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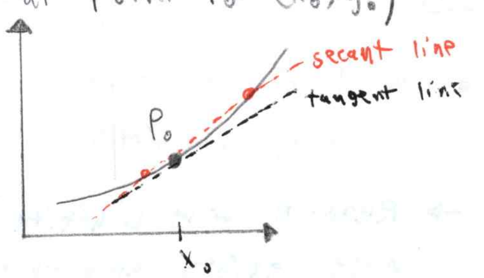
Lecture #1: Rate of Change

Differentiation

- What is a Derivative?
  - Geometric Intereretation
  - Physical Intereretation
- Importance in all Mesurements
- How to differentiate anything $\left(\frac{d}{dx} e^{x \arctan(x)}\right)$, etc.

Geometric Intereretation

- Find tangent line to $y=f(x)$ at Point $P_0=(x_0, y_0)$



$y - y_0 = m(x - x_0)$ (Eqn of the line)

Point $y_0 = f(x_0)$

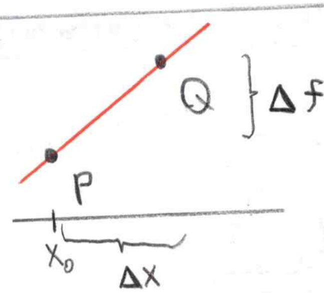
slope $m = f'(x_0)$

Definition

$f'(x_0)$, the derivative of f at x_0 is the slope of the tangent line to $y=f(x)$ at P.

Definition

- Tangent line = Limit of secant Lines PQ as $Q \to P$ (P is Fixed)



$$\boxed{\text{Slope} = \frac{\Delta f}{\Delta x}}$$

↳ of secant line

$$\boxed{\text{Slope} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}}$$

↳ of tangent line

Practical Formalization

$P_0: (x_0, f(x_0))$

$Q: (x_0 + \Delta x, f(x_0 + \Delta x))$

$$\boxed{f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}}$$ ***

$= m$

Example #1:

$f(x) = 1/x$

Difference Quotient

$$\frac{\frac{1}{x_0+\Delta x} - \frac{1}{x_0}}{\Delta x} = f'(x_0)$$

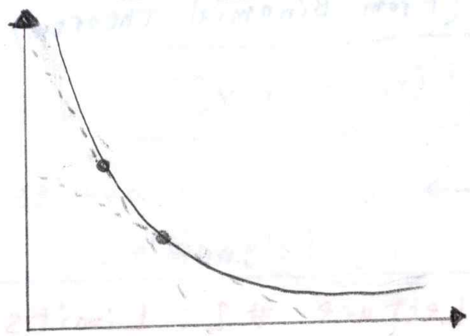
$$= \left(\frac{x_0}{x_0(x_0+\Delta x)} - \frac{(x_0+\Delta x)}{x_0(x_0+\Delta x)}\right) \cdot \frac{1}{\Delta x}$$

$$\boxed{= \lim_{\Delta x \to 0} -\frac{1}{x_0(x_0+\Delta x)}}$$

$$\boxed{= -\frac{1}{x_0^2}}$$

Example #2:

- Find Areas of Triangles enclosed by the tangent to $\boxed{y = 1/x}$



step 1: specify $(x_0, f(x_0))$ pair

step 2: Solve via Difference Quotient the slope of Tangent line

Step 3: solve for (x, y) axes intercepts (when $x=0$ or $y=0$)

step 4: Calculate area $\boxed{=2}$

* Keep in mind, can recycle x & y for multiple lines in consideration

More Notations

$y = f(x)$, $\Delta y = \Delta f$

$$\underset{\text{Newton's}}{f'} = \underbrace{\frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}f = \frac{d}{dx}y}_{\text{Liebniz}}$$

Examele # 3

$f(x) = x^n$

$$f'(x_0) = \frac{(x_0+\Delta x)^n - (x_0)^n}{\Delta x}$$

=> all terms cancel except x_0^{n-1}
(From Binomial Theorem)

$$\boxed{f'(x_0) = n\,x_0^{n-1}}$$

→ Same logic applies to Polynomials

Lecture #2: Limits

Rate of Change Intereretation

$$\underbrace{\frac{\Delta y}{\Delta x}}_{\text{average change}} \longrightarrow \underbrace{\frac{dy}{dx}}_{\text{Instantaneous Rate}}$$

ex.

q = charge, $\frac{dq}{dt}$ = current

s = distance, $\frac{ds}{dt}$ = speed

examele: Pumekin Drop

Height: $h = 80 - 5t^2$ meters

$(0,80) \to (4,0)$

Avg Speed: $\frac{\Delta h}{\Delta t} = \frac{0-80}{4-0} = \boxed{-20}$

Instananeon speed

$$\boxed{\frac{d}{dt}h = 0 - 10t}$$ (From $\frac{d}{dt}x^n$ derivation)

• T = Temperature, $\frac{dT}{dt}$ = Temp Gradient

• Sensitivity of Measurements

ex.

(h measured, ℓ calculated)

critical: Δh (uncertainty in h)

Limits + Continuity

① Easy Limits

→ $\lim_{x\to 4} \frac{x+3}{x^2+1} = \frac{4+3}{4^2+1} = \boxed{\frac{7}{17}}$

② Derivatives are always harder

$$\lim_{x\to x_0} \frac{f(x_0+\Delta x) - f(x_0)}{x - x_0}$$

→ always will need cancellation (otherwise $0/0$)

Right-Hand Limit { $\lim_{x\to x_0^+} f(x)$ $x > x_0$

Left-Hand Limit { $\lim_{x\to x_0^-} f(x)$ $x < x_0$

→ comes in handy for Disjoint Functions
(Do not need value of $f(0)$ to solve)

Definition

∘ f is continuous at x_0 means $\boxed{\lim_{x\to x_0} = f(x_0)}$

1. $\lim_{x\to x_0}$ has to exist ($\lim^+ = \lim^-$)
2. $f(x_0)$ is defined
3. then are equal

Dis Continuous Functions

1. Jump Discontinuity

• $\lim^+, \lim^-$ exist, BUT not equal

2. Removable Discontinuity

• $\lim^+, \lim^-$ are equal, but $f(x_0)$ not defined

→ Reason why L'hôpital's rule exists and that

$\boxed{\frac{d}{dx}\frac{\sin(x)}{x} = 1}$ $\boxed{\frac{d}{dx}\frac{1-\cos(x)}{x} = 0}$ is true

3. Infinite Discontinuity

• Your limits go to infinity

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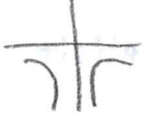
$y = 1/x$

both not continuous b/c not defined at $x=0$

$y' = 1/-x^2$



4. Other Discontinuities

ex. $y = \sin\left(\frac{1}{x}\right)$

Theorem (Differentiable $\Rightarrow$ continuous)

- If f is differentiable at x_0, then f is continuous at x_0

Proof $\lim_{x \to x_0} f(x) - f(x_0) = 0$

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}(x - x_0)$$

$$= f'(x_0) \cdot 0 = 0$$

* Proves implicit assumptions of continuity
* existence of differentiability

Lecture #3: Derivatives

Derivative Formulas

1) Specific $f'(x)$

2) General: $(u+v)' = u' + v'$

$(cu)' = cu'$, c = constant

* Need both for Polynomials

Trig Functions (specific)

$$\frac{d}{dx}\sin(x)$$

$$\frac{\sin(x+\Delta x) - \sin(x)}{\Delta x}$$

$$= \frac{\sin(x)\cos(\Delta x) + \cos(x)\sin(\Delta x) - \sin x}{\Delta x}$$

$$= \sin(x)\underbrace{\left[\frac{\cos(\Delta x) - 1}{\Delta x}\right]}_{(A) = 0} + \cos(x)\underbrace{\frac{\sin(\Delta x)}{\Delta x}}_{(B) = 1}$$

$$\boxed{= \cos(x)}$$

$$\frac{d}{dx}\cos(x)$$

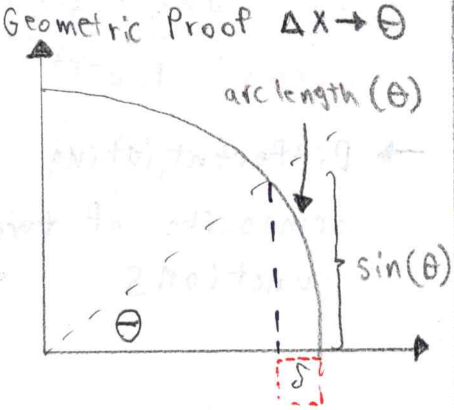
$$= \frac{\cos(x)\cos(\Delta x) - \sin(x)\sin(\Delta x) - \cos(x)}{\Delta x}$$

$$= \cos(x)\left[\frac{\cos \Delta x - 1}{\Delta x}\right] + (-\sin(x))\frac{\sin(\Delta x)}{x}$$

$$\boxed{= -\sin(x)}$$

Proofs of Results (A) + (B)

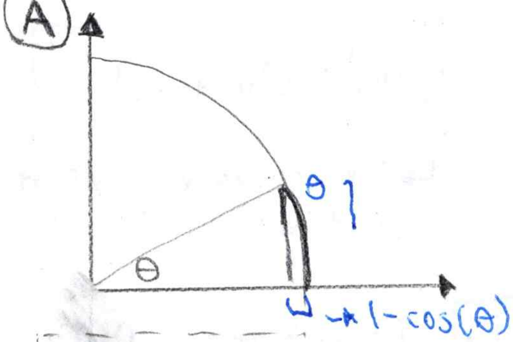
(B) Geometric Proof $\Delta x \to \theta$



as $\delta \to 0$ when $\theta \to 0$

$$\boxed{\sin(\theta) = \theta}$$

(A)

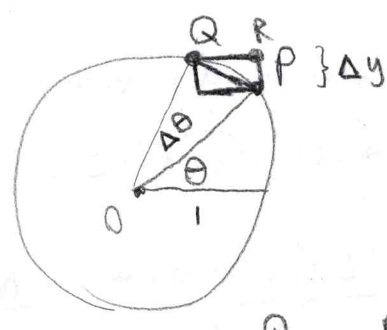


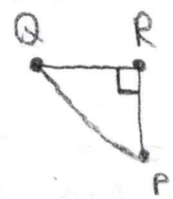
$$\lim_{\theta \to 0} \frac{1 - \cos(\theta)}{\theta} = 0$$

* because $(1 - \cos(\theta))$ tends toward 0 asymptotically faster

Proof that $\frac{d}{dx}\sin(x) = \cos(x)$

- $y = \sin(\theta)$, vertical position of circular motion





$\Delta y = PR$

$PQ \approx \Delta\theta$

$\angle QPR = \theta$

$PQ \perp\approx OP$; PR is vertical

$PR = (\Delta\theta)\cos(\theta)$

$$\frac{\Delta y}{\Delta \theta} \approx \cos\theta$$

$$\boxed{\lim_{\Delta\theta \to 0} \frac{\Delta y}{\Delta \theta} = \cos(\theta)}$$

General Rules

1) Product Rule

$$\boxed{(uv)' = u'v + uv'}$$

↳ like changing one at a time

2. Quotient Rule

$$\boxed{\left(u/v\right)' = \frac{u'v - uv'}{v^2}}$$

↳ $v \neq 0$

Lecture #4: Chain Rule

Proof of Product Rule:

$\Delta(uv)$

$= u(x+\Delta x)v(x+\Delta x) - u(x)v(x)$

$= \left(u(x+\Delta x) - u(x)\right)v(x+\Delta x) + u(x)\left[v(x+\Delta x) - v(x)\right]$

$= \frac{(\Delta u)\,v(x+\Delta x)}{\Delta x} + \frac{u(x)\Delta v}{\Delta x}$

$= \lim_{\Delta x \to 0}\left(\frac{\Delta u}{\Delta x} v(x+\Delta x) + u(x)\frac{\Delta v}{\Delta x}\right)$

$$\boxed{= u'(x)\,v(x) + u(x)\,v'(x)}$$

Proof: Quotient Rule

$\Delta\left(\frac{u}{v}\right) = \frac{u+\Delta u}{v+\Delta v} - \frac{u}{v}$

$= \frac{uv + \Delta u(v) - uv - u\Delta v}{(v+\Delta v)v}$

$= \frac{\Delta u v - u\Delta v}{(v+\Delta v)v}$

$\lim_{\Delta x \to 0}\left[\frac{\Delta u}{\Delta x}\frac{v}{v+\Delta v(v)} - \frac{u\Delta v}{(v+\Delta v)v\Delta x}\right]$

$$\boxed{= \frac{u'v - uv'}{v^2}}$$

3. Composition Rule

ex. $y = (\sin(t))^{10}$

Method: use new variable name

$x = \sin(t)$, $y = x^{10}$

Proof:

$\frac{\Delta y}{\Delta t} = \frac{\Delta y}{\Delta x} \cdot \frac{\Delta x}{\Delta t}$

$\Downarrow$ as $\Delta t \to 0$

$$\boxed{\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}}$$

(Chain Rule)

→ Differentiating composite of two functions

Higher Derivatives

$u = u(x) \to u' \to u''$

$\frac{du}{dx}$ $\quad \frac{d^2u}{dx^2}$

Lecture #5: Implicit Differentiation

Implicit Differentiation

ex. $\left.\frac{d}{dx}x^a = ax^{a-1}\right\}$ Know

Now: $a = m/n$ (m & n integers)

$y = x^{m/n}$

$y^n = x^m$ → Apply dy/dx

$\frac{d}{dx}y^n = \frac{d}{dx}x^m$

$\left(\frac{d}{dy}y^n\right)\frac{dy}{dx} = mx^{m-1}$

$ny^{n-1}\frac{dy}{dx} = mx^{m-1}$

$\frac{dy}{dx} = \frac{mx^{m-1}}{ny^{n-1}}$

$$= \boxed{\frac{dy}{dx} = \frac{mx^{m-1}}{nx^{m/n(n-1)}}}$$

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ex. $x^2+y^2=1$

→ want to know what is $\boxed{dy/dx}$ (chain Rule)

$$y=(1-x^2)^{1/2}$$

$$\boxed{\frac{dy}{dx}=-2x\left(\frac{1}{2}\right)(1-x^2)^{-1/2}}$$

↳ explicit

$$x^2+y^2=1$$

$$\frac{d}{dx}(x^2+y^2)=\frac{d}{dx}(1)$$

$$2x+2yy'=0$$

$$\boxed{y'=-\frac{x}{y}}$$ implicit

ex. $y^4+xy^2-2=0$

→ Quartic Roots get messy

Implicit:

$$4y^3y'+y^2+2xyy'=0$$

$$(4y^3+2xy)y'=-y^2$$

$$\boxed{y'=\frac{-y^2}{4y^3+2xy}}$$

→ FAST. Didn't have to differentiate explicit form, so can plug in y algebraicly

Inverse Functions

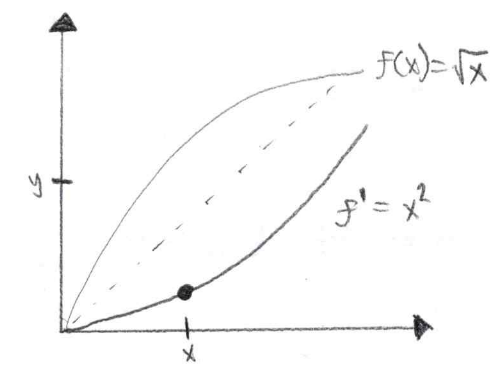
$y=\sqrt{x}$, $x>$, $y^2=x$

if $f(x)=\sqrt{x}$, $g(y)=x$

$$g(y)=y^2$$

→ $g(f(x))=x$ | $g=f^{-1}$, $f=g^{-1}$

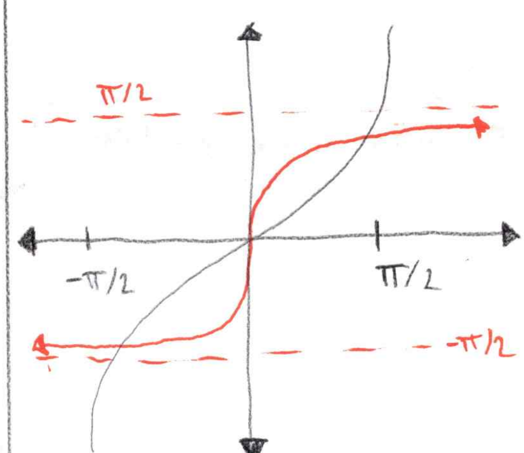
Graphically: f & f^{-1}



⇒ Implicit Differentiation allows for derivative of any inverse function provided we know Derivative of the function

$$y=\tan^{-1}(x)$$

use $\boxed{\tan(y)=x}$



$$\frac{d}{dy}\tan(y)=\frac{d}{dy}\frac{\sin y}{\cos y}$$

$$=\boxed{\frac{1}{\cos^2(y)}}$$

Now

$$\frac{d}{dx}(\tan(y)=x)$$

$$\underbrace{\left(\frac{d}{dy}\tan(y)\right)\frac{dy}{dx}}_{\text{via Chain Rule}}=1$$

$$\frac{1}{\cos^2}\frac{dy}{dx}=1$$

$$\frac{dy}{dx}=\cos^2(y)$$

$$\frac{d}{dx}\tan^{-1}(x)=\cos^2(\tan^{-1}(x))$$

↓ (using Right Triangle knowledge)

$$\boxed{\frac{d}{dx}\tan^{-1}(x)=\frac{1}{1+x^2}}$$

Lecture #6: Exponential & Log

Exponentials + Logarithms

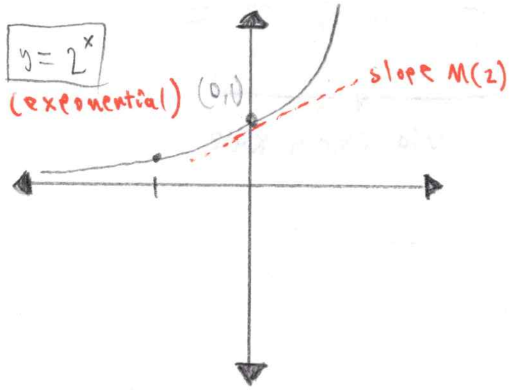
$a>0$ (base), $a^0=1$, $a^1=a$, $a^2=a\cdot a$

$$\boxed{a^{x_1+x_2}=a^{x_1}a^{x_2}}$$

$$\boxed{(a^{x_1})^{x_2}=a^{x_1\cdot x_2}}$$

$$a^{m/n}=\sqrt[n]{a^m}$$

$\therefore$ a^x defined for all x by "filling in" by continuity



Goal: $\frac{d}{dx}a^x$?

$$\lim_{\Delta x\to 0}\frac{a^{x+\Delta x}-a^x}{\Delta x}$$

$$" \quad " \quad \frac{a^x a^{\Delta x}-a^x}{\Delta x}$$

$$\frac{d}{dx}a^x = a^x\underbrace{\lim_{\Delta x\to 0}\frac{a^{\Delta x}-1}{\Delta x}}_{M(a)}$$

$$\boxed{\frac{d}{dx}a^x = M(a)\,a^x}$$

Figuring out M(a)

① Plug in $x=0$

$$\frac{d}{dx}a^x\Big|_{x=0}=M(a)$$

→ $M(a)$ is the slope of a^x at $x=0$

• Define a base $\boxed{e}$ so that $M(e)=1$

if the above is True

$$\boxed{\frac{d}{dx}e^x = e^x}$$

Why $\boxed{e}$ Exists

$f(x)=2^x$, $f'(0)=M(2)$

▲ Stretch by $\boxed{k}$

$$f(kx)=2^{kx}=(2^k)^x=b^x$$

→ you are re-scaling the x-axis and thus changing slope at $x=0$

$$\frac{d}{dx}b^x=\frac{d}{dx}f(kx)=k\,f'(x)$$

$$\frac{d}{dx}b^x\Big|_{x=0}=k\,M(2)$$

$b=e$ when $k=1/M(2)$

Natural Log (Ln)

$$\boxed{y=e^x \iff \ln(y)=x}$$

$\ln(x_1x_2)=\ln(x_1)+\ln(x_2)$

$\ln(1)=0$, $\ln(e)=1$

Finding Derivative of $\ln(x)$

$$w=\ln(x)$$

$$\frac{d}{dx}e^w=\frac{d}{dx}x=1$$

$$\frac{d}{dw}e^w\,\frac{dw}{dx}=1$$

$$e^w\frac{dw}{dx}=\frac{1}{e^w}$$

$$\boxed{\frac{dw}{dx}=\frac{1}{x}}$$

Differentiating Any Exponential

Method 1

⇒ Convert to base $\boxed{e}$

$$a^x=\left(e^{\ln(a)}\right)^x$$

$$\frac{d}{dx}a^x=\frac{d}{dx}e^{x\ln(a)}$$

$$\boxed{=\ln(a)\,e^{\ln(a)\,x}}$$

$$\boxed{M(a)=\ln(a)}$$ (Magic # revealed)

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Method 2: Logarithmic Differentiation

$$\frac{d}{dx}\ln(u)$$

$$= \frac{d\ln(u)}{du}\left(\frac{du}{dx}\right)$$

$$= \frac{du}{dx}\,\frac{1}{u}$$

$$= \boxed{u'/u}$$

$\frac{d}{dx}a^x = ?$ | $u = a^x$

$$\ln(u) = x\ln(a)$$

$$(\ln u)' = \ln(a)$$

$$\frac{u'}{u} = \ln(a)$$

$$\frac{da^x}{dx} = \ln(a)\,a^x$$

ex| $\frac{d}{dx} x^x$, let $u = x^x$

$$\frac{d}{dx}\ln(u) = \frac{u'}{u} = \Big[$$

$$\frac{\frac{d}{dx}x^x}{x^x} = 1 + \ln(x)$$

$$\boxed{\frac{dx^x}{dx} = (1+\ln(x))\,x^x}$$

ex| $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

$$\ln\left[\left(1+\frac{1}{n}\right)^n\right] = n\ln\left(1+\frac{1}{n}\right)$$

(rewrite) $\Downarrow$ as $n\to\infty$, $\Delta x \to 0$

$$\underbrace{\frac{1}{\Delta x}\left[\ln(1+\Delta x) - \underset{0}{\ln(1)}\right]}_{\text{takes form of Derivative @ } x=1}$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e^{\lim_{n\to}\ln\left(1+\frac{1}{n}\right)^n}, \quad \boxed{=1}$$

can approximate with large $\boxed{n}$; e

$\boxed{= e}$ (numerical approximation)

Lecture #9: Linear & Quad Approximations

Linear Approximations

$$f(x) = f(x_0) + f'(x)\big|_{x=x_0}(x - x_0)$$

$$\frac{\Delta f}{\Delta x} \approx f'(x_0) \left[\text{for small } \Delta x \text{ values}\right]$$

Near $x=0$, useful Approximations

$$\left.\begin{aligned} \sin(x) &= x \\ \cos(x) &= 1 \\ e^x &= 1+x \end{aligned}\right\} x \approx 0$$

$$\ln(1+x) \approx x$$

$$(1+x)^r \approx 1 + rx$$

ex| $\ln(1.1) \approx 1/10$

$\boxed{b/c}$ $\underbrace{\ln\left(1+\frac{1}{10}\right)}_{\text{hard, b/c we need to strategically choose}} \approx 1/10$

ex| Find Linear Approximation near $x=0$ of

$$e^{-3x}/\sqrt{1+x}$$

$$= \boxed{e^{-3x}(1+x)^{-1/2}}$$ (Know how to solve)

$= (1-3x)\left(1-\frac{1}{2}x\right) \Rightarrow$ get rid of higher power terms ($x^2 \to x^n$)

Quadratic Approximation

$$\boxed{f(x) \approx f(x_0) + \underbrace{f'(x-x_0)}_{\text{linear part}} + \underbrace{\frac{f''}{2}(x-x_0)^2}_{\text{Quadratic part}}}$$

* Note the $\boxed{1/2}$ that is because the quadratic geometric component must be differentiated twice to become a scalar to be applied

Lecture #10 : Curve Sketching

Quadratic Approximations (cont.)

o Use when linear not enough

Curve Sketching

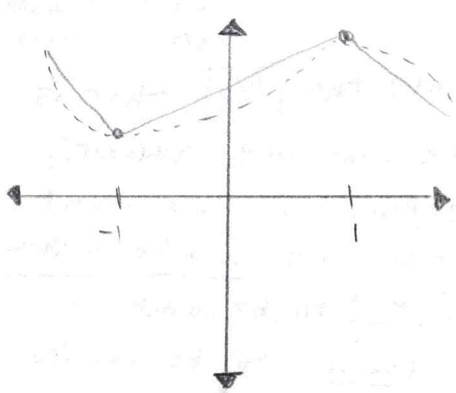
Goal: Graph $f(x)$ using f', f'' (positive/negative)

$f'>0$, f increase	$f''>0$, f' increase concave up
$f'<0$, f decrease	$f''<0$, f' decrease concave down

ex| $f(x) = 3x - x^3$

$$f'(x) = 3(1-x)(1+x)$$

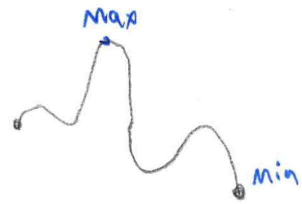
x	$f'(x)$	$f''(x)$
$-1<x<1$	increasing	
$1<x$	decrease	
$x<-1$	decrease	



* if $f'(x_0)=0$, x_0 is a **critical point**

Lecture #11 Max - Min

Goal: Given function find maxima & minima



o Easy w/ sketch

→ **use shortcuts**

Key Idea:

→ONLY Need to Look at

1) Critical Points
2) End Points
3) Discontinuities

Re-framing Max-Min in words

ex. Wire of length 1 cut into two pieces. Largest area enclosed?

Length_1 $= x$

Length_2 $= 1 - x$

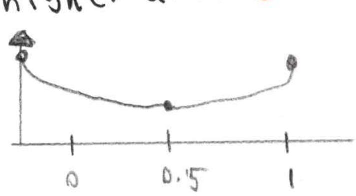
$$\text{Area} = \left(\frac{x}{4}\right)^2 + \left(\frac{1-x}{4}\right)^2$$

$$0 = \frac{(x^2)}{16} + \left(\frac{1-2x+x^2}{16}\right)$$

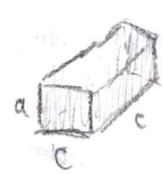
$$0 = 2x + 2x - 2$$

$$\boxed{x = 0.5}$$

BUT, End points yield higher area. Be careful!



ex 2| Find Box w/o a top with least surface area for fixed volume



Box volume: V

$$V = c^2 a$$

$$a = V/c^2$$

∴ Same as before

$$A = c^2 + 4c\left(\frac{V}{c^2}\right)$$

$$A' = 2c - \frac{4V}{c^2}$$

$$\boxed{c = 2^{1/3} V^{1/3}} \Rightarrow \text{Critical Point}$$

More Meaningful Answers

1. **Dimensionless variables**

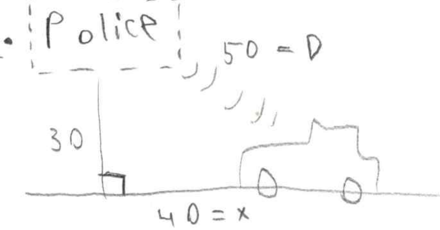
$$A/V^{2/3} = 3 \cdot 2^{1/3}$$

$$c/a = \boxed{2}$$

2. Solvable via Implicit Different

Lecture #12 : Related Rates

ex. Police



o 80 ft/second Approach

• x is horizontal distance

• D is radar/car distance

Q: Are you speeding if

$$\boxed{\frac{dD}{dt} = 80 \text{ ft/sec}}$$

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Know

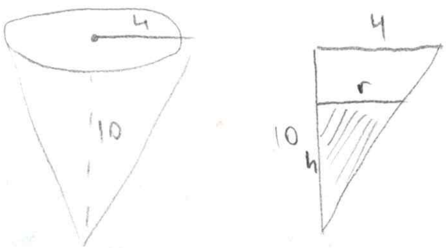
$x^2 + 30^2 = D^2, \quad \frac{dD}{dt} = -80 \text{ ft/s}$

→ Use implicit Differentiation

$$\boxed{2x\frac{dx}{dt} = 2D\frac{dD}{dt}}$$ (Plug in values of D/x to solve)

ex. 2 | A conical tank with top of radius 4 ft, depth of 10 ft, is being filled at 2 ft³/min

Q: How fast is water rising when at depth 5 ft?



Know

$$\frac{r}{h} = \frac{4}{10}$$ simil

$V = \frac{1}{3}\pi r^2 h, \quad \frac{dV}{dt} = 2$

$$2 = \frac{1}{3}\pi\left[\left(\frac{2}{5}\right)^2 3h^2 \frac{dh}{dt}\right]$$

⇓

$\boxed{\text{solve}}$

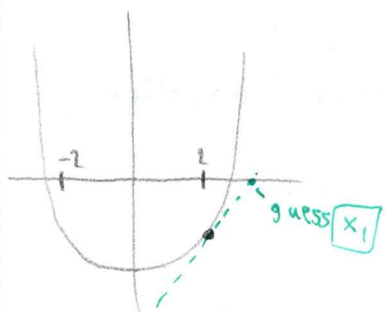
Suspension Bridge Problem

$$\sqrt{x^2+y^2} + \sqrt{(a-x)^2+(b-y)^2} = L$$

- $y = y(x)$ implicitly
  → bottom is where $y' = 0$

Newton's Method

- Solve $x^2 = 5$
  → $f(x) = x^2 - 5$, solve $f(x) = 0$



I start w/ initial guess of $x_0 = 2$

II x_1 is the x-intercept

$$0 - y_0 = m(x_1 - x_0)$$

$$\frac{-y_0}{m} = x_1 - x_0$$

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$ Newton's Method * * *

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

→ each iteration accuracy doubles (error ^(1/4))

Newtons method works well if

1) You start nearby (close initial x_0 guess)
2) $|f'|$ not small

Mean Value Theorem (MVT)

(*) $$\frac{f(b)-f(a)}{b-a} = f'(c)$$

for some $\boxed{c}$ $a < c < b$

Provided

1. f is differentiable in $a < x < b$
2. f is continuous $a \le x \le b$

Lecture # 15: Anti-Derivatives

Differentials

$y = f(x)$

Differential of f $\boxed{dy = f'(x)\,dx}$

Anti Derivatives

$$G(x) = \int g(x)\,dx$$

→ anti-derivative of g

Uniqueness of anti-Derivatives

If $F' = G'$

then $\boxed{F(x) = G(x) + c}$

Anti-Derivative using substitution

$$\int x^3(x^4+2)^5\,dx$$

$u = x^4 + 2, \quad du = 4x^3$ (sub back in $\boxed{u}$)

↓

$$\int \frac{1}{4}u^5\,du = \boxed{\frac{1}{24}u^6 + C}$$

Lecture #16: Differential Equations

Ex. $\frac{dy}{dx} = f(x)$

$y = \int f(x)\,dx \leftarrow$ solved

Methods to solve

1) substitution
2) Intelligent functional form guesswork

Seperation of Variables

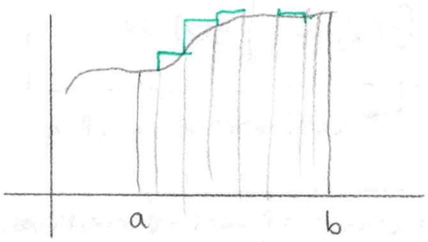
$$\frac{dy}{dx} = f(x)\,g(y)$$

$$\frac{dy}{g(y)} = f(x)\,dx$$

$$\int \frac{dy}{g(y)} = \int \frac{dx}{f(x)} + c$$

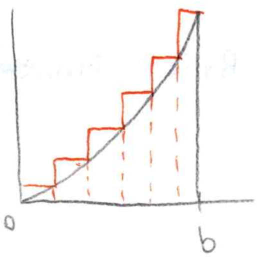
Lecture #18: Definite Integrals

Find Area under curve or cumulative sum



1. Divide into rectangles
2. Add up areas
3. Take limit as # rectangles $n \to \infty$

Example 1 $f(x) = x^2$; $a = 0$



Base Length: b/n

x	$f(x)$
b/n	$(b/n)^2$
$2b/n$	$(2b/n)^2$

Sum of areas

$$\left(\frac{b}{n}\right)\left(\frac{b}{n}\right)^2 + \left(\frac{b}{n}\right)\left(\frac{2b}{n}\right)^2 + \dots$$

$$= \left(\frac{b}{n}\right)^3 (1 + 2^2 + 3^2 + \dots n^2)$$

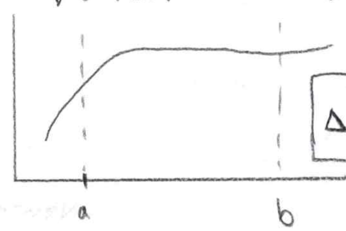
$$= \boxed{\frac{1}{3} b^3}$$

Summation Notation

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \dots a_n$$

Riemann Sums

→ General Procedure for Definite Integrals



$\boxed{\Delta x = \frac{b-a}{n}}$

→ Pick any height in each interval

$$\underbrace{\sum_{i=1}^{n} f(c_i)\Delta x}_{\text{Riemann Sum}} \xrightarrow{(\Delta x \to 0)} \int_a^b f(x)\,dx$$

Lecture #19: FFTC

Fundamental Thm of Calculus #1

if $F'(x) = f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Properties of Integrals

1. $\int_a^b (f(x) + g(x)) = \int_a^b f(x) + \int_a^b g(x)$
2. $\int_a^b c\,f(x) = c \int_a^b f(x)$
3. for $a < b < c$

   $\int_a^b f(x) + \int_b^c f(x) = \int_a^c f(x)$
4. $\int_a^a f(x) = 0$
5. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Change of Variables

$$\int_{u_1}^{u_2} g(u)\,du = \int_{x_1}^{x_2} g(u(x))\,u'(x)\,dx$$

$u = u(x)$ $\quad u_1 = u(x_1)$

$du = u'(x)\,dx$ $\quad u_2 = u(x_2)$

*NOTE: only works if u' DOESN'T change signs

Lecture #20: SFTC

Fundamental Thm of Calculus #2

- If f is continuous, and $G(x) = \int_a^x f(t)\,dt$, $a \leq t \leq x$

$$\boxed{G'(x) = f(x)}$$

$$\frac{d}{dx}\int_1^x f(u)\,du = \boxed{f(x)}$$

Transcendental → value originating or as of consequence outside realm of algebra

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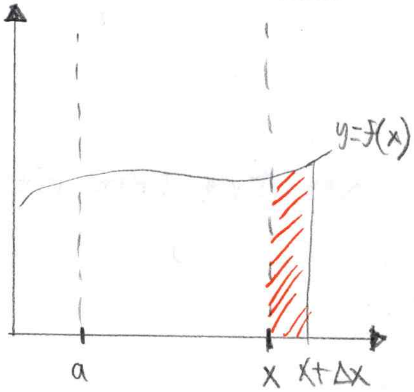
Graphical proof of FTC 2



$\Delta G = \Delta x \, f(x)$

$$\boxed{\lim_{\Delta x \to 0} \frac{\Delta G}{\Delta f(x)} = f(x)}$$

Lecture #21: Application to Logarithms

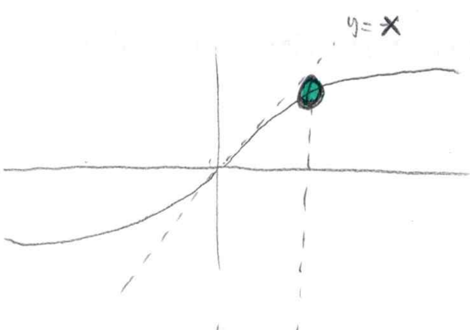
$$L(x) = \int_1^x \frac{dt}{t}$$

↳ Definition of LOG

Now,

$F(x) = \int_0^x e^{-t^2}\, dt$ (integration of standard normal)

$F(x) = e^{-x^2}$



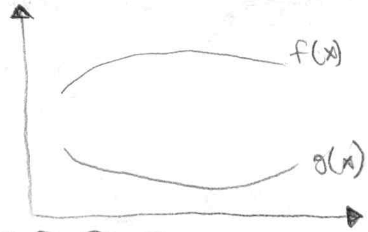
fact

$\lim_{x \to \infty} \boxed{F(x) = \sqrt{\pi}/2}$

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

(tables convert erf(x) to Prob)

Areas Between Curves

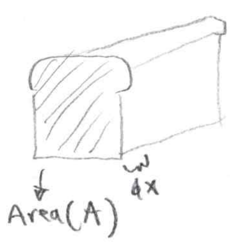


$$\text{Area} = \int_a^b \big(f(x) - g(x)\big)\, dx$$

→ If given complex curves, divide up the integral or switch co-ordinate axes or functional forms integrated over

Lecture #22: Volumes

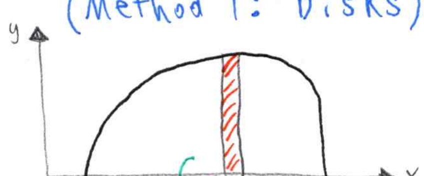
Volumes by Slicing



$dV = A(x)\, dx$

$V = \int A(x)\, dx$

Solids of Revolution (Method 1: Disks)

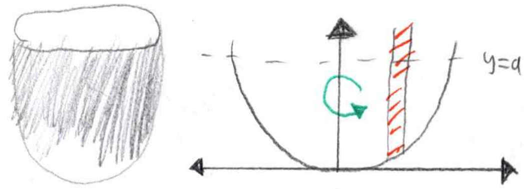


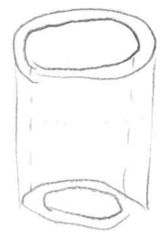
⇒ Revolves around x-axis

$$\boxed{dV = \pi (f(x))^2\, dx}$$

AND, with intelligent limits, can achieve other volumes

Method 2: Shells





Thickness = dx

height = a − y

Integrand = top − bottom

$$\boxed{dV = 2\pi x (a - f(x))\, dx}$$

Limits: $\{x: 0, \sqrt{a}\}$

Lecture #23: Work/Probability

Weighted Average

$$\boxed{\frac{\int_a^b f(x)\, w(x)\, dx}{\int_a^b w(x)\, dx}}$$

Probability

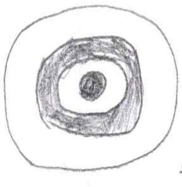
- $a \le x_1 < x_2 \le b$

want $P(x_1 < x < x_2)$

$$\boxed{= \frac{\int_{x_1}^{x_2} w(x)\, dx}{\int_a^b w(x)\, dx}}$$

Lecture #24: Numerical Integration

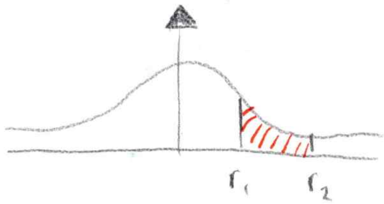
worked example



what is probability of getting hit

if $n_{hits} \approx c\,e^{-r^2}$

side view of e^{-r^2}



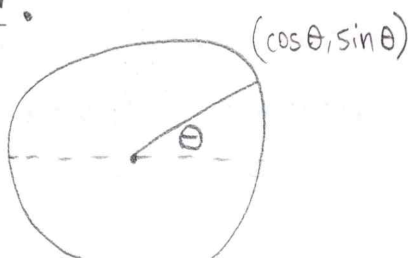
$$\int_{r_1}^{r_2} (2\pi r)\, e^{-r^2}\, dr$$

Numerical Integrations

1. Riemann Sums
2. Trapezoidal Rule
3. Simpson's Rule

Lecture #27: Trig Integrals

Review:



$$\sin^2\theta + \cos^2(\theta) = 1$$

$$\cos(2\theta) = \cos^2\theta - \sin^2(\theta)$$

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

Trig Integrals

$$d\,\sin(x) = \cos(x)$$

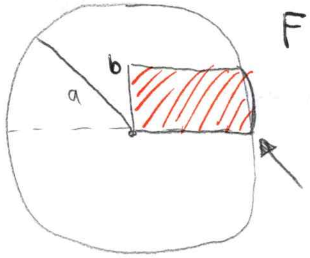
$$d\,\cos(x) = -\sin(x)\,dx$$

① $\int \sin^m(x)\cos^n(x)\,dx$

$m, n = 0, 1, 2, 3$

→ can tackle w/ substitution (or cleverly manipulate integrand by trig properties)

Sample Problem:



Find area of shaded section

$x = \sqrt{a^2 - y^2}$

$$\text{Area} = \int y\,dx \quad \underline{\text{or}} \quad \int x\,dy$$

Alternatively; Polar Coord

$y = a\sin(\theta)$

$dy = a\cos(\theta)$

$$\text{Area} = \int_0^b \sqrt{a^2 - y^2}\,dy$$

$$\Downarrow$$

$$\int_{\square}^{\square} (a\cos(\theta))(a\cos(\theta)\,d\theta)$$

- Solve for indefinite integral
- Then back substitute in terms of $\boxed{y}$

quite messy

Lecture # 28: Inverse substitution

Recall:

$\sec = \frac{1}{\cos}$	$\tan = \frac{\sin}{\cos}$
$\csc = \frac{1}{\sin}$	$\text{ctn} = \frac{\cos}{\sin}$

Key Trig Identities:

1. $\sec^2 = 1 + \tan^2$
2. $\tan' = \sec^2$
3. $\sec' = \sec\tan$

ex

$$\int \tan\,dx = \int \frac{\sin x}{\cos x}\,dx \quad \begin{array}{|l} u = \cos(x) \\ du = -\sin(x)dx \end{array}$$

$$= \int \frac{-du}{u} = \boxed{-\ln(u) + C}$$

ex

$\int \sec(x)\,dx$ → "logarithmic derivative form"

ex

$$\int \sec^4(x) = \int \sec^2(x)\left(1 + \tan^2(x)\right)$$

$\boxed{\begin{array}{l} u = \tan(x) \\ du = \sec^2(x) \end{array}}$ $\int (1 + u^2)\,du$

$u + \frac{1}{3}u^3 + c$

Trig Function Substitution strategy

$$\int \frac{dx}{x^2\sqrt{1+x^2}}$$

$x = \tan(\theta)$

$\sqrt{1+x^2} = \sec(\theta)$

$dx = \sec^2(\theta)\,d\theta$

$$\downarrow$$

$$\int \frac{\sec^2(\theta)\,d\theta}{(\tan^2\theta)\sec(\theta)}$$

in terms of $\sin(\theta)$ $\boxed{\&}$ $\cos(\theta)$

$$\int \frac{\cos(\theta)}{\sin^2(\theta)} = \Big(\text{Another Substitution}$$

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- Takeaway → Trig substitution offers approach to make integrals more manageable

Ex.

Direct substitution

$u = x+2$

$du = dx$

$x^2+4x = u^2-4$

$$\int \frac{dx}{\sqrt{x^2+4x}}$$

$u = 2\sec(\theta)$

$du = 2\sec(\theta)\tan(\theta)$

$$= \int \frac{du}{\sqrt{u^2-4}}$$

$$\int \frac{2\sec(\theta)\tan(\theta)}{2\tan(\theta)}$$

$$\boxed{= \ln(\sec(\theta)+\tan(\theta))+C}$$

Lecture #29: Partial Fractions

$\frac{P(x)}{Q(x)}$ → "Rational Function"

Def| Ratio of two polynomials $P(x)$ & $Q(x)$

Partial Fractions

→ Splits $\boxed{P}$ & $\boxed{Q}$ into easier pieces

① Factor Denominator

ex. $\frac{4x-1}{x^2+x-2} \rightarrow \frac{4x-1}{(x+2)(x-1)}$

② Write in Partial Form

$$= \frac{A}{x-1} + \frac{B}{x+2}$$

③ Solve for $\boxed{A}$ & $\boxed{B}$

Conditions to work|

1) Denominator $Q(x)$ has distinct linear factors

2) degree P < degree Q

Note|

- For repeated Linear roots need seperate term for EACH Factor

- For higher order Denominator Power Degree(numerator) = Deg(Denom) -1

ex| $\frac{Bx+C}{x^2-4}$

- If cubic root, haven't factored enough

- If deg(num) > deg(den) use long division first

Lecture #30: Integration By Parts

From Product Rule:

$$\boxed{(uv)' = u'v + uv'}$$

$$uv' = (uv)' - u'v$$

integration by parts (Indefinite)

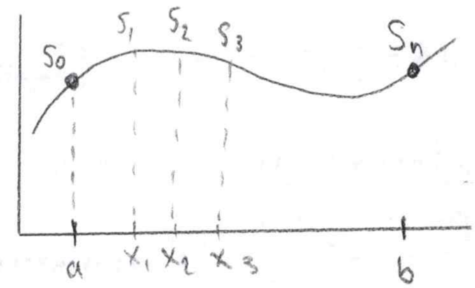
$$\boxed{\int uv'\,dx = uv - \int u'v\,dx}$$

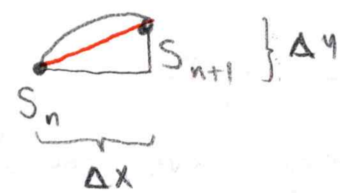
Integration by parts (Definite)

$$\boxed{\int_a^b uv' = uv\Big|_a^b - \int_a^b u'v\,dx}$$

Lecture #31: Parametric Equations

Arc Length





$$(\Delta S)^2 \approx (\Delta x)^2 + (\Delta y)^2$$

$$\boxed{(ds)^2 = (dx)^2 + (dy)^2}$$

Taking Square root

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

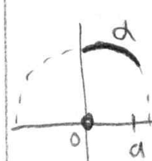
FORMULA

ARCLENGTH (Total $S_n - S_0$)

$$= \int_a^b \sqrt{1+(dy/dx)^2}\,dx$$

ex|

$y = \sqrt{1-x^2}$



$y' = \frac{-x}{\sqrt{1-x^2}}$

$ds = \sqrt{1+(y')^2}\,dx$

$ds = \frac{1}{1-x^2}\,dx$

Derivation of meaning of Radians!

$\alpha = \sin^{-1}\Big|_0^a$

$\boxed{\sin^{-1}(a)}$

Surface Area

ex| $y = x^2$, rotated about x-axis

$dA = (2\pi y)(ds)$ (carry out integration)

Parametric Curves

$x = x(t)$
$y = y(t)$

"t" → parameter

Lecture # 32: Polar Co-ordinates

Parametric Curves (cont.)

$x = a\cos(\theta)$, $x^2 + y^2 = a^2$

$y = a\sin(\theta)$

transformation of ds/dt

$$ds^2 = dx^2 + dy^2$$

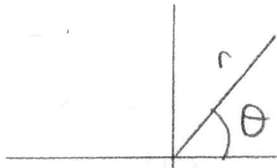
$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

→ substitute in Polar co-ordinate system

$$\boxed{ds = a\, dt}$$

* The key is you change the variable you integrate over (can pick, in this case t = radians)

Polar Coordinates



r = Distance to origin

θ = angle of rayform w/ horizontal axis

$r = \sqrt{x^2+y^2}$ $\theta = \tan^{-1}(y/x)$
$x = r\cos(\theta)$ $y = r\sin(\theta)$

Lecture # 35: Indeterminate Forms

L'hopital's Rule

→ convenient way to calculate limits

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(x)/(x-a)}{g(x)/f(x-a)}$$

$$\boxed{= \lim_{x\to a} \frac{f'(a)}{g'(a)}}$$

Indeterminate Forms: $\frac{0}{0}$, $\frac{\pm\infty}{\infty}$

ex| $\lim_{x\to 0^+} x\ln(x) = \frac{\ln(x)}{(1/x)} = \frac{(1/x)}{-(1/x^2)} = \boxed{0}$

KEY POINT: L'Hôpital's rule can show which function grows faster / slower

Lecture #36: Improper Integrals

- $\left\{\begin{array}{l} f(x) \to \infty \\ g(x) \to \infty \\ f'(x)/g'(x) \to L \end{array}\right\}$ as $x \to a$

Then $\frac{f(x)}{g(x)} \to L$ as $x \to a$

Rates of Growth

$f(x) \ll g(x)$ means $\frac{f(x)}{g(x)} \to 0$
$x \to \infty$

$\ln(x) \ll x^p \ll e^x \ll e^{x^2}$
(slow) (very fast)

Improper Integrals

DEF'N $\left\{ \int_a^\infty f(x)\,dx = \lim_{N\to\infty} \int_a^N f(x)\,dx \right.$

converges if limit exists
diverges if not

ex| $\int_{-\infty}^{\infty} e^{-x^2} = \boxed{\sqrt{\pi}}$ (important in probability)

Limit Comparison

- IF $f(x) \sim g(x)$ as $x \to \infty$

THEN $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$

either both converge or both diverge

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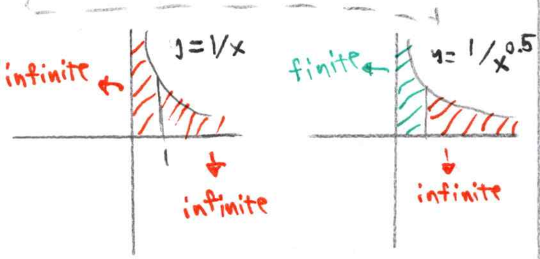
Lecture #37: Infinite Series

of the 2^{nd} Kind

$$\int_0^1 f(x)\,dx = \lim_{a \to 0^+} \int_a^1 f(x)\,dx$$



Infinite series

- $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 2$

Geometric Series

$$1 + a + a^2 + a^3 + \dots = \boxed{\frac{1}{1-a}}$$

Notation

$S_N = \sum_{n=0}^{N} a_n$ (Partial sum)

$S = \sum_{n=0}^{\infty} a_n = \lim_{n \to \infty} S_n$ (converges) (diverges)

Method of Integral comparison

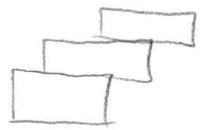
- If $f(x)$ is decreasing & positive

Then $\left| \sum_{n=1}^{\infty} f(n) - \int_1^{\infty} f(x)\,dx \right| < f(1)$

→ also they converge/diverge together

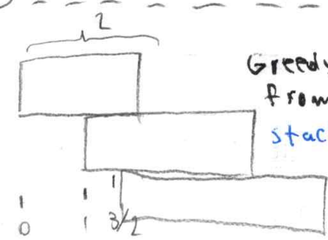
Lecture #38: Taylor Series

Generalized example w/ Block



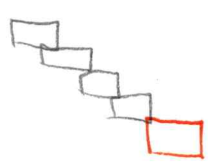
Does this fall down?

(1) Start w/ Top Block



Greedy Algorithm from top: stack at center of mass

Generalized



N Blocks

$(N+1)^{st}$ Block

↓

$\boxed{\text{center at } C_n + 1}$

$$C_{N+1} = \frac{N C_n + 1(C_n + 1)}{N+1}$$

$$\boxed{C_{N+1} = C_n + \frac{1}{N+1}}$$

- From Integral comparison, Distance spanned is Divergent

Power Series

- $\boxed{S}$ required to exist

General Power Series

$$a_0 + a_1 x + a_2 x^2 = \sum_{n=0}^{\infty} a_n x^n$$

$|x| < R$ → converges

KEY IDEA

★ Series are flexible enough to represent any function examined so far ★

- Rules for converging power series same as for polynomials

$\boxed{2}$ interesting properties

1) Differentiation

2) Integration

Taylor's Formula

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$