Assignment-17

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***Abstract*—This document contains the problem related to Eigenvalue and Eigenvectors (UGC-June-2017 Maths Q-78)**

Download the latex-file from

https://github.com/ankuraditya13/EE5609−Assignment17

1 Problem

Let T be the linear operator on $\mathbf{R}^3$ which is represented in the standard ordered basis by the matrix

$$\begin{pmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{pmatrix} \tag{1.0.1}$$

Prove that T is diagonalizable by exhibiting a basis for $\mathbf{R}^3$, each vector of which is a characteristic vector of T.

2 Theorem

2.1 Theorem 1

Let $\mathbf{T}$ be a linear operator on a finite-dimensional space $\mathbf{V}$. Let $c_1, \cdots, c_k$ be the distinct characteristic values of $\mathbf{T}$ and let $\mathbf{W}_i$ be the null space of $\mathbf{T} - c_i\mathbf{I}$. The following are equivalent.

(i) $\mathbf{T}$ is diagonalizable.

(ii) The characteristic polynomial for $\mathbf{T}$ is,

$$f = (x - c_1)_1^d \cdots (x - c_k)^{d_k} \tag{2.1.1}$$

and $\dim W_i = d_i, i = 1, \cdots, k$

(iii) $\dim W_1 + \cdots + \dim W_k = \dim V$

3 Solution

Now let,

$$\mathbf{A} = \begin{pmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{pmatrix} \tag{3.0.1}$$

Solving $\left|\lambda\mathbf{I} - \mathbf{A}\right| = 0$

$$\left|\lambda\mathbf{I} - \mathbf{A}\right| = \begin{vmatrix} \lambda+9 & -4 & -4 \\ 8 & \lambda-3 & -4 \\ 16 & -8 & \lambda-7 \end{vmatrix} \tag{3.0.2}$$

$$\xleftrightarrow{C_2 \leftarrow C_2 - C_3} \begin{vmatrix} \lambda+9 & 0 & -4 \\ 8 & \lambda+1 & -4 \\ 16 & -\lambda-1 & \lambda-7 \end{vmatrix} \tag{3.0.3}$$

$$\therefore \left|\lambda\mathbf{I} - \mathbf{A}\right| = (\lambda+1) \begin{vmatrix} \lambda+9 & 0 & -4 \\ 8 & 1 & -4 \\ 16 & -1 & \lambda-7 \end{vmatrix} \tag{3.0.4}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + R_2} (\lambda+1) \begin{vmatrix} \lambda+9 & 0 & -4 \\ 8 & 1 & -4 \\ 24 & 0 & \lambda-11 \end{vmatrix} \tag{3.0.5}$$

$$\implies (\lambda+1) \begin{vmatrix} \lambda+9 & -4 \\ 24 & \lambda-11 \end{vmatrix} \tag{3.0.6}$$

$$\implies \left|\lambda\mathbf{I} - \mathbf{A}\right| = (\lambda+1)^2(\lambda-3) = 0 \tag{3.0.7}$$

$$\implies \lambda_1 = -1, \lambda_2 = -1, \lambda_3 = 3 \tag{3.0.8}$$

Now at λ_1 and λ_3,

$$\mathbf{A} + \mathbf{I} = \begin{pmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{pmatrix} \tag{3.0.9}$$

$$\mathbf{A} - 3\mathbf{I} = \begin{pmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{pmatrix} \tag{3.0.10}$$

Now we know that $\mathbf{A} - 3\mathbf{I}$ is singular and rank$(\mathbf{A} - 3\mathbf{I}) \geq 2$. Therefore, rank$(\mathbf{A} - 3\mathbf{I}) = 2$. Hence from the theorem-1 (iii) it is evident that rank$(\mathbf{A} + \mathbf{I}) = 1$. Let X_1 and X_3 be the spaces of characteristic vectors associated with the characteristic values 1 and 3 respectively. We know from rank nullity theorem that $\dim X_1 = 2$ and $\dim X_3 = 1$. Hence by Theorem-2 (i) $\mathbf{T}$ is diagonalizable.
The null-space of $\mathbf{T} + \mathbf{I}$ is spanned by the vectors,

$$\alpha_1 = \begin{pmatrix} 1 & 0 & 2 \end{pmatrix} \tag{3.0.11}$$

$$\alpha_2 = \begin{pmatrix} 1 & 2 & 0 \end{pmatrix} \tag{3.0.12}$$

As both α_1 and α_2 are independent, hence they form basis for X_1. The null-space of $\mathbf{T} - \mathbf{3I}$ is spanned by the vector,

$$\alpha_3 = \begin{pmatrix} 1 & 1 & 2 \end{pmatrix} \tag{3.0.13}$$

Here α_3 is a characteristic vector and a basis for $\mathbf{X_3}$. Also the matrix P which enables us to change coordinates from the basis β to the standard basis is the matrix which has transposes of α_1, α_2 and α_3 as it's column vectors:

$$\mathbf{P} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 2 \end{pmatrix} \tag{3.0.14}$$