Assignment-15

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***Abstract*—This document contains the problem related to Linear Transformations (Hoffman:- Page-111,Q-1a)**

Download the latex-file from

https://github.com/ankuraditya13/EE5609−Assignment15

1 Problem

Let n be a positive integer and $\mathbf{F}$ a field. let $\mathbf{W}$ be the set of all vectors $\left(\mathbf{x_1}, \quad \mathbf{x_2}, \cdots, \mathbf{x_n}\right) \in \mathbf{F}^n$ such that $\mathbf{x_1}+\mathbf{x_2}+\mathbf{x_3}+\cdots\mathbf{x_n} = 0$. Prove that $\mathbf{W}^0$ consists of all linear functional f of the form

$$f\left(\mathbf{x_1}, \quad \mathbf{x_2} \quad, \cdots \quad, \mathbf{x_n}\right) = c\sum_{j=1}^{n}\mathbf{x_j} \tag{1.0.1}$$

2 Definitions

2.1 Definition 1

If the vector space $\mathbf{V}$ is finite dimensional (say dimension =n), the dimension of null-space $\mathbf{N}_f$ by rank nullity theorem is given by,

$$\left|\mathbf{N}_f\right| = |\mathbf{V}| - 1 = n - 1 \tag{2.1.1}$$

2.2 Definition 2

If $\mathbf{V}$ is a vector space over the field $\mathbf{F}$ and $\mathbf{S}$ is a subset of $\mathbf{V}$, the annihilator of $\mathbf{S}$ is the set $\mathbf{S}^0$ of linear functional f on $\mathbf{V}$ such that $f(\alpha) = 0$ for every α in $\mathbf{S}$.

3 Solution

Let h be the functional,

$$h\left(\mathbf{x_1}, \quad \mathbf{x_2}, \cdots, \mathbf{x_n}\right) = \mathbf{x_1} + \mathbf{x_2} + \cdots\mathbf{x_n} \tag{3.0.1}$$

$$\text{Let, } \mathbf{X} = \begin{pmatrix}\mathbf{x_1}\\ \mathbf{x_2}\\ \vdots\\ \mathbf{x_n}\end{pmatrix} \tag{3.0.2}$$

$$\implies h\left(\mathbf{x_1}, \quad \mathbf{x_2}, \cdots, \mathbf{x_n}\right) = 0 \tag{3.0.3}$$

Then $\mathbf{W}$ is in null-space of h. Hence by definition-1, the dimension of $\mathbf{W}$ is,

$$|W| = n - 1 \tag{3.0.4}$$

Now let,

$$a_j = \epsilon_1 - \epsilon_{i+1}, \text{ for } i = (1, \cdots, n-1) \tag{3.0.5}$$

Hence clearly $\{a_1, a_2, \cdots, a_{n-1}\}$ are linearly independent. Hence from (3.0.4) and above statement we can conclude that $\{a_1, a_2, \cdots, a_{n-1}\}$ are all in $\mathbf{W}$ so they must form basis for $\mathbf{W}$. Now, it is given that f is linear functional hence,

$$f\left(\mathbf{x_1}, \quad \mathbf{x_2}, \cdots, \mathbf{x_n}\right) = \sum_{j=1}^{n} c_j\mathbf{x_j} \tag{3.0.6}$$

$$\implies f\left(\mathbf{x_1}, \quad \mathbf{x_2}, \cdots, \mathbf{x_n}\right) = C^T\mathbf{X} \tag{3.0.7}$$

Where,

$$C = \begin{pmatrix}c_1\\ \vdots\\ c_n\end{pmatrix} \tag{3.0.8}$$

Now $f \in \mathbf{W}^0$ from definition-2 is given as,

$$f(a_1) = f(a_2) = \cdots = f(a_n) = 0 \text{ for every } a_j \in \mathbf{W} \tag{3.0.9}$$

$$\implies c_1 - c_i = 0 \forall i = 2\cdots n \tag{3.0.10}$$

$$\implies c_i = c_1 = c \forall i \tag{3.0.11}$$

$$\implies C = c\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} \tag{3.0.12}$$

Hence,

$$f\left(\mathbf{x_1}, \quad \mathbf{x_2}, \cdots, \mathbf{x_n}\right) = C^T\mathbf{X} \tag{3.0.13}$$

$$\implies f\left(\mathbf{x_1}, \quad \mathbf{x_2} \quad, \cdots \quad, \mathbf{x_n}\right) = c\sum_{j=1}^{n}\mathbf{x_j} \tag{3.0.14}$$

Hence, proved