Assignment-13

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***Abstract*—This document contains the problem related to computations concerning subspaces. (Hoffman:- Page-66,Q-7)**

Download the latex-file codes from

https://github.com/ankuraditya13/EE5609−Assignment13

1 Problem

Let **A** be an m x n matrix over the field **F**, and consider the system of equations $\mathbf{AX} = \mathbf{Y}$. Prove that this system of equations has a solutions if and only if the row rank of **A** is equal to the row rank of augmented matrix of the system.

2 Solution

Consider **A** as,

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}_{mxn} \tag{2.0.1}$$

Now to solve the equation $\mathbf{AX} = \mathbf{Y}$ for **X**, we first write its augmented matrix $\mathbf{A_Y}$ and then convert it into row reduced echelon form given as $\mathbf{R_Y}$. Here **R** is the row reduced form of matrix **A**. Also we know that, for $\mathbf{AX} = \mathbf{Y}$ to have a solution, **Y** should be in column space of **A**.

$$\mathbf{A_Y} = \begin{pmatrix} \mathbf{A} & \big| & \mathbf{Y} \end{pmatrix} \tag{2.0.2}$$

Assume that the last **k** rows of **R** are zero rows.This implies that we have **k** number of linear dependent rows in matrix **A**. Hence the row rank of matrix **A** is $\mathbf{r}$ = m-k. As there are m-k number of non-zero vectors in the row of **R**. Now,

$$\mathbf{R}_Y = \mathbf{EA}_Y \tag{2.0.3}$$

$$\implies \mathbf{C} \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & y_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & y_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & y_m \end{array}\right) \tag{2.0.4}$$

Here, **E** is product of matrix responsible for performing operation on matrix such as scaling, Row exchange etc, given as,

$$\mathbf{E} = E_i E_{i-1} \cdots E_1 \tag{2.0.5}$$

Hence row reduced echelon form, $\mathbf{R_Y}$ is given as,

$$\mathbf{R_Y} = E\mathbf{A_Y} = \begin{pmatrix} E\mathbf{A} & \big| & E\mathbf{Y} \end{pmatrix} \tag{2.0.6}$$

Now RREF can be represented in Block matrix as,

$$\implies \mathbf{R_Y} = \left(\begin{array}{cc|c} I & F & Y \\ 0 & 0 & \end{array}\right) \tag{2.0.7}$$

Also from equation (2.0.7) it can be observed that for $\mathbf{AX} = \mathbf{Y}$ to have a solution,

$$y'_{m-k} = y'_{m-k-1} = \cdots = y'_{m-1} = y'_m = 0 \tag{2.0.8}$$

Hence, the rank of $\mathbf{R'_Y}$ is, also **r**. This implies rank of augmented matrix $\mathbf{A_Y}$ is also **r**.

Hence Proved.

3 Example

Let **A** be a 3×4 matrix given as,

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 5 & 7 \\ 2 & 9 & 3 & 6 \\ 1 & 7 & 2 & 9 \end{pmatrix} \tag{3.0.1}$$

$$\mathbf{Y} = \begin{pmatrix} 4 \\ 8 \\ 14 \end{pmatrix} \tag{3.0.2}$$

For the time being it could be concluded that the following set of linear equation have a solution as $\mathbf{Y} \in$ Column-space(**A**) Hence to solve $\mathbf{AX} = \mathbf{Y}$ first the augmented matrix $\mathbf{A_Y}$ is given as,

$$\mathbf{A_Y} = \left(\begin{array}{cccc|c} 1 & 2 & 5 & 7 & 4 \\ 2 & 9 & 3 & 6 & 8 \\ 1 & 7 & 2 & 9 & 14 \end{array}\right) \tag{3.0.3}$$

Now reducing in RREF we get,

$$\mathbf{R_Y} = E\mathbf{A_Y} \tag{3.0.4}$$

Here E is elementary matrix given as,

$$E = \begin{pmatrix} -3/20 & 31/20 & -39/20 \\ -1/20 & -3/20 & 7/20 \\ 1/4 & -1/4 & 1/4 \end{pmatrix} \tag{3.0.5}$$

$$\implies \mathbf{R_Y} = E\mathbf{A_Y} = \begin{pmatrix} E\mathbf{A} & | & E\mathbf{Y} \end{pmatrix} \tag{3.0.6}$$

$$\implies \mathbf{R_Y} = \begin{pmatrix} I & F & | & Y \end{pmatrix} \tag{3.0.7}$$

$$\mathbf{I}_{3x3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.0.8}$$

$$\mathbf{F}_{3x1} = \begin{pmatrix} -93/10 \\ 19/10 \\ 5/2 \end{pmatrix} \tag{3.0.9}$$

$$\mathbf{Y}_{3x1} = \begin{pmatrix} -31/2 \\ 7/2 \\ 5/2 \end{pmatrix} \tag{3.0.10}$$

Hence, rank = 3 as there are 0 non zero rows in row reduced echelon form. Also it could be observed that both **R** and $\mathbf{R_Y}$ have non zero rows hence we have a solution and same rank.