# 量子力学

Anko

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## 1　ベクトル解析

**定義** (Kronecker のデルタ)**.**

$$\delta_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases} \tag{1}$$

**定義** (レビ・チビタの完全反対称テンソル (Levi-Civita antisymmetric tensor))**.**

$\epsilon_{ijk}$ は $1, -1, 0$ の値を取る. $\epsilon_{xyz} = 1$ であり, 任意の 2 つの添字の交換に対して符号を変え, また任意の 2 つの添字の値が等しければ 0 となる.

$$\epsilon_{\mu_1 \cdots \mu_k} := \begin{cases} \mathrm{sgn} \begin{pmatrix} 1 & \cdots & k \\ \mu_1 & \cdots & \mu_k \end{pmatrix} & (\mu_1 \cdots \mu_k \text{が順列のとき}) \\ 0 & (else) \end{cases} \tag{2}$$

$$= \begin{cases} 1 & (\mu_1 \cdots \mu_k \text{が偶置換のとき}) \\ -1 & (\mu_1 \cdots \mu_k \text{が奇置換のとき}) \\ 0 & (else) \end{cases} \tag{3}$$

**定義** (Einstein の縮約記法)**.**
同じ項で添字が重なる場合はその添字について和を取る.

$$A_iB_i = \sum_{i=x,y,z} A_iB_i \tag{4}$$

**定義.**
ベクトルについて内積と外積を定義する。

$$\boldsymbol{A}\cdot\boldsymbol{B} = A_xB_x + A_yB_y + A_zB_z = A_iB_i \tag{5}$$

$$\boldsymbol{A}\times\boldsymbol{B} = (A_yB_z - A_zB_y, A_zB_x - A_xB_z, A_xB_y - A_yB_x) = \boldsymbol{e}_i\epsilon_{ijk}A_jB_k \tag{6}$$

**定義.**

$$\boldsymbol{\nabla}\boldsymbol{A} = \left(\frac{\partial A_x}{\partial x}, \frac{\partial A_y}{\partial y}, \frac{\partial A_z}{\partial z}\right) \tag{7}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \partial_i A_i \tag{8}$$

$$\boldsymbol{\nabla}\times\boldsymbol{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) = \boldsymbol{e}_i\epsilon_{ijk}\partial_j A_k \tag{9}$$

**定理 1.**

$$\boldsymbol{\nabla}(f+g) = \boldsymbol{\nabla} f + \boldsymbol{\nabla} g \tag{10}$$

$$\boldsymbol{\nabla}(fg) = f\boldsymbol{\nabla} g + g\boldsymbol{\nabla} f \tag{11}$$

$$\boldsymbol{\nabla}(\boldsymbol{A}\cdot\boldsymbol{B}) = \boldsymbol{A}\times(\boldsymbol{\nabla}\times\boldsymbol{B}) + \boldsymbol{B}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) + (\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{B} + (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{A} \tag{12}$$

$$\boldsymbol{\nabla}\cdot(\boldsymbol{A}+\boldsymbol{B}) = \boldsymbol{\nabla}\cdot\boldsymbol{A} + \boldsymbol{\nabla}\cdot\boldsymbol{B} \tag{13}$$

$$\boldsymbol{\nabla}\cdot(f\boldsymbol{A}) = f(\boldsymbol{\nabla}\cdot\boldsymbol{A}) + \boldsymbol{A}\cdot(\boldsymbol{\nabla} f) \tag{14}$$

$$\boldsymbol{\nabla}\cdot(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A}) - \boldsymbol{A}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B}) \tag{15}$$

$$\boldsymbol{\nabla}\times(\boldsymbol{A}+\boldsymbol{B}) = \boldsymbol{\nabla}\times\boldsymbol{A} + \boldsymbol{\nabla}\times\boldsymbol{B} \tag{16}$$

$$\boldsymbol{\nabla}\times(f\boldsymbol{A}) = f(\boldsymbol{\nabla}\times\boldsymbol{A}) - \boldsymbol{A}\times(\boldsymbol{\nabla} f) \tag{17}$$

$$\boldsymbol{\nabla}\times(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{A}(\boldsymbol{\nabla}\cdot\boldsymbol{B}) - \boldsymbol{B}(\boldsymbol{\nabla}\cdot\boldsymbol{A}) + (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{A} - (\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{B} \tag{18}$$

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A}) = 0 \tag{19}$$

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla} f) = \boldsymbol{0} \tag{20}$$

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) = \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\boldsymbol{A}) - \nabla^2\boldsymbol{A} \tag{21}$$

◇

証明

ベクトルのときは各要素について考える。

$$(\boldsymbol{\nabla}(f+g))_i = \partial_i(f+g) = \partial_i f + \partial_i g = (\boldsymbol{\nabla} f + \boldsymbol{\nabla} g)_i \tag{22}$$

$$(\boldsymbol{\nabla}(fg))_i = \partial_i(fg) = f\partial_i g + g\partial_i f = (f\boldsymbol{\nabla} g + g\boldsymbol{\nabla} f)_i \tag{23}$$

$$\begin{aligned}
(\boldsymbol{\nabla}(\boldsymbol{A}\cdot\boldsymbol{B}))_i &= \partial_i(A_jB_j) \qquad (24)\\
&= (A_j\partial_iB_j + B_j\partial_iA_j) - (A_j\partial_jB_i + B_j\partial_jA_i) + (A_j\partial_jB_i + B_j\partial_jA_i) \qquad (25)\\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})(A_j\partial_lB_m + B_j\partial_lA_m) + (A_j\partial_jB_i + B_j\partial_jA_i) \qquad (26)\\
&= \epsilon_{kij}\epsilon_{klm}(A_j\partial_lB_m + B_j\partial_lA_m) + (A_j\partial_jB_i + B_j\partial_jA_i) \qquad (27)\\
&= \epsilon_{ijk}A_j\epsilon_{klm}\partial_lB_m + \epsilon_{ijk}B_j\epsilon_{klm}\partial_lA_m + A_j\partial_jB_i + B_j\partial_jA_i \qquad (28)\\
&= (\boldsymbol{A}\times(\boldsymbol{\nabla}\times\boldsymbol{B}) + \boldsymbol{B}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) + (\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{B} + (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{A})_i \qquad (29)
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\nabla}\cdot(\boldsymbol{A}+\boldsymbol{B}) &= \partial_i(A_i+B_i) = \partial_i A_i + \partial_i B_i && (30)\\
&= \boldsymbol{\nabla}\cdot\boldsymbol{A} + \boldsymbol{\nabla}\cdot\boldsymbol{B} && (31)\\
\boldsymbol{\nabla}\cdot(f\boldsymbol{A}) &= \partial_i(fA_i) = f\partial_i A_i + A_i\partial_i f && (32)\\
&= f(\boldsymbol{\nabla}\cdot\boldsymbol{A}) + \boldsymbol{A}\cdot(\boldsymbol{\nabla} f) && (33)\\
\boldsymbol{\nabla}\cdot(\boldsymbol{A}\times\boldsymbol{B}) &= \partial_i(\epsilon_{ijk}A_jB_k) && (34)\\
&= \epsilon_{ijk}(B_k\partial_i A_j + A_j\partial_i B_k) && (35)\\
&= B_k\epsilon_{kij}\partial_i A_j - A_j\epsilon_{jik}\partial_i B_k && (36)\\
&= \boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A}) - \boldsymbol{A}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B}) && (37)
\end{aligned}$$

$$\begin{aligned}
(\boldsymbol{\nabla}\times(\boldsymbol{A}+\boldsymbol{B}))_i &= \epsilon_{ijk}\partial_j(A_k+B_k) && (38)\\
&= \epsilon_{ijk}\partial_j A_k + \epsilon_{ijk}\partial_j B_k && (39)\\
&= \boldsymbol{\nabla}\times\boldsymbol{A} + \boldsymbol{\nabla}\times\boldsymbol{B} && (40)\\
(\boldsymbol{\nabla}\times(f\boldsymbol{A}))_i &= \epsilon_{ijk}\partial_j(fA_k) && (41)\\
&= f\epsilon_{ijk}\partial_j A_k + \epsilon_{ijk}A_k\partial_j f && (42)\\
&= f\epsilon_{ijk}\partial_j A_k - \epsilon_{ikj}A_k\partial_j f && (43)\\
&= (f(\boldsymbol{\nabla}\times\boldsymbol{A}) - \boldsymbol{A}\times(\boldsymbol{\nabla} f))_i && (44)\\
(\boldsymbol{\nabla}\times(\boldsymbol{A}\times\boldsymbol{B}))_i &= \epsilon_{ijk}\partial_j\epsilon_{klm}A_lB_m && (45)\\
&= \epsilon_{kij}\epsilon_{klm}(B_m\partial_j A_l + A_l\partial_j B_m) && (46)\\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})(B_m\partial_j A_l + A_l\partial_j B_m) && (47)\\
&= (B_j\partial_j A_i + A_i\partial_j B_j) - (B_i\partial_j A_j + A_j\partial_j B_i) && (48)\\
&= A_i\partial_j B_j - B_i\partial_j A_j + B_j\partial_j A_i - A_j\partial_j B_i && (49)\\
&= \boldsymbol{A}(\boldsymbol{\nabla}\cdot\boldsymbol{B}) + \boldsymbol{B}(\boldsymbol{\nabla}\cdot\boldsymbol{A}) + (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{A} - (\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{B} && (50)
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A}) &= \partial_i(\epsilon_{ijk}\partial_j A_k) = \epsilon_{ijk}\partial_i\partial_j A_k = 0 && (51)\\
(\boldsymbol{\nabla}\times(\boldsymbol{\nabla} f))_i &= \epsilon_{ijk}\partial_j\partial_k f = 0 && (52)\\
(\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A}))_i &= \epsilon_{ijk}\partial_j(\epsilon_{klm}\partial_l A_m) && (53)\\
&= \epsilon_{kij}\epsilon_{klm}\partial_j\partial_l A_m && (54)\\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\partial_j\partial_l A_m && (55)\\
&= \partial_j\partial_i A_j - \partial_j^2 A_i && (56)\\
&= (\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\boldsymbol{A}) - \nabla^2\boldsymbol{A})_i && (57)
\end{aligned}$$

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