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Differential Equations Formula Sheet

NB! WRITE DOMAIN AT END OF EACH QUESTION

First Order Differential Equations

Separation of Variables: $y' = a(x)y \rightarrow \frac{dy}{dx} = a(x)y \rightarrow \frac{dy}{y} = a(x)dx \rightarrow \int \frac{dy}{y} = \int a(x)dx \rightarrow y = \tilde{C}e^{-\int a(x)dx}$

Linear First Order Equation

$y' + p(x)y = q(x) \rightarrow$ Integrating Factor Method

Solution: $y(t) = \frac{1}{r(x)}\int r(x)\,q(x)dx \rightarrow$ Integrating Factor: $r(x) = e^{\int p(x)dx}$

$e^{+\ln D} = D^{+}$ $e^{-\ln D} = \frac{1}{D^{+}}$ $\int e^{D} \cdot D' dx = e^{D} + C$

*If homogeneous, $q(x) = 0$: Use separation of variables technique.

Variation of Parameters – 1st Order

$y' + a(t)y = b(t)$.   remember $\ln C$ $y_h = \frac{A}{x}$

Step 1) Solve homogenous equation by separation of variables => $y_h = C \cdot f(t)$ Step 2) Assume solution to be of form: $y_p = C(t) \cdot f(t)$

Step 3) $y_p' = C'(t) \cdot f(t) + C(t) \cdot f'(t)$ Step 4) Sub y and y' into original equation and solve for $C(t)$ Step 5) Solution: $y_p = C(t) \cdot f(t)$

Non-Linear First Order Equations

Bernoulli: $y' = a(t)y + b(t)y^n$ - divide by y^{-n} $\boxed{v = y^{1-n}}$ + $\boxed{\frac{v'}{1-n} = y^{-n} \cdot y'}$ $y = y_h + y_p$

Divide original equation by $y^n \rightarrow$ Substitute: $v = y^{1-n}$ and $v' = (1-n)y^{-n} \cdot y' \rightarrow$ Solve First Order Linear Equation $y^{-n}y' + a(t)y^{1-n} = b(t)$

$v' + ((1-n)p(x))v = (1-n)q(x)$ $\frac{1}{1-n}v' + a(t)v = 0$

Rikati: $y' = a(t)y^2 + b(t)y + c(t)$ ① given ... Y_p sub in

$\tilde{y}' = a(t)\tilde{y}^2 + \left(a(t) \cdot 2y_p + b(t)\right)\tilde{y} \rightarrow$ Bernouli Equation $(\tilde{y} = y - y_p) \rightarrow z = \tilde{y}^{-1}$

$z' = -a(t) - \left(b(t) + \left(a(t) \cdot 2y_p\right)\right)z \rightarrow$ Solve First order linear equation for z. Track back for y. (y_p *given or guessed*)

$y(x) = Y_p + \frac{1}{z(x)}$

Homogeneous Equations

$y' = F\left(\frac{y}{t}\right) = \frac{g(y,t)}{h(y,t)}$

1) Check equations are homogeneous of same order $\rightarrow$ Sub in $y = \lambda y$ and $x = \lambda x \rightarrow$ Receive: $\lambda^x \cdot g(y,t)$ and $\lambda^x \cdot h(y,t)$
2) If homogeneous: Sub-in $y = vt$ and $dy = dv \cdot t + v \cdot dt$ to original equation. Solve first order linear (separating variables)

if divided e.g. x to check for extra answer

sub in $v = \frac{y}{t}$ at end

Almost Homogeneous - Special Case

$y' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2} = \frac{g(x,y)}{h(x,y)} \Rightarrow h(x,y) \cdot dy = g(x,y) \cdot dx$

Case 1: $a_1 \cdot b_2 = a_2 \cdot b_1 \rightarrow \frac{dg}{dx} = a_1 + b_1 \cdot \frac{dy}{dx} \rightarrow dy = \frac{1}{b_1}(dg - a_1 \cdot dx)$ (1) $\rightarrow$ Find h in terms of g (2) $\rightarrow$ Using (1) and (2) sub into original equation and solve by separation of variables. $h(g) \cdot \frac{1}{b_1}(dg - a_1 \cdot dx) = g(x,y) \cdot dx$

Case 2: $a_1 \cdot b_2 \neq a_2 \cdot b_1 \rightarrow$ 1) Equate $g = 0$ and $h = 0$ and solve for x and y. 2) x=h, y=k 3) Sub into original: $x = X + h, dx = dX,\ y = Y + k, dy = dY$ 4) Solve as regular homogenous equation. $F\left(\frac{y}{x}\right)$.

Exact Equations

$M(x,y)dx + N(x,y)dy = 0 \rightarrow$ If $M_y = N_x$ it's an exact equation. Solution form: $F(x,y) = C$

To find $F(x,y)$: $F_x = M$ and $F_y = N \rightarrow F = \int M\,dx \rightarrow F = g(x,y) + c(y) \rightarrow F_y = g_y + c'(y) = N \rightarrow c'(y) = N - g_y \rightarrow c(y) = \int(N - g_y)dy \rightarrow$ Sub into F $\rightarrow$ Solution: $F(x,y) = C$ **OR** $F = \int N\,dy \rightarrow F = h(x,y) + c(x) \rightarrow F_x = h_x + c'(x) = M \rightarrow c'(x) = M - h_x \rightarrow c(x) = \int(M - h_x)dx \rightarrow$ Sub into F $\rightarrow$ Solution: Solution: $F(x,y) = C$

Non-Exact Equations

$M(x,y)dx + N(x,y)dy = 0$ **but** $M_y \neq N_x \rightarrow$ Solve using integrating factor.

I.F: $h(x) \rightarrow h(x) \cdot M(x,y)dx + h(x) \cdot N(x,y)dy = 0$ will be **exact** $\rightarrow h(x)$ given in question or we must find it $\rightarrow$ To find $h(x)$:

1) If $\frac{M_y - N_x}{N} = f(x)$ a function of **x only** $\rightarrow$ I.F = $h(x) = e^{\int f(x)dx}$ 2) If $\frac{M_y - N_x}{M} = g(y)$ a function of **y only** $\rightarrow$ I.F = $h(y) = e^{-\int g(y)dy}$

Existence and Uniqueness Theorem – First Order   for series

$y' = f(x,y) = a(x)y + b(x)$ $y'' + a(x)y' + b(x)y = 0$

1) If $f(x,y)$ is continuous in a block where $a < x < b$ and $c < y < d \rightarrow$ **exists** a solution in the interval $I \subseteq (a,b)$ at point (x_0, y_0)

2) If $f_y(x,y)$ is continuous in the block $\rightarrow$ the solution that exists is also **unique.**

*If a(x) and b(x) are continuous in interval (a,b) containing $x_0 \rightarrow f(x,y)$ and $f_y(x,y)$ are continuous in the same interval and exists a unique solution at the point (x_0, y_0) in the interval (a,b)   check if defined at 0

Second Order Differential Equations

Existence and Uniqueness Theorem – Second Order

$y'' = f(x,y,y') = a(x)y' + b(x)y + c(x)$

*If a(x), b(x) and c(x) are continuous in interval (a,b) containing $x_0 \rightarrow$ Exists a unique solution at the point (x_0, y_0) in the interval (a,b)

General: Solutions have two constants C_1 and C_2

Second Order Exclusion Equations:

1) $f(x,y',y'') = 0$ **No y term in equation** $\rightarrow$ Solve by: $y' = z(x)$ and $y'' = z'(x) \rightarrow$ 1st order equation: separating variables. (Reduce Order)
2) $f(y,y',y'') = 0$ **No x term in equation** $\rightarrow$ Solve by: $y' = z(y)$ and $y'' = z'(y) \cdot z(y) \rightarrow$ Solve first order equation
3) Given 1 solution: $y_1 \rightarrow y_2 = y_1 \cdot u(x) \rightarrow$ Differentiate twice $\rightarrow$ Sub-into equation $\rightarrow$ Find solution to 1st order linear homogeneous equation for u(x) to find y_2.

Linear Second (or higher) Order Equation – Homogeneous – Constant Co-Efficients

$ay'' + by' + cy = 0 \rightarrow$ Convert to polynomial: $y'' = k^2, y' = k, y = 1 \rightarrow$ Solve polynomial for roots:

$k_1 \neq k_2$	$y = C_1e^{k_1 \cdot x} + C_2e^{k_2 \cdot x}$
$k_1 = k_2 = k$	$y = C_1e^{k \cdot x} + C_2xe^{k \cdot x}$
$k_{1,2} = a \pm bi$	$y = e^{ax}[C_1\cos(b \cdot x) + C_2\sin(b \cdot x)]$

*Same solution types will work for higher order linear homogeneous equations but may have combination of different types.

Superposition Theorem - y_1 and y_2 solve: $y'' + p(x)y' + q(x)y = 0$ (homogeneous) $\rightarrow y = C_1y_1 + C_2y_2$ is also a solution.

<u>Theorem</u>: If y_1 and y_2 are linearly independent (DEF: they are not a scalar multiple of each other) → the general solution to the differential equation will be: $y = C_1y_1 + C_2y_2$

$Q(x) = x^3$ $\sin x$ xe^x [illegible]

<u>Linear Second (or higher) Order Equation – Non-Homogeneous – Constant Co-Efficients **(Guessing Method)**</u>

$ay'' + by' + cy = Q(x) \rightarrow Q(x)$ must be an addition/subtraction/multiplication of one of the following functions:

Q(X)	$y_p(x)$
n[th] order polynomial	$A + Bx + \cdots + Cx^n$
Ce^{ax}	Ae^{ax}
$C \cdot \sin(bx)/C \cdot \cos(bx)$	$A \cdot \sin(bx) + B \cdot \cos(bx)$
$Ce^{ax} \cdot \sin(bx)/Ce^{ax} \cdot \cos(bx)$	$e^{ax}(A \cdot \sin(bx) + B \cdot \cos(bx))$

$\frac{e^x}{x}$ [illegible] sin [illegible]

1) Solve for y_h (homogeneous form of the equation)
2) If any terms in y_p are the same as in y_h *or* y_p, multiply by x until it is independent
3) Once we find the form of y_p we find y_p' and y_p'', sub-into $ay_p'' + by_p' + cy_p$ = Q(x) and **compare variables** to find A,B,C...
*If $Q(x) = a(x) + b(x)$ we solve for y_{p1} and y_{p2} separately → $y_p = y_{p1} + y_{p2}$
4) $y = y_h + y_p$

<u>Linear Second (or higher) Order Equation – Non-Homogeneous – Constant Co-Efficients **(Variation of Parameters Method)**</u>

$ay'' + by' + cy = Q(x)$ **but** $Q(x)$ is not on the above table (derivatives do not repeat themselves at any point)

1) Solve the homogeneous equation $ay'' + by' + cy = 0 \rightarrow y_h = C_1y_1 + C_2y_2$
2) Now we solve for y_p: First we solve the Wronskian: W = $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$
3) Solve W_1 and W_2 → $W_1 = \begin{vmatrix} 0 & y_2 \\ Q(x) & y_2' \end{vmatrix}$ and $W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & Q(x) \end{vmatrix}$
4) Using Kramer's rule: $C_1'(x) = \frac{W_1}{W}$ and $C_2'(x) = \frac{W_2}{W} \rightarrow C_1(x) = \int \frac{W_1}{W}dx$ and $C_2(x) = \int \frac{W_2}{W}dx$
5) $y_p = C_1(x)y_1 + C_2(x)y_2$ 6) $y = y_h + y_p$

<u>Second (or higher) Order Linear Equation – Homogeneous – Non-Constant Co-Efficients **(Euler's Formula)**</u> $(a \neq 0, x \neq 0)$

Order 2: $ax^2 \cdot y'' + bx \cdot y' + c \cdot y = 0$ → 1) We solve for roots of k: $\boldsymbol{ak(k-1) + bk + c = 0}$

Order 3: $ax^3 \cdot y''' + bx^2 \cdot y'' + cx \cdot y' + d \cdot y = 0$ → We solve for roots of k: $\boldsymbol{ak(k-1)(k-2) + bk(k-1) + ck + d = 0}$

$k_1 \neq k_2$	$y = C_1\lvert x\rvert^{k_1} + C_2\lvert x\rvert^{k_2}$
$k_1 = k_2 = k$	$y = C_1\lvert x\rvert^{k} + C_2\lvert x\rvert^{k} \cdot \ln\lvert x\rvert$
$k_{1,2} = a \pm bi$	$y = \lvert x\rvert^a[C_1 \cdot \cos(b \cdot \ln\lvert x\rvert) + C_2 \cdot \sin(b \cdot \ln\lvert x\rvert)]$

<u>Second (or higher) Order Linear Equation – Non-Homogeneous – Non-Constant Co-Efficients **(Euler's Formula)**</u>

$ax^2y'' + bxy' + cy = Q(x)$ $(a \neq 0, x \neq 0)$

1) Solve the homogeneous Euler Equation → $y_h = C_1y_1 + C_2y_2$
2) We solve for y_p using Variation of Parameters method with $Q(x)$. **NB: Co-efficient of y'' must be 1 (divide all terms by co-efficient of y'')**
3) $y = y_h + y_p$

<u>Convert Euler Equation to Equivalent Differential Equation with Constant Co-Efficients</u>

1) $x = e^t$ and $e^t dt = dx$ $\frac{dt}{dx} = e^{-t}$
2) 2) $\frac{dy}{dx} = y' = \frac{dy}{dt} \cdot \frac{dt}{dx} \rightarrow y' = \frac{dy}{dt} \cdot e^{-t}$ and $y'' = \left(\frac{dy}{dt} \cdot e^{-t}\right) \cdot \frac{dt}{dx} \rightarrow y'' = \left(\frac{d^2y}{dt^2}(e^{-t}) - e^{-t} \cdot \frac{dy}{dt}\right) \cdot \left(\frac{dt}{dx}\right) \rightarrow y'' = e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$

<u>Systems of Linear Differential Equations</u>

<u>Eigenvectors and Eigenvalues</u> → How to find them given matrix A?

1) Find characteristic matrix: $(x \cdot I - A)$
2) Find characteristic polynomial: $|(x \cdot I - A)|$ (determinant of characteristic matrix) $P(x) = (x-A)(x-B)(x-c)$
3) Solve for roots of characteristic polynomial to get EIGENVALUES $P(x) = 0$
4) For each eigenvalue, sub-in $x = eigenvalue$ to characteristic matrix, solve for x,y and z by equating rows to 0

<u>Solving Homogeneous System of Linear Equations</u>

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{aligned} \rightarrow \begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Let's define: $\bar{X}'(t) = A \cdot \bar{X}(t)$, $\bar{X} = (x_1, x_2, \dots, x_n)$

*The solution to a system or linear differential equations is: $\bar{X} = (x_1, x_2, \dots, x_n)$

Method to Solve:

1) Find all eigenvalues and eigenvectors of A (λ *and* v) 2) Use table below to find solution:

Every eigenvalue has its own eigenvector	$\bar{X}(t) = C_1e^{\lambda_1 t} \cdot v_1 + C_2e^{\lambda_2 t} \cdot v_2 + \cdots + C_ne^{\lambda_n t} \cdot v_n$
If an eigenvalue repeats twice (one eigenvector repeated)	$\bar{X}(t) = C_1e^{\lambda_1 t} \cdot v_1 + C_2e^{\lambda_2 t} \cdot v_2 + C_3e^{\lambda_2 t} \cdot \left[t\begin{pmatrix} \vert \\ v_2 \\ \vert \end{pmatrix} + \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right] + \cdots C_ne^{\lambda_n t} \cdot v_n$ *To find x,y,z: $(A - \lambda_2 \cdot I) \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \vert \\ v_2 \\ \vert \end{pmatrix}$ → Solve system of equations
If an eigenvalue repeats three times (only one eigenvector)	$\bar{X}(t) = C_1e^{\lambda_1 t} \cdot v_1 + C_2e^{\lambda_1 t} \cdot \left[t\begin{pmatrix} \vert \\ v_1 \\ \vert \end{pmatrix} + \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right] + C_3e^{\lambda_1 t} \cdot \left[\frac{t^2}{2}\begin{pmatrix} \vert \\ v_1 \\ \vert \end{pmatrix} + t \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} p \\ q \\ r \end{pmatrix}\right] + \cdots C_ne^{\lambda_n t} \cdot v_n$ *Find x,y,z as before (with λ_1) *To find p,q,r: $(A - \lambda_1 \cdot I) \cdot \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ → Solve system of equations

If an eigenvalue repeats three times (two eigenvectors)	$\bar{X}(t) = C_1 e^{\lambda_1 t} \cdot v_1 + C_2 e^{\lambda_1 t} \cdot v_2 + C_3 e^{\lambda_1 t} \cdot \left[t \cdot \left[\alpha \cdot \begin{pmatrix} \vert \\ v_1 \\ \vert \end{pmatrix} + \beta \cdot \begin{pmatrix} \vert \\ v_2 \\ \vert \end{pmatrix} + \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right]\right] + \cdots C_n e^{\lambda_n t} \cdot v_n$ *To solve: $(A - \lambda_1 \cdot I) \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \alpha \cdot \begin{pmatrix} \vert \\ v_1 \\ \vert \end{pmatrix} + \beta \cdot \begin{pmatrix} \vert \\ v_2 \\ \vert \end{pmatrix}$ → Solve system (solution is not unique)

Solving Non-Homogeneous System of Linear Equations – Variation of Parameters

$$\begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$ Let's define: $\bar{X}'(t) = A \cdot \bar{X}(t) + \bar{b}(t)$, $\bar{X} = (x_1, x_2, \dots, x_n)$

How to solve?

1) Solve homogeneous system → $X_h(t) = C_1 e^{\lambda_1 t} \cdot v_1 + C_2 e^{\lambda_2 t} \cdot v_2 + \cdots + C_n e^{\lambda_n t} \cdot v_n$
2) $X_p(t) = C_1(t) \cdot e^{\lambda_1 t} \cdot v_1 + C_2(t) \cdot e^{\lambda_2 t} \cdot v_2 + \cdots + C_n(t) \cdot e^{\lambda_n t} \cdot v_n$ → Need to find $C_1(t), C_2(t) \dots C_n(t)$
   $X_p(t) = C_1(t) \cdot x_1 + C_2(t) \cdot x_2 + \cdots + C_n(t) \cdot x_n$
3) How to find them: $\begin{pmatrix} \vert & \cdots & \vert \\ x_1 & \cdots & x_n \\ \vert & \cdots & \vert \end{pmatrix} \cdot \begin{pmatrix} C_1' \\ C_2' \\ C_n' \end{pmatrix} = \begin{pmatrix} \vert \\ \bar{b}(t) \\ \vert \end{pmatrix}$

$$C_1'(t) = \frac{\begin{vmatrix} \vert & \vert & \vert \\ \bar{b}(t) & x_2 & x_3 \\ \vert & \vert & \vert \end{vmatrix}}{\begin{vmatrix} \vert & \vert & \vert \\ x_1 & x_2 & x_3 \\ \vert & \vert & \vert \end{vmatrix}}$$ → $C_1(t) = \int C_1'(t)dt$ *find all others in the same way, shifting the row of $\bar{b}(t)$ every time

Power Series – Analytic Co-Efficients

Form (order 2 as example): $y'' + p(x)y' + q(x)y = r(x)$ only need to copy until x^n e.g. xy until $n-1$, x^2y $n-2$, y n

substitute

$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots + a_{n-1}x^{n-1} + a_n x^n + a_{n+1}x^{n+1} + a_{n+2}x^{n+2}$

$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + \cdots + (n-1)a_{n-1}x^{n-2} + n \cdot a_n x^{n-1} + (n+1)a_{n+1}x^n + (n+2)a_{n+2}x^{n+1}$

$y'' = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \cdots + (n-1)(n-2)a_{n-1}x^{n-3} + n(n-1)a_n x^{n-2} + (n+1)n \cdot a_{n+1}x^{n-1} + (n+2)(n+1)a_{n+2}x^n$

$Ax^2 = 0$ $\therefore n \geq 2$

$n \geq$ from no equation $= 0$

a_{n+2} sub in n to

***$a_0 = y(0)$ and $a_1 = y'(0)$ if given**

***$x = 0$ is a regular point if both $p(x)$ and $q(x)$ are defined at $x = 0$**

*** We looking to solve the values of $a_0, a_1, a_2, \dots$ to the solution: $y = a_0 \cdot x + a_1 \cdot x^2 + a_2 \cdot x^3 + \cdots$**

*** When denominators have form: $2 \cdot 4 \cdot 6 \dots$ → it's equivalent to $2^n \cdot n!$**

*** When denominators have form: $1 \cdot 3 \cdot 5 \cdot 7 \dots$ → it's equivalent to $\frac{2^n \cdot n!}{(2n+1)!}$**

1) Solve for the common formula 2) Find the solution in the form: $y = a_0(\dots) + a_1(\dots) + a_2(\dots) + \cdots$ have to write up until n

Linear Differential Equations – Non-Homogeneous – Constant Co-Efficients (Laplace Transform) — $ay''(t) + by'(t) + cy(t) = g(t)$

- We use Laplace transform to solve equations where the **non-homogeneous** part is not differentiable. $y(0) = 0$ $y'(0) = y_1$
- We need to have starting conditions.

Characteristics of Laplace:

General Formula: $\int_0^\infty f(t) \cdot e^{-st}dt$ $\frac{e^{-sc}}{-s}$

Change from s to $(s-a)$: $\mathcal{L}\{e^{at} \cdot f(t)\}(s) = \mathcal{L}\{f\}(s-a)$

Linearity: $\mathcal{L}\{a \cdot f(y) + b \cdot g(y)\} = a\mathcal{L}\{f(y)\} + b\mathcal{L}\{g(y)\}$ take out [illegible]

$L[t^n g(t)] = (-1)^n (L[g(t)])^{(n)}$ tags

it seperated to cases $0<x<1$ $\int_0^1 + \int_1^\infty e^{sc}$

General expansions for equations of form:

$Y(s) = \mathcal{L}[y]$

Order 1: $ay'(t) + by(t) = g(t)$

$\mathcal{L}[ay'(t) + by(t)] = \mathcal{L}[g(t)]$ → $Y(s)[as + b] - y(0)[a] = G(s)$ → $Y(s) = \frac{G(s)+y(0)[a]}{as+b}$ → $y(t) = \mathcal{L}^{-1}[\frac{G(s)+y(0)[a]}{as+b}]$

Order 2: $ay''(t) + by'(t) + cy(t) = g(t)$

$\mathcal{L}[ay''(t) + by'(t) + cy(t)] = \mathcal{L}[g(t)]$ → $Y(s)[as^2 + bs + c] - y(0)[as + b] - y'(0)[a] = G(s)$ →

$Y(s) = \frac{G(s)+y(0)[as+b]+y'(0)[a]}{as^2+bs+c}$ → $y(t) = \mathcal{L}^{-1}[\frac{G(s)+y(0)[as+b]+y'(0)[a]}{as^2+bs+c}]$

Order 3: $ay'''(t) + by''(t) + cy'(t) + dy(t) = g(t)$

$\mathcal{L}[ay'''(t) + by''(t) + cy'(t) + dy(t)] = \mathcal{L}[g(t)]$ → $Y(s)[as^3 + bs^2 + cs + d] - y(0)[as^2 + bs + c] - y'(0)[as + b] - y''(0)[a] =$

$G(s)$ → $Y(s) = \frac{G(s)+y(0)[as^2+bs+c]+y'(0)[as+b]+y''(0)[a]}{as^3+bs^2+cs+d}$ → $y(t) = \mathcal{L}^{-1}[\frac{G(s)+y(0)[as^2+bs+c]+y'(0)[as+b]+y''(0)[a]}{as^3+bs^2+cs+d}]$

get $Y(s)$ then use partial fractions to get back to t

Integrals with e:

$\int e^x dx = e^x + c$ $\int e^{ax}dx = \frac{1}{a}e^{ax} + c$ $\int b \cdot e^{ax}dx = \frac{b}{a}e^{ax} + c$ $\int x \cdot e^x dx$ → Integration by parts → $(x-1) \cdot e^x + c$

$\int x^2 \cdot e^x dx$ → Integrate by parts twice → $(x^2 - 2x + 2) \cdot e^x + x$

Integration by Parts

$\int u \cdot dv = u \cdot v - \int v \cdot du$ Used for: $x \cdot e^x$, $\ln(x)$

ln Integrals

* $\int \frac{1}{x}dx = \ln|x| + c$ * $\int \frac{a}{x}dx = a \cdot \ln|x| + c$ * $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$ * $\int \ln(x)\,dx = x \cdot \ln(x) - x + c$

* $\int \frac{x}{x^2+a}dx$ → Use substitution: $u = x^2 + a$ * $\int \frac{x^2+a}{x}dx$ → Separate into 2 integrals * $\int \frac{ax}{(x-b)^2}dx$ → Substitute: $u = x - b$

$\int \frac{\cos x}{\sin x}dx$ → Substitute: $u = \sin x$ → $= \ln|\sin x| + c$ *

*If degree of numerator is higher than the denominator and we cannot reduce cancel out terms → Use long division!

$\int \sin^2 x\, dx = \frac{-\sin(2x)-2x}{4}+C$ $\int \frac{\cos(x)}{\sin(x)}dx = \ln(|\sin(x)|)+C$ $\int e^{x^2}\cdot 2x\,dx = e^{x^2}+C$

$\int \cos^2 x\, dx = \frac{\cos(x)\sin(x)+x}{2}+C$ $\int x\ln(x)dx = \frac{x^2(2\ln(x)-1)}{4}+C$ $\int e^{f(x)}\cdot f'(x)dx = e^{f(x)}+C$

תחום הגדרה

If we divide by X at any point in the equation: $X \neq 0$. If we need to sub-in starting conditions:

If $X_0 > 0 \Rightarrow$ תחום הגדרה: $X > 0$

If $X_0 < 0 \Rightarrow$ תחום הגדרה: $X < 0$.

נקודות סינגולריות

If we divide by y at any point, $y \neq 0$.
We sub y into original equation and see if it solves the מד"ר. If it does, we add it as a solution.

ln laws

$\ln 1 = 0$
$\ln e = 1$
$\ln x = y \leftrightarrow e^y = x$
$e^{\ln x} = x, x > 0$
$\log_e x = \ln x$
$\ln(e^x) = x, x \in \mathbb{R}$
$\ln(xy) = \ln x + \ln y$
$\ln\left(\frac{x}{y}\right) = \ln x - \ln y$
$\ln(x^r) = r \ln x$

Trig Identities

$\sin(-\alpha) = -\sin\alpha$
$\cos(-\alpha) = \cos\alpha$
$\tan(-\alpha) = -\tan\alpha$
$\sin(90-\alpha) = \cos\alpha$
$\cos(90-\alpha) = \sin\alpha$
$\tan(90-\alpha) = \cot\alpha$
$\cot(90-\alpha) = \tan\alpha$
$\sin(180-\alpha) = \sin\alpha$
$\cos(180-\alpha) = -\cos\alpha$
$\tan(180-\alpha) = -\tan\alpha$
$\tan\alpha = \sin\alpha / \cos\alpha$
$\tan\alpha \cdot \cot\alpha = 1$
$\sin^2\alpha + \cos^2\alpha = 1$
$1 + \tan^2\alpha = 1/\cos^2\alpha$
$1 + \cot^2\alpha = 1/\sin^2\alpha$
$\sin(2\alpha) = 2\sin\alpha\cos\alpha$
$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$
$\cos(2\alpha) = 2\cos^2\alpha - 1$

Common Integrals

מסקנות כלליות

$\sin(2\alpha) = 2\sin\alpha\cos\alpha$ $\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$	$\int \sin x\,dx = -\cos x + C$	$\int \frac{dx}{\cos^2 x} = \tan x + C$	$\cosh x = \frac{e^x + e^{-x}}{2}$
$\cos(2\alpha) = 2\cos^2\alpha - 1$ $\cos(2\alpha) = 1 - 2\sin^2\alpha$	$\int \tan x = -\ln(\cos x) + C$	$\int \frac{dx}{\sin^2 x} = -\cot x + C$	$\sinh x = \frac{e^x - e^{-x}}{2}$
$\sin^2\alpha + \cos^2\alpha = 1$	$\int \frac{dx}{x^2+1} = \arctan x + C$	$\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + C \quad \alpha \neq -1$	$\int \sinh x\,dx = \cosh x + C$ $\int \cosh x\,dx = \sinh x + C$
$\int \cos x\,dx = \sin x + C$	$\int a^x dx = \frac{a^x}{\ln a} + C \quad 0 < a \neq 1$	$\int e^x dx = e^x + C$	$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$

Laplace Example

2. ב

$$y'' + 3y' + 2y = \begin{cases} t & 0 \le t < 1 \\ 2-t & 1 \le t < 2 \\ 0 & t \ge 2 \end{cases}$$

$$y(0) = 0$$
$$y'(0) = 0$$

$$(s^2+3s+2)\mathcal{L}\{y\} = \mathcal{L}\{u_0(t)\cdot t + u_1(t)\cdot[2-2t] + u_2(t)\cdot[t-2]\}$$
$$= \frac{1}{s^2} + 2e^{-s}\left[-\frac{1}{s^2}\right] + e^{-2s}\frac{1}{s^2}$$

$$\mathcal{L}\{y\} = \frac{1}{s^2(s+1)(s+2)} + 2e^{-s}\left[-\frac{1}{s^2(s+1)(s+2)}\right] + e^{-2s}\frac{1}{s^2(s+1)(s+2)}$$

נבצע שברים חלקיים על

$$\frac{1}{s^2(s+1)(s+2)} = \frac{1}{2}\frac{1}{s^2} - \frac{3}{4}\frac{1}{s} + \frac{1}{s+1} - \frac{1}{4}\frac{1}{s+2}$$

ולכן

$$\mathcal{L}\{y\} = [1 - 2e^{-s} + e^{-2s}]\left(\frac{1}{2}\frac{1}{s^2} - \frac{3}{4}\frac{1}{s} + \frac{1}{s+1} - \frac{1}{4}\frac{1}{s+2}\right)$$
$$= [1 - 2e^{-s} + e^{-2s}]\,\mathcal{L}\left\{\frac{1}{2}t - \frac{3}{4} + e^{-t} - \frac{1}{4}e^{-2t}\right\}$$

$$y(t) = u_0(t)\cdot\left[\frac{1}{2}t - \frac{3}{4} + e^{-t} - \frac{1}{4}e^{-2t}\right] - 2u_1(t)\cdot\left[\frac{t-1}{2} - \frac{3}{4} + e^{-(t-1)} - \frac{1}{4}e^{-2(t-1)}\right] + u_2(t)\cdot\left[\frac{t-2}{2} - \frac{3}{4} + e^{-(t-2)} - \frac{1}{4}e^{-2(t-2)}\right]$$

Power Series Example

$$\begin{cases} y'' - xy' - y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$$

נציב $y = \sum_{n=0}^{\infty} a_n x^n$ ונקבל

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$
$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$
$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$
$$\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} - (n+1)a_n]x^n = 0$$

השוואת מקדמים תיתן

$$(n+2)(n+1)a_{n+2} - (n+1)a_n = 0$$
$$\Rightarrow a_{n+2} = \frac{a_n}{n+2}$$

נחשב את המקדמים:

$n=0$: $a_2 = \frac{a_0}{2}$
$n=1$: $a_3 = \frac{a_1}{3}$
$n=2$: $a_4 = \frac{a_2}{4} = \frac{a_0}{2\cdot 4}$
$n=3$: $a_5 = \frac{a_3}{5} = \frac{a_1}{3\cdot 5}$
$n=4$: $a_6 = \frac{a_4}{6} = \frac{a_0}{2\cdot 4\cdot 6}$

ולכן נסיק כי

$$a_{2n} = \frac{1}{2^n n!}a_0$$
$$a_{2n+1} = \frac{2^n n!}{(2n+1)!}a_1$$
$$\Rightarrow y = a_0\sum_{n=0}^{\infty}\frac{1}{2^n n!}x^{2n} + a_1\sum_{n=0}^{\infty}\frac{2^n n!}{(2n+1)!}x^{2n+1}$$

כעת נתייחס לתנאי ההתחלה. נשים לב כי ניתן לרשום את הביטוי שחישבנו עבור y באופן הבא

$$\Rightarrow y = a_0 + a_1 x + a_0\sum_{n=1}^{\infty}\frac{1}{2^n n!}x^{2n} + a_1\sum_{n=1}^{\infty}\frac{2^n n!}{(2n+1)!}x^{2n+1}$$

ניתן לראות מיידית כי $y(0) = a_0 = 1$ (הצבנו $x = 0$ והשתמשנו בתנאי ההתחלה).כדי להציב את תנאי ההתחלה השני נגזור את y

$$y' = a_1 + a_0\sum_{n=1}^{\infty}\frac{1}{2^n n!}\cdot 2n\cdot x^{2n-1} + a_1\sum_{n=1}^{\infty}\frac{2^n n!}{(2n+1)!}(2n+1)x^{2n}$$

ולכן תנאי ההתחלה על הנגזרת ייתן $y'(0) = a_1 = 0$.

לסיכום:

$$y = \sum_{n=0}^{\infty}\frac{1}{2^n n!}x^{2n}$$

Power Series

R	מקלורן	פונקציה
∞	$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\cdot x^{2n+1}$	sin sin (x
∞	$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\cdot x^{2n}$	cos cos (x)
1	$\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\cdot x^{2n+1}$	arctan arc
1	$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cdot x^n$	ln ln (1 +
∞	$\sum_{n=0}^{\infty}\frac{1}{n!}\cdot x^n$	e^x
1	$\sum_{n=0}^{\infty} x^n$	$\frac{1}{1-x}$

COMMON DERIVATIVES

$\frac{d}{dx}(x) = 1$
$\frac{d}{dx}(\sin x) = \cos x$
$\frac{d}{dx}(\cos x) = -\sin x$
$\frac{d}{dx}(\tan x) = \sec^2 x$
$\frac{d}{dx}(\sec x) = \sec x \tan x$
$\frac{d}{dx}(\csc x) = -\csc x \cot x$
$\frac{d}{dx}(\cot x) = -\csc^2 x$
$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
$\frac{d}{dx}(a^x) = a^x \ln(a)$
$\frac{d}{dx}(e^x) = e^x$
$\frac{d}{dx}(\ln(x)) = \frac{1}{x}, x > 0$
$\frac{d}{dx}(\ln|x|) = \frac{1}{x}$
$\frac{d}{dx}(\log_a(x)) = \frac{1}{x\ln(a)}$

$y'' - y' - 2y = 0$, $y(0) = 1$, $y'(0) = 0$

$\mathcal{L}(y'') - \mathcal{L}(y') - 2\mathcal{L}(y) = \mathcal{L}(0) = 0$

$[s^2 y(s) - s y(0) - y'(0)] - [s y(s) - y(0)] - 2y(s) = 0$

$\Rightarrow s^2 y(s) - s - 0 - s y(s) + 1 - 2y(s) = 0$

$y(s) = \frac{s-1}{s^2-s-2}$

אחרי חלקים

$e^{at} = \frac{1}{s-a}$

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