Assignment-6

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Abstract—**This document contains the procedure to find the equation of tangent to parabola.**

Download the python code from

https://github.com/ankuraditya13/EE5609−Assignment6

and latex-file codes from

https://github.com/ankuraditya13/EE5609−Assignment6

1 Problem

Find the equation of the tangent to the curve,

$$y = \sqrt{3x-2} \tag{1.0.1}$$

which is parallel to the line,

$$\begin{pmatrix} 4 & 2 \end{pmatrix} \mathbf{x} + 5 = 0 \tag{1.0.2}$$

2 Solution

The equation (1.0.1) can be written as,

$$y^2 - 3x + 2 = 0 \tag{2.0.1}$$

Comparing it with standard equation,

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.0.2}$$

$\therefore$ a = b = e = 0, d = $\frac{-3}{2}$, c = 1, f = 2.

$$\therefore \mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.3}$$

$$\therefore \mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} \frac{-3}{2} \\ 0 \end{pmatrix} \tag{2.0.4}$$

$$\text{Now,} \left|V\right| = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0 \tag{2.0.5}$$

$\implies$ that the curve is a parabola. Now, finding the eigen values corresponding to the $\mathbf{V}$,

$$\left|\mathbf{V} - \lambda \mathbf{I}\right| = 0 \tag{2.0.6}$$

$$\begin{vmatrix} -\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0 \tag{2.0.7}$$

$$\implies \lambda = 0, 1. \tag{2.0.8}$$

Calculating the eigenvectors corresponding to $\lambda = 0, 1$ respectively,

$$\mathbf{V}\mathbf{x} = \lambda \mathbf{x} \tag{2.0.9}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 0 \implies \mathbf{p_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.10}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \mathbf{x} \implies \mathbf{p_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.0.11}$$

Now by eigen decomposition on $\mathbf{V}$,

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^{\mathbf{T}} \tag{2.0.12}$$

$$\text{where,} \mathbf{P} = \begin{pmatrix} \mathbf{p_1} \mathbf{p_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.13}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.14}$$

Hence equation (2.0.12) becomes,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.15}$$

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.16}$$

Now the tangent to parabola is parallel to the line equation (1.0.2), Hence the direction vectors ($\mathbf{m}$) and normal ($\mathbf{n}$) vectors are,

$$\mathbf{m} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{2.0.17}$$

$$\mathbf{n} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{2.0.18}$$

Now, the equation for the point of contact for the parabola is given as,

$$\begin{pmatrix} \mathbf{u}^T + \kappa\mathbf{n}^T \\ \mathbf{V} \end{pmatrix} \mathbf{q} = \begin{pmatrix} -f \\ \kappa\mathbf{n} - \mathbf{u} \end{pmatrix} \tag{2.0.19}$$

$$\text{where, } \kappa = \frac{\mathbf{p_1}^T \mathbf{u}}{\mathbf{p_1}^T \mathbf{n}} = \frac{-3}{4} \tag{2.0.20}$$

Hence substituting the values of (2.0.20), (2.0.18), (2.0.12) and (2.0.4) in equation (2.0.19) we get,

$$\begin{pmatrix} -3 & \frac{-3}{4} \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{q} = \begin{pmatrix} -2 \\ 0 \\ \frac{-3}{4} \end{pmatrix} \tag{2.0.21}$$

Solving for $\mathbf{q}$ by removing the zero row and representing (2.0.21) as augmented matrix and then converting the matrix to echelon form,

$$\implies \begin{pmatrix} -3 & \frac{-3}{4} & -2 \\ 0 & 1 & \frac{-3}{4} \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \left(\frac{-R_1}{3}\right)} \begin{pmatrix} 1 & \frac{1}{4} & \frac{2}{3} \\ 0 & 1 & \frac{-3}{4} \end{pmatrix} \tag{2.0.22}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{1}{4}R_2} \begin{pmatrix} 1 & 0 & \frac{41}{48} \\ 0 & 1 & \frac{-3}{4} \end{pmatrix} \tag{2.0.23}$$

Hence from equation (2.0.23) it can be concluded that the point of contact is,

$$\mathbf{q} = \begin{pmatrix} \frac{41}{48} \\ \frac{-3}{4} \end{pmatrix} \tag{2.0.24}$$

Now $\mathbf{q}$ is a point on the tangent. Hence, the equation of the line can be expressed as

$$\mathbf{n}^T \mathbf{x} = c \tag{2.0.25}$$

where c is,

$$c = \mathbf{n}^T \mathbf{q} = \begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{41}{48} \\ \frac{-3}{4} \end{pmatrix} = \frac{23}{24} \tag{2.0.26}$$

Hence equation of tangent to the curve (1.0.1) parallel to (1.0.2) is given by substituting the value of c and $\mathbf{n}$ from equation (2.0.26) and (2.0.18) respectively to the equation (2.0.25),

$$\implies \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = \frac{23}{24} \tag{2.0.27}$$

Figure 0 verifies that the $\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = \frac{23}{24}$ is a tangent to parabola $y = \sqrt{3x - 2}$

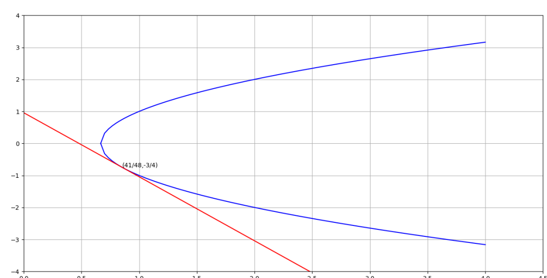

Fig. 0: Tangent to parabola $y = \sqrt{3x - 2}$