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Lecture → 18

Method of Variation of Parameters

$$\boxed{f(D)y = X(x)}$$

→ R.H.S will be little bit complicated

- $\tan x$, $\ln x$, x^{-1}, e^{x^2}, . . .

for a second order equation :

$$\underbrace{y'' + Py' + Qy}_{} = R(x).$$

1. → y_1, y_2.

2. Assume :

$$y = u(x)\, y_1 + v(x)\, y_2.$$

3. Solve.

$$u' y_1 + v' y_2 = 0$$
$$u' y_1' + v' y_2' = R(x)$$

4.) Find u', v'.

5) Final solution :

5) Final solution:

$$\boxed{y = uy_1 + vy_2 + CF}$$

Examples:

#1. $y'' - y = e^x$.

C.F

Auxiliary eqn:

$m^2 - 1 = 0$ $\quad$ C.F $= c_1 e^x + c_2 e^{-x}$

$\Rightarrow m = \pm 1$.

$y_1 = e^x$, $y_2 = e^{-x}$.

Assume:

$y = ue^x + ve^{-x}$.

Equations:

$u'e^x + v'e^{-x} = 0$ --- (i)

$u'e^x - v'e^{-x} = e^x$ --- (ii).

$2u'e^x = e^x$

$\Rightarrow u' = 1/2$ --- (iii)

$v' = -\frac{1}{2}e^{2x}$ --- (iv).

Integrate u' and v' —

Integrate u and v —

$$\left.\begin{aligned} u &= x/2 \\ v &= -\tfrac{1}{4}e^{2x} \end{aligned}\right\}.$$

$$y = \frac{x}{2}e^{x} - \frac{1}{4}e^{2x}\left(e^{-x}\right)$$

$$= \frac{x}{2}e^{x} - \frac{1}{4}e^{x}$$

$$\boxed{y = c_1 e^{x} + c_2 e^{-x} + \frac{x}{2}e^{x} - \frac{1}{4}e^{x}}$$

GS solution

#2 Solve:

$$y'' + y = \tan x$$

Step 1:

C.F $\quad m^2 + 1 = 0.$

$\Rightarrow m = \pm i.$

$y_1 = \cos x, \quad y_2 = \sin x.$

$\text{C.F} = c_1 \cos x + c_2 \sin x.$

Step 2:

$y = u\cos x + v\sin x.$

Step 3:

$$u'\cos x + v'\sin x = 0 \quad \text{--- (1)}$$

$$-u'\sin x + v'\cos x = \tan x \quad \text{--- (2)}.$$

$$-u'\sin x + v'\cos x = \tan x \quad (2).$$

eq (i) $\times \sin x$ || eq (ii) $\times \cos x$

$$\left.\begin{aligned} u' &= -\tan x \sin x = -\frac{\sin^2 x}{\cos x} \\ v' &= \tan x \cos x = \sin x. \end{aligned}\right\}.$$

Integrate u' and v'.

$$\frac{du}{dx} = -\frac{\sin^2 x}{\cos x}$$

$$\Rightarrow du = -\frac{\sin^2 x}{\cos x}\, dx$$

$$\Rightarrow du = -\left(\frac{1-\cos^2 x}{\cos x}\right) dx$$

$$\Rightarrow du = -(\sec x - \cos x)\, dx$$

$$\Rightarrow \int du = -\int (\sec x - \cos x)\, dx.$$

$$\Rightarrow u = -\left(\ln|\sec x + \tan x| - \sin x\right)$$

$$v = -\cos x.$$

Final:

$$\boxed{y = C.F + y}$$

Simultaneous Linear Differential equation.

$$\frac{dx}{dt} = ax + by$$

$$\frac{dx}{dt} = ax + by$$

$$\frac{dy}{dt} = cx + dy.$$

writting in a operator form $D = \frac{d}{dt}$.

$$Dx = ax + by$$

$$\Rightarrow (D-a)x - by = 0 \quad \text{--- (i)}.$$

$$-cx + (D-d)y = 0 \quad \text{--- (ii)}.$$

Examples.

#1. $\frac{dx}{dt} = x + y$

$\frac{dy}{dt} = x - y$.

writting in a operator form —.

$$Dx = x + y$$

$$\Rightarrow (D-1)x - y = 0 \quad \text{--- (i)}.$$

$$Dy + y - x = 0$$

$$\Rightarrow -x + (D+1)y = 0 \quad \text{--- (ii)}.$$

$$(D^2-1)x - (D+1)y = 0$$

$$-x + (D+1)y = 0$$

$$(D^2 - 1 - 1)x = 0$$

$$\Rightarrow (D^2 - 2)x = 0.$$

Auxiliary eq^n for m -

$$m^2 - 2 = 0$$

$$\Rightarrow m = \pm\sqrt{2}.$$

$$x = c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t}.$$

From eq - (ii) —

$$(D+1)y = x.$$

$$\Rightarrow y = x - Dy.$$

$$\Rightarrow \frac{dy}{dt} + y = x.$$

$$\Rightarrow \frac{dy}{dt} + y = c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t}$$

$$I.F = e^{\int 1\, dt} = e^{t}.$$

$$e^{t}\frac{dy}{dt} + e^{t} y = c_1 e^{(\sqrt{2}+1)t} + c_2 e^{(-\sqrt{2}+1)t}$$

$$\Rightarrow \frac{d}{dt}\left(y e^{t}\right) = c_1 e^{(\sqrt{2}+1)t} + c_2 e^{(-\sqrt{2}+1)t}$$

$$\Rightarrow y e^{t} = \frac{c_1}{\sqrt{2}+1} e^{(\sqrt{2}+1)t} + \frac{c_2}{(-\sqrt{2}+1)} e^{(-\sqrt{2}+1)t}$$

$$\Rightarrow y = \frac{c_1}{} e^{\sqrt{2}t} + \frac{c_2}{} e^{-\sqrt{2}t}$$

$$\Rightarrow \boxed{y = \frac{c_1}{\sqrt{2}+1} e^{\sqrt{2}t} + \frac{c_2}{1-\sqrt{2}} e^{-\sqrt{2}t}}$$

$$(D+1)y = x.$$

$$\Rightarrow y = \frac{1}{D+1} x.$$

$$\boxed{y = e^{-t} \int x e^{t} \, dt}$$

$$\boxed{\begin{aligned} &(D+a)y = f(t) \\ &\Rightarrow y = e^{-at} \int f(t) e^{at} \, dt. \end{aligned}}$$

Solve :

$$\left.\begin{aligned} \frac{dx}{dt} &= 3x + 4y. \\ \frac{dy}{dt} &= -4x + 3y \end{aligned}\right\}.$$

Step 1:

$$D = \frac{d}{dt}.$$

$$(D-3)x - 4y = 0 \quad \text{— (i)}$$

$$4x + (D-3)y = 0 \quad \text{— (ii)}.$$

Eliminate y.

$$(D-3)^2 x + 16x = 0$$

$$(D-3)\,x + 16x = 0$$

$$\Rightarrow (D^2 - 6D + 25)x = 0.$$

Auxiliary eq^n:

$$m^2 - 6m + 25 = 0.$$

$$m = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(25)}}{2 \cdot 1}$$

$$= \frac{6 \pm \sqrt{36 - 100}}{2}$$

$$= \frac{6 \pm 8i}{2}$$

$$= 3 \pm 4i \rightarrow a + ib.$$

$$x = e^{3t}(c_1 \cos 4t + c_2 \sin 4t).$$

$$4x + (D-3)y = 0.$$

$$\Rightarrow (D-3)y = -4x.$$

$$\Rightarrow y = \frac{1}{(D-3)}(-4x)$$

$$= e^{-3t}\int (-4x)\, e^{3t}\, dt$$

$$y = \ldots\ldots$$

Euler - Cauchy (Equidimensional) Differential equation

Differential ~ ~

D.E of the form :

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots$$

$$+ \dots + a_{n-1} x \frac{dy}{dx} + a_n y = 0. \quad \text{--- (1)}$$

Let assume : $y = x^m$.

Successive differentiation :

$$\frac{dy}{dx} = m x^{m-1}$$

$$\frac{d^2 y}{dx^2} = m(m-1) x^{m-2},$$

$$\frac{d^3 y}{dx^3} = m(m-1)(m-2) x^{m-3}$$

..

Examples :

Solve : $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - y = 0.$

→ This is an Euler-cauchy eqn

Assume : $y = x^m$.

$$\frac{dy}{dx} = m x^{m-1}$$

$$d^2 y \quad = m(m-1) x^{m-2}.$$

$$\frac{d^2y}{dx^2} = m(m-1)x^{m-2}.$$

$$x^2\left[m(m-1)x^{m-2}\right] + x\left(mx^{m-1}\right) - x^m = 0$$

$$\Rightarrow m(m-1)x^m + mx^m - x^m = 0.$$

$$\Rightarrow x^m\left[m(m-1) + m - 1\right] = 0$$

$$\Rightarrow x^m\left[m^2-1\right] = 0.$$

Solving the auxiliary eq^n:

$$m^2 - 1 = 0$$

$$\Rightarrow m = \pm 1.$$

$$y = c_1 x + c_2 x^{-1}.$$

Based on auxiliary eq^n roots:

Roots	Solution
Real & distinct	$y = C_1 x^{m_1} + C_2 x^{m_2}$
Equal roots	$y = C_1 x^m + C_2 x^m \ln x$
Complex roots $m = \alpha \pm i\beta$	$y = x^\alpha[C_1 \cos(\beta \ln x) + C_2 \sin(\beta \ln x)]$

H/W

Solve:

i) $x^2y'' - 3xy' + 4y = 0$

ii) $x^2y'' + xy' + y = 0$

Condition for exactness of Higher order Linear differential equations

General form:

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \dots + a_n(x)y = \phi(x)$$

$$\boxed{a_0 y^n + a_1 y^{(n-1)} + \dots + a_n y = \phi(x)}$$

$$\frac{d}{dx}\left[b_0(x)y^{n-1} + b_1(x)y^{(n-2)} + \dots + b_{n-1}(x)y\right] = \phi(x)$$

Expanding we get:

$$b_0 y^n + b_0' y^{n-1} + \dots$$

$$= \boxed{b_0 y^n + (b_0' + b_1)y^{(n-1)} + \dots (b_1' + b_2)y^{n-2} + \dots + b'_{n-1} y}$$

Comparing it with this eq^n:

$$a_0 y^n + a_1 y^{(n-1)} + \dots + a_n y = \phi(x)$$

$$\left.\begin{aligned} a_0 &= b_0 \\ a_1 &= b_0' + b_1 \\ a_2 &= b_1' + b_2 \\ &\vdots \end{aligned}\right\}$$

a

$\vdots$

$a_n = b'_{n-1}$).

$a_1 = \frac{d a_0}{dx}$, $a_2 = \frac{d a_1}{dx}$

$a_3 = \frac{d a_3}{dx}$, ... $a_n = \frac{d a_{n-1}}{dx}$.

$$a_K = \frac{d}{dx}(a_{K-1}) .$$

$x^2 y'' + 2xy' + 2y = 0$

$a_0 = x^2$

$a_1 = 2x = \frac{d}{dx}(x^2) = a_0'$

$a_2 = 2 = \frac{d}{dx}(2x)$

$$\frac{d}{dx}(x^2 y' + 2xy) = 0$$

$$x^2 y' + 2xy = \underline{c_1}$$

$\Rightarrow x^2 \frac{dy}{dx} + 2xy = \underline{c_1}$

$\Rightarrow \frac{dy}{dx} + \frac{2}{x} y = \frac{c_1}{x^2}$.

I.F. $= e^{\int 2/x \, dx} = x^2$.

$$I.F = e^{\int 2/x \, dx} = x^2.$$

$$\frac{d}{dx}(x^2 y) = c_1$$

$$\Rightarrow x^2 y = c_1 x + c_2.$$

$$\boxed{\Rightarrow y = \frac{c_1}{x} + \frac{c_2}{x^2}}$$

H/W

$$\left.\begin{array}{l} ①\ (1+x^2)y'' + 2xy' + 2y = 1. \\ ②\ (1+x)^2 y'' + 2(1+x)y' + 2y = 0 \end{array}\right\}$$