— x —

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Normal Subgroup : $aH = Ha \ \forall a \in G$

E.g : all subgroups of a commutative group

$H = \{h_1, h_2, h_3, \dots h_n\}$

$aH = \{ah_1, ah_2, \dots ah_n\}$

$Ha = \{h_1 a, h_2 a, \dots h_n a\}$

These 2 sets are same i.e. all the elements present in aH are also present in Ha.

→ that doesn't mean $ah_1 = h_1 a$

$ah_1 = h_3 a$ → possible

Def^n : A subgroup H of a group G is said to be normal if $aH = Ha \ \forall a \in G$. This is denoted by

$H \Delta G$

Proposition: Let G be a group & H be a subgroup of G, such that $[G:H] = 2$

→ no. of distinct (left/right) cosets of H in G is 2

Then H is a normal subgroup in G.

Proof: Since $[G:H]=2$ the 2 distinct cosets are eH (i.e H) & $(G-H)$.

Let, $a \in G$ → 2 cases arise.

Case ①: Let, $\boxed{a \in H}$

$$aH = H = Ha$$

$\therefore aH = Ha$ ~~(Proved : H is normal subgroup)~~

Case ②: Let, $\boxed{a \in G-H}$

$aH = G-H$, since $aH \cap H = \phi$

we have only 2 cosets of H in G → $(G-H)$ & H

Similarly $Ha = G-H$, by same logic

$\therefore Ha = aH$ ~~$\forall a \in G$~~

$\therefore Ha = aH \ \forall a \in G$

$\therefore$ H is normal subgroup

Prob: The centre of a group defined as

$$Z(G) = \{x \in G : xg = gx \ \forall g \in G\}$$

Prove that $Z(G)$ is a Normal subgroup of G.

$Z(G)$ can't be empty as the ~~only~~ element that can lie in $Z(G)$ is identity element, if not others.

~~$x \in Z(G)$ | $x \in G - Z(G)$.~~

~~$x . Z(G)$~~

Proof: Let, $a \in G$.

R.T.P. : $aZ(G) = Z(G) a \ \forall a \in G$

Let, $p \in aZ(G) \Rightarrow p = axg$, for some $x \in$

$p = g x_1 a$, for some $\boxed{x_1 \in G}$

$= x_1 g a$

$= Z(G).a.$

$\therefore p \in Z(G).a.$

$\therefore aZ(G) \subseteq Z(G).a \rightarrow (1)$

Let, $p_1 \in Z(G).a.$

$p_1 = g x a$ for some $x \in G$

$= g x_1 a$ for some $x_1 \in G$

$= x_1 g a$

$= a x_1 g$

$= a.Z(G).$

$\therefore p_1 \in aZ(G).$

$\therefore Z(G).a \subseteq aZ(G)$

$\therefore aZ(G) = Z(G).a$

Proved

Let $Z(G) = H$

Let, $p \in aH$

$p = ah$ for some $h \in H$

$= h_1 a$ for some $h_1 \in H$

$\in Ha$

$\therefore aH \subseteq Ha$ ...... |Same as previous|

Proposition : Let G be a group & H be a subgroup of G. Then H is a normal subgroup iff $h \in H$ & $x \in G$ implies $\boxed{xhx^{-1} \in H}$ $\rightarrow$ Necessary & sufficient condition for a subgroup to be a normal subgroup

Proof : Let H be a normal subgroup of G.

$\forall x \in G \Rightarrow \boxed{xH = Hx}$

These are sets

$\therefore x h_0 = h_1 x$ for some $\boxed{h_1, h_2 \in H}$

$\Rightarrow xhx^{-1} = h_1 \in H$

$\therefore \boxed{xhx^{-1} \in H}$

Conversely Let, $x \in G$ & $h \in H$, such that $\boxed{xhx^{-1} \in H}$

$\therefore xhx^{-1} = h_1$ for some $\boxed{h_1 \in H}$ $\Rightarrow \boxed{xh = h_1 x}$ $\forall x \in G$.

Let, $\boxed{p \in xH}$

$\therefore p = xh_2$, for some $h_2 \in H$

$= h_3 x$ ($\because xh = h_1 x$) for some $\boxed{h_3 \in H}$

$h_3 x \in Hx$

$\therefore \boxed{xH \subseteq Hx}$

Similarly $q \in Hx$

$\therefore q = h_4 x$ for some $h_4 \in H$

$= xh_5$ for some $h_5 \in H$

$\in xh_5 \in xH$

$\therefore \boxed{Hx \subseteq xH}$

$\therefore Hx = xH$ (Proved) H is a n.s.g. of G

Using this proposition we solve the previous prob

Let $x \in G$ & $h \in Z(G)$ ↳ denoted by H. $\therefore hx = xh$ ↳ defn of $Z(G)$

$\therefore xhx^{-1} = hx.x^{-1}$

$= h.e$

$= h.$

$\therefore \boxed{xhx^{-1} \in Z(G)}$

$\therefore Z(G)$ is a normal subgroup of G