Key Concepts and Formulas

1. Newton's law in vector form: $m\frac{d^2\vec{r}}{dt^2} = \vec{F}$

2. Degrees Of Freedom (DOF): No. of independent variables required to specify the position of a particle or a system of particles.

3. Constraints: A set of differential or algebraic relations imposed on a particle or system of particles.

4. An N-particle system in 3-dimension with K constraints has d.o.f = 3N−K.

5. A constraint given in the form of a differential equation which is integrable or in the form of an algebraic equation is called holonomic otherwise nonholonomic.

6. A constraint expressed in the form $f(x,y,z,t)=0$ is called scleronomic iff $\partial f/\partial t = 0$ otherwise rheonomic.

7. Virtual displacement: Difference between two possible displacements within the same time interval

8. Virtual Work principle (D'Alembert principle): For an N-particle system the sum of virtual work done by all the constraint forces is equal to zero

$$\sum_{\nu=1}^{N}\left(m_\nu \ddot{\vec{r}}_\nu - \vec{F}_\nu\right)\cdot\delta\vec{r}_\nu = 0$$

9. Generalized Co-ordinates: The independent co-ordinates necessary to specify the trajectory of a system in configuration space. A holonomic system with N-particle and K constraints has generalized co-ordinates

$$n = 3N - K = d.o.f.$$

10. Lagrange equation of 2nd kind:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \quad \Big| \; j=1\cdots n$$

T = Kinetic energy

$\{q_j | j=1\cdots n\} \equiv$ ~~no. of~~ generalized co-ordinates

$\{Q_j | j=1\cdots n\} \equiv$ components of generalized force

11. Lagrange equation of 2nd kind for potential forces:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0, \quad L = T - V, \text{ is called}$$

Lagrangian of the system and V is the potential energy and $Q_j = -\partial V/\partial q_j$ is called the j-th component of generalized force.

12. Generalized momenta : $\{ p_j \mid j = 1 \cdots n \}$ are called components of generalized momenta where

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \quad \Big| \, j = 1 \cdots n.$$

13. Cyclic co-ordinate: A co-ordinate q_j is called cyclic co-ordinate if $\frac{\partial L}{\partial q_j} = 0$

14. The momentum corresponding to a cyclic co-ordinate is a conserved quantity

15. If $\frac{\partial L}{\partial t} = 0$ $J = \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j - L$ is called the the Jacobi integral which is a conserved quantity

16. The Hamiltonian of a system is given by

$$H = \sum_{j=1}^{n} p_j \dot{q}_j - L$$

17. The Hamilton's equations of motions are given by

$$\dot{p}_j = -\frac{\partial H}{\partial q_j} \; ; \quad \dot{q}_j = \frac{\partial H}{\partial p_j} \quad \Big| \, j = 1 - n$$

18. $\frac{dH}{dt} = \frac{\partial H}{\partial t} = \frac{\partial L}{\partial t}$. If $\frac{\partial H}{\partial t} = 0$, $\frac{dH}{dt} = 0$. Hence H becomes a conserved quantity.

19. A system with Lagrangian L can have at all any Hamiltonian iff $\left| \partial^2 L / \partial \dot{q}_j \partial \dot{q}_k \right| \neq 0$

20. i) Dimension of phasespace $= 2n$, n being the d.o.f.
ii) Dimension of configuration space $= n$

• 0. Introduction:

Classical Mechanics is one of the few fundamental branches of physics that not only provides some of the basic building blocks of the subject itself but also introduces a plethora of formal techniques useful to achieve mathematical formulation of various phenomenons of nature and of our real life experiences. The development of classical mechanics and its various forms of application demands, therefore, a historical understanding of the subject both in terms of contextual issues as well as its usefulness in natural sciences, and ~~to~~ technological fields.

(•) Historical Issues: The motions of various objects starting from celestial to terrestial domain seems to have caught human attention right from the premitive ages of natural sciences. Among all relevant observations the periodic motion of celestial object (planets) and that of pendulum etc. provided

(i) A time scale to measure time.

(ii) A phenomenological understanding of the geometric form of the path of motion, later known as trajectory. (Kepler's law)

This endeavour was ~~further~~ facilitated ~~by~~ with the use of co-ordinate geometry. The subject called kinematics deals with the path of an object, its velocity and their time dependence. Finally, the cause of motion i.e.; the idea of force was introduced in mechanics with the development of differential calculus by Issac Newton and many others. The eventual realization of force ~~and~~ as proportional to the rate of change of momentum gave birth to modern dynamics where it is possible to express velocity and position as ~~a~~ functions of time through different stages of integration of Newton's force equation known as equation of motion. But the construction of equation of motion with force as a starting premise (or determined by geometrical or various arguments) came out to be more & more difficult and in some cases almost impossible leading to a growing discontent among Newton's successors like Euler, d'Alembert and Lagrange. The demand to avoid force (as a starting point) in favor of co-ordinates, energy. etc. ~~were~~ was felt giving rise to an algebraic formulation of motion in a suitably chosen mathematical space leads us to what is today known as classical mechanics.

Notes: Classical Mechanics

- **I. Aim of classical Mechanics:**
  (i) Calculation of trajectory.
  (ii) Conservation principle.

- **II Newton's law and its difficulties:**

(a) **Axiom-1:** The trajectory of a one particle system $S_{\{1\}} = \{m_1, x_1, y_1, z_1\}$ is a time parametrized vector $(x_1(t), y_1(t), z_1(t)) \in \mathbb{R}^3$ which is a solution of the differential eqn.

$$\frac{d^2\vec{r}_1}{dt^2} = \vec{F}_1/m_1 \quad \text{---- [1]}$$

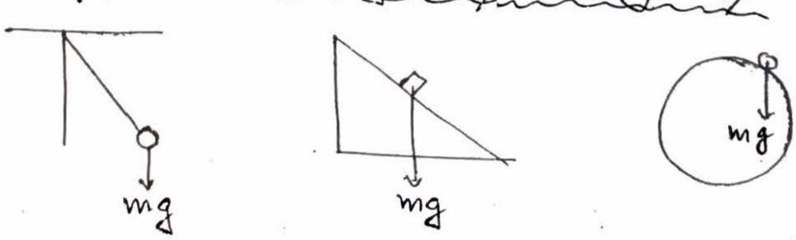
- **Remark:** 1. $\vec{F}_1$ is called the force on the particle, $\frac{d\vec{r}_1}{dt}$, the velocity, $\frac{d^2\vec{r}_1}{dt^2}$ is the acceleration and $m_1 d\vec{r}/dt$ is the momentum.

2. For an N-particle system $S_{\{\nu\}} = \{m_\nu; x_\nu, y_\nu, z_\nu \,|\, \nu = 1 \text{--} N\}$ we can write

$$\frac{d^2\vec{r}_\nu}{dt^2} = \frac{\vec{F}_\nu}{m_\nu} \Big| \nu = 1 \text{--} N \quad \text{---- } [1]^N$$

3. Eqn.-[1] is known as Newton's equation of motion.

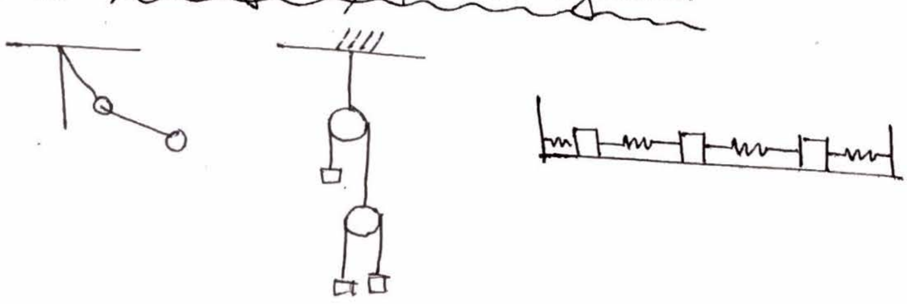
- (b) **Difficulties with Newton's law:**
  - (i) Influence of geometry on trajectory



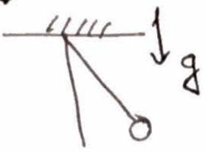
External forces are identical but trajectories are different. $[1]^N$ has to be modified. A new force has to be introduced.

$$m_\nu \frac{d^2\vec{r}_\nu}{dt^2} = \vec{F}_\nu^{ext} + \vec{R}_\nu \Big| \nu = 1 \text{--} N \quad [1']^N$$

- **Remark:** 1. $\{\vec{R}_\nu \,|\, \nu = 1 \text{--} N\}$ is called the set of reaction force.
  - (ii) Tackling many particle systems.



- (iii) Difficulties with non-interial system.



- III. Constraints/Classification

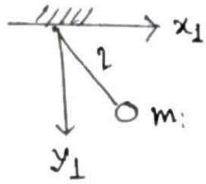
  - Definition–1: For a 1-particle system $S_{\{1\}} = \{m_1, x_1, y_1, z_1\}$ a constraint is expressed by either of the following relations:

    (i) A differential equation : $X_1\, dx_1 + Y_1\, dy_1 + Z_1\, dz_1 + T_1 dt = 0$

    or

    (ii) An algebraic equation : $f_1(x_1, y_1, z_1, t) = 0$, where $\{X_1, Y_1, Z_1, T_1\}$ are functions of (x_1, y_1, z_1, t)

  - Examples–1



static pendulum.

Constraint: $z_1 = 0$;
$x_1^2 + y_1^2 = l^2$

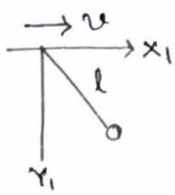
or

$dz_1 = 0$; $x_1\, dx_1 + y_1\, dy_1 = 0$

or

$\dot{z}_1 = 0$; $x_1 \dot{x}_1 + y_1 \dot{y}_1 = 0$

  - Example–2.



Pendulum moving with const velocity

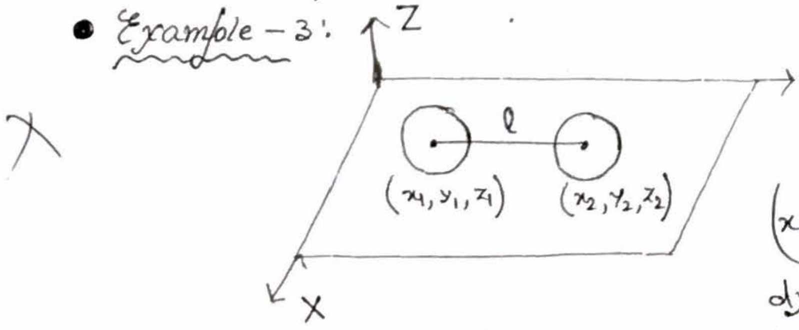
Constraint:
$z_1 = 0$;
$(x_1 - vt)^2 + y_1^2 = l^2$

or

$dz_1 = 0$;
$(x_1 - vt)\, dx_1 - v(x_1 - vt)\, dt + y_1\, dy_1 = 0$

  - Remark: 1. A differential constraint may or may not be reducible to algebraic one but the reverse is always true.

  - Example–3:



Bi-cycle

Constraint:
$z_1 = 0 = z_2$,
$(x_1 - x_2)^2 + (y_1 - y_2)^2 = l^2$,
$dy_1 (x_1 - x_2) = dx_1 (y_1 - y_2)$
(not reducible to algebraic relation)

- *Definition-2*: A differential constraint reducible to algebraic form is called holonomic, otherwise non-holonomic.

holonomic : both differential and algebraic form exist
non-holonomic : only differential form exist.

- *Definition-3*: A constraint $f \doteq f(x_1, y_1, z_1, t)$ is scleronomic iff $\frac{\partial f}{\partial t} = 0$, otherwise rheonomic.

- *Remark*: Example-1 is scleronomic and example-2 is rheonomic constraint.

- IV 1st Fundamental form (FFF) in classical mechanic

1. For scleronomic case $f(x, y, z, t) = 0$

$$\Rightarrow \vec{\nabla} f \cdot d\vec{r} = 0 \quad --- (1)$$

$$\left[\text{where } d\vec{r} = (dx, dy, dz) \text{ and } \vec{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\right]$$

$$\Rightarrow \vec{\nabla} f \perp d\vec{r}$$

2. For rheonomic case $f(x, y, z, t) = 0$

$$\Rightarrow \vec{\nabla} f \cdot d\vec{r} + \frac{\partial f}{\partial t} dt = 0 \quad -- (2)$$

$$\Rightarrow \vec{\nabla} f \not\perp d\vec{r}$$

Considering another displacement $d\vec{r}'$ within the same time interval dt eq^n. (1) & (2) becomes

$$\vec{\nabla} f \cdot d\vec{r}' = 0 \quad --- (1)' \qquad \vec{\nabla} f \cdot d\vec{r}' + \frac{\partial f}{\partial t} dt = 0 \quad -- (2')$$

(1) − (1′) yields.

$$\vec{\nabla} f \cdot (d\vec{r} - d\vec{r}') = 0$$

(2) − (2′) yields.

$$\vec{\nabla} f \cdot (d\vec{r} - d\vec{r}') = 0$$

- *Definition-4*: The difference $d\vec{r} - d\vec{r}' = \delta\vec{r}$ is called a virtual displacement.

- *Remark*: 1. As $\vec{\nabla} f \perp \delta\vec{r}$ and $\vec{R} \perp \delta\vec{r}$; $\vec{R} = \lambda \vec{\nabla} f$
R being the reaction force.

- *Principle of Virtual Work*: For $S_{\{\nu\}} = \{m_\nu ; x_\nu, y_\nu, z_\nu \mid \nu = 1 - N\}$ admitting N virtual displacements $\{\delta\vec{r}_\nu \mid \nu = 1 -- N\}$ and reaction forces $\{\vec{R}_\nu \mid \nu = 1 -- N\}$

$$\sum_{\nu=1}^{N} \vec{R}_\nu \cdot \delta\vec{r}_\nu = 0$$

which in view of equation [1'] gives us the 1st fundamental form (FFF)

$$\boxed{\sum_{\nu=1}^{N}\left(m_\nu \ddot{\vec{r}}_\nu - \vec{F}_\nu\right)\cdot \delta\vec{r}_\nu = 0 \quad \cdots\cdots [2]}$$

• V. Lagrange equation of 2nd kind.

• *Definition 5* : For a system $S_{\{\nu\}} = \{m_\nu, x_\nu, y_\nu, z_\nu \mid \nu = 1 \cdots N\}$ admitting K holonomic constraints $\{f_\alpha(x_\nu, y_\nu, z_\nu, t \mid \nu = 1 \cdots N) \mid \alpha = 1 \cdots K\}$, $K < 3N$, the degrees of freedom.

$$n = 3N - K.$$

• *Remark* : 1. The above equation can be argued in the light of the fact that had there been a system involving N- free particle we would have needed 3N independent variable in $\mathbb{R}^{3N}$. If K equations of constraints are imposed on it, K of 3N variables become dependent. Hence the number of free variables reduces to $3N - K$.

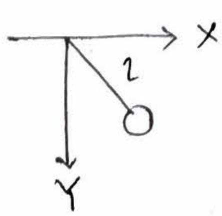
• Steps to determine dof of a holonomic system.

step-1 : Identify 3N variables $\{x_\nu, y_\nu, z_\nu \mid \nu = 1 \cdots N\}$ for an $S_{\{\nu\}} = \{m_\nu ; x_\nu, y_\nu, z_\nu \mid \nu = 1 \cdots N\}$

step-2 : Identify K relations among $\{x_\nu, y_\nu, z_\nu \mid \nu = 1 \cdots N\}$ imposed on $S_{\{\nu\}}$.

step-3 : dof $n = 3N - K$.

• *Example* : 1. $S_{\{1\}} = \{m_1, x_1, y_1, z_1\}$



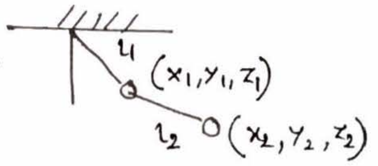
Variables : x_1, y_1, z_1 $\quad 3N = 3$

Equation imposed $z_1 = 0,\ x_1^2 + y_1^2 = l^2$

$\Rightarrow K = 2$

dof $n = 3 - 2 = 1$.

- **Example-2:** $S_{\{1,2\}} = \{ m_\nu, x_\nu, y_\nu, z_\nu \mid \nu = 1, 2 \}$



$N = 2 \Rightarrow 3N = 6$

Relation imposed:

$z_1 = 0 = z_2$, $x_1^2 + y_1^2 = l_1^2$ and

$(x_1 - x_2)^2 + (y_1 - y_2)^2 = l_2^2$

Hence $K = 4$. So dof $n = 3N - K = 6 - 4 = 2$

Let us consider an N-particle system

$$S_{\{\nu\}} = \{ m_\nu, x_\nu, y_\nu, z_\nu \mid \nu = 1 \cdots N \}$$

and effect a notational change

$$(x_\nu, y_\nu, z_\nu) \longrightarrow (x_{3\nu-2}, x_{3\nu-1}, x_{3\nu}) \; \forall \nu$$

so that

$$S_{\{\nu\}} \equiv S_{\{\alpha\}} = \{ m_\alpha ; x_\alpha \mid \alpha = 1 \cdots 3N \}$$

With this the 1st fundamental form reduces to

$$\sum_{\nu=1}^{N} \left(m_\nu \ddot{\vec{r}}_\nu - \vec{F}_\nu \right) \cdot \delta \vec{r}_\nu = 0$$

$$\equiv$$

$$\boxed{\sum_{\alpha=1}^{3N} \left(m_\alpha \ddot{x}_\alpha - F_\alpha \right) \delta x_\alpha = 0 \quad \cdots\cdots [3]}$$

If $\{ \delta x_\alpha \mid \alpha = 1 \cdots 3N \}$ are independent we would have got 3N Newton's law from eqn. [3]. But $\{ x_\alpha \mid \alpha = 1 \cdots 3N \}$ are not independent, the are related by constraint relation and hence $\{ \delta x_\alpha \mid \alpha = 1 \cdots 3N \}$. If the system admits K holonomic constraints $\{ f_\beta = 0 \mid \beta = 1 \cdots K \}$ i.e.;

$$f_\beta \left(x_\alpha \mid \alpha = 1 \cdots 3N, t \right) = 0$$

$$\text{or} \sum_{\alpha=1}^{3N} \frac{\partial f_\beta}{\partial x_\alpha} \delta x_\alpha = 0 \quad \cdots\cdots [4]$$

~~The case for $S_{\{1\}} = \{ m_1, x_1, x_2, x_3$~~

- Remark : 1. $m_1 = m_2 = m_3$.

$m_4 = m_5 = m_6$

$m_{3j-2} = m_{3j-1} = m_{3j}$

- 1-particle-1 constraint system
- 1st fundamental form: $\left(m_1\ddot{x}_1 - F_1\right)\delta x_1 + \left(m_2\ddot{x}_2 - F_2\right)\delta x_2 + \left(m_3\ddot{x}_3 - F_3\right)\delta x_3 = 0$

$$\Rightarrow \left(m_1\ddot{x}_1\delta x_1 + m_1\ddot{x}_2\delta x_2 + m_1\ddot{x}_3\delta x_3\right) - \left(F_1\delta x_1 + F_2\delta x_2 + F_3\delta x_3\right) = 0$$

$$\Rightarrow \quad M - N = 0$$

- Constraint relation: $f_1(x_1, x_2, x_3, t) = 0$

$$\Rightarrow \frac{\partial f_1}{\partial x_1}\delta x_1 + \frac{\partial f_1}{\partial x_2}\delta x_2 + \frac{\partial f_1}{\partial x_3}\delta x_3 = 0$$

$$\text{dof} = n = 3N - K = 3.1 - 1 = 2$$

Let's choose a co-ordinate system having exactly the same number of co-ordinate as the number of dof. Let, the co-ordinates be $\{q_1, q_2\}$ and

$$\left.\begin{aligned} x_1 &\doteq x_1(q_1, q_2, t) \\ x_2 &\doteq x_2(q_1, q_2, t) \\ x_3 &\doteq x_3(q_1, q_2, t) \end{aligned}\right]$$

With slight manipulation we get (see Appendix)

$$M = m_1\Big[\frac{d}{dt}\left(\dot{x}_1\frac{\partial \dot{x}_1}{\partial \dot{q}_1} + \dot{x}_2\frac{\partial \dot{x}_2}{\partial \dot{q}_1} + \dot{x}_3\frac{\partial \dot{x}_3}{\partial \dot{q}_1}\right)\delta q_1$$

$$+ \frac{d}{dt}\left(\qquad 2 \qquad\right)\delta q_2$$

$$\cancel{+ \frac{d}{dt}\left(\qquad 3 \qquad\right)\delta q_3}$$

$$- \left(\dot{x}_1\frac{\partial \dot{x}_1}{\partial q_1} + \dot{x}_2\frac{\partial \dot{x}_2}{\partial q_1} + \dot{x}_3\frac{\partial \dot{x}_3}{\partial q_1}\right)\delta q_1$$

$$- \left(\qquad 2 \qquad\right)\delta q_2$$

$$- \cancel{\left(\qquad 3 \qquad\right)\delta q_3}\Big]$$

$$N = \left(F_1\frac{\partial x_1}{\partial q_1} + F_2\frac{\partial x_2}{\partial q_1} + F_3\frac{\partial x_3}{\partial q_1}\right)\delta q_1$$

$$\left(\ 2\ \right)\delta q_2 + \cancel{\left(\ 3\ \right)\delta q_3} = Q_1\delta q_1 + Q_2\delta q_2 + \cancel{Q_3\delta q_3}$$

Which can further be simplified as.

$$M - N = \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1} - Q_1\right]\delta q_1$$
$$+ \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) - \frac{\partial T}{\partial q_2} - Q_2\right]\delta q_2 = 0$$

As. $\{q_1, q_2\}$ are independent the term inside the square bracket is zero. Hence,

$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1} - Q_1 = 0 \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) - \frac{\partial T}{\partial q_2} - Q_2 = 0 \end{aligned}\right\}$$

where $T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2)$ is called the kinetic energy of the system and $\{Q_j \mid j = 1, 2\}$ are called the components of generalized force.

For a system with n degrees of freedom, the equations

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - Q_j = 0 \,\Big|\, j = 1 \cdots n}$$

are called Lagrange equations of 2nd kind. We can conclude the following theorem —

- **Theorem-1:** For a system with n d.o.f. the equations of motion for n independent co-ordinates are given by

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - Q_j = 0 \,\Big|\, j = 1 \cdots n.$$

- **Definition-6:** The set of n independent variables $\{q_j \mid j = 1 \cdots n\}$ coming out as solutions (time parametrized) of LE-2 determin the trajectory in an n-dimensional system $\Lambda_{\{j\}} = \{0, q_j \mid j = 1 \cdots n\}$ known as configuration space. The set $\{q_j \mid j = 1 \cdots n\}$ is called the set of generalized co-ordinate for the system.

VVI

A system with N ~~generalized co-ordinate~~ particles and K holonomic constraints

The number of generalized co-ordinates
$= d\,of =$ dimension of configuration space
$= 3N - K = n$

- **Remark:** The set $\{\dot{q}_j \mid j = 1 \cdots n\}$ gives us the components of generalized velocities.

- *Definition-7*: Let $V \doteq V(q_j | j=1\cdots n, t)$. If $Q_j = -\frac{\partial V}{\partial q_j}\ \forall j$ then V is called the generalized potential for the system.

V.V.I

- *Remark*: For a system admitting generalized potential LE- can be written as

$$\left(\frac{d}{dt}\frac{\partial}{\partial \dot q_j} - \frac{\partial}{\partial q_j}\right)(T-V) = 0$$

$$\Rightarrow \left(\frac{d}{dt}\frac{\partial}{\partial \dot q_j} - \frac{\partial}{\partial q_j}\right)L = 0 \;;\; L = T - V \text{ is called}$$

the Lagrangian of the system.

V.V.I

- *Definition-8*: Let $L \doteq L(q_j, \dot q_j, t | j=1\cdots n)$ be the Lagrangian of a system. The set of n-independent quantities $\{ p_j = \frac{\partial L}{\partial \dot q_j} | j=1\cdots n\}$ gives us components of generalized momenta.

V.V.I

- *Definition-9*: Let $L \doteq L(q_j, \dot q_j, t | j=1\cdots n)$ be the Lagrangian of a system. If there exists a co-ordinate q_α such that $\frac{\partial L}{\partial q_\alpha} = 0$, the co-ordinate q_α is said to be cyclic or ignorable.

Conservation Principle

Let q_α be a cyclic co-ordinate corresponding to a Lagrangian $L \doteq L(q_j | j=1\cdots n t)$. Then $\frac{\partial L}{\partial q_\alpha} = 0$ by definition. So, from LE-2

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_\alpha}\right) = 0 \Rightarrow \frac{d}{dt} p_\alpha = 0 \Rightarrow p_\alpha = \text{Const}.$$

- *Theorem-2*: If a co-ordinate q_α is cyclic in L the corresponding generalized momentum p_α is conserved.

Now for an $L \doteq L(q_j | j=1\cdots n, t)$

$$\frac{dL}{dt} = \sum_{j=1}^{n}\left(\frac{\partial L}{\partial q_j}\dot q_j + \frac{\partial L}{\partial \dot q_j}\ddot q_j\right) + \frac{\partial L}{\partial t}$$

$$\Rightarrow \frac{dL}{dt} = \sum_{j=1}^{n}\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_j}\right)\dot q_j + \frac{\partial L}{\partial \dot q_j}\ddot q_j\right) + \frac{\partial L}{\partial t}.$$

$$\Rightarrow \frac{dL}{dt} = \frac{d}{dt}\sum_{j=1}^{n}\frac{\partial L}{\partial \dot q_j}\dot q_j + \frac{\partial L}{\partial t}$$

$$\Rightarrow \frac{d}{dt}\left(\sum_{j=1}^{n}\frac{\partial L}{\partial \dot q_j}\dot q_j - L\right) = -\frac{\partial L}{\partial t}$$

$$\Rightarrow \frac{dJ}{dt} = -\frac{\partial L}{\partial t}$$

If $\frac{\partial L}{\partial t}=0$ $\quad \frac{dJ}{dt}=0 \Rightarrow J = \text{Const}$. Hence the theorem

- ~~Theorem - 3~~: If for a system with Lagrangian L, $\frac{\partial L}{\partial t}=0$ ~~the quantity~~ $J=\sum_{j=1}^{n}\frac{\partial L}{\partial \dot{q}_j}\dot{q}_j - L$, called Jacobi Integral is a conserved ~~quantity~~.

Steps to construct Lagrange equation of 2nd kind

Step-1: Find the number of particles (N) and the number of constraint (k) and hence the

$$\text{dof } n = 3N - k.$$

Step-2: As the dof $= n =$ number of generalized co-ordinates, choose n generalized co-ordinate for the system and express $3N$ cartesian co-ordinates in terms of n generalized co-ordinat and time t.

Step-3: Express the ~~cartes~~ kinetic and potential energy of the system as functions of generalized co-ordinates. Construct the Lagrangian

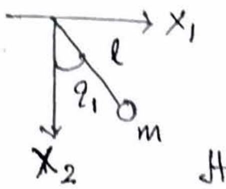
$$L = T - V$$

Step-4: Use the Lagrange equation of 2nd kind

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \,\Big|\, j = 1 \cdots n$$

for each of the co-ordinates $\{q_j \mid j = 1 \cdots n\}$ and find n equations of motion.

- Example-1:



Step-1: $N = 1$, $K = 2$

$x_3 = 0$, $x_1^2 + x_2^2 = l^2$

Hence dof $n = 3N - K = 3.1 - 2 = 1$

Step-2: Choose 1 generalized co-ordinate q_1, the angle w.r.t. the vertical. Hence

$$x_1 = l \sin q_1$$
$$x_2 = l \cos q_1$$

Step-3: $\dot{x}_1 = \frac{dx_1}{dt} = l\cos q_1 \frac{dq_1}{dt} = l \cos q_1\, \dot{q}_1$

$\dot{x}_2 = -l \sin q_1\, \dot{q}_1$

Hence $T = \frac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2\right) = \frac{1}{2} m l^2 \dot{q}_1^2$

$V = -mgx_2 = -mgl\cos q_1$

Hence $L = \frac{1}{2}ml^2\dot{q}_1^2 + mgl\cos q_1$

Step-4: Applying $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = 0$

$$\Rightarrow \frac{d}{dt}\left(ml^2 \dot{q}_1\right) + mgl \sin q_1 = 0$$

$$\Rightarrow \ddot{q}_1 = -\frac{g}{l}\sin q_1$$

$$\Rightarrow \ddot{\theta} = -\frac{g}{l}\sin\theta \text{, identifying } q_1 = \theta.$$

• VI Hamilton's equations of motion

From the definition p_j, $p_j = \frac{\partial L}{\partial \dot{q}_j}$, L being the Lagrangian. As L is a function of $\{q_j, \dot{q}_j \text{ and } t\}$, it is well known that all $\{\dot{q}_j\}$ are replacable provided the socalled Hessian $\left|\partial^2 L/\partial \dot{q}_j \partial \dot{q}_k\right| \neq 0$

• *Theorem-4:* Let's define a function $H \doteq H\left(q_j, p_j, t \,\middle|\, j=1 \cdots n\right)$ like the following, known as Legendre dual transformation

$$H\left(q_j, p_j, t \,\middle|\, j=1\cdots n\right) = \sum_{j=1}^{n} p_j \dot{q}_j - L \cdots [1]$$

Then,

$$\left.\begin{aligned} \dot{p}_j &= -\frac{\partial H}{\partial q_j} \\ \dot{q}_j &= \frac{\partial H}{\partial p_j} \end{aligned}\right\} j = 1 \cdots n$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}.$$

Taking ~~both~~ differentials on both sides of the equation - [1]

$$dH = d\left(\sum_{j=1}^{n} p_j \dot{q}_j - L\right)$$

$$\Rightarrow \sum_{j=1}^{n}\left(\frac{\partial H}{\partial q_j}dq_j + \frac{\partial H}{\partial p_j}dp_j + \frac{\partial H}{\partial t}dt\right)$$

$$= \sum_{j=1}^{n}\left(p_j d\dot{q}_j + \dot{q}_j dp_j\right) - \sum_{j=1}^{n}\left(\frac{\partial L}{\partial q_j}dq_j + \frac{\partial L}{\partial \dot{q}_j}d\dot{q}_j\right) - \frac{\partial L}{\partial t}dt$$

Equating the co-efficients of dp_j, dq_j and dt, we get

$$\frac{\partial H}{\partial q_j} = -\frac{\partial L}{\partial q_j} = -\dot{p}_j \quad \text{(by Lagrange eq}^n\text{.)}$$

$$\frac{\partial H}{\partial p_j} = \dot{q}_j \qquad \forall\, j$$

and

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

- *Remark*: 1. $H \doteq H(q_j, p_j, t \mid j=1 \cdots n)$ is called the Hamiltonian of the system.
  2. p_j and q_j are called canonically conjugate quantities
  - Conservation principle

In view of Hamilton's equations of motion

$$\frac{\partial H}{\partial q_j} = 0 \Rightarrow \dot{p}_j = 0 \Rightarrow p_j = \text{const.}$$ Hence the following theorem.

- *Theorem-6*: If q_j is cyclic in H, its corresponding momentum p_j is conserved.

- *Remark*: If q_j is cyclic in L, $\frac{\partial L}{\partial q_j} = 0 \Rightarrow \frac{\partial H}{\partial q_j} = 0$ i.e., it is also cyclic in H.

On the other hand,

$$\frac{dH}{dt} = \sum_{j=1}^{n} \left(\frac{\partial H}{\partial p_j} \dot{p}_j + \frac{\partial H}{\partial q_j} \dot{q}_j \right) + \frac{\partial H}{\partial t}$$

$$= \sum_{j=1}^{n} \left(\frac{\partial H}{\partial p_j} \left(-\frac{\partial H}{\partial q_j} \right) + \frac{\partial H}{\partial q_j} \left(\frac{\partial H}{\partial p_j} \right) \right) + \frac{\partial H}{\partial t}$$

$$= \frac{\partial H}{\partial t} \Rightarrow \text{if } \frac{\partial H}{\partial t} = 0, \frac{dH}{dt} = 0 \Rightarrow H = \text{const.}$$

Hence the following theorem

- *Theorem-7*: For a system with Hamiltonian H, $\frac{\partial H}{\partial t} = 0 \Rightarrow H = \text{const}$.

- *Definition-10*: The set of $2n$ independent variables $\{q_j, p_j \mid j=1 \cdots n\}$ coming out as a solutions (time parametrized) of Hamilton's equations motion determines the trajectory in a $2n$-dimensional system $\Gamma_{\{j\}} = \{0, q_j, p_j \mid j=1 \cdots n\}$ known as phase space. The set $\{q_j, p_j \mid j=1 \cdots n\}$ is known as the set of phase space variables.

- *Example-1*: Starting from the Lagrangian of a pendulum

$$L = \frac{1}{2} m l^2 \dot{q}_1^2 + mgl \cos q_1$$

$$p_1 = \frac{\partial L}{\partial \dot{q}_1} = m l^2 \dot{q}_1 \Rightarrow \dot{q}_1 = \frac{p_1}{m l^2}$$

Hence, the Hamiltonian

$$H = p_1 \dot{q}_1 - L = p_1 \frac{p_1}{m l^2} - \frac{1}{2} m l^2 \frac{p_1^2}{m^2 l^4} - mgl \cos q_1$$

$$= \frac{p_1^2}{2 m l^2} - mgl \cos q_1$$

Hence, the Hamilton's equations motion becomes.

$$\dot{p}_1 = -\frac{\partial H}{\partial q_1} = -mgl\sin q_1$$

$$\dot{q}_1 = \frac{\partial H}{\partial p_1} = \frac{p_1}{ml}$$

Identifying $q_1 = \theta$ and $p_1 = p_\theta$

$$\dot{p}_\theta = -mgl\sin\theta \quad \text{and} \quad \dot{\theta} = \frac{p_\theta}{ml}$$

• VII Comparative Study : Newtonian/Lagrangian/Hamiltonian mechanics:

Issue	Newtonian	Lagrangian	Hamiltonian
1. Starting premise	Momentum and Force	Co-ordinates and work.	Transformation of momentum and Lagrangian.
2. Number of equations	N 2nd order vector equation	n 2nd order scalar equation	2n 1st order scalar equation.
3. Trajectory	Time parametrized in orthogonal cartesian system	Time parametrized in configuration space	Time parametrized in phase space.
4. Status of dynamical variables	Dimensionally consistent and defined by usual notion	Not always dimensionally consistent and may not conform with usual notion	Not always dimensionally consistent and may not conform with usual notion.

• VIII Conclusion.

The idea of Lagrangian and Hamiltonian later would give birth to various new paradigms even quite less reconcilable in the domain of classical mechanics — like statistical mechanics, Quantum mechanics and continuum field theories. Not only that in some exotic systems (non-linear in particular) the qualitative idea of phase space dynamics would give birth to what is known as deterministic chaos, which is gaining much theoretical and experimental attention in recent time.