Part III: Dynamic Programming

Course: Design and Analysis of Algorithms
by Dr. Partha Basuchowdhuri

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March 1, 2017

Outline for Part I

Why do we need dynamic programming?

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- Characterize the structure of an optimal solution.
- Recursively define the value of an optimal solution.
- Compute the value of an optimal solution, typically in a bottom-up fashion.
- Construct an optimal solution from computed information.

Outline for Part II

Matrix Chain Multiplication

We are given *n* matrices in a sequence (say, A_1, A_2, . . . , A_n), which can be multiplied to generate the product.

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For example, if we have a sequence of matrices $A_1A_2A_3$ with dimensions 10 $\times$ 100, 100 $\times$ 5 and 5 $\times$ 50, there can be two possible orders of generating the product - $((A_1A_2)A_3)$ and $(A_1(A_2A_3))$.

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In $((A_1A_2)A_3)$, we perform $10 \times 100 \times 5 + 10 \times 5 \times 50 =$ 7,500 multiplications. (Preferred)
In $(A_1(A_2A_3))$, we perform $100 \times 5 \times 50 + 10 \times 100 \times 50 =$ 75,000 multiplications.

Step 1: Structure of an optimal parenthesization

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We can further divide the two optimal substructures to split them into smaller-sized optimal substructures until the substructures can be trivially split into optimal substructures (i.e., $i = j$).

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$$m[i,j] = m[i,k] + m[k+1,j] + p_{i-1}p_kp_j$$

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Recursive definition for the minimum cost of parenthesizing the product can be expressed as,

$$m[i,j] = \begin{cases} 0, \text{if } i = j \\ min_{i \leq k < j}\{m[i,k] + m[k+1,j] + p_{i-1}p_kp_j\}, \text{if } i < j \end{cases}$$

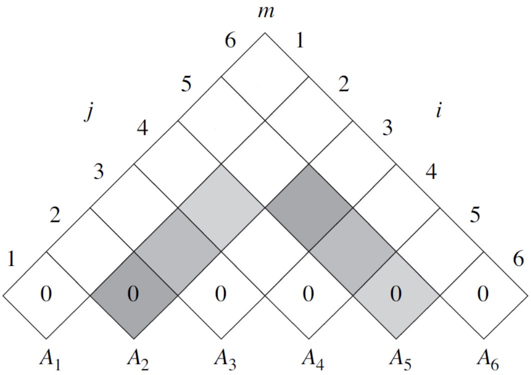
Step 3: Algorithm

Algorithm 1: MATRIX-CHAIN-ORDER(p))

Input : Sequence of matrix dimensions *p*
Output: Matrices *m*, *s*

$n \leftarrow p.length$ - 1
Let $m[1 \dots n$, 1 $\dots n]$ and $s[1 \dots (n$-1), 2 $\dots n]$ be new tables
for $i = 1$ **to** n **do**
  $m[i,i] = 0$
for $l = 2$ **to** n **do**
  for $i = 1$ **to** $n - l + 1$ **do**
    $j = i + l$-1
    $m[i,i] = \infty$
    for $k = i$ **to** j-1 **do**
      $q = m[i,k] + m[k+1,j] + p_{i-1}p_kp_j$
      if $q < m[i,j]$ **then**
        $m[i,j] = q$
        $s[i,j] = k$

Find optimal parenthesization for $A_1A_2A_3A_4A_5A_6$, where $p = [30,35,15,5,10,20,25]$



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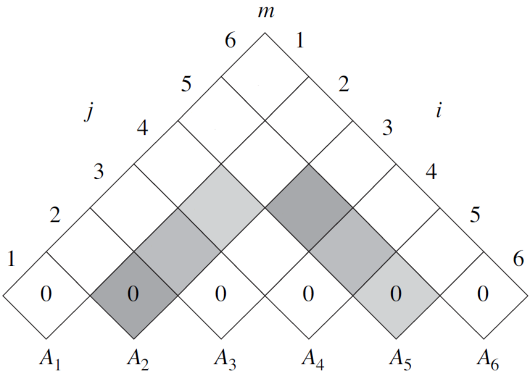
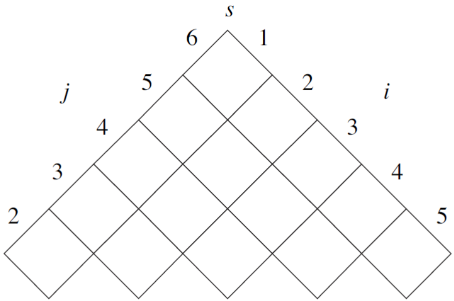
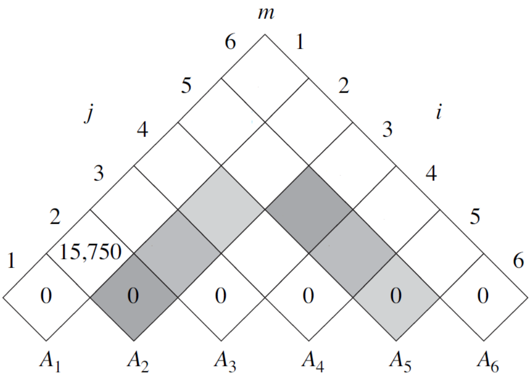




Initially, all the $m[i,i]$ values are set to zero.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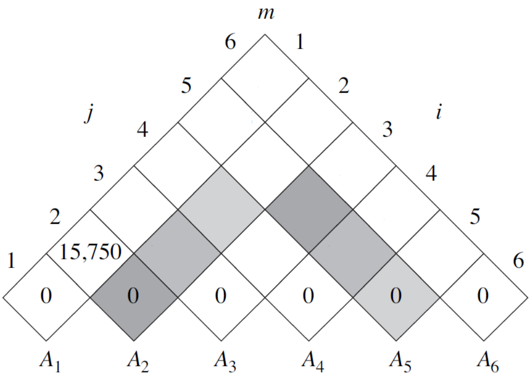
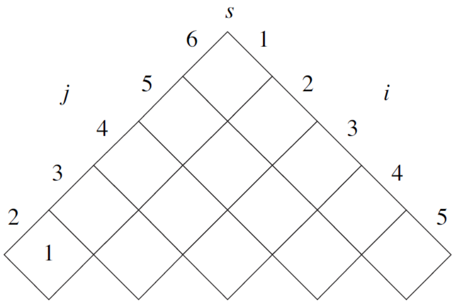
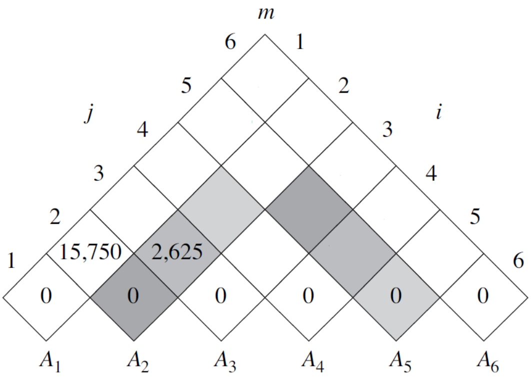




l=2, i=1, j=i+l-1=1+2-1=2

for k=1, m[1,2] = m[1,1] + m[2,2] + $p_0p_1p_2$ = 0 + 0 + 30 × 35 × 15 = 15,750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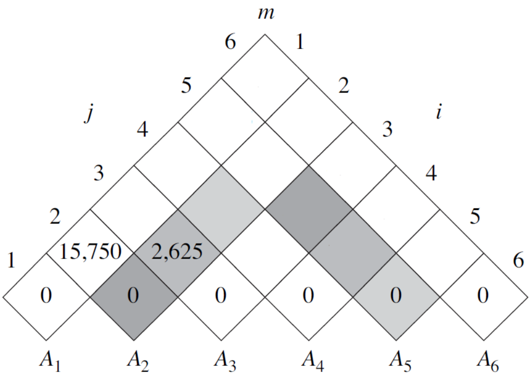
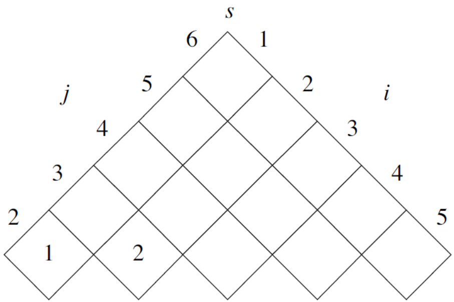
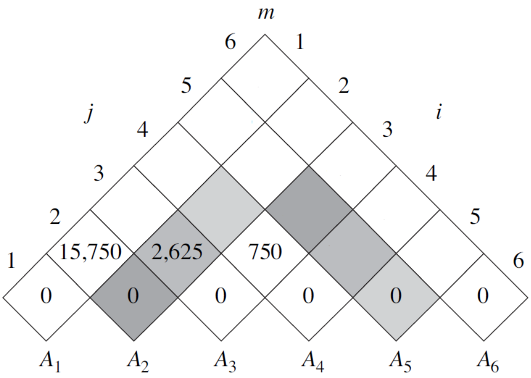




l=2, i=2, j=i+l-1=2+2-1=3

for k=2, m[2,3] = m[2,2] + m[3,3] + $p_1 p_2 p_3$ = 0 + 0 + 35 × 15 × 5 = 2,625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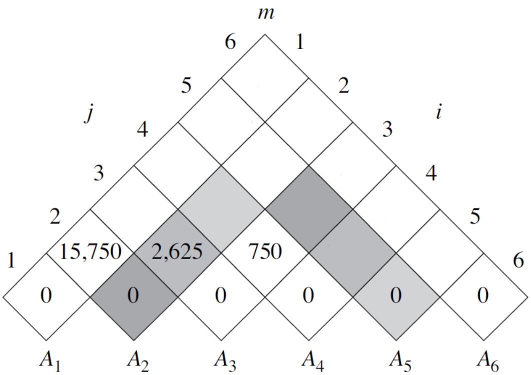
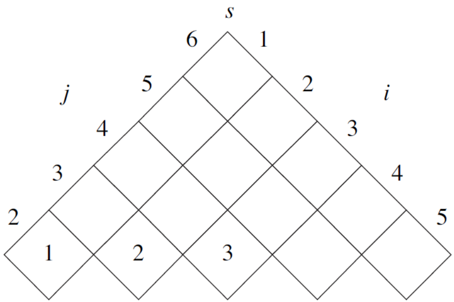
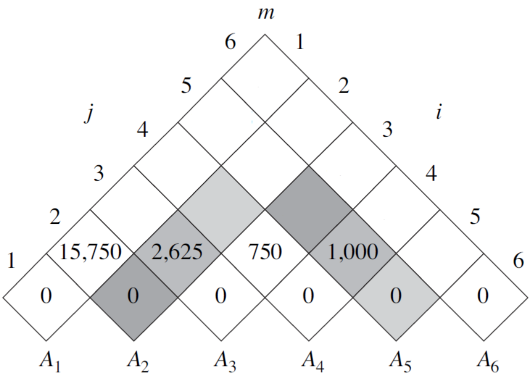




l=2, i=3, j=i+l-1=3+2-1=4

for k=3, m[3,4] = m[3,3] + m[4,4] + $p_2p_3p_4$ = 0 + 0 + 15 × 5 × 10 = 750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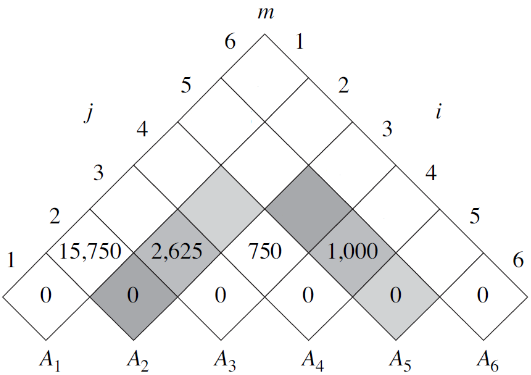
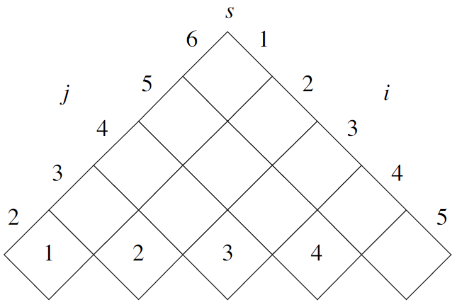
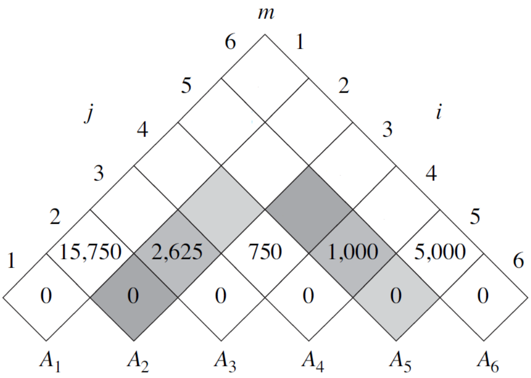




l=2, i=4, j=i+l-1=4+2-1=5

for k=4, m[4,5] = m[4,4] + m[5,5] + $p_3p_4p_5$ = 0 + 0 + 5 × 10 × 20 = 1,000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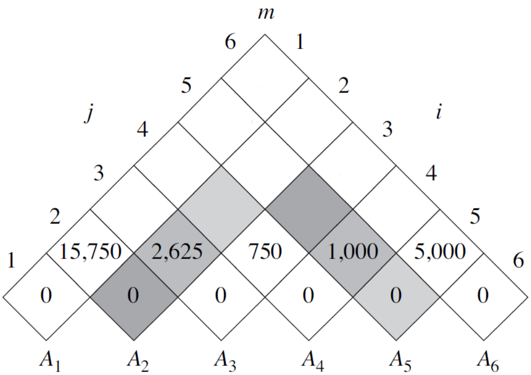
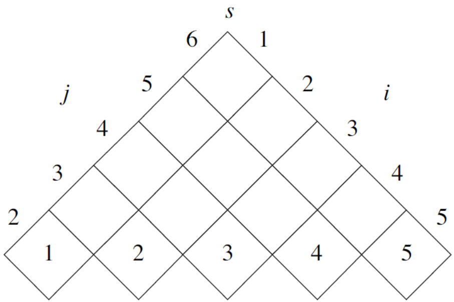
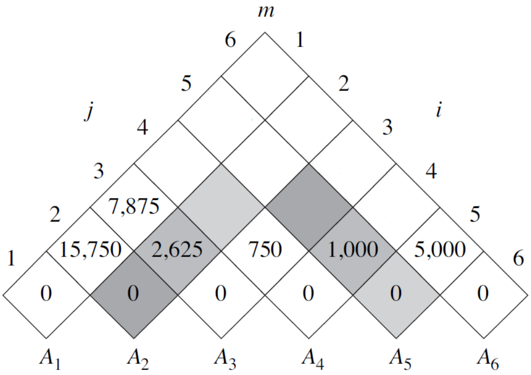




l=2, i=5, j=i+l-1=5+2-1=6

for k=5, m[5,6] = m[5,5] + m[6,6] + $p_4p_5p_6$ = 0 + 0 + 10 × 20 × 25 = 5,000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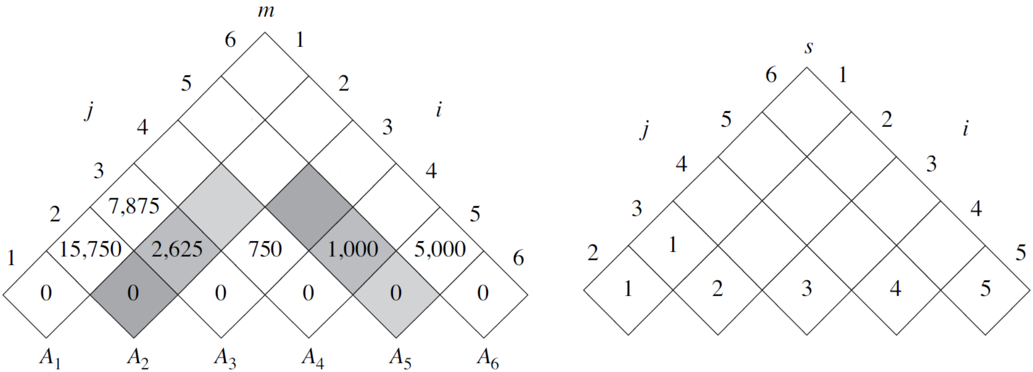
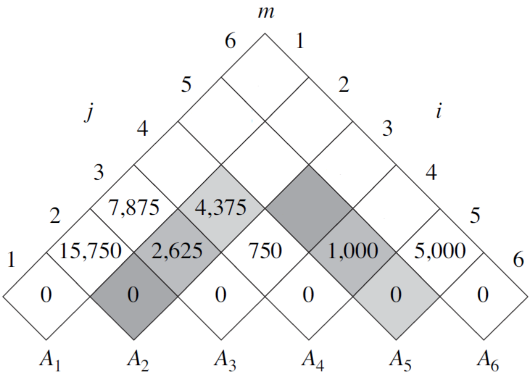


l=3, i=1, j=i+l-1=1+3-1=3

for k=1, m[1,3] = m[1,1] + m[2,3] + $p_0p_1p_3$ = 0 + 2625 + 30 × 35 × 5 = 7,875
for k=2, m[1,3] = m[1,2] + m[3,3] + $p_0p_2p_3$ = 15750 + 0 + 30 × 15 × 5 = 20,250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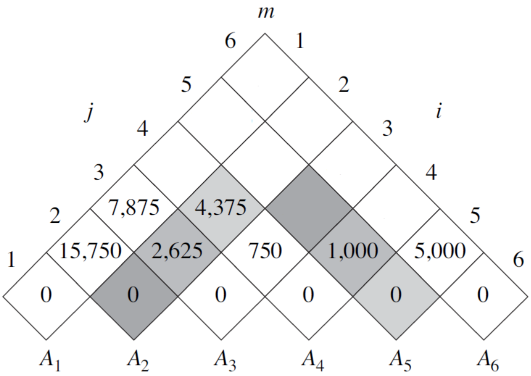

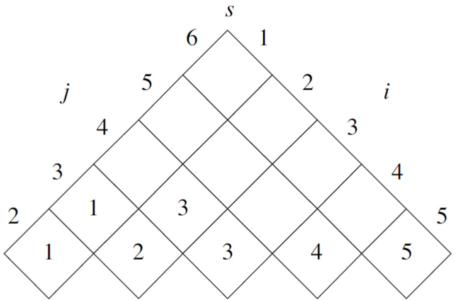

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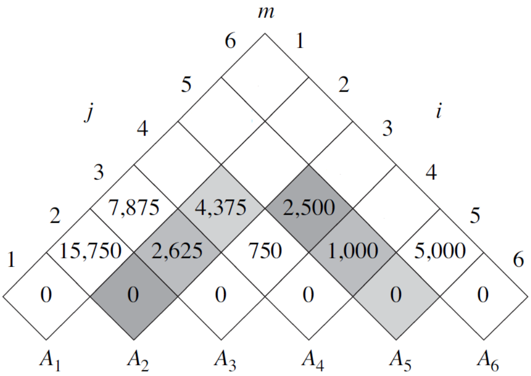
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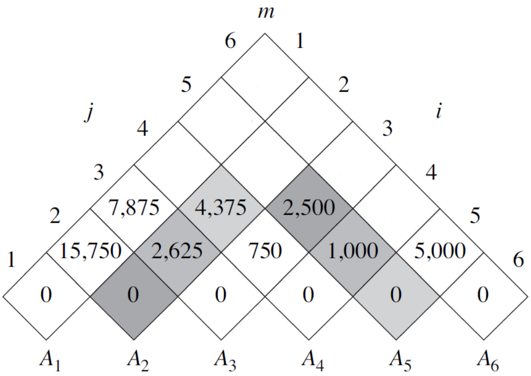

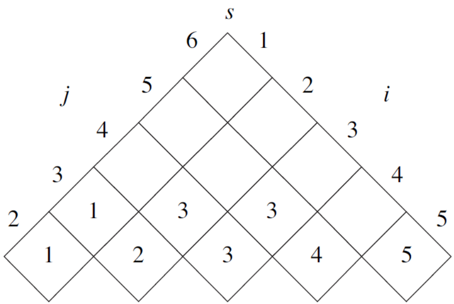

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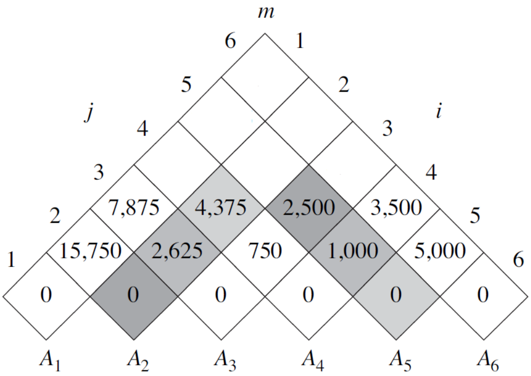
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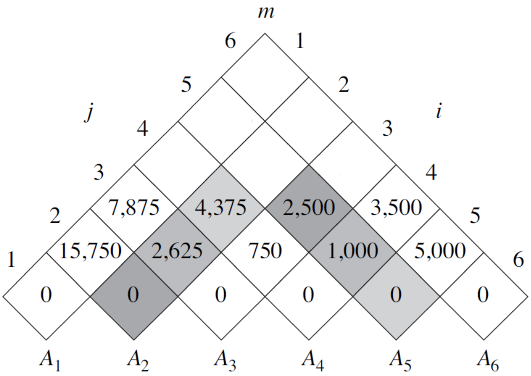

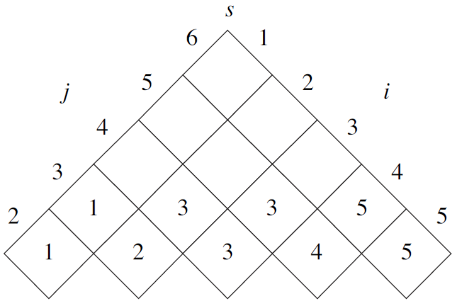

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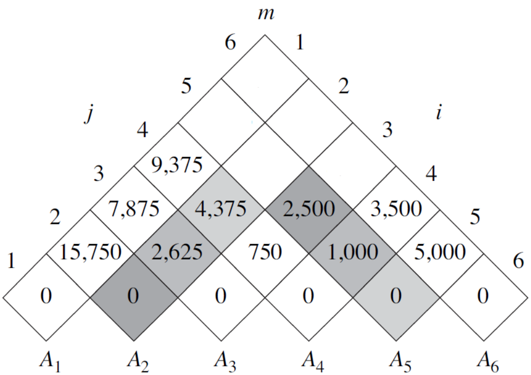
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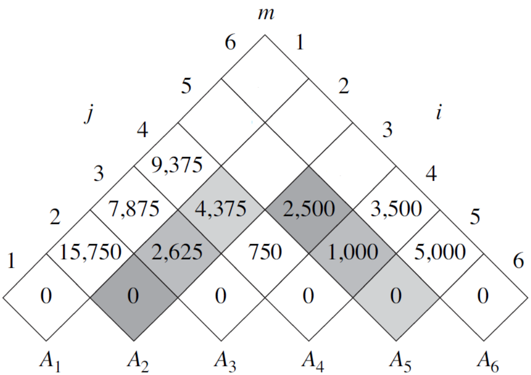
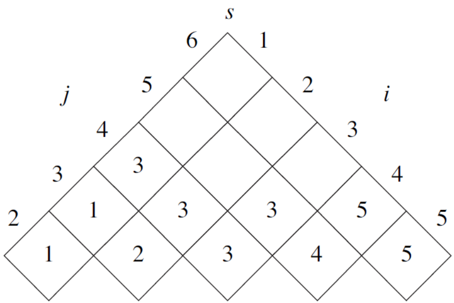


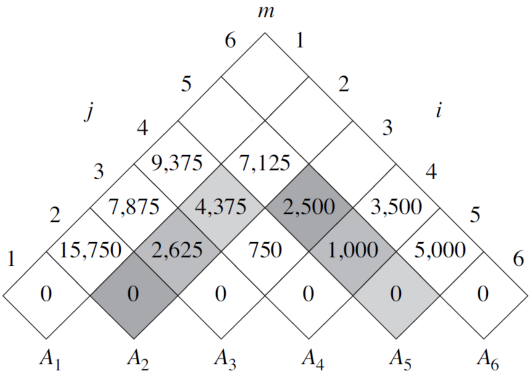


l=4, i=1, j=i+l-1=1+4-1=4

k=1, m[1,4] = m[1,1] + m[2,4] + $p_0p_1p_4$ = 0 + 4375 + 30 × 35 × 10 = 14,875

k=2, m[1,4] = m[1,2] + m[3,4] + $p_0p_2p_4$ = 15750 + 750 + 30 × 15 × 10 = 21,000

k=3, m[1,4] = m[1,3] + m[4,4] + $p_0p_3p_4$ = 7875 + 0 + 30 × 5 × 10 = 9,375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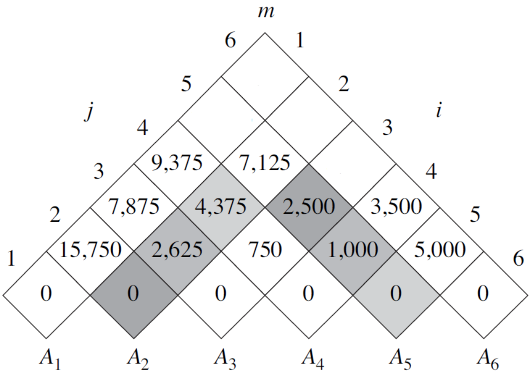
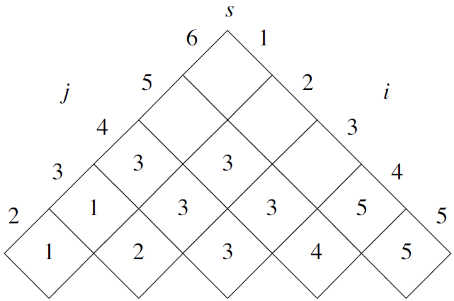
2,625
750
1,000
5,000
0
0
0
0
0
0
A_1
A_2
A_3
A_4
A_5
A_6



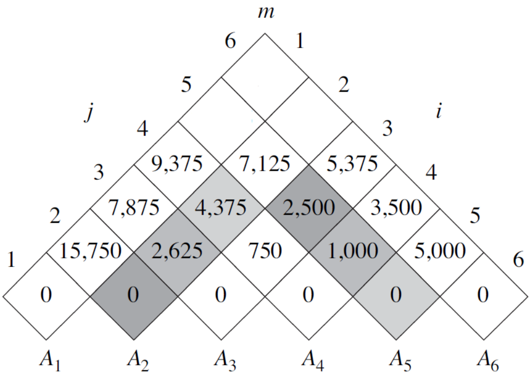


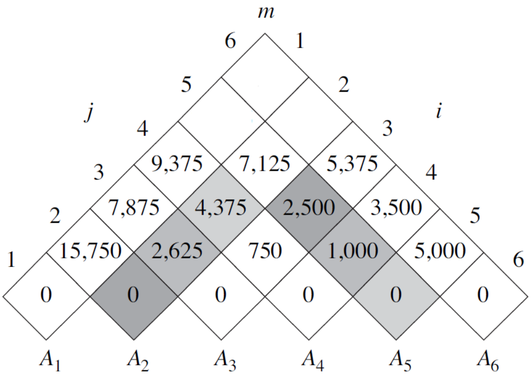
l=4, i=2, j=i+l-1=2+4-1=5

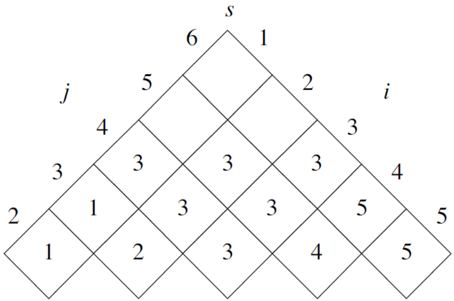
k=2, m[2,5] = m[2,2] + m[3,5] + $p_1p_2p_5$ = 0 + 2500 + 35 × 15 × 20 = 13,000

k=3, m[2,5] = m[2,3] + m[4,5] + $p_1p_3p_5$ = 2625 + 1000 + 35 × 5 × 20 = 7,125

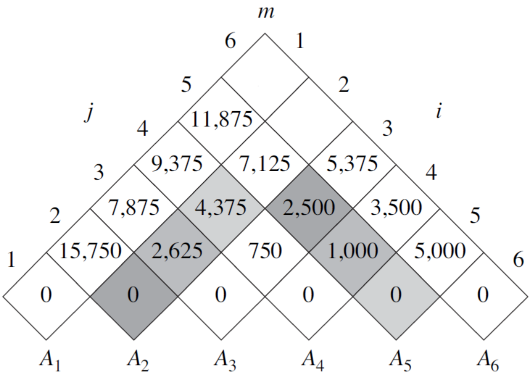
k=4, m[2,5] = m[2,4] + m[5,5] + $p_1p_4p_5$ = 4375 + 0 + 35 × 10 × 20 = 11,375

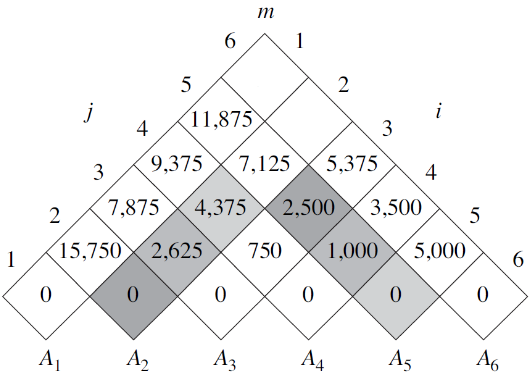

m
6
1
5
2
j
i
4
3
9,375
7,125
5,375
3
4
7,875
4,375
2,500
3,500
2
5
15,750
2,625
750
1,000
5,000
1
6
0
0
0
0
0
0
A_1
A_2
A_3
A_4
A_5
A_6

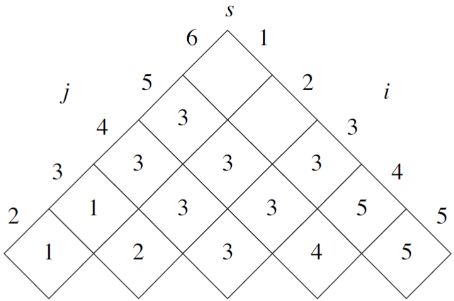




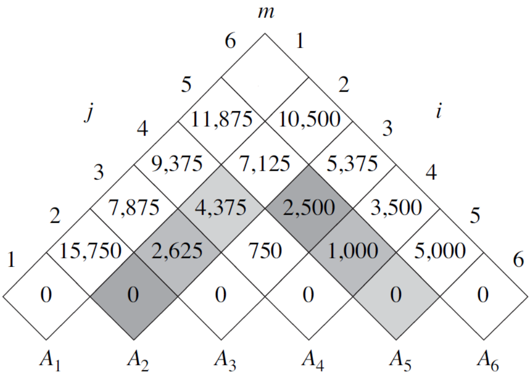
Do it yourself.

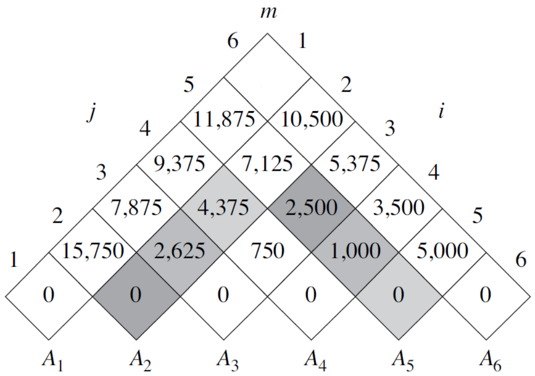

m
j
i
1
2
3
4
5
6
11,875
9,375
7,125
5,375
7,875
4,375
2,500
3,500
15,750
2,625
750
1,000
5,000
0
0
0
0
0
0
A_1
A_2
A_3
A_4
A_5
A_6

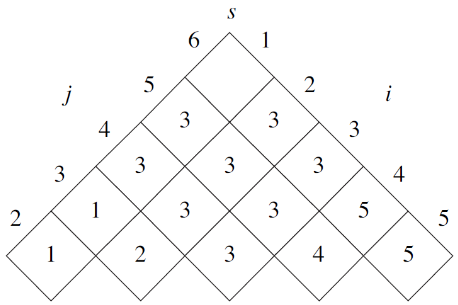




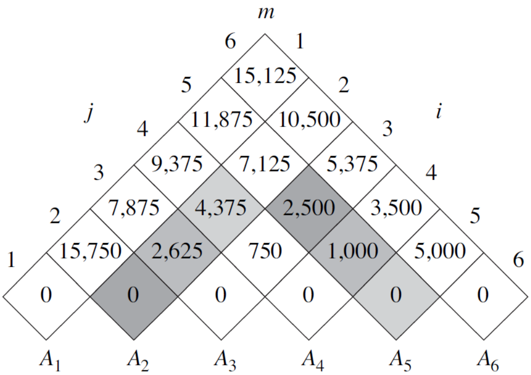
Do it yourself.

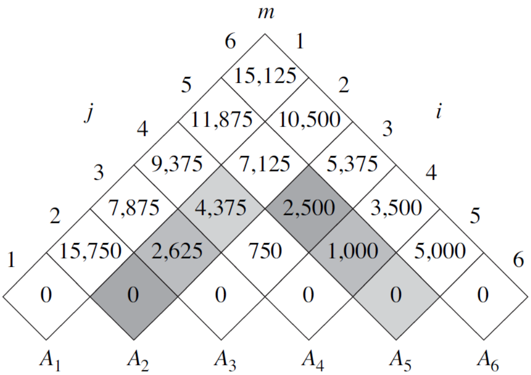

m
j
i
6
5
4
3
2
1
1
2
3
4
5
6
11,875
10,500
9,375
7,125
5,375
7,875
4,375
2,500
3,500
15,750
2,625
750
1,000
5,000
0
0
0
0
0
0
A1
A2
A3
A4
A5
A6

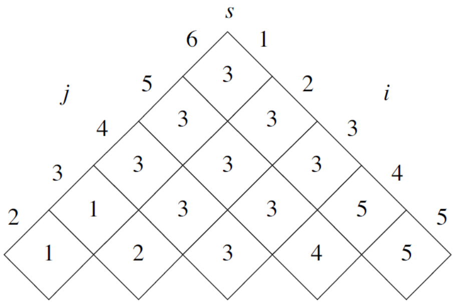




Do it yourself.


m
j
i
6 1
5 2
4 3
3 4
2 5
1 6
15,125
11,875 10,500
9,375 7,125 5,375
7,875 4,375 2,500 3,500
15,750 2,625 750 1,000 5,000
0 0 0 0 0 0
A1 A2 A3 A4 A5 A6





Do it yourself. $m[1,6] = 15{,}125$

Step 4: Constructing an optimal solution

Algorithm 2: PRINT-OPTIMAL-PARENS(s,i,j)

Input : Matrix *s*, Indices *i*,*j*
Output: Optimal parenthesization

```
begin
    if i == j then
        print "A"
    else
        print "("
        PRINT-OPTIMAL-PARENS(s,i,s[i,j])
        PRINT-OPTIMAL-PARENS(s,s[i,j]+1,j)
        print ")"
```

The optimal parenthesization for the example can be expressed as,
$((A_1(A_2A_3))((A_4A_5)A_6))$

Outline for Part III

Knapsack Problem - Informal Representation

The Problem

Knapsack Problem - Informal Representation

The Problem

Warning: The problem is meant for a mathematically sound thief!

Knapsack Problem - Informal Representation

The Problem

Warning: The problem is meant for a mathematically sound thief!

- A thief enters a store with a bag of capacity W.
- There are n items in the store with weights $\{w_1, w_2, \ldots, w_i, \ldots, w_n\}$.
- Valuation of those n items can be represented by a set $\{v_1, v_2, \ldots, v_i, \ldots, v_n\}$.

Knapsack Problem - Informal Representation

The Problem

Warning: The problem is meant for a mathematically sound thief!

- A thief enters a store with a bag of capacity W.
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Two variants of the problem -

1. **0-1 Knapsack**: Items can be accrued only in whole. Either you take an item or you don't. Example: Where items are indivisible and partial items are valueless.
2. **Fractional Knapsack**: Items can be accrued in part as well. You can take as much of an item you want to fill your bag. Example: Consider containers consisting of valuable metals such as gold, silver, copper, etc.

0-1 Knapsack Problem - Informal Representation

Camera
Weight: 1 kg
Value: 1000$

Laptop
Weight: 3 kg
Value: 2000$

Necklace
Weight: 4 kg
Value: 4000$

Vase
Weight: 5 kg
Value: 4500$

Knapsack
Capacity: 7 kg
Max value: ???

0-1 Knapsack Problem - Informal Representation

Capacity of the bag is 7 kgs.

Item 1 Camera: $w_1 = 1$ kg, $v_1 = 60$K

Item 2 Laptop: $w_2 = 3$ kg, $v_2 = 150$K

Item 3 Jewellery: $w_3 = 4$ kg, $v_3 = 300$K

Item 4 Collectible: $w_4 = 5$ kg, $v_4 = 400$K

0-1 Knapsack Problem - Formal Representation

Problem Definition

Given a knapsack with maximum capacity W, and a set S consisting of n items, where each item i has some weight w_i and benefit value b_i (all w_i, b_i and W are integer values). The objective is to find a $T \subseteq S$, such that

$$\sum_{i \in T} b_i \text{ is maximized, subject to } \sum_{i \in T} w_i \leq W.$$

0-1 Knapsack Problem - Formal Representation

Problem Definition

Given a knapsack with maximum capacity W, and a set S consisting of n items, where each item i has some weight w_i and benefit value b_i (all w_i, b_i and W are integer values). The objective is to find a $T \subseteq S$, such that

$\sum_{i \in T} b_i$ is maximized, subject to $\sum_{i \in T} w_i \leq W$.

How to solve 0-1 Knapsack

- Brute-force algorithm to find optimal solution.

0-1 Knapsack Problem - Formal Representation

Problem Definition

Given a knapsack with maximum capacity W, and a set S consisting of n items, where each item i has some weight w_i and benefit value b_i (all w_i, b_i and W are integer values). The objective is to find a $T \subseteq S$, such that

$$\sum_{i \in T} b_i \text{ is maximized, subject to } \sum_{i \in T} w_i \leq W.$$

How to solve 0-1 Knapsack

- Brute-force algorithm to find optimal solution.
- We go through all combinations and find the one with maximum value and with total weight less or equal to W. O(2^n)

0-1 Knapsack Problem - Formal Representation

Problem Definition

Given a knapsack with maximum capacity W, and a set S consisting of n items, where each item i has some weight w_i and benefit value b_i (all w_i, b_i and W are integer values). The objective is to find a $T \subseteq S$, such that

$$\sum_{i \in T} b_i \text{ is maximized, subject to } \sum_{i \in T} w_i \leq W.$$

How to solve 0-1 Knapsack

- Brute-force algorithm to find optimal solution.
- We go through all combinations and find the one with maximum value and with total weight less or equal to W. O(2^n)
- Can we find an easier solution applying dynamic programming ?

Item	w_i	b_i
1	2	3
2	3	4
3	4	5
4	5	8
5	9	10

- If items are labeled 1, . . ., n, then a subproblem would be to find an optimal solution for $S_k = \{$items labeled 1, 2, . . . $k\}$
- The question is: can we describe the final solution(S_n) in terms of subproblems(S_k)?

Item	w_i	b_i
1	2	3
2	3	4
3	4	5
4	5	8
5	9	10

- If items are labeled 1, . . ., n, then a subproblem would be to find an optimal solution for $S_k = \{$items labeled 1, 2, . . . $k\}$
- The question is: can we describe the final solution(S_n) in terms of subproblems(S_k)?
- $S_4 = \{1, 2, 3, 4\}$, $S_5 = \{1, 3, 4, 5\}$

Item	w_i	b_i
1	2	3
2	3	4
3	4	5
4	5	8
5	9	10

- If items are labeled 1, . . ., n, then a subproblem would be to find an optimal solution for $S_k = \{$items labeled 1, 2, . . . $k\}$
- The question is: can we describe the final solution(S_n) in terms of subproblems(S_k)?
- $S_4 = \{1, 2, 3, 4\}$, $S_5 = \{1, 3, 4, 5\}$
- Solution S_4 is not a part of solution S_5. Therefore, the framing of the dynamic programming solution is incorrect.

- Let us add another parameter w, which will represent the exact weight for each subset of items. The subproblem then will be to compute B[k,w].

- Let us add another parameter w, which will represent the exact weight for each subset of items. The subproblem then will be to compute B[k,w].
- The recursive formula may look like this,

$$B[k,w] = \begin{cases} B[k-1,w] & \text{if } w_k > w, \\ max\{B[k-1,w], B[k-1,w-w_k]+b_k\} & \text{otherwise} \end{cases}$$

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- It means, that the best subset of S_k that has total weight w is,
  1. the best subset of S_{k-1} that has total weight w, or
  2. the best subset of S_{k-1} that has total weight w-w_k plus the item k

$$B[k,w] = \begin{cases} B[k-1,w] & \text{if } w_k > w, \\ max\{B[k-1,w], B[k-1,w-w_k]+b_k\} & \text{otherwise} \end{cases}$$

$$B[k,w] = \begin{cases} B[k-1,w] & \text{if } w_k > w, \\ max\{B[k-1,w], B[k-1,w-w_k]+b_k\} & \text{otherwise} \end{cases}$$

- The best subset of S_k that has the total weight w, either contains item k or not.

$$B[k,w] = \begin{cases} B[k-1,w] & \text{if } w_k > w, \\ max\{B[k-1,w], B[k-1,w-w_k] + b_k\} & \text{otherwise} \end{cases}$$

- The best subset of S_k that has the total weight w, either contains item k or not.
- **First case:** $w_k > w$. Item k cannot be part of the solution, since if it was, the total weight would be $>$ w, which is unacceptable.

$$B[k,w] = \begin{cases} B[k-1,w] & \text{if } w_k > w, \\ max\{B[k-1,w], B[k-1,w-w_k]+b_k\} & \text{otherwise} \end{cases}$$

- The best subset of S_k that has the total weight w, either contains item k or not.
- **First case:** $w_k > w$. Item k cannot be part of the solution, since if it was, the total weight would be $>$ w, which is unacceptable.
- **Second case:** $w_k \leq w$. Then the item k can be in the solution, and we choose the case with greater value.

0-1 Knapsack Problem - Algorithm

Algorithm 3: 01-KNAPSACK(W, $\{x_1, \ldots, x_n\}$)

Input : W, $\{w_1, \ldots, w_n\}$, $\{b_1, \ldots, b_n\}$
Output: Maximum benefit (within W capacity)

```
for w ← 0 to W do
    B[0,w] = 0
for i ← 1 to n do
    B[i,0] = 0
for i ← 1 to n do
    for w ← 0 to W do
        if w_i ≤ w then
            if b_i + B[i-1, w − w_i] > B[i-1, w] then
                B[i, w] = b_i + B[i-1, w − w_i]
            else
                B[i, w] = B[i-1, w]
        else
            B[i,w] = B[i-1, w]
```

- Time complexity: O(nW)
- Let us solve the problem for n = 4, W = 5
- Elements (weight, benefit) : {(2, 3), (3, 4), (4, 5), (5, 6)}

0-1 Knapsack Problem - Example

Algorithm 4: 01-KNAPSACK(W, $\{x_1,..., x_n\}$)

Input : W, $\{w_1,..., w_n\}$, $\{b_1,..., b_n\}$
Output: Maximum benefit (within W capacity)

```
for w ← 0 to W do
    B[0,w] = 0
for i ← 1 to n do
    B[i,0] = 0
for i ← 1 to n do
    for w ← 0 to W do
        if w_i ≤ w then
            if b_i + B[ i-1, w − w_i] > B[i-1, w] then
                B[i, w] = b_i + B[ i-1, w − w_i]
            else
                B[i, w] = B[i-1, w]
        else
            B[i,w] = B[i-1, w]
```

- Solve the problem for n = 4, W = 5
- Elements (weight, benefit) : $\{(2, 3), (3, 4), (4, 5), (5, 6)\}$

w	0	1	2	3	4	5
i_0						
i_1						
i_2						
i_3						
i_4						

0-1 Knapsack Problem - Example

Algorithm 5: 01-KNAPSACK(W, $\{x_1, \ldots, x_n\}$)

Input : W, $\{w_1, \ldots, w_n\}$, $\{b_1, \ldots, b_n\}$
Output: Maximum benefit (within W capacity)

```
for w ← 0 to W do
    B[0,w] = 0
for i ← 1 to n do
    B[i,0] = 0
for i ← 1 to n do
    for w ← 0 to W do
        if w_i ≤ w then
            if b_i + B[ i-1, w − w_i] > B[i-1, w] then
                B[i, w] = b_i + B[ i-1, w − w_i]
            else
                B[i, w] = B[i-1, w]
        else
            B[i,w] = B[i-1, w]
```

- Solve the problem for n = 4, W = 5
- Elements (weight, benefit) : {(2, 3), (3, 4), (4, 5), (5, 6)}

w	0	1	2	3	4	5
i_0	0	0	0	0	0	0
i_1	0					
i_2	0					
i_3	0					
i_4	0					

0-1 Knapsack Problem - Example

Algorithm 6: 01-KNAPSACK(W, $\{x_1,\ldots, x_n\}$)

Input : W, $\{w_1,\ldots, w_n\}$, $\{b_1,\ldots, b_n\}$
Output: Maximum benefit (within W capacity)

```
for w ← 0 to W do
    B[0,w] = 0
for i ← 1 to n do
    B[i,0] = 0
for i ← 1 to n do
    for w ← 0 to W do
        if w_i ≤ w then
            if b_i + B[ i-1, w − w_i] > B[i-1, w] then
                B[i, w] = b_i + B[ i-1, w − w_i]
            else
                B[i, w] = B[i-1, w]
        else
            B[i,w] = B[i-1, w]
```

- Elements (weight, benefit) : $\{(2, 3), (3, 4), (4, 5), (5, 6)\}$
- $i = 1$, $w_i = 2$, $b_i = 3$, $w = 1$, $w - w_i =$ -1

w	0	1	2	3	4	5
i_0	0	0	0	0	0	0
i_1	0	0				
i_2	0					
i_3	0					
i_4	0					

0-1 Knapsack Problem - Example

Algorithm 7: 01-KNAPSACK(W, $\{x_1, \ldots, x_n\}$)

Input : W, $\{w_1, \ldots, w_n\}$, $\{b_1, \ldots, b_n\}$
Output: Maximum benefit (within W capacity)

```
for w ← 0 to W do
    B[0,w] = 0
for i ← 1 to n do
    B[i,0] = 0
for i ← 1 to n do
    for w ← 0 to W do
        if w_i ≤ w then
            if b_i + B[ i-1, w − w_i] > B[i-1, w] then
                B[i, w] = b_i + B[ i-1, w − w_i]
            else
                B[i, w] = B[i-1, w]
        else
            B[i,w] = B[i-1, w]
```

- Elements (weight, benefit) : $\{(2, 3), (3, 4), (4, 5), (5, 6)\}$
- $i = 1$, $w_i = 2$, $b_i = 3$, $w = 2$, $w - w_i = 0$

w	0	1	2	3	4	5
i_0	0	0	0	0	0	0
i_1	0	0	3			
i_2	0					
i_3	0					
i_4	0					

0-1 Knapsack Problem - Example

Algorithm 8: 01-KNAPSACK(W, $\{x_1,..., x_n\}$)

Input : W, $\{w_1,..., w_n\}$, $\{b_1,..., b_n\}$
Output: Maximum benefit (within W capacity)

for $w \leftarrow 0$ **to** W **do**
  B[0,w] = 0

for $i \leftarrow 1$ **to** n **do**
  B[i,0] = 0

for $i \leftarrow 1$ **to** n **do**
  for $w \leftarrow 0$ **to** W **do**
    if $w_i \leq w$ **then**
      if b_i+ B[i-1, $w - w_i$] > B[i-1, w] **then**
        B[i, w] = b_i+ B[i-1, $w - w_i$]
      else
        B[i, w] = B[i-1, w]
    else
      B[i,w] = B[i-1, w]

- Elements (weight, benefit) : {(2, 3), (3, 4), (4, 5), (5, 6)}
- $i = 1$, $w_i = 2$, $b_i = 3$, $w = 3$, $w - w_i = 1$

w	0	1	2	3	4	5
i_0	0	0	0	0	0	0
i_1	0	0	3	3		
i_2	0					
i_3	0					
i_4	0					

0-1 Knapsack Problem - Example

Algorithm 9: 01-KNAPSACK(W, $\{x_1, \ldots, x_n\}$)

Input : W, $\{w_1, \ldots, w_n\}$, $\{b_1, \ldots, b_n\}$
Output: Maximum benefit (within W capacity)

```
for w ← 0 to W do
    B[0,w] = 0
for i ← 1 to n do
    B[i,0] = 0
for i ← 1 to n do
    for w ← 0 to W do
        if w_i ≤ w then
            if b_i + B[i-1, w − w_i] > B[i-1, w] then
                B[i, w] = b_i + B[i-1, w − w_i]
            else
                B[i, w] = B[i-1, w]
        else
            B[i,w] = B[i-1, w]
```

- Elements (weight, benefit) : $\{(2, 3), (3, 4), (4, 5), (5, 6)\}$
- $i = 1$, $w_i = 2$, $b_i = 3$, $w = 4$, $w - w_i = 2$

w	0	1	2	3	4	5
i_0	0	0	0	0	0	0
i_1	0	0	3	3	3	
i_2	0					
i_3	0					
i_4	0					

0-1 Knapsack Problem - Example

Algorithm 10: 01-KNAPSACK(W, $\{x_1, \ldots, x_n\}$)

Input : W, $\{w_1, \ldots, w_n\}$, $\{b_1, \ldots, b_n\}$
Output: Maximum benefit (within W capacity)

```
for w ← 0 to W do
    B[0,w] = 0
for i ← 1 to n do
    B[i,0] = 0
for i ← 1 to n do
    for w ← 0 to W do
        if w_i ≤ w then
            if b_i + B[ i-1, w − w_i] > B[i-1, w] then
                B[i, w] = b_i + B[ i-1, w − w_i]
            else
                B[i, w] = B[i-1, w]
        else
            B[i,w] = B[i-1, w]
```

- Elements (weight, benefit) : $\{(2, 3), (3, 4), (4, 5), (5, 6)\}$
- $i = 1$, $w_i = 2$, $b_i = 3$, $w = 5$, $w - w_i = 3$

w	0	1	2	3	4	5
i_0	0	0	0	0	0	0
i_1	0	0	3	3	3	3
i_2	0					
i_3	0					
i_4	0					

0-1 Knapsack Problem - Example

Algorithm 11: 01-KNAPSACK(W, $\{x_1,..., x_n\}$)

Input : W, $\{w_1,..., w_n\}$, $\{b_1,..., b_n\}$
Output: Maximum benefit (within W capacity)

for $w \leftarrow 0$ **to** W **do**
  B[0,w] = 0
for $i \leftarrow 1$ **to** n **do**
  B[i,0] = 0
for $i \leftarrow 1$ **to** n **do**
  for $w \leftarrow 0$ **to** W **do**
    if $w_i \leq w$ **then**
      if b_i+ B[i-1, $w - w_i$] > B[i-1, w] **then**
        B[i, w] = b_i+ B[i-1, $w - w_i$]
      else
        B[i, w] = B[i-1, w]
    else
      B[i,w] = B[i-1, w]

- Elements (weight, benefit) : {(2, 3), (3, 4), (4, 5), (5, 6)}
- $i = 2$, $w_i = 3$, $b_i = 4$, $w = 1$, $w - w_i =$ -2

w	0	1	2	3	4	5
i_0	0	0	0	0	0	0
i_1	0	0	3	3	3	3
i_2	0	0				
i_3	0					
i_4	0					

0-1 Knapsack Problem - Example

Algorithm 12: 01-KNAPSACK(W, $\{x_1, \ldots, x_n\}$)

Input : W, $\{w_1, \ldots, w_n\}$, $\{b_1, \ldots, b_n\}$
Output: Maximum benefit (within W capacity)

```
for w ← 0 to W do
    B[0,w] = 0
for i ← 1 to n do
    B[i,0] = 0
for i ← 1 to n do
    for w ← 0 to W do
        if w_i ≤ w then
            if b_i + B[ i-1, w − w_i] > B[i-1, w] then
                B[i, w] = b_i + B[ i-1, w − w_i]
            else
                B[i, w] = B[i-1, w]
        else
            B[i,w] = B[i-1, w]
```

- Elements (weight, benefit) : $\{(2, 3), (3, 4), (4, 5), (5, 6)\}$
- $i = 2$, $w_i = 3$, $b_i = 4$, $w = 2$, $w - w_i =$ -1

w	0	1	2	3	4	5
i_0	0	0	0	0	0	0
i_1	0	0	3	3	3	3
i_2	0	0	3			
i_3	0					
i_4	0					

0-1 Knapsack Problem - Example

Algorithm 13: 01-KNAPSACK(W, $\{x_1,\ldots,x_n\}$)

Input : W, $\{w_1,\ldots,w_n\}$, $\{b_1,\ldots,b_n\}$
Output: Maximum benefit (within W capacity)

```
for w ← 0 to W do
    B[0,w] = 0
for i ← 1 to n do
    B[i,0] = 0
for i ← 1 to n do
    for w ← 0 to W do
        if w_i ≤ w then
            if b_i + B[ i-1, w − w_i] > B[i-1, w] then
                B[i, w] = b_i + B[ i-1, w − w_i]
            else
                B[i, w] = B[i-1, w]
        else
            B[i,w] = B[i-1, w]
```

- Elements (weight, benefit) : $\{(2, 3), (3, 4), (4, 5), (5, 6)\}$
- $i = 2$, $w_i = 3$, $b_i = 4$, $w = 3$, $w - w_i = 0$

w	0	1	2	3	4	5
i_0	0	0	0	0	0	0
i_1	0	0	3	3	3	3
i_2	0	0	3	4		
i_3	0					
i_4	0					

0-1 Knapsack Problem - Example

Algorithm 14: 01-KNAPSACK(W, $\{x_1,\ldots, x_n\}$)

Input : W, $\{w_1,\ldots, w_n\}$, $\{b_1,\ldots, b_n\}$
Output: Maximum benefit (within W capacity)

```
for w ← 0 to W do
    B[0,w] = 0
for i ← 1 to n do
    B[i,0] = 0
for i ← 1 to n do
    for w ← 0 to W do
        if w_i ≤ w then
            if b_i+ B[ i-1, w − w_i] > B[i-1, w] then
                B[i, w] = b_i+ B[ i-1, w − w_i]
            else
                B[i, w] = B[i-1, w]
        else
            B[i,w] = B[i-1, w]
```

- Elements (weight, benefit) : $\{(2, 3), (3, 4), (4, 5), (5, 6)\}$
- $i = 2$, $w_i = 3$, $b_i = 4$, $w = 4$, $w - w_i = 1$

w	0	1	2	3	4	5
i_0	0	0	0	0	0	0
i_1	0	0	3	3	3	3
i_2	0	0	3	4	4	
i_3	0					
i_4	0					

0-1 Knapsack Problem - Example

Algorithm 15: 01-KNAPSACK(W, $\{x_1, \ldots, x_n\}$)

Input : W, $\{w_1, \ldots, w_n\}$, $\{b_1, \ldots, b_n\}$
Output: Maximum benefit (within W capacity)

```
for w ← 0 to W do
    B[0,w] = 0
for i ← 1 to n do
    B[i,0] = 0
for i ← 1 to n do
    for w ← 0 to W do
        if w_i ≤ w then
            if b_i + B[ i-1, w − w_i] > B[i-1, w] then
                B[i, w] = b_i + B[ i-1, w − w_i]
            else
                B[i, w] = B[i-1, w]
        else
            B[i,w] = B[i-1, w]
```

- Elements (weight, benefit) : $\{(2, 3), (3, 4), (4, 5), (5, 6)\}$
- $i = 2$, $w_i = 3$, $b_i = 4$, $w = 5$, $w - w_i = 2$

w	0	1	2	3	4	5
i_0	0	0	0	0	0	0
i_1	0	0	3	3	3	3
i_2	0	0	3	4	4	7
i_3	0					
i_4	0					

0-1 Knapsack Problem - Example

Algorithm 16: 01-KNAPSACK(W, $\{x_1,\ldots, x_n\}$)

Input : W, $\{w_1,\ldots, w_n\}$, $\{b_1,\ldots, b_n\}$
Output: Maximum benefit (within W capacity)

```
for w ← 0 to W do
    B[0,w] = 0
for i ← 1 to n do
    B[i,0] = 0
for i ← 1 to n do
    for w ← 0 to W do
        if w_i ≤ w then
            if b_i + B[ i-1, w − w_i] > B[i-1, w] then
                B[i, w] = b_i + B[ i-1, w − w_i]
            else
                B[i, w] = B[i-1, w]
        else
            B[i,w] = B[i-1, w]
```

- Elements (weight, benefit) : $\{(2, 3), (3, 4), (4, 5), (5, 6)\}$
- $i = 3$, $w_i = 4$, $b_i = 5$, $w = 1$, $w - w_i = -3$

w	0	1	2	3	4	5
i_0	0	0	0	0	0	0
i_1	0	0	3	3	3	3
i_2	0	0	3	4	4	7
i_3	0	0				
i_4	0					

0-1 Knapsack Problem - Example

Algorithm 17: 01-KNAPSACK(W, $\{x_1,..., x_n\}$)

Input : W, $\{w_1,..., w_n\}$, $\{b_1,..., b_n\}$
Output: Maximum benefit (within W capacity)

```
for w ← 0 to W do
    B[0,w] = 0
for i ← 1 to n do
    B[i,0] = 0
for i ← 1 to n do
    for w ← 0 to W do
        if w_i ≤ w then
            if b_i + B[i-1, w − w_i] > B[i-1, w] then
                B[i, w] = b_i + B[i-1, w − w_i]
            else
                B[i, w] = B[i-1, w]
        else
            B[i,w] = B[i-1, w]
```

- Elements (weight, benefit) : $\{(2, 3), (3, 4), (4, 5), (5, 6)\}$
- $i = 3$, $w_i = 4$, $b_i = 5$, $w = 2$, $w - w_i =$ -2

w	0	1	2	3	4	5
i_0	0	0	0	0	0	0
i_1	0	0	3	3	3	3
i_2	0	0	3	4	4	7
i_3	0	0	3			
i_4	0					

0-1 Knapsack Problem - Example

Algorithm 18: 01-KNAPSACK(W, $\{x_1,..., x_n\}$)

Input : W, $\{w_1,..., w_n\}$, $\{b_1,..., b_n\}$
Output: Maximum benefit (within W capacity)

```
for w ← 0 to W do
    B[0,w] = 0
for i ← 1 to n do
    B[i,0] = 0
for i ← 1 to n do
    for w ← 0 to W do
        if w_i ≤ w then
            if b_i + B[ i-1, w − w_i] > B[i-1, w] then
                B[i, w] = b_i + B[ i-1, w − w_i]
            else
                B[i, w] = B[i-1, w]
        else
            B[i,w] = B[i-1, w]
```

- Elements (weight, benefit) : $\{(2, 3), (3, 4), (4, 5), (5, 6)\}$
- $i = 3$, $w_i = 4$, $b_i = 5$, $w = 3$, $w - w_i =$ -1

w	0	1	2	3	4	5
i_0	0	0	0	0	0	0
i_1	0	0	3	3	3	3
i_2	0	0	3	4	4	7
i_3	0	0	3	4		
i_4	0					

0-1 Knapsack Problem - Example

Algorithm 19: 01-KNAPSACK(W, $\{x_1, \ldots, x_n\}$)

Input : W, $\{w_1, \ldots, w_n\}$, $\{b_1, \ldots, b_n\}$
Output: Maximum benefit (within W capacity)

```
for w ← 0 to W do
    B[0,w] = 0
for i ← 1 to n do
    B[i,0] = 0
for i ← 1 to n do
    for w ← 0 to W do
        if w_i ≤ w then
            if b_i + B[ i-1, w − w_i] > B[i-1, w] then
                B[i, w] = b_i + B[ i-1, w − w_i]
            else
                B[i, w] = B[i-1, w]
        else
            B[i,w] = B[i-1, w]
```

- Elements (weight, benefit) : $\{(2, 3), (3, 4), (4, 5), (5, 6)\}$
- $i = 3$, $w_i = 4$, $b_i = 5$, $w = 4$, $w - w_i = 0$

w	0	1	2	3	4	5
i_0	0	0	0	0	0	0
i_1	0	0	3	3	3	3
i_2	0	0	3	4	4	7
i_3	0	0	3	4	5	
i_4	0					

0-1 Knapsack Problem - Example

Algorithm 20: 01-KNAPSACK(W, $\{x_1,\ldots, x_n\}$)

Input : W, $\{w_1,\ldots, w_n\}$, $\{b_1,\ldots, b_n\}$
Output: Maximum benefit (within W capacity)

```
for w ← 0 to W do
    B[0,w] = 0
for i ← 1 to n do
    B[i,0] = 0
for i ← 1 to n do
    for w ← 0 to W do
        if w_i ≤ w then
            if b_i + B[ i-1, w − w_i] > B[i-1, w] then
                B[i, w] = b_i + B[ i-1, w − w_i]
            else
                B[i, w] = B[i-1, w]
        else
            B[i,w] = B[i-1, w]
```

- Elements (weight, benefit) : $\{(2, 3), (3, 4), (4, 5), (5, 6)\}$
- $i = 3$, $w_i = 4$, $b_i = 5$, $w = 5$, $w - w_i = 1$

w	0	1	2	3	4	5
i_0	0	0	0	0	0	0
i_1	0	0	3	3	3	3
i_2	0	0	3	4	4	7
i_3	0	0	3	4	5	7
i_4	0					

0-1 Knapsack Problem - Example

Algorithm 21: 01-KNAPSACK(W, $\{x_1, \ldots, x_n\}$)

Input : W, $\{w_1, \ldots, w_n\}$, $\{b_1, \ldots, b_n\}$
Output: Maximum benefit (within W capacity)

```
for w ← 0 to W do
    B[0,w] = 0
for i ← 1 to n do
    B[i,0] = 0
for i ← 1 to n do
    for w ← 0 to W do
        if w_i ≤ w then
            if b_i + B[ i-1, w − w_i] > B[i-1, w] then
                B[i, w] = b_i + B[ i-1, w − w_i]
            else
                B[i, w] = B[i-1, w]
        else
            B[i,w] = B[i-1, w]
```

- Elements (weight, benefit) : $\{(2, 3), (3, 4), (4, 5), (5, 6)\}$
- $i = 4$, $w_i = 5$, $b_i = 6$,
  $w = 1, 2, 3, 4,$
  $w - w_i =$ -4, -3, -2, -1

w	0	1	2	3	4	5
i_0	0	0	0	0	0	0
i_1	0	0	3	3	3	3
i_2	0	0	3	4	4	7
i_3	0	0	3	4	5	7
i_4	0	0	3	4	5	

0-1 Knapsack Problem - Example

Algorithm 22: 01-KNAPSACK(W, $\{x_1, \ldots, x_n\}$)

Input : W, $\{w_1, \ldots, w_n\}$, $\{b_1, \ldots, b_n\}$
Output: Maximum benefit (within W capacity)

```
for w ← 0 to W do
    B[0,w] = 0
for i ← 1 to n do
    B[i,0] = 0
for i ← 1 to n do
    for w ← 0 to W do
        if w_i ≤ w then
            if b_i + B[ i-1, w − w_i] > B[i-1, w] then
                B[i, w] = b_i + B[ i-1, w − w_i]
            else
                B[i, w] = B[i-1, w]
        else
            B[i,w] = B[i-1, w]
```

- Elements (weight, benefit) : $\{(2, 3), (3, 4), (4, 5), (5, 6)\}$
- $i = 4$, $w_i = 5$, $b_i = 6$, $w = 5$, $w - w_i = 0$

w	0	1	2	3	4	5
i_0	0	0	0	0	0	0
i_1	0	0	3	3	3	3
i_2	0	0	3	4	4	7
i_3	0	0	3	4	5	7
i_4	0	0	3	4	5	7

0-1 Knapsack Problem - Example (Finding the items)

Algorithm 23: 01-KSP-FIND-ITEMS(B)

Input : Table with benefit values (B)
Output: Items in the Knapsack

```
i = n, k = W
while i, k > 0 do
    if B[ i, k] ≠ B[i-1, k] then
        Mark i^th item as in Knapsack
        i = i - 1
        k = k - w_i
    else
        i = i - 1
```

- Elements (weight, benefit) : $\{(2, 3), (3, 4), (4, 5), (5, 6)\}$
- $i = 4$, $k = 5$, $w_i = 5$, $b_i = 6$, $B[i,k] = 7$, $B[i-1,k] = 7$

w	0	1	2	3	4	5
i_0	0	0	0	0	0	0
i_1	0	0	3	3	3	3
i_2	0	0	3	4	4	7
i_3	0	0	3	4	5	7
i_4	0	0	3	4	5	7

0-1 Knapsack Problem - Example (Finding the items)

Algorithm 24: 01-KSP-FIND-ITEMS(B)

Input : Table with benefit values (B)
Output: Items in the Knapsack

```
i = n, k = W
while i, k > 0 do
    if B[ i, k] ≠ B[i-1, k] then
        Mark i^th item as in Knapsack
        i = i - 1
        k = k - w_i
    else
        i = i - 1
```

- Elements (weight, benefit) : {(2, 3), (3, 4), (4, 5), (5, 6)}
- $i = 4$, $k = 5$, $w_i = 5$, $b_i = 6$, $B[i,k] = 7$, $B[i-1,k] = 7$

w	0	1	2	3	4	5
i_0	0	0	0	0	0	0
i_1	0	0	3	3	3	3
i_2	0	0	3	4	4	7
i_3	0	0	3	4	5	7
i_4	0	0	3	4	5	7

0-1 Knapsack Problem - Example (Finding the items)

Algorithm 25: 01-KSP-FIND-ITEMS(B)

Input : Table with benefit values (B)
Output: Items in the Knapsack

```
i = n, k = W
while i, k > 0 do
    if B[ i, k] ≠ B[i-1, k] then
        Mark i^th item as in Knapsack
        i = i - 1
        k = k - w_i
    else
        i = i - 1
```

- Elements (weight, benefit) : $\{(2, 3), (3, 4), (4, 5), (5, 6)\}$
- $i = 3$, $k = 5$, $w_i = 4$, $b_i = 5$, $B[i,k] = 7$, $B[i-1,k] = 7$

w	0	1	2	3	4	5
i_0	0	0	0	0	0	0
i_1	0	0	3	3	3	3
i_2	0	0	3	4	4	7
i_3	0	0	3	4	5	7
i_4	0	0	3	4	5	7

0-1 Knapsack Problem - Example (Finding the items)

Algorithm 26: 01-KSP-FIND-ITEMS(B)

Input : Table with benefit values (B)
Output: Items in the Knapsack

```
i = n, k = W
while i, k > 0 do
    if B[ i, k] ≠ B[i-1, k] then
        Mark i^th item as in Knapsack
        i = i - 1
        k = k - w_i
    else
        i = i - 1
```

- Elements (weight, benefit) : $\{(2, 3), (3, 4), (4, 5), (5, 6)\}$
- $i = 2$, $k = 5$, $w_i = 3$, $b_i = 4$, $B[i,k] = 7$, $B[i-1,k] = 3$

w	0	1	2	3	4	5
i_0	0	0	0	0	0	0
i_1	0	0	3	3	3	3
i_2	0	0	3	4	4	7
i_3	0	0	3	4	5	7
i_4	0	0	3	4	5	7

0-1 Knapsack Problem - Example (Finding the items)

Algorithm 27: 01-KSP-FIND-ITEMS(B)

Input : Table with benefit values (B)
Output: Items in the Knapsack

i = n, k = W
while $i, k > 0$ **do**
  if $B[\ i,\ k] \neq B[i\text{-}1,\ k]$ **then**
    Mark i^{th} item as in Knapsack
    $i = i$ - 1
    $k = k$ - w_i
  else
    $i = i$ - 1

- Elements (weight, benefit) : {(2, 3), (3, 4), (4, 5), (5, 6)}
- $i = 1$, $k = 2$, $w_i = 2$, $b_i = 3$, $B[i,k] = 3$, $B[i-1,k] = 0$

w	0	1	2	3	4	5
i_0	0	0	0	0	0	0
i_1	0	0	3	3	3	3
i_2	0	0	3	4	4	7
i_3	0	0	3	4	5	7
i_4	0	0	3	4	5	7

01-Knapsack Problem & Optimal Substructure

- Both solutions exhibit optimal substructure.
- To show this for the 0-1 problem, consider the most valuable load weighing at most W pounds
  - If we remove item j from the load, what do we know about the remaining load?

01-Knapsack Problem & Optimal Substructure

- Both solutions exhibit optimal substructure.
- To show this for the 0-1 problem, consider the most valuable load weighing at most W pounds
  - If we remove item j from the load, what do we know about the remaining load?
  - Answer: Remainder must be the most valuable load weighing at most W - w_j that thief could take, excluding item j.