* The Order of a differential equation is the highest n such that the n^{th} derivative is present

Types of ODE's

1. A homogenous, linear ODE

- each summand is a multiple of (t) * ($y, \dot{y}, \ddot{y}$, respectively)

ex. $e^t \ddot{y} + 5\dot{y} + t^9 y = 0$

2. An inhomogenous, linear ODE

- exactly same except one term is a function of (t) only

ex. $e^t \ddot{y} + 5\dot{y} + t^9 y = 7\sin(t)$

* Any linear ODE will fit one of the 2 forms above

Standard Liner Form - Co-efficient of highest derivative is set to be one by dividing out coefficient preceding highest deriv

Non-Linear ODE

* For An ODE to be Non-Linear $y, \dot{y}, \ddot{y}$ must enter in more complicated fashion

1. Be raised to a power ($y^2, \dot{y}^3$)
2. Multiplied with eachother
3. Non-linear functions applied to them

ex. Modeling Problem

BASIC GROWTH EQN: $\dot{y} = ay$

BASIC DECAY EQN: $\dot{y} = -ay$

Solving 1st Order ODE

Seperation of Variables

• Certain 1st Order ODE's (linear & Non-linear) can be evaluated

Form for use: $\frac{dy}{dt} = g(t) f(y)$

ex.

$\dot{y} - 2ty = 0$

$\int \frac{\dot{y}}{y} = \int 2t$

$\ln(y) = t^2 + c$

$\boxed{y = Ce^{t^2}}$

ex.

$\dot{y} = (y-1)^2$

$= \int \frac{dy}{(y-1)^2} = \int dt$

$= -\frac{1}{y-1} = t + c$

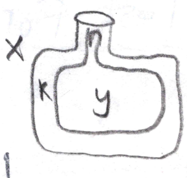
$\boxed{y = 1 - \frac{1}{t+c}}$

NEWTON'S LAW OF COOLING

k = heat conduction

x = external Temp.

y = internal Temp



X

$$\boxed{\dot{y} = k(x - y)} \text{ !!}$$

Variation of Parameters

• Used to Solve inhomogenous linear ODE's

ex. $t\dot{y} + 2y = t^5$

Step 1 (Find y_h)

$\dot{y} + \frac{2}{t} y = 0$

$\frac{\dot{y}}{y} = -\frac{2}{t}$

$\ln(y) = -2\ln(t) + c$

$y_h = Ct^{-2}$

Step 2 ($y = v y_h$) → Plug into original Problem

$t(\dot{v}t^{-2}) + v(-2t^{-3}) + 2vt^{-2} = t^5$

$\dot{v} t^{-1} = t^5$

$\dot{v} = t^6$

$v = \frac{t^7}{7} + C$

$\boxed{y = \frac{t^5}{7} + Ct^{-2}}$

Method by Integrating Factor

MIDTERM I

start with: $\dot{y} + p(t)y = q(t)$

1. Find anti-derivative of $p(t)$ [$P(t)$]

integrating Factor $= e^{P(t)}$

2. Multiply both sides of Eqn by this factor

$\dot{y}e^{P(t)} + p(t)ye^{P(t)} = q(t)e^{P(t)}$

$\frac{d}{dt}\left(ye^{P(t)}\right) = q(t)e^{P(t)}$

$ye^{P(t)} = R(t) + C$

$y = \frac{R(t)}{e^{P(t)}} + Ce^{-P(t)}$

Linear Combination

• A Linear combo of a list of functions is any function that can be built from them by scalar Multiplication & addition/subtraction

* terms don't necessarily have to appear to be involved

ex.

$5 = 0t^2 + 5(1)$ } linear combo of 1 & t^2

Super Position

$$y = y_p + y_h$$

y_h = general homogenous

y_p = particular Inhomogenous

y = General Inhomogenous

ex.

$\dot{x} + 2x = 0$ — $x = 1/2$

" " $= 1$ — $x_p = 1/2$

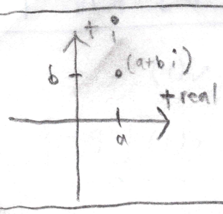
" " $= t$ — $x_p = t/2 - \frac{1}{4}$

" " $= e^{-2t}$ — $x_p = te^{-2t}$

Find General soln of: $\dot{x} + 2x = 5 + 6t - 7e^{-2t}$

Ans: $x = \frac{5}{2} + \left(3t - \frac{3}{2}\right) - 7\left(te^{-2t}\right) + ce^{-2t}$

Complex #'s



$z = a + bi$

Operations on Real #'s

Real Part: $Re(a+bi) = \boxed{a}$

Imaginary Part: $Im(a+bi) = \boxed{b}$

Complex conjugate: $\overline{a+bi} = \boxed{a-bi}$

* also addition/subtraction & scalar multiplication

* whenever you divide by a complex number, you must multiply top & bottom by its conjugate

For complex # $a+bi$

Modulus = length = $\boxed{\sqrt{a^2+b^2}}$

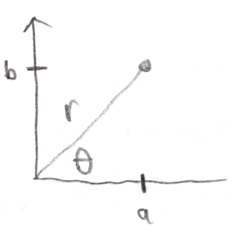
Argument = ANY angle it makes with + x-axis = $\boxed{\tan^{-1}\left(\frac{b}{a}\right)}$

POLAR FORM OF A COMPLEX

$a+bi$ $\quad a = r\cos(\theta)$

$b = r\sin(\theta)$

$$\boxed{a+bi = r(\cos(\theta) + i\sin(\theta))}$$



$a>0$; $\theta = \arctan(b/a)$

$a<0$; $b \geq 0 = \arctan(b/a) + \pi$

$a<0$; $b<0 = \arctan(b/a) - \pi$

Euler's Formula

*** *** ⟹

$$\boxed{e^{i\theta} = \cos(\theta) + i\sin\theta}$$

Rectangular → polar

$a+bi$ $\quad r = \sqrt{a^2+b^2}$

$\theta = \arctan(b/a) + \text{extra}$

$= \boxed{\sqrt{a^2+b^2}\, e^{i\arctan(b/a)}}$

Polar → Rectangular

$e^{i\theta} \to r\cos\theta + i\sin\theta$

visualize in the plane

Test For Equality of 2 Polar

$$r_1 e^{i(\theta_1)} = r_2 e^{i(\theta_2)}$$

ONLY IF $\quad r_1 = r_2$

$\theta_1 = \theta_2 + 2\pi k$

Arithmatic

$r_1 e^{i\theta_1} * r_2 e^{i\theta_2} = \boxed{r_1 r_2 e^{i(\theta_1+\theta_2)}}$

$\frac{1}{re^{i\theta}} = \boxed{\frac{e^{-i\theta}}{r}}$

$\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \boxed{\frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}}$

$(re^{i\theta})^n = \boxed{r^n e^{in\theta}}$

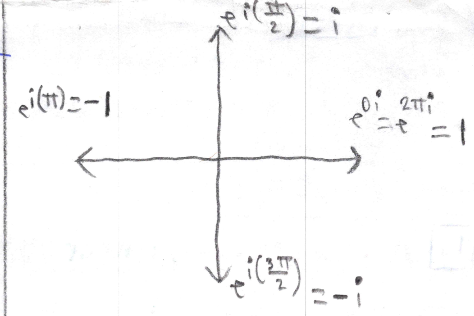
$\overline{re^{i\theta}} = \boxed{re^{-i\theta}}$

Complex Exponential (e^z)

$e^{a+ib} = \boxed{e^a(\cos(b) + i\sin(b))}$

$e^{z+w} = \boxed{e^z e^w}$

$(e^z)^h = \boxed{e^{zh}}$



Trajectory:

$e^{(a+bi)t} \to e^{at} e^{bit}$ can tell you about shape of the function

ex. $e^{(-5-2i)t} = e^{-5} e^{-2it}$

$e^{-5} \to$ goes to 0 as $t \to \infty$

$e^{-2it} \to$ spirals clockwise as $t \to \infty$

∴ Trajectory is it spins clockwise as it moves inwards

Complexifying Integral

$$\int e^{-x}\cos(x)$$

$$= Re \int e^{(-1+i)}$$

$$= Re\left(\frac{e^{(-1+i)x}}{-1+i} + C\right)$$

$$= Re\left(\frac{(-1-i)(e^{-x}[\cos(x) + i\sin(x)])}{2}\right)$$

$$= \boxed{\frac{e^{-x}[\cos(x) + \sin(x)]}{2}}$$

Complex Roots of Polynomials

- A polynomial of order $\boxed{n}$ has exactly n roots
- Can be factored into Real & complex Polynomials

Fundamental Thm of Algebra

- Every Degree n complex polynomial $\boxed{f(z)}$ has exactly n comelex roots, if including multiplicity

* FOR REAL POLYNOMIALS any comelex roots will appear with its conjugate

- whenever we want to find roots substitute $\boxed{z = re^{i\theta}}$

ex. $z^5 = -32$

$r^5 e^{5i\theta} = 32 e^{i\pi}$

$\boxed{r=2}$

$\boxed{\theta = \frac{\pi + 2\pi k}{5}}$

Un-dampenned Spring Eqn

$$\boxed{m\ddot{x} + kx = 0}$$

* Any linear combination of cosine & sine, and any $x(t)$ such that $\boxed{x(t) = c_1\cos(t) + c_2\sin(t)}$

$$\boxed{m\ddot{x} + b\dot{x} + kx = 0}$$

* This is a 2nd order, linear, homogenous with constant co-efficients

To solve: PLUG IN

$$\boxed{x = e^{rt}}$$

$m\ddot{x} + b\dot{x} + kx = 0$

characteristic polynomial: $\boxed{(mr^2 + br + k)}$ Find r

ex. $\ddot{y} + 5\dot{y} + 6y = 0$

$p(r) = r^2 + 5r + 6$

$(r+3)(r+2)$ $r = -3, -2$

$$\boxed{y = c_1 e^{-2t} + c_2 e^{-3t}}$$

- Suppose $\ddot{y} + A\dot{y} + By = 0$ has characteristic roots $\boxed{a \pm bi}$

$$\boxed{y(t) = c_1 e^{at}\cos(bt) + c_2 e^{at}\sin(bt)}$$

Sinusoidal Function

Recall: $m\ddot{x} + b\dot{x} + kx = 0$

where $b^2 < 4mk$, and produces $\boxed{2}$ complex roots

roots: $e^{\frac{-b \pm \sqrt{b^2 - 4mk}}{2m}}$

$$x(t) = e^{-b/2m}\left[\cos\left(\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}\right) + \sin('' '')\right]$$

$$\omega_d = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

$$\omega_{(Nodamp)} = \sqrt{\frac{k}{m}}$$

From Rectangular to Polar fo. Sinusoids

$a\cos(\theta) + b\sin(\theta) = A\cos(\theta - \phi)$

~~A~~ $A = \sqrt{a^2 + b^2}$

$\phi = \tan^{-1}\left(\frac{b}{a}\right)$

3 Forms of Sinusoid

A. Amplitude-Phase Form

$$\boxed{A\cos(\omega t - \phi)}$$

B. $\boxed{Re(ce^{i\omega t})}$ where c is complex #

C. Linear Combination

$$\boxed{a\cos(\omega t) + b\sin(\omega t)}$$

Easist to Graph sinusoid $A\cos(\omega t - \phi)$ in Polar

* A: How high graph rises above t-axis at maximum

Period $\left(\frac{2\pi}{\omega}\right)$: The time for one complete oscillation between Maxima

MAIN Diffence of Damped/Undamped

Damped solutions have exponential Factor

$\pm Ae^{(-b/2m)t}$ is Envelope of oscillation

Given $m\ddot{x} + b\dot{x} + kx = 0$

Case	Roots	Situation
$b = 0$	2 complex Roots $(\pm i\omega_n)$	undamped
$b^2 < 4mk$	2 complex roots $(-p \pm i\omega_d)$	underdamped
$b^2 = 4mk$	Repeated Real root $(-p, -p)$	critically damped
$b^2 > 4mk$	2 distinct negative roots	overdamped

Solving high order linear ODE

$\text{span}(f_1, f_2, \ldots f_n)$ = set of all linear combinations of these Functions

Vector Space = a span of a set of functions

!!!

$\boxed{\text{\# of Roots = Order of DE = \# of Function in Basis}}$

18.03 Cheat Sheet

Vector Spaces

- S is a set of vectors in R^n
- S is a vector space if
  - 1. The zero vector is in S
  - 2. Multiplying any vector in S gives another vector in S
  - 3. Adding 2 vectors in S gives a vector in S

Subspace of R^2	Subspace of R^3
1. Point (0,0)	1. The point (0,0)
2. Any line through 0	2. Any line through the origin
3. the whole R^2	3. Any Plane through the origin

* The span of a set of vectors is a set of all possible linear combinations

* When checking for Overlap in Span decompose into simplest vectors

Linear Matrix Systems & Row Operations

Standard Form: $\boxed{(A)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}}$

* a system is consistent if it has at least one solution

* a system is inconsistent if it has no solution

* This linear system is homogenous if $b = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

* This system is inhomogenous if $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

Given: $\begin{pmatrix} 1 & 0 & -1 & -1 \\ 2 & 1 & -2 & -3 \\ 0 & 2 & -1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Augmented Matrix $= \left(\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 1 \\ 2 & 1 & -2 & -3 & 0 \\ 0 & 2 & -1 & -1 & 1 \end{array}\right)$

Row Operations:

- Multiply a Row by a Non-zero Number
- Interchange any 2 Rows
- Adding a Multiple of one Row to another

- A zero Row is a Row consisting of all zeros

- A pivot is the 1^{st} non-zero entry of a row

Row Echelon Form

1. All zeros are Grouped at the bottom of the Matrix
2. Each pivot lies farther to the right than Pivots of higher Rows

Gaussian Elimination

* Add Multiples of Rows to one another to get Matrices in Row-Echelon Form

Reduced Row-Echelon Form

1. Is in Row-Echelon Form
2. All pivots are 1
3. All pivot column entries are zero except for 1

Back Substitution

- Solving for each variable in Reverse order

* Introduce a parameter for each variable not directly expressed in terms of later variables

# of Free Variables = # non-Pivot columns

Null space, Column Space, Determinant, Inverse Matrices

Nullspace – The set of all solutions in a vector space

$\dim(NS) = $ # non-pivot columns

Finding Nullspace:

1. Do Back substitution
2. Solution will be a linear combination of a list of vectors

* Only for Homogenous, linear systems

- For an inhomogenous linear system $\boxed{Ax = b}$ there are 2 possibilities

1. There are No Solutions
2. There exists a Solution
   - if so there exists an X_p from $Ax = b$ & X_h from $Ax = 0$

$\boxed{X = X_p + X_h}$

Column Space of a Matrix

is the span of its columns

- Since Cs(A) is a span its a Vector Space

* The Linear System

$\boxed{Ax = b}$ has a solution

if & only if b is in CS(A)

Steps to compute Basis of CS(A):

1. Identify the Pivot Columns of B (Row Echelon Matrix)
2. The Corresponding Columns of Matrix A form the Basis

$\boxed{\dim CS(A) = \#\text{ of Pivot columns}}$

Determinents, Diagonal, Inverses

Diagonal Matrix $\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$ * a_1, a_2, a_3 can be zero

Upper Triangular Matrix $\begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & 0 & a_6 \end{pmatrix}$

Properties of Determinents

- Interchanging 2 rows changes det sign
- Multiplying a row by scalar c multiplies det by c
- If a zero Row exist, det = 0

Inverse Matrix

$$\boxed{AA^{-1} = I}$$

2x2 $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$, $A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} a_4 & -a_2 \\ -a_3 & a_1 \end{pmatrix}$

or

1. Form Augmented Matrix $[A \mid I]$
2. Convert to RREF → $[I \mid B]$

Then $\boxed{A^{-1} = B}$

Properties of Invertible Matrices

1. $\det(A) \neq 0$
2. $NS(A) = 0$
3. $\text{Rank}(A) = n$
4. $CS(A) = \mathbb{R}^n$
5. For each vector b $Ax=b$ has only one solution
6. A^{-1} exists
7. $RREF(A) = I$

Properties of Singular Matrices

1. $\det(A) = 0$
2. $NS(A) > 0$
3. $\text{rank}(A) < n$
4. $CS(A)$ is smaller then $\mathbb{R}^n$
5. $Ax=b$ has either no solution or Infinitely many
6. A^{-1} doesn't exist
7. $RREF(A) \neq I$

Characteristic Polynomial of a Matrix

- Use λ to denote scalar Variable

$$\boxed{\det(I\lambda - A)}$$

* Set $\det(I\lambda - A) = 0$ to Find Eigenvalues(λ)

For 2x2 Matrix

$$p(\lambda) = \lambda^2 - tr(A)\lambda + \det(A)$$

Finding All Eigenvalues

1. Calculate $\det(I\lambda - A)$
2. The roots of this polynomial are the eigenvalues of A

Eigenspaces

- An Eigenspace is a set of all eigenvectors Belonging to a Particular EigenValue

To find Eigenvectors **

set

$$\begin{pmatrix} a_1 - \lambda_1 & a_2 \\ 0 & a_4 - \lambda \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = 0$$

$\boxed{\text{solve for } v_1, v_2}$

Dimension of Eigenspace

- The Dimension of the Eigenspace is the Max # of Linearly Independent Eigenvectors Found
- A Matrix is Deficient if one of its Eigenspaces is deficient
  1) Repeated EigenValue
  2) One vector a Multiple of another

* Fundamental Matrix

- Let $(x_1 \cdots x_n)$ be the set of Basis Vectors from the solution of $\dot{x} = Ax$. Basically Matrix of Eigen values * Eigen Vector

* You can use initial conditions to find exact solution setting $x(0) = (\text{Fundamental Matrix})\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

Matrix Exponential

- $e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$
- $e^0 = I$
- $\frac{d}{dt} e^{At} = A e^{At}$
- If $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, then $e^A = \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix}$

Solving $\dot{x} = Ax + q(t)$ by Decoupling

1. Find Eigenvalues of A & put them in Diagonal Matrix D
2. Compose a Matrix N containing All the Eigenspaces
3. Substitute $\boxed{x = Sy}$
   ↳ $\dot{y} = Dy + S^{-1} q(t)$
4. Solve for Each co-ordinate function of y
5. Compute Sy; the result is x

Solving $\dot{x} = AX + q(t)$ by variation of Parameters

1. Find the Bases of Solutions to the corresponding system

$\boxed{\dot{x} = Ax}$

2. Put these Bases together to Form a Matrix X

then substitute $x = Xu$

$\dot{x} = Ax + q$

$\dot{X}u + X\dot{u} = AXu + q$

$X\dot{u} = q$

* $\boxed{\dot{u} = X^{-1}q}$

3. Compute the Right side and integrate each component to find u

4. $\boxed{x = Xu}$; This is the general solution

co-ordinates

$$c_1\begin{pmatrix}2\\1\end{pmatrix} + c_2\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}2\\4\end{pmatrix}$$

* These c_1 & c_2 involved in the linear combination are called co-ordinates

* if basis vectors are ORTHAGONAL

$c_1V_1 + c_2V_2 = W$

* $\boxed{c_1 = \frac{W \cdot V_1}{V_1 \cdot V_1}}$

Periodic Functions

- A function $f(t)$ is Periodic of Period P if $f(t+P) = f(t)$
- BASE Period is the smallest common Period of a series of Functions

Fourier Series

- A Linear Combination of infinitely many $\cos(nt)$ & $\sin(nt)$

$$f(t) = \frac{a_0}{2} + a_1\cos(nt) + a_2(\cos(2t) + \dots$$
$$= b_1\sin(t) + b_2\sin(2t) + \dots$$

Inner Product

$$\boxed{\langle f, g \rangle = \int_{-\pi}^{\pi} f(t)g(t)\,dt}$$

FOURIER CO-Efficient Formulas

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt$$

Solving ODE's with Fourier series

- Suppose the input function $f(t)$ is an odd Periodic Function

USE ERF $\rightarrow \frac{1}{P(r)}\sin(nt)$

SO GIVEN: $b_1\sin(t) + b_2\sin(2t)$ as an input

$$\boxed{x(t) = \sum_{n\geq 1} \frac{1}{P_r(ni)} b_n \sin(nt)}$$

Near Resonance & Pure Resonance

Near Resonce that

- specific Fourier term will be non-Periodic

Complex Fourier series

Functions of Arbitrary Period