When making Math, you may mix the partials when...
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Our goal: To prove that mixed partial derivatives are equal when the partial derivatives exist within a neighborhood around a point (a, b) and the mixed partials are continuous at (a, b)

First we need to define $\Delta(h, k)$.

$$\Delta(h,k) = \frac{f(a+h,b+k) - F(a+h,b) - f(a,b+k) + f(a,b)}{hk}$$

Then prove that

$$\frac{\partial^2 f}{\partial y \partial x} = \lim_{k \to 0}(\lim_{h \to 0} \Delta(h,k))$$

and that

$$\frac{\partial^2 f}{\partial x \partial y} = \lim_{h \to 0}(\lim_{k \to 0} \Delta(h,k))$$

To start,

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(\frac{\partial}{\partial x} f(a,b)) = \frac{\partial}{\partial y}(\lim_{h \to 0} \frac{1}{h} f(a+h,b) - f(a,b)$$

Now we must take the derivative with respect to y to each of the two terms from above, giving:

$$\frac{\partial^2 f}{\partial y \partial x} = \lim_{k \to 0} \frac{1}{k}(\lim_{h \to 0} \frac{1}{h}[f(a+h,b+k) - f(a+h,b)] - [f(a,b+k) - f(a,b)]) = \lim_{k \to 0}(\lim_{h \to 0} \Delta(h,k))$$

If we swap the order in which we take the derivatives, giving $\frac{\partial^2 f}{\partial x \partial y}$, we find that the overall effect is to swap the order in which we take the limits, our desired result.

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If we define $F(x) = f(x, b+k) - f(x, b)$ so that

$$\Delta(h,k) = \frac{F(a+h) - F(a)}{hk}$$

This $\Delta(h, k)$ is equal to the one previously defined. Prove that there's a point a_1 between a and $a + k$ such that

$$\frac{F(a+h) - F(a)}{h} = F'(a_1) = \frac{\partial f}{\partial x}(a_1, b+k) - \frac{\partial f}{\partial x}(a_1, b)$$

The Mean Value Theorem states that

$$\frac{f(\vec{b}) - f(\vec{a})}{|\vec{b} - \vec{a}|} = f'(\vec{c}; \vec{u})$$

for some $\vec{c}$ between $\vec{a}$ and $\vec{b}$. We must also assume that $\vec{F}$ is a scalar field, the directional derivative at $\vec{a}$ exists in every direction, and $\vec{F}$ exists on every point between $\vec{a}$ and $\vec{b}$. This means that the derivative of $F(a_1)$, a function value between a and $a + h$ where

$$F(a_1) = f(a_1, b+k) - f(a_1, b)$$

is:

$$F'(a_1) = \frac{\partial f}{\partial x}(a_1, b+k) - \frac{\partial f}{\partial x}(a_1, b)$$

which is what we set out to prove.

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Next, show there's a point, b_1, between b and $b+k$ such that

$$\Delta(h,k) = \frac{F(a+h) - F(a)}{hk} = \frac{\partial^2 f}{\partial y \partial x}(a_1, b_1)$$

By the mean value theorem,

$$\frac{f(a+h,b) - f(a,b)}{a+h-a} = \frac{f(a+h,b) - f(a,b)}{h} = \frac{\partial f}{\partial x}(a_1)$$

and

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(a_1)\right)(b_1) = \frac{(f(a+h,b+k) - f(a+h,b)) - (f(a,b+k) - f(a,b))}{h(b+k-b)} = \Delta(h,k) = \frac{F(a+h) - F(a)}{hk}$$

which is what we set out to show.

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Using the three previous exercises, prove that if $\partial^2 f/\partial x \partial y$ and $\partial^2 f/\partial y \partial x$ exist for all points sufficiently close to (a,b) and if $\frac{\partial^2 f}{\partial y \partial x}$ is continuous at (a,b) then

$$\frac{\partial^2 f}{\partial y \partial x}(a,b) = \frac{\partial^2 f}{\partial x \partial y}(a,b)$$

The existence of a vector limit implies that the mixed partials are equal. If

$$\lim_{(h,k)\to(0,0)} \left(\frac{\partial^2 f}{\partial x \partial y}(a_1, b_1)\right) = \lim_{(a+h,b+k)\to(a,b)} \left(\frac{\partial^2 f}{\partial x \partial y}(a_1, b_1)\right)$$

exists, it implies continuity and means that the two partials are equal. This limit does exist, therefore the function is continuous at (a,b) and the mixed partials are equal.