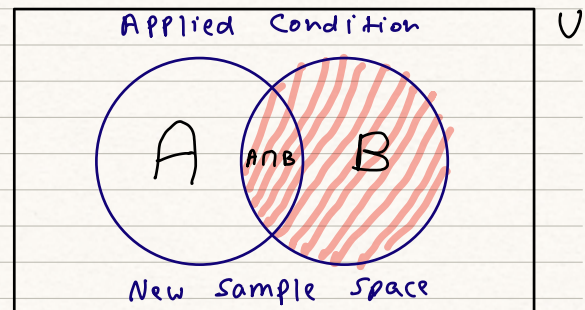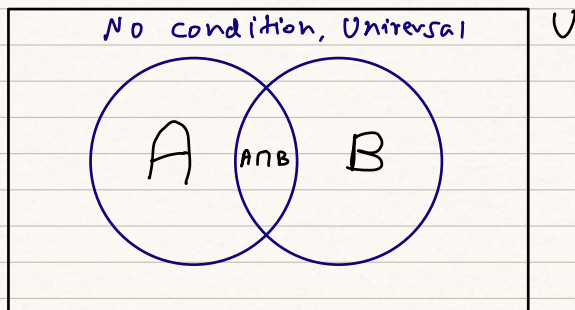# Conditional Probability

$$P(A/B) \xrightarrow{\text{given } B} = \frac{P(A \cap B)}{P(B) \rightarrow \text{event } B \text{ serving as condition}}$$

Here, we are finding the probability of event A given that, <mark>event B has already</mark> occured.

No condition, Universal — U

Applied Condition — U

New Sample space

We will find probability considering this new sample space

Final under consideration area is proper event B and A∩B.
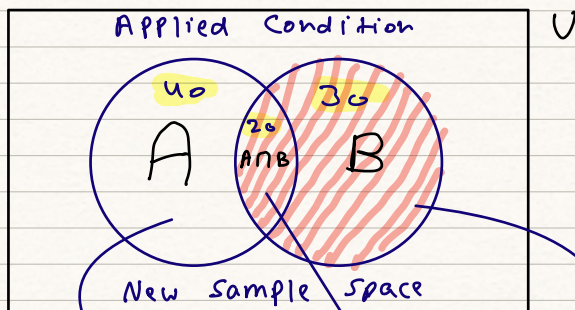
When initially no condition is applied, the sample space is universe (U).

As soon as you apply the condition, the sample space gets reduced. And with the help of new sample space, we are going to find probability of A.

Applied Condition — U

New Sample space

* Total 100 students

40 students likes apples

30 students likes oranges.

20 students likes both apples and oranges.

$$P(B) = \frac{30}{100} = 0.3$$

$$P(A \cap B) = \frac{20}{100} = 0.2$$

$$P(A|B) = \frac{0.2}{0.3} = \frac{2\emptyset}{3\emptyset} = 0.67$$

Independent Events :— In probability, independent events are those events whose occurrence is not dependent on any other event.

For example, you toss a coin, and it comes "heads" three times. What is the chance that the next toss will also be a " head" ?

The chance is simply 1/2 (or 0.5) just like any other toss of the coin.

Dependent Events :— In probability, two events are dependent if the outcome of the first event affects the outcome of the second event, so that the probability is changed.

For example, taking cards from a deck makes less cards each time.
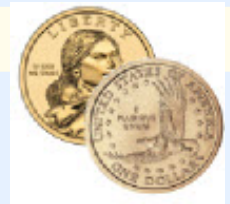
Conditional Probability

*How to handle **Dependent Events***

Life is full of random events! You need to get a "feel" for them to be a smart and successful person.

Independent Events

Events can be " Independent ", meaning each event is **not affected** by any other events.

> ### Example: Tossing a coin.
>
> Each toss of a coin is a perfect isolated thing.
>
> What it did in the past will not affect the current toss.
>
> The chance is simply 1-in-2, or 50%, just like ANY toss of the coin.
>
> So each toss is an **Independent Event**.

Dependent Events

But events can also be "dependent" ... which means they **can be affected by previous events** ...
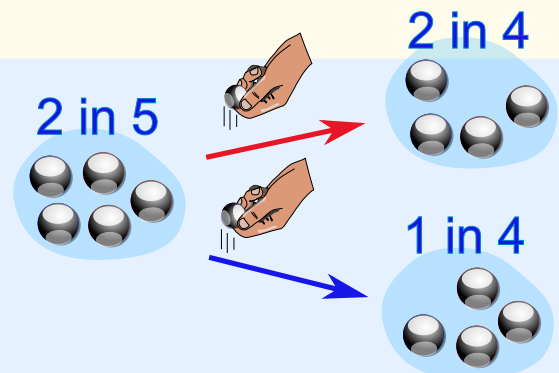
> ### Example: Marbles in a Bag
>
> 2 blue and 3 red marbles are in a bag.
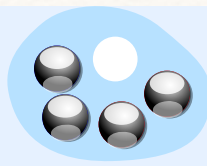>
> What are the chances of getting a blue marble?
>
> The chance is **2 in 5**
>
> 2 in 5
>
> 2 in 4
>
> 1 in 4
>
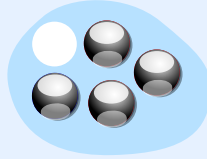> **But after taking one out** the chances change!
>
> So the next time:

if we got a **red** marble before, then the chance of a blue marble next is **2 in 4**

if we got a **blue** marble before, then the chance of a blue marble next is **1 in 4**

This is because we are **removing** marbles from the bag.

So the next event **depends on** what happened in the previous event, and is called **dependent**.

> Replacement
>
> Note: if we **replace** the marbles in the bag each time, then the chances do **not** change and the events are independent :
>
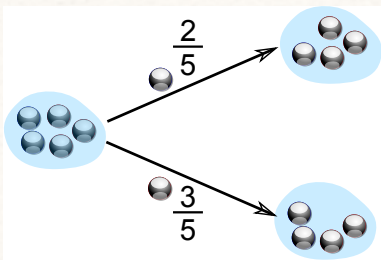> **With** Replacement: the events are **Independent** (the chances don't change)
>
> **Without** Replacement: the events are **Dependent** (the chances change)

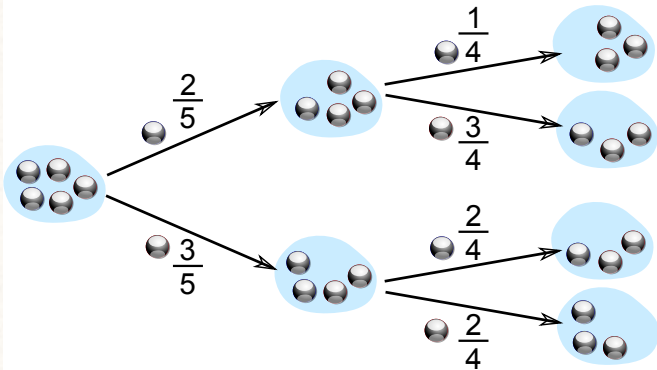**Dependent** events are what we look at here.

Tree Diagram

A Tree Diagram : is a wonderful way to picture what is going on, so let's build one for our marbles example.

There is a 2/5 chance of pulling out a Blue marble, and a 3/5 chance for Red:

We can go one step further and see what happens when we pick a second marble:

If a blue marble was selected first there is now a 1/4 chance of getting a blue marble and a 3/4 chance of getting a red marble.

If a red marble was selected first there is now a 2/4 chance of getting a blue marble and a 2/4 chance of getting a red marble.

Now we can answer questions like **"What are the chances of drawing 2 blue marbles?"**

Answer: it is a **2/5 chance** followed by a **1/4 chance**:

$$\frac{2}{5} \times \frac{1}{4} = \frac{2}{20} = \frac{1}{10}$$
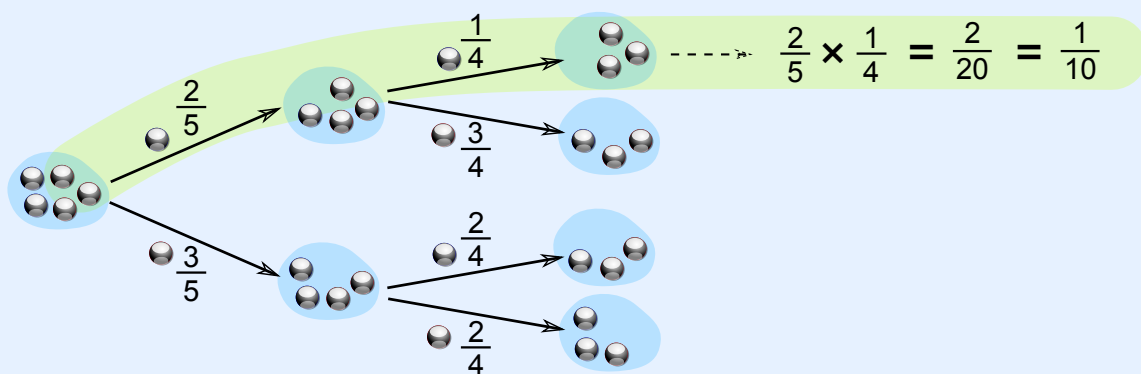
Did you see how we multiplied the chances? And got 1/10 as a result.

The chances of drawing 2 blue marbles is 1/10

Notation

We love notation in mathematics! It means we can then use the power of algebra to play around with the ideas. So here is the notation for probability:

P(A) means "Probability Of Event A"

In our marbles example Event A is "get a Blue Marble first" with a probability of 2/5:

P(A) = 2/5

And Event B is "get a Blue Marble second" ... but for that we have 2 choices:

If we got a **Blue Marble first** the chance is now **1/4**

If we got a **Red Marble first** the chance is now **2/4**

So we have to say **which one we want**, and use the symbol "|" to mean "given":

P(B|A) means "Event B **given** Event A"

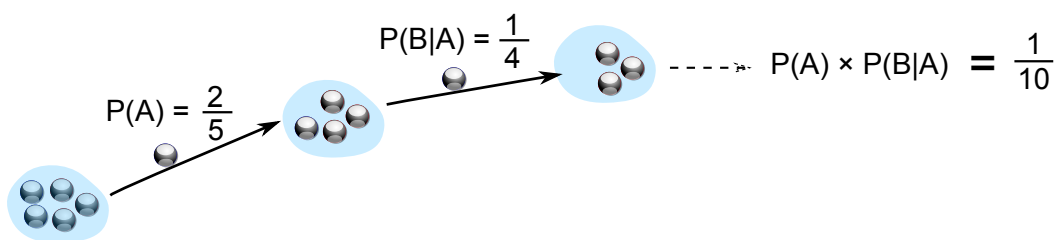In other words, event A has already happened, now what is the chance of event B?

P(B|A) is also called the "Conditional Probability" of B given A.

And in our case:

P(B|A) = 1/4

So the probability of getting **2 blue marbles** is:

$$P(B|A) = \frac{1}{4} \qquad P(A) \times P(B|A) = \frac{1}{10}$$

$$P(A) = \frac{2}{5}$$

And we write it as

$$\text{"Probability Of"} \qquad \text{"Given"}$$

$$P(\ A \text{ and } B\ ) = P(\ A\ ) \times P(\ B \mid A\ )$$

$$\text{Event A} \quad \text{Event B}$$

*"Probability of **event A and event B** equals*
*the probability of **event A** times the probability of **event B given event A**"*

Let's do the next example using only notation:

> ## Example: Drawing 2 Kings from a Deck
>
> **Event A** is drawing a King first, and **Event B** is drawing a King second.
>
> For the first card the chance of drawing a King is 4 out of 52 (there are 4 Kings in a deck of 52 cards):
>
> $$P(A) = 4/52$$
>
> But after removing a King from the deck the probability of the 2nd card drawn is **less** likely to be a King (only 3 of the 51 cards left are Kings):
>
> $$P(B|A) = 3/51$$
>
> And so:
>
> **P(A and B) = P(A) x P(B|A)** = (4/52) x (3/51) = 12/2652 = **1/221**
>
> So the chance of getting 2 Kings is 1 in 221, or about 0.5%

Finding Hidden Data

Using Algebra we can also "change the subject" of the formula, like this:

Start with:	P(A and B) = P(A) x P(B|A)
Swap sides:	P(A) x P(B|A) = P(A and B)
Divide by P(A):	P(B|A) = P(A and B) / P(A)

And we have another useful formula:

$$P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)}$$

*"The probability of **event B given event A** equals the probability of **event A and event B** divided by the probability of **event A***

Big Example: Soccer Game

You are off to soccer, and want to be the Goalkeeper, but that depends who is the Coach today:

with Coach Sam the probability of being Goalkeeper is **0.5**

with Coach Alex the probability of being Goalkeeper is **0.3**

Sam is Coach more often ... about 6 out of every 10 games (a probability of **0.6**).

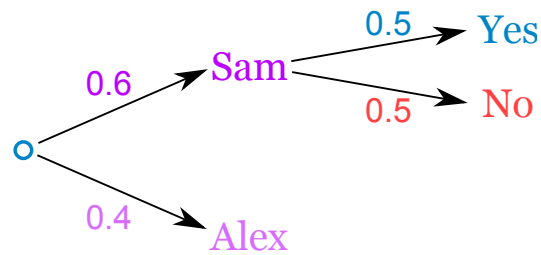So, what is the probability you will be a Goalkeeper today?

Let's build a tree diagram. First we show the two possible coaches: Sam or Alex:
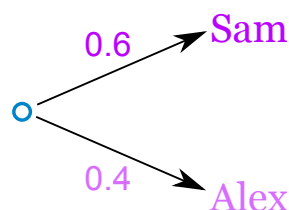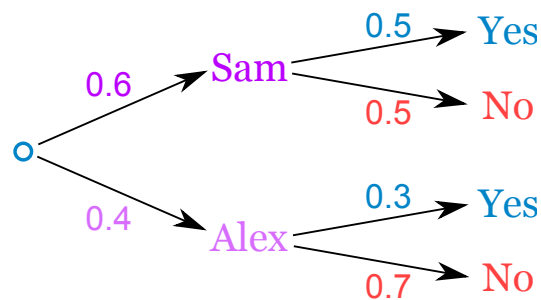

The probability of getting Sam is 0.6, so the probability of Alex must be 0.4 (together the probability is 1)

Now, if you get Sam, there is 0.5 probability of being Goalie (and 0.5 of not being Goalie):
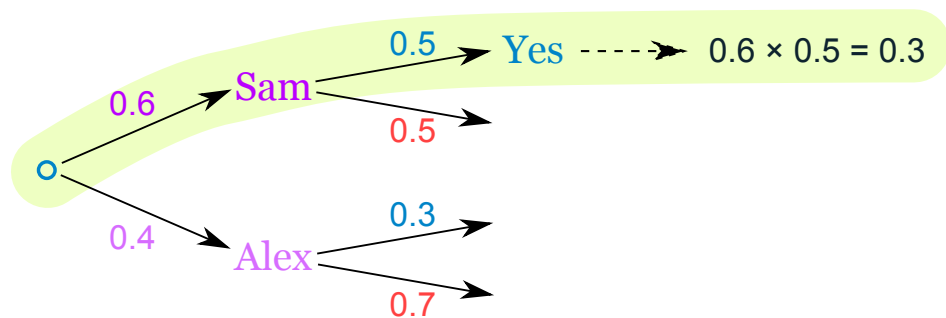

If you get Alex, there is 0.3 probability of being Goalie (and 0.7 not):

The tree diagram is complete, now let's calculate the overall probabilities. Remember that:
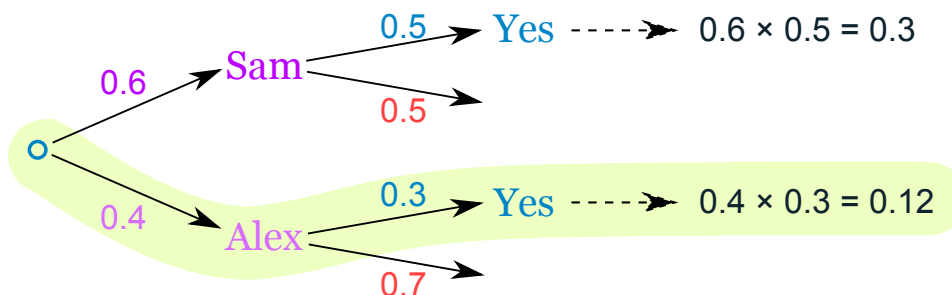
$$P(A \text{ and } B) = P(A) \times P(B|A)$$

Here is how to do it for the "Sam, Yes" branch:

(When we take the 0.6 chance of Sam being coach times the 0.5 chance that Sam will let you be Goalkeeper we end up with an 0.3 chance.)

But we are not done yet! We haven't included Alex as Coach:

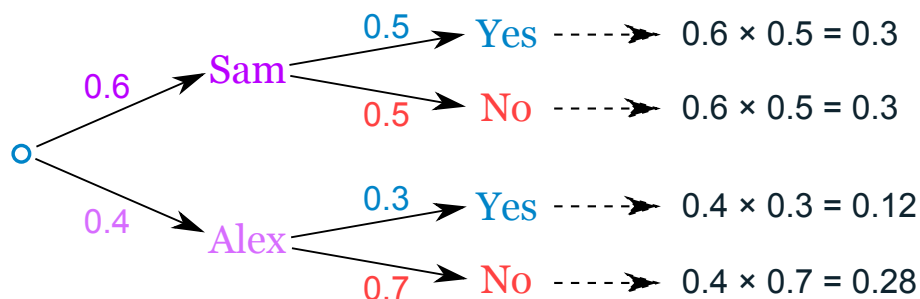An 0.4 chance of Alex as Coach, followed by an 0.3 chance gives 0.12

And the two "Yes" branches of the tree together make:

0.3 + 0.12 = **0.42 probability** of being a Goalkeeper today

(That is a 42% chance)

Check

One final step: complete the calculations and make sure they add to 1:

$$0.3 + 0.3 + 0.12 + 0.28 = 1$$

Yes, they add to **1**, so that looks right.

Friends and Random Numbers

Here is another quite different example of Conditional Probability.

4 friends (Alex, Blake, Chris and Dusty) each choose a random number between 1 and 5. What is the chance that any of them chose the same number?

Let's add our friends one at a time ...

First, what is the chance that Alex and Blake have the same number?

Blake compares his number to Alex's number. There is a 1 in 5 chance of a match.
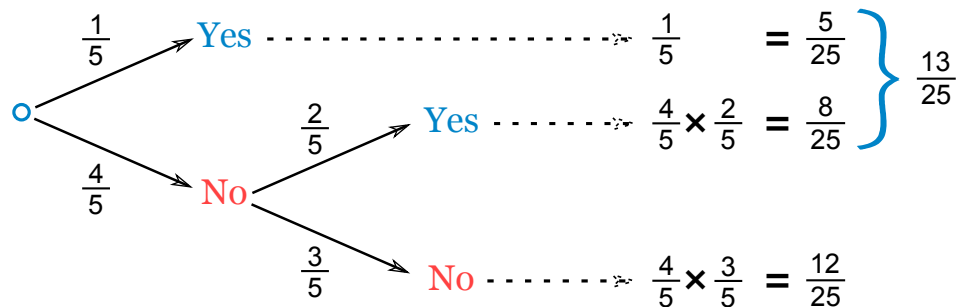
As a tree diagram :
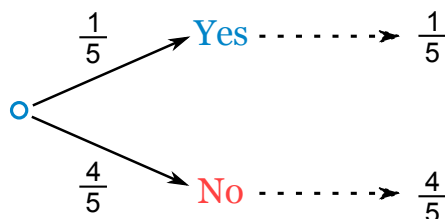
Now, let's include Chris ...

But there are now two cases to consider:

If Alex and Blake **did** match, then Chris has only **one number** to compare to.

But if Alex and Blake **did not** match then Chris has **two numbers** to compare to.

And we get this:

$$\frac{1}{5} \quad \text{Yes} \cdots\cdots\cdots\cdots \quad \frac{1}{5} \quad = \quad \frac{5}{25}$$

$$\frac{2}{5} \quad \text{Yes} \cdots\cdots \quad \frac{4}{5} \times \frac{2}{5} = \frac{8}{25} \quad \Biggr\} \quad \frac{13}{25}$$

$$\frac{4}{5} \quad \text{No}$$

$$\frac{3}{5} \quad \text{No} \cdots\cdots \quad \frac{4}{5} \times \frac{3}{5} = \frac{12}{25}$$

For the top line (Alex and Blake **did** match) we already have a match (a chance of 1/5).

But for the "Alex and Blake **did not** match" there is now a **2/5** chance of Chris matching (because Chris gets to match his number against both Alex and Blake).

And we can work out the combined chance by **multiplying the chances** it took to get there:

Following the "No, Yes" path ... there is a 4/5 chance of No, followed by a 2/5 chance of Yes:

➡ (4/5) × (2/5) = 8/25

Following the "No, No" path ... there is a 4/5 chance of No, followed by a 3/5 chance of No:
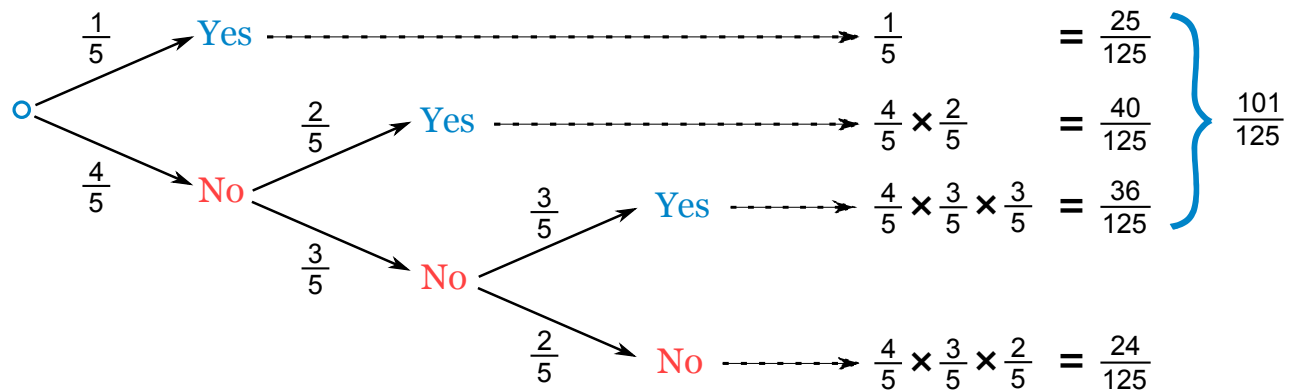
➡ (4/5) × (3/5) = 12/25

Also notice that when we add all chances together we still get 1 (a good check that we haven't

made a mistake):

$$(5/25) + (8/25) + (12/25) = 25/25 = 1$$

Now what happens when we include Dusty?

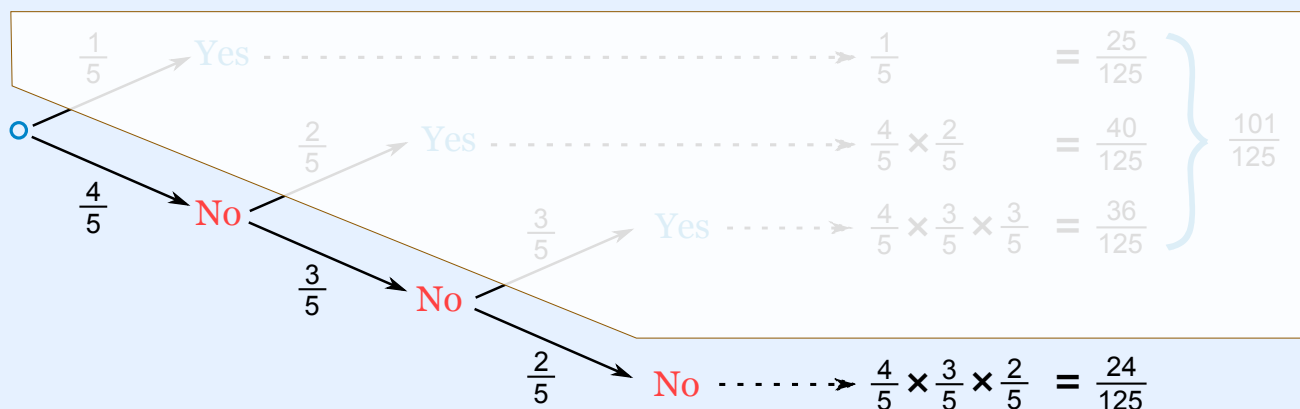It is the same idea, just more of it:

OK, that is all 4 friends, and the "Yes" chances together make 101/125:

<p style="text-align:center">Answer: 101/125</p>

But here is something interesting ... if we follow the "No" path we can **skip all the other calculations** and make our life easier:

The chances of **not matching** are:

$$(4/5) \times (3/5) \times (2/5) = \mathbf{24/125}$$

So the chances of **matching** are:

$$1 - (24/125) = \mathbf{101/125}$$