Assignment : Probability and Numerical Methods

Subject Code: MATH2202

Module-IV

Joint Probability Distribution

1. Three balls a,b,c are randomly distributed into three boxes (a box may contain any number of balls). Let X_1 and X_2 be the number of balls in box 1 and box 2 respectively. Find the joint p.m.f. of X_1 and X_2. Also compute $E(X_1 + X_2)$ and the marginal p.m.f's of X_1 and X_2.

2. The joint p.m.f of two random variables $X\ and\ Y$ is given by: $P(X = 0, Y = 1) = \frac{1}{3}$, $P(X = 1, Y = -1) = \frac{1}{3}$ and $P(X = 1, Y = 1) = \frac{1}{3}$. Find the marginal p.m.f. of $X\ and\ Y$. Also find the conditional p.m.f. of *X* given *Y=1*.

3. For the adjoining bivariate probability distribution of $X\ and\ Y$, find:
(i) $P(X \leq 1, Y = 2)$ (ii) $P(X \leq 1)$ (iii) $P(Y \leq 3)$ (iv) $P(X < 3, Y \leq 4)$

X \ Y	1	2	3	4	5	6
0	0	0	1/32	2/32	2/32	3/32
1	1/16	1/16	1/8	1/8	1/8	1/8
2	1/32	1/32	1/64	1/64	0	2/64

4. The random variables $X\ and\ Y$ have a joint probability mass function given by:
$$f(x, y) = \frac{2x + y}{27} \quad for\ x, y = 0, 1, 2$$
Find the marginal p.m.f. of of $X\ and\ Y$ and expected value for both the random variables $X\ and\ Y$.

5. A fair coin is tossed three times. Let X denote the number of heads in three tossings and Y denote the absolute difference between the number of heads and the number of tails. Find the joint p.m.f. of (X, Y). Also find the marginal p.m.f. of X and Y. Are these two random variables X and Y independent? Find the conditional p.m.f. of X, given Y=1.

6. The random variables $X\ and\ Y$ have a joint p.d.f.
$$f(x, y) = \begin{cases} \frac{1}{4}(1 + xy), & |x| < 1,\ |y| < 1 \\ 0 & ,\ otherwise \end{cases}$$

Find the marginal p.d.f. of of X *and* Y. Are they independent?

7. X *and* Y are two independent random variables with following individual p.m.f.'s $f_X(1) = \frac{1}{3}$, $f_X(2) = \frac{1}{3}$, $f_X(3) = \frac{1}{3}$, *and* $f_X(x) = 0$ for any other value of *x*. $f_Y(5) = \frac{1}{4}$, $f_Y(8) = \frac{3}{4}$, $f_Y(y) = 0$ for $y \neq 5, 8$. Find the joint p.m.f. of X *and* Y. How much is $E(XY)$?

8. X *and* Y are two random variables with joint p.d.f. $f(x, y) = \begin{cases} \frac{1}{8}(6 - x - y); 0 \leq x < 2, \ 2 \leq y < 4 \\ 0 \qquad , otherwise \end{cases}$

   Find $P(X + Y < 3)$. Also find (i) $P(X < 1, Y < 3)$, (ii) $P(X < 1 | Y < 3)$.

9. If X *and* Y have a joint p.d.f. $f(x, y) = \begin{cases} 6x^2y, 0 < x < 1, 0 < y < 1 \\ 0 \qquad , otherwise \end{cases}$

   Find $P(X + Y < 1)$ and $P(X > Y)$.

10. If X *and* Y have a joint p.d.f. $f(x, y) = \begin{cases} k \quad for\ x^2 + y^2 \leq 4 \\ 0 \qquad , otherwise \end{cases}$

    where k is a constant. Find the value of k and $P(x^2 + y^2 > 1)$. Also find the marginal p.d.f. of *Y*.

<u>Markov chain</u>

1. Two white balls and two black balls are distributed into two urns so that each urn contains two balls. Then one ball is randomly selected from each urn and their places are interchanged. This process of selecting balls and interchanging urns is repeated multiple times. Let X_n denote the number of white balls in the first urn after repeating this process n times. What is the state space of this markov chain? Find out the underlying transition probability matrix.

2. A box contains 4 balls. Every ball is either white or red. Two balls are randomly picked up and are randomly picked up and are replaced by two other balls of complementary colour. Let X_n denote the number of white balls in the box after repeating this process n times. What is the state space of this markov chain? Find out the underlying transition probability matrix.

3. Write down the transition probability matrix for the same markov chain as above with the change mechanism modified as:-
   (i) Case I:- each time one ball is picked up and colour complemented before replacement.
   (ii) Case-II each time three balls are picked up and colour complemented before replacement.

4. Suppose that the chance of rain tomorrow depends only on today weather condition. i.e. whether it is rainy or dry today. It is given that if it rains today then it will rain tomorrow with

probability 0.7; and if it does not rain today then it will rain tomorrow with probability 0.4. Treat the day wise weather condition as a markov chain with state space $\mathcal{S} = \{rainy, dry\}$. Write down its transition probability matrix. Also compute the following probabilities.

(i) If it rains today then the probability that it will rain day after tomorrow also

(ii) If it was dry on day before yesterday then the probability that it will rain day after tomorrow.

5. $X_1, X_2, X_3, \ldots$ form a markov chain where each X_i can only assume numerical values 0, 1, 2. The transition probability matrix of the markov chain is as follows:

$$P = \begin{bmatrix} 1/3 & 0 & 2/3 \\ 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

Find (i) $P(X_8 = 2|X_6 = 0)$ (ii) $P(X_5 = 0|X_3 = 2)$

(iii) If the markov chain starts from the state 0 i.e. $X_1 = 0$, then find $E(X_3)$ and $V(X_3)$.