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A GARCH model of the implied volatility of the Swiss market index from option prices

Michael Sabbatini, Oliver Linton*

Department of Economics, Yale University, New Haven CT 06520, USA

Abstract

This paper estimates the implied stochastic process of the volatility of the Swiss market index (SMI) from the prices of options written on it. A GARCH(1,1) model is shown to be a good parameterization of the process. Then, using the GARCH option pricing model of Duan (1991), the implied volatility process is estimated by a simulation minimization method from option price data. We find the persistence of volatility shocks implied by options on the SMI to be very close to that estimated from historical data on the index itself. Comparing the performances of the implied GARCH option pricing model to that of the Black and Scholes model it appears that the overall pricing performance of the former is superior. However, the large sample standard deviations of the out-of-sample pricing errors suggest that this result should be taken with caution. © 1998 Elsevier Science B.V.

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1. Introduction

A central hypothesis in the derivation of the Black and Scholes (1973) option pricing model is that the returns on the underlying asset are lognormally distributed with a mean and volatility that are constant through time. However, it has been widely recognized since Mandelbrot (1963, 1967) and Fama (1965) that asset returns possess both fat-tailed distributions and that squared returns appear to cluster. These characteristics are interpreted as evidence of the stochastic volatility of financial assets. The assumption of constant volatility has since been relaxed by numerous authors. For example, Merton (1976) examines the case of a financial asset whose price follows a mixed jump-diffusion process. Hull and White (1987) propose a more general stochastic volatility model in which the price of the underlying asset and its variance follow independent diffusion processes. The Rubinstein (1983) model postulates a positive relation between the price of the asset and its variance. Conversely, the CEV model (Constant Elasticity of Variance) of Cox and Ross (1976) assumes that the variance of the stock is negatively related to the evolution of its price. Finally, over the last fifteen years ARCH (Autoregressive Conditional Heteroskedasticity) models have been widely applied in the financial literature and specifically in the option literature. We owe the ARCH model to Engle (1982) and its generalized version called GARCH to Bollerslev (1986). These models assume conditionally normally distributed returns with a time-varying conditional variance, and appear to reconcile the

*Corresponding author. Tel.: +1203 432 3699; fax: +1203 432 6167; e-mail: linton@econ.yale.edu

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stylized facts of financial time series better than more ‘‘classic’’ time series models (such as ARMA models). The initial success of ARCH models in capturing these nonlinearities in the financial series has led to many extensions of the original model.[1] At the same time we witness a rewriting of the economic models, such as option pricing models, to incorporate this progress. The GARCH option pricing model by Duan (1991), that we use in this paper, is an example of this.

Volatility and its measurement are thus of great importance in finance and in option pricing in particular. In option pricing, two methods are now commonly used to obtain a good ex-ante measure of the volatility of returns. The first method is known as the direct method. On the basis of market returns a statistic such as the standard deviation is calculated and is then used as a forecast for the future volatility. The indirect method consists in using the prices of the contingent assets to extract a measure of the volatility. Latane and Rendleman (1976) were the first to propose this approach based on the Black and Scholes option pricing formula. The idea is to solve the pricing formula for the implied volatility given by the market price of the contingent asset.

In this paper, we examine a series of daily returns on the Swiss Market Index (SMI), which is a continuously computed stock market index that is market value weighted and contains the 23 stocks with the largest market values, and a series of call and put options on this index which were traded on the SOFFEX (Swiss Options and Financial Futures Exchange). The Swiss financial markets differ in various ways from the US markets on which most of the studies in applied finance are conducted. Aside from the obvious difference in size, the Swiss stock market is highly concentrated and its degree of trading activity is severely segmented by firm size as well as by ownership structure.[2] The Swiss stock market is also relatively young, for example, the SMI was not computed prior to July 1988. The Swiss options market is also very recent. The SOFFEX started trading American call and put options on 11 individual blue chip stocks on May 19, 1988. Contrary to the US experience, options on the index [which are considered as more complex instruments than index futures – especially if they are American] were launched first on December 7, 1988, followed by SMI futures on November 9, 1990. A thin initial trading activity and irrational early exercise policies led the SOFFEX to convert the options on the SMI to the European variety during the second half of 1990. We show that a GARCH(1,1) model is a good representation for the daily return series. In regard to results obtained on other markets, notably the US stock market, our estimations of the volatility by the direct method reflect a low persistence of shocks on the variance of returns on the SMI. We also apply the indirect method, in the framework of the GARCH option pricing model of Duan (1991), to options written on the SMI. The implied volatility model that results can then be used to test the validity of a GARCH specification and to evaluate European options. Duan’s GARCH option pricing formula does not allow for an analytical solution; therefore, we use simulated option prices to obtain an estimate of the parameters of the implied volatility process. The implied GARCH model is the one whose parameters minimize a loss function that measures the distance between the simulated prices and the observed market prices. The implied volatility process we obtain suggests a persistence in variance similar to that obtained by the direct method. Finally, comparing the out-of-sample pricing errors of the implied GARCH model to those of the Black and Scholes model, it appears that the former is superior in evaluating options on the SMI. However, the large sample standard deviations of the out-of-sample pricing errors suggest that this result should be taken with some caution.

The study we conduct in this paper is similar to that by Engle and Mustafa (1992) on the S&P500 index. Their results show that a GARCH specification is not superior in general to that of the Black and Scholes model. However, they advocate that for long-lived options their model provides better pricing. Concerning the Swiss market, Chesney et al. (1993), (1994) show that the Black and Scholes,

[1] See the excellent reviews by Bollerslev et al. (1992); Pagan (1993) for further discussion.

[2] Bruand and Gibson (1995) report that in July 1995, for example, the 10 largest companies accounted for more than 60% of its total capitalization. They also stress the fact that for large, genuinely public, companies, bid and ask spreads represent less than 0.5% of the stock’s bid price while they reach 3 to 4% of the bid price in small corporations, and that it is not uncommon, for some small firms’ stocks, to observe only one transaction per week.

CEV and stochastic volatility models are inadequate for a satisfactory pricing of European options on the SMI. Adjaoute (1993) obtains similar results for options on individual Swiss stocks.

The rest of this paper begins in Section 2 with a brief presentation of ARCH models. We define the concept of persistence in variance and consider the estimation of these models. We present the GARCH option pricing model of Duan in Section 3. Section 4 contains the empirical application of this model to options on the SMI and Section 5 some concluding comments.

2. ARCH models

We suppose that our dependent variable y_t is generated by

$$y_t = x_t' \gamma + \varepsilon_t \quad t = 1, \ldots, T, \tag{1}$$

where x_t is a $k \times 1$ vector of lagged endogenous variables and exogenous variables, and γ is an $k \times 1$ vector of parameters. The ARCH model characterizes the distribution of the stochastic error term ε_t conditionally on a set of lagged variables $\Psi_{t-1} = \{y_{t-1}, x_{t-1}, y_{t-2}, x_{t-2}, \ldots\}$. In his original model, Engle (1982) assumes a conditional normal distribution of the error term,

$$\varepsilon_t | \Psi_{t-1} \sim \mathcal{N}(0, h_t), \tag{2}$$

where

$$h_t = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-2}^2 + \cdots + \alpha_q \varepsilon_{t-q}^2, \tag{3}$$

with $\alpha_0 > 0$ and $\alpha_j \geq 0$ for $j = 1, \ldots, q$, in order to ensure a positive conditional variance. Here, q is the order of the ARCH process.

The appeal of model (1)–(3) is that the conditional variance h_t depends on the past Ψ_{t-1} and is a positive function of the size of past errors in absolute terms. Thus a large positive or negative error tends to be followed by a large (in absolute terms) error, and similarly a small error tends to be followed by a small error. The order q determines the length of the period during which a disturbance persists in conditioning the variance of the following disturbances. The larger is q, the longer the periods of volatility clustering. Another important property of ARCH processes is that the mixing produced by the changing conditional variance induces additional kurtosis in the unconditional distribution. In fact, the parameterization does not impose a priori the existence of unconditional moments, which allows the model to be consistent with Mandelbrot (1963) who found evidence that the distribution of financial asset returns may well have infinite variance.

It rapidly became apparent in applied work that the specification of the conditional variance as an ARCH(q) called for a large number of lags and therefore the estimation of numerous parameters subject to inequality constraints. In consequence, Bollerslev (1986) proposed a generalization of the ARCH model – which he termed GARCH – that allows for a parsimonious representation of a high-order ARCH model. The conditional variance function of a GARCH(p,q) model has the following form

$$\begin{aligned} h_t = {} & \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \cdots + \alpha_q \varepsilon_{t-q}^2 \\ & + \beta_1 h_{t-1} + \cdots + \beta_p h_{t-p} \end{aligned} \tag{4}$$

with

$$\begin{aligned} & \alpha_0 > 0 \\ & \alpha_j \geq 0 \text{ for } j = 1, \cdots, q \\ & \beta_k \geq 0 \text{ for } k = 1, \cdots, p, \end{aligned} \tag{5}$$

where the constraints (5) ensure a positive conditional variance.[3]

Engle and Bollerslev (1986) introduced the integrated GARCH process (IGARCH). This model is a GARCH process in which $\Sigma_{j=1}^{q} \alpha_j + \Sigma_{k=1}^{p} \beta_k = 1$. In this case, a contemporaneous shock persists indefinitely in future conditional variances. For agents in the options market, the degree of persistence of the shocks on the variance is an essential element.[4] In effect, they will be prepared to pay a higher price for long-lived options if they perceive that the shocks are large and sufficiently permanent relative to the life of the options.

[3] Nelson and Cao (1992) show that Bollerslev's original constraints (5) can be violated without, however, the conditional variance function ever being negative. They proposed less restrictive constraints that do not impose the positivity of each parameter of (4).

[4] Engle and Mustafa (1992) show that in the case of a GARCH(1,1) the degree of persistence depends essentially on the value of $\alpha_1 + \beta_1$.

The finding of a very high degree of persistence for financial data is not universal. Although Engle and Mustafa (1992) obtain a very high degree of persistence for several individual stocks of large firms on the US stock market and for the S&P500 index, the results of Engle and Gonzalez–Riviera (1991) suggest that the degree of persistence depends on the size of the firm, smaller firms exhibiting a lower degree of persistence than larger ones. Furthermore, according to Lamoureux and Lastrapes (1990), the high degree of persistence that is observed could be due to a misspecification of the conditional variance. They suggest that structural changes in the unconditional variance of the process produce volatility clusters that result in a high degree of persistence. Allowing the constant in the conditional variance function to vary on different subperiods of their sample, they obtain a lower degree of persistence as compared to that of a model without structural changes.

Estimation and testing of ARCH-type models can be performed using standard maximum likelihood methods. Assuming conditional normality as above in (2), the single period log likelihood function – leaving out the constant terms – is given by

$$l_t(\delta) = -\frac{1}{2}\log h_t - \frac{1}{2}\frac{\varepsilon_t^2}{h_t}, \tag{6}$$

where $\delta = (\alpha_0, \alpha_1, \ldots, \alpha_q, \beta_1, \ldots, \beta_p)'$ is the vector of unknown parameters. In the GARCH(p,q) model the first max (p, q) observations are used as starting values. Therefore the log likelihood function we maximize is (6) summed over the $T - \max(p, q)$ remaining observations. The excluded observations enter indirectly through h_t.

We use the BHHH algorithm to compute our estimates. Let $\delta^{(i)}$ be the parameter estimates given by the i^{th} iteration, then

$$\delta^{(i+1)} = \delta^{(i)} + \lambda_i \left(\sum_t \frac{\partial l_t}{\partial \delta}\frac{\partial l_t}{\partial \delta'} \right)^{-1} \sum_t \frac{\partial l_t}{\partial \delta}, \tag{7}$$

where $\partial l_t / \partial \delta$ is evaluated at $\delta^{(i)}$ and λ_i is a variable step-length.[5] The last iteration of BHHH yields a consistent estimate of the asymptotic variance–covariance matrix, given by $T^{-1}[\Sigma_t\ (\partial l_t/\partial \delta)(\partial l_t/\partial \delta')]^{-1}$.

[5] McCurdy and Morgan (1988) give the expressions of the partial derivatives of a more general model of type GARCH-M.

3. The GARCH option pricing model

To apply the indirect method to get a measure of the volatility of returns we need first to define an option pricing model. This model will provide us with a theoretical option pricing formula; that is, a functional relationship between the price of a contingent asset and the underlying asset's volatility. We can then solve this function for the implied volatility resulting from the market price of the option. The theoretical option pricing formula we use in this paper is that which follows from the GARCH(p,q) option pricing model proposed by Duan (1991).

Let X_t, in a discrete time economy, be the price of the financial asset at time t. Its return rate is assumed to be conditionally lognormally distributed, i.e.,

$$\log \frac{X_t}{X_{t-1}} = \mu_t + \varepsilon_t, \tag{8}$$

where ε_t has mean zero and conditional variance h_t. We assume that ε_t is a GARCH(p,q) process, i.e.,

$$\varepsilon_t | \Psi_{t-1} \sim \mathcal{N}(0, h_t) \tag{9}$$

with

$$h_t = \alpha_0 + \sum_{j=1}^{q} \alpha_j \varepsilon_{t-j}^2 + \sum_{k=1}^{p} \beta_k h_{t-k}. \tag{10}$$

Now suppose one would like to determine the price C of a call option written on X. Duan (1991) shows that if the representative agent of this economy maximizes expected utility and that his utility function is time separable, then under any of the following three conditions, the ''risk-neutral pricing'' approach [see below] is valid:

1. the utility function exhibits constant relative risk aversion and the variations of the logarithm of aggregate consumption follow a GARCH(p,q) process;
2. the utility function exhibits constant absolute risk aversion and the variations of aggregate consumption follow a GARCH(p,q) process;
3. the utility function is linear.

When any of (i)–(iii) hold, we can get an option pricing formula that does not depend on individual preferences.[6]

The risk-neutral pricing approach, which is central to modern option pricing theory, is applied in the context of two very different classes of models. The first assumes continuous time transactions and imposes no other constraint on the agent's preferences other than non-satiation. The second class of models, of which Duan's is part, assumes discrete time transactions and imposes stronger restrictions on the agent's preferences. In continuous time models, risk-neutral pricing relationships follow directly from the assumptions of non-satiation and the absence of arbitrage. In effect, when transactions take place continuously, the dynamics of the price of the underlying asset can be described by an Itô process, and the no-arbitrage condition implies a partial differential equation whose solution does not depend on the agent's preferences. Cox and Ross (1976) show that, when it is possible to construct a portfolio composed of the contingent asset and the underlying asset in such proportions that its instantaneous return is non-stochastic, then the pricing relationship that results is risk-neutral.

In discrete time models, it is, in general, not possible to construct such a portfolio. That is why stronger restrictions than non-satiation are required to obtain a risk-neutral pricing relationship. Rubinstein (1976); Brennan (1979) establish different sets of conditions on preferences and distributions of the underlying asset under which the risk-neutral pricing principal is valid. Conditions (i)–(iii) above are a generalization of their results to the case where the returns follow a GARCH process.

Under certain conditions, then, the risk-neutral pricing principal is valid. Duan shows that, in a risk-neutral world, the model (8)–(10) above becomes

$$\log \frac{X_t}{X_{t-1}} = r_c - \frac{1}{2} h_t^* + \xi_t \tag{11}$$

with

$$\xi_t | \Psi_{t-1} \sim \mathcal{N}(0, h_t^*). \tag{12}$$

$$h_t^* = \alpha_0 + \sum_{j=1}^{q} \alpha_j \xi_{t-j}^2 + \sum_{k=1}^{p} \beta_k h_{t-k}^*, \tag{13}$$

where r_c is the instantaneous risk-free rate. The direct consequence of a passage to a risk-neutral world is that the return rate of the financial asset no longer depends on μ_t.[7] From (11) we have that the price of the underlying asset at maturity T is

$$X_T = X_t \exp\left[(T-t) r_c - \frac{1}{2} \sum_{s=t+1}^{T} h_s^* + \sum_{s=t+1}^{T} \xi_s \right]. \tag{14}$$

It follows that the price of a European call, at time t, under a GARCH(p,q) specification is

$$C_t^G = e^{-(T-t) r_c} E^*[\max(X_T - K, 0) | \Psi_t], \tag{15}$$

where K is the strike-price, T is the time of maturity, and E^* the expectation in the risk neutral world.[8]

Duan's specification of the instantaneous return rate as conditionally lognormal is very important since it allows for the Black and Scholes model to be a particular case of the GARCH(p,q) option pricing model. To see this, set p and q equal to zero; the model then is simply the Black and Scholes model, with constant mean $\mu_t = \mu$ and constant variance $h_t = \alpha_0$. Another interesting property of this model is that the instantaneous expected rate of return of the underlying asset behaves itself as a GARCH(p,q)-M process $r_c - (1/2) h_t^*$, which is negatively correlated with the conditional variance h_t^*.

Finally, there exists no analytical solution for (15); that is why in Section 4.3 we appeal to simulated option prices to estimate the parameters of the implied volatility process.

[6] Notice that expression (9) is not necessarily independent of individual preferences because the expected return μ_t, which depends on the agent's degree of risk aversion, is present.

[7] Notice that the GARCH(p,q) parameters are invariant to the change of probability measure.

[8] The price of a European put, with otherwise the same characteristics as the call, can be derived through the put-call parity relationship.

4. Estimation of the implied volatility parameters

This section is concerned with the estimation of the parameters of the implied volatility process of the SMI. First, we discuss our data set in Section 4.1. Then we show, in Section 4.2, that a GARCH(1,1) model is a good representation of the SMI daily returns. Finally, on the basis of this specification, in Section 4.3 we estimate the volatility process of the returns on the SMI, implied by simulated option prices.

4.1. The data

Our data set covers a four-year period, running from January 1, 1992, through December 31, 1996. Over the entire period we have the daily closing values of the SMI which we use in our estimation of the volatility process by the direct method. The indirect estimation using option prices will be done on a two year sub-period of the data set, running from September 30, 1992, through September 29, 1994. Over this sub-period we have the following additional series: the daily closing prices of all options on the SMI, their respective maturities and strike prices, the daily trading volumes of each option and the Euro–Swiss–Franc interest rates for all relevant maturities.

4.2. Estimation of the volatility process

A prerequisite to applying the GARCH option pricing model is that the underlying asset's returns behave like a GARCH process. We therefore test for the presence of ARCH in the daily returns on the SMI. A simple way to test for ARCH, proposed by Bollerslev (1986), is to use the standard Box–Jenkins method, usually applied for the identification of the order of AR or ARMA processes, but here applied to the squared series or squared residuals rather than to the series itself. For an ARCH(q) model, the partial autocorrelation function ϕ_{kk} is of the form

$$\phi_{kk} \neq 0,\ k \leq q$$

$$\phi_{kk} = 0,\ k > q. \tag{16}$$

This is identical to the partial autocorrelation function of an AR(q) process for ε_t^2. Furthermore, a GARCH(p,q) process is an ARMA[max(p,q),p] for ε_t^2. It follows that the partial autocorrelation function of a GARCH process, as for an ARMA process, is in general non-zero but progressively decreasing.[9]

Let X_t be the closing value of the SMI on day t. We are interested in the daily return rate, $r_t = \log(X_t / X_{t-1})$, for which we have 1304 observations.[10] Fig. 1 summarizes the evolution of the daily returns on the SMI from January 1, 1992, to December 31, 1996. Notice that the mean of the series appears to be constant whereas the variance clearly changes over time. Also, some volatility clustering is present in the data.

As suggested above we first estimate an OLS model, i.e. we assume, as in the Black and Scholes model, that the volatility is time invariant and regress the daily return on a constant. Tables 1 and 2 present the results of these estimations on the historical data for the entire period covered by our data set and the sub-period respectively. In these tables, for each model, we give the estimates of the relevant parameters with the corresponding t-statistics in parentheses and the values of the maximized log likelihood denoted by log l.

The OLS estimates of the constant, which are of minor interest to us, are fairly similar in both samples with the exception that in the full period the parameter estimate is significantly different from zero at the 5% level, whereas it is not in the sub-period. In Tables 3 and 4 we present, for each of the models we estimate, the partial autocorrelations of the squared standardized residuals up to the fifth lag and the skewness and kurtosis coefficients of the standardized residuals $\hat{\eta}_t = \hat{\varepsilon}_t / \hat{h}_t^{1/2}$.

In both samples, for the OLS specification, only the partial autocorrelations at the two first lags are larger than twice their asymptotic standard error. The skewness coefficients are significantly non-zero, thus the unconditional empirical distributions are

[9] The autocorrelation function of GARCH processes is also related to the autocorrelation function of ARMA models. The results with the autocorrelation function were essentially the same and are not reported.

[10] The sub-period, from September 30, 1992, to September 29, 1994, contains 522 observations.

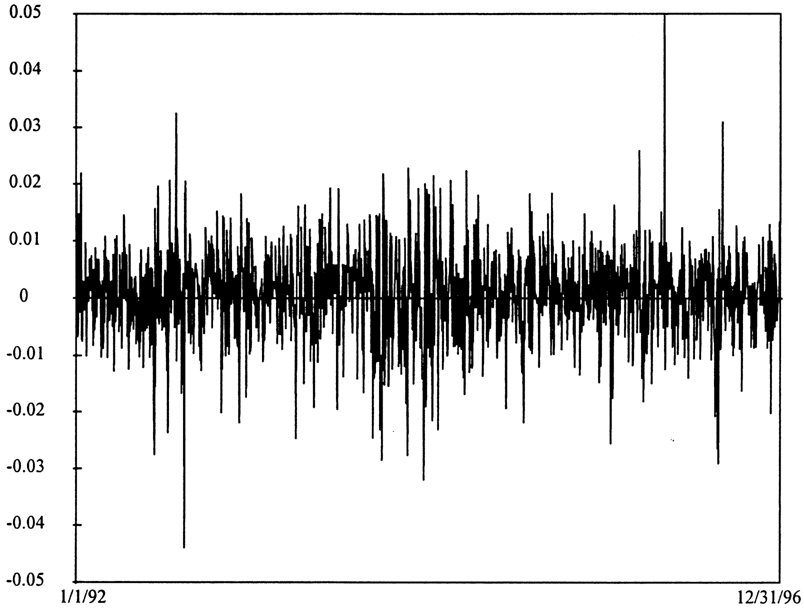

Fig. 1. Evolution of the daily returns of the SMI.

Table 1
Estimation results for the period from 1/1/92 to 12/31/96

Model	Coefficients					log l
	constant	α_0	α_1	α_2	β_1	
OLS	0.000659 (2.95)					4437.3
ARCH(1)	0.000744 (3.28)	0.000055 (25.91)	0.166698 (5.23)			4451.1
ARCH(2)	0.000876 (3.89)	0.000050 (22.98)	0.128877 (4.40)	0.107255 (3.32)		4461.1
GARCH(1,1)	0.000804 (3.52)	0.000007 (4.15)	0.091542 (4.79)		0.800903 (21.14)	4471.4

Table 2
Estimation results for the period from 9/30/92 to 9/29/94

Model	Coefficients					log l
	constant	α_0	α_1	α_2	β_1	
OLS	0.000590 (1.51)					1723.8
ARCH(1)	0.000912 (2.32)	0.000060 (12.57)	0.262118 (5.23)			1729.8
ARCH(2)	0.000966 (2.42)	0.000053 (11.52)	0.213308 (3.82)	0.122377 (1.89)		1738.7
GARCH(1,1)	0.000970 (2.49)	0.000007 (3.10)	0.103326 (3.72)		0.796601 (17.88)	1747.6

Table 3
Partial autocorrelations and skewness and kurtosis coefficients of the residuals for the period from 1/1/92 to 12/31/96

Model	Lags					Skewness	Kurtosis
	1	2	3	4	5		
OLS	0.091	0.110	0.032	0.044	0.014	−0.21	5.62
ARCH(1)	−0.25	0.093	0.029	0.045	0.026	−0.16	5.49
ARCH(2)	−0.015	−0.005	0.016	0.040	0.026	−0.16	5.69
GARCH(1,1)	−0.002	0.004	−0.018	0.004	−0.006	−0.19	5.69

The asymptotic standard errors of the partial autocorrelations and the skewness and kurtosis coefficients are respectively 0.028, 0.068 and 0.136.

Table 4
Partial autocorrelations and skewness and kurtosis coefficients of the residuals for the period from 9/30/92 to 9/29/94

Model	Lags					Skewness	Kurtosis
	1	2	3	4	5		
OLS	0.144	0.164	0.016	0.081	0.002	−0.55	4.45
ARCH(1)	−0.038	0.128	0.039	0.072	0.032	−0.50	4.02
ARCH(2)	−0.021	−0.007	0.004	0.059	0.017	−0.59	4.31
GARCH(1,1)	0.004	0.025	−0.009	0.016	−0.019	−0.50	3.53

For the sub-period, the asymptotic standard errors of the partial autocorrelations and the skewness and kurtosis coefficients are respectively 0.044, 0.107 and 0.214.

asymmetric. The kurtosis coefficients are clearly larger than 3, which indicates a fat-tailed empirical distribution of the returns over both the entire period under consideration and the sub-period.[11] These properties suggest the presence of ARCH in the data. The fact that only the two first partial autocorrelations are significantly different from zero suggests that an ARCH(2) model is an adequate representation of the errors. The estimation of both an ARCH(1) and ARCH(2) specification confirms these indications.

In the case of an ARCH(1) model, the estimated ARCH parameter, α_1, is highly significant in both samples. It is somewhat larger for the sub-period than for the entire sample, whereas the constant in the conditional variance function, α_0, is similar in both. The statistics on the squared standardized residuals, $\hat{\eta}_t^2 = \hat{\varepsilon}_t^2/\hat{h}_t$, given in Tables 3 and 4, indicate that, in both samples, the partial autocorrelations at the first lag are no longer significant at any reasonable level. However, both tables show that the partial autocorrelations corresponding to the second lag remain larger than twice their asymptotic standard errors. Both the skewness and kurtosis coefficients are lower in magnitude as compared to those of the OLS residuals but they still betray a non-normal unconditional distribution of the residuals.[12] This suggests that an ARCH(1) specification does not entirely capture the nonlinearities present in the data.

We now turn to the estimates of the ARCH(2) specification. In both the samples covering the entire period and the sub-period, the estimates of α_0 and α_1

[11] A Jarque–Bera normality test, based on the skewness and kurtosis coefficients, rejects, at any reasonable level, the null hypothesis of normally distributed residuals in both samples. See Jarque and Bera (1980).

[12] Again a Jarque–Bera normality test leads to the rejection of the null at any reasonable level.

have varied only slightly from those obtained in the ARCH(1) model. The results indicate that these parameters remain significantly different from zero in each sample. The estimates of the second ARCH parameter, α_2, are quite similar in size in both samples, that of the entire period being significantly different from zero at the 5% level, whereas that corresponding to the sub-period is only significant at a slightly higher level. For both periods under consideration, a likelihood ratio test rejects comfortably the null hypothesis of a model with homoskedastic, independent and normally distributed errors against the ARCH(2) alternative. Also the same test rejects easily, in both samples, the null of an ARCH(1) model against an ARCH(2) alternative. The statistics on the squared standardized residuals indicate that no partial autocorrelations are larger than twice their asymptotic standard error in either sample. Surprisingly, for both periods under consideration the skewness and kurtosis coefficients have either remained unchanged or increased in magnitude, although not significantly. However we argue that this specification captures satisfactorily the presence of ARCH in the daily returns on the SMI.[13]

The GARCH(1,1) model we estimate does even better. Tables 1 and 2 indicate that all the parameters of this model are highly significant. The constants in the mean equations of both periods remain similar to those estimated in the previous specifications. The constants in the conditional variance equations are much smaller than in the OLS or ARCH specifications, and are equal in both samples. Finally, both the ARCH parameter, α_1, and the GARCH parameter, β_1, are similar across samples. In consequence, the degrees of persistence in each period under consideration are very similar and indicate a fair amount of persistence of volatility shocks on the daily returns of the SMI. However, the persistence remains lower than that observed on the US stock market.[14] The statistics on the standardized residuals resulting from the estimated GARCH(1,1) model confirm that it is a good representation of the volatility process of the daily returns on the SMI. None of the partial autocorrelations are significantly different from zero at any reasonable level, in either sample. The skewness and kurtosis coefficients still point to a non-normal unconditional distribution of the residuals on both the entire period and the sub-period and again this is confirmed by a Jarque–Bera normality test. However, for the sub-period, the kurtosis is now significantly reduced in comparison to the three alternative models we estimate and, taking into account the value of the corresponding asymptotic standard error, is relatively close to 3, the kurtosis coefficient of a normal distribution. This suggests that a GARCH(1,1) specification captures the nonlinearities in the data better than the ARCH(2) model. Finally, Tables 1 and 2 indicate that the values of the maximized GARCH(1,1) log likelihoods are substantially greater than those of the ARCH(2) log likelihoods, therefore, since both models demand the estimation of the same number of parameters, any model selection criteria would select the GARCH(1,1) specification over the ARCH(2) specification. Thus, on the basis of this representation of the daily returns on the SMI, we will in Section 4.3 use Duan's GARCH option pricing formula to estimate the parameters of the implied volatility process.

To conclude this section on the representation of the historical data, it is interesting to compare the evolution of the estimated conditional variances of the GARCH(1,1) model in Fig. 2 to that of the daily returns presented in Fig. 1.

4.3. Estimation of the parameters of the implied volatility process

We established in Section 4.2 that a GARCH(1,1) model is a good representation of the daily returns on the SMI. Thus, following Duan (1991) we can apply a GARCH(1,1) pricing model of the options on the

[13] It is well documented in the applied finance literature that ARCH-type specifications assuming conditionally normally distributed errors often do not capture entirely the excess kurtosis in the data. In consequence, several authors have suggested that the assumption of t-distributed errors together with a GARCH model should be preferred. See, for example, Baillie and DeGennaro (1990).

[14] Engle and Mustafa (1992) obtain, for the S&P500 index, $\alpha_1 + \beta_1 = 0.998$.

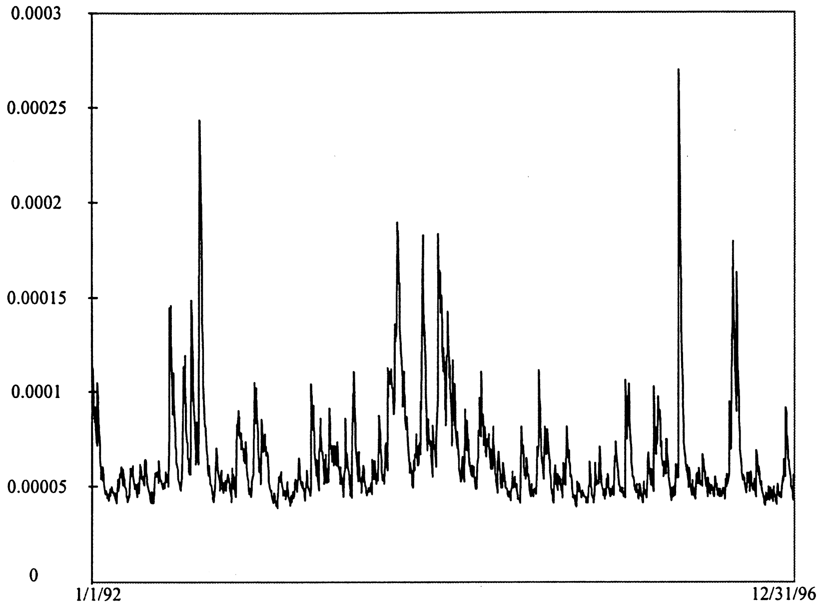

Fig. 2. Evolution of the conditional variance of the daily returns on the SMI.

SMI. This ''risk-neutral'' model is given by equations (11–15) above. Now let Q_t be the observed market price of an option at time t and let $Q_t^G(X_T, K, \theta)$, where $\theta = (\alpha_0, \alpha_1, \beta_1)' \in \Theta \subseteq \mathbb{R}^3$, be the theoretical price for this same option, i.e. (15) for a call option. Consider the nonlinear least squares estimator $\hat{\theta}_T$ that minimizes the objective function

$$L_T(Q, X, \theta) = \frac{1}{T} \sum_{t=1}^{T} \{Q_t - Q_t^G(X_T, K, \theta)\}^2. \tag{17}$$

Unfortunately, in our model there exists no analytical solution for $Q_t^G(X_T, K, \theta)$, so that (17) cannot be computed. Therefore, we are going to approximate the theoretical options prices numerically using Monte Carlo sampling. This type of estimation procedure using simulation methods has been widely discussed in the econometrics literature [see for example, McFadden (1989); Pakes and Pollard (1989); Gourieroux and Monfort (1991), (1993); Laroque and Salanié (1989); Laffont et al. (1995)]. Since $Q_t^G(X_T, K, \theta)$ is not readily available, we replace it with an unbiased simulator $\hat{Q}_t^G(X_{T,n}, K, \theta)$, i.e., a simulator for which $E[\hat{Q}_t^G(X_{T,n}, K, \theta)] = Q_t^G(X_T, K, \theta)$. We construct the simulators for the GARCH(1,1) option prices as follows: using the current Euro–Swiss–Franc interest rate for r_t and the estimate of h_t from the historical fit of the GARCH(1,1) model, we simulate the value of the underlying asset at maturity for each option under consideration using Eqs. (11)–(13), by taking random draws of the variable ξ_t.[15] This procedure is repeated n times to obtain the simulated theoretical price of the option using

$$\hat{C}_t^G(X_{T,n}, K, \theta) = \frac{1}{n} e^{-(T-t)r_t} \sum_{i=1}^{n} \max(X_{T,i} - K, 0) \tag{18}$$

for the calls, and

$$\hat{P}_t^G(X_{T,n}, K, \theta) = \frac{1}{n} e^{-(T-t)r_t} \sum_{i=1}^{n} \max(K - X_{T,i}, 0) \tag{19}$$

for the puts, where n is the number of simulations, chosen very large. Clearly, as n goes to infinity the simulators $\hat{C}_t^G(X_{T,n}, K, \theta)$ and $\hat{P}_t^G(X_{T,n}, K, \theta)$

[15] Invoking the theoretical invariance of the parameters of the GARCH(1,1) process when applying the risk-neutral valuation principal, we assume that the series of estimated conditional variances $\hat{h}_t$ provide us with good starting values for the simulation of the underlying asset.

converge to $C_t^{\,G}$ and $P_t^{\,G}$, the theoretical prices of a call and put option respectively.

Replacing $Q_t^{\,G}(X_T, K, \theta)$ in (17) by these simulators we construct an approximate objective function

$$L_T(Q, X, \theta) = \frac{1}{T}\sum_{t=1}^{T}\{Q_t - \hat{Q}_t^{\,G}(X_{T,n}, K, \theta)\}^2. \tag{20}$$

The simulated nonlinear least squares estimator $\hat{\theta}_{nT}$ is the value of θ that minimizes (20). Note that we use different drawings $\xi_{it}\}$, $i=1,\ldots,n$, for different observations t. However, in the estimation procedure, when θ changes the same drawings ξ_{it}, $i=1,\ldots,n$, and $t=1,\ldots, T$ are used for the computation of (20). As the number of simulations n goes to infinity (20) converges to (17) with probability one. Furthermore, the simulated nonlinear least squares estimator behaves like the nonlinear least squares estimator as n approaches infinity, i.e. $\hat{\theta}_{nT}$ is consistent and asymptotically normal at rate $\sqrt{T}$. Our approach is similar to that of Laroque and Salanié (1989), who estimate Fix–Price models using a simulated nonlinear least squares estimator. Their model presents some problematic features that are similar to ours, in particular their simulator is not continuously differentiable with respect to θ – as is the case in expressions (18) and (19), since θ appears inside the ''max''. They show that, under the standard assumption that Θ is compact and continuity requirements on the objective functions (17) and (20) that are satisfied in our Monte Carlo procedure described above, $\hat{\theta}_{nT}$ is consistent and that the following convergence in distribution obtains for any sequence of approximate estimators $\hat{\theta}_{nT}$:

$$\sqrt{T}(\hat{\theta}_{nT} - \theta_0) \Rightarrow \mathcal{N}(0, J_0^{-1} I_0 J_0^{-1}) \tag{21}$$

as first $n \to \infty$ and then $T \to \infty$, and

$$I_0 = E_0\left[\frac{\partial L_1}{\partial \theta}(Q, X, \theta_0)\frac{\partial L_1}{\partial \theta'}(Q, X, \theta_0)\right]$$

$$J_0 = E_0\left[-\frac{\partial^2 L_1}{\partial \theta\, \partial \theta'}(Q, X, \theta_0)\right],$$

where L_1 is defined by (17) for $T=1$, and θ_0 denotes the true value of the parameter.

Notice that minimizing the objective function (20) with respect to θ produces an inconsistent estimator for any fixed number of simulations. However we argue that with the number of simulations we use our results are robust to the simulation error.[16]

For the estimation of the parameters of the implied volatility process we use a set of put and call European options on the SMI, of different maturities and strike prices. The value at maturity of the underlying asset is simulated $n=500$ times for each option. In order to reduce computation time, we kept only 1006 options in our sample; these are all near the money.[17]

The objective function (20) is minimized using a BHHH algorithm with numerically computed first derivatives. The estimated parameter values are given below with the corresponding estimated asymptotic t-statistics in parentheses:

$$\alpha_0 = \underset{(3.43)}{0.000009}, \quad \alpha_1 = \underset{(2.43)}{0.087068}, \quad \beta_1 = \underset{(18.67)}{0.804530}.$$

Table 5 gives the estimated asymptotic correlation matrix of the parameters. The estimated parameters

[16]Laroque and Salanié (1989) use $n=50$ and $n=100$ respectively for their Fix–Price model with and without the introduction of micro markets.
Several authors have proposed estimators that are consistent under a fixed number of simulations. Gourieroux and Monfort (1991), (1993) propose an estimator based on simulating the first-order conditions, using an additional independent set of random draws ξ^*_{it}, $i=1,\ldots,n$ to simulate the partial derivatives $\partial Q_t^{\,G}(X_T, K, \theta)/\partial\theta$. However, they assume that their simulator is twice continuously differentiable with respect to θ. Another drawback of this method is its computational burden. First, simulating the partial derivatives using a different set of random draws increases computation time. Second, the estimator is the value of θ that sets this set of simulated first-order conditions to zero. Finding the root of a set of simulated equations is not an easy task especially when the derivative of these equations has to be evaluated numerically. Laffont et al. (1995) propose a different estimator that is consistent under a fixed number of simulations. Their estimator is the value of θ that minimizes a simulated nonlinear least squares objective function adjusted by the sample variance of the simulator. However they also assume that the simulator is twice continuously differentiable and use importance sampling to achieve this required property.

[17]The observations we include in our sample are randomly drawn out of a subset of over 44 000 observations. Two observations, a put and a call, were drawn for each trading day between 9/30/92 and 9/29/94. The subset included only options actually traded on the day in question, that differed by 10% or less from par and with a period to maturity of at least 15 days. Thus we are assured of only considering observations with sufficient relevant information content.

Table 5
Estimated asymptotic correlation matrix

	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\beta}_1$
$\hat{\alpha}_0$	1	−0.007	−0.661
$\hat{\alpha}_1$	−0.007	1	−0.742
$\hat{\beta}_1$	−0.661	−0.742	1

Table 7
Sample standard deviation of the average relative valuation errors of the GARCH(1,1)

	Total	Near maturity	Mid maturity	Long maturity
Calls	0.421	0.578	0.380	0.245
Puts	0.226	0.248	0.217	0.187

Table 6
Average relative valuation errors of the GARCH(1,1) model

	Total	Near maturity	Mid maturity	Long maturity
Calls	0.134	0.072	0.155	0.154
Puts	−0.109	−0.211	−0.097	−0.036

of the implied GARCH(1,1) volatility process are very similar to those obtained by the direct estimation on historical data. Both the constant, α_0, and the GARCH parameter, β_1, are slightly larger whereas the ARCH parameter, α_1, is smaller. Also, the asymptotic standard errors we obtain are similar to those obtained above and the parameters are clearly significantly different from zero at a 5% level. This result indicates that the persistence of volatility shocks on the daily SMI returns implied by option prices is very close to that obtained from historical data on the index itself.

To get an idea of performance of the implied GARCH(1,1) option pricing model we turn now to the resulting average relative valuation errors given in Table 6.[18] In general, our model overvalues call options and underestimates the prices of put options. Overall it appears to have more difficulty with calls. Engle and Mustafa (1992) obtain a similar result. For calls the smallest biases are for the near maturity options whereas for puts longer maturity options are better priced. The sample standard deviation of the average relative valuation errors of the GARCH(1,1) model are given in Table 7.

At this stage, a further analysis of our model necessitates the consideration of another model as reference. The Black and Scholes model is a natural candidate. To measure the performance of our model, we calculated the average relative valuation errors of both our implied GARCH(1,1) model and the Black and Scholes model on a sample of 10 728 options that were not included in the estimation sample.[19] To price an option at time t, the volatility used in the pricing formula is the estimate of h_t from the historical fit of the GARCH(1,1) model. These errors and the corresponding sample standard deviations are given respectively in Tables 8 and 9 for the implied GARCH(1,1) model and Tables 10 and 11 for the Black and Scholes model.[20]

Overall our implied GARCH(1,1) model undervalues both classes of options. Contrary to the results in the estimation sample, the GARCH(1,1) model does better overall for call options in the out-of-sample test. The undervaluation of put options appears to be a more robust result than that of calls. All long-lived calls and those of mid maturity that are out of the money appear to be overvalued, whereas only short-lived in the money puts are slightly overpriced by our model.

The Black and Scholes model overprices the bulk of call options and underprices puts, whereas our implied GARCH(1,1) model undervalues both

[18] Relative valuation error = (theoretical price − market price)/market price. A referee has pointed out that use of relative valuation error may lead to a serious misrepresentation of the valuation error for deep in-the-money options. Nevertheless, this is the standard approach used by Engle and Mustafa (1992), for example.
We define near maturity options as those with a month or less to maturity, mid maturity as those with between one and three months to maturity, the remainder being considered as long maturity options.

[19] The observations included in this out-of-sample test were also chosen such that the option had a maturity of at least 15 days and was actually traded on the day in question.

[20] We define near the money options as those within 10% of par, the remainder being either in or out of the money.

Table 8
Out-of-sample average relative valuation errors of the GARCH(1,1) model

	Total	Near maturity	Mid maturity	Long maturity
Calls	−0.028	−0.058	−0.029	0.023
Out of the money	0.079	−0.442	0.268	0.068
Near the money	−0.033	−0.055	−0.037	0.017
In the money	−0.007	−0.010	−0.007	0.002
Puts	−0.231	−0.302	−0.220	−0.168
Out of the money	−0.720	−0.960	−0.710	−0.475
Near the money	−0.194	−0.245	−0.191	−0.136
In the money	−0.032	0.007	−0.033	−0.054

Table 9
Sample standard deviation of the out-of-sample average relative valuation errors of the GARCH(1,1) model

	Total	Near maturity	Mid maturity	Long maturity
Calls	0.234	0.255	0.239	0.166
Out of the money	0.669	0.921	0.964	0.264
Near the money	0.202	0.237	0.195	0.149
In the money	0.033	0.016	0.032	0.063
Puts	0.242	0.310	0.217	0.177
Out of the money	0.257	0.071	0.225	0.189
Near the money	0.194	0.247	0.175	0.139
In the money	0.062	0.021	0.059	0.072

Table 10
Out-of-sample average relative valuation errors of the Black and Scholes model

	Total	Near maturity	Mid maturity	Long maturity
Calls	0.038	−0.020	0.045	0.107
Out of the money	0.512	−0.093	1.061	0.322
Near the money	0.021	−0.020	0.024	0.082
In the money	−0.008	−0.010	−0.007	−0.005
Puts	−0.258	−0.330	−0.236	−0.223
Out of the money	−0.776	−0.979	−0.725	−0.633
Near the money	−0.219	−0.274	−0.207	−0.182
In the money	−0.018	0.008	−0.027	−0.023

Table 11
Sample standard deviation of the out-of-sample average relative valuation errors of the Black and Scholes model

	Total	Near maturity	Mid maturity	Long maturity
Calls	0.578	0.321	0.719	0.389
Out of the money	2.461	0.803	4.239	0.704
Near the money	0.340	0.318	0.354	0.321
In the money	0.033	0.015	0.030	0.066
Puts	0.306	0.352	0.286	0.279
Out of the money	0.300	0.053	0.335	0.283
Near the money	0.267	0.304	0.253	0.240
In the money	0.074	0.021	0.064	0.099

classes of options. Overall, the Black and Scholes model does better with call options than puts. Here again the undervaluation of puts seems a more robust result than the overpricing of call options. Comparing the average relative pricing errors of the two models, our results suggest that overall the GARCH(1,1) model performs better than the Black and Scholes model. The latter typically prices call options higher and puts lower than our implied GARCH(1,1) model, except for in the money options in both classes. One would expect that the valuation errors due to stochastic volatility would be more pronounced for long-lived options. Our results for call options tend to confirm this belief, although both models have more difficulty with the put variety. However, these results should be taken with caution due to the relatively large corresponding sample standard deviations, especially for out of the money calls.

5. Conclusion

We showed that the daily returns on the SMI admit a GARCH(1,1) representation. The estimated parameter values we obtain directly from the historical SMI series imply a substantially higher degree of persistence than those obtained by Grünbichler and Schwartz (1993) for the daily returns on the SMI for the period from January 1989 through October 1991. They show that the best model for these returns is a GARCH(1,1) with $\alpha_1 = 0.17352$ and $\beta_1 = 0.54805$ and, therefore, conclude in a low persistence of volatility. The degrees of persistence implied by our results from the full period from January 1, 1992, through December 31, 1996, and the sub-period are essentially identical, but are lower than that observed by Engle and Mustafa (1992) on the S&P500 index.

Using a simulation minimization method we estimated an implied GARCH(1,1) option pricing model from the prices of options written on the SMI. The estimated model indicates that the persistence of shocks on the conditional variance function of the index is reflected in the prices of options written on it. The valuation errors resulting from the estimation indicate that the model tends to overvalue call options and underprice put options in the estimation sample.

The out-of-sample test we conduct suggests that the overall performance of our implied GARCH(1,1) option pricing model is superior to that of the Black and Scholes model in evaluating European options written on the SMI. In the test sample the implied GARCH(1,1) model undervalues both classes of options whereas the Black and Scholes model typically prices call options higher and puts lower and thus, overprices call options and undervalues puts. However, the large sample standard deviations of the average relative valuation errors in the out-of-sample test suggest that these results should be taken with caution.

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Biographies: Oliver LINTON is Associate Professor in the Department of Economics at Yale University. He received his doctorate from the University of California at Berkeley in 1991. His research is about estimating models for financial time series and semiparametric estimation. He has published more than twenty articles including some in Econometrica, the Journal of Econometrics, the Annals of Statistics, Biometrika, and the Journal of the American Statistical Association.

Michael SABBATINI is doctoral candidate at Yale University.