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Dynamics of quantum correlations for three-qubit states in a noisy environment

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Girolami and coworkers have proposed measures of quantum correlations and weaving [Girolami *et al.*, *Phys. Rev. Lett.* **119**, 140505 (2017)]. This work derives the analytic time-evolution of such measures and weaving for two kinds of initial states in three qubits under an amplitude-damping and a dephasing noisy environment. It is shown that the 2-partite correlation is forever frozen, which is dependent on an initial state and the property of noise.

Keywords: Quantum correlations; three qubits; noisy environment.

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1. Introduction

Quantum correlation becomes a key resource in quantum information science, where two kinds of quantum correlation have been extensively and intensively studied.[1,2] The first one is the entanglement that describes the inseparability of quantum states, and the corresponding measures include concurrence,[3] negativity,[4] and so forth. Concurrence has the analytic form for two-qubit states,[3] while negativity is the trace norm of the partial transpose of a density matrix.[4] Those two measures of entanglement have been extended to tripartite systems,[5,6] and applied to explore dynamics of entanglement for three-qubit states in a noisy environment.[7] It has been shown that three-qubit entanglement in such environment may suddenly die or decrease asymptotically.[7] The entanglement was investigated for various states

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in spin chains, the Tavis–Cummings model, and triatomic molecules.[8] Moreover, classification and entanglement for pure three-qubit states were analyzed in terms of the group of local unitary transformation.[9] Entanglement and nonlocality were considerably studied for two- and three-qubit states.[10] Recently, the dynamics of nonlocality and entanglement has been explored for three- and four-qubit states under noisy environments and weak measurements.[11] It reveals that maximal nonlocality may not coincide with maximal entanglement for the GHZ states, which depends on a noisy and weak measurement parameter.[11]

The second is the correlation induced by measurements, with various measures such as quantum discord[12] and geometric discord[13] being proposed. It has been shown that quantum discord and geometric discord in two qubits do not display the phenomenon of sudden death, and they are more robust than entanglement.[14] Such correlations in two qubits under decoherence reveals that they can remain frozen for an interval of time under the specified dynamical conditions.[15] Moreover, quantifying such correlation was presented for multipartite quantum states,[16,17] and characterizing tripartite discord[17] was carried out in three-qubit states without decoherence. However, the calculation of such correlations has to tackle the cumbersome optimization problem for higher-dimensional states.[12,13,16,17]

Recently, Girolami *et al.*[18] have introduced novel measures of multipartite correlations. Moreover, the concept of weaving[18] is proposed to classify quantum states that exhibit different correlation patterns. Therefore, multipartite states with equal total correlation or highest order of correlations have different weaving values. The weaving of a quantum state is defined by the weighted sum of multipartite correlations of any order, inheriting the properties of correlation indicators.[18] If quantum relative entropy is taken to describe the distance between two states, the characterization of such measures for an arbitrary state avoids the difficult optimization problem. Consequently, analytic forms of k-partite correlations and weaving are provided for several typical N-qubit states $(2 \leq k \leq N-1)$.[18] However, the behavior of those correlations with decoherence is unclear. In this work, we will study the dynamics of those correlations[18] for three-qubit states in a noisy environment.

The objective of this work is threefold. When examining the dynamics of a system, one must consider decoherence because of unavoidable disturbance of environments. There are several approaches to describe a decoherence process. In particular, decoherence in a noisy environment is governed by the Kraus operators.[1,7,11] Recently, theoretical and experimental efforts have been made to characterize entanglement or nonlocality of qubits in noisy environments.[7,11,19,20] It is thus of importance to explore the effects of decoherence processes on novel measures of multipartite correlations.[18] On the other hand, we choose three qubits under two kinds of noisy environments as our model, since they can be experimentally realized by the platform of linear optics[19] and a nuclear magnetic resonance system.[20] Therefore, it is interesting and necessary to investigate the dynamical properties of 3- and 2-partite correlations and weaving[18] for three-qubit states in two noisy environments. Moreover, such investigation can motivate further studies of quantum correlations in mulitiqubits with decoherence. More importantly, quantum correlations are always useful for different information processing tasks. For example, bipartite correlations have been widely applied to those tasks such as quantum computing, cloning, and teleportation.[1] Moreover, tripartite correlations have been used to increase the security of quantum cryptography[21] and the efficiency of quantum cloning.[22] Therefore, the dynamical properties of the correlation in the model of interest are significant for quantum information processing tasks.

The paper is organized as follows. Section 2 presents the definitions of novel measures of multipartite correlations.[18] Section 3 studies the dynamics of correlations for three-qubit states in two noisy environments, where two kinds of initial states are taken into account. We derive the explicit time-evolution of 3- and 2-partite correlations and the weaving.[18] Moreover, we demonstrate that the 2-partite correlation can be forever frozen under a suitable condition. Section 4 concludes the paper with discussions.

2. Quantum Correlations

There are various measures of quantum correlations in a state. We are interested in novel measures of multipartite correlations,[18] since they are easy to compute for arbitrary multipartite states. For N partite state ρ, its k-partite quantum correlation is given by[18]

$$D^k(\rho) = D^{k-1\to N}(\rho) - D^{k\to N}(\rho), \tag{1}$$

where

$$D^{k\to N}(\rho) = \min_{\sigma\in P_k} D(\rho,\sigma), \tag{2}$$

with $P_k = \{\sigma_N = \bigotimes_{j=1}^m \sigma_{k_j}, \sum_{j=1}^m k_j = N, k = \max\{k_j\}\}$ being the set of product states, and D being the distance between states ρ and σ. Such correlation satisfies desirable properties of quantum information theory.[18]

Moreover, inspired by ideas in complexity science, the weaving is proposed as[18]

$$W(\rho) = \sum_{l=1}^{N-1} \Omega_l D^{l\to N}(\rho). \tag{3}$$

For the convenience of calculation, Ω_l is taken to be 1 for all l.[18] Weaving ranks classical and quantum multipartite states with a single index, and classifies states that display various correlation patterns. It is expected that weaving can link quantum information science and complexity science. However, it is desirable to explain the operational meaning of weaving.

It is worth mentioning that the minimization of Eq. (2) is very complicated for a generic distance D. Reference 18 thus suggests that once the relative entropy, $S(\rho\,\|\,\sigma) = \mathrm{Tr}[\rho\log_2(\rho/\sigma)]$, is chosen to be the distance measure, the calculation of Eq. (2) will be significantly simplified. Taking a 3-partite quantum state as an example, we have two values of k, 1 and 2. Therefore, the 2-partite correlation is given by

$$S^2(\rho) = S^{1\to 3}(\rho) - S^{2\to 3}(\rho), \tag{4}$$

where $S^{1\to 3}(\rho)$ is the total correlation, and $S^{2\to 3}(\rho)$ is the 3-partite correlation. In this case, the weaving by Eq. (3) reads

$$W(\rho) = S^{1\to 3}(\rho) + S^{2\to 3}(\rho), \tag{5}$$

where

$$S^{1\to 3}(\rho) = S(\rho \,\|\, \rho^a \otimes \rho^b \otimes \rho^c), \tag{6}$$

$$\begin{aligned} S^{2\to 3}(\rho) = \min\{&S(\rho \,\|\, \rho^{ab} \otimes \rho^c), S(\rho \,\|\, \rho^{ac} \otimes \rho^b),\\ &S(\rho \,\|\, \rho^{bc} \otimes \rho^a), S(\rho \,\|\, \rho^a \otimes \rho^b \otimes \rho^c)\}, \end{aligned} \tag{7}$$

with ρ^α being the state of the αth subsystem ($\alpha = a, b, c, ab, ac, bc$), which is reduced from system state ρ, such as $\rho^a = \mathrm{Tr}_{bc}\rho$.[18] Thanks to the relative entropy as a distance measure, static and analytic forms of S^k and W[18] were presented for several states in multipartite qubits. In next section, we study the dynamics of those correlations for three-qubit states with decoherence.

3. Dynamics of Correlations

We consider the three-qubit system that is subject to a dephasing or amplitude damping noisy environment, which is accessible in recent experiments.[19,20] If $\rho(0)$ is an initial state of the system, its evolution with time is given by

$$\rho(t) = \sum_{i,j,k=1}^{2} (G_i^a \otimes G_j^b \otimes G_k^c)\rho(0)(G_i^a \otimes G_j^b \otimes G_k^c)^\dagger, \tag{8}$$

where G_i^α is the αth ($\alpha = a, b, c$) Kraus operators for describing the dynamical property of particular quantum noise. The corresponding operators for a dephasing and an amplitude damping noise are respectively given by[1,7]

$$G_1^\alpha = \begin{pmatrix} \sqrt{p} & 0 \\ 0 & 1 \end{pmatrix}, \quad G_2^\alpha = \begin{pmatrix} \sqrt{1-p} & 0 \\ 0 & 0 \end{pmatrix}, \tag{9}$$

$$G_1^\alpha = \begin{pmatrix} \sqrt{p} & 0 \\ 0 & 1 \end{pmatrix}, \quad G_2^\alpha = \begin{pmatrix} 0 & 0 \\ \sqrt{1-p} & 0 \end{pmatrix}, \tag{10}$$

where variable $p = e^{-\gamma t}$ represents the time-dependent probability of the qubit that remains unchanged, and three qubits are assumed to have the same decoherence rate (γ), as done in Ref. 7.

We are concerned with the generalized W and the slice state[9] as an initial state, since the slice state includes the generalized GHZ state. It is worthwhile to stress that the W and the GHZ state represent two inequivalent ways of entanglement[23] and have important applications in quantum information processing.[24] Moreover, they are experimentally accessible.[19,20,25] The generalized W and the slice state of interest are, respectively, given by

$$|\psi\rangle_w = \sin\phi\sin\theta|100\rangle + \cos\phi\sin\theta|010\rangle + \cos\theta|001\rangle, \tag{11}$$

$$|\psi\rangle_s = \sin\varphi\sin\Theta|000\rangle + \sin\varphi\cos\Theta|001\rangle + \cos\varphi|111\rangle. \tag{12}$$

Thus, we have the following four cases to study total correlation $S^{1\to3}(\rho)$ and 3-partite correlation $S^{2\to3}(\rho)$. Now that those two quantities are at hands, it is easy to obtain the explicit forms of 2-partite correlation $S^2(\rho)$ and weaving $W(\rho)$ with Eqs. (4) and (5).

Case 1. When the system starts with the generalized W state in an amplitude damping noise, we calculate the total correlation, given by

$$S^{1\to3}(\rho) = h(p\chi_1) + h(p\chi_2) + h(p\chi_3) - h(p), \tag{13}$$

with $h(x) = -x\log_2 x - (1-x)\log_2(1-x)$ being the binary entropy function, $\chi_1 = \cos^2\theta, \chi_2 = \sin^2\phi\sin^2\theta$, and $\chi_3 = \cos^2\phi\sin^2\theta$. The 3-partite correlation $S^{2\to3}(\rho)$ is available via Eq. (7), where

$$\begin{aligned} S(\rho\,\|\,\rho^{bc}\otimes\rho^a) &= h(p\sin^2\theta) + h(p\chi_1) - h(p),\\ S(\rho\,\|\,\rho^{ac}\otimes\rho^b) &= h\left(\tfrac{p}{8}(6 + 2\cos 2\phi + 2\cos 2\theta - \cos 2(\phi-\theta) - \cos 2(\phi+\theta)\right)\\ &\quad + h(p\chi_2) - h(p),\\ S(\rho\,\|\,\rho^{ab}\otimes\rho^c) &= h\left(\tfrac{p}{8}(6 - 2\cos 2\phi + 2\cos 2\theta + \cos 2(\phi-\theta) + \cos 2(\phi+\theta)\right)\\ &\quad + h(p\chi_3) - h(p). \end{aligned} \tag{14}$$

If the time of evolution is infinite, $S^{1\to3}(\rho) = S^{2\to3}(\rho) = 0$.

Case 2. When the generalized W state is taken as an initial state for the system in a dephasing noise, the total correlation explicitly reads

$$S^{1\to3}(\rho) = h(\chi_1) + h(\chi_2) + h(\chi_3) + \mathrm{Tr}\rho_d\log_2\rho_d. \tag{15}$$

The 3-partite correlation needs the following expressions:

$$\begin{aligned} S(\rho\,\|\,\rho^{bc}\otimes\rho^a) &= h(\chi_1) - \mathrm{Tr}\rho_e\log_2\rho_e + \mathrm{Tr}\rho_d\log_2\rho_d,\\ S(\rho\,\|\,\rho^{ac}\otimes\rho^b) &= h(\chi_2) - \mathrm{Tr}\rho_f\log_2\rho_f + \mathrm{Tr}\rho_d\log_2\rho_d,\\ S(\rho\,\|\,\rho^{ab}\otimes\rho^c) &= h(\chi_3) - \mathrm{Tr}\rho_g\log_2\rho_g + \mathrm{Tr}\rho_d\log_2\rho_d, \end{aligned} \tag{16}$$

where $\rho_j (j = d, e, f, g)$ is the matrix given in the Appendix.

If time is infinite,

$$S^{1\to 3}(\rho) = h(\chi_1) + h(\chi_2) + h(\chi_3) + \sum_{i=1}^{3} \chi_i \log_2 \chi_i, \tag{17}$$

while $S(\rho \,\|\, \rho^{bc} \otimes \rho^a) = h(\chi_1)$, $S(\rho \,\|\, \rho^{ac} \otimes \rho^b) = h(\chi_2)$, and $S(\rho \,\|\, \rho^{ab} \otimes \rho^c) = h(\chi_3)$.

Case 3. When the system is set out from the slice state in the amplitude damping noise, the total correlation is given by

$$\begin{aligned} S^{1\to 3}(\rho) = h(\zeta_1) + 2h(p\kappa_1) + \mathrm{Tr}\rho_m \log_2 \rho_m + 2(1-p)p^2\kappa_1 \log_2[(1-p)p^2\kappa_1] \\ + 2p(1-p)^2\kappa_1 \log_2[p(1-p)^2\kappa_1], \end{aligned} \tag{18}$$

where $\zeta_1 = \frac{1}{4}[2 + [4 - 7p + 6p^2 + 8(1-p)p\cos(2\varphi) + p(2p-1)\cos(4\varphi) + 2p\cos(2\Theta)\sin^2(2\varphi)]^{\frac{1}{2}}]$, $\kappa_1 = \sin^2\varphi \sin^2\Theta$, and ρ_m is the matrix given in the Appendix. The 3-partite correlation calls for the following expressions:

$$\begin{aligned} S(\rho \,\|\, \rho^{bc} \otimes \rho^a) &= h(\zeta_1) - \mathrm{Tr}\rho_n \log_2 \rho_n + \mathrm{Tr}\rho_m \log_2 \rho_m \\ &\quad + 2(1-p)p^2\kappa_1 \log_2[(1-p)p^2\kappa_1] \\ &\quad + 2p(1-p)^2\kappa_1 \log_2[p(1-p)^2\kappa_1], \\ S(\rho \,\|\, \rho^{ac} \otimes \rho^b) &= h(p\kappa_1) - \mathrm{Tr}\rho_r \log_2 \rho_r + \mathrm{Tr}\rho_m \log_2 \rho_m \\ &\quad + 2(1-p)p^2\kappa_1 \log_2[(1-p)p^2\kappa_1] \\ &\quad + 2p(1-p)^2\kappa_1 \log_2[p(1-p)^2\kappa_1], \end{aligned} \tag{19}$$

where $S(\rho \,\|\, \rho^{ab} \otimes \rho^c = S(\rho \,\|\, \rho^{ac} \otimes \rho^b)$ and ρ_n, ρ_r is given in the Appendix.

In the case of $\Theta = \frac{\pi}{2}$, the slice state reduces to the generalized GHZ state. In this case, we have $\rho^a = \rho^b = \rho^c$ and $\rho^{ab} = \rho^{ac} = \rho^{bc}$ due to the symmetry of three qubits. The total correlation is

$$S^{1\to 3}(\rho) = 3h(p\sin^2\varphi) + \sum_{i=1}^{8} \lambda_i \log_2 \lambda_i, \tag{20}$$

with $\lambda_1 = \frac{1}{2}(1 - 3p\sin^2\varphi + 3p^2\sin^2\varphi + \sigma)$, $\lambda_2 = \frac{1}{2}(1 - 3p\sin^2\varphi + 3p^2\sin^2\varphi - \sigma)$, $\lambda_3 = \lambda_4 = \lambda_5 = (p-1)p^2\sin^2\varphi$, and $\lambda_6 = \lambda_7 = \lambda_8 = p(p-1)^2\sin^2\varphi$, where $\sigma = \sqrt{(p^3\sin^2\varphi + (p-1)^3\sin^2\varphi - \cos^2\varphi)^2 + p^3\sin^2(2\varphi)}$. The formula for the 3-partite correlation is

$$S(\rho \,\|\, \rho^{bc} \otimes \rho^a) = h(p\sin^2\varphi) - \sum_{i=1}^{4} \xi_i \log_2 \xi_i + \sum_{i=1}^{8} \lambda_i \log_2 \lambda_i, \tag{21}$$

where $\xi_1 = p^2\sin^2\varphi, \xi_2 = \xi_3 = p(p-1)\sin^2\varphi$, and $\xi_4 = 1 - \xi_1 - \xi_2 - \xi_3$. If $t \to \infty$, $S^{1\to 3}(\rho) = S^{2\to 3}(\rho) = 0$.

Case 4. When the slice state is assumed to be the initial state for three qubits in the dephasing noise, the total correlation clearly reads

$$S^{1\to 3}(\rho) = h(\zeta_2) + 2h(\kappa_1) + \mathrm{Tr}\rho_u \log_2 \rho_u, \tag{22}$$

where $\zeta_2 = \frac{1}{4}[2 + \sqrt{2 + p + (2-p)\cos(4\varphi) + 2p\cos(2\Theta)\sin^2(2\varphi)}]$, $\kappa_1 = \sin^2\varphi \sin^2\Theta$, and ρ_u is the matrix given in the Appendix. The 3-partite correlation needs the following expressions:

$$\begin{aligned} S(\rho \,\|\, \rho^{bc} \otimes \rho^a) &= h(\zeta_2) + h(\zeta_3) + \mathrm{Tr}\rho_u \log_2 \rho_u, \\ S(\rho \,\|\, \rho^{ac} \otimes \rho^b) &= h(\kappa_1) - \mathrm{Tr}\rho_v \log_2 \rho_v + \mathrm{Tr}\rho_u \log_2 \rho_u, \end{aligned} \tag{23}$$

where $\zeta_3 = \frac{1}{8}[4 + [9 + 3p^2 + 4(1-p^2)\cos(2\varphi) + (3+p^2)\cos(4\varphi) + 8(1-p^2)\cos(4\Theta)\sin^4\varphi + 8\cos(2\Theta)\sin^2(2\varphi)]^{\frac{1}{2}}]$, $S(\rho \,\|\, \rho^{ab} \otimes \rho^c = S(\rho \,\|\, \rho^{ac} \otimes \rho^b)$, and ρ_v is given in the Appendix.

In the case of $\Theta = \frac{\pi}{2}$, we have $\rho^a = \rho^b = \rho^c$, $\rho^{ab} = \rho^{ac} = \rho^{bc}$, and the total correlation as follows:

$$S^{1\to 3}(\rho) = 3h(\sin^2\varphi) - h(\tau), \tag{24}$$

where $\tau = \frac{1}{4}[2 + \sqrt{2 + 2p^3 + 2(1-p^3)\cos(4\varphi)}]$. The 3-partite correlation reads

$$S^{2\to 3}(\rho) = 2h(\sin^2\varphi) - h(\tau). \tag{25}$$

According to Eq. (4), we arrive at the conclusion that 2-partite correlation $S^2(\rho) = h(\sin^2\varphi)$ is constant, which is dependent on the initial condition φ in Eq. (12). This remarkable property is owing to the generalized GHZ state and the dephasing noise. Such correlation is freezing forever, which differs in a very drastic manner from the correlation induced via measurements,[15] where such correlation is freezing within a finite-time interval. Those and earlier findings[15] clearly show that freezing of quantum correlations is explicitly dependent not only on the choice of correlation indicators, but also on an initial state and a noise. Such behaviors of the 2-partite correlation may be useful for quantum information tasks such as teleportation and superdense coding.[1,2,10]

If time t is infinite,

$$S^{1\to 3}(\rho) = 3h(\sin^2\varphi) - h(\tau_0), \tag{26}$$

$$S^{2\to 3}(\rho) = 2h(\sin^2\varphi) - h(\tau_0), \tag{27}$$

where $\tau_0 = \frac{1}{4}[2 + \sqrt{2 + 2\cos(4\varphi)}]$. The 2-partite correlation is still constant, given by $S^2(\rho) = h(\sin^2\varphi)$.

In order to visualize the dynamical behaviors of quantum correlations, as an example, we take parameters $\phi = \frac{\pi}{3}$, $\theta = \frac{\pi}{6}$, $\varphi = \frac{\pi}{3}$, and $\Theta = \frac{\pi}{2}$ in Eqs. (11) and (12), respectively. Figure 1 shows the evolution of 2-partite correlation S^2 (solid line), 3-partite correlation $S^{2\to 3}$ (dashed line), and weaving W (dotted line) with γt for above four cases. Indeed, 2-partite correlation is forever freezing for the generalized GHZ state in the dephasing noise, shown in Fig. 1 (Case 4). It is seen that when γt increases, those three quantities in the amplitude-damping noise asymptotically decrease to zero, while they decrease to the constants that have been respectively given by Eqs. (17), (24), and (25) in the dephasing noise, which is saturated. Such a

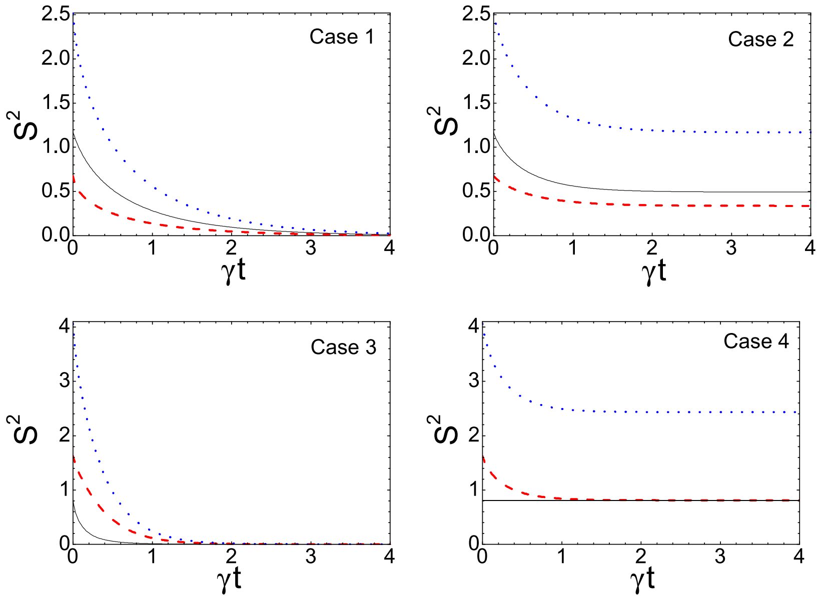

Fig. 1. (Color online) Evolution of 2-partite correlation S^2(solid line), 3-partite correlation $S^{2\to 3}$(dotted line), and weaving W(dashed line) for initial state $|\psi\rangle_w$ with $\phi = \frac{\pi}{3}, \theta = \frac{\pi}{6}$ for Cases 1 and 2, and $|\psi\rangle_s$ with $\varphi = \frac{\pi}{3}, \Theta = \frac{\pi}{2}$ for Cases 3 and 4, where two noisy environments are used.

saturation is desirable so that quantum information processing and quantum computing have enough time to be performed. For the amplitude-damping noise, those correlations can be protected via weak measurement and quantum measurement reversal.[11,19] It is worthwhile to remark that those constant correlations are classical, since in the case of $t \to \infty$ the initially-correlated state in a noisy environment reduces to the classical. It is not surprising that the studied measures describe both quantum and classical correlations.[18] Using the ideas of Ref. 13, one can define indicators of quantum and classical correlations, respectively. In contrast to previous findings, focusing on quantum entanglement,[7] we mention that entanglement can exhibit a sudden death while the studied correlations do not have such phenomenon, since they are different measures of quantum correlation. Nevertheless, we believe that the present work can trigger more theoretical and experimental investigations to examine the properties of such correlation,[18] quantum discords,[12,13,16,17] entanglement,[5–8] and nonlocality[9–11] for multipartite states, so as to establish a proper and genuine measure of quantum correlations.

4. Conclusion

In summary, we have explicitly presented the dynamical forms of 2-partite correlation, 3-partite correlation, and weaving for two kinds of three-qubit states in two noisy environments. It is demonstrated that those three quantities asymptotically decrease to zero in an amplitude-damping noise, while they decrease to saturated values in a dephasing noise. Remarkably, the 2-partite correlation is forever frozen for the generalized GHZ state in a dephasing noise. Such saturated or frozen correlations sustain long enough so that information processing tasks could be accomplished.

It is highly desirable to consider the possible application of the studied correlations in quantum coding and teleportation.[1,2,10] Since quantum correlation is always fragile in an environment, it is also desirable to discuss the preservation of correlation with weak measurement and measurement reversal.[11,19] It is possible to explore the behaviors of other measures of quantum correlations[5–13,16–18] for high-dimensional states in multipartite systems,[26] and the results will be discussed elsewhere.

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Appendix

In this Appendix, we list four matrices to be needed in Case 2, which are

$$\rho_d = \begin{pmatrix} \chi_1 & \frac{p}{2}\sin\phi\sin(2\theta) & \frac{p}{2}\cos\phi\sin(2\theta) \\ \frac{p}{2}\sin\phi\sin(2\theta) & \chi_2 & \frac{p}{2}\sin(2\phi)\sin^2\theta \\ \frac{p}{2}\cos\phi\sin(2\theta) & \frac{p}{2}\sin(2\phi)\sin^2\theta & \chi_3 \end{pmatrix},$$

$$\rho_e = \begin{pmatrix} \chi_2 & \frac{p}{2}\sin(2\phi)\sin^2\theta & 0 \\ \frac{p}{2}\sin(2\phi)\sin^2\theta & \chi_3 & 0 \\ 0 & 0 & \chi_1 \end{pmatrix},$$

$$\rho_f = \begin{pmatrix} \chi_1 & \frac{p}{2}\cos\phi\sin(2\theta) & 0 \\ \frac{p}{2}\cos\phi\sin(2\theta) & \chi_3 & 0 \\ 0 & 0 & \chi_2 \end{pmatrix},$$

$$\rho_g = \begin{pmatrix} \chi_1 & \frac{p}{2}\sin\phi\sin(2\theta) & 0 \\ \frac{p}{2}\sin\phi\sin(2\theta) & \chi_2 & 0 \\ 0 & 0 & \chi_3 \end{pmatrix},$$

where $p = e^{-\gamma t}$, $\chi_1 = \cos^2\theta, \chi_2 = \sin^2\phi\sin^2\theta$ and $\chi_3 = \cos^2\phi\sin^2\theta$.

Those matrices in Cases 3 and 4 are given by

$$\rho_m = \begin{pmatrix} p^3\kappa_1 & \frac{p^2}{2}\sin^2\varphi\sin(2\Theta) & 0 & \frac{1}{2}p^{\frac{3}{2}}\sin(2\varphi)\sin\Theta \\ \frac{p^2}{2}\sin^2\varphi\sin(2\Theta) & p\kappa_2 + p(p-1)^2\kappa_1 & 0 & \frac{\sqrt{p}}{2}\sin(2\varphi)\cos\Theta \\ 0 & 0 & (1-p)p^2\kappa_1 & \frac{1}{2}p(1-p)\sin^2\varphi\sin(2\Theta) \\ \frac{1}{2}p^{\frac{3}{2}}\sin(2\varphi)\sin\Theta & \frac{\sqrt{p}}{2}\sin(2\varphi)\cos\Theta & \frac{1}{2}p(1-p)\sin^2\varphi\sin(2\Theta) & \kappa_3 + (1-p)\kappa_2 + (1-p)^3\kappa_1 \end{pmatrix},$$

$$\rho_n = \begin{pmatrix} p^2\kappa_1 & 0 & 0 & \frac{p}{2}\sin(2\Theta)\sin^2\varphi \\ 0 & p(1-p)\kappa_1 & 0 & 0 \\ 0 & 0 & p(1-p)\kappa_1 & 0 \\ \frac{p}{2}\sin(2\Theta)\sin^2\varphi & 0 & 0 & \kappa_3 + \kappa_2 + (p-1)^2\kappa_1 \end{pmatrix},$$

$$\rho_r = \begin{pmatrix} p^2\kappa_1 & 0 & 0 & 0 \\ 0 & \frac{p}{2}[2-p+p\cos(2\Theta)]\sin^2) & 0 & \frac{\sqrt{p}}{2}\cos\Theta\sin(2\varphi) \\ 0 & 0 & p(1-p)\kappa_1 & 0 \\ 0 & \frac{\sqrt{p}}{2}\cos\Theta\sin(2\varphi) & 0 & \kappa_3 + \frac{1}{2}(1-p)[2-p+p\cos(2\Theta)]\sin^2\varphi \end{pmatrix},$$

$$\rho_u = \begin{pmatrix} \kappa_1 & \frac{p}{2}\sin(2\Theta)\sin^2\varphi & \frac{1}{2}p^{\frac{3}{2}}\sin\Theta\sin(2\varphi) \\ \frac{p}{2}\sin(2\Theta)\sin^2\varphi & \kappa_2 & \frac{\sqrt{p}}{2}\cos\Theta\sin(2\varphi) \\ \frac{1}{2}p^{\frac{3}{2}}\sin\Theta\sin(2\varphi) & \frac{\sqrt{p}}{2}\cos\Theta\sin(2\varphi) & \kappa_3 \end{pmatrix},$$

$$\rho_v = \begin{pmatrix} \kappa_1 & 0 & 0 \\ 0 & \kappa_2 & \frac{\sqrt{p}}{2}\cos\Theta\sin(2\varphi) \\ 0 & \frac{\sqrt{p}}{2}\cos\Theta\sin(2\varphi) & \kappa_3 \end{pmatrix},$$

where $\kappa_1 = \sin^2\varphi\sin^2\Theta$, $\kappa_2 = \sin^2\varphi\cos^2\Theta$ and $\kappa_3 = \cos^2\varphi$.

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