Assignment-18

Ankur Aditya - EE20RESCH11010

***Abstract*—This document contains the problem related to characteristic and minimal polynomial (Hoffman Page-198, Q-2)**

Download the latex-file from

https://github.com/ankuraditya13/EE5609−Assignment18

1 Problem

Let a, b and c be the elements of a field $\mathbf{F}$, and let $\mathbf{A}$ be the following $3\times$ 3 matrix over $\mathbf{F}$

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & c \\ 1 & 0 & b \\ 0 & 1 & a \end{pmatrix} \tag{1.0.1}$$

Prove that the characteristic polynomial for $\mathbf{A}$ is $x^3 - ax^2 - bx - c$ and that this is also minimal polynomial for $\mathbf{A}$.

2 Definitions

Minimal polynomial of $\mathbf{A}$ is a polynomial which satisfies,

1) P($\mathbf{A}$) = 0
2) P(x) is monic.
3) It there is some other annihilating polynomial q(x) such that, q($\mathbf{A}$) = 0, then q does not divide p.

3 Solution

The characteristic polynomial is calculated by solving $\left|\mathbf{A} - \lambda\mathbf{I}\right| = 0$

$$\implies \left|\mathbf{A} - \lambda\mathbf{I}\right| = \begin{vmatrix} -\lambda & 0 & c \\ 1 & -\lambda & b \\ 0 & 1 & a-\lambda \end{vmatrix} \tag{3.0.1}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + \lambda R_3} \begin{vmatrix} -\lambda & 0 & c \\ 1 & 0 & b + a\lambda - \lambda^2 \\ 0 & 1 & a-\lambda \end{vmatrix} \tag{3.0.2}$$

$$\implies \left|\mathbf{A} - \lambda\mathbf{I}\right| = 1 \begin{vmatrix} -\lambda & c \\ 1 & b + a\lambda - \lambda^2 \end{vmatrix} \tag{3.0.3}$$

$$\implies \left|\mathbf{A} - \lambda\mathbf{I}\right| = (-\lambda)(b + a\lambda - \lambda^2) - c \tag{3.0.4}$$

Hence the characteristic polynomial of $\mathbf{A}$ is,

$$\lambda^3 - a\lambda^2 - b\lambda - c \tag{3.0.5}$$

Now for any r,s $\in \mathbf{F}$ and considering the annihilating polynomial f with degree 2.

$$f(\mathbf{A}) = \mathbf{A}^2 + r\mathbf{A} + s \tag{3.0.6}$$

$$\implies \begin{pmatrix} 0 & c & ac \\ 0 & b & c+ba \\ 1 & a & b+a^2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & rc \\ r & 0 & rb \\ 0 & r & ra \end{pmatrix} + \begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} \tag{3.0.7}$$

$$\therefore f(\mathbf{A}) = \mathbf{A}^2 + r\mathbf{A} + s = \begin{pmatrix} s & c & ac + rc \\ r & b+s & c + ba + br \\ 1 & a+r & b + a^2 + ra + s \end{pmatrix} \neq 0 \tag{3.0.8}$$

Element positioned at row-3 and column-1 is non-zero, hence for any $r, s \in \mathbf{F} \implies f(\mathbf{A}) \neq 0 \forall f \in \mathbf{F}$, such that degree($\mathbf{F}$) = 2. Hence minimal polynomial cannot have degree 2. Hence degree of minimal polynomial is 3. Also $x^3 - ax^2 - bx - c$ divides f. Hence from definition-1 we can conclude that,

$$p(x) = x^3 - ax^2 - bx - c \tag{3.0.9}$$

is a minimal polynomial.

4 Example

Let a = 0,b = 0,c = 0 $\in \mathbf{F}$. Hence,

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{4.0.1}$$

Now finding characteristic polynomial by substituting the values of a,b, and c in equation (3.0.5) we get,

$$\lambda^3 = 0 \tag{4.0.2}$$

Now let r=0, s=0 $\in$ **F**, Hence f(**A**) is given by using the equation (3.0.8),

$$\implies f(\mathbf{A}) = \mathbf{A}^2 + 0.\mathbf{A} + 0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \neq 0 \tag{4.0.3}$$

Hence, $f(\mathbf{A}) \neq 0$, Hence degree of minimal polynomial is 3 and is equal to,

$$p(x) = x^3 \tag{4.0.4}$$

Verification by calculating p(**A**),

$$p(\mathbf{A}) = \mathbf{A}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{4.0.5}$$

$$\implies \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{4.0.6}$$

$$\implies f(\mathbf{A}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0 \tag{4.0.7}$$

Hence, from definition-1 it can be concluded that p(x) = x^3 is a minimal polynomial.

5 Conclusion

For the **A** given by equation (1.0.1) ,characteristic and minimal polynomial is given by,

$$x^3 - ax^2 - bx - c \tag{5.0.1}$$

Hence, Proved