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MATH 2201

Group Theory.

Assignment - 1.

1. Let S be a set having exactly one element. How many different binary operations can be defined on S? Answer the question if S has exactly 2 elements; 3 elements; n elements.

2. How many different commutative binary operations can be defined on a set of 2 elements? on a set of 3 elements? on a set of n elements?

3. Determine whether $*$ defined as follows gives a binary operation on the set or not. If not justify.
   (a) On $\mathbb{Z}^+$, define $*$ by $a*b=a-b$
   (b) On $\mathbb{Z}^+$, define $*$ by $a*b=a^b$
   (c) On $\mathbb{R}$, define $*$ by $a*b=a-b$.
   (d) On $\mathbb{R}$, define $*$ by $a*b=|a|+|b|$
   (e) On $\mathbb{Z}$, define $*$ by $a*b=|a|b$
   (f) On $\mathbb{Q}$, define $*$ by $a*b=ab+3$.

   For the binary operations above determine whether they are associative or commutative? Find the identity elements in each of the structures above if they exist.

4. Either prove the following statement or give a counterexample:
   (a) Every binary operation on a set consisting of a single element is both commutative and associative.
   (b) Every commutative binary operation on a set having just two elements is associative.

5. In the following cases determine whether the binary $*$ gives a group structure on the given set or not. If not, justify.
   (i) $\langle \mathbb{Z}, * \rangle$, $*$ given by $a*b=ab$
   (ii) $\langle 2\mathbb{Z}, * \rangle$, $*$ given by $a*b=a+b$

(iii) $\langle \mathbb{R}^+, * \rangle$, $*$ given by $a*b = \sqrt{ab}$

(iv) $\langle \mathbb{R}^*, * \rangle$, $*$ given by $a*b = \frac{a}{b}$.

(v) $\langle \mathbb{C}, * \rangle$, $*$ given by $a*b = |ab|$

(vi) $\langle Q[\sqrt{2}], + \rangle$ where $Q[\sqrt{2}] = \{a + b\sqrt{2};\ a, b \in Q\}$.

(vii) $\langle P(X), \Delta \rangle$ where $P(x)$ is the power set of x and Δ is the symmetric difference.

(viii) $\langle Q[\sqrt{2}] - \{0\}, * \rangle$, '$*$' is the usual product.

(ix) $\langle G, * \rangle$, where $G = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in \mathbb{R} - \{0\} \right\}$ and $*$ is the matrix multiplication.

6. Give an example of an abelian group G where G has exactly 1000 elements.

7. Let G be a group with a finite number of elements. Show that for any $a \in G$, there exists an $n \in \mathbb{Z}^+$ such that $a^n = e$.

8. Suppose that a group G has an element x such that $ax = x$ for all $a \in G$. Show that G contains only identity element.

9. Let G be a group, $a, b \in G$. Show that $(aba^{-1})^n = aba^{-1}$ iff $b = b^n$.

10. An element $a \in G$ is called idempotent if $a^2 = a$. Show that the only idempotent element in G is the unit element.

11. Find a solution of the equation $ax = b$ in S_3, where $a = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$.

12. If G is a group such that $a^2 = e$ for every $a \in G$. Show that G is abelian. Is it true if $a^3 = e$, $\forall a \in G$.

13. Show that G is abelian iff $(ab)^2 = a^2b^2$. $\forall a, b \in G$.

14. Let G be a finite group with even number of elements. Show that there is at least one $a \in G$ such that $a^2 = e$.

15. Give an example to show that union of two subgroups may not be a subgroup.

16. If K is a subgroup of H and H is a subgroup of G, show that K is a subgroup of G.

17. If G is an abelian group, show that $H = \{a : a \in G, a^2 = e\}$ is a subgroup of G.

18. Show that a group can not be expressed as a union of two proper subgroups.

19. Give an example of a group which is not cyclic but every proper subgroup of which is cyclic.

20. Let $a, b \in G$ such that $b = x a x^{-1}$ for some $x \in G$. Show that $o(a) = o(b)$.

21. Let $a, b \in G$. Show that $o(ab) = o(ba)$.

22. Write all complex roots of $x^6 = 1$. Show that they form a group under the usual complex multiplication.

23. Let $G = \{a \in \mathbb{R}, -1 < a < 1\}$. Define a binary operation $*$ on G by $a * b = \frac{a+b}{1+ab}$, $\forall a, b \in G$. Show that $(G, *)$ is a group.

24. Let $(G, *)$ be a group and $a, b \in G$. Suppose that $a^2 = e$, and $a * b * a = b^7$. Prove that $b^{48} = e$.

25. Let $(G, *)$ be a group such that $(a * b)^{-1} = a^{-1} * b^{-1}$ $\forall a, b \in G$, show that G is a commutative group.

26. Prove that a group $(G, *)$ is commutative if $(a * b)^n = a^n * b^n$, for any three consecutive integer n and for all $a, b \in G$.