Mean-Variance Analysis

Wang Wei Mun

Lee Kong Chian School of Business
Singapore Management University

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Asset Allocation vs Asset Pricing

- **Asset allocation** (or **portfolio choice**) is theory of how investor allocates wealth amongst multiple financial assets
- For simplicity, assume investor is "price taker": i.e., scale of investment is small enough that it doesn't affect prices
- Also for simplicity, assume perfect "frictionless" financial market with no taxes or transaction costs, etc.
- Asset allocation goes hand-in-hand with asset pricing: if we know how all investors choose to allocate their wealth, we can then find equilibrium prices to balance supply and demand
- Earliest known formal theory of asset allocation is theory of **mean-variance-efficient frontier**, developed by Harry Markowitz and published in 1952

Investment Environment

- Financial market consists of $n \geq 2$ risky tradable assets with normal returns (and no riskless asset)
- Let $\mathbf{R} = (R_1, \ldots, R_n)'$ be $n \times 1$ vector of expected returns
- Let $\mathbf{V}$ be $n \times n$ covariance matrix of returns, which consists of variances on diagonal and covariances on off-diagonal
- $\mathbf{V}$ must be **symmetric**: $\mathbf{V}' = \mathbf{V}$ and **positive definite**:

$$\mathbf{z}'\mathbf{V}\mathbf{z} > 0 \text{ for any } \mathbf{z} \neq 0$$

- Assume no redundant assets, so returns must be **linearly independent** and covariance matrix must be **invertible**:

$$\exists \ \mathbf{V}^{-1} \text{ such that } \mathbf{V}^{-1}\mathbf{V} = \mathbf{I}$$

Portfolio Weights

- Let $\mathbf{w} = (w_1, \ldots, w_n)'$ be $n \times 1$ vector of portfolio weights, which represents proportion of investor's wealth allocated to each tradable financial asset
- No restriction on individual portfolio weights: positive weight indicates normal investment (or "long position") while negative weight indicates short-selling (or "short position")
- Only restriction is that portfolio weights must sum to one: $\mathbf{w}'\mathbf{e} = 1$, where $\mathbf{e} = (1, \ldots, 1)'$ is $n \times 1$ unit vector
- Investor can allocate more than available wealth into any individual financial asset, but cannot allocate more than available wealth in aggregate
- Notice that $\mathbf{w}'\mathbf{R}$ is expected return for investor's portfolio, and $\mathbf{w}'\mathbf{V}\mathbf{w} > 0$ is variance of return for investor's portfolio

Asset Allocation Problem

- Investor aims to create portfolio with fixed expected return of R_p and lowest possible variance of return
- Use Lagrangian to represent objective function for investor's asset allocation (or portfolio choice) problem:

$$\min_{\{\mathbf{w},\lambda,\gamma\}} \mathcal{L} = \frac{1}{2}\mathbf{w}'\mathbf{V}\mathbf{w} + \lambda\left(R_p - \mathbf{w}'\mathbf{R}\right) + \gamma\left(1 - \mathbf{w}'\mathbf{e}\right)$$

- Here λ and γ are **Lagrange multipliers**, which ensure that constraints on mean return and portfolio weights are satisfied
- Objective function is convex (since $\mathbf{V}$ is positive definite), so solution is guaranteed to be global minimum
- "Dual problem" is to maximise expected return for fixed variance of return, but more difficult to solve

Optimal Portfolio Weights – Part 1

- Set partial derivative w.r.t. $\mathbf{w}$ equal to zero, then pre-multiply by $\mathbf{V}^{-1}$ and rearrange to find optimal portfolio weights:

$$\mathbf{V}\mathbf{w}^* - \lambda\mathbf{R} - \gamma\mathbf{e} = 0 \quad \Longrightarrow \quad \mathbf{w}^* = \lambda\mathbf{V}^{-1}\mathbf{R} + \gamma\mathbf{V}^{-1}\mathbf{e}$$

- Pre-multiply by $\mathbf{R}'$ and apply constraint for expected return:

$$\mathbf{R}'\mathbf{w}^* = \lambda\mathbf{R}'\mathbf{V}^{-1}\mathbf{R} + \gamma\mathbf{R}'\mathbf{V}^{-1}\mathbf{e} = R_p$$

- Pre-multiply by $\mathbf{e}'$ and apply constraint for portfolio weights:

$$\mathbf{e}'\mathbf{w}^* = \lambda\mathbf{e}'\mathbf{V}^{-1}\mathbf{R} + \gamma\mathbf{e}'\mathbf{V}^{-1}\mathbf{e} = 1$$

Optimal Portfolio Weights – Part 2

- Solve simultaneous equations to find Lagrange multipliers:

$$\lambda = \frac{\delta R_p - \alpha}{\zeta\delta - \alpha^2}; \qquad \gamma = \frac{\zeta - \alpha R_p}{\zeta\delta - \alpha^2}$$

- Here α is scalar, while ζ and δ are strictly positive scalars:

$$\alpha = \mathbf{R}'\mathbf{V}^{-1}\mathbf{e}; \qquad \zeta = \mathbf{R}'\mathbf{V}^{-1}\mathbf{R}; \qquad \delta = \mathbf{e}'\mathbf{V}^{-1}\mathbf{e}$$

- Confirm that denominator is strictly positive:

$$(\alpha\mathbf{R} - \zeta\mathbf{e})'\,\mathbf{V}^{-1}\,(\alpha\mathbf{R} - \zeta\mathbf{e}) = \zeta\left(\zeta\delta - \alpha^2\right) > 0$$

Optimal Portfolio Weights – Part 3

- Replace λ and γ to find portfolio weights corresponding to expected return of R_p and lowest possible variance of return:

$$\mathbf{w}^* = \left(\frac{\delta R_p - \alpha}{\zeta\delta - \alpha^2}\right)\mathbf{V}^{-1}\mathbf{R} + \left(\frac{\zeta - \alpha R_p}{\zeta\delta - \alpha^2}\right)\mathbf{V}^{-1}\mathbf{e}$$

- Rearrange to get linear relationship: $\mathbf{w}^* = \mathbf{a} + \mathbf{b}R_p$, where:

$$\mathbf{a} = \frac{\zeta\mathbf{V}^{-1}\mathbf{e} - \alpha\mathbf{V}^{-1}\mathbf{R}}{\zeta\delta - \alpha^2}; \qquad \mathbf{b} = \frac{\delta\mathbf{V}^{-1}\mathbf{R} - \alpha\mathbf{V}^{-1}\mathbf{e}}{\zeta\delta - \alpha^2}$$

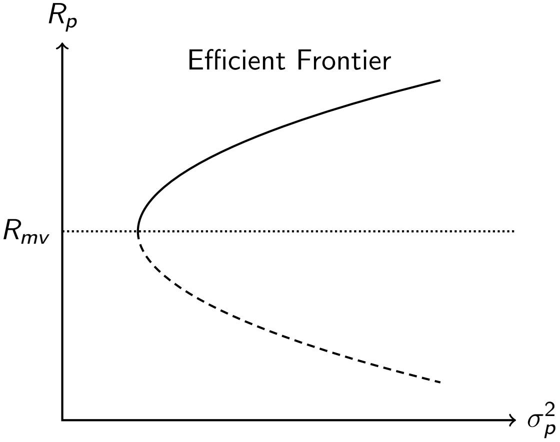
- **Minimum-variance frontier** consists of portfolios with lowest possible variance of return, for different values of R_p

Portfolio Frontier – Part 1

- Variance of return for portfolio on minimum-variance frontier:

$$\begin{aligned}\sigma_p^2 = \mathbf{w}'\mathbf{V}\mathbf{w} &= (\mathbf{a} + \mathbf{b}R_p)'\,\mathbf{V}\,(\mathbf{a} + \mathbf{b}R_p) \\ &= \frac{\delta R_p^2 - 2\alpha R_p + \zeta}{\zeta\delta - \alpha^2} \\ &= \frac{1}{\delta} + \frac{\delta}{\zeta\delta - \alpha^2}\,(R_p - R_{mv})^2\end{aligned}$$

- $R_{mv} = \frac{\alpha}{\delta}$ is mean return for global minimum variance portfolio
- Minimum-variance frontier is parabola when plotted with variance of return on y-axis and expected return on x-axis
- Standard practice to flip axes, as shown on next slide

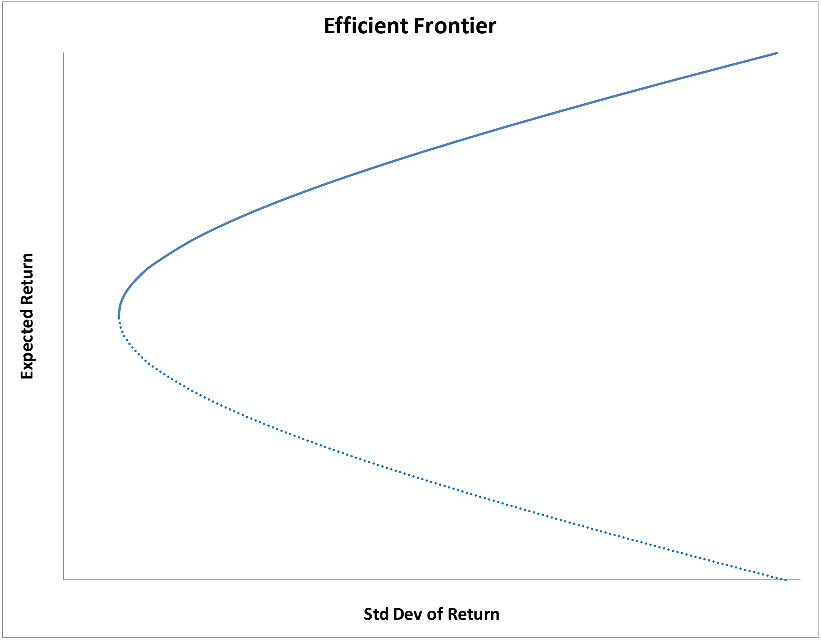

R_p
Efficient Frontier
R_{mv}
σ_p^2

Portfolio Frontier – Part 2

- Top half of minimum-variance frontier (where $R_p \geq R_{mv}$) is known as **efficient frontier**, consisting of portfolios with highest expected return for different values of σ_p^2
- If switch from variance of return to standard deviation of return on x-axis, then minimum-variance frontier is hyperbola with center at $(0, R_{mv})$ and asymptotes:

$$R_p = R_{mv} \pm \left(\zeta - \frac{\alpha^2}{\delta}\right)^{\frac{1}{2}} \sigma_p$$

- In order to maximise expected utility of final wealth, risk-averse investor should choose portfolio where (convex) indifference curve is tangent to efficient frontier


Efficient Frontier
Expected Return
Std Dev of Return

Portfolio Separation

- Let p_1 and p_2 be any two portfolios on minimum-variance fronter, and let q be any other frontier portfolio
- Can always find θ such that $R_q = \theta R_{p_1} + (1-\theta) R_{p_2}$, so can invest θ in p_1 and $1-\theta$ in p_2 to replicate q:

$$\begin{aligned}
\theta \mathbf{w}_{\mathbf{p_1}} + (1-\theta) \mathbf{w}_{\mathbf{p_2}} &= \theta (\mathbf{a} + \mathbf{b} R_{p_1}) + (1-\theta) (\mathbf{a} + \mathbf{b} R_{p_2}) \\
&= \mathbf{a} + \mathbf{b} \left(\theta R_{p_1} + (1-\theta) R_{p_2} \right) \\
&= \mathbf{a} + \mathbf{b} R_q \\
&= \mathbf{w}_{\mathbf{q}}
\end{aligned}$$

- Can combine any two frontier portfolios in different ways to generate entire minimum-variance frontier

Orthogonal Frontier Portfolios – Part 1

- Covariance of return between any two frontier portfolios:

$$\begin{aligned}\mathbf{w}_{\mathbf{p_1}}'\mathbf{V}\mathbf{w}_{\mathbf{p_2}} &= (\mathbf{a}+\mathbf{b}R_{p_1})'\,\mathbf{V}\,(\mathbf{a}+\mathbf{b}R_{p_2}) \\ &= \frac{1}{\delta}+\frac{\delta}{\zeta\delta-\alpha^2}\,(R_{p_1}-R_{mv})\,(R_{p_2}-R_{mv})\end{aligned}$$

- For given R_{p_1}, set covariance to zero and solve for R_{p_2}:

$$R_{p_2} = R_{mv} - \frac{\zeta\delta-\alpha^2}{\delta^2\,(R_{p_1}-R_{mv})}$$

- Hence if p_1 is on efficient frontier, then (orthogonal portfolio) p_2 must be on inefficient frontier (and vice versa)

Orthogonal Frontier Portfolios – Part 2

- Can measure slope at any point of minimum-variance frontier:

$$\frac{\partial R_p}{\partial \sigma_p} = \frac{\zeta\delta - \alpha^2}{\delta\left(R_p - R_{mv}\right)}\sigma_p$$

- Evaluate at (σ_{p_1}, R_{p_1}) to get slope of minimum-variance frontier at point corresponding to p_1
- Hence equation for line (with unknown y-intercept of R_0) that is tangent to frontier at point corresponding to p_1:

$$R_p = R_0 + \left[\frac{\zeta\delta - \alpha^2}{\delta\left(R_{p_1} - R_{mv}\right)}\sigma_{p_1}\right]\sigma_p$$

Orthogonal Frontier Portfolios – Part 3

- Evaluate at (σ_{p_1}, R_{p_1}) and solve for y-intercept:

$$
\begin{aligned}
R_0 &= R_{p_1} - \frac{\zeta\delta - \alpha^2}{\delta\left(R_{p_1} - R_{mv}\right)}\sigma_{p_1}^2 \\
&= R_{p_1} - \frac{\zeta\delta - \alpha^2}{\delta\left(R_{p_1} - R_{mv}\right)}\left[\frac{1}{\delta} + \frac{\delta}{\zeta\delta - \alpha^2}\left(R_{p_1} - R_{mv}\right)^2\right] \\
&= R_{mv} - \frac{\zeta\delta - \alpha^2}{\delta^2\left(R_{p_1} - R_{mv}\right)} \\
&= R_{p_2}
\end{aligned}
$$

- Hence y-intercept for line that is tangent to frontier at p_1 shows mean return for portfolio that is orthogonal to p_1

Investment Environment with Riskless Asset

- Financial market consists of $n \geq 2$ risky assets (with normal returns) and riskless asset with risk-free rate of R_f
- Let $\mathbf{w}$ be vector of portfolio weights for risky assets, so that $1 - \mathbf{w}'\mathbf{e}$ is proportion of wealth invested in riskless asset
- If $\mathbf{w}'\mathbf{e} < 1$, then investor is lending money (to other investors, through bank) at risk-free rate
- If $\mathbf{w}'\mathbf{e} > 1$, then investor is borrowing money (from other investors, through bank) at risk-free rate
- Expected return for investor's portfolio:

$$R_p = \mathbf{w}'\mathbf{R} + \left(1 - \mathbf{w}'\mathbf{e}\right) R_f = R_f + \mathbf{w}' \left(\mathbf{R} - R_f \mathbf{e}\right)$$

- Here $\mathbf{R} - R_f \mathbf{e}$ represents $n \times 1$ vector of **risk premiums**

Asset Allocation with Riskless Asset – Part 1

- Lagrangian for investor's asset allocation problem:

$$\min_{\{\mathbf{w},\lambda\}} \mathcal{L} = \frac{1}{2}\mathbf{w}'\mathbf{V}\mathbf{w} + \lambda\left\{R_p - R_f - \mathbf{w}'\left(\mathbf{R} - R_f\mathbf{e}\right)\right\}$$

- Use first-order condition to find optimal portfolio weights:

$$\mathbf{V}\mathbf{w}^* - \lambda\left(\mathbf{R} - R_f\mathbf{e}\right) = 0 \quad \Longrightarrow \quad \mathbf{w}^* = \lambda\mathbf{V}^{-1}\left(\mathbf{R} - R_f\mathbf{e}\right)$$

- Solve for Lagrange multiplier:

$$\begin{aligned} R_p &= R_f + \lambda\left(\mathbf{R} - R_f\mathbf{e}\right)'\mathbf{V}^{-1}\left(\mathbf{R} - R_f\mathbf{e}\right) \\ &= R_f + \lambda\left(\zeta - 2\alpha R_f + \delta R_f^2\right) \Longrightarrow \lambda = \frac{R_p - R_f}{\zeta - 2\alpha R_f + \delta R_f^2} \end{aligned}$$

Asset Allocation with Riskless Asset – Part 2

- Variance of return for portfolio on minimum-variance frontier:

$$\sigma_p^2 = \mathbf{w}'\mathbf{V}\mathbf{w} = \lambda^2 \left(\mathbf{R} - R_f\mathbf{e}\right)' \mathbf{V}^{-1} \left(\mathbf{R} - R_f\mathbf{e}\right) = \frac{\left(R_p - R_f\right)^2}{\zeta - 2\alpha R_f + \delta R_f^2}$$

- Minimum-variance frontier is linear in mean–std dev space:

$$R_p = R_f \pm \left(\zeta - 2\alpha R_f + \delta R_f^2\right)^{\frac{1}{2}} \sigma_p$$

- Efficient frontier consists of straight line with y-intercept of R_f and positive slope of $\left(\zeta - 2\alpha R_f + \delta R_f^2\right)^{\frac{1}{2}}$

Portfolio Separation with Riskless Asset – Part 1

- If $R_f < R_{mv} = \frac{\alpha}{\delta}$, then efficient frontier (with riskless asset) is tangent to efficient portion of risky-asset-only frontier
- If tangent line for risky-asset-only frontier has y-intercept of R_f, then expected return for "tangency" portfolio:

$$R_{tg} = R_{mv} - \frac{\zeta\delta - \alpha^2}{\delta^2 (R_f - R_{mv})} = \frac{\alpha R_f - \zeta}{\delta R_f - \alpha}$$

- Risk premium for tangency portfolio:

$$R_{tg} - R_f = \frac{\alpha R_f - \zeta}{\delta R_f - \alpha} - R_f = -\frac{\zeta - 2\alpha R_f + \delta R_f^2}{\delta (R_f - R_{mv})}$$

Portfolio Separation with Riskless Asset – Part 2

- Variance of return for tangency portfolio:

$$\sigma_{tg}^2 = \frac{1}{\delta} + \frac{\delta\left(R_{tg} - R_{mv}\right)^2}{\zeta\delta - \alpha^2} = \frac{1}{\delta} + \frac{\zeta\delta - \alpha^2}{\delta^3\left(R_f - R_{mv}\right)^2}$$
$$= \frac{1}{\delta}\left[1 + \frac{\zeta\delta - \alpha^2}{\left(\delta R_f - \alpha\right)^2}\right] = \frac{\zeta - 2\alpha R_f + \delta R_f^2}{\delta^2\left(R_f - R_{mv}\right)^2}$$

- Choose square root such that σ_{tg} is positive:

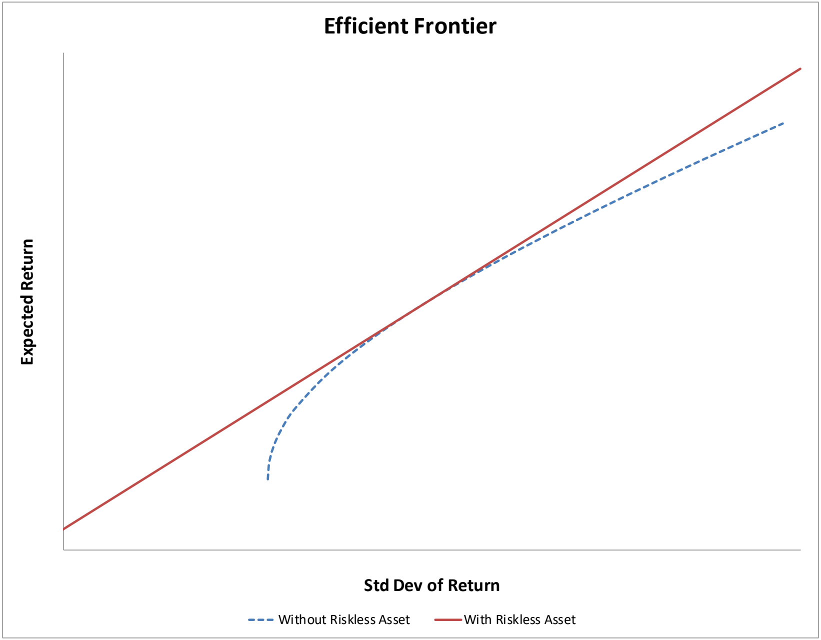
$$\sigma_{tg} = -\frac{\left(\zeta - 2\alpha R_f + \delta R_f^2\right)^{\frac{1}{2}}}{\delta\left(R_f - R_{mv}\right)}$$

Portfolio Separation with Riskless Asset – Part 3

- Sharpe ratio for tangency portfolio:

$$\begin{aligned} \frac{R_{tg} - R_f}{\sigma_{tg}} &= \left[-\frac{\zeta - 2\alpha R_f + \delta R_f^2}{\delta \left(R_f - R_{mv} \right)} \right] \left[-\frac{\delta \left(R_f - R_{mv} \right)}{\left(\zeta - 2\alpha R_f + \delta R_f^2 \right)^{\frac{1}{2}}} \right] \\ &= \left(\zeta - 2\alpha R_f + \delta R_f^2 \right)^{\frac{1}{2}} \end{aligned}$$

- Hence tangency portfolio lies on efficient frontier (with riskless asset), and has highest Sharpe ratio out of all risky portfolios
- Similar result for $R_f > R_{mv}$, except that tangency portfolio lies on inefficient portion of risky-asset-only frontier (and has lowest Sharpe ratio out of all risky portfolios)


Efficient Frontier
Expected Return
Std Dev of Return
Without Riskless Asset
With Riskless Asset

CARA Utility: Economic Environment

- Financial market consists of $n \geq 2$ risky assets (with normal returns) and riskless asset with risk-free rate of R_f
- Let $\tilde{\mathbf{R}}$ be $n \times 1$ vector of asset returns, so portfolio return:

$$\tilde{R}_p = R_f + \mathbf{w}' \left(\tilde{\mathbf{R}} - R_f \mathbf{e} \right)$$

- Let $b_r = bW_0 > 0$ be investor's coefficient of relative risk aversion, measured at initial wealth
- Hence investor's utility of (random) final wealth:

$$U\left(\tilde{W}\right) = -e^{-b\tilde{W}} = -e^{-b_r \frac{\tilde{W}}{W_0}} = -e^{-b_r \tilde{R}_p}$$

CARA Utility: Asset Allocation

- Asset returns have joint normal distribution, so utility of final wealth has lognormal distribution:

$$
\begin{aligned}
E\left[U\left(\tilde{W}\right)\right] &= E\left[-e^{-b_r \tilde{R}_p}\right] \\
&= -e^{-b_r[R_f + \mathbf{w}'(\mathbf{R} - R_f \mathbf{e})] + \frac{1}{2} b_r^2 \mathbf{w}' \mathbf{V} \mathbf{w}}
\end{aligned}
$$

- Exponential function is monotonically increasing, so minimise (negative) exponent to maximise expected utility:

$$
\max_{\mathbf{w}} \left\{ \mathbf{w}' \left(\mathbf{R} - R_f \mathbf{e} \right) - \frac{1}{2} b_r \mathbf{w}' \mathbf{V} \mathbf{w} \right\}
$$

CARA Utility: Optimal Portfolio

- Use first-order condition to find optimal portfolio weights:

$$\mathbf{R} - R_f\mathbf{e} - b_r\mathbf{V}\mathbf{w}^* = 0 \quad \Longrightarrow \quad \mathbf{w}^* = \frac{1}{b_r}\mathbf{V}^{-1}\left(\mathbf{R} - R_f\mathbf{e}\right)$$

- Pre-multiply by $W_0\mathbf{e}'$ to find absolute (dollar) amount of wealth invested in risky assets:

$$W_0\mathbf{e}'\mathbf{w}^* = \frac{1}{b}\left(\alpha - \delta R_f\right)$$

- RHS is not dependent on W_0, so all investors with same degree of absolute risk aversion will invest same (dollar) amount in risky assets (regardless of initial wealth)