Assignment-8

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***Abstract*—This document contains the procedure to find the foot of the perpendicular from a point to the plane.**

Download the python code from

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https://github.com/ankuraditya13/EE5609−Assignment8
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and latex-file codes from

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https://github.com/ankuraditya13/EE5609−Assignment8
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1 Problem

Find the foot of the perpendicular from,

$$\mathbf{A} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \tag{1.0.1}$$

to the plane,

$$\begin{pmatrix} 2 & -3 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{1.0.2}$$

2 Solution

The equation of plane is given as,

$$\mathbf{n}^T \mathbf{x} = c \tag{2.0.1}$$

Hence the normal vector $\mathbf{n}$ is,

$$\mathbf{n} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \tag{2.0.2}$$

Let, the normal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the normal vector $\mathbf{n}$ be,

$$\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{2.0.3}$$

$$\text{then, } \mathbf{m}^T \mathbf{n} = 0 \tag{2.0.4}$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0 \tag{2.0.5}$$

Let, a=0 and b=1 we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \tag{2.0.6}$$

Let, a=1 and b=0,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \tag{2.0.7}$$

Now solving the equation,

$$\mathbf{Mx} = \mathbf{b} \tag{2.0.8}$$

Where,

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \tag{2.0.9}$$

$$\text{and, } \mathbf{b} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \tag{2.0.10}$$

To solve (2.0.8) we perform singular value decomposition on M given by,

$$\mathbf{M} = \mathbf{USV}^T \tag{2.0.11}$$

substituting the value of $\mathbf{M}$ from equation (2.0.11) to (2.0.8),

$$\implies \mathbf{USV}^T \mathbf{x} = \mathbf{b} \tag{2.0.12}$$

$$\implies \mathbf{x} = \mathbf{VS_+U}^T \mathbf{b} \tag{2.0.13}$$

where, $\mathbf{S_+}$ is Moore-Pen-rose Pseudo-Inverse of $\mathbf{S}$. Columns of $\mathbf{U}$ are eigenvectors of $\mathbf{MM}^T$, columns of $\mathbf{V}$ are eigenvectors of $\mathbf{M}^T\mathbf{M}$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$. First calculating the eigenvectors corresponding to $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \tag{2.0.14}$$

Eigenvalues corresponding to $\mathbf{M}^T\mathbf{M}$ is,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{2.0.15}$$

$$\implies \begin{pmatrix} 5-\lambda & -6 \\ -6 & 10-\lambda \end{pmatrix} \tag{2.0.16}$$

$$\implies (\lambda - 14)(\lambda - 1) = 0 \tag{2.0.17}$$

$$\therefore \lambda_1 = 14 \tag{2.0.18}$$

$$\lambda_2 = 1 \tag{2.0.19}$$

Hence the eigenvectors corresponding to λ_1 and λ_2 respectively is,

$$\mathbf{v_1} = \begin{pmatrix} \frac{-2}{3} \\ 1 \end{pmatrix} \tag{2.0.20}$$

$$\mathbf{v_2} = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} \tag{2.0.21}$$

Normalizing the eigenvectors we get,

$$\mathbf{v_1} = \frac{1}{\sqrt{13}}\begin{pmatrix} -2 \\ 3 \end{pmatrix} \tag{2.0.22}$$

$$\mathbf{v_2} = \frac{1}{\sqrt{13}}\begin{pmatrix} 3 \\ 2 \end{pmatrix} \tag{2.0.23}$$

$$\implies \mathbf{V} = \frac{1}{\sqrt{13}}\begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix} \tag{2.0.24}$$

Now calculating the eigenvectors corresponding to $\mathbf{MM}^T$

$$\mathbf{MM}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \tag{2.0.25}$$

$$\implies \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ -2 & 3 & 13 \end{pmatrix} \tag{2.0.26}$$

Eigenvalues corresponding to $\mathbf{MM}^T$ is,

$$\left|\mathbf{MM}^T - \lambda\mathbf{I}\right| = 0 \tag{2.0.27}$$

$$\implies \begin{pmatrix} 1-\lambda & 0 & -2 \\ 0 & 1-\lambda & 3 \\ -2 & 3 & 13-\lambda \end{pmatrix} \tag{2.0.28}$$

$$\implies -\lambda^3 + 15\lambda^2 - 14\lambda = 0 \tag{2.0.29}$$

$$\implies -\lambda(\lambda - 1)(\lambda - 14) = 0 \tag{2.0.30}$$

$$\therefore \lambda_3 = 14 \tag{2.0.31}$$

$$\lambda_4 = 1 \tag{2.0.32}$$

$$\lambda_5 = 0 \tag{2.0.33}$$

Hence the eigenvectors corresponding to λ_3, λ_4 and λ_5 respectively is,

$$\mathbf{v_3} = \begin{pmatrix} \frac{-2}{13} \\ \frac{3}{13} \\ 1 \end{pmatrix} \tag{2.0.34}$$

$$\mathbf{v_4} = \begin{pmatrix} \frac{3}{2} \\ 1 \\ 0 \end{pmatrix} \tag{2.0.35}$$

$$\mathbf{v_5} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \tag{2.0.36}$$

Normalizing the eigenvectors we get,

$$\mathbf{v_3} = \frac{1}{\sqrt{182}}\begin{pmatrix} -2 \\ 3 \\ 13 \end{pmatrix} = \begin{pmatrix} -\sqrt{\frac{2}{91}} \\ \frac{3}{\sqrt{182}} \\ \sqrt{\frac{13}{14}} \end{pmatrix} \tag{2.0.37}$$

$$\mathbf{v_4} = \frac{1}{\sqrt{13}}\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \\ 0 \end{pmatrix} \tag{2.0.38}$$

$$\mathbf{v_5} = \frac{1}{\sqrt{14}}\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{2}{7}} \\ -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{1}{14}} \end{pmatrix} \tag{2.0.39}$$

$$\implies \mathbf{U} = \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix} \tag{2.0.40}$$

Now $\mathbf{S}$ corresponding to eigenvalues λ_3, λ_4 and λ_5 is as follows,

$$\mathbf{S} = \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.0.41}$$

Now, Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{14}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.0.42}$$

Hence we get singular value decomposition of $\mathbf{M}$ as,

$$\mathbf{M} = \frac{1}{\sqrt{13}} \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix} \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix}^T \tag{2.0.43}$$

Now substituting the values of (2.0.24), (2.0.42), (2.0.40) and (2.0.10) in (2.0.13),

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix}^T \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \tag{2.0.44}$$

$$\implies \mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{-29}{\sqrt{182}} \\ \frac{11}{\sqrt{13}} \\ \frac{-13}{\sqrt{14}} \end{pmatrix} \tag{2.0.45}$$

$$\mathbf{VS}_+ = \frac{1}{\sqrt{13}} \begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{14}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.0.46}$$

$$\implies \mathbf{VS}_+ = \frac{1}{\sqrt{13}\sqrt{14}} \begin{pmatrix} -2 & 3\sqrt{14} & 0 \\ 3 & 2\sqrt{14} & 0 \end{pmatrix} \tag{2.0.47}$$

$\therefore$ from equation (2.0.13),

$$\mathbf{x} = \frac{1}{\sqrt{13}\sqrt{14}} \begin{pmatrix} -2 & 3\sqrt{14} & 0 \\ 3 & 2\sqrt{14} & 0 \end{pmatrix} \begin{pmatrix} \frac{-29}{\sqrt{182}} \\ \frac{11}{\sqrt{13}} \\ \frac{-13}{\sqrt{14}} \end{pmatrix} \tag{2.0.48}$$

$$\implies \mathbf{x} = \begin{pmatrix} \frac{20}{7} \\ \frac{17}{14} \end{pmatrix} \tag{2.0.49}$$

Verifying the solution using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{2.0.50}$$

$$\implies \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \tag{2.0.51}$$

$$\implies \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 7 \\ -5 \end{pmatrix} \tag{2.0.52}$$

Solving the augmented matrix we get,

$$\begin{pmatrix} 5 & -6 & 7 \\ -6 & 10 & -5 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{5}} \begin{pmatrix} 1 & -\frac{6}{5} & \frac{7}{5} \\ -6 & 10 & -5 \end{pmatrix} \tag{2.0.53}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + 6R_1} \begin{pmatrix} 1 & -\frac{6}{5} & \frac{7}{5} \\ 0 & \frac{14}{5} & \frac{17}{5} \end{pmatrix} \tag{2.0.54}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{5}{14}R_2} \begin{pmatrix} 1 & -\frac{6}{5} & \frac{7}{5} \\ 0 & 1 & \frac{17}{14} \end{pmatrix} \tag{2.0.55}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + \frac{6}{5}R_2} \begin{pmatrix} 1 & 0 & \frac{20}{7} \\ 0 & 1 & \frac{17}{14} \end{pmatrix} \tag{2.0.56}$$

$$\implies \mathbf{x} = \begin{pmatrix} \frac{20}{7} \\ \frac{17}{14} \end{pmatrix} \tag{2.0.57}$$

Hence from equations (2.0.49) and (2.0.57) we conclude that the solution is verified.