5. Let u and v be vectors in $\mathbb{R}^n$. Form the matrix

$$G = \begin{pmatrix} u^T u & u^T v \\ v^T u & v^T v \end{pmatrix}$$

Show that u and v are linearly independent if and only if G is invertible.

Proof. We first prove that G is invertible implies that u and v are linearly independent. We do this using proof by contraposition. If u and v are not linearly independent, then $u = kv$ for some $k \in \mathbb{R}$. We can then write G as follows:

$$G = \begin{pmatrix} (kv) \cdot (kv) & (kv) \cdot v \\ v \cdot (kv) & v \cdot v \end{pmatrix}$$
$$\det(G) = k^2(v \cdot v)^2 - k^2(v \cdot v)^2 = 0$$

Therefore u, v not linearly independent $\implies G$ not invertible, or equivalently, G invertible $\implies u, v$ linearly independent.

Next we prove that u and v being linearly independent implies that G is invertible. From the original matrix, $\det(G) = (u \cdot u)(v \cdot v) - (u \cdot v)^2$.

We can use the Cauchy–Schwarz inequality to prove that $\det(G) > 0$. It states that $u \cdot v \leq |u||v|$. But if u and v are linearly independent, then we can strengthen this assertion to state that $u \cdot v < |u||v|$. Equivalently, $u \cdot v < \sqrt{(u \cdot u)(v \cdot v)}$. Squaring both sides, $(u \cdot v)^2 < (u \cdot u)(v \cdot v)$.

Therefore $\det(G) = (u \cdot u)(v \cdot v) - (u \cdot v)^2 > 0$, so G must be invertible if u and v are linearly independent. This completes the proof that u, v linearly independent $\iff G$ invertible. □