Comparison of Signals and Vectors:-

Signals, defined for only a finite number of time instants (say N), can be written as a vectors (of dimension N). Thus consider a signal $g(t)$ defined over a closed time interval $[a,b]$. N uniform points on time interval $[a,b]$ such that

$$t_1 = a, \quad t_2 = a+\epsilon, \ldots, \quad t_N = a+(N-1)\epsilon = b,$$

$$\epsilon = \frac{b-a}{(N-1)}$$

So, signal vector g can be written as an N-dimensional vector $g = [g(t_1), \ldots, g(t_N)]$

As $N \to \infty$, the signal values will form a vector of infinitely long dimension. because $\epsilon \to 0$, the signal vector g will trans-form into the continious-time signal $g(t)$ defined over the interval $[a,b]$.

So, $\lim_{N \to \infty} g = g(t), \quad t \in [a,b]$

This shows that continious time signals are straight-forward generalization of finite dimension vector.

suppose, g & x are two real valued vectors two vectors.

Then inner product of two vectors

$$\langle g, x \rangle = \|g\| \cdot \|x\| \cos\theta \quad \ldots \ldots (i)$$

$\theta \to$ angle between vectors g & x.

Therefore, $\langle x, x \rangle = \|x\|^2 \quad \ldots \ldots (ii)$

where $\|x\| \to$ length, or norm of vector x. This defined a normed vector space.

Component of a Vector Along Another Vector

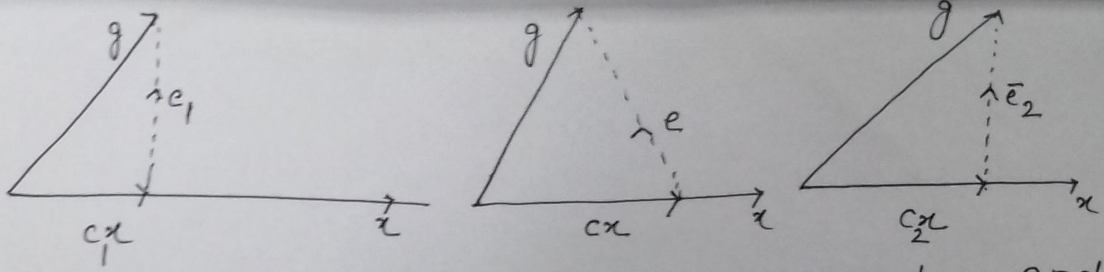

Fig: component of a vector along another vector and approximation of vector in terms of another vector.

$$g = cx + e \quad \text{or} \quad g = c_1 x + e_1 \quad \text{or} \quad g = c_2 x + e_2 \quad \text{.....(iii)}$$

signal g, can be approximated as sum of component of g along x + error vector.

Now, the goal is to approximate g by cx, $g \simeq \hat{g} = cx$ ......(iv)

+ to chose c such that magnitude of error vector $e = g - cx$ can be minimized.

Since, projection of g along x is $\|g\|\cos\theta$

+ ~~$\|e\|\,x$~~

Therefore, $c\|x\| = \|g\|\cos\theta$

multiplying both sides by $\|x\|$, gives.

$c\|x\|^2 = \|g\|\|x\|\cos\theta = \langle g, x\rangle$

and $c = \dfrac{\langle g, x\rangle}{\langle x, x\rangle} = \dfrac{1}{\|x\|^2}\langle g, x\rangle$ .....(v)

if g and x are perpendicular to each other, then, component of g along x is zero, also, $c = 0$. Also, two vectors are orthogonal to each other if their inner product is zero.

Decomposition of a Signal and Signal components:

Consider the problem of approximating a real signal $g(t)$ in terms of another real signal $x(t)$ over an interval $[t_1, t_2]$:

$$g(t) \simeq c\, x(t) \quad ; \quad t_1 \le t \le t_2 \qquad \ldots (vi)$$

The error $e(t)$ in this approximation is

$$e(t) = \begin{cases} g(t) - c\, x(t) & ; \ t_1 \le t \le t_2 \\ 0 & ; \ \text{otherwise} \end{cases}$$

For best approximation, we need to minimize norm of $e(t)$.

$$\therefore \quad E_e = \int_{t_1}^{t_2} e^2(t)\, dt$$

$$= \int_{t_1}^{t_2} \{g(t) - c\, x(t)\}^2 dt \qquad \ldots (vii)$$

Right hand side is a a definite integral with dummy variable t, hence E_e is function of c, (not t). and e is minimum for some choice of c.

Therefore necessary condition is

$$\frac{dE_e}{dc} = 0 \qquad \ldots (viii)$$

or, $$\frac{d}{dc}\left[\int_{t_1}^{t_2} [g(t) - cx(t)]^2 dt\right] = 0$$

$$\Rightarrow \frac{d}{dc}\left[\int_{t_1}^{t_2} [g^2(t) + c^2 x^2(t) - c2g(t)x(t)]\, dt\right] = 0$$

$$= \frac{d}{dc}\left[\int_{t_1}^{t_2} g^2(t)\,dt\right] - \frac{d}{dc}\left[2c\int_{t_1}^{t_2} g(t)\,x(t)\,dt\right]$$

$$+ \frac{d}{dc}\left[c^2\int_{t_1}^{t_2} x^2(t)\,dt\right] = 0$$

from which we obtain

$$-2\int_{t_1}^{t_2} g(t)\,x(t)\,dt + 2c\int_{t_1}^{t_2} x^2(t)\,dt = 0$$

and $$c = \frac{\int_{t_1}^{t_2} g(t)\,x(t)\,dt}{\int_{t_1}^{t_2} x^2(t)\,dt} \qquad \text{.......} \quad (ix)$$

$$= \frac{1}{E_x}\int_{t_1}^{t_2} g(t)\,x(t)\,dt \qquad \text{.......} \quad (x)$$

So, if a signal $g(t)$ is approximated by another signal $x(t)$ as $g(t) \simeq c\,x(t)$, then optimum value of c that minimizes the energy of the error signal in this approximation is given by

$$c = \frac{1}{E_x}\int_{t_1}^{t_2} g(t)x(t)\,dt$$

With respect to real-valued signals, two signals $x(t)$ and $g(t)$ are at orthogonal when there is zero contribution from one signal to the other (i.e. $c=0$). Thus $x(t)$ and $g(t)$ are orthogonal if and only if

$$\int_{t_1}^{t_2} g(t)\,x(t)\,dt = 0 \qquad \text{......} \quad (xi)$$

Therefore, we can say that two signals are ortho-
-gonal if and only if their inner product
is zero. So, the equation (xi) is closely related to the concept of an inner product between two vectors.

The definition of the inner product of two N- dimensional vectors g and x.

$$\langle g, x \rangle = \sum_{i=1}^{N} g_i x_i$$

is identical to the integration of eq^n (xi). Therefore inner product of two (real-valued) signals $g(t)$ and $x(t)$, both defined over a time interval $[t_1, t_2]$, as.

$$\langle g(t), x(t) \rangle = \int_{t_1}^{t_2} g(t)\, x(t)\, dt \quad \ldots (xii)$$

So, norm of ~~the~~ a signal $g(t)$ may be define as,

$$\|g(t)\| = \sqrt{\langle g(t), g(t) \rangle} \quad \ldots\ldots (xii)$$

which is the square root of the signal energy in the time interval. So, it is clear that norm of a signal is analogous to the length of a finite dimensional vector. Time domain $[t_1, t_2]$ may be represented as θ.

Complex Signal Space and Orthogonality:—

Consider $g(t)$ and $x(t)$ are complex valued functions over an interval $(t_1 \le t \le t_2)$

Approximating a signal $g(t)$ by a signal $x(t)$ over an interval $(t_1 \le t \le t_2)$

$$g(t) \simeq c\, x(t) \quad \cdots\cdots\cdots \text{(xiii)}$$

$$\& \quad e(t) = g(t) - c\,x(t) \quad \cdots\cdots \text{(xiv)}$$

Energy E_x of the complex signal $x(t)$ over an interval $[t_1, t_2]$ is $\quad E_x = \int_{t_1}^{t_2} |x(t)|^2\, dt$

For the best approximation, we need to choose c that minimizes E_e, the energy of the error signal $e(t)$ is given by

$$E_e = \int_{t_1}^{t_2} |g(t) - c\,x(t)|^2\, dt \quad \cdots\cdots \text{(xv)}$$

$$\because \quad |u+v|^2 = (u+v)(u^* + v^*) = |u|^2 + |v|^2 + v^* u + u^* v \quad \cdots\cdots \text{(xvi)}$$

From eqn (xv), we can write

$$E_e = \int_{t_1}^{t_2} |g(t)|^2\, dt - \left| \frac{1}{\sqrt{E_x}} \int_{t_1}^{t_2} g(t)\, x^*(t)\, dt \right|^2 + \left| c\sqrt{E_x} - \frac{1}{\sqrt{E_x}} \int_{t_1}^{t_2} g(t)\, x^*(t)\, dt \right|^2$$

The first two terms of above expression is independent of c. So, E_e is minimized by choosing c such that the third term is zero. This yields the optimum coefficient

$$C = \frac{1}{E_x}\int_{t_1}^{t_2} g(t)\,x^*(t)\,dt \quad \text{...... (xvii)}$$

So, from the above discussions we can say that complex signals $x_1(t)$ and $x_2(t)$ are ortogonal over an interval $(t_1 \leq t \leq t_2)$ as long as:

$$\int_{t_1}^{t_2} x_1(t)\,x_2^*(t)\,dt = 0 \quad \text{or,} \quad \int_{t_1}^{t_2} x_1^*(t)\,x_2(t)\,dt = 0$$

This is general definition of ortogonality, (xviii) which reduces to eqn (xi), when the functions are real.

Similarly, the the inner product for complex signals over a time domain θ can be modified as,

$$\langle g(t), x(t)\rangle = \int_{\{t:\,t\in\theta\}} g(t)\,x^*(t)\,dt \quad \text{...... (xix)}$$

So, the norm of a signal $g(t)$ ~~is simply~~ may be given as.

$$\|g(t)\| = \left[\int_{\{t:\,t\in\theta\}} |g(t)|^2\,dt\right]^{1/2} \quad \text{...... (xx)}$$

<u>Energy of the Sum of Orthogonal Signals:</u>

For two orthogonal vectors x & y, if $z = x+y$, then

$$\|z\|^2 = \|x\|^2 + \|y\|^2$$

Similarly if signals $x(t)$ and $y(t)$ are orthogonal over an interval $[t_1, t_2]$ and if $z(t) = x(t) + y(t)$, then

$$E_z = E_x + E_y$$

if, $x(t)$ & $y(t)$ are complex, then from eqn (xvi),

$$\int_{t_1}^{t_2} |x(t)+y(t)|^2\,dt = \int_{t_1}^{t_2} |x(t)|^2\,dt + \int_{t_1}^{t_2} |y(t)|^2\,dt + \underbrace{\int_{t_1}^{t_2} x(t)\,y^*(t)\,dt}_{=0} + \underbrace{\int_{t_1}^{t_2} x^*(t)\,y(t)\,dt}_{=0}$$

(for orthogonal signals)

$$= \int_{t_1}^{t_2} |x(t)|^2 dt + \int_{t_1}^{t_2} |y(t)|^2 dt$$

This result can be extended to sum of any number of mutually orthogonal signals.

Thus, $$\int_{t_1}^{t_2} |x(t) + y(t)|^2 dt = \int_{t_1}^{t_2} |x(t)|^2 dt + \int_{t_1}^{t_2} |y(t)|^2 dt$$

Correlation of Signals:—

Similarity between two vectors is indicated by the angle θ between the vectors.

The smaller θ, the larger the similarity, and vice versa. The amount of similarity can be measured by $\cos\theta$. The larger the $\cos\theta$, the larger the similarity between two vectors. So, the, a suitable measure would be $\rho = \cos\theta = \dfrac{\langle g, x \rangle}{\|g\| \|x\|}$ ------(a)

This measure is independent of the lengths of g & x.

This similarity measure ρ is known as the correlation coefficient.

$\because \quad -1 \leq \rho \leq 1$ ------(b)

$\rho = 1$ (maximum value of ρ) $\rightarrow$ when two vectors are aligned in same direction.

$\rho = -1$. (minimum value of ρ) $\rightarrow$ when two vectors are aligned in opposite directions.

$\rho = 0 \rightarrow$ when two vectors are orthogonal.

Similarly for signals, the similarity index may be given as, (from eqn b)

$$\rho = \frac{1}{\sqrt{E_g E_x}} \int_{-\infty}^{\infty} g(t)\, x(t)\, dt \qquad \text{....... (c)}$$

where $x(t)$ & $g(t)$ are real valued signals over entire time interval $-\infty$ to $+\infty$.

ρ is independent of size or of energies of signals $x(t)$ & $g(t)$.

Using Cauchy-Schwarz inequality, it can be shown that $-1 \le \rho \le 1$ .... (d)

* Cauchy-Schwarz Inequality state that for two real signals $x(t)$ & $g(t)$, $\left(\int_{-\infty}^{\infty} g(t)\, x(t)\, dt\right)^2 \le E_g E_x$ with equality if and only if $x(t) = K g(t)$, where K is an arbitrary constant.

The above inequality is also valid for complex signals.

Correlation functions:-

Cross-correllation function of two complex signals $g(t)$ and $z(t)$, may be defined as -

$$\psi_{gz}(\tau) \equiv \int_{-\infty}^{\infty} z(t)\, g^*(t-\tau)\, dt = \int_{-\infty}^{\infty} z(t+\tau)\, g^*(t)\, dt \qquad \text{....... (e)}$$

$\psi_{gz}(\tau)$ is an indication of similarity (correlation) of $g(t)$ with $z(t)$ advanced (left-shifted) by τ seconds.

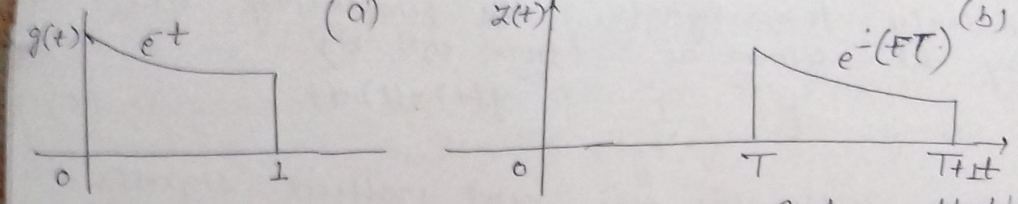

Fig: Physical Representation of the Autocorellation function

Autocorrelation function:

Correlation of a signal with itself is called the autocorrelation. The autocorrelation function $\psi_g(\tau)$ of a real signal $g(t)$ is defined as-

$$\psi_g(\tau) \equiv \int_{-\infty}^{\infty} g(t)\, g(t+\tau)\, dt \quad \cdots\cdots (e)$$

It measures the similarity of the signal $g(t)$ with its own displaced version. Autocorrelation function provides valuable spectral information about the signal.

Orthogonal Signal Space: Orthogonality of a signal set $x_1(t), x_2(t), \ldots, x_N(t)$ over a time domain Θ [may be an interval $[t_1, t_2]$] may be defined as:

$$\int_{t\in\Theta} x_m(t)\, x_n^*(t)\, dt = \begin{cases} 0 & ; m \neq n \\ E_n & ; m = n \end{cases} \quad \cdots (f)$$

iff $E_n = 1$, for all $n \in \{1, \ldots, N\}$, then the set is normalized and is called an 'orthonormal set'. ~~Four~~ An orthogonal set can always be normalized by minimizer dividing $x_n(t)$ by $\sqrt{E_n}$ for all n, Now consider the problem of approximating a signal $g(t)$ over Θ by a set of N mutually orthogonal signals: $x_1(t), x_2(t), \ldots, x_N(t)$:

$$g(t) \simeq c_1 x_1(t) + c_2 x_2(t) + \cdots + c_N x_N(t) \quad \cdots (f)$$

$$= \sum_{n=1}^{N} c_n x_n(t) \quad ; \quad t \in \Theta \quad \cdots (g)$$

Since the energy of the error signal $e(t)$ in this approximation, can be minimized if we choose

$$c_n = \frac{\int_{t\in\Theta} g(t) x_n^*(t)\, dt}{\int_{t\in\Theta} |x_n(t)|^2\, dt} = \frac{1}{E_n}\int_{\Theta} g(t) x_n^*(t)\, dt$$

$$n = 1, 2, \ldots, N \quad \cdots (h)$$

If the orthogonal set is complete, then the error energy $E_e \to 0$, and the representation in (f) is no longer approximation, but an equality. Now, let the N-term approximation error be defined by

$$e_N(t) = g(t) - c_1 x_1(t) + c_2 x_2(t) + \cdots + c_N x_N(t)$$

$$= g(t) - \sum_{n=1}^{N} c_n x_n(t) \quad ; \quad t \in \Theta \quad \cdots (i)$$

If the ortogonal basis is complete, then the error signal energy converges to zero, i.e.

$$\lim_{N\to\infty} \int_{t\in\Theta} |e_N(t)|^2\, dt = 0 \quad \cdots (j)$$

So, for all practical purposes we consider that signals are continious for all t, and the equality (j) states that error signal has zero energy as $N \to \infty$. Thus for $N \to \infty$, from eqn (f)

$$g(t) = c_1 x_1(t) + c_2 x_2(t) + \cdots + c_n x_n(t) + \cdots$$

$$= \sum_{n=1}^{\infty} c_n x_n(t) \quad ; \; t \in \theta \qquad \cdots\cdots (k)$$

where coefficients c_n are given by eq^n (h)

since the error signals energy approaches zero, it follows that the energy of $g(t)$ is now equal to the sum of energies of its orthogonal components.

The right hand side series of eq^n(k) is called the generalized fourier series of $g(t)$ with respect to the set $\{x_n(t)\}$.

When for the set $\{x_n(t)\}$, error energy $E_n \to 0$, as $N \to \infty$, for every member of some particular class, then $\{x_n(t)\}$ is complete on $\{t : \theta\}$ for that class of $g(t)$, and the set $\{x_n(t)\}$ is called a set of basis functions or basis signals. The class of (finite) energy signals over θ is denoted as $L^2\{\theta\}$.

<u>Parseval's Theorem:</u>

Since the energy of the sum of ortogonal signals is equal to the sum of their energies. Hence, ~~the~~ ~~from eqn (k)~~, the energy of the right-hand side of eq^n (k) is the sum of the energies of the individual orthogonal components. The energy of a component $C_n x_n(t)$ is $C_n^2 E_n$. Equating the energies of two sides of eqn (k) yields -

$$E_g = C_1^2 E_1 + C_2^2 E_2 + C_3 E_3 + \cdots$$
$$= \sum_n C_n^2 E_n \qquad \text{------} \quad (l)$$

The above results is known as Parseval's Theorem.

Spectral Analysis :

Time domain representations of a signal are often inconvenient in communication system analysis. The frequency domain represen- -tation of a given signal $g(t)$ in the time domain is called the spectrum. The purpose of spectral analysis is to resolve a given signal $g(t)$ into its constituent frequency components with appropriate amplitude + phase. Fourier series and fourier transforms are two powerful mathematical tools for such analysis. fourier series analysis is only applicable for periodic signals, while fourier transform can be analysed for spectral analysis of non-periodic or aperio- -dic signals. Thus fourier transform is more general in application than fourier series.

FOURIER SERIES :
Here we describe different forms of fourier series representation of a periodic signal.

1. Sinusoidal form :

Let $g_p(t)$ is a periodic signal having fundamental period T_0. The signal can be represented as an infinite sum of sinusoidal waveform using Fourier Series. The function $g_p(t)$ can be represented as —

$$g_p(t) = A_0 + 2\sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{2\pi n t}{T_0}\right) + B_n \sin\left(\frac{2\pi n t}{T_0}\right)\right] \quad (1)$$

where. A_0 & B_0 are unknown amplitudes of the cosine and sine terms, respectively.

$n/T_0 \rightarrow$ nth harmonic of the fundamental frequency, $f_0 = 1/T_0$.

$A_0 \rightarrow$ average value of $g_p(t)$ given by,

$$A_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g_p(t)\, dt \quad \cdots\cdots (2)$$

The coefficients A_n and B_n are given by,

$$A_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g_p(t) \cos\left(\frac{2\pi n t}{T_0}\right) dt \quad \cdots (3)$$

and.

$$B_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g_p(t) \sin\left(\frac{2\pi n t}{T_0}\right) dt \quad \cdots\cdots (4)$$

where, $n = 1, 2, 3 \ldots\ldots$

To apply the fourier series Representation it is sufficient that the function $g_p(t)$ satisfies the following conditions -

i) The function $g_p(t)$ is single valued within the interval T_0.

ii) The function $g_p(t)$ has at most a finite number of discontinuities and a finite number of maxima and minima in the interval T_0.

iii) The function $g_p(t)$ is absolutely integrable that is,

$$\int_{-T_0/2}^{T_0/2} |g_p(t)|\,dt < \infty$$

These conditions are known as Dirichlet's conditions, which are ~~functions of~~ satisfied by periodic signals, encountered in communication systems.

2. <u>Exponential form</u>:

The fourier series of equation (1) can be expressed in a more elegant form using the following relations -

$$\cos\left(\frac{2\pi nt}{T_0}\right) = \frac{1}{2}\left[\exp\left(j\frac{2\pi nt}{T_0}\right) + \exp\left(-j\frac{2\pi nt}{T_0}\right)\right] \quad \ldots 5(a)$$

$$\sin\left(\frac{2\pi nt}{T_0}\right) = \frac{1}{2j}\left[\exp\left(j\frac{2\pi nt}{T_0}\right) - \exp\left(-j\frac{2\pi nt}{T_0}\right)\right] \quad \ldots 5(b)$$

So, Thus the eqn (1) can be written as,

$$g_p(t) = \sum_{n=-\infty}^{\infty} c_n \exp\left(j\frac{2\pi n t}{T_0}\right) \quad \cdots\cdots (6)$$

where, $C_n = A_n - jB_n$, for $n > 0$

$= A_0$, for $n = 0$

$= A_n + jB_n$, for $n < 0$

The constant C_n can be evaluated as -

$$C_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g_p(t)\exp\left(-j\frac{2\pi n t}{T_0}\right)dt \quad \cdots (7)$$

where $n = 0, \pm 1, \pm 2, \cdots\cdots$

The series expression (6) is the complex exponential fourier series and C_n's are called complex fourier coefficients.

Eqn (6) shows that periodic signal contains both positive and negative frequencies which are harmonically related to the fundamental frequency $f_0 = 1/T_0$. For practical purposes the frequency is only positive. The negative frequency arises in the mathe--matical representation because of the use of complex valued basis functions e.g., $\exp\left(\pm\frac{j2\pi n t}{T_0}\right)$. which have no real physical meaning.

Discrete Spectrum:

The fourier series representation of a periodic signal $g_p(t)$ is equivalent to the resolution of the signal into various harmonic components of frequencies $0, \pm f_0, \pm 2f_0, \pm 3f_0, \ldots\ldots,$ where $f_0 = \frac{1}{T_0}$ is the fundamental frequency. The frequency domain description of the signal is called spectrum.

In general, fourier coefficient C_n is a complex number and can be expressed as-

$$C_n = |C_n| \exp[j \arg(C_n)] \quad \text{------} (8)$$

where $|C_n| \rightarrow$ amplitude of the nth harmonic component of $g_p(t)$ and $\arg(C_n) \rightarrow$ phase of the nth harmonic component. A plot of $|C_n|$ versus frequency yields amplitude spectrum & plot of $\arg(C_n)$ versus frequency yields the discrete phase spectrum of the signal. The spectra are called discrete because both amplitude and phase of C_n have non-zero values only for discrete frequencies that are integer (both positive and negative) multiples of fundamental freqn, $f_0 = \frac{1}{T_0}$, for real valued periodic function $g_p(t)$, from eqn (7),

$$C_{-n} = C_n^* \quad \ldots\ldots\ldots (9)$$

where, C_n^* is the complex conjugate of C_n.

Therefore, we have,

$$|c_{-n}| = |c_n| \qquad \cdots (10)$$

and $$\arg(c_{-n}) = -\arg(c_n) \qquad \cdots (11)$$

Thus the amp. spectrum of a real valued periodic signal is symmetrical i.e. an even function of n where as the phase spectrum is antisymmetrical or and odd function of n about the vertical axis passing through the origin.

Example : find the fourier series representation for the periodic train of rectangular pulse described over one period, $-\frac{T_0}{2} \le t \le \frac{T_0}{t}$, as $g_p(t) = A \quad ; -\frac{\tau}{2} \le t \le \frac{\tau}{2}$

$= 0 \quad$; for the rest of the period.

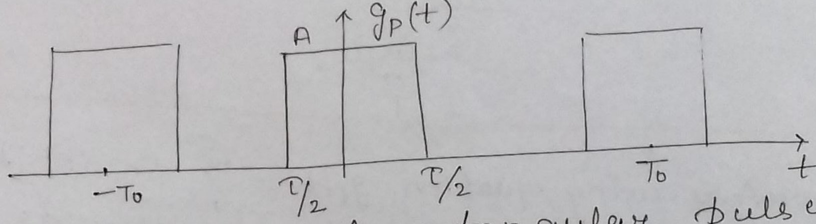

Fig: Periodic train of rectangular pulses of amplitude A.

Solution : from equation $c_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} g_p(t) \exp\left(-j\frac{2\pi nt}{T_0}\right) dt$

where $n = 0, \pm 1, \pm 2, \ldots$ can be calculated the complex fourier coefficient can be calculated

as $c_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} A \exp\left(-j\frac{2\pi nt}{T_0}\right) dt$

$= \frac{1}{T_0} \int_{-\tau/2}^{\tau/2} A \exp\left(-j\frac{2\pi nt}{T_0}\right) dt$

$= \frac{A}{n\pi} \sin\left(\frac{n\pi\tau}{T_0}\right) \quad ; \quad n = 0, \pm 1, \pm 2, \ldots$

$= \frac{A\tau}{T_0} \operatorname{sinc}\left(\frac{\pi\tau}{T_0}\right)$

where the sinc function is defined by

$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

~~2 | 200 | 100, 2 | 100 | 50, 2 | 50 | 25, 2 | 25 | 12, 2 | 12 | 6, 2 | 6 | 3, 2 | 3 | 1~~

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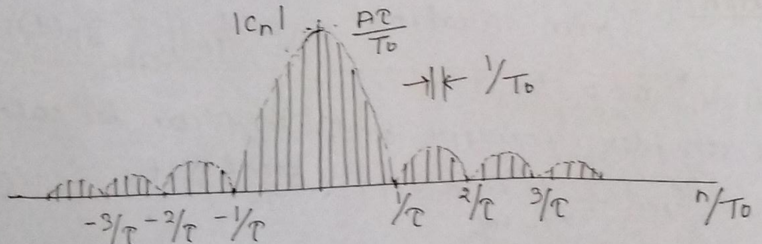
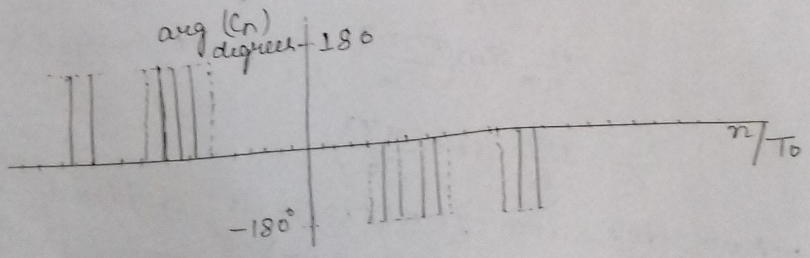
Therefore using equation $g_p(t) = \sum_{n=-\infty}^{\infty} c_n \exp\left(j\frac{2\pi n t}{T}\right)$, the function $g_p(t)$ can be written as

$$g_p(t) = \frac{A\tau}{T_0} \sum_{n=-\infty}^{\infty} \operatorname{sinc}\left(\frac{n\tau}{T_0}\right) \exp\left(j\frac{2\pi n t}{T_0}\right)$$



a) Amplitude spectrum.



b) phase spectrum of the periodic pulse train of duty cycle 0.2.

Fourier Transforms: Fourier transform is a mathematical tool which enables one to do frequency domain analysis of non-periodic signals.

Consider a signal $g(t)$ defined arbitrarily as a function of time t. As shown in below figure. Consider a signal $g_p(t)$ with period T_0 & signal $g(t)$ represents one cycle of $g_p(t)$.

Thus,
$$g(t) = \lim_{T_0 \to \infty} g_p(t) \quad \text{......(8)}$$

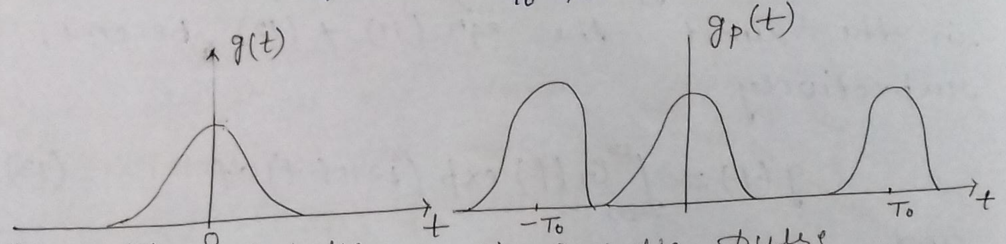


Figure (a) single pulse b) periodic pulse

Now, $g_p(t)$ being periodic function can be expressed in the complex exponential form of eqn (7) as –

$$g_p(t) = \sum_{n=-\infty}^{\infty} c_n \exp\left(j\frac{2\pi n t}{T_0}\right) \quad \text{.......(9)}$$

$$\text{where, } c_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g_p(t)\left(-j\frac{2\pi n t}{T_0}\right)dt \quad \text{......(10)}$$

Defining $\Delta f = 1/T_0$, $f_n = n/T_0$

and $G(f_n) = c_n T_0$

we obtain from eqn (9) + (10) the following,

$$g_p(t) = \sum_{n=-\infty}^{\infty} G(f_n)\, e(j2\pi f_n t)\,\Delta f \quad \text{------ (11)}$$

and. $$G(f_n) = \int_{-T_0/2}^{T_0/2} g_p(t) \exp(-j2\pi f_n t)\, dt \quad \text{---- (12)}$$

As $T_0 \to \infty$ or Δf approaches zero, then in the limit the discrete frequency f_n becomes a continious frequency f and the discrete sum in equation (11) can be replaced by integration. Also as T_0 tends to infinity, the function $g_p(t)$ approaches $g(t)$. Thus in the limit the eqn (11) & (12) become, respectively,

$$g(t) = \int_{-\infty}^{\infty} G(f) \exp(j2\pi f t)\, df \quad \text{--- (13)}$$

and

$$G(f) = \int_{-\infty}^{\infty} g(t) \exp(-j2\pi f t)\, dt \quad \text{--- (14)}$$

$g(t) \to$ time domain representation of signal

$G(f) \to$ frequency domain representation of signal.

$G(f)$ is called fourier series transform of $g(t)$ & $g(t)$ is called inverse fourier transform of $G(f)$. The functions $g(t)$ and $G(f)$ are said to constitute a fourier Transform pair, and one is called the mate of each other. We write -

$$g(t) \leftrightarrow G(f)$$

$$F[g(t)] = G(f)$$

$$G(f) = G^*(-f) \qquad \cdots (16)$$

So, if $g(t)$ is real valued function of time then

$$|G(-f)| = |G(f)| \qquad \cdots (17)$$

and

$$\theta(-f) = -\theta(f) \qquad \cdots (18)$$

That is, the amplitude spectrum $|G(f)|$ of a real-valued signal is a even function of f while the phase spectrum $\theta(f)$ is an odd function of f.

Some Properties of Fourier Transforms :

These properties enable us to know the effect in one domain caused by certain operation in the other domain. Some of the important properties are-

(1) Linearity (Superposition)

Let $g_1(t) \leftrightarrow G_1(f)$ and $g_2(t) \leftrightarrow G_2(f)$, we have then the constants a_1 and a_2,

Then

$$a_1 g_1(t) + a_2 g_2(t) \leftrightarrow a_1 G_1(f) + a_2 G_2(f)$$

2) Duality (Symmetry)

If $g(t) \leftrightarrow G(f)$, then

$$G(t) \leftrightarrow g(-f)$$

Proof The function $g(t)$ can be written in terms of $G(f)$ as

$$g(t) = F^{-1}[G(f)]$$

For a signal $g(t)$ to be Fourier transformable, it is sufficient that $g(t)$ satisfies the Dirichlet's conditions:

(i) The functions must be single-valued with a finite number of maxima and a finite number of discontinuities in any finite time interval.

ii) The function must be absolutely integrable that is $\int_{\infty}^{\infty} |g(t)|\, dt < \infty$

<u>Continious Spectrum</u>: An aperiodic signal $g(t)$ of finite energy can then be expressed as a continious sum of exponential functions in the frequency interval $-\infty$ to $+\infty$.

In general. the fourier transform $G(f)$ is a complex function of frequency f, so that we may write,

$$G(f) = |G(f)| \exp[j \arg(G(f))] \quad \cdots\cdots (15)$$

$$= |G(f)| \exp[j\theta(f)] \quad \cdots\cdots (16)$$

where,
$|G(f)|$ is continious amplitude spectrum of $g(t)$ and $\theta(f)$ is continious phase spectrum of $g(t)$. These spectra are continious because both amplitude and phase of $G(f)$ are defined for all frequencies. For a real valued function $g(t)$, we have,

$$g(t) = \int_{-\infty}^{\infty} G(f) \exp(+j2\pi f t)\, df$$

replacing dummy variable f in above integral by x and t by $-t$, we get.

$$g(-t) = \int_{-\infty}^{\infty} G(x) \exp(-j2\pi x t)\, dx$$

Replacing t by f we get,

$$g(-f) = \int_{-\infty}^{\infty} G(x) \exp(-j2\pi f x)\, dx$$

Again by replacing dummy variable by another variable t, we get

$$g(-f) = \int_{-\infty}^{\infty} G(t) \exp(-j2\pi f t)\, dt$$

$$= F[G(t)]$$

Thus, $G(t) \leftrightarrow g(-f)$

If $g(t)$ is an even function, the symmetry will be perfect. For an even function $g(-t) = g(t)$ and the duality property becomes

$$G(t) \leftrightarrow g(f) \qquad \text{---} (19)$$

3) <u>Scaling</u>

If $g(t) \leftrightarrow G(f)$, then

$g(at) \leftrightarrow \frac{1}{|a|} G\left(\frac{f}{a}\right)$; a is a real constt., +ive or -ive.

<u>Proof</u> Fourier transform of $g(at)$ is

$$F[g(at)] = \int_{-\infty}^{\infty} g(at) \exp(-j2\pi f t)\, dt$$

let $at = x$, the for $a > 0$, we get.

$$F[g(at)] = \frac{1}{a}\int_{-\infty}^{\infty} g(x) \exp\left[-j2\pi\left(\frac{f}{a}\right)x\right] dx$$
$$= \frac{1}{a} G(f/a)$$

On the other hand, if $a < 0$, then

$$F[g(at)] = \frac{1}{a}\int_{-\infty}^{\infty} g(x) \exp\left[-j2\pi\left(\frac{f}{a}\right)x\right] dx$$
$$= -\frac{1}{a} G(f/a)$$

Thus we get, $g(at) \leftrightarrow \frac{1}{|a|} G(f/a)$ ..... (20

Scaling property states that the compression of a signal by in the time domain is equivalent to expansion of its fourier spectrum in the frequency domain and vice versa.

4) Frequency-shifting (frequency translation)

If $g(t) \leftrightarrow G(f)$, then

$$g(t)\exp(j2\pi f_0 t) \leftrightarrow G(f-f_0) \quad \text{.....(21)}$$

5) Time shifting

$g(t) \leftrightarrow G(f)$, then

$$g(t-t_0) \leftrightarrow G(f)\exp(-j2\pi f t_0)\,dt \quad \text{...(22)}$$

6) Differentiation in the time domain:

If $g(t) \leftrightarrow G(f)$ and the first derivative of $g(t)$ is fourier transformable, then

$$\frac{d\,g(t)}{dt} \leftrightarrow j2\pi f\, G(f) \quad \text{........(23)}$$

for the nth derivative of $g(t)$

$$\frac{d^n g(t)}{dt^n} \leftrightarrow (j2\pi f)^n G(f)$$

if $\frac{d^n g(t)}{dt^n}$ is fourier transformable.

7) Integration in the time domain :

if $g(t) \leftrightarrow G(f)$, then

$$\int_{-\infty}^{t} g(\tau)\,d\tau \leftrightarrow \frac{1}{j2\pi f} G(f),$$ provided

that $\left(\frac{G(f)}{f}\right)$ is bounded at $f=0$,

that is $G(0) = 0$.

8) frequency differentiation:

If $g(t) \leftrightarrow G(f)$, then

$$-j2\pi t\, g(t) \leftrightarrow \frac{dG(f)}{df}$$

9) Conjugate function:

If $g(t) \leftrightarrow G(f)$, then for a complex valued

time function $g(t)$, $g^*(t) \leftrightarrow G^*(-f)$

where * denotes, the complex conjugate operator.

10) ~~Multiplication in time & frequency domain:~~

Fourier transforms of some simple functions:—

1) Single-sided exponential signal:—

For, a one-sided decaying exponential, the function $g(t)$ can be written as $g(t) = \exp(-at)u(t)$

where, $u(t) = 1$, for $t > 0$

$u(t) = 0$ for $t < 0$

The fourier transform of the signal is

$$G(f) = \int_{-\infty}^{\infty} \exp(-at)\, u(t) \exp(-j2\pi f t)\, dt$$

$$= \int_{0}^{\infty} \exp(-a(a+j2\pi f)t)\, dt$$

$$= \frac{1}{a+j2\pi f}$$

$$= \frac{1}{\sqrt{a^2+(2\pi f)^2}} \exp\left(-j \tan^{-1}\left(2\pi\left(\frac{f}{a}\right)\right)\right)$$

where, $|G(f)| = \dfrac{1}{\sqrt{a^2+(2\pi f)^2}}$

and the phase angle

$$G(f) = -\tan^{-1}\left(2\pi\left(\frac{f}{a}\right)\right)$$

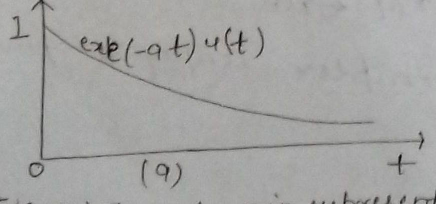
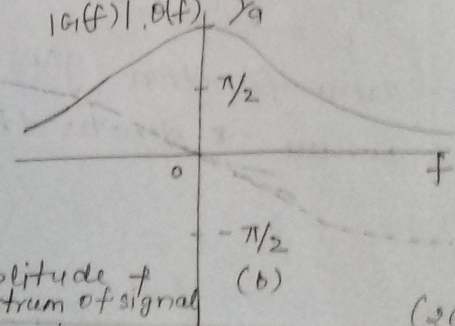


Fig: (a) Time domain representation of signal



(b) Amplitude & phase spectrum of signal

(26)