

# Assignment-18

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**Abstract**—This document contains the problem related to characteristic and minimal polynomial (Hoffman Page-198, Q-2)

Download the latex-file from

<https://github.com/ankuraditya13/EE5609–Assignment18>

## 1 PROBLEM

Let  $a, b$  and  $c$  be the elements of a field  $\mathbf{F}$ , and let  $\mathbf{A}$  be the following  $3 \times 3$  matrix over  $\mathbf{F}$

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & c \\ 1 & 0 & b \\ 0 & 1 & a \end{pmatrix} \quad (1.0.1)$$

Prove that the characteristic polynomial for  $\mathbf{A}$  is  $x^3 - ax^2 - bx - c$  and that this is also minimal polynomial for  $\mathbf{A}$ .

## 2 DEFINITIONS

Minimal polynomial of  $\mathbf{A}$  is a polynomial which satisfies,

- 1)  $P(\mathbf{A}) = 0$
- 2)  $P(x)$  is monic.
- 3) If there is some other annihilating polynomial  $q(x)$  such that,  $q(\mathbf{A}) = 0$ , then  $q$  does not divide  $p$ .

## 3 SOLUTION

The characteristic polynomial is calculated by solving  $|\mathbf{A} - \lambda \mathbf{I}| = 0$

$$\Rightarrow |\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} -\lambda & 0 & c \\ 1 & -\lambda & b \\ 0 & 1 & a - \lambda \end{vmatrix} \quad (3.0.1)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + \lambda R_3} \begin{vmatrix} -\lambda & 0 & c \\ 1 & 0 & b + a\lambda - \lambda^2 \\ 0 & 1 & a - \lambda \end{vmatrix} \quad (3.0.2)$$

$$\Rightarrow |\mathbf{A} - \lambda \mathbf{I}| = 1 \begin{vmatrix} -\lambda & c \\ 1 & b + a\lambda - \lambda^2 \end{vmatrix} \quad (3.0.3)$$

$$\Rightarrow |\mathbf{A} - \lambda \mathbf{I}| = (-\lambda)(b + a\lambda - \lambda^2) - c \quad (3.0.4)$$

Hence the characteristic polynomial of  $\mathbf{A}$  is,

$$\lambda^3 - a\lambda^2 - b\lambda - c \quad (3.0.5)$$

Now for any  $r, s \in \mathbf{F}$  and considering the annihilating polynomial  $f$  with degree 2.

$$f(\mathbf{A}) = \mathbf{A}^2 + r\mathbf{A} + s \quad (3.0.6)$$

$$\Rightarrow \begin{pmatrix} 0 & c & ac \\ 0 & b & c + ba \\ 1 & a & b + a^2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & rc \\ r & 0 & rb \\ 0 & r & ra \end{pmatrix} + \begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} \quad (3.0.7)$$

$$\therefore f(\mathbf{A}) = \mathbf{A}^2 + r\mathbf{A} + s = \begin{pmatrix} s & c & ac + rc \\ r & b + s & c + ba + br \\ 1 & a + r & b + a^2 + ra + s \end{pmatrix} \neq 0 \quad (3.0.8)$$

Element positioned at row-3 and column-1 is non-zero, hence for any  $r, s \in \mathbf{F} \Rightarrow f(\mathbf{A}) \neq 0 \forall f \in \mathbf{F}$ . Hence minimal polynomial cannot have degree 2. Hence degree of minimal polynomial is 3. Also  $x^3 - ax^2 - bx - c$  divides  $f$ . Hence from definition-1 we can conclude that,

$$p(x) = x^3 - ax^2 - bx - c \quad (3.0.9)$$

is a minimal polynomial.

## 4 EXAMPLE

Let  $a = 0, b = 0, c = 0 \in \mathbf{F}$ . Hence,

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (4.0.1)$$

Now finding characteristic polynomial by substituting the values of  $a, b$ , and  $c$  in equation (3.0.5) we get,

$$\lambda^3 = 0 \quad (4.0.2)$$

Now let  $r=0, s=0 \in \mathbf{F}$ , Hence  $f(\mathbf{A})$  is given by using the equation (3.0.8),

$$\Rightarrow f(\mathbf{A}) = \mathbf{A}^2 + 0.\mathbf{A} + 0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \neq 0 \quad (4.0.3)$$

Hence,  $f(\mathbf{A}) \neq 0$ , Hence degree of minimal polynomial is 3 and is equal to,

$$p(x) = x^3 \quad (4.0.4)$$

Verification by calculating  $p(\mathbf{A})$ ,

$$p(\mathbf{A}) = \mathbf{A}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (4.0.5)$$

$$\Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (4.0.6)$$

$$\Rightarrow f(\mathbf{A}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0 \quad (4.0.7)$$

Hence, from definition-1 it can be concluded that  $p(x) = x^3$  is a minimal polynomial.

## 5 CONCLUSION

For the  $\mathbf{A}$  given by equation (1.0.1), characteristic and minimal polynomial is given by,

$$x^3 - ax^2 - bx - c \quad (5.0.1)$$

**Hence, Proved**