Subgroups

Defn: Let $(G, \circ)$ be a group and 'H' be a "non-empty" subset of G. H is said to be closed under $\circ$ if $a \in H$, $b \in H \Rightarrow a \circ b \in H$.

$(H, \circ)$ is a subgroup of the group $(G, \circ)$.

Examples:

$(G, \circ) \rightarrow$ group G being the subset of G.

$(G, \circ) \rightarrow$ improper group.

Let $e \rightarrow$ identity element of G.

$\{e\}, \circ$ $(\{e\}, \circ) \rightarrow$ trivial subgroup

The subgroups other than $(G, \circ)$ and $(\{e\}, \circ)$ are said to be non-trivial proper subgroup of $(G, \circ)$.

$(\mathbb{Q}, +)$ is a group. $\mathbb{Z} \subseteq \mathbb{Q}$

So, $(\mathbb{Z}, +) \rightarrow$ subgroup of $(\mathbb{Q}, +)$.

$(\mathbb{Q}, +) \rightarrow$ group. $\mathbb{Q}^* = \mathbb{Q} - \{0\}$.

$$\mathbb{Q}^* \subseteq \mathbb{Q}, \quad (\mathbb{Q}^*, \cdot)$$

$(\mathbb{Q}^*, \cdot) \rightarrow$ group $\quad \hookrightarrow_{\text{subgroup}} (\mathbb{Q}, +)$?

$$\mathbb{Q}^* \subseteq \mathbb{Q}$$

So, $(\mathbb{Q}^*, \cdot)$ is not a subgroup of the group $(\mathbb{Q}, +)$.

$(G, \circ) \rightarrow$ commutative

$(H, \circ) \xrightarrow{\text{subgroup}}$ commutative. on H.

Th^m: Let $(G, \circ)$ be a group and $(H, \circ)$ be a subgroup of $(G, \circ)$. Then.

i) the identity element of the subgroup $(H, \circ)$ is the identity element of group $(G, \circ)$.

ii) If $a \in H$, then the inverse of a in the subgroup $(H, \circ)$ is same as the inverse of a in the group $(G, \circ)$.

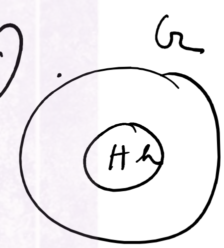
<u>Proof</u>: Let $e_H \rightarrow$ identity element in $(H, \circ)$.

$e_G \rightarrow$ identity element in $(G, \circ)$.

$h \in H$, $\quad e_H \circ h = h \circ e_H = h \; \cdots (i) \; \forall h \in H.$

Also, $\quad e_G \circ h = h \circ e_G = h \cdots (ii)$

$\forall h \in G.$



$h \circ e_H = h \circ e_G$ in G.

$\Rightarrow e_H = e_G$ (By left cancellation).

iii) Try it by yourself.

Thm: Let $(G, \circ) \rightarrow$ group. A non-empty subset H of G forms a subgroup of $(G, \circ)$ if and only if

i) $a \in H, b \in H \Rightarrow a \circ b \in H.$

ii) $a^{-1} \in H$

i) $a \in H, b \in H \ldots$

ii) $a \in H \Rightarrow a^{-1} \in H.$

Proof: Let $(H, \circ)$ is a subgroup of $(G, \circ)$.

As, $(H, \circ)$ is a group

(i) and (ii) are evident.

Conversely, Let H be a non-empty subset of G satisfying (i) and (ii).

From -(i) $\rightarrow$ H is closed under '$\circ$'.

$H \subset G$, '$\circ$' associative in G.

$\rightarrow$ '$\circ$' " " H.

Let, $a \in H$ from -(ii), $a^{-1} \in H$.

$\rightarrow \in G$.

$a \in H$ implies that the inverse of a belongs to H.

$$a \circ a^{-1} = e \in H.$$

So, $(H, \circ)$ is a group, $H \subseteq G$,

$(H, \circ)$ subgroup of $(G, \circ)$.

Th^m: $(G, \circ) \rightarrow$ group. H is a non-empty subset . . .

$$\underline{H \subseteq G \neq \phi}.$$

it forms a subgroup if and only if

$a \in H,\ b \in H \Rightarrow a \circ b^{-1} \in H.$

Proof: Let $\underline{(H, \circ)}$ be a subgroup of $(G, \circ)$.

$a, b \in H.$

$b \in H \Rightarrow b^{-1} \in H.$

$a \in H,\ b^{-1} \in H,\ \Rightarrow a \circ b^{-1} \in H.$ (From above thm).

Converse Let, $H \subseteq G\ (\neq \phi)$.

$$\boxed{a \in H,\quad b \in H \Rightarrow a \circ b^{-1} \in H.}$$

Let $\underline{a \in H}$, $\underbrace{a \circ a^{-1}}_{e} \in H$ So, $e \in H$.

$\therefore$ H contains the identity element! (Identity)

$\therefore$ H contains the identity element ! (Identity)

$e \in H$, $a \in H \Rightarrow e \circ a^{-1} \in H$ by the condition.

i.e $a^{-1} \in H$.

So, $a \in H \Rightarrow a^{-1} \in H$. (Inverse)

Let $a \in H$, $b \in H$, Then $a \in H$ and $b^{-1} \in H$.

$a \circ (b^{-1})^{-1} \in H$.

$\Rightarrow a \circ b \in H$. (closure).

Associative property is trivial !

$\text{'}\circ\text{'} \xrightarrow{G}$ associative. (associative)

$H \subseteq G \Rightarrow \text{'}\circ\text{'} \xrightarrow[H]{}$ associative.

So, $(H, \circ)$ is a subgroup of $(G, \circ)$.

Th^m: Let $(G, \circ)$ be a group and H be a non empty finite subset of G. Then $(H, \circ)$ is a subgroup of $(G, \circ)$ if and only if

$$a \in H,\ b \in H \Rightarrow a \circ b \in H.$$

Thm: $(G, \circ) \rightarrow$ Group

$(H, \circ)$ and $(K, \circ) \rightarrow$ subgroup of G.

Then $H \cap K$ is a subgroup of $(G, \circ)$.

Is this same work for union of groups?

$G = (\mathbb{Z}, +)$

$H = (2\mathbb{Z}, +)$ $\quad K = (3\mathbb{Z}, +)$. $\quad$ $H \cup K$

$2, 4, -2, -4$
$- - - - - -$
$3, 6, 9, -3,$
$-6, -9$
$- \cdots$

$2 \in H \cup K$, $\quad 3 \in H \cup K$.

$2 + 3 \notin H \cup K$.

So, $H \cup K$ is not closed.

Thm. $H \cup K \rightarrow$ subgroup if and only if either $H \subset K$ or $K \subset H$.

$H = (2\mathbb{Z}, +)$ $\quad K = (4\mathbb{Z}, +)$

$K \subseteq H$.

$H \cup K = K$.

Thm: H, K subgroup of G. Then HK is a subgroup of G if and only if

$HK = KH.$

Thm: $H, K \rightarrow$ finite subgroup.

$|HK| \rightarrow$ no. of elements in the set HK.

$$|HK| = \frac{o(H) \cdot o(K)}{o(H \cap K)}.$$

Examples

$H \rightarrow 2\times 2$ real matrices.

$$\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 \right\}.$$

Is H a subgroup of $GL(2, R)$, the group all non-singular matrices of order 2 w.r.t matrix multiplication.

Proof: H is non-empty since $I_2 \in H$

$$\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \neq 0.$$

Let, $A \in H$, $B \in H$ $\qquad \det A = 1$

$\det B = 1.$

$$\det(AB) = \det(A).\det(B).$$
$$= \underline{1}.$$

So, $AB \in H$. (Hence closure property satisfied) ... (i).

Let $A \in H$. Then A is non-singular and $\det A = 1$.

So, A^{-1} exist $\det(A^{-1}) = \dfrac{1}{\det A} = 1$.

So, $A^{-1} \in H$.

$$A A^{-1} = A^{-1} A = I_2.$$

A^{-1} is the inverse of A in H.

$\therefore$ $A \in H \Rightarrow A^{-1} \in H$. ... (ii).

H is a subgroup of $GL(2, \mathbb{R})$.

$\rightarrow SL(2, \mathbb{R})$.

Some important subgroups of a group

1.) <u>Center of a Group</u>.

$H \leq G$ ($G \rightarrow$ group). 'o'

$H = \{x \in G : x o g = g o x \ \forall g \in G\}$.

P.T. (H, o) is a subgroup of (G, o).

P.T $(H, \circ)$ is a subgroup of $(G, \circ)$.

Proof Non-empty is trivial $(e \in H)$.

Let, $p \in H$, $q \in H$.

$$p \circ g = g \circ p \quad \forall g \in G.$$

$$q \circ g = g \circ q$$

$$(p \circ q) \circ g = p \circ (q \circ g) = p \circ (g \circ q)$$
$$= (p \circ g) \circ q$$
$$= (g \circ p) \circ q$$
$$= g \circ (p \circ q) \quad \forall g \in G.$$

$(p \circ q) \in H$.

$\therefore$ $p \in H$, $q \in H \Rightarrow (p \circ q) \in H$. (closure).

$p \in H \Rightarrow p^{-1} \in H$.

$$p \circ g = g \circ p \quad \forall g \in G.$$

$$p^{-1} \circ (p \circ g) \circ p^{-1} = p^{-1} \circ (g \circ p) \circ p^{-1} \quad \forall g \in G.$$

$$\Rightarrow e \circ g \circ p^{-1} = p^{-1} \circ g \circ e$$

$\Rightarrow \quad \underline{g \circ p^{-1} = p^{-1} \circ g} \qquad p^{-1} \in H.$

'o' is associative.

So, $(H, \circ)$ is a subgroup of $(G, \circ)$.

$Z(G) \rightarrow$ center of group

Centralizer

Let $(G, \circ)$ be a group and let $a \in G$. Let H be the subset of G defined by

$$H = \{x \in G : x \circ a = a \circ x\}.$$

P.T : $(H, \circ)$ is a subgroup of $(G, \circ)$.

Proof : $\underline{a \circ a = a \circ a} \rightarrow \underline{a \in H}$

So, H is non-empty subset of G.

$p \in H, \ q \in H.$ $\qquad p \circ a = a \circ p$

$\qquad q \circ a = a \circ q$

$\hookrightarrow p \circ q \in H \qquad (H|W)$

Same like the previous proof!