POSETS & LATTICE

① Let S be the set of all lines in 3D space.
A relation ρ is defined on S by "lρm if and only if 'l' lies on the plane of m" for l, m ∈ S.
Examine if ρ is an equivalence relation.

Ans:-
Let l, m ∈ S be two line from all lines in 3D space.

Reflexive:- If a line 'l' exists on a plane then it must lie in its own plane.

Anti Symmetric:- Let l, m ∈ S.
If 'l' lies on the plane of 'm',
then 'm' have to lie on the plane of 'l'.

Transitive:- Let l, m, n ∈ S.
If 'l' lies on the plane of 'm'.
and 'm' lies on the plane of 'n'.
Then 'l' may not lie on the plane of 'n'.

eg:- If 'm' is the line of intersection of planes containing 'l' and 'n'.
Then transitive property does not hold.

∴ It is not equivalence.

(2) Define a relation R on a set Z of all integers by mRn, if only if $m^2 = n^2$. Is R a poset?

Ans:- let $m, n \in Z$.

We have check whether R is a poset. Then we will show whether it satifies the 3 properties.

Reflexive:- If ~~...~~ $m = m$ where $m \in Z$. then obviously $m^2 = n^2$ will be true as $m^2 = m^2$ will always be true.

Anti Symmetric:- let $m, n \in Z$

Now if mRn, then we will have to show nRm.

If $m^2 = n^2$ is true, then $n^2 = m^2$

Since '=' is commutative and $m, n \in Z$

Transitive:- let $m, n, r \in Z$

Now if mRn, nRr, then we will show than mRr is true

Now $m^2 = n^2$, $n^2 = r^2$ is true.

$\therefore$ m^2 must be equal to r^2

~~$\therefore$ m = ±n ≠ ±r.~~

$\therefore m^2 = r^2$ $\therefore$ Transitivity holds

$\therefore$ It is a partial order (POSET)

(3) Define a relation ρ on $\mathbb{C}$ by "$(a+ib)\rho(c+id)$ iff $a \leq c$ and $b \leq d$" for $(a+ib), (c+id) \in \mathbb{C}$. Show that ρ is a partial order.

Ans:- Let $(a+ib)$ and $(c+id) \in \mathbb{C}$

Reflexive:-

$(a+ib)\rho(a+ib)$

$\because$ $a \leq a$ and $b \leq b$ is true. (obvious reasons)
It is reflexive.

Anti-Symmetric:-

$(a+ib)\rho(c+id)$

If $a \leq c$ & $b \leq d$. is true, then.
we have to check.

$c \leq a$. & $d \leq b$. is true.
which implies.

$c = a$, $d = b$.

$\therefore$ Anti-Symmetric.

Transitive: Let $(a+ib)$, $(c+id)$, $(m+in)$ $\in \mathbb{C}$
If $(a+ib)\rho(c+id)$ & $(c+id)\rho(m+in)$ is true, then it means.

$a \leq c$ & $b \leq d$.
$c \leq m$ & $d \leq n$.

$\therefore$ $a \leq c \leq m$ & $b \leq d \leq n$
$\therefore$ $a \leq m$ & $b \leq n$.

$\therefore$

$(a+ib)\rho(m+in)$

$\therefore$ Transitive property holds.
$\therefore$ Proved it is a poset.

4) Let D_{30} be the set of all positive divisors of 30. Define a relation "$\leq$" on D_{30} by "$x \leq y$ iff x divides y" for $x, y \in D_{30}$. Prove that D_{30} is a poset. Hence draw the hasse diagram of the poset.

4

$D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$

Let $x, y \in D_{30}$.

Reflexive:- $x \in D_{30}$

$x \mid x \Rightarrow x = 1 \cdot x$ which is true.

$\therefore$ Reflexive holds.

Anti Symmetric:- $x, y \in D_{30}$

$x \mid y \Rightarrow y = ax$ where $a, b \in \mathbb{Z}^+$

& $y \mid x \Rightarrow x = by = bax = (ba)x$

$\therefore ba = 1$ now since $b, a \in \mathbb{Z}^+$

$\therefore b = a = 1$.

$\therefore x = y$. (implies)

Transitive:- Let $x, y, z \in D_{30}$.

$x \mid y$ & $y \mid z$ holds

$\therefore y = ax$ —①, $z = by$ —② where $a, b \in \mathbb{Z}^+$

$\therefore$ from ① & ②

~~$z = by = a \cdot b \cdot z = (ab)z$~~

~~$\therefore (ab)$ is a +ve integer.~~

$\therefore z$

$\therefore$ from (2) & (1)

$z = by$

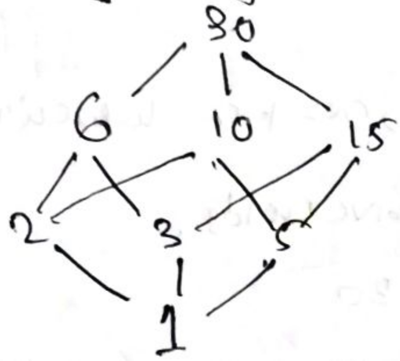
$= b(ax)$

$= abx$

$\therefore x | z$ $\therefore$ Transitive.

$\therefore$ It is a poset

The Hasse Diagram of D_{30}



(5) Let S be the set of all positive divisors of 72. Define a relation $\leq$ on S by "$x \leq y$ iff x is a divisor of y". for $x, y \in S$. Prove that $(S, \leq)$ is a poset. Draw the covering diagram of the poset.

Ans

$D_{72} = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72\}$

$x \leq y$ is x is a divisor of y.

Reflexive:
$x \in D_{72}$.

$x | x$

$\Rightarrow x = 1 \cdot x$ True

$\therefore$ Reflexive holds.

Anti Symmetric :- $\forall x, y \in D_{72}$

To show $x \leq y$, $y \leq x$ holds.

i.e. $x | y$ and $y | x$

$\Rightarrow y = ax$ — (1) $\quad x = by$ — (2)

from $y = ax$ where $a, b \in D_{72}$

$= a \cdot by$

$= aby$.

$\therefore a, b$ are integers and $ab = 1$, $\therefore a = b = 1$.

$\therefore$ I implies $x = y$.

Anti-Symmetric holds.

Transitive :- $x, y, z \in D_{72}$

$x \leq y$, $y \leq z$ to show $x \leq z$

$x | y$ & $y | z$

$\Rightarrow y = ax$ — (1) $\quad z = by$ — (2)

from $y = ax$

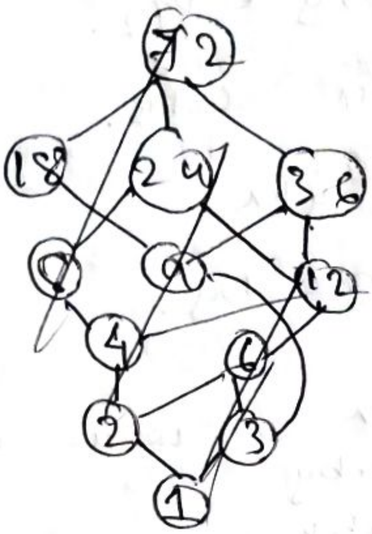
$\Rightarrow \frac{z}{b} = ax$

$\Rightarrow z = abx$

$\therefore x | z$ (proved) It is transitive.

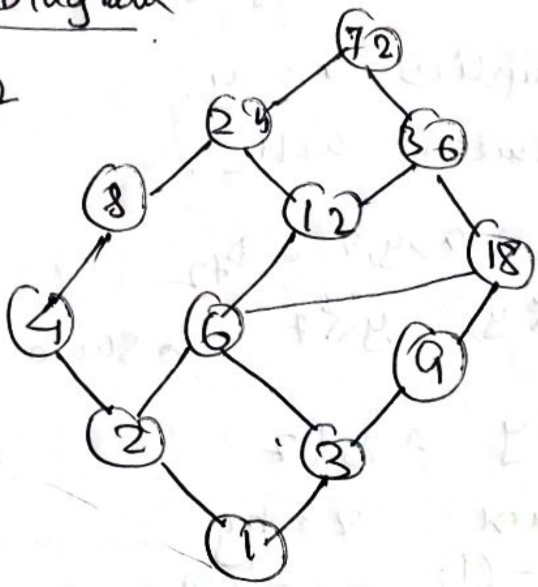
$\therefore$ It is a poset (proved)

P.T.O

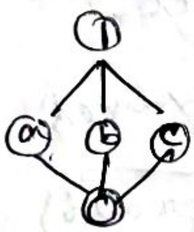


Hasse Diagram

of D_{72}



(6) Define distributive lattice. Give an example. Is the lattice with following Hasse Diagram distributive? Justify.



Ans For a distributive lattice

$a, b, c \in$ Set S.

Then $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

where $\vee \to$ gcd

$\wedge \to$ lub.

∴ lets say $a \vee b = 0$

$a \vee c = 0$

∴ $(a \vee b) \wedge (a \vee c) = 0$

$a \vee (b \wedge c) = a \vee (I)$

$= a$

∴ $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

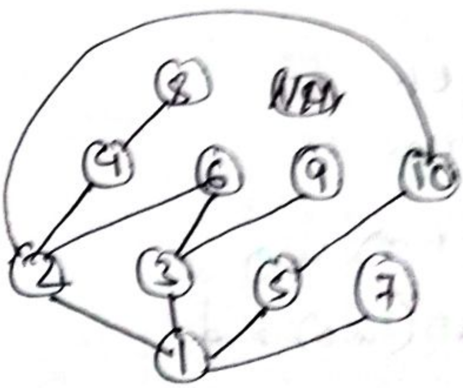
∴ It is not a distributive lattice.

(7) Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

let $x \leq y$ means x is a divisor of y.

Find the maximal element/elements in the poset $(S; \leq)$

Ans

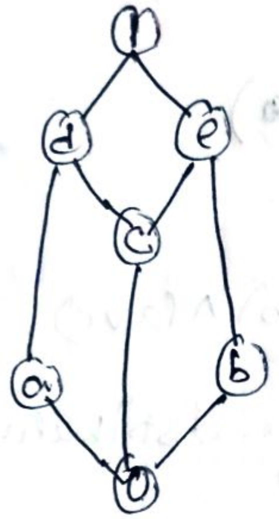


~~The maximal element is 8~~

The above Hasse diagram shows that every element $a \leq b$ for b in $S' \{4, 6, 7, 9, 10\}$ ~~greatest~~

∴ There is the maximal elements/element in $S \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ is $\{4, 6, 7, 9, 10\}$

(8) Show that the poset given by the following Hasse is a lattice. I$ if a distributive and complemented. Justify your answer.



Ans Let The set be S. Here lb = 0

∴ a, b, c ∈ S. ub = 1

For distributive we have to show $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee$

where $\wedge \rightarrow$ glb

$\vee \rightarrow$ lub.

$b \wedge c = glb(b, c) = 0$

$a \vee (b \wedge c) = lub(a, 0) = d$

Also

$(a \vee b) = lub(a, b) = 1$

$(a \vee c) = lub(a, c) = d.$

$\therefore (a \vee b) \wedge (a \vee c) = (1 \wedge d)$

$= d.$

∴ LHS = RHS ∴ Distributive.

For complemented lattice

There must be an element a'

so that $a \wedge a' = 0$ & $a \vee a' = 1$

for all element $a \in S$.

a has its complement b.

e has its complement d

But c has no complement.

$\therefore$ It is not a complemented lattice.