15.4371 Cheat Sheet

What is a Derivative

→ security whose payoffs are tied by contract to an underlying variable

- ∘ Price of Asset
- ∘ Property of Asset

Types & Features of Derivative

[1] Objective & Observable

[2] outcomes have economic importance

Types:

- ∘ forwards
- ∘ futures
- ∘ swaps
- ∘ options
- ∘ exotic

Forward Contract

∘ An agreement between * two parties to trade a pre-specified amount of goods/securities at future date $\boxed{T}$, for price $\boxed{F}$

Attributes:

- Does not cost anything to enter forward contract
- Price $\boxed{F}$ set to make contract value = 0

→ Party who buys has, Long Position

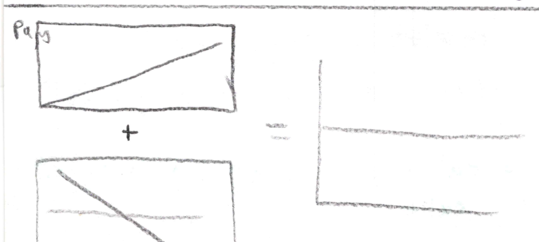
→ Party who sells has, Short Position

P/L on Forward Contract

$$P/L_{Long} = N*(S_T - F)$$

$$P/L_{Short} = N*(F - S_T)$$

Graph expression of Cash Hedging



Futures Contract

→ type of forward contract

Features

- ∘ Traded on exchange
- ∘ contracts standardized
- * Profits/Losses marked-to-Market (MTM) Daily
  - ↳ daily settlement

Futures contracts specifications

- ∘ Asset
- ∘ Delivery time
- ∘ Contract size
- ∘ Price Quote
- ∘ Delivery/cash settlement
- ∘ Price/Position Limits

Cross Hedging, Basis Risk, Hedge Ratio

Basis = difference between spot & forward price of security

* No Arbitrage → Basis goes to zero

cross-hedging = engaging in contract to hedge asset not what you want to hedge

Hedge Ratio

⇒ Relative Number of Futures contracts to units of security to Maximize effectiveness

Goal:

$$\frac{N_s}{N_F} = \frac{E[dF]}{E[dS]}$$

No Arbitrage

∘ An Arbitrage opportunity

1) yields positive profit today
2) costs nothing today, & yields positive cash flows in future

No Arbitrage ≠ No Market Frictions

Law of One Price

* Securities w/ identical payoffs must have same price

Cash'n'Carry for non-Dividend stock

t=0 → t=T

short-stock forwards → Receive $\boxed{F}$, Deliver share

Buy stock, Borrow $P_{s,0}$ → use Delivery to repay interest

$$P/L = F - P_{s,0}e^{rT}$$

Reverse Cash'n'Carry

t=0 → t=T

long stock forward → receive share

short stock for $P_{s,0}$, lend out $P_{s,0}$ → use to cover short

$$P/L = P_{s,0}(e^{rT}) - F$$

Forward Price w/ No Arbitrage

No-Dividend

$$F = P_{s,0}e^{rt}$$

w/ Dividend payed $(0 < t < T)$

$$F = [P_{s,0} - De^{-rt}]e^{rT}$$

Valuing Forward contract over time

→ initial forward price $\boxed{f_0} = 0$

value of long forward (time = t)

$$(F_t - K)e^{-r(T-t)}$$

value of short forward (time = t)

$$(K - F_t)e^{-r(T-t)}$$

Bond Basics

◦ Price = PV[promised payments]

Note: Market Rate $\boxed{r_T}$ stated on continuous basis

$$P = e^{-r_T T} Z$$

Cash 'n' carry on zero-coupon Bonds

• Forward contract on zero coupon bond w/ maturity $\boxed{T_2}$, delivered $\boxed{T_1}$, Face value $\boxed{Z}$

t=0 — t=T_1

short forward

Receive F
Deliver zero Bond

Buy Bond

Borrow $P_{B,0}$

$$P/L = F - P_{B,0}\, e^{r_T T_1}$$

No-Arbitrage Pricing: zero coupon Bond

current bond price: $P_{B,0} = e^{-r_{T_2} T_2} Z$

No Arbitrage: $F = e^{r_{T_1} T_1} P_{B,0}$

$$\Rightarrow F = Z e^{(r_{T_1} T_1 - r_{T_2} T_2)}$$

Note: existence of forward price, implies forward rate

which solves

$$F = Z e^{-f(0,T_1,T_2)(T_2 - T_1)}$$

Forward Rate: $f(0, T_1, T_2) \rightarrow$ return locked in once bond delivered at T_1

Cash & Carry: Currencies

t=0 — t=T

• Short $\boxed{X}$ euros

. Borrow \$ $X e^{-r_E T} S_0$

• convert

• invest @ $r_\$$

$$P/L = FX - (\$X e^{-r_E T}) e^{r_\$ T}$$

No Arbitrage: $F = S_0 e^{(r_\$ - r_E)T}$

Swap Basics

• A swap is a contract calling for exchange of payments, on one or more future dates.

Interest Rate Swaps

→ contract to periodically exchange a floating interest payment for a fixed interest rate payment

Specifications:

1) Notional Principal
2) Basis for Float rate
3) fixed rate
4) Payment Frequency
5) Maturity

uses: hedge against interest rate changes

Key: opportunity arises when they can offer one another better rates

Currency Swaps

◦ Agree to exchange payment for one currency for another.

Example:

− US to receive 5 mil euros in 5 equal installments, every 6 mo for 2.5 yrs

Facts

$S_0 = 1.2673$

$r_\$ = 5\%$

$r_E = 3\%$

Maturity	Fwd Rate
0.5	1.28
1	1.2929
1.5	1.3059
2	1.3190
2.5	1.3323

Q: How to establish no arbitrage rate $\boxed{K}$

$$K = w_{.5}F_{.5} + \dots + w_{2.5}F_{2.5}$$

value of swap after initiation

$$V_t^{swap} = \sum_{i=1}^{n} e^{-r_\$(T_i - t)}\left(K - F_{t,T_i}\right)$$

Commodity Swap

→ an agreement to exchange pre-specified fixed payment for payment tied to price of market commodity

ex

$F_{1\text{-}yr} = \$110$, $F_{2\text{-}yr} = \$111$

Fixed Annual payment = \$110.483

How to find fair Fixed Payment

step 1) Find PDV of Payment of Futures Prices

step 2) solve for fixed payment $\boxed{X}$ that solves for same PDV

• one can consider a swap a forward contract (w/ borrowing/lending)

→ One party is said to receive a loan early on from another party (ΔP)

+ calculate implied interest rate

Options

- European Option → right (not obligation) to Buy/sell security commodity at strike price at time $\boxed{T}$

- using option → "exercising"
- selling option → "writing"

- Buyer premium, usually charged to write option

- American Option → same except can be exercised at any time

Call Option Payoff

- If at maturity $\boxed{S_T < K}$

→ holder walks away, payoff = 0

- if at maturity $\boxed{S_T > K}$

→ holder earns $S_T - K$

$$\boxed{\text{Payoff of call} = \max(0, S_T - K)}$$

Put Option Payoff

same logic

$$\boxed{\text{Payoff of Put} = \max(0, K - S_T)}$$

Payoff versus Profit

$$\boxed{\text{Profit} = \text{Payoff} \pm \text{option premium}}$$

Put - Call Parity

→ "Long Put" + "short call" equivalent to short forward contract w/ Delivery $\boxed{K}$

so

$$\boxed{\text{Put} - \text{Call} = e^{-rT}(K - F_{0,T})}$$

recall that for Forward Price

$$\boxed{F_{0,T} = S_0 e^{rT}}$$ if no dividends

Takeaways

* ONLY applies to European options

* Given Price of call, can price Put (and vice versa)

Concept: Hedging w/ options

- Firm A is long in stock, worried about price Decline

(1) Enter short forwards w Delivery K

- Locked in payoff
- No cost to enter

(2) enter Put option w/ strike price K

- only upside on payoff
- costs option premium to enter

→ which is better depends on probability curve

Option Premiums

⇒ How do we compute?

- Can be valued using no-arbitrage

⇒ 3 Methods to determine Premiums

- Binomial trees
- Black Scholes

Options Premiums: No-arbitrage Bounds

[1.] Call never worth more than stock

$$\boxed{\text{Call} \leq S_t}$$

[2] Put option worth no more than strike Price

$$\boxed{\text{Put} \leq K}$$

[3] Options never have negative value

[4] Value of put (call) always greater than value of long (short) forward contract (Delivery Price = strike price)

Option Strategies

* Review + understand all structures

→ know their usefulness

Binomial Trees

→ useful in Risk-Neutral Pricing

- adaptable to complicated scenarios

advantage: Avoids need to explicitly identify cost of capital

Six Factors Needed:

1. current stock price $\boxed{S_0}$
2. Strike Price $\boxed{K}$
3. Time to Expiration $\boxed{T}$
4. Volatility of Price $\boxed{\sigma}$
5. Risk-free rate $\boxed{r}$
6. Expected Dividends $\boxed{E[D]}$

One-Step Binomial Trees

- Today at $t = 0$

$$\left\{\begin{matrix} S_{1,u} = 70, & p = 0.7 \\ S_{1,d} = 35, & p = 0.3 \end{matrix}\right\}$$

- If market expects $\boxed{r = 19\%}$

$$S_0 = \frac{E[S_1]}{(1+R)} = \boxed{50}$$

Expected Return $\left\{ E\left(\frac{S_1}{S_0}\right) \right.$

Variance $\left\{ E\left\{\left[\frac{S_1}{S_0} - E\left(\frac{S_1}{S_0}\right)\right]^2\right\} \right.$

Option Prices on Binomial Tree

step 1: Figure out cash flows from call at each node

$C_{1,u} = 20$	$C_{1,d} = 0$

Problem: know probabilities, but not cost of capital

"Replicating Portfolio"

① Long Position in Stock

→ $\boxed{\Delta}$ units

② short position in bonds

→ $\boxed{B_0}$

Solve two eqs of two vars to replicate up/down payoffs

No-Arbitrage condition:

$$C_0 = V_0 = (\Delta \times S_0) + B_0$$

where does portfolio come from

→ arbitrageur buys call option & shorts stock

Portfolio cost $\{ \Pi_0 = C_0 - \Delta \cdot S_0$

Delta-Hedging → select Δ s.t portfolio value is constant

$$\boxed{C_{1,u} - \Delta S_{1,u} = C_{1,d} - \Delta S_{1,d}}$$

implies

$$\Delta = \frac{C_{1,u} - C_{1,d}}{S_{1,u} - S_{1,d}}$$

Δ = sensitivity of call price to change in stock price

(continued)

$\boxed{S0}$

$$\Pi_0 = e^{-rT} \Pi_{1,u}$$

$$\Pi_0 = C_0 - \Delta S_0 \left(\text{by definition} \right)$$

$$\Downarrow$$

$$C_0 = \Delta S_0 + \Pi_0$$

$$\boxed{B_0 = e^{-rT}(C_{1,u} - \Delta S_{1,u})}$$

Replicating Portfolio (summary)

① Define Delta

$$\Delta = \frac{V_{1,u} - V_{1,d}}{S_{1,u} - S_{1,d}}$$

② Compute Bond Amount

$$B_0 = e^{-rT}(V_{1,u} - \Delta S_{1,u})$$

③ Compute security value

$$V_0 = \Delta S_0 + B_0$$

where do Probabilities go?

- Option prices dependent on $\boxed{S_0}$, so probabilities implicitly priced in

Risk-Neutral Pricing

⇒ Given $S_0, S_{1,u}, S_{1,d}$, we can choose fake probability q^*

→ choose $\boxed{q^*}$ s.t asset earns risk-~~free~~ free rate

condition

$$q^* \cdot S_{1,u} \cdot e^{-rT} + (1 - q^*) \cdot S_{1,d} e^{-rT}$$

$$\parallel$$

$$S_0$$

$$q^* = \frac{S_0 e^{+rT} - S_{1,d}}{S_{1,u} - S_{1,d}}$$

Implications:

Stock Price $\{ \; S_0 = E^*[e^{-rT} S_T]$

Price of Derivative Security $\{ \; \text{Price} = E^*\left[e^{-rT}\left(\text{Derivative Payoff}\right)\right]$

Risk-Neutral Pricing: A recipe

Price of Derivative Security $\{ \; = E^*\left[e^{-rT} * \left(\text{Payoff at } T\right)\right]$

Risk-Neutral Pricing & discount Rates

→ a convenient pricing device, but does not imply market participants are risk-neutral

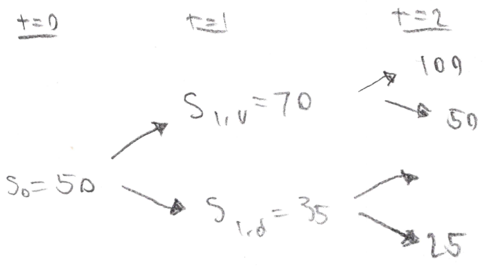
⇒ they are Risk-Averse

2 ways to account for risk-aversion

① Add risk premium to cost of capital

② Distort Probabilities toward Bad states

2-step Binomial Trees



→ want to price option w/ maturity $\boxed{T=2}$ & $\boxed{K = 50}$

Method 1|

∘ Solve backwards each 1-step branch

Method 2|

∘ Risk-neutral (a lot easier)

Risk-Neutral Pricing (Multi-step)

• Denote by (i,j) node

→ (time, node)

• $i \to 0,1,2 \ \{ \ j \to \{u, uu, ud\}$

$$\boxed{V_{i,j} = e^{-rxh} \cdot E^*[V_{i+1}]}$$

Key Point (knowing probs):

$$Pr^*(S_{2,uu}) = q_0^* \cdot q_{1,u}^*$$

$$Pr^*(S_{2,ud}) = q_0^*(1-q_{1,u}^*) + (1-q_0^*)\cdot q_{1,d}^*$$

⋮

Multi-Step: Pr(Distribution, S_T)

∘ What is the probability distribution of price at maturity $\boxed{S_T}$

→ consider n-step tree

∘ contains $\boxed{n+1}$ nodes at Maturity

$$Pr\left(\begin{matrix}\text{node}=j \\ @\ T=n\times h\end{matrix}\right) = \left(\frac{n!}{j!(n-j)!}\cdot (q^*)^{n-j}\cdot (1-q)^{j}\right)$$

Black-Scholes Model

Assumptions:

1. Markets are frictionless
2. Borrowing = Lending Rates
3. No dividends
4. stock prices conform to log-normal model

Replicating Dynamic trading strategy

* we can show rebalancing pay-off of stock & risk-free security, you get pay off of option

→ stock position always positive

$$\boxed{\Delta_c = \frac{\partial C}{\partial S}}$$ (fraction of stock in Replicating Portfolio)

Black-Scholes Merton formula

$\boxed{S}$ = current stock price

$\boxed{K}$ = Exercise Price

$\boxed{T-t}$ = time to expiration

$\boxed{r}$ = risk free rate

$\boxed{\sigma}$ = annualized std. dev of returns

BSM Formula

$$C(S,K,T-t,r,\sigma) = S\cdot N(d_1) - Ke^{-r(T-t)}N(d_2)$$

→ N = CDF of standard Normal

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r+\frac{1}{2}\sigma^2\right)T-t}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

For Put Option:

$$\boxed{P(S,K,T-t,r,\sigma) = Ke^{-r(T-t)}N(-d_2) - S\,N(-d_1)}$$

BSM Formula interpretation

• $N(d_1)$ is fraction of share held in replicating portfolio today

• $Ke^{-r(T-t)}N(d_2)$ amount of initial borrowing in replicating portfolio

Risk in options & the greeks

• Options have sensitivity to various factors

→ these factors known as "the greeks"

$\boxed{1.}$ Delta

$$\Delta = \frac{d\ \text{Option Price}}{d\ S} = \begin{cases} N(d_1), & \text{calls} \\ -N(-d_1), & \text{put}\end{cases}$$

$\boxed{2.}$ Gamma

$$\Gamma = \frac{d\Delta}{d\ S} = \frac{N'(d_1)}{S\sigma\sqrt{T}}$$

→ curvature of option price w/ respect to price $\boxed{S}$

3 Theta

→ sensitivity of option to passage of time

$$\boxed{\Theta = \frac{d\ \text{Option Price}}{dt}}$$

Supplemental Info from Practice Problems

Put-Call Parity w/ Div yield

$$\text{Put} - \text{Call} = e^{-rT}\left(K - S e^{(r-\delta)T}\right)$$

$$= -S_0 e^{-\delta T} + K e^{-rT}$$

Upper/Low option Price Bounds w/ Div yield

Put-Call Parity (American)

*its an Inequality

$$\boxed{S_0 - Ke^{-rT} \geq C - P \geq S_0 - K}$$

3 Guiding Eqns of Multi-Step trees

① $u = e^{\sigma} \times \sqrt{h}$

② $d = 1/u$

$\boxed{\uparrow}$ goes on e^{σ}, not u

③ ** $\boxed{q_r = \dfrac{e^{r \times h} - d}{u - d}}$

Multi-Step tree: Derivative Pricing

* Apply $\boxed{q_r}$ regressively backwards along nodes

The Greeks

1) Delta (Δ)

- sensitivity of option to changes in asset price

$$\Delta = \frac{d\ \text{Option}}{d\ S}$$

2) Gamma (Γ)

- sensitivity of Delta to Price

3) Theta (Θ)

- sensitivity to passage of time

$$\Theta = \frac{d\ \text{Option Price}}{d\ t}$$

4) Rho (ρ)

- change in option with r/

High/Low Theory Option Bounds

Call

Low	High
$\max(0, S_t - Ke^{-r(T-t)})$	S_t

Put

Low	High
$\max(0, Ke^{-r(T-t)} - S_t)$	$e^{-r(T-t)} K$

15.4371 Cheat Sheet (Part 2)

The Greeks

① Delta

→ sensitivity of option price to changes in underlying asset

$$\Delta = \frac{d\ \text{Option Price}}{dS} = \begin{cases} N(d_1) & \text{calls} \\ -N(d_1) & \text{puts} \end{cases}$$

② Gamma

→ Curvature of price w/ respect to $\boxed{S}$

$$\Gamma = \frac{d\Delta}{dS}$$

③ Theta

→ sensitivity of option to passage of time

$$\Theta = \frac{d\ \text{Option Price}}{dt}$$

④ Rho

→ Change in option price due to change in interest rate $\boxed{r}$

$$\rho = \frac{d\ \text{Option Price}}{dr}$$

$$= \begin{cases} KTe^{-rT}N(d_2), & \text{calls} \\ -KTe^{-rT}N(-d_2), & \text{puts} \end{cases}$$

⑤ Vega

→ Change in option price due to volatility

$$\nu = \frac{d\ \text{OptionPrice}}{d\sigma}$$

Notes on Greeks

• Theta

$$\text{Put} \begin{cases} \Theta > 0, & \text{for low } S \\ \Theta < 0, & \text{for high } S \end{cases}$$

→ for high S, payoff is zero, but price positive (must decrease, to reflect less time-space of value being realized)

* also arbitrage balancing

• Rho

→ depends whether holder will pay K (call) or receive K. value of K ↓ ar r ↑

∘ Payment smaller for long put or call

Using options for Financial engineering

• investors like principal protection, but enjoying in upside

=> capital-protected note

Portfolio Insurance

→ advised clients on how to dynamically allocate assets to insure a drop in value

Problems w/ Delta Hedging

① Need to rebalance frequently incurring transaction cost

② Hedge breaks down in large price swings

Delta-Gamma Hedging

→ consider portfolio $\boxed{i}$ which is short in $\boxed{T}$ dated calls, long N stock, long in N^c, T_1 dated calls

$$i = -\text{Call}(S,T) + (N * S) + N^c \cdot \text{Call}(S,T_1)$$

what we want:

$$\frac{di}{dS} = \frac{d^2 i}{dS^2} = 0$$

This requires the following:

Delta Hedging

$$\frac{di}{dS} = 0 = -\frac{d\ \text{Call}(S,T)}{dS} + N + N^c\frac{d\ \text{Call}(S,T_1)}{dS}$$

Gamma Hedging

$$\frac{d^2 i}{dS^2} = 0 = -\frac{d^2\ \text{Call}(S,T)}{dS^2} + N^c\left(\frac{d^2\ \text{Call}(S,T_1)}{dS^2}\right)$$

Result

$$N^c = \Gamma(S,T)/\Gamma(S,T_1)$$

$$N = \Delta(S,T) - N^c\Delta(S,T)$$

Black-Sholes Shortcomings

Terminology:

"Moneyness" is ratio of strike price to current stock price $\boxed{K/S}$

∘ For low K/S, BS underprices both

∘ For high K/S, BS overvalues both

Why?

→ interest and dividends observable, ~~hard~~ not variable

★ Volatility is a major culprit

other major Plot-Holes:

(1) stock-prices are not always Log-Normal

(2) Volatility is stochastic

(3) Non-continuous Pricing

BSM Implied Volatility

→ Level of volatility σ_{imp} that when inserted into Black-Sholes delivers true market value of option

• every option has potentially different implied volatility

Models that are new

(1) Deterministic & stochastic volatility models

(2) Models w/ Price jumps

(3) Implied tree models

Deterministic Volatility Models

• assume $\boxed{\sigma}$ depends on $\boxed{S}$

constant elasticity of variance model

$$\boxed{\sigma(S) = \Sigma \cdot S^{\alpha}}$$

So:

$$\boxed{\log\left(\frac{S_{t,t+h}}{S_t}\right) = \mu \cdot h + \underbrace{\left(S_t^{\alpha} \cdot \Sigma\right)}_{\text{price factor}}}$$

• assume $\boxed{\sigma_t}$ moves over time

→ Heston Model $\left\{ \; \boxed{\sigma_t = \sqrt{V_t}} \right.$

$$\boxed{R_{t,t+h} = (\mu \cdot h) + \sqrt{V_t} \cdot \epsilon_t}$$

$$\boxed{(V_{t+h} - V_t) = k \cdot (\bar{V} - V_t) \cdot h + \Sigma \cdot \sqrt{V_t}}$$

Implied Tree: Example

→ Previously we saw how to find option prices given a tree

★ w/ implied trees we begin w/ observed prices, and **calibrate tree to be consistent with them**

Key: we choose $\boxed{S_{1,u}}$ to price option correctly

set $\boxed{u = S_{1,u}/S_0}\ \boxed{S_{1,d} = S_0/u}$

Option Betas & Expected Returns

• What is $\boxed{\beta}$ sensitivity of option to % change in $\boxed{S}$

Recall

$$\boxed{Call = \Delta \cdot S + B}$$

$$\boxed{R_{call} = w \cdot R_S + (1-w) \cdot R_B}$$

$$w = \frac{\text{Position in stock}}{\text{Value of Port}} > 1$$

So

$$\boxed{R_C - R_B = \beta \cdot (R_S - R_B)}$$

Exotic Options

• Non-Standard, formulated to tweak positions

(1) why exotic options?

(2) Can it be replicated by portfolio of regular options?

(3) Is it relatively cheaper or expensive

Non-Standard American

→ between European & American

• strike price can change over life of option

Binary Options

(1) Cash-or-nothing

Call: pays 1 if $S_T > K$, 0 else

Put: pays 1 if $S_T < K$, 0 else

↓

see Notes for valuations

Asian Options

• Payoff based on **Average-Price** over some time period

→ path dependency

why useful?

→ price at single point subject to manipulation

→ price swings substantial

Two types: Arithmetic or Geometric mean-based

American Options

→ Not only *if* to exercise but **when**

No-Arbitrage Bounds

1. American at least as valuable as European

$$\boxed{C^A(S,K,t,T) \geq CE(S,K,t,T)}$$

2. Americans w/ longer time to maturity, at least as valuable as same option with shorter time to maturity. $\boxed{T_2 > T_1}$

Americans: Early Exercise

→ On Non-Dividend paying stock, American option always worth more alive than dead

⇒ what if Dividends are paid though?

If we exercise call early

(Good) Get Dividends paid between now & maturity

(Bad) we pay K today instead of future, Lose interest on $\boxed{K}$

(Bad) Pay K today, even though could be worth less at maturity

→ In any case, waiting reflects the negation of risk of not realizing right price

Americans: Dividend Rules

$D^+ = \max PV[\text{Dividends}]$

$D^- = \min PV[\text{Dividends}]$

calls

- Never exercise if

$$D^+ < K(1-e^{-rT})$$

Puts

- Never exercise if

$$D^- > K(1-e^{-rT})$$

IMPORTANT

- Only optimal to exercise American at Maturity, or before dividend

Americans: Binomial Tree Framework

- Exercise in current Period only if

$$\boxed{K - S > \text{Discounted value of future distributions of payoff}}$$

1) Start w/ Binomial Tree of European

2) ** at each node evaluate whether

$$S_i - K > e^{-rh}\left[E[P_{i+1}|S_{i0}]\right]$$

or

$K - S$

→ Replace payoff at each node w/ which is higher

* work backwards to find price at $\boxed{t=0}$, from immediate-exercise validated payoffs

Monte Carlo: Risk-Neutral Trees

- Likewise can simulate up or down movements on computer

⇒ use Rand() to sample [0, 1]

- if Rand() > q^*, DOWN STEP
- if Rand() < q^*, UP STEP

$$\boxed{\hat{V}_0 = \frac{1}{N}\sum_{i=1}^{N} e^{-rT} V(S_i)}$$

Why Monte Carlo?

* Useful for Path dependent options

Interest Rate Options

Types

1. Treasury Bond Options
2. Swaptions
   → option to enter a swap
3. Caps & Floors
   → securities that pay when rates go up (caps)
   Rates go Down (Floor)

See LN for Cap & Caplet valuation Procedure

Barrier Options

→ Payoff depends on whether over it's life the underlying price hits certain barrier

1. Knock Out → go out of existence if price exceeds/falls below threshold

2. Knock-in options → comes into existence

3. Rebate Options → fixed payment if underlying price falls or rises above price

Look back Options

1) Floating Look back Call

→ allows buyer to buy at lowest observed price

$S_T - S_{min}$

2) Floating Look back Put

→ allows buyer to sell at highest observed price

$S_{max} - S_T$

3) Fixed Look back Call

$\max(S_{max} - K, 0)$

4) Fixed Look back Put

$\max(K - S_{min}, 0)$

Exchange Options

→ Pass off only if the underlying asset outperforms some other asset

Payoff: $\boxed{\max(0, S_T - N_T)}$

• Valu formula & parameters in "exotic options Notes"

Compound Options

→ Option to buy or sell an option

Δ Priced by Backwards induction on binomial Tree

Gap Options

• Pays $\boxed{S - k_1}$ when $\boxed{S > k_2}$

→ valuation in "exotic options notes"

BSM: Stock w/ Dividend yield

→ Dividend stream depresses price growth

→ relies on observation that price distribution same at $\boxed{T}$

∘ Stock starts at S_0 & pays a

∘ stock starts at $S_0 e^{-aT}$, no dividend

$$c = S_0 e^{-aT} N(d_1) - Ke^{-rT} N(d_2)$$

$$d_1 = \frac{\ln(S_0/K) + (r - a + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

* often used to value of options on stocks

Put-call Parity

$$\boxed{c + Ke^{-rT} = p + S_0 e^{-aT}}$$

BSM: Euro Currency options

• replace $\boxed{r_f}$ w/ $\boxed{a}$ & you get BSM Equations

Futures Options

• Futures-Call

Payoff = $\max(F - K, 0)$

• Futures Put

Payoff = $\max(K - F, 0)$

Put-call Parity (Futures options)

$$c + Ke^{-rT} = p + F_0 e^{-rT}$$

→ referred to by delivery month of underlying futures contract (rather than by expiration)

$$c = e^{-rT}[F_0 N(d_1) - K N(d_2)]$$

$$p = e^{-rT}[K N(-d_2) - F_0 N(-d_1)]$$

$$d_1 = \frac{\ln(F_0/K) + (\sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Price interest & Bond options on Binomial Trees

- Interest Rates have well-characterized statistical Distribution

=> Bonds & interest rate derivatives can be valued using pricing model of evolution of interest rates

The Ho-Lee Model

$$\boxed{r_{i+1} = r_i + \theta(i)\Delta \pm \sigma\sqrt{\Delta}}$$

↳ Δ is time step

↳ $\theta(i)$ function of time, to match current prices

↳ $\pm\sqrt{\Delta}$ occurs w/ 1/2 probability

$$r_{1,u} = r_0 + \theta(0)\Delta + \sigma\sqrt{\Delta}$$
$$r_{1,d} = r_0 + \theta(0)\Delta - \sigma\sqrt{\Delta}$$
$$r_{2,uu} = r_{1,u} + \theta(1)\Delta + \sigma\sqrt{\Delta}$$
$$\vdots$$

so that $r_{2u,d} = r_{2,du} = r_0 + (\theta(0)+\theta(1))\Delta$

Calibrating Ho-Lee Model

Inputs

(a) Time series of short-term interest rates to find $\boxed{\hat{\sigma}}$

(b) compute $\theta(i)$ recursively

→ use bond maturing $i=2$ $\}$ $\theta(0)$

→ use bond maturing $i=3$ $\}$ $\theta(1)$

Ho-Lee Procedure

1) Use time-series to calculate $\boxed{\sigma}$
2) Backwards impute & solve for $\theta(i)$
3) use $\theta(i)$ to calculate the $\boxed{r}$ at different time periods to Maturity
4) Backwards calculate Value of Bond-related Derivatives

Treatment of Intermediate cash flow

$$P_{i,j} = e^{-r_{i,j}\Delta}\left(\frac{1}{2}P_1 + \frac{1}{2}P_2 + CF_{i+1}\right)$$

→ see notes for coupon bond treatment

Ho-Lee Pricing: Callable Bonds

- at any node, holder can choose to exercise or wait so

$$Call^{wait}_{i,j} = e^{-r_{i,j}\Delta} E^*[Call_{i+1}]$$

$$Call^{exercise}_{i,j} = \boxed{P^*_{i,j} - 100}$$

=> so at each node, it will be Max[ ] of these two choices,

Ex: Pricing Callable Bond

- Buyer is Buying a Bond

→ selling call to issuer

$$P_0^{call}(3) = P_0^{No\,Call}(3) - C_0(3)$$

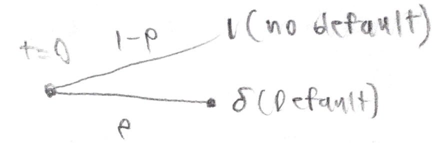
Credit Derivatives

- structural models derive default & recovery rates modelling structure of assets & liabilities

exemple

- zero-coupon bond w/ maturity T=1

=> if default Payoff = $\delta \times 1$



→ how do we price?

Merton Model

- Today is t=0, w/ $V_0 = E_0 + D_0$
- Market value is log-normal

Two possibilities

1) $V_T > F$, firm sells assets to pay debt holders. Equity gets the rest

2) $V_T < F$, Debt holders get V_T, company defaults

Key: Payoff to equity holder is call option

$$Payoff_{Equity} = (V_T - F, 0)$$

- can value equity component using Black Sholes

$$\boxed{E_0 = call(V_0, F, r, T, \sigma)}$$

* see credit derivs packet for other formulas

Merton: Volatility of Levered Equity

$$\sigma_e = \left(\frac{V\,N(d_1)}{V\,N(d_1) - Ke^{-rT} N(d_2)} \right) \cdot \sigma$$

→ as value of equity relative to assets falls, Volatility increases

Merton: Value of Debt

$$\text{Payoff}_{\text{Debt}} = \min(V_T, F)$$
$$= V_T - \max(V_T - F, 0)$$

Also

$$D_0 = V_0 - E_0 = V_0 - \text{Call}(V_0, F, r, T, \sigma)$$

$$\boxed{D_0 = Fe^{-rT} - \text{Put}(V_0, F, r, T, \sigma)}$$

(+) risk-free value of debt

(−) Put option representing risk-adjusted expected losses due to default

Merton: computing credit spread

$$D_0 = e^{-yT} \cdot F$$
$$\Downarrow$$
$$e^{-yT} F = Fe^{-rT} - \text{Put}(V_0, F, r, T, \sigma)$$

$$e^{-yT} = e^{-rT} - \text{Put}\left(\tfrac{V_0}{F}, 1, r, T, \sigma\right)$$

$$\text{Credit spread} = y - r = -\frac{1}{T} \ln(*)$$

$$* = \left[1 - e^{rT}\,\text{Put}\left(\tfrac{V_0}{F}, 1, r, T, \sigma\right)\right]$$

Credit Derivatives

Exec Comp, Real Options

Why?

- Most Real decisions involve choices that have a nature similar to that of options

improvements relative to NPV

1) recognize effect of state-contingent future decisions

2) Explicitly recognize optionality implies time-varying cost of risk

Example: Real options

- Project undertaken in two phases

Phase 1: Upfront cost of 125,000

Phase 2: if sucessful, \$1 mil for plant, 250K each period after

- initial probability of sucess $\boxed{50\%}$
- Discount Rate $\boxed{r}$ is 25%

Naïve NPV

$$NPV = -125 + \frac{500}{1.25} - \sum_{2}^{\infty} \frac{125}{1.25^t}$$

BUT

the two phases have very different levels of risk

$$\text{real value of Project} \left\{ -125 + \frac{750}{1+r} \right.$$

→ There is a valuable abandonment option

Example 2: waiting for resolution of uncertainty

- Firm has option to produce one widget per year forever

→ \$800 to build factory

→ $P_0 = 100$, could be 150 or 50

compare NPV of waiting 1-yr vs. acting now

$$NPV_0 = -800 + \sum_{t=0}^{\infty} \frac{100}{(1.1)^t} = 300$$

$$NPV_1 = \boxed{\$386}$$

option value of waiting $\boxed{\$86}$

Think about it this way

- 1-yr call opti