Solutions to Problem Set 4

Note: This problem set contains a number of questions that will require you to write proofs. The goal is not only to have correct proofs, but also to make sure they are clear, orderly, and well-presented.

Problem 1. The EECS department has six committees that meet once a month. How many different meeting times must be used to ensure that no one is scheduled to be at two meetings at the same time? The committees are $C_1 = \{\text{Abelson, Meyer, Karger}\}$, $C_2 = \{\text{Meyer, Leiserson, Rivest}\}$, $C_3 = \{\text{Abelson, Rivest, Karger}\}$, $C_4 = \{\text{Leiserson, Rivest, Karger}\}$, $C_5 = \{\text{Abelson, Meyer}\}$, $C_6 = \{\text{Meyer, Rivest, Karger}\}$. Explain how you arrived at the answer.

Solution. 5 meeting times are required.

Draw and color a graph with six nodes, each representing a different committee. Connect two nodes with an edge iff there is a conflict (overlapping members) between the two committees. This is almost a complete graph on the six nodes, with only the edge between C_4 and C_5 missing. Now find the minimum coloring. Since nodes 4 and 5 are not connected, they may be colored the same. But every other node must receive a different color since they are all connected to each other. ■

Problem 2. Let G be an undirected graph. Prove that there is a path from every vertex of odd degree to some other vertex of odd degree in G.

Solution. *Proof.* For some graph G, pick an arbitrary vertex v of odd degree. Take the subset of the graph that is connected to v via some path i.e. the connected component of G that contains v; call this G_v. Since G_v is itself a graph, it must have total degree even, which means there must be another vertex w of odd degree in G_v. Since G_v is connected, that means there must exist a path between v and another odd degree vertex. You can do this for any/every odd degree vertex in G. □

Proof. This is an alternate proof by induction on the number of edges, k.

Let $P(k)$ be the proposition that there is a path from every vertex of odd degree to some other vertex of odd degree in G, where G has k edges.

The base case of $k = 0$ is vacuously true, since all nodes are of even degree zero.

Given graph, G, with $k + 1$ edges, there are three cases:

- There are no odd degree nodes in G; $P(k + 1)$ is vacuously true

- There is an edge, g, from an odd-degree node, N, to an even-degree node, M.

  Remove g to form a k-edge graph H. By induction, all odd-degree nodes in H have paths to other odd degree nodes in H. Note that all nodes besides M and N have the same degree in G and in H, while N and M switch to having even and odd degree, respectively, in H. Hence, any path from an odd-degree node in H to another odd-degree node in H will also be a path between these nodes in G. So except possibly for N, all odd-degree nodes in G have paths to other odd-degree nodes in G.

  But M has odd degree in H, and so it has a path in H to an odd-degree node, O, in H. Of course O =/= N, since N is even-degree in H. So there is a path in G from N following edge g to M to O.

  Hence, every odd-degree node in G has a path to another odd-degree node.

- There is an edge, g, from an odd-degree node, N, to another odd-degree node, M.

  As in the previous case, all odd-degree nodes except possibly M and N have paths to other odd-degree nodes besides M and N. But the edge g itself is a length-one path between the odd-degree nodes M and N. So in fact every odd-degree node in G has a path to another odd-degree node.

□

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Problem 3. A 2-colorable graph is called *bipartite* (two pieces).

(a) Prove by induction that every tree is bipartite.

Solution. We first prove the following claim:

Lemma. *By a* leaf *in a graph, we mean a vertex of degree* 1*. We prove that every tree on at least* 2 *nodes has a leaf.*

Proof. Let T be a tree, and let P be a simple path in T of maximum length. (Because we disallow repeated vertices, the length of any simple path is upper-bounded by the number of vertices in T. This implies there must be at least one simple path with maximum length.)

Suppose that $P = \{v_1, v_2 \ldots , v_k\}$ in this order. We claim that v_1 is a leaf. For if not, then there is an edge (v, v_1) in T, where $v \neq v_2$. Now if $v \notin P$, we can attach v to the end of P to obtain a longer path — a contradiction. But if $v \in P$, then since $v \neq v_2$, this will give a cycle in T — also a contradiction. □

Given this fact, we prove the theorem by induction on n.

Proof. Base case: $n = 1$. We can color the single node anything we want, thereby using at most two colors.

Inductive step. Suppose that we know that every tree on n nodes is bipartite, for some $n \geq 1$. Let T be a tree on $n+1$ nodes; we need to show how to two-color it. Since $n+1 \geq 2$, we know that T has a leaf v. Let w be the unique neighbor of v. Let $T' = T - \{v\}$. Note that T' is a tree, since it is still connected and has no cycles. Thus, by the induction hypothesis, we can two-color it.

Now we define the following two-coloring of T. If u is a vertex of T not equal to v, give u the color that it had in our two-coloring of T'. Then give v whichever color we didn't give to its neighbor w. As this is a valid two-coloring of T, this completes the inductive step and the proof. □

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(b) Prove that a graph is bipartite *if and only if* every simple cycle is of even length.[1]

Solution. Assume a graph has a simple cycle of odd length. Let the cycle be $v_1, v_2, \ldots, v_n, v_1$, where n is odd. Suppose that two colors (red and blue) sufficed to color the graph containing this circuit. Without loss of generality let the color of v_1 be red. Then v_2 must be blue, v_3 must be red, and so on, until finally v_n must be red (since n is odd). But this is a contradiction, since v_n is adjacent to v_1. Therefore at least three colors are needed. Hence a graph is 2-colorable only if every simple cycle is of even length.

Conversely, assume the graph has no simple cycle of odd length. We show that it is possible to color the graph using two colors. Without loss of generality, assume that the graph is connected. (As otherwise we can repeat the argument for each connected component of the graph). Select a node, say m, in the graph and color it red. Color all the nodes adjacent to m blue. Call the set of all such nodes S_1. Now color all the nodes that are adjacent to the nodes in S_1 but have not yet been colored, red. Call the set of all such nodes S_2. Continue this process until you have colored all the nodes. We now show that no two nodes in S_i are connected by an edge, for all i. This is sufficient to establish that our coloring is valid, since two adjacent nodes are either in the same set S_i — which we are going to show can't happen — or are in two consecutive sets S_i and S_{i+1} that are colored differently. This is true because if one vertex gets included in S_i before its neighbor is colored, then the neighbor will be colored in the next step.

Assume that we can find two nodes u and w in some set S_i that are adjacent in the graph. If u and w are adjacent to a node in S_{i-1} then we have a simple cycle of length 3, which is impossible. So let $u_{i-1} \neq w_{i-1} \in S_{i-1}$ be nodes adjacent to u and w respectively. Now if u_{i-1} and w_{i-1} are adjacent to a single node in S_{i-2}, say c, then $u, u_{i-1}, c, w_{i-1}, w, u$ is a simple cycle of length 5, which is impossible. So let $u_{i-2} \neq w_{i-2} \in S_{i-2}$ be nodes adjacent to u_{i-1}, and w_{i-1} respectively. Continuing this way we arrive at nodes $u_1 \neq w_1 \in S_1$. However these are adjacent to m. This gives us a simple cycle $m, u_1, u_2, \ldots, u_{i-1}, u, w, w_{i-1}, w_{i-2} \ldots, w_1, m$, where $u_j, w_j \in S_j$. But this cycle has length $2i+1$, which is odd, hence a contradiction. Therefore, for all i we do not have two nodes in S_i that are connected by an edge. Therefore our coloring is valid. ■

[1]Reminder: for *iff* you need to prove both directions of the implication.
Hint: Can v have both an even and an odd length path to w?

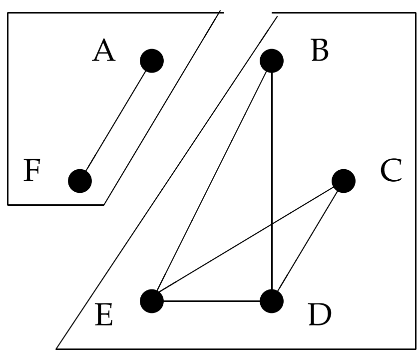
(c) Explain why (a) follows immediately from (b).

Solution. Since a tree has no simple cycles, it has no simple cycles of odd length, and thus (a) follows directly from (b). ■

Problem 4. The *complement* of an undirected graph $G = (V, E)$ is the graph $G^c = (V, E^c)$. That is, G^c contains exactly those edges that are not in G.

(a) Let the graph $H = \big(\{A, B, C, D, E, F\}, \big\{\{A, F\}, \{B, D\}, \{B, E\}, \{C, D\}, \{C, E\}, \{D, E\}\big\}\big)$. Draw the graph H and circle the connected components.

Solution. There are two connected components circled below.



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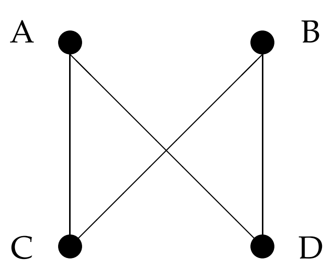
(b) Let the graph $J = \big(\{A, B, C, D\}, \big\{\{A, B\}, \{C, D\}\big\}\big)$. Draw the complement of J.

Solution. ■



(c) Prove the following theorem:

Theorem 4.1. *If an undirected graph G is not connected then G^c is connected.*

Solution. *Proof.* Suppose $G = (V, E)$ is not connected. Let a, b be any two vertices in V. Consider the following two cases:

Case 1: If nodes a and b are in different connected components of G, then they are connected in G^c. Then there is no edge in E connecting a and b. Thus G^c contains an edge connecting a and b by definition.

Case 2: If nodes a and b are in the same connected component, then they are connected in G^c.

Let c be a vertex in a different connected component. We are guranteed that such a vertex exists because we know that G is disconnected therefore it must contain more than one connected components. By case 1, a and c are connected, as are b and c. Thus a and b are connected through c.

Thus G^c contains a path from a to b for any pair of vertices a, b, and so is connected. □

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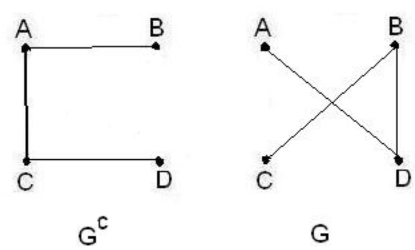
(d) However the converse of this theorem is not true: if G^c is connected, then G is not necessarily disconnected. Give an example of a connected graph whose complement is also connected.

Solution. Many solutions are possible. One is shown below. Another is that G^c is a graph on 4 nodes arranged as a square with three edges forming the letter Z. Then G has its three edges forming the letter N.



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Problem 5. The Touring Knight A knight is a chess piece that can move either two spaces horizontally and one space vertically or one space horizontally and two spaces vertically. That is, a knight on square (x, y) can move to any of the eight squares $(x \pm 2, y \pm 1), (x \pm 1, y \pm 2)$, if these squares exist on the chessboard (which is normally 8×8).

A knight's tour is a sequence of legal moves by a knight starting at some square and visiting each square exactly once. A knight's tour is called reentrant if there is a legal move that takes the knight from the last square of the tour back to where the tour began. We can model knight's tours using the graph that has a vertex for each square on the board,with an edge connecting two vertices if a knight can legally move between the squares represented by these vertices.

(a) Show that finding a reentrant knight's tour on an $m \times n$ chessboard is equivalent to finding a Hamilton circuit on the corresponding graph.

Solution. In a Hamiltonian circuit, we need to visit each vertex once moving along the edges and returning to the starting point. A reentrant knight's tour is precisely such a circuit, since we visit each square once, making legal moves, and return to the starting point. ■

(b) Show that the graph representing the legal moves of a knight on a $m \times n$ chessboard, wherever m and n are positive integers, is *bipartite*.

Solution. Each square of the board can be thought of as a pair of integers (x, y). Let A be the set of squares for which $x + y$ is odd, and let B be the set of square for which $x + y$ is even. This partitions the vertex set of the graph representing the legal moves of a knight on the board into two parts. Now every move of the knight changes $x + y$ by an odd number – either $1 + 2 = 3$, $2 - 1 = 1$, $1 - 2 = -1$, or $-1 - 2 = -3$. Therefore every edge in this graph joins a vertex in A to a vertex in B. Thus the graph is bipartite. ■

(c) Deduce that there is no reentrant knight's tour on an $m \times n$ chessboard when m and n are both odd.

Solution. Since there are mn squares on the $m \times n$ board, if both m and n are odd, there are an odd number of squares. Since by part (b), the corresponding graph is bipartite, from PS 4, problem 3, we know that it has no cycle of odd length, *i.e.* it has no Hamilton circuit (since there are an odd number of nodes). Hence, there is no reentrant knight's tour. ■