Lecture Notes on Probability

Nirman Ganguly

Chapter 1

Sample Space, Events and Definition of Probability

1.1 Sample Space

In a random experiment,the collection of all possible outcomes is known as the sample space and is usually denoted by the letter S. Consider now the examples given below:

- Consider that a single coin is tossed.The sample space in this case is $S = \{H, T\}$.
- When one throws a single die,the sample space is given by $S = \{1, 2, 3, 4, 5, 6\}$.
- The sample space when two coins are thrown is $S = \{HH, HT, TH, TT\}$.
- When one chooses to throw two dice,the sample space is $\{(1, 1), (1, 2), ..., (6, 6)\}$.
- Consider that one needs to calculate the number of α particles emitted from a radioactive source. Here the sample space is $S = \{0, 1, 2, 3....\}$.
- If one needs to keep a measure of the distance travelled by a new tyre then the sample space is $S = \{x : 0 \leq x \leq 10000\}$. Here we assume that the maximum distance travelled by the tyre is 10000 km.

The first four examples constitute finite sample spaces.They are discrete in nature.The fifth one is an example of a discrete but infinite sample space.It is however *countably infinite*.The last one exemplifies a continuous sample,one which is *uncountably infinite*.

Chapter 2

Probability Distributions and Expectation

Problems:

1. Let X be a random variable which can be discrete or continuous. Then, prove the following two properties:
(a) $E(aX + b) = aE(X) + b$
(b) $Var(aX + b) = a^2 Var(X)$

2. In 4 tosses of a coin,let X be the number of heads. Find $E(X)$ and $Var(X)$.

3. Let a variate X have the distribution
$P(X = 0) = P(X = 2) = p, P(X = 1) = 1 - 2p$, where $0 \leq p \leq 0.5$. For what value is the $Var(X)$ a maximum.

4. Let X be a random variable with pdf as given below:
$X : 0 \ \ 1 \ \ 2 \ \ 3$
$p : 1/3 \ \ 1/2 \ \ 1/24 \ \ 1/8$
Find $E(Y)$ if $Y = (X - 1)^2$.

5. The radius of a circle has continuous distribution given by pdf :
$f(x) = 1, 1 \leq x \leq 2$,
$f(x) = 0$ otherwise.
Find the mean and variance of the area.

6.The diameter of an electric cable, say X is assumed to be a continuous random variable with pdf $f(x) = 6x(1 - x), 0 \leq x \leq 1$.
(a) Check that above is a pdf.
(b) Determine b, such that $P(X < b) = P(X \geq b)$.

7. Let X be a continuous random variable with pdf

$$\begin{aligned} f(x) = & \quad ax, 0 \leq x \leq 1 \\ = & \quad a, 1 \leq x \leq 2 \\ = & \quad -ax + 3a, 2 \leq x \leq 3 \\ = & \quad 0, \quad elsewhere \end{aligned} \tag{2.1}$$

(i) Determine the constant a.
(ii) Compute $P(X \leq 1.5)$

8. From a lot of 10 items containing 3 defectives, a sample of 4 items is drawn at random. Let the random variable X denote the number of defective items in the sample.Find the probability distribution of X.

9. The mileage C in thousands of miles which car owners get with a certain kind of tyre is a random variable having pdf
$f(x) = \frac{1}{2}e^{-x/20}, \quad x > 0.$
Find the probabilities that one of these tires will last
(i) at most 10000 miles
(ii) anywhere from 16000 to 24000 miles
(iii) at least 30000 miles

10. A random variable X has the pdf
$f(x) = 2x, 0 < x < 1$ and zero otherwise.
Find $P(X > \frac{3}{4}|X > \frac{1}{2})$

11. If X denotes the number of failures preceding the first success with probability of success p, then find expectation of X.

Chapter 3

Special Discrete Distributions

Problems on Binomial and Poisson Distribution:

1. It is known that any item produced by a certain machine will be defective with probability 0.1 independently of any other item. What is the probability that in a sample of 3 items,at most 1 is defective.

2. A and B play a game in hich their chance of winning is of ratio 3:2. Find A's chance of winning at least 3 out of 5 games played.

3. In a binomial distribution consisting of 5 independent trials,the probability of 1 and 2 successes are 0.4096 and 0.2048 respectively. Find the parameter p of the distribution.

4. Suppose that an airplane engine will fail when in flight with probability $(1-p)$ independently with respect to any other engine. Suppose that the airplane will make successful flight if at least 50% of its engines remain operative. For what values of p is a 4-engine plane preferable to a 2-engine plane.

5. An airline knows that 5% of the people making reservation on a certain flight will not show up. Consequently,their policy is to sell 52 tickets for a flight that can hold only 50 passengers.What is the probability that a seat will be available for every passenger who shows up?

6. A manufacturer of copper pins knows that 5% of the products is defective. If he sells copper pins in boxes of 100 and guarantees that not more than 10 pins will be defective,what is the probability that a box will fail to meet the guaranteed quality.

7. A radioactive source emits on the average 2.5 particles per second. Calculate the probability that 2 or more particles will be emitted in an interval of 4 seconds.

8. How long a series of random digits has to be in order so that the probability of the digit '7' appearing is at least 9/10.

9. Alice finds that when she is developing a program module,syntax errors are discovered on the compiler on 60% of the runs she makes.Furthermore, this percentage is independent of the number of runs made on the same module. How many runs does she need to make of one module ,on the average, to get a run with no syntax errors? What is the probability that more than 4 runs will be required?

10. A supermarket has 4 checkouts in operation. A customer is in a hurry and leaves without making a purchase if all the checkouts are busy. At that time of day the probability of each checkout being free is 0.25.Assume that whether or not a checkout is busy is independent of any other checkout. What is the probability that the customer will make a purchase?

11. A data center has 10000 disk drives . Suppose that a disk drive fails in a given day with probability $10^{-}3$.
(a) Find the probability that there are no failures on a given day.
(b) Find the probability that there are fewer than 10 failures in 2 days.

Chapter 4

Special Continuous Distributions

Problems on Uniform , Exponential and Normal Distribution

1. A random variable X follows normal distribution with mean=4.35 and standard deviation 0.59. Find
(a) $P(4 < X < 5)$
(b) $P(X \geq 5.5)$
Ans:(a) 0.5867 (b) 0.0256

2. The time for a superglue to set can be treated as a random variable having normal distribution with mean 40 secs. Find its standard deviation if the probability is 0.16 that it will take on a value greater than 50 secs.
Ans: 10.

3. The average monthly sales of 5000 firms are normally distributed with mean Rs. 36000 and standard deviation Rs. 10000. Find
(a) number of firms with sales over Rs. 40000
(b) percentage of firms with sales between Rs. 38500 and Rs. 41000.
Given, $P(0 \leq z \leq 0.4) = 0.1554, P(0 \leq z \leq 0.5) = 0.1915, P(0 \leq z \leq 0.25) = 0.0987$
Ans:(a) 1723 (b) 9.28%

4. The local authorities in a certain city installed 2000 electric lamps in a street of the city. If the lamps have an average life of 1000 burning hours with a standard deviation of 200 hours, then
(i) What number of the lamps might be expected to fail in the first 700 burning hours?
(ii) After what period of burning hours would we expect that 10% of the lamps

would have failed?
(iii) After what periods of burning hours would we expect 10% of the lamps to be still burning?
Ans: (i) 134 (ii) 744 (iii) 1256

5. Steel rods are manufactured to be 3cm in diameter but they are acceptable if they are inside the limits 2.99 cms and 3.01 cms. It is observed that 5% are rejected as oversize and 5% are rejected as undersize. Assuming that the diameters are normally distributed, find the mean and standard deviation of the distribution. Hence calculate, what would be the proportion of rejected rods if the permissible limits were widened to 2.985cms and 3.015cms?
Ans: $\mu = 3, \sigma = 1/165, 1.34\%$

6. Two chips are being considered for use in a certain system. The lifetime of chip 1 is modeled by a normal random variable with mean 20,000 hours and standard deviation 5000 hours. (The probability of negative lifetime is negligible.) The lifetime of chip 2 is also a normal random variable but with mean 22,000 hours and standard deviation 1000 hours. Which chip is preferred if the target lifetime of the system is 20,000 hours? 24,000 hours?

7. Find the mean and variance of the uniform distribution.

8. Find the mean and variance of the exponential distribution.

9. Buses arrive at a specified stop at 15 minutes interval starting at 7 am. That is, they arrive at 7,7:15,7:30 and so on. If a passenger arrives at the stop at a time that is uniformly distributed between 7 and 7:30, find the probability that he waits
(a) less than 5 minutes for a bus
(b) at least 12 minutes for a bus
Ans: (a) 1/3 (b) 1/5

10. A communication system accepts a positive voltage V as input and outputs a voltage $Y = aV + N$, where $a = 10^{-2}$ and N is a Gaussian random variable with parameters $\mu = 0, \sigma = 2$. Find the value of V that gives $P(Y < 0) = 0.1$.

11. Suppose that during rainy season, on a tropical island, the length of shower has an exponential distribution with average length of shower $\frac{1}{2}$ mins. What is the probability that a shower will last more than three minutes? If a shower has already lasted for 2 minutes, what is the probability that it will last for at least one more minute?

12. A point P is chosen at random on a straight line segment AB of length $2a$. Find the probability that the area of the rectangle AP, PB will exceed $\frac{a^2}{2}$.

13. The length of life in years,T, of a heavily used terminal in a student computer laboratory is exponentially distributed with $\lambda = 0.5 years$. Find the reliability of t, i.e., $P(X > t)$. [Reliability measures the probability that the terminal will last more than t years]. Hence,find Reliability(1).Plot the graph of the reliability against time in years for the failure rate $\lambda = 0.5 years$
Ans: 0.607