ENPM667 – Control of Robotic Systems

Final Project Report

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ENPM667

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Introduction

- As depicted in figure 1 we are given a system in which two masses are suspended to a crane. In this project we calculate the equations of motion using Euler-Lagrange method.
- As the system is non-linear, we linearize it using Jacobian Linearization around an equilibrium point. State-space representation of the linearized system is given.
- Then the conditions on M, m_1, m_2 , l_1, l_2 are obtained for the linearized system to be controllable.
- For specific values of M, m_1, m_2 , l_1, l_2 we checked the controllability of the system. Designed an LQR controller to control the system, obtained the response of both linearized and original nonlinear system to initial conditions. The stability of the close loop system is checked using Lyapunov's indirect method.
- For given set of output vector, determined for which vectors the linearized system is observable.
- Optimal Luenberger observer is obtained for each output vector for which the system is observable and simulated its response for both linearized and original systems to initial conditions and unit step input.
- Designed an output feedback controller for the smallest output vector. Used the LQG method to apply the resulting output feedback controller to the original nonlinear system.

First Component

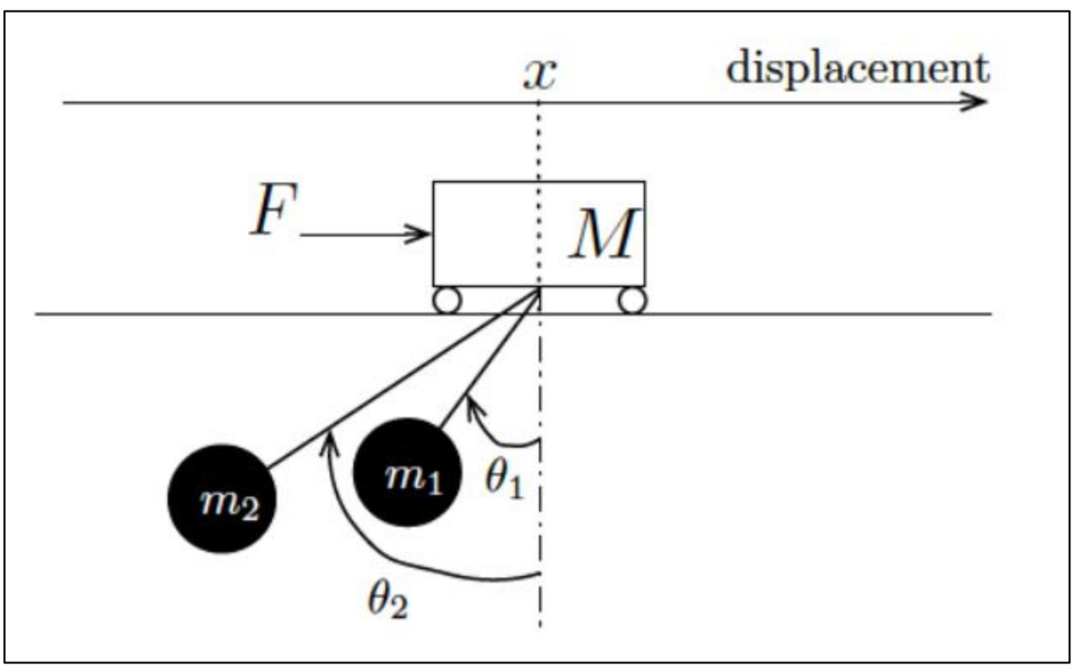


Figure 1: Crane with two suspended loads

A. Equations of motions

The equation of motions for the cart and two suspended load system are represented below.

The assumption of the equation of motion are as follows –

1. The cart-track interface is frictionless.
2. The cart moves only in one dimensional motion in x direction

Finding the position of mass $m1$:

$$x_{m1} = \left(x - l_1 \sin\left(\theta_1\right)\right)\hat{\imath} + \left(-l_1 \cos\left(\theta_1\right)\right)\hat{\jmath}$$

Differentiating the above equation w.r.t time gives us the velocity of $m1$:

$$v_{m1} = \left(\dot{x} - l_1 \cos\left(\theta_1\right)\dot{\theta_1}\right)\hat{\imath} + l_1 \sin\left(\theta_1\right)\dot{\theta_1}\hat{\jmath}$$

Similarly, finding the position of mass $m2$:

$$x_{m2} = \left(x - l_2 \sin\left(\theta_2\right)\right)\hat{\imath} + \left(-l_2 \cos\left(\theta_2\right)\right)\hat{\jmath}$$

Differentiating the above equation w.r.t time gives us the velocity of $m2$:

$$v_{m2} = \left(\dot{x} - l_2 \cos\left(\theta_2\right)\dot{\theta_2}\right)\hat{\imath} + l_2 \sin\left(\theta_2\right)\dot{\theta_2}\hat{\jmath}$$

From the above 2 velocity equations, we can calculate the Kinetic energy (K.E) of the system as follows:

- K.E of the system = (K.E) of mass M + (K.E) of mass $m1$ + (K.E) of mass $m2$

$$\text{K.E } = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m_1\left(\dot{x} - l_1\dot{\theta}_1 \cos\left(\theta_1\right)\right)^2 + \frac{1}{2}m_1\left(l_1\dot{\theta}_1\left(\sin\left(\theta_1\right)\right)\right)^2 + \frac{1}{2}m_2(\dot{x} - \dot{\theta}_2 l_2 \cos(\theta_2))^2 + \frac{1}{2}m_2\left(l_2\dot{\theta}_2(\sin(\theta_2))\right)^2 \quad \ldots\ldots\ldots(\mathbf{1})$$

- Potential Energy (P.E) of the system = (P.E) of mass $m1$ + (P.E) of mass $m2$

$$\text{P.E } = -m_1 g l_1 \cos\left(\theta_1\right) - m_2 g l_2 \cos\left(\theta_2\right) = -g[m_1 l_1 \cos\left(\theta_1\right) + m_2 g l_2 \cos\left(\theta_2\right) \quad \ldots\ldots\ldots \textbf{(2)}$$

We know the Lagrange equation:

L = K.E – P.E

$$\mathrm{L} = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2\cos^2(\theta_1) - m_1 l_1\dot{\theta}_1\dot{x}\cos(\theta_1) + \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2\sin^2(\theta_1)$$
$$+\frac{1}{2}m_2\dot{x}^2 + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2\cos^2(\theta_2) - m_2 l_2\dot{\theta}_2\dot{x}\cos(\theta_2) + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2\sin^2(\theta_1) + g[m_1 l_1\cos(\theta_1) +$$
$$m_2 l_2\cos(\theta_2)]$$

Simplifying the above equation, we get:

$$\mathbf{L} = \boldsymbol{\frac{1}{2}M\dot{x}^2 + \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 -}$$
$$\boldsymbol{\dot{x}\left(m_1 l_1\dot{\theta}_1\cos(\theta_1) + m_2 l_2\dot{\theta}_2\cos(\theta_2)\right) + g[m_1 l_1\cos(\theta_1) + m_2 l_2\cos(\theta_2)]} \quad \dots\dots\dots(\mathbf{3})$$

Using Euler- Lagrange's method to calculate the equations of the motion for the system:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = F \qquad \dots\dots\dots\dots(4)$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \left(\frac{\partial L}{\partial \theta_1}\right) = 0 \qquad \dots\dots\dots\dots(5)$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \left(\frac{\partial L}{\partial \theta_2}\right) = 0 \qquad \dots\dots\dots\dots(6)$$

Calculating above equations:

For Equation (4):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = F$$

First, calculating the partial differential of L w.r.t $\dot{x}$, we get:

$$\frac{\partial L}{\partial \dot{x}} = M\dot{x} + (m_1+m_2)\dot{x} - m_1 l_1\dot{\theta}_1\cos(\theta_1) - m_2 l_2\dot{\theta}_2\cos(\theta_2)$$

Differentiating the above term w.r.t time, we get:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = M\ddot{x} + (m_1+m_2)\ddot{x} - \left[m_1 l_1\ddot{\theta}_1\cos(\theta_1) - m_1 l_1\dot{\theta}_1^2\sin(\theta_1)\right]$$
$$-\left[m_2 l_2\ddot{\theta}_2\cos(\theta_2) - m_2 l_2\dot{\theta}_2^2\sin(\theta_2)\right]\cdot \mathrm{dequatic}$$

In equation (4) the differential of L w.r.t x is zero:

$$\frac{\partial L}{\partial x} = 0$$

Substituting the above values in equation (4), we get:

$$[\boldsymbol{M} + \boldsymbol{m_1} + \boldsymbol{m_2}]\ddot{x} - \boldsymbol{m_1 l_1 \ddot{\theta}_1} \cos(\boldsymbol{\theta_1}) + \boldsymbol{m_1 l_1 \dot{\theta}_1}^2 \sin(\boldsymbol{\theta 1}) - \boldsymbol{m_2 l_2 \ddot{\theta}_2} \cos(\boldsymbol{\theta_2}) + \boldsymbol{m_2 l_2 \dot{\theta}_2}^2 \sin(\boldsymbol{\theta_2}) = \boldsymbol{F}$$

Let the above equation be equation (7)

For Equation (5):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \left(\frac{\partial L}{\partial \theta_1}\right) = 0$$

First, calculating the partial differential of L w.r.t $\dot{\theta}_1$, we get:

$$\frac{\partial L}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1 - m_1 \dot{x} l_1 \cos(\theta_1)$$

Differentiating the above term w.r.t time, we get:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = m_1 l_1^2 \ddot{\theta}_1 - \left[m_1 l_1 \ddot{x} \cos(\theta_1) - m_1 \dot{x} l_1 \dot{\theta}_1 \sin(\theta_1)\right]$$

Calculating differential of L w.r.t θ_1:

$$\frac{\partial L}{\partial \theta_1} = m_1 l_1 \dot{\theta}_1 \dot{x} \sin(\theta_1) - m_1 l_1 g \sin(\theta_1)$$

Substituting the above values in equation (5), we get:

$$m_1 l_1^2 \ddot{\theta}_1 - m_1 \ddot{x} l_1 \cos(\theta_1) + m_1 \dot{\theta}_1 \dot{x}_1 l_1 \sin(\theta_1) - m_1 \dot{\theta}_1 \dot{x} l_1 \sin(\theta_1) + m_1 l_1 g \sin(\theta_1) = 0$$

Cancelling out the equivalent terms, we get the following equation:

$$\boldsymbol{m_1 l_1^2 \ddot{\theta}_1 - m_1 \ddot{x}_1 \cos(\theta_1) + m_1 l_1 g \sin(\theta_1) = 0} \quad \textbf{... ... ... (8)}$$

Now, for Equation (6):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \left(\frac{\partial L}{\partial \theta_2}\right) = 0$$

First, calculating the partial differential of L w.r.t $\dot{\theta}_2$, we get:

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 - m_2 \dot{x} l_2 \cos(\theta_2)$$

Differentiating the above term w.r.t time, we get:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = m_2 l_2^2 \ddot{\theta}_2 - \left[m_2 \ddot{x} l_2 \cos(\theta_2) - m_2 \dot{\theta}_2 \dot{x} l_2 \sin(\theta_2)\right]$$

Calculating differential of L w.r.t θ_1:

$$\left(\frac{\partial L}{\partial \theta_2}\right) = m_2 \dot{x} l_2 \dot{\theta}_2 \sin(\theta_2) - m_2 l_2 g \sin(\theta_2)$$

Substituting the above values in equation (6), we get:

$$m_2 l_2^2 \ddot{\theta}_2 - m_2 l_2 \ddot{x} \cos(\theta_2) + m_2 \dot{\theta}_2 \dot{x} l_2 \sin(\theta_2) - m_2 \dot{\theta}_2 \dot{x} l_2 \sin(\theta_2) + m_2 g l_2 \sin(\theta_2) \ = 0$$

Simplifying the above equation, we get:

$$\boldsymbol{m_2 l_2^2 \ddot{\theta}_2 - m_2 l_2 \ddot{x} \cos(\theta_2) + m_2 g l_2 \sin(\theta_2) = 0} \quad \textbf{... ... ... (9)}$$

Upon simplifying and rearranging we get the value of $\ddot{x}$:

$$\boldsymbol{\ddot{x} = \frac{F - m_1\left(g\sin(\theta_1)\cos(\theta_1) + l_1 \sin(\theta_1)\dot{\theta}_1^2\right) - m_2\left(g\sin(\theta_2)\cos(\theta_2) + l_2 \sin(\theta_2)\dot{\theta}_2^2\right)}{\left(M + m_1\left(\sin(\theta_1)\right)^2 + m_2\left(\sin(\theta_2)\right)^2\right)}}$$

Substituting $\ddot{x}\ to\ get\ \ddot{\theta}_1\ and\ \ddot{\theta}_2$:

$$\boldsymbol{\ddot{\theta}_1 = \frac{1}{l_1}[\ddot{x}\cos(\theta_1) - g\sin(\theta_1)]}$$

$$\boldsymbol{\ddot{\theta}_2 = \frac{1}{l_1}[\ddot{x}\cos(\theta_2) - g\sin(\theta_2)]}$$

Hence, we got all the equations of motion for our system.

The state space variable matrix for the system in consideration is:

$$[x] = \begin{bmatrix} x \\ \dot{x} \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix}$$

The state-space representation for our nonlinear system is:

$$[\dot{x}] = \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \dot{\mathrm{x}} \\ \dfrac{\mathrm{F}-\mathrm{m}_1\left(\mathrm{g}\sin(\theta_1)\cos(\theta_1)+\mathrm{l}_1\sin(\theta_1)\dot{\theta}_1^2\right)-\mathrm{m}_2\left(\mathrm{g}\sin(\theta_2)\cos(\theta_2)+\mathrm{l}_2\sin(\theta_2)\dot{\theta}_2^2\right)}{\left(\mathrm{M}+\mathrm{m}_1\left(\sin(\theta_1)\right)^2+\mathrm{m}_2\left(\sin(\theta_2)\right)^2\right)} \\ \dot{\theta}_1 \\ \dfrac{\cos(\theta_1)}{\mathrm{l}_1}\left[\dfrac{\mathrm{F}-\mathrm{m}_1\left(\mathrm{g}\sin(\theta_1)\cos(\theta_1)+\mathrm{l}_1\sin(\theta_1)\dot{\theta}_1^2\right)-\mathrm{m}_2\left(\mathrm{g}\sin(\theta_2)\cos(\theta_2)+\mathrm{l}_2\sin(\theta_2)\dot{\theta}_2^2\right)}{\left(\mathrm{M}+\mathrm{m}_1\left(\sin(\theta_1)\right)^2+\mathrm{m}_2\left(\sin(\theta_2)\right)^2\right)}\right]-\dfrac{\mathrm{g}\sin(\theta_1)}{\mathrm{l}_1} \\ \dot{\theta}_2 \\ \dfrac{\cos(\theta_2)}{l_2}\left[\dfrac{F-m_1\left(g\sin(\theta_1)\cos(\theta_1)+l_1\sin(\theta_1)\dot{\theta}_1^2\right)-m_2\left(g\sin(\theta_2)\cos(\theta_2)+l_2\sin(\theta_2)\dot{\theta}_2^2\right)}{\left(M+m_1\left(\sin(\theta_1)\right)^2+m_2\left(\sin(\theta_2)\right)^2\right)}\right]-\dfrac{g\sin(\theta_2)}{l_2} \end{bmatrix}$$

Above is the required representation of the system. We can see that the system is nonlinear as it has $\sin(\theta_i)$ and $\cos(\theta_i)$ terms

B. Linearization of the system:

We use Jacobian linearization around an equilibrium point to linearize the system.

Equilibrium point is given by x = 0 and θ1 = θ2 = 0.

The Jacobians are given by:

$$A = \begin{bmatrix} \frac{\delta f_1}{\delta X_1} & \frac{\delta f_1}{\delta X_2} & \frac{\delta f_1}{\delta X_3} & \frac{\delta f_1}{\delta X_4} & \frac{\delta f_1}{\delta X_5} & \frac{\delta f_1}{\delta X_6} \\ \frac{\delta f_2}{\delta X_1} & \frac{\delta f_2}{\delta X_2} & \frac{\delta f_2}{\delta X_3} & \frac{\delta f_2}{\delta X_4} & \frac{\delta f_2}{\delta X_5} & \frac{\delta f_2}{\delta X_6} \\ \frac{\delta f_3}{\delta X_1} & \frac{\delta f_3}{\delta X_2} & \frac{\delta f_3}{\delta X_3} & \frac{\delta f_3}{\delta X_4} & \frac{\delta f_3}{\delta X_5} & \frac{\delta f_3}{\delta X_6} \\ \frac{\delta f_4}{\delta X_1} & \frac{\delta f_4}{\delta X_2} & \frac{\delta f_4}{\delta X_3} & \frac{\delta f_4}{\delta X_4} & \frac{\delta f_4}{\delta X_5} & \frac{\delta f_4}{\delta X_6} \\ \frac{\delta f_5}{\delta X_1} & \frac{\delta f_5}{\delta X_2} & \frac{\delta f_5}{\delta X_3} & \frac{\delta f_5}{\delta X_4} & \frac{\delta f_5}{\delta X_5} & \frac{\delta f_5}{\delta X_6} \\ \frac{\delta f_6}{\delta X_1} & \frac{\delta f_6}{\delta X_2} & \frac{\delta f_6}{\delta X_3} & \frac{\delta f_6}{\delta X_4} & \frac{\delta f_6}{\delta X_5} & \frac{\delta f_6}{\delta X_6} \end{bmatrix} \quad B = \begin{bmatrix} \frac{\delta f_1}{\delta u} \\ \frac{\delta f_1}{\delta u} \\ \frac{\delta f_1}{\delta u} \\ \frac{\delta f_1}{\delta u} \\ \frac{\delta f_1}{\delta u} \\ \frac{\delta f_1}{\delta u} \end{bmatrix}$$

Using MATLAB to calculate the Jacobian, we get:

```
syms M m1 m2 l1 l2 g F
syms x theta1 theta2
syms x_dot theta1_dot theta2_dot
syms x_double_dot theta1_double_dot theta2_double_dot


A = [ 0 1 0 0 0 0;
      0 0 -g*m1/M 0 -g*m2/M 0;
      0 0 0 1 0 0;
      0 0 -g*(M+m1)/(M*l1) 0 -g*m2/(M*l1) 0;
      0 0 0 0 0 1;
      0 0 -g*m1/(M*l2) 0 -g*(M+m2)/(M*l2) 0
    ]

B = [0; 1/M; 0; 1/(M*l1); 0; 1/(M*l2)]
```

Matrix A as:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-gm_1}{M} & 0 & \frac{-gm_2}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-g(M+m_1)}{Ml_1} & 0 & \frac{-gm_2}{Ml_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-gm_1}{Ml_2} & 0 & \frac{-g(M+m_2)}{Ml_2} & 0 \end{bmatrix}$$

Matrix B as:

$$B = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{Ml_1} \\ 0 \\ \frac{1}{Ml_2} \end{bmatrix}$$

Substituting values of A and B to get the state space equation of the linearized system as:

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-gm_1}{M} & 0 & \frac{-gm_2}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-g(M+m_1)}{Ml_1} & 0 & \frac{-gm_2}{Ml_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-gm_1}{Ml_2} & 0 & \frac{-g(M+m_2)}{Ml_2} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{Ml_1} \\ 0 \\ \frac{1}{Ml_2} \end{bmatrix} F$$

C. Conditions for Controllable Linearized system

To check the controllability of the system we calculate the controllability matric C and check its rank. If the rank of C matrix is 6, then the system is controllable. The determinant of C matrix should not be zero for the system to be controllable. We will use this condition to find the conditions on M, m_1, m_2, l_1, l_2.

```
disp('Conditions on M,m1,m2,l1,l2 for which the linearized system is controllable \n')


FinalProject_PartB

%Cm = ctrb(A, B) // Since ctrb only uses numeric inputs, the
%controllability matrix is developed manually from equations
Cm = [B A*B A*A*B A*A*A*B A*A*A*A*B A*A*A*A*A*B];

%det(Cm)
Det = simplify(det(Cm))
```

```
Det =

-(g^6*(l1 - l2)^2)/(M^6*l1^6*l2^6)
```

From the displayed determinant we can see that it becomes zero when $l_1 = l_2$.

Therefore, the system is controllable for all values of M, m_1, m_2, l_1, l_2 except when $l_1 = l_2$. The system is uncontrollable when the lengths of the pendulums are equal.

D. Design of LQR Controller

Given M = 1000 kg, $m_1 = m_2 = 100\ kg, l_1 = 20\ m\ and\ l_2 = 10\ m.$

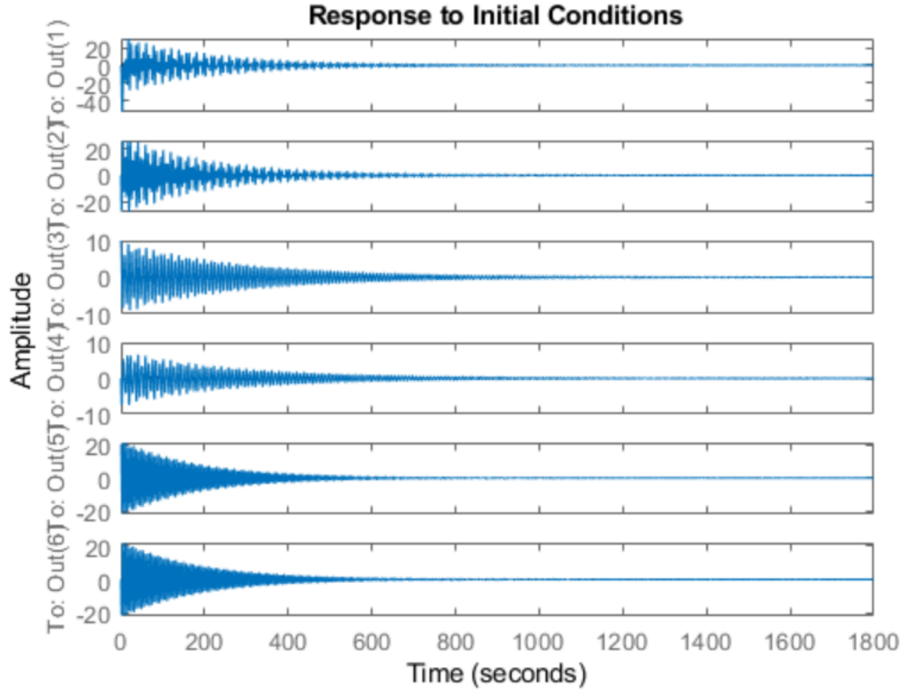
In this case we can see that l_1 is not equal to l_2, hence from the condition we obtained in section C, we can say that the system is controllable.

```
--------------------------------------------------------------------------------
Controllability Test using rank condition\n
--------------------------------------------------------------------------------
Rank of Controllability matrix is : 6
System is Controllable.\n

Q =

         100           0           0           0           0           0
           0           0           0           0           0           0
           0           0        1000           0           0           0
           0           0           0           0           0           0
           0           0           0           0        1000           0
           0           0           0           0           0           0


K =

   31.6228  275.2266  -16.9297 -295.8750   -5.5388 -145.7754
```

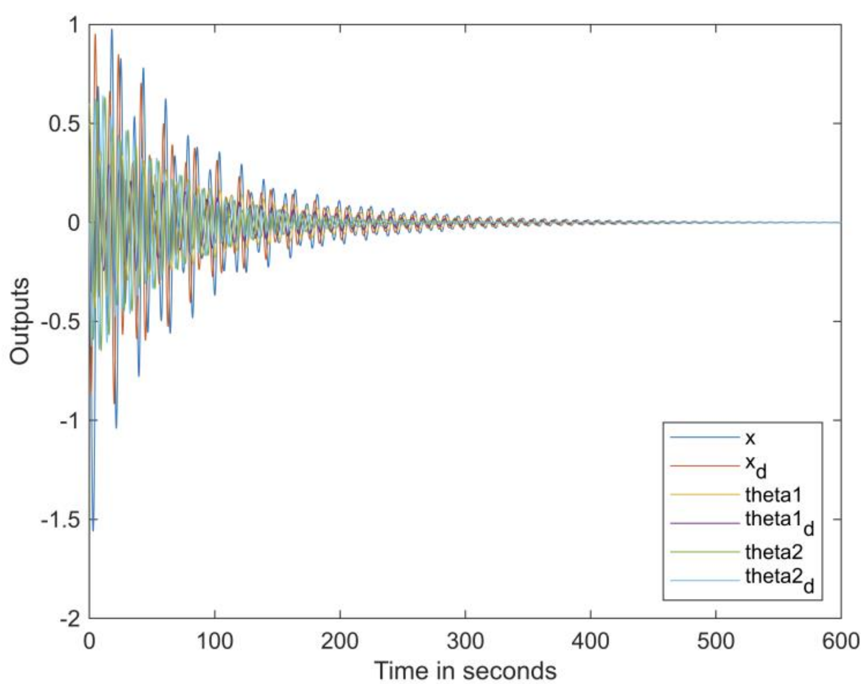
Here you can see that the rank of the controllability matrix is 6, hence the system is controllable.

```
disp('Given Parameters of the System\n')

fprintf('M : %i, m1:%i, m2:%i, l1:%i, l2:%i', 1000, 100, 100, 20, 10)

FinalProject_PartB

A = double(subs(subs(A), {M m1 m2 l1 l2 g},{1000 100 100 20 10 9.8}))
B = double(subs(subs(B), {M m1 m2 l1 l2 g},{1000 100 100 20 10 9.8}))

% fprintf(['Eigen values of A :\n'])
% eigs(A)


Cm = ctrb(A, B);

disp('--------------------------------------------------------------------------------')
disp('Controllability Test using rank condition\n')
disp('--------------------------------------------------------------------------------')
fprintf(['Rank of Controllability matrix ' ...
    'is : %i\n'], rank(ctrb(A, B)))

if rank(ctrb(A, B)) == 6
    disp('System is Controllable.\n')
else
    disp('System is Uncontrollable.\n')
end

Q = [100 0 0 0 0 0;
    0 0 0 0 0 0;
    0 0 1000 0 0 0;
    0 0 0 0 0 0;
    0 0 0 0 1000 0;
    0 0 0 0 0 0]
R = 0.1 ;
K = lqr(A,B,Q,R)

Af = A-B*K
Bf = B
Cf = C
Df = D

sysf = ss(Af,Bf,Cf,Df);

x0 = [0;0;10;0;20;0];
initial(sysf,x0)
```

Response to initial conditions when the controller is applied to the linearized system:



Response to initial conditions when the controller is applied to the non linearized system:



Using Lyapunov's indirect method to certify stability:

The eigen values of (A + BK) are as follows:

```
Eigen values of A :

ans =

  -0.0540 +10.4245i
  -0.0540 -10.4245i
  -0.0311 + 7.2812i
  -0.0311 - 7.2812i
  -1.1441 + 1.1518i
  -1.1441 - 1.1518i
```

Here you can see that the real part of the eigen values is negative, hence the closed loop system is stable.

Second Component

E. Check for Observable system for given vectors

To determine the observability, we check the rank of observability matrix. If the rank is 6, then the system is observable.

```
C1=[1 0 0 0 0 0];                                    % C matrix for X as output
C2=[0 0 1 0 0 0; 0 0 0 0 1 0];                       % C matrix for theta1,theta2 as output
C3=[1 0 0 0 0 0; 0 0 0 0 1 0];                       % C matrix for X,theta2 as output
C4=[1 0 0 0 0 0; 0 0 1 0 0 0; 0 0 0 0 1 0];          % C matrix for X,theta1,theta2 as output
```

Check for Observability of output - X

```
O1 =[C1 ;C1*A; C1*A^2; C1*A^3; C1*A^4; C1*A^5];
rank(O1)
% As a result, because rank is 6, it can be seen.
```

```
ans =

     6
```

In the above case the rank is 6, hence observable.

Check for Observability of output - theta1,theta2

```
O2 =[C2 ;C2*A; C2*A^2; C2*A^3; C2*A^4; C2*A^5];
rank(O2)
%Thus it is not observable since rank is 4.
```

```
ans =

     4
```

In the above case the rank is 4, hence not observable.

Check for Observability of output - X,theta2

```
O3 =[C3 ;C3*A; C3*A^2; C3*A^3; C3*A^4; C3*A^5];
rank(O3)

%Thus it is observable since rank is 6.
```

```
ans =

     6
```

In the above case the rank is 6, hence observable.

Check for Observability of output - X,theta1,theta2

```
O4=[C4 ;C4*A; C4*A^2; C4*A^3; C4*A^4; C4*A^5];
rank(O4)
%Thus it is observable since rank is 6.
```

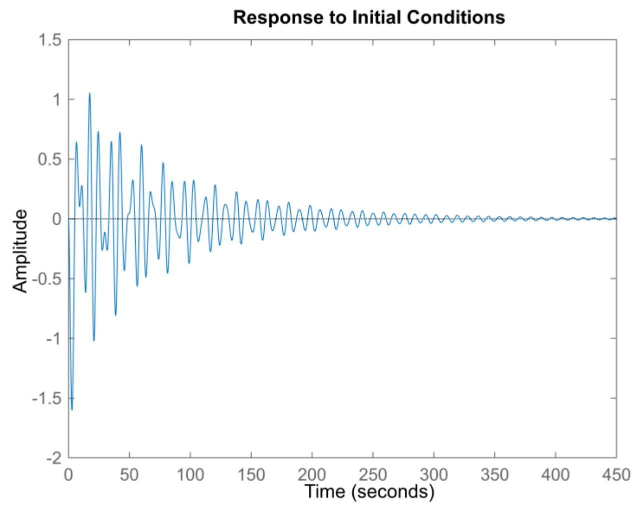
```
ans =

     6
```

In the above case the rank is 6, hence observable.

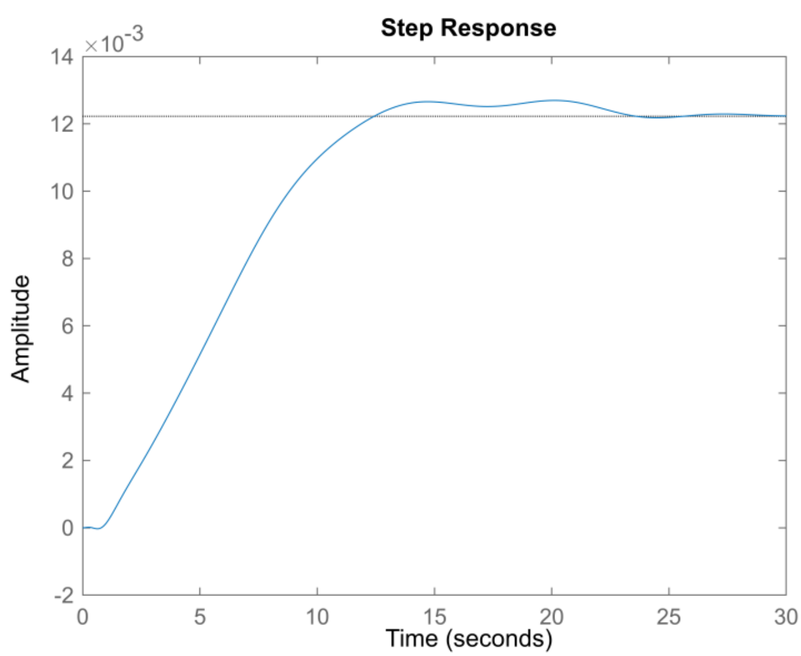
F. Luenberger Observer for observable system

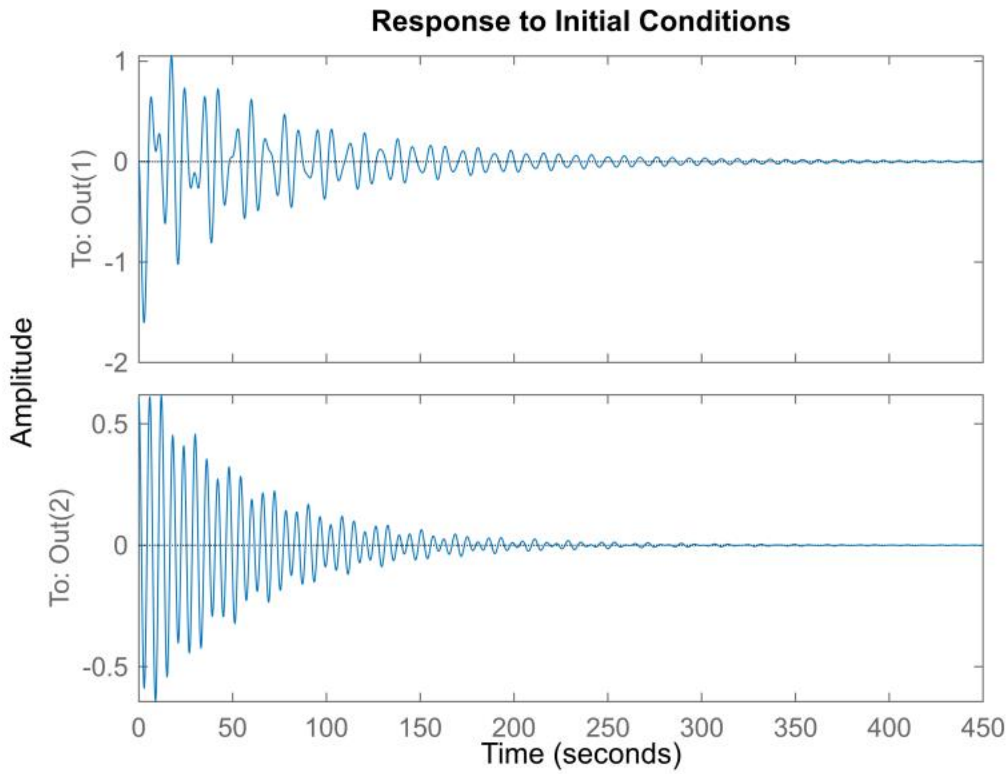
Obtaining best Luenberger observer for each one of the output vectors for which the system is observable and simulating its response to initial conditions and unit step input.

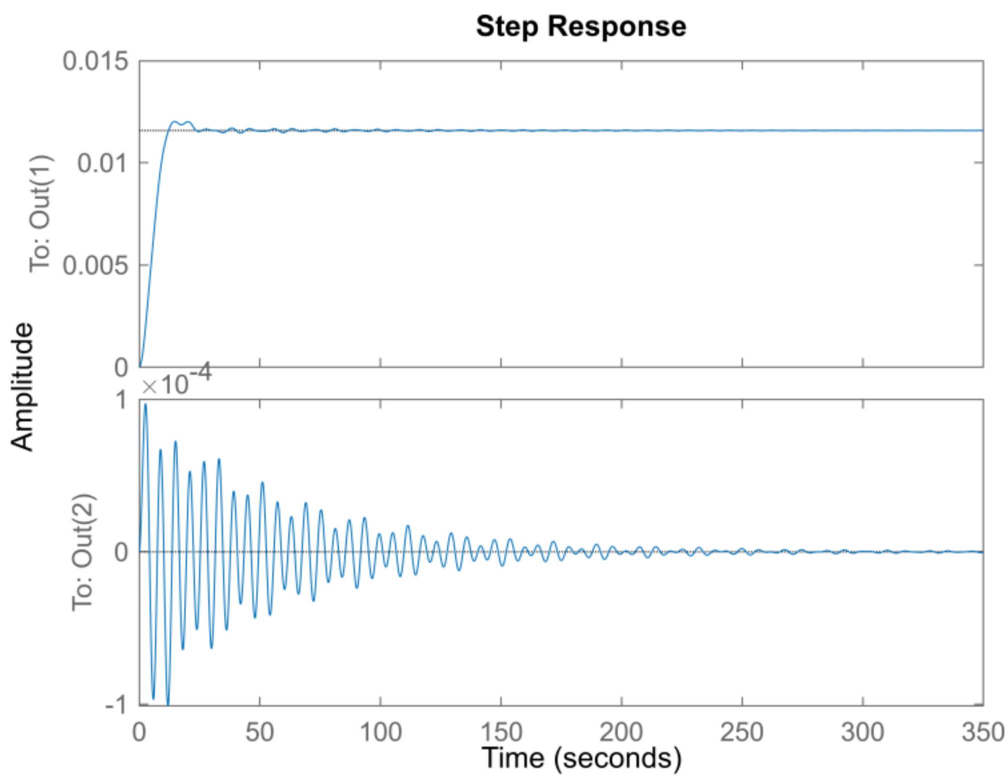
For Linearized system:

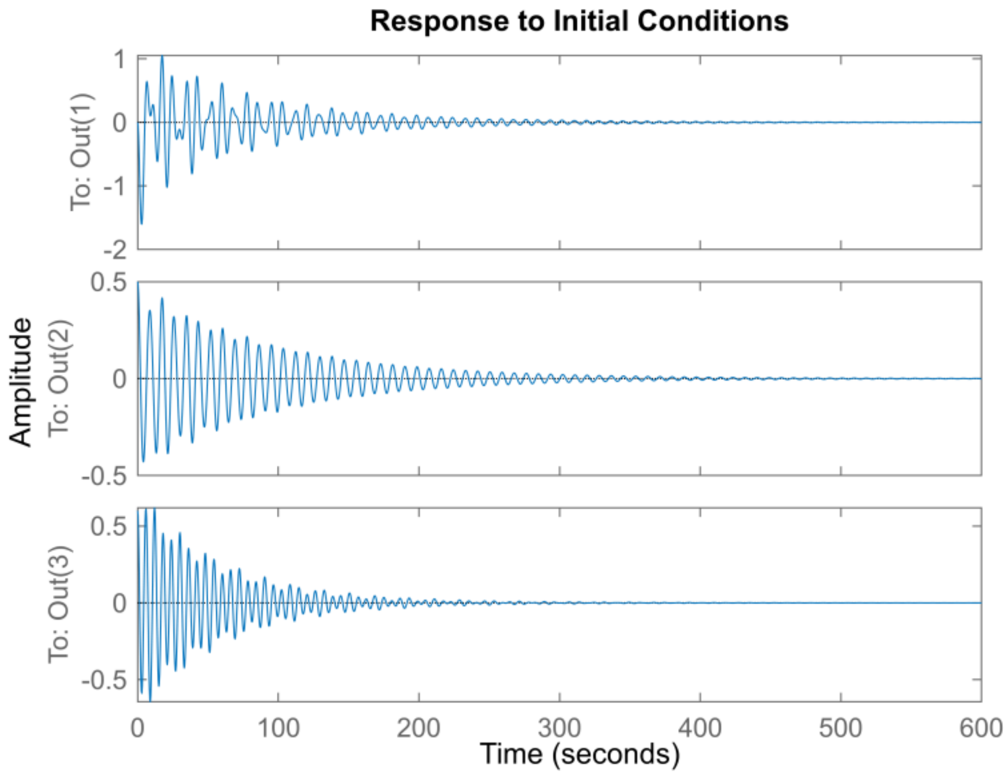


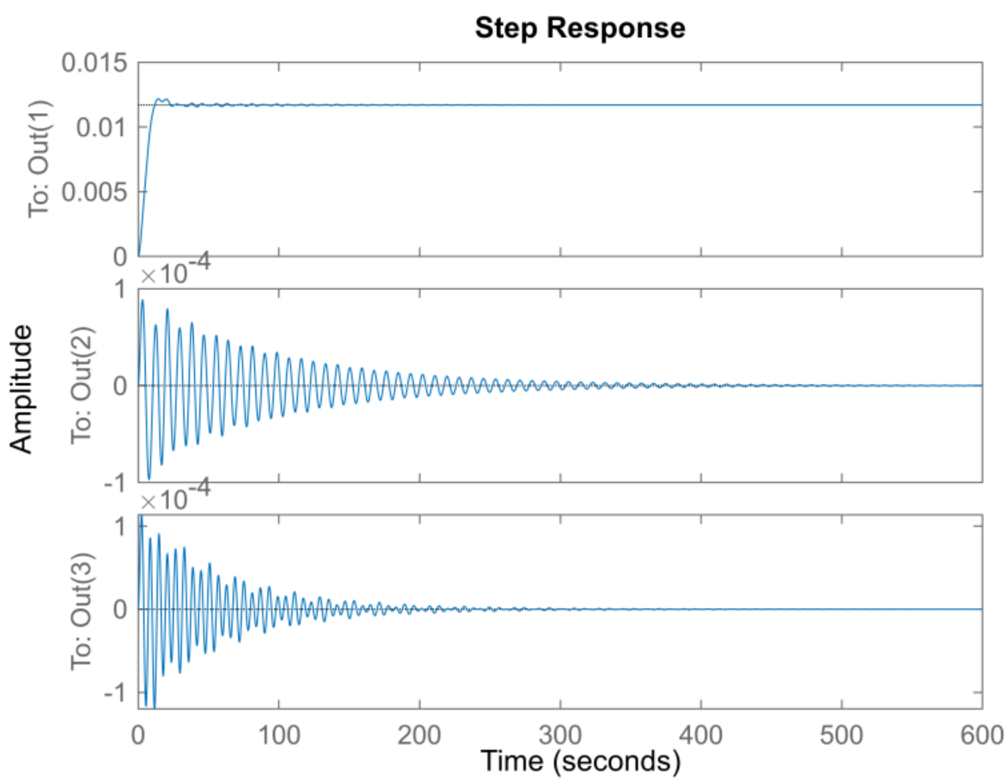

```
% Response of system when unit step is given as input
step(sys_ob1)
```



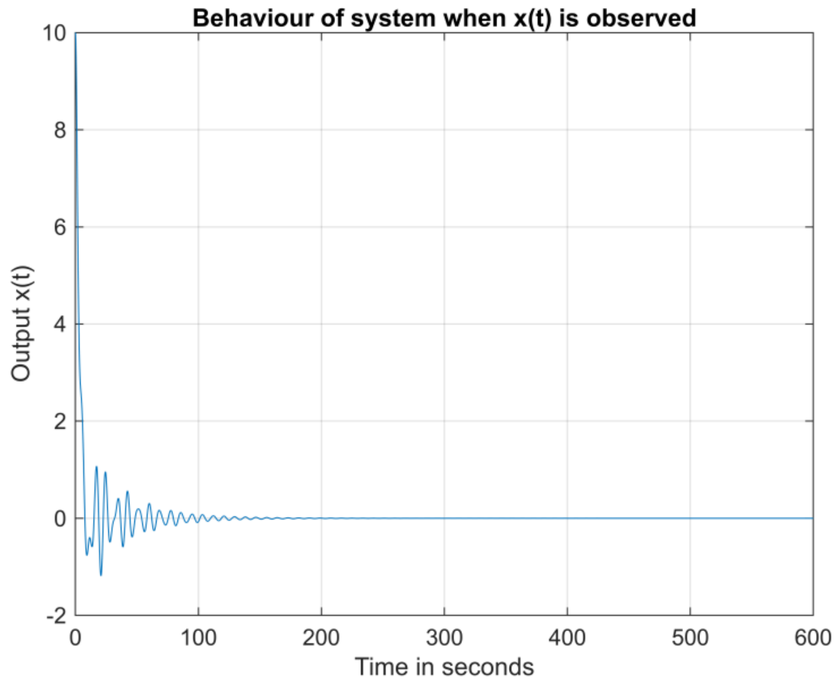

Response to Initial Conditions
Amplitude
To: Out(1)
To: Out(2)
1
0
-1
-2
0.5
0
-0.5
0 50 100 150 200 250 300 350 400 450
Time (seconds)

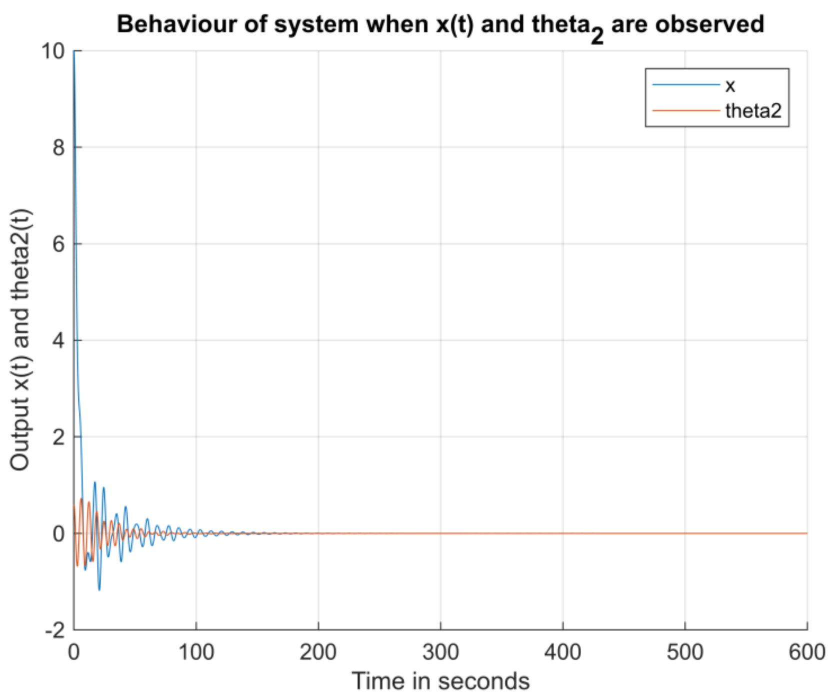

Step Response
Amplitude
To: Out(1)
To: Out(2)
0.015
0.01
0.005
0
×10-4
1
0
-1
0 50 100 150 200 250 300 350
Time (seconds)

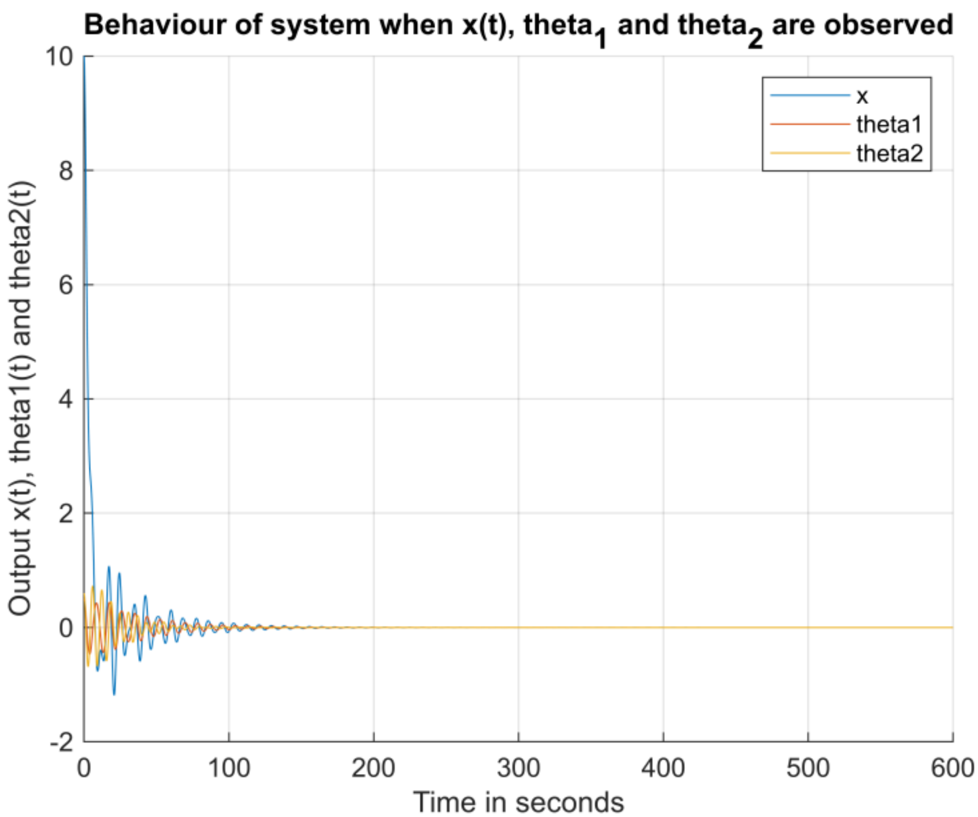

Response to Initial Conditions
Amplitude
To: Out(1)
To: Out(2)
To: Out(3)
1
0
-1
-2
0.5
0
-0.5
0.5
0
-0.5
0
100
200
300
400
500
600
Time (seconds)


Step Response
Amplitude
To: Out(1)
To: Out(2)
To: Out(3)
0.015
0.01
0.005
0
×10^-4
1
0
-1
×10^-4
1
0
-1
0
100
200
300
400
500
600
Time (seconds)

For original nonlinear system:






Behaviour of system when x(t), theta$_1$ and theta$_2$ are observed
x
theta1
theta2
Output x(t), theta1(t) and theta2(t)
Time in seconds
10
8
6
4
2
0
-2
0
100
200
300
400
500
600

G.LQG Controller design

Output feedback controller for the smallest output vector:

```
% Declaring variables to derive the state space representation of system
% M is the Mass of cart
% m1 is the mass attached to pendulum 1
% m2 is the mass attached to pendulum 2
% l1 is the length of pendulum 1
% l2 is the length of pendulum 2
% g is the accleration due to gravity
syms M m1 m2 l1 l2 g;

% Declaring the values of system variables
M_val = 1000;
m1_val = 100;
m2_val = 100;
l1_val = 20;
l2_val = 10;

A_val = [0 1 0 0 0 0;
         0 0 -(m1_val*9.81)/M_val 0 -(m2_val*9.81)/M_val 0;
         0 0 0 1 0 0;
         0 0 -((M_val+m1_val)*9.81/(M_val*l1_val)) 0 -(m2_val*9.81)/(M_val*l1_val) 0;
         0 0 0 0 0 1;
         0 0 -(m1_val*9.81)/(M_val*l2_val) 0 -((M_val+m2_val)*9.81/(M_val*l2_val)) 0];

% Declaring the B matrix of the system
B_val = [0; 1/M_val; 0; 1/(M_val*l1_val); 0; 1/(M_val*l2_val)];

% Defining the Q and R matrices for LQR controller
Q = [100  0  0  0  0  0;
     0  0  0  0  0  0;
     0   0 1000 0  0  0;
     0   0  0  0  0  0;
     0   0  0  0 1000 0;
     0   0  0  0  0  0];

R = 0.01;

% Finding the optimal closed-loop feedback gain using LQR
[K, R_soln, poles] = lqr(A_val,B_val,Q,R)
```

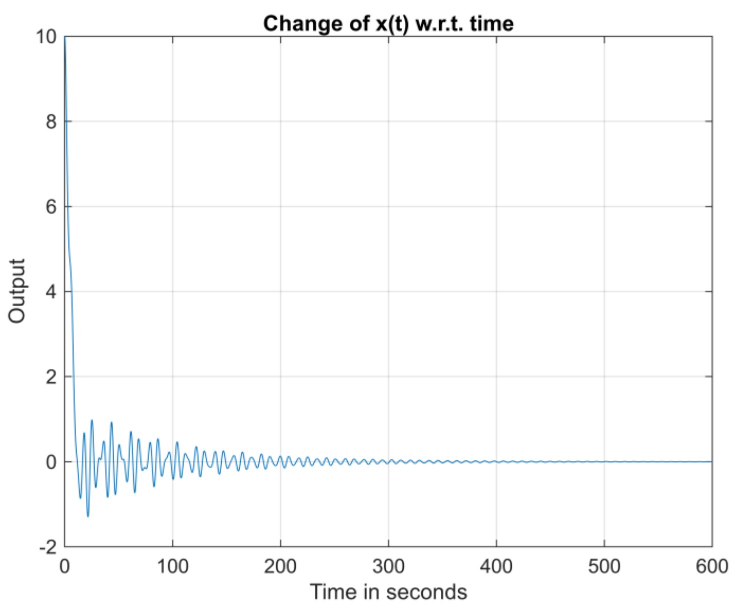
```
K = 1×6
  100.0000  491.1132   21.2712 -343.3167   37.2030 -161.6165
R_soln = 6×6
10^5 ×
    0.0049    0.0121   -0.0034   -0.0204   -0.0016   -0.0104
    0.0121    0.0597    0.0036   -0.1074    0.0024   -0.0526
   -0.0034    0.0036    0.6444    0.0059    0.0026   -0.0174
   -0.0204   -0.1074    0.0059    1.3839    0.0230    0.0382
   -0.0016    0.0024    0.0026    0.0230    0.3225    0.0013
   -0.0104   -0.0526   -0.0174    0.0382    0.0013    0.3455
poles = 6×1 complex
  -0.0097 + 0.7280i
  -0.0097 - 0.7280i
```

```
  -0.0170 + 1.0427i
  -0.0170 - 1.0427i
  -0.2021 + 0.2064i
  -0.2021 - 0.2064i
```

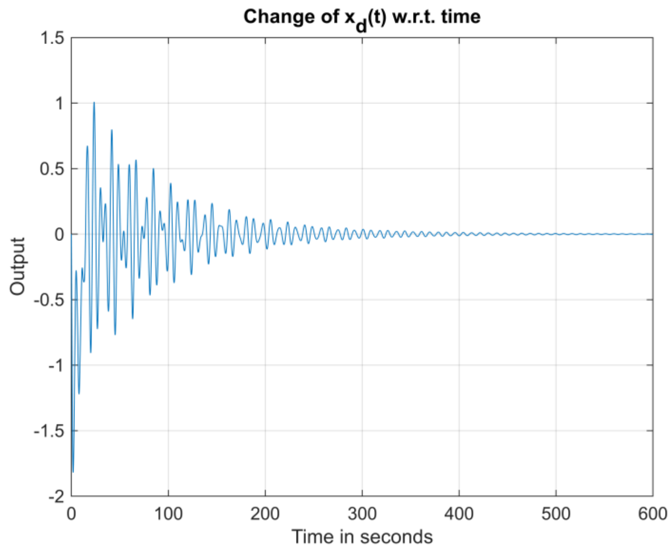
```
C1 = [1 0 0 0 0 0];

D = 0;

% Defining the process and measurement noise
Vd = 0.2*eye(6);
Vn = 1;

% Finding the optimal observer gains for each observer
L1 = lqr(A_val', C1', Vd, Vn)
```

```
L1 = 1×6
    2.1429    2.1960   -0.2843    0.5589   -0.3758    0.3826
```

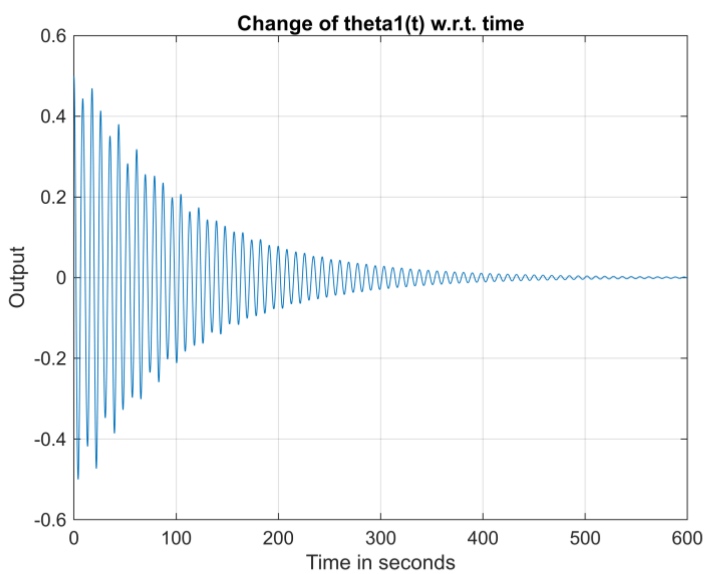
```
A_c1 = [(A_val - B_val*K) B_val*K; zeros(size(A_val)) (A_val - L1'*C1)];
B_c1 = [B_val; B_val];
C_c1 = [C1 zeros(size(C1))];

x0_lqg = [10;0;0.5;0;0.6;0];

% Plotting x(t) for non-linear system
t_span = 0:0.01:600;
[ts,x_dots] = ode45(@(t,x)non_lin_sys(t,x,-K*x,L1,C1),t_span,x0_lqg);
plot(ts,x_dots(:,1))
grid
xlabel('Time in seconds')
ylabel('Output')
title('Change of x(t) w.r.t. time')
```



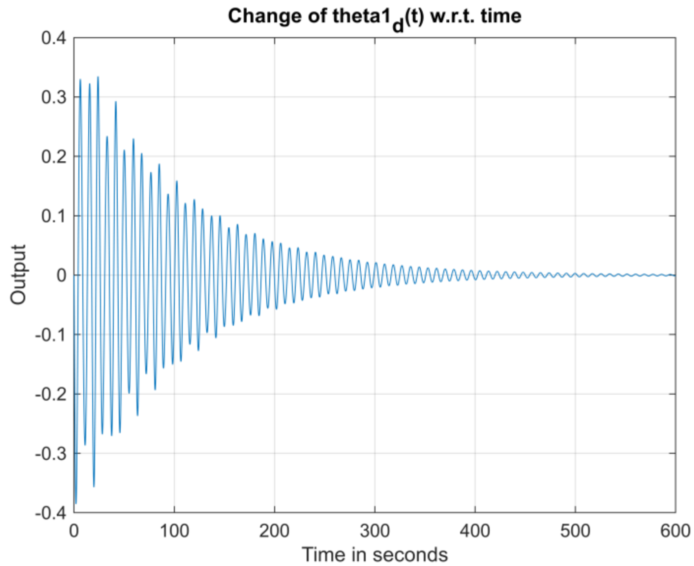

```
% Plotting x_d(t) for non-linear system
plot(ts,x_dots(:,2))
grid
xlabel('Time in seconds')
ylabel('Output')
title('Change of x_d(t) w.r.t. time')
```



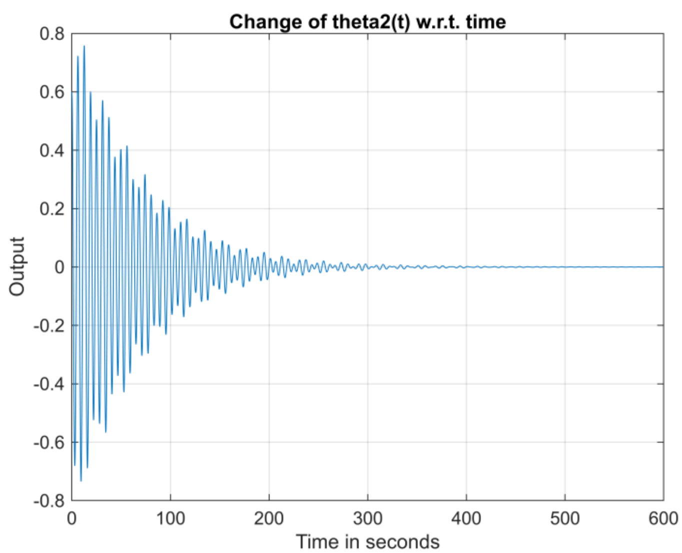

```
% Plotting theta1(t) for non-linear system
plot(ts,x_dots(:,3))
grid
xlabel('Time in seconds')
ylabel('Output')
title('Change of theta1(t) w.r.t. time')
```



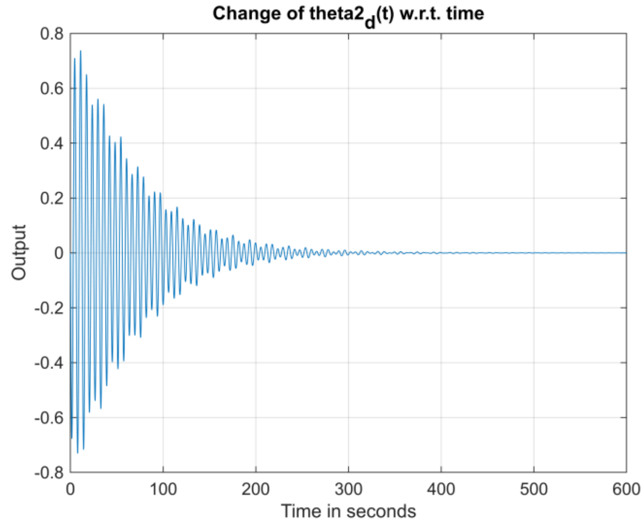

```
% Plotting theta1_dot(t) for non-linear system
plot(ts,x_dots(:,4))
grid
xlabel('Time in seconds')
ylabel('Output')
title('Change of theta1_d(t) w.r.t. time')
```




```
% Plotting theta2(t) for non-linear system
plot(ts,x_dots(:,5))
grid
xlabel('Time in seconds')
ylabel('Output')
title('Change of theta2(t) w.r.t. time')
```




```
% Plotting theta2_dot(t) for non-linear system
plot(ts,x_dots(:,6))
grid
xlabel('Time in seconds')
ylabel('Output')
title('Change of theta2_d(t) w.r.t. time')
```




```
function x_dot = non_lin_sys(t,X,F,L,C)
x_dot = zeros(6,1);
% Declaring the values of system variables
M_val = 1000;
m1_val = 100;
m2_val = 100;
l1_val = 20;
l2_val = 10;
g_val = 9.81;
x = X(1);
x_d = X(2);
theta1 = X(3);
theta1_d = X(4);
theta2 = X(5);
theta2_d = X(6);
obs = L*(x-C*X);
x_dot(1) = x_d + obs(1);
x_dot(2) = (F-((m1_val*sin(theta1)*cos(theta1))+(m2_val*sin(theta2)*cos(theta2)))*g_val - (
x_dot(3) = theta1_d+obs(3);
x_dot(4) = ((cos(theta1)*x_dot(2)-g_val*sin(theta1))/l1_val) + obs(4);
x_dot(5) = theta2_d + obs(5);
x_dot(6) = (cos(theta2)*x_dot(2)-g_val*sin(theta2))/l2_val + obs(6);
end
```

The desired constant reference tracking is accomplished by minimizing the LQR cost function.

$$\int_0^{\infty} (\overrightarrow{X(t)} - \overrightarrow{X_d})^T Q (\overrightarrow{X(t)} - \overrightarrow{X_d}) - (\overrightarrow{U_k(t)} - \overrightarrow{U_\infty})^T R (\overrightarrow{U_k(t)} - \overrightarrow{U_\infty})\, dt$$

If there is U_∞ such that:

$$A\overrightarrow{X_d} + B_k\overrightarrow{U_\infty} = 0$$

The designed controller will accommodate the constant force disturbances given that they are gaussian in nature.