Second Year Calculus

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1 F = ma

Kepler's Laws

1. A planetary orbit sweeps out equal area in equal time
2. A planetary orbit is an ellipse with the sun at one focus
3. The square of the period of the orbit is directly proportional to the cuve of the mean distance(the average of the closest and farthest distances from the physical focus)

Newton's Laws

1. Every body continues in its state of rest or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it.
2. The change of motion in proportional to the motive force impressed, and is made in the direction of the right line in which that force is impressed
3. To every action there is always an opposed an equal reaction; or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

Newton's second law is more commonly known as $F = ma$, though Newton used $F = \frac{\partial p}{\partial t}$ where p is momentum.

Newton's First Proposition Let S be a fixes point and let P be a moving particle such that the only forces acting on P at any given time lie in the direction of the line connecting S ot P at that mmoment. Then, the path followed by P will lie in a single plane, and the area swept out by the line connecting S to P will be the same for any equal length of time.

Newton's Second Proposition Let S be a fixes point and let P be a moving particle that stays in a fixed plane containing S and sweeps out equal area in equal time; then, the only forces acting on P are radial forces from S.

Newton's Eleventh Proposition If a particle P moves along an ellipse in such manner that its only acceleration is always directed along the line from P to the focus F_1, then the magnitude of that acceleration, and thus the magnitude of the accelerative force, is inversely proportional to the square of the distance between P and F_1.

2 Vector Algebra

Planes A plane containing the origin is uniquely determined by two nonparallel vectors $\vec{r}$ and $\vec{s}$. Specifically, it is the set of all linear combinations of $\vec{r}$ and $\vec{s}$.

$$P = \{a_0\vec{r} + a_1\vec{s} \;|\; a_0, a_1 \in \mathbb{R}\}$$

The plane can be parameterized, and those parameterizations reduced to one equation from which any two nonparallel vectors which satisfy that equation define the same plane as $\vec{r}$ and $\vec{s}$.

Dot Product The sum of componentwise multiplication.
For $\vec{r}$ and $\vec{s}$,

$$\vec{r} \cdot \vec{s} = |\vec{r}||\vec{s}| \cos\theta$$

If

$$\vec{r} = \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix}, \vec{s} = \begin{pmatrix} s_x \\ s_y \\ s_z \end{pmatrix}$$

$$\vec{r} \cdot \vec{s} = r_x s_x + r_y s_y + r_z s_z$$

Vector Decomposition
A unit vector, $\vec{u}$ in the direction of $\vec{v}$ is found by dividing $\vec{v}$ by it's magnitude.
Vector $\vec{r}$ can be decomposed into a piece in the direction of $\vec{u}$ and a piece perpendicular.

$$\vec{r} = \vec{r}_{\vec{u}} + \vec{r}_{\perp\vec{u}}$$

$$\vec{r}_{\vec{u}} = |\vec{r}| \cos\theta\vec{u} = (\vec{r} \cdot \vec{u})\vec{u}$$

Cross Product
Vector product, anticommutative, gives signed area of parallelogram formed by $\vec{r}$ and $\vec{s}$

$$\vec{r} \times \vec{s} = |\vec{r}||\vec{s}| \sin\theta$$

For

$$\vec{r} = \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix}, \vec{s} = \begin{pmatrix} s_x \\ s_y \\ s_z \end{pmatrix}$$

$$\vec{r} \times \vec{s} = det \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ r_x & r_y & r_z \\ s_x & s_y & s_z \end{vmatrix}$$

3 Celestial Mechanics

Orbital Mechanics

Theorem - If $\vec{r}(t)$ is position at time t, $\vec{v}(t)$ is velocity, and $\vec{a}(t)$ is acceleration, if $\vec{a}(t)$ is radial, $\vec{r} \times \vec{v} = \vec{K}$ where $\vec{K}$ is a constant vector of magnitude

$$K = |\vec{K}| = 2\frac{\partial A}{\partial t} = rv \sin \phi$$

where ϕ is the angle between $\vec{r}$ and $\vec{s}$, and $\frac{\partial A}{\partial t}$ is the rate at which area is being swept out.

The force of gravity, $\vec{F}_G$ is given by:

$$\vec{F}_G = -G\frac{M_1 M_2}{r^2}\vec{u}_r \Rightarrow \vec{a} = -\frac{GM_1}{r^2}\vec{u}_r$$

Apogee, the farthest distance from the physical focus in an orbit, is given by:

$$\frac{\gamma}{1 - |\epsilon|}$$

and Perigee, the nearest distance to the physical focus in an orbit, is given by:

$$\frac{\gamma}{1 + |\epsilon|}$$

where $\gamma = \frac{K^2}{GM}$ and ϵ is the eccentricity of the orbit. If ϵ is zero, the orbit is circular. If $0 < \epsilon < 1$, the orbit is elliptical, and if $\epsilon = 1$, the orbit is parabolic. Finally, any orbiting object with $\epsilon < 1$ is in a hyperbolic orbit.

The mean distance, a, is defined by:

$$a = \frac{\gamma}{1 - \epsilon^2}$$

Finally, eccentricity, ϵ, can be found in many ways.

$$\epsilon^2 = \frac{r^2 v^4 \sin^2 \phi}{G^2 M^2} - \frac{2rv^2 sin^2\phi}{GM} + 1$$

$$\epsilon^2 = 1 + \frac{rv^2 sin^2\phi}{G^2 M^2}\left(rv^2 - 2GM\right)$$

$$\epsilon^2 = sin^2\phi\left(1 - \frac{rv^2}{GM}\right)^2 + cos^2\phi$$