Module I : POSETS and Lattice (MATH 2201)

1 Relations and Their Properties

Relations are natural way to associate objects of various sets. For example, we can take the set, say A, of people in a town and the set, say B, of business in that town. We can say that an element a of A ia related to an element b of B if a is an employee of b.

A binary relation or simply a relation R from the set A into a set B is a subset of $A \times B$.

Examples

1. Let $S = \{2, 3, 4, 5\}$ and $T = \{11, 12, 13, 14\}$. A relation ρ between S and T is defined by an element of S is a divisor of any element of T. Then $\rho = \{(2, 12), (2, 14), (3, 12), (4, 12)\}$, whereas $(2, 11) \notin \rho$.

2. Let $S = \{1, 2, 3, 4, 5, 6\}$. Let R be defined by all $a, b \in S$, aRb if and only if a, b are relatively prime. Then R is a relation on S.

3. $R = \{(A, B) : A, B \in \mathscr{P}(U) \text{ and } A \subseteq B\}$ where $\mathscr{P}(U)$ is the power set of U.

4. $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 1\}$ is a relation on $\mathbb{R}$. Here R is the set of points on the unit circle.

Let ρ be a relation between sets A and B. The domain of ρ is $D(\rho) = \{a : a \in A, (a, b) \in \rho\}$ and the image or, range is $Im(\rho) = \{b : b \in B, (a, b) \in \rho\}$. Inverse of ρ is denoted by ρ^{-1} as $\rho^{-1} = \{(b, a) : (a, b) \in \rho\}$. Let ρ be a relation between the sets A and B and σ be a relation between the sets B and C. Then the composite relation $\sigma \circ \rho = \{(a, c) : a \in A, c \in C\}$ if there exists an element $b \in B$ such that $(a, b) \in \rho$ and $(b, c) \in \sigma$.

Equivalence relation: Let S be a non-empty set and ρ be a binary relation on S. The relation ρ is an equivalence relation on S if ρ is

(i) **reflexive :** if for all $a \in S$, $a\rho a$

(ii) **symmetric :** if for all $a, b \in S$, $a\rho b \Rightarrow b\rho a$

(iii) **transitive :** if for all $a, b, c \in S$, $a\rho b$ and $b\rho c \Rightarrow a\rho c$

Example: A congruence relation is an equivalence relation.

Result: Let ρ be a relation on a set S. Then ρ is symmetric if and only if $\rho^{-1} = \rho$.

Result: Let ρ be a relation on a set S. Then ρ is transitive if and only if $\rho \circ \rho \subset \rho$.

Partial Order relation: Let S be a non-empty set and ρ be a binary relation on S. The relation ρ is an partial Order relation on S if ρ is

(i) **reflexive :** if for all $a \in S$, $a\rho a$

(ii) **antisymmetric :** if for all $a, b \in S$, $a\rho b, b\rho a \Rightarrow a = b$

(iii) **transitive :** if for all $a, b, c \in S$, $a\rho b$ and $b\rho c \Rightarrow a\rho c$

A relation of partial order is often denoted by "$\leq$" even if it is not of usual meaning. A non-empty set S together with a relation of partial order $\leq$ on S is called a **Poset**.

Example:

1. The set $\mathbb{Z}$ together with the usual "less than or equal to", $\leq$ is a poset.

2. Let X be a non-empty set and $\mathscr{P}(X)$ be the power set of X. Then $(\mathscr{P}(X), \subseteq)$ is a poset.

3. $(\mathbb{N}, \leq)$ where $\leq$ is the relation "is a divisor of" is a poset.

24. Let S be a finite set with n elements. Prove that the number of symmetric relations on S is $2^{n(n+1)/2}$ and the number of relations that are both reflexive and symmetric is $2^{n(n-1)/2}$.
25. Find the reflexive closures of the following relations R on $A = \{1, 2, 3, 4\}$.
    a. $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$
    b. $R = \{(1, 3), (1, 2), (2, 1)\}$
26. Find the symmetric closures of the following relations R on $A = \{1, 2, 3, 4\}$.
    a. $R = \{(1, 1), (2, 2), (3, 3), (1, 3), (2, 1)\}$
    b. $R = \{(1, 3), (1, 2), (2, 1)\}$
27. Find the transitive closures of the following relations R on $A = \{1, 2, 3, 4\}$.
    a. $R = \{((2, 2), (3, 3), (1, 3), (2, 1)\}$
    b. $R = \{(1, 3), (3, 2), (2, 1)\}$
28. Prove that a relation R on a set S is transitive if and only if $R^n \subseteq R$ for all $n = 1, 2, 3, \ldots$.
29. If the following assertions are true, prove them; otherwise give counterexamples to disprove them.
    a. If R is a reflexive relation on a set A, then so is R^{-1}.
    b. If R is a transitive relation on a set A, then so is R^{-1}.
    c. If R_1 and R_2 are transitive relations on a set A, then so is $R_1 \circ R_2$.
    d. If a relation is symmetric and transitive, then it is reflexive.
    e. Every relation must either be symmetric or antisymmetric.
    f. Let R be a reflexive and transitive relation on a set A. Then $R \cap R^{-1}$ is an equivalence relation.

3.2 Partially Ordered Sets

In the first section of this chapter, we defined binary relations and studied their basic properties. More specifically, we discussed reflexive, symmetric, and transitive relations. In this section, we consider binary relations, which are reflexive and transitive and satisfy a new property, called the antisymmetric property. We begin with the following definition.

Definition 3.2.1 ▶ A relation R on a set S is called **antisymmetric** if for all $a, b \in S$, $a\,R\,b$ and $b\,R\,a$, then $a = b$.

On the set of all integers, the usual "less than or equal to," $\leq$ relation is an antisymmetric relation, because if a and b are integers such that $a \leq b$ and $b \leq a$, then $a = b$.

If T is the set of subsets of a set A, then the inclusion relation $\subseteq$ is an antisymmetric relation, because for subsets X and Y of A such that $X \subseteq Y$ and $Y \subseteq X$, we have $X = Y$.

Definition 3.2.2 ▶ A relation R on a set A is called a **partial order** on A if R is reflexive, antisymmetric, and transitive. In other words, if R satisfies the following conditions:

(i) $a\,R\,a$ for all $a \in A$ (i.e., R is reflexive).
(ii) For all $a, b \in A$ if $a\,R\,b$ and $b\,R\,a$, then $a = b$ (i.e., R is antisymmetric).
(iii) For all $a, b, c \in A$, if $a\,R\,b$ and $b\,R\,c$, then $a\,R\,c$ (i.e., R is transitive).

A set A together with a partial order relation R is called a **partially ordered set**, or simply **poset**, and we denote this poset by (A, R).

Let (A, R) be a poset. If there is no confusion about the partial order, we may refer to this poset simply by A.

Example 3.2.3 The set $\mathbb{Z}$, together with the usual "less than or equal to," $\leq$ relation is a poset.

Note that the relation $<$ (only less than) is not a partial order relation on $\mathbb{Z}$ because the relation $<$ is not reflexive as $1 \not< 1$.

EXAMPLE 3.2.4 Consider $\mathbb{N}$, the set of all natural numbers, and the divisibility relation, R, on $\mathbb{N}$. That is, for all $a, b \in \mathbb{N}$, $a \, R \, b$ if $a \mid b$ (i.e., $a \, R \, b$ if there exists a positive integer c such that $b = ac$).

We show that R is a partial order relation on $\mathbb{N}$. That is, we show that the divisibility relation is reflexive, antisymmetric, and transitive. For simplicity, we write R as $|$.

Reflexive: Let $a \in \mathbb{N}$. Because $a = 1a$, we have $a \mid a$.

Antisymmetric: Let $a \mid b$ and $b \mid a$. Then $b = ad$ and $a = bc$ for some positive integers c and d. Therefore, $a = bc = adc$, so $1 = cd$. Because c and d are positive integers and $cd = 1$, it follows that $c = d = 1$. Hence, $a = b$.

Transitive: Let $a \mid b$ and $b \mid c$ in $\mathbb{N}$. Then $b = an$ and $c = bm$ for some positive integers m and n. This implies that $c = bm = anm$, and because m and n are positive integers, nm is a positive integer. Thus, $a \mid c$ in $\mathbb{N}$.

Consequently, the divisibility relation is a partial order on $\mathbb{N}$, and hence $(\mathbb{N}, |)$ is a poset.

REMARK 3.2.5 ▶ Though the divisibility relation is a partial order relation on the set of all positive integers, it is not so on the set of all nonzero integers. For example, $4 = (-1)(-4)$ and $-4 = (-1)4$ imply that $4 \mid -4$ and $-4 \mid 4$, but $4 \neq -4$.

EXAMPLE 3.2.6 Let S be a set and $\mathcal{T}$ be the set of some subsets of S. Let R be a relation on $\mathcal{T}$ given by $R = \{(A, B) \in \mathcal{T} \times \mathcal{T} \mid A \subseteq B\}$. We show that R is a partial order on $\mathcal{T}$.

Because $A \subseteq A$ for all $A \in \mathcal{T}$, we find that the relation R is reflexive.

To show that R is antisymmetric, let $(A, B), (B, A) \in R$. Then, by the definition of R, $A \subseteq B$ and $B \subseteq A$, so $A = B$. Thus, R is antisymmetric.

To show that R is transitive, let $(A, B), (B, C) \in R$. Then $A \subseteq B$ and $B \subseteq C$, so $A \subseteq C$. Hence, R is transitive. Consequently, R is a partial order on $\mathcal{T}$.

Let R be a partial order on a set S. Then R^{-1} is also a partial order relation on S. Therefore, if (S, R) is a poset, then (S, R^{-1}) is a poset. The poset (S, R^{-1}) is called the **dual** of (S, R).

For example, the relation $\geq$ (the usual "greater than or equal to") on the set $\mathbb{Z}$ is the inverse relation of $\leq$, and hence the poset $(\mathbb{Z}, \geq)$ is the dual of the poset $(\mathbb{Z}, \leq)$.

Notation 3.2.7: Let R be a partial order on a set A; i.e., (A, R) is a poset. We usually denote R by $\leq_A$. If the set A is understood, then we write $\leq_A$ as $\leq$. If A is a partially ordered set with a partial order $\leq$, then we denote this by $(A, \leq_A)$ or $(A, \leq)$.

Let $(A, \leq)$ be a poset and $a, b \in A$. If $a \leq b$ and $a \neq b$, then we write $a < b$. Note that here $\leq$ means any relation, not the usual less than or equal to. Similarly, $<$ means related but not equal.

DEFINITION 3.2.8 ▶ Let $(S, \leq)$ be a poset and $a, b \in S$. If either $a \leq b$ or $b \leq a$, then we say that a and b are **comparable**. The poset $(S, \leq)$ is called a **linearly ordered set**, or a **totally ordered set**, or a **chain**, if for all $a, b \in S$ either $a \leq b$ or $b \leq a$.

Thus, a linearly ordered set, or a totally ordered set, or a chain, is a poset in which any two elements are comparable.

EXAMPLE 3.2.9

(i) Consider the poset $(\mathbb{Z}, \leq)$ of Example 3.2.3. For any two integers a and b, $a < b$, or $a = b$, or $a > b$. Thus, any two integers with respect to the partial order $\leq$ are comparable. Hence, $(\mathbb{Z}, \leq)$ is a chain.

(ii) Consider the poset $(\mathbb{N}, \leq)$ of Example 3.2.4. Notice that here the relation $\leq$ is the divisibility relation. That is, $a \leq b$ means $a \mid b$. Now, 3 does not divide 5 and 5 does not divide 3. Therefore, 3 and 5 are not comparable. Hence, $(\mathbb{N}, \leq)$ is not a chain.

(iii) Let A be a set with more than one element. Consider $\mathcal{P}(A)$, the power set of A, together with the set inclusion relation. Then, as in Example 3.2.6, $(\mathcal{P}(A), \leq)$ is a poset. (Notice that here $\leq$ means $\subseteq$). Let a and b be distinct elements of A. Then $\{a\}$ is a not a subset of $\{b\}$ and $\{b\}$ is a not subset of $\{a\}$; i.e., $\{a\}$ and $\{b\}$ are not comparable. It follows that $(\mathcal{P}(A), \leq)$ is not a chain.

Lexicographic Order

Let $(A, \leq)$ and $(B, \leq)$ be two posets. Define a relation R on the set $A \times B$ by $(a, b)\, R\, (c, d)$ if $a \leq c$ and $b \leq d$ for all $(a, b), (c, d) \in A \times B$. This relation R is a partial order and it is called the **product partial order**.

There is another partial order relation, denoted by $\preceq$, on $A \times B$, which is defined as follows:

$$(a, b) \preceq (c, d) \quad \text{if and only if} \quad a < c \quad \text{or} \quad a = c \quad \text{and} \quad b \leq d.$$

This partial order is called **lexicographic order**.

We can extend lexicographic order from the Cartesian product of two sets to, say n sets, as follows. Let $A_1, A_2, \ldots, A_n$, $n \geq 1$, be partially ordered sets; i.e., $(A_i, \leq)$ is a poset for all $i = 1, 2, \ldots, n$. Define the relation $\preceq$ on $A_1 \times A_2 \times \cdots \times A_{n-1} \times A_n$ as follows: Let $(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n) \in A_1 \times A_2 \times \cdots \times A_{n-1} \times A_n$. Then

$$(a_1, a_2, \ldots, a_n) \preceq (b_1, b_2, \ldots, b_n)$$

if and only if

$$\begin{aligned}
&a_1 < b_1 \quad \text{or}\\
&a_1 = b_1 \quad \text{and} \quad a_2 < b_2 \quad \text{or}\\
&a_1 = b_1,\ a_2 = b_2 \quad \text{and} \quad a_3 < b_3 \quad \text{or}\\
&\quad \vdots\\
&a_1 = b_1,\ a_2 = b_2,\ a_3 = b_3, \ldots, a_{n-1} = b_{n-1} \quad \text{and} \quad a_n < b_n \quad \text{or}\\
&a_1 = b_1,\ a_2 = b_2,\ a_3 = b_3, \ldots, a_{n-1} = b_{n-1} \quad \text{and} \quad a_n = b_n.
\end{aligned}$$

EXAMPLE 3.2.10

Consider the poset $\mathbb{R}$ of all real numbers under partial order $\leq$ (usual "less than or equal to") relation. Then $(\mathbb{R} \times \mathbb{R}, \preceq)$ is a poset under the lexicographic order relation.

In this poset,

$$\begin{aligned}
(2, 8) &\preceq (3, 0), && \text{because } 2 < 3,\\
(5, 1) &\preceq (5, 3), && \text{because } 5 = 5, \text{ but } 1 < 3,\\
(5, 3) &\not\preceq (1, 0), && \text{because } 5 > 1.
\end{aligned}$$

We show that this poset is a linearly ordered set.

Let (a, b) and (c, d) be two elements of $\mathbb{R} \times \mathbb{R}$. Now for the real numbers a and c, either $a < c$ or $a = c$ or $c < a$. Therefore,

$$\begin{aligned}
&\text{if } a < c, \text{ then } (a, b) \preceq (c, d),\\
&\text{if } a = c \text{ and } b < d, \text{ then } (a, b) \preceq (c, d),\\
&\text{if } a = c \text{ and } b > d, \text{ then } (c, d) \preceq (a, b),\\
&\text{if } a = c \text{ and } b = d, \text{ then } (a, b) = (c, d),\\
&\text{if } a > c, \text{ then } (c, d) \preceq (a, b).
\end{aligned}$$

Thus, we find that $(\mathbb{R} \times \mathbb{R}, \preceq)$ is a linearly ordered set.

Let us look at the ordering that is used to arrange the words in an English dictionary. Let A be the set of all 26 letters, $a, b, c, d, e, \ldots, x, y, z$. We define an ordering on A as follows: a is the first element, b is the second element, c is the third element, ..., x is the 24th element, y is the 25th element, and z is the 26th element.

Let a_i, a_j denote the ith and jth elements, respectively, where $i, j \in \{1, 2, 3, \ldots, 24, 25, 26\}$. Define $a_i \leq a_j$ if and only if $i \leq j$. Then A is a poset under this relation.

We denote the Cartesian products $A \times A \times A \times \cdots \times A$ of n A's by A^n. Then $(A^n, \preceq)$ is a poset under the lexicographic order relation $\preceq$.

Consider two words $a_1 a_2 \cdots a_n$ and $b_1 b_2 \cdots b_m$ over A. Let r be the minimum of m and n. Define the relation R on the set of all English words on A as follows:

$$a_1 a_2 \cdots a_n \, R \, b_1 b_2 \cdots b_m$$

if and only if

$(a_1, a_2, \ldots, a_r) \neq (b_1, b_2, \ldots, b_r)$ and $(a_1, a_2, \ldots, a_r) \preceq (b_1, b_2, \ldots, b_r)$ in the poset $(A^r, \preceq)$ or

$(a_1, a_2, \ldots, a_r) = (b_1, b_2, \ldots, b_r)$ and $m > n$.

One can verify that R is a partial order relation. This relation is called **dictionary order**, and it also is denoted by $\preceq$.

EXAMPLE 3.2.11

In the dictionary, the word *mango* comes before *money* because of the letters in the second position: $a < o$. Likewise, *grass* comes before *grit*; here we have to compare the first three letters, we see that the first two letters are the same and in the third position, $a < i$. To compare the words *earth* and *earthquake*, we have to compare the first five letters. Because the first five letters are the same, our next step is to compare the length of the words. As the word *earthquake* is longer than word *earth*, it follows that *earth* comes before *earthquake* in the dictionary.

Digraphs of Posets

Because any partial order is also a relation, we can give a digraph representation of partial order.

EXAMPLE 3.2.12

On the set $S = \{a, b, c\}$, consider the relation

$$R = \{(a, a), (b, b), (c, c), (a, b)\}.$$

The digraph of R is shown in Figure 3.19.

From the directed graph it follows that the given relation is reflexive and transitive. This relation is also antisymmetric because there is a directed edge

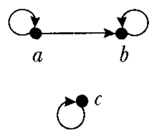

FIGURE 3.19
Digraph of R

from a to b, but there is no directed edge from b to a. Again, in the graph we notice that there are two distinct vertices a and c such that there are no directed edges from a to c and from c to a.

In a digraph of a partial order, one can see that if there is a directed edge from a vertex a to a different vertex b, then there is no directed edge from b to a.

EXAMPLE 3.2.13 Let $S = \{1, 2, 3, 4, 6, 12\}$. Consider the divisibility relation on S, which is a partial order. A digraph of this poset is as shown in Figure 3.20.

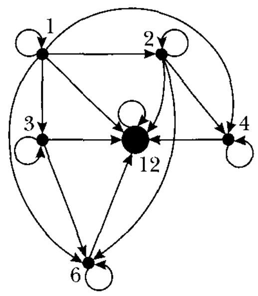

FIGURE 3.20 Digraph of a relation on S

A digraph representation of a partial order suggests the following theorem.

Theorem 3.2.14: A digraph of a partial order relation R cannot contain a closed directed path other than loops. (A path $a_1, a_2, \ldots, a_n$ in the digraph is **closed** if $a_1\,R\,a_2$, $a_2\,R\,a_3, \ldots, a_n\,R\,a_1$.)

Proof: Let $a_1, a_2, \ldots, a_n, n \neq 1$, be a closed directed path of distinct vertices a_1, $a_2, \ldots, a_n$. Then $a_1\,R\,a_2, a_2\,R\,a_3, \ldots, a_n\,R\,a_1$. Now R is transitive, and $a_1\,R\,a_2$, $a_2\,R\,a_3, \ldots, a_{n-1}\,R\,a_n$. Therefore, $a_1\,R\,a_n$. Also, we have $a_n\,R\,a_1$. Now, $a_1\,R\,a_n$ and $a_n\,R\,a_1$, and so by the antisymmetric property of R, it follows that $a_1 = a_n$. This contradicts our assumption that $a_1, a_2, \ldots$, and a_n are distinct. Consequently, there is no directed closed path other than loops. ∎

By Theorem 3.2.14, it follows that if a digraph of a relation contains a closed path other than loops, then the corresponding relation is not a partial order.

EXAMPLE 3.2.15 On the set $S = \{a, b, c\}$ consider the relation

$$R = \{(a, a), (b, b), (c, c), (a, b), (b, c), (c, a)\}.$$

The digraph of this relation is given in Figure 3.21.

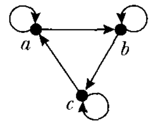

FIGURE 3.21
Digraph of R

In this digraph, we see that a, b, c, a form a closed path. Hence, the given relation is not a partial order relation.

Hasse Diagram

Another visual device used in the study of posets is the Hasse diagram (named after Helmut Hasse, a twentieth-century German number theorist). Before we discuss how to draw Hasse diagrams, however, we need to define a few terms.

Let $(S, \leq)$ be a poset and $x, y \in S$. We say that y **covers** x, if $x \leq y$, $x \neq y$, and there are no elements $z \in S$ such that $x < z < y$.

We draw a diagram using the elements of S as follows: We represent the elements of S in the diagram by the elements themselves such that if $x \leq y$, then y is placed above x. We connect x with y by a line segment if and only if y covers x. The resulting diagram is called the **Hasse diagram** of $(S, \leq)$.

The following example illustrates how to draw Hasse diagrams.

EXAMPLE 3.2.16 Let $S = \{1, 2, 3\}$. Then

$$\mathcal{P}(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, S\}.$$

Now $(\mathcal{P}(S), \leq)$ is a poset, where $\leq$ denotes the set inclusion relation. The poset diagram of $(\mathcal{P}(S), \leq)$ is shown in Figure 3.22.

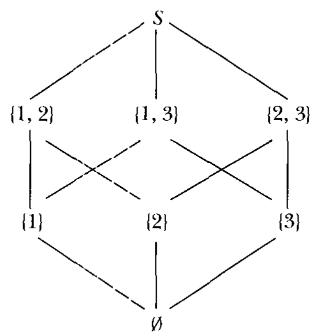

FIGURE 3.22 Hasse diagram of $(\mathcal{P}(S), \leq)$

Historical Notes

Helmut Hasse
(1898–1979)

Helmut Hasse was born in Germany and attended the Fichte-Gymnasium in Berlin for two years. At the age of 15, he volunteered for naval service during World War I. Upon leaving the navy, he entered the University of Göttingen.

In 1920, Hasse studied p-adic numbers under the tutelage of Hensel and discovered the Hasse principle as part of his dissertation. This principle states that the representability of a number by a given form and whether two forms are equivalent can be decided using only local information. Beginning in 1922, Hasse began working at the University of Kiel in the area of field theory.

During World War II, Hasse worked for the navy on ballistics but was denied membership in the Nazi Party because of his Jewish ancestry. He finished his career at Hamburg (1950–1966), writing a textbook containing an introduction to algebraic number theory.

A Hasse diagram of a poset $(S, \leq)$ of finite elements can also be obtained from the digraph of the poset $(S, \leq)$. To do this, we use the following steps.

1. In the digraph, we place vertex a above vertex b if a covers b in the poset $(S, \leq)$.
2. We delete all loops from the digraph. (Because the relation is reflexive, there is a loop at each vertex, so it is not necessary to show the loop).
3. We delete all the directed edges that are implied by the transitive property. For example, suppose $a \leq b$, $b \leq c$. Then $a \leq c$, so we omit the edges from a to c.
4. We omit the arrow signs from the directed edges (because we draw the directed edges following the condition stated in step (1).)

In the following example, we show how we can construct the Hasse diagram of the poset of Example 3.2.16 from the digraph of the partial order.

EXAMPLE 3.2.17 Let $S = \{1, 2, 3\}$. Then

$$\mathcal{P}(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, S\}.$$

Now $(\mathcal{P}(S), \leq)$ is a poset, where $\leq$ denotes the set inclusion relation.

Let us draw the digraph of this inclusion relation (see Figure 3.23). Place the vertex A above vertex B if $B \subset A$. Now follow steps (2), (3), and (4). Thus, we obtain the Hasse diagram as shown in Figure 3.22.

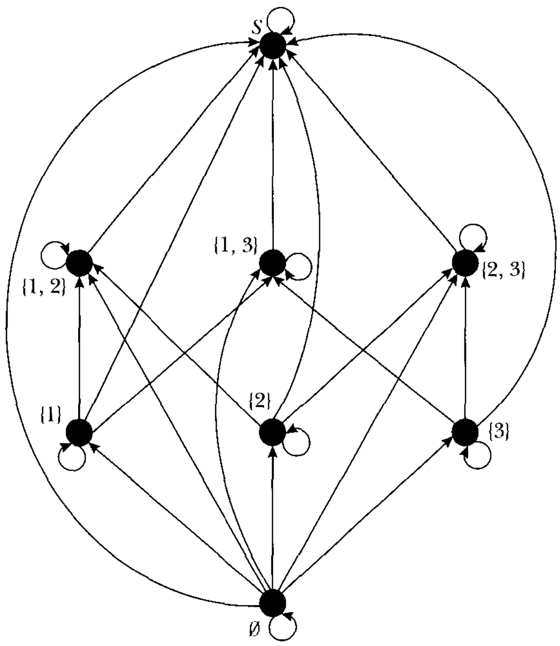

FIGURE 3.23 Digraph of $(\mathcal{P}(S), \leq)$

Minimal and Maximal Elements

Let us now define some special elements in a poset.

DEFINITION 3.2.18 ▶ Let $(S, \leq)$ be a poset. An element $a \in S$ is called

(i) a **minimal element** if there is no element $b \in S$ such that $b < a$,
(ii) a **maximal element** if there is no element $b \in S$ such that $a < b$,
(iii) a **greatest element** if $b \leq a$ for all $b \in S$,
(iv) a **least element** if $a \leq b$ for all $b \in S$.

EXAMPLE 3.2.19 In this example, we consider the poset $(S, \leq)$, where

$$S = \{2, 4, 5, 10, 15, 20\}$$

and the partial order $\leq$ is the divisibility relation.

In this poset, there is no element $b \in S$ such that $b \neq 5$ and b divides 5. (That is, 5 is not divisible by any other element of S except 5). Hence, 5 is a minimal element. Similarly, 2 is a minimal element.

Now, 10 is not a minimal element because $2 \in S$ and 2 divides 10. That is, there exists an element $b \in S$ such that $b < 10$. Similarly, 4, 15, and 20 are not minimal elements.

We find that 2 and 5 are the only minimal elements of this poset. Notice that 2 does not divide 5. Therefore, it is not true that $2 \leq b$, for all $b \in S$, and so 2 is not a least element in $(S, \leq)$. Similarly, 5 is not a least element. Actually, we can show that this poset has no least element.

There is no element $b \in S$ such that $b \neq 15$, $b > 15$, and 15 divides b. That is, there is no element $b \in S$ such that $15 < b$. Thus, 15 is a maximal element. Similarly, 20 is a maximal element.

Notice that 10 is not a maximal element because $20 \in S$ and 10 divides 20. That is, there exists an element $b \in S$ such that $10 < b$. Similarly, 4 is not a maximal element.

We find that 20 and 15 are the only maximal elements of this poset.

We also notice that 10 does not divide 15, hence it is not true that $b \leq 15$, for all $b \in S$, and so 15 is not a greatest element in $(S, \leq)$. Actually, we can show that this poset has no greatest element. Let us draw the Hasse diagram of this poset. (See Figure 3.24.) The Hasse diagram shows the maximal and minimal elements.

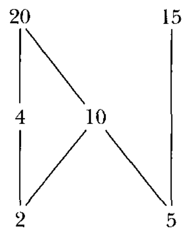

FIGURE 3.24
Hasse diagram

The following lemma ensures that every finite poset contains a minimal element. In this connection, we point out that a poset with an infinite number of elements may not have a minimal element. For example, the poset $(\mathbb{Z}, \leq)$ of all integers under the usual 'less than or equal to" relation has no minimal and maximal elements.

Lemma 3.2.20: Let $(S, \leq)$ be a poset such that S is a finite nonempty set. Then this poset has a minimal element.

Proof: Let a_1 be an element of S. If a_1 is a minimal element, then we are done. Suppose a_1 is not a minimal element. Then there exists $a_2 \in S$ such that $a_2 < a_1$. If a_2 is a minimal element, then we are done, otherwise there exists $a_3 \in S$ such that $a_3 < a_2$. If a_3 is not a minimal element, we repeat this process. Now $a_3 < a_2 < a_1$ shows that a_3, a_2, a_1 are distinct elements. Because S is finite, after a finite number of steps we get an element $a_n \in S$ such that a_n is a minimal element. ∎

REMARK 3.2.21 ▶ Let $(S, \leq)$ be a poset such that S is a finite nonempty set. Then S has minimal and maximal elements, but S may not have the least and the greatest elements.

DEFINITION 3.2.22 ▶ Let S be a set and let $\leq_1$ and $\leq_2$ be two partial order relations on S. The relation $\leq_2$ is said to be **compatible** with the relation $\leq_1$ if $a \leq_1 b$ implies $a \leq_2 b$.

It is interesting to note that given a finite nonempty set, say S, we can define a linear order on it as follows.

Because S is nonempty, S has at least one element. Choose an element from S, and call it the first element, a_1. Let $S_1 = S - \{a_1\}$. If S_1 is not empty, then from S_1 choose an element a_2. Let $S_2 = S - \{a_1, a_2\}$. If S_2 is not empty, then from S_2 choose an element a_3. Let $S_3 = S - \{a_1, a_2, a_3\}$. If S_3 is not empty, continue the process. Because S is a finite set, this process must stop after a finite number of steps. Hence, there exists a positive integer n such that $S_n = S - \{a_1, a_2, \ldots, a_n\}$ is empty, where a_n is an element of $S_{n-1} = S - \{a_1, a_2, \ldots, a_{n-1}\}$. We now define a partial order $\leq_1$ on S by $a_1 \leq_1 a_2 \leq_1 \cdots \leq_1 a_n$. This means that $a_i \leq_1 a_j$ if and only if either $i = j$ or $i < j$, where $i, j \in \{1, 2, \ldots, n\}$. It follows that this is a linear order.

Now suppose that not only S is a finite nonempty set, but S also has a partial order $\leq$. Can we define a linear order $\leq_1$ on S that is compatible with the partial order $\leq$? The following theorem proves that there exists such a linear order.

Theorem 3.2.23: Let $(S, \leq)$ be a finite poset. There exists a linear order $\leq_1$ on S which is compatible with the relation $\leq$.

Proof: Because $(S, \leq)$ is a finite poset, by Lemma 3.2.20 there exists a minimal element, say a_1, in S. Let $S_1 = S - \{a_1\}$. If S_1 is not empty, then S_1 is also a poset under the partial order relation induced by the given partial order relation on S; i.e., for all $a, b \in S_1$, $a \leq b$ in S_1 if and only if $a \leq b$ in S. By Lemma 3.2.20, S_1 has a minimal element, say a_2.

Let $S_2 = S - \{a_1, a_2\}$. If S_2 is not empty, then S_2 is also a poset under the partial order relation induced by the given partial order on S. By Lemma 3.2.20, S_2 has a minimal element, say a_3. Let $S_3 = S - \{a_1, a_2, a_3\}$. If S_3 is not empty, repeat this process.

Because S is a finite set, the above process must stop after a finite number of steps. Hence, there exists a positive integer n such that $S_n = S - \{a_1, a_2, \ldots, a_n\}$ is empty, where a_n is a minimal element in $S_{n-1} = S - \{a_1, a_2, \ldots, a_{n-1}\}$.

We now define a partial order $\leq_1$ on S by considering $a_1 \leq_1 a_2 \leq_1 \cdots \leq_1 a_n$. It follows that this is a linear order. We now show that $\leq_1$ is compatible with $\leq$.

Let $a, b \in S$ be such that $a < b$. Because $S = \{a_1, a_2, \ldots, a_n\}$, there exist i and j such that $a = a_i$ and $b = a_j$. Now $a < b$ implies that $i < j$. Hence, we find that $a_i \leq_1 \cdots \leq_1 a_j$. By the transitive property, we can conclude that $a_i \leq_1 a_j$. Therefore, it follows that the linear order $\leq_1$ on S is compatible with the relation $\leq$. ∎

The following example clarifies the construction of the compatible relation in Theorem 3.2.23.

EXAMPLE 3.2.24 Consider the poset $(S, \leq)$ of the Example 3.2.19. In this poset, $S = \{2, 4, 5, 10, 15, 20\}$ and the partial order relation is the divisibility relation.

As shown before, 5 is a minimal element of this poset. Let $a_1 = 5$ and $S_1 = S - \{5\} = \{2, 4, 10, 15, 20\}$. Then S_1 is also a poset under the divisibility relation. Also, S_1 has a minimal element $a_2 = 2$. Let

$$S_2 = S - \{2, 5\} = \{4, 10, 15, 20\}.$$

Now, S_2 has a minimal element $a_3 = 4$. Let

$$S_3 = S - \{2, 5, 4\} = \{10, 15, 20\}.$$

S_3 has a minimal element $a_4 = 10$. Let

$$S_4 = S - \{2, 5, 4, 10\} = \{15, 20\}.$$

$a_5 = 15$ is a minimal element of $\{15, 20\}$. Let

$$S_5 = S - \{2, 5, 4, 10, 15\} = \{20\}.$$

Finally, $a_6 = 20$ is a minimal element of $\{20\}$. We now define a partial order $\leq_1$ on S by $5 \leq_1 2 \leq_1 4 \leq_1 10 \leq_1 15 \leq_1 20$. It follows that this is a linear order.

We now show that $\leq_1$ is compatible with $\leq$.

Now, $5 \leq 15$ because 5 divides 15. In the relation $\leq_1$, we have

$$5 \leq_1 2 \leq_1 4 \leq_1 10 \leq_1 15.$$

By the transitivity of $\leq_1$, we have $5 \leq_1 15$. Similarly, we can verify that the compatibility holds for other elements. So it follows that the linear order $\leq_1$ on S is compatible with the relation $\leq$.

We also note that the relation $\leq_1$ on S is not the only linear order that is compatible with the relation $\leq$. In the construction of the linear order $\leq_1$, we started with the minimal element with 5. Now, 2 is another minimal element, so we can use it as the first element and construct the linear order

$$2 \leq_2 5 \leq_2 4 \leq_2 10 \leq_2 15 \leq_2 20.$$

We can also construct the linear order

$$2 \leq_3 5 \leq_3 4 \leq_3 15 \leq_3 10 \leq_3 20.$$

Both linear orders $\leq_2$ and $\leq_3$ are compatible with the relation $\leq$.

DEFINITION 3.2.25 ▶ The process of constructing a linear order $\leq_1$ on a poset $(S, \leq)$ that is compatible with the partial order relation $\leq$ is called **topological ordering**.

REMARK 3.2.26 ▶ We discuss topological ordering in Chapter 10 Graph Theory.

Lattices

DEFINITION 3.2.27 ▶ Let $(S, \leq)$ be a poset and let $\{a, b\}$ be a subset of S. An element $c \in S$ is called an **upper bound** of $\{a, b\}$ if $a \leq c$ and $b \leq c$.

An element $d \in S$ is called a **least upper bound (lub)** of $\{a, b\}$ if

(i) d is an upper bound of $\{a, b\}$; and
(ii) if $c \in S$ is an upper bound of $\{a, b\}$, then $d \leq c$.

EXAMPLE 3.2.28

(i) Consider the set $\mathbb{N}$ together with the divisibility relation of Example 3.2.4. For all $a, b \in \mathbb{N}$, $a \leq b$ if and only if a divides b.

Consider the subset $\{12, 8\}$. We see that 24, 48, and 72 are all common multiples of 12 and 8. Hence, $12 \leq 24$ *and* $8 \leq 24$; $12 \leq 48$ *and* $8 \leq 48$; and $12 \leq 72$ *and* $8 \leq 72$. Therefore, 24, 48, and 72 are upper bounds of $\{12, 8\}$. However, 24 is the least upper bound of $\{12, 8\}$. Notice that $24 \notin \{12, 8\}$.

(ii) Consider the set $\mathbb{Z}$, together with the usual "less than or equal to," $\leq$, relation of Example 3.2.3. Consider the subset $\{5, 7\}$. We see that $7, 8, 9, \ldots$ are all upper bounds of $\{5, 7\}$. However, 7 is the least upper bound of $\{5, 7\}$. Notice that $7 \in \{5, 7\}$.

(iii) Let $S = \{1, 2, 3\}$. Let $\leq$ denote the set inclusion relation. Then $(\mathcal{P}(S), \leq)$ is a poset. Let $A = \{1, 2\}$ and $B = \{1, 3\}$. Then $A \cup B = \{1, 2, 3\}$ is a least upper bound of $\{A, B\}$. Notice that $\{1, 2, 3\} = A \cup B \notin \{A, B\}$.

The following theorem shows that the lub of a subset, if it exists, is unique.

Theorem 3.2.29: In a poset $(S, \leq)$, if a subset $\{a, b\}$ of S has a lub, then this lub is unique.

Proof: Let $a, b \in S$ and a lub of $\{a, b\}$ exists. Suppose $c, d \in S$ are two lubs of $\{a, b\}$. Then c and d are upper bounds of $\{a, b\}$. Because c is a lub of $\{a, b\}$ and d is an upper bound of $\{a, b\}$, $c \leq d$. Similarly, $d \leq c$. Thus, we have $c \leq d$ and $d \leq c$. Therefore, by the antisymmetric property of the relation $\leq$, it follows that $c = d$. Hence, the lub is unique. ∎

Notation 3.2.30: The lub of $\{a, b\}$ in $(S, \leq)$, if it exists, is denoted by $a \vee b$.

DEFINITION 3.2.31 ▶ Let $(S, \leq)$ be a poset and let $\{a, b\}$ be a subset of S. An element $c \in S$ is called a **lower bound** of $\{a, b\}$ if $c \leq a$ and $c \leq b$.

An element $d \in S$ is called a **greatest lower bound (glb)** of $\{a, b\}$ if

(i) d is a lower bound of $\{a, b\}$; and
(ii) if $c \in S$ is a lower bound of $\{a, b\}$, then $c \leq d$.

Proceeding as in the proof of the Theorem 3.2.29, we can prove the following theorem.

Theorem 3.2.32: In a poset $(S, \leq)$, if a subset $\{a, b\}$ of S has a glb, then this glb is unique.

Notation 3.2.33: The glb of $\{a, b\}$ in $(S, \leq)$, if it exists, is denoted by $a \wedge b$.

We have seen several examples of posets in which lub (glb) need not exist. Next, we discuss those posets for which lub and glb exist.

DEFINITION 3.2.34 ▶ A poset $(L, \leq)$ is called a **lattice** if $a \wedge b$ and $a \vee b$ exist in L for all $a, b \in L$.

EXAMPLE 3.2.35

Let L be the set of all nonnonnegative real numbers. Then $(L, \leq)$ is a poset, where $\leq$ denotes the usual "less than or equal to" relation. Let $a, b \in L$. Now $\max\{a, b\} \in L$ and $\min\{a, b\} \in L$. It is easy to see that $\max\{a, b\}$ is the lub of $\{a, b\}$ and $\min\{a, b\}$ is the glb of $\{a, b\}$. For example, $\max\{2, 6\} = 6 = 2 \vee 6$ and $\min\{2, 6\} = 2 = 2 \wedge 6$. Hence, $(L, \leq)$ is a lattice.

EXAMPLE 3.2.36

Let S be a set. Then $(\mathcal{P}(S), \leq)$ is a poset, where $\leq$ is the set inclusion relation. For $A, B \in \mathcal{P}(S)$, we can show that $A \vee B = A \cup B$ and $A \wedge B = A \cap B$. Hence, $(\mathcal{P}(S), \leq)$ is a lattice.

In the following theorem, we collect several useful properties of lattices.

Theorem 3.2.37: Let $(L, \leq)$ be a lattice and $a, b, c \in L$. Then

(L1) $a \vee b = b \vee a$, $a \wedge b = b \wedge a$ (commutative laws),

(L2) $a \vee (b \vee c) = (a \vee b) \vee c$, $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ (associative laws),

(L3) $a \vee a = a$, $a \wedge a = a$ (idempotent laws),

(L4) $a \vee (a \wedge b) = a$, $a \wedge (a \vee b) = a$ (absorption laws).

Proof: We only prove (L1) and (L4) and leave others as exercises.

(L1): $a \vee b =$ lub of $\{a, b\} =$ lub of $\{b, a\} = b \vee a$. Note that the proof follows from the fact that the set $\{a, b\}$ is the same as the set $\{b, a\}$.

(L4): Now, $a \leq a$ and $a \wedge b \leq a$. Hence, a is an upper bound of $\{a, a \wedge b\}$. Thus, by the definition of least upper bound, $a \vee (a \wedge b) \leq a$. Because $a \vee (a \wedge b)$ is the lub of $\{a, a \wedge b\}$, we have $a \leq a \vee (a \wedge b)$. Hence, $a = a \vee (a \wedge b)$ because $\leq$ is antisymmetric. ■

The proof of the following result is left as an exercise.

Theorem 3.2.38: Let $(S, \leq)$ be a poset and $a, b \in S$. Then the following conditions are equivalent.

(i) $a \leq b$
(ii) $a \vee b = b$
(iii) $a \wedge b = a$

DEFINITION 3.2.39 ▶ A lattice $(L, \leq)$ is called **distributive** if it satisfies

$$\text{(D1)} \quad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

for all $a, b, c \in L$.

The lattices defined in Examples 3.2.35 and 3.2.36 are distributive lattices.

EXAMPLE 3.2.40 Consider the lattice given in Figure 3.25.

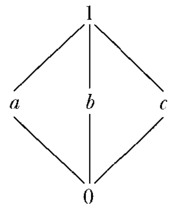

FIGURE 3.25
Nondistributive lattice

Because $a \wedge (b \vee c) = a \wedge 1 = a \neq 0 = 0 \vee 0 = (a \wedge b) \vee (a \wedge c)$, this is not a distributive lattice.

Theorem 3.2.41: A lattice $(L, \leq)$ is distributive if and only if

$$(D2)\ a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

for all $a, b, c \in L$.

Proof: Suppose $(L, \leq)$ is distributive. Let $a, b, c \in L$. Then

$$\begin{aligned}
(a \vee b) \wedge (a \vee c) &= ((a \vee b) \wedge a) \vee ((a \vee b) \wedge c) && \text{by D1} \\
&= (a \wedge (a \vee b)) \vee ((a \vee b) \wedge c) && \text{by L1} \\
&= a \vee ((a \vee b) \wedge c) && \text{by L4} \\
&= a \vee (c \wedge (a \vee b)) && \text{by L1} \\
&= a \vee ((c \wedge a) \vee (c \wedge b)) && \text{by D1} \\
&= (a \vee (c \wedge a)) \vee (c \wedge b) && \text{by L2} \\
&= (a \vee (c \wedge a)) \vee (b \wedge c) && \text{by L1} \\
&= a \vee (b \wedge c) && \text{by L4.}
\end{aligned}$$

Hence, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$. Similarly, $D2 \Rightarrow D1$. ■

Theorem 3.2.42: In a distributive lattice $(L, \leq)$,

$$a \wedge b = a \wedge c \quad \text{and} \quad a \vee b = a \vee c \quad \text{imply that} \quad b = c$$

for all $a, b, c \in L$.

Proof: Now, $b = b \wedge (a \vee b) = b \wedge (a \vee c) = (b \wedge a) \vee (b \wedge c) = (a \wedge c) \vee (b \wedge c) = (c \wedge a) \vee (c \wedge b) = c \wedge (a \vee b) = c \wedge (a \vee c) = c$. ■

REMARK 3.2.43 ▶ Note that a poset $(L, \leq)$ may not contain a greatest element, but from the antisymmetric property of the relation $\leq$, it can be shown that if there exists a greatest element in a poset, then it is unique. Similarly, a poset may contain at most one least element.

We denote the greatest element of a poset, if it exists, by 1 and the least element, if it exists, by 0.

Notice that here 1 and 0 are merely notations; these are not necessarily the integers 1 and 0. For example, for the poset $(\mathcal{P}(A), \leq)$, where A is a set, the greatest element is A and the least element is $\emptyset$. Thus in notation we can write for the poset $(\mathcal{P}(A), \leq)$, $1 = A$ and $0 = \emptyset$.

DEFINITION 3.2.44 ▶ Let $(L, \leq)$ be a lattice with 1 and 0. If $a \in L$, then an element $b \in L$ is said to be a **complement** of a if $a \vee b = 1$ and $a \wedge b = 0$.

EXAMPLE 3.2.45 Let D_{30} denote the set of all positive divisors of 30. Then

$$D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}.$$

Now, $(D_{30}, \leq)$ is a poset where $a \leq b$ if and only if a divides b ($\leq$ is the divisibility relation). Because 1 divides all elements of D_{30}, it follows that $1 \leq m$, for all $m \in D_{30}$. Therefore, 1 is the least element of this poset. Again, every member of D_{30} divides 30. Hence, $m \leq 30$ for all $m \in D_{30}$. This shows that 30 is the greatest element of this poset. The Hasse diagram of D_{30} is given in Figure 3.26.

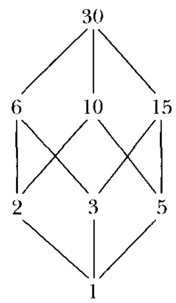

FIGURE 3.26
Hasse diagram of D_{30}

Let $a, b \in D_{30}$. Let $d = \gcd\{a, b\}$ and $m = \text{lcm}\{a, b\}$. Now $d \mid a$ and $d \mid b$. Hence, $d \leq a$ and $d \leq b$. This shows that d is a lower bound of $\{a, b\}$. Let $c \in D_{30}$ and $c \leq a, c \leq b$. Then, $c \mid a$ and $c \mid b$ and because $d = \gcd\{a, b\}$, it follows that $c \mid d$, so $c \leq d$. Thus, we find that $d = \text{glb}\{a, b\}$. Because all the positive divisors of a, b are also divisors of 30, $d \in D_{30}$, so $d = a \wedge b$. Similarly, we can show that

$$m \in D_{30}$$

and $m = a \vee b$.

Hence, $(D_{30}, \leq)$ is a lattice with the least element integer 1 and the greatest element 30.

Now for any element $a \in D_{30}$, $\frac{30}{a} \in D_{30}$. Using the properties of gcd and lcm, we can show that $a \wedge \frac{30}{a} = 1$ and $a \vee \frac{30}{a} = 30$. For example,

$$10 \wedge \frac{30}{10} = \gcd\{10, 3\} = 1$$

and

$$10 \vee \frac{30}{10} = \text{lcm}\{10, 3\} = 30.$$

Hence, 3 is a complement of 10 in this lattice.

REMARK 3.2.46 ▶ For any positive integer n, we can construct the lattice $(D_n, \leq)$, where D_n is the set of all positive divisors of n, $a \leq b$ if and only if a divides b. In the lattice, $a \vee b = \text{lub}\{a, b\} = \text{lcm}\{a, b\}$ and $a \wedge b = \text{glb}\{a, b\} = \text{gcd}\{a, b\}$ for any $a, b \in D_n$.

Theorem 3.2.47: In a distributive lattice $(L, \leq)$ with 1 and 0, every element has at most one complement.

Proof: Let $a \in L$. Suppose b, c are two complements of a in L. Then $a \vee b = 1$, $a \wedge b = 0$, $a \vee c = 1$, and $a \wedge c = 0$. Hence, $a \vee b = a \vee c$ and $a \wedge b = a \wedge c$. Then, by Theorem 3.2.42, it follows that $b = c$. ■

We now introduce the definition of *Boolean algebra*, named after the famous mathematician George Boole (1813–1864). Boole tried to formalize the process of logical reasoning using symbols instead of words. There are several equivalent definitions of Boolean algebra. Here we define Boolean algebra with the help of a lattice.

DEFINITION 3.2.48 ▶ A distributive lattice $(L, \leq)$ with the greatest element 1 and the least element 0 is called a **Boolean algebra** if every element has a complement in L.

From the above theorem, it follows that in a Boolean algebra $(L, \leq)$ every element $a \in L$ has a unique complement. The complement of a in L is denoted by a'.

EXAMPLE 3.2.49 Let $P(S)$ be the set of all subsets of a nonempty set S. Then $(P(S), \leq)$ is a poset, where $A \leq B$ if and only if $A \subseteq B$, for all $A, B \in P(S)$. This poset is a lattice, where $A \vee B = A \cup B$ and $A \wedge B = A \cap B$, for all $A, B \in P(S)$. The subset S is the greatest element 1, and the empty subset $\emptyset$ is the least element 0. Also, for each $A \in P(S)$, the set complement of A in S is the complement of A in this lattice. Hence, the lattice $(P(S), \leq)$ is a Boolean algebra. We will discuss more about Boolean algebra in Chapter 12.

WORKED-OUT EXERCISES

Exercise 1: For each of the following relations, draw the digraph. Determine which are antisymmetric. Also determine which are partial orders.

(a) (S, R), where $S = \{2, 6, 8, 10, 20\}$ and R denotes the divisibility relation

(b) (S, R), where $S = \{1, 5, 6, 8, 10\}$ and R denotes the relation

$$R = \{(1,1), (5,5), (6,6), (8,8), (10,10), (1,5), (5,6), (1,6)\}$$

(c) (S, R), where $S = \{1, 5, 6, 8, 10\}$ and R denotes the relation

$$R = \{(1,1), (5,5), (6,6), (8,8), (10,10), (1,6)(8,6), (6,1)\}$$

Solution:

(a) Here $R = \{(2,2), (6,6), (8,8), (10,10), (20,20), (2,6), (2,8), (2,10), (2,20), (10,20)\}$. Because $(a, a) \in R$, for all $a \in S$, the relation is reflexive. There are no distinct elements $a, b \in S$ such that $(a, b) \in R$ and $(b, a) \in R$. Hence, the relation is antisymmetric. The relation is also transitive, because if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for all $a, b, c \in S$. Hence, the relation R is a partial order. The digraph of this relation is shown in Figure 3.27(a).

(b) Because $(a, a) \in R$, for all $a \in S$, the relation is reflexive. There are no distinct elements $a, b \in S$ such that $(a, b) \in R$ and $(b, a) \in R$. Hence, the relation is antisymmetric. The relation is also transitive, because if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for all $a, b, c \in S$. Hence, the relation R is a partial order. The digraph of this relation is shown in Figure 3.27(b).

(c) Because $(a, a) \in R$, for all $a \in S$, the relation is reflexive. This relation is not antisymmetric, because $(1, 6), (6, 1) \in R$, but $1 \neq 6$. Hence, this relation is also not a partial order. The digraph of this relation is shown in Figure 3.27(c).

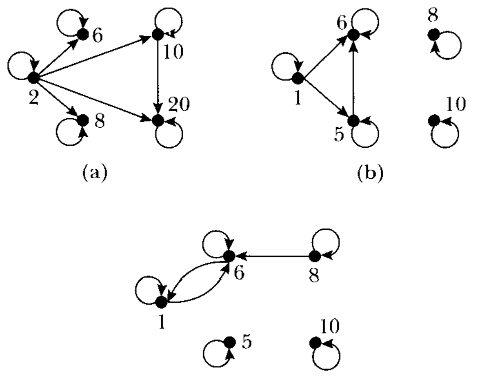

FIGURE 3.27 Digraphs

Exercise 2: Let $S = \{1, 2, 3\}$ and T be the set of all proper nonempty subsets of S. In the poset $(T, \leq)$, where $\leq$ is the set inclusion relation. Draw the digraph of the relation $\leq$ and the Hasse diagram of the poset. Find the maximal and minimal elements.

Solution: The digraph is shown in the Figure 3.28(a), and the Hasse diagram is shown in Figure 3.28(b).

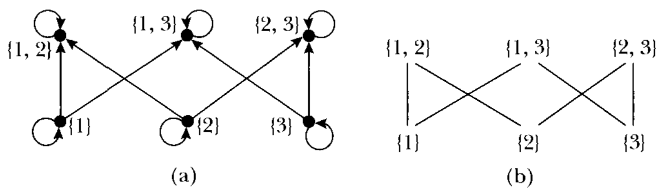

FIGURE 3.28 Digraph and Hasse diagram of $S = \{1, 2, 3\}$

In this poset, $\{1\}, \{2\}$, and $\{3\}$ are minimal elements, and $\{1, 2\}, \{1, 3\}, \{2, 3\}$ are maximal elements.

Exercise 3: Draw the digraph of the divisibility relation and the Hasse Diagram of the poset $(D_{20}, \leq)$.

Solution: We have

$$D_{20} = \{1, 2, 4, 5, 10, 20\}.$$

The digraph is shown in Figure 3.29(a), and the Hasse diagram is shown in Figure 3.29(b).

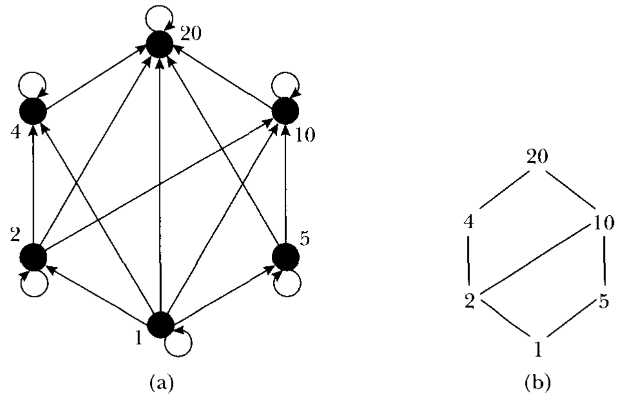

FIGURE 3.29 Digraph and Hasse diagram of D_{20}

Exercise 4: Define a relation R on the set $\mathbb{Z}$ of all integers by $m \, R \, n$ if and only if $m^2 = n^2$. Is R a partial order?

Solution: Because $m^2 = m^2$ for all $m \in \mathbb{Z}$, it follows that the relation is reflexive. Now $(2)^2 = (-2)^2$ implies that $2 \, R \, (-2)$ and $(-2) \, R \, 2$, but $-2 \neq 2$. Hence, R is not antisymmetric, and therefore R is not a partial order.

Exercise 5: Let $(S_1, \leq_1)$ and $(S_2, \leq_2)$ be two posets, where $S_1 = \{1, 2, 4\}$, $S_2 = \{1, 2, 3, 6\}$, and both the relations $\leq_1, \leq_2$ are divisibility relations. With respect to lexicographic order on $S_1 \times S_2$, find all pairs $(a, b) \in S_1 \times S_2$ such that $(a, b) \preceq (2, 3)$.

Solution: Note that $(a, b) \preceq (c, d)$ if and only if $a <_1 c$ or $a = c$ and $b \leq_2 d$. We find those pairs $(a, b) \in S_1 \times S_2$ such that $(a, b) \preceq (2, 3)$. The pairs are $(1, b) \in S_1 \times S_2$, $b \in S_2$ and $(2, b) \in S_1 \times S_2$ such that b divides 3. Hence, the pairs are $(1, 1), (1, 2), (1, 3), (1, 6), (2, 1)$, and $(2, 3)$.

Exercise 6: Consider the poset $(S, \leq)$, where $S = \{2, 4, 3, 6, 12\}$ and the partial order is the divisibility relation. Find a linear order on S compatible with the given partial order.

Solution: 2 is a minimal element of S. Let $a_1 = 2$ and $S_1 = S - \{2\} = \{4, 3, 6, 12\}$. Then S_1 is also a poset under the divisibility relation. Also, S_1 has a minimal element $a_2 = 3$. Let

$$S_2 = S - \{2, 3\} = \{4, 6, 12\}.$$

Now S_2 has a minimal element $a_3 = 4$. Let

$$S_3 = S - \{2, 3, 4\} = \{6, 12\}.$$

S_3 has a minimal element $a_4 = 6$. Let

$$S_4 = S - \{2, 3, 4, 6\} = \{12\}.$$

Finally, $a_5 = 12$ is a minimal element of $\{12\}$. We now define the partial order $\leq_1$ on S by $2 \leq_1 3 \leq_1 4 \leq_1 6 \leq_1 12$. It follows that this is a linear order.

Notice that $4 \leq 12$ because 4 divides 12. In the relation $\leq_1$, we have

$$4 \leq_1 6 \leq_1 12,$$

which implies that $4 \leq_1 12$ by the property of transitivity. Similarly, we can verify that the compatibility holds for other elements. So it follows that the linear order $\leq_1$ on S is compatible with the relation $\leq$.

Exercise 7: Show that every chain is a distributive lattice.

Solution: Let $(L, \leq)$ be a chain and $a, b, c \in L$. Because L is a chain, either $a \leq b$ or $b \leq a$. If $a \leq b$, then $a \vee b = b$ and $a \wedge b = a$. If $b \leq a$, then $a \vee b = a$ and $a \wedge b = b$. Hence, for any two elements $a, b \in L$, $a \wedge b$ and $a \vee b$ exist in L. Suppose $a \leq b$.

Case 1: $b \leq c$
Now $a \wedge (b \vee c) = a \wedge c = a$ and $(a \wedge b) \vee (a \wedge c) = a \vee a = a$. Hence, we have

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

Case 2: $c \leq b$

Subcase 2a: $a \leq c$
In this case, we have $a \leq c \leq b$. Now $a \wedge (b \vee c) = a \wedge b = a$ and

$$(a \wedge b) \vee (a \wedge c) = a \vee a = a.$$

Hence,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

Subcase 2b: $c \leq a$
In this case, we have $c \leq a \leq b$. Now $a \wedge (b \vee c) = a \wedge b = a$ and $(a \wedge b) \vee (a \wedge c) = a \vee c = a$. Hence,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

Similarly, if $b \leq a$, then $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

Exercise 8: In a lattice $(L, \leq)$, prove that $(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee (a \wedge c))$ for all $a, b, c \in L$.

Solution: Now $a \wedge b \leq a$, $a \wedge c \leq a$. Therefore, $(a \wedge b) \vee (a \wedge c) \leq a$. Again, $a \wedge b \leq b$ implies

$$(a \wedge b) \vee (a \wedge c) \leq b \vee (a \wedge c).$$

Thus, we find that $(a \wedge b) \vee (a \wedge c)$ is a lower bound of $\{a, b \vee (a \wedge c)\}$. But $a \wedge (b \vee (a \wedge c))$ is the glb of $\{a, b \vee (a \wedge c)\}$. Hence,

$$(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee (a \wedge c)).$$

Exercise 9: Consider the lattice $(D_{20}, \leq)$, where $\leq$ denotes the divisibility relation. Find $4 \wedge (5 \vee 10)$ and $(2 \vee (2 \wedge 5)) \vee 4$. Is this lattice a Boolean algebra?

Solution: $D_{20} = \{1, 2, 4, 5, 10, 20\}$. Now

$$\begin{aligned} 4 \wedge (5 \vee 10) &= 4 \wedge 10 && \text{because } 5 \vee 10 = \text{lcm}\{5, 10\} = 10 \\ &= 2 && \text{because } 4 \wedge 10 = \gcd\{4, 10\} = 2. \end{aligned}$$

Also,

$$\begin{aligned} (2 \vee (2 \wedge 5)) \vee 4 &= (2 \vee 1) \vee 4 && \text{because } 2 \wedge 5 = \gcd\{2, 5\} = 1 \\ &= 2 \vee 4 && \text{because } 2 \vee 1 = \text{lcm}\{2, 1\} = 2 \\ &= 4 && \text{because } 2 \vee 4 = \text{lcm}\{2, 4\} = 4. \end{aligned}$$

The Hasse diagram of D_{20} is shown in Figure 3.29. In this lattice, the least element is 1 and the greatest element is 20. Now,

$$\begin{aligned} &2 \wedge 1 = 1, \quad && 2 \vee 1 \neq 20, \quad && 2 \wedge 4 \neq 1, \\ &2 \wedge 5 = 1, \quad && 2 \vee 5 \neq 20, \quad && 2 \wedge 10 \neq 1. \end{aligned}$$

Hence, 2 has no complement in D_{20}. Therefore, this is not a Boolean algebra.

Exercise 10: Consider the lattice $(S, \leq)$, where $S = \{1, 2, 4, 5, 8, 9)$ and $\leq$ denotes the usual "less than or equality" relation. Find $4 \wedge (5 \vee 9)$ and $(2 \vee (2 \wedge 8)) \vee 4$. Is this lattice a Boolean algebra?

Solution: In this lattice, $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. The Hasse diagram of S is the given in Figure 3.30.

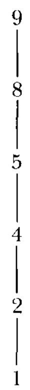

FIGURE 3.30
Hasse diagram of S

Now,

$$\begin{aligned} &4 \wedge (5 \vee 9) \\ &= 4 \wedge \max\{5, 9\} \\ &= 4 \wedge 9 \\ &= \min\{4, 9\} \\ &= 4 \end{aligned}$$

and

$$\begin{aligned} &(2 \vee (2 \wedge 8)) \vee 4 \\ &= (2 \vee \min\{2, 8\}) \vee 4 \\ &= (2 \vee 2) \vee 4 \\ &= \max\{2, 2\} \vee 4 \\ &= \max\{2, 4\} \\ &= 4. \end{aligned}$$

This is a chain. Hence, S is a distributive lattice with the greatest element 9 and the least element 1. In this lattice, suppose there exists an element b such that $2 \vee b = 9$ and $2 \wedge b = 1$. Then $\max\{2, b\} = 9$ implies that $b = 9$. On the other hand, $\min\{2, b\} = 1$ implies that $b = 1$. Thus, we find that 2 has no complement in this lattice. Hence, S is not a Boolean algebra.