Review

- State space $\boxed{S}$ indexes potential outcomes of states s
  - ex: $S=\{1,2,3,4,5,6\}$ for dice
- Probability distribution function $\mathbb{P}$ that maps subsets $\boxed{A \subseteq S}$ to $[0,1]$ such that
  1. $P(S)=1$
  2. $\mathbb{P}(A \cup B)=\mathbb{P}(A)+\mathbb{P}(B)$

  * assuming A & B disjoint
- we denote $p_s = \mathbb{P}(\{s\})$
- A Random Variable (X) is any function on the state space

ex. I bet \$1 my die roll is at least 5
payoff = Random Variable

$\{-1,1\} \to$ x's of state function X

Expectation

- The Expectation of random variable $\boxed{X}$ w/ state space $\boxed{S}$

$$E[X] = \sum_{s \in S} p_s X(s)$$

ex. expected payoff of die roll
$= \frac{4}{6}(-1) + \frac{2}{6}(1) = \boxed{-\frac{1}{3}}$

- Expectation of a function $\Phi(X)$

$$E[\Phi(X)] = \sum p_s \Phi(X(s))$$

- Expectations are **Linear Functions**

Cumulative Distribution Function

- (c.d.f) of one Dimensional Random Variable $\boxed{X}$

$$F(x)=\mathbb{P}(\{s \mid X(s) \leq x\})$$

- $F(x) \in [0,1]$

- **cdf is differentiable** and gives Probability Density Function $\boxed{f(x)}$

14.07 Exam I

- Expectation of Function $\boxed{\Phi}$ of $\boxed{X}$

$$E[\varphi(X)] = \int_{x \in S} \phi(x) f(x)\,dx$$

Variance & higher moments

- $\mathrm{Var}(X) = E[X^2] - (E[X])^2$
  - Gives Measure of **Risk**
- Standard Deviation $(\sigma(X))$
  - $\sigma(X) = \sqrt{\mathrm{Var}(X)}$

Joint Distributions

- $p(x,y) = \mathbb{P}[\{X=x, Y=y\}]$
- Marginal Distribution of $\boxed{X}$

$$p_X(x) = \sum_{y \in Y} p(x,y)$$

- $E[X] = \sum_{x} p_X(x)\, X$
- **If X & Y are independant**

$$p(x,y) = p_X(x)\, p_Y(y)$$

Covariance, Correlation

- Covariance of Random variables $\boxed{X}$ & $\boxed{Y}$

$$\mathrm{Cov}(X,Y) = \sum_{X,Y} p(x,y)(x-E[X])(y-E[Y])$$

- if $\mathrm{Cov}(x,y) > 0$, we expect them

$\boxed{\text{correlation coefficient}}$

$$\rho_{XY} = \frac{\mathrm{Cov}(X,Y)}{\sigma(X)\,\sigma(Y)}$$

Lecture #1: Assets

- Real Assets → Require an input today for output later
- Financial Assets → claims to output produced by real assets

Arbitrage

Arbitrage → what do we know about asset prices by assuming no risk-free opportunities

Optimality → what is the best **combination** of assets given returns & preferences

Equilibrium → what is the nature of **ASSET PRICES**

when,
1. Investors optimize
2. Markets clear

Stylized Model

- $\boxed{2}$ Periods $t \in (0,1)$ [Now & Then]
- $\boxed{S}$ states may occur in the future
  - $\{1, \ldots, s\}$
- $\boxed{J}$ assets are available
  - $j \in \{1, \ldots, j\}$
- **states capture uncertainty**

* Asset $\boxed{j}$ pays out $\boxed{X_{js}}$ in state $\boxed{S}$
  - Forms a payout vector across all states

Portfolio

- Denoted $[w_j]_{j=1}^{J}$ (buy w_j units of asset j)

Payoff of Portfolio is (vector)

$$\left[\sum_{j=1}^{J} w_j X_{j1}, \ldots, \sum_{j=1}^{J} w_j X_{JS}\right]$$

- across all states $\boxed{S}$

- **A Portfolio is Just a New Asset**

* Arrow-Debrau Securities *

- A portfolio that pays \$1 in state s & 0 elsewhere
- **That is an arrow-debrau security for state s**
- We write its price as $\boxed{q_s}$
  - captures how much it costs "now" to get paid if s happens

Complete Markets

- If there exists $\boxed{S}$ Arrow-Debrau securities, then we can obtain any payoff by combining AD securities

↳ If any assets payoff can be constructed from securities (Markets complete)

Real-World

- Start with existing assets payoff $[X_{j1}, \ldots, X_{jS}]$
- Consider one AD security
- If a portfolio $[w_j]\big|_{j=1}^{J}$ exists such that,

$$\left.\begin{aligned} \sum_{j=1}^{J} w_j X_{j1} &= 1 \\ &\cdots \\ \sum_{j=1}^{J} w_j X_{jS} &= 0 \end{aligned}\right\} \text{systems of equations can be solved}$$

Markets are complete when there are AT LEAST as many assets as States

Theory of Arbitrage

- An Investment Strategy That
  1. Never requires a cash outflow
  2. Generates Inflow now or later

* We assume that arbitrage opportunities do not exist

Law of One Price

- Two assets with the same payoffs in every future state must be priced the same
- AD securities must have positive state price

↳ If Negative, BUY

↳ If zero, BUY

* To Price any Asset j w/ payoff vector $[X_{j1}, \ldots, X_{jS}]$

$$P_j = \sum_{s=1}^{S} X_{js} q_s$$

↳ "Happy Meal Thorem" (sum of Prices)

Measuring Return

- Measure of how well an investment performs

Gross Return:

$$1 + R_{t+1} = \frac{D_{t+1} + P_{t+1}}{P_{t+1}}$$

Net Return:

$$R_{t+1} = \frac{D_{t+1}}{P_t} + \frac{P_{t+1} - P_t}{P_t}$$

Compounding Return

- By convention Return is an annual Metric
- To Normalize annually Raise the Return to a Power

ex. If an invest doubles money of decade, whats annual?

↳ $2^{1/10} \rightarrow$ 7.2 %

Measuring portfolio returns

- Return of portfolio is weighted average of returns of Individual Assets

$$R_{port, t+1} = \sum_{j=1}^{J} w_j R_{j,t+1}$$

↳ w_j = weights of asset

Indexes

- Portfolios Representative of Market

Weighting Schemes:

1. Equally-weighted $(1/J)$
2. Price-weighted $\left(w_j = P_j / (P_1 \ldots P_j)\right)$
3. Value-weighted $\left(w_j = V_j / (V_1 \ldots V_J)\right)$

↳ value = price x # of shares

Portfolio weights can be Negative

1. Borrowing

↳ Borrow Money now

↳ commit to pay in future

2. Shorting

↳ w_j can be negative for risky asset to reflect positive gain from shorting

Leverage: Buying on Margin

- suppose you have \$1 & there is risky asset (j) & risk-free (f)
- Fraction of purchased asset value contributed by investor is the margin
- Usually borrow from Broker at risk-free rate

if $m = 0.5$

$$R = R_F + 2(R_j - R_f)$$

↳ Buying on Margin amplifies fluctuations

Short-Selling in Practice

1. Borrow shares & sell for cash
2. Post Money as collateral
3. If Price goes down you Profit

$$R_{Short} = R_f + \boxed{}\,(R_f - R_j)$$

Leverage Ratio

Lecture 3: Risk

- An Investor has a utility function $U(c)$

↳ Increasing

↳ Twice Differentiable

- Investors choose assets which determine wealth $\boxed{W_1}$ & $\boxed{C_1}$ (consumption)
- In 2-Period Model $\boxed{C_1 = W_1}$
- Investor Maximizes Expected Utility
- Given $S \in \{H, L\}$

$$E[U(C_1)] = \pi_H U(C_H) + \pi_L U(C_L)$$

Risk aversion

- An investor stands to gain +/- $h with 0.5 Probability each
- Risk averse if subject Rejects (zero-Mean Gamble)

Inequality:

$$U(C_1)-U(C_1-h) > U(C_1+h)-U(C_1)$$

↳ $U''(C) < 0$, concave-Down utility Function

Measuring Risk-Aversion

1. Absolute Risk $A(C)$

↳ $\boxed{-U''(C)/U'(C)}$

↳ Risk Tolerance is Inverse

2. Relative Risk $R(C)$

↳ $\boxed{= C\,A(C)}$

Insurance Premium / Quantifying Risk

- Consider Random variable $\boxed{Z}$ w/ mean variance σ^2
- Insurance Premium (π) is amount one pays to avoid risk

* $$U(C_1-\pi) = E[U(C_1+Z)]$$

For Small Risks:

$$\boxed{\pi(\varepsilon) \simeq \tfrac{1}{2}\,var(\varepsilon Z)\,A(C_1)}$$

↳ Random variable (Z)

↳ Absolute Risk

Relative Risk Aversion → Share of consumption you'd forgo to be risk-free

Common Used Utility Function

Linear: $U(C) = a + bC$

↳ Risk-Neutral

Quadratic: $U(C) = a + bC - cC^2$

Negative Exponential: $U(C) = -\exp(-aC)$

↳ constant absolute risk-averse

Power Utility:

$$U(C) = \frac{C^{1-\gamma}-1}{1-\gamma}$$

↳ constant Relative Risk-Averse

Arithmetic & Geometric Returns

1. Arithmetic Average Return

$\boxed{A = \text{average Return}}$

$$A = \frac{1}{T}\sum_{t=1}^{T} R_t$$

2. Geometric Average Return

$\boxed{G = \text{Average Return}}$

$$1+G = \left[\prod_{t=1}^{T}(1+R_t)\right]^{1/T}$$

Properties:

1. $A > G$
2. $A = G + \frac{1}{2}(\text{variance of Returns})$

Log Gross Returns

- $\log(1+R_t)$
- Returns over long horizons are additive in Log Scale
- log returns over long time period tend to be Normal

Central Limit Theorem → sum of independent Random Variables approaches Normal Distribution

* When Log Returns are Normal, G is Median

Portfolio Choice Problem

- 2 Periods & states $s \in S$ at t=1
- 2 States (H, L)
- There are two AD securities

↳ one for each state

Starting wealth $= W_0(1+R_0)$

* Chooses how much to consume (C_0) & save (W_1)

↳ consume all b/c 2 period

- Recall state Prices $\boxed{q_H}$ & $\boxed{q_L}$

Budget constraints:

$$C_0 + W_1 = W_0(1+R_0)$$

$$W_1 = C_H q_H + C_L q_L$$

Preferences:

- Investor values future consumption

$$E[U(C_1)] = \pi_H U(C_H) + \pi_L U(C_L)$$

- Investor ALSO values current consumption

$$\boxed{U(C_0)}$$

- So Net Utility

$$U(C_0) + \beta E[U(C_1)]$$

β → Discount Factor

Investors Problem

$$\max_{C_0, W_1, C_H, C_L} U(C_0) + \beta(\pi_H U(C_H) + \pi_L U(C_L))$$

↳ $C_0 + W_1 = W_0(1+R_0)$

↳ $W_1 = C_H q_H + C_L q_L$

solving

1. assume we start at optimum
2. we buy x more units of AD_H
3. C_0 reduced by $q_H x$

C_H increased by $q_H x$

$$\boxed{U(C_0 - q_H x) + \beta \pi_H U(C_H + x)}$$

↳ * take 1st order condition & set $x = 0$ *

$$\frac{\beta U'(C_H)}{U'(C_0)} = \frac{q_H}{\pi_H}$$

$$\frac{\beta U'(C_L)}{U'(C_0)} = \frac{q_L}{\pi_L}$$

- if $q_H \uparrow$, Id want Less AD_H
- if $\pi_H \uparrow$, Id want More AD_H

Additional Derived Relations

$$\frac{U'(c_H)}{U'(c_L)} = \frac{q_H/\pi_H}{q_L/\pi_L}$$

$$\frac{\beta[\pi_H U'(c_H) + \pi_L U'(c_L)]}{U'(c_0)} = q_H + q_L$$

- 4 Variables
- 2 constraints
- 2 optimality conditions

* can solve

Asset Pricing Implications

- we observe $\boxed{C_0}$ as well as $\boxed{C_H}$ & $\boxed{C_L}$
- Portfolio optimality determines Relative State Prices

$$\frac{q_H}{q_L} = \frac{\pi_H}{\pi_L} * \frac{U'(C_H)}{U'(C_L)}$$

- state prices determined by
  - ↳ probabilities
  - ↳ marginal utilities

Special Cases

- Risk-Neutrality $\boxed{\frac{q_H}{q_L} = \frac{\pi_H}{\pi_L}}$
  - Probable payments are expensive
- with $U(c) = \frac{c^{1-\gamma} - 1}{1-\gamma}$

$$\frac{q_H}{q_L} = \frac{\pi_H}{\pi_L} \cdot \left(\frac{C_H}{C_L}\right)^{-\gamma}$$

- Payments in Unpleasant states are expensive

State Price / Asset Price / Returns

Recall: $P_j = q_H X_{jH} + q_L X_{jL}$

Also

$$E[1+R_j] = \frac{E[X_j]}{P_j} = \frac{\pi_H X_{jH} + \pi_L X_{jL}}{q_H X_{jH} + q_L X_{jL}}$$

* payoff in bad state is more valuable

SDF Approach to asset Pricing

Recall: $P_j = q_H X_{jH} + q_L X_{jL}$

$$P_j = \pi_H (M_H) X_{jH} + \pi_L (M_L) X_{jL}$$

$$\boxed{M_S = \frac{q_S}{\pi_S}}$$ Stochastic discount Factor

Properties:

$\boxed{1.}$ Different across states
The same for all assets

$\boxed{2.}$ Assets w/ higher SDF more valuable

Intro to Mean-Variance Analysis

- Assume investors judge by the
  - $\boxed{1.}$ mean
  - $\boxed{2.}$ Variance

  } of its return $(\tilde{R}_j)$

- Recall investor saves W_1 & allocates across securities (C_s)

Complete Portfolio Returns

$$1 + R_{CH} = \frac{C_H}{W_1}$$

$$1 + R_{CL} = \frac{C_L}{W_1}$$

* Investor chooses their interest rates by picking C

so investors problem becomes

$$\max_{R_{cs},\, s\in S} E[U(W_1(1+R_{cs}))]$$

$$\hookrightarrow \sum^{S} (1+R_{cs}) q_s = 1$$

⇓

Equivalent to

Maximizing $\boxed{\bar{R}_c - \frac{1}{2}(RRA)\sigma_c^2}$

↳ $(\bar{R}_c) = E[R_{cs}]$ (Mean of Portfolio Return)

↳ $\sigma_c^2 = E\left[(R_{cs} - \bar{R}_c)^2\right]$ (Variance of Portfolio Return)

Mean-Variance: Applied

$\boxed{1.}$ One risky & One Risk Free Asset

- Risk-Free Asset $\boxed{f}$
- risky asset $\boxed{p}$

* consider p a portfolio (diverse) of risky stocks

Mean:	$\bar{R}_p = E[R_p]$
Variance:	$E[(R_{ps} - \bar{R}_p)^2]$

- w_p = fraction of wealth invested in p

Complete Portfolio Return: $R_c = R_f + w_p(R_p - R_f)$

Likewise

Mean of Portfolio Return: $\bar{R}_c = R_f + w_c(\bar{R}_p - R_f)$

* in Terms of Risk Premium

$$\bar{R}_c - R_F = w_p(\bar{R}_p - R_f)$$

Risk Premium on c — Risk premium on r

Variance of whole Portfolio: $\sigma_c^2 = w_p^2 \sigma_p^2$

Std. Dev of whole Portfolio: $\sigma_c = w_p \sigma_p$

Relation between Std. Dev & Mean

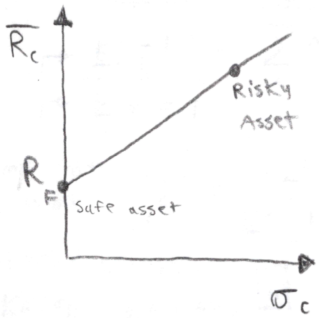
$$w_p = \frac{\sigma_c}{\sigma_p}$$

⇓ Plugging in

$$\bar{R}_c - R_f = \frac{\sigma_c}{\sigma_p}(\bar{R}_p - R_f)$$

↳ Capital Allocation Line

CAL: Graphical Representation



"risk-reward ratio of asset"

- How much more return per unit std. dev

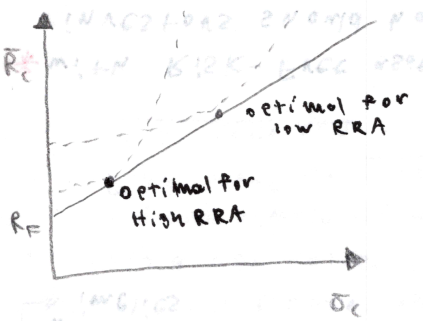
Slope of Line = *Sharpe's Ratio = $\frac{\bar{R}_p - R_f}{\sigma_p}$

Risk Evaluation: Indifference Curves

Recall: $\bar{R}_c - \frac{1}{2}(RRA)\,\sigma_c^2$

↳ used to express investor preferences

↳ Represented as indifference curves on $\bar{R}_c - \sigma_c$ diagram



↑ RRA = Steeper curves

Analytical solution

$$\max_{w_p} \ \bar{R}_c - \frac{1}{2}(RRA)\,\sigma_c^2$$

$$= R_f + w_p(\bar{R}_p - R_f) - \frac{1}{2}(RRA)\sigma_p^2 w_p^2$$

⇓ (First-order condition)

$$\bar{R}_p - R_f - RRA * w_p \sigma_p^2 = 0$$

$$w_p = \frac{\bar{R}_p - R_f}{RRA * \sigma_p^2} = \frac{S_p}{RRA * \sigma_p}$$

↳ optimal share in risky asset | $\frac{\text{Risk Premium}}{\text{Risk Aversion} * \text{Var}}$

Diversification

- How do we combine many risky stocks to get portfolio $\boxed{P}$

↳ Diversification

Model

- 2 Risky assets exist w/ same Expected Return

$$R_1 = \bar{R} + \varepsilon_1$$
$$R_2 = \bar{R} + \varepsilon_2$$

- ε_1 & ε_2 are risks that are zero-mean & uncorrelated

↳ have same variance σ^2

Properties

1. Returns each have their own specific Risk
2. Returns have same Mean & variance

Investing

- You split your wealth equally between the 2 assets

$$\frac{R_1 + R_2}{2} = \bar{R} + \frac{\varepsilon_1 + \varepsilon_2}{2}$$

↳ $Var\left(\frac{\varepsilon_1 + \varepsilon_2}{2}\right) = \boxed{\frac{\sigma^2}{2}}$

* Diversification reduces variance while providing same expected Return

↳ For J assets

$$Var\left(\sum^{J} \frac{\varepsilon_j}{J}\right) = \boxed{\frac{\sigma^2}{J}}$$

* Large J eliminates spontaneous Risk

Existence of systematic risk (ε_{sys})

$$R_j = \bar{R} + \varepsilon_{sys} + \varepsilon_j$$

↳ $cov(\varepsilon_{sys}, \varepsilon_j) = 0$

↳ w/ variance $\boxed{\sigma_{sy}^2}$

* Diversification doesn't eliminate systematic risk

Mean-Variance w/ 1 risk free & many risky

$\boxed{F}$ → risk-free asset

risky assets: $j \in \{1, \dots, J\}$

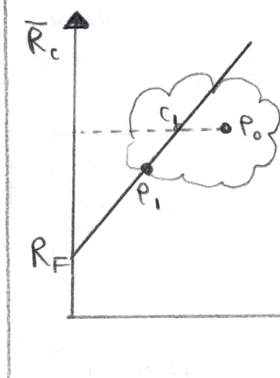
Consider configurations of expected return

$$\bar{R}_c = w_1 \bar{R}_1 + \dots + w_J \bar{R}_J$$

$$\sigma_c = \sqrt{var(w_1 R_1 + \dots + w_j R_j}$$

↳ s.t $\sum_{j=1}^{J} w_j = 1$

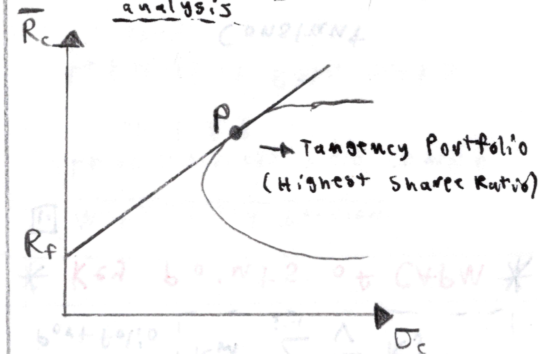
Graphical Interpretation of solns



* P_0 dominated by C_1 which is combo of $\boxed{P_1}$ & $\boxed{F}$

- Reward-Risk combos can be obtained by forming portfolios of risky assets only

↳ Including Risk-Free Analysis simplifies analysis



↳ Best Risk Reward combos w/ free asset is called The Efficient Frontier

Portfolio Optimization

1. Find Risky Portfolio (P) w/ Highest Sharpe Ratio
2. Choose Portfolio on CAL corresponding to $\boxed{P}$

Step 1: Portfolio w/ highest S_p

$$\max_{w_1,\dots,w_J} S_p = \frac{\bar{R}_p - R_f}{\sigma_p}$$

↳ $\sum^J w_1 + \dots w_J = 1$

Solution Straightforward if

1. Expected Return of Assets $[\bar{R}_1, \bar{R}_2, \dots, \bar{R}_J]$

2. Variance-covariance Matrix of assets

$$\begin{bmatrix} var(R_1) & \cdots Cov(R_j,R_1) & \cdots Cov(R_J,R_1) \\ Cov(R_1,R_j) & \cdots Var(R_j) & \cdots Cov(R_J,R_j) \\ Cov(R_1,R_J) & \cdots Cov(R_j,R_J) & \cdots Var(R_J) \end{bmatrix}$$

Asset Allocation Puzzle

- Given Mean-Variance Preferences they hold combos of $\boxed{f}$ & $\boxed{p}$

↳* implies investment advisers should not adjust portfolios beyond simple choice of fund to reflect client Risk aversion

* with Risk-Free asset, investors should hold risky assets in same proportion

Asset Allocation Puzzle: Borrowing constraints

* Does Not Hold in Practice

↳ Due to Borrowing constraint

- They'll take higher Percentage of stock to increase Risk but not be maximizing Sharpe's

CAPM & Empirical Anomalies

Recall Efficient $\boxed{P}$ Risky Portfolio

$$\max_{w_1,\dots,w_J} S_p = \frac{\bar{R}_p - R_f}{\sigma_p}$$

↳ $\sum^J w_i = 1$

↳ $\bar{R}_p - R_f = \sum^J w_j(\bar{R}_j - R_f)$

↳ $\sigma_p = \sqrt{Var(R_p)} = \sqrt{\sum^J \sum^{J'} w_j w_{j'} Cov(R_j, R_{j'})}$

*Note: adding Asset $\boxed{f}$ doesn't change Sharpe Ratio(S_c)

Beta-Representation Theorem

- Suppose there exists a risk-free asset, For any risky Asset j

$$\bar{R}_j - R_f = \beta_{jc}(\bar{R}_c - R_f)$$

where

↳ $$\beta_{jc} = \frac{Cov(R_j, R_c)}{Var(R_c)}$$

Given sets of Means, variances, & Covariances Mean return on every asset j is explained with Portfolio c

Beta theorem: Example

- suppose safe asset has R_f
- $R_j = \bar{R} + \varepsilon_{sys} + \varepsilon_j$

↳ $Var(\varepsilon_j) = \sigma^2$

↳ $Var(\varepsilon_{sys}) = \sigma_{sys}^2$

- Due to symmetry

↳ $\boxed{R_p = \sum^J \frac{1}{J} R_j}$

- $Var(R_p) = \sigma_{sys}^2 + \frac{\sigma^2}{J}$

↳ $\boxed{\beta_{jp} = 1}$

Beta Theorem: Sketch Proof

- 1st order condition

$$\frac{d(\bar{R}_c - R_f)}{d w_j} - (\bar{R}_c - R_f)\frac{d\sigma_c/dw_j}{\sigma_c} = 0$$

Marginal Reward ⇓

$$\underbrace{\bar{R}_j - R_f}_{\text{Marginal Reward}} = \underbrace{Cov(R_j, R_c)}_{\substack{\text{marginal cost} \\ \text{(covariance)}}} \left[\frac{\bar{R}_c - R_f}{Var(R_c)}\right]$$

Beta Theorem Graphically

$$R_{js} - R_f = \alpha_{jc} + \beta_{jc}(R_{cs} - R_f) + \varepsilon_{js}$$

↳ $$\beta_{jc} = \frac{Cov(R_j, R_c)}{Var(R_c)}$$

*efficient Portfolio Alpha = 0

Capital Asset Pricing Model

Assumptions:

1. investors take price as given
2. * All Investors are Mean-Variance optimizers
3. All investors perceive same means, variances, & covariances of Returns

Market Portfolio → value-weighted index that contains all risky assets

Return on Market Portfolio: $$R_m = \sum_{j=1}^J \frac{V_j}{V} R_j$$

* Key Points of CAPM *

1. Mutual Fund Theorem

↳ All investors hold combo of f & p

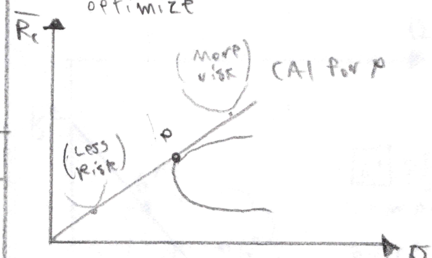
↳ Ratios of Risky assets Stays Constant

2. MAIN RESULT OF CAPM

- Market Portfolio is the optimal risky Portfolio

Assumptions:

1. All investors mean-variance optimize



Per CAPM: $$\frac{w_i}{w_j} = \frac{V_i}{V_j} = \frac{\text{total invest in } i}{\text{total invest in } j}$$

- If V_x changes weight on that asset must change as well

Forces that Equalize CAPM:

1. if more dollars chase i than value of shares: ↳ Price of $\boxed{i}$ ↑
2. Increase in Price lowers return $\bar{R}_i$, shifting weight away ↓ w_i

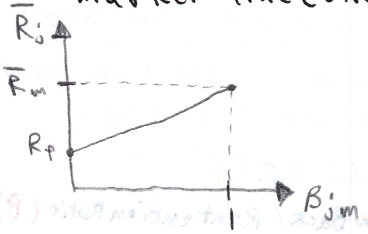
Alpha, Beta, & CAPM

- Combining CAPM w/ Beta Theorem

$$\bar{R}_j - R_f = \beta_j(\bar{R}_m - R_f)$$

Beta of Asset w/ respect to market: $\beta_j = \dfrac{Cov(R_j, R_m)}{Var(R_m)}$

↳ one creates security market-line (SML)



- Capital Allocation line (CAL)
  - ↳ connects risk-free rate to that of Risky portfolio
- Capital Market Line (CML)
  - ↳ special case where risky portfolio is market portfolio
- Security Market Line
  - ↳ connects R_f to R_m in Mean-Beta Diagram

Real-world Results

Qualitative success	High-Beta stocks deliver higher return to compensate for risk
Quantitative failure	Additional return per β not as high as expected

Anomalies in CAPM

1. Value Effect
   - ↳ Stocks with higher Accounting-Market value Ratios had higher returns
2. Momentum Effect
   - ↳ over short time horizons past returns predict future ones

Arbitrage Pricing Theory

- we run Regression

$$R_{jt} - R_f = \alpha_j + \beta_{jm}(R_{mt} - R_f) + \varepsilon_{jt}$$

↳ called market model or single-index

Key assumptions:

1. Errors in this equation uncorrelated across stock

$$E[\varepsilon_{it}\varepsilon_{jt}] = 0$$

↳ residual risk is idiosyncratic (unrelated to all others)

Arbitrage Pricing Theory (APT): if many assets are available & each pair satisfies condition above, α_j will be low

Understanding APT

- consider Portfolio of J assets

$$R_{pt} - R_f = \alpha_p + \beta_{pm}(R_{mt} - R_f) + \varepsilon_{pt}$$

where,

$$\alpha_p = \sum_{j=1}^{J} w_j \alpha_j \qquad \beta_{pm} = \sum_{j=1}^{J} w_j \beta_{jm} \qquad \varepsilon_{pt} = \sum_{j=1}^{J} w_j \varepsilon_{jt}$$

- we neglect ε_{pt}
- $\alpha_{pt} = 0$, or there is arbitrage opportunity
- if $\alpha_p > 0$, there is an arbitrag
  - ↳ long 1 unit of portfolio & short β_{pm} of the Market
  - Returns α
- if $\alpha_p < 0$, long β_{pm} units of Market, short one asset j

* so diversified portfolios w/ positive alpha increase return w/ no change in risk

why most α_j are near 0:

$$\lim_{J \to \infty} \frac{1}{J} \sum_{j=1}^{J} \alpha_j^2 = 0$$

* Assumptions: APT vs. CAPM *

- CAPM → Mean-Variance optimality
- APT → Absence of Arbitrage (weaker assumption)
- APT → statistical structure on returns (stronger)

Multi-Factor Models

- There exist Industry Effects
- To handle these distint risks we outline the Multi-Factor Model

- If there are K portfolios capturing K sources of risk

$$R_{jt} - R_f = \alpha_j + \sum_{k=1}^{K} \beta_{jk}(R_{kt} - R_f) + \varepsilon_{jt}$$

New key assumption: Residual of Regression uncorrelated across stocks

$$\alpha_j = 0, \text{ for most stocks}$$

- Theory only useful if $K \ll N$

Refining APT

- K could be LARGE, but we only need consider factors investors care about

Factors investors likely consider:

1. interest rate
2. Inflation
3. Industrial Production

Factors investors seem to:

1. Size & Value

Connection to SDF Approach

Recall: $M = \dfrac{q_j}{\pi_j}$

$$P_j = E[MX_j]$$

$$1 = E[M(1+R_j)]$$

Market Efficiency

- where security prices "reflect all available information" markets are efficient
- if information set Φ were to be released & prices dont change,

Formalization:

- Expected return on asset: R_j

$$P_j = \frac{1}{1+R_j} E[X_j \mid I]$$

↳ It's conditional on info set I

Market Efficiency: Basic issues

1. What info do we consider
2. Returns vs. Prices
3. Adjusting Returns for risk

Strong Form EMH implausible

- Unless info is free strong form Efficiency impossible to achieve
  - ↳ If info is costly, investors will only gather when **benefit outweighs cost**

Paradox: If efficient no one gathers info because no one can profits, BUT if no one gathers it can't be efficient

implications:

- Prices will reflect some but not all info

Price Reaction to New info

- Assume $\bar{R}_j = 0$

Price = expectation of future payoff given I

- Take 2 info sets
  - ↳ knowing I is to know H **But not vice-versa**

$$E[X_j|H] = E\left[E[X_j|I]\,|H\right]$$

$$\Downarrow$$

$$E\left[P_{j|I} - P_{j|H}\,|H\right] = 0$$

Relating to Time:

$$E\left[E_{t+1}[X_j] - E_t[X_j]\right] = 0$$

Example: Coin-Toss Example

Slide 40, Deck 8

Prediction Markets

1. Winner-Take-All
   - ↳ Al gore wins presidency
   - ↳ contract costs $p(y)$
   - ↳ pays \$1
   - (Reveals probability y occurs)
2. Index
   - ↳ pays \$1 for each percent point of vote won

3. Spread
   - ↳ Finds Median

Cumulative Abnormal Return (CAR)

How to Measure:

1. Subtract Real return - Expected Return
2. Record over several days
3. Average across firms involved
4. Compound over days involved to get car

Real-world observations:

- Prices move rapidly
  - ↳ consistent w/ efficiency
- Price sometimes moves before info released
  - ↳ not consistent

Notable: **Post-Earnings Drift**

Dividend Discount Model

- Suppose $E_t[R_{t+1}] = \boxed{R}$
  - ↳ R = constant Discount Rate

$$\boxed{P_t = E_t\left[\frac{D_{t+1} + P_{t+1}}{1+R}\right]}$$

More generally, considering $\boxed{K}$ periods

$$P_t = E_t\left[\sum_{i=1}^{K}\left(\frac{1}{1+R}\right)^i D_{t+i}\right] + E_t\left[\left(\frac{1}{1+R}\right)^K P_{t+K}\right]$$

$\Downarrow$ we assume Prices can't out pace interest Rate Forever

$$\boxed{P_t = E_t\left[\sum_{i=1}^{\infty}\left(\frac{1}{1+R}\right)^i D_{t+i}\right]}$$

↳ **Expected Discount value of Dividends**

Gordon Growth Model

- Assume Dividends Grow constantly at rate $\boxed{G}$

&

$$E_t[D_{t+1}] = D_{t+1} \quad (\text{is Known})$$

So, $$P_t = \left(\frac{1}{1+R}\right)\sum_{i=0}^{\infty}\left(\frac{1+G}{1+R}\right)^i D_{t+1}$$

$$= \left(\frac{1}{1+R}\right)\left(\frac{1}{1-\frac{1+G}{1+R}}\right) D_{t+1}$$

$$\frac{D_{t+1}}{R-G} = P_{t+1}$$

Gordon Growth Model

Implications of Gordon Growth

D/P is down

Means, $\uparrow G$ or $\downarrow R$ or combo of two

From Dividends to Earnings

- At steady state Dividends, earnings, & prices all grow at rate $\boxed{G}$

we calculate G dependent on:

1. How much does firm invest?
2. How much will the firm earn on these investments

key values:

1. PlowBack/Retention Ratio (B)
   - ↳ firm invests $\boxed{B}$ dollars for every dollar
2. Return on Equity (ROE)
   - ↳ # of dollars earned per dollar invested

$$\boxed{G = B * ROE}$$

Earning to Dividends

$$D = E * (1-B)$$

↳ More investments means less today

BUT increases growth rate for future dividends

Price-to Earnings Ratio
$$\frac{P}{E} = \frac{1-B}{R - ROE * B}$$

Formula Rewritten + Implications

$$\boxed{R = (1-B)\frac{E}{P} + B * ROE}$$

↳ Rate of Return is weighted average of Earning Yield & Profitability

- if ROE < R; firm invests less
- if ROE > R; firms invest more
- if Potential for growth is larger $\boxed{\uparrow G}$ P/E is larger

Consumption-Savings Problem

• assume no uncertainty $(S=1)$

• Denote this one-state AD Security

$$1+R^f = \frac{1}{q^f} \qquad \text{Price} = q^f,\ \text{Return} = R^f$$

Budget constraints:

$$C_0 + W_1 = W_0$$
$$C_1 = W_1(1+R_f)$$

Consumption-Savings Labor Income

Budget constraint w/ income

$$C_0 + A_1 = Y_0^L + A_0 \qquad C_1 = Y_1^L + A_1(1+R_f)$$

Lifetime Budget:

$$C_0 + \frac{C_1}{1+R^f} = D_0 + \frac{D_1}{1+R^f} + Y_0^L + \frac{Y_1^L}{1+R_f}$$

↳ Define

$$W_0 = D_0 + \frac{D_1}{1+R^f} + Y_0^L + \frac{Y_1^L}{1+R^f}$$

Problem to solve

$$\max\ U(C_0) + \frac{1}{1+\rho} U(C_1)$$

$$C_0 + \frac{C_1}{1+R_f} = W_0$$

ρ = Discount Factor

C_0 = consumption $t=0$

C_1 = consumption $t=1$

where → $W_0 = D_0 + \frac{D_1}{1+R_f} + Y_0^L + \frac{Y_1^L}{1+R^f}$

* Optimality Condition: Euler Condition *

$$U'(C_0) = \frac{1+R_f}{1+\rho} U'(C_1)$$

where, $C_0 = W_0 - \frac{C_1}{1+R_f}$

↳ Euler Equation

intuition:

↑ ρ, you prefer to spend more at $t=0$

↑ R_f, prefer to spend more at $t=1$

w/ CRRA utility Function

if $U(c) = \frac{c^{1-\gamma}-1}{1-\gamma}$ → $U'(c) = c^{-\gamma}$

⇓ Euler Condition

$$\frac{C_1}{C_0} = \left(\frac{1+R_f}{1+\rho}\right)^{1/\gamma}$$

$\boxed{1/\gamma}$ → how fast consumption responds to shocks

↳ "Elasticity of intertemporal substitution"

14.07 Summary Sheet #2

• Solving Equations

$$C_0 = W_0 \Big/ \left(1 + \frac{\left(\frac{1+R_f}{1+\rho}\right)^{1/\gamma}}{(1+R_f)}\right)$$

Substitution + Income effects

1. Substitution Effect

• Higher R_f makes future consumption relatively cheaper

↓ C_0 ↑ C_1

2. Income Effect

• But R_f increases C_0 as well b/c the agent will feel richer

Special case: Log Utility

$$U(c) = \log C$$

$$C_0 = W_0 \Big/ \left(1 + \frac{X}{1+R_f}\right)$$

$$C_0 = \frac{1+\rho}{2+\rho} W_0$$

The Wealth Effect

• Higher R_f reduces W_0

↳ Reducing W_0

* With log utility wealth effect remains

Consumption with Uncertainty

• suppose $\boxed{J}$ risky assets w/ returns $\{1+R_{js}\}_{s \in S}$

• Also safe asset R_f

Problem:

$$\max_{C_0, W_1, \{w_j\}} \log(C_0) + \frac{1}{1+\rho} E[\log(C_s)]$$
$$C_0 + W_1 = W_0$$
$$C_s = W_1(1+R_{ps})$$
$$R_{ps} = R_f + \sum w_j (R_{ms} - R_f)$$

⇓ same solution

$$C_0 = \frac{1+\rho}{2+\rho} W_0$$

&

$$W_0 = Y_0 + \frac{E[Y_s]}{1+R_f+(\bar{R}_m - R_f)}$$

↓ R_f but not $(\bar{R}_m - R_f)$ ↑ W_0 ↑ C_0

↑ $(\bar{R}_m - R_f)$ ↑ W_0 ↑ C_0

Wealth Effect over long Horizons

• agent lives in $T+1$ periods w/ utility $\sum_0^T \beta^t \log(C_t)$

$$C_t = \left(\frac{1+R_f}{1+\rho}\right)^t C_0$$

Solving for C_0

$$C_0 = \frac{1 - \frac{1}{1+\rho}}{1 - \left(\frac{1}{1+\rho}\right)^{T+1}} W_0$$

↳ w/ lower T shocks to consumption from change in wealth MORE Drastic

Features of Macro Environment

• 2 Periods • log utility

• Output from capital only

Potential output: Y_0^* & Y_s^* if state s realized

so $\boxed{Y_0 = C_0}$ & $\boxed{Y_s = C_s}$

• If $C_0 < Y_0^*$ recession

• If $C_0 > Y_0^*$ economy boom

Consumer-Investor Problem

$$\max_{C_0, W_1, w_m} \log(C_0) + \frac{1}{1+\rho} \sum_{s=1}^{S} \pi_s \log(C_s)$$
$$C_0 + W_1 = W_0 \equiv Y_0 + P_m$$
$$C_s = W_1\big(w_m(1+R_{ms}) + (1-w_m)R_f\big)$$

at state s

$$1 + R_{ms} = \frac{Y_s^*}{P_m}$$

Definition of Equilibrium

• Returns & Prices (P_m, R_{ms}, R_f)

• Allocations $(C_0, W_1, w_m, (C_s)_s)$

• initial output, Y_0

Givens of Eq. Problem:

• $\rho, Y_0^*, (Y_s^*, \pi_s)_s$

Growth rate

$$\bar{g} = E[g_s] = \sum \pi_s g_s$$

Output-Price Relation

Recall: $C_0 = \frac{1+\rho}{2+\rho}(Y_0 + P_m)$

↳ $\boxed{Y_0 = C_0}$

Output-Price Relation: $\boxed{P_m = \frac{1}{1+\rho} Y_0}$

Determinants of Market risk premium

• with $w_m = 1$

$$\boxed{\bar{R}_m - R_f = RRA * \sigma_m^2}$$

↳ 1. volatility
2. Risk Premium

Equity Premium Puzzle

$$\boxed{1+R_{ms} = \frac{1+\bar{R}_m}{1+\bar{g}}(1+\bar{g}_s)}$$

↳ Risk-Premium is lower than expected

Asset Pricing Relation

$$\bar{R}_m - R_f = RRA * \sigma_g^2$$

↳ $P_m = \frac{E[Y_s^*]}{1+\bar{R}_m}$ & $E[Y_s^*] = Y_0^*(1+\bar{g})$

Asset-Pricing Relation: $\boxed{P_m = \frac{Y_0^*(1+\bar{g})}{1+R_f+RRA*\sigma_g^2}}$

↑R_f ↓P_m (lower Demand for assets in market portfolio)

Interest Rates & Fisher Eqn

$$(1+R_f) = (1+R_f^{Nominal})/(1+\pi)$$

↳ $\boxed{\pi = \frac{P_1}{P_0} - 1}$ inflation

or

$$\boxed{\bar{\pi} = \frac{\bar{P}_1}{P_0} - 1}$$

$\boxed{R_f = R_f^n - \bar{\pi}}$ For small amounts

Monetary Policy & Interest Rate

• Money is a medium of exchange

↳ offers liquidity services

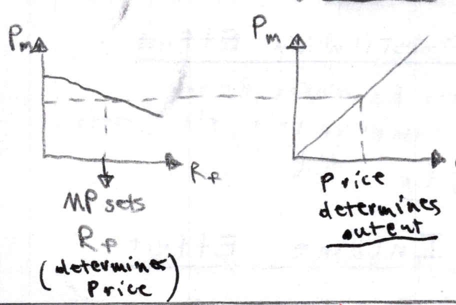
* # of Money available determines interest rate

* PUTTING IT ALL TOGETHER *

(1) $P_m = \frac{1}{1+\rho} Y_0$

(2) $P_m = \frac{Y_0^*(1+\bar{g})}{1+R_f+RRA*\sigma_g^2}$

(3) Monetary Policy sets R_f



Determinants of Natural Interest Rate

Madates: 1. Keep inflation low
2. Keep output/Employment at potential

• Level of R_f such that ($Y_0 = Y_0^*$)

$\boxed{R_f^*}$

$$R_f^* = (1+\bar{g})(1+\rho) - 1 - RRA*\sigma_g^2$$

↓

$$\boxed{R_f^* \simeq \bar{g} + \rho - RRA*\sigma_g^2}$$

Results/Intuition behind R_f

↑$\bar{g}$ ↑R_f	↑ρ ↑R_f	↑RRA or σ_g^2 ↓R

1. Slowdown in expected growth
   • $\bar{g}$ declines to $\bar{g}_{new}$
   • If R_f doesn't change ↓P_m
   • R should Decrease
2. Increase in Propensity to save
   • $\rho_{new} < \rho$
   • R should Decrease

Productivity & r star in recent times

Causes of Potential output Decline?

(1) Decline in Population growth

(2) Decline in labor participation

(3) Slowdown in Productivity Growth

Growth Accounting: $\boxed{Y_t^* = A_t (K_t^*)^{\alpha} (L_t^*)^{1-\alpha}}$

Causes of Low Interest Rates

* higher propensity to save

↳ but R_f^* reflects aggregate savings which is a weighted average

• Rich save more than the poor

• rising inequality captured w/ lower ρ

Global saving Glut & low R-star

• Global trend towards saving

Constraints on Interest Rate

1. Lower Bound Restraint

Recall: $R_f = R_f^n - \pi^e$

zero low bound: $\boxed{R_f^n \geq 0}$ (otherwise no one would invest in Deposits or risk-free assets)

when $R_f^n = 0$, → Liquidity trap

* Also → implied Bound on Real Rate
  ↳ b/c inflation sticky

2. Currency Unions

• One R_f (on Euro) cannot respond to individual country shocks

• ∴ upward & downward constraint

3. Monetary Policy

• Could place downward constraint on R_f if trying to preserve exchange rate

Secular Stagnation & fragility

• what happens when $\boxed{R^* < R_f}$

↳ central gov cannot reduce R_f sufficiently

Secular Stagnation & Fragility

• What happens when $R^* < R_f$

• For example take increase in Propensity to save (↑e)

1. This sets $R^* < R_f$
2. so $R = R_f$ $\quad Y_0 < Y_0^*$

↳ Recession | secular stagnation if permanent

Risk-Centric Recessions

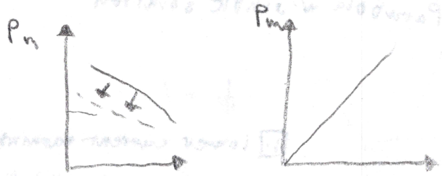
• Recall stock/housing prices fluctuate w/o real changes

↳ work into our framework

$$P_m = \frac{Y_0^*(1+\bar{g})}{1+R_f+RRA*\sigma_g^2}$$

↳ $\bar{g}^*$ = correct growth rate

↳ $\bar{g} = \bar{g}^* + \boxed{\delta}$ ← captures pessimism/optimism



2. Time-Varying Risk-Premium

• No Mistakes in g $\boxed{\bar{g} = \bar{g}^*}$

BUT,

$$R_m - R_f = RRA*\sigma_g^2 = RRA*\sigma_g^{*2} - \delta$$

• $\boxed{\delta}$ captures change in RRA or σ_g^2

Time Varying Risk Premium & recession

• Stock prices collapsed

• Fed Reduced R_f but possibly hit R_f

case 1: Eurozone

• House & stock price decline in Eurozone followed by worse recessions

↳ b/c of flexibility to alter their policy interest rate

case 2: Household Deleveraging

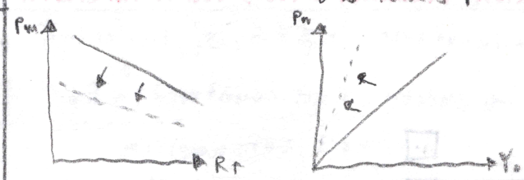
• Most household debt was collateralized by their house

• When bubble collapsed households had to deleverage $\boxed{\downarrow P}$

* strengthened wealth effect

* House prices can effect $\boxed{W_0}$, $\boxed{P_0}$, & $\boxed{P}$

• shock that reduces δ & lowers P



Fed Put

• Low Stocks Returns, & Decrease in Price that follows

↳ Followed by Decrease in R_f

* Investment & Asset Prices: Q theory

• Investment, though smaller than consumption, is part of output

setup:

1. 2 Periods
2. various states $s \in S$ w/ π_s
3. Firm operates initial capital $\boxed{K_0}$

* earnings per unit of capital

t=0	t=1
e_0	e_1

evolution of Capital

$$K_1 = K_0(1-\delta) + i_0$$

↳ δ = depreciation

Investment costs

• to invest i_0 units of Machines, firm must spend

$$\left(\left(\frac{i_0}{K_0}\right) + \frac{1}{2}\phi\left(\frac{i_0}{K_0}\right)^2\right)K_0$$

↳ φ = severity of adjustment penalty

• costly for $i_0 > 0$ & $i_0 < 0$

Firm Dividends & tradeoff

Firm's Dividends:

$$D_0 = e_0K_0 - \left(\frac{i_0}{K_0} + \frac{1}{2}\phi\left(\frac{i_0}{K_0}\right)^2\right)K_0$$

• investment reduces dividends at t=0

• But raises K_1, raising future Dividends

Firm's optimization Problem

$$\max \; D_0 + \sum_{s \in S} \pi_s M_s D_s$$

1. $D_0 = e_0K_0 - \left(\frac{i_0}{K_0} + \frac{1}{2}\phi\left(\frac{i_0}{K_0}\right)^2\right)K_0$
2. $D_s = e_s K_1$
3. $K_1 = K_0(1-\delta) + i_0$

* Note we discount future earnings w/ SDF

- Generating payoff in high M_s states more valuable

Tobin's Q

* Known as Marginal Value of capital

$$Q = \sum_s \pi_s M_s e_s$$

Firm's Problem: Solution

$$\max_{i_0} \underbrace{e_0K_0}_{\text{initial}} - \underbrace{\left(\frac{i_0}{K_0} + \frac{1}{2}\phi\left(\frac{i_0}{K_0}\right)^2\right)}_{\text{investment}} + \underbrace{(K_0(1-\delta)+i_0)Q}_{\text{future Dividend}}$$

⇓

$$1 + \phi\frac{i_0}{K_0} = Q$$

Marginal cost = Marginal Benefit

Q: Main Result & Interpretation

$$\frac{i_0}{K_0} = \frac{Q-1}{\phi} \qquad Q = \sum_s \pi_s M_s e_s$$

• Optimal investment increasing in $\boxed{Q}$

IF

• Q>1, firm invests

• Q<1, firm divests

• size response $\left|\frac{i_0}{K_0}\right|$ depends on φ

Q theory: Expected Return

$$1 + \bar{R}_K = \frac{\bar{e}}{Q} \qquad Q = \frac{\bar{e}}{1+\bar{R}_K}$$

• When $Q > 1$, $\bar{e} > 1 + \bar{R}_K$

$\bar{R}_K$ = Hurdle Rate

Hurdle Rate & Pricing Theory

• When SDF is of Form $M_s = a - bR_{ms}$

⇓

CAPM Result: $\boxed{\bar{R}_K - R_f = \beta_K(\bar{R}_m - R_f)}$

* w CAPM, firm calculates expected return by considering the β of its investments w/ the market

Q theory w/ CAPM risk pricing

$$\frac{i_0}{K_0} = \frac{Q-1}{\phi}$$

$$Q = \frac{\bar{e}}{1+\bar{R}_K}$$

$$\bar{R}_K - R_f = \beta_K(\bar{R}_m - R_K)$$

$\downarrow \bar{e}$ $\quad \downarrow Q$ (lower earnings make less viable)

$\uparrow(\bar{R}_m - R_f)$ $\quad \downarrow Q$ (less valuable if pays out in good state)

$\uparrow R_f$ $\quad \downarrow Q$

* Q provides summary statistic for variety of shocks

Relating Marginal Q to Average Q

- $\boxed{P}$ = total market value

$$P = \sum_s \pi_s M_s D_s$$

↳ $$\boxed{Q^{AV} = \frac{P}{K_1}}$$

- Model Average & Marginal Q the same thing

Evidence for Q theory in aggregate data

- Is Aggregate Q correlated to Aggregate Investment?
  - ↳ $\boxed{\text{Yes}}$
- Is relationship tight
  - ↳ $\boxed{\text{No}}$ • other factors effect investment
    - Borrowing constraints

Q theory & risk-centric Demand

- Firm receives a constant fraction of output in period 1

$$\boxed{e_s = \frac{1}{K_1} Y_s^*}$$

$$Q = \frac{P_m}{K_1} = \frac{E\left[\frac{1}{K_1} Y_s^*\right]}{1+R_m}$$

- In this Model two factors for Demand

Risk Premium shocks reduce investment

- increase in RRA, σ_g^2, declining
  - ↳ shocks if not countered by R_f
    1. Shock consumption
    2. Shock investment

* Reasoning: $\bar{R}_m - R_f$ is higher
  - ↳ HURDLE RATE is Larger

Implications:

1. Projects failing to exceed greater hurdle rate will not be pursued *
2. Projects that clear Hurdle receive less investment

"Risk Premium Shock" & feedback loops

- If $Y_s \leq Y_s^*$ & $e_s \leq e_s^*$
  - ↳ Lowers P_m → Lowers Q

Recap: Q-theory

① $\frac{i_0}{K_0} = \frac{Q-1}{\phi}$

② $Q = Q_{av} = \frac{P}{K_1} = \frac{\bar{e}}{1+\bar{R}_K}$

③ $\bar{R}_K = R_f + \beta_K(\bar{R}_m - R_f)$

Test of Q theory w/ firm level data

- Not Great
  1. using aggregate Price

Investment w/ Borrowing constraints

- Firm Budget constraint

$$D_0 + \left(\frac{i_0}{K_0} + \frac{1}{2}\phi\left(\frac{i_0}{K_0}\right)^2\right)K_0 = e_0 K_0$$

- What if current earnings below desired spending ($e_0 = 0$?)
- D_0 will be negative as investors carry only about future probability
  - ↳ UNREALISTIC
- In Practice firm will borrow money

Borrowing Constraint

- Say that $\tilde{D}_0 \geq 0$ at All times

Exogenous constraint

$$B_0 \leq \bar{B}_0$$

Firm Problem w/ Borrowing constraint

$$\max_{i_0} D_0 + Q K_1$$

where $D_0 + \left(\frac{i_0}{K_0} + \frac{1}{2}\phi\left(\frac{i_0}{K_0}\right)^2\right)K_0 = e_0 K_0$

$$K_1 = K_0(1-\delta) + i_0$$

$$D_0 \geq -\bar{B}_0$$

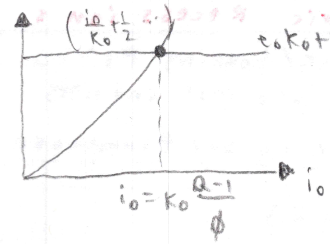
Solving case:

Scenario 1: Sufficient Funds

$$\frac{i_0}{K_0} = \frac{Q-1}{\phi}$$

Firms spends: $\left(\frac{i_0}{K_0} + \frac{1}{2}\phi\left(\frac{i_0}{K_0}\right)^2\right)K_0$

if " " $< e_0 K_0 + \bar{B}_0$

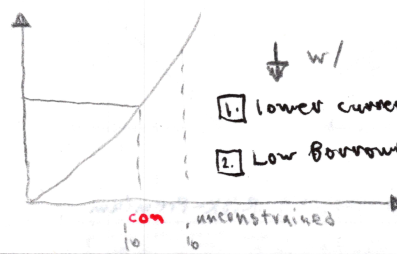


- investment determined by Q

Scenario 2: Insufficient Funds

$$\left(\frac{i_0}{K_0}^{\text{unconstrained}} + \frac{1}{2}\phi\left(\frac{i_0}{K_0}\right)^2\right)K_0 = e_0 K_0 + \bar{B}$$

↳ Parabola w/ single solution



$\downarrow$ w/

1. lower current earnings
2. Low Borrowing Limit

Q vs. Cash Flows

Regression Ran $\left(\frac{i}{K}\right)_j = \beta_Q Q_j + \beta_{cash}\left(\frac{\text{cash flow}}{K}\right)_j$

Findings: $\beta_{cash} > 0$, $\beta_Q \simeq 0$

* strong evidence for Borrow constraint theory

Item 2: Cash windfalls w/ no effect on Q

↳ result favors BC Theory

Info Frictions & Borrowing constraints

- So far it was assumed $\boxed{D_0 \geq -\bar{B}_0}$
- Information Frictions can lessen assurance of Pay back to lenders
  - ↳ IDEALLY if $Q > 1$, money invested

Examples:

1. Lenders might perceive Q to be lower
2. Similar Firms with low Q ask for money
3. Firm could invest in wrong project — Moral Hazard
4. * Firm might waste payoff

Model of Information Frictions

- Suppose $\phi = 0, \delta = 1, K_1 = i_0$
- "good Project" is absolutely safe $\boxed{e_1 > 1 + R_f}$

Problem Firm Solves:

$$\max_{K_1} D_0 + e_1 K_1 \qquad D_0 + K_1 = e_0 K_0$$

$e_1 > 0$ firm will invest as much as possible

Moral Hazard } Firms insiders might misbehave

- Firm either succeeds yielding $\boxed{e_1}$ per unit $\boxed{K_1}$ or fails & yields 0

Project	Good	Bad
Private Benefit	0	$cK_1 > 0$
Prob. of success	1	$I < 1$

↳ so $\boxed{Ie_1 + c < 1}$ (if bad, should not invest)

Borrowing contract to Deal w/ Moral Hazard

Friction: Financiers dont witness whether good or bad chosen

Borrowing Contract:

1. Prescribes actions to be taken
2. Divides Payoff: $e_1 K_1 = X^F + X^{firm}$

Two Objectives:

(1) Financiers Break even

$$\boxed{X^F = B_0}$$

(2) Bank insiders prefer good project over bad

$$X^{firm} \geq I X^{firm} + cK_1$$

The Incentive Constraint

$$\boxed{X^{Firm} \geq \frac{cK_1}{1-I}}$$

* For good Management, firm must have skin in the game

⇓

$$\boxed{X^{Financial} \leq \overline{X}^{Fin} = e_1 K_1 - \frac{cK_1}{1-I}}$$

$X^F = B_0(1+R_f)$ ⇓ creates ~~Budget~~ Borrowing constraint through Break-through condition

$$\boxed{B_0 \leq \overline{B}_0 = K_1 \overline{p}}$$

where $\overline{p}$ is $e_1 - \frac{c}{1-I}$

Optimal investment w/ Borrowing constraint

- subbing $D_0 = -\overline{B}_0$

$$K_1 = e_0 K_0 + \overline{B}_0$$

$$\boxed{K_1 = \frac{e_0 K_0}{1 - \overline{p}}}$$

depends on cash flows e_0K_0 and how much it can borrow $\overline{p}$

Limited Pledgeability

- Firm can Only Pledge up to a level $\boxed{\overline{p} < e_1}$, Beyond this are Information Frictions Problematic

$$e_1 = \underbrace{\overline{p}}_{\text{less risky}} + \underbrace{e_1 - \overline{p}}_{\text{more risky}}$$

} $\overline{p}$ is collateralized or guaranteed part of return

Determinants of corporate Borrowing

1. Asset Based lending

↳ $\overline{p}$ tied specifically to collateral

2. Cash-Flow Based Lending

↳ Claim on future earnings

Earnings-Based Borrowing constraints

$$\boxed{\overline{p} = \phi e_0}$$

Implications:

1. Doesn't hold for unconstrained firms
2. Doesn't hold for constrained firms with Asset based lending

Role of Financial Institutions

- investment specialists/experts
- Banks borrow from financiers & invest in loans
- bank investment raises firms borrowing

Information Frictions & demand for Financing

- consider this model

$$\delta = 1 \quad K_1 = i_0 \quad \phi = 0$$

- $e_1 > 1 + R_f \Rightarrow$ firm wants to borrow

* New agent: Banks with funds to lend

Financiers (F) → Banks → Firms

* Bank Capital is costlier, requiring $(1+s)$ return

Bank Mitigates Moral Hazard Problem

Project	Good	Bad-F	Bad-Bank
Private Benefit	0	$c^F K_1$	$c^{Bank} K_1$
Prob. of success	1	$I < 1$	$I < 1$

- Though $\boxed{c^B < c^F}$

Case 1: Firm Borrows from Financier

$$\overline{p}^F = e_1 - \frac{c^F}{1-I} \qquad \overline{B}_0^F = \overline{p}^F K_1$$

Firm invests

$$\boxed{K_1 = \frac{e_0 K_0}{1 - \overline{p}^F}}$$

Firm Payoff

$$\boxed{U_1^F = e_1 K_1 - \overline{B}_0^F}$$

⇓

Firm Payoff

$$U_1^F = \left(1 + \frac{e_1 - 1}{1 - \overline{p}^F}\right) e_0 K_0$$

Implications:

1. $\frac{1}{1-\overline{p}_F}$ = leverage ratio
2. b/c $e_1 > 1$ firm want to borrow as much as possible to earn as much as possible

Case 2: Firm chooses to borrow from bank

$$\overline{p}_{bank} = e_1 - \frac{c^{bank}}{1-I}$$

↳ Bank monitors more; reduces private Benefit

- Bank receives $(1+s)$ return

$$\overline{B}_0^{Bank} = \frac{\overline{p}^{Bank}}{(1+s)} K_1$$

- Firm Borrows up to Max

$$K_1 = \frac{e_0 K_0}{1 - \frac{\overline{p}^{Bank}}{1+s}}$$

- Firm Payoff w/ Bank Borrowing

$$U_1^{Bank} = e_1 K_1 - (1+s)\overline{B}_0^{Bank}$$

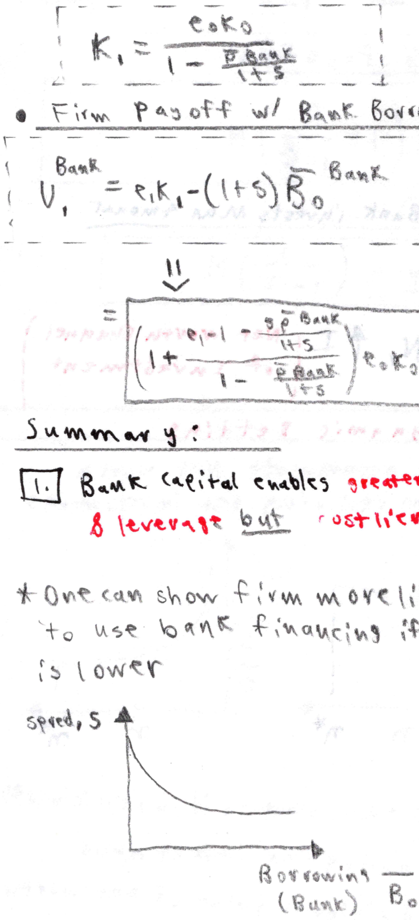
⇓

$$= \boxed{\left(1 + \frac{e_1 - 1 - \frac{s\overline{p}^{Bank}}{1+s}}{1 - \frac{\overline{p}^{Bank}}{1+s}}\right) e_0 K_0}$$

Summary:

1. Bank capital enables greater Borrowing & leverage but costlier

* One can show firm more likely to use bank financing if s is lower



Bank Net worth & supply of Bank Financing

Supply of Bank Funds Model

- Banks start w/ internal funds $\boxed{N}$
- Banks Investment $\boxed{I}$

↳ $\boxed{I}$ determines $\bar{B}_0^{Bank}$ *

- Bank's investment opportunity generates $(1+R)$ per dollar with $R > 0$
- $\bar{\rho}$ denotes pledgeable return to financiers

Bank's Optimization Problem

- Bank chooses how much to borrow per asset $(\rho \leq \bar{\rho})$

$$\max_{I,\rho} (1+R)I - \rho I$$

$$I = N + \rho I$$

$$\rho \leq \bar{\rho} < 1$$

- Bank rely on Asset-Lending much more

Banks Budget constraint

$$I = \frac{1}{1-\rho} N$$

Bank's Payoff:

$$U = \left(1 + \frac{R}{1-\bar{\rho}}\right) N$$

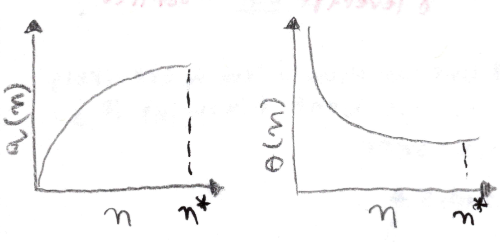
Bank invests Max Amount

$$I = \left(\frac{1}{1-\bar{\rho}}\right) N$$

* ↑ N, ↑ I | Net worth Channel of Investment

Dynamic setting

- Bank accumulate Net worth from realizing past investments



- $\boxed{n}$ is equivalent to N in static model
- $\boxed{q}$ is analog for investment
- $\boxed{\theta}$ additional value of Net worth

* as ↑ n, banks run into Diminishing returns

- when $n = n^*$, Banks pays dividends
- when $m \leq n^*$ Two Regimes

$\boxed{1.}$ n remains around n^*

↳ sensitivity of Investment Low

$\boxed{2.}$ n remains much lower than m

↳ VERY SENSITIVE

** Net worth channel aplies especially when N is low

Credit Crunch

- Financial shocks that lower bank's net worth, lowers their investment
- so ↓ N ↓ I ↑ s ↓ $\bar{B}_0^{bank}$

* effects investment & Economic activity

Evidence for Credit Crunch

Case 1	Inter-Bank lending

- s rises in 2007 & 2008

↳ Bank N fell → I falls → spread rises

case 2	Syndicated Loans

- Fell by 79%
- Fell 47% at 2008 Q4

case 3	Bank-to-Bank comparison

- US Bancorp firms 4x more likely to get loan

Results:

$\boxed{1.}$ Shock to net worth N will lower investment

$\boxed{2}$ Firm can't easily switch banks

Crises & amplification Mechanisms

- Losses in sub-prime crisis seemed very small, however damage on asset prices & output was large

Amplification From Leverage

Recall Bank Model

Invest	Borrowing
$I_0 = \frac{N_0}{1-\bar{\rho}_0}$	$\bar{\rho}_0 I_0$

- Realization $(1+R_1)$ determines N_1

Example:

$$N_1 = (1+R_1)20 - 19$$

* 1% change in R_1 has 20% effect on N_1

Leverage Amplifies Loss

$$N_1 = \left(1 + \frac{R_1}{1-\bar{\rho}_0}\right) N_0$$

* This drop in Net worth reduces investment I_1

Result: $N_1 = (1+R_1)20 - 19$
1% drop

$$I_1 = \frac{1}{1-\bar{\rho}_1} N_1$$

20% drop ↓ 20% drop

Amplication from Pro-cyclical Leverage

- Debt limits seem to be Procyclical

$\bar{\rho}$ is HIGH in Good times

$\bar{\rho}$ is LOW in Bad times

Example:

- suppose $\bar{\rho}_1 = 0.9 < 0.95$ after the loss

Fire Sales & Amplification from asset Price

- Recall Return on Financial Asset

$$1 + R_1 = \frac{D_1 + P_1}{P_0}$$

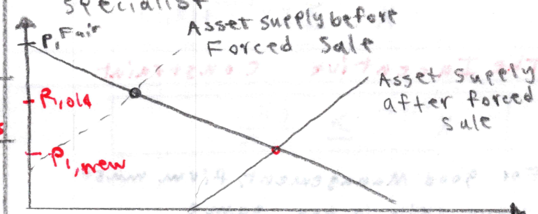
- Price can depend on Supply & Demand in the short Run
- These are FIRE SALES

Forced Sales Lower Price

- To buy loans & bonds usually go through specialist

* when you sell asset you'll receive $P_1 < P_1^{fair}$

- $(P_1^{fair} - P_1)$ = liquidity premium that compensates market specialist



* specialists will hold more but only at greater discount

Example

- Farmer has low cash flows
- cannot reschedule debt
- Must liquidate ~~debt~~ farm to pay debt

Potential Buyers

High value: Fellow Farmer

Low Value: Public field

* if Neighbor is similarly stressed

Evidence of Existence → convertible Bonds

Fire sales as Amplification Mechanism

$I_1 = \frac{1}{1-\bar{p}_1} N_1$ where, $N_1 = (1+R_1)N_0 - \bar{B}$

• suppose a shock to fair price which lowers R_1

• This induces Banks to sell assets, Further lowering Price

Frame work:

$$P_1 = P_1^{Fair} + P_0 G(I_1)$$

↳ G is increasing Function

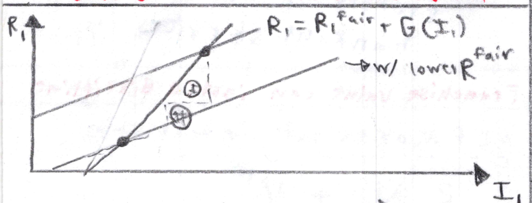
$$1+R_1 = \frac{D_1+P_1}{P_0} = \underbrace{1+R_1^{Fair}}_{\text{Fair Return}} + G(I_1)$$

$$N_1 = \left(\overbrace{(1+R_1^{Fair})}^{1\% \text{ drop}} + G(I_1) \right) 20 - 19$$

more than 1% drop

more than 20% drop

Fire Sale & Net Worth triggers spiral



$I_1 = 1/(1-\bar{p}_1)((1+R_1)20-19))$

(I) Net worth channel

(II) Fire sales channel

Summary of Amplification Mechanisms

$$I_1 = \frac{1}{1-p_1} N_1 = \frac{1}{1-p_1(R_1)}\left((1+R^{fair}+G(I_1))20-19\right)$$

1. Leverage amplifies gains/losses
2. Procyclical Leverage $\boxed{p_1(R_1)}$ generates further leverage
3. Fire Sales $\boxed{G(I_1)}$ generates further amplification

Maturity Mismatch & Illiquidity

Bank Balance sheets feature 2 mismatch

1. Risk Mismatch → Relatively risky assets, Relatively safe Liabilities
2. Maturity Mismatch: Longer Maturity Assets, but Short-term Liabilities

Liquidity is low for Bank assets

1. Why do the banks feature a mismatch to begin with
2. How does this create fragility/runs/panics
3. What should government policy do about this

Model of Banks & Financiers

• Number of Banks $\boxed{B}$ & Financiers $\boxed{F}$

• 3 Periods

↳ Date 0: Investments are made

↳ Dates {1,2} returns might be realized

• Each $\boxed{F}$ has 1 dollar at $t=0$, but nowhere else

• B's have no resources

F subject to Liquidity shock

$C_0 + C_1 + C_2$ if no liquidity shock

$C_0 + C_1 + C_2$ if shock, but $C_1 \geq \bar{C}_1$

$C_0 + C_1 + C_2 - L$ if shock & $C_1 < \bar{C}_1$

• each investor experiences Liquidity shock w/ probability $\boxed{\lambda}$

• Shocks are $\boxed{F}$'s Private Information

Project choice: Liquidity trade-off

2 types of Projects

1. Liquid: 1 dollar yields 1 tomorrow
2. Illiquid: Investing 1
   • yields $R>1$ at $t=2$
   • $L<1$ at $t=1$ if liquidated

What should we hold?

Without Coordination: Liquidity wasted

• On their own each $\boxed{F}$ would put $\bar{C}_1$ into cash, & rest into loans $(1-\bar{C}_1)$

* left to their own devices, no one would invest in loans! wasted liquidity

Ideal arrangement: Liquidity Pooling

• Invest $\boxed{\lambda \bar{C}_1}$ in cash

• Invest remaining 0.9 in loans

How Banks implement this outcome

1. Each F pays in their $1 to bank at t=0
   • agent can withdraw $\bar{C}_1 = 1$ at $t=1$
   • can wait & withdraw $C_2 = 1.5$ at $t=2$
2. Each F accepts a bank contract
3. F's only withdraw money if liquidity shock

Bank Liabilities & Liquidity pooling

• This contract represents various types of banking: savings deposits, term deposits

Equilibrium relied on

4. F's withdraw only if they experience liquidity shock

↳ There exists "Panic withdrawls"

Model w/ Panic withdrawls

• Since bank doesn't know who faces liquidity shock

↳ can't distinguish fundamental & panic withdrawls

The Model

• Let $\tilde\lambda \geq \lambda$ denote total withdrawls

↳ λ = true withdawl

↳ $\tilde\lambda - \lambda$ = Panic withdrawl

• Bank promises $C_1 = 1$ & $C_2 = 1.5$

• But may be unable to make those if $\tilde\lambda$ sufficiently high

• Let $(C_1(\tilde\lambda), C_2(\tilde\lambda))$ denote actual payoffs conditional on $\tilde\lambda$

Bank's decision after Panic withdrawl

• Bank has just enough cash to meet $\boxed{\lambda}$

↳ But can liquidate loans to get $I = 0.5 < 1$

• Bank can still pay everyone who wants to withdraw an early amount

$$C_1(\tilde\lambda) = \bar{C}_1 = 1$$

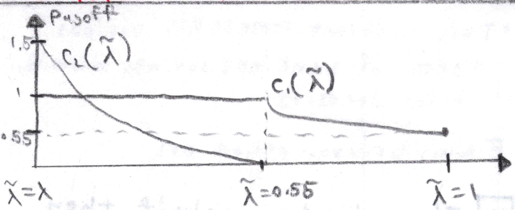
• * But has to liquidate $\boxed{2(\tilde\lambda - \lambda)}$ to do so

Return of Not Panicking

$$C_2(\tilde\lambda) = \frac{1}{1-\tilde\lambda} R(1-\lambda-2(\tilde\lambda-\lambda))$$

• in extreme cases, non-panicking depositors receive nothing

Pictorial Representation



- $C_1(\tilde\lambda) - C_2(\tilde\lambda)$ capture incentive for F to panic
- Panic withdrawals are complementary

* This leads to Multiple equilibria

Bad Equilibrium → Bank Run

Solutions to Bank Runs

$\boxed{1.}$ Close the bank

- Bank temporarily shuts down after $\boxed{\lambda}$ withdrawls

Problems

- Bank doesn't know true $\boxed{\lambda}$
- There is no clear distinction of t=1 & t=2

$\boxed{2.}$ Lender of Last Resort

- Central Bank will lend at rate $\boxed{r}$ such that $1+r \in [0,R]$

* Bank now borrows from Fed instead of liquidating assets

$$C_2(\tilde\lambda) = \frac{R(1-\lambda) - (1+r)(\tilde\lambda - \lambda)}{1-\tilde\lambda} = 1.5$$

- less costly because of no liquidation & full returns

LLR eliminates Bank Run Equilibrium

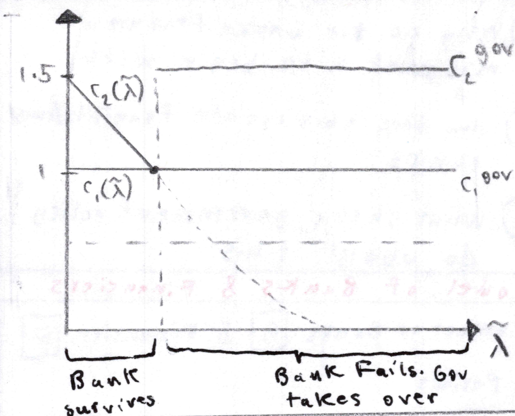
- Presence of LLR ensures that $\boxed{F}$'s do not panic
- Main Role of LLR is preventitive. Knowing there exists LLR ensures depositors don't panic

Deposit Insurance

- If Bank "Fails" ($C_1, C_2 < 0$)

↳ government takes over & redeems its original promise to Depositers

Deposit Insurance: Picture



Deposit Insurance, LLR, & Moral Hazard

- Deposit insurance creates Moral Hazard
- If bank has $\boxed{R<1}$, & FDIC absorbs the losses

~~Bank Runs in Modern Financial Markets~~

Why do financial institutes make losses

$\boxed{1.}$ Mistakes: Optimism & neglect risk

$\boxed{2.}$ Moral Hazard *

Forms of Moral Hazard

$\boxed{1.}$ compensation contracts

$\boxed{2.}$ Borrowing contracts

$\boxed{3.}$ Government gaurentees

↳ 3 harder to pin down

Framework: Moral Hazard

- Suppose $\bar p = 0$ (Bank only invests it's money)
- Bank starts w/ $\boxed{N_0}$

	Safe	Risky
High State	$\bar R_1$	$R_1^H > \bar R_1$
Low state	$\bar R_1$	$R_1^L = 0$

Probability of Low State $\boxed{\pi}$

Key Assumption:

$$\bar R_1 > (1-\pi) R_1^H$$

- absent intervention it wouldn't be profitable for risky stuff

Strict No-Bail-out Policy?

- Could prevent reckless investing
- But if $\bar R_1 < (1-\pi) R_1^H$
- Bank would take risk even in absence of government bail out

* Testing For Optimism vs. Moral Hazard

Moral Hazard: - High π - Deliberate risk

Mistake: - low π - neglected risk

Was Housing Bubble Obvious?

* Bubbles really only recognized in hindsight

Litmus test for confidence in π | ARE THE INSIDERS ON BOARD?

Results

- Securitization managers were at least as optimistic as general public about Returns
- The more "on-board" CEO's were, the greater their losses

Problems with basic Moral Hazard

$\boxed{1.}$ owners/shareholders dont always profit from bailout

$\boxed{2.}$ Franchise Value → Stable profits from future banking services $\boxed{V_F}$

Franchise value can instill discipline

- If makes safe investment

$$\bar R_1 N_0 + V^F$$

- If risky

$$(1-\pi)(R_1^H N_0 + V^F) + \pi(R_1^{L,bail} N_0)$$