1 Pillai's Trace as an M-estimator

Given two sets of residuals $\boldsymbol{R^x} = \{R_1^x, \cdots, R_p^x\}$ and $\boldsymbol{R^y} = \{R_1^y, \cdots, R_q^y\}$.

$$\begin{aligned} \boldsymbol{\theta} &= (\boldsymbol{\theta^x}, \boldsymbol{\theta^y}, \boldsymbol{\theta^{xy}}) \\ \boldsymbol{\theta^x} &= (\theta_1^x, \cdots, \theta_p^x) \\ \boldsymbol{\theta^y} &= (\theta_1^y, \cdots, \theta_q^y) \\ \boldsymbol{\theta^{xy}} &= (\theta_{11}^{xy}, \cdots, \theta_{pq}^{xy}) \end{aligned} \tag{1}$$

$$\Psi = \begin{cases} R_i^x - \theta_i^x & \forall i \in \{1, \cdots, p\} \\ R_i^y - \theta_i^y & \forall i \in \{1, \cdots, q\} \\ R_i^x R_j^x - \theta_{ij}^{xx} & \forall i, j \in \{1, \cdots, p\} \\ R_i^y R_j^y - \theta_{ij}^{yy} & \forall i, j \in \{1, \cdots, q\} \\ R_i^x R_j^y - \theta_{ij}^{xy} & \forall i \in \{1, \cdots, p\}, j \in \{1, \cdots, q\} \end{cases} \tag{2}$$

$$\begin{aligned} \hat{\theta}_i^x &= \mathbb{E}[R_i^x] \\ \hat{\theta}_i^y &= \mathbb{E}[R_i^y] \\ \hat{\theta}_{ij}^{xx} &= \mathbb{E}[R_i^x R_i^x] \\ \hat{\theta}_{ij}^{yy} &= \mathbb{E}[R_i^y R_i^y] \\ \hat{\theta}_{ij}^{xy} &= \mathbb{E}[R_i^x R_j^y] \end{aligned} \tag{3}$$

$$\begin{aligned} \Sigma_{ij}^x &= \mathbb{E}[R_i^x R_j^x] - \mathbb{E}[R_i^x]\mathbb{E}[R_j^x] = \mathbb{E}[R_i^x R_j^x] \;\; \textit{(under null)} \\ \Sigma_{ij}^y &= \mathbb{E}[R_i^y R_j^y] - \mathbb{E}[R_i^y]\mathbb{E}[R_j^y] = \mathbb{E}[R_i^y R_j^y] \;\; \textit{(under null)} \\ \Sigma_{ij}^{xy} &= \mathbb{E}[R_i^x R_j^y] - \mathbb{E}[R_i^x]\mathbb{E}[R_j^y] = 0 \;\; \textit{(under null as no cross association)} \\ \Lambda &= (\Sigma^x)^{-\frac{1}{2}} \Sigma^{xy} (\Sigma^y)^{-\frac{1}{2}} \\ g(\boldsymbol{\theta}) &= \sum_{i=\{1,\cdots,p\}, j=\{1,\cdots,q\}} \Lambda_{i,j}^2 \quad \textit{elementwise-square} \end{aligned} \tag{4}$$

$$\begin{aligned} \kappa^x &= (\Sigma^x)^{-\frac{1}{2}} \\ \kappa^y &= (\Sigma^y)^{-\frac{1}{2}} \\ \Lambda_{i,j} &= \sum_{k=1}^{q} \left[\left(\sum_{l=1}^{p} \kappa_{il}^x \Sigma_{lk}^{xy} \right) \kappa_{kj}^y \right] \end{aligned} \tag{5}$$

Under null all residual expectations are 0.

$$\begin{aligned} A(\boldsymbol{\theta}) &= \mathbb{E}[-\frac{\partial}{\partial \boldsymbol{\theta}} \Psi_i(\boldsymbol{\theta})] = \mathbb{I}_{2(p+q)(pq)} \\ B(\boldsymbol{\theta}) &= \mathbb{E}[\Psi_i(\boldsymbol{\theta})\Psi_i(\boldsymbol{\theta})'] \\ V(\boldsymbol{\theta}) &= A(\boldsymbol{\theta})B(\boldsymbol{\theta})A(\boldsymbol{\theta})' = B(\boldsymbol{\theta}) \end{aligned} \tag{6}$$

$$\begin{aligned} \sqrt{n}(g(\hat{\boldsymbol{\theta}}) - g(\boldsymbol{\theta})) &\to \mathcal{N}(\mathbf{0}, \boldsymbol{\sigma^2}) \\ \sqrt{n}(g(\hat{\boldsymbol{\theta}})) &\to \mathcal{N}(\mathbf{0}, \boldsymbol{\sigma^2}) \\ \sigma^2 = B \frac{\partial}{\partial \boldsymbol{\theta}} g(\boldsymbol{\theta}) B' & \end{aligned} \tag{7}$$

2 Q1 as an M-Estimator

Given two sets of residuals, $(\boldsymbol{X}, \boldsymbol{Y})$, trying to write:

- Canonical correlation: $\frac{\mathbb{E}[XY]}{\mathbb{E}[XY]-\mathbb{E}[X]\mathbb{E}[Y]}$
- Q1: $\frac{\mathbb{E}[XY]}{\mathbb{E}[(XY)^2]-\mathbb{E}[XY]^2}$