Assignment-11

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***Abstract*—This document contains the the solution of problem related to subspaces.(Hoffman Page-40, Question-5)**

Download latex-file codes from

https://github.com/ankuraditya13/EE5609−Assignment11

1 Problem

Let **F** be a field and let n be a positive integer (n≥2). Let **V** be the vector space of all n×n matrices over **F**. Which of the following set of matrices **A** in **V** are subspaces of **V**?

1) all invertible **A**;
2) all non-invertible **A**;
3) all **A** such that **AB** = **BA**, where **B** is some fixed matrix in **V**;
4) all **A** such that $\mathbf{A}^2 = \mathbf{A}$.

2 Solutions

1) Let the matrices **A** and **B** ∈ **V**, be set of invertible matrix. For them to be a subspace they need to be closed under addition. Let,

$$\mathbf{A} = \mathbf{I} \tag{2.0.1}$$
$$\mathbf{B} = -\mathbf{I} \tag{2.0.2}$$

It could be easily proven that both matrices **A** and **B** are invertible as,

$$rank(\mathbf{I}_{nxn}) = rank\left(\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}_{nxn}\right) \tag{2.0.3}$$
$$\implies rank(-\mathbf{I}_{nxn}) = rank(\mathbf{I}_{nxn}) = n \tag{2.0.4}$$

or it is a full rank matrix as there are n pivots.

$$\therefore \mathbf{A} + \mathbf{B} = \mathbf{0}. \tag{2.0.5}$$

But the zero matrix **0** is non-invertible as,

$$rank(\mathbf{0}_{nxn}) = rank\left(\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{nxn}\right) \tag{2.0.6}$$
$$\implies rank(\mathbf{0}_{nxn}) = 0 \tag{2.0.7}$$

∴ the set of invertible matrices are not closed under addition. Hence not a subspace of V.

2) Let the matrices $\mathbf{A_1}, \mathbf{A_2}, \cdots, \mathbf{A_n} \in \mathbf{V}$, be set of non-invertible matrix. For them to be a subspace they need to be closed under addition. Let,

$$\mathbf{A_1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{nxn} \tag{2.0.8}$$
$$\mathbf{A_2} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{nxn} \tag{2.0.9}$$
$$\mathbf{A_n} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}_{nxn} \tag{2.0.10}$$
$$\tag{2.0.11}$$

It could be proven that matrices $\mathbf{A_1}, \mathbf{A_2}, \cdots, \mathbf{A_n}$ are non-invertible as,

$$rank(\mathbf{A_1}) = rank\left(\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}\right) \tag{2.0.12}$$
$$\implies rank(\mathbf{A_1}) = 1 \tag{2.0.13}$$

or there is only one pivot hence rank is 1.

$$\implies \mathbf{A_1} + \mathbf{A_2} + \mathbf{A_3} + \cdots \mathbf{A_n} = \mathbf{I}_{nxn} \tag{2.0.14}$$

Now the identity matrix $\mathbf{I}$ is invertible as shown in equation (2.0.4). $\therefore$ **the set of non-invertible matrices are not closed under addition. Hence not a subspace of V.**

3) **Theorem 1:**. A non-empty subset W of V is a subspace of V if and only if for each pair of vectors α, β in W and each scalar c $\in$ F, the vector $c\alpha + \beta \in$ W.
Let the matrices $\mathbf{A_1}$ and $\mathbf{A_2}$ satisfy,

$$\mathbf{A_1B} = \mathbf{BA_1} \tag{2.0.15}$$
$$\mathbf{A_2B} = \mathbf{BA_2} \tag{2.0.16}$$

Let, c$\in$ $\mathbf{F}$ be any constant.

$$\therefore (c\mathbf{A_1} + \mathbf{A_2})\,\mathbf{B} = c\mathbf{A_1B} + \mathbf{A_2B} \tag{2.0.17}$$

Substituting from equations (2.0.15) and (2.0.16) to (2.0.17),

$$\implies (c\mathbf{A_1} + \mathbf{A_2})\,\mathbf{B} = c\mathbf{BA_1} + \mathbf{BA_2} \tag{2.0.18}$$
$$\implies \mathbf{B}c\mathbf{A_1} + \mathbf{BA_2} \tag{2.0.19}$$
$$\implies \mathbf{B}\,(c\mathbf{A_1} + \mathbf{A_2}) \tag{2.0.20}$$

Thus, $(c\mathbf{A_1} + \mathbf{A_2})$ **satisfy the criteria and from Theorem-1 it can be seen that the set is a subspace of V.**

4) Let $\mathbf{A}$ and $\mathbf{B} \in \mathbf{V}$ be set of matrices such that,

$$\mathbf{A^2} = \mathbf{A} \tag{2.0.21}$$
$$\mathbf{B^2} = \mathbf{B} \tag{2.0.22}$$

Now for them to be closed under addition,

$$(\mathbf{A} + \mathbf{B})^2 = \mathbf{A} + \mathbf{B} \tag{2.0.23}$$

Which is not always same. Example let,

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.0.24}$$
$$\mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.25}$$

Clearly,

$$\mathbf{A}^2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \mathbf{A} \tag{2.0.26}$$
$$\mathbf{B}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{B} \tag{2.0.27}$$

Now,

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{2.0.28}$$
$$\implies (\mathbf{A} + \mathbf{B})^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \tag{2.0.29}$$

Hence, clearly from equations (2.0.28) and (2.0.29),

$$(\mathbf{A} + \mathbf{B})^2 \neq \mathbf{A} + \mathbf{B} \tag{2.0.30}$$

$\therefore$ **the set of all A such that** $\mathbf{A}^2 = \mathbf{A}$ **is not closed under addition. Hence, not a subspace of V.**