Assignment-11

Ankur Aditya - EE20RESCH11010

***Abstract*—This document contains the the solution of problem related to subspaces.(Hoffman Page-40, Question-5)**

Download latex-file codes from

https://github.com/ankuraditya13/EE5609−Assignment11

1 Problem

Let $\mathbf{F}$ be a field and let n be a positive integer (n≥2). Let $\mathbf{V}$ be the vector space of all n×n matrices over $\mathbf{F}$. Which of the following set of matrices $\mathbf{A}$ in $\mathbf{V}$ are subspaces of $\mathbf{V}$?

1) all invertible $\mathbf{A}$;
2) all non-invertible $\mathbf{A}$;
3) all $\mathbf{A}$ such that $\mathbf{AB} = \mathbf{BA}$, where $\mathbf{B}$ is some fixed matrix in $\mathbf{V}$;
4) all $\mathbf{A}$ such that $\mathbf{A}^2 = \mathbf{A}$.

2 Solution 1

Let the matrices $\mathbf{A}$ and $\mathbf{B} \in \mathbf{V}$, be set of invertible matrix. For them to be a subspace they need to be closed under addition. Let,

$$\mathbf{A} = \mathbf{I} \tag{2.0.1}$$

$$\mathbf{B} = -\mathbf{I} \tag{2.0.2}$$

It could be easily proven that both matrices $\mathbf{A}$ and $\mathbf{B}$ are invertible as their determinant ≠ 0. Now,

$$\mathbf{A} + \mathbf{B} = \mathbf{0}. \tag{2.0.3}$$

Now the zero matrix $\mathbf{0}$ is non-invertible as,

$$rank(\mathbf{0}_{nxn}) = rank\left(\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{nxn}\right) = 0 \tag{2.0.4}$$

∴ **the set of invertible matrices are not closed under addition. Hence not a subspace of V.**

3 Solution 2

Let the matrices $\mathbf{A_1}, \mathbf{A_2}.....\mathbf{A_n} \in \mathbf{V}$, be set of non-invertible matrix. For them to be a subspace they need to be closed under addition. Let,

$$\mathbf{A_1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{nxn} \tag{3.0.1}$$

$$\mathbf{A_2} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{nxn} \tag{3.0.2}$$

$$\mathbf{A_n} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}_{nxn} \tag{3.0.3}$$

$$\tag{3.0.4}$$

It could be easily proven that matrices $\mathbf{A_1}, \mathbf{A_2}.....\mathbf{A_n}$ are non-invertible as their determinant = 0. Now,

$$\mathbf{A_1} + \mathbf{A_2} + \mathbf{A_3} + \cdots \mathbf{A_n} = \mathbf{I}_{nxn} \tag{3.0.5}$$

Now the identity matrix $\mathbf{I}$ is invertible as,

$$rank(\mathbf{I}_{nxn}) = rank\left(\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}_{nxn}\right) = n \tag{3.0.6}$$

or it is a full rank matrix as there are n pivots. ∴ **the set of non-invertible matrices are not closed under addition. Hence not a subspace of V.**

4 Solution 3

Theorem 1:. A non-empty subset W of V is a subspace of V if and only if for each pair of vectors α, β in W and each scalar c ∈ F, the vector $c\alpha + \beta \in$

W.
Let the matrices $\mathbf{A_1}$ and $\mathbf{A_2}$ satisfy,

$$\mathbf{A_1B} = \mathbf{BA_1} \tag{4.0.1}$$
$$\mathbf{A_2B} = \mathbf{BA_2} \tag{4.0.2}$$

Let, c$\in$ $\mathbf{F}$ be any constant.

$$\therefore (c\mathbf{A_1} + \mathbf{A_2})\,\mathbf{B} = c\mathbf{A_1B} + \mathbf{A_2B} \tag{4.0.3}$$

Substituting from equations (4.0.1) and (4.0.2) to (4.0.3),

$$\implies (c\mathbf{A_1} + \mathbf{A_2})\,\mathbf{B} = c\mathbf{BA_1} + \mathbf{BA_2} \tag{4.0.4}$$
$$\implies \mathbf{B}c\mathbf{A_1} + \mathbf{BA_2} \tag{4.0.5}$$
$$\implies \mathbf{B}\,(c\mathbf{A_1} + \mathbf{A_2}) \tag{4.0.6}$$

Thus, $(c\mathbf{A_1} + \mathbf{A_2})$ satisfy the criteria and from Theorem-1 it can be seen that the set is a subspace.

5 Solution 4

Let $\mathbf{A}$ and $\mathbf{B} \in \mathbf{V}$ be set of matrices such that,

$$\mathbf{A^2} = \mathbf{A} \tag{5.0.1}$$
$$\mathbf{B^2} = \mathbf{B} \tag{5.0.2}$$

Now for them to be closed under addition,

$$(\mathbf{A} + \mathbf{B})^2 = \mathbf{A} + \mathbf{B} \tag{5.0.3}$$

Which is not always same. Example let,

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \tag{5.0.4}$$
$$\mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.0.5}$$

Clearly,

$$\mathbf{A}^2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \mathbf{A} \tag{5.0.6}$$
$$\mathbf{B}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{B} \tag{5.0.7}$$

Now,

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{5.0.8}$$
$$\implies (\mathbf{A} + \mathbf{B})^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \tag{5.0.9}$$

Hence, clearly from equations (5.0.8) and (5.0.9),

$$(\mathbf{A} + \mathbf{B})^2 \neq \mathbf{A} + \mathbf{B} \tag{5.0.10}$$

$\therefore$ the set of all A such that $\mathbf{A}^2 = \mathbf{A}$ is not closed under addition. Hence not a subspace in V.