Fall 2006 Math 510 HW3 Solutions

Section 3.6

10. A committee of 5 is to be chosen from a club that boast a membership of 10 men and 12 women. How many ways can the committee be formed if it has to have at least 2 women? How many ways if, in addition, one particular man and one particular woman who are members of the club, refuse to serve together on the committee.?

Solution. The total number of possible committees can be formed without any restrict is $\binom{22}{5}$. Among them, the number of committees without any women is $\binom{10}{5}$ and the number of the committee with exactly one woman is $12 \cdot \binom{10}{4}$. Thus the number of committees with at least two women is $\binom{22}{5} - \binom{10}{5} - 12 \cdot \binom{10}{4}$.

If A is the particular man and B is the particular woman who can not serve together. Then the total number of committees with both A and B on the committee is $\binom{20}{3}$ without any other restrictions. But the number of such committees with exactly one woman is $\binom{9}{3}$. Thus $\binom{20}{3} - \binom{9}{3}$ is the number of committees with at least two women and A and B are both on. This number should be excluded from the earlier answer. Thus the number of committees with at least two women and not A and B both on the committee is $\binom{22}{5} - \binom{10}{5} - 12 \cdot \binom{10}{4} - (\binom{20}{3} - \binom{9}{3})$.

12. A football team of 11 players is to be selected from a set of players, 5 of whom can play only in the backfield, 8 of them can only play on the line, 2 of them can play either in the backfield, or on the line. Assuming a football team has 7 men on the line and 4 men in the backfield, determine the number of football teams possible.

Solution. We divide the formation of the team into the following situations:

(1) None of those two players (A and B) who can player in the backfield or on the line is on the team. In this case, there are $\binom{8}{7} \cdot \binom{5}{4} = 40$.

(2) Exactly one of A and B is on the team. There two choices. For each choices, there are two situations: (a) This person plays in the back field and there are $\binom{5}{3}\binom{8}{7} = 160$ many possible. (b) This person is plays on the line, there are $\binom{5}{4}\binom{8}{6} = 5 \cdots 28 = 140$ possible teams. Thus the total number of teams when only one of A and B plays is $2(160 + 140) = 600$.

(3) Both A and B in the team. Now there are four different cases.

(a) Both A and B are on line, there are $\binom{5}{4} \cdot \binom{8}{5}$ ways.

(b) Both A and B are in the back field, there are $\binom{5}{2} \cdot \binom{8}{7}$ ways.

(c) One of the A and B is on the line and the other is in the backfield. There are $2 \cdot \binom{5}{3} \cdot \binom{8}{6}$ ways.

Finally the total number of ways is the sum of the numbers above.

13. There are 100 students at a school and three dormitories, A, B, and C, with capacities 25, 35, and 40 respectively.

(a). How many ways are there to fill the dormitories?

(b). Suppose that of the 100 students, 50 are men and 50 are women and that A is an all-men's dorm, B is an all women's dorm, and C is co-ed. How many ways are there to fill the dormitories?

Solution. Note that we are not in the positions to assign the room numbers students.

(a). This is a questions of permutations of set multiset with three types of objects and repetition numbers 25, 35, and 40. Then there are $\frac{100!}{25! \cdot 35! \cdot 40!}$.

(b). The selection can be completed in two stages. First, accommodate 50 women by deciding who goes to dorm B and who goes to dorm C. There are $\binom{50}{35}$ ways to fill dorm B (the remaining 15 women had to go to dorm C). Similarly there are $\binom{50}{25}$ ways to accommodate 50 man by choosing 25 men for dorm A and the remaining 25 men go to dorm C. Then by multiplication principle, the total number of ways to fill in the dorms is $\binom{50}{35}\binom{50}{25}$.

17. In how many ways can 6 indistinguishable rooks be placed on a 6-by-6 board so that no two rooks can attack one another? In how many ways if there are 2 red and 4 blue rooks?

Solution. (a) Since all rooks are indistinguishable, there is only one way place the 6 rooks on 6 rows. Then for the rooks on each to be placed in columns in that so that no two rooks will be in one column. This is an permutation and there are 6! ways to do so.

(b) Since red rooks and blue rooks are distinguishable, the first step is to decide which two rows will have the two red rooks. There are $\binom{6}{2} = 15$ way to choose. For each placement of rooks in rows, for each rook placed in each row, move them to a square in the row determined by the column numbers so that no two rooks from different rows should be in the same column. There are 6 rows and thus there 6! ways to place them. Thus the total number of ways to place 2 red rooks and 6 blue rooks is $15 \cdot 6!$.

20. Determine the number of circular permutations of $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ in which 0 and 9 are not opposite.

Solution. Let S be the set of all circular permutations of the above set and A be the set of all circular permutations that have 0 and 9 in opposite positions. Then $|S| = P(10, 10)/10 = 10!/10 = 9!$. To count the number of elements of A, one first seat 0 and 9 to make them sit in opposite positions. There is only one way to do so. Then one puts the remaining eight numbers into the remaining 8 positions. There are $P(8, 8) = 8!$ ways to do so. Hence by the multiplication principle, $|A| = 8!$. Hence the number of circular permutations of the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ 0 and 9 not seating in the opposite position is $|\overline{B}| = |S| - |A| = 9! - 8! = 8 \cdot 8!$.

25. A ferris wheel has 5 cars, each containing 4 seats in a row. There are 20 people ready for ride. In how many ways can the ride begin? Same question but now a certain two people want to sit in different cars.

Solution. The question was not clear whether the different seating of four people in a car will be different. Since it a row of seats, then seating in a car should be treated differently (as any one in the middle or this side of that side. In any case, students should make an assumption and compute the numbers based on the assumption.

First of all divide the 20 people into 5 named groups of 4 each. There are $\frac{20!}{4!\cdot 4!\cdot 4!\cdot 4!\cdot 4!}$ ways to do so (Thm 3.4.3). Now send the 5 named groups to 5 cars. Since there this a circular permutations, there are $\frac{5!}{5} = 4!$. Then arrange the 4 people in each car to seat. There should be 4! ways to seat four people in each of the five cars if different seatings are regarded different. Hence by multiplication principle, there are totally $\frac{20!}{4!\cdot 4!\cdot 4!\cdot 4!\cdot 4!} \cdot 4! \cdot (4!)^5 = 20! \cdot 4!$.

Now let A and B to the two individuals who don't want to be in the same car. First county the number of ways for A and B to be in the same group. There are five named groups for A and B to be in. For that group there are $\binom{18}{2}$ to complete the group A and B are in. For the remaining 16 people to be in 4 named groups of 4, there are $\frac{(16)!}{4!\cdot 4!\cdot 4!\cdot 4!}$ ways. Thus there are $5 \cdot \binom{18}{2} \cdot \frac{(16)!}{4!\cdot 4!\cdot 4!\cdot 4!}$ to divide the 20 people into 5 named groups and with A and B in the same group. Thus there are

$$\frac{20!}{4!\cdot 4!\cdot 4!\cdot 4!\cdot 4!}-5\cdot\binom{18}{2}\cdot\frac{(16)!}{4!\cdot 4!\cdot 4!\cdot 4!}=\frac{(16)!}{4!\cdot 4!\cdot 4!\cdot 4!}\left(\frac{20\cdot 19\cdot 18\cdot 17}{4!}-\frac{5\cdot 18\cdot 17}{2!}\right)=\frac{(18)!}{(4!)^5}\cdot 20\cdot 16.$$

The rest will be the same as before: arrange the five groups to cars (using circular permutations) and then seat the people in each car. So the total is

$$\frac{(18)!}{(4!)^5}\cdot 20\cdot 16\cdot 4!\cdot (4!)^5 = 18!\cdot 4!\cdot 20\cdot 16.$$

30. We are to seat 5 men, 5 women, and 1 dog in a circular arrangement around a table. In how many ways can this be done if no two man is to sit next to a man and no woman is to sit next to a woman?

Solution. Step 1: Let 5 men sit around the table. There are $5!/5 = 4!$ ways to seat them. Step 2: for each seating of men in step 1, ask 5 women to take their chairs and sit in the 5 places between the 5 men. There are 5! ways to do this. Step 3: Let the dog sit at places that are between two people (there are 10 such places). Thus the total number of ways seating the group around a table is $4!\cdot 5!\cdot 10 = 2\cdot (5!)^2$.

34. Determine the number of 11-permutations of the multiset

$$\{3\cdot a, 3\cdot b, 3\cdot c, 3\cdot d\}.$$

Solution. Each $n-1$ permutation of a multiset S of size n corresponds to a unique n-permutation of S by placing the only "unused" object to the last and any n-permutation arises this way from a unique $n-1$-permutation of S (by removing the last object). Hence the number of $n-1$-permutation is the same as the number of n-permutations. Thus the answer to this problem is $\frac{11!}{3!\cdot 3!\cdot 3!\cdot 3!}$.

41. In how many ways can 12 indistinguishable apples and 1 orange be distributed among three children in such a way that each child gets at least one piece of fruit.

Solution. *Step 1:* distribute the orange and two apples to three children with child getting one piece of fruit. This is 3-permutation of the multiset $\{2\cdot A, 1\cdot O\}$ and number of such permutations is $\dfrac{3!}{2!\cdot 1!}=3$.

Step 2: For each out come of step 1 (assuring each child has one piece of fruit), distribute the remaining 10 apples to three children without any restriction. Let x_1 be the number of apples the first child gets, x_2 be the number of apples the second child gets, and x_3 be the number of apples the third child gets. The number of ways of distributing the apples is the same as the number of integral solutions of

$$x_1+x_2+x_3=10 \quad \text{with} \quad x_1\geq 0, x_2\geq 0, x_3\geq 0.$$

That is the number of 10-combinations of the multiset with the 3 types and infinite repetitions, i.e., $\binom{10+3-1}{3-1}=\binom{12}{2}=66$.

Using multiplication principle, the total number of ways to distribute the fruits to three children is $3\times 66 = 198$.