Hamilton's formulation.

Let's take a system with Lagrangian

$$L \doteq L(q_j, \dot{q}_j, t \mid j = 1 \cdots n)$$

n being the degrees of freedom.

Then the Lagrange equation of 2nd kind is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \;\Big|\; j = 1 \cdots n \quad \cdots (1)$$

where $\frac{\partial L}{\partial \dot{q}_j} = p_j$ represents the j-th component of generalized momentum.

Let's define a function $H \doteq H(p_j, q_j, t \mid j = 1 \cdots n)$ as

$$H = \sum_{j=1}^{n} p_j \dot{q}_j - L$$

$$\Rightarrow \quad dH = \sum_{j=1}^{n} \left(p_j \, d\dot{q}_j + \dot{q}_j \, dp_j\right) - dL$$

$$\Rightarrow \sum_{j=1}^{n} \left(\frac{\partial H}{\partial q_j} dq_j + \frac{\partial H}{\partial p_j} dp_j\right) + \frac{\partial H}{\partial t} dt$$

$$= \sum_{j=1}^{n} \left(p_j \, d\dot{q}_j + \dot{q}_j \, dp_j\right) - \sum_{j=1}^{n} \left(\frac{\partial L}{\partial q_j} dq_j + \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j\right) - \frac{\partial L}{\partial t} dt.$$

Comparing the coefficient of

$$\left.\begin{aligned} dq_j : \Rightarrow \quad \frac{\partial H}{\partial q_j} &= -\frac{\partial L}{\partial q_j} \\ dp_j : \Rightarrow \quad \frac{\partial H}{\partial p_j} &= \dot{q}_j \end{aligned}\right\} \quad \cdots (2\,a;\,b)$$

$$dt : \Rightarrow \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

Eqn. (2a) can be written as

$$\frac{\partial H}{\partial q_j} = -\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) \quad (\text{from eq}^n (1))$$

$$\Rightarrow \quad \frac{\partial H}{\partial q_j} = -\dot{p}_j \quad \cdots (3)$$

Eqn. (2b) and (3) gives us Hamilton's eqns of motion

$$\left.\begin{aligned} \dot{p}_j &= -\frac{\partial H}{\partial q_j} \\ \dot{q}_j &= \frac{\partial H}{\partial p_j} \end{aligned}\right\} \quad j = 1 \cdots n \qquad \cdots (4)$$

Remark : (i) Hamilton's eqn determines the trajectory in a 2n dimensional system $\{q_j, p_j \mid j = 1 \cdots\}$ known as phase space.

(ii) p_j & q_j are called canonically conjugate variables.

(iii) If H is independent of any variable ~~in~~ is the corresponding conjugate variable is independent of time.

Now $\quad H \doteq H(q_j, p_j, t \mid j = 1 \cdots n)$

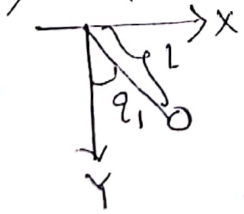
$$\Rightarrow \frac{dH}{dt} = \sum_{j=1}^{n} \left(\frac{\partial H}{\partial q_j} \dot{q}_j + \frac{\partial H}{\partial p_j} \dot{p}_j \right) + \frac{\partial H}{\partial t}.$$

$$\Rightarrow \frac{dH}{dt} = \sum_{j=1}^{n} \left(\cancel{\frac{\partial H}{\partial q_j} \frac{\partial H}{\partial p_j}} + \cancel{\frac{\partial H}{\partial p_j} \left(- \frac{\partial H}{\partial q_j} \right)} \right) + \frac{\partial H}{\partial t}.$$

$$\Rightarrow \frac{dH}{dt} = \frac{\partial H}{\partial t}.$$

Hence if H is not an explicit function of time (i.e.; $\frac{\partial H}{\partial t} = 0$) then $\frac{dH}{dt} = 0$ i.e.; H is conserved.

Example : Pendulum.



The Lagrangian $L = \frac{1}{2} m l \dot{q}_1^2 + mgl\cos q_1$

$$p_1 = \frac{\partial L}{\partial \dot{q}_1} = m l^2 \dot{q}_1 \Rightarrow \dot{q}_1 = \frac{p_1}{ml^2}$$

$$\text{Hence } H = p_1 \dot{q}_1 - L = p_1 \frac{p_1}{ml^2} - \frac{1}{2} m l^2 \left(\frac{p_1}{ml^2}\right)^2 - mgl\cos q_1$$

$$\Rightarrow H = \frac{p^2}{2ml^2} - mgl\cos q_1$$

So $\dot{p}_1 = -\frac{\partial H}{\partial q_1} = -mg\sin q_1$ & $\dot{q}_1 = \frac{p_1}{ml^2}$

Problem : Find the Hamilton's eqns of motion for (a) Particle on inclined plane (b) Atwood machine.