Assignment : Probability and Numerical Methods

Subject Code: MATH2202

Module-I

1. Use **Bisection** Method to solve the following equations:
   (a) $f(x) = x + \log_e x - 2 = 0$, correct 3 significant figures. [Ans: 1.56]
   (b) $3x - \sqrt{1 - \sin x} = 0$ correct upto 2 decimal places. [Ans: 10.39]

2. Use **Regula-Falsi** Method to solve the following equations:
   (a) $f(x) = 2x - \log_{10} x - 7 = 0$ correct upto 3 decimal places. [Ans: 3.789]
   (b) $f(x) = \sin x + \cos x - 1 = 0,$ correct upto 4 significant figures. [Ans: 1.571]

3. Use **Newton-Rapshson** Method to solve the following equations:
   (a) $f(x) = \log_e x - \cos x = 0$, correct upto 3 decimal places. [Ans: 1.303]
   (b) Evaluate $\sqrt[5]{3}$ correct upto 5 significant figures.

4. Solve the following system of linear equations by **Gauss Elimination** method:

   (a) $$\begin{aligned} 2x + y + z &= 10 \\ 3x + 2y + 3z &= 18 \\ x + 4y + 9z &= 16 \end{aligned}$$
   [Ans: x=7, y=-9, z=5]

   (b) $$\begin{aligned} 2x - y + 3z &= 4 \\ x \quad\;\; + z &= 2 \\ 2y + z &= 3 \end{aligned}$$
   [Ans: x=1, y=1, z=1]

5. Solve the following system of linear equations by **Gauss- Seidel** Method:

   (a) $$\begin{aligned} 2x + 10y + z &= 13 \\ 10x + y + z &= 12 \\ 2x + 2y + 10z &= 14 \end{aligned}$$ correct upto 2 decimal places. $\left[Ans: \begin{aligned} x &= 0.99 \\ y &= 0.99 \\ z &= 1.00 \end{aligned} \right]$

   (b) $$\begin{aligned} x + 4y + 2z &= 17 \\ x + 2y + 4z &= 16 \\ 6x - y + 4z &= 13 \end{aligned}$$ correct upto 3 decimal places. $\left[Ans: \begin{aligned} x &= 1.091 \\ y &= 2.818 \\ z &= 2.318 \end{aligned} \right]$

6. Solve the following system of linear equations by **LU-Factorization Method** :

   (a) $$\begin{aligned} 3x - y + 2z &= 1 \\ 2x + 4y - z &= 3 \\ 7x + y + z &= 3 \end{aligned}$$
   [Ans: x=1/4, y=3/4, z=1/2]

   (b) $$\begin{aligned} 3x + 4y + 2z &= 15 \\ 5x + 2y + z &= 18 \\ 2x + 3y + 2z &= 10 \end{aligned}$$
   [Ans: x=3, y=2, z=-1]

7. Find the **missing term** from the table by using the difference table

x	0	1	2	3	4
Y	1	3	9	--	81

[Ans. 31]

8. Find **missing terms** from the following table by using the difference table

x	2.0	2.1	2.2	2.3	2.4	2.5	2.6
y=f(x)	0.135	--	0.111	0.100	--	0.082	0.024

[Ans. f(2.1)=0.132, f(2.4)=0.90]

9. Find $\tan 0.12$, $\tan 0.26$ and $\tan 0.35$, $\tan 0.5$ from the following table

x	0.10	0.15	0.20	0.25	0.30
y=$\tan x$	0.1003	0.1511	0.2027	0.2553	0.3093

[Ans. $\tan 0.12 = 0.1205, \tan 0.26 = 0.2662, \quad \tan 0.35 = 0.365300$, $\tan 0.5 = 0.5543$]

11. Write down the interpolating polynomial expression using the following data and hence find f(0.5)

x	-1	0	1	2
y=f(x)	1	1	1	-3

[Ans. $-\frac{1}{3}(2x^3 - 2x - 3)$]

12. Find the **Lagrange interpolating** polynomial of degree 2 approximating the function $y = \ln x$ defined by the tabular values. Hence find $\ln 2.7$.

x	2	2.5	3.0
$y = \ln x$	0.69315	0.91629	1.09861

. [Ans. $f(x) = -0.08164x^2 + 0.81366x - 0.60761$, $\ln 2.7$ =0.9941164]

13. Evaluate $\int_0^1 \sqrt{1-x^2}dx$ using **Trapezoidal and Simpson's 1/3 rule** for n=6.

[Ans. 0.765496, 0.777532]

14. Find from the table, the area under the curve & the x-axis from x=7.47 to x=7.52

x	7.47	7.48	7.49	7.50	7.51	7.52
f(x)	1.93	1.95	1.98	2.01	2.03	2.06

[Ans. 0.0996]

15. Evaluate $\int_1^5 log_{10} x \; dx$ taking n=8 by using suitable numerical method.

[Ans. 1.750505025]

17. Use **Euler's method** to compute $y(0.2)$ & take $h = 0.05$, $\frac{dy}{dx} = x^2 + 4y$, $y(0) = 1$.

[Ans. $y(0.5) = 1.82524, y(0.1) = 1.0933, y(0.15) = 1.7286, \; y(0.2) = 2.0754$]

18. Find $y(0.2)$, $y(0.5)$ by **Modified Euler's method** for $\frac{dy}{dx} = \log_{10}(x+y)$, $y(0) = 1$, take $h = 0.2$.

[Ans. $y(0.2) = 1.0082,\ y(0.5) = 1.0490$]

19. Use **RK method** to find $y(0.5)$ and $y(1)$ for $\frac{dy}{dx} = \frac{1}{x+y},\ y(0) = 1$ by taking h=0.5.

[Ans. y(0.5)=1.357, y(1)=1.584]

Assignment: MATH 2202
(MODULES II AND III)

FUNDAMENTALS OF PROBABILITY

1.The probabilities of X, Y and Z becoming the principal of a college are respectively 0.3, 0.5 and 0.2. The probabilities that "student aid fund" will be introduced in the college if X, Y and Z become principal are 0.4, 0.6 and 0.1 respectively. Given that student aid fund is introduced, find the probability that Y has been appointed as the principal.

2. Three urns contain respectively 1 white and 2 black balls, 2 white and 1 black ball, 2 white and 2 black balls. One ball is transferred from the first to the second urn; then one ball is transferred from the second to the third urn; finally one ball is drawn from the third urn. Find the probability that the ball is white.What will be the probability if the ball is black?

3. Suppose each of three persons tosses a coin. If the outcome of one of the tosses differs from the other outcomes, then the game ends. If not, then the persons start over and retoss their coins. Assuming fair coins, what is the probability that the game will end with first round of tosses? If all three coins are biased and have probability $1/4$ of landing heads, what is the probability that the game will end at the first round?

4. If $P(A \cap B \cap C) = 0$, then show that $P(X|C) = P(A|C) + P(B|C)$. Here $X = A \cup B$ and it is given that $P(C) > 0$.

5. A system consists of a controller and three peripheral units. The system is said to be "up" if the controller and at least two of the peripherals are functioning. Find the probability that the system is "up", assuming that all components fail independently. [You can assume probability of controller functioning as a and the probability that each peripheral fails as b]

6. A communication system transmits binary information over a channel that introduces random bit errors with probability $\epsilon = 10^{-3}$. The transmitter transmits each information bit three times,and a decoder takes a majority vote of the received bits to decide on what the transmitted bit was. Find the probability that the receiver will make an incorrect decision.

7. A student needs eight chips of a certain type to build a circuit. It is known that 5% of these chips are defective. How many chips should he buy for there to be a greater than 90% probability of having enough chips for the circuit?

PROBABILITY DISTRIBUTIONS

8. If a random variable X has the following probability mass function

X	1	2	3	4	5	6	7
$P(X = x_i)$	K	$2K$	$2K$	$3K$	K^2	$2K^2$	$7K^2 + K$

(i) Find K.
(ii) Evaluate $P(X < 6)$,$P(X \geq 6)$, $P(X > 7)$and $P(0 < X < 5)$.
(iii) If $P(X \leq a) > \frac{1}{2}$, find the minimum value of a, and
(iv) Determine the distribution function of X
(v) Find the mean and variance of X.

9. Show that

$$f(x) = \begin{cases} x & ; 0 < x < 1 \\ k - x & ; 1 < x < 2 \\ 0 & ; \text{otherwise} \end{cases} \tag{1}$$

is a density function for a suitable value of k? Calculate the distribution function of the random variable X. Also calculate $P(\frac{1}{2} < X < \frac{3}{2})$. Find the mean and variance of X.

10. If the probability density function $f(x)$ of a random variable X is defined by

$$f(x) = ce^{-|x|}, -\infty < x < \infty$$

Find c. Evaluate the mean and variance.

11. A system uses triple redundancy for reliability: Three microprocessors are installed and the system is designed so that it operates as long as one

microprocessor is still functional. Suppose that the probability that a microprocessor is still active after t seconds is $p = e^{-\lambda t}$. Find the probability that the system is still operating after t seconds.

12. The number of page requests that arrive at a Web server is a Poisson random variable with an average of 6000 requests per minute.
(a) Find the probability that there are no requests in a 100 ms period.
(b) Find the probability that there are between 5 and 10 requests in a 100 ms period.

13. A data center has 10,000 disk drives. Suppose that a disk drive fails in a given day with probability 10^{-3} .
(a) Find the probability that there are no failures in a given day.
(b) Find the probability that there are fewer than 10 failures in two days.
(c) Find the number of spare disk drives that should be available so that all failures in a day can be replaced with probability 99%.

14. A communication system accepts a positive voltage V as input and outputs a voltage $Y = aV + N$, where $a = 10^{-2}$ and N is a Gaussian(Normal) random variable with mean as zero and standard deviation 2.Find the value of V that gives $P[Y < 0] = 10^{-6}$.

15. Suppose that during rainy season, on a tropical island, the length of shower has an exponential distribution with average length of shower $\frac{1}{2}$ mins. What is the probability that a shower will last more than three minutes? If a shower has already lasted for 2 minutes, what is the probability that it will last for at least one more minute?

16. A point P is chosen at random on a straight line segment AB of length $2a$. Find the probability that the area of the rectangle AP, PB will exceed $\frac{a^2}{2}$.

STATISTICS

17. If $x = 4y + 5$ and $y = kx + 4$ are two regression equations of x on y and y on x respectively, obtain the interval in which k lies.

18. For two variables x and y the equations of two regression lines are $x + 4y + 3 = 0$ and $4x + 9y + 5 = 0$. Identity which one is of "y on x". Find the means of x and y. Find the correlation coefficient between x and y. Estimate the values of x when $y = 1.5$.

19. Find the missing frequency of the following data if its mode is 25:

Class	0-10	10-20	20-30	30-40	40-50	50-60	60-70
Frequency	14	22	27	?	23	20	15

20. A frequency distribution is provided below:

Class	Below 10	Below 20	Below 30	Below 40	Below 50
Frequency	3	8	17	20	22

Find the median and mode of the distribution.

MISCELLANEOUS

21. The Pareto random variable arises in the study of the distribution of wealth where it has been found to model the tendency for a small portion of the population to own a large portion of the wealth. Recently the Pareto distribution has been found to cap- ture the behavior of many quantities of interest in the study of Internet behavior, e.g., sizes of files, packet delays, audio and video title preferences, session times in peer-to-peer networks, etc. The pdf of the Pareto random variable is given by:

$$f(x) = \begin{cases} 0 & ; x < x_m \\ \alpha \frac{x_m^\alpha}{\alpha+1} & ; x \geq x_m \end{cases} \tag{2}$$

Find the mean and variance of the distribution.

22. Show that the exponential distribution posseses *the memoryless* property.

23. A fair coin is tossed 400 times. using normal approximation to binomial distribution, find the probability of obtaining
(i) exactly 200 heads
(ii)between 190 and 210 heads both inclusive.

24. A coin, having probability p of coming up heads, is to be successively flipped until the first head appears. What is the expected number of flips required?

25. A miner is trapped in a mine containing three doors. The first door

leads to a tunnel that takes him to safety after two hours of travel. The second door leads to a tunnel that returns him to the mine after three hours of travel. The third door leads to a tunnel that returns him to his mine after five hours. Assuming that the miner is at all times equally likely to choose any one of the doors, what is the expected length of time until the miner reaches safety?

1. Three balls a,b,c are randomly distributed into three boxes (a box may contain any number of balls). Let X_1 and X_2 be the number of balls in box 1 and box 2 respectively. Find the joint p.m.f. of X_1 and X_2. Also compute $E(X_1 + X_2)$ and the marginal p.m.f's of X_1 and X_2.

2. The joint p.m.f of two random variables $X\ and\ Y$ is given by: $P(X = 0, Y = 1) = \frac{1}{3}$, $P(X = 1, Y = -1) = \frac{1}{3}$ and $P(X = 1, Y = 1) = \frac{1}{3}$. Find the marginal p.m.f. of $X\ and\ Y$. Also find the conditional p.m.f. of *X* given *Y=1*.

3. For the adjoining bivariate probability distribution of $X\ and\ Y$, find:
   (i) $P(X \leq 1, Y = 2)$ (ii) $P(X \leq 1)$ (iii) $P(Y \leq 3)$ (iv) $P(X < 3, Y \leq 4)$

Y \ X	1	2	3	4	5	6
0	0	0	1/32	2/32	2/32	3/32
1	1/16	1/16	1/8	1/8	1/8	1/8
2	1/32	1/32	1/64	1/64	0	2/64

4. The random variables $X\ and\ Y$ have a joint probability mass function given by:
$$P(X = x, Y = y) = \frac{x^2 + y^2}{32} \quad for\ x = 0, 1, 2, 3\ and\ y = 0, 1$$
Find the marginal p.m.f. of of $X\ and\ Y$. Also find $P(X \leq 2, Y \geq 1)$.

5. The random variables $X\ and\ Y$ have a joint probability mass function given by:
$$f(x, y) = \frac{2x + y}{27} \quad for\ x, y = 0, 1, 2$$
Find the marginal p.m.f. of of $X\ and\ Y$ and expected value for both the random variables $X\ and\ Y$.

6. The random variables $X\ and\ Y$ have a joint probability mass function given by:
$$f(x, y) = \frac{xy + 2x + y + 2}{15} \quad for\ x = 0, 1, y = 0,1$$
Check if $X\ and\ Y$ are independent.

7. The random variables $X\ and\ Y$ have a joint probability mass function

X / Y	1	2	3
1	0.1	0.1	0.2
2	0.2	0.3	0.1

   find the conditional p.m.f. of *X* given *Y=1*. Also find $E(X|Y = 1)$ and $V(X|Y = 1)$. What's the value of $P(X + Y < 4)$.

8. A fair coin is tossed three times. Let X denote the number of heads in three tossings and Y denote the absolute difference between the number of heads and the number of tails. Find the joint p.m.f. of (X, Y). Also find the marginal p.m.f. of X and Y. Are these two random variables X and Y independent? Find the conditional p.m.f. of X, given Y=1.

9. The random variables $X\ and\ Y$ have a joint p.d.f.

$$f(x,y) = \begin{cases} \frac{1}{4}(1+xy)\,, & |x|<1,\ \ |y|<1 \\ 0 & ,\ \ otherwise \end{cases}$$

Find the marginal p.d.f. of of $X\ and\ Y$. Are they independent?

10. $X\ and\ Y$ are two independent random variables with following individual p.m.f.'s $f_X(1) = \frac{1}{3}$, $f_X(2) = \frac{2}{3}$, $f_X(3) = \frac{1}{3}$, $and\ f_X(x) = 0$ for any other value of *x*. $f_Y(5) = \frac{1}{4}$, $f_Y(8) = \frac{3}{4}$, $f_Y(y) = 0$ for $y \neq 5, 8$. Find the joint p.m.f. of $X\ and\ Y$. How much is $E(XY)$?

11. If $X\ and\ Y$ have a joint p.d.f. $f(x,y) = \begin{cases} 6x^2y\,, 0<x<1\,, 0<y<1 \\ \quad 0 \qquad\qquad ,otherwise \end{cases}$

    Find $P(X+Y<1)$ and $P(X>Y)$.

12. The joint p.d.f. of $X\ and\ Y$ is $f(x,y) = \begin{cases} 2\,, 0<x<1\,, 0<y<x \\ 0 \qquad\qquad ,otherwise \end{cases}$

    Find the marginal p.d.f. of $X\ and\ Y$. are they independent?

13. The joint p.d.f. of $X\ and\ Y$ is $f(x,y) = \begin{cases} e^{-(x+y)}\,, x\geq 0\,, y\geq 0 \\ 0 \qquad\qquad ,otherwise \end{cases}$

    Find $P(X>1)$ and $P(1<X+Y<2)$.

14. If $X\ and\ Y$ have a joint p.d.f. $f(x,y) = \begin{cases} k \quad for\ \ x^2+y^2 \leq 4 \\ 0 \qquad\qquad ,otherwise \end{cases}$

    where k is a constant. Find the value of k and $P(x^2+y^2>1)$. Also find the marginal p.d.f. of *Y*.

15. The joint distribution of X and Y is given by

$$\mathrm{f(x,y)} = \begin{cases} \frac{1}{2}\,, & \mathrm{if\ (x,y)} \in \mathcal{R} \\ 0\ , & \mathrm{otherwise} \end{cases}$$

Where $\mathcal{R}$ is the interior of the triangle of area 2 square units having vertices $(0,0), (2,0)$ and $(1,2)$. Find the marginal density function of X and Y. Find $\mathrm{P(X \leq 1, Y \leq 1)}$ and $\mathrm{P(X \leq 1 |\ Y \leq 1)}$.

Markov chain

1. Two white balls and two black balls are distributed into two urns so that each urn contains two balls. Then one ball is randomly selected from each urn and their places are interchanged. This process of selecting balls and interchanging urns is repeated multiple times. Let X_n denote the number of white balls in the first urn after repeating this process n times. What is the state space of this markov chain? Find out the underlying transition probability matrix.
2. A box contains 4 balls. Every ball is either white or red. Two balls are randomly picked up and are randomly picked up and are replaced by two other balls of complementary colour. Let X_n denote the number of white balls in the first urn

after repeating this process n times. What is the state space of this markov chain? Find out the underlying transition probability matrix.

3. Write down the transition probability matrix for the same markov chain as above with the change mechanism modified as:-
   (i) caseI:- each time one ball is picked up and colour complemented before replacement.
   (ii) Case-II each time three balls are picked up and colour complemented before replacement.
4. $X_1, X_2, X_3, \dots$ form a markov chain where each X_i can only assume numerical values 0, 1, 2. The transition probability matrix of the markov chain is as follows:

$$P = \begin{bmatrix} 1/3 & 0 & 2/3 \\ 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

   Find (i) $P(X_8 = 2|X_6 = 0)$ (ii) $P(X_5 = 0|X_3 = 2)$

   (i) If the markov chain starts from the state 0 i.e. $X_1 = 0$, then find $E(X_3)$ and $V(X_3)$.
5. A fair dice is tossed repeatedly. If X_n denotes the maximum of the numbers occurring in the first n tosses, find the transition probability matrix P of the Markov chain $\{X_n\}$. Find also P^2 and $P(X_2 = 6)$.
6. There are 2 white marbles in urn A and 3 red marbles in urn B. At each step of the process, a marble is selected from each urn and the 2 marbles selected are interchanged. Let the state a_i of the system be the number of red marbles in A after i changes. What is the probability that there are 2 red marbles in A after 3 steps? In the long run, what is the probability that there are 2 red marbles in urn A?