Cosets

Let G be a group and H a subgroup of G. Define a **left coset** of H with representative $g \in G$ to be the set

$$gH = \{gh : h \in H\}.$$

Right cosets can be defined similarly by

$$Hg = \{hg : h \in H\}.$$

If left and right cosets coincide or if it is clear from the context to which type of coset that we are referring, we will use the word coset without specifying left or right.

Example 1. Let H be the subgroup of Z_6 consisting of the elements 0 and 3. The cosets are

$$0 + H = 3 + H = \{0, 3\}$$

$$1 + H = 4 + H = \{1, 4\}$$

$$2 + H = 5 + H = \{2, 5\}.$$

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Example 2. Let H be the subgroup of S_3 defined by the permutations {(1), (123), (132)}. The left cosets of H are

$$(1)H = (123)H = (132)H = \{(1), (123), (132)\}$$

$$(12)H = (13)H = (23)H = \{(12), (13), (23)\}.$$

The right cosets of H are exactly the same as the left cosets:

$$H(1) = H(123) = H(132) = \{(1),(123),(132)\}$$

$$H(12) = H(13) = H(23) = \{(12),(13),(23)\}.$$

It is not always the case that a left coset is the same as a right coset. Let K be the subgroup of S_3 defined by the permutations {(1), (12)}. Then the left cosets of K are

$$(1)K = (12)K = \{(1), (12)\}$$

$$(13)K = (123)K = \{(13), (123)\}$$

$$(23)K = (132)K = \{(23), (132)\};$$

however, the right cosets of K are

$$K(1) = K(12) = \{(1), (12)\}$$

$$K(13) = K(132) = \{(13), (132)\}$$

$$K(23) = K(123) = \{(23), (123)\}.$$

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Lemma 6.1 Let H be a subgroup of a group G and suppose that g_1, g_2 $\in$ G. The following conditions are equivalent.

1. $g_1H = g_2H$;
2. $Hg_1^{-1} = Hg_2^{-1}$;
3. $g_1H \subseteq g_2H$;
4. $g_2 \in g_1H$;
5. g_1^{-1} g2 $\in$ H.

In all of our examples the cosets of a subgroup H partition the larger group G. The following theorem proclaims that this will always be the case.

Theorem 6.2 Let H be a subgroup of a group G. Then the left cosets of H in G partition G. That is, the group G is the disjoint union of the left cosets of H in G.

Proof. Let g_1H and g_2H be two cosets of H in G. We must show that either $g_1H \cap g_2H = \varnothing$ or $g_1H = g_2H$. Suppose that $g_1H \cap g_2H \neq \varnothing$ and a $\in g_1H \cap g_2H$. Then by the definition of a left coset, a = $g_1h_1 = g_2h_2$ for some elements h_1 and h_2 in H. Hence, $g_1 = g_2\, h_2\, h^{-1}$ or $g_1 \in g_2$ H. By Lemma 6.1 $g_1H=g_2H$.

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Remark. There is nothing special in this theorem about left cosets. Right cosets also partition G; the proof of this fact is exactly the same as the proof for left cosets except that all group multiplications are done on the opposite side of H.

Let G be a group and H be a subgroup of G. Define the **index** of H in G to be the number of left cosets of H in G. We will denote the index by [G : H].

Example 3. Let $G = Z_6$ and H = {0,3}. Then [G : H] = 3.

Example 4. Suppose that $G = S_3$, H = {(1),(123),(132)}, and K = {(1),(12)}. Then[G:H]=2 and [G:K]=3.

Theorem 6.3 Let H be a subgroup of a group G. The number of left cosets Of H in G is the same as the number of right cosets of H in G.

Proof. Let L_H and R_H denote the set of left and right cosets of H in G, respectively. If we can define a bijective map $\phi : L_H \to R_H$, then the theorem will be proved. If $gH \in L_H$, let $\phi(gH) = Hg^{-1}$. By Lemma 6.1, the map ϕ is well-defined; that is, if $g_1H = g_2H$, then $Hg_1^{-1} = Hg_2^{-1}$. To show that ϕ is one-to-one, suppose that

$$Hg_1^{-1} = \phi(g_1H) = \phi(g_2H) = Hg_2^{-1}.$$

Again by Lemma 6.1, $g_1H = g_2H$. The map ϕ is onto since $\phi(g^{-1}H) = Hg$.

6.2 Lagrange's Theorem

Proposition 6.4 Let H be a subgroup of G with $g \in G$ and define a map $\phi : H \to gH$ by $\phi(h) = gh$. The map ϕ is bijective; hence, the number of elements in H is the same as the number of elements in gH.

Proof. We first show that the map ϕ is one-to-one. Suppose that $\phi(h_1) = \phi(h_2)$ for elements $h_1, h_2 \in H$. We must show that $h_1 = h_2$, but $\phi(h_1) = gh_1$ and $\phi(h_2) = gh_2$. So $gh_1 = gh_2$, and by left cancellation $h_1 = h_2$. To show that ϕ is onto is easy. By definition every element of gH is of the form gh for some $h \in H$ and $\phi(h)=gh$.

Theorem 6.5 (Lagrange) Let G be a finite group and let H be a subgroup of G. Then $|G|/|H| = [G : H]$ is the number of distinct left cosets of H in G. In particular, the number of elements in H must divide the number of elements in G.

Proof. The group G is partitioned into $[G : H]$ distinct left cosets. Each left coset has $|H|$ elements; therefore, $|G| = [G : H]|H|$.

Corollary 6.6 Suppose that G is a finite group and $g \in G$. Then the order of g must divide the number of elements in G.

Corollary 6.7 Let $|G| = p$ with p a prime number. Then G is cyclic and any $g \in G$ such that $g \neq e$ is a generator.

Proof. Let g be in G such that $g \neq e$. Then by Corollary 6.6, the order of g must divide the order of the group. Since $|\langle g \rangle| > 1$, it must be p. Hence, g generates G.

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