Clairaut's form

A diff eq^n of the form

$$\boxed{y = px + f(p)} \qquad p = \frac{dy}{dx}.$$

→ Clairaut's eq^n.

It's complete primitive is

$$y = cx + f(c).$$

Diff the above eq^n w.r.t x

$$p = p + \left\{x + f'(p)\right\}\frac{dp}{dx}.$$

$$\Rightarrow \left\{x + f'(p)\right\}\frac{dp}{dx} = 0.$$

either, $\frac{dp}{dx} = 0 \longrightarrow p = \text{constant}(c)$.

OR, $x + f'(p) = 0$.

$$\boxed{y = cx + f(c)}$$

Geometrically this represent family of st. lines.

Geometrically this represents family of ...

$$x + f'(p) = 0.$$

$$\Rightarrow x = -f'(p).$$

$$y = p(-f'(p)) + f(p).$$

__Problems.__

Solve: $y = px + \frac{a}{p}$.

Diff w.r.t x –

$$p = p + x\frac{dp}{dx} - \frac{a}{p^2}\cdot\frac{dp}{dx}.$$

or, $x\frac{dp}{dx} - \frac{a}{p^2}\cdot\frac{dp}{dx} = 0.$

or, $\frac{dp}{dx}\left(x - a/p^2\right) = 0.$

__either,__ $\frac{dp}{dx} = 0 \longrightarrow p = c.$

OR, $x - \frac{a}{p^2} = 0$.

The gen sol^n —

$$y = cx + a/c.$$

$$x - \frac{a}{p^2} = 0$$

$$\Rightarrow p^2 = \frac{a}{x}$$

$$\Rightarrow x = a/p^2.$$

$$y = px + a/p$$

$$= p \cdot \frac{a}{p^2} + \frac{a}{p}.$$

$$= \frac{a}{p} + \frac{a}{p}$$

$$= \frac{2a}{p}.$$

$$x = a/p^2$$

$$y = 2a/p.$$

$$\Rightarrow p = 2a/y$$

$$x = \frac{a}{(2a/y)^2} \Rightarrow x = \frac{a}{\frac{4a^2}{y^2}}$$

$$\Rightarrow y^2 = 4ax.$$

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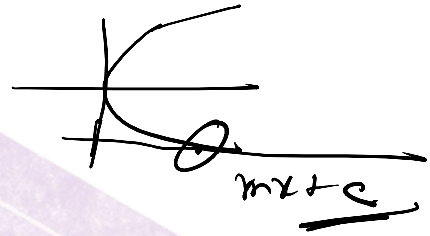
$$\boxed{y = cx + \frac{a}{c}} \qquad \boxed{y^2 = 4ax}$$



#2 Solve: $x^2(y - px) = p^2 y$.

Sol^n: Put $x^2 = u$, $y^2 = v$.

$\hookrightarrow 2x\,dx = du \quad \Big| \quad 2y\,dy = dv$.

$$p = \frac{dy}{dx} = \frac{\frac{1}{2y}\,dv}{\frac{1}{2x}\,du} = \frac{x}{y}\cdot\frac{dv}{du}.$$

$$x^2(y - px) = p^2 y.$$

$$\Rightarrow x^2\left(y - \frac{x^2}{y}\frac{dv}{du}\right) = \left(\frac{x}{y}\frac{dv}{du}\right)^2 y.$$

$$\Rightarrow y^2 - x^2\frac{dv}{du} = \left(\frac{dv}{du}\right)^2.$$

$$\Rightarrow \underbrace{V - u\frac{dv}{du} = \left(\frac{dv}{du}\right)^2}_{2}. \qquad p = px + f(p)$$

$$\Rightarrow \boxed{V = u\frac{dv}{du} + \left(\frac{dv}{du}\right)^2}.$$

$\hookrightarrow$ which is in clairaut's form.

Diff w.r.t $\underline{u}$