Tutorial-01, B.Tech. Semester-I
(Leibnitz's rule, Partial Differentiation)

Problem 1: Find the n^{th} derivative of the following functions:

(a) $\tan^{-1}\frac{2x}{1-x^2}$ (b) $e^x \sin 4x \cos 6x$ **Ans:** (a)$(-1)^{n-1}(n-1)!\, sin^n\theta . \sin n\theta$

(b) $e^x \frac{(101)^{\frac{n}{2}}}{2} sin(10x + n\, tan^{-1} 10) - e^x \frac{(5)^{\frac{n}{2}}}{2} sin(2x + n \tan^{-1} 10)$

Problem 2: If $I_n = \frac{d^n}{dx^n}(x^n \ \log x)$, prove that $I_n = nI_{n-1} + (\,n-1)!$ and hence show that

$$I_n = n!\,\{\log x + 1 + \frac{1}{2} + \frac{1}{3} + \quad \dots \quad + \frac{1}{n}\}$$

Problem 3: If $y = (x^2-1)^n$, use Leibnitz's theorem to show that

$$(1-x^2)y_{n+2} - 2x\, y_{n+1} + n(n+1)y_n = 0.$$

Problem 4: If $y = \sin(m \sin^{-1} x)$, show that

$(1-x^2)y_{n+2} - (2n+1)x\, y_{n+1} + (\,m^2 - n^2)y_n = 0$ and hence evaluate $(y_n)_0$.

Problem 5: If $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$, prove that $x^2 y_{n+2} + (2n+1)x\, y_{n+1} + 2n^2 y_n = 0.$

Problem 6: If $y = [x - \sqrt{(x^2-1)}]^m$,

prove that $(1-x^2)y_{n+2} - \ (2n+1)xy_{n+1} + (\,m^2 - n^2)y_n = 0.$

Problem 7:(a) If u=x^y establish the relation that $u_{xy} = u_{yx}$.

(b) If u=$(x^2+y^2+z^2)^{-1/2}$, show that $x.u_x + y\,.u_y + zu_z = -u$ and $u_{xx} + u_{yy} + u_{zz} = 0.$

(c) If u=$\varphi\,(y-ax) + \phi(y+ax)$, show that $u_{xx} - a^2 u_{yy} = 0.$

(d) If u= $\sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$, show that $x.u_x + y\,.u_y \ = 0.$

(e) If u=$\begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$, then show that $u_x + u_y \ + u_z = 0.$

(f) If u=e^{xyz} , show that $u_{xyz} = (1 + 3xyz + x^2y^2z^2)e^{xyz}$.

(g) If u=$\log(x^3+y^3+z^3-3xyz)$, then prove that

$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}.$

Problem 8:(a) If $u = f(r)$ and r= $\sqrt{x^2+y^2}$, then $u_{xx} + u_{yy} = f''(r) + \frac{1}{r}\, f'(r)$.

(b) If k = $x^x y^y z^z$, show that at $= y = z$, $z_{xy} = -\frac{1}{(x \log ex)}$.

(c) If $\frac{x^2}{u+a^2} + \frac{y^2}{u+b^2} + \frac{z^2}{u+c^2} = 1$ prove that

$$(u_x)^2 + \left(u_y\right)^2 + (u_z)^2 = 2(\ x.u_x + y\,.u_y \ + z.u_z)$$

Problem 9:(a) If $\theta = t^n e^{-\frac{r^2}{4t}}$, find the value of n will make $\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial\theta}{\partial r})$= $\frac{\partial\theta}{\partial t}$ Ans: $n = -\frac{3}{2}$

(b) If u = $x^3 + y^3$, where $x = a\cos t$, $y = b \sin t$ find $\frac{du}{dt}$?

Ans:$3(b^3 sin^2 t\ \cos t\ - a^3 cos^2 t \sin t)$

(c) If $u = f(x,y)$, where $x = r\cos\theta$, $y = r\sin\theta$ find the value of $u_x\ and\ u_y$.

Then prove that $(u_x)^2 + \left(u_y\right)^2 = (u_r)^2 + \frac{1}{r^2}(u_\theta)^2$.

(d) If $x^y + y^x = c$, find the value of $\frac{dy}{dx}$. **Ans:** $\frac{-[y^x \log y + yx^{y-1}]}{x^y \log x + xy^{x-1}}$

(e) If $V = f(x-y, y-z, z-x)$, then prove that $\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial V}{\partial z} = 0.$

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Problem 10: State and prove Euler's theorem for a homogeneous function $f(x,y)$ of degree n in two variables x and y. Also deduce that

(i) $x.u_{xx} + y.u_{yy} = (n-1)u_x$

(ii) $x.u_{xy} + y.u_{yy} = (n-1)u_y$

(iii) $x^2.u_{xx} + y^2.u_{yy} + 2xy\,u_{xy} = n(n-1)u$

Problem 11: Prove the following results:

u	Result
$f(y/x)$	$x.u_x + y.u_y = 0.$
$\tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$	$x.u_x + y.u_y = \sin 2u$
$\sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$	$x.u_x + y.u_y = \tan u$
$\sin^{-1}\left(\sqrt{\frac{x+y}{x^2+y^2}}\right)$	$x.u_x + y.u_y = -\frac{1}{2}\tan u$
$\log\left(\frac{x^4+y^4}{x+y}\right)$	$x.u_x + y.u_y = 3$
x $\sin^{-1}\left(\frac{y}{x}\right)$	$x^2.u_{xx} + y^2.u_{yy} + 2xy\,u_{xy} = 0$
$\frac{x^2y^2}{x^2+y^2}$	$x^2.u_{xx} + y^2.u_{yy} + 2xy\,u_{xy} = 2u$
$\cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$	$x.u_x + y.u_y + \frac{1}{2}\cot u = 0$

Problem 12: If $y = e^{a\sin^{-1}x}$, Establish the relation

$(1-x^2)y_{n+2} - (2n+1)x\,y_{n+1} - (n^2+a^2)y_n = 0$ and hence evaluate $(y_n)_0$

Problem 13: If $y = \sin mx + \cos mx$ prove that $y_n = \mathrm{m}^{\mathrm{n}}\sqrt{\{1+(-1)^{\mathrm{n}}\sin 2\mathrm{mx}\}}$

Problem 14: If $y = [x + \sqrt{(x^2+1)}]^m$,prove that $(1+x^2)y_2 + xy_1 - m^2y = 0$ and hence evaluate $(y_n)_0$.

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Tutorial-02, B.Tech Semester-I

(Expansions of functions of several variables and Curve Tracing)

Problem 1: Find the equation of the tangent plane and the normal to the surface $z^2 = 4(1+x^2+y^2)$ at $(2,2,6)$ **Ans:** $4x+4y-3z=-2$; $\frac{x-2}{4}=\frac{y-2}{4}=\frac{z-6}{-3}$

Problem 2: Expand $e^x \cos y$ near the point $(1,\frac{\pi}{4})$ by Taylor's theorem.

Ans: $\frac{e}{\sqrt{2}}\left[1+(x-1)-\left(y-\frac{\pi}{4}\right)+\frac{(x-1)^2}{2}-(x-1)\left(y-\frac{\pi}{4}\right)-\left(y-\frac{\pi}{4}\right)^2+\cdots\ldots\ldots\right]$

Problem 3: Obtain Taylor's expansion of $\tan^{-1}\frac{y}{x}$ about (1,1) upto and including the second degree terms. Hence compute $f(1.1,0.9)$.

Ans: $\tan^{-1}\frac{y}{x}=\frac{\pi}{4}-\frac{1}{2}(x-1)+\frac{1}{2}(y-1)+\frac{1}{4}(x-1)^2-\frac{1}{4}(y-1)^2+\ldots$ and 0.7862

Problem 4: Expand x^y in powers of $(x-1)$ and $(y-1)$upto the third degree terms.

Ans: $x^y=1+(x-1)+(x-1)(y-1)+\frac{1}{2}(x-1)^2(y-1)+\ldots\ldots\ldots$

Problem 5: Expand $e^{ax}\sin by$ in powers of x and y upto the third degree terms.

Ans: $by+abxy+\frac{(3a^2bx^2y-b^3y^3)}{3!}+\cdots\ldots\ldots\ldots$

Problem 6: Trace the following curves

1. $a^2y=x^3$; Cubical parabola
2. $xy^2=4a^2(2a-x)$;
3. $y^2(2a-x)=x^3$ *Cissoid*
4. $x^5+y^5=5ax^2y^2$
5. $y=a\cosh\frac{x}{a}$ *Catenary*
6. $y(x^2-1)=(x^2+1)$
7. $y(x^2+4a^2)=8a^3$
8. $a^2x^2=y^3(2a-y)$
9. $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ or $x=a\cos^3\theta\ \ y=a\sin^3\theta$ Astroid

10.. $x=a(\theta-\sin\theta), y=a(1-\cos\theta)$ Cycloid

11. $r=a+b\cos\theta$, $a>b$
12. $r=a(1+\cos\theta)$ Cardiod
13. $r=a\sin 3\theta$ Three leaves rose
14. $r=a(1-\cos\theta)$ Cardiod
15. $r=a\cos 2\theta$ Four leaves rose
16. $x=a(\theta+\sin\theta), y=a(1+\cos\theta)$ Cycloid
17. $y^2=x^5(2a-x)$
18. $r=a+b\cos\theta$, $a<b$ Limecon
19. $r^2=a^2\cos 2\theta$
20. $r(1-\cos\theta)=2a$

21.. $r(1+\cos\theta)=2a$

22. $r^2\cos 2\theta=a^2$ Hyperbola
23. $y^2=ax^3$ Semi- cubical parabola
24. $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$ or $x=a\cos^3\theta\ \ y=b\sin^3\theta$ Hypocycloid
25. $x=(y-1)(y-2)(y-3)$
26. $x=a\cos t+\frac{1}{2}a\log\tan^2\left(\frac{t}{2}\right), y=a\ \sin t$ { Tractrix }
27. $x=\frac{3at}{1+t^3}$; $y=\frac{3at^2}{1+t^3}$
28. $x=\frac{1-t^2}{1+t^2}$; $y=\frac{2t}{1+t^2}$
29. $x=t^2$, $y=t-\frac{1}{3}t^3$
30. $x=a(\theta+\sin\theta), y=a(1+\cos\theta)$ Cycloid
31. $x=a(\theta-\sin\theta), y=a(1+\cos\theta)$ Cycloid
32. $x=a(\theta+\sin\theta), y=a(1-\cos\theta)$ Cycloid

R.K. Soni's

Tutorial-3 B.Tech. Semester-I

(Double integrals and their applications)

1. Find the area of the loop of the curve $x^3 + y^3 = 3axy$. Also find the area bounded between the curve and its asymptote. **Ans:** $\frac{3a^2}{2}$ **and** $\frac{3a^2}{2}$.

2. Evaluate $\iint_A (x^2 + y^2)\,dxdy$ over the area A enclosed by the curves $y = 4x, x + y = 3, y = 0$ and $y = 2$. **Ans:** $9\frac{31}{48}$

3. Evaluate the following double integrals

(i) $\int_0^2 \int_0^{x^2} e^{\frac{y}{x}}\,dydx$ (ii) $\int_0^{2a} \int_0^{\sqrt{2ax - x^2}} x^2\,dydx$ (iii) $\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}}\,dydx$

(iv) $\int_1^0 \int_0^1 (x + y)\,dydx$ (v) $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2 - x^2 - y^2}\,dydx$

Ans: (i) $e^2 - 1$.(ii) $\frac{5\pi a^4}{8}$ (iii) $\frac{a^3}{3}\log(\sqrt{2} + 1)$ (iv)-1 (v) $\frac{\pi a^3}{6}$

4. Show that $\int_1^2 \int_3^4 (xy + e^y)\,dxdy = \int_3^4 \int_1^2 (xy + e^y)\,dydx$

5. Show that $\int_1^2 \int_0^{\frac{y}{2}} y\,dy\,dx = \int_1^2 \int_0^{\frac{x}{2}} x\,dx\,dy$

6. Show that $\int_0^1 \int_0^1 \frac{1}{\sqrt{\{1-x^2\}\{1-y^2\}}}\,dy\,dx = \int_0^1 \int_0^1 \frac{1}{\sqrt{\{1-x^2\}\{1-y^2\}}}\,dxdy$

7. Show that $\int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3}\,dy \neq \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3}\,dx$, also find the values of two integrals.

Ans: $L.H.S = \frac{1}{2}$, $R.H.S = -\frac{1}{2}$ Give a conclusion on the basis of the results in Q. 4 to 7.

8. Evaluate $\iint (x^2y^2)\,dxdy$ over the region $x^2 + y^2 \leq 1$. **Ans:** $\frac{\pi}{24}$

9. Evaluate $\iint_A (x^2y^2)\,dxdy$ over the region A bounded by the curves $x = 0, y = 0,$ and $x^2 + y^2 = 1$. **Ans:** $\frac{\pi}{96}$

10. Evaluate $\iint r^2\,d\theta dr$ over the area of the circle $r = a\cos\theta$ **Ans:** $\frac{4a^3}{9}$

11. Find by double integration the area lying inside the circle $r = a\sin\theta$ and outside the parabola $r(1 + \cos\theta) = a$ **Ans:** $\frac{(9\pi+16)}{12}$

12. Change the order of the following double integrations:

(i) $\int_1^\infty \int_1^x x e^{\frac{-x^2}{y}}\,dydx$ **Ans:** $\int_1^\infty \int_y^\infty x e^{\frac{-x^2}{y}}\,dxdy$

(ii) $\int_0^{2a} \int_0^{\sqrt{2ax-x^2}} f(x, y)\,dydx$ **Ans:** $\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} f(x, y)\,dxdy$

(iii) $\int_0^{a\cos\alpha} \int_{x\tan\alpha}^{\sqrt{a^2-x^2}} f(x, y)\,dy\,dx$ **Ans:** $\int_0^{a\sin\alpha} \int_0^{y\cot\alpha} f(x, y)\,dxdy + \int_{a\sin\alpha}^{a} \int_0^{\sqrt{a^2-y^2}} f(x, y)\,dx\,dy$

(iv) $\int_0^a \int_{mx}^{lx} f(x, y)\,dydx$ **Ans:** $\int_0^{am} \int_{\frac{y}{l}}^{\frac{y}{m}} f(x, y)\,dxdy + \int_{am}^{al} \int_{\frac{y}{l}}^{a} f(x, y)\,dx\,dy$

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(v) $\int_0^{a} \int_{\frac{x^2}{a}}^{2a-x} \mathbf{xy\ dydx}$ **Ans:** $\int_0^{a} \int_0^{\sqrt{ay}} \mathbf{xy\ dxdy} + \int_a^{2a} \int_0^{2a-y} \mathbf{xy\ dx\ dy}$

13. Express as single integral and evaluate: $\int_0^{\frac{a}{\sqrt{2}}} \int_0^{x} \mathbf{x\ dxdy} + \int_{\frac{a}{\sqrt{2}}}^{a} \int_0^{\sqrt{a^2-x^2}} \mathbf{x\ dxdy}$

Ans: $\int_0^{\frac{a}{\sqrt{2}}} \mathbf{dy} \int_y^{\sqrt{a^2-y^2}} \mathbf{x\ dx;}$ $\frac{5a^3}{6\sqrt{2}}$

14. Convert into polar co-ordinates $\int_0^{2a} \int_0^{2ax-x^2} \mathbf{dydx}$ **Ans:** $\int_0^{\frac{\pi}{2}} \int_0^{2a\cos\theta} \mathbf{rd\theta\ dr}$

15. Using transformation $x + y = u, y = vu$ show that $\int_0^{1} \int_0^{1-x} e^{\frac{y}{x+y}} \mathbf{dydx} = \frac{e-1}{2}$

16. Using transformation $x - y = u, x + y = v$ show that $\iint_R \cos\frac{x-y}{x+y} \mathbf{dxdy} = \frac{\sin 1}{2}$ Where R is the region bounded by $x = 0, y = 0, x + y = 1$.

17. Find the whole area of the curve $a^2x^2 = y^3(2a - y)$ by double integration. **Ans:** πa^2

18. Find the area enclosed by the curve $r = 3 + 2\cos\theta$ by double integration. **Ans:** 11π

19. Find the volume of the torus generated by revolving the circle $x^2 + y^2 = 4$ about the line $x = 3$. **Ans:** $24\pi^2$

20. Find the Center of gravity of the area bounded by the parabola $y^2 = x$ and the line $x + y = 2$. **Ans:** $(\frac{8}{5}, -\frac{1}{2})$

21. Find the Center of gravity of the loop of the curve $r^2 = a^2 \cos 2\theta$. **Ans:** $(\frac{\pi a\sqrt{2}}{8}, 0)$

22. Find the Center of gravity of an arc of the curve $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ in the positive quadrant. **Ans:** $[a(\pi - \frac{4}{3}, \frac{2a}{3}]$

R.K. Soni's

Semester–I, Tutorial 4
Error, Approximations and Maxima and Minima

1. Compute an approximate value of $(1.04)^{3.01}$ **Ans: 1.12**

2. If $u = x^2y^3/z$, find the maximum percentage error in u if the percentage error in x, y, z are $2, 3$ and 4 respectively. **Ans: 17**

3. If the sides and angles of a plane triangle vary in such a way that its circum radius remains constant prove that $\frac{da}{cosA} + \frac{db}{cosB} + \frac{dc}{cosC} = 0$ where da, db and dc are small increments in the sides a, b and c respectively.

4. Find the shortest distance from origin to the surface $xyz^2 = 2$. Ans: 2

5. Find the dimensions of a rectangular box, with open top and given volume V, so that the total surface area of the box is a minimum. **Ans**: Length=breadth=2 height=$(2V)^{(13)}$

6. Find the Minimum value of $x^2 + y^2 + z^2$ subject to $1/x + 1/y + 1/z = 1$

7. Find the absolute extrema of $f = 3x^2 + 2y^2 - 4y$ in the region bounded by $y = x^2$ and $y = 4$. **Ans**: Abs max$= 28$ at $(\pm 2, 4)$; Abs. min$= -2$ at $(0, 1)$

8. Find Min $f = x^2 + y^2$ subject to $x + y = 10$. Ans: 50 at $(5, 5)$

9. Find extremum of $f = x + y + z$ subject to $x^2 + y^2 + z^2 = 12$. **Ans**: Max $f(2, 2, 2) = 6$, Min $f(-2, -2, -2) = -6$.

10. Find extrema of $u = a^2x^2 + b^2y^2 + c^2z^2$ subject to $x^2 + y^2 + z^2 = 1$ and $lx + my + nz = 0$. **Ans** : The extremum can be obtained by the equation $\frac{l^2}{u-a^2} + \frac{m^2}{u-b^2} + \frac{n^2}{u-c^2} = 0$.

11. Find the extrema of $u = x^2+y^2+z^2$ where $ax^2+by^2+cz^2+2fyz+2gzx+2hxy = 1$. **Ans** : The roots of .

12. Find the extreme values of $f(x, y, z) = 2x + 3y + z$ subject to $x^2 + y^2 = 1$ and $x + z = 1$. **Ans with Hint**: for $\lambda_2 = -1, \lambda_1 = \frac{1}{\sqrt{2}}, f = 1 - 5\sqrt{2}$ while for $\lambda_2 = -1, \lambda_1 = -\frac{1}{\sqrt{2}}, f = 1 + 5\sqrt{2}$

    Find the local extrema for the following functions:

13. $f = -22x^2 + 22xy - 11y^2 + 110x - 44y - 23$ **Ans**: $f = 120$ is local min at $(3, 1)$

14. $f = x^3 + y^3 - 3xy$ **Ans**: $f = -1$ is local min at $(1, 1)$; $(0, 0)$ is saddle point.

15. $f = xy + \frac{9}{x} + \frac{3}{y}$ **Ans**: $f = 9$ is local min at $(3, 1)$.

Tutorial-05, B. Tech Sem-I
Jacobians

1. If $u = x + y + z + t, v = x + y - z - t, w = xy - zt, r = x^2 + y^2 - z^2 - t^2$ show that $\frac{\partial(u,v,w,r)}{\partial(x,y,z,t)} = 0$ and hence find a relation between u, v, w and r. Ans: $uv = r + 2w$

2. Prove that the following functions are not independent. Find the relation between them

(i) $u = x + y + z, v = xy + yz + zx, w = x^3 + y^3 + z^3 - 3xyz$. Ans: $u^3 = 3uv + w$

(ii) $u = x^2 + y^2 + z^2, v = x + y + z, w = xy + yz + zx$. Ans: $v^2 = u + 2w$

3. If x, y, z are connected by a functional relation $f(x, y, z) = 0$, show that $\frac{\partial(y,z)}{\partial(x,z)} = \left(\frac{\partial y}{\partial x}\right)_{z=\text{constt}}$.

4. If λ, μ, ν are the roots of the equation in k, $\frac{x}{a+k} + \frac{y}{b+k} + \frac{z}{c+k} = 1$, prove that $\frac{\partial(x,y,z)}{\partial(\lambda,\mu,\nu)} = -\frac{(\mu-\nu)(\nu-\lambda)(\lambda-\mu)}{(a-b)(b-c)(c-a)}$.

5. If $u = x(1 - r^2)^{\frac{-1}{2}}, v = y(1 - r^2)^{\frac{-1}{2}}, w = z(1 - r^2)^{\frac{-1}{2}}$, where $r^2 = x^2 + y^2 + z^2$ show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = (1 - r^2)^{\frac{-5}{2}}$.

6. If $u^3 = xyz, \frac{1}{v} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}, w^2 = x^2 + y^2 + z^2$, show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = -\frac{v(y-z)(z-x)(x-y)(x+y+z)}{3u^2\, w(xy+yz+zx)}$.

7. Find the Jacobian of $y_1, y_2, y_3, \ldots, y_n$ being given $y_1 = x_1(1 - x_2)$, $y_2 = x_1x_2(1 - x_3)$, ….., $y_{n-1} = x_1x_2 \ldots\ldots x_{n-1}(1 - x_n)$, $y_n = x_1x_2 \ldots\ldots x_n$ find $J(y_1, y_2, .., y_n)$

{**Hint:** $y_1 + y_2 + y_3 + \cdots .. + y_n = x_1$, Ans: $x_1^{n-1} x_2^{n-2} \ldots \ldots x_{n-1}$}.

8. Find the Jacobian $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ being given $x = r\cos\theta\cos\phi$, $y = r\sin\theta\sqrt{1 - m^2\sin^2\phi}$, $z = r\sin\phi\sqrt{(1 - n^2\sin^2\theta)}$ where $m^2 + n^2 = 1$. {Hint: $r^2 = x^2 + y^2 + z^2$, Ans: $\frac{r^2(n^2\cos^2\theta + m^2\cos^2\phi)}{\sqrt{(1-n^2\sin^2\theta)(1-m^2\cos^2\phi)}}$}.

9. Prove that $JJ'=1$

10. If $u = xyz, v = x + y - z, w = x + y + z$, Find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ Ans: $\frac{x}{2(2x^2-y^2)}$

11. If $f(0) = 0$ and $f'(x) = \frac{1}{1+x^2}$, prove without using the method of integration, that $f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$. {**Hint**: Let $u = f(x) + f(y)$ and $v = \frac{x+y}{1-xy}$ the find $J(u, v)$}

R.K. Soni's

Tutorial-6, B. Tech. Sem-I,
(Triple integrals and their application)

Problem 1: Evaluate the triple integrals $\int_0^1\int_1^{1-x}\int_0^{x+y} \mathbf{e^z\,dzdydx}$ **Ans:** $\frac{1}{2}$

Problem 2: Evaluate $\iiint (\mathbf{x+y+z})^9\,\mathbf{dxdydz}$ over the region bounded by $\boldsymbol{x \geq 0, y \geq 0, z \geq 0, x+y+z \leq 1}$. **Ans:** $\boldsymbol{\frac{1}{24}}$

Problem 3: Evaluate the following triple integrals

(i) $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(\mathbf{x^2+y^2+z^2})\quad \mathbf{dzdydx}$ **(ii)** $\int_0^1\int_{y^2}^1\int_0^{1-x} \mathbf{x\,dzdxdy}$

(iii) $\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} \mathbf{e^{x+y+z}\,dzdydx}$

(iv) $\int_0^4\int_0^{2\sqrt{z}}\int_0^{\sqrt{4z-x^2}} \mathbf{dydxdz}$ **(v)** $\int_0^2\int_0^2\int_0^{4-x^2}(\mathbf{2x+y})\mathbf{dzdydx}$

Ans: (i) $\boldsymbol{\frac{8abc(a^2+b^2+c^2)}{3}}$.**(ii)**$\boldsymbol{\frac{4}{35}}$ **(iii)**$\boldsymbol{\frac{8}{3}\log(2)-\frac{19}{9}}$ **(iv)**$\boldsymbol{8\pi}$ **(v)** $\boldsymbol{\frac{80}{3}}$

Problem 4: Evaluate the triple integral $\iiint_{\mathbf{R}} (\mathbf{x-2y+z})\,\mathbf{dxdydz}$, where R is the region determined by $\boldsymbol{0\leq x \leq 1}$, $\boldsymbol{0\leq, y \leq x^2 \,and\; 0\leq z \leq x+y}$. **Ans:** $\boldsymbol{\frac{29}{105}}$

Problem 5: Find the volume of the region bounded by the surface $\boldsymbol{y = x^2, x = y^2}$ and the planes $\boldsymbol{z = 0, z = 3}$. **Ans: 1**

Problem 6: Find the volume of the tetrahedron bounded by the co-ordinate planes and the plane $\boldsymbol{\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1}$. **{** Hint: $\int_0^a\int_0^{b(1-\frac{x}{a})}\int_0^{c(1-\frac{x}{a}-\frac{y}{b})} \mathbf{dzdydx}$**}** **Ans:** $\boldsymbol{\frac{abc}{6}}$

Problem 7: Find the volume common to the cylinder $x^2+y^2 = a^2$ and $x^2+z^2 = a^2$.

{ Hint: $\int_{-a}^{a}\int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}}\int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} \mathbf{dzdydx}$ **}** **Ans:** $\boldsymbol{\frac{16a^3}{3}}$

Problem 8: Find the mass of the tetrahedron bounded by the co-ordinate planes and the plane $\boldsymbol{\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1}$ the variable density $\rho = kxyz$

{ Hint: $\int_0^a\int_0^{b(1-\frac{x}{a})}\int_0^{c(1-\frac{x}{a}-\frac{y}{b})} \boldsymbol{kxyz}\,\mathbf{dzdydx}$ **}** **Ans:** $\boldsymbol{\frac{2ka^3}{3}(3\sqrt{3}-\frac{\pi}{3})}$

Problem 9: Find the moment of inertia of the solid about its major axes generated by revolving the ellipse $\mathbf{\frac{a}{r} = \frac{2-\cos\theta}{2}}$ about minor axes**.**

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(Vector Calculus)

Note: In this exercise bold face letters (say) **F** represents vector $\vec{f}$ and i,j,k represents unit vectors $\hat{i},\hat{j},\hat{k}$ respectively.

Problem1: Evaluate the following (i) $\nabla\cdot(r^3\mathbf{r})$ (ii) $\nabla\cdot(r\,\nabla(\frac{1}{r^3}))$ (iii) $\nabla^2\,(\nabla\cdot(\mathbf{r}\,\frac{1}{r^2}))$(iv) grad Div$(\frac{\mathbf{r}}{r})$

Ans: (i)$6r^3$, (ii)$3r^{-4}$, (iii)$2r^{-4}$, (iv) $-\frac{2\mathbf{r}}{r^3}$

Problem2: If **A**= $2yz\,i - x^2y\,j + xz^2k$, **B**= $x^2i + yz\,j - xy\,k$ and $\varphi=2x^2yz^3$, then find

(i) $(\boldsymbol{A}\cdot\varphi)\,\boldsymbol{B}$; **Ans:**

(ii) $\boldsymbol{A}\cdot\nabla\varphi$;

(iii) $(\boldsymbol{B}\cdot\nabla)\,\boldsymbol{A}$;

(iv) $(\boldsymbol{A}\times\nabla)\varphi$;

(v) $\boldsymbol{A}\times(\nabla\varphi)$

Problem3: If **A** and **B** are differentiable vector functions, ϕ and φ are differentiable scalar functions of position (x, y, z), then prove the following results:

(i)$\nabla\cdot(\boldsymbol{A}+\boldsymbol{B}) = \nabla\cdot\boldsymbol{A} + \nabla\cdot\boldsymbol{B}$

(ii) $\nabla\cdot(\varphi\boldsymbol{A}) = \varphi\,(\nabla\cdot\boldsymbol{A}) + \boldsymbol{A}\cdot\boldsymbol{A}\times$

(iii) $\nabla\times(\varphi\boldsymbol{A}) = \varphi(\nabla\times\boldsymbol{A}) + \boldsymbol{A}\times\nabla\varphi$

(iv) $\nabla\cdot(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{B}\cdot(\nabla\times\boldsymbol{A}) - \boldsymbol{A}\cdot(\nabla\times\boldsymbol{B})$

(v) $\nabla(\boldsymbol{A}\cdot\boldsymbol{B}) = (\boldsymbol{B}\cdot\nabla)\,\boldsymbol{A} + (\boldsymbol{A}\cdot\nabla)\,\boldsymbol{B} + \boldsymbol{B}\times(\nabla\times\boldsymbol{A}) + \boldsymbol{A}\times(\nabla\times\boldsymbol{B})$

(vi) $\nabla\times(\nabla\varphi) = 0$

Problem4: Evaluate the grad. of $\log|\mathbf{r}|$ Ans: $\frac{\mathbf{r}}{r^2}$

Problem5: Show that $\nabla\varphi$ is a vector perpendicular to the surface $\varphi(x, y, z)$= constt.

Problem6: Find the directional derivative of $\varphi(x, y, z)=x^2yz + 4xz^2$ at $(1, -2, -1)$ in the direction $2i - j - 2k$. Ans:37/3

Problem7: Prove that the vector **A**= $3y^4z^2i + 4x^3z^2\,j - 3x^2y^2k$ is solenoidal.

Problem8: Prove following identities

(i) $\text{Div}(\nabla\varphi\times\nabla\psi)=0$

(ii) If **A** and **B** are irrotational then **A**×**B** is solenoidal.

(iii) $\nabla\times(\varphi\,\nabla\Phi)=0$

(iv) Div (f ∇g) = $f\,\nabla^2g+\nabla f\cdot\nabla g$

(v) $\mathbf{b}\cdot\nabla(\mathbf{a}\cdot\nabla\frac{1}{r}) = \frac{3(\mathbf{a}\cdot\mathbf{r})(\mathbf{b}\cdot\mathbf{r})}{r^5} - \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{r^3}$ where **a** and **b** are constt. Vectors.

(vi) $\nabla\cdot(U\nabla V - V\nabla U)=U\nabla^2U - V\,\nabla^2U$

(vii) $\nabla r^n = nr^{n-2}\boldsymbol{r}$

Problem9: (a) Prove that $\boldsymbol{F} = (y^2cosx + z^3)i + (2ysinx - 4)j + (3xz^2 + 2)k$ is a conservative force field.

(b)Find the scalar potential for **F**. Ans: $(y^2sinx + xz^3 - 4y + 2z + c)$

(c)Find the work done in moving an object in this field from $(0,1,-1)$ to $(\frac{\pi}{2}, -1, 2)$.

Ans: 15+4π

Problem10: Show that **V**=$2xyz\,i + (x^2z + 2y)j + x^2yk$ is irrotational. Express **V** as gradient of a scalar function φ.

Problem11: Evaluate$\int_{(0,0)}^{(2,1)}(10\,x^4 - 2xy^3)dx - 3x^2y^2dy$ along the path $x^4 - 6xy^3 = 4y^2$

Ans: 60. {Hint: Use Exact differential}

Problem12: Use Green's theorem to evaluate $\oint_C\,(cosx\,siny - xy)dx + \sin x\cos y\,dy$,where

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$C \equiv \mathbf{x^2 + y^2 = 1}$. **Ans**: 0 .

Problem13: Verify Green's theorem in the plane for $\oint_C (xy + y^2)\,dx + x^2 dy$,where C is the closed curve of the region bdd. by the line $y = x$ and curve $y = x^2$.

Problem14: Find the work done in moving a particle once around a circle C in the xy-plane, if circle has center at the origin and radius 3, Force field is given by $\boldsymbol{F} = (2x - y + z)i + (x + y - z^2)j + (3x - 2y + 4z)k$ Ans: 18π

Problem15: State and prove Green's theorem.

Problem16: Prove that the area bounded by a simple closed curve C is given by $\frac{1}{2}\oint_C xdy - ydx$ {Hint: Use Green's theorem }

Problem16: Find the constants a,b,c such that **V**= $(x + 2y + az)i + (bx - 3y - z)j + (4x + cy + 2z)k$ is irrotational. Express V as gradient of a scalar function φ.

Problem18: State Green's theorem and hence evaluate $\oint_C (\cos y)dx + (x - x\sin y)\,dy$,where C is the closed curve $\mathbf{x^2 + y^2 = 1.}$

Problem19: Use Green's theorem to evaluate $\oint_C (x^2 + xy)dx + (x^2 + y^2)\,dy$,where C is the square formed by the lines $y = \pm 1, x = \pm 1$.

Problem20: Use Stoke's theorem to evaluate $\oint_C (x + 2y)dx + (x - z)\,dy + (y - z)dz$,where C is the boundary of the Δ with the vertices (2,0,0),(0,3,0),(0,0,6) oriented in the anticlockwise direction.

Problem21: Verify Stoke's theorem for **F**= $x^2y\,k - y\,j + xzi$ and S is the surface of the region bounded by $x = 0, y = 0, z = 0, 2x + y + 2z = 8$, which is not included in the xz-plane .

Problem22: Evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot \hat{n}\;ds$,where **F**= $y\,i + (x - 2xz)\,j - xy\,k$ and S is the surface of the sphere $\mathbf{x^2 + y^2 + z^2 = a^2}$ above the xy-plane.
Ans: Zero.

Problem23: Evaluate $\iint_S \mathbf{F} \cdot \hat{n}\;ds$,where **F**= $4xz\,i - y^2\,j + yz\,k$ and S is the surface of the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.
Ans: $\frac{3}{2}$.

Problem24: If **F**= $(x^2 + y - 4)i + 3xy\,j + (2xz + z^2)\,k$,evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot \hat{n}\;ds$, and S is the surface of the sphere $\mathbf{x^2 + y^2 + z^2 = 16}$ above the xy-plane.
Ans: -16π

Problem25: Evaluate $\iint (y^2z^2 i + x^2z^2\,j + x^2y^2 k) \cdot \hat{n}\,ds$,where S is the part of the surface of the sphere $\mathbf{x^2 + y^2 + z^2 = 1}$ above xy-plane.

Problem26: If V is the volume enclosed by the surface S, Find the value of $\int_S \mathbf{r} \cdot \hat{n}\,ds$
Ans: 3V

Problem27: State and prove Gauss Divergence theorm.

Problem28: Evaluate $\iint (ax\,i + by\,j + cz\,k) \cdot \hat{n}\,ds$,where S is the surface of the sphere $\mathbf{x^2 + y^2 + z^2 = 1.}$ [P.U. 2004]

Problem29: Evaluate $\iint_S \mathbf{A} \cdot \hat{n}\;ds$,where **A**= $18z\,i - 12\,j + 3y\,k$ and S is the part of the plane $2x + 3y + 6z = 12$ which is located in the first octant. **Ans**: 24

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Problem30: Evaluate $\iint_S \mathbf{A}\cdot\hat{n}\ ds$,where $\mathbf{A}= z\,i + x\,j - 3y^2 z\,k$ and S is the surface of the cylinder $\mathbf{x^2 + y^2 = 16}$ included in the first octant between $z = 0$ and $z = 5$.
Ans: 90

Problem 31: Prove that (i) $\iint_S \frac{\mathbf{r}\cdot\hat{n}}{r^2}\,ds = \iiint_V \frac{dv}{r^2}$; (ii) $\iint_S \frac{r^5\cdot\hat{n}}{}\,ds = \iiint_V 5r^3\mathbf{r}\,dv$

(iii) $\iint_S (\nabla\times\mathbf{F})\cdot\hat{n}\ ds = 0$ for any closed surface S.

Problem 32: Use divergence theorem to evaluate $\iint_S \mathbf{A}\cdot\overrightarrow{ds}$,where $\mathbf{A}= 4x\,i - 2y^2 j + z^2 k$ and S is the surface of the cylinder $\mathbf{x^2 + y^2 = 4}$ bdd. between $z = 0$ and $z = 3$.

Problem 33: Verify Stoke's theorem for $\mathbf{F}= (x^2 + y^2)\,i - 2xy\,j$ taken around rectangle bdd. by the lines $x = \pm a; y = 0, b$.

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