Magnetic Properties

1. The Magnetic Dipole Moment (μ_m)

Rectangular Current carrying loop PQRS carrying a (steady) current 'I' as shown in Figure 1, with PQ (=RS) = a & QR (= SP) = b. It is placed in a uniform magnetic field $\vec{B}$ so that the sides QR and SP are perpendicular to $\vec{B}$.

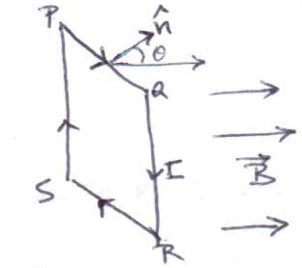

Figure 1.

$\hat{n}$ - unit normal to the surface PQRS as shown. Wrap your fingers in the direction of current flow - your extended thumb points towards $\hat{n}$.

Angle between $\hat{n}$ and $\vec{B}$ - θ.

Since I is steady and $\vec{B}$ is uniform, the net force on the loop is

$$\vec{F} = \int I\,(d\vec{l} \times \vec{B}) \equiv I\left(\oint d\vec{l}\right) \times \vec{B} = 0 \qquad (1)$$

But there is a torque acting on the loop.

$$\left.\begin{array}{l} \text{Force on side QR} : \vec{F}_{QR} = I\,\vec{QR} \times \vec{B} \rightarrow \perp \text{ to page coming out} \\ \text{Force on side SP} : \vec{F}_{SP} = I\,\vec{SP} \times \vec{B} \left(= -I\,\vec{QR} \times \vec{B}\right) \rightarrow \perp \text{ to page going in} \end{array}\right\} \quad (2)$$

$\vec{F}_{QR}$ and $\vec{F}_{SP}$ (with $\vec{F}_{QR} + \vec{F}_{SP} = 0$) form a couple.

(• Students: Verify that $\vec{F}_{PQ} + \vec{F}_{RS} = 0$ but $\vec{F}_{PQ}$ and $\vec{F}_{RS}$ do NOT form a couple.)

The torque on the rectangular loop due to $\vec{F}_{QR}$ and $\vec{F}_{SP}$ will try to rotate the loop clockwise (when viewed along the axis of the loop from the top) so as to reduce θ and bring $\hat{n}$ along $\vec{B}$. (At $\theta = 0$, lines of action of $\vec{F}_{QR}$ and $\vec{F}_{SP}$ match and they no longer form a couple.)

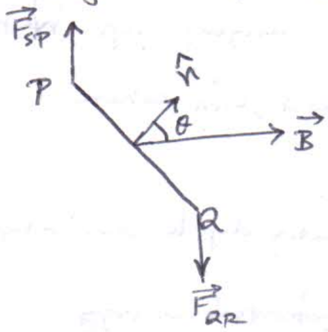

Figure 2.

When viewed along the axis of the loop from the top, $\vec{F}_{SP}$ and $\vec{F}_{QR}$ are as shown in Figure 2.

$$|\vec{F}_{SP}| = |\vec{F}_{QR}| = IbB\sin 90^\circ = IbB \qquad (3)$$

Torque on the current carrying loop due to $\vec{B}$ is

$$\vec{\tau}_B = \vec{PQ} \times \vec{F}_{QR} \quad (\text{or } \vec{QP} \times \vec{F}_{SP}) \qquad (4)$$

Direction of $\vec{\tau}_B$ - along $\vec{QR}$ in Figure 1
- perpendicular to the page and pointing into the page in Figure 2

• Students: Make sure that you understand this.

Then,

$$\tau_B = |\vec{\tau}_B| = |\vec{PQ}||\vec{F}_{QR}| \sin\theta = a \cdot IbB \sin\theta$$

$$\Rightarrow \quad \tau_B = I\,(ab)\,B \sin\theta = IAB\sin\theta \qquad (5)$$

where,

$A = a.b$ is the area of the loop.

(For N turns in the loops, each turn carrying a current i, the equation becomes $\tau = NiAB\sin\theta$ $\Rightarrow \tau = IAB\sin\theta$, $I = Ni$)

We now <u>define</u> a vector $\vec{\mu}_m$ such that

$$\vec{\mu}_m = IA\,\hat{n}, \qquad (6)$$

then

$$\mu_m = |\vec{\mu}_m| = IA, \qquad (7)$$

and Eq.(5) becomes,

$$\tau_B = \mu_m B \sin\theta \qquad (8)$$

In vector form, Eq.(8) can be written as,

$$\vec{\tau}_B = \vec{\mu}_m \times \vec{B}, \qquad (9)$$

as it is clear that the cross product $\vec{\mu}_m \times \vec{B}$ consistently gives <u>both</u> the magnitude (see Eq.(8)) <u>and</u> the direction of the torque.

Recall that the torque on an electric dipole of dipole moment $\vec{p}$ in an electric field $\vec{E}$ is given by

$$\vec{\tau}_E = \vec{p} \times \vec{E} \qquad (10)$$

We call $\vec{\mu}_m$ as the <u>magnetic dipole moment</u> of the current carrying loop. In other words, we refer to the current carrying loop as a magnetic dipole of moment $\vec{\mu}_m$ where $\mu_m = IA$.*

The effect of the torque $\vec{\tau}_B$ is to rotate the magnetic dipole and align it with $\vec{B}$. Consequently the dipole has an orientational potential energy given by

$$U = -\vec{\mu}_m \cdot \vec{B} \qquad (11)$$

- Students – Prove Eq.(11). Where is the 'zero' of the potential energy? Does this choice matter? Why or why not?

* In general, it can be shown that $\vec{\mu}_m = I\vec{A}$ for a loop carrying current I and having the "vector area" $\vec{A}$. If the loop is <u>flat</u> (as considered in Figure 1), $\vec{A}$ is the ordinary area enclosed, with the direction assigned by the usual right-hand rule (fingers in the direction of the current).

2. Orbital and Spin Magnetic Dipole Moments (μ_m^L and μ_m^S respectively):

The current loops in an atom are composed of rotating electrons. In that case we can establish a simple relation between the magnetic dipole moment $\vec{\mu}_m$ that results from a rotating electron and its orbital angular momentum $\vec{L}$. Our derivation of this relationship will be based on classical physics. Later on, results from quantum mechanics will be used to further develop the ideas. This procedure is justified by the fact that the final results agree with those of completely quantum mechanical treatments.

Consider an electron of mass 'm' and charge '$-e$' moving with velocity of magnitude 'v' in a circular Bohr orbit of radius 'r' as shown in Figure 3. The charge circulating in a loop constitutes a current of magnitude 'I', where

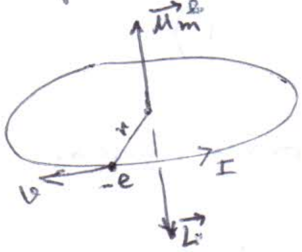

Figure 3

$$I = \frac{e}{T}, \tag{1}$$

where T – orbital period of the electron. Then,

$$vT = 2\pi r, \text{ i.e., } T = \frac{2\pi r}{v} \tag{2}$$

$$\therefore I = \frac{e}{T} = \frac{ev}{2\pi r} \tag{3}$$

The area of the loop is

$$A = \pi r^2 \tag{4}$$

Then the magnitude of the magnetic moment due to orbital motion μ_m^l of the equivalent magnetic dipole is [See Eq. (7), Section 1],

$$\mu_m^l = I.A = \frac{ev}{2\cancel{\pi}\cancel{r}} \times \cancel{\pi} r^{\cancel{2}} = \frac{evr}{2} \tag{5}$$

Because the electron has a negative charge, its orbital magnetic dipole moment $\vec{\mu}_m^l$ is antiparallel to its orbital angular momentum $\vec{L}$, whose magnitude is given by

$$L = mvr \tag{6}$$

and whose direction is illustrated by Figure 3. From (5) and (6),

$$\frac{\mu_m^l}{L} = \frac{e\cancel{v}\cancel{r}}{2} \cdot \frac{1}{m\cancel{v}\cancel{r}} = \frac{e}{2m}, \tag{7}$$

which is a combination of universal constants. In vector form,

$$\vec{\mu}_m^l = -\frac{e}{2m}\vec{L} \tag{8}$$

Now, from (7),

$$\mu_m^l = \frac{e}{2m} L_l \qquad (9)$$

The quantum mechanical expression for the orbital angular momentum is given by,

$$L_l = \sqrt{l(l+1)}\,\hbar \qquad (10)$$

where l – orbital angular momentum quantum number of the electron, and

$\hbar = \frac{h}{2\pi}$, h – Planck's constant.

From (9) and (10),

$$\mu_m^l = \frac{e}{2m}\sqrt{l(l+1)}\,\hbar, \qquad (11)$$

which is exactly the expression of μ_m^l that is obtained from a rigorous quantum mechanical treatment. We write (11) as,

$$\mu_m^l = \frac{e\hbar}{2m}\sqrt{l(l+1)} \equiv \mu_B\sqrt{l(l+1)} \qquad (12)$$

where,

$$\mu_B = \frac{e\hbar}{2m} = 0.927\times10^{-23}\ \text{ampere-m}^2 \qquad (13)$$

The quantity μ_B forms a natural unit for the measurement of atomic magnetic dipole moments, and is called the Bohr magneton. Equation (11) has, of course, been backed by experimental evidence.

Experimental results also lead to the conclusion that an electron has an intrinsic (built-in) magnetic dipole moment $\vec{\mu}_m^s$, due to the fact that it has an intrinsic angular momentum $\vec{S}$ called its <u>spin</u>. The magnitude S of the <u>spin angular momentum</u> is given by the quantization relation:

$$S = \sqrt{s(s+1)}\,\hbar \qquad (14)$$

where $s = \frac{1}{2}$ is known as the spin quantum number. The <u>spin magnetic dipole moment</u> and the spin angular momentum are related as follows:

$$\vec{\mu}_m^s = -\frac{e}{m}\vec{S} \qquad (15)$$

Recall from Eq. (8) that $\vec{\mu}_m^l = -\frac{e}{2m}\vec{L}_l$. This equation was obtained using classical means. Spin, however, has no classical analogue. Its origin is completely quantum mechanical. From (14) and (15), the magnitude μ_m^s is given by,

$$\mu_m^s = \frac{e}{m}\sqrt{s(s+1)}\,\hbar = 2\left(\frac{e\hbar}{2m}\right)\sqrt{s(s+1)}\,\hbar = 2\mu_B\sqrt{s(s+1)} \quad (16)$$

Note: The z-components of $\vec{L}$ and $\vec{S}$ are also quantized; they are respectively given by,

$$L_z = m_l\hbar \quad \& \quad S_z = m_s\hbar \quad (17)$$

For a given l, m_l goes from $-l$ to $+l$ in integral steps; $m_s = \pm\frac{1}{2}$. Thus the z-components of the orbital and spin magnetic dipole moments are also quantized - using Eqns. (8) and (15), their magnitudes are respectively given by

$$\mu_m^{lz} = \frac{e}{2m}L_z = \frac{e\hbar}{2m}m_l = \mu_B m_l \quad \& \quad \mu_m^{sz} = \frac{e}{m}S_z = 2\cdot\frac{e\hbar}{2m}\cdot m_s = 2\mu_B m_s \quad (18)$$

For further details and a more exact treatment, see Resnick & Eisberg, Chapter 8.

More precisely, Eqs. (11) and (16) are respectively written as,

$$\mu_m^l = g_l\,\mu_B\sqrt{l(l+1)} \quad (19)$$

&

$$\mu_m^s = g_s\,\mu_B\sqrt{s(s+1)} \quad (20)$$

where

$$g_l = 1, \quad (21)$$

is known as the orbital g factor; and

$$g_s = 2, \quad (22)$$

is known as the spin g factor. (Experiments have shown that the actual value of the spin g factor is $g_s = 2.00232$, but (22) is adequate for most purposes.)

The orbital angular momentum and spin may be combined vectorially to give the total angular momentum $\vec{J}$, i.e.,

$$\vec{J} = \vec{L} + \vec{S} \quad (23)$$

The magnitude of $\vec{J}$ is also quantized according to the usual condition,

$$J = \sqrt{j(j+1)}\,\hbar, \quad (24)$$

where 'j' is the total angular momentum quantum number of the electron. If the electron has a certain 'l', then j can accept the values $l \pm s$, i.e.,

$$j = l + \frac{1}{2} \text{ or } l - \frac{1}{2} \quad (25)$$

(since $s = \frac{1}{2}$). We next consider multi-electron atoms.

Consider an atom containing a number of electrons — typically such an atom contains a core of completely filled subshells surrounding the nucleus, plus several electrons in a partially filled outer subshell. The orbital angular momentum vectors of all the electrons may be combined to form a resultant $\vec{L}$, and the spin angular momentum vectors of all the electrons may be combined to form a resultant $\vec{S}$ — this is known as LS coupling; it is the only type of coupling that we shall consider. The resultant $\vec{L}$ and $\vec{S}$ then combine to form the total angular momentum $\vec{J}$ of the whole electron system of the atom. Now it follows from Pauli's exclusion principle that when a substell is completely filled, the only allowed state is one in which the total orbital angular momentum, the total spin angular momentum, and the total angular momentum are all zero for this subshell. So the core of completely filled subshells do not contribute to $\vec{L}$, $\vec{S}$, and $\vec{J}$; as a consequence a completely filled subshell has no net magnetic dipole moment. Therefore, only the few electrons in an atom that are not in filled subshells contribute to $\vec{L}$, $\vec{S}$, and $\vec{J}$ and hence to the magnetic moment of the atom. Thus the magnetic moment in atoms must result from incompletely filled shells.

So if $\vec{S}_1, \vec{S}_2, \vec{S}_3, \ldots$ are the individual spin angular momenta of the electrons in the unfilled subshells of the atom, they combine to form a total $\vec{S}$, where

$$\vec{S} = \vec{S}_1 + \vec{S}_2 + \vec{S}_3 + \cdots \qquad (26)$$

Also the individual orbital angular momenta $\vec{L}_1, \vec{L}_2, \vec{L}_3, \ldots$ of the electrons in the unfilled subshells of the atom combine to form a total $\vec{L}$, where

$$\vec{L} = \vec{L}_1 + \vec{L}_2 + \vec{L}_3 + \cdots \qquad (27)$$

$\vec{L}$ and $\vec{S}$ subsequently combine to give the total angular momentum $\vec{J}$:

$$\vec{J} = \vec{L} + \vec{S} \qquad (28)$$

Now according to quantum mechanics, $\vec{S}$ has a constant magnitude given by.

$$S = \sqrt{S(S+1)}\,\hbar, \qquad (29)$$

where S is the total spin quantum number of the atom as a whole.

Next, $\vec{L}$ also has a constant magnitude satisfying the quantization condition

$$L = \sqrt{L(L+1)}\,\hbar \qquad (30)$$

where L is the total orbital angular momentum quantum number (of all the electrons) of the atom as a whole.

Finally, $\vec{J}$ also has a constant magnitude and is quantized according to

$$J = \sqrt{J(J+1)}\,\hbar \qquad (31)$$

where J is the total angular momentum quantum number (of all the electrons) of the atom as a whole.

Analogous to Eqs. (19) and (20), the magnitudes of the orbital and spin magnetic dipole moments are given by,

$$\mu_m^L = g_\ell \mu_B \sqrt{L(L+1)}, \qquad \left[\vec{\mu}_m^L = -g_\ell \mu_B \frac{\vec{L}}{\hbar}\right] \qquad (32)$$

&

$$\mu_m^S = g_s \mu_B \sqrt{S(S+1)}, \qquad \left[\vec{\mu}_m^S = -g_s \mu_B \frac{\vec{S}}{\hbar}\right] \qquad (33)$$

where $g_\ell = 1$ and $g_s = 2$.

The average component of the total magnetic dipole moment along $\vec{J}$ is given by,

$$\mu_m^J = g \mu_B \sqrt{J(J+1)}, \qquad \left[\vec{\mu}_m^J = -g \mu_B \frac{\vec{J}}{\hbar}\right] \qquad (34)$$

where

$$g = 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)} \qquad (35)$$

'g' is called the Landé g factor.

Note: $\mu_m^J = g\sqrt{J(J+1)}\,\mu_B \equiv p_{eff}\,\mu_B$ (34a)
where $p_{eff} = g\sqrt{J(J+1)}$ can be calculated from susceptibility measurements.

p_{eff} is known as the effective number of Bohr magnetons.

The values of L, S, and J for a given atom can be found using a combination of Pauli's exclusion principle and Hund's rules.
The purpose of the above discussion is to convince the students that the origin of magnetic dipole moments in atoms is quantum mechanical in nature.

References: i). Solid State Physics by A J Dekker, ii) Quantum Physics by Resnick & Eisberg