# Assignment-14

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***Abstract*—This document contains the problem related to Linear Transformations (Hoffman:- Page-106,Q-9)**

Download the latex-file from

https://github.com/ankuraditya13/EE5609−Assignment14

## 1 Problem

Let $\mathbf{V}$ be the vector space of all $2 \times 2$ matrices over the field of real numbers, and let

$$\mathbf{B} = \begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix} \tag{1.0.1}$$

Let $\mathbf{W}$ be the subspace of $\mathbf{V}$ consisting of all $\mathbf{A}$ such that $\mathbf{AB}$=0. Let f be a linear functional on $\mathbf{V}$ which is an annihilator of $\mathbf{W}$. Suppose that f($\mathbf{I}$) = 0 and f($\mathbf{C}$) = 3, where $\mathbf{I}$ is the $2 \times 2$ identity matrix and,

$$\mathbf{C} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.0.2}$$

Find f($\mathbf{B}$)?

## 2 solution

The general Linear functional f on $\mathbf{V}$ is of the form,

$$f(\mathbf{A}) = aA_{11} + bA_{12} + cA_{21} + dA_{22} \tag{2.0.1}$$

$$\implies f(\mathbf{A}) = tr(\mathbf{x}^T\mathbf{y}) \tag{2.0.2}$$

$$\mathbf{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbf{y} = \begin{pmatrix} \mathbf{A_{11}} & \mathbf{A_{12}} \\ \mathbf{A_{21}} & \mathbf{A_{22}} \end{pmatrix} \tag{2.0.3}$$

for a,b,c,d $\in \mathbf{R}$ Let $\mathbf{A} \in \mathbf{W}$ be,

$$\mathbf{A} = \begin{pmatrix} p & q \\ q & s \end{pmatrix} \tag{2.0.4}$$

$$\because \mathbf{AB} = 0 \tag{2.0.5}$$

$$\implies \begin{pmatrix} p & q \\ q & s \end{pmatrix} \begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix} = 0 \tag{2.0.6}$$

$$\implies \begin{pmatrix} 2p - q & -2p + q \\ 2q - s & -2q + s \end{pmatrix} = 0 \tag{2.0.7}$$

$\therefore$ q=2p and s=2q. Hence $\mathbf{W}$ consists of all matrices of the form

$$\begin{pmatrix} p & 2p \\ q & 2q \end{pmatrix} \tag{2.0.8}$$

Now V is an annihilator of W. Hence, f$\in \mathbf{W}^0$

$$\implies f\begin{pmatrix} p & 2p \\ q & 2q \end{pmatrix} = 0 \forall p, q \in \mathbf{R} \tag{2.0.9}$$

from equation (2.0.1)

$$\implies tr\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^T \begin{pmatrix} p & 2p \\ q & 2q \end{pmatrix}\right) = 0, \forall p, q \in \mathbf{R} \tag{2.0.10}$$

$$\implies tr\begin{pmatrix} ap + cq & 2ap + 2cq \\ bp + dq & 2bp + 2dq \end{pmatrix} \tag{2.0.11}$$

$$\implies (a + 2b)p + (c + 2d)q = 0, \forall p, q \in \mathbf{R} \tag{2.0.12}$$

Hence, b=$\frac{-1}{2}$a and d=$\frac{-1}{2}$c. Hence x of equation (2.0.3) is now given as,

$$\mathbf{x} = \begin{pmatrix} a & \frac{-1}{2}a \\ c & \frac{-1}{2}c \end{pmatrix} \tag{2.0.13}$$

General $f \in \mathbf{W}^0$ is of the form,

$$f(\mathbf{A}) = tr(\mathbf{x}^T\mathbf{y}) = tr\begin{pmatrix} a & c \\ \frac{-1}{2}a & \frac{-1}{2}c \end{pmatrix} \begin{pmatrix} \mathbf{A_{11}} & \mathbf{A_{12}} \\ \mathbf{A_{21}} & \mathbf{A_{22}} \end{pmatrix} \tag{2.0.14}$$

Now, f($\mathbf{C}$) = 3 $\implies$ d =3 $\implies$ c=-6. Also given that, f($\mathbf{I}$) = 0 $\implies$ a-$\frac{1}{2}$c =0 $\implies$ a=-3. Substituting the above parameters in equation (2.0.14) we get,

$$\therefore f(\mathbf{A}) = tr\begin{pmatrix} -3 & -6 \\ \frac{3}{2} & 3 \end{pmatrix} \begin{pmatrix} \mathbf{A_{11}} & \mathbf{A_{12}} \\ \mathbf{A_{21}} & \mathbf{A_{22}} \end{pmatrix} \tag{2.0.15}$$

$$\text{Now, } f(\mathbf{B}) = f\begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix} \tag{2.0.16}$$

$$\implies f(\mathbf{B}) = tr\begin{pmatrix} -3 & -6 \\ \frac{3}{2} & 3 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix} \tag{2.0.17}$$

$$\implies f(\mathbf{B}) = -3(2) + \frac{3}{2}(-2) - 6(-1) + 3(1) = 0 \tag{2.0.18}$$