partial \phi}{\partial c} = 0$$

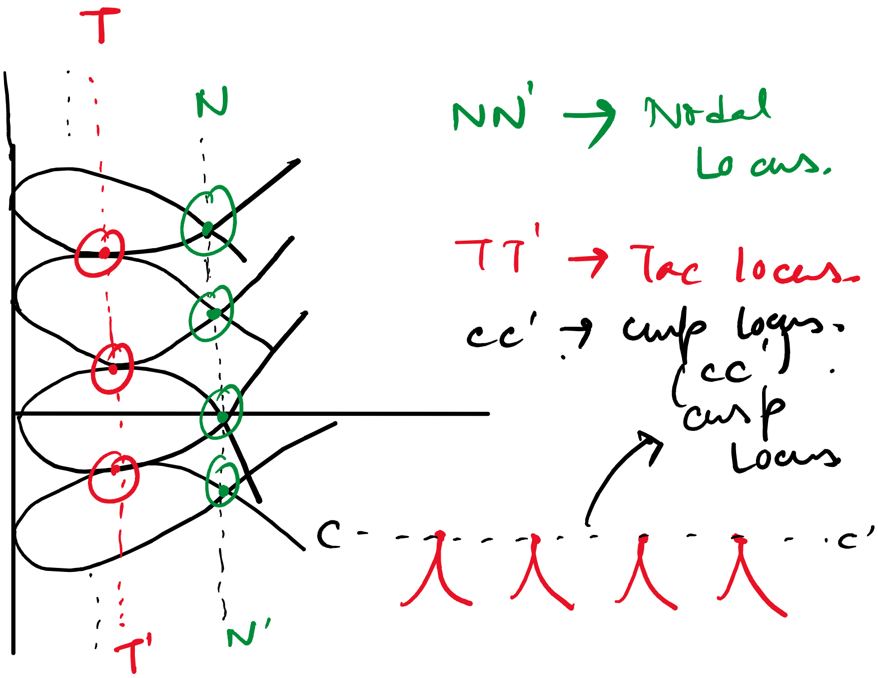
|| Extraneous Loci

* Tac - locus.
* Nodal Locus
* Cuspidal Locus

* cuspidal Locus
* Envelope locus

Tac - Locus





Discriminat relation to Locus :

P - discriminant

c - discriminant .

P - discriminat is expected to contain the following eq^n :

1. Envelope Locus. (E) .

1. Envelope Locus. (E).
2. Tac-locus squared (T^2).
3. cuspidal locus (C).

$$\boxed{p\text{-discriminant} \cong \boxed{ET^2C}}$$

C - discriminant contains:

i) Envelope (E)
ii) Nodal locus sq. (N^2)
iii) cuspidal locus cube (C^3).

$$\boxed{C\text{-discri.} \cong EN^2C^3}$$

Example - 1.

$4xp^2 - (3x-1)^2 = 0$ ↙.

$f(x, y, p) = 0$

Solve for p.

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{}$$

$$p = \frac{-b \pm \sqrt{b - 4ac}}{2a}$$

$$p = \frac{\pm\sqrt{-4(4x)\left[-(3x-1)^2\right]}}{2 \cdot 4x}$$

$$= \frac{\pm\sqrt{16x\,(3x-1)^2}}{8x}.$$

$$= \pm\frac{4(3x-1)\sqrt{x}}{8x}$$

$$= \pm\frac{(3x-1)\sqrt{x}}{2x}.$$

$$p = \pm\left(\frac{3}{2}\sqrt{x} - \frac{1}{2\sqrt{x}}\right).$$

$$p = \frac{dy}{dx}.$$

$$\frac{dy}{dx} = \frac{3\sqrt{x}}{2} - \frac{1}{2\sqrt{x}}.$$

$$\Rightarrow dy = \frac{3\sqrt{x}}{2}dx - \frac{1}{2\sqrt{x}}dx.$$

$$\Rightarrow \int dy = \int \frac{3\sqrt{x}}{2}dx - \int \frac{1}{2\sqrt{x}}dx.$$

$$\Rightarrow y+c = \frac{3}{2}x^{3/2}\cdot\frac{2}{3} - \frac{1}{2}x^{1/2}\cdot\left(\frac{2}{1}\right)$$

$$\Rightarrow y+c = x^{3/2} - x^{1/2}$$

$c_1 - c_2 = c$

$$\Rightarrow y+c = x^{1/2}(x-1).$$

$$\Rightarrow \boxed{(y+c)^2 = x(x-1)^2}.$$

general solution.

P - discriminant

$4xp^2 - (3x-1)^2 = 0$

$B^2 - 4ac = 0.$

p - discriminant

$4(4x)(3x-1)^2 = 0$

$4(4x)(3x-1)$

$$\Rightarrow \boxed{x(3x-1)^2 = 0}$$

c - discriminant

$$(y+c)^2 = x(x-1)^2.$$

Diff w.r.t c

$$2(y+c) = 0.$$

$$\Rightarrow c = -y$$

$$\boxed{x(x-1)^2 = 0.}$$

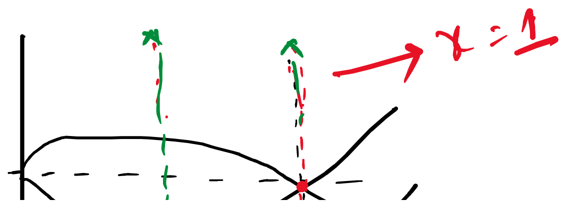
P - discriminant : $x(3x-1)^2 = 0$

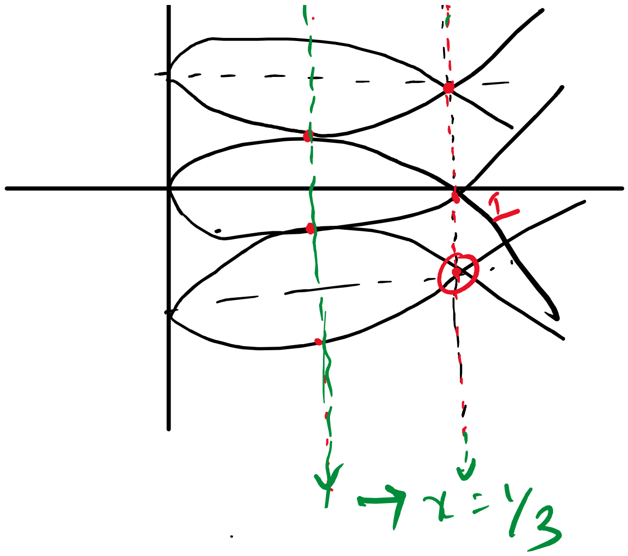
C - discriminant : $x(x-1)^2 = 0$

$E\,C\,T^2$ $\qquad$ $E\,N^2\,C^3$

T : $3x - 1 = 0$ $\rightarrow$ Tac Locus

N : $x - 1 = 0$ $\rightarrow$ Nodal Locus





Discuss the Loci of the differential equation:

$$xp^2 = (x-a)^2.$$

$$p = \pm \frac{\sqrt{4(x)(x-a)^2}}{2x}$$

$$= \pm \frac{2(x-a)\sqrt{x}}{2x}$$

$$= \pm \frac{(x-a)}{x}\sqrt{x}.$$

$$\frac{dy}{dx} = \frac{x-a}{x}\sqrt{x}.$$

$$\Rightarrow dy = \left(1 - a/x\right)\sqrt{x}\ dx$$

$$\Rightarrow dy = \left(\sqrt{x} - ax^{-1/2}\right)dx$$

$$\Rightarrow dy = \left(\sqrt{x} - a x^{-1/2}\right) dx$$

$$\Rightarrow y + c = \frac{2}{3} x^{3/2} - 2a x^{1/2}.$$

$$\Rightarrow y + c = x^{1/2}\left(\tfrac{2}{3} x - 2a\right)$$

$$\Rightarrow (y+c)^2 = x\left(\frac{2}{3} x - 2a\right)^2.$$

$$\Rightarrow (y+c)^2 = \frac{4}{9} x (x - 3a)^2,$$

$$\Rightarrow \boxed{\frac{9}{4}(y+c)^2 = x(x-3a)^2.}$$

$\hookrightarrow$ general solution.

P - discriminant

$$B^2 - 4AC = 0.$$

$$\Rightarrow 4x(x-a)^2 = 0$$

$$\Rightarrow \boxed{x(x-a)^2 = 0}$$

$x = 0$ $\quad$ $x - a = 0$

C - discriminant

Diff the gen sol^n w.r.t c :

$$\frac{9}{4} \times 2(y+c) = 0$$

$\Rightarrow y = -c.$

$\Rightarrow c = -y.$

Put, $c = -y$ in gen sol^n:

$$\boxed{x(x-3a)^2 = 0}. \rightarrow c\text{-discriminant}$$

$x = 0$
$x - 3a = 0$

P discriminant $\rightarrow$ ECT^2

Tac locus: $x - a = 0.$

C-discriminant $\rightarrow$ EN^2C^3

Nodal locus: $x - 3a = 0$

Envelope locus: $x = 0.$

Method of Undetermined coeff.

It is used to find PI. of non-homogeneous linear D.E.

$$f(D)y = X(x)$$

$X(x) \rightarrow$ Polynomial, Exponential, sine / cosine.

Step by Step:

$f(D)y = 0$

1. $f(D)y = 0$.

2. Assume a trial form for PI based on R.H.S.

R.H.S $[X(x)]$	Assume PI
e^{ax}	Ae^{ax}
x^n	$Ax^n + Bx^{n-1} + \ldots$
$\sin ax, \cos ax$	$A\sin ax + B\cos ax$

Example:

$$(D^2 - 3D + 2)y = e^x.$$

Auxiliary eqn

$$m^2 - 3m + 2 = 0$$

$$\Rightarrow (m-1)(m-2) = 0$$

$$CF = c_1 e^x + c_2 e^{2x}$$

$$PI = Axe^x.$$

Just substitute and find A

Solve:

$$(D^2 - 5D + 6)y = e^{2x}.$$

$$\Rightarrow y'' - 5y' + 6y = e^{2x} \quad \ldots (1)$$

Find CF

Auxilliary eq^n:

$$m^2 - 5m + 6 = 0.$$

$$\Rightarrow (m-2)(m-3) = 0$$

$$\Rightarrow m = 2, 3$$

$$C.F = c_1 e^{2x} + c_2 e^{3x}.$$

Calculate PI

$$R.H.S = e^{2x}. \quad (e^{2x} \text{ already in C.F})$$

$$y_p = Ae^{2x} x$$

$$Axe^{2x}.$$

Differentiate.

$$y_p = Axe^{2x}$$

$$y_p' = Ae^{2x} + 2Axe^{2x}.$$

$$y_p'' = 2Ae^{2x} + 2Ae^{2x} + 4Axe^{2x}.$$

$$y_p'' = 4Ae^{2x} + 4Axe^{2x}.$$

$$y'' - 5y' + 6y = e^{2x}$$

$$(4Ae^{2x} + 4Axe^{2x}) - 5(Ae^{2x} + 2Axe^{2x}) + 6(Axe^{2x}) = e^{2x}$$

- Grouping the term.
- compare

$A = ?$

$A = -1$.

$$PI = Axe^{2x} = -xe^{2x}.$$

$$y = CF + PI = c_1e^{2x} + c_2e^{3x} - xe^{2x}$$

$= c_1 e^{2x} + c_2 e^{5x} - xe$

A/W

i) Solve $(D^2 - 3D + 2)y = x^2$

ii) Solve $(D^2 + 4)y = \sin 2x$

iii) Solve $(D^2 - 2D + 1)y = e^x + x$.

Remember this chart:

RHS	Assume PI
e^{ax}	Ae^{ax}
x^n	Polynomial
$\sin ax, \cos ax$	$A \sin ax + B \cos ax$
Already in CF	Multiply by x
Repeated	Multiply by x^2