# Quantum Mechanics Problems

1. Infinite square well · 1D i). Calculate $P(x,t)[\equiv P(x)]$ : $P(x) = A\frac{2}{a}\sin^2\left(\frac{n\pi x}{a}\right)$ ii). For what values is this maximum?

iii). Given $\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$ & $\psi_m(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{m\pi x}{a}\right)$. Show that $\int_0^a \psi_m^* \psi_n \, dx = 0$.

iv). Given $\psi_n(x) = A\sin\left(\frac{n\pi x}{a}\right)$, find $A$. What is the probability of finding the particle between $x=0$ & $x=\frac{a}{4}$, when it is in its ground state?

v). Given $\psi(x) = C(a^2 - x^2)$ for $-a < x < a$
$= 0$, otherwise. Find $C$.

2. The energy eigenvalue and the corresponding eigenfunction for a particle of mass 'm' in a 1-D potential $V(x)$ are $E=0$ & $\psi(x) = \frac{A}{x^2+a^2}$ respectively (A - constant). Determine $V(x)$.

3. A system has two energy ~~eigenstates~~ eigenvalues, $E_0$ and $3E_0$. $\psi_1(x,t)$ and $\psi_2(x,t)$ are the corresponding normalized eigenfunctions, or eigenstates which form a complete orthonormal set. At some time the total w.f. of the system is given by

$$\psi(x,t) = c_1\psi_1(x,t) \text{ and } c_2\psi_2(x,t), \text{ with } c_1 = \frac{1}{\sqrt{2}}.$$

i). Assuming $\psi(x,t)$ is normalized, find $c_2$.

ii). What is the probability that an energy measurement will yield $3E_0$?

iii). Find out the energy expectation value.

4. Find the expression of the momentum eigenfunctions. $\rightarrow$ Sol. of $-i\hbar\frac{\partial}{\partial x}\psi(x) = p\psi(x)$

5. For the particle in the box, find $\langle x\rangle$, $\langle p_x\rangle$, $\langle x^2\rangle$, $\langle p_x^2\rangle$.

6. a) $\psi_n(x) = \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}$ {for 1D infinite well from 0 to a}. Find $\langle x\rangle$, $\langle p_x\rangle$ for $n=1$

Also find $\langle x^2\rangle$, $\langle p_x^2\rangle$.

b). For what value of $x$, $|\psi_1(x)|^2$ is a maximum?

7. In class, for $V \equiv V(x)$ only, we stated that the general solution, $\psi(x,t)$, of the time-dependent Schrödinger equation can be written as a linear combination of the wave functions corresponding to different eigenstates, $\psi_n(x,t) = \psi_n(x)e^{-iE_nt/\hbar}$, where $\psi_n(x)$ - $n$th eigenfunction of the Hamiltonian with energy eigenvalue $E_n$. That is,

$$\psi(x,t) = \sum_{n=1}^{\infty} c_n\psi_n(x)e^{-iE_nt/\hbar}, \quad c_n \rightarrow \text{constants}.$$

(i). Verify that $\psi(x,t)$ is indeed a solution of the time-dependent Schrödinger equation.

(ii). Show that $\sum_{n=1}^{\infty}|c_n|^2 = 1$. You may assume that $\psi(x,t)$ is normalized. What is the physical significance of $|c_n|^2$?

8. Given an ~~normal~~ arbitrary wave function $\psi(x,t)$, how will you normalize it? What does normalization mean physically?

9. ~~The ground state and~~ Suppose a quantum system has two eigenstates $\psi_0$ (ground state) and $\psi_1$ (excited state) with corresponding energy eigenvalues being $E_0$ and $E_1$ respectively. If there is a 40% chance of finding the system in its ground state and 60% in its excited state,

1) what is the general wave function of the atom? $|c_1|^2 = 0.4$ & $|c_2|^2 = 0.6$ $\therefore \psi = \sqrt{0.4}\,\psi_0 + \sqrt{0.6}\,\psi_1$

2) What is the average energy of the system? $\langle E\rangle = \int(\sqrt{0.4}\,\psi_0 + \sqrt{0.6}\,\psi_1)\hat{H}(\sqrt{0.4}\,\psi_0 + \sqrt{0.6}\,\psi_1)\,dx$

$= \int(\sqrt{0.4}\,\psi_0 + \sqrt{0.6}\,\psi_1)(\sqrt{0.4}\,E_0\psi_0 + \sqrt{0.6}\,E_1\psi_1)\,dx$

$= 0.4E_0 + 0.6E_1$.

10. A system has two energy eigenstates $\psi_1$ and $\psi_2$ with corresponding energy eigenvalues $E_0$ and $3E_0$ respectively. The general wave function of the system may then be written as,

$$\psi = c_1\psi_1 + c_2\psi_2 . \qquad \text{Given: } c_2 = \frac{1}{\sqrt{2}}$$

W i). Find $c_1$ if $\psi$ is normalized.

ii). What is the probability that an energy measurement will yield the value $3E_0$?

iii). Find out the energy expectation value.

11. The wave function $\psi(x,t)$ of a particle at $t=0$ is given by

$$\psi(x,0) = \begin{cases} Ax(x-1), & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases} \qquad A\text{ - constant}$$

i). Calculate the probability of finding the particle in the region $0 \le x \le 0.5$ at

ii). What is the average position of the particle at $t=0$?

iii) ~~What will be the maximum probability~~ At what position will the probability of finding the particle at $t=0$ be maximum?

12. A particle is limited to the x-axis has the wave function $\psi(x) = ax$ between $x=0$ and $x=1$; $\psi(x)=0$ elsewhere. Find the probability that the particle can be found between $x=0.45$ and $x=0.55$.

13. Show that (i) $[\hat{x}, \hat{p}_x] = i\hbar$, (ii) $[\hat{x}, \hat{p}_x^n] = i\hbar n \hat{p}_x^{n-1}$, (iii) $[\hat{x}^n, \hat{p}_x] = n i\hbar \hat{x}^{n-1}$
(iv) $[\hat{x}, \hat{p}_y] = 0$, (v) $[\hat{y}, \hat{p}_y] = i\hbar$, (vi) $[\hat{z}, \hat{p}_z] = i\hbar$

14. Show that (i) $[\hat{x}, \hat{L}_x] = 0$, (ii) $[\hat{x}, \hat{L}_y] = i\hbar\hat{z}$, (iii) $[\hat{y}, \hat{L}_z] = i\hbar\hat{x}$, (iv) $[\hat{z}, \hat{L}_x] = i\hbar\hat{y}$
(iv) $[\hat{y}, \hat{L}_y] = 0$, (v) $[\hat{z}, \hat{L}_z] = 0$, (vi) $[\hat{p}_x, \hat{p}_y] = 0$

15. Given that $[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$, $[\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x$, $[\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y$. show that
(i). $[\hat{L}_x, \hat{L}^2] = 0$, (ii) $[\hat{L}_y, \hat{L}^2] = 0$, (iii) $[\hat{L}_z, \hat{L}^2] = 0$ where $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$.

16. Find $[\hat{A}, \hat{B}]$ where $\hat{A} = \hat{x}^3$ and $\hat{B} \equiv \hat{x}\hat{p}_x$.

17. Can the following functions be called wave functions? Explain. (i) $\psi = \sin^{-1}x$, (ii) $\psi = \frac{\sin x}{x}$
(iii) $\psi(x) = e^{x}$, (iv) $\psi(x) = e^{-x^2}$

18. Show that $[\hat{p}_z, \hat{L}_x] = i\hbar\hat{p}_y$. Calculate $[\hat{p}_x, \hat{L}_y]$, $[\hat{p}_y, \hat{L}_z]$.

19. Calculate i). $[\hat{p}_x, \hat{L}_x]$ ii) $[\hat{p}_y, \hat{L}_y]$ iii) $[\hat{p}_z, \hat{L}_z]$.

20. Particle in 1-D box: $\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$, $n = 1, 2, 3, \ldots$ Is $\psi_n(x)$ an eigenfunction of the momentum operator $\hat{p}_x$? of $\hat{p}_x^2$?