Chapter 2

Representations

In this chapter we will discuss the most important ways of parameterizing and representing a quantum channel: (i) in terms of a bipartite quantum state, leading to the state-channel duality introduced by Jamiolkowski and Choi, (ii) as the reduced dynamics of a larger (unitarily evolving) system as expressed by the representation theorems of Kraus, Stinespring, and Neumark (for POVMs) and (iii) as a linear map represented in terms of a 'transfer matrix'. This will be supplemented with a brief discussion about normal forms.

2.1 Jamiolkowski and Choi

We saw in Prop.1.2 that complete positivity of a linear map T is equivalent to positivity of the operator $\tau := (T \otimes \mathrm{id})(|\Omega\rangle\langle\Omega|)$ which is obtained by letting T act on half of a maximally entangled state Ω. In fact, the operator τ obtained in this way encodes not only complete positivity but every property of T. This reflects the simple equivalence $\mathcal{B}(\mathcal{M}_d, \mathcal{M}_{d'}) \simeq \mathcal{M}_{dd'}$ which was certainly observed and used at various places, but in the context of quantum channels the credit goes to Choi and Jamiolkowski. $d\tau$ is often called *Choi matrix*, and if T is a trace-preserving quantum channel, then τ is called the corresponding *Jamiolkowski state*.

Proposition 2.1 (Choi-Jamiolkowski representation of maps) *The following provides a one-to-one correspondence between linear maps $T \in \mathcal{B}(\mathcal{M}_d, \mathcal{M}_{d'})$ and operators $\tau \in \mathcal{B}(\mathbb{C}^{d'} \otimes \mathbb{C}^d)$:*

$$\tau = (T \otimes \mathrm{id}_d)(|\Omega\rangle\langle\Omega|), \qquad \mathrm{tr}\left[A T(B)\right] = d\,\mathrm{tr}\left[\tau A \otimes B^T\right], \tag{2.1}$$

for all $A \in \mathcal{M}_{d'}, B \in \mathcal{M}_d$ and $\Omega \in \mathbb{C}^d \otimes \mathbb{C}^d$ being a maximally entangled state as in Eq.(1.11). The maps $T \mapsto \tau$ and $\tau \mapsto T$ defined by (2.1) are mutual inverses and lead to the following correspondences:

- Hermiticity: $\tau = \tau^\dagger$ *iff* $T(B^\dagger) = T(B)^\dagger$ *for all* $B \in \mathcal{M}_d$,[1]
- Complete positivity: T *is completely positive iff* $\tau \geq 0$,
- Doubly-stochastic: $T(\mathbb{1}) \propto \mathbb{1}$ *and* $T^*(\mathbb{1}) \propto \mathbb{1}$ *iff* $\mathrm{tr}_A[\tau] \propto \mathbb{1}$ *and* $\mathrm{tr}_B[\tau] \propto \mathbb{1}$.
- Unitality: $T(\mathbb{1}) = \mathbb{1}$ *iff* $\mathrm{tr}_B[\tau] = \mathbb{1}_{d'}/d$.
- Preservation of the trace: $\mathrm{tr}_A[\tau] = T^*(\mathbb{1})^T/d$, *i.e.*, $T^*(\mathbb{1}) = \mathbb{1}$ *iff* $\mathrm{tr}_A[\tau] = \mathbb{1}_d/d$.
- Normalization: $\mathrm{tr}\,[\tau] = \mathrm{tr}\,[T^*(\mathbb{1})]\,/d$.

Proof The complete positivity correspondence was proven in Prop.1.2 and the other equivalences follow from Eqs.(2.1) by direct inspection. So, the only piece to prove is that the two relations in (2.1) are mutual inverses which then establishes the claimed one-to-one correspondence. To this end let us start with the rightmost expression in (2.1) and insert $\tau = (T \otimes \mathrm{id})(|\Omega\rangle\langle\Omega|)$:

$$d\ \mathrm{tr}\left[\tau A \otimes B^T\right] = \mathrm{tr}\left[\mathbb{F}^{T_B} T^*(A) \otimes B^T\right] \tag{2.2}$$
$$= \mathrm{tr}\left[\mathbb{F}\ T^*(A) \otimes B\right] = \mathrm{tr}\left[A T(B)\right], \tag{2.3}$$

where we have used the basic tools from Example 1.2, in particular that $d|\Omega\rangle\langle\Omega| = \mathbb{F}^{T_B}$. Eqs.(2.2,2.3) show that indeed $T \to \tau \to T$. So the two maps in (2.1) are mutual inverses if $T \to \tau$ is surjective. This is, however, easily seen by decomposing τ into a linear combination of rank-one operators $|\psi\rangle\langle\psi'|$ and using $|\psi\rangle = (X \otimes \mathbb{1})|\Omega\rangle$ from Eq.(1.13). □

When writing out the mapping $\tau \to T$ from Eq.(2.1) in terms of a product basis we get

$$T(B) = d \sum_{ijkl} \langle ij|\tau|kl\rangle\ |i\rangle\langle j|\ B\ |l\rangle\langle k|. \tag{2.4}$$

If T is a quantum channel in the Schrödinger picture, then τ is a density operator with reduced density matrix $\tau_B = \mathbb{1}/d$. This means that the set of quantum channels corresponds one-to-one to the set of bipartite quantum state which have one reduced density matrix maximally mixed. By replacing Ω in the construction of τ by any other pure state we can easily establish a similar *state-channel-duality* with respect to states with different reduced density matrices. Such a correspondence will be one-to-one iff the respective reduced state has full rank.

If d is countable infinite then Ω looses its meaning. However, we can partly restore the above correspondence by using either its unnormalized counterpart $\sum_{ij} |ii\rangle\langle jj|$ or any non-maximally entangled state which has reduced density matrix with full rank instead.

The correspondence between T and τ enables us to show that every linear map admits a decomposition into at most four completely positive maps:

[1] Note that this is in turn equivalent to T being a Hermiticity preserving map, i.e., $T(X) = T(X)^\dagger$ for all $X = X^\dagger$.

Proposition 2.2 (Decomposition into completely positive maps) *Every linear map $T \in \mathcal{B}(\mathcal{M}_d, \mathcal{M}_{d'})$ can be written as a complex linear combination of four completely positive maps. If T is Hermitian (i.e., $T(B^\dagger) = T(B)^\dagger$ for all B), it can be written as a real linear combination of two of them.*

Proof We use that the mapping $T \leftrightarrow \tau$ in Prop.2.1 is one-to-one and linear so that it suffices to decompose τ. On this level the existence of such a decomposition is rather obvious: we can always decompose τ into a Hermitian and anti-Hermitian part and each of them into a positive and negative part. More explicitly, $\tau = (\tau+\tau^\dagger)/2 + i(i\tau^\dagger - i\tau)/2$ provides a complex linear combination of two Hermitian parts and each of the latter can, by invoking the spectral decomposition, be written as a real linear combination of two positive semi-definite matrices. $\square$

As a first simple application of this (extremely useful and later extensively used) correspondence we show the following:

Proposition 2.3 (No information without disturbance) *Consider an instrument represented by a set of completely positive maps $\{T_\alpha : \mathcal{M}_d \to \mathcal{M}_d\}$. If there is no disturbance on average, i.e., $T = \sum_\alpha T_\alpha$ satisfies $T = \mathrm{id}$, then $T_\alpha \propto \mathrm{id}$ for every α and the probability for obtaining an outcome α (given by $\mathrm{tr}\,[T_\alpha(\rho)]$) is independent of the input ρ (hence, no information gain).*

Proof On the level of the Jamiolkowski states the decomposition $\mathrm{id} = \sum_\alpha T_\alpha$ reads $|\Omega\rangle\langle\Omega| = \sum_\alpha \tau_\alpha$. Since $\tau_\alpha \geq 0$ (due to complete positivity of T_α) this corresponds to a convex decomposition of the density operator for the maximally entangled state. Since the latter is pure there is only the trivial decomposition, so that $\tau_\alpha = c_\alpha |\Omega\rangle\langle\Omega|$ for some constants $c_\alpha \geq 0$. Consequently, $T_\alpha = c_\alpha \mathrm{id}$ so that $\mathrm{tr}\,[T_\alpha(\rho)] = c_\alpha$ independent of ρ. $\square$

Implementation via teleportation If T is a quantum channel then its Jamiolkowski state τ can operationally be obtained by letting the channel act on a maximally entangled state. What about the converse? If τ is given, does this help us to implement T as an action on an arbitrary input ρ? The answer to this involves some form of *teleportation*: assume that the bipartite state τ is shared by two parties, Alice and Bob, so that Bob has the maximally mixed reduced state. Suppose that Bob has an additional state ρ and that he performs a measurement on his composite system using a POVM which contains the maximally entangled state $\omega := |\Omega\rangle\langle\Omega|$ as an effect operator. We claim now that Alice's state is given by $T(\rho)$ whenever Bob has obtained a measurement outcome corresponding to ω. In order to show this denote Alice's reduce density matrix (in case of Bob's success) by ρ_A, the success probability by p and compute the expectation value with an arbitrary operator A:

$$
\begin{aligned}
p\,\mathrm{tr}\,[A\rho_A] &= \mathrm{tr}\,[(\tau\otimes\rho)(A\otimes\omega)] && (2.5)\\
&= \mathrm{tr}\left[(\tau\otimes\rho^T)(A\otimes\mathbb{F})\right]/d && (2.6)\\
&= \mathrm{tr}\left[\tau(A\otimes\rho^T)\right]/d \;=\; \mathrm{tr}\,[AT(\rho)]/d^2, && (2.7)
\end{aligned}
$$

where we used the basic ingredients from Example1.2 again and, in the last step, the r.h.s. of Eq.(2.1). This shows that the described protocol is indeed successful with probability $p = 1/d^2$. In some cases Alice and Bob can, however, do much better: assume that there is a set of local unitaries $\{V_i \otimes U_i\}_{i=1..N}$ w.r.t. which $\tau = (V_i \otimes U_i)\tau(V_i \otimes U_i)^\dagger$ is invariant and which are orthogonal in the sense that $\mathrm{tr}\left[U_i U_j^\dagger\right] = d\delta_{ij}$. Due to the latter condition Bob can use a POVM which contains all $(\mathbb{1} \otimes U_i)^\dagger \omega (\mathbb{1} \otimes U_i)$ as effect operators. When Bob obtains one of the corresponding outcomes i and he reports this to Alice, she can 'undo' the action of the unitary by applying V_i with the consequence that $\rho_A = T(\rho)$ with probability $p = N/d^2$. So if the U_i's form a basis (i.e., $N = d^2$) the two protagonists can implement T with unit probability by 'teleporting' through τ. If $T = \mathrm{id}$ and therefore $\tau = |\Omega\rangle\langle\Omega|$ this reduces to the standard protocol for entanglement-assisted teleportation.

Note that by linearity the implemented action of T does not only hold for uncorrelated states but it also works properly if ρ is part of a larger system.

Problem 6 (Implementation via teleportation) *Suppose Alice and Bob are given the Jamiolkowski state τ corresponding to a given quantum channel T. Which is the largest probability at which Bob can teleport an unknown state ρ to Alice (using τ, local operations and classical communication as resource) so that Alice ends up with the state $T(\rho)$?*

2.2 Kraus, Stinespring and Neumark

The Choi-Jamiolkowski state-channel duality allows us to translate between properties of bibipartite states and quantum channels. One immediate implication is a more specific and very useful representation of quantum channels which corresponds, on the level of the Jamiolkowski state, to a convex (or spectral) decomposition into rank-one operators:

Theorem 2.1 (Kraus representation) *A linear map $T \in \mathcal{B}(\mathcal{M}_d, \mathcal{M}_{d'})$ is completely positive iff it admits a representation of the form*

$$T(A) = \sum_{j=1}^{r} K_j A K_j^\dagger. \tag{2.8}$$

This decomposition has the following properties:

1. *Normalization: T is trace preserving iff $\sum_j K_j^\dagger K_j = \mathbb{1}$ and unital iff $\sum_j K_j K_j^\dagger = \mathbb{1}$.*

2. Kraus rank:[2] *The minimal number of* Kraus operators $\{K_j \in \mathcal{B}(\mathbb{C}^d, \mathbb{C}^{d'})\}_{j=1..r}$ *is $r = \mathrm{rank}(\tau) \leq dd'$.*

[2]We call $r = \mathrm{rank}(\tau)$ *Kraus rank* or *Choi rank* in order not to confuse it with the rank of T as a linear map. Take for instance $T = \mathrm{id}$ the ideal channel. As this is obviously invertible it has full rank as a linear map. However, its Kraus rank is $r = 1$. $T(B) = \mathrm{tr}[B]$ instead has rank one but Kraus rank equal to d, the dimension of the input space.

3. Orthogonality: *There is always a representation with* $r = \text{rank}(\tau)$ *Hilbert-Schmidt orthogonal Kraus operators (i.e.,* $\text{tr}\left[K_i^\dagger K_j\right] \propto \delta_{ij}$*).*

4. Freedom: *Two sets of Kraus operators* $\{K_j\}$ *and* $\{\tilde{K}_l\}$ *represent the same map* T *iff there is a unitary* U *so that* $K_j = \sum_l U_{kl}\tilde{K}_l$ *(where the smaller set is padded with zeros).*

Proof Assume T is completely positive. By Prop.2.1 this is equivalent to saying that $\tau \geq 0$ which allows for a decomposition of the form

$$\tau = \sum_{j=1}^{r} |\psi_j\rangle\langle\psi_j| = \sum_{j=1}^{r}(K_j \otimes \mathbb{1})|\Omega\rangle\langle\Omega|(K_j \otimes \mathbb{1})^\dagger, \tag{2.9}$$

where the first step uses $\tau \geq 0$ and the second Eq.(1.13). Comparing the r.h.s. of Eq.(2.9) with the definition $\tau := (T \otimes \text{id})(|\Omega\rangle\langle\Omega|)$ and recalling that $T \leftrightarrow \tau$ is one-to-one leads to the desired decomposition in (2.8). It also shows that $r \geq \text{rank}(\tau)$ where equality can be achieved and if the ψ_j's are in addition chosen to be orthogonal, then the Kraus operators are orthogonal (w.r.t. the Hilbert-Schmidt inner product) as well.

Conversely, if a map is of the form in (2.8) then $\tau \geq 0$ which implies complete positivity. The conditions for unitality and preservation of the trace are straight forward. It remains to show that the freedom in the representation is precisely given by unitary linear combinations. This is a direct consequence of Eq.(2.9) and the subsequent proposition. □

Proposition 2.4 (Equivalence of ensembles) *Two ensembles of (not necessarily normalized) vectors* $\{\psi_j\}$ *and* $\{\tilde{\psi}_l\}$ *satisfy*

$$\sum_j |\psi_j\rangle\langle\psi_j| = \sum_l |\tilde{\psi}_l\rangle\langle\tilde{\psi}_l| \tag{2.10}$$

iff there is a unitary U *such that* $|\psi_j\rangle = \sum_l U_{jl}|\tilde{\psi}_l\rangle$ *(where the smaller set is padded with zero vectors).*

Proof W.l.o.g. we may think of the mixture in (2.10) as a given density matrix ρ. From both ensembles we can construct a purification of $\rho = \text{tr}_B[|\Psi\rangle\langle\Psi|]$, of the form $|\Psi\rangle = \sum_j |\psi_j\rangle\otimes|j\rangle$ where $\{|j\rangle\}$ is an orthonormal basis which we use as well in $|\tilde{\Psi}\rangle = \sum_l |\tilde{\psi}_l\rangle\otimes|l\rangle$. It follows from the Schmidt decomposition (Prop.1.1) that two purifications differ by a unitary (or isometry if the dimensions do not match), i.e., $|\Psi\rangle = (\mathbb{1}\otimes U)|\tilde{\Psi}\rangle$. By taking the scalar product with a basis vector $\langle j|$ on the second tensor factor this leads to $|\psi_j\rangle = \sum_l U_{jl}|\tilde{\psi}_l\rangle$ which proves necessity of the condition.[3] Sufficiency follows by inspection from unitarity of U. □

[3] If the U relating the two purifications is an isometry it can always be embedded into a unitary, just by completing the set of orthonormal row or column vectors to an orthonormal basis.

Note that the number of linearly independent Kraus operators is $r = \mathrm{rank}(\tau)$ independent of the representation. The Kraus representations of a completely positive map T and its dual T^* are related via interchanging $K_j \leftrightarrow K_j^\dagger$. Moreover, $T^* = T$ iff there is a representation with Hermitian Kraus operators (a simple exercise using the unitary freedom).

Stinespring and the open-system point of view A common perspective is to regard a quantum channel as a description of 'open system dynamics' which originates from considering only parts of a unitarily evolving system. Using the Kraus representation we will now see that indeed every quantum channel can be represented as arising in this way. Before we discuss this in the Schrödinger picture (which might be the more familiar version for physicists), we will state the equivalent theorem in the Heisenberg picture (more common to operator algebraists):

Theorem 2.2 (Stinespring representation) *Let $T : \mathcal{M}_d \to \mathcal{M}_{d'}$ be a completely positive linear map. Then for every $r \geq \mathrm{rank}(\tau)$ (recall that $\mathrm{rank}(\tau) \leq dd'$) there is a $V : \mathbb{C}^d \to \mathbb{C}^{d'} \otimes \mathbb{C}^r$ such that*

$$T^*(A) = V^\dagger (A \otimes \mathbb{1}_r) V, \qquad \forall A \in \mathcal{M}_{d'}. \tag{2.11}$$

V is an isometry (i.e., $V^\dagger V = \mathbb{1}_d$) iff T is trace preserving.

Proof This is a simple consequence of the Kraus representation $T^*(A) = \sum_{j=1}^r K_j^\dagger A K_j$: construct $V := \sum_j K_j \otimes |j\rangle$ with $\{|j\rangle\}$ any orthonormal basis in $\mathbb{C}^r$. Then Eq.(2.11) holds by construction, r is at least the Kraus rank (hence minimally $r = \mathrm{rank}(\tau)$, see Thm.2.1) and $V^\dagger V = \sum_j K_j^\dagger K_j = \mathbb{1}$ just reflects the trace preserving condition. □

Note from the proof that a representation of the form (2.11) exists for T as well (and not only for T^*). However, the trace-preserving condition is easier expressed in terms of T^* where it becomes unitality.

The ancillary space $\mathbb{C}^r$ is usually called *dilation space*. In the same way in which we constructed V from the set of Kraus operators, we can obtain the latter from V as $K_j = (\mathbb{1}_{d'} \otimes \langle j|)V$. As $r = \mathrm{rank}(\tau)$ is the smallest number of Kraus operators, it is also the least possible dimension for a representation of the form (2.11). Dilations with $r = \mathrm{rank}(\tau)$ are called *minimal*. From the unitary freedom in the choice of Kraus operators (Thm.2.1) we obtain that for minimal dilations V is unique up to the obvious unitary freedom $V \to (\mathbb{1}_{d'} \otimes U)V$. All other dilations can be obtained by an isometry U.

An alternative, but equivalent, way of characterizing minimal dilations is the identity

$$\mathbb{C}^{d'} \otimes \mathbb{C}^r = \{(A \otimes \mathbb{1}_r)V|\psi\rangle \mid A \in \mathcal{M}_{d'},\ \psi \in \mathbb{C}^d\}, \tag{2.12}$$

i.e., the requirement that the set on the r.h.s. spans the entire space.

There is a natural partial order in the set of completely positive maps: we write $T_2 \geq T_1$ iff $T_2 - T_1$ is completely positive. Due to the Choi-Jamiolkowski representation (Prop.2.1) this is equivalent to $\tau_2 \geq \tau_1$. Such an order relation is reflected in the Stinespring representation in the following way:

Theorem 2.3 (Relation between ordered cp-maps) *Let $T_i : \mathcal{M}_{d'} \to \mathcal{M}_d$, $i = 1, 2$ be two completely positive linear maps with $T_2 \geq T_1$. If $V_i : \mathbb{C}^d \to \mathbb{C}^{d'} \otimes \mathbb{C}^{r_i}$ provide Stinespring representations $T_i(A) = V_i^\dagger (A \otimes \mathbb{1}_{r_i}) V_i$, then there is a contraction $C : \mathbb{C}^{r_2} \to \mathbb{C}^{r_1}$ such that $V_1 = (\mathbb{1}_{d'} \otimes C) V_2$. If V_2 belongs to a minimal dilation, then C is unique (for given V_1 and V_2).*

Proof We exploit the fact that $T_2 \geq T_1$ is equivalent to $\tau_2 \geq \tau_1$, i.e., the analogous order relation between the corresponding Choi-Jamiolkowski operators. Define $W_i := (\mathbb{1}_{r_i} \otimes \langle\Omega|)(V_i \otimes \mathbb{1}_{d'}) \in \mathcal{B}(\mathbb{C}^d \otimes \mathbb{C}^{d'}, \mathbb{C}^{r_i})$. Then for all $\psi \in \mathbb{C}^d \otimes \mathbb{C}^{d'}$

$$||W_2|\psi\rangle||^2 = \langle\psi|\tau_2|\psi\rangle \geq \langle\psi|\tau_1|\psi\rangle = ||W_1|\psi\rangle||^2. \qquad (2.13)$$

Hence, there is a contraction (meaning $C^\dagger C \leq \mathbb{1}$) $C : \mathbb{C}^{r_2} \to \mathbb{C}^{r_1}$ such that $W_1 = C W_2$. Since the map $V_i \to W_i$ is one-to-one[4] this implies $V_1 = (\mathbb{1}_{d'} \otimes C) V_2$. Moreover, if $r_2 = \text{rank}(\tau_2)$ (i.e., $\mathbb{C}^{r_2}$ is a minimal dilation space), then W_2 must be surjective so that C becomes uniquely defined. □

As a simple corollary of this result one obtains a Radon-Nikodym type theorem for instruments:

Theorem 2.4 (Radon-Nikodym for quantum instruments) *Let $\{T_i\}$ be a set of completely positive linear maps such that $\sum_i T_i = T \in \mathcal{B}(\mathcal{M}_{d'}, \mathcal{M}_d)$ with Stinespring representation $T(A) = V^\dagger (A \otimes \mathbb{1}_r) V$. Then there is a set of positive operators $P_i \in \mathcal{M}_r$ wich sum up to $\mathbb{1}_r$ such that $T_i(A) = V^\dagger (A \otimes P_i) V$.*

Now let us turn to the Schrödinger picture again and see how to represent a quantum channel from a system-plus-environment point of view:

Theorem 2.5 (Open-system representation) *Let $T : \mathcal{M}_d \to \mathcal{M}_{d'}$ be a completely positive and trace-preserving linear map. Then there is a unitary $U \in \mathcal{M}_{dd'^2}$ and a normalized vector $\varphi \in \mathbb{C}^{d'} \otimes \mathbb{C}^{d'}$ such that*

$$T(\rho) = \text{tr}_E \left[U(\rho \otimes |\varphi\rangle\langle\varphi|) U^\dagger \right], \qquad (2.14)$$

where tr_E denotes the partial trace over the first two tensor factors of the involved Hilbert space $\mathbb{C}^d \otimes \mathbb{C}^{d'} \otimes \mathbb{C}^{d'}$.

Proof By expressing Stinespring's representation theorem in the Schrödinger picture we get $T(\rho) = \text{tr}_{\mathbb{C}^r}[V \rho V^\dagger]$. Let us choose a dilation space of dimension $r = dd'$. In this way we can embed V into a unitary which acts on a tensor product and write $V = U(\mathbb{1}_d \otimes |\varphi\rangle)$ for some $\varphi \in \mathbb{C}^{d'} \otimes \mathbb{C}^{d'}$ so that Eq.(2.14) follows. □

Notice the possible departure from the dimension of the minimal dilation space in Thm.2.5: if d is not a factor of $d'\text{rank}(\tau)$, then the construction fails

[4] The inverse is $V_i = d'^2 (W_i \otimes \mathbb{1}_{d'})(\mathbb{1}_d \otimes |\Omega\rangle)$.

and we have to use a space of larger dimension (e.g., $r = dd'$ which is always possible). In this case the type of freedom in the representation is less obvious.

The physical interpretation of Thm.2.5 is clear: we couple a system to an environment, which is initially in a state φ, let them evolve jointly according to a unitary U and finally disregard (i.e., trace out) environmental degrees of freedom. This way of representing quantum channels nicely reminds us of the fundamental assumption used when we express an evolution in terms of a linear, completely positive, and trace preserving map: the initial state of the system has to be independent of the 'environment'—in other words T itself must not depend on the input ρ.

Let us finally revisit Thm.2.4 from the system-plus-environment point of view:

Proposition 2.5 (Environment induced instruments) *Let $T : \mathcal{M}_d \to \mathcal{M}_{d'}$ be a completely positive and trace-preserving linear with a system-plus-environment representation as in Eq.(2.14). For every decomposition of the form $T = \sum_\alpha T_i$ into completely positive maps T_i there is a POVM $\{P_i \in \mathcal{M}_{dd'}\}$ such that*

$$T_i(\rho) = \mathrm{tr}_E\big[(P_i \otimes \mathbb{1}_{d'})U(\rho \otimes |\varphi\rangle\langle\varphi|)U^\dagger\big]. \tag{2.15}$$

The Kraus rank k_i of T_i satisfies $k_i \leq \mathrm{rank}(P_i)$.

Proof .. wouldn't be necessary, as this can essentially be considered a rewriting of Thm.2.4. However, we want to provide a simple alternative proof which is not based on Thm.2.4. Consider the Jamiolkowski state τ of T and its purification $|\psi\rangle := (\mathbb{1}_d \otimes U)(|\Omega\rangle \otimes |\varphi\rangle)$ so that $\tau = \mathrm{tr}_E[|\psi\rangle\langle\psi|]$. The decomposition of T corresponds to a decomposition $\tau = \sum_i \tau_i$ on the level of Choi-Jamiolkowski operators. As this is a convex decomposition of the reduced density matrix of ψ we can invoke the quantum steering result in Prop.1.3 (together with the one-to-one correspondence between T_i and τ_i) in order to arrive at Eq.(2.15). From this the bound on the Kraus rank follows by utilizing $k_i = \mathrm{rank}(\tau_i)$. □

Neumark's representation of POVMs In the same way as Stinespring's theorem enables us to regard a quantum channel as part of larger, unitarily evolving system, one can represent a POVM as a von Neumann measurement performed in an extended space. To simplify matters let us assume that the effect operators all have rank one (which can always be achieved by spectral decomposition).

Theorem 2.6 (Neumark's theorem) [5] *Consider a POVM with effect operators $\{|\psi_i\rangle\langle\psi_i|\}_{i=1..n}$ acting on $\mathbb{C}^d$, i.e., $\sum_i |\psi_i\rangle\langle\psi_i| = \mathbb{1}_d$. There exists an orthonormal basis $\{\phi_i\}_{i=1..n}$ in $\mathbb{C}^n \supseteq \mathbb{C}^d$ such that each ψ_i is the restriction of ϕ_i to $\mathbb{C}^d$.*

[5] As always we restrict ourselves to the case of finite dimensions and finite outcomes. Neumark's theorem holds, however, similarly if the set of measurement outcomes is characterized by any regular, positive, $\mathcal{B}(\mathcal{H})$-valued measure on a compact Hausdorff space. The general theorem can be viewed as a corollary of Stinespring's representation theorem.

PROOF Take an orthonormal basis $\{|j\rangle \in \mathbb{C}^n\}_{j=1..n}$ so that $\mathbb{C}^d$ is embedded as a subspace spanned by the vectors $|1\rangle, \ldots, |d\rangle$. Then the matrix $\Psi \in \mathcal{M}_{n,d}$ with components $\Psi_{i,j} := \langle j|\psi_i\rangle$ satisfies $\Psi^\dagger \Psi = \mathbb{1}_d$. That is, Ψ is an isometry and can be extended to a unitary in $\mathcal{M}_n$ by completing the set of orthonormal column vectors to an orthonormal basis. The vectors ϕ_j are then obtained from the extended $\Psi \in \mathcal{M}_n$ by $\langle j|\phi_i\rangle := \Psi_{i,j}$. □

Note that in order to implement a general POVM with effect operators $\{P_i\}_{i=1..n}$ as von Neumann measurement the construction leading to Thm.2.6 requires a space of dimension $\sum_i \mathrm{rank}(P_i)$. Sometimes we can, however, do better by using rank-one projections which appear in the decomposition of more than one P_i.

Problem 7 (Minimal Neumark dilation) *Which is the minimal dimension required to implement a given POVM as von Neumann measurement in an extended space if post-processing of the measurement outcomes (such as introducing randomness) is allowed?*

Neumark's theorem shows that one can always implement a POVM (at least in principle) as a von Neumann/sharp measurement in a larger space. That is, we embed the system which is, say, described by a density matrix ρ into a larger space where the additional levels are just not populated so that the state becomes a direct sum (i.e., block diagonal) $\rho \oplus 0$ and we obtain the identity

$$\langle\psi_i|\rho|\psi_i\rangle = \langle\phi_i|\rho \oplus 0|\phi_i\rangle. \tag{2.16}$$

For actual practical implementations a tensor product would be more convenient than a direct sum structure, since this would allow a realization by coupling the system to an ancillary system. Such an 'ancilla representation' of a POVM can easily be obtained from Neumark's theorem by further enlarging the space (if necessary). Let us extend the space until the dimension is a multiple of d, say $dd' \geq n$. As the zero-block now has dimension $(d'-1)d$, this enables us to write $\rho \oplus 0 = \rho \otimes |1\rangle\langle 1|$ (with $|1\rangle$ an element of the computational basis in $\mathbb{C}^{d'}$). Embedding the vectors ϕ_i into this space, such that $|\Phi_i\rangle := |\phi_i\rangle \oplus 0$ becomes part of an orthonormal basis in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$, we get

$$\langle\psi_i|\rho|\psi_i\rangle = \langle\Phi_i|(\rho \otimes |1\rangle\langle 1|)|\Phi_i\rangle. \tag{2.17}$$

2.3 Linear maps as matrices

The last representation, which we discuss, is the simplest one—linear maps represented as matrices in a way that concatenation of maps corresponds to multiplication of the respective matrices.

To this end, we use that the space $\mathcal{M}_{d,d'}(\mathbb{C})$ of complex valued $d \times d'$ matrices is a vector space, i.e., it is closed under linear combinations and scalar multiplication. As a vector space $\mathcal{M}_{d,d'}(\mathbb{C})$ is isomorphic to $\mathbb{C}^{dd'}$. We can upgrade it to a Hilbert space by equipping it with a scalar product. A common choice is

$$\langle A, B\rangle := \mathrm{tr}\left[PA^\dagger B\right], \qquad A, B \in \mathcal{M}_{d,d'}, \tag{2.18}$$

where $P \in \mathcal{M}_{d'}$ is any positive definite operator. Each set of dd' operators which is orthonormal w.r.t. this scalar product ($\langle A_i, A_j \rangle = \delta_{ij}, i, j = 1, \ldots, dd'$) forms a basis of $\mathcal{M}_{d,d'}$. Such orthonormal bases lead to simple completely positive maps of the form

$$\sum_{i=1}^{dd'} A_i^\dagger \rho A_i = \mathrm{tr}\,[\rho]\, P^{-1}, \qquad \forall \rho \in \mathcal{M}_d. \tag{2.19}$$

If $P = \mathbb{1}$ in Eq.(2.18) the scalar product is called *Hilbert-Schmidt inner product* and the respective space *Hilbert-Schmidt Hilbert space*. Unless otherwise stated we will in the following use the Hilbert-Schmidt inner product.

The space of linear maps of the form $T : \mathcal{M}_d \to \mathcal{M}_{d'}$ is isomorphic to $\mathcal{M}_{d'^2,d^2}$. A concrete matrix representation, which we denote by $\hat{T} \in \mathcal{M}_{d'^2,d^2}$ and refer to as *transfer matrix*, is then obtained in given orthonormal bases $\{G_\beta \in \mathcal{M}_d\}_{\beta=1..d^2}$ and $\{F_\alpha \in \mathcal{M}_{d'}\}_{\alpha=1..d'^2}$ via

$$\hat{T}_{\alpha,\beta} := \mathrm{tr}\left[F_\alpha^\dagger T(G_\beta)\right]. \tag{2.20}$$

By construction a concatenation of maps, e.g., $T''(\rho) := T'\big(T(\rho)\big)$, then corresponds to a matrix multiplication $\hat{T}'' = \hat{T}'\hat{T}$. With some abuse of notation we usually write the composed map as a formal product $T'' = T'T$ as well.

From Eq.(2.20) we can see that the matrix representation of T^* is given by the Hermitian conjugate matrix $\hat{T}^\dagger$ if T is a Hermitian map (i.e., $T(X^\dagger) = T(X)^\dagger$) and by the transpose matrix $\hat{T}^T$ if the bases are Hermitian.

Proposition 2.6 (Self-dual channels) *Let $T : \mathcal{M}_d \to \mathcal{M}_d$ be a completely positive linear map. The following are equivalent:*

1. *$T = T^*$,*
2. *$\hat{T} = \hat{T}^\dagger$ if identical bases $F_\alpha = G_\alpha$ are chosen,*
3. *there exists a set of Hermitian Kraus operators for T.*

Proof $1 \leftrightarrow 2$ follows from direct inspection. Also $3 \to 1$ is obvious as the Kraus operators of T and T^* differ by Hermitian conjugation. In order to show $1 \to 3$ we can thus write $T(X) = \sum_j K_j X K_j^\dagger = \frac{1}{2}\sum_j (K_j X K_j^\dagger + K_j^\dagger X K_j)$. Every pair $(K_j, K_j^\dagger)/\sqrt{2}$ of Kraus operators in the last representation can, however, be transformed into a pair of Hermitian operators $(K_j + K_j^\dagger, i(K_j - K_j^\dagger))/2$ by a unitary linear combination. This provides a different representation of the same map (see Prop.2.1, item 4.) by Hermitian Kraus operators. □

Let us in the following have a closer look at different operator bases. The simplest basis which is orthonormal w.r.t. the Hilbert-Schmidt inner product (i.e., $\mathrm{tr}\left[G_\beta^\dagger G_{\beta'}\right] = \delta_{\beta,\beta'}$) is given by *matrix units* of the form $G_\beta = |k\rangle\langle l|$ where β is identified with the index pair $(k, l) =: \beta$ and $k, l = 1, \ldots, d$. Making this identification explicit leads to the isomorphism $\mathcal{M}_d \simeq \mathbb{C}^{d^2}$ via the correspondence $|k\rangle\langle l| \leftrightarrow |k,l\rangle$. When applied to an arbitrary linear map represented as $T(\cdot) = \sum_j L_j \cdot R_j$ this gives

$$\hat{T} = \sum_j L_j \otimes R_j^T. \tag{2.21}$$

That is, in particular for completely positive maps with Kraus representation $T(\cdot) = \sum_j K_j \cdot K_j^\dagger$ we obtain the simple expression $\hat{T} = \sum_j K_j \otimes \bar{K}_j$, which is independent of the particular choice of Kraus operators (as the unitaries, relating the different sets of Kraus operators, cancel). Another advantage of using matrix units as a basis is a simple mapping between $\hat{T}$ and the Choi-Jamiolkowski operator $\tau = (T \otimes \mathrm{id})(|\Omega\rangle\langle\Omega|)$ for arbitrary linear maps:

$$\hat{T} = d\,\tau^\Gamma, \tag{2.22}$$

where $\tau \mapsto \tau^\Gamma$ is an involution defined by $\langle m,n|\tau^\Gamma|k,l\rangle := \langle m,k|\tau|n,l\rangle$.

More operator bases Despite the useful relations in Eqs.(2.21,2.22) it is sometimes advantageous to use operator bases other than matrix units. The most common ones can be regarded as generalization of the 2×2 Pauli matrices (incl. identity matrix) to higher dimensions. Since the two-dimensional case is rather special one has to drop some of the nice properties—for instance generalize to Hermitian *or* unitary bases.

Problem 8 (Generalizing Pauli matrices) *Construct Hilbert-Schmidt orthogonal bases of operators in $\mathcal{M}_d$ (different from tensor products of Pauli matrices) which are unitary* and *Hermitian.*

Hermitian operator bases: The set of Hermitian matrices forms a real vector space. Thus, an orthonormal basis of Hermitian operators helps to have this property nicely reflected in a concrete representation. A simple example for such a basis can be constructed by embedding normalized Pauli matrices $\sigma_x/\sqrt{2}$ and $\sigma_y/\sqrt{2}$ as 2×2 principal submatrices into $\mathcal{M}_d$ (so that only two entries are non-zero). This leads to $d^2 - d$ orthonormal matrices. In order to complete the basis we add d diagonal matrices which we construct from any orthogonal matrix $M \in \mathcal{M}_d(\mathbb{R})$ by choosing the diagonal entries of the k'th diagonal matrix equal to the k'th column vector of M. This yields a complete Hilbert-Schmidt orthonormal basis. If in addition one column of M leads to $\mathbb{1}_d$, then the other $d^2 - 1$ matrices are traceless and generate the Lie-algebra $su(d)$, i.e., they provide a complete set of infinitesimal generators of $SU(d)$. A popular example of this kind are the 8 Gell-Mann matrices in $\mathcal{M}_3$ (and, of course, the 3 Pauli matrices in $\mathcal{M}_2$).

Unitary operator bases play an important role in quantum information theory as they correspond one-to-one to bases of maximally entangled states (see example 1.2). In $\mathcal{M}_2$ all orthogonal bases of unitaries are of the form

$$U_k = e^{i\varphi_k} V_1 \sigma_k V_2, \qquad k = 0, \ldots, 3, \tag{2.23}$$

where V_1 and V_2 are unitaries and σ_k denote the Pauli matrices (with $\sigma_0 = \mathbb{1}$). That is, in $\mathcal{M}_2$ all orthogonal bases of unitaries are essentially equivalent to the set of Pauli matrices. In higher dimensions a similar characterization of all unitary operator bases is not known. A priori, it is quite remarkable that such orthogonal bases exist in every dimension. Note that this depends crucially on the chosen scalar product: if we choose the A_i's in Eq.(2.19) proportional to unitaries (and thus $d = d'$), then $\rho = \mathbb{1}$ shows that they can only form an orthogonal basis if $P \propto \mathbb{1}$, corresponding to the Hilbert-Schmidt scalar product.

Example 2.1 (Discrete Weyl system and GHZ bases) *One of the most important unitary operator bases and also a very inspiring example regarding the construction of others comes from a discrete version of the Heisenberg-Weyl group. Consider a set $\{U_{kl} \in \mathcal{M}_d\}_{k,l=0..d-1}$ of d^2 unitaries defined by*

$$U_{kl} := \sum_{r=0}^{d-1} \eta^{rl} |k+r\rangle\langle r|, \quad \eta := e^{\frac{2\pi i}{d}}, \tag{2.24}$$

where addition inside the ket is modulo d. This set has the following nice properties:

- *It forms a basis of operators in $\mathcal{M}_d$ which is orthogonal w.r.t. the Hilbert-Schmidt scalar product, i.e., $\mathrm{tr}\left[U_{ij}^\dagger U_{kl}\right] = d\ \delta_{ik}\delta_{jl}$. Since $U_{00} = \mathbb{1}$ we have in particular $\mathrm{tr}\,[U_{kl}] = 0$ for all $(k,l) \neq (0,0)$.*
- *It is a* discrete Weyl system *since $U_{ij}U_{kl} = \eta^{jk} U_{i+k,j+l}$ (with addition modulo d). Thus $U_{kl}^{-1} = \eta^{kl} U_{-k,-l}$.*
- *The set is generated by U_{01} and U_{10} via $U_{kl} = (U_{10})^k (U_{01})^l$.*
- *If d is odd, then $U_{kl} \in SU(d)$. For d even $\det U_{kl} = (-1)^{k+l}$.*
- *For $d = 2$ the set reduces to the set of Pauli matrices with identity, i.e., $(\sigma_x, \sigma_y, \sigma_z) = (U_{10}, iU_{11}, U_{01})$.*
- *The group generated by the unitaries U_{kl} is isomorphic to the discrete Heisenberg-Weyl group*

$$\left\{ \begin{pmatrix} 1 & l & m \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \Big|\, k,l,m \in \mathbb{Z}_d \right\}. \tag{2.25}$$

Among the many applications of this set we will briefly discuss the construction of bases of entangled states. As indicated in example 1.2, $(U_{kl} \otimes \mathbb{1})|\Omega\rangle$ is an orthonormal basis of maximally entangled states in $\mathbb{C}^d \otimes \mathbb{C}^d$. For more than two, say n, parties a similar construction can be made by exploiting some group structure. Consider the goup of local unitaries

$$G = \left\{ \bigotimes_{k=1}^{n} U_{i,j_k} \Big| \sum_{k=1}^{n} j_k \mathrm{mod}\, d = 0 \right\}, \tag{2.26}$$

where $i \in \mathbb{Z}_d$, $j \in \mathbb{Z}_d^n$ and one of the components of j, say j_1, depends on the others by the additional constraint. Utilizing the above properties of the set $\{U_{kl}\}$ it is readily *verified that G is an abelian Group with d^n elements and that it spans its own commutant G'. Since the algebra G' is therefore abelian as well, it forms a simplex, i.e., each element in G' has a unique convex decomposition into minimal projections. The latter turn out to be one-dimensional and can be parametrized by $\omega \in \mathbb{Z}_d^n$ such that they correspond to vectors of the form*

$$|\Psi_\omega\rangle := \frac{1}{\sqrt{d}} \sum_{l=0}^{d-1} \eta^{l\omega_1} \bigotimes_{k=1}^{n} |\omega_k + l\rangle. \tag{2.27}$$

These vectors form an orthonormal basis of the entire Hilbert space $\mathbb{C}^{d\otimes n}$ and they can all be obtained by local unitaries from the GHZ state $|\Psi_0\rangle = \sum_k |k\rangle^{\otimes n}/\sqrt{d}$. For $n = d = 2$ we get $G = \{\mathbb{1}, \sigma_x \otimes \sigma_x, -\sigma_y \otimes \sigma_y, \sigma_z \otimes \sigma_z\}$ (which is the Klein four group[6]*) and the states in Eq.(2.27) become the four* Bell states.

Example 2.1 exhibits two properties which suggest possible starting points for the construction of other unitary operator bases: (i) the group structure of the set, and (ii) the property that every element is generated by two elementary actions: shifting the computational basis and multiplying with phases. The latter approach leads to a fairly general construction scheme. Bases of unitaries obtained in this way are said to be of *shift-and-multiply* type, and they are of the general form

$$U_{kl} = \sum_{r=0}^{d-1} H_{kr}^{(l)} \; |\Lambda_{lr}\rangle\langle r|, \tag{2.28}$$

where $\{H^{(l)} \in \mathcal{M}_d\}$ is a set of (not necessarily distinct) complex Hadamard matrices and $\Lambda \in \mathcal{M}_d(\mathbb{Z}_d)$ is a Latin square. That is, $|H_{kr}^{(l)}| = 1$ are phases so that $H^{(l)} H^{(l)\dagger} = d\mathbb{1}$, and Λ is such that each row and column contains every number from 0 to $d-1$ exactly once. Obviously, the set in Eq.(2.24) is just the simplest example of a shift-and-multiply basis. Its group property (the fact, that the elements form a group of order d^2, up to phases) is not shared by all shift-and-multiply bases. For small dimensions (at least for $d < 6$), every unitary basis with such a group structure (sometimes called *nice error basis*) is of shift-and-multiply type. For higher dimensions ($d = 165$ being the first known counter example) this is no longer valid.

Overcomplete sets, frames and SIC-POVMs We have seen that operator bases for $\mathcal{M}_d$ can be constructed from unitary as well as from Hermitian operators. What about positive semidefinite operators or Hermitian rank-one projections? This turns out not to be possible if one demands orthogonality w.r.t. the Hilbert-Schmidt scalar product (see Prop.2.7). However, relaxing this condition leads to bases with remarkable properties—and our discussion to the framework of *frames*. A frame of a vector space, say $\mathbb{C}^d$, is a set of vectors $\{\phi_i\}_{i=1..n}$ for which there are constants $0 < a \le b < \infty$ such that for all $\psi \in \mathbb{C}^d$

$$a||\psi||^2 \le \sum_i |\langle\psi|\phi_i\rangle|^2 \le b||\psi||^2. \tag{2.29}$$

[6] ok, 'Vierergruppe' sounds better

Hence, the ϕ_i's have to span the entire vector space. If $a = b$, the frame is called *tight* and if in addition $n = d$ it is nothing but an orthonormal basis (with norm equal to $\sqrt{a}$). Since for tight frames $\sum_i |\phi_i\rangle\langle\phi_i| = a\mathbb{1}$, they correspond one-to-one to rank-one POVMs.

A POVM is called *informationally complete* if its measurement outcomes allow for a complete reconstruction of an arbitrary state. The existence of such POVMs is fairly obvious: just use properly normalized spectral projections of a unitary or Hermitian operator basis as effect operators. A simple physical implementation in $\mathbb{C}^d$ would be to first draw a random number from $1, \ldots, d^2$ and then measure the corresponding element of a Hermitian operator basis. Clearly, a POVM acting on $\mathbb{C}^d$ is informationally complete iff it contains d^2 linearly independent effect operators.

Now let us try to construct a positive semidefinite operator basis for $\mathcal{M}_d$ which is 'as orthogonal as possible' and see how this relates to tight frames and informationally complete POVMs:

Proposition 2.7 (SIC POVMs) *Let $\{P_i\}_{i=1..n}$ be a set of positive semidefinite operators in $\mathcal{M}_d$, $d \leq n$, which are normalized w.r.t. the Hilbert-Schmidt scalar product, i.e., $\langle P_i, P_i\rangle = \mathrm{tr}\left[P_i^2\right] = 1$. Then*

$$\sum_{i\neq j} |\langle P_i, P_j\rangle|^2 \geq \frac{(n-d)^2 n}{(n-1)\, d^2}, \tag{2.30}$$

with equality iff all P_i are rank-one projections fulfilling $\langle P_i, P_j\rangle = \frac{n-d}{(n-1)d}$ for all $i \neq j$, and $\sum_i P_i = \frac{n}{d}\mathbb{1}$. In case of equality the P_i's are linearly independent.

Proof Define $Q := \sum_i P_i$. Invoking Cauchy-Schwarz inequality twice we get

$$\begin{aligned}\sum_{i\neq j} |\langle P_i, P_j\rangle|^2 \quad &\geq \quad \frac{1}{n^2-n}\Big(\sum_{i\neq j}\langle P_i, P_j\rangle\Big)^2 && (2.31)\\ &= \quad \frac{1}{n^2-n}\big(\mathrm{tr}\left[Q^2\right]-n\big)^2 \geq \frac{\big(\mathrm{tr}\left[Q\right]^2/d-n\big)^2}{n^2-n}, && (2.32)\end{aligned}$$

where the first inequality holds iff $\langle P_i, P_j\rangle = \mathit{const}$ for all $i \neq j$ and the second one iff $Q \propto \mathbb{1}$. Since due to normalization $\mathrm{tr}\left[P_i\right] \geq 1$ and thus $\mathrm{tr}\left[Q\right] \geq n$ with equality iff P_i is a rank-one projection we can further bound (2.32) from below leading to the r.h.s. of (2.30). Collecting the conditions for equality completes the proof of the first part. In order to show linear independence assume $\sum_j c_j P_j = 0$ and set $c := \langle P_i, P_j\rangle$ for $i \neq j$. Then for all i: $0 = \sum_j c_j\langle P_i, P_j\rangle = c_i(1-c) + c\sum_j c_j$, so c_i is independent of i and therefore $c_i = 0$. □

As claimed before, Prop.2.7 shows that positive semidefinite operators cannot form a Hilbert-Schmidt orthogonal operator basis (for that the l.h.s. of Eq.(2.30) would have to be zero, which is in conflict with $n = d^2$). Whether the conditions for equality in Prop.2.7 are vacuous or actually have a solution depends on n and d. Whenever there exists a solution, it forms a tight frame.

While for $n = d$ the solutions are exactly all orthonormal bases of $\mathbb{C}^d$, for $n > d^2$ no solution exists (as the P_i's become necessarily linear dependent then).

Remarkably, there exist solutions for some $n = d^2$, i.e., tight frames which are informationally complete POVMs with the additional symmetry $\mathrm{tr}\,[P_i P_j] = (d+1)^{-1}$ for all $i \neq j$. These are called *symmetric informationally complete* (SIC) POVMs. For $d = 2$ all SIC POVMs are obtained by choosing points on the Bloch sphere which correspond to vertices of a tetrahedron. For $d = 3$ an example can be constructed from the shift-and-multiply basis in Eq.(2.24) by taking projections onto the vectors $|\xi_{kl}\rangle := U_{kl}|\xi_{00}\rangle$ with $|\xi_{00}\rangle := (|0\rangle + |1\rangle)/\sqrt{2}$. In other words, in this example the vectors onto which the P_i's project are *generalized coherent states* w.r.t. the discrete Heisenberg-Weyl group. Analogous to continuous coherent states, all SIC POVMs provide (due to the operator basis property) a representation of arbitrary operators $\rho \in \mathcal{M}_d$:

$$\rho = \frac{1}{d}\sum_i \Big((d+1)\mathrm{tr}\,[P_i\rho] - \mathrm{tr}\,[\rho]\Big)\, P_i. \tag{2.33}$$

In the context of coherent states this is called *diagonal representation* or (in quantum optics for the continuous Heisenberg-Weyl group) *P-function* representation.

Using the effect operators of a SIC-POVM as Kraus operators we get a quantum channel of the form

$$\frac{1}{d}\sum_{i=1}^{d^2} P_i\rho P_i = \frac{\mathbb{1}\mathrm{tr}\,[\rho] + \rho}{d+1}, \qquad \forall \rho \in \mathcal{M}_d. \tag{2.34}$$

Both Eq.(2.33) and Eq.(2.34) are readily verified by using the properties (in particular the operator basis property) of the P_i's.

Problem 9 (SIC POVMs) *Determine the pairs of (n, d) for which equality in Eq.(2.30) can be achieved.*

2.4 Normal forms

It is sometimes useful to cut some trees in order to get a better overview of the forest. Depending on the situation will later encounter various 'normal forms' for the representations discussed previously—in particular, decompositions based on spectral, convex or semi-group properties. This section is devoted to normal forms of completely positive maps w.r.t. invertible maps with Kraus rank one. More specifically, given a completely positive map $T : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2}$ we are interested in in simple representatives of the equivalence class

$$T \sim \Phi_2 T \Phi_1, \tag{2.35}$$

where $\Phi_i \in \mathcal{B}(\mathcal{M}_{d_i})$ is of the form $\Phi_i(\cdot) := X_i \cdot X_i^\dagger$ with $X_i \in SL(d_i, \mathbb{C})$, i.e., complex valued matrices with unit determinant. Such maps often run under the name *filtering operations*. On the level of Choi-Jamiolkowski operators $\tau \in \mathcal{B}(\mathbb{C}^{d_2} \otimes \mathbb{C}^{d_1})$ the transformation in (2.35) corresponds to $\tau \mapsto \tau' := (X_2 \otimes X_1)\tau(X_2 \otimes X_1)^\dagger$ (up to an irrelevant transposition for X_1). In order to construct a normal form w.r.t. such transformations we utilize the following optimization problem:

$$p := \inf_{X_i \in SL(d_i,\mathbb{C})} \operatorname{tr}[\tau'] \,. \tag{2.36}$$

Recall that we assume complete positivity which by Prop.2.1 means $\tau \geq 0$. Clearly, $p \leq \operatorname{tr}[\tau]$ (as we can choose $X_i = \mathbb{1}$) and $p \geq ||X_i||_\infty^2 \lambda_{\min}(\tau)$ with $\lambda_{\min}(\tau)$ the smallest eigenvalue of τ. Our aim is to choose as a representative of the equivalence class one which minimizes (2.36):

Proposition 2.8 (Normal form for generic τ) *Let $\tau \in \mathcal{B}(\mathbb{C}^{d_2} \otimes \mathbb{C}^{d_1})$ be positive definite. Then there exist $X_i \in SL(d_i,\mathbb{C})$ which attain the infimum in (2.36) so that the respective optimal $\tau' = (X_2 \otimes X_1)\tau(X_2 \otimes X_1)^\dagger$ is such that both partial traces are proportional to the identity matrix.*

PROOF From the upper and lower bound on p we know that we can w.l.o.g. restrict to $||X_i||_\infty^2 \leq \operatorname{tr}[\tau]/\lambda_{\min}(\tau)$. These X_i form a compact set (since $\tau > 0$ by assumption) so that the infimum is indeed attained.

Eq.(2.36) can now be minimized by a simple iteration in X_1 and X_2: denote by $\tau_1 = \operatorname{tr}_2[\tau]$ a partial trace of τ so that $\tau_1' := X_1 \tau_1 X_1^\dagger$ is the corresponding one for $\tau' := (\mathbb{1} \otimes X_1)\tau(\mathbb{1} \otimes X_1)^\dagger$. Choosing $X_1 := \det(\tau_1)^{1/2d_1}\tau_1^{-1/2}$ will lead to $\tau_1' \propto \mathbb{1}$. Moreover, the arithmetic-geometric mean inequality gives

$$\operatorname{tr}[\tau'] = \det(\tau_1)^{1/d_1} d_1 \leq \operatorname{tr}[\tau_1] = \operatorname{tr}[\tau] \,, \tag{2.37}$$

with equality iff $\tau_1 \propto \mathbb{1}$. Iterating this step w.r.t. X_1 and X_2 will thus decrease the trace while both reduced density matrices converge to something proportional to the identity. As this holds true in particular for the optimal τ', it has to have $\tau_i' \propto \mathbb{1}$. □

Together with the Choi-Jamiolkowski correspondence in Prop.2.1 this has an immediate corollary on the level of completely positive maps:

Proposition 2.9 (Normal form for generic cp maps) *Let $T : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2}$ be a completely positive map with full Kraus rank. There exist invertible completely positive maps $\Phi_i \in \mathcal{B}(\mathcal{M}_{d_i})$ with Kraus rank one such that $T' := \Phi_2 T \Phi_1$ is* doubly-stochastic, *i.e., both $T(\mathbb{1})$ and $T^*(\mathbb{1})$ are proportional to the identity matrix.*

The normal form in Prop.2.8 is unique[7] up to local unitaries $\tau' \mapsto (U_2 \otimes U_1)\tau'(U_2 \otimes U_1)$, which corresponds to unitary channels $\Phi_i(\cdot) = U_i \cdot U_i^\dagger$ in Prop.2.9.

[7]This is an exercise about non-negative matrices: assume $\tilde{\tau} := (D_2 \otimes D_1)\tau'(D_2 \otimes D_1)$ is a second normal form with D_i positive diagonal matrices (we can always use the unitary freedom to choose them like this). Then $M_{ij} := \langle ij|\tau'|ij\rangle$ (and analogously $\tilde{M}$) is a doubly stochastic non-negative matrix, i.e., $M_{ij} \geq 0$ and all rows as well as well as all columns have the same sum. Moreover, $\tilde{M} = M * H$ is a Hadamard product with $H_{ij} := [D_2]_{ii}[D_1]_{jj}$. The only H which achieves such a mapping is, however, a multiple of $H_{ij} = 1$.

This implies that the algorithmic procedure used in the proof of Prop.2.8 in order to obtain the normal form by minimizing $\mathrm{tr}\,[\tau']$ eventually converges to the global optimum.

In cases where τ has a kernel the infimum in (2.36) may either be zero or not be attained for finite X_i. An exhaustive investigation of general normal forms w.r.t. the equivalence class (2.35) has been performed for qubit channels.

Qubit maps We continue the discussion about normal forms w.r.t. Kraus-rank one operations for completely positive maps $T : \mathcal{M}_2 \to \mathcal{M}_2$. Choosing normalized Pauli matrices as operator basis (with $\sigma_0 = \mathbb{1}$) we can represent T as a 4×4 matrix $\hat{T}_{ij} := \mathrm{tr}\,[\sigma_i T(\sigma_j)]\,/2$. If T is Hermitian, then $\hat{T}$ is real, and if T is trace preserving, then $(T_{1,j}) = (1, 0, 0, 0)$. A quantum channel in the Schrödinger picture is thus represented by

$$\hat{T} = \begin{pmatrix} 1 & 0 \\ v & \Delta \end{pmatrix}, \tag{2.38}$$

where Δ is a real 3×3 matrix and $v \in \mathbb{R}^3$. The corresponding Jamiolkowski state is given by $\tau = \frac{1}{4}\sum_{ij} \hat{T}_{ij}\sigma_i \otimes \sigma_j^T$ so that v describes its reduced density operator and Δ its correlations. The conditions for $\hat{T}$ to correspond to a completely positive map have to be read off from $\tau \geq 0$. The complexity of this condition is the drawback of this representation. One of its advantages is a nice geometric interpretation: parameterizing a density operator via $\rho = (\mathbb{1} + \sum_{k=1}^{3} x_k\sigma_k)/2$, i.e., in terms of a vector $x \in \mathbb{R}^3$ within the Bloch ball $||x|| \leq 1$ (see example1.1), the action of T is a simple affine transformation

$$x \mapsto v + \Delta x. \tag{2.39}$$

From here conditions for T to be positive are readily derived (as the vector has to stay within the Bloch ball), and we will discuss them in greater detail in ... . The following is a useful proposition in this context:

Proposition 2.10 *Let $T : \mathcal{M}_2(\mathbb{C}) \to \mathcal{M}_2(\mathbb{C})$ be a Hermiticity preserving linear map with $\Delta_{ij} := \mathrm{tr}\,[\sigma_i T(\sigma_j)]\,/2$ the lower-right submatrix (i.e., $i, j = 1, 2, 3$) of the matrix representation $\hat{T}$. Then $\hat{T}' := \hat{T}_{00} \oplus \Delta$ represents a (completely) positive map iff $\hat{T}$ does so.*

Proof We use that $D := \mathrm{diag}(1, -1, -1, -1)$ is the matrix representation of time-reversal (see Sec.3.3), i.e., matrix transposition in some basis. Following the relation in Eq.(1.24) the map $D\hat{T}D$ is (completely) positive if $\hat{T}$ is. The same holds thus for the convex combination $(\hat{T} + D\hat{T}D)/2 = \hat{T}'$. □

Suppose now that we act with a unitary before and another one after applying the channel T so that the overall action is $\rho \mapsto U_2 T(U_1 \rho U_1^\dagger) U_2^\dagger$. The transfer matrix corresponding to this concatenation is then given by the product $(1 \oplus O_2)\hat{T}(1 \oplus O_1)$, where $O_i \in SO(3)$ are real rotations of the Bloch sphere. This reflects the two-to-one group homomorphism $SU(2) \to SO(3)$ (see examples 1.1 and 2.2). We may use this in order to diagonalize $\Delta \to \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$

so that $\lambda_1 \geq \lambda_2 \geq |\lambda_3|$ and for positive, trace-preserving maps necessarily $1 \geq \lambda_1$. Up to a possibly remaining sign this diagonalization is nothing but a real singular value decomposition. Expressed in terms of the λ_i's a necessary condition for complete positivity is that

$$\lambda_1 + \lambda_2 \leq 1 + \lambda_3. \tag{2.40}$$

This becomes sufficient if the map is unital, i.e., if $v = 0$ (see also Exp.2.3).

Extending the freedom in the transformations from $SU(2)$ to $SL(2)$ enables us to further simplify $\hat{T}$ and to bring it to a normal form (as stated in Prop.2.9 for the full rank case):

Proposition 2.11 (Lorentz normal form) *For every qubit channel T there exist invertible completely positive maps Φ_1, Φ_2, both of Kraus rank one, such that the concatenation $\Phi_2 T \Phi_1 =: T'$ (characterized by v and Δ) is a qubit channel of one of the following three forms:*

1. Diagonal: *T' is unital ($v = 0$) with Δ diagonal. This is the generic case (proved in Prop.2.9).*
2. Non-diagonal: *T' has $\Delta = \mathrm{diag}(x/\sqrt{3}, x/\sqrt{3}, 1/3)$, $0 \leq x \leq 1$ and $v = (0, 0, 2/3)$. These channels have Kraus rank 3 for $x < 1$ and Kraus rank 2 for $x = 1$.*
3. Singular: *T' has $\Delta = 0$ and $v = (0, 0, 1)$. This channel has Kraus rank 2 and is singular in the sense that it maps everything onto the same output.*

Example 2.2 (Lorentz group and spinor representation) *Our aim is to understand the action of $T \to \Phi_2 T \Phi_1$ as an equivalence transformation on $\hat{T}$ by elements from the Lorentz group. In order to see how this arises, consider the space $\mathcal{M}_2^\dagger(\mathbb{C})$ of complex Hermitian 2×2 matrices. Every such matrix can be expanded as $M = \sum_{i=0}^{3} x_i \sigma_i$ with $x \in \mathbb{R}^4$. Since $\det(M) = x_0^2 - x_1^2 - x_2^2 - x_3^2 = \langle x, \eta x\rangle$ with $\eta := \mathrm{diag}(1, -1, -1, -1)$ we can identify $\mathcal{M}_2^\dagger(\mathbb{C})$ with* Minkowski space *such that the determinant provides the Minkowski metric. If $X \in SL(2, \mathbb{C})$, the map*

$$M \mapsto X M X^\dagger \tag{2.41}$$

in this way becomes a linear isometry in Minkowski space, i.e., a Lorentz transformation. *In fact, the group $SL(2, \mathbb{C})$ is a double cover of the special orthochronous Lorentz group*

$$SO^+(1,3) := \{L \in \mathcal{M}_4(\mathbb{R}) \mid \det(L) = 1,\ L\eta L^T = \eta,\ L_{00} > 0\} \tag{2.42}$$

in very much the same way as $SU(2)$ is a double cover of $SO(3)$. The map $SL(2, \mathbb{C}) \to SO^+(1,3)$ constructed above is sometimes called spinor map. *It is two-to-one since $\pm X$ have the same effect in (2.41). Due to this equivalence the transfer matrix of the channel $\Phi_2 T \Phi_1$ becomes*

$$L_2 \hat{T} L_1, \quad \textit{with} \quad L_i \in SO^+(1,3). \tag{2.43}$$

The normal form in Prop.2.11 can thus be regarded as a normal form w.r.t. special ortochronous Lorentz transformations.

Every $L \in SO^+(1,3)$ *can be decomposed into a 'boost' and a spatial rotation. This decomposition can be obtained from the* polar decomposition $X = PU$ *in* $SL(2,\mathbb{C})$*, where* $P > 0$ *and* $UU^\dagger = \mathbb{1}$*. In order to make this more explicit, define generators of spatial rotations by* $R_i := \sum_{jk=1}^3 \epsilon_{ijk}|k\rangle\langle j| \in \mathcal{M}_4$*, and generators of boosts by* $B_i := |0\rangle\langle i| + |i\rangle\langle 0| \in \mathcal{M}_4$ *with* $i = 1,2,3$*. Then with* $\vec{n},\vec{m} \in \mathbb{R}^3$ *the mapping* $SL(2,\mathbb{C}) \to SO^+(1,3)$ *takes the form*

$$U = e^{-i\vec{n}\cdot\vec{\sigma}/2} \to e^{-\vec{n}\cdot\vec{R}}, \qquad P = e^{\vec{m}\cdot\vec{\sigma}/2} \to e^{\vec{m}\cdot\vec{B}}. \tag{2.44}$$

Decomposing P *further according to the spectral decomposition leads to a diagonal matrix which corresponds to a boost in* z*-direction (i.e., it acts in the 0-3–plane as* $\mathbb{1}\cosh m_3 + \sigma_x \sinh m_3$*).*

Example 2.3 (Bell diagonal states and Pauli channels) *...*

2.5 Literature