Topics to cover today :

i) Symmetric Groups. (S_n)

ii) S_3 (problem)

iii) Alternating group (A_n)

iv) Klein's - 4 - group.

v) Dihedral group.

Symmetric Group (S_n)

Let S be the set of all permutations on the set $\{1, 2, 3, \dots n\}$.

Let's check whether this forms a group w.r.t multiplication of permutation.

* Let f, g be two 'perm' set $\{1, 2, \dots n\}$

$f \cdot g \longrightarrow$ 'perm' set on $\{1, 2, \dots n\}$.

$f \rightarrow (2, 3, 1, 4, \dots n)$

$g \longrightarrow (5, 6, 1, 2, 3, 4, 7, \dots n)$

$\therefore f \in S, \; g \in S \Rightarrow f \cdot g \in S$ [closure property] ✓

ii) 'f, g' $\quad f \cdot g \rightarrow$ Composition g f^n

So, it is also associative.

iii) $i = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix} \in S$ [identity].

$f \in S \qquad i \cdot f = f \cdot i = f \quad \forall f \in S.$

$$f = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 3 & 4 & \cdots & 1 \end{pmatrix}$$

iv) $f = \begin{pmatrix} 1 & 2 & \cdots & n \\ f(1) & f(2) & \cdots & f(n) \end{pmatrix} \in S.$

$$g = \begin{pmatrix} f(1) & f(2) & \cdots & f(n) \\ 1 & 2 & \cdots & n \end{pmatrix} \in S.$$

So, 'g' is the inverse of 'f'.

$$g \cdot f = f \cdot g = i.$$

Now tell me is it commutative?

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 2 & 1 & 3 & 4 & \cdots & n \end{pmatrix} = (1, 2)$$

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 3 & 2 & 1 & 4 & \cdots & n \end{pmatrix} = (1, 3)$$

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 3 & 2 & 1 & 4 & \cdots & n \end{pmatrix} = (1, 3)$$

$$f \cdot g = (1, 2)\,(\underline{1, \ 3})$$

$$= \underline{(1, 3, 2)}.$$

$$g \cdot f = (1, 3)(1, 2)$$

$$= (\underline{1, 2, 3})$$

1 2 1 3

$$f \cdot g \neq g \cdot f$$

So, $(S, \cdot)$ is non-commutative group.

<u>Example</u>

$S_3 \rightarrow$ Group of permutation on a set with 3 elements.

Permutation of $\{1, 2, 3\} \rightarrow 3!$ ways.

1 2 3
2 1 3
3 1 2
1 3 2
2 3 1
3 2 1

$\rightarrow$ 6 ways.

$$P_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \qquad P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (1, 2, 3)$$

$$P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (1,2,3) \qquad P_3 = \begin{pmatrix} \boxed{1} & 2 & 3 \\ \boxed{1} & 3 & 2 \end{pmatrix} = (2, 3)$$

$$P_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (1, 3) \qquad P_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (1, 2)$$

f g

$$\begin{matrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 3 & 1 \end{matrix} \quad * \quad \begin{matrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 3 & 1 & 2 \end{matrix}$$

$f \cdot g$

$f(1) = 2$

$f(2) = 3$

$f(3) = 1$

$g(1) = 3$

$g(2) = \underline{1}$

$g(3) = 2$

$f \circ g(1) = f(3) = \underline{1}.$

$f \circ g(2) = f(1) = 2$

$f \circ g(3) = f(2) = 3.$

$(1, 2, 3) \quad (1, 3, 2)$

$P_1 \cdot P_2 = P_1(P_2(1)) = 1 \qquad P_1 P_2(2) \quad P_1(3) = 2$

$$P_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \qquad P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \underline{(1, 2, 3)}$$

$$P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \underline{(1, 3, 2)} \qquad P_3 = \begin{pmatrix} \boxed{\begin{matrix} 1 \\ 1 \end{matrix}} & \begin{matrix} 2 \\ 3 \end{matrix} & \begin{matrix} 3 \\ 2 \end{matrix} \end{pmatrix} = (2, 3)$$

$$P_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (1, 3) \qquad P_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (1, 2)$$

$\rho_4 \cdot \begin{pmatrix} 3 & 2 & 1 \end{pmatrix} = (1, 3)$ $\qquad$ $(\;\;) : (1, 2)$

	ρ_0	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5
ρ_0	ρ_0	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5
ρ_1	ρ_1					
ρ_2						
ρ_3						
ρ_4						
ρ_5						

$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (1, 2, 3)$$

$\rho_1 \cdot \rho_1 (1) = \rho_1(2) = 3.$

$\rho_1 \cdot \rho_1 (2) = \rho_1(3) = 1$

$(3, 1, 2)$

$\rho_1 \cdot \rho_1 (3) = \rho_1(1) = 2.$

$(1\ 2,\ 3)$

Multiplication table

We portray elements as permutations on the set $\{1, 2, 3\}$ using the cycle decomposition. The *row element is multiplied on the left and the column element on the right*, with the assumption of *functions written on the left. This means that the column element is applied first and the row element is applied next.*

f :

Element	()	(1, 2)	(2, 3)	(1, 3)	(1, 2, 3)	(1, 3, 2)

We portray elements as permutations on the set $\{1,2,3\}$ using the cycle decomposition. The *row element is multiplied on the left and the column element on the right*, with the assumption of *functions written on the left. This means that the column element is applied first and the row element is applied next.*

$f:$

Element	()	(1, 2)	(2, 3)	(1, 3)	(1, 2, 3)	(1, 3, 2)
()	()	(1, 2)	(2, 3)	(1, 3)	(1, 2, 3)	(1, 3, 2)
(1, 2)	(1, 2)	()	(1, 2, 3)	(1, 3, 2)	(2, 3)	(1, 3)
(2, 3)	(2, 3)	(1, 3, 2)	()	(1, 2, 3)	(1, 3)	(1, 2)
(1, 3)	(1, 3)	(1, 2, 3)	(1, 3, 2)	()	(1, 2)	(2, 3)
(1, 2, 3)	(1, 2, 3)	(1, 3)	(1, 2)	(2, 3)	(1, 3, 2)	()
(1, 3, 2)	(1, 3, 2)	(2, 3)	(1, 3)	(1, 2)	()	(1, 2, 3)

$\leftarrow S\{1, 2, 3\}$

→ Not symmetric.

1 2 3

(1 3 2)

(1, 2) (2 1)

If we used the opposite convention (i.e., functions written on the right), the row element is to be multiplied on the right and the column element on the left.

Here is the multiplication table where we use the one-line notation for permutations, where, as in the previous multiplication table, the column permutation is applied first and then the row permutation. Thus, with the left action convention, the row element is multiplied on the left and the column element on the right:

i) So, S is closed, I can say.

ii) Bijective mapping, their composition is associative.

iii) Identity element exist

iv)

$$\begin{matrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ (2 & 1 & 3) \end{matrix} \qquad \begin{matrix} (2 & 1 & 3) \\ \downarrow & \downarrow & \downarrow \\ (1 & 2 & 3) \end{matrix}$$

From the table:

Element	Inverse
e	e
(12)	(12)
(13)	(13)
(23)	(23)
(123)	(132)
(132)	(123)

$$\boxed{(1\ 2\ 3)\cdot(1\ 3\ 2) = e}$$

As it is not commutative, so they are not symmetric.

Alternating Group (A_n).

↳ even permutation on the set $\{1, 2, \dots n\}$ forms a group w.r.t multiplication of permutation.

$$A_n \rightarrow \frac{n!}{2} \text{ elms.}$$

$$A_3 \rightarrow \frac{3!}{2} = 3.$$

$$P_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

$$P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

$$\left.\begin{aligned} (1)(1\ 2\ 3) &= (1\ 2\ 3) \\ (1\ 2\ 3)(1) &= (1\ 2\ 3) \end{aligned}\right\} \checkmark$$

$$\left.\begin{aligned} (1\ 2\ 3)(1\ 3\ 2) &= (1). \\ (1\ 3\ 2)(1\ 2\ 3) &= (1) \end{aligned}\right\} \checkmark.$$

$$(1\ 2\ 3)(1\ 2\ 3) = (1\ 3\ 2).$$

Recall:

$$A_3 = \{e, (123), (132)\}$$

Let's use the operation table:

$*$	e	(123)	(132)
e	e	(123)	(132)
(123)	(123)	(132)	e
(132)	(132)	e	(123)

Check symmetry:

- $(123) * (132) = e$
- $(132) * (123) = e$

✅ All entries symmetric about diagonal.

So, it is a commutative group

Klein - 4 - group (K_4).

$S = \{e, a, b, c\}.$

'o' $\longrightarrow$ binary composition.

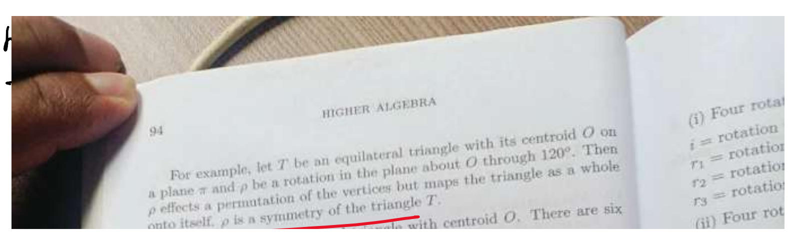
$e \circ a = a \circ e = a$

$e \circ e = a \circ a = b \circ b = c \circ c = e.$

$a \circ b = b \circ a = c$ $\qquad$ $b \circ c = c \circ b = a$

$a \circ c = c \circ a = b.$

Di


HIGHER ALGEBRA

94

For example, let T be an equilateral triangle with its centroid O on a plane π and ρ be a rotation in the plane about O through 120°. Then ρ effects a permutation of the vertices but maps the triangle as a whole onto itself. ρ is a symmetry of the triangle T.

... with centroid O. There are six

(i) Four rotat

i = rotation

r_1 = rotation

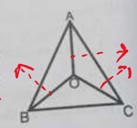
r_2 = rotation

r_3 = rotation

(ii) Four rot

For example, let T be an equilateral triangle with its centroid O on a plane π and ρ be a rotation in the plane about O through 120^o. Then ρ effects a permutation of the vertices but maps the triangle as a whole onto itself. ρ is a symmetry of the triangle T.

Let ABC be an equilateral triangle with centroid O. There are six symmetries of the triangle.



i = rotation in the plane about O through 0^o;

r_1 = rotation in the plane about O through 120^o;

r_2 = rotation in the plane about O through 240^o;

a = reflection about AO;

b = reflection about BO;

c = reflection about CO.

Let $\circ$ stand for the composition of mappings. Then $r_1 \circ r_1 = r_2, a \circ b = r_1, b \circ a = r_2$ etc.

Let $S = \{i, r_1, r_2, a, b, c\}$. Taking $\circ$ as the binary composition on S, the composition table is given below.

$\circ$	i	r_1	r_2	a	b	c
i	i	r_1	r_2	a	b	c
r_1	r_1	r_2	i	c	a	b
r_2	r_2	i	r_1	b	c	a
a	a	b	c	i	r_1	r_2
b	b	c	a	r_2	i	r_1
c	c	a	b	r_1	r_2	i

D_3

The six symmetries of an equilateral triangle form a non-commutative group. This group is called the *dihedral group* D_3.

Note. The three rotations correspond to the following permutations of the vertices.

$$i = \begin{pmatrix} A & B & C \\ A & B & C \end{pmatrix}, r_1 = \begin{pmatrix} A & B & C \\ B & C & A \end{pmatrix}, r_2 = \begin{pmatrix} A & B & C \\ C & A & B \end{pmatrix}.$$

The three reflections correspond to the following permutations of the vertices. $a = \begin{pmatrix} A & B & C \\ A & C & B \end{pmatrix}, b = \begin{pmatrix} A & B & C \\ C & B & A \end{pmatrix}, c = \begin{pmatrix} A & B & C \\ B & A & C \end{pmatrix}$.

The dihedral group D_3 is same as the symmetric group S_3.

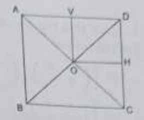
13. Symmetries of a square.

Let us consider a square $ABCD$ with centre at O. There are eight symmetries of the square:-

A symmetry of a geometrical fig in a euclidean space is an isometry that keeps the figure as a whole unchanged.

(i) Four rotations in the plane about O.

i = rotation through $0°$;
r_1 = rotation through $90°$;
r_2 = rotation through $180°$;
r_3 = rotation through $270°$.



(ii) Four rotations out of the plane.

h = rotation about the horizontal line OH;
v = rotation about the vertical line OV;
d = rotation about the (principal) diagonal OA;
d' = rotation about the (other) diagonal OB.

Let $\circ$ stand for the composition of mappings.

Let $S = \{i, r_1, r_2, r_3, h, v, d, d'\}$. Taking $\circ$ as the binary composition on the set S, the composition table is given below:

$\circ$	i	r_1	r_2	r_3	h	v	d	d'
i	i	r_1	r_2	r_3	h	v	d	d'
r_1	r_1	r_2	r_3	i	d'	d	h	v
r_2	r_2	r_3	i	r_1	v	h	d'	d
r_3	r_3	i	r_1	r_2	d	d'	v	h
h	h	d	v	d'	i	r_2	r_1	r_3
v	v	d'	h	d	r_2	i	r_3	r_1
d	d	v	d'	h	r_3	r_1	i	r_2
d'	d'	h	d	v	r_1	r_3	r_2	i

D_4

The eight symmetries of a square form a non-commutative group. This group is called the *octic group* or the *dihedral group* D_4.

Note 1. The symmetries correspond to the following permutations of the vertices.

$$i = \begin{pmatrix} A & B & C & D \\ A & B & C & D \end{pmatrix}, \quad r_1 = \begin{pmatrix} A & B & C & D \\ B & C & D & A \end{pmatrix} = (ABCD),$$

$$r_2 = \begin{pmatrix} A & B & C & D \\ C & D & A & B \end{pmatrix} = (AC)(BD), \quad r_3 = \begin{pmatrix} A & B & C & D \\ D & A & B & C \end{pmatrix} = (ADCB),$$

$$h = \begin{pmatrix} A & B & C & D \\ B & A & D & C \end{pmatrix} = (AB)(CD), \quad v = \begin{pmatrix} A & B & C & D \\ D & C & B & A \end{pmatrix} = (AD)(BC),$$

$$d = \begin{pmatrix} A & B & C & D \\ A & D & C & B \end{pmatrix} = (BD), \quad d' = \begin{pmatrix} A & B & C & D \\ C & B & A & D \end{pmatrix} = (AC).$$

Note 2. The symmetries of a regular pentagon form a non-commutative group of order 10, called the dihedral group D_5. They are five rotations about the centre and five reflections about the right bisectors of the sides.

The symmetries of a regular n-gon form a non-commutative group of order $2n$, called the dihedral group D_n.