MATH 154 Homework 1 Solutions

Due October 5, 2012 Version September 23, 2012

Assigned questions to hand in:

(1) If G is a graph of order n, what is the maximum number of edges in G?
HHM 1.1.2.1 p. 9

Solution: The maximum number of edges is realized when there is an edge between every pair of vertices. An example from lecture (handshakes between n people) is analogous. We computed that the number of pairs is $\frac{n(n-1)}{2}$. Recall two proofs of this formula:

- By induction: The base cases are $n = 0, 1$. Here, the graph can't have any edges and, indeed $\frac{n(n-1)}{2} = 0$. For the induction hypothesis, suppose that any graph of order n has at most $\frac{n(n-1)}{2}$ edges. Consider, now, a collection of $n+1$ vertices. To put maximally many edges on these vertices, we can connect one vertex to all n others, and then put maximally many edges between those n remaining vertices. Thus,

$$\text{Max edges for } n+1 \text{ vertices} = n + (\text{Max edges for } n \text{ vertices})$$
$$\stackrel{IH}{=} n + \frac{n(n-1)}{2} = \frac{2n + n^2 - n}{2} = \frac{n(n+1)}{2}.$$

- Summation formula: The maximum number of edges on n vertices can be realized by placing an edge between one vertex and all $n-1$ others, then an edge between another vertex and the $n-2$ others, etc. Thus, we get the sum

$$\text{Max edges for } n \text{ vertices} = \sum_{i=1}^{n}(n-i) = n^2 - \sum_{i=1}^{n} i = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}.$$

(2) Is it true that finite graphs having exactly two vertices of odd degree must contain a path from one to the other? Give a proof or counterexample.
HHM 1.1.2.4 p. 9

Solution: Yes, this is true. To prove the statement, we proceed by contradiction. Suppose there is a finite graph with exactly two vertices of odd degree and such that there is no path between these vertices. Let v be one of these vertices. Consider the maximal connected component of G which contains v. It is itself a graph; call it G'. Moreover, note that the degree of a vertex $x \in V(G')$ is the same as the degree of that vertex when calculated with respect to the edges in G. Since v is has odd degree in G and the only other vertex of odd degree in G is not in G', v is the only vertex of odd degree in G'. But, by Theorem 1.1, the number of vertices with odd degree in G" is even, a contradiction.

(3) Determine whether K_4 is a subgraph of $K_{4,4}$. If yes, then exhibit it. If no, then explain why not.
HHM 1.1.3.3 p. 16

Solution: K_4 is not a subgraph of $K_{4,4}$. To prove this, denote by X, Y the two parts of $K_{4,4}$. For each subgraph H of $K_{4,4}$ with four vertices, some number of its vertices are in X and the rest are in Y. We have the following options:

- $V(H) \subseteq X$ or $V(H) \subseteq Y$. Then H must have no edges because a bipartite graph has no edges both of whose endpoints are in X (respectively, Y). So H is not K_4.
- Three vertices from H are in X and one is is Y (or vice versa). Then at most one of the vertices in H has degree at most 3 and the rest of the vertices have degree at most 1. But, the degree sequence of K_4 is $3, 3, 3, 3$. So, H is not K_4 in this case either.
- Two vertices from H are in X and two are in Y. Then the maximum degree of a vertex in H is 2, and H is not K_4.

Since we considered all possible subgraphs of $K_{4,4}$ with four vertices and none of them could be K_4, K_4 is not a subgraph of $K_{4,4}$.

(4) Prove that if graphs G and H are isomorphic, then their complements $\bar{G}$ and $\bar{H}$ are also isomorphic.
HHM 1.1.3.8 p. 17

Solution: Let G and H be isomorphic graphs, witnessed by the isomorphism $f : V(G) \to V(H)$. We will argue that f also witnesses that $\bar{G}$ and $\bar{H}$ are isomorphic. Recall (p. 11) that the complement of a graph is the one with the same vertex set and whose edge set consists of all edges that are not in the original graph. Since f is a one-to-one correspondence between $V(G)$ and $V(H)$, we need only show that for each pair of vertices $x, y \in V(\bar{G})$, $xy \in E(\bar{G})$ if and only if $f(x)f(y) \in E(\bar{H})$. Let $x, y \in V(\bar{G})$. Then $x, y \in V(G)$. Since f witnesses the isomorphism between G and H,

$$xy \in E(\bar{G}) \quad \text{iff} \quad xy \notin E(G) \quad \text{iff} \quad f(x)f(y) \notin E(H) \quad \text{iff} \quad f(x)f(y) \in E(\bar{H}).$$

(5) Prove that the two graphs in Figure 1.24 are not isomorphic.
HHM 1.1.3.9 p. 17

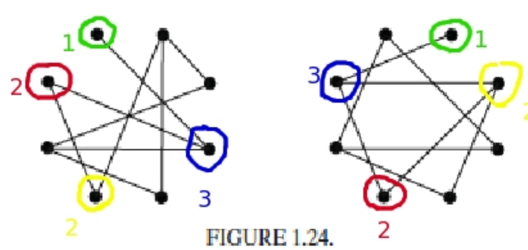

FIGURE 1.24.

Both graphs have degree sequence $3, 3, 3, 2, 2, 2, 2, 1$. An isomorphism must map a vertex to another vertex of the same degree. Since there is only one vertex of degree 1 (circled in green) in each graph these must be matched up by any isomorphism. Then, the degree 3 vertex (circled in blue) adjacent to the degree 1 vertex in each of the graphs must be matched. These vertices have one degree 3 neighbor and one degree 2 neighbor (circled in red) so each of these match to the corresponding one. Now, the remaining neighbors of these degree 2 vertices (circled in yellow) get matched. But, in the LHS graph, this yellow vertex has degree 2 whereas the yellow-circled vertex in the RHS graph has degree 3. This contradicts the existence of an isomorphism between the graphs.