Effects of Relativity on Quadrupole Oscillations of Compact Objects

Abhijit Gupta, North Carolina School of Science and Mathematics

Motivation

- Asteroseismology correlates observed periodic luminosity fluctuations to mass- and composition- dependent characteristics of pulsating stars
- In the age of space-based photometry, observed pulsation frequencies are accurate to one part in 10^7 [1] (See Figure 1 below)
- For full effect, equally precise theoretical models are required. For white dwarfs and neutron stars, this requires the consideration of relativity
- We seek to quantify the effects of relativity on compact object pulsations

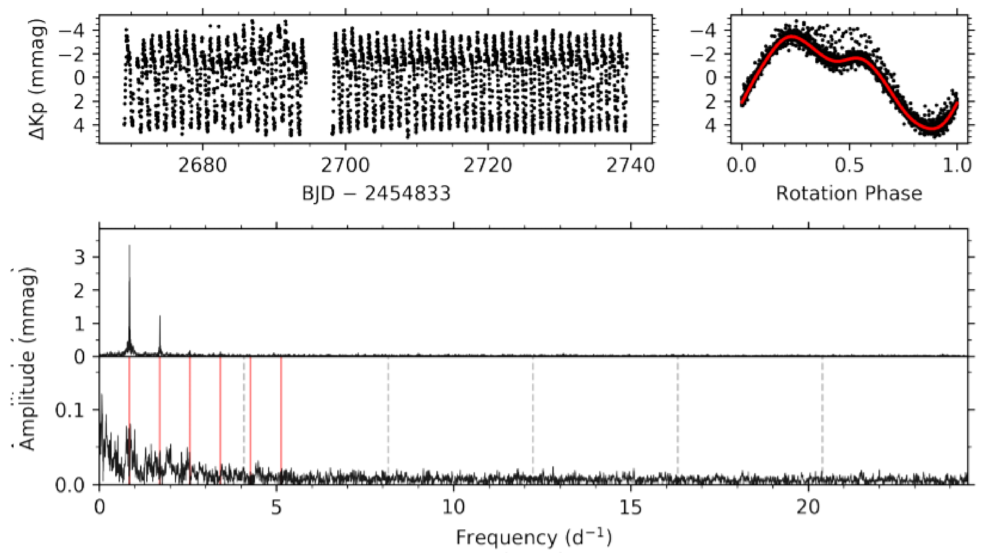


Figure 1: Experimental results from the K2 mission. The top panels show the standard and phase-folded light curves. The bottom panel shows the amplitude and residual amplitude spectrum after the pulsation frequencies are removed . Red vertical lines indicate observed pulsation frequencies [2]

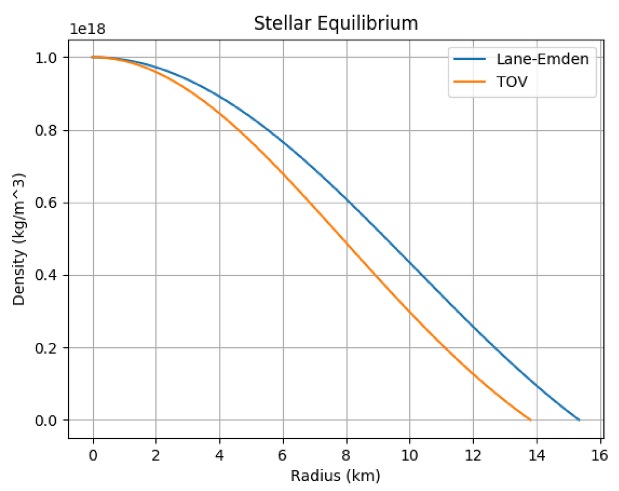
Stellar Equilibrium

- Compact objects are well modelled with the polytropic equation of state

$$P = K\rho^{(n+1)/n}$$

Equation 1: polytropic equation of state

- The Lane-Emden equation and Tolman-Oppenheimer-Volkoff (TOV) equation relate the equilibrium radius-dependent pressure and density in flat and curved spacetime respectively [3]
- Figure 2 compares these models for a neutron star equilibrium state state



$$\frac{dP}{dr} = -\frac{Gm}{r^2}\rho\left(1+\frac{P}{\rho c^2}\right)\left(1+\frac{4\pi r^3 P}{mc^2}\right)\left(1-\frac{2Gm}{rc^2}\right)^{-1}$$

Equation 2: TOV equation

Figure 2: Comparison of Lane-Emden and TOV predictions for n=1 polytrope corresponding to a typical neutron star. The TOV equation predicts a smaller radius.

Stellar Pulsations

- Each pulsation has a spherical harmonic degree *l*. For each *l*, there are multiple energy eigenmodes with ascending mode number *k*
- **Pressure modes (p-modes)** are high frequency modes whose deviations from equilibrium are counteracted by pressure changes in the convective zone
- **Gravity modes (g-modes)** are low frequency modes, counteracted by mass movement in the radiative zone
- In this research, we focus on p-modes, although our methods apply to g-modes as well

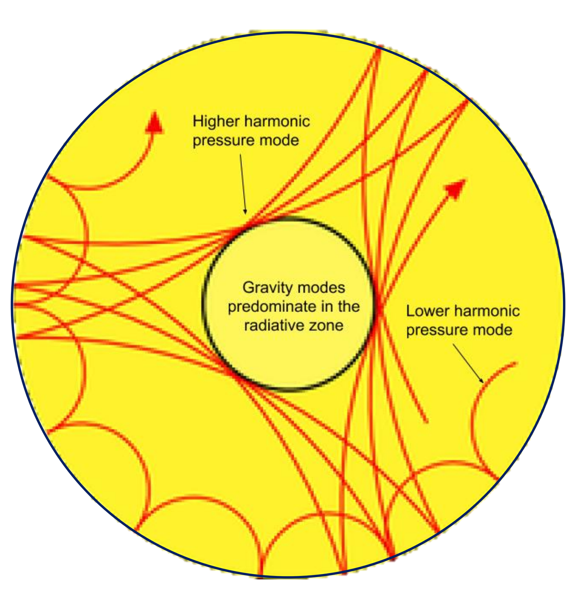


Figure 3: p-mode propagation for two distinct harmonics. The number of reflections is the degree of a component. The resonant modes result from a superposition of component waves travelling in opposite directions [4]

Newtonian Nonradial Oscillations

- We use the Lane-Emden equation to model equilibrium radius-dependent pressure and density
- Four coupled differential equations govern perturbations of radial displacement, pressure, gravitational potential, and gravitational field. In dimensionless form, they are labelled y_1 to y_4
- For given equilibrium conditions and spherical harmonic *l*, we search for eigenfrequencies where the perturbation variables are continuous throughout the stellar interior using Equation 3

$$\frac{d}{d\ln(x)}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} V_g - 3 & \frac{l(l+1)}{c_1\omega^2} - V_g & V_g & 0 \\ c_1\omega^2 - A_* & A_* - U + 1 & -A_* & 0 \\ 0 & 0 & 1 - U & 1 \\ UA_* & UV_g & l(l+1) - UV_g & -U \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

Equation 3: Matrix representation of four perturbation differential equations.

- x is a dimensionless radius, ω is frequency, *l* is spherical harmonic degree, A_*, U, V_g, and c_1 are all common dimensionless stellar variables, defined in [5]

Newtonian Model Results

- Our Newtonian results compare favorably to literature values. Example results for n=3 polytrope (l=2 pressure modes) shown in Table 1

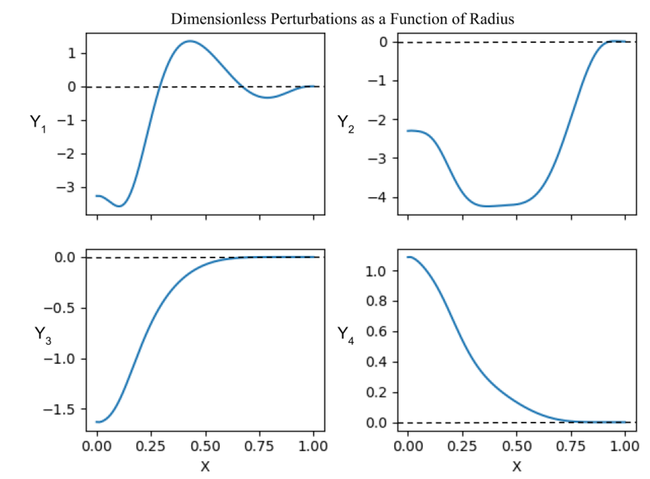


Figure 4: Newtonian perturbations of n=3 polytrope (l=2 2nd harmonic pressure mode)

Harmonic	Literature	Calculated	Rel Error
Fundamental	3.90687	3.90687	$1.2491 * 10^{-7}$
1st Harmonic	5.169468	5.169469	$7.6588 * 10^{-8}$
2nd Harmonic	6.439991	6.439990	$4.5185 * 10^{-8}$
3rd Harmonic	7.708951	7.708951	$1.8080 * 10^{-10}$
4th Harmonic	8.975891	8.975891	$3.1879 * 10^{-8}$

Table 1: Comparison of dimensionless eigenfrequencies for low harmonic Newtonian pressure-modes to JCD-DJM paper [6]

- For slightly relativistic stars, this model acts as a point of comparison to relativistic results. More relativistic stars (neutron stars) deviate further in behavior and eigenfrequencies

Relativistic Nonradial Oscillations

- We use the TOV equation to model the equilibrium state radius-dependent pressure and density
- Pulsations affect the surrounding spacetime of the star, perturbing the Schwarzschild metric

$$ds^2 = \begin{pmatrix} -e^{\nu}(1+\mu^l H_0 Y) & -i\omega(\mu^l H_1 Y) & 0 & 0 \\ -i\omega(\mu^l H_1 Y) & e^{\lambda}(1-\mu^l H_0 Y) & 0 & 0 \\ 0 & 0 & r^2(1-\mu^l KY) & 0 \\ 0 & 0 & 0 & r^2\sin^2(\theta)(1-\mu^l KY) \end{pmatrix}$$

Equation 4: Even-parity Regge-Wheeler gauge perturbation metric to the Schwarzschild spacetime. H_0, H_1, K are perturbation variables, Y represents time and angle dependencies.

- Four coupled first-order differential equations govern metric perturbations (H_0, K) and fluid perturbations (W, X), derived from linear perturbations to the Schwarzschild metric and equation for hydrostatic equilibrium [7]
- Once again, we search for acceptable solutions (continuous) in the stellar interior. Unlike the Newtonian model, for a given spherical harmonic and frequency, one solution always exists
- Each frequency produces gravitational radiation, calculated using the Zerilli equation [6]
- At spatial infinity, the Zerilli function is decomposed into ingoing and outgoing components Z_+ and Z_-
- We search for **quasinormal modes,** frequencies with no ingoing radiation, representing resonant modes

$$\left[\frac{d^2}{dr_*^2} + \omega^2 - V_Z(r)\right]Z = 0$$

Equation 5: Zerilli equation. V_Z is the effective potential, Z is the Zerilli function

Relativistic Model Results

- To compare the relativistic model against the Newtonian, perturbations for the n=3 polytrope (l=2 pressure mode) are computed (Shown in Figure 5)
- Strong similarities between the shapes of the Newtonian and relativistic perturbation functions inside the star indicate agreement between the Newtonian and relativistic models

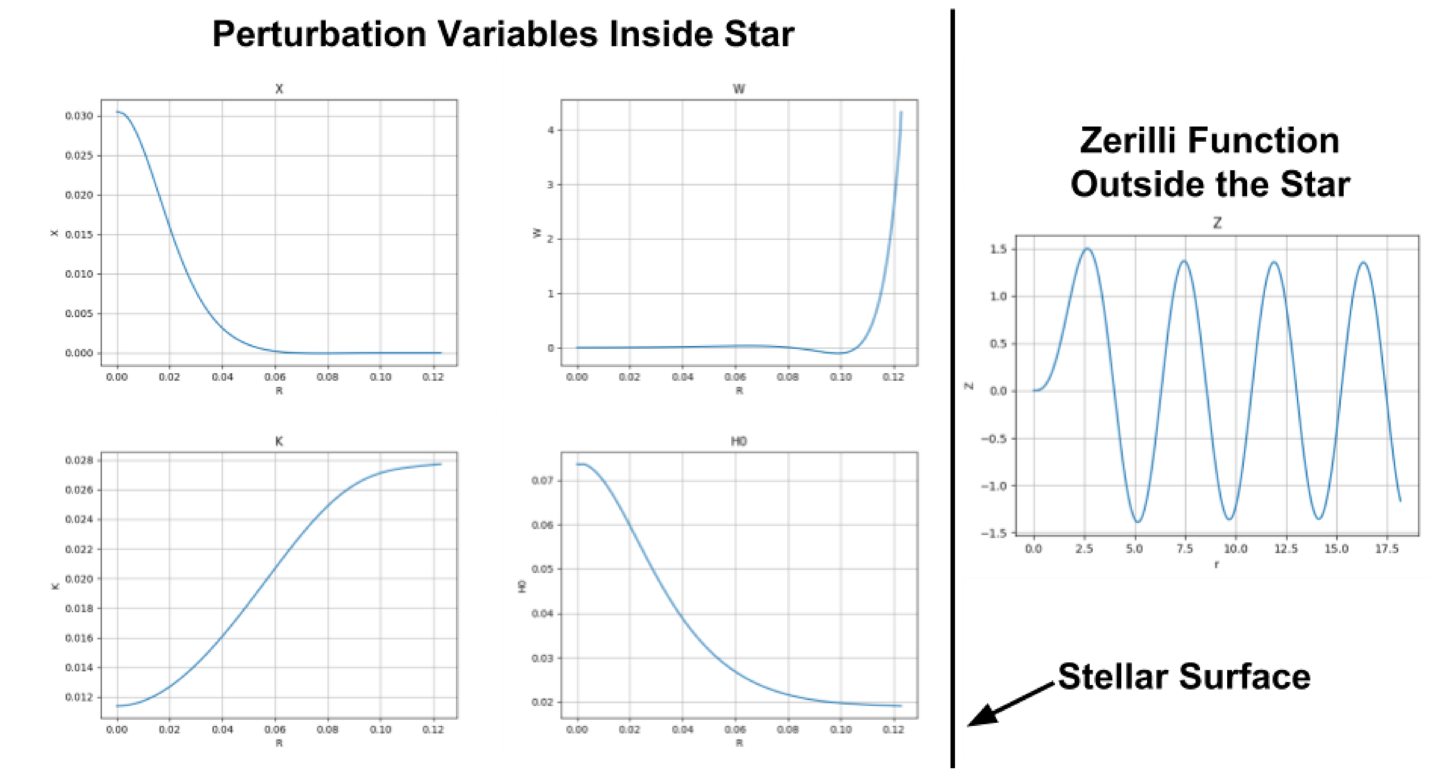


Figure 5: Perturbation variables calculated for a specific pulsation, corresponding Zerilli variable integrated using the Zerilli equation

Conclusions

- We find a strong correlation between calculated Newtonian pressure-mode eigenfrequencies and literature values. We implemented computational improvements to previous published algorithms
- Further testing is required to understand the relativistic nonradial oscillations, including locating the quasinormal modes
- Once the relativistic model is completed, we will compare corresponding frequencies in flat and curved spacetime to better understand the impacts of general relativity on stellar oscillations
- Future research can be conducted in improving the relativistic model accuracy, including handling the Zerilli equation integration outside the stellar surface– one alternative approach is the WKB approximation [8]
- This research can be applied to test additional families of modes, or extended to slowly rotating stars and non-polytropic equilibria states
- This research has applications in experimental relativistic asteroseismology, to make use of data such as Figure 1

References

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