17) Prove that $Z(G)$ is a normal subgroup of G.

Proof) The set $Z(G)$ is given by —

$$Z(G) = \{x \in G : xg = gx, \forall g \in G\}$$

Let us choose an element $p \in Z(G)$.

$$\Rightarrow pg = gp \quad \forall g \in G$$

$$\Rightarrow pgg^{-1} = gpg^{-1}$$

$$\Rightarrow p = gpg^{-1} \in Z(G)$$

Since for any $p \in Z(G)$ —

$$gpg^{-1} \in Z(G) \quad g \in G.$$

Then $Z(G)$ is a normal Subgroup of G.

---

18) Let H and K be finite subgroups of a group G. ~~Then prove that~~ Then if HK be a subgroup, then prove that

$$|HK| = \frac{|H|\,|K|}{|H \cap K|}$$

Let $H = \{h_1, h_2, \ldots, h_m\}$ and $O(H) = m$

$K = \{k_1, k_2, \ldots, k_n\}$ and $O(K) = n$

and $H \cap K = \{t_1, t_2, \ldots, t_p\}$

i.e $O(H \cap K) = p$

Now $HK = \{ h_i k_j : 1 \le i \le m, 1 \le j \le n \}$.

All the elements $h_i k_j$ may not be distinct.

Then we have to find how many times an element appears in the list.

Let $h_1 k_1 = h_r k_s$ for some $\begin{pmatrix} 1 \le r \le m \\ 1 \le s \le n \end{pmatrix}$

$\Rightarrow h_r^{-1} h_1 = k_s k_1^{-1} = t \in H \cap K$

$\Rightarrow h_1 = t h_r$ and $k_s = t k_1$

$\Rightarrow h_r^{-1} h_1 = k_s k_1^{-1} = t \in H \cap K$

$\Rightarrow h_1 = t h_r$ and $k_1 = t^{-1} k_s$

Let, $h_1 k_1 = h_r k_s$

$\Rightarrow h_1^{-1} h_r = k_1 k_s^{-1} = t \in H \cap K$

$\Rightarrow h_r = h_1 t$ and $k_s = t^{-1} k_1$

Thus any $h_r k_s$ which is equal to $h_1 k_1$ is of the form $(h_1 t)(t^{-1} k_1)$ for some $t \in H \cap K$

Now for all $t \in H \cap K$

$h_1 t = h_r$ is distinct

and, $t^{-1} k_1 = k_s$ is distinct

And, $(h_1 t)(t^{-1} k_1) = h_1 k_1$

Then, h_1k_1 appears p times.

This is true for any arbitrary $hk \in HK$.

Then every element of HK appears p times

in the list, $L = \{h_1k_1, h_2k_2, \ldots, h_rk_s, \ldots, h_mk_n\}$

Then, $$|HK| = \frac{mn}{p}$$

$$\Rightarrow |HK| = \frac{O(H)\,O(K)}{O(H \cap K)} = \frac{|H|\,|K|}{|H \cap K|}$$

(proved)

---

(13) Let H be a subgroup of a group G such that $[G:H] = 2$. Then prove that H is a normal subgroup of G.

(Ans) The index of H in G is given, $[G:H] = 2$

i.e H has two distinct coset in G.

One coset is H itself.

Then the other one is $(G-H)$.

$\boxed{\text{Case I}}$ Let $a \in H$.

then $aH = H = Ha$

Case II

$a \in G - H$

Let, $p \in aH$

$\Rightarrow p = ah_1$ for some $h_1 \in H$

Now p must belong to $G - H$, else if it is possible that $p \in H$

$\Rightarrow p = ah_1 = h_2 \quad (h_2 \in H)$

$\Rightarrow a = h_2 h_1^{-1} \in H$

~~which is contradiction. Since we assumed $a \in G-$~~

Which is a contradiction, since we have assumed $a \in G - H$

$\Rightarrow p = ah_1 \in G - H$

Now aH is a left coset of H in G.

$(G-H)$ is also a left coset of H in G.

Since they have a element $p = ah_1$ in common.

$\Rightarrow aH = (G-H)$

Similarly we can show that —

$Ha = (G-H)$

$\therefore aH = Ha$

$\therefore$ Considering both the cases, H is a normal subgroup of G.

Then, ~~H_1k_1~~ h_1k_1 appears p times.

(20) Find all subgroups of S_3. Show that union of any two non-trivial distinct subgroups of S_3 is not a subgroup of S_3.

(Ans) Let $S_3 = (e_1, e_2, e_3, e_4, e_5, e_6)$

$$e_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = i = (1,2)(1,2)$$

$$e_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = (2,3)$$

$$e_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (1,3)$$

$$e_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (1,3,2) = (1,2)(1,3)$$

$$e_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (1,2,3)$$

$$e_6 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (1,2)$$

The subgroups are —

$H_1 = \{e_1\}$

$H_2 = \{e_1, e_2\}$

$H_3 = \{e_1, e_3\}$

$H_4 = \{e_1, e_6\}$

$H_5 = \{e_1, e_4, e_5\}$

$H_6 = \{e_1, e_2, e_3, e_4, e_5, e_6\}$

Here, we can see ~~...~~ H_2, H_3, H_4, H_5 are distinct non-trivial subgroups of S_3.

Here from we can clearly see that the union of any two of them are not forming a subgroup.

---

(11) Let G be a group of order 28. Show that G has a non-trivial subgroup.

(Ans) According to Cauchy theorem, if there exists a finite group G of order $O(G)$ and p be the prime divisor of $O(G)$, then there exists an element of order p.

Here $7 | 28$ and 7 is a prime.

Let there exists a element g (say) $\in G$ of order 7.

Then, $g, g^2, g^3, \ldots, g^7 (=e)$ all are distinct.

Let S be a set, $S = \{g, g^2, \ldots, g^7(=e)\}$

within this set S, identity $g^7 = e$ exists.

If we pick up any two elements and operate them, then resulting elements also lies in the group.

Let, $p = g^m$ and $q = g^n$ $1 \leq m, n \leq 7$

Then, $pq = g^m \cdot g^n$

$$= g^{m+n} = g^{x7+r} \quad \left(\begin{array}{l} x \in \mathbb{Z} \\ 1 \leq r \leq 7 \end{array} \right)$$

$\therefore$ the elements of the set S is bounded.

for any arbitary element, $p = g^m$ the inverse of it $p^{-1} = g^{7-m}$ exists in S.

$\therefore$ S forms a subgroup.

G has a non-trivial proper subgroup.