Normal Subgroups

A subgroup H of a group G is **normal** in G if gH = Hg for all g $\in$ G. That is, a normal subgroup of a group G is one in which the right and left cosets are precisely the same.

Example 1. Let G be an abelian group. Every subgroup H of G is a normal subgroup. Since gh=hg for all g$\in$G and h$\in$H, it will always be the case that gH = Hg.

Example 2. Let H be the subgroup of S_3 consisting of elements (1) and (12). Since

$$(123)H = \{(123), (13)\} \text{ and } H\,(123) = \{(123), (23)\},$$

H cannot be a normal subgroup of S_3. However, the subgroup N, consisting of the permutations (1), (123), and (132), is normal since the cosets of N are

$$N = \{(1), (123), (132)\}$$

$$(12)N = N\,(12) = \{(12), (13), (23)\}$$

Sandip Chatterjee, Department of Mathematics

The following theorem is fundamental to our understanding of normal subgroups.

Theorem 10.1 Let G be a group and N be a subgroup of G. Then the following statements are equivalent.

1. The subgroup N is normal in G.
2. For all $g \in G$, $gNg^{-1} \subset N$.
3. For all $g \in G$, $gNg^{-1} = N$.

Proof. (1) $\Rightarrow$ (2). Since N is normal in G, $gN = Ng$ for all $g \in G$. Hence, for a given $g \in G$ and $n \in N$, there exists an n' in N such that $gn=n'g$. Therefore, $gng^{-1} = n' \in N$ or $gNg^{-1} \subset N$.

(2)$\Rightarrow$(3). Let $g \in G$. Since $gNg^{-1} \subset N$, we need only to show $N \subset gNg^{-1}$. For $n \in N$, $g^{-1}ng = g^{-1}n(g^{-1})^{-1} \in N$. Hence, $g^{-1}ng = n'$ for some $n' \in N$. Therefore, $n = gn'g^{-1}$ is in gNg^{-1}.

(3)$\Rightarrow$(1). Suppose that $gNg^{-1} =N$ for all $g \in G$. Then for any $n \in N$ there exists an $n' \in N$ such that $gng^{-1} = n'$. Consequently, $gn = n'g$ or $gN \subset Ng$. Similarly, $Ng \subset gN$.

Sandip Chatterjee, Department of Mathematics

Factor Groups

If N is a normal subgroup of a group G, then the cosets of N in G form a group G/N under the operation (aN)(bN) = abN. This group is called the factor or quotient group of G and N. Our first task is to prove that G/N is indeed a group.

Theorem 10.2 Let N be a normal subgroup of a group G. The cosets of N in G form a group G/N of order [G:N].

Proof. The group operation on G/N is (aN)(bN) = abN. This operation must be shown to be well-defined; that is, group multiplication must be independent of the choice of coset representative. Let aN = bN and cN = dN . We must show that

(aN)(cN) = acN = bdN = (bN)(dN).

Then $a=bn_1$ and $c=dn_2$ for some n_1 and n_2 in N. Hence,

$$
\begin{aligned}
acN &= bn_1dn_2N \\
&= bn_1dN \\
&= bn_1Nd \\
&= bNd \\
&= bdN.
\end{aligned}
$$

The remainder of the theorem is easy: eN = N is the identity and $g^{-1}N$ is the inverse of gN. The order of G/N is, of course, the number of cosets of N in G.

It is very important to remember that the elements in a factor group are sets of elements in the original group.

Example 3. Consider the normal subgroup of S_3, N = {(1), (123), (132)}. The cosets of N in S_3 are N and (12)N. The factor group S_3/ N has the following multiplication table.

	N	(12)N
N	N	(12)N
(12)	N	(12)N N

This group is isomorphic to Z_2. At first, multiplying cosets seems both complicated and strange; however, notice that S_3/N is a smaller group. The factor group displays a certain amount of information about S_3. Actually, N = A_3, the group of even permutations, and (12)N = {(12), (13), (23)} is the set of odd permutations. The information captured in G/N is parity; that is, multiplying two even or two odd permutations results in an even permutation, whereas multiplying an odd permutation by an even permutation yields an odd permutation.

Consider the normal subgroup 3Z of Z. The cosets of 3Z in Z are

$0 + 3Z = \{. . . , −3, 0, 3, 6, . . .\}$

$1 + 3Z = \{. . . , −2, 1, 4, 7, . . .\}$

$2 + 3Z = \{. . . , −1, 2, 5, 8, . . .\}$.

The group Z/3Z is given by the multiplication table below.

+	0+3Z	1+3Z	2+3Z
0+3Z	0+3Z	1+3Z	2+3Z
1+3Z	1+3Z	2+3Z	0+3Z
2+3Z	2+3Z	0+3Z	1+3Z

In general, the subgroup nZ of Z is normal. The cosets of Z/nZ are

nZ

1+nZ

2+nZ

..

(n−1)+nZ.

The sum of the cosets k+Z and l+Z is k+l+Z. Notice that we have written our cosets additively, because the group operation is integer addition.