Assignment-13

Ankur Aditya - EE20RESCH11010

***Abstract*—This document contains the problem related to computations concerning subspaces. (Hoffman:- Page-66,Q-7)**

Download the latex-file codes from

https://github.com/ankuraditya13/EE5609–Assignment13

1 Problem

Let $\mathbf{A}$ be an m x n matrix over the field $\mathbf{F}$, and consider the system of equations $\mathbf{AX} = \mathbf{Y}$. Prove that this system of equations has a solutions if and only if the row rank of $\mathbf{A}$ is equal to the row rank of augmented matrix of the system.

2 Solution

Consider $\mathbf{A}$ as,

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}_{mxn} \tag{2.0.1}$$

Now to solve the equation $\mathbf{AX} = \mathbf{Y}$ for $\mathbf{X}$, we first write its augmented matrix $\mathbf{A_Y}$ and then convert it into row reduced echelon form given as $\mathbf{R'_Y}$. Here $\mathbf{R}$ is the row reduced form of matrix $\mathbf{A}$. Also we know that, for $\mathbf{AX} = \mathbf{Y}$ to have a solution, $\mathbf{Y}$ should be in column space of $\mathbf{A}$.

$$\mathbf{A_Y} = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & y_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & y_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & y_m \end{array}\right) \tag{2.0.2}$$

Assume that the last $\mathbf{k}$ rows of $\mathbf{R}$ are zero rows.This implies that we have $\mathbf{k}$ number of linear dependent rows in matrix $\mathbf{A}$. Hence the row rank of matrix $\mathbf{A}$ is $\mathbf{r}$ = m-k. As there are m-k number of non-zero vectors in the row of $\mathbf{R}$. Now,

$$\mathbf{R} = \mathbf{CA} = \mathbf{C}\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \tag{2.0.3}$$

Here, $\mathbf{C}$ is a matrix which converts $\mathbf{A}$ to $\mathbf{R}$

$$\therefore \mathbf{R'_Y} = \left(\begin{array}{cccc|c} 1 & a'_{12} & \cdots & a'_{1n} & y'_1 \\ 0 & 1 & \cdots & a_{2n} & y'_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & y'_{m-k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & y'_{m-1} \\ 0 & 0 & \cdots & 0 & y'_m \end{array}\right) \tag{2.0.4}$$

Also from equation (2.0.4) it can be observed that for $\mathbf{AX} = \mathbf{Y}$ to have a solution,

$$y'_{m-k} = y'_{m-k-1} = \cdots = y'_{m-1} = y'_m = 0 \tag{2.0.5}$$

Hence, the rank of $\mathbf{R'_Y}$ is, also $\mathbf{r}$. This implies rank of augmented matrix $\mathbf{A_Y}$ is also $\mathbf{r}$.

Hence Proved.